Representative Family

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Alice VS Bob

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$$\mathcal{F} = \{\{a, b, c\},\ \{a, b, d\},\ \{a, d, e\}\}$$

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Rules of the Game

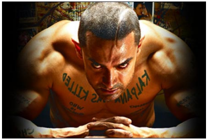
Board: universe of size n

All Alice's sets have size p

Bob picks a set B of size q

Alice wins if she has a set disjoint from B

Short memory of Alice



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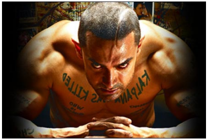

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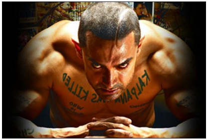

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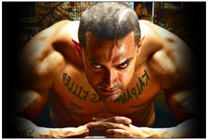

Alice can not remember all those sets.

Alice hates losing to Bob.

Can she forget a set A from $\mathcal{F}$, and be sure this will not make the difference between winning and losing?

Irrelevant sets

$A \in \mathcal{F}$ is called irrelevant if :

for every set B of size q such that $A \cap B = \emptyset$,
then there is a set $\widehat{A} \in \mathcal{F}$ such that $\widehat{A} \cap B = \emptyset$.

Alice may forget the irrelevant sets.

Relevant Sets

$$F = \{A_1,\ A_2,\ A_3, \cdots, A_m\}$$

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Bollabás' Lemma [1966]

Let $A_1, A_2, \cdots, A_m$ be sets of size of p and

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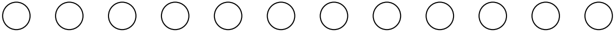
Can $m \leq f(p, q)$?

Proof of Bollabás' Lemma



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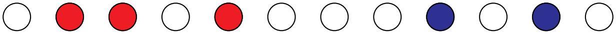

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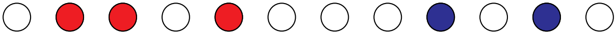

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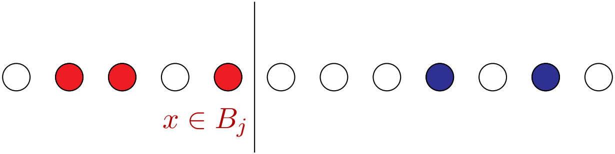

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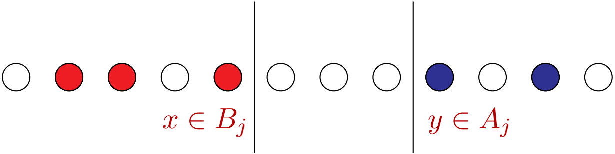

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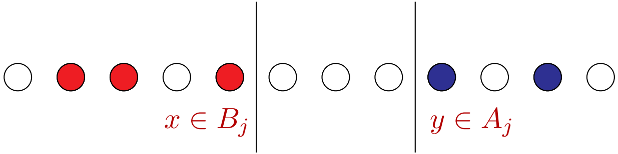

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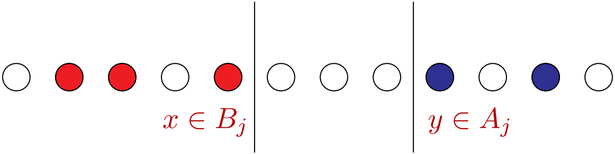

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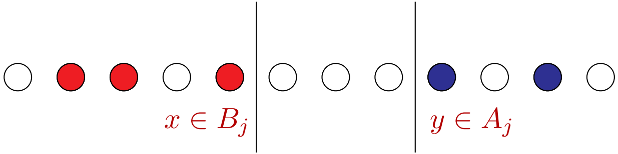

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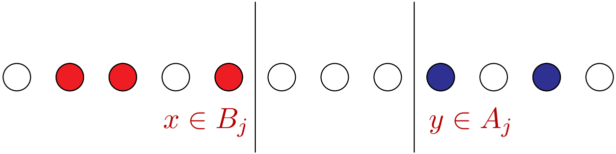

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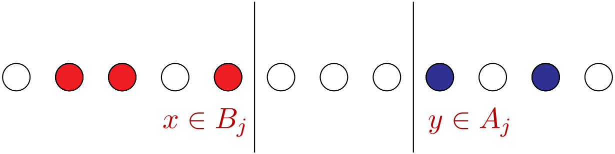

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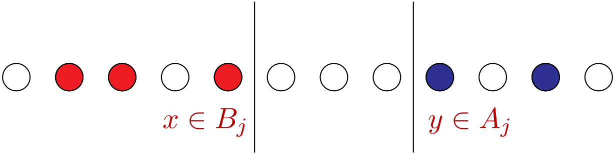

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Hence $m \leq \binom{p+q}{p}$ No dependence on Universe size n

Bollabás' Lemma immediately implies

that Alice only needs to remember at

most $\binom{p+q}{p}$ sets.

