Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth

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Source

- Joint work with:
  - Daniel Lokshtanov,
  - Marcin Pilipczuk, and
  - Saket Saurabh.
- Presented at FOCS 2014.
- Check out arxiv.org/abs/1404.0818

Graph Isomorphism

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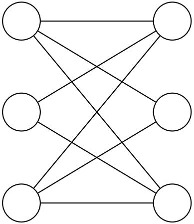

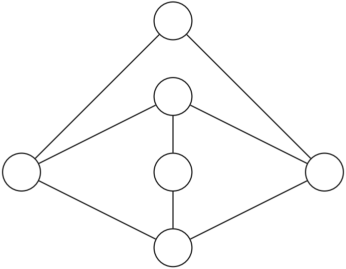

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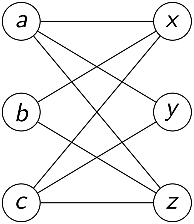

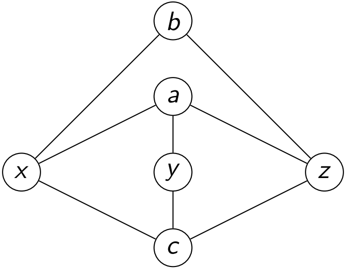

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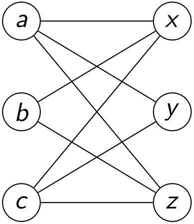

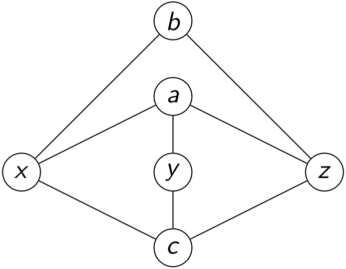

Graph Isomorphism

Input: Graphs G_1 and G_2
Question: Is there a bijection $\phi \colon V(G_1) \to V(G_2)$ s.t. $uv \in E(G_1)$ iff $\phi(u)\phi(v) \in E(G_2)$?

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  - trees [Kelly'57]
  - planar graphs [Hopcroft-Wong'74]
  - interval graphs [Booth-Lueker'79]
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  - bounded genus graphs [Miller'80], [Filotti-Mayer'80]
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  - bounded treewidth graphs [Bodlaender'90]
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- In almost all the relevant cases above, these are **XP** algorithms.

- Root G_1 in r_1 and guess its image r_2.

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- Computation at one step boils down to a matching problem.
- **Hard exercise**: Do it in linear time.

Planar Graph Isomorphism

Theorem (Hopcroft-Tarjan'73)

Given a graph G, one can in linear time compute its decomposition into 3-connected components. Moreover, the decomposition is isomorphism-invariant.

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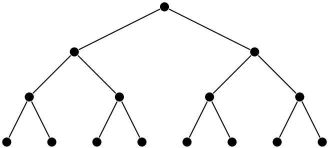
Theorem (Whitney)

A 3-connected planar graph has unique planar embedding.

It is easy to compare *embedded* graphs.

Bounded treewidth graphs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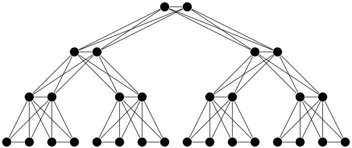
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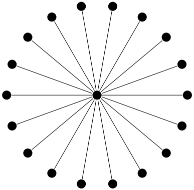

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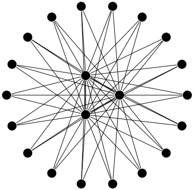

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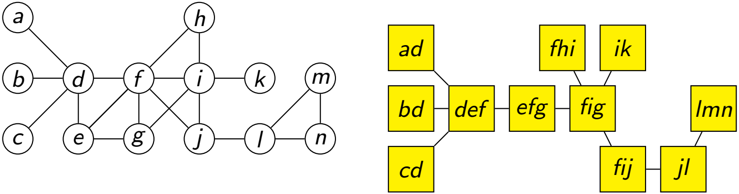

Definition

A *tree decomposition* of a graph G is a pair (T, β) where T is a tree and $\beta : V(T) \to 2^{V(G)}$ satisfying:

1. $\{t : v \in \beta(t)\}$ is nonempty and connected for every $v \in V(G)$;
2. for every $uv \in E(G)$ there exists $t \in V(T)$ such that $u, v \in \beta(t)$.

