Matching Theory

1 Introduction

A **matching** M of a graph is a subset of E such that no two edges in M share a vertex; edges which have this property are called **independent** edges. A matching M is said to cover a subset $U \subseteq V$ of vertices if every vertex in U is incident with an edge in M. The vertices covered by M are called **matched** and those not covered are called **unmatched**. A matching is said to be **perfect** if it mathces all the vertices of the graph. For the first part, we will be interested in matchings for bipartite graphs. Throughout we will assume that our graphs are connected. A concept from set theory that will be useful to us is the notion of **symmetric difference**, $A \ominus B$, of two sets: $A \ominus B$ is the set of elements in A and B but not in both, i.e., the set $(A \cup B) - (A \cap B)$.

A more general notion than matching is that of a factor. More precisely, a k**-factor** of G is a k-regular subgraph of G that contains all vertices of G. Thus a perfect matching is a 1-factor of a graph. A graph that has a 1-factor is called **factorizable**, otherwise it is called **non-factorizable**; since we are only interested in 1-factors here, we can use such a definition. A 2-factor is a Hamiltonian cycle; clearly, a graph which has all vertices of even degree has a 2-factor.

2 Matching in Bipartite Graphs

A graph is r**-partite** if the set of vertices can be partitioned into r classes such that all the edges in the graph are between different classes, i.e., vertices in the same class are not adjacent to each other. Bipartite graphs are 2-partite graphs. In the remaining part of this section, we focus only on bipartite graphs. We will represent bipartite graphs using the customary notation $G = (A, B)$.

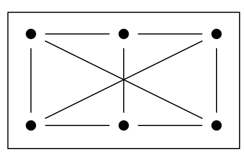

Figure 1: Is this graph bipartite?

The following result characterizes bipartite graphs.

THEOREM 1. *A graph is bipartite iff it contains no odd cycles.*

Proof. Clearly, if G is bipartite then it cannot contain any cycle of odd length, or odd cycles.

Now we show a constructive way to show the converse: we will construct a two-partition for a graph containing no odd cycles. We know that G has a spanning tree T. Fix a vertex in T as the root r, assign to A all the vertices at an even distance from r in T and to B all the vertices at an odd distance from r in T. We claim that A, B form a bi-partition of G. Consider an edge xy in G. If $xy \in T$ then clearly, x and y are in different sets. If $e \notin T$ then $T + e$ has a cycle $x \leftrightsquigarrow r \leftrightsquigarrow y - x$. Since the cycle has even length, the path $x \leftrightsquigarrow r \leftrightsquigarrow y$ must have odd length, thus one of x/y is at an even distance from r and the other is at an odd distance, and so x and y are in different sets in the bi-partition. **Q.E.D.**

The cardinality of a matching M is the number of edges in it. A **maximum matching** is one that has the largest number of edges in it. A **vertex cover** for a graph G is a subset $U \subseteq V$ such that every edge in

G is incident on some vertex in V. The following theorem of König is one amongst the equivalent versions of what may be called **the fundamental theorem of bipartite matching**:

THEOREM 2 (König 1931). *In a bipartite graph, the maximal cardinality of a matching is equal to the minimum cardinality of a vertex cover.*

The easier implication: since the edges in M are independent, every vertex cover has to use at least $|M|$ vertices to cover M. Thus the minimum cardinality of a vertex cover is not smaller than the maximal cardinality of a matching in any graph, not just bipartite. To show the converse, and later for algorithmic purposes, we will need the following fundamental concept in matching theory: given a matching M, an **alternating path** w.r.t. M is a path that alternates between the edges of $E \backslash M$ and M. An M**-augmenting** path is an alternating path that starts and ends in *unmatched vertices*. From the definition it follows that an augmenting path has odd length; thus if it starts in A then it has to end in B, and all edges from B to A in the augmenting path are edges in M.

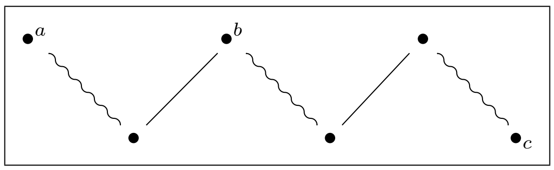

Figure 2: Alternating path (b to c) and Augmenting Path (a to c).

