Planarity

1 Introduction

A notion of drawing a graph in the plane has led to some of the most deep results in graph theory. Vaguely speaking by a drawing or embedding of a graph G in the plane we mean a topological realization of G in the plane such that no two edges intersect except at their endpoints. A graph in the plane that has this property is called a **plane graphs** and abstract graphs that can be embedded in the plane as a plane graph are called **planar graphs**. The question whether G has a planar embedding is thus a topological one, and was one of the first results that used basic tools from topology. Some of the fundamental results in this area are: Euler's invariant for plane graphs, Kuratowski's result showing the fundamental nature of K_5 and $K_{3,3}$ in non-planar graphs, Whitney's duality characterization of planar graphs, and algorithms for efficiently testing planarity. In these notes, we will study these results.

The study of planarity originated from puzzles, and one such puzzle is by Möbius: Suppose we are given five cities and ten roads and we want to connect every pair of cities using the ten roads without using any bridge or tunnel and such that no two roads intersect. Can we do this? In graph theoretic terms, does K_5 have a planar embedding? Another puzzle involving planarity was included in a book of *Amusements in mathematics* compiled by H.E. Dudeney and published in 1917: given a water, gas and electricity station, and three houses, is there a way to lay pipes from each station to all the three houses such that no pipe crosses another. Again, in graph theoretic terms the puzzle is asking whether $K_{3,3}$ is a planar graph or not. The answer to both these problems is no, and these two graphs play a fundamental nature in some of the first characterizations of planarity.[1]

We start with basic topological results required for formalizing the notion of planar embedding, the most fundamental result being Jordan's curve theorem. Then we study the structural properties of plane graphs, followed by characterizations of planarity and duality results, and ending with algorithms for planarity testing.

2 Topological Results

¶1. Basic Definitions:

i A **Jordan arc** or simply an arc is a continuous injective map of the unit interval $[0, 1]$ into $\mathbb{R}^2$.

ii A **Jordan curve** or simply curve is a continuous injective map of the unit circle into $\mathbb{R}^2$.

iii A **region** in $\mathbb{R}^2$ is a set such that any pair of points is connected by an arc contained in the set.

iv The **frontier** of a set $X \in \mathbb{R}^2$ is the set of points y such that every neighbourhood of y intersects both X and $\mathbb{R}^2 \setminus X$.

It is easy to see that a graph without cycles is planar since it is a tree. So the interesting graphs from the viewpoint of planarity are those that have cycles. Embedding these cycles in the plane naturally leads us to the following fundamental result from topology.

[1] We can further ask if it is possible to embed these two graphs on a sphere in $\mathbb{R}^3$? It turns out that this is also not possible. But both graphs can be embedded on a torus. However, for $K_{3,3}$ we can even embed it on the Möbius strip, a non-orientable surface unlike the torus. The embedding of graphs on surfaces is a very rich topic and underlies the fundamental results of Robertson-Seymour. In general, any graph can be embedded in $\mathbb{R}^3$. The number of "loops" or "handles" in a surface in $\mathbb{R}^3$ that embeds a graph is the genus of the graph. Surfaces can be characterized by the graphs embeddable on them. Euler's formula then gives an invariant for the surface. Two orientable closed surfaces can thus be distinguished based upon this invariant.

Theorem 1 (Jordan's Curve Theorem). *For every curve $C \in \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus C$ has exactly two regions, of which exactly one is unbounded. Each region has the entire curve as its frontier.*

A computational formulation of the above theorem is as follows: Given a Jordan curve C and three points in $\mathbb{R}^2 \setminus C$, two of them can be connected by a path not intersecting C.

As an immediate application of Jordan's theorem we have:

Theorem 2. *K_5 and $K_{3,3}$ are not planar.*

Proof. Let's show the proof for K_5. We know that K_5 contains a triangle. Consider an embedding of this triangle on the plane. There are two cases to consider depending upon whether the fourth vertex v is either inside or outside the triangle.

1. Suppose v is inside $\triangle$. Connecting it to each of the vertices of $\triangle$ partitions $\triangle$ into three faces. If the fifth vertex w is outside $\triangle$, then clearly any $v - w$ arc has to cross $\triangle$. If, however, w is one of the faces in the partition of $\triangle$ formed by the arcs connecting v to the vertices of $\triangle$, then the arc from w to the vertex of $\triangle$ not contained in the face containing w crosses the edges of the face.
2. The case when v is outside the $\triangle$ is similar to the argument above.

Q.E.D.

3 Plane Graphs

Now that we have the tools, a **plane graph** G is a pair (V, E) of finite sets such that

i the vertex set $V \subseteq \mathbb{R}^2$,

ii every edge is an arc between two vertices,

iii there are no multiedges, and

iv the interior of an edge contains no vertex and no point from any other edge.

Strictly speaking, such a graph is a *simple plane graph* as we have disallowed multiedges. The **faces** of G are the regions in the set $\mathbb{R}^2 \setminus G$. Since G is bounded (i.e., is contained inside a sufficiently large disc D) exactly one of its faces is unbounded. We call this face the **outer face** and the remaining faces as the **inner faces**. The set of faces of G is denoted by $F(G)$. If $H \subseteq G$ is a subgraph of G then for every face $f \in F(G)$ there is a face $f' \in F(H)$ s.t. $f \subseteq f'$. The frontier of a face f will be denoted by ∂f.

The following claims are intuitively clear:

Lemma 3. *Let e be an edge in a plane graph G.*

1. *The frontier ∂f of a face f of G either $e \in \partial f$ or the interior of e does not intersect ∂f.*
2. *If e lies on a cycle $C \subseteq G$ then e lies on the frontier of exactly two faces of G and these are contained in the two distinct faces of C.*
3. *If e is a bridge then e lies on the frontier of exactly the outer face of G.*

As a consequence it follows that the frontier of a face f of G is a subgraph of G, and is called the **boundary** $b[f]$ of f; thus $b[f] \subseteq G$ is said to bound f. The degree of a face $d(f)$ is the number of edges in ∂f, where every bridge is counted twice.

Lemma 4. *A tree has exactly one face.*

Proof. Since every edge in a tree is a bridge, deleting it does not change the number of faces. **Q.E.D.**

Lemma 5. *If a plane graph has two distinct faces with the same boundary then the graph is a cycle.*

Proof. Let f_1, f_2 be the two distinct faces and $H \subseteq G$ be their boundary. Since f_1, f_2 are faces of H, we know that H must have a cycle C. Moreover, from Lemma 3(2) it follows that the two faces must belong to the two distinct faces of C . Also, $H = C$, because any further edge or vertex in $H \setminus C$ must be in one of the two faces of C and hence cannot belong to both f_1, f_2. Thus f_1 and f_2 are the distinct faces of C, i.e., $f_1 \cup f_2 \cup C = \mathbb{R}^2$, and hence $G = C$. **Q.E.D.**

Another way to interpret the result above is that two distinct internal faces cannot have the same boundary; if two faces have the same boundary then they are resp. the internal and external faces of a cycle.

