

WEIGHTLESS COHOMOLOGY OF ALGEBRAIC VARIETIES

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ABSTRACT. Using Morel's weight truncations in categories of mixed sheaves, we attach to varieties over finite fields or the complex numbers a series of groups called the weightless cohomology groups. These lie between the usual cohomology and intersection cohomology, have natural ring structures, and are functorial for certain morphisms.

1. INTRODUCTION

1.1. Let k be the field $\mathbb{C}$ of complex numbers or a finite field. If k is finite let l be a prime different from the characteristic of k . For a variety X/k we let $H^*(X)$ denote the singular cohomology $H^*(X, \mathbb{Q})$ if $k = \mathbb{C}$ or the étale cohomology $H^*(X_{\bar{k}}, \mathbb{Q}_l)$ if k is finite. Similarly, $IH^*(X)$, $H_*^{BM}(X)$, etc. denote the appropriate intersection cohomology, Borel-Moore homology, etc. Let $D^b(X)$ denote the derived category of mixed Hodge modules [18] in the case $k = \mathbb{C}$ or the derived category of mixed l -adic complexes [4] if k is finite. $(\cdot)$ denotes the Tate twist.

Using truncation functors introduced by S. Morel [11], we attach to a variety X/k a complex

$$EC_X \in D^b(X)$$

which we call the *weightless complex* of X . Its hypercohomology groups

$$EH^*(X) := \mathbb{H}^*(X, EC_X)$$

are the *weightless cohomology* groups of X . These are finite-dimensional $\mathbb{Q}$ -vector spaces (if $k = \mathbb{C}$) or $\mathbb{Q}_l$ -vector spaces (if k is finite) with the following properties (Theorem 4.2.1):

- (i) If X is of dimension d then $EH^i(X) \neq 0$ only if $i \in [0, 2d]$. If $\hat{X} \rightarrow X$ is the normalization then $EH^*(\hat{X}) = EH^*(X)$. If X is smooth then $EH^*(X) = H^*(X)$.
- (ii) $EH^*(X)$ is a graded-commutative $\mathbb{Q}$ -algebra (if $k = \mathbb{C}$) or $\mathbb{Q}_l$ -algebra (if k is finite).
- (iii) $EH^*(X \times Y) \cong EH^*(X) \otimes EH^*(Y)$
- (iv) $EH^*(X)$ lives between the usual cohomology $H^*(X)$, and the intersection cohomology $IH^*(X)$, that is, there are homomorphisms

$$H^*(X) \rightarrow EH^*(X) \rightarrow IH^*(X)$$

factoring the natural homomorphism $H^*(X) \rightarrow IH^*(X)$. The first morphism is a ring homomorphism and the second is an $H^*(X)$ -module homomorphism. In general, these homomorphisms are neither injective nor surjective.

- (v) If $f : Y \rightarrow X$ is a morphism of varieties such that the image of each irreducible component of f meets the smooth locus of X then there is a natural ring homomorphism $f^* : EH^*(X) \rightarrow EH^*(Y)$ compatible with $H^*(X) \rightarrow H^*(Y)$. If $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ are morphisms such that f, g , and $f \circ g$ satisfy this condition (that the image of each irreducible component of the domain meets the smooth locus of the codomain) then $(f \circ g)^* = g^* \circ f^*$.
- (vi) If $d = \dim X$ then $EH^{2d}(X) = IH^{2d}(X)$ and $EH^{2d-1}(X) \twoheadrightarrow IH^{2d-1}(X)$.
- (vii) There is a module action of $EH^*(X)$ on $IH^*(X)$ factoring the action of $H^*(X)$ on $IH^*(X)$.

These statements reflect properties of the complex EC_X in $D^b(X)$.

If $k = \mathbb{C}$ then $EH^*(X)$ carries a (rational) mixed Hodge structure. If k is finite then $EH^*(X)$ has an action of $\text{Gal}(\bar{k}/k)$ and hence a weight filtration coming from the action of Frobenius. The ring structure in (ii), the homomorphisms in (iii)–(vi), and the action in (vii) respect these structures. Furthermore, the weights of $EH^*(X)$ satisfy the following constraints (Corollary 4.3.4):

- (viii) $Gr_j^W EH^i(X) \neq 0$ only if $j \in [0, 2i] \cap [2i - 2 \dim X, 2 \dim X]$
- (ix) If X is proper then $Gr_j^W EH^i(X) = 0$ for $j > i$ and

$$Gr_i^W EH^i(X) = \text{im}(EH^i(X) \rightarrow IH^i(X)).$$

If X is smooth then $Gr_j^W EH^i(X) = 0$ for $j < i$.

Thus $EH^*(X)$ looks very much like the cohomology of an algebraic variety.

Since $EH^*(X)$ is defined via a complex EC_X in $D^b(X)$, it has a compactly supported variant $EH_c^*(X) = \mathbb{H}_c^*(X, EC_X)$, and the natural map $EH_c^*(X) \rightarrow EH^*(X)$ is an isomorphism when X is proper. There are also Poincaré dual homology theories defined by $EH_i^{BM}(X) = \mathbb{H}^{-i}(X, \mathbb{D}EC_X(-d))$ and $EH_*(X) = \mathbb{H}^{-*}(X, \mathbb{D}EC_X(-d))$ where $\mathbb{D}$ is the Verdier duality functor on $D^b(X)$ and $d = \dim X$. The fundamental class homomorphism then factors as

$$H^*(X) \rightarrow EH^*(X) \rightarrow IH^*(X) = IH_{2d-*}^{BM}(X)(-d) \rightarrow EH_{2d-*}^{BM}(X)(-d) \rightarrow H_{2d-*}^{BM}(X)(-d)$$

$EH_*^{BM}(X)$ and $EH^*(X)$ are graded modules for $EH^*(X)$ and satisfy properties dual to those of $EH^*(X)$ and $EH_c^*(X)$ above. The functoriality property (iii) implies that subvarieties of X meeting the smooth locus have a canonical cycle class in $EH_*^{BM}(X)$ refining the cycle class in Borel-Moore homology.

1.2. In the recent paper [2] of Ayoub and Zucker, the authors construct, for any variety X over a field of characteristic zero, a ring object in a category of motivic sheaves on X . This should, presumably, realize to our EC_X under an appropriate realization functor to Saito's mixed Hodge modules and would give some of the properties above (specifically, (i)–(iii) and (v) for dominant morphisms). To our knowledge, such a realization has not yet been constructed. (A Hodge realization is available after pushing forward to $\text{Spec}(k)$, so that one should be able to recover $EH^*(X)$ this way.) The construction of EC_X here is relatively elementary and the additional properties (specifically, (iv), (vi), and (ix)) here are necessary for some applications in [13]. ⁽¹⁾ In [21] an alternate construction of the motive of Ayoub and

¹The construction of EC_X and the proofs of its properties were worked out by the authors in 2009, before the first version of [2] became available.

Zucker is given along the lines suggested by the construction of EC_X and properties analogous to those above are established, again in characteristic zero.

1.3. The theory $EH^*(X)$ is, in general, closer to ordinary cohomology than to intersection cohomology. Indeed, $H^*(X)$ and $EH^*(X)$ tend to differ only for varieties with rather complicated singularities (we do not attempt to make this precise). For example, they coincide for toric varieties (in any characteristic) or for complex algebraic varieties with isolated rational singularities (by a result of [1], cf. Ex. 5.0.6)).

On the other hand, compactified Hilbert-Blumenthal varieties (of arbitrary dimension) give examples where $EC_X = IC_X$ and hence $EH^*(X)$ and $IH^*(X)$ coincide (and differ from $H^*(X)$ if $\dim X > 2$). Note that when $EH^*(X)$ and $IH^*(X)$ coincide then intersection cohomology has a natural ring structure; it would be interesting to see whether in other examples where $IH^*(X)$ is known (or expected) to have a ring structure the underlying explanation is that $EC_X = IC_X$.

An interesting example of the construction EC_X , and in fact the one that motivated the results here, is when X is the Baily-Borel compactification of a locally symmetric variety. In this case $EH^*(X)$ is the cohomology of the reductive Borel-Serre compactification [14], cf. also [2]. Some results of this paper are used in [13] to prove new results about the cohomology of the reductive Borel-Serre compactification. (These examples and others are discussed in §5.)

1.4. The main technical tools in the construction of the weightless complex EC_X and in the proof of its properties are Morel's truncation functors [11]. Given a stratification $X = \sqcup_i S_i$ of a variety X and a function $\underline{a}$ from the set of strata to $\mathbb{Z} \cup \{\pm\infty\}$, Morel defines a t -structure $({}^w D^{\leq \underline{a}}, {}^w D^{> \underline{a}})$, and hence truncation functors $w_{\leq \underline{a}}, w_{> \underline{a}}$ on $D^b(X)$. The complex EC_X is defined using the truncation given by the function $\underline{dim}$ with $\underline{dim}(S_i) := \dim S_i$. Assume that the stratification is such that $U = S_0$ is smooth, open, and dense in X . Let $j : U \rightarrow X$ be the inclusion; assume further that $Rj_* \mathbb{Q}_U$ and its Verdier dual are constructible with respect to the stratification. (For example, if $k = \mathbb{C}$ this holds if the stratification is Whitney. Here $\mathbb{Q}_U$ means the constant l -adic sheaf if k is finite and the analogous object if $k = \mathbb{C}$.) Then EC_X is defined by

$$EC_X := w_{\leq \underline{dim}} Rj_* \mathbb{Q}_U.$$

We call EC_X the *weightless complex* of X for the following reason: If $x \in X$ is a closed point then the local stalk cohomology groups $H^i(EC_X)_x$ are of weight zero for all i . If X is equidimensional of dimension d and the intersection complex $IC_X = (j_! \mathbb{Q}_U[d])[-d]$ is constructible with respect to the stratification one also has

$$EC_X = w_{\leq \underline{dim}} IC_X$$

as a consequence of Morel's remarkable formula $IC_X = w_{\leq d} Rj_* \mathbb{Q}_U$ where $w_{\leq d}$ is the truncation functor for the constant function $d = \dim X$. Finally, for a surjective birational morphism $\pi : Y \rightarrow X$ from a smooth variety we have a natural isomorphism

$$EC_X = w_{\leq \underline{dim}} R\pi_* \mathbb{Q}_Y$$

(assuming that $R\pi_* \mathbb{Q}_Y$ and its Verdier dual are constructible for the stratification). This implies that the pullback map $H^*(X) \rightarrow H^*(Y)$ factors through $EH^*(X)$, suggesting that $EH^*(X)$ should be thought of as the cohomology of a genuine

space which behaves like a “partial resolution” of X such that all other resolutions of X factor through it. Note, however, that even if X does have such a resolution $Y \rightarrow X$ it need not be the case that $EH^*(X) = H^*(Y)$.

In the body of the paper we will only be interested in $\underline{a}$ which depend only on the dimensions of the strata, so that $\underline{a}$ is replaced by a function $D : \mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}$. Then $\underline{dim}$ corresponds to the identity function Id . We show that, much as in the case of t -structures associated with perversities, for certain choices of D (monotone step functions, see 3.1.2), one can define t -structures $({}^w D^{\leq D}, {}^w D^{> D})$ on the appropriate derived category which are independent of the stratification. The associated truncation functors $w_{\leq D}, w_{> D}$ are then defined and the special case $D = \text{Id}$ gives the earlier construction of EC_X . The Verdier dual of EC_X , used to define the groups $EH_*^{BM}(X)$, is also given by a suitable truncation $w_{\leq D}$.

1.5. The definition of EC_X and its main properties generalize immediately to the situation with coefficients in a pure lisse $\mathbb{Q}_l$ -sheaf (when k is finite) or pure smooth Hodge module (equivalently, a polarizable variation of Hodge structure, when $k = \mathbb{C}$) on a smooth open dense subset of X . (Purity is essential for Morel’s truncation theory.) The construction also makes sense in any suitable theory of mixed sheaves (e.g. in the sense of Saito [16]), and the properties outlined above for $EH^*(X)$ will hold in any such theory, with the same proofs. In particular, a theory of mixed l -adic complexes is now available for varieties over a number field [12]; the weightless cohomology groups are then representations of the absolute Galois group. More generally, the construction works over any field which is finitely generated over the prime field, or a totally inseparable extension of such a field.

1.6. The properties of $EH^*(X)$ strongly suggest (when $k = \mathbb{C}$) the following picture, which we have found a useful heuristic: There is a nice topological space $\bar{X}$ with a proper mapping $p : \bar{X} \rightarrow X$ which is an isomorphism over the nonsingular locus of X and such that $Rp_* \mathbb{Q}_{\bar{X}} = EC_X$. The space $\bar{X}$ should be constructed out of any resolution of singularities of X but should be independent of the choice up to simple homotopy equivalence in a “stratumwise” manner. (This independence statement is not elementary. For example, if X has isolated singularities one would expect the complement of the smooth locus in $\bar{X}$ to be the dual graph of components in a resolution of the singularity. This is, up to homotopy, independent of the choice of resolution (cf. [19, 20, 1, 15], but this fact depends on the weak factorization theorem of birational geometry.) We call such a space $\bar{X}$ a *weightless blowup* of X . The properties of weightless cohomology should reflect properties of weightless blowups. We plan to construct these spaces and study their properties in a sequel to this paper. This would, in particular, give an integral structure to $EH^*(X)$, and we have other applications in mind for $\bar{X}$. (Note that one nontrivial case is known: When X is the Baily-Borel compactification of a locally symmetric variety there is a natural $\bar{X}$, namely the reductive Borel-Serre compactification.)

1.7. The outline of this paper is as follows: In Section 2 we discuss preliminaries on triangulated categories. Subsection 2.1 is devoted to general facts about t -structures, while 2.2 is devoted to Morel’s construction. In Section 3 we construct the required t -structures $({}^w D^{\leq D}, {}^w D^{> D})$ and discuss some of their basic properties. We also introduce the complexes EC_X^D on any scheme X and study their properties. In Section 4 we construct pullbacks and establish the other main properties of $EH^*(X)$. Section 5 is devoted to examples.

Notation. All schemes will be reduced, separated, of finite type over a fixed field k , which will be either a finite field or $\mathbb{C}$ (except possibly in Section 2.1). We will let $D^b(X)$ denote the bounded derived category of mixed complexes $D_m^b(X, \mathbb{Q}_l)$ if k is finite with $\text{char}(k) \neq l$, or the derived category of mixed Hodge modules $D^b\text{MHM}(X)$ (in the sense of Saito, see [17, 18], especially §4 of the latter) if $k = \mathbb{C}$. In the latter case an assertion like “ $\mathcal{F}$ is a local system” should be interpreted as the corresponding statement for its constructible realization $\text{rat}(\mathcal{F})$ (see [17, 1.3]).

Treating the cases k finite and $k = \mathbb{C}$ simultaneously leads to one notational problem: If $k = \mathbb{C}$ the notation ${}^p H^i$ is to be read as the cohomology functor for the “standard” t -structure on mixed Hodge modules, which is usually simply denoted H^i . If $k = \mathbb{C}$ the notation H^i is to be read as ${}^c H^i$, the cohomology functor corresponding to the analogue of the classical t -structure on $D^b\text{MHM}(X)$ [18, 4.6].

Finally, by the constant sheaf $\mathbb{Q}_X$, we will mean the constant l -adic sheaf $\mathbb{Q}_l$ on X if k is finite, or the object $\mathbb{Q}_X^H := a_X^* \mathbb{Q}^H$ where $a_X : X \rightarrow \text{Spec } \mathbb{C}$ and $\mathbb{Q}^H$ is the constant object, if $k = \mathbb{C}$.

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2. PRELIMINARIES

2.1. Triangulated categories and t -structures. In this section, we recall some basic properties of gluing of t -structures in the form we will apply them later. The standard reference for t -structures is [4].

2.1.1. Notation. We use the following notation:

$t = ({}^t D^{\leq 0}, {}^t D^{> 0})$	A t -structure ([4, Section 1]).
$t[-r] = ({}^t D^{\leq -r}, {}^t D^{> -r})$	The t -structure $({}^t D^{\leq 0}[-r], {}^t D^{> 0}[-r])$
$({}^t \tau_{\leq r}, {}^t \tau_{> r})$	Truncation functors for the t -structure $t[r]$
$+\infty$	The t -structure with ${}^t D^{\leq 0}$ being the full triangulated category and ${}^t D^{> 0} = \{0\}$
$-\infty$	The t -structure with ${}^t D^{\leq 0} = \{0\}$ and ${}^t D^{> 0}$ being the full triangulated category

We will also replace a t -structure by the corresponding index (e.g. use ${}^i \tau_{\leq 0}$ instead of ${}^i \tau_{\leq 0}$) if it simplifies notation.

If the triangulated category is not specified, the t -structure should be thought of as on the category $D^b(X)$ for an appropriate X . If we want to stress the scheme X in such cases, we will use notation $({}^t D^{\leq 0}(X), {}^t D^{> 0}(X))$ for the t -structure.

