Lower bounds based on ETH and SETH

Michał Pilipczuk

Institute of Informatics, University of Warsaw, Poland

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- Assumption P $\neq$ NP seems too week to answer these questions.

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- Current record: $\mathcal{O}^\star(1.308^n)$ (Paturi, Pudlák, Saks, Zane; Hertli)

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ETH and SETH, first try

- 3-SAT cannot be solved in time $\mathcal{O}^\star(2^{o(n)})$.
- General CNF-SAT cannot be solved in time $\mathcal{O}^\star(c^n)$ for any $c < 2$.

Actual statements

$$\delta_q = \inf\{c : \text{ There is an } \mathcal{O}^\star(2^{cn}) \text{ algorithm for } q\text{-SAT}\}$$

Exponential Time Hypothesis, ETH

$$\delta_3 > 0.$$

There is a $c > 0$ such that 3-SAT cannot be solved in $\mathcal{O}^\star(2^{cn})$ time.

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- SETH $\Rightarrow$ ETH (nontrivial)

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Vertex Cover

Input: A graph G and an integer k
Question: Is there a set $X \subseteq V(G)$ with $|X| \leq k$ such that every edge of G has at least one endpoint in X?

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Let us inspect the standard reduction from 3-SAT to Vertex Cover.

clauses

variables

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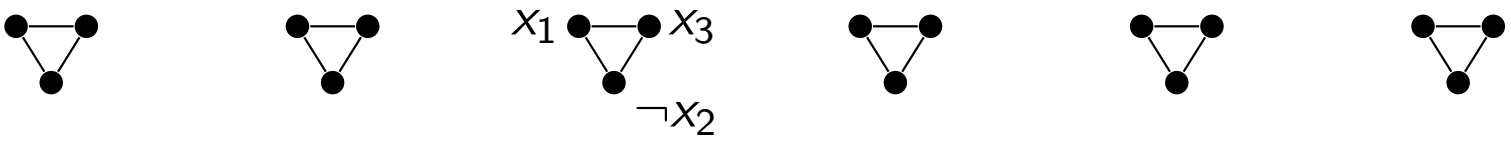

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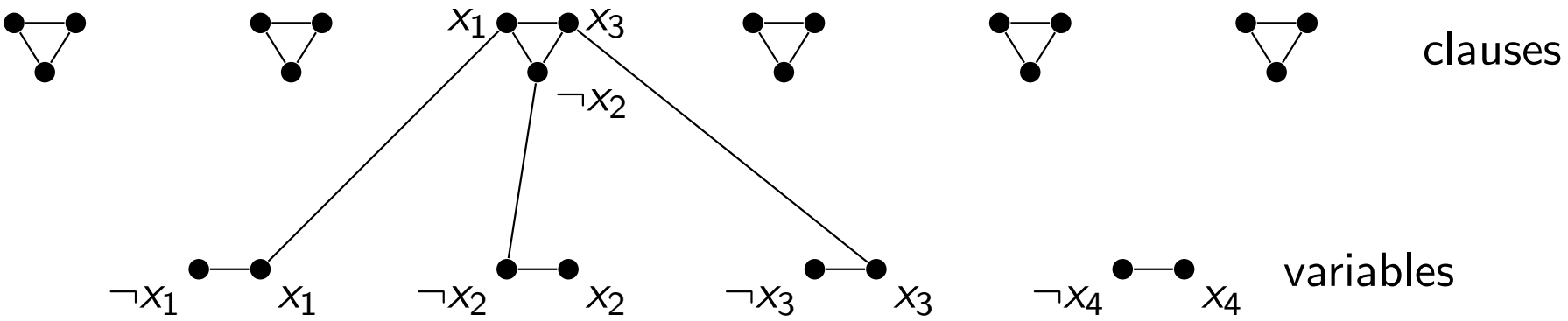

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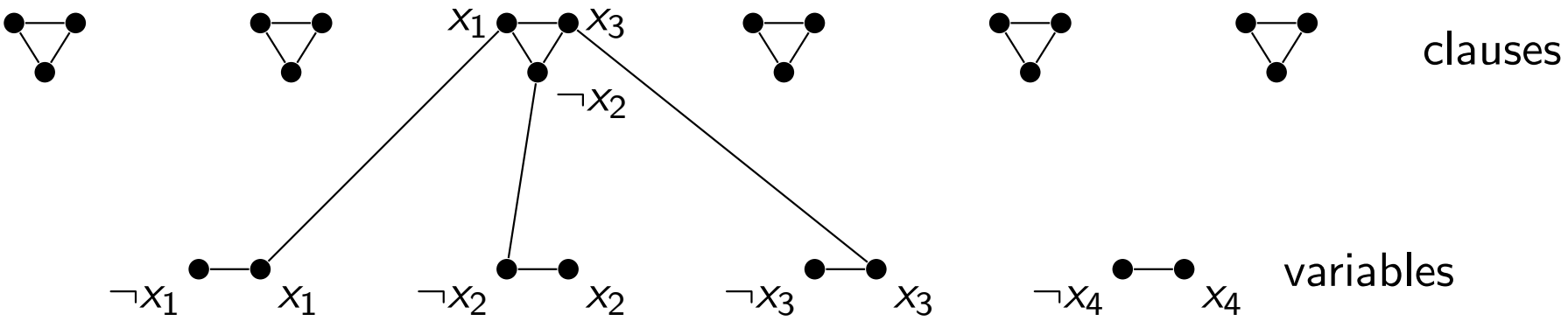

The formula is satisfiable iff the created graph has vertex cover of size $n + 2m$.

Vertex Cover reduction, analysis

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- But we know no algorithm with running time $\mathcal{O}^\star(2^{o(N)})$...
- If we started with an instance of $3\text{-}\mathrm{SAT}$ that is *sparse*, i.e., $m = \mathcal{O}(n)$, then a $\mathcal{O}^\star(2^{o(N)})$ lower bound would follow.

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Sparsification Lemma [IPZ, 2001]

For any $q \geq 3$ and $\varepsilon > 0$ there is a constant $C = C(q, \varepsilon)$ such that any q-CNF formula φ can be expressed as $\varphi = \bigvee_{i=1}^{t} \psi_i$, where $t \leq 2^{\varepsilon n}$ and each ψ_i is a q-CNF formula with the same set of variables and at most $C \cdot n$ clauses. Such disjunction can be computed in time $\mathcal{O}^\star(2^{\varepsilon n})$.

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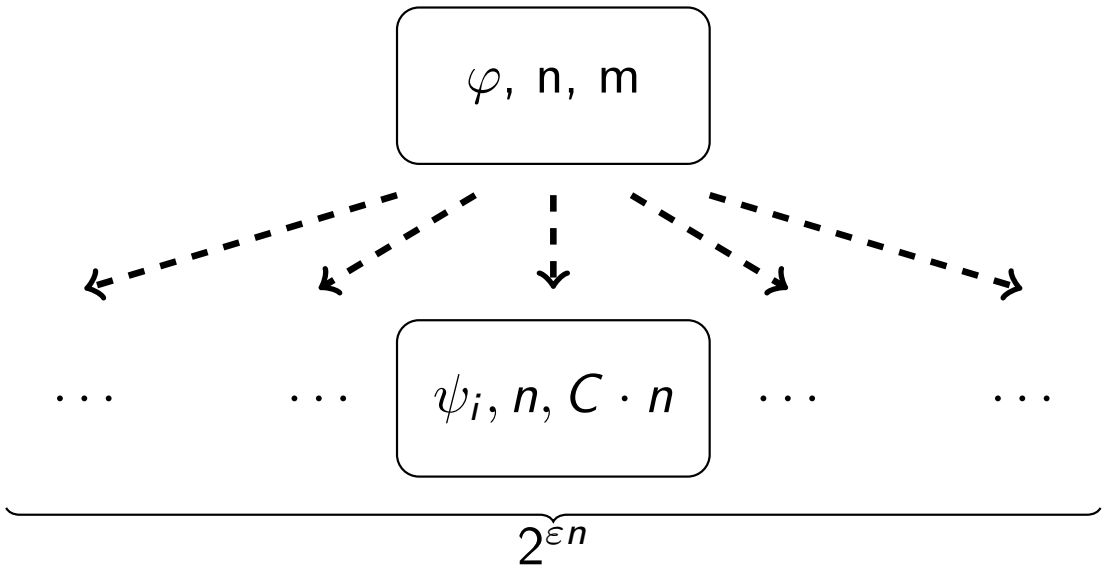

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Proof by contradiction:

- Suppose that for every $c > 0$ there is an algorithm $\mathcal{A}_c$ solving 3-SAT in time $\mathcal{O}^\star(2^{c(n+m)})$.
- Consider any $d > 0$. We want to solve 3-SAT in time $\mathcal{O}^\star(2^{dn})$.
- Use Sparsification Lemma for $\varepsilon = d/2$. Denote $C = C(3, \varepsilon)$.
- Solve each ψ_i by $\mathcal{A}_{c'}$, where $c' = \frac{d}{2(C+1)}$.
- The total running time is $\mathcal{O}^\star(2^{\varepsilon n}) + \mathcal{O}^\star(2^{\varepsilon n} \cdot 2^{\frac{d}{2(C+1)} \cdot (C+1)n}) = \mathcal{O}^\star(2^{dn})$.

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Corollaries

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- Other problems that also admit such reductions:
  - Feedback Vertex Set,
  - Dominating Set,
  - 3-Coloring,
  - Hamiltonian Cycle,
  - and many others.

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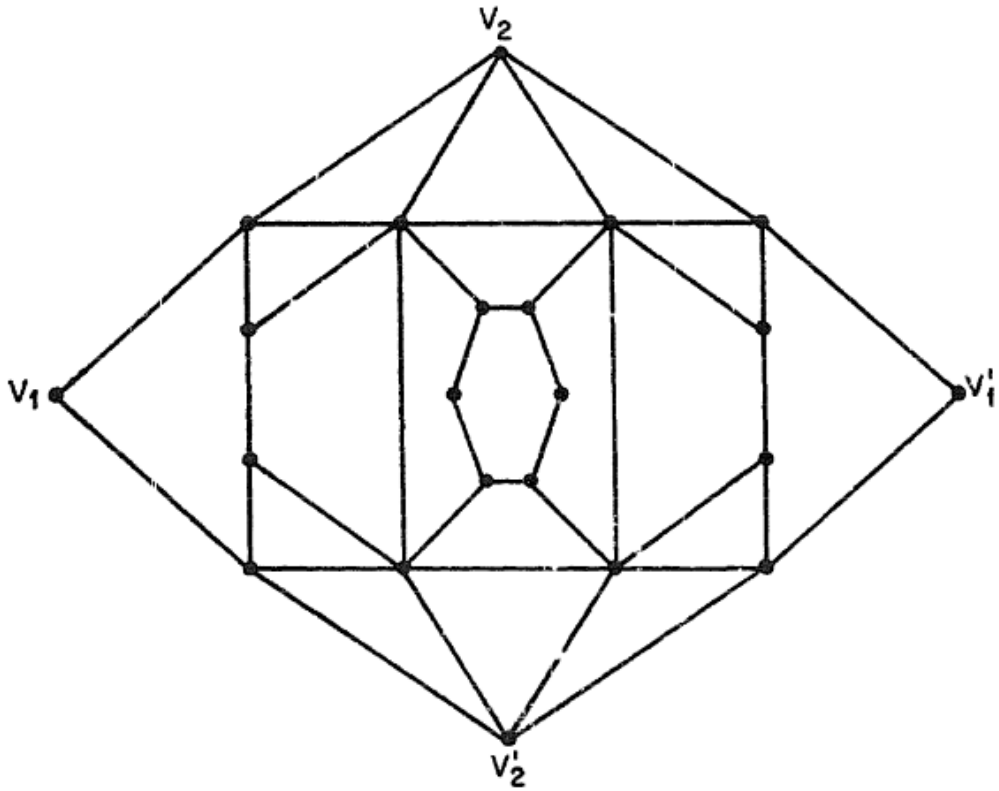

Fig. 11. Crossover H for Theorem 2.7.