A plane graph is called **maximally plane**, or just maximal, if we cannot add a new edge, while maintaining the vertex set, to form a plane graph. Examples are the triangle graph, tetrahedral graph, octahedral graph and so on. Observe that in all these graphs the faces are bounded by triangles. Is this is a way to characterize maximally plane graphs? It turns out yes. We call G a **plane triangulation** if every face of G (including the outer face) is bounded by a triangle. The following proposition gives shows the equivalence of the two concepts.

PROPOSITION 6. *A plane graph with at least three vertices is maximally plane iff it is a plane triangulation.*

Proof. The easier direction is to verify that every plane triangulation is maximal. If not, then the extra edge starts from a vertex on the boundary ∂f of a face f and ends on ∂f. But as the faces are all triangular the starting and ending vertices are already adjacent, and what we've constructed is a multiedge; however, our definition of plane graphs is simple, so we've a contradiction.

The converse intuitively follows from the observation that if any face of a maximal plane graph is not a triangle, but a polygon with more than three vertices then we can triangulate the interior *and the exterior*. Let's be more precise. Suppose f is a non-triangular face of G and $H \subseteq G$ is its boundary; thus $|H| \geq 4$.

We first show that H contains a cycle C. If H is a tree, then f must be the outer face of G; but this would imply G is a tree, which cannot be since it is a maximal plane graph. Since $C \subseteq H$, f is either in the interior or exterior of C; basically, H is a cycle C with possibly smaller cycles or trees hanging from vertices in C. Suppose $v_1, \ldots, v_4$ are four vertices on C, as shown in Figure 1. Since G is maximal the edges (v_1, v_3) and (v_2, v_4) are in the outer face wrt C; but this means that the edges intersect in their interiors, which cannot be. Therefore, f must be bounded by a triangle. **Q.E.D.**

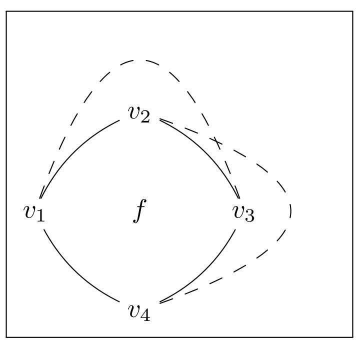

Figure 1: A face with more than four vertices in a maximal plane graph

THEOREM 7 (Euler's Formula). *Let G be a connected plane graph with n vertices, m edges and ℓ faces. Then*

$$n - m + \ell = 2.$$

Euler's original result was for convex polyhedron: he mapped such a polyhedron to the plane by removing a face of the polyhedron and stretching the remaining polyhedron onto the plane in a bijective manner; the embedding gives a plane graph that is in bijection with the original polyhedron; the outer face of the plane graph is in bijection with the removed face. Hence the formula for planar graphs above translates immediately to a formula for vertices, edges and faces of a convex polyhedron.

There are various proofs for this formula (Eppstein has collected nineteen such proofs). The most basic proofs are by induction on vertices, edges and faces. We give one such proof by induction on edges.

Proof. Base case: If there are no edges then we have a single vertex, and one face and hence the theorem holds.

Otherwise, we pick an edge and contract it. The resulting graph has $n-1$ vertices, $m-1$ edges, but still has ℓ faces. By the induction hypothesis $(n-1)-(m-1)+\ell = 2$, and so we've the desired result. If the edge is part of a triangle, the contracting it will yield multiedges, which we avoid; in this situation, it is better to delete the edge, which reduces edges by one and number of faces by one. **Q.E.D.**

As a consequence of Euler's formula, a plane graph cannot have many edges.

Corollary 8. *1. A plane graph with $n > 3$ vertices has at most $3n-6$ edges, equality being obtained for planar triangulations.*

2. A planar graph with $n > 3$ vertices has a vertex with degree ≤ 5.

Proof.

1. Since planar triangulations are maximally planar, we only show the second part. Any face in a triangulation is bounded by exactly three edges. We also know that every edge lies on the boundary of exactly two faces. Thus by a counting argument it follows that $2m = 3\ell$. Substituting this in Euler's formula we obtain $\ell \leq 3(n-2)$.

2. Suppose all vertices have degree ≥ 6. Then the number of edges is $\sum_{v\in V} d_v/2 \geq 3|V|$. However, from the first part we know that $|E| < 3|V|$.

Q.E.D.

It is clear that that K_5 and $K_{3,3}$ cannot occur as a subgraph of a plane graph. However, the inductive proof Theorem 7 suggests more. A **subdivision** of a graph G is a graph G' obtained by adding vertices on the edges of G, i.e., •——• is replaced with •——•——•. Another way to state subdivision is that an edge vw in G is replaced by an independent path between v and w with a new set of vertices. The converse operation of subdividing an edge is **smoothing** an edge: pick a vertex with degree two, remove the vertex and connect its neighbours to each other. The crucial observation is that subdivision maintains planarity and non-planarity, whereas smoothing maintains planarity (non-planarity?). The proof suggests that a planar graph cannot contain subdivisions of K_5 and $K_{3,3}$. What is surprising is that the converse also holds: every non-planar graph contains a subdivisions of K_5 or $K_{3,3}$. This is Kuratowski's result, which we study next.

4 Characterizations of Planarity

In this section we give various ways to characterize planar graphs. We start with some basic definitions. A **combinatorial minor**, or just minor, of a graph G is a graph X that can be obtained from G by deleting certain vertices and edges, and performing edge contractions (where while contracting we delete any self loops and multiedges). We usually represent this as $X \preceq G$. Clearly, the relation $\preceq$ is transitive and forms a partial ordering on graphs.[2]

A **topological minor** of a graph G is a graph obtained from G after deleting some vertices and edges, and performing some smoothing operations. The following proposition gives the relation between a minor and a topological minor:

PROPOSITION 9. *A topological minor is always a combinatorial minor. Conversely, a combinatorial minor with degree at most three is a topological minor.*

Proof. The first claim is straightforward: clearly, smoothing is a special case of contracting an edge.
The second claim: **Q.E.D.**

4.1 Kuratowski's Result

Assume that the graph is connected throughout.

THEOREM 10 (Kuratowski 1930). *A graph G is planar iff it does not contain a subdivision of K_5 or $K_{3,3}$.*

Remark: Since both K_5 and $K_{3,3}$ have degree at most three, the notion of minor and topological minor is the same, and we could have equally stated: A graph G is planar iff K_5 and $K_{3,3}$ are not its minor. This formulation is by Wagner (1937). We have already shown the necessary condition. Here we will show that it is also sufficient.

Let G be a non-planar graph. We know that it must contain a cycle C. Consider the components in $G \setminus C$. We call these components **bridges**[3]. See Figure 2 for examples. The study of bridges is crucial because it is how they interact with each other that determines the non-planarity of G.