We recall the following basic facts about t -structures:

2.1.2. Lemma. *If t, t' are two t -structures on the same underlying triangulated category such that, ${}^t D^{\leq 0} \subset {}^{t'} D^{\leq 0}$ and ${}^t D^{> 0} \supset {}^{t'} D^{> 0}$, then for all objects $\mathcal{F}$ the following natural maps are isomorphisms:*

$$\begin{aligned} {}^t \tau_{\leq 0} {}^{t'} \tau_{\leq 0} \mathcal{F} &\xrightarrow{\cong} {}^t \tau_{\leq 0} \mathcal{F} \xleftarrow{\cong} {}^{t'} \tau_{\leq 0} {}^t \tau_{\leq 0} \mathcal{F} \\ {}^t \tau_{> 0} {}^{t'} \tau_{> 0} \mathcal{F} &\xleftarrow{\cong} {}^t \tau_{> 0} \mathcal{F} \xrightarrow{\cong} {}^{t'} \tau_{> 0} {}^t \tau_{> 0} \mathcal{F} \end{aligned}$$

Proof. See [4, 1.3.4], where this lemma is stated in the case $t = t'[n]$ for some $n > 0$. Even though we need this slightly more general case for use with Morel's t -structures, the proof remains the same. $\square$

2.1.3. Notation. By a stratification of a scheme X we mean a collection of locally closed subschemes $S_0, S_1, \dots, S_r$ with S_i open in $X - \bigcup_{0 \leq j < i} S_j$, such that closure of any stratum is a union of strata. We will also denote S_0 by U . We will always assume each stratum to be irreducible. We will often denote the stratification $(S_0, S_1, \dots, S_r)$ by S .

We also let $U_k := \bigcup_{j \leq k} S_j$ be the open subschemes, and $Z_k := X - U_k$ be their closed complements. We let $j_k : U_k \rightarrow X$, $i_k : Z_k \rightarrow X$ denote the corresponding immersions. Let $f_k : S_k \rightarrow X$ denote the locally closed immersion.

Further let $j_{i|k} : U_i \rightarrow U_k$, $i_{i|k} : (U_k - U_i) \rightarrow U_k$ and $f_{i|k} : S_i \rightarrow U_k$ denote the natural maps for $i \leq k$.

The following lemma summarizes results on gluing of t -structures from [4].

2.1.4. Lemma. *We assume the situation of 2.1.3. Let there be t -structures $t_i = (D^{\leq 0}(S_i), D^{> 0}(S_i))$ on S_i , and the corresponding truncations be $({}^i\tau_{\leq 0}, {}^i\tau_{> 0})$. Define:*

$$\begin{aligned} D^{\leq 0}(X) &= \{\mathcal{F} | f_i^* \mathcal{F} \in D^{\leq 0}(S_i), \forall i\} \\ D^{> 0}(X) &= \{\mathcal{F} | f_i^! \mathcal{F} \in D^{\leq 0}(S_i), \forall i\} \end{aligned}$$

Then:

- (1) $(D^{\leq 0}(X), D^{> 0}(X))$ is a t -structure. We denote it by $[t_0, t_1, \dots, t_r]$.
- (2) The order of gluing is irrelevant. (e.g. $[t_0, \dots, t_r] = [[t_0, t_1], t_2, \dots, t_r]$ etc.).
- (3) Let $(\tau_{\leq 0}, \tau_{> 0})$ denote the truncation functors for the glued t -structures. One can consider t -structures $[\dots, +\infty, t_i, +\infty, \dots]$ and let the corresponding truncation functors be $\sigma_{\leq 0}^i$. Then $\tau_{\leq 0} = \sigma_{\leq 0}^r \dots \sigma_{\leq 0}^0$.
Similarly consider t -structures $[\dots, -\infty, t_i, -\infty, \dots]$ and let the corresponding truncation functors be $\sigma_{> 0}^i$. Then $\tau_{> 0} = \sigma_{> 0}^r \dots \sigma_{> 0}^0$.

2.1.5. Lemma. *Let the notation be as in the last lemma. Then for all i we have the triangles:*

$$\begin{aligned} \sigma_{\leq 0}^i \mathcal{F} &\longrightarrow \mathcal{F} \longrightarrow Rf_{i*} {}^i\tau_{> 0} f_i^* \mathcal{F} \longrightarrow \\ Rf_{i!} {}^i\tau_{\leq 0} f_i^! \mathcal{F} &\longrightarrow \mathcal{F} \longrightarrow \sigma_{> 0}^i \mathcal{F} \longrightarrow \end{aligned}$$

We also have the following isomorphisms:

$$\begin{aligned} Rf_{i*} {}^i\tau_{\leq 0} \mathcal{K} &\cong \sigma_{\leq 0}^i Rf_{i*} \mathcal{K} \\ Rf_{i!} {}^i\tau_{> 0} \mathcal{K} &\cong \sigma_{> 0}^i Rf_{i!} \mathcal{K} \end{aligned}$$

Proof. The first two triangles are proved as in [11, 3.3.4]. For the last two isomorphisms consider the triangle:

$$Rf_{i*} {}^i\tau_{\leq 0} \mathcal{K} \longrightarrow Rf_{i*} \mathcal{K} \longrightarrow Rf_{i*} {}^i\tau_{> 0} \mathcal{K} \longrightarrow$$

obtained by applying triangle preserving Rf_{i*} to standard triangle for t_i . Let $t' := [\dots, +\infty, t_i, +\infty, \dots]$. Therefore $\sigma_{\leq 0}^i = {}^{t'}\tau_{\leq 0}$ by definition.

Then, since $f_i^* Rf_{i*} {}^i\tau_{\leq 0} \mathcal{K} = {}^i\tau_{\leq 0} \mathcal{K} \in {}^iD^{\leq 0}$, first term is in ${}^{t'}D^{\leq 0}$. Also for the last term, $f_j^! Rf_{i*} {}^i\tau_{> 0} \mathcal{K} = 0$ for $j \neq i$, while for $j = i$, $f_i^! Rf_{i*} {}^i\tau_{> 0} \mathcal{K} = {}^i\tau_{> 0} \mathcal{K} \in {}^iD^{> 0}$, and hence last term is in ${}^{t'}D^{> 0}$. Therefore, we must have $Rf_{i*} {}^i\tau_{\leq 0} \mathcal{K} \cong {}^{t'}\tau_{\leq 0} Rf_{i*} \mathcal{K}$ as required. Second isomorphism is just the dual statement. $\square$

2.1.6. Lemma. *Let $X = U \sqcup Z$ with U open and let t be a t -structure on Z . Then*

$$\tau_{\leq [+ \infty, t]} Rj_* \mathcal{F} \cong \tau_{> [- \infty, t]} j_! \mathcal{F}$$

Proof. By the previous lemma, we have the triangle:

$$\tau_{\leq [+ \infty, t]} Rj_* \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* \tau_{> t} i^* Rj_* \mathcal{F} \longrightarrow$$

But now the natural morphism $j_! \mathcal{F} \rightarrow Rj_* \mathcal{F}$ must factor through the first term since $\text{Hom}(j_! \mathcal{F}, i_* \mathcal{G}) = 0$ for any $\mathcal{G}$. Assume that the third term in the triangle is $\mathcal{T}$, that is we have a triangle:

$$\mathcal{T} \rightarrow j_! \mathcal{F} \rightarrow \tau_{\leq [+ \infty, t]} Rj_* \mathcal{F} \rightarrow$$

Recall we also have the triangle:

$$j_! \mathcal{F} \rightarrow Rj_* \mathcal{F} \rightarrow i_* i^* Rj_* \mathcal{F} \rightarrow$$

Using the octahedral axiom we will get a triangle:

$$\mathcal{T}[1] \rightarrow i_* i^* Rj_* \mathcal{F} \rightarrow i_* \tau_{> t} i^* Rj_* \mathcal{F} \rightarrow$$

Hence we must have a natural isomorphism $\mathcal{T}[1] \cong i_* \tau_{\leq t} i^* Rj_* \mathcal{F}$. But now using, that $i^* Rj_* \mathcal{F} \cong i^! j_! \mathcal{F}[1]$, the triangle defining $\mathcal{T}$ can be rewritten as:

$$i_* \tau_{\leq t} i^! j_! \mathcal{F} \rightarrow j_! \mathcal{F} \rightarrow \tau_{\leq [+ \infty, t]} Rj_* \mathcal{F} \rightarrow$$

By the previous lemma, we get an isomorphism $\tau_{\leq [+ \infty, t]} Rj_* \mathcal{F} \cong \tau_{> [- \infty, t]} j_! \mathcal{F}$. $\square$

2.2. Morel's t -structures and weights. In [11, Section 3], S. Morel defines certain t -structures on the category of mixed l -adic complexes $D_m^b(X, \mathbb{Q}_l)$ on a scheme X over a finite field k using the notion of weights. We recall her key definitions below, and develop some properties of these t -structures which we will need in the sequel. Morel's construction is quite general and immediately carries over to any reasonable category of mixed sheaves ([16]), for example the derived category of mixed Hodge modules of Saito. Thus we will use the analogues of Morel's results for $k = \mathbb{C}$ as well.

2.2.1. Notation. As is standard, we let $({}^pD^{\leq 0}, {}^pD^{> 0})$ denote the perverse t -structure for middle perversity ([4, Section 2]), and we let ${}^pH^i$ denote the cohomology functors with respect to this perverse t -structure. We also let $(D_{\leq a}^b, D_{> a}^b)$ denote bounded complexes of weight $\leq a$ (resp. $> a$) in the sense of [4, Section 5]. We also let $(D^{\leq 0}, D^{> 0})$ denote the standard t -structure whose core is the abelian category of sheaves. ⁽²⁾

²We remind the reader that in the case $k = \mathbb{C}$ with $D^b(X) = D^b\text{MHM}(X)$, $({}^pD^{\leq 0}, {}^pD^{> 0})$ is the usual t -structure, ${}^pH^0$ is the usual cohomology functor and $(D^{\leq 0}, D^{> 0})$ is the analogue of the classical t -structure, cf. [18, 4.6].

For a fixed scheme X and any $a \in \mathbb{Z}$, we define:

${}^w D^{\leq a}(X) :=$ full subcategory of $D^b(X)$ with objects $\mathcal{F}$ s.t. $\forall i, {}^p H^i(\mathcal{F}) \in D_{\leq a}^b$

${}^w D^{\geq a}(X) :=$ full subcategory of $D^b(X)$ with objects $\mathcal{F}$ s.t. $\forall i, {}^p H^i(\mathcal{F}) \in D_{\geq a}^b$

The central result of Morel's construction is:

2.2.2. Proposition. *For any X , and any integer a , we have:*

- (1) ${}^w D^{\leq a}$ and ${}^w D^{\geq a}$ are subcategories stable under shifts and extensions inside $D^b(X)$. In particular they are triangulated subcategories of $D^b(X)$.
- (2) If (1) is the tate twist, ${}^w D^{\leq a}(1) = {}^w D^{\leq a-2}$ and ${}^w D^{\geq a}(1) = {}^w D^{\geq a-2}$.
- (3) If $\mathcal{F} \in {}^w D^{\leq a}$ and $\mathcal{G} \in {}^w D^{\geq a+1}$, then $R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.
- (4) ${}^w D^{\leq a}$ and ${}^w D^{\geq a+1}$ form a t -structure on $D^b(X)$.

Proof. See [11, Proposition 3.1.1]. □

2.2.3. Definition. A complex of sheaves $\mathcal{F} \in D^b(X)$ is *lisce* on X if the cohomology sheaves $H^i(X)$ are local systems for each i .

2.2.4. Lemma. *Let $f : T \rightarrow S$ be a morphism of schemes:*

- (1) *If f has fibers of dimension less than or same as d . Then, we have:*

$$Rf_!({}^w D^{\leq a}(T)) \subset {}^w D^{\leq a+d}(S)$$

$$f^*({}^w D^{\leq a}(S)) \subset {}^w D^{\leq a+d}(T)$$

$$Rf_*({}^w D^{> a}(T)) \subset {}^w D^{> a-d}(S)$$

$$f^!({}^w D^{> a}(S)) \subset {}^w D^{> a-d}(T)$$

- (2) *Assume S, T are smooth irreducible of dimension d_S, d_T . If a complex $\mathcal{L}$ is lisce over T , then*

$$\mathcal{L} \in {}^w D^{\leq a}(S) \Rightarrow f^* \mathcal{L} \in {}^w D^{\leq a-d_S+d_T}(T)$$

$$\mathcal{L} \in {}^w D^{> a}(S) \Rightarrow f^! \mathcal{L} \in {}^w D^{> a-d_S+d_T}(T)$$

$$\mathcal{L} \in {}^w D^{> a}(S) \Rightarrow f^* \mathcal{L} \in {}^w D^{> a-d_S+d_T}(T)$$

$$\mathcal{L} \in {}^w D^{\leq a}(S) \Rightarrow f^! \mathcal{L} \in {}^w D^{\leq a-d_T+d_S}(T)$$

Proof. (1) is contained in [11, 3.1.3(v)].

- (2) It is enough to argue for local systems (since ${}^w D^{\leq 0}$ etc. are triangulated categories, ${}^p H^{i+d_S} \mathcal{L} = H^i \mathcal{L}[d_S]$ and we can use induction on number of nonzero H^i). Now for the first implication:

$\mathcal{L} \in {}^w D^{\leq a}(S)$ and is a local system

$$\Rightarrow \mathcal{L}[d_S] \in D_{\leq a}^b(S) \text{ (as } \mathcal{L}[d_S] \text{ is perverse, containment by definition)}$$

$$\Rightarrow \mathcal{L} \in D_{\leq a-d_S}^b(S)$$

$$\Rightarrow f^* \mathcal{L} \in D_{\leq a-d_S}^b(T) \text{ (by [4, 5.1.14])}$$

$$\Rightarrow f^* \mathcal{L}[d_T] \in D_{\leq a-d_S+d_T}^b(T)$$

$$\Rightarrow f^* \mathcal{L}[d_T] \in {}^w D^{\leq a-d_S+d_T}(T) \text{ (as } f^* \mathcal{L}[d_T] \text{ is perverse)}$$

$$\Rightarrow f^* \mathcal{L} \in {}^w D^{\leq a-d_S+d_T}(T)$$

as required. The second implication follows from first using duality. The third implication follows in the same way as the first noting that in our situation, if $\mathcal{L}$ is pure of weight a , so is $f^*\mathcal{L}$. ([6, 1.2.5]). Therefore if $\mathcal{L}$ has weights $> a$, so has $f^*\mathcal{L}$. The final implication is dual to the third. $\square$

2.2.5. Notation. If we fix a stratification $S = (S_0, \dots, S_n)$ of a scheme X , and let $\bar{d}^S = (d_0, \dots, d_n)$ (or simply $\bar{d}$) stand for an $n+1$ -tuple of integers indexed by the strata of S (or simply $\bar{d}$). Therefore, we get a t -structure on each stratum $({}^wD^{\leq d_i}, {}^wD^{\geq d_i+1})$. We can in fact allow $\bar{d}$ to take values in $\mathbb{Z} \cup \{+\infty, -\infty\}$, by using notations of 2.1.1 for t -structures $\pm\infty$.

These can then be glued using 2.1.4, to obtain a t -structure $({}^wD^{\leq \bar{d}}, {}^wD^{> \bar{d}})$. We let $({}^wD^{\leq \bar{d}}, {}^wD^{> \bar{d}})$ stand for truncations with respect to this t -structure.

2.2.6. Lemma. *If $\bar{d} = (d, d, \dots, d)$, then ${}^wD^{\leq \bar{d}} = {}^wD^{\leq d}$ and ${}^wD^{> \bar{d}} = {}^wD^{> d}$.*

Proof. See [11, 3.3.3]. $\square$

2.2.7. Lemma. *Let S, T be stratifications of schemes X, Y . Also let $\bar{d}^S, \bar{d}^T$ be a tuple of integers indexed by the stratifications T and S respectively. Then,*

$$\begin{aligned} \mathcal{F} \in {}^wD^{\leq \bar{d}^S}, \mathcal{G} \in {}^wD^{\leq \bar{d}^T} &\Rightarrow \mathcal{F} \boxtimes \mathcal{G} \in {}^wD^{\leq \bar{d}^S \boxtimes \bar{d}^T} \\ \mathcal{F} \in {}^wD^{> \bar{d}^S}, \mathcal{G} \in {}^wD^{> \bar{d}^T} &\Rightarrow \mathcal{F} \boxtimes \mathcal{G} \in {}^wD^{> \bar{d}^S \boxtimes \bar{d}^T} \end{aligned}$$

where by $\bar{d}^S \boxtimes \bar{d}^T$ we mean the tuple subordinate to stratification $\{S_i \times T_j | i, j\}$ on $X \times Y$ which takes value $d_i^S + d_j^T$ on stratum $S_i \times T_j$.