The following result makes it clear why such paths *augment* M:

LEMMA 3 (Berge's Result). *A matching M is maximum iff there is no M-augmenting path in G.*

Proof. If M is a maximum matching then there cannot be an M-augmenting path P, since the symmetric difference $P \ominus M$ (i.e., edges contained in exactly one of M or P) forms a matching in G. Moreover, $|P \ominus M| = |M| + 1$, which gives us a contradiction; the extra edge come from the flipping of the matched and unmatched edges in P; the edges of M not in P remain unaffected.

Conversely, suppose M is not maximum and M' is a maximum matching (thus $|M'| > |M|$) then we want to show that there is an M-augmenting path. Consider the symmetric difference $M' \ominus M$, i.e., the set of edges in $(M \cup M') \setminus (M \cap M')$. Let H be the subgraph corresponding to $M' \ominus M$ in G. Then every vertex in H has degree at most two, as it is incident to either an edge in M or M'. Thus the connected components in H are either alternating paths or even cycles with edges of M and M' alternating. Since $|M'| > |M|$ there must be more edges of M' in H than of M, which implies there must be a maximum alternating path P that starts and ends with an M' edge. As the starting and ending of P are matched in M' in H they are unmatched by M in G. Thus we have an M-augmenting path. **Q.E.D.**

Corollary 4. *Let M', M be matchings such that $|M'| \geq |M|$. Then in $M' \ominus M$ there are $|M'| - |M|$ vertex-disjoint M-augmenting paths. Moreover, the shortest path has length at most $|V|/(|M'| - |M|)$.*

Proof. The proof of the first claim is similar to the lemma above. Since any M-augmenting path in $M' \ominus M$ has an extra edge from M' the number of such paths are $|M'| - |M|$. Moreover, the vertices in the path have degree at most two, so any two M-augmenting paths are vertex-disjoint.

The second claim follows by the Pigeonhole Pricinple: if all the $|M'| - |M|$ vertex-disjoint paths have length $> |V|/(|M'| - |M|)$ then we have more than $|V|$ vertices, which is a contradiction. So the length of the shortest path is $\leq |V|/(|M'| - |M|)$. **Q.E.D.**

To prove König's theorem, we will construct a vertex cover for G from a maximal matching M. The key idea is to use the alternating path.

Proof. Let M be a maximum matching in a bipartite graph $G = (A, B)$, F_A be the set of unmatched vertices in A and F_B the unmatched vertices in B. We construct a vertex cover $\mathcal{C}$ for G from M. Suppose one of the sets F_A, F_B is non-empty; if both are empty then, M is a perfect matching and we can take either of the sets A or B as our vertex cover $\mathcal{C}$. Suppose $F_A \neq \emptyset$; if not, the argument can be modified appropriately. Then let T be the set F_A and all the vertices reachable from F_A by an alternating path. Define

$$\mathcal{C} := (A \setminus T) \cup (B \cap T).$$

We claim that $\mathcal{C}$ is a vertex cover and $|\mathcal{C}| = |M|$.

Let ab be an edge with $a \in A$ and $b \in B$. To prove our claim we have to show that either $a \notin T$ or $b \in T$. We consider two cases: if $a \notin T$ then $a \in \mathcal{C}$ and hence ab is covered. So suppose $a \in T$, then we want to show that $b \in T$ as well. Since $a \in T$, either $a \in F_A$ or a is at the end of an alternating path P. Let's consider the latter case first: if $b \in P$ then b is at the end of an alternating sub-path and hence in T (the edge ab may or may not be matched); otherwise (the edge ab is not matched), we can extend P to Pb, via a, to get an alternating path that ends in b, and so $b \in T$. If $a \in F_A$, then b is clearly at the end of the alternating path ab and hence in T. In both cases, $b \in T$ and hence the edge ab is covered.

