Pigeonhole Principle and Ramsey Theory

The Pigeonhole Principle (PP) has often been termed as one of the most fundamental principles in combinatorics. The familiar statement is that if we have n pigeonholes and more than n pigeons, then there must be a pigeonhole with more than one pigeon.[1]. More formally, a function f that maps a set X, $|X| = m$, to a set Y, $|Y| = n$, where $m > n$, cannot be injective, i.e., there is a $y \in Y$ such that $|f^{-1}(y)| > 1$. But this is not the complete picture. The stronger implication is that there are two elements y, z such that $|f^{-1}(y)| \geq m/n \geq |f^{-1}(z)|$.

Though the principle is very simple to state, proofs involving the principle are usually considered ingenious, since finding the "pigeonholes" and the "pigeons" is non-trivial. In this lecture, we give some interesting applications of the principle.

The principle is a special case of the more general theory is that was developed by Ramsey, namely a large structure should satisfy some property. For instance, for any given n, if we pick sufficiently many points in the plane (no three collinear) then there will be a subset amongst them that form a convex polygon with n vertices (5 points for a quadrilateral, 9 for a pentagon). Or, given two numbers a, b there is a number n such that a two-coloring of K_n either contains a monochromatic K_a or a monochromatic K_b. We will subsequently study some results from the general theory, where the existence of such numbers and bounds on them are derived.[2]

¶1. Initiating Examples: Given the numbers $1, \ldots, 2n$, let $f(n)$ be the number such that any subset of $[2n]$ of size $f(n)$ contains two numbers that are relatively prime. Formulated in this way the solution is not evident. But if we find two numbers that are consecutive, then we know that they are relatively prime. Clearly, $f(n) > n$, since we can pick the n even numbers. So we guess $f(n) = n + 1$, and indeed that is the case, since in any subset of $n + 1$ numbers two must be consecutive. To formulate in terms of pigeonhole principle, let $x_1, \ldots, x_{n+1}$ be the numbers; $x_0 := 1$. Let g_i, $i = 1, \ldots, n$, be the number of elements remaining between x_i and x_{i+1}. Then $\sum_{i=1}^{n} g_i = n - 1$. Thus there must be a g_i that is zero, i.e., two elements x_i and x_{i+1} must be consecutive.

Now let's consider the complement property: let $f(n)$ be the number such that any subset of $[2n]$ of size $f(n)$ contains two numbers such that one divides the other. Again $f(n) > n$, since in the set $\{n+1, \ldots, 2n\}$ no number divides another. What is surprising is $f(n) = n + 1$ again, i.e., any subset of size $n + 1$ has two numbers that are relatively prime and two numbers such that one divides the other. The proof via pigeonhole principle is tricky and is based upon the observation that any number in $[2n]$ can be expressed in the form $2^k m$, where m is an odd number. Since there are only n odd numbers in $[2n]$, in any subset of size $n + 1$ there must be two numbers that have the same odd part, and hence one divides the other. This result already shows the ingenuity needed to apply pigeonhole principle.

1 Dirichlet's Application

One of the earliest non-trivial applications of pigeonhole principle was by Dirichlet in Diophantine Approximation, and basically says that every irrational real number can be approximated quite well with rationals.

[1] Dijkstra's remarks: The Strange Case of the Pigeonhole Principle

[2] The complement problem is, How large can a structure be such that it avoids a certain property. For instance, how many edges can a graph on n vertices have have such that we do not have a cycle of length 4? Such problems are called *extremal problems.*

More precisely, let α be an irrational number, then for all $N \in \mathbb{N}$, there exists p, q, $1 \leq q \leq N$, such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}. \tag{1}$$

This implies that there are infinitely many rationals p/q for which the above holds. Also, there is at most one rational with a fixed denominator q that satisfies this inequality (any two rationals with the same denominator q differ by $1/q$).

We will show the stronger claim:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q(N+1)},$$

or equivalently

$$|q\alpha - p| < \frac{1}{(N+1)}. \tag{2}$$

The above inequality suggests that p must be the integer nearest to $q\alpha$, and since $1 \leq q \leq N$, it makes sense to define $\alpha_i := i\alpha - \lfloor i\alpha \rfloor$, $i = 1, \ldots, N$. Then $\alpha_i \in (0, 1)$, and α_i are irrationals (otherwise α will be a rational). Consider the partition of $(0, 1)$ into $N + 1$ open intervals of the form $I_j := (j/(N+1), (j+1)/(N+1))$, $j = 0, \ldots, N$. There are three cases to consider. In all the cases, we will show that there exists p, q s.t. $q \leq N$ and they satisfy (2).