Proof. It is enough to show that $f_{ij}^*(\mathcal{F} \boxtimes \mathcal{G}) = f_i^*\mathcal{F} \boxtimes f_j^*\mathcal{G} \in {}^wD^{\leq d_i^S + d_j^T}$. We let $\mathcal{F}_i := f_i^*\mathcal{F}$, $\mathcal{G}_j := f_j^*\mathcal{G}$. Now since our complexes are bounded, there is a triangle, for some large n :

$${}^pH^{-n}\mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow \mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow {}^p\tau_{>-n}\mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow$$

Now by induction on the number of nonzero perverse cohomology objects and using that ${}^wD^{\leq a}$ is closed under taking triangles and perverse truncations for any integer a , it is enough to assume that $\mathcal{F}_i, \mathcal{G}_j$ are perverse, and hence, by definition of Morel's t -structure, in $D_{\leq d_i^S}^b$ (resp. $D_{\leq d_j^T}^b$). But then $\mathcal{F}_i \boxtimes \mathcal{G}_j$ is perverse as well ([4, 4.2.8]), and since $\mathcal{F}_i \boxtimes \mathcal{G}_j \in D_{\leq d_i^S + d_j^T}^b$ ([4, 5.1.14]), we have $\mathcal{F}_i \boxtimes \mathcal{G}_j \in {}^wD^{\leq d_i^S + d_j^T}$ as required. The other statement follows by duality. $\square$

2.2.8. Lemma. *Let S be a stratification on a scheme X . Let $\bar{d}, \bar{d}'$ be tuple of integers indexed by the stratification S . Assume in addition that for all strata $f_i : S_i \hookrightarrow X$ in S , S_i is smooth and $f_i^*\mathcal{F}$, $f_i^!\mathcal{F}$, as well as $f_i^*\mathcal{G}$ and $f_i^!\mathcal{G}$, are lisse. Then*

$$\begin{aligned} \mathcal{F} \in {}^wD^{\leq \bar{d}}, \mathcal{G} \in {}^wD^{\leq \bar{d}'} &\Rightarrow \mathcal{F} \otimes^L \mathcal{G} \in {}^wD^{\leq \bar{d} \otimes \bar{d}'} \\ \mathcal{F} \in {}^wD^{> \bar{d}}, \mathcal{G} \in {}^wD^{> \bar{d}'} &\Rightarrow \mathcal{F} \otimes^L \mathcal{G} \in {}^wD^{> \bar{d} \otimes \bar{d}'} \end{aligned}$$

where by $\bar{d} \otimes \bar{d}'$ we mean the tuple which takes value $d_i + d'_i - \dim(S_i)$ on the stratum S_i .

Proof. It is enough to show that $f_i^*(\mathcal{F} \otimes^L \mathcal{G}) = f_i^* \mathcal{F} \otimes^L f_i^* \mathcal{G} \in {}^w D^{\leq d_i + d'_i - \dim S_i}$. We let $\mathcal{F}_i := f_i^* \mathcal{F}$, $\mathcal{G}_i := f_i^* \mathcal{G}$. Now since our complexes are bounded, there is a triangle, for some large n :

$$\mathcal{H}^{-n} \mathcal{F}_i \otimes^L \mathcal{G}_i \rightarrow \mathcal{F}_i \otimes^L \mathcal{G}_i \rightarrow \tau_{>-n} \mathcal{F}_i \otimes^L \mathcal{G}_i \rightarrow$$

Now by induction on the number of nonzero perverse cohomology objects and using that ${}^w D^{\leq a}$ is closed under taking triangles, perverse truncations and shifts, and that for the complex $\mathcal{F}_i$, $\mathcal{H}^{-n} \mathcal{F}_i = ({}^p H^n \mathcal{F}_i)[-n]$ (since all the cohomologies are given to be local systems on the smooth variety S_i) it is enough to assume that $\mathcal{F}_i, \mathcal{G}_i$ are local systems, and hence, by definition of Morel's t -structure, in $D_{\leq d_i - \dim S_i}^b$ (resp. $D_{\leq d'_i - \dim S_i}^b$). But then $\mathcal{F}_i \otimes^L \mathcal{G}_i$ is a local system as well, and since $\mathcal{F}_i \otimes^L \mathcal{G}_i \in D_{\leq d_i + d'_i - 2 \dim S_i}^b$ ([4, 5.1.14]), we have $\mathcal{F}_i \otimes^L \mathcal{G}_i \in {}^w D^{\leq d_i + d'_i - \dim S_i}$ as required. The other statement follows by duality noting that dual of ([4, 5.1.14]) holds for local systems on smooth varieties. $\square$

2.2.9. Lemma. *For a stratification $S = (S_0, \dots, S_i, \dots, S_n)$ the functor $w_{\leq (+\infty, \dots, +\infty, d, +\infty, \dots, +\infty)}$ is left exact with respect to the standard t -structure, i.e. it takes $D^{\geq 0}$ to $D^{\geq 0}$.*

Proof. We will use Noetherian induction on the underlying scheme X . Let $f : S_i \hookrightarrow X$ be a stratum. Then for any complex $\mathcal{F} \in D^{\geq 0}$ we have the triangle:

$$w_{\leq (+\infty, \dots, +\infty, d, +\infty, \dots, +\infty)} \mathcal{F} \rightarrow \mathcal{F} \rightarrow Rf_* w_{>d} f^* \mathcal{F} \rightarrow$$

It is enough to show that $Rf_* w_{>d} f^* \mathcal{F} \in D^{\geq -1}$. Since f^* is exact while Rf_* is left exact, it is enough to show that $w_{>d}(D^{\geq 0}) \subset D^{\geq -1}$ over S_i . For any complex $\mathcal{F}$ on S_i , using the triangle:

$$w_{\leq d} \mathcal{F} \rightarrow \mathcal{F} \rightarrow w_{>d} \mathcal{F} \rightarrow$$

it is enough to show that $w_{\leq d}$ is left exact. If $\bar{S}_i$ is a proper subset of X this follows from induction. If $\bar{S}_i = X$, we restratify $S_i = U \sqcup Z$, with the property that $\mathcal{F}|_U$ is lisse and U is smooth.

Now $w_{\leq d} = w_{\leq (d, d)} = w_{\leq (\infty, d)} w_{\leq (d, \infty)}$ on S_i by 2.2.6 and $w_{\leq (\infty, d)}$ is left exact by Noetherian induction. If $j : U \rightarrow S_i$ denotes the open immersion, as before we have the triangle:

$$w_{\leq (d, \infty)} \mathcal{F} \rightarrow \mathcal{F} \rightarrow Rj_* w_{>d} j^* \mathcal{F} \rightarrow$$

But now $j^* \mathcal{F}$ is lisse, and hence we have $H^i(\mathcal{F}) = {}^p H^{i + \dim S_i}(\mathcal{F})[-\dim S_i]$. Over the smooth set U , $w_{>d}$ is perverse exact and takes local systems to local systems, therefore $w_{>d} j^* \mathcal{F} \subset D^{\geq 0}$ as well, as required. $\square$

2.2.10. Lemma. *Let us fix a stratification $S = (U, Z)$ of X . Then the functor $w_{\leq (+\infty, d)}$ is perverse right exact, i.e. $w_{\leq (+\infty, d)} {}^p D^{\leq 0} \subset {}^p D^{\leq 0}$.*

Proof. Fix $\mathcal{F} \in {}^p D^{\leq 0}$ and let $\mathcal{E} = w_{\leq (+\infty, d)} \mathcal{F}$. Consider the triangle of 2.1.5:

$$\mathcal{E} \rightarrow \mathcal{F} \rightarrow i_* w_{>d} i^* \mathcal{F} \rightarrow$$

Applying j^* we see that $j^* \mathcal{E} = j^* \mathcal{F} \in {}^p D^{\leq 0}$. Applying i^* we see that $i^* \mathcal{E} \cong w_{\leq d} i^* \mathcal{F}$. But $i^* \mathcal{F} \in {}^p D^{\leq 0}$, while $w_{\leq d}$ is perverse exact. Hence $i^* \mathcal{E} \in {}^p D^{\leq 0}$ as well. Hence by definition of the perverse t -structure as a glued t -structure, we see that $\mathcal{E} \in {}^p D^{\leq 0}$. $\square$

2.2.11. Lemma. *Let $S = (U, Z)$ be a stratification of X . Then for any $a \in \mathbb{Z}$ the functor $w_{\leq(+\infty, b)}$ preserves the subcategory of complexes of weight $\leq a$.*

Proof. If $\mathcal{F} \in D^b(X)$ has weights $\leq a$ then, since $i^*w_{\leq(+\infty, b)}\mathcal{F} = w_{\leq b}i^*\mathcal{F}$ we see that ${}^pH^k(i^*w_{\leq(+\infty, b)}\mathcal{F}) = w_{\leq b}{}^pH^k(i^*\mathcal{F})$ has weights $\leq \min(b, k + a) \leq k + a$ for all k (since $i^*\mathcal{F}$ has weights $\leq a$). So $i^*w_{\leq(+\infty, b)}\mathcal{F}$ has weights $\leq a$. Since $j^*w_{\leq(+\infty, b)}\mathcal{F} = j^*\mathcal{F}$ also has weights $\leq a$, the pointwise criterion for weights (cf. [4, 5.1.9] or [18, 4.6.1]) implies that $w_{\leq(+\infty, b)}\mathcal{F}$ has weights $\leq a$. $\square$

2.2.12. Lemma. *Let X be a scheme and let $\mathcal{F}$ denote an irreducible perverse sheaf, pure of weight d , whose support is the whole of X . Then on any proper closed subscheme $i : Z \hookrightarrow X$, $i^*\mathcal{F} \in {}^wD^{\leq d-1}$.*

Proof. We have $i^*\mathcal{F} \in {}^pD^{\leq 0}$, hence for $i > 0$, ${}^pH^i(i^*\mathcal{F}) = 0$. Also $i^*\mathcal{F} \in D_{\leq d}^b$. Hence, ${}^pH^i(i^*\mathcal{F}) \in D_{\leq d+i}^b$, and for $i < 0$, $D_{\leq d+i}^b \subset D_{\leq d-1}^b$. Hence for $i \neq 0$, ${}^pH^i(i^*\mathcal{F}) \in D_{\leq d-1}^b$, and it only remains to verify the same for ${}^pH^0(i^*\mathcal{F})$. But for a perverse sheaf, just as in the previous lemma, we have a surjection:

$$\mathcal{F} = {}^pH^0\mathcal{F} \twoheadrightarrow {}^pH^0(i_*i^*\mathcal{F}) = i_*{}^pH^0(i^*\mathcal{F})$$

However since $\mathcal{F}$ is irreducible and has support on whole of X and Z is proper, this implies that $i_*{}^pH^0(i^*\mathcal{F}) = 0$. Hence also ${}^pH^0(i^*\mathcal{F}) = 0$ as required. Therefore by definition of Morel's t -structure, $i^*\mathcal{F} \in {}^wD^{\leq d-1}$ and we are done. $\square$

2.2.13. Lemma. *Let X be a scheme with $j : U \hookrightarrow X$ open. Let $\mathcal{F} \in {}^wD^{\leq a}(U) \cap {}^wD^{\geq a}(U)$ and let $S = (S_0, \dots, S_r)$ be a stratification with $S_0 = U$. Then for any $r+1$ -tuple $\bar{d} = (a, d_1, \dots, d_r)$ we have the equality:*

$$w_{\leq(a, d_1, \dots, d_r)}Rj_*\mathcal{F} \cong w_{\geq(a, d_1+1, \dots, d_r+1)}j_!\mathcal{F}$$

Proof. The claim is proved in [11, 3.4.2]. It also follows immediately using 2.1.6. $\square$

3. WEIGHT-TRUNCATED COMPLEXES

3.1. Definition of the weight-truncated complexes EC_X^D .

3.1.1. Notation. We let $\mathcal{S} := \{S \mid S = (S_0, \dots, S_r) \text{ is a stratification of } X \text{ in smooth locally closed subschemes}\}$ be partially ordered by $\preceq$, where

$$S \preceq S' \text{ if } S' \text{ is a refinement of } S$$

We also use the notations of 2.1.3 with superscript to indicate the relevant stratification (e.g. ${}^S w_{\leq \bar{d}}$). Superscripts will also be used to specify the scheme under consideration. All our stratifications will be in smooth subschemes unless specified otherwise.

Further, we let $D : \{0, \dots, \dim X\} \rightarrow \mathbb{Z}$ denote any function, we will impose additional restrictions later. Let ${}^S w_{\leq D}$ (respectively ${}^S w_{> D}$) denote the functor $w_{\leq(D(d_0), \dots, D(d_r))}$ (respectively $w_{>(D(d_0), \dots, D(d_r))}$) where d_i is the dimension of S_i . By abuse of notation we will also use $D(Y)$ for $D(\dim Y)$.

3.1.2. Notation. For any $d \in \mathbb{Z}$ we say a function $D : \{0, \dots, d\} \rightarrow \mathbb{Z}$ is a *monotone step function* if D is monotone, and $|D(j) - D(j-1)| \leq 1$ for all $1 < j \leq d$. In particular, we say D is *non-increasing* (resp. *non-decreasing*) *step function* if it is non-increasing (resp. non-decreasing).

3.1.3. Lemma. *For any scheme X , if $D : \{1, \dots, \dim X\} \rightarrow \mathbb{Z}$ is a non-decreasing step function then for any $\mathcal{G} \in D^b(X)$:*

$$w_{\leq D} \mathcal{G} := \lim_{\substack{\leftarrow \\ S \in \mathcal{S}}} {}^S w_{\leq D} \mathcal{G}$$

is well defined. In fact $w_{\leq D} \mathcal{G} = {}^S w_{\leq D} \mathcal{G}$ for all S sufficiently fine. Similarly if D is a non-increasing step function, then:

$$w_{> D} \mathcal{G} := \lim_{\substack{\rightarrow \\ S \in \mathcal{S}}} {}^S w_{> D} \mathcal{G}$$

and $w_{> D} \mathcal{G} = {}^S w_{> D} \mathcal{G}$ for all S sufficiently fine as before.

Proof. Consider D non-decreasing. Let S be a stratification such that $f_i^* \mathcal{G}$ is lisse on stratum S_i where $f_i : S_i \hookrightarrow X$ is the locally closed inclusion. Such a stratification exists since $\mathcal{G}$ is bounded constructible. It is enough to show that if $T \succeq S$ is another stratification in smooth subschemes, then ${}^S w_{\leq D} \mathcal{G} = {}^T w_{\leq D} \mathcal{G}$, under the natural maps as described below. It follows from 2.2.6

$${}^S w_{\leq D} = w_{\leq (D(d_0^S), D(d_1^S), \dots; D(d_1^S), D(d_2^S), \dots; \dots)}$$

where truncation on the right is with respect to T and we use $D(d_j^S)$ on T_i , if $T_i \subset S_j$.

Now since, in the above situation, $\dim(T_i) \leq d_j^S$, hence $D(T_i) \leq D(d_j^S)$ and hence we have ${}^T w_{\leq D} = {}^T w_{\leq D} {}^S w_{\leq D}$ (2.1.2), and this gives the morphisms with respect to which the limit is taken. It is enough to show that

$${}^S w_{\leq D} \mathcal{G} \in {}^w D^{\leq (D(d_0^T), D(d_1^T), \dots)}$$

To show this, we only need to show that, $f_i^{T*}({}^S w_{\leq D} \mathcal{G}) \in {}^w D^{\leq D(d_i^T)}$. Now $f_i^{T*} = f_{ij}^{TS*} f_j^{S*}$, where f_{ij}^{TS} denotes the inclusion $T_i \rightarrow S_j$. Since by definition of ${}^S w_{\leq D}$, $f_j^{S*} {}^S w_{\leq D} \mathcal{G} \in {}^w D^{\leq D(d_j^S)}$, we have by 2.2.4, $f_i^{T*}({}^S w_{\leq D} \mathcal{G}) \in {}^w D^{\leq D(d_i^S) - d_i^S + d_j^T} \subset {}^w D^{\leq D(d_j^T)}$ by restrictions on the function D as required.

The case for ${}^S w_{> D}$ is similar. $\square$

3.1.4. Definition. Fix a $\mathcal{G}$. For a non-decreasing step function we have from the proof above that $w_{\leq D} \mathcal{G} = {}^S w_{\leq D} \mathcal{G}$ for some stratification S . We define $w_{> D} \mathcal{G} := {}^S w_{> D} \mathcal{G}$. If T is a refinement of S , using the morphism of triangles

$$\begin{array}{ccccc} {}^T w_{\leq D} \mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & {}^T w_{> D} \mathcal{G} \longrightarrow \\ \cong \downarrow & & \text{Id} \downarrow & & \downarrow \\ {}^S w_{\leq D} \mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & {}^S w_{> D} \mathcal{G} \longrightarrow \end{array}$$

we see that the last vertical map exists and must be an isomorphism as well. Hence the definition does not depend on choice of the stratification (a priori upto non-canonical isomorphism, but see 3.1.7). Similarly we can work with non-increasing step functions D as well, and define $w_{\leq D}$ for such functions.

3.1.5. Notation. We will let $\text{Id} := m \mapsto m$. Note that for any integer a Morel's truncation $w_{\leq a}$ coincides with $w_{\leq \bar{a}}$ with $\bar{a}$ being the constant function using 2.2.6. Hence we can also let an integer a stand for the constant function $\bar{a}$ and this causes no confusion.