¶2. Bridges:

1. A bridge B intersects C at certain vertices called the **vertices of attachment**. The set of these vertices will be denoted by $\mathcal{A}(B)$, and B is called a type-$|\mathcal{A}(B)|$ bridge; so B_1 in Figure 2 is a type-3 bridge. Every bridge has at least one vertex of attachment, in a connected graph; in a 2-connected graph, there are at least two vertices of attachment. We will only be interested in the vertices of attachment of B, and so we can collapse the edges and vertices in $V(B) \setminus \mathcal{A}(B)$, i.e., the non-attached vertices, into a single vertex and connect it to all the vertices in $\mathcal{A}(B)$. Thus a bridge can be viewed either as a star-shaped graph, or a chord of C; this especially applies in plane graphs, where collapsing edges does not affect planarity; for instance, in Figure 2, the bridges B_1, B_3, B_6 are in the simplified form, and the bridge B_{11} could be simplified to a single vertex connected to C with three edges.

2. Two bridges with same vertices of attachment are called **equivalent**; e.g., B_1 and B_2 in Figure 2. How can equivalent bridges connect with C? If B_1, B_2 are equivalent and have one vertex of attachment then they can be in the same face of C; this is valid even when B_1, B_2 have two vertices of attachment (e.g., B_9, B_{10} in Figure 2); however, if they share three or more vertices then they cannot be in the same face without their edges intersecting. Also, since our graph is simple, two chords cannot be equivalent to each other.

[2]One fundamental breakthrough result in the study of minors was by Robertson-Seymour showing that the ordering is a well-quasi-ordering, i.e., in any infinite sequence of graphs there are always two graphs such that one is a minor of the other. Thus the set of non-planar graphs under the ordering $\preceq$ have a finite set. This set characterizes planar graph, and is precisely $\{K_5, K_{3,3}\}$.

[3]An unfortunate clash with notation from connectivity

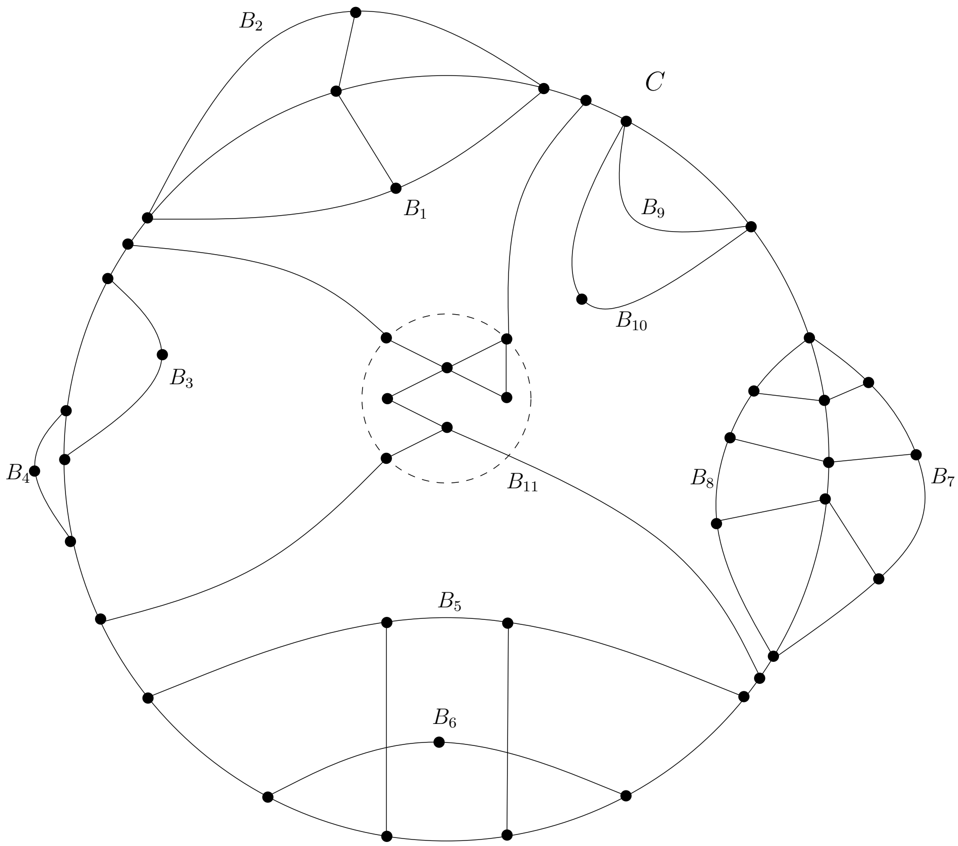

Figure 2: Bridges in a graph

3. We observe that the vertices of attachment of a bridge partition C into disjoint **segments**; strictly speaking we should mention whether a segment is closed, open, or half-open, however, in the statements below it should be understood that there is a way to choose the segments as a combination of these three types and satisfy the claim of the statement. Two bridges **avoid** one another if the attachment vertices of one is contained in exactly one segment of the other; otherwise, they **overlap**; note that the definition is independent of the which face of C the bridges are. For instance, B_9 and B_{10} avoid one another in Figure 2, where the segment considered could either be the closed and the smaller one, or the closed and the longer one; B_{11} avoids all the bridges; both the pairs (B_7, B_8) and (B_5, B_6) overlap, though the first pair doesn't intersect.

4. Two bridges B_1, B_2 are **skew** if there are four vertices u, v, u', v' of C such that $u, v \in \mathcal{A}(B_1)$, $u', v' \in \mathcal{A}(B_2)$, and they appear in the cyclic order u, u', v, v' in C. For example, the following pairs are skew in Figure 2: (B_3, B_4), (B_5, B_6), and (B_7, B_8).

The following lemma is immediate:

Lemma 11. *Two bridges overlap iff they are either skew or they are equivalent* 3*-bridges.*

Proof. The sufficient condition is clear, so we only show the necessary part. If B and B' overlap they must have at least two vertices of attachment. If both are type-2 bridges then they must be skew (as in B_3, B_4 in Figure 2). So assume that both B and B' have at least three vertices of attachment. Two cases to consider:

1. B and B' are not equivalent. Since they overlap there must be two distinct vertices of B in two distinct segments of B', and hence they are skew.

2. They are equivalent type-k bridges , $k \geq 3$. If $k \geq 4$ then they are skew, otherwise they are equivalent.

Q.E.D.

Now let's look at bridges in plane graphs. The bridges in a plane graph are contained in exactly one face of C. If they are in the interior we call them **inner brides**; otherwise we call them **outer bridges**.

LEMMA 12. *Inner (resp. outer) bridges of a plane graph G avoid one another.*

Proof. If two inner bridges overlap, then they are either skew or equivalent type-3 bridges. Since bridges are connected components and are edge independent, it follows that in either case the two bridges must intersect. **Q.E.D.**

An inner bridge B w.r.t. C is said to be **transferable** if there exists a planar embedding $\widetilde{G}$ of G that is identical to G in every respect except B is an outer bridge of C in $\widetilde{G}$; so B_{11} in Figure 2 is transferable, as shown in Figure 3.