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- The $\sqrt{N}$ term in the exponent is not a coincidence!

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  - It is W[1]-hard.

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Lower bounds for parameterized problems under ETH

If L admits a reduction from $3\text{-}\mathrm{SAT}$ where $k = f(n+m)$, then L does not admit an $\mathcal{O}^\star(2^{o(f^{-1}(k))})$ algorithm unless ETH fails.

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- $\mathcal{O}^\star(2^{o(\sqrt{k})})$ algorithm for PlVC would be also a $\mathcal{O}^\star(2^{o(\sqrt{n})})$ algorithm, contradiction.
  - **Remark**: $\mathcal{O}^\star(2^{\mathcal{O}(\sqrt{k})})$ possible using *bidimensionality*.

Lower bounds for parameterized problems under ETH

If L admits a reduction from $3\text{-}\mathrm{SAT}$ where $k = f(n+m)$, then L does not admit an $\mathcal{O}^\star(2^{o(f^{-1}(k))})$ algorithm unless ETH fails.

- The parameter blow-up governs the strength of the lower bound.

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- Let's look at some more exotic running times.

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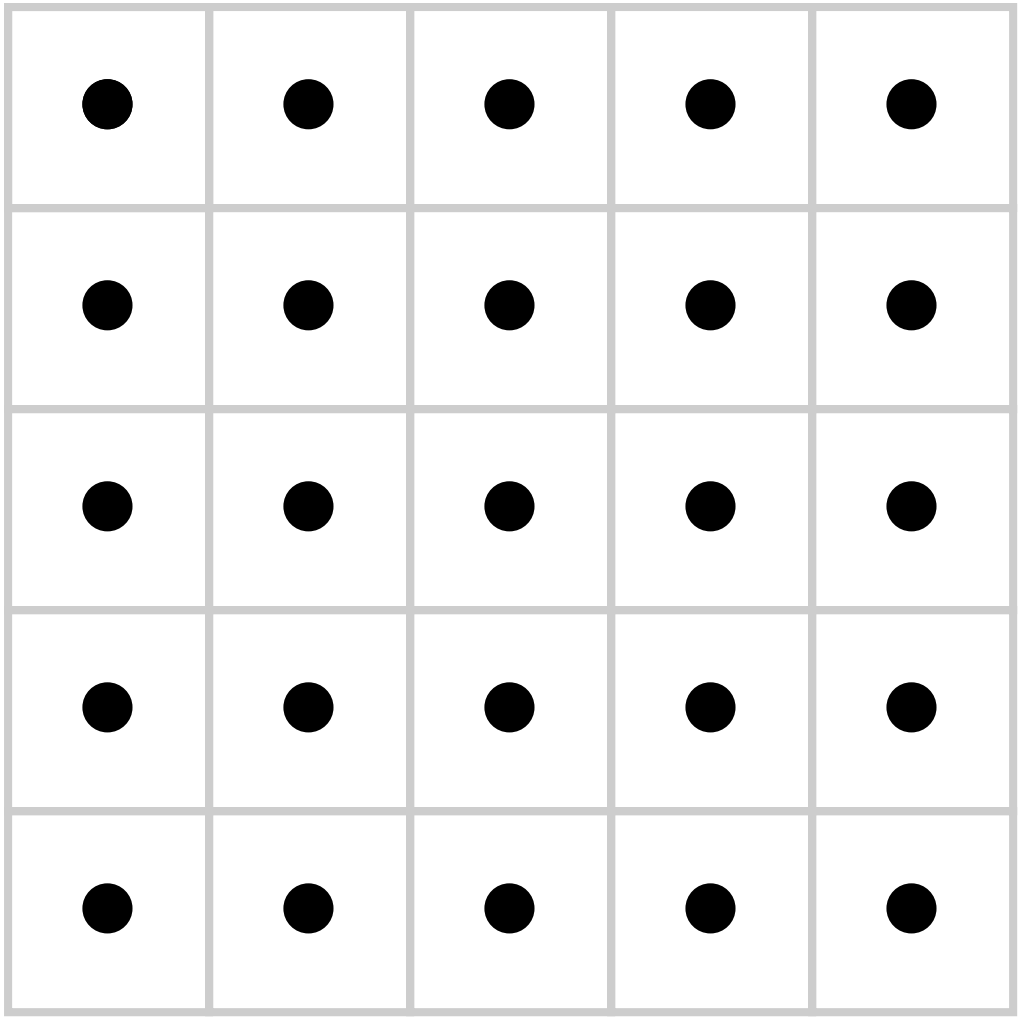
- Results by Lokshtanov, Marx, and Saurabh (2011).

$k \times k$-Clique

Input: A graph H on vertex set $[k] \times [k]$
Question: Is there a k-clique in H that contains exactly one vertex from each row?

- $[k] = \{1, 2, \ldots, k\}$.

On a picture



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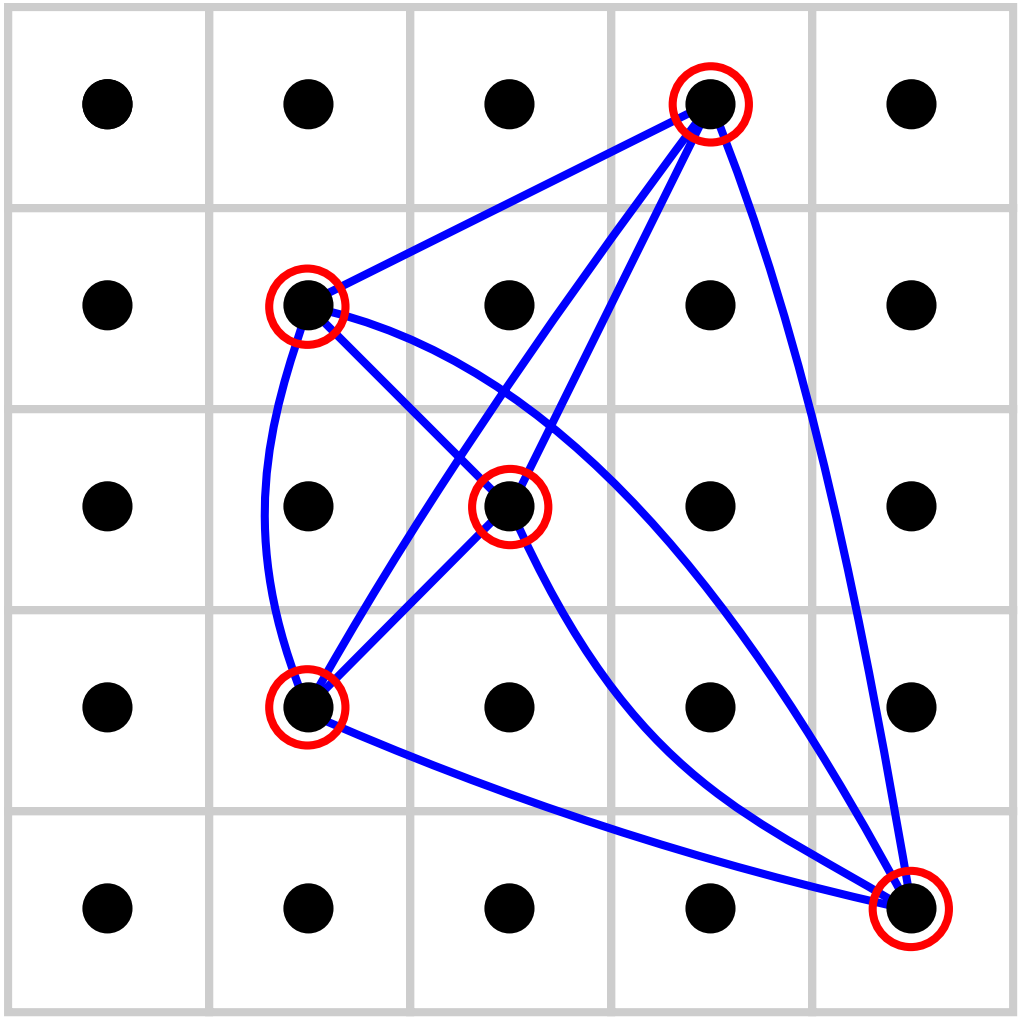

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- Trivial $\mathcal{O}^\star(k^k)$ algorithm: verify all the choices.
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- **Now**: $k \times k$-Clique does not admit an $\mathcal{O}^\star(2^{o(k \log k)})$ algorithm unless ETH fails.

Lower bound for $k \times k$-Clique

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- Finally, fill the rows with isolated vertices up to size k.

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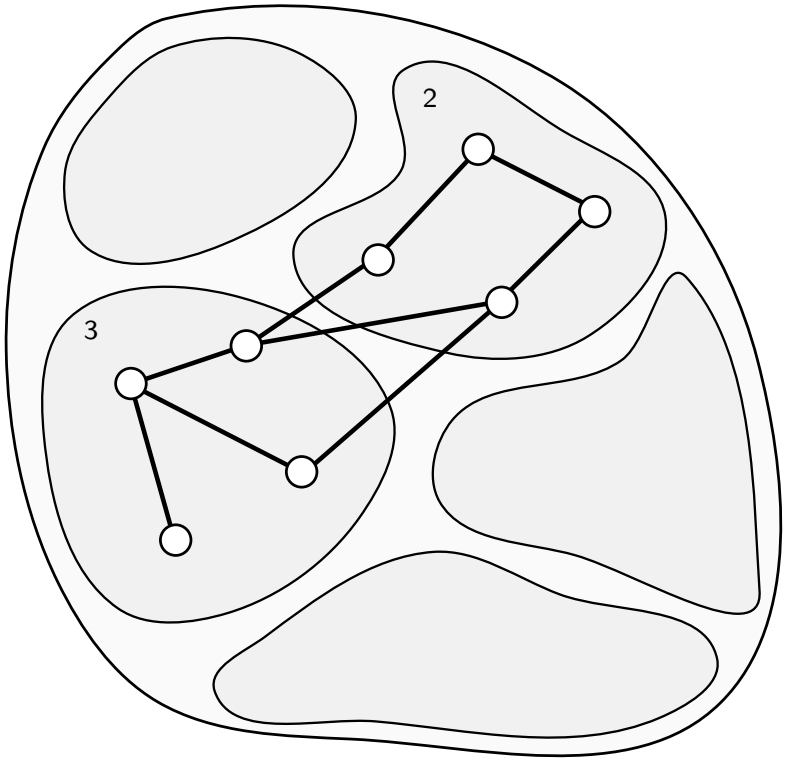