To complete our argument, we have to further show that $|\mathcal{C}| = |M|$. The argument above shows that $|\mathcal{C}| \geq |M|$, since $\mathcal{C}$ is a vertex cover. Thus we only need to show that every vertex in $\mathcal{C}$ is an endpoint of an edge in M, and no two vertices in $\mathcal{C}$ share the same edge in M. Any vertex $a \in \mathcal{C} \cap A$ is actually in $A \setminus T$, and hence not in F_A (since $F_A \subseteq T$) and thus matched; any vertex in $b \in \mathcal{C} \cap B = B \cap T$ has to be matched, since otherwise we have an M-augmenting path starting from F_A and ending in an unmatched vertex b; but this cannot be as M is a maximum matching. Moreover, if two vertices, $a \in \mathcal{C} \cap A$ and $b \in B \cap T$ form an edge $ab \in M$, then $a \in T$, because we know that there is an alternating path P from F_A to b (which has to end in an edge in $E \setminus M$), and hence P can be extended to a, which would imply that $a \in T$, a contradiction. Thus $|M| \geq |\mathcal{C}|$. **Q.E.D.**

The proof above almost suggests a way to find a maximum matching. We describe below a method that finds a maximum matching in a bipartite graph. It is clear that to extend a matching M, we need to find an M-augmenting path, and the algorithm does precisely that.

2.1 Hopcroft-Karp – Finding Maximum Matchings

Let G, A, B, F_A and F_B be as defined above. We conceptually construct a directed graph H from G as follows: All the edges in M are directed $A \to B$ and edges not in M are directed from $B \to A$. The following claim is evident: G has an M-augmenting path iff H has a path from F_B to F_A. The algorithm then does the following: Given a matching M, do a DFS in H from F_B and stop as soon as we reach a vertex of F_A; the path so obtained is an M-augmenting path, so we augment our matching, and pick a vertex from F_B and continue till we cannot find a DFS path from F_B to F_A; note we may repeat an already tried vertex from F_B since augmenting may lead to alternating paths where none existed earlier (example?).

Let $|V| = n$ and $|E| = m$. The running-time analysis of the algorithm is as follows:

1. Given an M-augmenting path, augmenting M takes $O(m)$ time.
2. Assume $|B| \leq |A|$; otherwise swap their roles. Then $|F_B| \leq |B| \leq n/2$. The number of times we find an augmenting path is bounded by $n/2$, since every time we find an M-augmenting path from F_B to F_A, we match a previously unmatched vertex of F_B. Also, augmenting keeps the matched vertices.
3. At each stage, we have to do a DFS from all the vertices of F_B, until we succeed or exhaust F_B. We know that each DFS takes $O(n + m)$ time.

Thus the overall complexity is bounded by the number of stages, $\leq n/2$, times the number of DFS's and augmenting (if any) done at each stage, which is $O(|F_B|(n+m)+m)$. Since $|F_B| \leq n/2$ the overall complexity is $O(n^2(n + m))$. We can save upon the failed DFS's by just doing one DFS starting from a super-vertex β connected to all the vertices in F_B. In this case the overall complexity is $n/2$ stages times a DFS at each stage, which is $O(n(m + n))$. Since in the worst case a bipartite graph can have of $m = (n^2/4)$ edges, the worst-case complexity is $O(n^3)$. Hopcroft-Karp shave a factor of $\sqrt{n}$ by finding more than one augmenting path at each stage.

The Hopcroft-Karp algorithm proceeds in stages, where by the ith stage we have constructed a matching M_{i-1} ($M_0 := \emptyset$), and at the ith stage we do the following: Find the largest set $\mathcal{F}$ of vertex-disjoint M_{i-1}-augmenting paths of shortest length and define $M_i := M_{i-1} \ominus \mathcal{F}$.

Given the sets A, B, a matching M, and the set of unmatched vertices F_A, F_B we find the set $\mathcal{F}$ as follows: Starting from $L_0 := F_B$ we do a BFS that constructs layers of vertices $L_0, L_1, \ldots, L_t$, where the edges from even to odd layers are in $E \setminus M$, and the edges from odd to even layers are restricted to M; we do this till either we reach the first level of vertices L_t containing vertices from F_A or we have exhausted all the vertices in the graph (in the latter case, we know that M is maximum since there are no M-augmenting paths). Now, from each vertex in $u \in L_t \cap F_A$ we try to construct a vertex-disjoint M-augmenting path back to F_B. This can be done by using a DFS from u where we alternate the edges between $E \setminus M$ and M, making sure we never visit a vertex already used in some earlier DFS. The set of all vertex-disjoint paths so obtained are M-augmenting and the shortest in length, and is our set $\mathcal{F}$.

