BACKDOORS TO SATISFIABILITY

NEW DEVELOPMENTS IN EXACT ALGORITHMS AND LOWER BOUNDS

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Outline

• Motivation

• 2 perspectives on backdoors

• Parameterized algorithms for SAT via backdoors
Satisfiability
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- **SATISFIABILITY**: Is a given propositional formula satisfiable?
Satisfiability

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- Best known algorithm for 3-SAT — $1.308^n$ (Hertli, FOCS 2011)
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- Resulting instances often have up to a million variables and several million clauses.

- Even for 300 variables, worst case bounds exceed age of the universe.
That’s all well and good in practice, but how does it work in theory?
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The instances arising in practice must have some structure!
Modern SAT solvers
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• ‘Complete’ SAT solvers are variants of the DPLL algorithm.
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\[ \text{DPLL}= \text{Davis-Putnam-Logemann-Loveland} \]
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\[
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Horn formulas: formulas with at most one positive literal in every clause, solved just by unit propagation.
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The DPLL algorithm

• Select a variable (based on some heuristic) and explore both assignments.

\[ F[x=1] \quad F[x=0] \]
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Modern Sat solvers

- Build on DPLL by better variable selection heuristics.
- Better backtracking strategies.
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• ‘Learning’ clauses.
Modern Sat solvers

- Build on DPLL by better variable selection heuristics.
- Better backtracking strategies.
- ‘Learning’ clauses.
- ‘Watching’ literals for fast unit propagations.
Variable-Variable Graph

Red nodes: Unit Propagation

Mini-SAT
variable dependencies
variable dependencies

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• Lots of real world instances seem to have a small set on which the remaining variables are dependent.

• Can we capture the structure of an instance through this small set of variables?
Backdoor sets

Introduced by
Williams, Gomes, Selman (IJCAI 2003)
and
Crama, Ekin, Hammer (D. A. M. 1997)

Informally, a set of variables whose instantiation results in a significantly simplified formula.
Subsolvers

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For every tractable base class for SAT, we have a sub-solver that solves instances in this class and rejects the rest.

- Subsolver1: Solve all 2cnf formulas and reject the rest.
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Subsolvers
Subsolvers

Base class = sub-solver
Backdoors to SAT
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- Weak Backdoor to $C$
Backdoors to SAT

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Some assignment leads to a satisfiable instance in the base class $C$
Backdoors to SAT

- Weak Backdoor to $C$
- Strong Backdoor to $C$

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All assignments lead to an instance in the base class $C$. 
Strong vs weak Backdoors
Strong vs weak Backdoors
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If the instance is **satisfiable** then every strong backdoor is also a weak backdoor!
If the instance is *satisfiable* then every strong backdoor is also a weak backdoor!

\[ \text{wbd} \leq \text{sbd} \]
2 Perspectives on backdoor sets
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- (a) (Williams, Gomes, Selman) the presence of small backdoor sets provides a good explanation for the performance of SAT solvers, the success of random restarts etc.

- (b) (Crama, Ekin, Hammer) backdoor sets provide an excellent framework to extend tractability results for SAT.

eg. SAT is in P for 2-cnf formulas—> SAT is in P for formulas with a strong backdoor of size 10 to 2-cnf.
Islands of Tractability
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- Think of the base classes as `Islands of tractability'.
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- An instance with a `small` backdoor to one of these base classes is `close` to an island of tractability.

- **Objective**: If an instance is close to an island of tractability, then we can solve it efficiently.
Research Agenda

- Instances with a backdoor of size $\log^2 n$ to an island
- Instances with a backdoor of size $\log n$ to an island
- Instances with a backdoor of size $c$ to an island
How to extend tractability results?
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Finding Backdoor sets
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- How do we detect that an instance is `close' to an island of tractability?
Finding Backdoor sets

• How do we detect that an instance is `close' to an island of tractability?

• For which islands can we do this detection efficiently (in polynomial time)?
Finding Backdoor sets
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- For any reasonable island of tractability, detecting if an instance is `close' to this island is NP-complete.
Finding Backdoor sets

• For any reasonable island of tractability, detecting if an instance is `close' to this island is NP-complete.

