Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth

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Joint work with:
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**Graph Isomorphism**

**Input:** Graphs $G_1$ and $G_2$

**Question:** Is there a bijection $\phi: V(G_1) \rightarrow V(G_2)$ s.t. $uv \in E(G_1)$ iff $\phi(u)\phi(v) \in E(G_2)$?
Graph Isomorphism

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- Some believe it’s an intermediate problem.
- Many polynomial-time algorithms for special graph classes:
  - trees [Kelly’57]
  - planar graphs [Hopcroft-Wong’74]
  - interval graphs [Booth-Lueker’79]
  - permutation graphs [Colbourn’81]
  - bounded genus graphs [Miller’80], [Filotti-Mayer’80]
  - bounded degree graphs [Luks’82]
  - graphs with bounded eigenvalue multiplicity [Babai-Grigoryev-Mount’82]
  - bounded treewidth graphs [Bodlaender’90]
  - graphs excluding a fixed minor [Ponomarenko’91]
  - graphs excluding a fixed topological minor [Grohe-Marx’12]
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  - graphs excluding a fixed topological minor [Grohe-Marx’12]
- In almost all the relevant cases above, these are **XP** algorithms.
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Run a bottom-up dynamic programming with the following table:

For $u \in V(G_1)$ and $v \in V(G_2)$, $T[u,v]$ is the answer to the question:

*Is the subtree rooted at $u$ isomorphic to the subtree rooted at $v$?*
Graph Isomorphism of trees

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- Fill $T[\cdot, \cdot]$ in a bottom-up manner. $T[r_1, r_2]$ is the answer.

Computation at one step boils down to a matching problem. Hard exercise: Do it in linear time.
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- Computation at one step boils down to a matching problem.
- **Hard exercise**: Do it in linear time.
Theorem (Hopcroft-Tarjan’73)

Given a graph $G$, one can in linear time compute its decomposition into 3-connected components. Moreover, the decomposition is isomorphism-invariant.

Theorem (Whitney)

A 3-connected planar graph has unique planar embedding.

It is easy to compare embedded graphs.
Bounded treewidth graphs

Tree

Treelike
Bounded treewidth graphs

Tree

Treelike
Definition

A tree decomposition of a graph $G$ is a pair $(T, \beta)$ where $T$ is a tree and $\beta : V(T) \rightarrow 2^{V(G)}$ satisfying:

1. $\{ t : v \in \beta(t) \}$ is nonempty and connected for every $v \in V(G)$;
2. for every $uv \in E(G)$ there exists $t \in V(T)$ such that $u, v \in \beta(t)$.

Width of the decomposition is $\max_{t \in V(T)} |\beta(t)| - 1$.

Treewidth of $G$ is minimum possible width of tree decomposition of $G$. 
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**Theorem (Bodlaender’90)**

Isomorphism of two $n$-vertex graphs of treewidth at most $k$ can be tested in time $n^{O(k)}$.
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Theorem (Our result)

There is an algorithm, that given a graph $G$ and integer $k$, runs in $2^{O(k^5 \log k)} \cdot n^5$ and either concludes that $\text{tw}(G) > k$, or labels the vertices with numbers $1, 2, \ldots, n$ such that two isomorphic graphs receive labelings certifying the isomorphism.
Comparing tree decompositions

- In the planar case, it was easy to compare two graphs decomposed into 3-connected components.
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Similarly, it is not hard to compare pairs \((G_1, (T_1, \beta_1))\) and \((G_2, (T_2, \beta_3))\).
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- Guess matching roots of tree decompositions.
- Do bottom-up dynamic programming, computing isomorphic subtrees with labeled vertices in the top bags.
- I.e., for every \(v_1 \in V(T_1), v_2 \in V(T_2)\), and every bijection \(\pi : \beta_1(v_1) \to \beta_2(v_2)\), compute if \(G_i\) restricted to the subtree rooted in \(v_i\) are isomorphic consistently with \(\pi\).
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Thus, in some sense, we look for an isomorphic-invariant way to compute a (near-)optimal tree decomposition.
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- Thus, in some sense, we look for an isomorphic-invariant way to compute a (near-)optimal tree decomposition.
  - We can have some preliminary guessing, like guess one matched pairs of vertices etc.
  - More formally, we can generate \(f(k)n^{O(1)}\) candidate decompositions, and compare every pair.
Recall: in the planar case, we could assume the graph is 3-connected.
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Assumption 1: no clique separators.
- A decomposition by clique separators with unique set of bags. [Tarjan’85]
- In particular, 2-connected.
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  - A decomposition by clique separators with unique set of bags.
    - [Tarjan’85]
    - In particular, 2-connected.

