The Parameterized Complexity of Geometric Graph Isomorphism

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Outline

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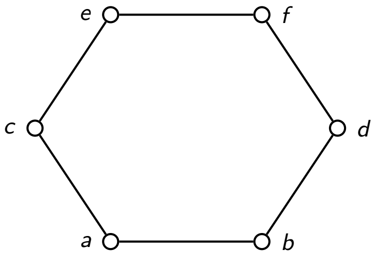
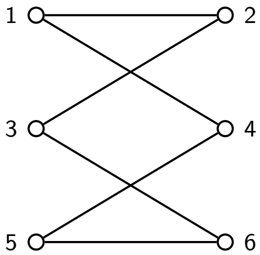
Graph Isomorphism

GI

Input: Graphs G and H

Question: Is G isomorphic to H? I.e., is there an adjacency-preserving bijection between the vertex sets of G and H?





Complexity of GI

Best known algorithm for GI runs in time $2^{\mathcal{O}(\sqrt{n \log n})}$ [Babai, Luks 1983]. Polynomial time algorithms known for restricted graph classes

- bounded genus graphs — Miller 1980
- bounded degree graphs — Luks 1982
- bounded eigenvalue multiplicity graphs — Babai et al. 1982
- bounded treewidth graphs — Bodlaender 1990
- graphs with excluded topological minors — Grohe, Marx 2012

$\mathrm{GI} \in$ FPT, parameterized by

- eigenvalue multiplicity — Evdikomov et al. 1997
- treewidth — Lokshtanov et al. 2014

Approaches: Graph-theoretic, group-theoretic, combinatorial, geometrical . . .

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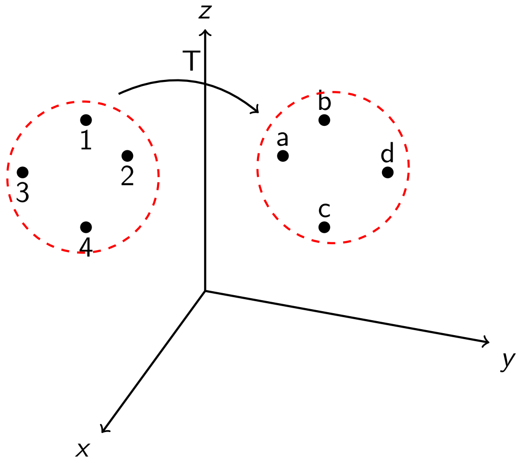
Geometric Graph Isomorphism

GEOM-GI
Input: Point sets $A, B \subseteq Q^k$ of size n
Parameter: k, dimension of the host vector space
Question: Is there is a distance preserving bijection from A to B?

Geometric Graph Isomorphism



Geometric Graph Isomorphism

(Equivalent Formulation:) Does there exist a transformation T of the host space ($T : \mathbb{R}^k \to \mathbb{R}^k$) such that $TA = B$?

- T preserves lengths: $\|Tx\| = \|x\|$ and $\|Tx - Ty\| = \|x - y\|$
- T preserves dot product: $(Tx)^t(Ty) = x^t y$ for all $x, y \in \mathbb{R}^k$

Formally, we call such a T to be an *orthogonal* transformation. We call A and B to be *geomterically-isomorphic via* T.

Fixed Parameter Tractability of GEOM-GI

Dimension of the host space is an important parameter.

Lemma

GEOM-GI *can be solved in polynomial time for bounded dimension k.*

Lemma

$\mathrm{GI} \leq_p \mathrm{GEOM\text{-}GI}_n$.

The reduction maps graphs on n vertices to point-sets in n dimensional host space.

Fixed Parameter Tractability of Geom-GI

Parameterized Algorithms?

- $\mathcal{O}^*(2^{\mathcal{O}(k^4)})$ time complexity, uses cellular algebras [Evdikomov, Ponomarenko]

Theorem (Arvind, R.)

Given point-sets $A, B \in \mathbb{Q}^k$, there is a deterministic $\mathcal{O}^*(k^{\mathcal{O}(k)})$ time algorithm which decides whether A is isomorphic to B.

P time algorithm for bounded dimension

For a k dimensional space, the transformation T can be uniquely described by its action on k independent vectors.