Representative Sets

Let $\mathcal{F} = \{S_1, \ldots, S_t\}$ be a family of p-sized sets. $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ is a q-representative family for $\mathcal{F}$ (denote by $\widehat{\mathcal{F}} \subseteq_{rep}^{q} \mathcal{F}$), if :

For any set Y of size q,

if there is a set $X \in \mathcal{F}$ such that $X \cap Y = \emptyset$
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Corollary of Bollabás' Lemma. For every $\mathcal{F}$, there is q-representative family $\widehat{\mathcal{F}}$ for $\mathcal{F}$ of size at most $\binom{p+q}{p}$.

Computational Problem

Input: A Family $\mathcal{F}$, of sets of size p and an integer q.
Output: $\widehat{\mathcal{F}} \subseteq_{rep}^{q} \mathcal{F}$ of size $\binom{p+q}{p}$.

Computing Representative Sets

Will show: We can compute $\widehat{\mathcal{F}} \subseteq^q_{rep} \mathcal{F}$ of size $\binom{p+q}{p}$ in time

$$\mathcal{O}\left(|\mathcal{F}|\binom{p+q}{p}^{\omega-1}\right)$$

where ω is the matrix multiplication constant ≤ 2.373.

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But first – An easy application.

d-Hitting Set

Input: A Family $\mathcal{F} = \{S_1, \ldots, S_m\}$, of sets of size d over a universe U and an integer k
Question: Does there exists a $X \subseteq U$ of size k such that $\forall i, S_i \cap X \neq \emptyset$?

- It has an easy d^k branching algorithm
- We will show that it has a kernel of size $\mathcal{O}(k^d)$.

d-HITTING SET as a Game

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Is $\{d,e\}$ is a hitting set

d-Hitting Set as a Game

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No, since
$\{a, b, c\}$

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Suppose not, then there is $X \in \mathcal{F}$ s.t $X \cap Y = \emptyset$.
This implies $\hat{X} \in \widehat{\mathcal{F}}$ s.t $\hat{X} \cap Y = \emptyset$
contradiction!

Playing on a matroid

Matroid

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a matroid if it satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.

(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists\ e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

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Uniform Matroid: A pair $M = (E, \mathcal{I})$ over an n-element ground set E is called a uniform matroid (denoted by $U_{n,k}$) if

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rank($U_{n,k}$)=k.

Linear Matroid

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. Given A we define the matroid $M = (E, \mathcal{I})$ as follows.
A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are linearly independent over $\mathbb{F}$.

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} * \text{ are elements of } \mathbb{F}$$

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The matroids that can be defined by such a construction are called linear matroids.

Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.
Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{\ell \times m} \end{array}$$

Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.
Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\
\begin{bmatrix}
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & \cdots & *
\end{bmatrix}_{\ell \times m}
\end{array}
$$

A matroid $M = (E, \mathcal{I})$ is called representable or linear if it is representable over some field $\mathbb{F}$.

Representation of Uniform matroid

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1} \end{bmatrix}_{k \times n} \end{array}$$

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Determinant of any $k \times k$ sub matrix is non zero

Representative Sets on Matroids

Let $M = (E, \mathcal{I})$ be a matroid.

Let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a family of p-sized subsets from $\mathcal{I}$.