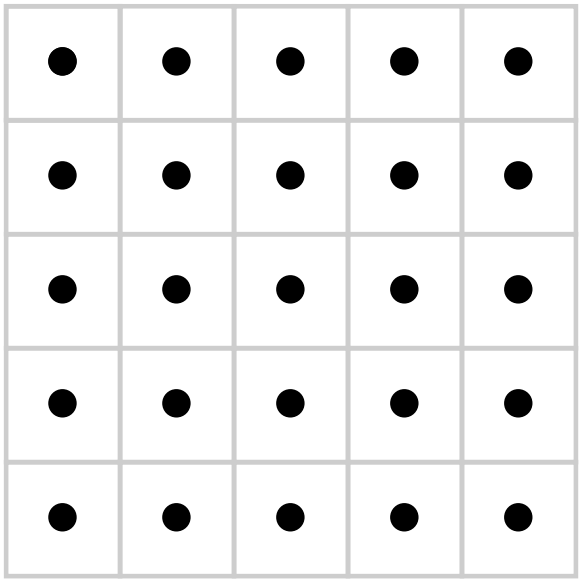

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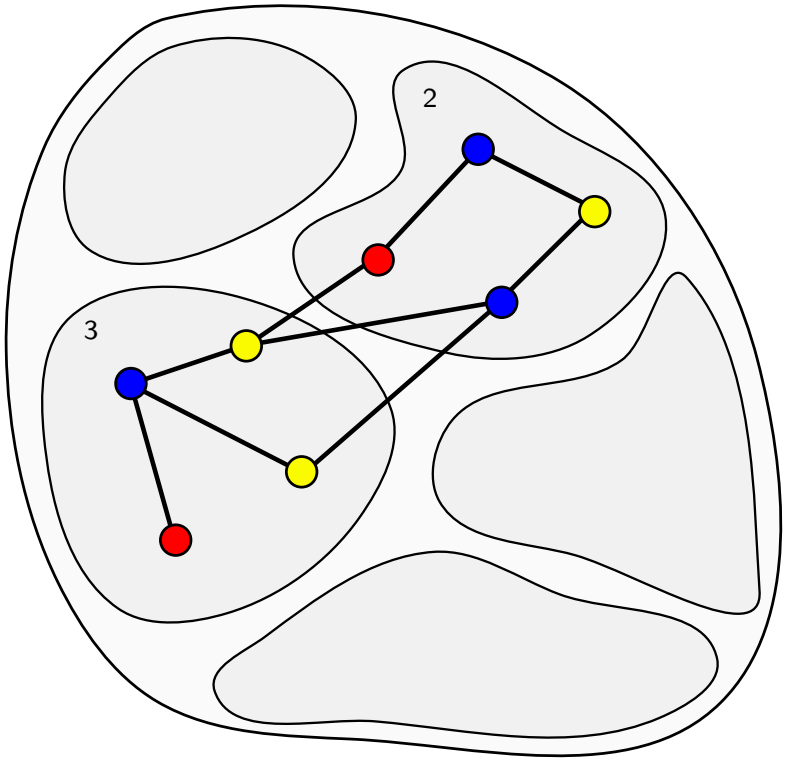

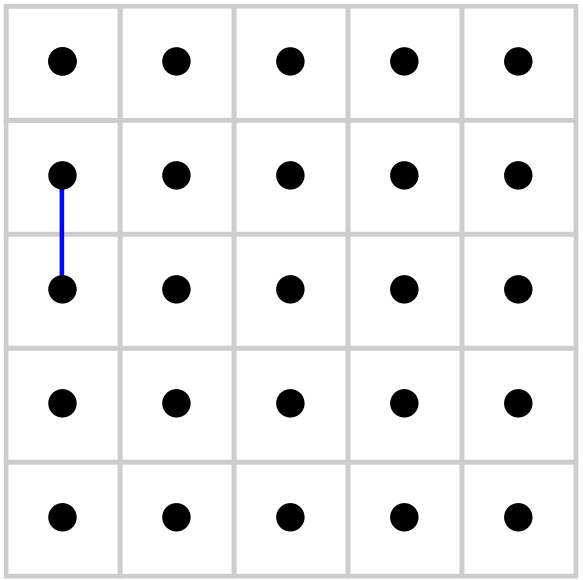

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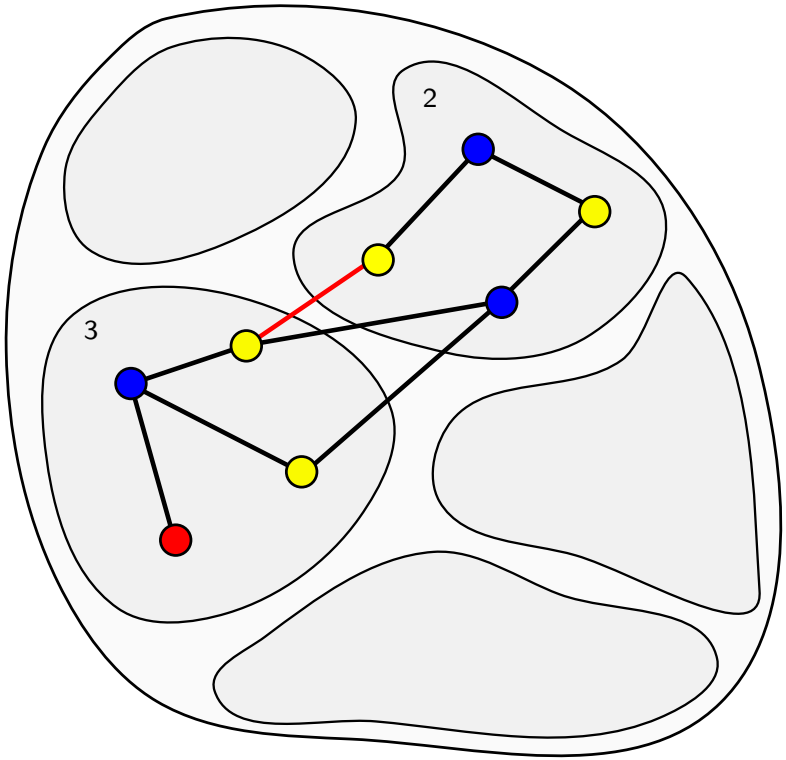

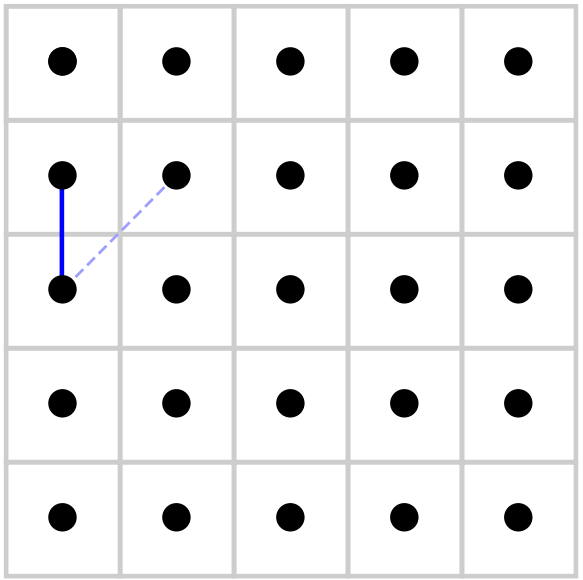

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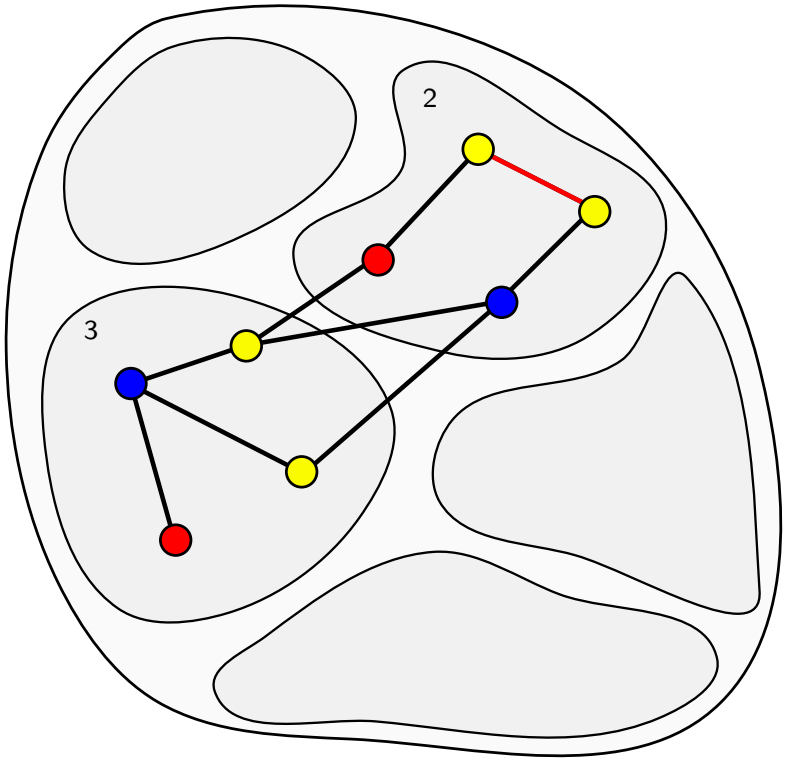

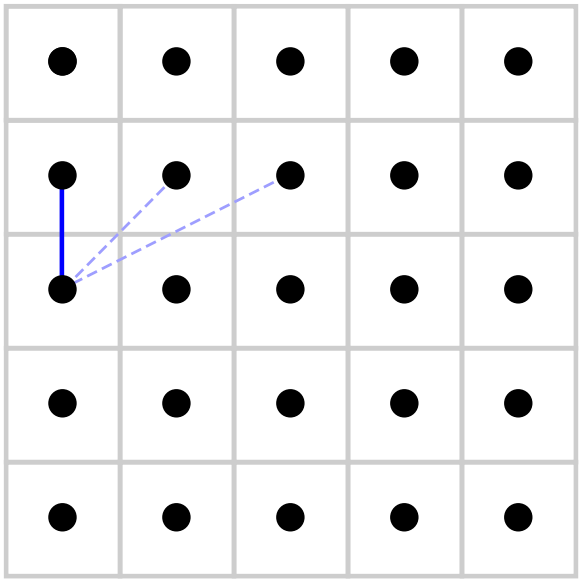

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- And we are done.

Further corollaries

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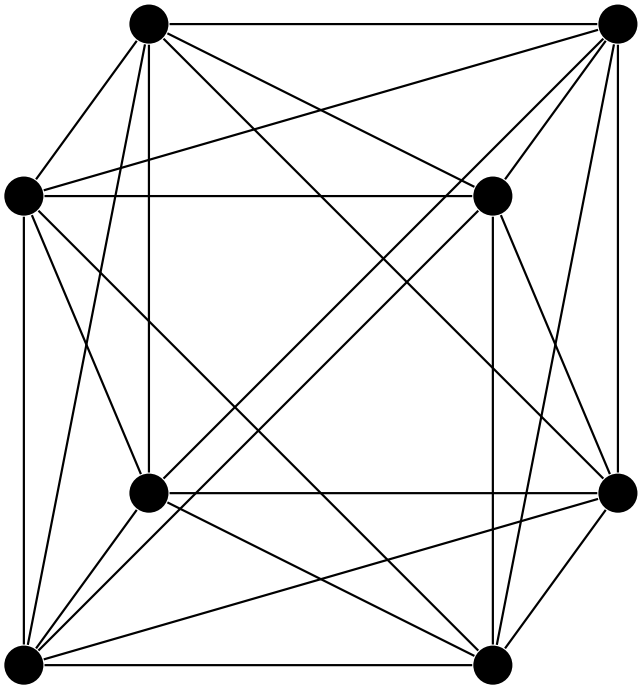
Input: Graph G, integer k
Question: Does there exists a collection of k cliques $C_1, C_2, \ldots, C_k$ in G, such that $E(G) = \bigcup_{i=1}^{k} E(C_i)$?