3.1.6. Definition. We say a stratification of a scheme X is *adapted to* $\mathcal{G} \in D^b(X)$ if the sheaf $\mathcal{G}$ and its Verdier dual are both lisse on each stratum. Equivalently, if $f_i : S_i \hookrightarrow X$ denote the locally closed immersions of strata, $f_i^* \mathcal{G}$ as well as $f_i^! \mathcal{G}$ are lisse on S_i for all i . Clearly, given any finite collection of bounded complexes, there is a stratification adapted to all of them (taking common refinements).

Thus for a stratification S adapted to $\mathcal{G}$ and any monotone step function D , we have the identity ${}^S w_{\leq D} \mathcal{G} = w_{\leq D} \mathcal{G}$ and ${}^S w_{> D} \mathcal{G} = w_{> D} \mathcal{G}$.

We also say a stratification is *adapted to a scheme* X if it is adapted to IC_X in the above sense (where IC_X denotes the intersection complex on X).

3.1.7. Proposition. *If D is a monotone step function then $w_{\leq D}, w_{> D}$ are truncations with respect to a t -structure which we denote by $({}^w D^{\leq D}, {}^w D^{> D})$.*

Proof. We let ${}^w D^{\leq D} := \{w_{\leq D} \mathcal{F} \mid \mathcal{F} \in D^b\}$ and ${}^w D^{> D} := \{w_{> D} \mathcal{F} \mid \mathcal{F} \in D^b\}$ (i.e. full subcategories with these objects). By working at a stratification S adapted to both $\mathcal{F}$ and $\mathcal{G}$ it easily follows that $\text{Hom}(w_{\leq D} \mathcal{F}, w_{> D} \mathcal{G}) = 0$ (since both truncations are with respect to an actual glued t -structure relative to S). The other axioms for a t -structure are almost immediate by definition. $\square$

Let us record a few propositions which will be useful later:

3.1.8. Proposition. *Let $\pi : Y \rightarrow X$ be a finite morphism. Let D be any monotone step function. Then:*

$$\begin{aligned} \mathcal{G} \in {}^w D^{\leq D}(Y) &\implies R\pi_* \mathcal{G} \in {}^w D^{\leq D}(X) \\ \mathcal{G} \in {}^w D^{> D}(Y) &\implies R\pi_* \mathcal{G} \in {}^w D^{> D}(X) \end{aligned}$$

Proof. Fix a $\mathcal{G} \in {}^w D^{\leq D}(Y)$. By 3.1.12 we can find a smooth stratification S^Y of Y adapted to $\mathcal{G}$ and S^X of X adapted to $R\pi_* \mathcal{G}$ such that each stratum of Y surjects onto some stratum of X . This image must be of the same dimension because π is finite.

But then by dimension reasons, for each i , $\pi^{-1} S_i^X = \sqcup S_j^Y$ the right hand side being a disjoint finite union of some set of strata depending on i .

Therefore, if $f_i : S_i^X \hookrightarrow X$ and $g_j : S_j^Y \hookrightarrow Y$ are the inclusions,

$$f_i^* R\pi_* \mathcal{G} \cong \bigoplus_j R\pi_{ji*} g_j^* \mathcal{G}$$

where $\pi_{ji} : S_j^Y \rightarrow S_i^X$ is the restriction of π to S_j^Y , and hence also finite. Since $\mathcal{G} \in {}^w D^{\leq D}(Y)$, $g_j^* \mathcal{G} \in {}^w D^{\leq D(\dim S_j)}(S_j^Y)$. Hence using 2.2.4, we see that $f_i^* R\pi_* \mathcal{G} \in {}^w D^{\leq D(\dim S_i)}(S_i^X)$. Since $\dim S_i = \dim S_j$, we conclude that $R\pi_* \mathcal{G} \in {}^w D^{\leq D}(X)$.

The other case follows by duality since π is proper. $\square$

3.1.9. Proposition. *Let $\pi : Y \rightarrow X$ be a smooth morphism. Then:*

$$\begin{aligned} \mathcal{G} \in {}^w D^{\leq \text{Id}}(X) &\implies \pi^* \mathcal{G} \in {}^w D^{\leq \text{Id}}(Y) \\ \mathcal{G} \in {}^w D^{> \text{Id}}(X) &\implies \pi^* \mathcal{G} \in {}^w D^{> \text{Id}}(Y) \end{aligned}$$

Proof. Fix $\mathcal{G} \in {}^w D^{\leq \text{Id}}(X)$ and consider a stratification S of X adapted to $\mathcal{G}$. Consider the stratification of Y with each stratum T_j being some irreducible component of $\pi^{-1} S_i$ for some i .

Since $\mathcal{G}|_{S_i}$ is lisse on S_i , so is $\pi^* \mathcal{F}|_{T_j}$ on T_j . Since S_i is smooth so are T_j . Hence, in particular, this stratification is adapted to $\pi^* \mathcal{F}$.

Therefore for $\pi|_{T_j} : T_j \rightarrow S_i$ 2.2.4 implies that $\pi^* \mathcal{G}|_{T_j} \in {}^w D^{\leq \dim T_j}$ if $\mathcal{G}|_{S_i} \in {}^w D^{\leq \dim S_i}$, and hence the first assertion. The proof for ${}^w D^{> \text{Id}}$ is the same. $\square$

The next proposition is the key input to construction of pullbacks:

3.1.10. Proposition. *Let $\pi : Y \rightarrow X$ be an arbitrary morphism. Then:*

$$\begin{aligned} \mathcal{G} \in {}^w D^{>\text{Id}}(Y) &\implies R\pi_* \mathcal{G} \in {}^w D^{>\text{Id}}(X) \\ \mathcal{G} \in {}^w D^{\leq \text{Id}}(X) &\implies \pi^* \mathcal{G} \in {}^w D^{\leq \text{Id}}(Y) \end{aligned}$$

Proof. The second statement follows using adjunction from the first, since $({}^w D^{\leq \text{Id}}, {}^w D^{>\text{Id}})$ forms a t -structure. We now prove the first statement.

Suppose first that we have stratifications of S^Y, S^X adapted to $\mathcal{G}, R\pi_* \mathcal{G}$ such that:

- Each stratum of S^Y surjects onto a stratum of S^X
- Restricting π to a single stratum of S^Y , each fiber is of same dimension.

Then, we write $S_{ij}^Y \twoheadrightarrow S_i^X$, for some finitely many j for each i (i.e. S_i^Y is a stratum of X and S_{ij}^Y as j varies are the finitely many strata of $\pi^{-1}S_i^X$). Hence $\mathcal{F} \in {}^w D^{>d_{ij}^Y}(S_{ij}^Y) \implies R\pi_* \mathcal{F} \in {}^w D^{>d_i^X}(S_i^X)$ (using 2.2.4, since fibers must be of dimension $d_{ij}^Y - d_i^X$). Using it for $\mathcal{F} = f_{ij}^{Y!} \mathcal{G}$, we note that $R\pi_* f_{ij}^{Y!} \mathcal{G} \in {}^w D^{>d_i^X}$.

Let $S_i^Y := \sqcup_j S_{ij}^Y = \pi^{-1}S_i^X$. We will use an induction on the number of j to show that $f_i^{X!} R\pi_* \mathcal{G} \in {}^w D^{>d_i^X}$. If $j = 1$, we have that $R\pi_* f_{ij}^{Y!} \mathcal{G} = f_i^{X!} R\pi_* \mathcal{G}$ (by base change), and this is in ${}^w D^{>d_i^X}$ by above.

If $j > 1$, let $Z \xrightarrow{i} S_i^Y \xleftarrow{j} U$ denote the closed stratum and its open complement. Let $\mathcal{G}' = f_i^{Y!} \mathcal{G} \in {}^w D^{>\text{Id}}$. Then we have a triangle:

$$R\pi_* i_* i^! \mathcal{G}' \rightarrow R\pi_* \mathcal{G}' \rightarrow R\pi_* j_* j^! \mathcal{G}' \rightarrow$$

applying $R\pi_*$ to the triangle for open and closed immersions. Now the first term is in ${}^w D^{>d_i^X}$, because it is a projection from one stratum and so is the last term, by the induction hypothesis. Hence the fact that middle term is also contained ${}^w D^{>\text{Id}}$ follows from the fact that for Morel's t -structure these are triangulated subcategories. Thus, $R\pi_* \mathcal{G}' \cong f_i^{X!} R\pi_* \mathcal{G} \in {}^w D^{>d_i^X}$ for all i as required and we are done.

It remains to construct the required stratification which is done in 3.1.12 below. $\square$

3.1.11. Remark. In the argument above all that is needed is that $D(n+1) - D(n) = 1$ for the function D . This gives for example, $Rf_*({}^w D^{\geq \text{Id}}) \subset {}^w D^{\geq \text{Id}}$ as well.

3.1.12. Lemma. *Let $\pi : Y \rightarrow X$ be any morphism. Then given stratifications S^{Y0}, S^{X0} of Y, X , there are refinements S^Y, S^X such that:*

- Each stratum of S^Y surjects onto a stratum of S^X
- Restricting π to a single stratum of S^Y , each fiber is of same dimension.

Proof. We reduce to the case where π is proper as follows: we can extend the map π to a proper map $\bar{\pi}$ from a scheme $\bar{Y}$ such that $Y \hookrightarrow \bar{Y}$ is an open immersion. Then beginning with a stratification of $\bar{Y}$ obtained by adjoining the complement of Y , and the given stratification on X we can recover the required stratification on Y by simply discarding those strata which do not lie in Y .

For the proper case, we use induction on the dimension and number of irreducible components of Y, X , the base case being trivial. In the following we will also denote by S^X, S^Y the stratification constructed at any intermediate stage.

- (1) Since S_i^Y is locally closed, the image is constructible and hence the union of finitely many locally closed subsets. Hence refining the stratification of X we can assume it is a union of finitely many strata of X , that is to say $S_i^Y \twoheadrightarrow \sqcup_j S_{ij}^X$. Also let $\pi|_{S_i^Y} = \pi_i$. Now we consider the stratification on Y given by $\{\pi_i^{-1}S_{ij}^X | i, j\}$ (these may not be smooth or irreducible). For this new stratification, we clearly have $S_k^Y \twoheadrightarrow S_{f(k)}^X$ for some mapping between the indexing sets f .
- (2) Now replacing X by S_k^X , we can assume that $S^X = (X)$, X is smooth irreducible, and π restricted to each strata of Y is surjective. (Finally we will take union of stratifications constructed for each $S_k^X, \pi^{-1}S_k^X$.)
- (3) In characteristic 0, since our schemes are of finite type over the base field, there is an open set U over which each of the maps $\pi|_{S_i^Y}$ is smooth. Since X is smooth, so are U and $(\pi|_{S_i^Y})^{-1}U$. Hence we stratify Y by irreducible components of $(\pi|_{S_i^Y})^{-1}U$ and its complement. For the complement, Z , $\pi|(Y - \pi^{-1}Z)$ can be stratified satisfying the given conditions by induction on $\dim X$. In the end all we have to do is take union of strata.
- (4) In characteristic p , we can still assume that there is an open set over which fibers have constant dimension (follows from eg. [9, II, Ex.3.22]). Since our schemes are reduced, they are generically smooth, and we can choose an open set V_i inside $(\pi|_{S_i^Y})^{-1}U$ which is smooth. The image $\pi(S_i^Y - V_i)$ is proper, closed subset for each i (since the map is proper) and since X is irreducible, so is their union. Now the same argument as above with Z as union of the images gives what we want.

□

Now we make the key definitions:

3.1.13. Definition. Let X be an equidimensional scheme of dimension d_X . Let D be a non-decreasing step function. Further assume $0 \leq D \leq d_X$ with $D(d_X) = d_X$. For a smooth, open and dense subset $j : U \hookrightarrow X$ define:

$$EC_X^D := w_{\leq D} Rj_* \mathbb{Q}_U$$

Using [11, 3.1.4] we see that there is a natural isomorphism $IC_X \cong w_{\leq d_X} Rj_* \mathbb{Q}_U$ for any such $U \subset X$ and hence by 2.1.2 we also have:

$$EC_X^D \cong w_{\leq D} IC_X.$$

In particular EC_X^D is independent of U .

3.1.14. Remark. Notice that if $D > d_X$, we would not be able to compare with the intersection complex and the definition may depend on U . If $D < 0$ then, since D is a step function, $D(d_X) < d_X$ and hence $w_{\leq D} IC_X = w_{\leq D} w_{< d_X} IC_X = 0$. This explains the constraints.

The equidimensionality assumption can be avoided in one special case of interest:

3.1.15. Definition. For any X we define

$$EC_X := w_{\leq \text{Id}} Rj_* \mathbb{Q}_U$$

for $j : U \hookrightarrow X$ smooth, open and dense. Thus for an equidimensional X we have that $EC_X = EC_X^{\text{Id}}$. In any case it is independent of U by the following lemma:

3.1.16. Lemma. *Let X be any arbitrary scheme and $j : U \hookrightarrow X$ be smooth open dense. Then:*

$$w_{\leq \text{Id}} Rj_* \mathbb{Q}_U = w_{\leq \text{Id}} IC_X$$

Proof. Let X_i be the irreducible components of X , then by definition $IC_X = \oplus IC_{X_i}$. Now $\mathbb{Q}_U = \oplus Rj_{i*} \mathbb{Q}_{U_i}$, where $\mathbb{Q}_{U_i}$ are constant sheaves supported on irreducible components $j_i : U_i \hookrightarrow U$ and $X_i = \bar{U}_i$. But then, using [11, 3.1.4] and 2.1.2 we see that:

$$w_{\leq \text{Id}} Rj_{i*} \mathbb{Q}_{U_i} \cong w_{\leq \text{Id}} w_{\leq \dim U_i} Rj_{i*} \mathbb{Q}_{U_i} \cong w_{\leq \text{Id}} IC_{X_i}$$

as required. $\square$

3.1.17. Definition. Let X be equidimensional and D satisfying the constraints of 3.1.13 or X be arbitrary and $D = \text{Id}$. Then we define:

$$EH^{D,i}(X) := \mathbb{H}^i(X, EC_X^D).$$

In particular,

$$EH^i(X) := EH^{\text{Id},i}(X) = \mathbb{H}^i(X, EC_X).$$

Notice that if X is equidimensional $EH^{*,d_X}(X) = IH^*(X)$ (where d_X stands for the constant function with value $\dim X$).

3.2. Properties of the weight-truncated complexes EC_X^D .

3.2.1. Proposition. *Let X be equidimensional and D satisfying the constraints of 3.1.13. Define $E(i) = D(i) + 1$ if $D(i) < \dim X$ and $E(i) = \dim X$ otherwise. Then E also satisfies the constraints of 3.1.13 and we have the equality:*

$$EC_X^D \cong w_{\geq E} Rj_* \mathbb{Q}_U$$

for $j : U \hookrightarrow X$ smooth open dense. If X is arbitrary, then in any case:

$$EC_X \in {}^w D^{\geq \text{Id}}$$

Proof. The proof is immediate using 2.2.13. For the second part note that if $U = \sqcup_i U_i$ in irreducible components, $EC_X = \oplus w_{\leq \text{Id}} Rj_* \mathbb{Q}_{U_i}$ and we can work with one U_i at a time. $\square$

3.2.2. Proposition. *Let X be equidimensional and D satisfying the constraints of 3.1.13 or X be arbitrary and $D = \text{Id}$. Then:*

- (1) *The natural map $EC_X^D \rightarrow IC_X$ is an isomorphism for $D = \dim X$, the constant function.*
- (2) *There is a natural map $\mathbb{Q}_X \rightarrow EC_X^D$ which is an isomorphism if X is smooth.*
- (3) *Let $j : U \hookrightarrow X$ be an open dense immersion, then $EC_X^D \cong w_{\leq \text{Id}} Rj_* EC_U^D$. In particular $j^* EC_X^D \cong EC_U^D$.*
- (4) *Let $\pi : Y \rightarrow X$ be a smooth map. Then, there is a natural isomorphism $\pi^* EC_X \cong EC_Y$.*

Proof. (1) This is [11, 3.1.4].

- (2) Consider any smooth stratification $S = (S_0, \dots, S_r)$. Now $\mathbb{Q}_X|_{S_i} \cong \mathbb{Q}_{S_i} = (\mathbb{Q}_{S_i}[\dim S_i])[-\dim S_i]$ and hence is in ${}^w D^{\leq \dim S_i}$ since for S_i smooth $\mathbb{Q}_{S_i}[\dim S_i]$ is perverse of weight S_i . Hence $\mathbb{Q}_X \in {}^w D^{\leq \text{Id}} \subset {}^w D^{\leq D}$.

We now obtain the required map by applying $w_{\leq D}$ on the adjunction map $\mathbb{Q}_X \rightarrow Rj_* \mathbb{Q}_U$ for a smooth open dense U . In case X is smooth, we can choose $U = X$ and the isomorphism is immediate.

- (3) Fix $j' : V \hookrightarrow U$ open dense smooth. Consider the triangle:

$$EC_U^D \rightarrow Rj'_* \mathbb{Q}_V \rightarrow w_{>D} Rj'_* \mathbb{Q}_V \rightarrow$$

Now apply the functor $w_{\leq D} Rj'_*$. The second term defines EC_X^D and it is enough to show that the third term vanishes. Choose a stratification of X adapted to $Rj'_* w_{>D} Rj'_* \mathbb{Q}_V$ such that V is a union of strata, and we immediately conclude that $Rj'_* w_{>D} Rj'_* \mathbb{Q}_V \subset {}^w D^{>D}$ completing the argument.