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Exhaustive Search Algorithm: On input sets A and B in $\mathbb{Q}^k$, size n,

1. Fix k linearly independent vectors $\{a_1, \ldots, a_k\}$ in A.
2. Branch on the possible images $\{b_1, \ldots, b_k\}$ inside B.
3. Let T be the unique transformation which sends $\{a_i\}$ to $\{b_i\}$.
4. Check if T sends A to B in a distance preserving manner.

The algorithm runs in $\mathcal{O}(n^k)$ time.

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Improving over exhaustive search

Build sets S_A and S_B with the following properties

- $|S_A| = |S_B| = f(k)$ *(small-sized)*.
- If A and B are isomorphic via an orthogonal T, then S_A and S_B are also isomorphic via T *(isomorphism-invariant)*.

Improvement: Would imply a $(f(k))^k$ branching, instead of n^k branching.

Lattices

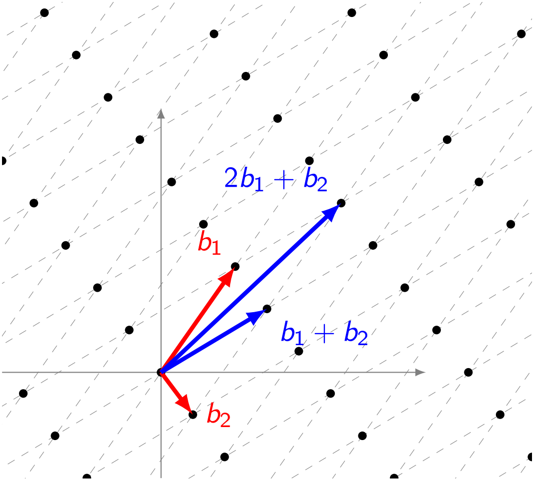

Figure: Integer lattice generated by vectors b_1 and b_2.

Lattices

Definition

The *lattice* generated by a set $A \in \mathbb{R}^k$, denoted by $\mathcal{L}_A$, is the set of all *integer* linear combinations of the set A.

$$\mathcal{L}_A = \{\alpha_1 a_1 + \cdots + \alpha_n a_n \,|\, \alpha_i \in \mathbb{Z}\}$$

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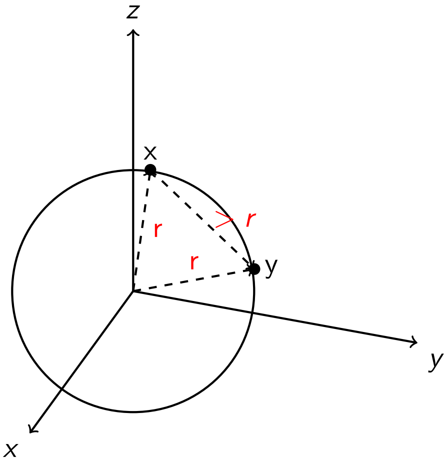
Claim

Let S_A be the set of shortest vectors in $\mathcal{L}_A$. Then, S_A is isomorphism-invariant.

Proof.

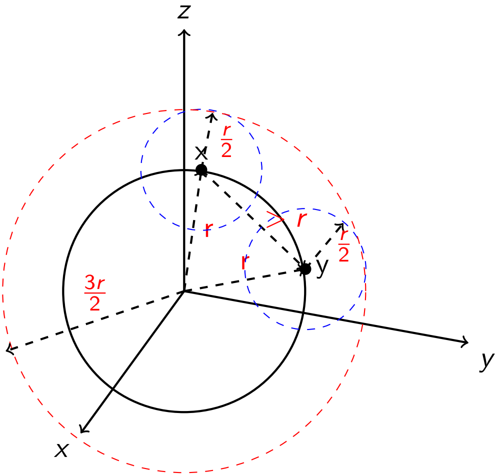
Suppose A and B are isomorphic via T, i.e. $TA = B$. Then, $T(\mathcal{L}_A) = \mathcal{L}_B$. Since T preserves lengths, $T(S_A) = S_B$. □

Shortest vectors in lattices



$$x, y \in \mathcal{L} \Rightarrow x - y \in \mathcal{L}$$

Shortest vectors in lattices



$$x, y \in \mathcal{L} \Rightarrow x - y \in \mathcal{L}$$

$$f(k) \leq \frac{\mathcal{O}\left(\left(\frac{3}{2}r\right)^k\right)}{\mathcal{O}\left(\left(\frac{r}{2}\right)^k\right)} = 3^k$$