• For which islands can we do this detection efficiently (for a relaxed notion of efficient)?
Finding Backdoors
Finding Backdoors

- How to develop and analyze `efficient' algorithms to detect small backdoors?
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Fixed-Parameter Algorithms!
FPT algorithms
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• Running time $f(k) |x|^c$ implies that for $k$ bounded by $f^{-1}(\text{poly}(n))$, we have a poly time algorithm.
FPT algorithms

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• Corresponding hardness theory.
FPT algorithms

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• $W$-hierarchy: $\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq \text{XP}$
Rest of this talk
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- Recent advances in FPT algorithms for computing backdoors to some base classes (q-Horn, tw-SAT, composite classes)
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• Recent advances in FPT algorithms for computing backdoors to some base classes (q-Horn, tw-SAT, composite classes)

• Discuss some interesting connections between the 2 perspectives.
Classical sub-solvers
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Schaefer Classes
Classical sub-solvers

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• Horn (at most one positive lit in each clause)
Classical sub-solvers

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- Dual-Horn (at most one negative lit in each clause)
Classical sub-solvers

Schaefer Classes

- **Horn** (at most one positive lit in each clause)
- **Dual-Horn** (at most one negative lit in each clause)
- **2-cnf** (at most 2 literals in each clause)
Classical sub-solvers

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- **0/1-valid** (satisfied by the all-0/all-1 assignment)
Classical sub-solvers

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Nishimura, Ragde, Szeider SAT 2004
Classical sub-solvers

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Weak</td>
<td>W[2]-hard (FPT for 3-cnf)</td>
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<tr>
<td>Strong</td>
<td>FPT</td>
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Why is weak backdoor detection hard?
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- Allows fairly straightforward encodings from Hitting Set/Set Cover, both $W[2]$-hard parameterized by the size of the solution (the hitting set or the set cover).
Why is weak backdoor detection hard?

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- Allows fairly straightforward encodings from **Hitting Set/Set Cover**, both **W[2]-hard** parameterized by the size of the solution (the hitting set or the set cover).

- Can change if restricted to 3-cnf formulas.
Backdoors to $q$-Horn
Backdoors to q-Horn
Backdoors to q-Horn

- quadratic-Horn (q-Horn) (Boros, Crama, Hammer 1990)
Backdoors to \(q\)-Horn

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- \(F\) is \(q\)-Horn if there is a weight function
  \(w: \text{lit}(F)\to\{0,1/2,1\}\) s.t

  \[w(x)+w(\neg x)=1\text{ and for every clause }C, w(C)\leq 1.\]
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• q-Horn generalizes Horn and 2-cnf.

$w(x) = 1$ and $w(\neg x) = 0$ for all $x$ \hspace{1cm} w(x) = w(\neg x) = 1/2$ for all $x$
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- q-Horn generalizes Horn and 2-cnf. SAT is in P for q-Horn formulas!
Backdoors to q-Horn
Backdoors to $q$-Horn

$\begin{array}{c}
\text{Weak: } W[2]\text{-hard} \\
\text{Strong: } \text{FPT}
\end{array}$
Backdoors to q-Horn

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  - [Gaspers, Ordyniak, R., Saurabh, Szeider STACS 2013]

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- 2-cnf
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Backdoors to q-Horn
Backdoors to $q$-Horn

• How can we extend tractability results if strong backdoor detection is also $W$-hard?
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• What other notions of `distance` can we have?
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• What other notions of `distance' can we have?

• What about distance through deletion instead of instantiation?
Deletion Backdoors
A deletion backdoor set of $F$ to the base class $C$ is a set of variables $S$ such that $F-S$ is in $C$. 
Deletion Backdoors

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\[(x_1 \lor x_2) \land (\neg x_1 \lor x_3 \lor x_5) \land (x_2 \lor \neg x_3 \lor x_4)\]
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Deleting $x_2$
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$x_3 = 0$

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$x_3 = 0$ $(x_1 \lor x_2) \land (\neg x_1 \lor x_5)$

$(x_1 \lor x_2) \land (\neg x_1 \lor x_5) \land (x_2 \lor x_4)$

$x_3 = 1$ $(x_1 \lor x_2) \land (x_2 \lor x_4)$

Deleting $x_3$

$x_3$ is a deletion backdoor into 2-cnf.
Deletion Backdoors
Deletion Backdoors
If the base class $C$ is 'clause induced' then $S$ is also a strong backdoor to $C$. If $F$ is in $C$, every subformula of $F$ induced by a subset of clauses is in $C$. 
Deletion Backdoors

If F is in C, every subformula of F induced by a subset of clauses is in C.