1. If there is an i s.t. $\alpha_i \in I_0$. Then

   $$0 < i\alpha - \lfloor i\alpha \rfloor < \frac{1}{N+1},$$

   and so we can choose $p := \lfloor i\alpha \rfloor$ and $q := i$.

2. If there is an i s.t. $\alpha_i \in I_{N+1}$. Then

   $$\frac{N}{N+1} < i\alpha - \lfloor i\alpha \rfloor < 1.$$

   Subtracting one from the inequality we get

   $$\frac{-1}{N+1} < i\alpha - \lfloor i\alpha \rfloor - 1 < 0,$$

   which implies

   $$|i\alpha - \lfloor i\alpha \rfloor - 1| < \frac{1}{N+1}.$$

   Thus in this case we choose $p := \lfloor i\alpha \rfloor + 1$ and $q := i$.

3. If there is no i falling in the first two cases, then the N numbers α_i must be contained in $N-1$ intervals $I_1, \ldots, I_{N-1}$. Thus by pigeonhole principle there are two indices i, j (say $i < j$) s.t. α_i and α_j are in the same interval I_k, $k = 1, \ldots, N-1$, i.e.,

   $$\frac{k}{N+1} < i\alpha - \lfloor i\alpha \rfloor < \frac{(k+1)}{N+1}$$

   and

   $$\frac{k}{N+1} < j\alpha - \lfloor j\alpha \rfloor < \frac{(k+1)}{N+1}.$$

   Therefore,

   $$|j\alpha - \lfloor j\alpha \rfloor - (i\alpha - \lfloor i\alpha \rfloor)| < \frac{1}{N+1},$$

   which implies

   $$|(j-i)\alpha - \lfloor j\alpha \rfloor + \lfloor i\alpha \rfloor)| < \frac{1}{N+1}.$$

   So we can choose $q := (j - i)$ and $p := \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$.

Thue-Siegel-Roth theorem states that there are numbers for which (1) is in some sense the best, namely irrational algebraic numbers cannot be approximated by infinitely many rationals better than what (1) suggests, i.e., with 2 replaced by $2+\epsilon$, for some $\epsilon > 0$. This property is very useful in numerical computations with algebraic numbers.

2 Erdös-Szekeres: Monotone Sequences

Given N numbers $a_1, \dots, a_N$, an increasing subsequence of length k is a set of k indices, $i_1 < \dots < i_k$, such that $a_{i_1} < \dots < a_{i_k}$; similarly define a decreasing subsequence.

THEOREM 1. *Any set of $mn+1$ distinct real numbers $a_0, \dots, a_{mn}$ either contains an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.*

Proof 1 (PTB): Let t_i, $i = 0, \dots, mn+1$, be the length of a longest increasing subsequence starting from a_i, and let f be this map, i.e., $f(a_i) = t_i$. If there is a $t_i \geq m+1$ then we are done. So assume all $t_i \leq m$. Since there are only m possible values of t_i and $mn+1$ numbers are mapped to these values, there must be a value, say $t \leq m$, and $n+1$ numbers $a_{i_0}, \dots, a_{i_n}$ such that $f(a_{i_0}) = f(a_{i_1}) = \dots = f(a_{i_n}) = t$. We claim that these $n+1$ numbers form a decreasing subsequece; if $a_{i_j} < a_{i_{j+1}}$, for some $j \in [0, \dots, n-1]$, then we have an increasing subsequence of length $t+1$ starting from a_{i_j}, namely the one obtained by prefixing a_{i_j} to the increasing subsequence starting from $a_{i_{j+1}}$, which is a contradiction.

Proof 2 (Seiderling): The fact that there are $mn+1$ numbers suggests us that we should try to map then into a matrix of size mn. Instead of assigning a single number, we assign a pair with each number: Let s_i be the length of a longest decreasing subsequence starting from a_i, and t_i be the length of a longest increasing subsequence starting from a_i. Let f be this map. If there exists an i, for which either $t_i > m$ or $s_i > n$ then we are done. So suppose for all i, $1 \leq t_i, s_i \leq m$. Thus f maps $mn+1$ numbers into mn pairs, thus by pigeonhole principle two numbers must have the same pair associated with them. But this cannot be, since if $a_i < a_j$ then $t_i > t_j$, and if $a_i \geq a_j$ then $s_i > s_j$, giving us a contradiction.