- (4) Let $j : U \rightarrow X$ be open immersion with U smooth dense. Then if $\tilde{U} = \pi^{-1}U$, $\tilde{U}$ is also smooth (since π is) dense. Now consider the triangle:

$$EC_X \rightarrow Rj_* \mathbb{Q}_U \rightarrow w_{>\text{Id}} Rj_* \mathbb{Q}_U \rightarrow$$

Pulling back using π and using smooth base change, we get a triangle of complexes over Y :

$$\pi^*(EC_X) \rightarrow Rj_* \mathbb{Q}_{\tilde{U}} \rightarrow \pi^*(w_{>\text{Id}} Rj_* \mathbb{Q}_U) \rightarrow$$

Now using 3.1.9, we see that the first term is in ${}^w D^{\leq \text{Id}}$ and last in ${}^w D^{>\text{Id}}$, and hence we must have the equality

$$\pi^*(EC_X) \cong w_{\leq \text{Id}} Rj_* \mathbb{Q}_{\tilde{U}} = EC_Y$$

□

3.2.3. Lemma. *Let $\mathcal{F} \in D^b(X), \mathcal{G} \in D^b(Y)$.*

- (1) *Let A, B, C be monotone step functions such that*

$$A(i) + B(j) \leq C(i+j) \text{ for all } 0 \leq i \leq \dim X, 0 \leq j \leq \dim Y$$

then if $\mathcal{F} \in {}^w D^{\leq A}(X), \mathcal{G} \in {}^w D^{\leq B}(Y)$, then $\mathcal{F} \boxtimes \mathcal{G} \in {}^w D^{\leq C}(X \times Y)$.

Dually, if C is such that

$$A(i) + B(j) \geq C(i+j) \text{ for all } 0 \leq i \leq \dim X, 0 \leq j \leq \dim Y$$

then if $\mathcal{F} \in {}^w D^{\geq A}(X), \mathcal{G} \in {}^w D^{\geq B}(Y)$, then $\mathcal{F} \boxtimes \mathcal{G} \in {}^w D^{\geq C}(X \times Y)$.

- (2) *Assume $X = Y$. Let A, B, C be monotone step functions such that*

$$A(i) + B(i) \leq C(i) + i \text{ for all } 0 \leq i \leq \dim X$$

then if $\mathcal{F} \in {}^w D^{\leq A}, \mathcal{G} \in {}^w D^{\leq B}$, then $\mathcal{F} \overset{L}{\otimes} \mathcal{G} \in {}^w D^{\leq C}$.

Dually, if C is such that

$$A(i) + B(i) \geq C(i) + i \text{ for all } 0 \leq i \leq \dim X$$

then if $\mathcal{F} \in {}^w D^{\leq A}, \mathcal{G} \in {}^w D^{\leq B}$, then $\mathcal{F} \overset{L}{\otimes} \mathcal{G} \in {}^w D^{\leq C}$.

Proof. (1) Let us fix a stratification S adapted to $\mathcal{F}$, and T adapted to $\mathcal{G}$. Then stratification $S \times T := \{S_i \times T_j | i, j\}$ is adapted to $\mathcal{F} \boxtimes \mathcal{G}$. Now $f_{ij}^*(\mathcal{F} \boxtimes \mathcal{G}) = f_i^* \mathcal{F} \boxtimes f_j^* \mathcal{G}$ is in ${}^w D^{\leq A(\dim(S_i)) + B(\dim(T_j))}$ by 2.2.7. Therefore it is enough to show that $A(\dim S_i) + B(\dim T_j) \leq C(\dim S_i \times T_j)$. But $\dim S_i \times T_j = \dim S_i + \dim T_j$, and the required inequality $C(\dim S_i + \dim T_j) \geq A(\dim S_i) + B(\dim T_j)$ holds by assumption.

- (2) Let us fix a stratification S adapted to $\mathcal{F}, \mathcal{G}, \mathcal{F} \overset{L}{\otimes} \mathcal{G}$. Now the claim follows from 2.2.8.

□

3.2.4. Corollary. *Let X be equidimensional and D satisfying the constraints of 3.1.13 or X be arbitrary and $D = \text{Id}$. Then:*

- (1) *For any scheme Y , $EC_{X \times Y} \cong EC_X \boxtimes EC_Y$.*
- (2) *There is a natural map $\phi_D : EC_X \otimes^L EC_X^D \rightarrow EC_X^D$ such that we have a commutative diagram:*

$$\begin{array}{ccc}
 \mathbb{Q}_X \otimes^L \mathbb{Q}_X & \xrightarrow{\cong} & \mathbb{Q}_X \\
 \downarrow & & \downarrow \\
 \mathbb{Q}_X \otimes^L EC_X^D & \xrightarrow{\cong} & EC_X^D \\
 \downarrow & & \downarrow = \\
 EC_X \otimes^L EC_X^D & \xrightarrow{\phi_D} & EC_X^D
 \end{array}$$

- (3) *We have an isomorphism of the compositions*

$$\phi_D \circ (1 \otimes^L \phi_D) \cong \phi_D \circ (\phi_{\text{Id}} \otimes^L 1)$$

(where 1 denotes the identify map, both for EC_X and EC_X^D) which are morphisms from $EC_X \otimes^L EC_X \otimes^L EC_X^D \rightarrow EC_X^D$.

(in particular EC_X is a ring object under ϕ_{Id} while EC_X^D is a module object over EC_X under ϕ_D)

Proof. (1) Fix $\mathcal{F} = Rj_*\mathbb{Q}_U$ and $\mathcal{G} = Rj_*\mathbb{Q}_V$ for $U \subset X$ and $V \subset Y$ smooth open dense. Consider the triangle:

$$w_{\leq \text{Id}}\mathcal{F} \boxtimes w_{\leq \text{Id}}\mathcal{G} \rightarrow w_{\leq \text{Id}}\mathcal{F} \boxtimes \mathcal{G} \rightarrow w_{\leq \text{Id}}\mathcal{F} \boxtimes w_{> \text{Id}}\mathcal{G}$$

The first term is in ${}^w D^{\leq \text{Id}}$ by 3.2.3 while the last term is in ${}^w D^{> \text{Id}}$ using the same lemma since $EC_X \in {}^w D^{\geq \text{Id}}$ using 3.2.1. Therefore we have that $EC_X \boxtimes EC_Y \cong w_{\leq \text{Id}}(\mathcal{F} \boxtimes \mathcal{G}) \cong EC_{X \times Y}$ by definition.

- (2) Let $j : U \rightarrow X$ be smooth open dense subset of X . Consider the natural morphism:

$$w_{\leq \text{Id}}Rj_*\mathbb{Q}_U \otimes^L w_{\leq D}Rj_*\mathbb{Q}_U \xrightarrow{f} Rj_*\mathbb{Q}_U \otimes^L Rj_*\mathbb{Q}_U \xrightarrow{g} Rj_*\mathbb{Q}_U$$

where f is obtained by functoriality of $\otimes^L$, and g is obtained via adjunction of natural morphism $j^*(Rj_*\mathbb{Q}_U \otimes^L Rj_*\mathbb{Q}_U) \cong j^*Rj_*\mathbb{Q}_U \otimes^L j^*Rj_*\mathbb{Q}_U \cong \mathbb{Q}_U \xrightarrow{\text{Id}} \mathbb{Q}_U$.

Now by 3.2.3 we have that $w_{\leq \text{Id}}Rj_*\mathbb{Q}_U \otimes^L w_{\leq D}Rj_*\mathbb{Q}_U \in {}^w D^{\leq D}$, and hence the composition gf factors through a natural map:

$$w_{\leq \text{Id}}Rj_*\mathbb{Q}_U \otimes^L w_{\leq D}Rj_*\mathbb{Q}_U \rightarrow w_{\leq D}Rj_*\mathbb{Q}_U$$

and we constructed the required morphism.

To see the commutativity properties, note that in the natural diagram:

$$\begin{array}{ccc}
\mathbb{Q}_X \otimes^L \mathbb{Q}_X & \xrightarrow{m} & \mathbb{Q}_X \\
\text{Id} \otimes^L a_D \downarrow & & a_D \downarrow \\
\mathbb{Q}_X \otimes^L EC_X^D & \xrightarrow{n} & EC_X^D \\
a \otimes^L \text{Id} \downarrow & & \downarrow = \\
EC_X \otimes^L EC_X^D & \xrightarrow{\phi_D} & EC_X^D \\
f \downarrow & & \beta \downarrow \\
Rj_* \mathbb{Q}_U \otimes^L Rj_* \mathbb{Q}_U & \xrightarrow{g} & Rj_* \mathbb{Q}_U
\end{array}$$

the outer square commutes. But cone of β lies in ${}^w D^{>D}$ while $\mathbb{Q}_X \in {}^w D^{\leq \text{Id}} \subset {}^w D^{\leq D}$, hence we must have that $\phi_D \circ (a \otimes^L \text{Id}) \circ (\text{Id} \otimes^L a_D) = a_D \circ m$ as required.

- (3) Tracing the definitions, both compositions are also obtained in the following way – consider the natural map which is composition of:

$$w_{\leq \text{Id}} Rj_* \mathbb{Q}_U \otimes^L w_{\leq \text{Id}} Rj_* \mathbb{Q}_U \otimes^L w_{\leq D} Rj_* \mathbb{Q}_U \rightarrow Rj_* \mathbb{Q}_U \otimes^L Rj_* \mathbb{Q}_U \otimes^L Rj_* \mathbb{Q}_U \rightarrow Rj_* \mathbb{Q}_U$$

(first map coming from truncation, and second adjunction as before). Since the first term lies in ${}^w D^{\leq D}$, this factors through a map

$$w_{\leq \text{Id}} Rj_* \mathbb{Q}_U \otimes^L w_{\leq \text{Id}} Rj_* \mathbb{Q}_U \otimes^L w_{\leq D} Rj_* \mathbb{Q}_U \rightarrow w_{\leq D} Rj_* \mathbb{Q}_U$$

which is the required composition. $\square$

3.2.5. Lemma. *Let X be a scheme and D be a monotone step function. Then $w_{\leq D}$ is left exact.*

Proof. Let $\mathcal{F} \in D^{\geq 0}$. We fix a stratification S adapted to X and hence we are reduced to computing ${}^S w_{\leq (d_0, d_1, \dots, d_r)} \mathcal{F}$. Now use 2.1.4 and 2.2.9. $\square$

3.2.6. Lemma. *Let X be a scheme and D be a non-decreasing step function. Then $w_{\leq D}$ is perverse right exact.*

Proof. Let $\mathcal{F} \in {}^p D^{\leq 0}$. Let us first fix a stratification S adapted to $\mathcal{F}$. Thus we have to compute ${}^S w_{\leq (d_0, d_1, \dots, d_r)} \mathcal{F}$, where $d_i \leq d_{i-1}$. We omit S from the notation for simplicity. We have:

$$\begin{aligned}
& w_{\leq (d_0, d_1, \dots, d_r)} \mathcal{F} \\
&= w_{\leq (d_0, d_1, \dots, d_r)} w_{\leq (d_0, d_1, \dots, d_{r-1}, d_{r-1})} \cdots w_{\leq (d_0, d_1, \dots, d_1)} w_{\leq (d_0, d_0, \dots, d_0)} \mathcal{F} \\
&= w_{\leq (+\infty, +\infty, \dots, d_r)} w_{\leq (+\infty, +\infty, \dots, d_{r-1}, d_{r-1})} \cdots w_{\leq (+\infty, d_1, \dots, d_1)} w_{\leq (d_0, d_0, \dots, d_0)} \mathcal{F}
\end{aligned}$$

(The first equality holds using 2.1.2. The second equality holds using further that $w_{\leq (d_0, \dots, d_i, +\infty, \dots)}$ is the identity on objects in ${}^w D^{\leq (d_0, \dots, d_i, \dots, d_i)}$, as is seen using 2.1.2). Thus noting that $w_{\leq (+\infty, \dots, +\infty, d, \dots, d)} = w_{\leq (+\infty, d)}$ for an appropriate re-stratification, the result follows from 2.2.10. $\square$

3.2.7. Corollary. *If X is equidimensional and D satisfies the constraints of 3.1.13, or if X is arbitrary and $D = \text{Id}$, then*

$$EC_X^D \in {}^pD^{\leq \dim X}(X) \text{ and } EC_X^D \in D^{\geq 0}(X)$$

Proof. Use 2.2.9 and 2.2.10 on the two isomorphisms $EC_X^D \cong w_{\leq D}IC_X$ (or $w_{\leq \text{Id}}IC_X$ in the latter case) and $EC_X^D \cong w_{\leq D}Rj_*\mathbb{Q}_U$ respectively. $\square$

3.2.8. Proposition. *For X equidimensional of dimension d_X and D satisfying the constraints of 3.1.13 we have an isomorphism*

$$EH^{D, 2d_X}(X) \cong IH^{2d_X}(X)$$

and a surjection

$$EH^{D, 2d_X-1}(X) \twoheadrightarrow IH^{2d_X-1}(X).$$

Proof. Consider the triangle:

$$EC_X^D \longrightarrow IC_X \longrightarrow w_{>D}IC_X \longrightarrow$$

Since for some open dense U , $j^*IC_X = \mathbb{Q}_U \in {}^wD^{\leq D}$, by 2.1.4 and 2.1.5 we have $w_{>D}IC_X = i_*w_{>D}i^*IC_X$.

Now consider the induced long exact sequence :

$$\rightarrow {}^pH^i(EC_X^D) \rightarrow {}^pH^i(IC_X) \rightarrow {}^pH^i(w_{>D}IC_X) \rightarrow {}^pH^{i+1}(EC_X^D) \rightarrow$$

Since $EC_X^D \in {}^pD^{\leq d_X}$ by 3.2.7, therefore ${}^pH^i(w_{>D}IC_X) = 0$ for $i \geq d_X + 1$.

Further for $i = d_X$ we get a surjection:

$${}^pH^{d_X}(IC_X) \rightarrow {}^pH^{d_X}(w_{>D}IC_X) \rightarrow 0$$

But ${}^pH^{d_X}(w_{>D}IC_X) = i_*{}^pH^i(w_{>D}i^*IC_X)$ has strictly proper support while ${}^pH^{d_X}(IC_X) = IC_X[d_X]$ has full support. Hence ${}^pH^{d_X}(w_{>D}IC_X)$ vanishes as well and $w_{>D}IC_X \in {}^pD^{\leq d_X-1}$.

Since $w_{>D}IC_X$ is supported on a strictly proper subvariety (of dimension $\leq d_X - 1$) we have $\mathbb{H}^i(X, w_{>D}IC_X) = 0$ for $i > 2d_X - 2$. Now the long exact sequence of cohomology for the above triangle yields:

$$\rightarrow \mathbb{H}^{i-1}(X, w_{>D}IC_X) \rightarrow EH^{D, i}(X) \rightarrow IH^i(X) \rightarrow \mathbb{H}^i(X, w_{>D}IC_X) \rightarrow$$

For $i = 2d_X$ we see that $EH^{D, 2d_X}(X) \cong IH^{2d_X}(X)$ and for $i = 2d_X - 1$ we get a surjection $EH^{D, 2d_X-1}(X) \twoheadrightarrow IH^{2d_X-1}(X)$ as required. $\square$

4. PULLBACKS AND OTHER PROPERTIES

4.1. Pullbacks.

4.1.1. Lemma. *Let $\pi : Y \rightarrow X$ be any morphism, and $j_X : U \hookrightarrow X$ be open. Let $j_Y : U_Y = \pi^{-1}U \hookrightarrow Y$ denote the base change. If $\mathcal{G}$ is in ${}^wD^{\geq \dim Y}$ and $U_Y \subset Y$ is dense, the following natural map is an isomorphism:*

$$w_{\leq \text{Id}}R\pi_*\mathcal{G} \xrightarrow{\cong} w_{\leq \text{Id}}Rj_{X*}j_X^*R\pi_*\mathcal{G} \cong w_{\leq \text{Id}}Rj_{X*}R\pi_*j_Y^*\mathcal{G}$$

(the second isomorphism is due to smooth base change, here π also denotes $\pi|_U$).