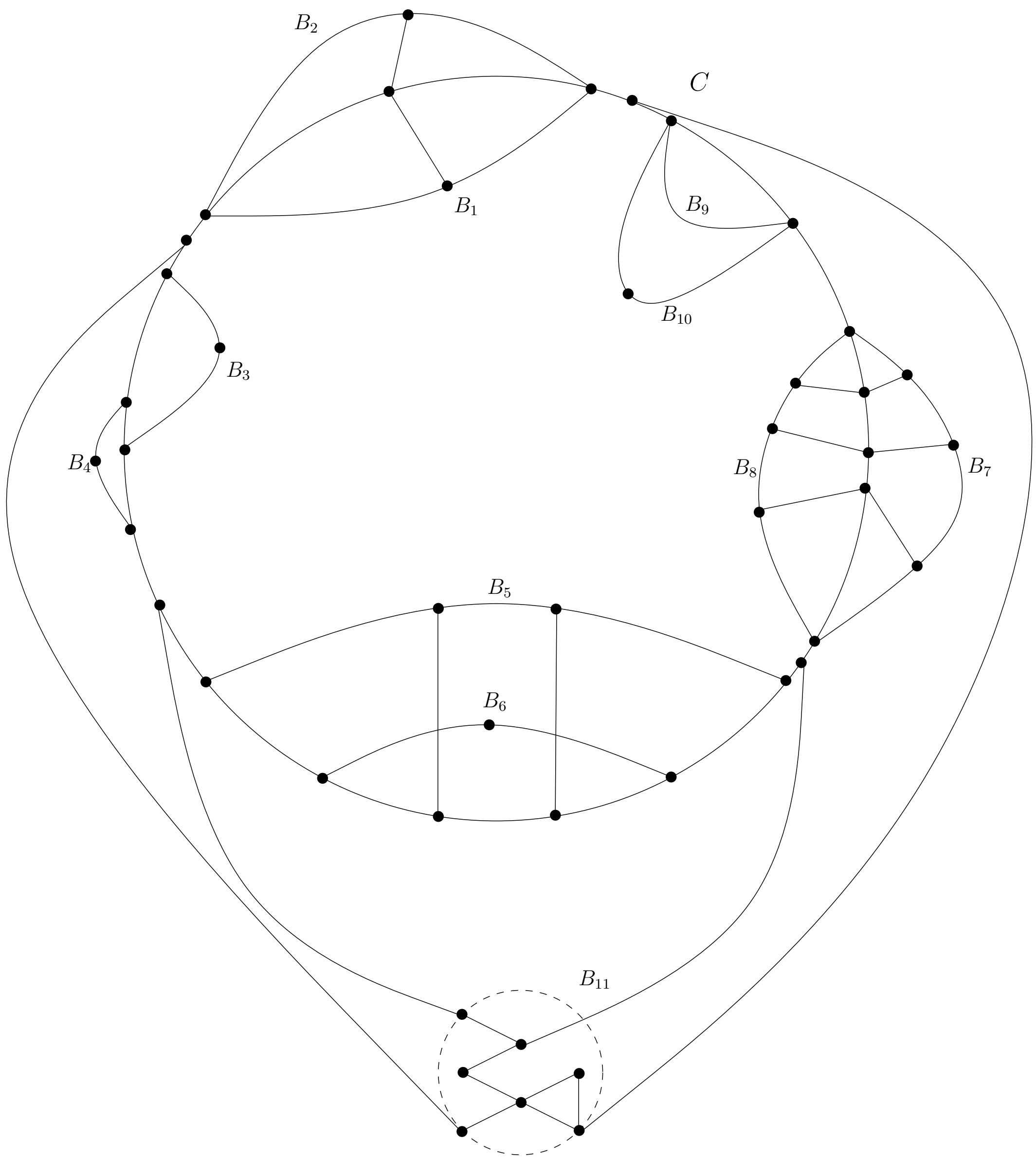

Figure 3: Transferring an inner bridge to an outer bridge. Note the reflection of B_{11} across C.

LEMMA 13. *An inner bridge that avoids every outer bridge is transferable.*

The proof is straightforward: we can move the non-attached vertices of the bridge to the exterior face of C, and since the bridge avoids all outer bridges, we can connect them to the attached vertices without overlapping any outer bridge. Note that, as show in Figure 3, it is important to reflect the bridge across C when transferring it outside, to be able to connect without intersecting

¶3. Proof of Kuratowski's result: Given a non-planar graph G we will show that it contains a subdivision of either K_5 or $K_{3,3}$. The proof is by a contradiction and is an extremal proof, i.e., we can assume that G is edge-minimal graph such that deleting any further edge makes it planar.

Step 1: From G we construct a graph G' by deleting edges while maintaining non-planarity. Clearly, if G' contains a subdivision of K_5 or $K_{3,3}$ then so does G. So from now on we focus on G'. How does a edge-minimal non-planar graph G' look like? The following step shows that it must be a simple block.

Step 2: A graph H is planar iff all its blocks are planar. This follows from the fact that if we construct a graph H' from H where the vertices in H' correspond to maximal blocks of H, and two vertices in H' are adjacent iff the corresponding maximum blocks in H are connected by exactly one edge. Since the resulting graph H' is a tree, it is planar. So if all the blocks of H are planar then H is planar. This implies that if a graph is non-planar then their is a block in the graph that witnesses this non-planarity. Hence a minimal nonplanar graph is a simple block, i.e., G' in step one is a simple block.

Step 3: If a non-planar simple block G' contains subdivision of K_5 or $K_{3,3}$ then we are done. However, if a non-planar simple block G' does not contain a subdivision of K_5 or $K_{3,3}$, then we claim it is a simple 3-connected graph. Suppose G' is not 3-connected; let $\{u, v\}$ be a 2-vertex cut for G' (since it is 2-connected there must be two such vertices). Thus there are at least two components in $G' - \{u, v\}$, and the components are all edge-disjoint. From these components we can form two subgraphs $G_1, G_2 \subseteq G$ such that their common vertices are only u, v, and the edge uv, if present in G', is in one of them, say G_1. Then it is clear that $G_1 \cup G_2 = G'$. In G_1 and G_2 join u, v by a new edge e to get graphs H_1 and H_2 resp. Then one of H_1 or H_2 has to be non-planar, otherwise $H_1 \cup H_2 - e$ is a planar embedding for G'; say H_1 is non-planar. Since u, v are connected by at least two edges to G_2 in G', H_1 has at least one fewer edge than G'; moreover, as it is non-planar, by the edge-minimality of G' it follows that H_1 must contain a subdivision K of K_5 or $K_{3,3}$. By assumption, K cannot be a subgraph of G', so $e \in K$. However, we know that in H_2 there is a path P connecting u and v. So if we replace e in K by P then we get a subgraph of G' that is a subdivision of either K_5 or $K_{3,3}$, which is a contradiction. Hence G' is 3-connected.

Step 4: We now claim that a non-planar 3-connected graph G' must contain a subdivision of K_5 or $K_{3,3}$. The proof is by contradiction. Suppose G' does not contain a subdivision of K_5 or $K_{3,3}$. Since G' is edge-minimal, we can delete an edge uv from G' to get a planar subgraph H. Since G' is 3-connected, H is at least 2-connected, and hence u, v are contained in a cycle C in H. Let C be a cycle of H that contains u, v and whose interior contains the largest number of edges. We will now consider the bridges of H w.r.t. C. The edge uv is a type-2 bridge w.r.t. C, and in the following steps we will use it in that sense as well.