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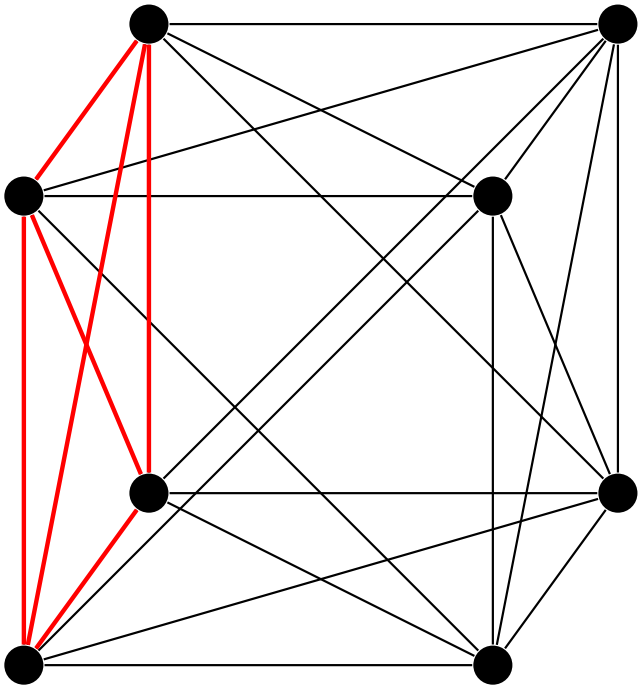

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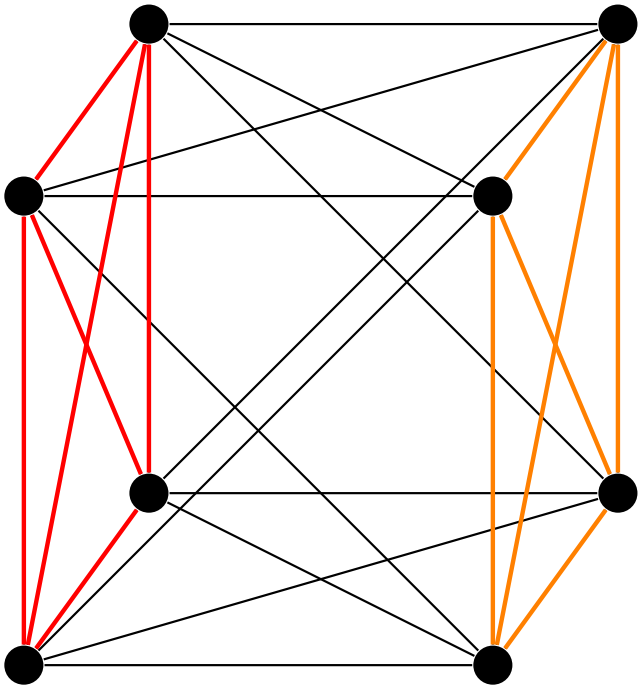

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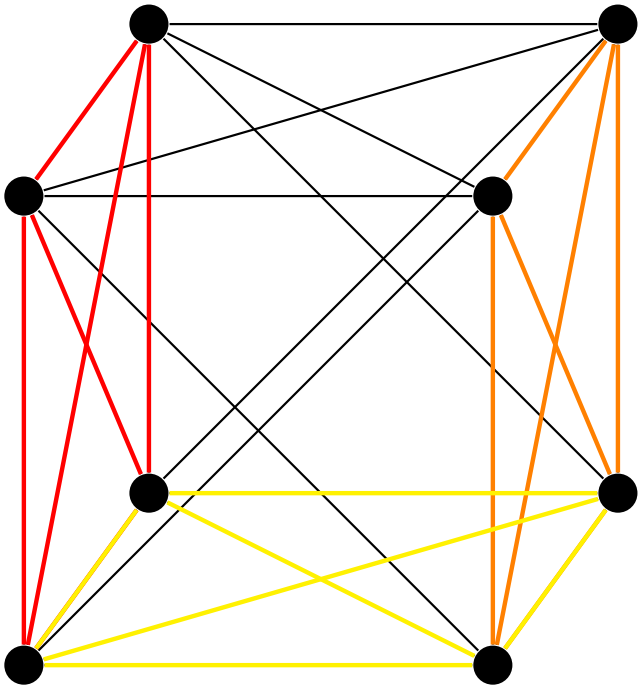

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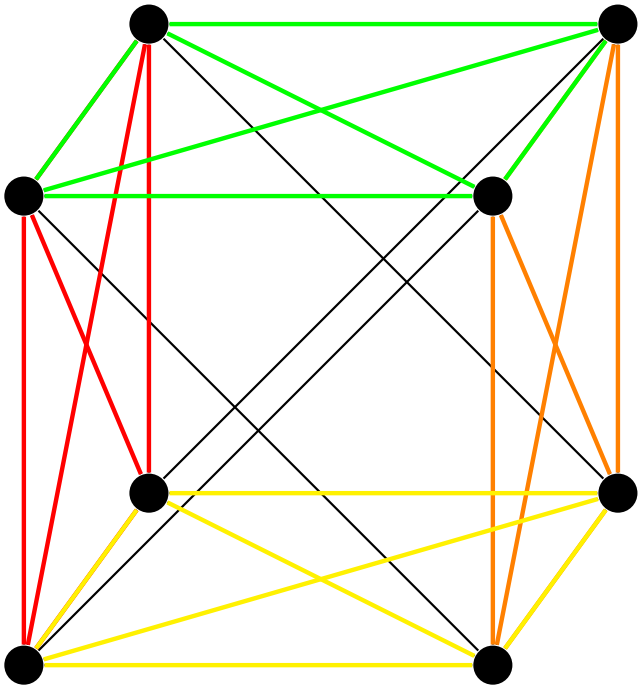

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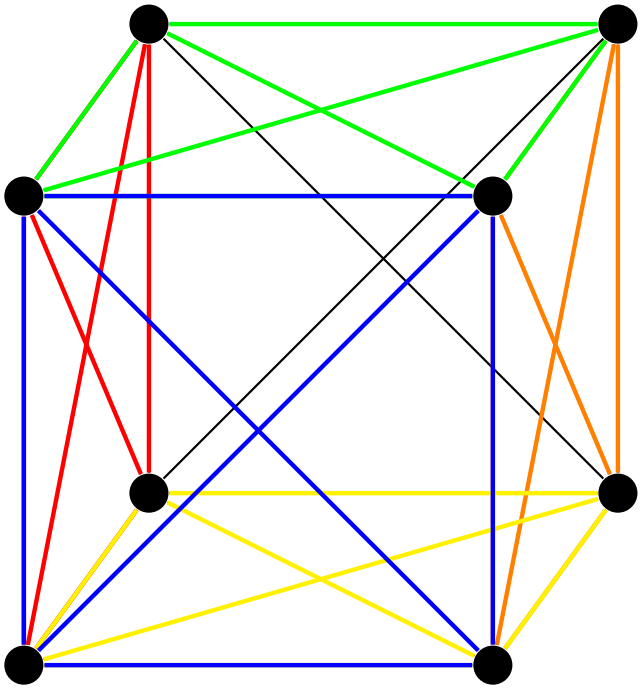

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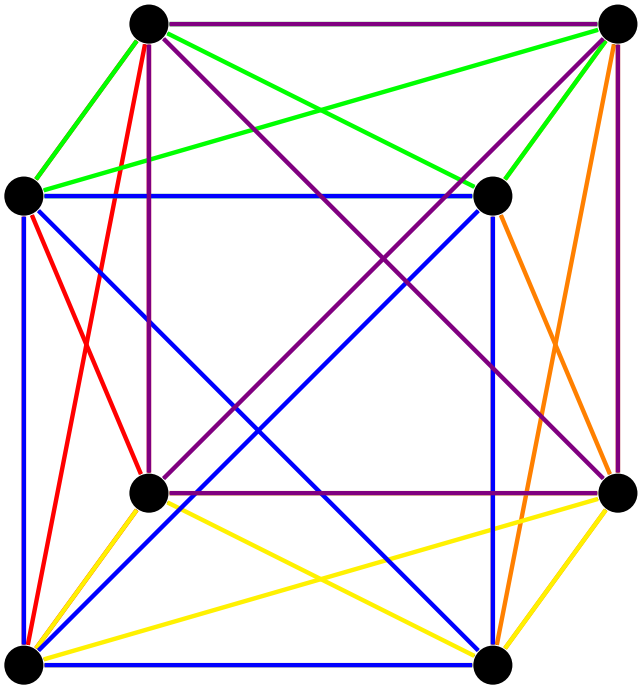

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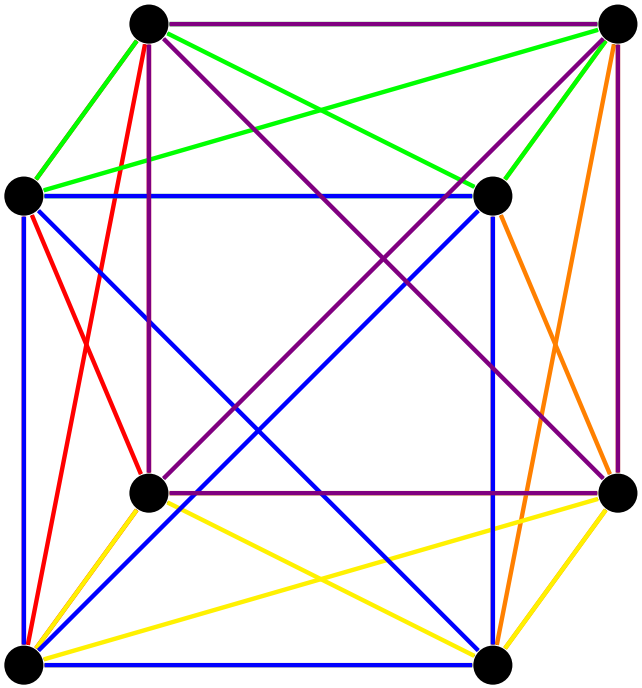

6 cliques, but not optimal

puzzle: find a solution with 5 cliques

Bound for ECC

Theorem (Gramm, Guo, Hüffner, Niedermeier)

Edge Clique Cover admits a kernel with at most 2^k vertices. That is, one can preprocess the instance in polynomial time to ensure that the number of vertices in at most 2^k.

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Theorem (Cygan, Pilipczuk, P)

There is a reduction from 3-SAT to ECC that outputs an instance with $k = \mathcal{O}(\log(n+m))$. Consequently, there is no $\mathcal{O}^\star(2^{2^{o(k)}})$ time algorithm for ECC unless ETH fails.

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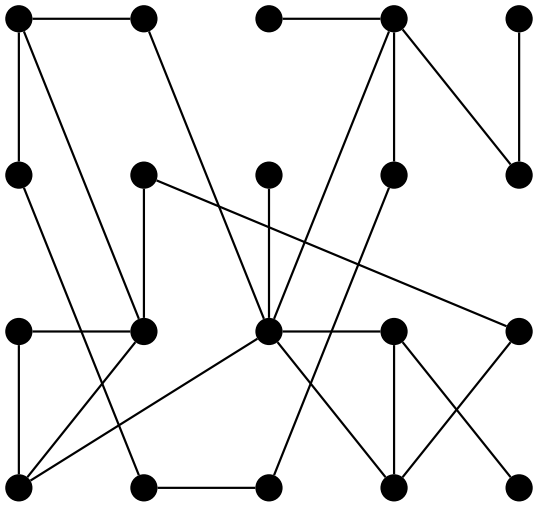
- **Corollary**: ETH $\Rightarrow \mathrm{W}[1] \neq \mathrm{FPT}$

Hardness for Clique



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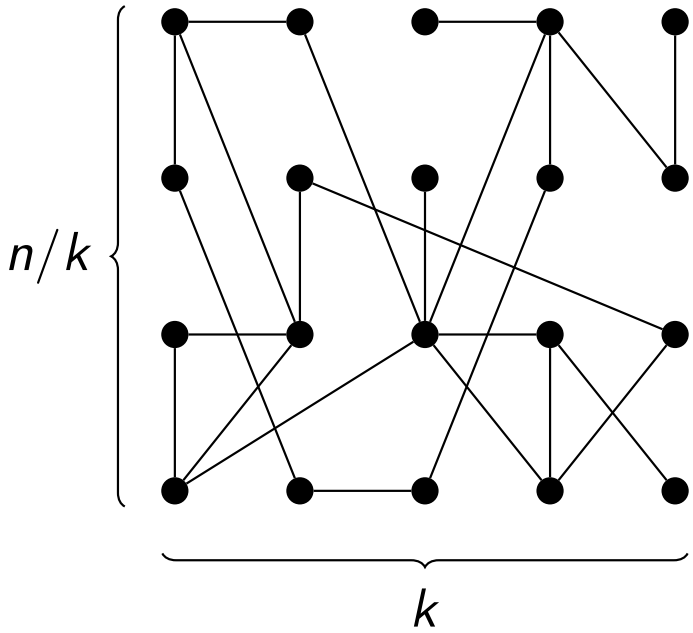