Proof. Let $i_Y : Z_Y \hookrightarrow Y$ denote the inclusion of the closed complement. Consider the triangle:

$$i_{Y*}i_Y^!\mathcal{G} \longrightarrow \mathcal{G} \longrightarrow Rj_{Y*}j_Y^*\mathcal{G} \longrightarrow$$

If $\mathcal{G} \in {}^w D^{\geq \dim Y}$, $i^! \mathcal{G} \in {}^w D^{\geq \dim Y} \subset {}^w D^{> \text{Id}}$ (because $\dim Z_Y < \dim Y$ as U_Y is dense). Hence $w_{\leq \text{Id}} R\pi_*$ applied to the first term is trivial by 3.1.10. In particular $w_{\leq \text{Id}} R\pi_* \mathcal{G} \cong w_{\leq \text{Id}} R\pi_* Rj_{Y*} j_Y^* \mathcal{G} \cong w_{\leq \text{Id}} Rj_{X*} R\pi_* j_Y^* \mathcal{G}$. $\square$

4.1.2. Proposition. *Let $\pi : Y \rightarrow X$ be generically proper with each generic fiber geometrically connected (i.e. assume that there is an open dense $U \subset Y$ such that $\pi|_U$ is proper with geometrically connected fibers). Also assume that Y is smooth over $\text{Spec } k$ and π is such that each irreducible component of Y dominates some component of X . Then $EC_X \cong w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y$.*

Proof. For simplicity assume X is irreducible, the proof remains similar otherwise. By the previous lemma with $\mathcal{G} = \mathbb{Q}_Y$,

$$w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y \cong w_{\leq \text{Id}} Rj_* R\pi_* \mathbb{Q}_{\pi^{-1}U}$$

Now by 3.1.10, right hand side is also same as $w_{\leq \text{Id}} Rj_* w_{\leq \text{Id}} R\pi_* \mathbb{Q}_U$. But by choosing U small enough, $R\pi_* \mathbb{Q}_{\pi^{-1}U}$ is lisse on U , and therefore we have an equality $w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{\pi^{-1}U} = w_{\leq \dim U} R\pi_* \mathbb{Q}_{\pi^{-1}U}$. But by the decomposition theorem on U we see immediately that $w_{\leq \dim U} R\pi_* \mathbb{Q}_{\pi^{-1}U} = R^0 \pi_* \mathbb{Q}_{\pi^{-1}U} = \mathbb{Q}_U$ (the last equality because the generic fiber is geometrically connected). Notice that the identification only uses the natural morphism $\pi_* \mathbb{Q}_{\pi^{-1}U} \rightarrow R\pi_* \mathbb{Q}_{\pi^{-1}U}$ and hence is canonical. Now the result follows immediately from definition of EC_X . $\square$

4.1.3. Remark. If the generic fibre is not geometrically connected, assuming that there is an open dense set over which π is proper, we have that $\mathbb{Q}_U$ is a summand of $\pi_* \mathbb{Q}_{\pi^{-1}U}$ for small enough U . Then EC_X is a summand of $w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y$.

4.1.4. Proposition. *For any morphism $f : Y \rightarrow X$ such that the closed image of each irreducible component of Y meets the smooth locus of X , there is an induced morphism $f^\# : EC_X \rightarrow Rf_* EC_Y$.*

If $g : Z \rightarrow Y$ is another morphism such that both g and $f \circ g$ satisfy this condition (i.e. the closed image of each irreducible component of the domain meets the codomain), then there is a natural identification $(f \circ g)^\# = f^\# \circ Rf_ g^\#$.*

Proof. Consider the open immersion $\pi_X : \tilde{X} \rightarrow X$ where $\tilde{X}$ is the smooth locus of X . Let $\tilde{Y}$ be the smooth locus of the fibered product $\tilde{X} \times_X Y$ (which is nonempty since the image of Y meets the smooth locus of X), and let the composite $\tilde{Y} \rightarrow Y$ be denoted π_Y .

Now consider the following diagram:

$$\begin{array}{ccccc} EC_X & \xrightarrow{\cong} & w_{\leq \text{Id}} R\pi_{X*} \mathbb{Q}_{\tilde{X}} & \overset{\phi}{\dashrightarrow} & Rf_* w_{\leq \text{Id}} R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \xrightarrow{\cong} Rf_* EC_Y \\ & & \downarrow & & \downarrow \\ & & R\pi_{X*} \mathbb{Q}_{\tilde{X}} & \xrightarrow{R\pi_{X*}(\text{adj.})} & Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \end{array}$$

We denote the composition $EC_X \rightarrow Rf_* EC_Y$ by $f^\#$. (Here the dotted arrow comes from the fact that $\text{Hom}({}^w D^{\leq \text{Id}}, {}^w D^{> \text{Id}}) = 0$ and using the long exact sequence of Hom's along with invariance of Morel's t -structure under shifts.)

To see that $(f \circ g)^\# = f^\# \circ Rf_* g^\#$, one only needs to arrange a tower $\tilde{Z} \rightarrow \tilde{Y} \rightarrow \tilde{X}$, here $\tilde{Z}$ is smooth locus of the fiber product $\tilde{Y} \times_Y Z$. It is enough to show that the fiber product is nonempty. Let $W \subset Y$ be the preimage of the smooth locus of X under f and let $U \subset Y$ be the smooth locus of Y . Then $\tilde{Y} = U \cap W$. If the image of some irreducible component does not meet $\tilde{Y}$, since it meets $U - W$ and $W - U$

(because of our assumptions on $f \circ g$ and g respectively), it is not irreducible, a contradiction. $\square$

4.1.5. Proposition. *Let $f : Y \rightarrow X$ be such that the closed image of each irreducible component of Y meets the smooth locus of X . Then $f^\#$ above is a morphism of ring objects. The natural diagram*

$$\begin{array}{ccc} \mathbb{Q}_X & \longrightarrow & Rf_* \mathbb{Q}_Y \\ \downarrow & & \downarrow \\ EC_X & \xrightarrow{f^\#} & Rf_* EC_Y \end{array}$$

commutes.

Proof. Take $\pi_X : \tilde{X} \rightarrow X$ to be the smooth locus of X and $\pi_Y : \tilde{Y} \rightarrow Y$ to be the smooth locus of the fiber product. By 3.2.2, the natural map $\mathbb{Q}_X \rightarrow R\pi_{X*} \mathbb{Q}_{\tilde{X}}$ factors through a map $g_X : \mathbb{Q}_X \rightarrow EC_X$ (similarly for Y), and hence we get a commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}_X & \xrightarrow{a} & Rf_* \mathbb{Q}_Y \\ g_X \downarrow & & \downarrow Rf_* g_Y \\ EC_X & \xrightarrow{f^\#} & Rf_* EC_Y \\ \downarrow & & \downarrow b \\ R\pi_{X*} \mathbb{Q}_{\tilde{X}} & \xrightarrow{R\pi_{X*}(adj.)} & Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \end{array}$$

where the topmost arrow is the natural adjunction.

Since the outer square and the bottom square commute, we see that $b \circ f^\# \circ g_X = b \circ Rf_* g_Y \circ a$. But $\text{Cone}(b) = Rf_* w_{>\text{Id}} R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \in {}^w D^{>\text{Id}}$, while $\mathbb{Q}_X \in {}^w D^{\leq \text{Id}}$. Hence we must have $f^\# \circ g_X = Rf_* g_Y \circ a$ as required.

To see that $f^\#$ is a morphism of ring objects, consider the natural diagram:

$$\begin{array}{ccccc} EC_X \otimes^L EC_X & \xrightarrow{f^\# \otimes f^\#} & R\pi_{X*} \mathbb{Q}_{\tilde{X}} \otimes^L R\pi_{X*} \mathbb{Q}_{\tilde{X}} & & \\ \downarrow m_X & \searrow & \downarrow & \searrow & \\ & Rf_* EC_Y \otimes^L Rf_* EC_Y & \xrightarrow{\quad} & Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \otimes^L Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} & \\ & \downarrow m_Y & \downarrow & \downarrow & \\ EC_X & \xrightarrow{f^\#} & R\pi_{X*} \mathbb{Q}_{\tilde{X}} & \xrightarrow{\quad} & Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} \\ & \searrow & \searrow & \searrow & \\ & Rf_* EC_Y & \xrightarrow{\alpha} & Rf_* R\pi_{Y*} \mathbb{Q}_{\tilde{Y}} & \end{array}$$

The bottom and top face commute by definition of $f^\#$. The front and back face commute as in proof of 3.2.4. The rightmost face commutes because ordinary cohomology is morphism of ring objects. Hence we must have the left face composed with α must commute, that is:

$$\alpha \circ f^\# \circ m_X = \alpha \circ m_Y \circ (f^\# \otimes^L f^\#).$$

Now notice that the cone of α is $Rf_*w_{>\text{Id}}R\pi_{Y*}\mathbb{Q}_{\hat{Y}} \in {}^wD^{>\text{Id}}$ while $EC_X \overset{L}{\otimes} EC_X \in {}^wD^{\leq \text{Id}}$. Hence we must already have $f^\# \circ m_X = m_Y \circ (f^\# \overset{L}{\otimes} f^\#)$ as required. $\square$

4.1.6. Remark. It is possible to define canonical pullbacks for arbitrary morphisms, however they do not seem to satisfy the composition law as above nor do they seem to be morphisms of ring objects in general. The key observation one uses for the construction of general pullbacks is that even if the generic fiber of $\pi : Y \rightarrow X$ is not geometrically connected, but π is generically proper, dominant and Y smooth, EC_X is canonically a summand of $w_{\leq \text{Id}}R\pi_*\mathbb{Q}_Y$. See [22] for more details.

4.2. Other properties. The following theorem summarizes the properties of the cohomology groups $EH^{D,*}(X)$:

4.2.1. Theorem. *Let X be an equidimensional variety and D a monotone non-decreasing step function satisfying the constraints of 3.1.13 or X be arbitrary and $D = \text{Id}$. The graded group $EH^{D,*}(X)$ satisfies:*

- (1) *If X is of dimension d then $EH^{D,i} \neq 0$ only if $i \in [0, 2d]$. If $\nu : \hat{X} \rightarrow X$ is the normalization then $EH^{D,*}(X) \cong EH^{D,*}(\hat{X})$. If X is smooth then $H^*(X) \cong EH^{D,*}(X)$.*
For $D = \dim X$, the constant function, $EH^{\dim X,i}(X) \cong IH^i(X)$.
- (2) *$EH^*(X)$ is a graded-commutative ring. The natural map $H^*(X) \rightarrow EH^*(X)$ is a ring map.*
- (3) *$EH^*(X)$ has ring pullbacks for morphisms with the image of each component meeting the smooth locus of the codomain.*
- (4) *$EH^*(X \times Y) \cong EH^*(X) \otimes EH^*(Y)$.*
- (5) *$EH^{D,*}$ live between the usual cohomology and the intersection cohomology, that is for all i there are natural maps:*

$$H^i(X) \rightarrow EH^{D,i}(X) \rightarrow IH^i(X)$$

- (6) *If $d = \dim X$, the second map induces:*

$$\begin{aligned} EH^{D,2d} &\xrightarrow{\cong} IH^{2d} \\ EH^{D,2d-1} &\twoheadrightarrow IH^{2d-1} \end{aligned}$$

- (7) *$EH^{D,*}(X)$ are graded modules over $EH^*(X)$, and in particular $IH^*(X)$ is a graded module over $EH^*(X)$. Furthermore, the natural action of the ring $H^*(X)$ on $EH^{D,*}(X)$ factors through that of $EH^*(X)$.*

Proof. (1) The bounds follows from 3.2.7 since for $\mathcal{F} \in D^{\geq 0}$ we have $\mathbb{H}^i(X, \mathcal{F}) = 0$ for $i < 0$, and if $\mathcal{F} \in {}^pD^{\leq \dim X}$ we have $\mathbb{H}^i(X, \mathcal{F}) = 0$ for $i > 2 \dim X$.

For the second statement, let $\hat{j} : U \hookrightarrow \hat{X}$ denote a dense open smooth subset of $\hat{X}$ mapping isomorphically into X . We let $j = \nu \circ \hat{j} : U \hookrightarrow X$ denote the inclusion. But then, 3.1.8 implies that

$$R\nu_*EC_{\hat{X}} = R\nu_*w_{\leq \text{Id}}R\hat{j}_*\mathbb{Q}_U = w_{\leq \text{Id}}Rj_*\mathbb{Q}_U = EC_X$$

which is what we require. The last two statements follows from 3.2.2.

- (2) This follows from 3.2.3 for $D = \text{Id}$. It is clear that we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}_X \overset{L}{\otimes} \mathbb{Q}_X & \xrightarrow{\cong} & \mathbb{Q}_X \\ \downarrow & & \downarrow \\ EC_X \overset{L}{\otimes} EC_X & \rightarrow & EC_X \end{array}$$

which tells us that the natural map $H^i(X) \rightarrow EH^i(X)$ is a ring map.

- (3) This follows from 4.1.4 and 4.1.5.
 (4) This follows from 3.2.4.
 (5) The second map comes from the map $EC_X \rightarrow IC_X$ of 3.1.16, while the first map comes from the map $\mathbb{Q}_X \rightarrow EC_X$ of 3.2.2.
 (6) This follows from 3.2.8.
 (7) This again follows from 3.2.3 taking D other than Id . We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}_X \overset{L}{\otimes} EC_X^D & \xrightarrow{\cong} & EC_X^D \\ \downarrow & & \downarrow \\ EC_X \overset{L}{\otimes} EC_X^D & \rightarrow & EC_X^D \end{array}$$

which tells us that the action of $H^*(X)$ on $EH^{D,*}(X)$ factors through the action of EH_X^* , in particular for $D = \dim X$ we recover the statement on intersection cohomology.

□

4.3. Weights. In this subsection we prove properties of the weight filtration on $EH^*(X)$ (including (viii) and (ix) of the introduction).

Consider first the special case when X is proper.

4.3.1. Lemma. *If X is proper then $Gr_j^W EH^i(X) = 0$ for $j > i$ and the map $EH^*(X) \rightarrow IH^*(X)$ induces a natural isomorphism*

$$Gr_i^W EH^i(X) = \text{im}(EH^i(X) \rightarrow IH^i(X)).$$

Proof. For the first assertion it is enough to show that the complex EC_X has weights ≤ 0 . Let $X = \sqcup_i S_i$ be a stratification with $U = S_0$ open and dense, for which $Rj_*\mathbb{Q}_U$, $j!\mathbb{Q}_U$, and IC_X are constructible. Let $d_i := \dim S_i$ and let $i_k : S_k \hookrightarrow X$ be the inclusions. Then $EC_X = w_{\leq(d_0, \dots, d_r)} Rj_*\mathbb{Q}_U = w_{\leq(d_0, \dots, d_r)} IC_X$. Since $d_0 \geq \dots \geq d_r$, one has

$$w_{\leq(d_0, \dots, d_r)} = w_{\leq(+\infty, \dots, +\infty, d_r)} \circ \dots \circ w_{\leq(+\infty, d_1, \dots, d_1)} \circ w_{\leq(d_0, \dots, d_0)}$$

(as in the proof of 3.2.6). Each term on the right is of the form $w_{\leq(\infty, b)}$ for a stratification with two strata, so respects the condition of having weights ≤ 0 by 2.2.11. Since IC_X is pure of weight zero, EC_X has weights ≤ 0 .

For the second assertion we use the triangle

$$EC_X \rightarrow IC_X \rightarrow w_{>(d_0, \dots, d_r)} IC_X \rightarrow .$$

By the fact that EC_X has weights ≤ 0 and the hypercohomology long exact sequence of this triangle, it is enough to show that $\mathcal{K} := w_{>(d_0, \dots, d_r)} IC_X$ has weights ≤ 0 . For this we will use the pointwise criterion for weights: $\mathcal{K} \in D^b(X)$ has weights ≤ 0 if, and only if, for every closed point $i_x : \{x\} \hookrightarrow X$, $H^i(i_x^* \mathcal{K})$ has

weights $\leq i$. If $x \in U = X_0$ this obviously holds, so assume $x \in S_k$ for $k \geq 1$. Then $i_k^* EC_X = i_k^* w_{\leq (d_0, \dots, d_r)} Rj_* \mathbb{Q}_U$ belongs to ${}^w D^{\leq d_k}$, i.e. ${}^p H^i(i_k^* EC_X)$ has weights $\leq d_k$ for all i . Since EC_X is constructible for the stratification,

$${}^p H^i(i_k^* EC_X) = H^{i-d_k}(i_k^* EC_X)[d_k],$$

so that $H^i(i_k^* EC_X)$ has weights $\leq d_k - d_k = 0$ for all i . In the long exact sequence

$$\cdots \rightarrow H^i(i_x^* IC_X) \rightarrow H^i(i_x^* \mathcal{K}) \rightarrow H^{i+1}(i_x^* EC_X) \rightarrow \cdots$$

the first group has weights $\leq i$ (because IC_X has weight 0) and the last group has weights ≤ 0 . So $H^i(i_x^* \mathcal{K})$ has weights $\leq i$ for all i , proving that $\mathcal{K} = w_{>(d_0, \dots, d_r)} IC_X$ is of weight ≤ 0 . $\square$

4.3.2. Remark. The lemma implies the following weak version of the hard Lefschetz theorem: If $e \in H^2(X)$ is the first Chern class of an ample line bundle then $\cdot e : Gr_i^W EH^i(X) \rightarrow Gr_{i+2}^W EH^{i+2}(X)$ is injective if $i < \dim X$. This is used in [13].

The bounds on the weights of $EH^*(X)$ in the general case will follow from the following proposition (which also contains another proof of the previous lemma).