Width of the decomposition is $\max_{t \in V(T)} |\beta(t)| - 1$.
Treewidth of G is minimum possible width of tree decomposition of G.

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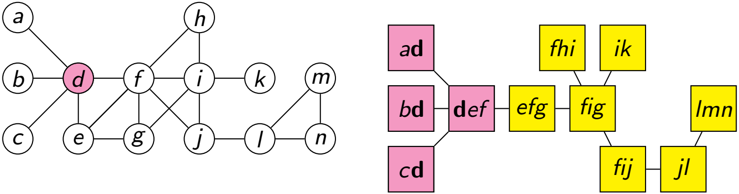

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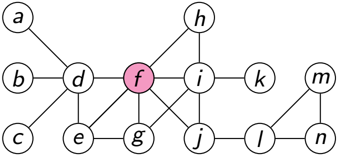

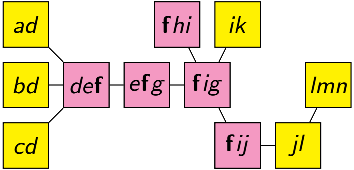

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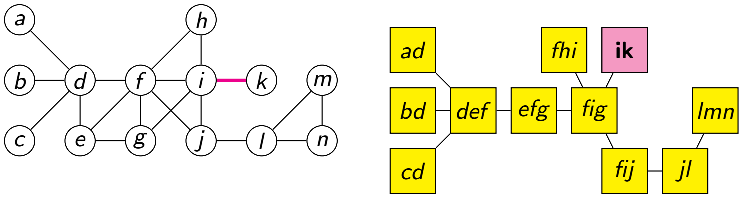

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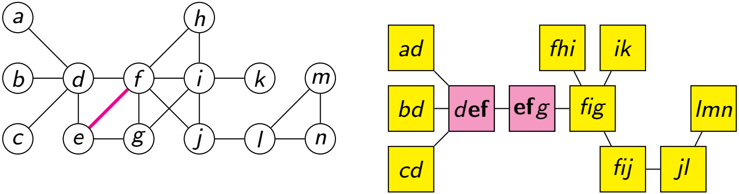

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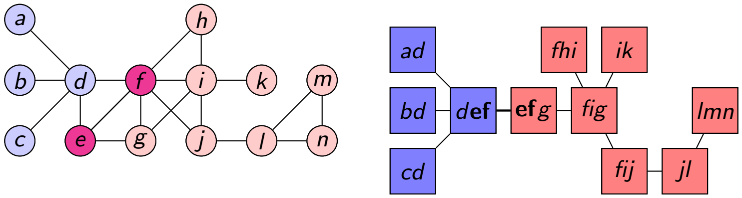

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There is an algorithm, that given a graph G and integer k, runs in $2^{\mathcal{O}(k^5 \log k)} \cdot n^5$ and either concludes that $\mathbf{tw}(G) > k$, or labels the vertices with numbers $1, 2, \ldots, n$ such that two isomorphic graphs receive labelings certifying the isomorphism.

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  - Guess matching roots of tree decompositions.
  - Do bottom-up dynamic programming, computing isomorphic subtrees with labeled vertices in the top bags.
  - I.e., for every $v_1 \in V(T_1)$, $v_2 \in V(T_2)$, and every bijection $\pi : \beta_1(v_1) \to \beta_2(v_2)$, compute if G_i restricted to the subtree rooted in v_i are isomorphic consistently with π.

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- Thus, in some sense, we look for an isomorphic-invariant way to compute a (near-)optimal tree decomposition.
  - We can have some preliminary guessing, like guess one matched pairs of vertices etc.
  - More formally, we can generate $f(k)n^{\mathcal{O}(1)}$ candidate decompositions, and compare every pair.

Simplifications

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  - Due to unique decomposition into 3-connected components.
- **Assumption 1**: no clique separators.
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  - In particular, 2-connected.
- **Assumption 2**: $\forall uv \notin E(G)$, there is a u-v vertex cut of size $\leq k$.
  - If not true for some uv, add edge uv. (So-called *improved graph*.)
  - Isomorphism-invariant operation if done at once for all such uv.
  - Maintains assumption $\mathbf{tw}(G) \leq k$.