Proof 3 (Hammersley): This is a constructive proof, and instead of assigning a pair with each number we try to fit them in a matrix of size mn; clearly, there will either be a row of length $n+1$ or a column of length $m+1$; the construction additionally ensures that the rows and columns are ordered subsequences. Arrange the $mn+1$ numbers in a column/stack as follows: place x_1 in the first column; if at any given stage we have place $x_1, \dots, x_{i-1}$ into som columns, then place x_i at the top of the first column that has the topmost entry smaller than x_i; if no such column exists then place x_i at the starting of a new column. Let k be the number of columns obtained. The crucial observation is that entries in a column form an increasing subsequence, and the topmost entries from the first to the kth column form a decreasing subsequence. If $k > n$ then we have a decreasing subsequence of length $n+1$. So suppose $k \leq n$. By pigeonhole principle we know that there is a column that has length at least $mn/k+1$. Since $k \leq n$, the length of this column is at least $m+1$, and so we have an increasing subsequence of the desired length.

Proof 4 (Erdös-Szekeres): By induction.

The theorem is tight as shown by the following sequence of mn numbers:

$$m, m-1, \dots, 1, 2m, 2m-1, \dots, m+1, 3m, 3m-1, \dots, 2m+1, \dots, nm, nm-1, \dots, (n-1)m+1.$$

Note that in proving Theorem 1 we have not used the fact that the numbers are real numbers. A more general statement is the following.

Corollary 2. *Given an ordered set S containing $mn+1$ elements, and a linear order π on these elements, there is an ordered subset T of S that is monotone wrt π. Note that T preserves the ordering of S as well as the ordering imposed by π.*

We now given an application of this generalization.

A set of linear orders $\pi_1, \dots, \pi_m$ on $[n]$ is said to **realize** K_n if for all $i, j \in [n]$ and $k \in [n] - \{i, j\}$ there exists a order π_i such that i, j precede k; express this as $i, j \prec k$. The **order dimension** of K_n is the size of the smallest set of linear orders that realize K_n. So $\dim(K_3) = 3$. It is also clear that $\dim(K_{n+1}) \geq \dim(K_n)$, since in any set of linear orders realizing K_{n+1} if we delete $n+1$ we get a linear order realizing K_n. Thus $\dim(K_4) \geq 3$, and it is 3 as the following set shows:

$$(1,2,3,4),\ (2,4,3,1),\ (1,4,3,2).$$

We claim that $\dim(K_n) \geq \log\log n$, and it suffices to verify it for $n = 2^{2^m}+1$, i.e., in this special case $\dim(K_n) \geq m+1$. Suppose not, and let $\pi_1, \dots, \pi_m$ be a set of linear orders over $[n]$ realizing K_n. From Corollary 2, we know that π_1 contains a monotone subset A_1 of length $2^{2^{m-1}}$. Consider the set A_1 in π_2, then it contains a monotone subset A_2 of length $2^{2^{m-2}}+1$ (the indices of the elements of A_2 are ordered wrt the indices in A, therefore, A_2 is monotone in π_1). Continuing in this manner, we will eventually get that π_n contains an ordered monotone subset $A_m \subseteq A_{m-1}$ of length $2^{2^{m-m}}+1=3$. Let $A_m=(x_i,x_j,x_k)$, where $i<j<k$ are the indices of the elements in A_1. Then what we've shown is that x_i,x_j,x_k form a monotone subsequence in *all the linear orders* $\pi_1,\ldots,\pi_m$. That is, either $x_i<x_j<x_k$ or $x_i>x_j>x_k$ in all the linear orders, which implies that there is no linear order in which x_i,x_k are dominated by x_j, which is a contradiction since $\pi_1,\ldots,\pi_n$ realize K_n. J. Spencer showed that this bound is tight, namely

$$\dim(K_n)=\log\log n+o(\log\log\log n).$$

3 Ramsey Theory

In this section we study a generalization of pigeonhole principle. One way to state pigeonhole principle is that given n objects and $m < n$ colors, in any coloring of the n objects there will be two objects that have the same color. Instead of coloring objects, what if we color pairs of objects, i.e., subsets of $[n]^2$? What will be the analogue of the pigeonhole principle? Let's start with a standard puzzle: How many people do we need in a room such that we are sure that either there is a triplet that are mutual friends, or mutual strangers? We assume that friendship is mutual (or symmetric), but not transitive. If we had asked for a pair of friends or strangers, then the answer is trivially two. As Figure 1 shows, even five is not sufficient. However, we next show that six is sufficient. This is the first non-trivial illustration of Ramsey theory.