What's the complexity of the algorithm? The BFS and DFS each take $O(m)$ time. The crucial observation is that the number of stages are $O(\sqrt{n})$, and so the overall complexity is $O(m\sqrt{n}) = O(n^{2.5})$. The upper bound on the number of stages depends upon the following claim:

LEMMA 5. *Given a matching M, let $\mathcal{F} = \{P_1, \ldots, P_k\}$ be the largest set of vertex-disjoint shortest M-augmenting paths of length ℓ. Then the length of any $(M \ominus \mathcal{F})$-augmenting path is at least $\ell + 2$.*

Before we prove the lemma above, let's see why does it imply the upper bound on the number of stages. Let M be the matching after $\sqrt{n}$ iterations, and M' is a maximum matching. By Lemma 5 we know that at each stage the length of the shortest augmenting paths increases by at least two, the length of the shortest M-augmenting path is at least $2\sqrt{n} - 1 \geq \sqrt{n}$. By Corollary 4 we know that the length of the shortest augmenting path is at most $n/(|M'| - |M|)$. Thus the number of vertex-disjoint paths in $M' \ominus M$ is at most $\sqrt{n}$. So even if at each stage we just augment one more path, we need at most $\sqrt{n}$ more stages. Thus the number of stages is at most $O(\sqrt{n})$.

To prove Lemma 5, we need the following two lemmas.

LEMMA 6. *Let M be a matching of G, and P be an M-augmenting path of shortest length. If P' is an $(M \ominus P)$-augmenting path then $|P'| \geq |P| + |P \cap P'|$, where the cardinalities here count the number of edges in the paths.*

Proof. Consider the matching $M' := (M \ominus P) \ominus P'$. Then $|M'| = |M| + 2$ and hence from Corollary 4 we know that $M' \ominus M$ contains two vertex-disjoint M augmenting paths, say P_1, P_2. Moreover, $M' \ominus M = P \ominus P'$. Thus $|P \ominus P'| \geq |P_1| + |P_2| \geq 2|P|$, since both P_1, P_2 are M-augmenting paths and of length at least $|P|$. From the definition of symmetric difference we know that $|P \ominus P'| = |P| + |P'| - |P \cap P'|$, which combined with the lower bound of $2|P|$ gives us $|P'| \geq |P| + |P \cap P'|$. **Q.E.D.**

LEMMA 7. *Consider a sequence of matchings for G, $\emptyset = M_0, M_1, M_2, \ldots,$ where M_{i+1} is obtained from M_i by augmenting P_i, a shortest M_i-augmenting path (i.e., $M_{i+1} := M_i \ominus P_i$). Then for $i < j$, $|P_i| \leq |P_j|$ and equality implies that P_i and P_j are vertex-disjoint.*

Proof. The claim that for $i < j$, $|P_i| \leq |P_j|$ follows from Lemma 6 since P_{i+1} is a $(M_i \ominus P_i)$-augmenting path and P_i is a shortest M_i-augmenting path we know that $|P_{i+1}| \geq |P_i|$, and so $|P_j| \geq |P_i|$.

So suppose $|P_i| = |P_j|$ and they are not vertex-disjoint; this implies all the intermediate paths have the same length as well. Let $k \geq i$ and $\ell \leq j$ be the indices such that P_k and P_ℓ are not vertex disjoint but all the intermediate paths P_m are vertex disjoint from *both* P_k and P_ℓ. Since P_m are all vertex disjoint from P_k and P_ℓ, augmenting by these path does not affect the matched edges in P_ℓ, or in other words we could have augmented by P_ℓ first and then by P_m's without affecting the final matching. Thus P_ℓ is an M_{k+1}-augmenting path and hence $|P_\ell| \geq |P_k| + |P_k \cap P_\ell|$. But as $|P_\ell| = |P_k|$ it follows that $P_k \cap P_\ell = \emptyset$. That is P_k and P_ℓ do not share an edge but share a vertex, say v. Since P_ℓ is an M_{k+1}-augmenting path this means one of the edges incident on v in P_k is an edge in P_ℓ, which gives us a contradiction since P_k and P_ℓ do not share any edge. **Q.E.D.**