Step 5: Since H is simple and 2-connected, each bridge of C must have at least two vertices of attachment. Moreover, we claim that all outer bridges of C must be type-2 bridges that are skew with uv, since otherwise if a type-k bridge, $k \geq 2$, avoided $[u, v]$ then we can form a cycle C' containing u, v and having more edges in its interior compared to C, which is a contradiction; Figure 4 illustrates these two situations, where the larger circle containing C is drawn with a fat-line. Thus every outer bridge is a type-2 bridge. Moreover, it is a chord, since if it contained a vertex besides the two vertices of attachment then G' will have a 2-vertex cut, but G' is 3-connected.

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Step 6: Since H is planar, no two inner bridges of H overlap. Now some inner bridge B must be skew with $[uv]$ and must overlap some outer bridge; otherwise, by Lemma 13 we know that we can transfer all the inner bridges to outer bridges and then draw the edge uv in H to get a planar embedding of G', which is a contradiction. Therefore, there exists a bridge B that is both skew to uv and to some outer bridge xy; note that all outer bridges are type-2 bridges, so overlap is tantamount to being skew. We now have two cases to consider depending upon whether B has a vertex of attachment different from u, v, x, and y. In all cases, we will show that either $K_{3,3}$ or K_5 has a subdivision inside G, which will be a contradiction.

Step 7: Suppose B has a vertex of attachment different from u, v, x, and y. Let's assume that one of its vertices of attachment v_1 is in the segment (xu). There must be another vertex of attachment v_2 either in the segment (yv) or in (uy). All the remaining cases, are symmetric to these two sub-cases.

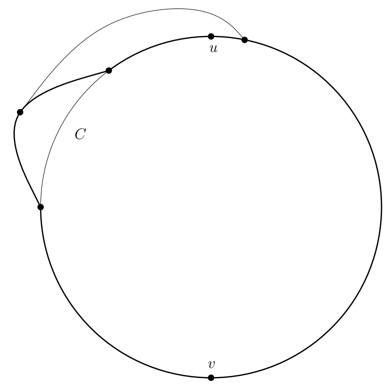

Figure 4: All outer bridges in H are chords

1. If $v_2 \in (yv)$, then it is clear that the subgraph $C \cup \{v_1v_2\} \cup \{uv\} \cup \{xy\}$ is a subdivision of $K_{3,3}$ in G, which is a contradiction. See Figure 5(a).

2. If $v_2 \notin (yv)$ but in $(uy]$ then for B to be skew with xy it must have a third vertex of attachment in $v_3 \in [vx)$. Let v_B be the star-vertex of B. In this case the subgraph

$$C \cup \{v_Bv_1\} \cup \{v_Bv_2\} \cup \{v_Bv_3\} \cup \{uv\} \cup \{xy\} \cup \{v - v_3 - v_1 - v_2 - y\}$$

is a subdivision of $K_{3,3}$ contained in G (smoothen out y and v), which is a contradiction. Figure 5(b) shows the subgraph containing that is a subdivision of $K_{3,3}$ in bold.

Strictly speaking, we should not work with the star-graph for B. However, it is easy to see that in the first subcase there is a path P between v_1 and v_2 in B that is disjoint from C and so we should replace the edge v_1v_2 by P; the subdivision property still holds. In the second subcase, we can construct three mutually edge-disjoint paths in B from a vertex v_B to v_1, v_2, v_3, and replace the three edges in the subgraph with these three paths; again the subdivision property holds.

Step 8: If u, v, x, and y are all vertices of attachment for B. In this case, than looking at just the star-graph of B is slightly misleading. But it is easy to show that there is a $u - v$ path $P \in B$ and $x - y$ path Q in B, such that $|V(P) \cap V(Q)| \geq 1$. We consider two cases, depending on whether P and Q share exactly one vertex or not.

1. If $|V(P) \cap V(Q)| = 1$, and let this vertex be v_B, then the subgraph

$$C \cup \{v_BPv\} \cup \{v_BPu\} \cup \{v_BQx\} \cup \{v_BQy\} \cup \{uv\} \cup \{xy\}$$

is a subdivision of K_5 contained in G, which is a contradiction. If we had considered B as a star-graph then we would have only considered this case. See Figure 5(c).

2. If $|V(P) \cap V(Q)| > 1$ then let u', v' be the first and last vertices of P on Q when going from u to v. Then the subgraph

$$C \cup \{uPu'\} \cup \{v'Pv\} \cup Q \cup \{uv\} \cup \{xy\}$$

is a subdivision of $K_{3,3}$, which is a contradiction; this subgraph has been illustrated in Figure 5(d) in bold.

4.2 Mac Lane's Cycle Space Basis

The characterization is an algebraic one, i.e., we associate a certain vector space with the graph G and the graph is planar iff a subspace of the vector space has a nice basis. To formulate this more precisely, we need to view our graph in terms of tools from linear algebra, so we first develop these notions.

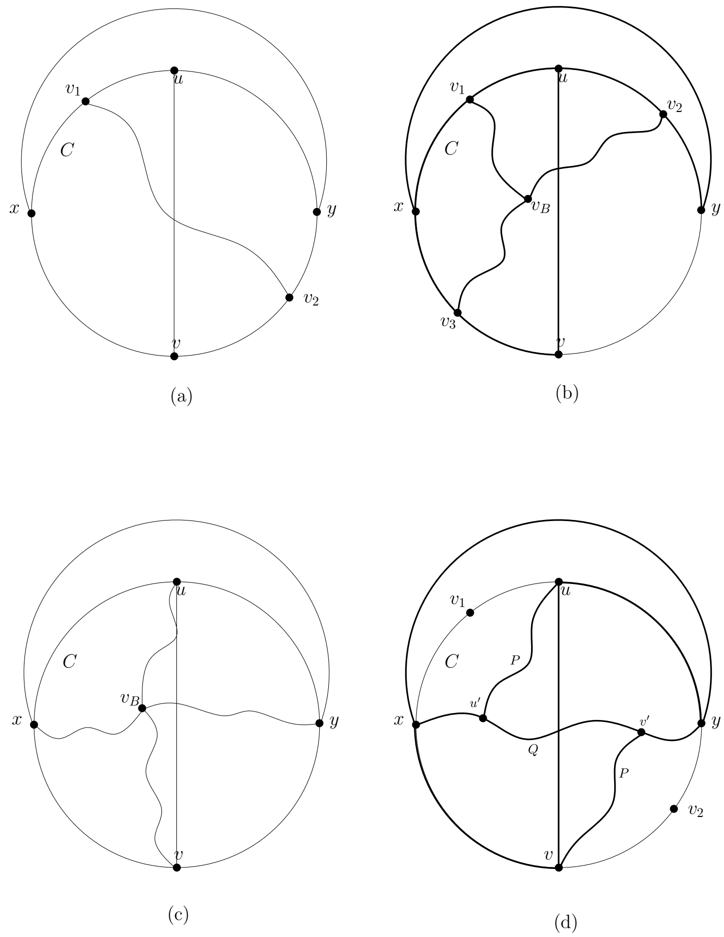

Figure 5: The four subcases showing the subdivision of either K_5 or $K_{3,3}$ in G'