If the base class $C$ is ‘clause induced’ then $S$ is also a strong backdoor to $C$.
Deletion Backdoors
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Deletion Backdoors

- The detection of strong backdoors being W-hard is not a dead end.
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If we can detect deletion backdoors then we can still extend the tractable region for SAT.
Backdoors to q-Horn
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Backdoors to $q$-Horn

- $q$-Horn is clause induced. Can we find a deletion backdoor to $q$-Horn in FPT time?
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- In $O(6^k mn)$ time, we can either conclude no deletion backdoor of size $k$ or compute a deletion backdoor of size at most $2k^2$. 
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SAT parameterized by size of deletion backdoor to q-Horn can be solved in time $2^{O(k^2)} mn$. 
Backdoors to q-Horn

SAT is in P for instances with a del. backdoor of size $O(\sqrt{\log n})$ to q-Horn.

SAT is in P for q-Horn
Backdoors to $q$-Horn

Deletion backdoor set detection to $q$-Horn can be solved in time $O(12^k m)$.

[R. and Saurabh, SODA 2014]
Backdoors to $q$-Horn

Deletion backdoor set detection to $q$-Horn can be solved in time $O(12^k m)$.

[S. and Saurabh, SODA 2014]

SAT parameterized by size of deletion backdoor to $q$-Horn can be solved in time $2^{O(k) m}$. 
Backdoors to q-Horn

SAT is in P for instances with a del. backdoor of size $O(\log n)$ to q-Horn [R. and Saurabh 2014].

SAT is in P for instances with a del. backdoor of size $O(\sqrt{\log n})$ to q-Horn [Gaspers, Ordyniak, R., Saurabh, Szeider 2014].

SAT is in P for q-Horn
Backdoors to q-Horn
Backdoors to q-Horn

- A linear time algorithm for SAT instances `close' to being q-Horn.
Backdoors to q-Horn

- A linear time algorithm for SAT instances `close’ to being q-Horn.

- **Corollary:** Deletion backdoor detection for RHorn can be done in time $O(4^k m)$. 
Backdoors to q-Horn

- A linear time algorithm for SAT instances `close' to being q-Horn.

- **Corollary:** Deletion backdoor detection for RHorn can be done in time $O(4^k m)$.

- A further consequence of this algorithm: the first linear time FPT algorithm for *Odd Cycle Transversal* (open problem of Reed, Smith and Vetta, 2003).
Backdoors to Bounded Treewidth SAT
Backdoors to acyclic SAT

Modeling CNF-formulas as graphs

Incidences Graph

○ : Clauses
●: Variables
If the Incidence graph is a forest then SAT is in P (Fischer, Makowsky, Ravve 2008).

Backdoors to acyclic SAT

Acyclic SAT

○ : Clauses
● : Variables
What about formulas with small backdoors to Acyclic SAT? Is SAT tractable on these formulas?
Backdoors to acyclic SAT

Gaspers and Szeider (ICALP 2012):

- Weak backdoor detection to acyclic SAT is \( W[2]-hard \).
- Weak backdoor detection to acyclic 3-SAT is FPT.
- Strong backdoor detection to acyclic SAT is FPT-approximable.
Backdoors to acyclic SAT

Gaspers and Szeider (ICALP 2012):

- weak backdoor detection to acyclic SAT is \( \text{W}[2]\)-hard.
- weak backdoor detection to acyclic 3-SAT is FPT.
- strong backdoor detection to acyclic SAT is FPT-approximable.

In FPT time, either conclude there is no strong backdoor of size \( k \) or compute a strong backdoor of size \( 2^k \).
Backdoors to bounded tw SAT

If the Incidence graph is tree-like then SAT is in P (Fischer, Makowsky, Ravve 2008).
Backdoors to bounded tw SAT

Gaspers and Szeider (FOCS 2013):

Strong backdoor detection to tw SAT is FPT-approximable.
Backdoors to bounded tw SAT

Gaspers and Szeider (FOCS 2013):

Strong backdoor detection to tw SAT is FPT-approximable.