To complete the proof of Lemma 5:
Proof. Let $\mathcal{F} = \{P_1, \ldots, P_r\}$ and P' be a $(M \ominus \mathcal{F})$-augmenting path. Then we claim that $|P'| > |P_1| = |P_2| = \cdots = |P_r| = \ell$. If not then Lemma 7 implies that P' is vertex-disjoint from $P_1, \ldots, P_r$ and is of the same length. So we can add P' to $\mathcal{F}$, which is a contradiction since $\mathcal{F}$ is the largest set of shortest M-augmenting paths. Moreover, $|P'|$ is an augmenting path, so its length must be odd, thus $|P'| \geq \ell + 2$. **Q.E.D.**

2.2 Existence of Matchings

The section above describes an algorithm to find a maximum matching for a bipartite graph $G = (A, B)$. What if we are only interested in knowing whether G has a matching for A? Suppose we know that there is a matching M for A. What can we say about the set B? Clearly, for all $S \subseteq A$, the neighbourhood $N_G(S) \subseteq B$ of S must have cadinality at least the cardinality of $|S|$, i.e., $|N_G(S)| \geq |S|$ (we will write $N(S)$ instead of $N_G(S)$, the neighbourhood of S in G, if G is understood). It turns out that this condition is also sufficient, and that is Hall's theorem.

THEOREM 8 (Hall's Theorem 1935). *A bipartite graph $G = (A, B)$ has a matching of A into B iff for all $S \subseteq A$, $|N(S)| \geq |S|$.*

Proof. We have already seen that the existence of a matching of G implies that Hall's conditions are satisfied. We now show the converse, namely if G has no matching for A then there must be a subset S of A for which $|N(S)| < |S|$.

If G has no matching for A, then the size of the maximum matching is $< |A|$. Thus a vertex cover $U = A' \cup B'$, where $A' \subseteq A$ and $B' \subseteq B$, of minimum cardinality must satisfy $|U| < |A|$. That is $|U| = |A'| + |B'| < |A|$, or $|B'| < |A| - |A'| = |A \setminus A'|$. Now the neighbourhood of the set $A \setminus A'$ must be contained in B', since if there is an edge between $A \setminus A'$ and $B \setminus B'$ then that edge is not covered by U, which is a contradiction as U is a vertex cover. Thus $|N(A \setminus A')| \leq |B'| < |A \setminus A'|$, which gives us a subset for which Hall's condition fails. **Q.E.D.**

A proof of Hall's theorem from first principles:

Proof. Proof is by induction on $|A|$: for $|A| = 1$ the theorem holds because $|N(A)| \geq 1$ implies that there is a matching for A. So given $|A| \geq 2$, assume that the induction hypothesis holds for all $G' = (A', B)$, where $A' \subset A$.

There are two cases to consider: first, if for all subsets $S \subset A$, $|S| < |N_G(S)|$, and second if there exists a subset $S \subset A$, s.t. $|S| = |N(S)|$. In both cases, we will inductively construct a matching for A.

Case 1: Pick an edge ab, $a \in A$, $b \in B$ and consider the graph G' obtained from G by deleting the vertices a, b and all the edges incident on them. Let $A' := A \setminus \{a\}$. Then the neighbourhood of $S \subseteq A'$ in G' satisfies

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|.$$

Thus G' satisfies Hall's condition and hence there is a matching M of A' in G'. Then $M \cup ab$ forms a matching for A in G.

Case 2: Let $S \subset A$ be such that $|N_G(S)| = |S|$. Then by the induction hypothesis applied to the induced subgraph $G' := G[S \cup N_G(S)]$, we know that there is a matching (in fact a perfect matching), say M', of S in G'. Consider the graph $G - G'$. Then this subgraph of G also satisfies Hall's condition; otherwise, if there is a set $T \subseteq A \setminus S$ such that $|N_{G-G'}(T)| < |T|$ then $|N_G(T \cup S)| < |T \cup S|$, which is a contradiction since $T \cup S \subseteq A$ and hence must satisfy Hall's condition. Thus $A \setminus S$ has a matching M in $G - G'$. Thus $M' \cup M$ is a matching for A in G. **Q.E.D.**

As an immediate corollary we have the following earlier result of Frobenius:

Corollary 9 (Frobenius 1917)**.** *A bipartite graph $G = (A, B)$ has a perfect matching of A into B iff for all $S \subseteq A$, $|N(S)| \geq |S|$ and $|A| = |B|$.*

We next study matchings in general graphs and give a generalization of the corollary above that gives a characterization for the existence of perfect-matchings in general graphs.