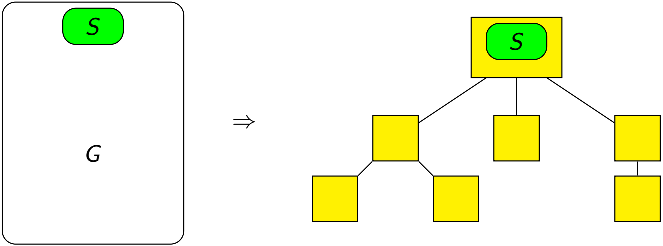
Task summary

Assumptions:

- 2-connected graph G, no clique separators.
- For every $uv \notin E(G)$, there is a u-v vertex cut of size at most k.

Task:
Compute isomorphism-invariant tree decomposition of G of width $\sim k$.
(Possibly after some small preliminary guessing.)

Robertson-Seymour approximation



Task in the recursion:

- given a graph G and a set $S \subseteq V(G)$, $|S| \leq 10k$,
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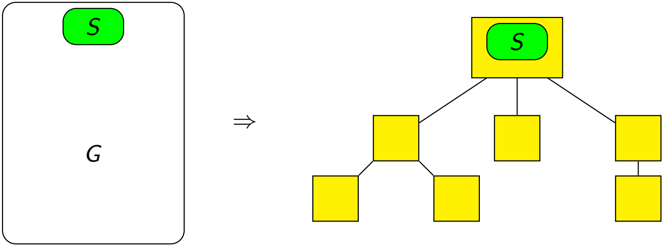

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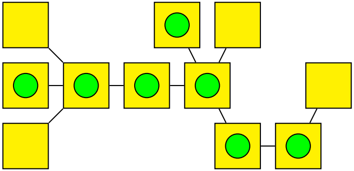
Step 1: If $S = V(G)$, return single bag S.
Step 2: If $|S| < 10k$, then add an arbitrary vertex to S and recurse.

Robertson-Seymour approximation

optimum decomposition of G:



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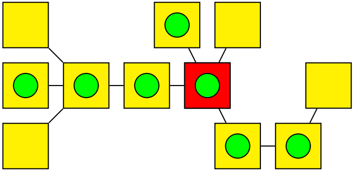

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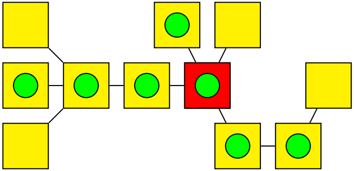

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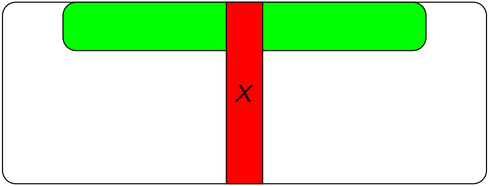
- There is a partition $S = S_1 \uplus S_2$ with $|S_1|, |S_2| \le 2|S|/3$ s.t. the minimum S_1-S_2 cut has size at most $k+1$.
- Iterate through all such partitions and let X be the found mincut.
- Pick $X \cup S$ as the root bag.
- Recurse on every connected component C of $G - (S \cup X)$ with graph $G[N[C]]$ and $S := N(C)$.



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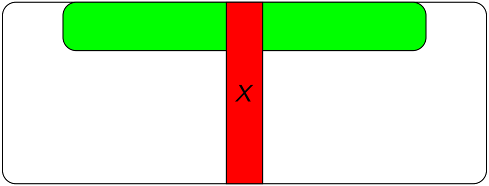

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- Find X by checking mincut for every balanced partition $S = S_1 \uplus S_2$.
- Pick $X \cup S$ as a root bag.
  - Size $\leq 10k + k + 1$.
- Recurse on every connected component C of $G - (S \cup X)$ with graph $G[N[C]]$ and $S := N(C)$.
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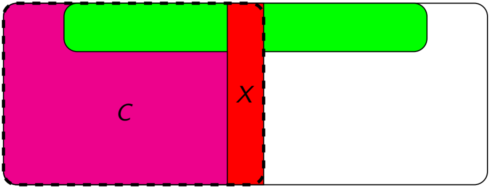