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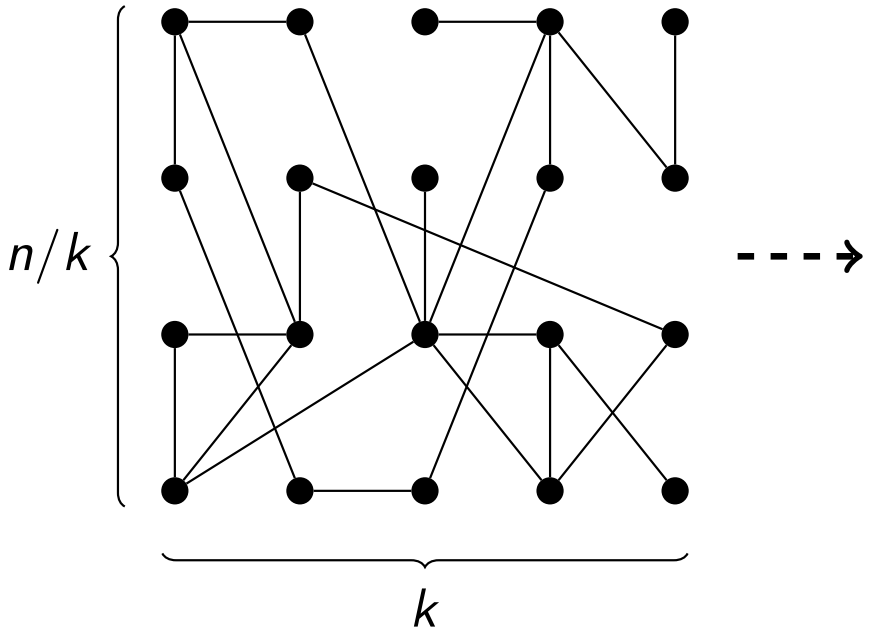

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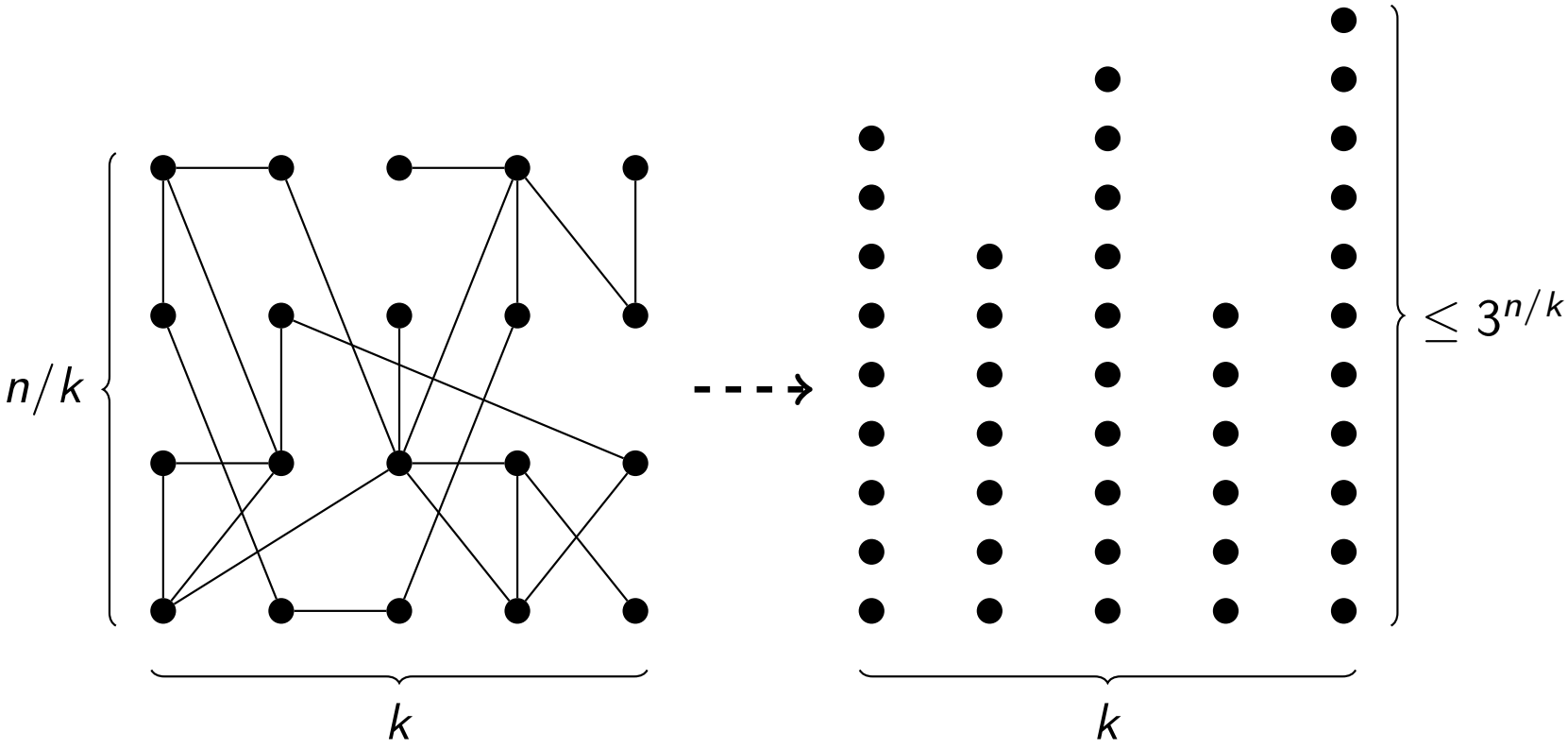

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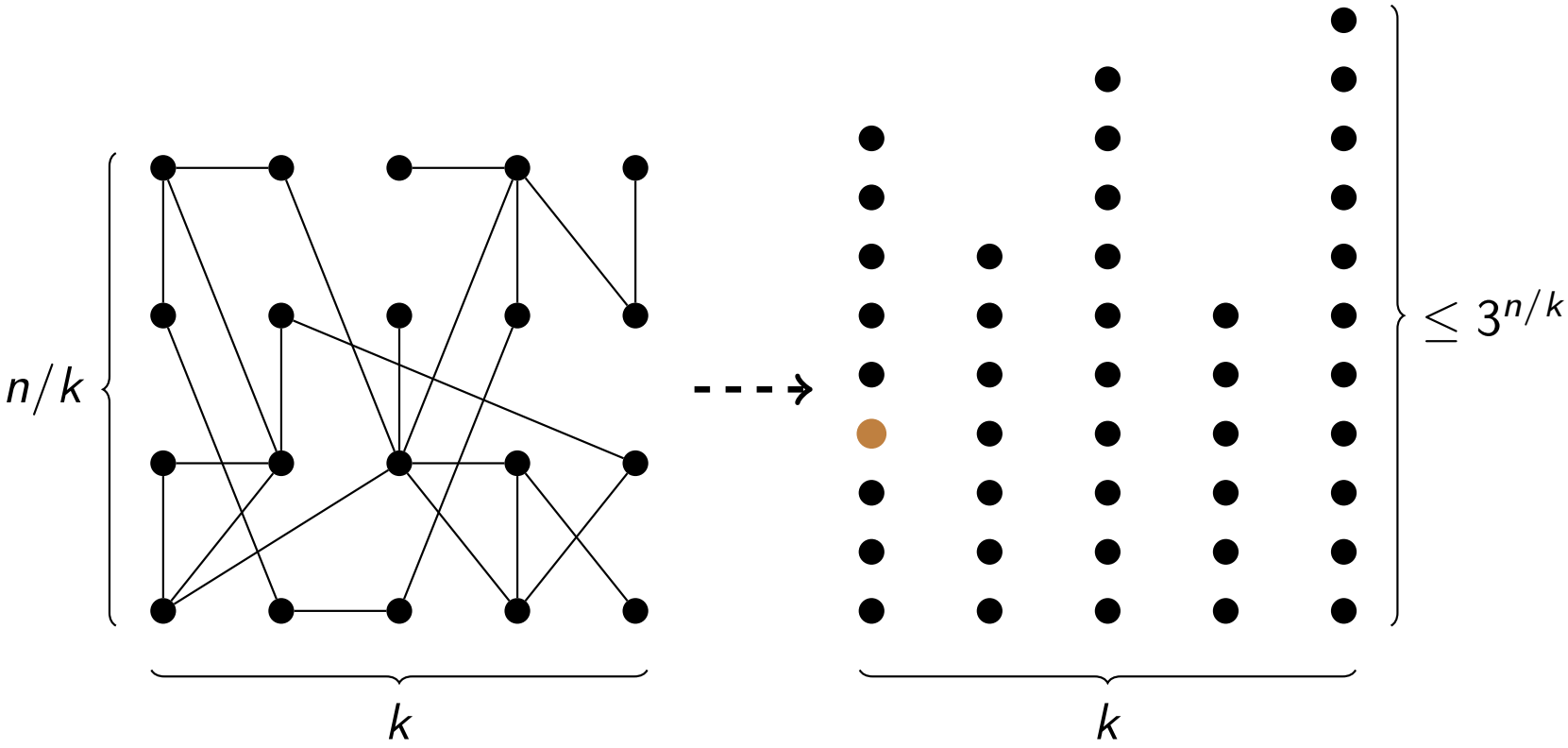

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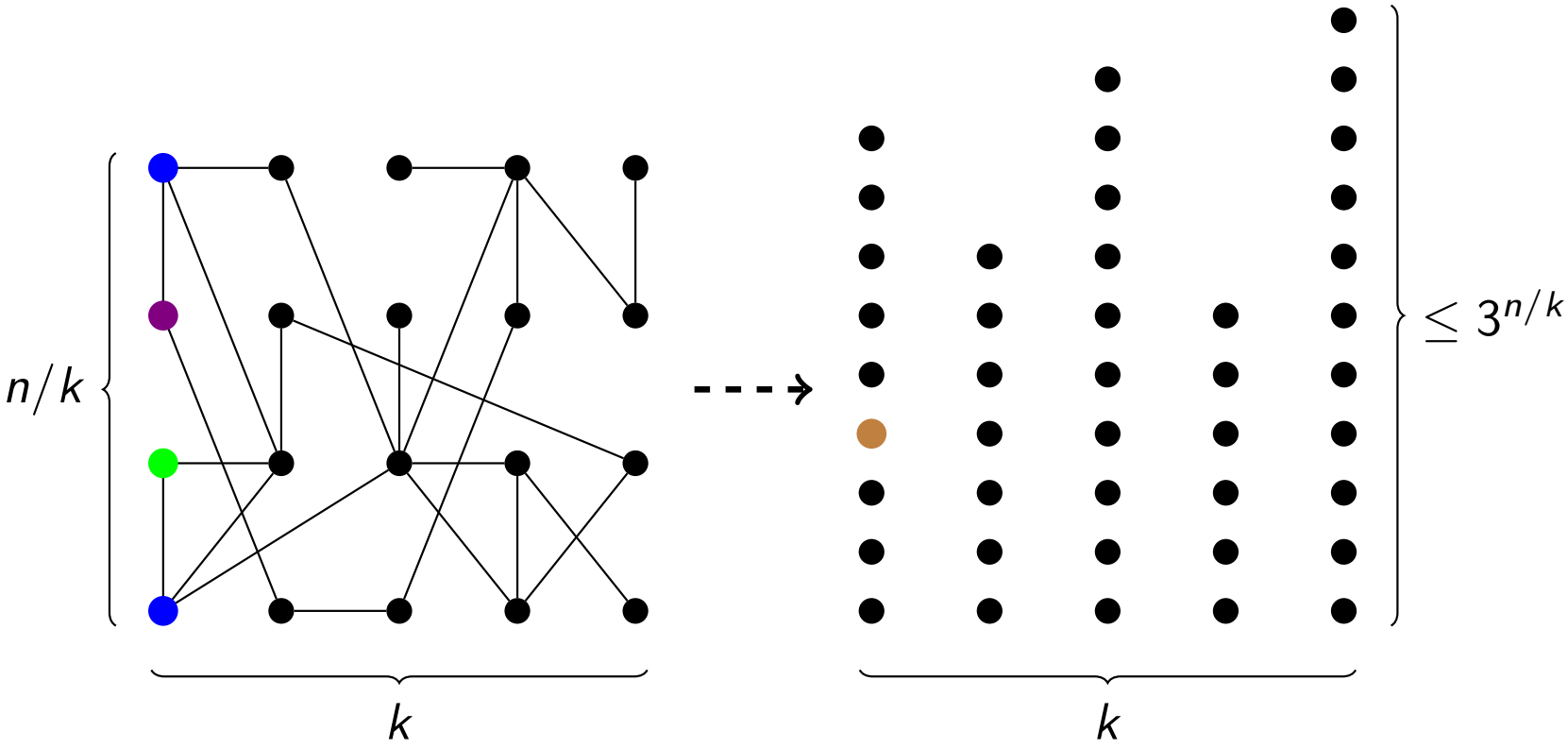