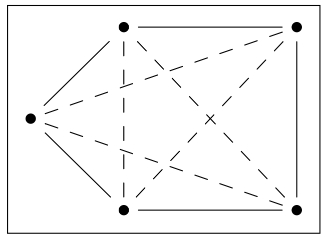

Figure 1: Five people do not necessarily have 3 friends or strangers; bold edges represent friendship and dashes represent strangers.

Let A, B, C, D, E, F be the six people. Now A either is friends with three people or stranger to three people; if neither of this is true, then A is friends with at most two people and stranger to at most two people, which only accounts for four out of the remaining five, which can't be. Suppose A is friends with B, C, D (the argument is similar when A is stranger to them). There are two cases to consider:

1. if amongst B, C, D there are two friends, say B, C, then A, B, C are mutual friends;

2. B, C, D are mutual strangers, in which case we are done.

How many people do we need to ensure that there are four mutual friends or strangers? Perhaps it's easier to ask the following question: How many people do we need to ensure that there are either four mutual friends or three mutual strangers? That is, we can ask mixed questions as well. It can be verified that ten is sufficient, but this is not tight. The argument is similar to above. A either knows at least 6 or doesn't know at least 4 people (WHY?). If he knows 6, then within the six there are either three friends or three strangers; in the former case, the three friends along with A give us four mutual friends, and in the latter we have three strangers. If A doesn't know four people, then there are two cases: if all the four know each other then we are done, otherwise there is a pair that don't know each other, and along with A we get three people that are mutual strangers. The inductive approach in the first case will be useful later on.

In general, we can ask given some ℓ how many people do we need to ensure that there are ℓ mutual friends or strangers. The existence of such a number is not even clear a priori. A special case of Ramsey's theory shows that such a number indeed exists for every ℓ. Before we proceed we formalize the setting using graph theoretic terms. What we have shown is that given a coloring of K_6 using two colors there always exists a monochromatic triangle, or K_3. The question on ten people shows that any two-coloring of K_{10} contains either a monochromatic K_4 or K_3.

Given $(\ell_1, \dots, \ell_r) \in \mathbb{N}^r$, define the **Ramsey function** $R(\ell_1, \dots, \ell_r)$ as the smallest number n such that in all colorings of K_n using at most r colors there will always be a monochromatic K_{ℓ_i}, for some color i. This is usually represented as

$$n \to (\ell_1, \dots, \ell_r). \tag{3}$$

If $\ell_1 = \ell_2 = \dots = \ell_r = \ell$, then we succinctly write $n \to (\ell)_r$ and the Ramsey function as $R(\ell; r)$. Thus the puzzles above show that $6 \to (3)$, and $10 \to (4, 3)$. The key result of Ramsey was to show that such a function is well-defined. Before we proceed further, we show some properties of the function.

P1. If $\ell'_i \leq \ell_i$, $i = 1, \ldots, r$, then $n \to (\ell_1, \ldots, \ell_r)$ implies $n \to (\ell'_1, \ldots, \ell'_r)$. Clearly, if there is a monochromatic K_{ℓ_i}, then all induced subgraphs of it of size ℓ'_i are monochromatic as well.

P2. If $m \geq n$ and $n \to (\ell_1, \ldots, \ell_r)$ then $m \to (\ell_1, \ldots, \ell_r)$. This is obvious, since any r-coloring of K_m contains an r-coloring of K_n, which contains a monochromatic K_{ℓ_i}.

P3. For any permutation $\pi : [r] \to [r]$, $n \to (\ell_1, \ldots, \ell_r)$ iff $n \to (\ell_{\pi(1)}, \ldots, \ell_{\pi(r)})$. Intuitively, this statement says that permuting the colors doesn't matter. More precisely, there is a monochromatic K_{ℓ_i} iff there is a monochromatic $K_{\ell_{\pi(j)}}$, where $j := \pi^{-1}(i)$.

P4. $n \to (\ell_1, \ldots, \ell_r)$ iff $n \to (\ell_1, \ldots, \ell_r, 2)$. The necessary part follows, since if we use r colors then there is a monochromatic K_{ℓ_i} in K_n still holds when we increase the number of colors, since the additional color may not be used in the coloring. For the sufficient part, if $n \to (\ell_1, \ldots, \ell_r, 2)$ then we know that in any $(r+1)$-coloring, where we only use the first r colors, we must have a monochromatic K_{ℓ_i}, for some i, therefore $n \to (\ell_1, \ldots, \ell_r)$. Note that the following is trivially true $n \to (2)_r$, for $n \geq 2$, and $n \to (n, 2)$, for any n; thus $R(n, 2) = R(2, n) = n$.