3 Matching in Non-Bipartite Graphs

The first non-trivial result on matchings in non-bipartite graphs was Tutte's (1947) characterization for the existence of perfect matchings in such graphs. However, algorithms to find perfect or maximum matchings were only discovered about two decades later in the seminal work of Edmond's (1965). Tutte's result is thus a generalization of Frobenius's result almost three decades ago. As was the case with Hall's/Frobenius's theorem the necessary condition turns out to be sufficient as well. But what is the necessary condition?

Consider a graph $G = (V, E)$, , not necessarily connected. An **odd component** of G is a connected component with odd number of vertices; similarly, define an **even component**. Let

$$\mathcal{O}(G) := \#\{\text{odd components of } G\}.$$

Suppose M is a *perfect matching* for G. Consider a subset $S \subseteq V$, and the connected components in the graph $G - S$. The edges of M matches an even number of vertices in any component. Thus in every odd component of $G - S$ there must be at least one vertex v which is connected to S by a matched edge (it has to be a matched edge because M is a perfect matching, and v cannot be matched within its component and so it has to be matched with a vertex in S). Thus the number of odd components in $G - S$ must be smaller than $|S|$, i.e., $\mathcal{O}(G - S) \leq |S|$. This is a necessary condition, and as will be the case, also sufficient.

THEOREM 10 (Tutte 1947). *A graph G has a perfect matching iff for all $S \subseteq V$, $\mathcal{O}(G - S) \leq |S|$.*

We will show that if G has no perfect matching then there is a witness set S violating Tutte's condition, i.e., $(\mathcal{O})(\mathcal{G} - \mathcal{S}) > \mathcal{S}$. A simple case is when $|V|$ is odd, in which case there cannot be a perfect matching and the witness set is $\emptyset$. So from now on assume that $|V|$ is even and G has no perfect matching.

The proof is an extremal graph theory proof, i.e., we will consider a graph that has some "extreme" property. The proof proceeds in steps: the first step shows that if a certain edge-maximal graph G' obtained from G has no perfect matching, then G' has a violating set and hence so does G; the second step explains the structure of the violating set S in G'; the third step shows the converse, that is, if G' has no perfect matching and has a set S with the structure described in step two then S must be a witness set (thus step two and three give us a characterization of witness sets of G'); the final step shows that a certain set in G' is a witness set.

¶1. Step 1: Let $G' = (V, E')$ be the edge-maximal graph obtained from G by adding edges while ensuring G' has no perfect matching. Such a graph is called **saturated non-factorizable** graph since adding one more edge gives us a graph that has a perfect matching. If S is a witness for G' then S is also a witness for G because the odd components in $G' - S$ are obtained by union of components of $G - S$ and so on deleting the extra edges in G' each odd component of $G' - S$ must give rise to at least one odd component in $G - S$, i.e., $\mathcal{O}(G - S) \geq \mathcal{O}(G' - S) > |S|$. So from now on we will find a witness set for G'.

¶2. Step 2: How can a witness S for G' look like? If S is a witness for G' then we know that $\mathcal{O}(G' - S) > |S|$. But for G' to be edge maximal it must be that S and the components of $G' - S$ are complete graphs (otherwise we can keep adding edges to them and Tutte's condition is still violated). But since we only desire that the components in $G' - S$ do not coalesce, there must also be edges from every vertex in S to all the remaining vertices of G'. To recap, S must have the following two properties:

P1. every vertex in S is connected to all the vertices in the graph (including S), and

P2. the components in $G' - S$ are complete graphs.