Assumption 2: \( \forall uv \not\in E(G), \) there is a \( u-v \) vertex cut of size \( \leq k \).
  - If not true for some \( uv \), add edge \( uv \). (So-called improved graph.)
  - Isomorphism-invariant operation if done at once for all such \( uv \).
  - Maintains assumption \( tw(G) \leq k \).
Task summary

Assumptions:

- 2-connected graph $G$, no clique separators.
- For every $uv \notin E(G)$, there is a $u$-$v$ vertex cut of size at most $k$.

Task:
Compute isomorphism-invariant tree decomposition of $G$ of width $\sim k$. (Possibly after some small preliminary guessing.)
Task in the recursion:
- given a graph $G$ and a set $S \subseteq V(G)$, $|S| \leq 10k$,
- compute a tree decomposition of $G$ of width $O(k)$ with $S$ in the top bag.
Robertson-Seymour approximation

Task in the recursion:
- given a graph $G$ and a set $S \subseteq V(G)$, $|S| \leq 10k$,
- compute a tree decomposition of $G$ of width $O(k)$ with $S$ in the top bag.

Step 1: If $S = V(G)$, return single bag $S$.
Step 2: If $|S| < 10k$, then add an arbitrary vertex to $S$ and recurse.
Robertson-Seymour approximation

optimum decomposition of $G$:

**Step 3**: Assume then $|S| = 10k$ and $\text{tw}(G) \leq k$. 
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Lemma

There exists $Y \subseteq V(G)$, $|Y| \leq k + 1$, such that every connected component of $G - Y$ contains at most $|S|/2$ vertices of $S$. 
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Lemma

There exists $Y \subseteq V(G)$, $|Y| \leq k + 1$, such that every connected component of $G - Y$ contains at most $|S|/2$ vertices of $S$.

- There is a partition $S = S_1 \uplus S_2$ with $|S_1|, |S_2| \leq 2|S|/3$ s.t. the minimum $S_1$-$S_2$ cut has size at most $k + 1$.
- Iterate through all such partitions and let $X$ be the found mincut.
- Pick $X \uplus S$ as the root bag.
- Recurse on every connected component $C$ of $G - (S \uplus X)$ with graph $G[N[C]]$ and $S := N(C)$. 
Lemma

There exists a partition $S = S_1 \cup S_2$ with $|S_1|, |S_2| \leq 2|S|/3$ such that there is $X \subseteq V(G)$ with $|X| \leq k + 1$ that separates $S_1$ from $S_2$. 
**Lemma**

There exists a partition \( S = S_1 \uplus S_2 \) with \( |S_1|, |S_2| \leq 2|S|/3 \) such that there is \( X \subseteq V(G) \) with \( |X| \leq k + 1 \) that separates \( S_1 \) from \( S_2 \).

- Find \( X \) by checking mincut for every balanced partition \( S = S_1 \uplus S_2 \).
- Pick \( X \cup S \) as a root bag.
  - Size \( \leq 10k + k + 1 \).
- Recurse on every connected component \( C \) of \( G - (S \cup X) \) with graph \( G[N[C]] \) and \( S \coloneqq N(C) \).
  - \( |N(C)| \leq 2|S|/3 + |X| < 10k \).
Lemma

There exists a partition $S = S_1 \cup S_2$ with $|S_1|, |S_2| \leq 2|S|/3$ such that there is $X \subseteq V(G)$ with $|X| \leq k + 1$ that separates $S_1$ from $S_2$.