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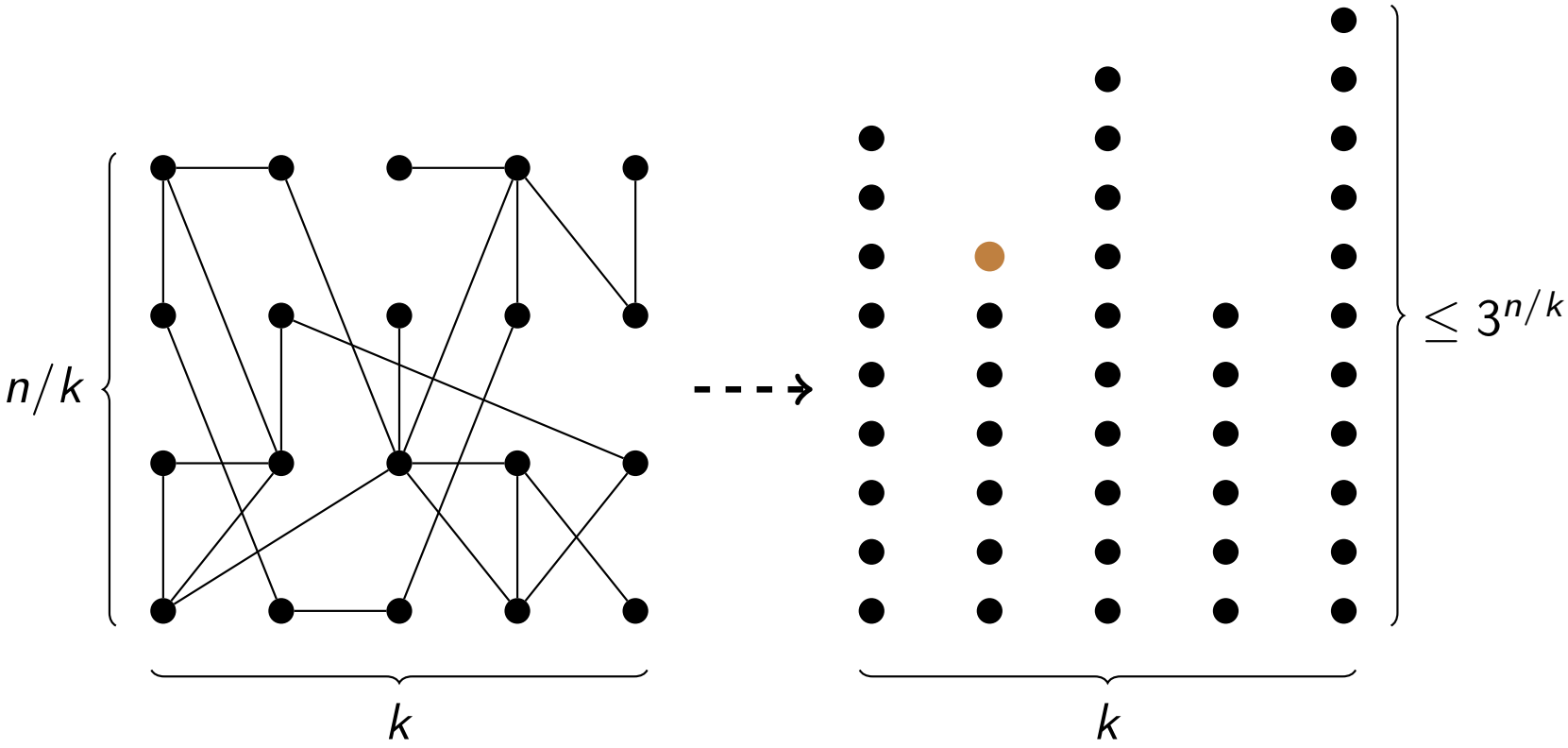

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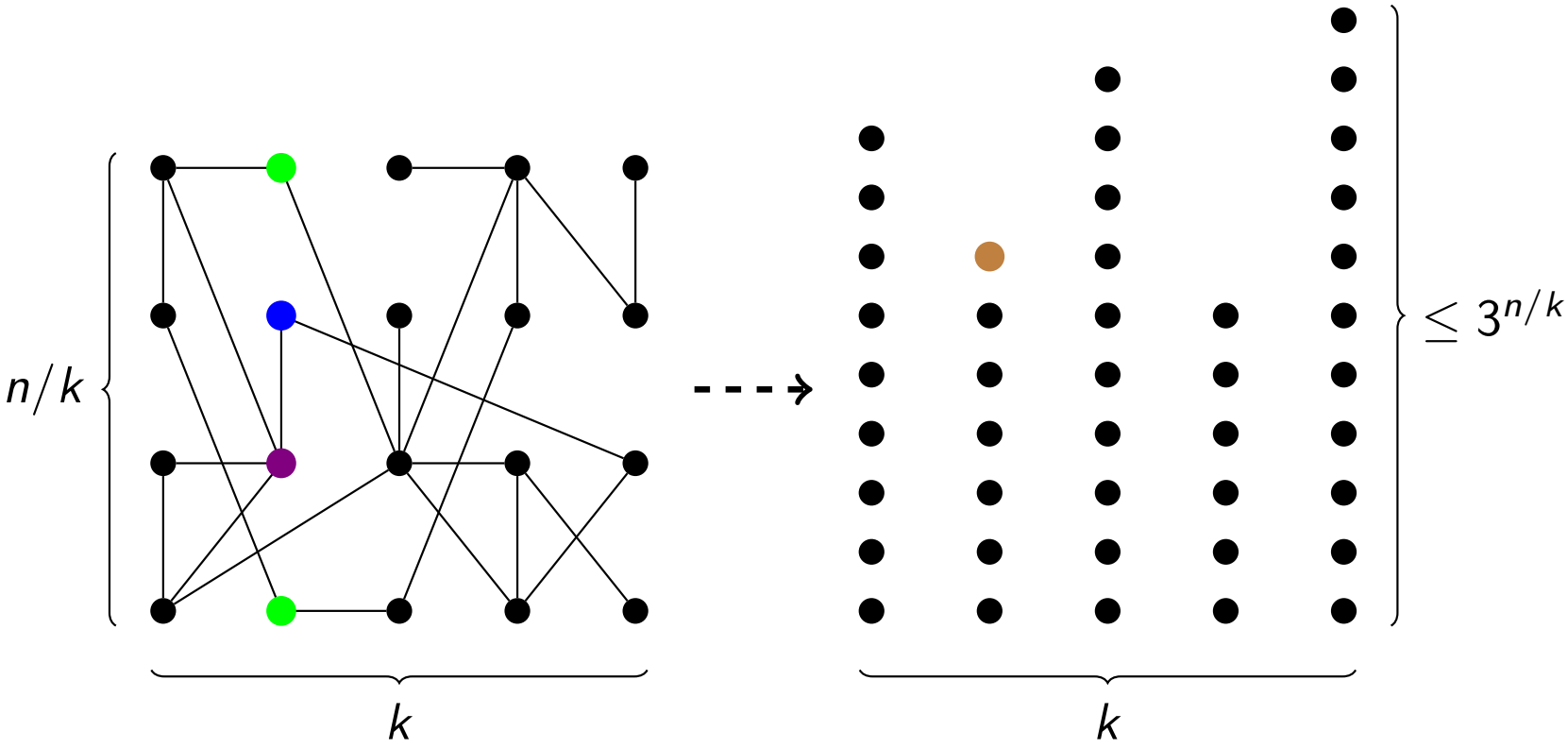

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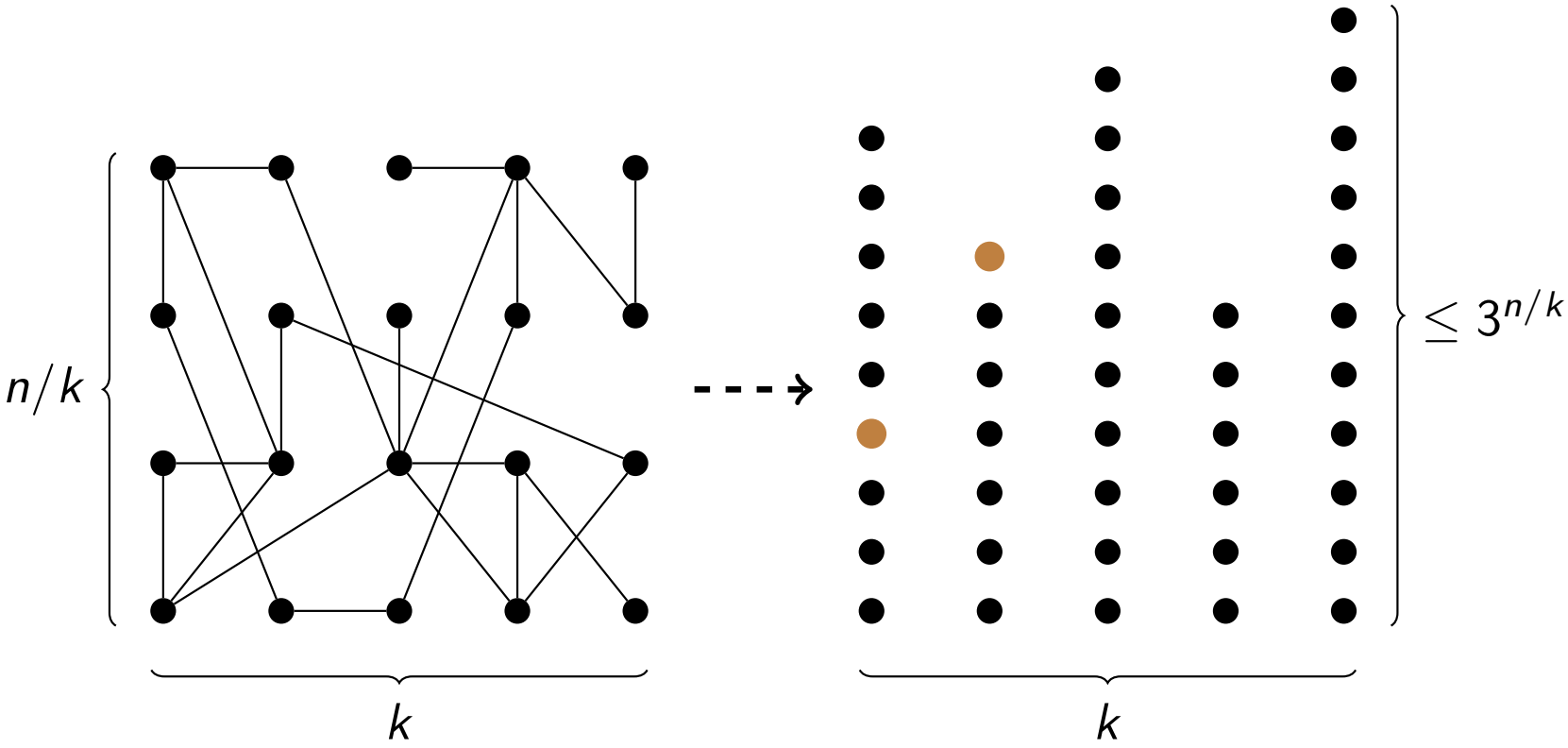

