Assignment 1

Assigned Date: 20 Aug 2015

Due Date: 14 Sept 2015 (in class)

Please be precise, concise, and legible in your answers; a good idea may be to submit a pdf file with your answers. Discussion is encouraged amongst yourselves, but for each problem you should acknowledge the people involved. If you are using a resource (book, website, notes, etc.) then please cite it in your solution. Most importantly: *Please write your own solutions; copying of solutions will not be accepted.*

1. Analogous to the definition of BFS for LP in equation form, we may define a BFS for

   $$P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$$

   as the points $\mathbf{x}$ for which there are n linearly independent inequalities that are satisfied as equalities. (A is an $m \times n$ matrix; $m \geq n$.) Show that in this setting it is also the case that $\mathbf{x}$ is a BFS for P if and only if $\mathbf{x}$ is an extreme feasible solution in P.

2. Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, i.e., there is a Taylor series expansion of f around every point $\mathbf{x}$, or more formally, for all $\mathbf{y}, \mathbf{x}$ in a convex set S

   $$f(\mathbf{y}) - f(\mathbf{x}) = \langle \nabla f(\eta), (\mathbf{y} - \mathbf{x}) \rangle$$

   where ∇ is the gradient operator where the ith entry is $\partial f / \partial x_i$ evaluated at $\mathbf{x}$, and $\eta \in S$ is a point on the line segment joining $\mathbf{x}$ and $\mathbf{y}$. Show that f is convex iff for all $\mathbf{x}, \mathbf{y} \in S$, we have

   $$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), (\mathbf{y} - \mathbf{x}) \rangle.$$

   What does it mean for f if we have equality (instead of inequality) above?

3. In class we saw that if an LP has a well-defined optimum, then the optimum is achieved at a BFS. This allows us to prune our search space from $\mathbb{R}^n$ to just the basic solutions. But how can we decide whether the LP has a well-defined optimum? Consider the following two problems:

   (a) `Bounded-Opt`: Given an LP in equation form, decide which of the following holds:

      (i) The LP is infeasible.

      (ii) The LP is feasible but does not have a well-defined optimum.

      (iii) The LP is feasible and has a well-defined optimum.

   (b) `Feasible`: Given an LP in equation form, decide whether the feasible set is non-empty.

      Using the result shown in class (that an opt is achieved at BFS), show that `Bounded-Opt` reduces to `Feasible`. The reduction need not be poly time (something in terms of $\binom{n}{m}$ is acceptable).

4. Does a non-degenerate step in the simplex method moves us along an edge of the polyhedron of feasible solutions? Give rigorous arguments to support your answer.

5. (a) Show that every linear program in equational form can be reduced to one of the following form: maximize $\langle \mathbf{c}, \mathbf{x} \rangle$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}$ is on the regular simplex $\left\{\mathbf{x} | \sum_{j=1}^{n} x_j = 1, \mathbf{x} \geq 0\right\}$.

(b) Given an lp in the form above, define the following quantities for the tth step of the simplex method: $\mathbf{x}_t$ be the bfs, z_t be the value of the objective function, and $\theta_t > 0$ be the value of the incoming variable, which is selected using Dantzig's rule, i.e., corresponding to the maximum coefficient in the $\mathbf{r}$ vector in the tableau (note we are assuming no degeneracies). Let z^* be the opt value of the objective function, and $\Delta_t := z^* - z^t$. Show that

$$\Delta_t/\Delta_0 \leq \prod_{i=1}^{t}(1 - \theta_i).$$

On the average we expect $\theta_t > 1/m$ (why?), then bound the number of iterations we need to get the ratio on the LHS below some $\epsilon < 1$? What is so interesting about this bound?

6. Let G be an undirected connected graph on $n + 1$ vertices, and $Q(G)$ be its Laplacian matrix (i.e., $Q(G) = D - A$, where D is the diagonal matrix containing the degree of the ith vertex, and A is the adjacency matrix). Let $\Delta(G)$ be the convex hull of the rows of $Q(G)$. Show that $\Delta(G)$ is an n-dimensional simplex whose centroid is at the origin.