Assignment 1

Due Date: 28 Sept 2018

Please be precise, concise, and legible in your answers; a good idea may be to submit a pdf file with your answers. Discussion is encouraged, and so is its acknowledgement. If you are using a reference then cite it in your solution. *Please write your own solutions; copying of solutions will not be accepted.*

1. Give combinatorial proofs for the following identities.

   (a)
   $$n\binom{n+m-1}{m-1} = m\binom{n+m-1}{m}.$$

   (b)
   $$\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}.$$

   (c)
   $$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

   (d) For a prime p,
   $$\binom{pa}{pb} \equiv \binom{a}{b} \mod p^2.$$

   (e) Show that the number of pairs (σ, π) of permutations of $[n]$ such that they have a total of $n+1$ cycles, and their composition $\sigma\pi$ is the permutation $(1, 2, \ldots, n)$ (the permutation with one cycle where i is mapped to $i+1 \mod n$) is the nth Catalan number.

   (f) Let $\left[{n \atop k}\right]$ be the number of permutations of $[n]$ with *exactly* k cycles. Give a combinatorial proof (i.e., show that the coefficient on the RHS is the same as that on the LHS)
   $$\sum_{n}(-1)^{n-k}\left[{n \atop k}\right]\frac{x^n}{n!} = \frac{(\ln(1+x))^k}{k!}.$$

   (g) Show that the nth Bell number
   $$B_n = \sum_{\substack{k_1,k_2,\ldots,k_n \geq 0\\ k_1+2k_2+\cdots+nk_n=n}} \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\ldots k_n!(n!)^{k_n}}.$$

2. Use Generating Functions for the following.

   (a) Prove the identities (1)(a-d) using generating functions.

   (b) Derive the ogf for $\left[{n \atop k}\right]$ defined as in (1)(f) above (first derive a recurrence).

   (c) Using the calculus of generating functions show that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

3. When the determinant of a symmetric $n \times n$ matrix is evaluated, let $M(n)$ denote the number of distinct monomials which appear. For instance, $M(2) = 2$, $M(3) = 5$.

   (a) What is the egf for $M(n)$?

   (b) Given an asymptotic estimate for $M(n)$?

4. Let r_k be the number of ways to place k non-attacking rooks on an $n \times n$ board. Show that the sequence $(r_1, \ldots, r_n)$ is unimodal.