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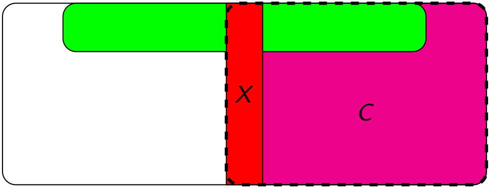

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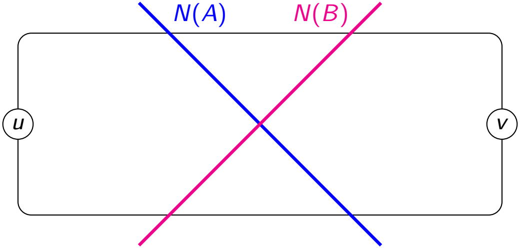
  - $|N(C)| \leq 2|S|/3 + |X| < 10k$.

Robertson-Seymour approximation

Two arbitrary decisions:

- **Step 2**: If $|S| < 10k$, then add an arbitrary vertex to S and recurse.
  - Which vertex to choose?
- **Step 3**: Pick any separator X that splits S well.
  - Which separator to choose?

Submodularity of cuts



- Let $N(A)$ and $N(B)$ be two minimum uv separators.

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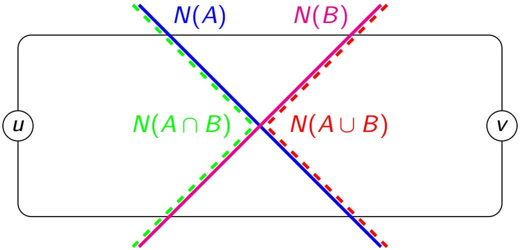

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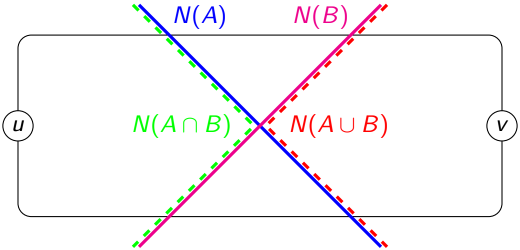
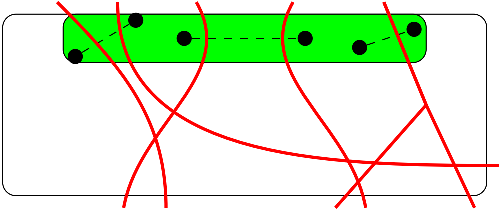


- Let $N(A)$ and $N(B)$ be two minimum uv separators.
- Then $N(A \cap B)$ and $N(A \cup B)$ are also minimum uv separators.
- Therefore there is a notion of minimum uv separator *closest to u* and *closest to v*.
  - Unique minimum separators that leaves inclusion-wise minimal and maximal set of vertices reachable from u.

Solution to Step 2

- **Step 2**: If $|S| < 10k$, then add an arbitrary vertex to S and recurse.

Solution to Step 2



- **Step 2**: If $|S| < 10k$, then add an arbitrary vertex to S and recurse.
- Suppose $|S| < 10k$. For every $u, v \in S$, $uv \notin E(G)$, $u \neq v$, let $X_{u,v}$ be the minimum uv separator closest to u.
  Pick root bag

$$B := S \cup \bigcup_{u,v \in S, uv \notin E(G), u \neq v} X_{u,v}.$$

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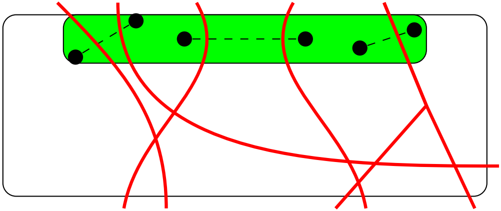

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- Again, recurse on $(G[N[C]], N(C))$ for C being connected components of $G - B$.
  - Definition is isomorphism invariant, and $|B| = \mathcal{O}(k|S|^2)$.
  - $N(C)$ can be as big as $\mathcal{O}(k|S|^2)$.

