Homework 2

Assigned: 29th Feb; Due Date: 16th March

Discussion is encouraged, but please acknowledge it and write your answers independently. Also, cite any sources that you have referred to.

Throughout this assignment, $A(x) = \sum_{i=0}^{n} a_i x^i$, is an integer polynomial with L-bit coefficients, and roots $\alpha_1, \ldots, \alpha_n$.

Q.1 Prove the Descartes's rule of signs.

Q.2 Let α be a root of A, and $N(\alpha)$ be another root of A closest to α. Show the following:

(a) $\prod_{i=1}^{n} |\alpha_i - N(\alpha_i)| = 2^{-O(nL+n^2)}$.

(b) Use this observation to simplify the analysis of the Descartes method (a brief argument will suffice).

(c) Let α' be a root of the derivative A' closest to a root α of A. Then

$$|\alpha - \alpha'| \geq \frac{|\alpha - N(\alpha)|}{n}.$$

Hints: (a) How many roots can be nearest to a given root? (c) Use the logarithmic-derivative.

Q.3 Show that the Budan-Fourier bound on the number of roots of A in an interval $[a, b]$ always exceeds the bound obtained by Jacobi's "little observation" by an even number, i.e.,

$$\mathrm{Var}(A(x+a)) - \mathrm{Var}(A(x+b)) = \mathrm{Var}(A; a, b) + \text{ non-negative even number }.$$

Q.4 Let $a_0, \ldots, a_n \in \mathbb{R}$, and $\alpha_i, \beta_i \in \mathbb{R}_{>0}$, $i = 0, \ldots, n-1$. Show that

$$\mathrm{Var}(a_0, \ldots, a_n) = \mathrm{Var}(a_0, \alpha_0 a_0 + \beta_0 a_1, \alpha_1 a_1 + \beta_1 a_2, \ldots, \alpha_{n-1} a_{n-1} + \beta_{n-1} a_n, a_n) + \text{ even number.}$$

Q.5 Given an integer polynomial $A(x) \in \mathbb{Z}[x]$, define $t(A) := \deg(A) + \|A\|_\infty$. Show that for two relatively prime polynomials $A(x), B(x)$, for all $x \in \mathbb{C}$

$$\max\{|A(x)|, |B(x)|\} \geq e^{-2t(A)t(B)}.$$