Improved Algorithm for GEOM-GI

Given point sets $A, B \subset \mathbb{Q}^k$ as input,

1. Compute sets S_A, S_B of shortest vectors in $\mathcal{L}_A$, $\mathcal{L}_B$ using the SVP algorithm of [MV10].

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   - Fix basis in S_A, branch in S_B, ...
3. If $dim(Span(S_A)) = k_1 < k$, *recursively construct the isomorphism*
   - Construct all $T_1 : Span(S_A) \to Span(S_B)$ enumeratively.
   - *Project sets* A *and* B *out of* $Span(A)$ *and* $Span(B)$.
   - Construct all $T_2 : Span(S_A)^{\perp} \to Span(S_B)^{\perp}$ recursively.
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Branching: $B(k) = (3^{k_1})^{k_1} \cdot B(k - k_1)$, which yields a $\mathcal{O}^*(2^{\mathcal{O}(k^2)})$ branching.

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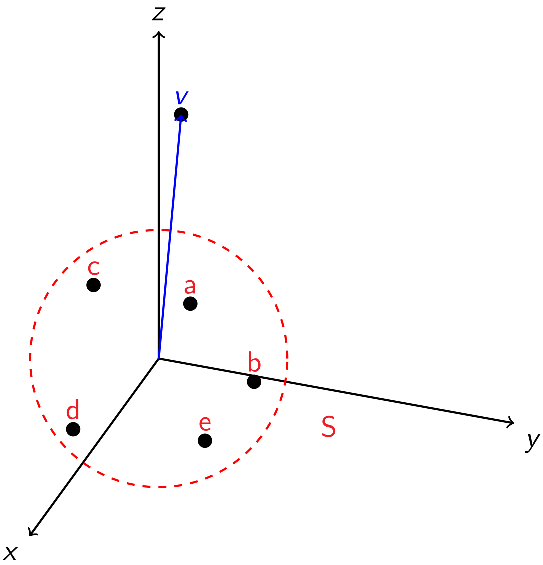
Exploiting the structure of the basis

We are searching for an isomorphism T which maps A to B

- length-preserving, i.e. $\|Tx\| = \|x\|$
- preserves dot product, i.e. $(Tx)^t(Ty) = x^t y$

Improvement: An isomorphism preserves the geometry of the basis set. (e.g. consider orthogonal bases ...)

Exploiting the structure of the basis



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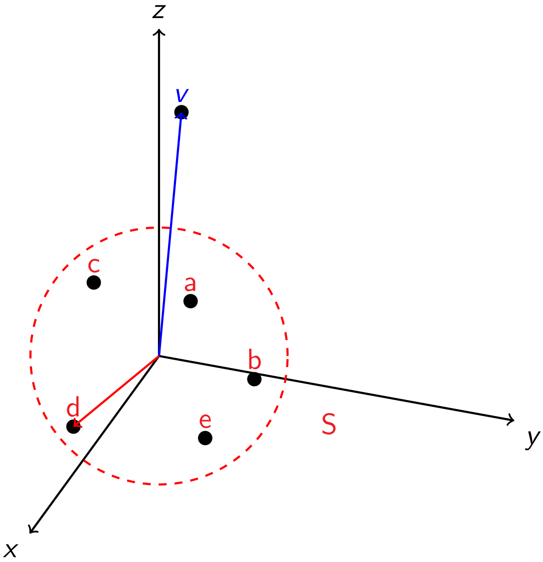

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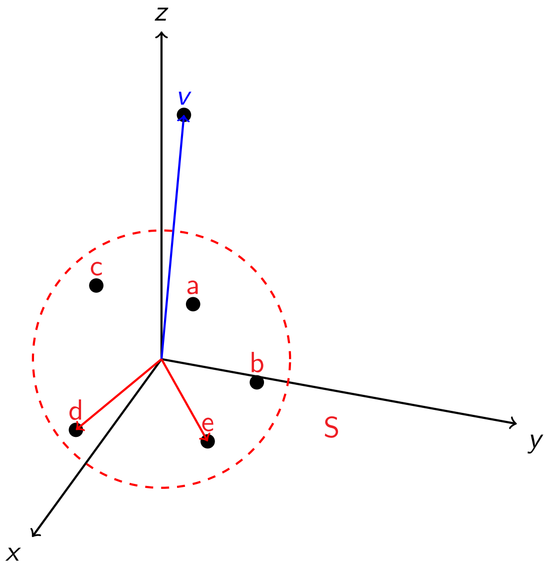