SAT parameterized by size of sbd to tw SAT is FPT.

Running time: $2^{2^k} n^3$
Backdoors to bounded tw SAT

SAT is in P for instances with sbd of size $O(\log \log n)$ to tw SAT.

tw SAT is in P.
Backdoors to bounded tw SAT

Fomin, Lokshtanov, Misra, R., Saurabh (SODA 2015)
Backdoors to bounded tw SAT

Fomin, Lokshtanov, Misra, R., Saurabh (SODA 2015)

3-SAT parameterized by \( k = \min \{ \text{sbd}, \text{wbd} \} \) to tw 3-SAT can be solved in time \( 2^{O(k)} \) \( m \).

This running time is optimal both w.r.t parameter and input-size.
Backdoors to bounded tw SAT

3-SAT is in P for instances with a s/w backdoor of size $O(\log n)$ to tw 3-SAT.

tw 3-SAT is in P.

This region cannot be extended.

Some new features in this algorithm!
Combining the perspectives
Combining the perspectives

- Williams et al. proposed that SAT solvers encounter backdoor sets without actually searching for them.
Combining the perspectives

- Williams et al. proposed that SAT solvers encounter backdoor sets without actually searching for them.

- This algorithm: revisit this perspective.
Backdoors to bounded tw SAT

DPLL
Backdoors to bounded tw SAT

DPLL

- Apply UP and PLE
Backdoors to bounded tw SAT

DPLL

- Apply UP and PLE
- Select a variable \( x \) u.a.r
Backdoors to bounded tw SAT

DPLL

- Apply UP and PLE
- Select a variable \( x \) u.a.r
- Branch on \( x \)
Backdoors to bounded tw SAT

DPLL

DPLL’

• Apply UP and PLE
• Select a variable \( x \) u.a.r
• Branch on \( x \)
Backdoors to bounded tw SAT

DPLL

- Apply UP and PLE
- Select a variable $x$ u.a.r
- Branch on $x$

DPLL'

- If formula has constant tw, then solve in poly time.
Backdoors to bounded tw SAT

DPLL

• Apply UP and PLE
• Select a variable \( x \) u.a.r
• Branch on \( x \)

DPLL’

• If formula has constant tw, then solve in poly time.
• Reduce all `protrusions`.
### Backdoors to bounded tw SAT

<table>
<thead>
<tr>
<th><strong>DPLL</strong></th>
<th><strong>DPLL’</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>• Apply UP and PLE</td>
<td>• If formula has constant</td>
</tr>
<tr>
<td></td>
<td>tw, then solve in poly</td>
</tr>
<tr>
<td></td>
<td>time.</td>
</tr>
<tr>
<td>• Select a variable (x)</td>
<td>• Reduce all <code>protrusions</code>.</td>
</tr>
<tr>
<td>u.a.r</td>
<td>• Select a variable (x)</td>
</tr>
<tr>
<td></td>
<td>u.a.r.</td>
</tr>
<tr>
<td>• Branch on (x)</td>
<td></td>
</tr>
</tbody>
</table>
Backdoors to bounded tw SAT

**DPLL**
- Apply UP and PLE
- Select a variable $x$ u.a.r
- Branch on $x$

**DPLL’**
- If formula has constant tw, then solve in poly time.
- Reduce all `protrusions`.
- Select a variable $x$ u.a.r
- Branch on $x$
Backdoors to bounded tw SAT

DPLL

- Apply UP and PLE
- Select a variable \(x\) u.a.r
- Branch on \(x\)

DPLL’

- If formula has constant \(tw\), then solve in poly time.
- Reduce all `protrusions`.
- Select a variable \(x\) u.a.r
- Branch on \(x\)

DPLL’ is an FPT algorithm for 3-SAT parameterized by \(\text{min}\{\text{sbd}, \text{wbd}\}\) to tw 3-SAT.
Backdoors to bounded tw SAT
Backdoors to bounded tw SAT

- Protrusion replacement takes the place of UP and PLE.
Backdoors to bounded tw SAT

• Protrusion replacement takes the place of UP and PLE.

