Homework 3

Assigned: 13th April; Due Date: 27th April

Discussion is encouraged, but please acknowledge it and write your answers independently. Also, cite any sources that you have referred to.

Q.1 Let $A, B \in \mathbb{R}[x]$. Show that $\deg(\mathtt{GCD}(A, B)) \geq k$ iff all the principal subresultant coefficients from $0, \ldots, k-1$ are zero.

Q.2 Let $a_0, \ldots, a_k$ be a sequence of $k+1$ real numbers such that

$$0 \leq a_0, a_1 \geq 0, a_2 \geq 0, \ldots, a_{k-1} > 0, a_k \leq 0.$$

Consider the $k-1$ numbers

$$b_j := a_k - 2\cos\theta a_{k-1} + a_{k-2},\ j = 2, \ldots, k.$$

Show that if $\theta \in (\frac{\pi}{k-1}, \frac{2\pi}{k}]$ then at least one of b_j's is negative.

Hint: Prove by contradiction. What if all b_j's are positive?

Q.3 Given $f \in \mathbb{Z}[x, y]$, show that $\deg(\mathtt{GCD}(f, f_y)) > 0$ iff $\deg(\mathtt{GCD}(f, f_x)) > 0$.

Q.4 Show that $f \in \mathbb{Z}[x, y]$, with degree d, is homogeneous iff $f(\lambda x, \lambda y) = \lambda^d f(x, y)$, for a non-zero constant λ.

Q.5 Given two polynomials $f, g \in \mathbb{Z}[x, y]$, with $\deg(f) = m$ and $\deg(g) = n$ and non-zero constant terms, define the homogeneous polynomials $F(x, y, z) := z^m f(x/z, y/z)$ and $G(x, y, z) := z^n g(x/z, y/z)$. Show that $\mathtt{res}_z(F, G)$ is a homogeneous polynomial of degree mn.

Q.6 Given a point $\mathbf{p} \in \mathbb{R}^2$ the **central reflexion** of a point $(x, y) \in \mathbb{R}^2$ w.r.t. $\mathbf{p}$ is the point (x', y') such that $\mathbf{p}$ is the midpoint of the line segment joining (x, y) and (x', y'). A point $\mathbf{p}$ is called **center** of a curve f, if for all points $(x, y) \in \mathbb{R}^2$, $f(x, y) = \lambda f(x', y')$, where (x', y') is the central reflexion of (x, y) w.r.t. $\mathbf{p}$ and $\lambda \neq 0$.

(a) What is a center for a line in the plane?

(b) Show that the concept of a center is affinely invariant.

(c) Give a characterization for origin to be a center of a curve f.

Q.7 Argue that if a curve f has a vertical asymptote at $x = \alpha$ then α is a root of $\mathtt{lead}_y(f)$.