- Find $X$ by checking mincut for every balanced partition $S = S_1 \cup S_2$.
- Pick $X \cup S$ as a root bag.
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- Recurse on every connected component $C$ of $G - (S \cup X)$ with graph $G[N[C]]$ and $S := N(C)$.
  - $|N(C)| \leq 2|S|/3 + |X| < 10k$. 
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  - Size $\leq 10k + k + 1$.
- Recurse on every connected component $C$ of $G - (S \cup X)$ with graph $G[N[C]]$ and $S := N(C)$.
  - $|N(C)| \leq 2|S|/3 + |X| < 10k$. 
Two arbitrary decisions:

- **Step 2:** If $|S| < 10k$, then add an arbitrary vertex to $S$ and recurse.
  - Which vertex to choose?
- **Step 3:** Pick any separator $X$ that splits $S$ well.
  - Which separator to choose?
Let $N(A)$ and $N(B)$ be two minimum $uv$ separators.
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Then \( N(A \cap B) \) and \( N(A \cup B) \) are also minimum \( uv \) separators.
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Then $N(A \cap B)$ and $N(A \cup B)$ are also minimum $uv$ separators.

Therefore there is a notion of minimum $uv$ separator closest to $u$ and closest to $v$.

Unique minimum separators that leaves inclusion-wise minimal and maximal set of vertices reachable from $u$. 
• **Step 2**: If $|S| < 10k$, then add an arbitrary vertex to $S$ and recurse.
Solution to Step 2

- **Step 2**: If $|S| < 10k$, then add an arbitrary vertex to $S$ and recurse.
- Suppose $|S| < 10k$. For every $u, v \in S$, $uv \notin E(G)$, $u \neq v$, let $X_{u,v}$ be the minimum $uv$ separator closest to $u$. Pick root bag

$$B := S \cup \bigcup_{u,v \in S, uv \notin E(G), u \neq v} X_{u,v}.$$
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- Again, recurse on \((G[N[C]], N(C))\) for \(C\) being connected components of \(G - B\).
  - Definition is isomorphism invariant, and \(|B| = \mathcal{O}(k|S|^2)\).
  - \(N(C)\) can be as big as \(\mathcal{O}(k|S|^2)\).
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  - Definition is isomorphism invariant, and \(|B| = O(k|S|^2)\).
  - \(N(C)\) can be as big as \(O(k|S|^2)\).
- **Issue:** do we always make progress?
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- We will always recurse on instances of the form \((G[N[C]], S := N(C))\) for some connected \(C\).
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- We will always recurse on instances of the form 
  \((G[N[C]], S := N(C))\) for some connected \(C\).

- Hence \(S = N(C)\) is always a separator, and thus never a clique.
  - We need to hack it at the beginning of the recursion, but we can use preliminary guessing for that, e.g., guess a mapping on one non-edge.
Step 3: We have $|S| \geq 10k$ and $\text{tw}(G) \leq k$.

**Lemma**

*There exists a partition $S = S_1 \cup S_2$ with $|S_1|, |S_2| \leq 2|S|/3$ such that there is $X \subseteq V(G)$ with $|X| \leq k + 1$ that separates $S_1$ from $S_2$.***
For every $P, Q \subseteq S$, $P \cap Q = \emptyset$, $|P| = |Q| = k + 2$, if there exists a $PQ$ separator of size at most $k + 1$, let $X_{P,Q}$ be the minimum one closest to $P$. 

Pick root bag $B := S \cup \bigcup P, Q$ as above $X_{P,Q}$.

Recurse as previously on all $\left(G \setminus N[C], N(C)\right)$ for $C$ being connected components of $G \setminus B$. 