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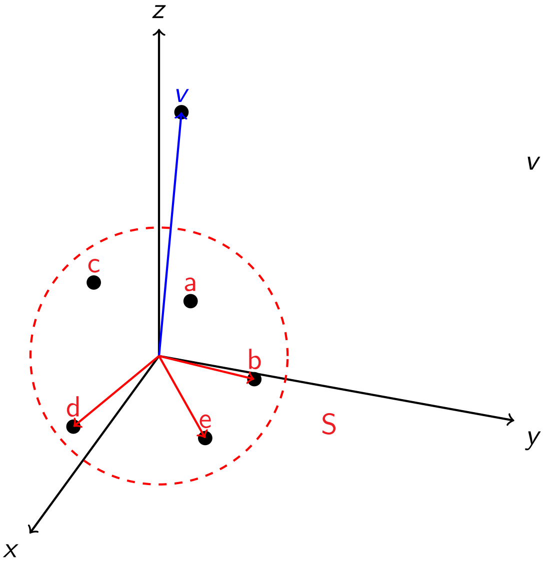

$$v^t d < v^t e < v^t b$$

Exploiting the structure of the basis

Let S be a set of vectors.

Definition (Haviv Regev)

A vector v defines a *chain* of length k in S if for some lin. ind. vectors $a_1, \ldots, a_k \in S$

- a_1 uniquely minimizes $x^t v$ over all $x \in S$
- a_2 uniquely minimizes $x^t v$ over all $x \in S \setminus Span(a_1)$
- and in general, a_i uniquely minimizes $x^t v$ over all $x \in S \setminus Span(a_1, \ldots, a_{i-1})$.

I.e. $v^t a_1 < \cdots < v^t a_k$

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I.e. $v^t a_1 < \cdots < v^t a_k$ *Remark:* A random v defines a chain of length 1 w.h.p. (Isolation Lemma [MVV]).

Chain isolation in lattices

Given a lattice $\mathcal{L}$, call a vector *dual* if

- it has *integral* dot product with every vector in $\mathcal{L}$.

The set of all dual vectors forms a lattice, the dual lattice $\mathcal{L}_A^*$.

Chain isolation in lattices

Given a lattice $\mathcal{L}$, call a vector *dual* if

- it has *integral* dot product with every vector in $\mathcal{L}$.

The set of all dual vectors forms a lattice, the dual lattice $\mathcal{L}_A^*$.

There are short dual vectors in $\mathcal{L}_A^*$ which define chains inside the set S_A of shortest vectors.

Chain isolation in lattices

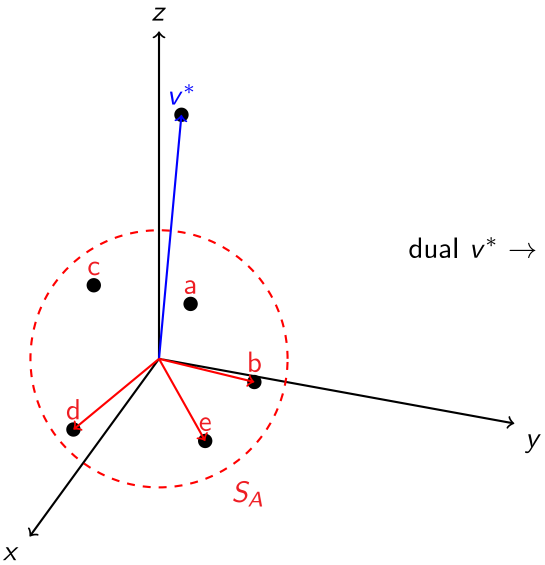
Theorem (Haviv, Regev 14)

There exists a dual vector $v \in \mathcal{L}_A^*$ *such that*

- *(defines chain)* v *defines a chain of length* k *in* S_A *and*
- *(small length)* $\|v\| \leq k^{\mathcal{O}(1)} \cdot \lambda(\mathcal{L}^*)$

Moreover, the set of all such *isolating* vectors has size at most $\mathcal{O}^*(k^{\mathcal{O}(k)})$, and can be computed in time $\mathcal{O}^*(k^{\mathcal{O}(k)})$.