• Since the base class is more complex, the preprocessing is also involved.
Backdoors to bounded tw SAT

- Protrusion replacement takes the place of UP and PLE.
- Since the base class is more complex, the preprocessing is also involved.
- But intuition remains the same: Remove ‘irrelevant’ parts of the formula or at the very least replace them with a `small’ equivalent formula.
Backdoors to bounded tw SAT
Backdoors to bounded tw SAT

- First FPT algorithm for SAT which does not depend on computing a backdoor set first.
Backdoors to bounded tw SAT

- First FPT algorithm for SAT which does not depend on computing a backdoor set first.

- Optimal running time (parameter and i/p size)
Backdoors to bounded tw SAT

- First FPT algorithm for SAT which does not depend on computing a backdoor set first.

- Optimal running time (parameter and i/p size)

- Again, techniques developed here have other applications: improving several kernelization and FPT algorithms to linear time.
Composite Base Classes
Heterogenous backdoors
Heterogenous backdoors

Consider the following formula.
Heterogenous backdoors

Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \\
(x \lor \neg b_1 \lor \neg c_1) \land (x \lor \neg b_2 \lor \neg c_2) \ldots \land (x \lor \neg b_n \lor \neg c_n)\]
Heterogenous backdoors

Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n)\]

What is the size of a smallest strong backdoor set into Horn?
Heterogenous backdoors

Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \]
\[(-x \lor b_1 \lor c_1) \land (-x \lor b_2 \lor c_2) \ldots \land (-x \lor b_n \lor c_n)\]

What is the size of a smallest strong backdoor set into Horn? at least \(n\)
**Heterogenous backdoors**

Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n)\]

What is the size of a smallest strong backdoor set into Horn? \(\text{at least } n\)

What is the size of a smallest strong backdoor set into 2-cnf?
Consider the following formula.

\[
(x \lor \neg a_1 \lor \neg a_2 \lor \ldots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \land \ldots \land (\neg x \lor b_n \lor c_n)
\]

What is the size of a smallest strong backdoor set into Horn? at least \( n \)

What is the size of a smallest strong backdoor set into 2-cnf? at least \( n-1 \)
Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \]
\[\land \neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n)\]
Heterogenous backdoors

Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n)\]

Consider \(F[x=0]\) \quad (\neg a_1 \lor \neg a_2 \ldots \lor \neg a_n)
Consider the following formula.

\[ (x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \]

Consider \( F[x=0] \)  \[ (\neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \]

Consider \( F[x=1] \)  \[ (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n) \]
Consider the following formula.

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \land \ldots \land (\neg x \lor b_n \lor c_n)\]

Consider \(F[x=0]\) 

\[(-a_1 \lor -a_2 \ldots \lor -a_n)\]  Horn

Consider \(F[x=1]\) 

\[(b_1 \lor c_1) \land (b_2 \lor c_2) \land \ldots \land (b_n \lor c_n)\]
Consider the following formula.

\[
(x \lor \neg a_1 \lor \neg a_2 \lor \cdots \lor \neg a_n) \land \\
(\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \land \cdots \land (\neg x \lor b_n \lor c_n)
\]

Consider \( F[x=0] \) \( (-a_1 \lor \neg a_2 \lor \cdots \lor \neg a_n) \) - Horn

Consider \( F[x=1] \) \( (b_1 \lor c_1) \land (b_2 \lor c_2) \land \cdots \land (b_n \lor c_n) \) - 2-cnf
Heterogenous backdoors

$C_1, C_2$: Horn, 2-cnf

\[
F = \begin{cases} 
F[x=0] & \text{Run sub-solver } C_1 \\
F[x=1] & \text{Run sub-solver } C_2
\end{cases}
\]
Heterogenous backdoors

Let $C_1,..,C_r$ be islands of tractability.

$X$ is a **heterogenous backdoor** into $C_1,..,C_r$ if for every assignment of $X$, the reduced formula is in some $C_i$. 
Heterogenous backdoors
Heterogenous backdoors

- Heterogenous backdoors can be arbitrarily smaller than normal strong backdoors.
Heterogenous backdoors

- Heterogenous backdoors can be arbitrarily smaller than normal strong backdoors.

