Recurrences

1 The Domain and Range Transformation Approach

The **fundamental recurrence** can be defined as follows:

$$S(n) = S(n-1) + d(n) \tag{1}$$

where $S(0) = 0$, and $d(n)$ is called the **driving function**. The reason we consider this recurrence as fundamental is that we immediately have the "solution" that $S(n) = \sum_{i=1}^{n} d(i)$. So, if $d(n)$ can be summed easily, then we have a closed form for S_n. For instance, if $d(n) = n$ then $S(n) = n(n+1)/2$. Moreover, we know powerful techniques, such as perturbation method, finite calculus etc., for summations.

Remark: The result above is just a reformulation of the fundamental theorem of finite calculus, since $\Delta S(n) = d(n)$.

Thus our aim should be *to reduce an aribtrary recurrence to the fundamental recurrence*. Let's see two such approaches for doing reductions.

¶1. Range Transformation Suppose our recurrence is of the form

$$R(n) = aR(n-1) + d(n).$$

How do we reduce such a recurrence to a fundamental recurrence. Let's define $S(n) := R(n)/a^n$; thus $S(0) = R(0)$. Then dividing both sides of the equation above by a^n, we obtain

$$S(n) = S(n-1) + \frac{d(n)}{a^n}.$$

Thus we get that

$$S(n) = R(0) + \sum_{i=1}^{n} \frac{d(i)}{a^i}$$

and consequently

$$R(n) = a^n R(0) + \sum_{i=1}^{n} d(i) a^{n-i}.$$

The above approach is called a "range" transformation since $S(n)$ is a transformation of the *range* of $R(n)$; their domain is the same.

¶2. Domain Transformation Suppose our recurrence is of the form

$$T(n) = T(n/b) + d(n).$$

How do we reduce such a recurrence to a fundamental form? Let's suppose $n = b^k$. Then let's define $S(k) := T(b^k)$; thus $S(0) = T(1)$. From this it follows that

$$S(k) = S(k-1) + d(b^k)$$

and hence $S(k) = T(1) + \sum_{k=1}^{\log_b n} d(b^k)$. Therefore,

$$T(n) = \sum_{k=1}^{\log_b n} d(n/b^{k-1}) + T(1).$$

Clearly, in this situation we had transformed the domain.

What if we have the general "divide and conquer" recurrence:

$$T(n) = aT(n/b) + d(n).$$

How should we proceed? Which of the two transformations should we apply first? If we start with a range transformation then we are in trouble since we obtain $S(n) = T(n/b)/a^{n-1} + d(n)/a^n$. But now applying the domain transformation does not give us the fundamental recurrence. So let's try the other way round. Applying domain transformation first we get

$$R(k) = aR(k-1) + d(b^k).$$

Clearly, this is in a form suitable to apply a range transformation, by which we obtain

$$S(k) = S(k-1) + \frac{d(b^k)}{a^k}.$$

This we know how to solve, and we can "propogate" the solution back to the original recurrence.

Let's apply what we've learnt to a recurrence coming from Karatsuba's algorithm:

$$T(n) = 3T(n/2) + n.$$

By Domain transformation, we've $R(k) = 3R(k-1) + 2^k$. By range transformation we further have $S(k) = S(k-1) + (2/3)^k$. Thus $S(k) = \sum_{i=0}^{k}(2/3)^i$ and hence $R(k) = \sum_{i=0}^{k} 3^{k-i}2^i$, and

$$T(n) \leq \sum_{i=0}^{\log n} 3^{\log n}(2/3)^i \leq n^{\log 3} \sum_{i \geq 0}(2/3)^i = 3n^{\log 3}.$$