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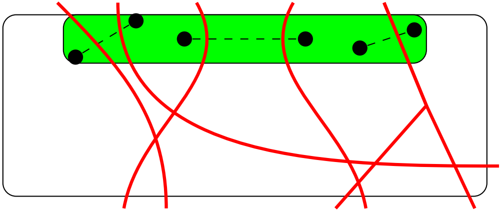

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- Issue: do we always make progress?

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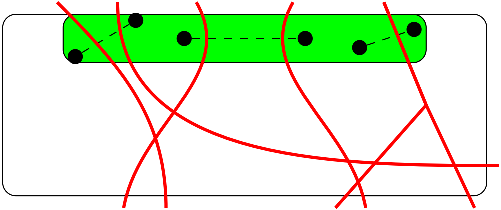

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- We will always recurse on instances of the form $(G[N[C]], S := N(C))$ for some connected C.

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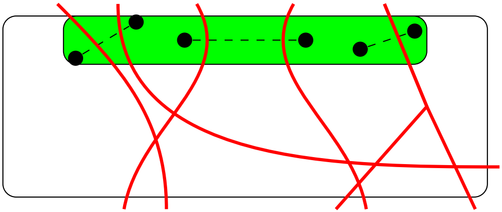

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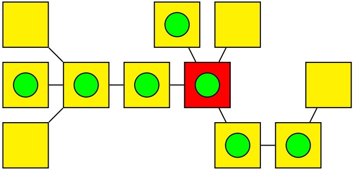
- We will always recurse on instances of the form $(G[N[C]], S := N(C))$ for some connected C.
- Hence $S = N(C)$ is always a separator, and thus never a clique.
  - We need to hack it at the begining of the recursion, but we can use preliminary guessing for that, e.g., guess a mapping on one non-edge.

Solution to Step 3

optimum
decomposition of G:

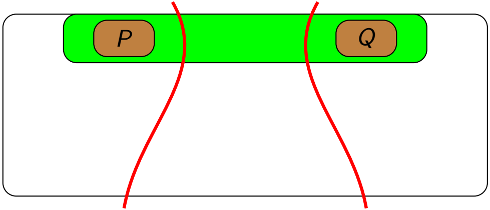


Step 3: We have $|S| \geq 10k$ and $\mathbf{tw}(G) \leq k$.

Lemma

There exists a partition $S = S_1 \uplus S_2$ with $|S_1|, |S_2| \leq 2|S|/3$ such that there is $X \subseteq V(G)$ with $|X| \leq k+1$ that separates S_1 from S_2.

Solution to Step 3



- For every $P, Q \subseteq S$, $P \cap Q = \emptyset$, $|P| = |Q| = k + 2$, if there exists a PQ separator of size at most $k + 1$, let $X_{P,Q}$ be the minimum one closest to P.

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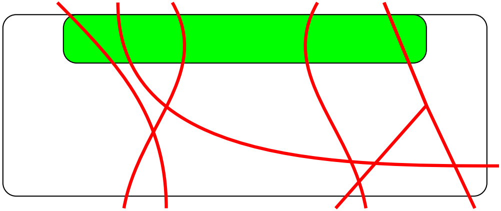

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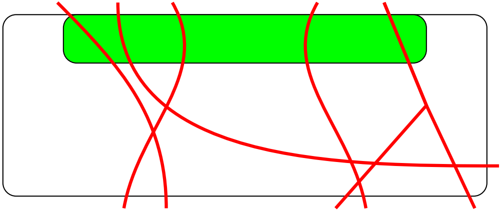

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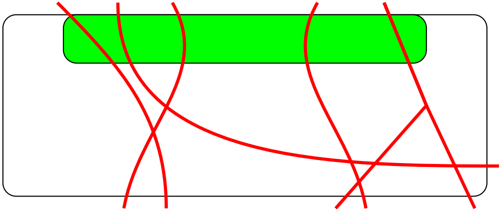
- Recurse as previously on all $(G[N[C]], N(C))$ for C being connected components of $G - B$.