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Graph Isomorphism parameterized by treewidth
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\[ B := S \cup \bigcup_{P,Q \text{ as before}} X_{P,Q}. \]

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- We have a bound \( |B| = \mathcal{O}(k|S|^{2k+4}). \)
- But the main question is: how big can be \( N(C) \) for \( C \) being a connected component of \( G - B \).
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- We have a bound \( |B| = O(k|S|^{2k+4}). \)
- But the main question is: how big can be \( N(C) \) for \( C \) being a connected component of \( G - B \).

**Lemma (The crux)**

*For every connected component \( C \) of \( G - B \) we have \( |N(C)| \leq |S| \).*
Solution to Step 3

Lemma (The crux)

For every connected component $C$ of $G - B$ we have $|N(C)| \leq |S|$.

- We analyze adding sets $X_{P,Q}$ to $B$ one-by-one, and analyze sizes of $N(C)$ for intermediate connected components of $G - B$. 
Lemma (The crux)

For every connected component $C$ of $G - B$ we have $|N(C)| \leq |S|$.

- We analyze adding sets $X_{P,Q}$ to $B$ one-by-one, and analyze sizes of $N(C)$ for intermediate connected components of $G - B$.
- Initially, every component of $G - S$ has neighborhood contained in $S$. 
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- Bag of size $\mathcal{O}(k^3)$.
- Blow up to $|S| = \mathcal{O}(k^3)$ in the subcalls.
Summary

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- Bag of size $\mathcal{O}(k^3)$.
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**Step 3:** If $|S| \geq 10k$, add a minimum cut of size $\leq k + 1$ for every pair $(P, Q)$ of sets of size $k + 2$.
- Bag of size $\mathcal{O}(k|S|^{2k+4})$.
- The crux: does not blow up $|S|$ in the subcalls.
- Thus, $|S| = \mathcal{O}(k^3)$ all the time and bags size is bounded by $2^{\mathcal{O}(k \log k)}$.
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**Theorem**

*For a graph $G$ of treewidth $\leq k$, we can obtain an isomorphism-invariant family of at most $n^2$ tree decompositions with bags of size $2^{\mathcal{O}(k \log k)}$ and adhesions of size $\mathcal{O}(k^3)$.***
Theorem

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Now: a quick sketch how to reduce dependency on $k$ to $2^{k^{O(1)}}$. 

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Graph Isomorphism parameterized by treewidth  
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Formally, we require to capture at least one full decomposition in the family of bags.

Later, we can use DP on tuples (bag $B$, a connected component of $G - B$, labeling of $B$) and compare them.

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Getting single-exponential running time

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- Every our bag $B$ can be further decomposed with width $\leq k$.
- We output every subset of size $O(k^4)$ of every bag in our decompositions, and this is guaranteed to capture some decomposition of width $O(k^4)$. 
For a graph $G$ of treewidth $\leq k$, we can output an isomorphism-invariant family $B$ of size $2^{O(k^5 \log k)} \cdot n^2$, where every element of $B$ is a subset of $V(G)$ of size $O(k^4)$ and $B$ contains all bags of some tree decomposition of $G$. 

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Isomorphism of two $n$-vertex graphs of treewidth at most $k$ can be tested in time $2^{O(k^5 \log k)} \cdot n^5$. 
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There is an algorithm, that given a graph $G$ and integer $k$, runs in $2^{O(k^5 \log k)} \cdot n^5$ and either concludes that $\text{tw}(G) > k$, or labels the vertices with numbers $1, 2, \ldots, n$ such that two isomorphic graphs receive labelings certifying the isomorphism.
Conclusions

**Theorem**

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*Isomorphism of two n-vertex graphs of treewidth at most k can be tested in time $2^{\mathcal{O}(k^5 \log k)} \cdot n^5$.*

**Open problems:**

- What about FPT algorithm for graph isomorphism parameterized by the maximum degree?
  - Luks’ algorithm has running time $\mathcal{O}(n^{f(\Delta)})$.
- What about FPT algorithm for graph isomorphism parameterized by the size of an excluded minor?
  - Ponomarenko’s algorithm has running time $\mathcal{O}(n^{f(|H|)})$. 