4.2.1 Graphs and Vector Spaces

Given a finite set S, its power set 2^S can be viewed as a vector space $\mathcal{V}(S)$ over the field $\mathbb{F}_2$: with each subset $A \subseteq S$ the vector associated v_A is the *indicator vector/function* corresponding to A (i.e., the vector that has zero for all elements in $S \setminus A$ and one otherwise). For two subsets $A, B \subseteq S$, $v_A \oplus v_B := v_{A \oplus B}$, where $A \oplus B$ is the symmetric difference of A and B; another way to think of $v_A \oplus v_B$ is that it is the standard vector addition, except that the coordinate-wise sum is done in $\mathbb{F}_2$. With this notation, it follows that for all $A \subseteq S$, the inverse of v_A is v_A itself, and $v_\emptyset$ is the all zeros vector. For convenience, however, we will often use A instead of v_A. The **dimension** of a subspace of $\mathcal{V}(S)$ is the size of its basis. The vectors corresponding to the elements of S form a **standard basis** for $\mathcal{V}(S)$, and $\dim(\mathcal{V}(S)) = |S| = n$. Two vectors A, B are **linearly independent** iff $v_A \oplus v_B \neq 0$, i.e., at least one of them contains an element not present in the other.

Given two vectors $A, B \in \mathcal{V}(S)$, let $v_A := (\lambda_1, \ldots, \lambda_n)$ and $v_B := (\lambda'_1, \ldots, \lambda'_n)$ be their representation in the standard basis. Then we can define the inner-product operator

$$\langle A, B \rangle := \lambda_1 \lambda'_1 + \lambda_2 \lambda'_2 + \cdots + \lambda_n \lambda'_n \tag{1}$$

where the addition is done in $\mathbb{F}_2$. Thus $\mathcal{V}(S)$ is an inner-product space, i.e., it satisfies, symmetry, linearity, and positive semi-definiteness; note that $\langle A, A \rangle$ can be zero even though $A \neq 0$, e.g., for $A = (1, 1, 0)$. Another way to characterize orthogonality is the following equivalence:

$$\langle A, B \rangle = 0 \iff |A \cap B| \text{ is an even number.} \tag{2}$$

Given a subspace $\mathcal{X} \subseteq \mathcal{V}(S)$, define the **orthogonal space**

$$\mathcal{X}^{\perp} := \{Y \in \mathcal{V}(S) | \langle F, Y \rangle = 0 \text{ for all } F \in \mathcal{X}\}.$$

By the linearity of the inner-product, it follows that $\mathcal{X}^{\perp}$ is itself a subspace. Moreover,

$$\dim(\mathcal{X}) + \dim(\mathcal{X}^{\perp}) = |S|.$$

Given a graph G, let $\mathcal{E}(G) := \mathcal{V}(E(G))$ be the **edge space** associated with G; the dimension of this space is the number of edges m. The subspace of $\mathcal{E}(G)$ spanned by all the cycles in G is called **cycle space**, $\mathcal{C}(G)$, of G. Of special interest are **induced cycles**, i.e., those cycles in $\mathcal{C}(G)$ that are induced subgraphs of G, or in other words, they do not contain a chord. Induced cycles are interesting because of the following property.

LEMMA 14. *The induced cycles in a graph G form a basis for $\mathcal{C}(G)$.*

Proof. By induction on $|C|$. We show that every cycle C with a chord e is generated by induced cycles. Consider the two cycles in $C + e$. Since each of them has cardinality smaller than $|C|$ we can apply the induction hypothesis to them, and their symmetric difference gives us C.

For cycles C without a chord, but containing a path between two vertices of the cycle, we can replace the path with a chord and argue as above. **Q.E.D.**

LEMMA 15. *An edge set $X \subseteq E$ is in $\mathcal{C}(G)$ iff every vertex of the graph (V, X) has even degree.*

Proof. By induction on the number of cycles generating X in $\mathcal{C}(G)$. If $X \in \mathcal{C}(G)$, then either it is a cycle C (in which case the implication is clear) or it is of the form $C_1 \oplus C_2$, for two cycles C_1, C_2. If $C_1 \cap C_2 = \emptyset$ then the result again follows, so assume $|C_1 \cap C_2| \geq 1$. Note that one cycle cannot be a subset of the other (WHY?). So let $e \in C_1 \setminus C_2$. Pick a direction from e and traverse around C_1. In this traversal let v be the first and v' be the last vertex common to both cycles. Let $P_1, \ldots, P_k \in E(G)$ be the connected components in $C_1 \cap C_2$ ordered in the direction of the traversal. Then in $C_1 \oplus C_2$ the path between v and v' contains cycles connecting v_1 to the first vertex of P_1 using edge-disjoint paths from C_1 and C_2; the last vertex of P_1 to the first vertex of P_1 again using edge-disjoint paths from C_1 and C_2; and so on, the last vertex of P_k with v' using edge-disjoint paths from C_1 and C_2. Finally there is the cycle formed by the two edge-disjoint paths between v and v' along C_1 and C_2. From this cycle decomposition of $C_1 \oplus C_2$ it follows that all the vertices in X have even degree.

Conversely, by induction on the size of X. Construct a path P as follows: start from a vertex $v \in V$ and pick one of its neighbors v' connected by X; from v' do the same; continue doing this until we reach the first vertex w that has already been visited. This implies that there must be a subpath C of P that forms a cycle; moreover, every vertex on C must have two distinct edges of X incident on it. Delete this edges from X to get $X' \subset X$; the degree of the vertices remains even. By induction, the edges in X' are in $\mathcal{C}(G)$. Since the edges in C are disjoint from those in X', it follows that X is also in $\mathcal{C}(G)$. **Q.E.D.**

If V_1, V_2 is a partition of V then the set of all edges $E(V_1, V_2)$ *crossing* this partition is called a **cut**; for a single vertex v, the set of edges incident on it $E(v)$ is the cut. From the definition it is clear that the cut is the minimal set of such edges.

PROPOSITION 16. *The cuts in G, together with $\emptyset$, for a subspace $\mathcal{C}^*(G)$ of $\mathcal{E}(G)$. Moreover, the sets $E(v)$ form a* simple basis *for C^*.*

Proof. Let D, D' be two cuts. We have to show that $D \oplus D' \in \mathcal{C}^*$. Since $D \oplus D'$ is the union of disjoint sets, we can assume that D and D' are disjoint. Let V_1, V_2 and V_1', V_2' be the corresponding partitions of V. Then $D \oplus D'$ contains all the edges which cross one of these partitions but not the other; see Figure 6. But these are precisely the edges between $(V_1 \cap V_1') \cup (V_2 \cap V_2')$ and $(V_1 \cap V_2') \cup (V_2 \cap V_1')$, two sets that form a partition of V. Thus $D \oplus D' \in \mathcal{C}^*$.