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For any $Y \subseteq E$ of size q,

if there is a set $X \in \mathcal{S}$ such that $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$,
then there is a set $\widehat{X} \in \widehat{\mathcal{S}}$ such that $\widehat{X} \cap Y = \emptyset$ and $\widehat{X} \cup Y \in \mathcal{I}$

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(**for a uniform matroid** $U_{n,p+q}$)

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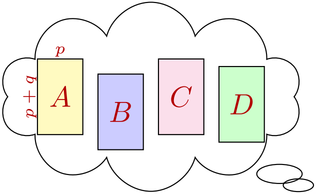
Playing on Linear Matroid

Playing on Linear Matroid



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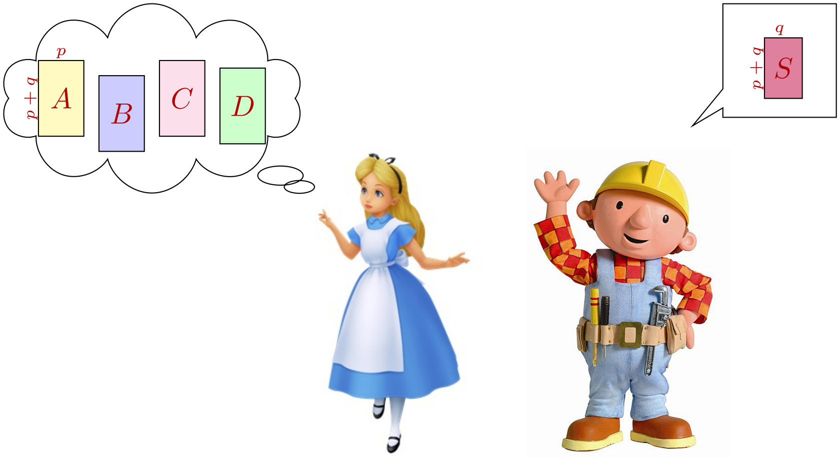

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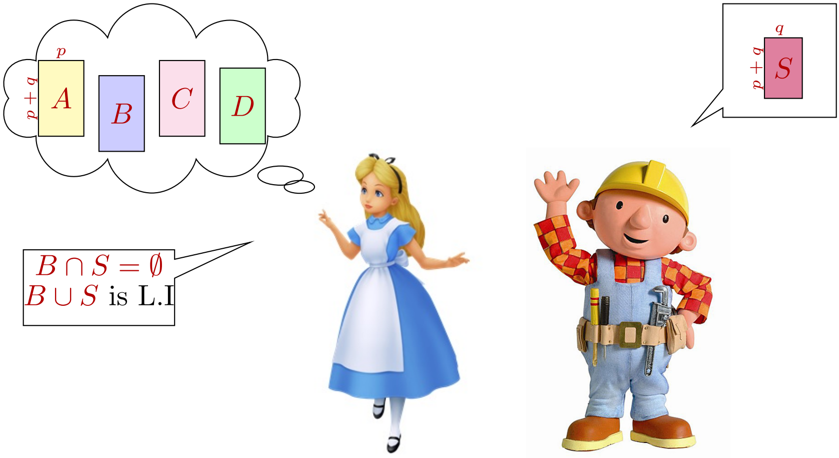

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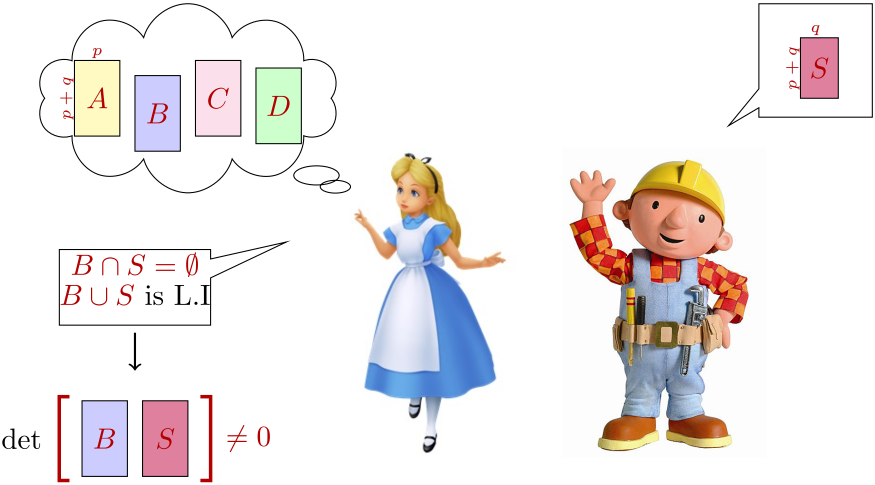