4.3.3. Proposition. *Let X be an irreducible scheme and let $\pi : Y \rightarrow X$ be proper with Y smooth. Then:*

$$Gr_j^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) = 0 \text{ if } j < 2i - 2 \dim Y \text{ or } j > 2i \text{ or } j > 2 \dim Y$$

If, in addition, X is proper then:

$$Gr_j^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) = 0 \text{ if } j > i$$

$$Gr_i^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \cong \text{im}(\mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \rightarrow \mathbb{H}^i(X, R\pi_* \mathbb{Q}_Y) = H^i(Y))$$

(Here the Gr_^W are the graded with respect to the weight filtration.)*

Proof. We will prove the fact using Noetherian induction on Y . The proof of all the statements is very similar, and we prove them together.

Reduction to Y irreducible: $Y = \sqcup Y_i$ with Y_i smooth. Then $\dim Y_i \leq \dim Y$. Hence vanishings for $j < 2i - 2 \dim Y$ (resp. $j > 2 \dim Y$) imply $j < 2i - \dim Y_i$ (resp. $j > 2 \dim Y_i$) and we see that the statement for Y_i implies that for Y .

Reduction to π surjective: Since π is proper, its image is closed. Let $i : X' \hookrightarrow X$ be the closed image of π and $\pi' : Y \rightarrow X'$ be the restriction of π . Then $\pi = i\pi'$. We also have:

$$w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y \cong w_{\leq \text{Id}} i_* R\pi'_* \mathbb{Q}_Y \cong i_* w_{\leq \text{Id}} R\pi'_* \mathbb{Q}_Y$$

But $\mathbb{H}^i(X, i_* F) = \mathbb{H}^i(X', F)$ for any complex of sheaves, and hence we can replace π by π' and X by X' , thereby assuming that π is surjective.

Base case of induction: Since Y is smooth of dimension zero and irreducible, Y is $\text{Spec } k$. Since π is surjective, the underlying space of X is a point as well. Now the claims are obvious.

Induction Step:

Case 1: π is cohomologically lisse outside an snc divisor: Assume first that there is some $j : U \hookrightarrow X$ open immersion of a smooth set such that $j^* R\pi_* \mathbb{Q}_Y$ is lisse on U , and if $i : Z \hookrightarrow X$ denotes its closed complement, $Y_Z := \pi^{-1} Z$ is a simple normal crossing divisor.

Step 1: Reducing to a condition on $\mathcal{T} = w_{> \text{Id}} {}^p \tau_{\leq d_X} R\pi_ \mathbb{Q}_Y$:*

Consider the triangle:

$${}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y \rightarrow R\pi_* \mathbb{Q}_Y \rightarrow {}^p\tau_{> d_X} R\pi_* \mathbb{Q}_Y \rightarrow$$

Now $\mathbb{Q}_Y$ is a sheaf of weight 0. Since π is proper, ${}^pH^i(R\pi_* \mathbb{Q}_Y)$ is of weight i . Hence ${}^p\tau_{> d_X} R\pi_* \mathbb{Q}_Y$ is of weight $> d_X > \text{Id}$. Hence the last term is in ${}^wD^{>\text{Id}}(X)$. Hence

$$\text{Hom}(w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y, {}^p\tau_{> d_X} R\pi_* \mathbb{Q}_Y[i]) = 0$$

for all i and using the long exact sequence of Hom 's for the above triangle, we get that the natural map $w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y \rightarrow R\pi_* \mathbb{Q}_Y$ uniquely lifts to a map $w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y \rightarrow {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y$. We consider the octahedral diagram arising out of this situation:

$$\begin{array}{ccccc} & & \mathcal{T} & & \\ & \nearrow & & \searrow & \\ {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y & & R\pi_* \mathbb{Q}_Y & & w_{> \text{Id}} R\pi_* \mathbb{Q}_Y \\ \nearrow & & \nwarrow & & \searrow \\ w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y & & & & {}^p\tau_{> d_X} R\pi_* \mathbb{Q}_Y \end{array}$$

(Here $\mathcal{T}$ is defined by the left most triangle.) Now since ${}^p\tau_{> d_X} R\pi_* \mathbb{Q}_Y \in {}^wD^{>\text{Id}}$ and hence applying $w_{\leq \text{Id}}$ to the triangle above, we also have an identification,

$$w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y \cong w_{\leq \text{Id}} {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y$$

and hence

$$\mathcal{T} \cong w_{> \text{Id}} {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y.$$

Consider the induced long exact sequence from top left triangle:

$$\rightarrow Gr_j^W \mathbb{H}^{i-1}(X, \mathcal{T}) \rightarrow Gr_j^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \rightarrow Gr_j^W \mathbb{H}^i(X, {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \rightarrow$$

By the decomposition theorem, the last term is a summand of the $Gr_j^W \mathbb{H}^i(X, R\pi_* \mathbb{Q}_Y) = Gr_j^W H^i(Y)$ and hence vanishes for

- (i) $j < 2i - 2 \dim Y$
- (ii) $j > 2i$
- (iii) $j > 2 \dim Y$
- (iv) $j > i$ for X (and hence Y) proper

by Deligne's theory of weights. Hence it is enough to show that the first term $Gr_j^W \mathbb{H}^{i-1}(X, \mathcal{T})$ vanishes for the same range of j , to obtain vanishing of the middle term.

Claim: If X is proper, the vanishing of $Gr_i^W \mathbb{H}^{i-1}(X, \mathcal{T})$ implies that $Gr_i^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) = \text{im}(\mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \rightarrow H^i(Y))$.

Proof of the claim: The vanishing implies an injection:

$$Gr_i^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \hookrightarrow Gr_i^W \mathbb{H}^i(X, {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

The second term is a direct summand of $H^i(Y)$, and hence the morphism

$$Gr_i^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \rightarrow Gr_i^W H^i(Y) = H^i(Y)$$

is injective. Since $H^i(Y)$ is pure of weight i , the image $\text{im}(\mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y) \rightarrow H^i(Y))$ is pure of weight i , and hence a subquotient of the i -th graded. But by injectivity of the above map, it must be precisely the i -th graded, as required.

Step 2: Reduction to the divisor in the boundary

Recall that $j : U \hookrightarrow X$ is such that $j^* R\pi_* \mathbb{Q}_Y$ is lisse. Therefore

$$j^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \cong {}^p\tau_{\leq d_X} R\pi_{U*} \mathbb{Q}_{Y_U}$$

where $\pi_U = \pi|_U : Y_U = \pi^{-1}U \rightarrow U$. Since $j^* R\pi_* \mathbb{Q}_Y = R\pi_{U*} \mathbb{Q}_{Y_U}$ is lisse, we further have

$${}^p\tau_{\leq d_X} R\pi_{U*} \mathbb{Q}_{Y_U} \cong \tau_{\leq 0} R\pi_{U*} \mathbb{Q}_{Y_U}$$

and this is pure of weight 0.

Now considering Id as the glued t -structure $[\text{Id}; \text{Id}]$ on strata (U, Z) , we have an isomorphism:

$$\mathcal{T} \cong w_{>(\text{Id}, \text{Id})}({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \cong w_{>(+\infty, \text{Id})} w_{>(\text{Id}, +\infty)}({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

Now consider the triangle:

$$j_! w_{\leq \text{Id}} j^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \rightarrow {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y \rightarrow w_{>(\text{Id}, +\infty)}({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

But, by the above, the first term is nothing but:

$$j_! w_{\leq \text{Id}} j^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \cong j_! w_{\leq \text{Id}} R^0 \pi_{U*} \mathbb{Q}_{Y_U} \cong j_! R^0 \pi_{U*} \mathbb{Q}_{Y_U} \cong j_! j^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

and comparing with the localization triangle, we have an isomorphism:

$$w_{>(\text{Id}, +\infty)}({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y) \cong i_* i^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

Consequently, we also have an isomorphism:

$$\mathcal{T} \cong w_{> \text{Id}} {}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y \cong w_{> \text{Id}} i_* i^*({}^p\tau_{\leq d_X} R\pi_* \mathbb{Q}_Y)$$

But then by the decomposition theorem, the last term is a summand of $w_{> \text{Id}} i_* i^*(R\pi_* \mathbb{Q}_Y) = i_* w_{> \text{Id}} R\pi_* \mathbb{Q}_{Y_Z}$ and it is enough to show that the graded

$$Gr_j^W \mathbb{H}^{i-1}(X, i_* w_{> \text{Id}}(R\pi_* \mathbb{Q}_{Y_Z})) \cong Gr_j^W \mathbb{H}^{i-1}(Z, w_{> \text{Id}}(R\pi_* \mathbb{Q}_{Y_Z}))$$

vanishes in the said range.

Step 3: Proving the required vanishing in the boundary

Let $Y_Z = \cup Y_i$, where Y_i are smooth irreducible divisors. Let $i_k : A_k \rightarrow Y_Z$ for $k \geq 0$ be disjoint union of the k -fold intersections (hence $A_0 = \sqcup Y_i$ and A_k is of dimension $\dim Y - k - 1$). Then there is a resolution:

$$0 \rightarrow \mathbb{Q}_{Y_Z} \rightarrow i_{0*} \mathbb{Q}_{A_0} \rightarrow i_{1*} \mathbb{Q}_{A_1} \rightarrow \dots$$

and this yields an E_1 spectral sequence:

$$E_1^{pq} = Gr_j^W \mathbb{H}^p(Z, w_{> \text{Id}}(R\pi_* \mathbb{Q}_{A_q})) \Rightarrow Gr_j^W \mathbb{H}^{p+q}(Z, w_{> \text{Id}}(R\pi_* \mathbb{Q}_{Y_Z}))$$

Now fix a q and consider the triangle on Z :

$$w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{A_q} \rightarrow R\pi_* \mathbb{Q}_{A_q} \rightarrow w_{> \text{Id}} R\pi_* \mathbb{Q}_{A_q} \rightarrow$$

giving rise to the long exact sequence:

$$\rightarrow Gr_j^W \mathbb{H}^p(Z, R\pi_* \mathbb{Q}_{A_q}) \rightarrow Gr_j^W \mathbb{H}^p(Z, w_{> \text{Id}}(R\pi_* \mathbb{Q}_{A_q})) \rightarrow Gr_j^W \mathbb{H}^{p+1}(Z, w_{\leq \text{Id}}(R\pi_* \mathbb{Q}_{A_q})) \rightarrow \quad (**)$$

Since A_q is smooth, the first term in $(**)$ vanishes for $j > 2p$ or $j > 2 \dim A_q = 2(\dim Y - q - 1)$ or $j < p$ or, in case X is proper for $j > p$. By Noetherian induction, the last term vanishes for

- (i.) $j < 2(p + 1 - \dim A_q) = 2(p + q + 2 - \dim Y)$
- (ii.) $j > 2p + 2$

(iii.) $j > 2 \dim A_q = 2(\dim Y - q - 1)$

(iv.) In case X is proper, can assume that the last term vanishes for $j > p$.

To see this, vanishing is clear for $j > p + 1$, by induction hypothesis. Further since $Gr_{p+1}^W \mathbb{H}^{p+1}(Z, w_{\leq \text{Id}}(R\pi_* \mathbb{Q}_{A_q}))$ maps isomorphically onto its image, the second map is 0. In particular, we can replace the last term in (**) by 0 for $j > p$.

Therefore for $0 \leq p, q$ such that $p + q = i - 1$, the middle term in (**) vanishes for

(i.) $j < 2i - 2 \dim Y$.

To see this, note that since $j < 2i - 2 \dim Y = 2(p + q + 1 - \dim Y) < 2(p + q + 2 - \dim Y)$ hence the last term vanishes. If $j < p$ we saw that the first term vanishes. If $j \geq p$, we get that $p \leq j < 2(p + q + 1 - \dim Y)$ and hence $p > 2(\dim Y - q - 1) = 2 \dim A_q$, and hence it vanishes as before.

(ii.) $j > 2i$.

Since $i \geq p + 1$, $j > 2i$ implies $j > 2p + 2 > 2p$ hence the first and the last term vanish.

(iii.) $j > 2 \dim Y$.

Since $\dim Y > \dim Y_Z \geq \dim A_q$, $j > 2 \dim Y$ imply that the first and the last term vanish as before.

(iv.) If X is proper, for $j \geq i$.

Since $j \geq i > p$, the vanishings are immediate.

Hence, using the spectral sequence above we get the required vanishing of the graded pieces $Gr_j^W \mathbb{H}^{i-1}(Z, w_{> \text{Id}}(R\pi_* \mathbb{Q}_{Y_Z}))$ as well, and we are done.

Case 2: Removing the simple normal crossing condition

In case $\pi : Y \rightarrow X$ does not satisfy that Y_Z is simple normal crossing, we can always find a $\pi' : Y' \rightarrow Y$ such that the $\pi \circ \pi'$ satisfies required properties and $\dim Y = \dim Y'$, by either using resolution of singularities or de Jong's alterations. Now note that by the decomposition theorem $Gr_j^W \mathbb{H}^i(X, w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y)$ is a summand of $Gr_j^W \mathbb{H}^i(X, w_{\leq \text{Id}} R(\pi \circ \pi')_* \mathbb{Q}_{Y'})$, and hence vanishes when the latter does. $\square$

4.3.4. Corollary. *Let X be an equidimensional scheme of dimension d_X . Then $Gr_j^W EH^i(X)$ vanishes if $j \notin [0, 2i]$ (if $i \leq d_X$) or $j \notin [2(i - d_X), 2d_X]$ (if $i \geq d_X$). If X is proper, it also vanishes for $j > i$ and if X is smooth it vanishes for $j < i$.*

Proof. For smooth X , $EH^i(X) = H^i(X)$ and hence its graded vanishes for $j < i$.

We can always find $Y \rightarrow X$ proper surjective with $\dim Y = \dim X$ and Y smooth by either resolution of singularities or de Jong's alterations. All the claims now follow from previous lemma (except for the case $j < 0$), noting that EC_X is then a summand of $w_{\leq \text{Id}} R\pi_* \mathbb{Q}_Y$ (see 4.1.3).

The case $j < 0$ is immediate from 3.2.1 and 3.1.11 noting that over $\text{Spec } k$ we have $w_{D \geq \text{Id}} = w_{D \geq 0}$. $\square$

4.4. Homology and cycles. Since EH^* is defined using an object in $D^b(X)$ it has a natural Verdier dual homology theory of Borel-Moore type defined by

$$EH_i^{BM}(X) := \mathbb{H}^{-i}(X, \mathbb{D}EC_X(-d))$$

where $d = \dim X$. There is also a homology theory defined by

$$EH_i(X) := \mathbb{H}_c^{-i}(X, \mathbb{D}EC_X(-d))$$

with a natural map $EH_*(X) \rightarrow EH_*^{BM}(X)$ which is an isomorphism if X is proper. By 2.2.13 and (7) of 4.2.1 there are natural module actions of $EH^*(X)$ on $EH_*(X)$ and $EH_*^{BM}(X)$ generalizing (and compatible with) the actions of $H^*(X)$ on $H_*(X)$ and $H_*^{BM}(X)$ by cap product.

If $f : Y \rightarrow X$ is proper with image meeting the smooth locus then applying duality to the homomorphism $EC_X \rightarrow Rf_*EC_Y$ gives a natural pushforward $f_* : EH_*^{BM}(Y) \rightarrow EH_*^{BM}(X)$. The projection formula

$$f_*(a \cdot f^*(b)) = f_*(a) \cdot b \quad (a \in EH^*(Y), b \in EH_*^{BM}(X))$$

follows formally from duality and the fact that f^* is a ring homomorphism.

The fundamental class homomorphism

$$\mathbb{Q}_X \rightarrow \mathbb{D}\mathbb{Q}_X[-2d](-d)$$

factors through a unique homomorphism

$$EC_X \rightarrow \mathbb{D}EC_X[-2d](-d).$$

This can be used to define a cycle class for certain subvarieties as follows. For an irreducible k -dimensional subvariety $i : Z \hookrightarrow X$ meeting the smooth locus of X , the composition

$$\mathbb{Q}_X \rightarrow i_*\mathbb{Q}_Z \rightarrow i_*EC_Z \rightarrow i_*\mathbb{D}EC_Z[-2k](-k) \rightarrow \mathbb{D}EC_X[-2k](-k).$$

gives a canonical class in $EH_{2k}^{BM}(X)$ lifting the usual cycle class in $H_{2k}^{BM}(X)$. For proper morphisms f with image meeting the smooth locus these cycle classes will have functoriality properties under the pushforward f_* .

4.4.1. Remark. In [3] the authors show that the cycle class of a subvariety $Z \subset X$ can be lifted to $IH^{2k}(X) = IH_{2 \dim X - 2k}^{BM}(X)$, but this lift is not canonical even if Z meets the smooth locus of X .

5. EXAMPLES

5.0.1. Example (Curves). For a curve C , the invariance of EH^* under normalization implies that the natural map $EH^*(C) \rightarrow IH^*(C)$ is an isomorphism. Thus $H^*(C) \rightarrow EH^*(C)$ is not an isomorphism if C is singular e.g. if C is the nodal cubic.