A faster $\mathcal{O}^*(k^{\mathcal{O}(k)})$ isomorphism algorithm



dual $v^* \to$ chain d-e-b in S_A

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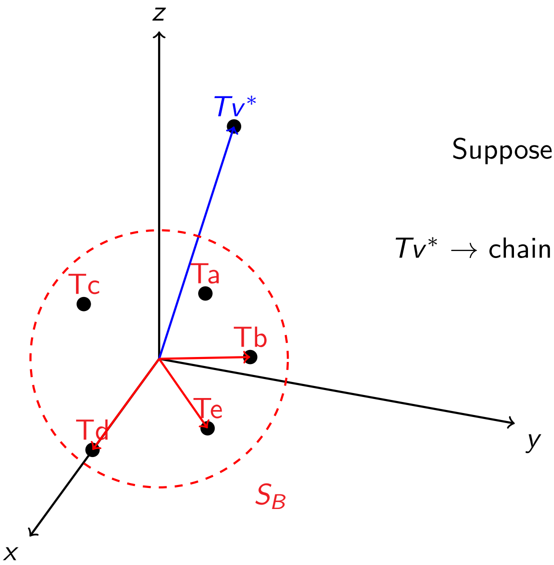

Suppose $A \equiv B$ via T

$Tv^* \to$ chain Td-Te-Tb in TS_A

Outline of $\mathcal{O}^*(k^{\mathcal{O}(k)})$ isomorphism algorithm

Algorithm: On input sets $A, B \in \mathbb{Q}^k$,

1. Compute the set of shortest vectors S_A, S_B in $\mathcal{L}_A$, $\mathcal{L}_B$.
2. Compute the set Γ_A, Γ_B of $\mathcal{O}^*(k^{\mathcal{O}(k)})$ dual vectors which induce chains.

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3. Pick a dual vector $u \in \Gamma_A$. Let $\{a_1, \ldots, a_k\}$ be the chain-basis defined by u inside S_A.
4. For every dual vector $v \in \Gamma_B$,
    - Let $\{b_1, \ldots, b_k\}$ be the chain-basis defined by v inside S_B.
    - Check if $T : \{a_i\} \to \{b_i\}$ is an orthogonal map which sends A to B.

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   - Check if $T : \{a_i\} \to \{b_i\}$ is an orthogonal map which sends A to B.
5. In case S_A is not k-dimensional, recurse ...

Outline of $\mathcal{O}^*(k^{\mathcal{O}(k)})$ canonization algorithm

Given a point set $A \subset \mathbb{Q}^k$, $f : \mathbb{Q}^k \to \mathbb{Q}^k$ is a *canonizing function* if it has the following properties

- $f(A)$ is isomorphic to A
- if A is isomorphic to B, then $f(A) = f(B)$.

Computing such a f is least as hard as the isomorphism problem.

Outline of $\mathcal{O}^*(k^{\mathcal{O}(k)})$ canonization algorithm

Algorithm: On input set $A \subset \mathbb{Q}^k$,

- Compute the set of shortest vectors S_A in $\mathcal{L}_A$.
- Compute a set Γ_A of $\mathcal{O}^*(k^{\mathcal{O}(k)})$ special bases inside S_A.
- For each basis $J = \{u_1, \ldots, u_k\} \in \Gamma_A$,
  - compute the Gram matrix $G_{i,j} = u_i^t u_j$.
  - compute the coordinates $C(a_i)$ of every point $a_i \in A$ in the basis J.
- Output the lexicographically least description $\sigma = (G, K)$ obtained above.

  If $A \equiv_B T$, then TJ generates same description for B as J generates for A.

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  If $A \equiv_B T$, then TJ generates same description for B as J generates for A.
- Compute a canonical A^* using σ as follows.
  - Find unique lower triangular matrix L such that $LL^T = G$.
  - Use row vectors of L to get a basis $J^* = \{v_1, \ldots, v_k\}$.
  - Compute the point set A^* s.t. a_i^* is $K(a_i)$-linear combination of J^*.

Future Directions

- A $\mathcal{O}^*(2^{\mathcal{O}(k)})$ algorithm for geometric graph isomorphism and canonization in Euclidean metric?
  - Faster canonization would give a $2^{\mathcal{O}(n)}$ algorithm for hypergraph canonization
- GEOM-GI in other l_p metrics?
  - Linear algebra breaks down for non-Euclidean metrices; combinatorial algorithms work.
  - Two-dimensional case is polynomial time.
  - Reductions between GEOM-GI for various metrics, similar to embeddings.

Thank you!