Ramsey's theorem, in its most simplified form, states the following:

Theorem 3 (Ramsey Theorem Weak Form). *The Ramsey function is well defined, i.e., given $(\ell_1, \ldots, \ell_r)$ there exists an n satisfying (3).*

We start with $r = 2$ and give two proofs: one an inductive argument, and another an explicit upper bound on $R(\ell; 2)$. We want to show that given (ℓ_1, ℓ_2), $R(\ell_1, \ell_2)$ exists.

Proof 1. From P4 we know that $R(\ell, 2) = R(2, \ell) = \ell$. Inductively, assume that $R(\ell_1 - 1, \ell_2)$ and $R(\ell_1, \ell_2 - 1)$ are well-defined. We claim that

$$n := R(\ell_1, \ell_2 - 1) + R(\ell_1 - 1, \ell_2) \to (\ell_1, \ell_2).$$

Pick a vertex $x \in [n]$, and consider the edges from x to the remaining $n - 1$ vertices. In any two-coloring of K_n, say by red and green, one of the following must hold true: either the number of red edges from x are greater than $R(\ell_1 - 1, \ell_2)$, or the number of green edges are greater than $R(\ell_1, \ell_2 - 1)$; if either condition does not hold, then we have only accounted for $< n - 1$ neighbors of x. In the first case, either there is a green K_{ℓ_2} or a red $K_{\ell_1 - 1}$, which along with x gives us a red K_{ℓ_1}. In the second case, we similarly get either a red K_{ℓ_1} or a green K_{ℓ_2} containing x. Note that the formula above explains $(3, 3) + (4, 2) = 10 \to (4, 3)$. In general for r colors we should choose $n := 2 + \sum_{i=1}^{r} R(\ell_1, \ldots, \ell_i - 1, \ldots, \ell_r) - 1$.

Proof 2. The second proof derives shows that $R(\ell, \ell) \leq n := 2^{2l-1} - 1$. Pick an $x_1 \in S_1 := [n]$. Consider a two-coloring χ of K_n; let the colors be R and G. Consider the edges from x to the remaining $n - 1$ vertices. The set of $n - 1$ vertices that are connected to x are partitioned into two classes depending on the color of the connecting edge; let S_2 be the larger of these two sets; clearly $|S_2| \geq (|S_1| - 1)/2 = 2^{2l-2} - 1$. Pick an $x_2 \in S_2$ arbitrarily, and again look at the edges from x_2 to the remaining elements in S_2; let S_3 be the larger set in the partitioning of S_2 induced by the color of edges emanating from x_2; then $|S_3| \geq 2^{2l-3} - 1$. Continue in this manner defining S_{i+1} from S_i always satisfying $|S_{i+1}| \geq (|S_i| - 1)/2$. In this way we can construct $S_1, S_2, , \ldots, S_{2\ell-1}$, since in general $|S_i| \geq 2^{2\ell-i} - 1$, and elements $x_1, \ldots, x_{2\ell-1}$, where $x_i \in S_i$. Note that x_i is connected to $x_{i+1}, \ldots, x_{2\ell-1}$, $i = 1, \ldots, 2\ell - 2$, with the same color. Let the dominating color of x_i be the color connecting it to all the vertices in S_{i+1}. Let T_R be the set of those x_i that have dominating color R; similarly, define T_G. One of T_R or T_G is of size $\geq \ell$. We claim that this is the monochromatic set K_ℓ that we are looking for. In general for r colors we should choose $n := r^{(\ell-1)r+1} - 1$.

The stronger form of Ramsey's theorem applies to the k-uniform hypergraph on n vertices, i.e., hypergraphs where all edges are sets of size k, i.e., in $\binom{[n]}{k}$. Given $(\ell_1, \ldots, \ell_r)$, the Ramsey function $R_k(\ell_1, \ldots, \ell_r)$ is defined as the smallest number n such that in any r-coloring of the k-uniform hypergraph on $[n]$ there exists a subset T of vertices of size ℓ_i such that all edges in $\binom{T}{k}$ are monochromatic in color i. This is usually represented as

$$n \to (\ell_1, \dots, \ell_r)^k.$$

The weak form states that $R_2(\ell_1, \dots, \ell_r)$ is well-defined, but how about $R_1(\ell_1, \dots, \ell_r)$? Now we are looking at r-coloring of vertices. How large n should be to ensure than in any r-coloring of $[n]$ some ℓ_i vertices have the same color. We claim that $n := \sum_{i=1}^{r}(\ell_i - 1) + 1$ suffices. This is the pigeonhole principle, and this is the reason why Ramsey theory is considered a generalization of pigeonhole principle. The stronger form of Ramsey's theorem states that