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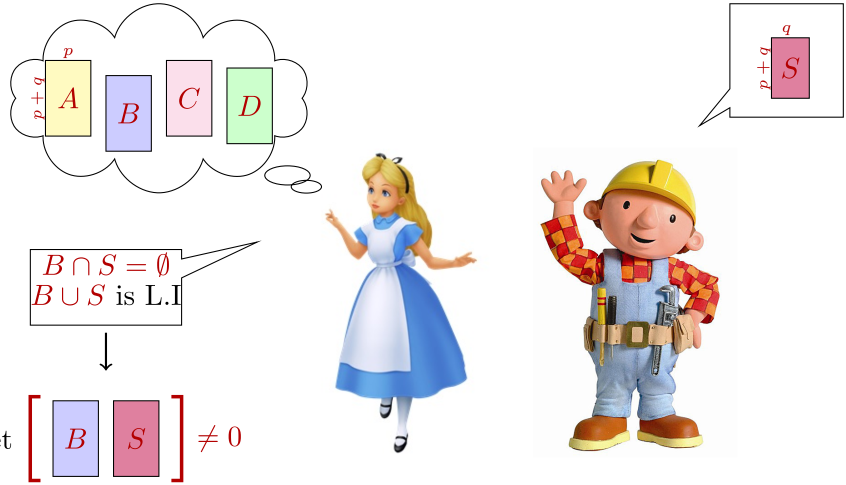

Note: This game generalizes the first game because Uniform matroid has a linear Representation.

Generalized Laplace Expansion

$$M = \begin{bmatrix} B & S \end{bmatrix}$$

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where $a = (p+1) + \ldots + (p+q)$

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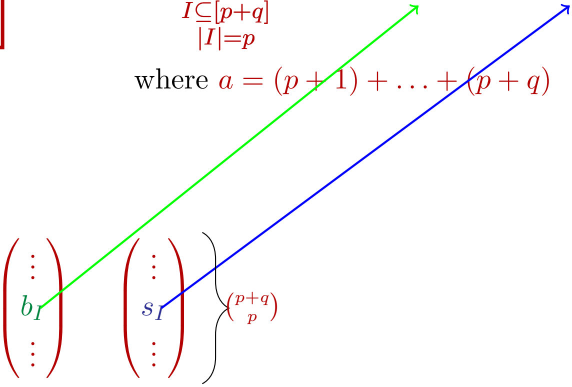

Generalized Laplace Expansion

$$M = \begin{bmatrix} B & S \end{bmatrix}$$

$$det(M) = \sum_{\substack{I\subseteq[p+q] \\ |I|=p}} (-1)^{a+\sum I} det(B_I) \cdot det(S_{\overline{I}})$$

where $a = (p+1) + \ldots + (p+q)$

$$det(M) = \begin{pmatrix} \vdots \\ b_I \\ \vdots \end{pmatrix} \bullet \left.\begin{pmatrix} \vdots \\ s_I \\ \vdots \end{pmatrix}\right\} \binom{p+q}{p}$$

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where $a = (p+1) + \ldots + (p+q)$

$$det(M) = \underbrace{\begin{pmatrix} \vdots \\ b_I \\ \vdots \end{pmatrix}}_{v_B} \bullet \underbrace{\begin{pmatrix} \vdots \\ s_I \\ \vdots \end{pmatrix}}_{u_S} \Bigg\} \binom{p+q}{p}$$

Giant Vector Game

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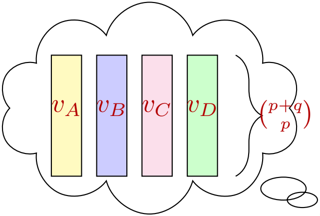

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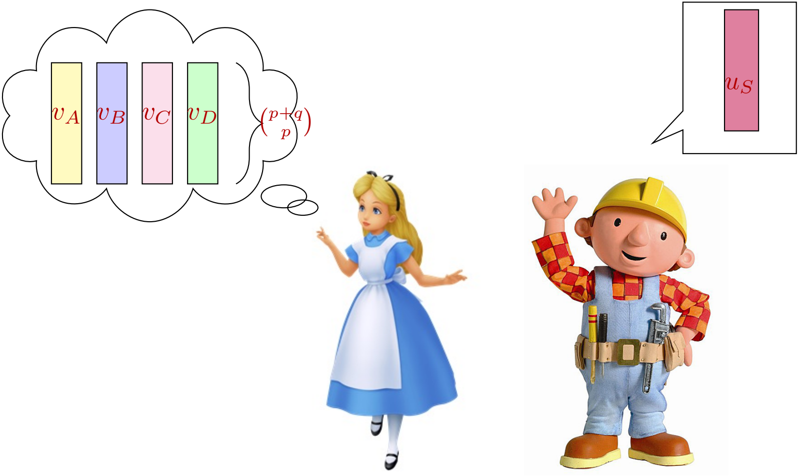