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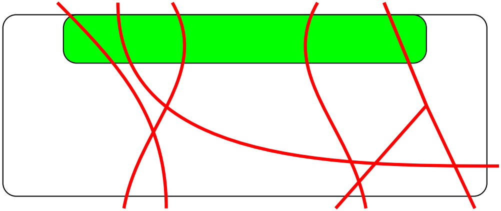

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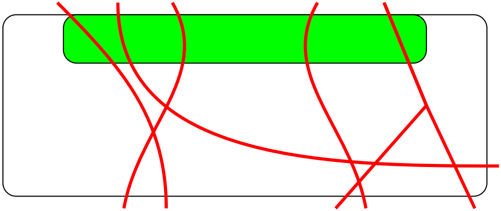

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For every connected component C of $G - B$ we have $|N(C)| \leq |S|$.

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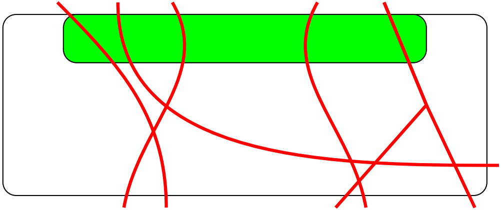

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- We analyze adding sets $X_{P,Q}$ to B one-by-one, and analyze sizes of $N(C)$ for intermediate connected components of $G - B$.
- Initially, every component of $G - S$ has neighborhood contained in S.

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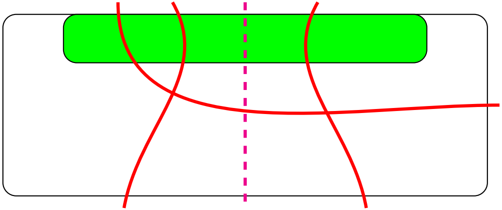

- Now consider adding one new pink cut $X_{P,Q}$.

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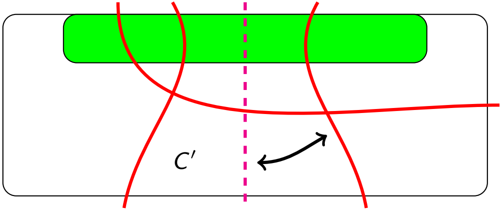

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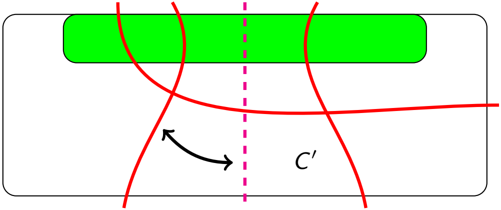

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Theorem

For a graph G of treewidth $\leq k$, we can obtain an isomorphism-invariant family of at most n^2 tree decompositions with bags of size $2^{\mathcal{O}(k \log k)}$ and adhesions of size $\mathcal{O}(k^3)$.

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Now: a quick sketch how to reduce dependency on k to $2^{k^{\mathcal{O}(1)}}$.

Getting single-exponential running time

- Instead of isomorphism-invariant tree decomposition, we want only *isomorphism-invariant family of candidate bags*.
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- We output every subset of size $\mathcal{O}(k^4)$ of every bag in our decompositions, and this is guaranteed to capture some decomposition of width $\mathcal{O}(k^4)$.

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Theorem

For a graph G of treewidth $\leq k$, we can output an isomorphism-invariant family $\mathcal{B}$ of size $2^{\mathcal{O}(k^5 \log k)} \cdot n^2$, where every element of $\mathcal{B}$ is a subset of $V(G)$ of size $\mathcal{O}(k^4)$ and $\mathcal{B}$ contains all bags of some tree decomposition of G.

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There is an algorithm, that given a graph G and integer k, runs in $2^{\mathcal{O}(k^5 \log k)} \cdot n^5$ and either concludes that $\mathbf{tw}(G) > k$, or labels the vertices with numbers $1, 2, \ldots, n$ such that two isomorphic graphs receive labelings certifying the isomorphism.

Conclusions

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Open problems:

- What about FPT algorithm for graph isomorphism parameterized by the maximum degree?
  - Luks' algorithm has running time $\mathcal{O}(n^{f(\Delta)})$.
- What about FPT algorithm for graph isomorphism parameterized by the size of an excluded minor?
  - Ponomarenko's algorithm has running time $\mathcal{O}(n^{f(|H|)})$.