- Class of instances with small heterogenous backdoors is a much larger class than instances with small strong backdoor.
Islands of Tractability

Heterogenous backdoors

Heterogenous backdoor of size $k$ to some Islands

Strong backdoor of size $k$ to some Island of Tractability

Islands of Tractability
Heterogenous backdoors
Heterogenous backdoors

Gaspers, Ordyniak, Misra, Szeider, Zivny (AAAI 2014)
Heterogenous backdoors

Gaspers, Ordyniak, Misra, Szeider, Zivny (AAAI 2014)

1. If $H = \text{Horn/dual-Horn} \cup \text{2CNF}$ then detecting heterogenous backdoors to $H$ is FPT

2. For every other combination of Schaefer classes, detecting heterogenous backdoors to $H$ is W[2]-hard.

but FPT for 3-cnf formulas
Archipelagos of tractability
Archipelagos of tractability

\((x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \ldots \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\)
Archipelagos of tractability

\[(x \lor -a_1 \lor -a_2 \ldots \lor -a_n) \land (-x \lor -p_1 \lor -p_2 \ldots \lor -p_n) \land (-x \lor b_1 \lor c_1) \land (-x \lor b_2 \lor c_2) \ldots \land (-x \lor b_n \lor c_n) \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

What is the size of a smallest heterogenous backdoor set into Horn \(\lor\) 2-cnfs?
Archipelagos of tractability

$$(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (-x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n)$$

$$\land$$

$$(-x \lor b_1 \lor c_1) \land (-x \lor b_2 \lor c_2) \ldots \land (-x \lor b_n \lor c_n)$$

$$\land$$

$$(x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)$$

What is the size of a smallest heterogenous backdoor set into Horn $\cup$ 2-cnf? at least 2n
Archipelagos of tractability
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \ldots \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (-x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n)\]
\[\land\]
\[(-x \lor b_1 \lor c_1) \land (-x \lor b_2 \lor c_2) \ldots \land (-x \lor b_n \lor c_n)\]
\[\land\]
\[(x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider $F[x=0]$

\[(-a_1 \lor -a_2 \ldots \lor -a_n) \land\]
\[(q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\]
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n)\]
\[\land (\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n)\]
\[\land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider F[x=0]

\[(-a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\]

Consider F[x=1]

\[(-p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)\]
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider \( F[x=0] \)

\[-(a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n) \]

Consider \( F[x=1] \)

\[-(p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n) \]
Consider $F[x=0]$

$\neg a_1 \lor \neg a_2 \ldots \lor \neg a_n \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)$

Consider $F[x=1]$

$\neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)$

**Archipelagos of tractability**

\[\begin{align*}
(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) & \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \\
& \land \\
(\neg x \lor b_1 \lor c_1) & \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \\
& \land \\
(x \lor q_1 \lor r_1) & \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)
\end{align*}\]
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider \(F[x=0]\)

\[(-a_1 \lor -a_2 \ldots \lor -a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\]

Consider \(F[x=1]\)

\[(-p_1 \lor -p_2 \ldots \lor -p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)\]
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (\neg x \lor b_1 \lor c_1) \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]
Archipelagos of tractability

\[
\begin{align*}
(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) & \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \\
& \land \\
(\neg x \lor b_1 \lor c_1) & \land (\neg x \lor b_2 \lor c_2) \ldots \land (\neg x \lor b_n \lor c_n) \\
& \land \\
(x \lor q_1 \lor r_1) & \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)
\end{align*}
\]

Consider \( F[x=0] \)

\[
\begin{align*}
(\neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) & \land \\
(q_1 \lor r_1) & \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)
\end{align*}
\]

Consider \( F[x=1] \)

\[
\begin{align*}
(\neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) & \land \\
(b_1 \lor c_1) & \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)
\end{align*}
\]
Archipelagos of tractability

\((x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)

Consider F[x=0]

\((x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)

Consider F[x=1]

\((\neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)