To show the second part, we observe that $E(V_1, V_2) = \boxplus_{v \in V_1} E(v)$, where $\boxplus$ denotes symmetric difference of a sequence of sets; note all edges except those between V_1 and V_2 are counted twice. The simplicity follows from the fact that an edge vw is shared only by the sets corresponding to its endpoints $E(v)$, $E(w)$. **Q.E.D.**

The subspace $\mathcal{C}^*(G)$ is called the **cut space** of the graph.

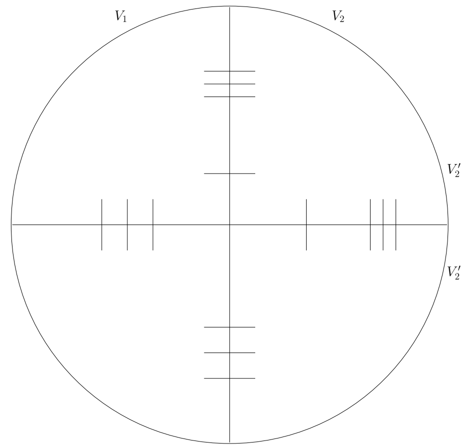

Figure 6: Partition corresponding to $D \oplus D'$

LEMMA 17. *The cycle space and cut space of a graph are mutually orthogonal spaces.*

Proof. We claim that every cycle C is orthogonal to $\mathcal{C}^*$. For any partition V_1, V_2 of V either $C \cap E(V_1, V_2) = \emptyset$ or it contains an even number of edges since C is a cycle. Thus from (2) it follows that C is orthogonal to all partitions, thus $\mathcal{C} \subseteq (\mathcal{C}^*)^\perp$. To show equality we show that if $X \subseteq \mathcal{E}(G)$ is not in $\mathcal{C}$ then $X \notin (\mathcal{C}^*)^\perp$. Since $X \notin \mathcal{C}$, there exists a vertex v with odd degree in (V, X). Thus $\langle E(v), X\rangle = 1$. But $E(v) \in \mathcal{C}^*$, thus X is not orthogonal to $\mathcal{C}^*$. **Q.E.D.**

From the lemma above it follows that $\dim(\mathcal{C}) + \dim(\mathcal{C}^*) = m = |E(G)|$. The following theorem gives an explicit formula for the dimensions of the two subspaces.

THEOREM 18. *Let G be a connected graph with n vertices and m edges. Then*

$$\dim \mathcal{C} = m - n + 1 \text{ and } \dim \mathcal{C}^* = n - 1.$$

Proof. We will show that there is a basis for both the subspaces of the corresponding sizes. Since the two dimensions add up to m, neither subspace could have a larger dimension than what is given in the theorem.

Let T be a spanning tree for G. We know that it has $n-1$ edges. Adding any of the remaining $m-n+1$ edges $e \in E \setminus E(T)$ to T forms a cycle C_e. Moreover, C_e does not contain any edge $e' \in E \setminus E(T) \cup \{e\}$. Thus two such cycles are linearly independent; such cycles are called **fundamental cycles** w.r.t. T. Thus $\mathcal{C}$ has dimension at least $m - n + 1$.

Each of the $n - 1$ edges $e \in E(T)$ gives rise to a partition (V_1, V_2) of V in $T - e$, and the set of edges D_e between V_1 and V_2 is a cut. Since all $e' \in E(T) \setminus \{e\}$ are not contained in D_e, these cuts are linearly independent and hence $\dim \mathcal{C}^*$ is at least $n - 1$. The cuts D_e are called the **fundamental cuts** w.r.t. T. **Q.E.D.**

So far all our statements above apply to planar and non-planar graphs. In order to say something special about planar graphs, we need the following definition: A subset $\mathcal{X} \subseteq \mathcal{E}(G)$ is called **simple** if every edge in G belongs to *at most* two sets in $\mathcal{X}$; the empty set is always considered simple. For instance, the cut space $\mathcal{C}^*(G)$ has a simple basis. What about the cycle space? We now state and prove another characterization of planar graphs.

THEOREM 19 (Mac Lane 1937). *A graph is planar iff its cycle space has a simple basis.*

Proof. We consider two cases: G is 2-connected or not, i.e., $\kappa(G) \le 1$ or $\kappa(G) > 1$. In both cases, the proof is by induction on $|G|$.

Case 1: If $\kappa(G) \le 1$, let v be a vertex cut of G. Then G can be expressed as union of two subgraphs G_1, G_2 where $G_1 \cap G_2 = \{v\}$. By induction hypothesis both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ have a simple basis, and since $\mathcal{C}(G)$ is their union, it also has a simple basis. For the converse, from the induction hypothesis it follows that since both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ have a simple basis G_1 and G_2 are planar, and hence G is planar.

Case 2: If $\kappa(G) > 1$, then G is 2-connected. If G is planar, then we claim that the boundaries of the faces of G form a basis for $\mathcal{C}(G)$. That the boundaries of the faces are cycles, follows from the ear-decomposition theorem for two connected graphs and induction: $G = H \cup P$, where P is an H-path and H is 2-connected; by induction, all faces in H have are bounded by cycles; any $f \in F(G)$ is either a face in $F(H)$, in which case we are done, or is contained in f', the outermost face of H; if C is the cycle bounding f' then $C \cup P$ forms two cycles and f is one of the two new faces in $C \cup P$. We will now show that the boundaries of the faces form a basis for $\mathcal{C}(G)$; the basis is simple, because by Lemma 3(b) we know that every edge in a face is shared by exactly one another face. Let $C \in \mathcal{C}(G)$. The proof is by induction on $|C|$. If C is a face-boundary then we are done. So let e be a chord of C. Then $C = C_1 \oplus C_2$, where $|C_1|, |C_2| < |C|$, and by induction hypothesis they are symmetric difference of the face boundaries, and so is C.

To show the sufficiency, we first show that if cycle space of G has a simple basis, then every subgraph of G has a cycle space with a simple basis; then we show that the cycle space of any subdivision of K_5 or $K_{3,3}$ is not simple; thus G cannot contain a subdivision of K_5 or $K_{3,3}$ as a subgraph, and, therefore, is planar by Kuratowski's theorem.

Let $C_1, \ldots, C_k$ be a simple basis for $\mathcal{C}(G)$. Then for every edge e, $\mathcal{C}(G - e)$ is simple as well: if e is a bridge then deleting it does not affect $\mathcal{C}(G)$; if $e \in C_i$, say $i = 1$, then the basis is $C_2, \ldots, C_k$; otherwise, if e is common to two basis elements, say C_1 and C_2, then the basis is $C_1 \oplus C_2, C_3, \ldots, C_k$. Since any subgraph of G can be obtained from G by deleting edges, and isolated vertices, it follows that the cycle space of any subgraph G has a simple basis. We now show that the cycle space of K_5 and $K_{3,3}$ is not simple.

Consider K_5 first. We know from Theorem 18 that $\dim \mathcal{C}(K_5) = 6$. Let $C_1, \ldots, C_6$ be a *simple basis*, and define $C_0 := C_1 \oplus \cdots \oplus C_6$; from the simplicity of the basis it follows that every edge of C_0 is in exactly one set C_i. Since $C_1, \ldots, C_6$ are linearly independent, none of the sets $C_0, \ldots, C_6$ are empty, and hence contain at least three edges. Thus

$$|C_1| + |C_2| + \cdots + |C_6| \geq 6 \cdot 3.$$

The sum $|C_1| + |C_2| + \cdots + |C_6|$ counts all the edges in the interior of C_0 twice and the edges in C_0 once, thus

$$|C_1| + |C_2| + \cdots + |C_6| \leq 2|E(K_5)| - |C_0| \leq 20 - 3 = 17$$

which gives us a contradiction.

