Linear Algebraic Approach to Unimodality

We know that giving combinatorial proofs of equalities $a = b$ involves setting up two sets A and B with caridinalities a and b, respectively, and then setting up a bijection between the two sets. Similarly, showing that $a \leq b$, we can set up two sets A and B with the respective cardinalities and give an injection from A to B. However, there is a more linear algebraic way if doing this: Set up two vectors spaces $\mathcal{A}$ and $\mathcal{B}$ such that $\dim(\mathcal{A}) = a$ and $\dim(\mathcal{B}) = b$ and either show a linear transformation $T : \mathcal{A} \to \mathcal{B}$ that is injective, or equivalently give an injective map from a basis of $\mathcal{A}$ to a basis of $\mathcal{B}$. Let's see this approach for the case of binomial coefficients; we had earlier showed log-concavity by arguing about the non-positiveness of the roots of its generating coefficients.

Since we want a vector space with dimension $\binom{n}{k}$, we consider the vector space V_k of all formal sums over subsets of size k of $[n]$, namely

$$\sum_{|S|=k:S\subseteq[n]} a_S S$$

where the scalars a_S are from some large enough field, say rationals. The product of the scalar-zero with any formal sum is the zero element of V_k. Note that V_k is not an inner-product space. The map that we intend to show as an injection is [1]

$$H(S) := \sum_{j\notin S} S \cup j$$

i.e., $H : V_k \to V_{k+1}$ will be an injection as long as $k < n/2$. The map extends linearly to all formal sums. In order to prove that H is injective, we need another companion operator $F : V_k \to V_{k-1}$ defined as [2]

$$F(S) := \sum_{j\in S} S \setminus j.$$

Our first claim is the following:
Claim:: The linear map $HF - FH : V_k \to V_k$ satisfies the following for any set S of size k

$$HF(S) - FH(S) = (2k - n)S =: \mu(k)S.$$

Proof. The sum

$$HF(S) = \sum_{j\in S} \sum_{i\notin S\setminus j} (S \setminus j) \cup i$$

and

$$FH(S) = \sum_{i\notin S} \sum_{j\in S\cup i} (S \cup i) \cup j.$$

If i and j are different in the first summation then they cancel out with the corresponding term in the second summation. The only terms left uncalled are when i and j are the same in both the sums. In the first sum, there are k such occurences of S for each element in S and in the second sum there are $(n - k)$ such occurences of S for each element not in S. Therefore, we get the right hand side. **Q.E.D.**

Next claim is about the commutativity of repeated firings with single hirings:
Claim:: $HF^r - F^r H = (\mu(k) + \cdots + \mu(k - r + 1))F^{r-1}$.
Proof. The proof is by induction over r. The term

$$HF^{r+1} - F^{r+1}H = HF^{r+1} - F^r HF + F^r HF - F^{r+1}H = (HF^r - F^r H)F + F^r(HF - FH).$$

[1]Borrowing Zeilberger's analogy, H is for hiring-a-new-faculty.
[2]In Zeilberger's analogy, F is for firing-an-existing-faculty.

Q.E.D.

To complete the proof we have to show that if $Hv = 0$ then $v = 0$ for $v \in V_k$. The proof is by contradiction. For such a v and $r \geq 1$ we have from the second claim above that

$$HF^r v = (\mu(k) + \cdots + \mu(k - r + 1))F^{r-1}v.$$

Applying H^{r-1} on both sides we get

$$H^r F^r v = (\mu(k) + \cdots + \mu(k - r + 1))H^{r-1}F^{r-1}v,$$

and by repetition we obtain that

$$H^r F^r v = (\mu(k) + \cdots + \mu(k - r + 1))(\mu(k) + \cdots + \mu(k - r + 2)) \cdots \mu(k)v.$$

Taking $r = k + 1$ we obtain that the left hand size is zero, since $v \in V_k$, $L^k v$ is a multiple of the empty-set and $L\emptyset = 0$ as the summation over an empty set. However, teh right hand side above is a non-zero multiple of v as long as $2k < n$, which gives us a contradiction. Therefore, $v = 0$ as desired.

0.1 A Proof Along Proctor's Line

Zeilberger's proof is a simplification of Proctor's proof. However, Proctor's proof is interesting because it directly shows unimodality without appealing to the mirror symmetry of the sequence; this is useful on occasions where it's not easy to prove symmetry in the absence of explicit formulas. The idea is to "take advantage of one of nice features of lienar algebra, the ability to change basis." This means that instead of showing that the kernel of the map H is trivial, it constructs a newer set of basis for each of the vector spaces V_k iteratively; the construction ensures some mirror symmetry around $n/2$.

Let's start with the vector space V_0 containing only one basis element, namely the zero-element. For consistency in exposition, let's denote it by v_0^0. What happens to this vector when we apply the map H? We get the vector

$$v_1^0 := \sum_{i \in [n]} \{i\}$$

in V_1. One more application gives us

$$v_2^0 := \sum_{\{i,j\} \subseteq [n] : i \neq j} \{i, j\}$$

in V_2, and so on we get a set of n vectors in each of the spaces $V_0, \ldots, V_n$; note that V_n is a one-dimensional space. Chose a vector $v_1^1 \in V_1$ that is not in the space generated by v_1^0; there's such a vector since $\dim(V_1) = n$, and we can take it to be $\{1\}$. Again consider the sequence $v_i^1 := H^i v_1^1$, $i = 1, \ldots,$ of vectors obtained by lifting v_1^0 under H. We make the following claim:

Lemma 1 *The vector v_i^1 is not in the space spanned by v_i^0, for $i = 1, \ldots, n - 1$.*

Proof. Intuitively, this is the case because the ith lift of v_0^1 will be formal sum of over all sets of size i that always contain the element 1; whereas the ith lift of v_1^0 is the sum over all sets of size i.

Since eventually all the lifts end with a vector in the space V_n, the claim holds true with $n - 1$ vectors.

Q.E.D.

Note that the vectors at the ith level form a subspace of V_i. The overall iterative procedure is as follows: At the ℓth iteration, we select the smallest level, say k, for which we still don't have a basis (in our case it is still $k = 1$), pick an element v_k^ℓ that is not spanned by elements $v_k^0, \ldots, v_k^{\ell-1}$, and lift the element iteratively using H till we reach V_n. The claim is the same as above, the lift preserves independence till the $(n - k)$th level, and hence we get new basis elements. We keep doing this until we have spanned all the spaces $V_0, \ldots, V_n$. Since we start an independent vector at level k and end with an independent vector at level $(n - k)$, the symmetry is always preserved.