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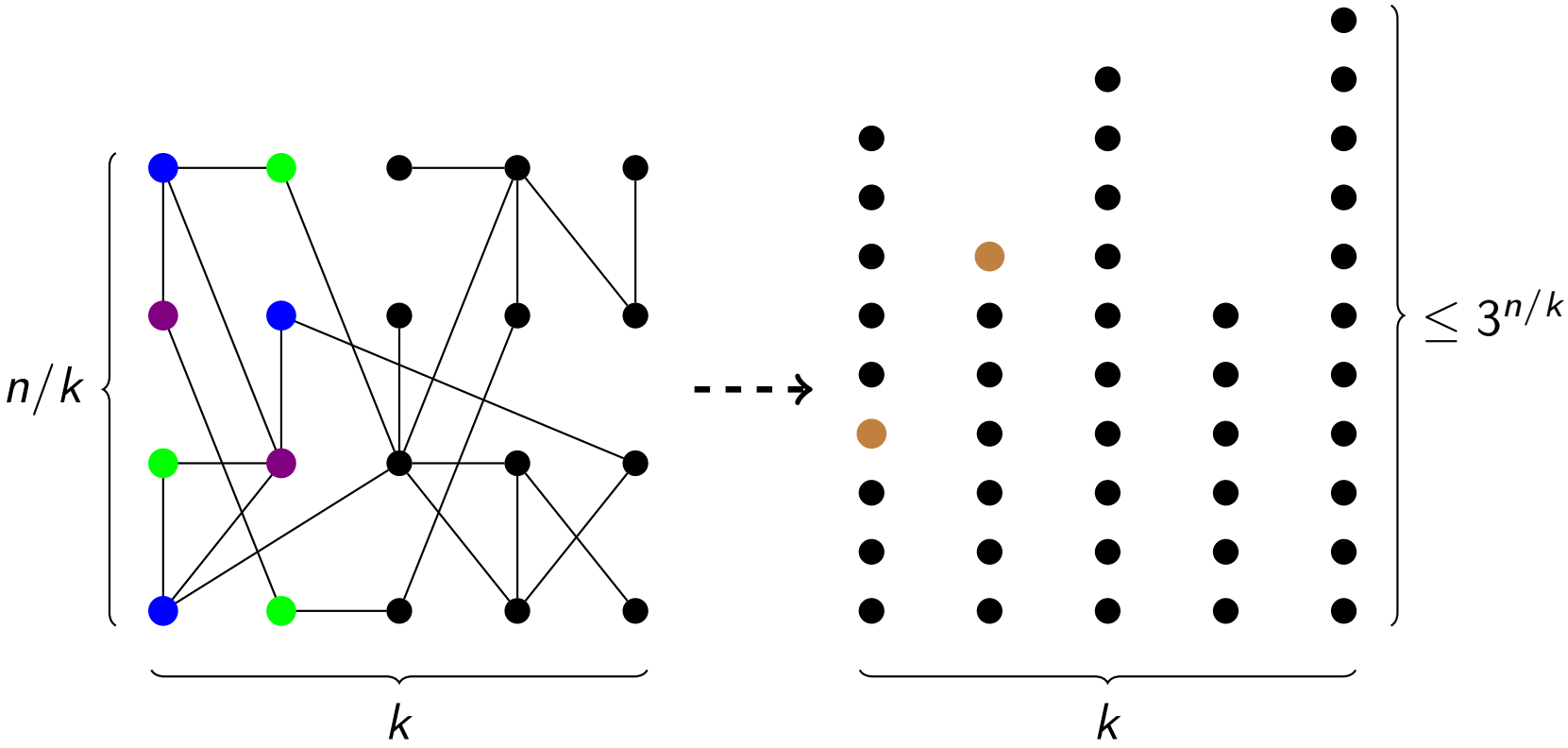

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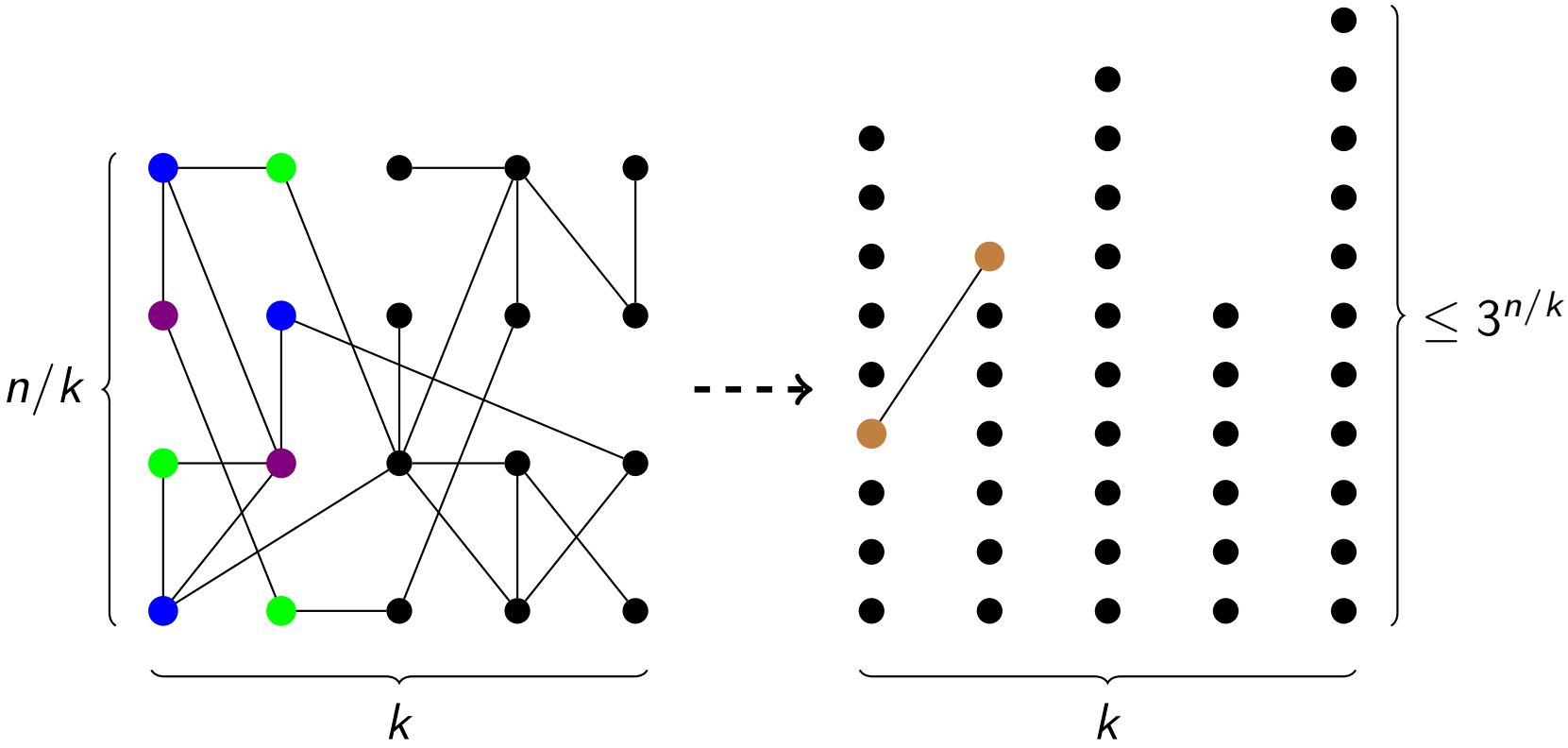

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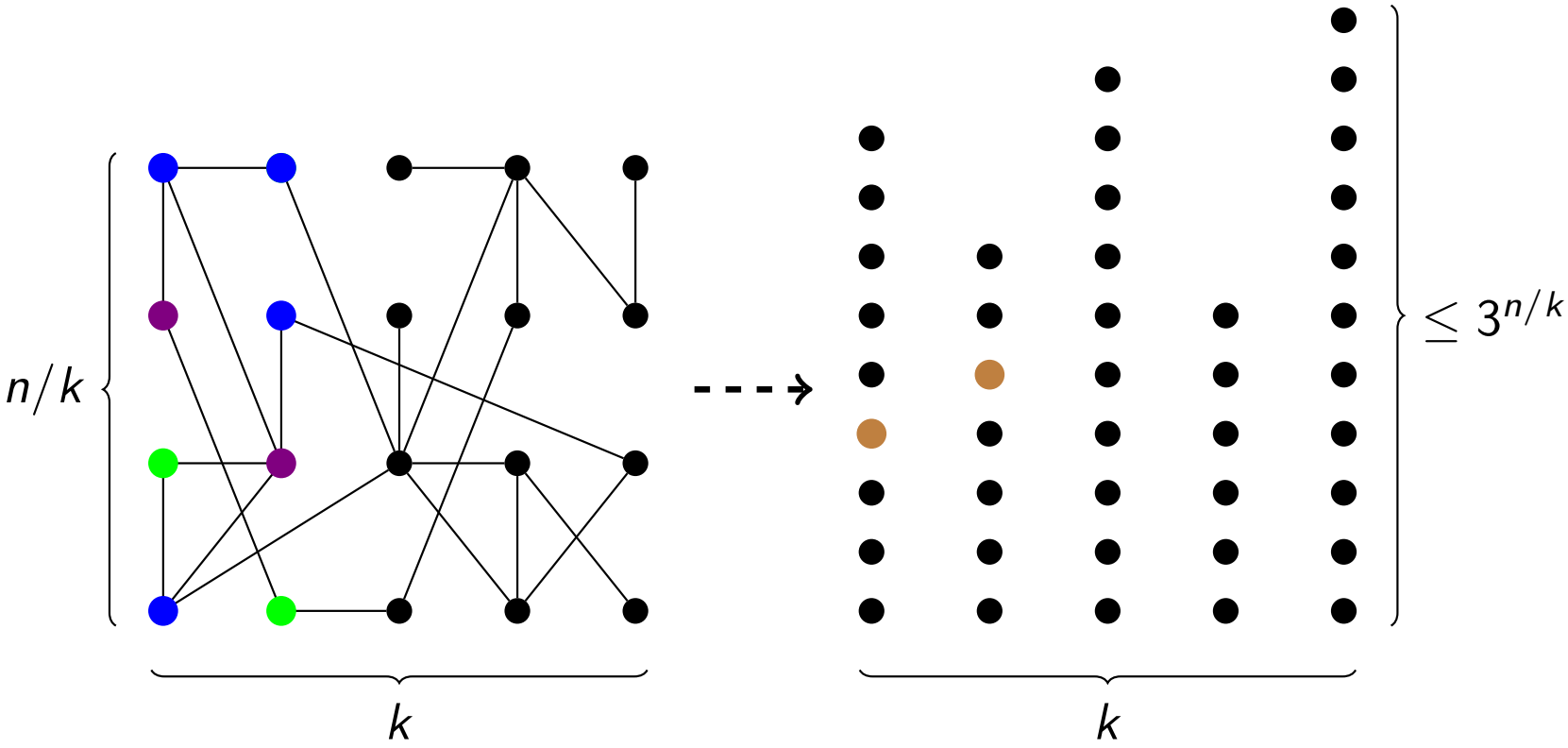