Horn
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n)\]
\[\land \]
\[(-x \lor b_1 \lor c_1) \land (-x \lor b_2 \lor c_2) \ldots \land (-x \lor b_n \lor c_n)\]
\[\land \]
\[(x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider $F[x=0]$

\[(-a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land \]
\[\land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\]

Consider $F[x=1]$

\[(-p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \]
\[\land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)\]
Archipelagos of tractability

\((x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \ldots \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\)

Consider F\([x=0]\)

\((-a_1 \lor -a_2 \ldots \lor -a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\)

Consider F\([x=1]\)

\((-p_1 \lor -p_2 \ldots \lor -p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)\)
Archipelagos of tractability

\[(x \lor \neg a_1 \lor \neg a_2 \ldots \lor \neg a_n) \land (\neg x \lor \neg p_1 \lor \neg p_2 \ldots \lor \neg p_n) \land \ldots \land (x \lor q_1 \lor r_1) \land (x \lor q_2 \lor r_2) \ldots \land (x \lor q_n \lor r_n)\]

Consider \(F[x=0]\)

\[(-a_1 \lor -a_2 \ldots \lor -a_n) \land (q_1 \lor r_1) \land (q_2 \lor r_2) \ldots \land (q_n \lor r_n)\]

Consider \(F[x=1]\)

\[(-p_1 \lor -p_2 \ldots \lor -p_n) \land (b_1 \lor c_1) \land (b_2 \lor c_2) \ldots \land (b_n \lor c_n)\]
Archipelagos of tractability

\[ F[x=0] \quad F[x=1] \]

Run appropriate sub-solver \( C_i \) on each part variable-disjoint from the rest

\[ C_1, C_2: \text{ Horn, 2-cnf} \]
Split backdoors

Let $C_1, \ldots, C_r$ be islands of tractability.

$X$ is a split backdoor into $C_1, \ldots, C_r$ if for every assignment of $X$, every connected component of the reduced formula is in some $C_i$. 
Split backdoors

Let $C_1, \ldots, C_r$ be islands of tractability.

X is a split backdoor into $C_1, \ldots, C_r$ if for every assignment of $X$, every connected component of the reduced formula is in some $C_i$.

A minimal set of clauses which is variable-disjoint from the remaining clauses.
Split backdoors
Split backdoors

- Split backdoors can be arbitrarily smaller than heterogenous backdoors.
Split backdoors

- Split backdoors can be arbitrarily smaller than heterogenous backdoors.

- Class of instances with small split backdoors is a much larger class than class of instances with small heterogenous backdoor.
Split backdoors

- Split backdoor of size $k$ to some Islands
- Heterogeneous backdoor of size $k$ to some Islands
- Strong backdoor of size $k$ to some Island of Tractability
- Islands of Tractability
If $H$ is a finite set of finite constraint languages, then detecting split-backdoors of the given CSP to $H$ is FPT.

Ganian, R., Szeider (2014):

Builds on a combination of traditional FPT tools and new graph separation tools like important separators, sequences and CSP based pattern replacements.
Summing up
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- We have seen how backdoors and fixed parameter tractability provide a framework to extend tractability results for SAT based on the `distance' of instances to islands of tractability.
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• Stronger definitions of sub-solvers and base classes allowing us to prove tractability for larger classes of instances.
Summing up

- We have seen how backdoors and fixed parameter tractability provide a framework to extend tractability results for SAT based on the `distance' of instances to islands of tractability.

- Stronger definitions of sub-solvers and base classes allowing us to prove tractability for larger classes of instances.

- Several other variants of backdoors have been proposed, eg. backdoor trees (Samer and Szeider AAAI 2008), learning sensitive backdoors (Dilkina, Gomes, Sabharwal SAT 2009).
Future research
Future research

• So far backdoor sets and variants have provided the best and theoretically most robust explanation for the performances of SAT solvers.
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- What other structural properties of instances are correlated to the computation time and can be effectively formalized in theory?
Future research
Future research

- So far, 'small' backdoors treated as certificates for closeness.

- Better measures than size?

- i.e. backdoors of potentially unbounded size but with some structure.
Future research
Future research

- Analysis of existing SAT algorithms in terms of FPT parameterized by backdoors.
Thank you for your attention!
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