Now consider $K_{3,3}$. Again, from Theorem 18, we know that $\dim \mathcal{C}(K_{3,3}) = 4$; let $C_1, \ldots, C_4$ be a simple basis for its cycle space, and let $C_0 := C_1 \oplus \cdots \oplus C_4$. Since the smallest cycle in $K_{3,3}$ has four edges, we have from the same argument as above, that

$$4 \cdot 4 \leq |C_1| + \cdots + |C_4| \leq 2|E(K_{3,3})| - |C_0| = 14$$

a contradiction.

Why can't a subdivision of K_5 or $K_{3,3}$ have a simple basis for their cycle space? The dimension of the cycle space remains invariant under subdivision, since subdividing an edge increases the number of edges and vertices by exactly one.

Since subdivision does not change the simplicity of a basis, it follows that any sub **Q.E.D.**

4.3 Whitney's Abstract Duality

For a plane graph G, its **plane dual** G^* is obtained as follows: each vertex $v^*(f)$ corresponds to a face $f \in F(G)$; two vertices $v^*(f_1)$ and $v^*(f_2)$ form an edge e^* iff f_1 and f_2 share an edge e in G; if an edge e is shared by one face f, then $v^*(f)$ has a self-loop denoted by the edge e^*. Thus with every edge e in G, we have associated an edge e^* in G^*. Note that G^* is a multigraph; every bridge contributes a self-loop, and every cycle contributes parallel edges. Also, G^* is always connected, because any two faces in disjoint components of G are connected to the outermost face, which implies in G^* there is a path from the vertex corresponding to the outermost face to the vertex corresponding to any other face.

The cut space of G^* and the cycle space of G (with multiedges) has an interesting relation.

Proposition 20. *A subset of edges $X \subseteq E(G)$ forms a cycle in G iff $X^* := \{e^* | e \in X\}$ forms a minimal cut in G^*.*

Proof. Let $X \subseteq E(G)$ and X^* be its image in G^*. Two vertices $v^*(f_1)$, $v^*(f_2)$ in V^* are in the same component of $G^* - X^*$ iff f_1 and f_2 are in the same region of $\mathbb{R}^2 \setminus X$: every path between the two vertices is an arc connecting f_1 and f_2 in $\mathbb{R}^2 \setminus X$, and conversely every such arc defines a *walk* between the two vertices.

Let X form a cycle in G. Then by the Jordan curve theorem and by the correspondence above, $G^* - X^*$ has exactly two components and X^* is the minimal number of edges that need to deleted to form the cut.

Conversely, if X^* is a minimal cut then there are two components in $G^* - X^*$; take two vertices, one in each component, then $G - X$ has two faces that are not in the same region of $\mathbb{R}^2 \setminus X$; so X must contain a cycle, otherwise if X does not contain a cycle then $G - X$ contains a single face. Moreover, since X^* is minimal, X cannot contain any edges besides the cycle. **Q.E.D.**

Just as the cycle space is mapped to the cut space, by the same argument the cut space of G is in bijection with the cycle space of G^*: If D is a cut in G and D^* is its image in G^*, then since D shares an even number of edges with all the cycles in G, D^* shares an even number of edges with all the cut sets in G^*, and hence is a cycle in G^*.

Whitney suggested a generalization of the planar dual for planar graphs to a **combinatorial dual**[4]: A graph G^* is a combinatorial dual of a graph G, if there exists a bijection $\phi : E(G) \to E(G^*)$ such that ϕ is also a bijection between the cycle space of G and the cut space of G^*, i.e., every cycle in G is mapped to a cut set of the same size in G^* and vice versa, every cut set in G^* has a cycle in G associated with it.

THEOREM 21 (Whitney 1932). *A graph is planar iff it has a combinatorial dual.*

Proof. The lemma above gives us one part of the proof, namely for every planar graph has the plane dual is its combinatorial dual.

The converse proof proceeds similar to Mac Lane's proof. We first show that K_5 and $K_{3,3}$ do not have combinatorial duals; duality is preserved under deletion of edges, and hence if a graph has a dual then so do all its subgraphs; subdivision also preserves duality; finally, we show that if G has a dual the it cannot contain a subdivision of either K_5 or $K_{3,3}$, and hence by Kuratowski's theorem it is planar.

Step 1: Let G^* be the dual of $K_{3,3}$. Since $K_{3,3}$ has 9 edges, no cut sets of size two, and cycles of length 4 and 6 only, the dual G^* has 9 edges, no cycles of length two, and degree of every vertex is at least 4. Now G^* has at least 5 vertices (since K_4 has only 6 edges) and hence the number of edges in G^* is at least $5 * 4/2 > 9$, a contradiction.

Suppose K_5 had a dual graph G^*. Since K_5 has 10 edges, no cycles of length two, and cut-sets of size 4 and 6, the dual graph G^* has 10 edges, the degree of all vertices is greater than two, and cycles of length 4 and 6. Thus G^* is a bipartite graph having ten edges. How many vertice does it have? A bipartite graph with 6 vertice has at most 9 edges, thus G^* must have at least 7 vertices. Since the degree of every vertex is at least three, G^* has at least $7 \times 3/2 > 10$ edges, a contradiction.

Step 2: Deleting an edge e in G preserves duality. Let e^* be the image of e in G^*. We claim that the dual graph is obtained by *contracting* e^* in G^*. There are two cases to consider:

1. If e is a bridge, then e^* cannot be part of any cut, which implies that e^* is a self-loop; hence contracting e^* gives us a dual of $G - e$.

2. If e is part of a cycle C, then e^* must be part of some cut A, B of G^*. Since the edges in $C - e$ do not form a cycle, deleting the edges in $E(A, B) - e^*$ does not form a cut between A and B; contracting the edge ensures this.

Since any subgraph of G can be obtained from G by edge deletions and removing isolated vertices, if G has a dual so do all its subgraphs. It can be verified that the duality is maintained.

Step 3: Duality is preserved under subdivision and smoothing. We claim that if we subdivide e, then the corresponding dual can be obtained by adding a parallel edge to e^*. Again, there are two cases to consider.

1. If e is a bridge, then subdividing it does not change the structure of the cycle space, hence the new edge intorduced must not yield to a cut in G^*; this can be obtained by adding a self-loop parallel to e^* in G^*.

2. If e is part of a cycle.

[4]Sometimes called an algebraic dual.

smoothing is tantamount to deleting a parallel edge. Again, we can verify that the duality is maintained.

Step 4: If G has a dual then all its subgraphs have duals, and all their smoothings have duals. Thus no subdivision of either K_5 or $K_{3,3}$ can occur in G. By Kuratowski's theorem it follows that G is planar.

Q.E.D.