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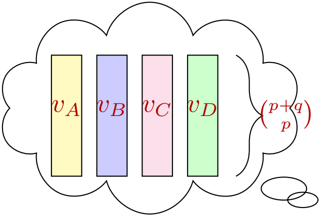

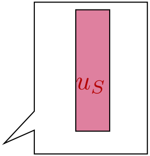

$$v_B \bullet u_S \neq 0$$

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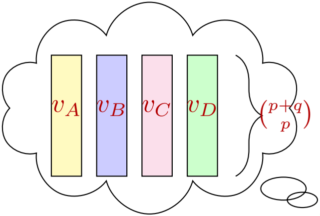

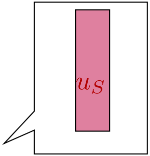

$$v_B \bullet u_S \neq 0$$

Can Alice forget some vectors from her collection?

Basis

Let $\mathcal{F} = \{v_A, v_B, \ldots,\}$ be Alice's collection of vectors.

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Basis of $\mathcal{F}$ can be computed in time

$$\mathcal{O}\left(|\mathcal{F}|\binom{p+q}{p}^{\omega-1}\right)$$

Wrap up

Alice has family of p-sized sets

$\rightarrow$ family of $(p+q) \times p$ matrices

$\rightarrow$ family of $\binom{p+q}{p}$-dimensional vectors

Keep linearly independent vectors

$\rightarrow$ Keep the corresponding sets

Computing Representative Family

Theorem[Fomin, Lokshtanov, Saurabh (2013)]
Let $M = (E, \mathcal{I})$ be linear matroid of rank $p+q$. Let $\mathcal{F}$ be a family of p-sized sets in $\mathcal{I}$. Then a q-representative family for $\mathcal{F}$ of size $\binom{p+q}{p}$ can be computed deterministically in time

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Theorem[Lokshtanov, Misra, P., Saurabh(2014)].
Let $M = (E, \mathcal{I})$ be linear matroid of rank $n \gg p + q$. Let $\mathcal{F}$ be a family of p-sized sets in $\mathcal{I}$. Then a q-representative family for $\mathcal{F}$ of size $\binom{p+q}{p}$ can be computed deterministically in time

$$\mathcal{O}\left(|\mathcal{F}|\binom{p+q}{p}^{\omega-1} n^{\mathcal{O}(1)}\right).$$

Application

- Steiner Tree and FVS can be solved in time $\mathcal{O}((1+2^{\omega-1}3)^{\mathbf{tw}}\mathbf{tw}^{\mathcal{O}(1)}n)$

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- Representative Sets is used to design polynomial kernels for OCT, Almost 2-SAT, a variant of Multiway Cut, etc.

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Is it possible to get faster algorithm for representative sets for special matroids like Uniform matroids, Graphic matroids, etc.

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Theorem[Fomin,Lokshtanov,P.,Saurabh(2014)]
Given family $\mathcal{F}$ (each set has size p) and $0 < x < 1$.
Then $\widehat{\mathcal{F}} \subseteq^q_{rep} \mathcal{F}$ of size at most

$$x^{-p}(1-x)^{-q} \cdot 2^{o(p+q)}$$

can be computed in time

$$\mathcal{O}((1-x)^{-q} \cdot 2^{o(p+q)} \cdot |\mathcal{F}| \cdot \log n)$$

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- k-MlD over $\mathbb{Z}^+$ can be solved in time $3.8408^k n^{\mathcal{O}(1)}$.

Open Probelms

- Can we get faster representative sets computation in graphic matroid, transversal matroid and gammoid, better than the computation in linear matroid?

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- Can we find Representative sets in transversal matroids and gammoids deterministically?

Thank You.