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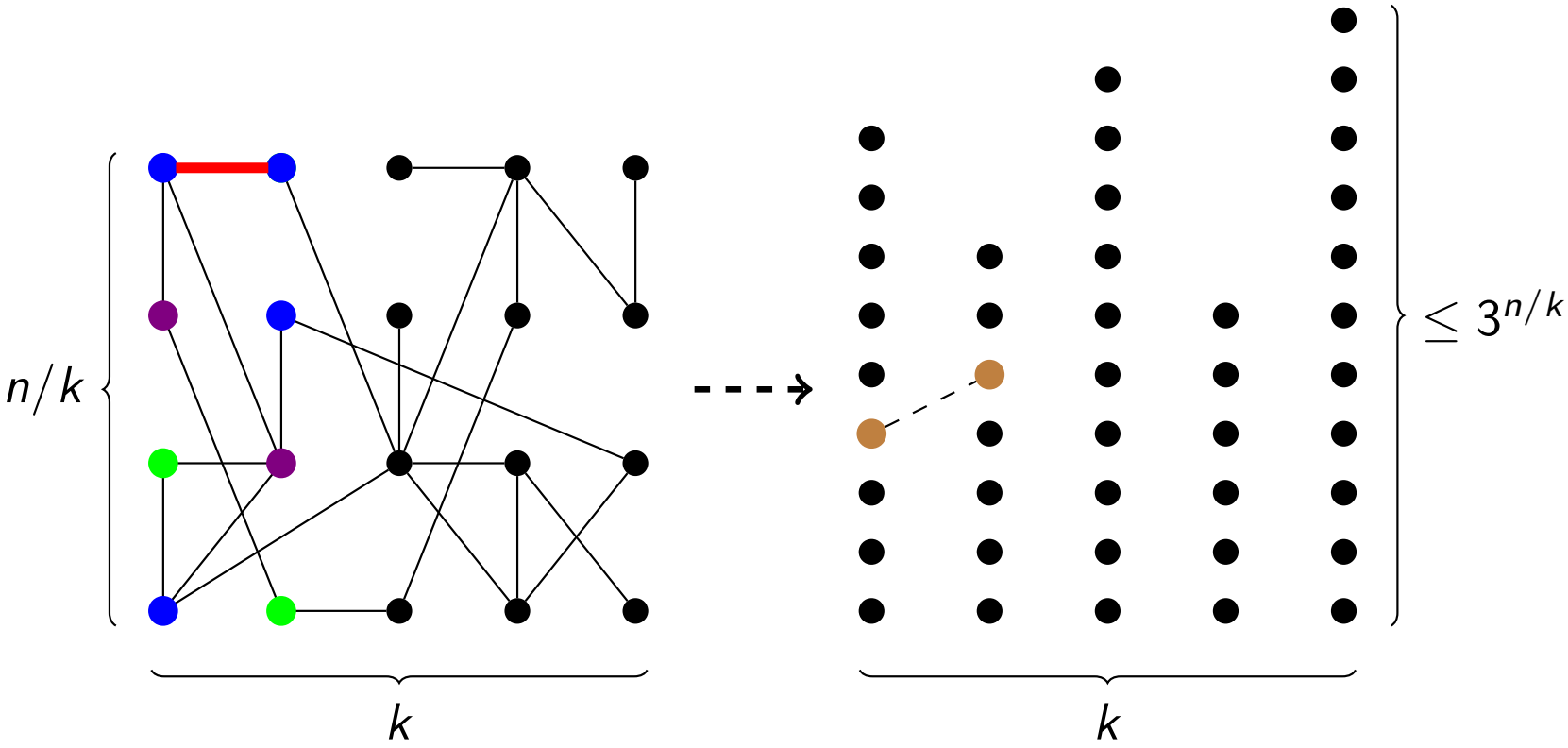

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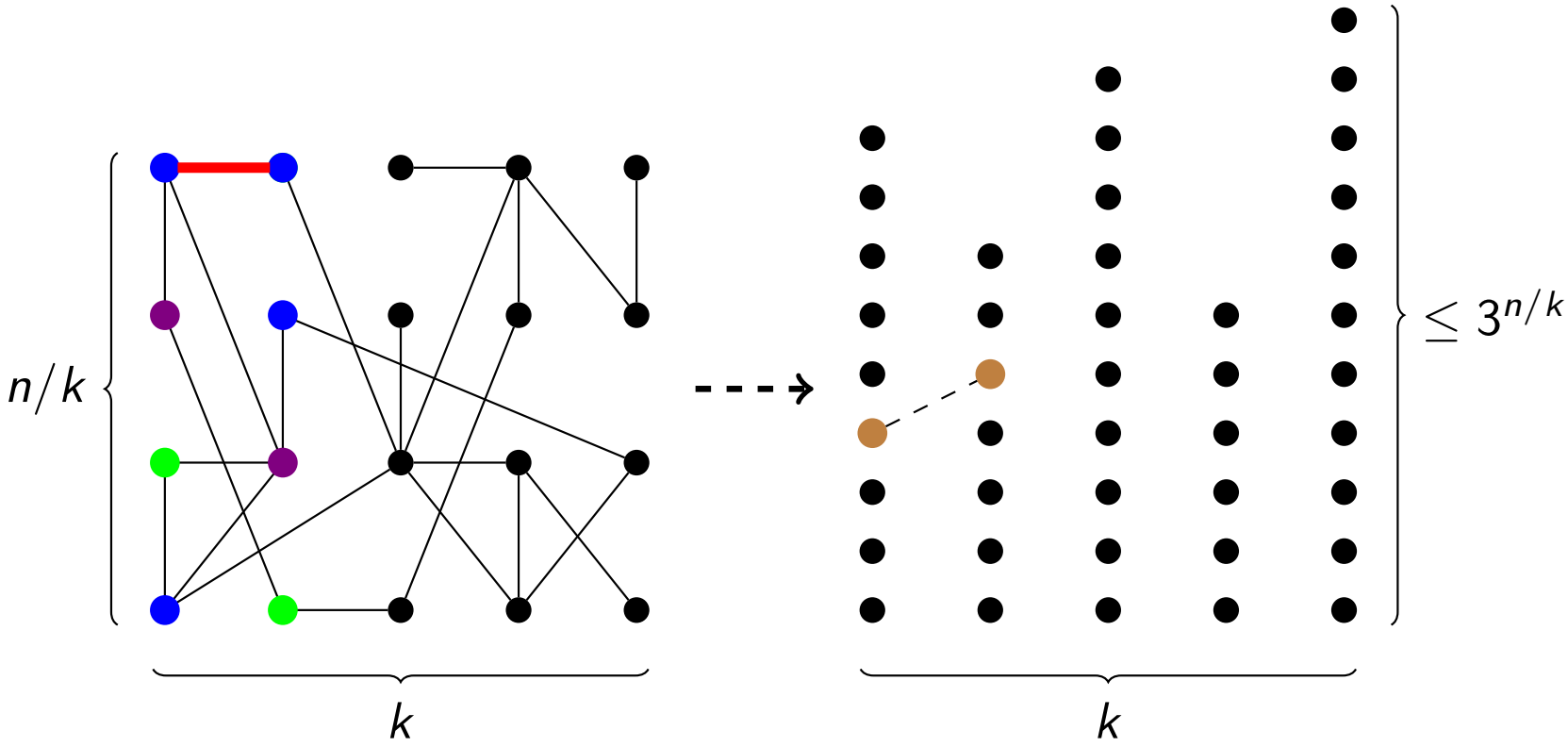

Left graph admits 3-coloring iff right graph contains k-clique.

Hardness for Clique

Theorem (Chen, Huang, Kanj, Xia)

Unless ETH fails, Clique cannot be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f.

Proof continued:

- Constructed graph has $N \leq k \cdot 3^{n/k}$ vertices.

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- Similarly $k = \log \log n$ implies no $2^{2^k} \cdot N^{o(k)}$ time algorithm.
- To exclude all computable $f(k)$, one needs roughly $k = f^{-1}(n)$ (technical difficulties omitted).

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  - the nonexistence of an $f(k) \cdot n^{o(\sqrt{k})}$ algorithm under ETH.
- **Example**: Planar d-Scattered Set — given an edge-weighted planar graph G, a real d and an integer k, verify whether there are k points in pairwise distance d from each other.
  (Marx, P)

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- Therefore, lower bounds under SETH are much more delicate and much scarcer.
- Probably more during Ryan's talk in the morning;
  now a quick parameterized perspective.

SETH and treewidth DPs

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- Space of states is formed by all relevant interactions between the solution and the bag.
- These results are tight.

SETH and treewidth DPs

Theorem (LMS+CNPPvRW)

Assume that CNF-SAT cannot be solved in time $\mathcal{O}^\star(c^n)$ for any $c < 2$. Then for every $\varepsilon > 0$ the following holds (p/t is the width of a given path/tree decomposition of the input graph):

- Independent Set cannot be solved in time $\mathcal{O}^\star((2-\varepsilon)^p)$;
- Dominating Set cannot be solved in time $\mathcal{O}^\star((3-\varepsilon)^p)$;
- Odd Cycle Traversal cannot be solved in time $\mathcal{O}^\star((3-\varepsilon)^p)$;
- Steiner Tree cannot be solved in time $\mathcal{O}^\star((3-\varepsilon)^p)$;
- Feedback Vertex Set cannot be solved in time $\mathcal{O}^\star((3-\varepsilon)^p)$;
- Connected Vertex Cover cannot be solved in time $\mathcal{O}^\star((3-\varepsilon)^p)$.

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Theorem (Cygan, Kratsch, Nederlof)

Hamiltonian Path can be solved in time $\mathcal{O}^\star((2+\sqrt{2})^p)$, but the existence of an algorithm with running time $\mathcal{O}^\star((2+\sqrt{2}-\varepsilon)^p)$ would yield an algorithm for CNF-SAT with running time $\mathcal{O}^\star(c^n)$ for some $c < 2$.

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- **Open**: The algorithmic result **does not** work for treewidth.

SETH and covering problems

Set Cover

Input: Universe U, set family $\mathcal{F} \subseteq 2^U$, integer k
Question: Is there a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| \leq k$ s.t. $\bigcup \mathcal{G} = U$?

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- Could any of these be improved?

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Set Cover Conjecture

Let λ_q be the infinimum of the set of constants c such that q-Set Cover can be solved in time $\mathcal{O}^\star(2^{cn})$, where n is the size of the universe. Then $\lim_{q\to\infty} \lambda_q = 1$. In particular, there is no algorithm for the general Set Cover problem that runs in $\mathcal{O}^\star((2-\varepsilon)^n)$ for any $\varepsilon > 0$.

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- SETH gives a precise bound on the base of the exponent, but is less plausible and less applicable.
- ETH and SETH are robust assumptions, under which for many problems we can pinpoint the precise influence of a parameter on the complexity of the problem.
- **Optimality program**: Finding matching lower and upper bounds on the parameterized complexity of various problems.

Commercial break!

This, and many more in the new book

Parameterized algorithms

by Cygan, Fomin, Kowalik,
Lokshtanov, Marx, Pilipczuk, P., and Saurabh.
Out in early 2015.