5.0.2. Example (Varieties with isolated singularities). Let X be a variety with an isolated singularity $i : \{\infty\} \hookrightarrow X$. Let $\pi : Y \rightarrow X$ denote any resolution of the singularity such that $\pi^{-1}(\infty) = \cup_i Y_i$ is a variety with simple normal crossings. Let $j : U \hookrightarrow X$ denote the smooth complement. Then we have a triangle:

$$j!\mathbb{Q}_U \rightarrow R\pi_*\mathbb{Q}_Y \rightarrow i_*R\pi_*\mathbb{Q}_{\pi^{-1}(\infty)} \rightarrow$$

Since $j!\mathbb{Q}_U \in {}^wD^{\leq \text{Id}}$ it follows that

$$i^*EC_X \cong i^*w_{\leq \text{Id}}R\pi_*\mathbb{Q}_Y \cong w_{\leq \text{Id}}R\pi_*\mathbb{Q}_{\pi^{-1}(\infty)} \cong W_0H^*(\pi^{-1}(\infty)).$$

Now since $\pi^{-1}(\infty)$ has normal crossings it follows that $W_0H^*(\pi^{-1}(\infty))$ is the cohomology of the dual simplicial complex, i.e. the nerve of the covering by closed irreducible components. Thus i^*EC_X is the cohomology of the dual simplicial complex at infinity. (In particular this cohomology independent of the choice of resolution. In fact, the simple homotopy type of the dual simplicial complex is already independent of the choice, cf. [19], [15].) Alternatively, we can also describe i^*EC_X as the weight zero part of the link, i.e. as $W_0H^*(i^*Rj_*\mathbb{Q}_U)$.

We remark that [1] shows that if X is Cohen-Macaulay the homotopy type of the dual simplicial complex of $\pi^{-1}(\infty)$ is that of a bouquet of spheres of dimension $\dim X - 1$. For other restrictions on the stalk i^*EC_X we refer to [1, 19, 15].

More generally, suppose that Z and $U = X - Z$ are smooth and $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are the inclusions. If $i^*Rj_*\mathbb{Q}_U$ is locally constant along Z (e.g. if $k = \mathbb{C}$ this is true if $X = U \sqcup Z$ is a Whitney stratification) then the complex $(i^*EC_X)_\infty$ can be identified with the weight zero part of the cohomology of the link, i.e. $(i^*EC_X)_\infty = W_0H^*((i^*Rj_*\mathbb{Q}_U)_\infty)$. There is also a statement in terms of fibres of a resolution of singularities if it is chosen to have simple normal crossings and such that each irreducible component maps smoothly onto Z .

Still more generally, if $Z \subset X$ is such that $X - Z$ is smooth then $W_0\mathbb{H}^*(Z, i^*EC_X)$ can be identified with the cohomology of the dual simplicial complex of the preimage of Z in any resolution $\pi : Y \rightarrow X$ if it is of simple normal crossing type. However, in general the hypercohomology $\mathbb{H}^*(Z, i^*EC_X)$ is not entirely of weight zero.

5.0.3. Example (Surfaces). Let X be a normal surface. If the exceptional fibre in a resolution of X has contractible dual graph then the previous example shows that $\mathbb{Q}_X = EC_X$. This is the case, for example, if X is the cone over a nonsingular curve. If the curve is of genus > 0 then $EC_X \rightarrow IC_X$ is not an isomorphism.

On the other hand, if the components of the exceptional fibre in a resolution of singularities of X are rational curves and the dual graph contains cycles then one sees that $EC_X \rightarrow IC_X$ is an isomorphism but $\mathbb{Q}_X \rightarrow EC_X$ is not. This is the case, for example, for Hilbert modular surfaces.

5.0.4. Example (Toric varieties). For a toric variety X the natural map $\mathbb{Q}_X \rightarrow EC_X$ is an isomorphism, so that $H^*(X) = EH^*(X)$.

To see this we will use induction on the dimension of $X = X(\Delta)$ where Δ is a fan inside $N_{\mathbb{R}} = \mathbb{R}^n$, the base case being easy.

By [7, 2.6] there is a toric resolution $\pi : Y \rightarrow X$ induced by another fan Δ_Y in $N_{\mathbb{R}}$. We let π also denote the morphism $\Delta_Y \rightarrow \Delta$ thought of as partitions of $N_{\mathbb{R}}$. Since this is a toric map, $\pi^{-1}\sigma = \cup_i \tau_i$ for some $\tau_i \in \Delta_Y$.

Let $j : U \hookrightarrow X$ denote the dense torus and $i : Z \hookrightarrow X$ denote the closed complement. For $Y_Z = \pi^{-1}Z$, we have a triangle:

$$j_!\mathbb{Q}_U \rightarrow R\pi_*\mathbb{Q}_Y \rightarrow i_*R\pi_*\mathbb{Q}_{Y_Z} \rightarrow$$

Since the first term lies in ${}^wD^{\leq \text{Id}}$, to show that the natural map $\mathbb{Q}_X \rightarrow w_{\leq \text{Id}}R\pi_*\mathbb{Q}_Y$ is an isomorphism, it is enough to show that the natural map $\mathbb{Q}_Z \rightarrow w_{\leq \text{Id}}R\pi_*\mathbb{Q}_{Y_Z}$ induces an isomorphism.

Let $Y_Z = \cup Y_i$ where Y_i are irreducible. Let W denote intersection of some of the Y_i 's. Then W is smooth and irreducible and its image $\pi(W)$ is toric, and therefore by the induction hypothesis, $w_{\leq \text{Id}}R\pi_*\mathbb{Q}_W \cong \mathbb{Q}_{\pi(W)}$ under the natural map.

Let $i_k : A_k \rightarrow Y_Z$ for $k \geq 0$ be the disjoint union of the k -fold intersections of components of Y_Z . Then we have a resolution:

$$0 \rightarrow \mathbb{Q}_{Y_Z} \rightarrow i_{0*}\mathbb{Q}_{A_0} \rightarrow i_{1*}\mathbb{Q}_{A_1} \rightarrow \cdots$$

Consider a torus orbit $i_\sigma : X(\sigma) \hookrightarrow X$ corresponding to some $\sigma \in \Delta$. Let $\pi^{-1}\sigma^\circ = \cup_i \tau_i^\circ$. Then $\pi^{-1}X(\sigma) = \cup_i Y(\tau_i)$. Now $i_\sigma^*\mathbb{Q}_{X(\sigma')} \neq 0$ iff $\bar{\sigma} \supset \sigma'$, and hence if $j_\sigma : U(\sigma) \hookrightarrow X(\sigma)$ be the relative interior of the orbit $X(\sigma)$ then $j_\sigma^*R\pi_*\mathbb{Q}_{Y(\tau)} \neq 0$ only if $\tau \subset \pi^{-1}\sigma$, and in this case $j_\sigma^*w_{\leq \text{Id}}R\pi_*\mathbb{Q}_{Y(\tau)} \cong j_\sigma^*\mathbb{Q}_{Y(\sigma')} \cong \mathbb{Q}_{U(\sigma)}$ for some σ' as above.

For $x \in U(\sigma)$, applying the (cohomological) functor $H^0 \circ i_x^* w_{\leq \text{Id}} R\pi_*$ to the quasi-filtered object (cf. [18, 5.2.17]) coming from the resolution of $\mathbb{Q}_{Y_Z}$ above gives a spectral sequence with

$$E_1^{p,q} = H^q(i_x^* w_{\leq \text{Id}} R\pi_* i_{p*} \mathbb{Q}_{A_p}) \Rightarrow H^{p+q}(i_x^* w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{Y_Z}).$$

The $E_1^{p,q}$ term vanishes for $q > 0$ (by the induction hypothesis) and thus the spectral sequence is identified with the complex computing the cohomology of the cone σ in terms of the subdivision $\pi^{-1}\sigma = \cup \tau_i$ by simplicial cones. (Since Y is smooth each cone of Δ_Y is simplicial.) Since σ is contractible we deduce that $H^*(i_x^* w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{Y_Z}) = \mathbb{Q}[0]$.

Thus the natural map $\mathbb{Q}_Z \rightarrow w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{Y_Z}$ is an isomorphism on stalks and hence an isomorphism, completing the argument. (Note that the argument works equally well for varieties which are locally toric.)

5.0.5. Example (Atiyah flop). The previous example also tells us that the natural map $EH^*(X) \rightarrow IH^*(X)$ may fail to be an isomorphism simply because there are toric varieties for which the natural map $H^*(X) \rightarrow IH^*(X)$ is not an isomorphism.

To consider a specific case let X be the cone over a quadric, that is X is given by the equation $xy - zw = 0$ in $\mathbb{A}^4$. If Y' is the blowup of this cone at the origin, the exceptional fiber is isomorphic to a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ in which either of the rulings can be contracted to yield two different small resolutions of X . Say $\pi : Y \rightarrow X$ is one of them. The resolution is small hence $IH^*(X) = H^*(Y)$, and the map $EH^*(X) \rightarrow IH^*(X)$ reduces to the map $H^*(X) \rightarrow H^*(Y)$. Using the localization sequence and noticing that the birational locus is an $\mathbb{A}^1$ fibration over the quadric, it is easy to see that this map is not an isomorphism.

Notice that in this case the two blow downs can be used to give different ring structures on $IH^*(X)$; the natural ring structure on $EH^*(X)$ is compatible with both.

5.0.6. Example (Isolated rational singularities). For a complex variety X with isolated rational singularities we have $EC_X = \mathbb{Q}_X$ and hence $EH^*(X) = H^*(X)$. This follows from a result of Arapura, Bakhtary, and Włodarczyk [1, Cor. 3.2] which says that if $\pi : Y \rightarrow X$ is a resolution then $W_0 H^*(\pi^{-1}(x)) = H^0(\pi^{-1}(x)) = \mathbb{Q}$ for any $x \in X$. The assertion then follows from Ex. 5.0.2.

Payne's example [15, §8] of an isolated singularity in which the dual complex of a resolution has the homotopy type of $\mathbb{R}P^2$ suggests that this may not be true integrally (although the integral version of $EH^*(X)$ is yet to be constructed).

5.0.7. Example (Locally symmetric varieties). Let M be a (noncompact) locally symmetric variety, i.e. the quotient of a Hermitian symmetric domain by a neat arithmetic group of automorphisms and let M^* be its Baily-Borel compactification, which is a normal complex variety. In this case there is a natural isomorphism $EH^*(M^*) = H^*(\overline{M}^{rbs}, \mathbb{Q})$ where $\overline{M}^{rbs}$ is the reductive Borel-Serre compactification. This follows from the identity

$$Rp_* \mathbb{Q}_{\overline{M}^{rbs}} = \text{rat}(EC_{M^*})$$

where $p : \overline{M}^{rbs} \rightarrow M^*$ extends the identity and rat is the realization functor on mixed Hodge modules. The identity is proved in [14] (or can also be deduced from [2], at least after further pushforward to a point). Let $\pi : \widetilde{M} \rightarrow M^*$ be any

resolution of singularities (for example, a toroidal compactification of M). We then have the alternate formula

$$Rp_* \mathbb{Q}_{\overline{M}^{rbs}} = rat(w_{\leq \text{Id}} R\pi_* \mathbb{Q}_{\widetilde{M}})$$

Thus the pullback $\pi^* : H^*(M^*) \rightarrow H^*(\widetilde{M})$ factors through $H^*(\overline{M}^{rbs})$. This is a result of [8] for toroidal compactifications.

The singularities of M^* are log canonical ([10, 3.7.1]), and hence du Bois. In general, $H^*(M^*) \rightarrow H^*(\overline{M}^{rbs})$ is not an isomorphism (not even if the singularities are isolated), so that $EH^*(M^*)$ differs from the ordinary cohomology of M^* in this case.

5.0.8. Example (Hilbert-Blumenthal varieties). An interesting locally symmetric example is that of a Hilbert-Blumenthal variety M and its Baily-Borel compactification M^* , which has isolated singularities at $M^* - M$. In this case one can show using the explicit computation of the local intersection cohomology that $EC_{M^*} \rightarrow IC_{M^*}$ is an isomorphism. (Since the singularity is isolated this amounts to the statement that the local cohomology of IC_{M^*} is entirely of weight zero.) It follows that $EH^*(M^*) = IH^*(M^*)$ so that $IH^*(M^*)$ is a ring.

5.0.9. Example (Picard modular varieties). If M is a Picard modular variety and M^* its Baily-Borel compactification as above, then there is a canonical smooth resolution which is minimal. However, since $\overline{M}^{rbs} = M^*$ in this case, $EC_{M^*} = \mathbb{Q}_{M^*}$ as one sees from the discussion above; so $EH^*(M^*) = H^*(M^*)$.

5.0.10. Example (Siegel modular threefolds). We give an example to show that the stalk of EC_X at $x \in X$ is not simply the weight zero piece of the fibre over x in a resolution. We use a locally symmetric example as the availability of the space $\overline{M}^{rbs}$ simplifies the discussion. The boundary of the Baily-Borel compactification of a Siegel modular threefold M is naturally stratified as the union of (open) modular curves and cusps. Let c be a cusp. In a toroidal resolution $\pi : \widetilde{M} \rightarrow M^*$ the weight zero piece $W_0 H^*(\pi^{-1}(c))$ is the cohomology of a certain (open) modular curve C . (The dual graph of $\pi^{-1}(c)$ is homeomorphic to a retract of C onto a certain compact submanifold-with-boundary.) On the other hand, the fibre of $p : \overline{M}^{rbs} \rightarrow M^*$ over c is homeomorphic to the compactified modular curve C^* . Thus $H^*(EC_{M^*})_c = H^*(C^*)$ while $W_0 H^*(\pi^{-1}(c)) = H^*(C)$.

5.0.11. Example (Locally symmetric subvarieties). Suppose that M is a locally symmetric variety and $f : M_H \rightarrow M$ is a morphism of locally symmetric varieties coming from a totally geodesic embedding of Hermitian symmetric domains. Thus $f : M_H \rightarrow M$ is finite onto a closed subvariety. Then f extends to $f : M_H^* \rightarrow M^*$ and the image of f meets the smooth locus M . In [13] we study the cycle class $cl(f(M_H^*)) \in EH_*^{BM}(M^*) = EH_*(M^*)$ and use it to prove injectivity of pullbacks $EH^*(M^*) \rightarrow EH^*(M_H^*)$. These results depend in an essential way on the de Rham model available for $EH^*(M^*) = H^*(\overline{M}^{rbs})$, which allow us to use arguments about automorphic forms.

REFERENCES

- [1] D. Arapura, P. Bakhtary, and J. Włodarczyk, Weights on cohomology, invariants of singularities, and dual complexes, arXiv:math/11024370 (2011).

- [2] J. Ayoub and S. Zucker, Relative Artin motives and the reductive Borel–Serre compactification of a locally symmetric variety, *Invent. Math.* 188 (2012), 277–427.
- [3] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber, and L. Kaup, Relèvement de cycles algébriques et homomorphismes associés en homologie d’intersection, *Ann. of Math.* 141 (1995), 147–179.
- [4] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, *Analysis and topology on singular spaces, I (Luminy, 1981)*, *Astérisque* 100, Soc. Math. France, Paris (1982), 5–171.
- [5] A. J. de Jong, Smoothness, semi-stability and alterations, *Publ. Math. de l’IHÉS* 83 (1996), 51–93.
- [6] P. Deligne, La conjecture de Weil II, *Publ. Math. de l’IHÉS* 52 (1980), 137–252.
- [7] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies 131, Princeton Univ. Press, Princeton, NJ (1993).
- [8] M. Goresky and Y.-S. Tai, Toroidal and reductive Borel–Serre compactifications of locally symmetric spaces, *Amer. J. Math.* 121 (1999), 1095–1151.
- [9] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York (1977).
- [10] J. Kollar, Singularities of pairs, *Algebraic geometry – Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc. (1997), Providence, RI, 221–287.
- [11] S. Morel, Complexes pondérés sur les compactifications de Baily–Borel: le cas des variétés de Siegel, *J. Amer. Math. Soc.* 21 (2008), 23–61.
- [12] S. Morel, Complexes mixtes sur un schéma de type fini sur $\mathbb{Q}$, preprint, available at <http://www.math.harvard.edu/~morel/> (2011).
- [13] A. N. Nair, On the cohomology of reductive Borel–Serre compactifications, preprint (2010).
- [14] A. N. Nair, Mixed structures in Shimura varieties and automorphic forms, preprint (2012).
- [15] S. Payne, Boundary complexes and weight filtrations, arXiv:math/1109.4286 (2011).
- [16] M. Saito, On the formalism of mixed sheaves, arXiv:math/0611597 (2006).
- [17] M. Saito, Introduction to mixed Hodge modules, Actes du Colloque de Théorie de Hodge (Luminy, 1987), *Astérisque* 179–180 (1989), 145–162.
- [18] M. Saito, Mixed Hodge modules, *Publ. Res. Inst. Math. Sci.* 26 (1990), 221–333.
- [19] D. A. Stepanov, A remark on the dual complex of a resolution of singularities, *Uspekhi Mat. Nauk* 61(1(367)) (2006), 185–186.
- [20] A. Thuillier, Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels, *Manuscripta Math.* 123 (2007) 381–451.
- [21] V. Vaish, Remarks on the weightless cohomology and a motive of Ayoub and Zucker, in preparation.
- [22] V. Vaish, Weight-truncated cohomology of algebraic varieties, Ph. D. thesis, TIFR (2012).

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