Theorem 4 (Ramsey Theorem Strong Form). *Given $k, \ell_1, \dots, \ell_r$ the function $R_k(\ell_1, \dots, \ell_r)$ is well-defined.*

3.1 Applications

¶2. Monotone Subsequences: Given m, n, we claim that there exists a function $f(m, n)$ such that any sequence $x_0, \dots, x_{f(mn)}$ of real numbers contains a either an increasing subsequence of length $m+1$ or decreasing subsequence of length $n+1$. We claim that $N := f(m, n) := R_2(m+1, n+1) - 1$ does the job. The key question is how do we 2-color the edges of the complete graph on K_N? Let's say the edge between x_i and x_j is colored R if $x_i < x_j$ and B if $x_i > x_j$. We know that any 2-coloring of K_{N+1} contains either a R K_{m+1} or a B K_{n+1}; in particular, this holds for the coloring we introduced; say we have an R K_{m+1}. What does it mean? Let the vertices be $x_{i_0}, \dots, x_{i_m}$, where $i_0 < \cdots < i_m$. Then a red edge between x_{i_j} and $x_{i_{j+1}}$, $j = 0, \dots, m$, implies that $x_{i_0} < x_{i_1} < \cdots < x_{i_m}$ as desired; a similar argument shows that a B K_{n+1} implies a decreasing subsequence of length $n+1$. The above argument does not give us an explicit value of the function, as was the case earlier.

¶3. Convex Polygons: [3] Given a $k > 2$, how many points $n(k)$ do we need in the plane such that we are sure they contain a convex polygon on k vertices, where points are in general position, i.e., no three points are collinear? If $k = 3$ then it is clear that three points suffice, since the three points are not collinear, they must form a triangle. How about $k = 4$? Do four points suffice? Claim $n(4) = 5$.

We start with a characterization of convex k polygons. Given k vertices of a convex polygon, it is clear that any four must form a quadrilateral; for if a point is contained inside a triangle formed by the remaining three, then that point cannot occur as a vertex of the k-gon. Is the converse also true, i.e., if k points in the plane in general position are such that all sets of four points form a convex quadrilateral then the k points form a k-gon? We show that if a set of k points in general position do not form a k-gon then there must be a point that is contained in a triangle formed by some other three points. Consider a triangulation of the convex hull of the k-points. Clearly, one of the k points must be inside some triangle in this triangulation; moreover, it cannot be on the boundary of the triangle since points are in general position. How do we use this result to show the existence of $n(k)$?

We claim that $n(k) := R_4(k, 5)$ points in general position must contain a k convex gon. Consider the following coloring of $\binom{[n]}{4}$, i.e., the set of sets of size four of $[n]$: if a $T \in \binom{[n]}{4}$ forms a convex quadrilateral then color T red, otherwise color it blue. By definition of $n(k)$ there is either a subset of size k such that all sets in $\binom{[k]}{4}$ are colored red, which by our earlier assumption implies that these k points form a convex polygon; the other case is if all sets in $\binom{[5]}{4}$ are colored blue, i.e., there are five points such that any subset of four points do not form a convex quadrilateral, but this cannot be the case since $n(4) = 5$. Therefore, $R_4(k, 5)$ points in general position in the plane must contain k points that form a convex k polygon.

¶4. Schur's Result: Given r, there exists $n(r) \in \mathbb{N}$ such that for any r-coloring of $1, \dots, n$, there exists three monochromatic $1 \le x, y, z \le n$ such that

$$x + y = z.$$

Claim is $n := R_2(3; r) - 1$. Given a coloring of $1, \dots, n$, we color the edge between the vertices i, j in K_{n+1} with the *color of* $1 \le |i - j| \le n$. Thus we know that in this coloring of K_{n+1} there must be a monochromatic

[3] The Happy Ending problem, since it led to the marriage of Esther Klein and George Szekeres.

triangle K_3, say between the vertices i, j, k. Suppose $i < j < k$, then $x := j - i$, $y := k - j$, and $z := k - i$. Since the edges of the triangle have the same color, it follows that x, y, z have the same color and clearly, $x + y = z$.