¶3. Step 3: We claim that the convers is also true, i.e., if a set S in G' satisfies P1 and P2, then it must be a witness to the non-factorizability of G'. We will prove this by contradiction. Suppose S satisfies the two properties above but is not a witness to non-factorizability of G', i.e., $\mathcal{O}(G - S) \leq |S|$, then we will show that G' has a perfect matching, which is a contradiction. If S is not a witness set then $\mathcal{O}(G' - S) \leq |S|$. Therefore, in an odd component of $G' - S$, we can pair all the vertices amongst themselves, except one vertex which can be paired with a vertex in S. Since $|V|$ is even, we're left with an even number of vertices, which can be paired as follows: vertices in S left after pairing with a vertex from each of the odd components are even in number and hence can be paired amongst themselves; every even component of $G' - S$, if any, can be paired within itself. Thus, in this manner, we can construct a perfect matching for G', which is a contradiction. So a set S having the properties P1 and P2 must be a witness set.

¶4. Step 4: We claim that the *largest* set S in G' satisfying P1 must necessarily satisfy P2, and therefore is our witness set. So suppose there is a component in $G' - S$ that is not a complete graph. Then this component must have three vertices a, b, c such that $ab, bc \in E'$ but $ac \notin E'$. Moreover, as $b \notin S$ there must be a vertex d not neighbouring b in G', i.e., $bd \notin E'$. Since G' is an edge-maximal graph that has no perfect matching, if we add the edge ac to G' then we get a perfect matching M_1 for G', and if we add bd to G' then we get another perfect matching M_2 for G'; moreover, ac must be an edge in M_1 and bd an edge in M_2. Let P be the longest path in G' from d with edges alternating between M_1 and M_2 and ending in a vertex v, or in other words, the longest path from d in $M_1 \ominus M_2$. Why is there such a path? Since M_1 is a perfect matching, we know that there is an M_1 edge from d, say de. Now there must be an M_2 edge from e, say ef, distinct from de because d is matched in M_2 by the added edge bd; similarly, there is an M_1 edge from f distinct from ef because e is already matched in M_1 by de. Continuing in this manner, if the last edge on P is from M_1 then $v = b$, since all other vertices in G' have an M_2 edge in G'. Thuc $C := dPbd$ is an alternating cycle and clearly of even length. If the last edge on P is from M_2, then $v \in \{a, c\}$, since every other vertex has an M_1 edge in G', say $v = a$. Then $C := dPabd$ is an even cycle. In both cases, the only edge in C that is not in G' is bd, and every alternate edge is in M_2. So if we consider $C \ominus M_2$, we get a perfect matching of G' that contains edges from E', but this is a contradiction. So our initial assumption that $G' - S$ has a component that is not a complete graph is false. This argument is illustrated in Figure 3.

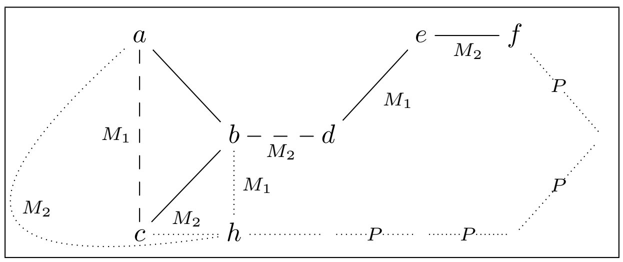

Figure 3: Constructing a perfect matching in G' using an alternating path P. The dashed edges ac and bd are added to G'.

As a consequence of Tutte's theorem, we can check apriori whether certain graphs have a perfect matching or not.

Corollary 11 (Petersen 1891)**.** *Every bridgeless cubic graph has a perfect matching.*

Proof. Let $S \subseteq V$ and consider an odd component C in $G - S$. The sum of the degrees of the vertices in C, when viewed as a subset of G, is $3|C|$, which is clearly an odd number. Since every edge in C is counted twice in this sum, we get that

$$3|C| = 2(\text{number of edges in } C) + (\text{edges between } C \text{ and } S).$$

Therefore, the number of edges between C and S must be an odd number. It cannot be one, as G is bridgeless, so it must be at least three. Thus the number of edges between S and the odd components in $G - S$ is at least $3\mathcal{O}(G - S)$. But as the graph is cubic, the total number of edges from vertices in S is at most $3|S|$. Thus $\mathcal{O}(G - S) \leq |S|$ for all $S \subseteq V$, and hence G satisfies Tutte's conditions and therefore has a perfect matching. **Q.E.D.**