Lecture 3 – Infinite Sums and Series

Read Euler, read Euler. The master of us all. – Laplace

1 Introduction

We have seen some approaches to compute closed expressions for sums of finite terms. In this setting it is but natural to expect that there is a closed form for the sum (assuming the summands are well-defined). However, such notions break down in the setting of infinite sums. In fact, infinite series are at the heart of some of the earliest paradoxes that we know of, e.g. Zeno's Dichotomy Paradox [1]:

> That which is in locomotion must arrive at the half-way stage before it arrives at the goal. – Aristotle, Physics.

More elaborately, suppose Zeno wanted to travel a distance x, then he would first have to travel $x/2$, and to do that he would have to travel $x/4$, and so on indefinitely $\ldots$. Zeno says that there is no "first distance" to be travelled, since to travel that he would have to travel half the distance first, and so the trip cannot even begin. Presumably, Zeno never travelled. On hearing this argument, Diogenes the Cynic, stood up and walked (thus showing the falsity of the conclusion, but not revealing the flaw in the argument). We, with our knowledge of geometric series, know that the argument is wrong because an infinite sum of ever decreasing terms can be substantial:[2]

$$R := \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

How do we prove this? Multiplying both sides by 2 and subtracting one we get that $2R - 1 = R$, or $R = 1$. Let's apply the same trick to the sum of even numbers $S := 1 + 2 + 4 + 8 + \cdots$. We obtain that $2S + 1 = S$, or $S = -1$. But this is absurd – how can adding positive quantities give us something negative? Let's see another infinite sum proposed by Guido Grandi in 1703:

$$T := 1 - 1 + 1 - 1 + 1 + \cdots$$

the sum where positive one and negative one alternate. If we pair up as follows

$$T = (1 - 1) + (1 - 1) + \cdots$$

then $T = 0$, but if we pair up as follows

$$T = 1 + (-1 + 1) + (-1 + 1) + \cdots$$

then we get $T = 1$. Now we also observe that $T = 1 - T$, thus $T = 1/2$, a third value? We can derive the same result by substituting $x = 1$ in the well know formula for geometric series

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-x)^i.$$

What is the correct value of T? Both Euler and Fourier believed that $T = 1/2$ was the correct value. Leibniz, predating both of them, even gave a "probabilistic argument": if we "stop" the infinite series it is possible to obtain both zero or one with equal probability, so the expected value is the average, that is $1/2$.[3]

[1] Aristotle considered it to be an equivalent formulation of the Achilleus-Tortoise Paradox.

[2] Archimedes derived a similar result for the Quadrature of a Parabola using the method of exhaustion, a method that is at the foundation of integral calculus, to show that the geometric series $\sum_{i \geq 0} 4^{-i}$ is bounded.

[3] Leibniz conceded that his statement was more metaphysical than mathematical, but went on to say that there was more metaphysical truth in mathematics than was generally recognized (Kline-Vol.2, p. 446).

What is wrong with the arguments above? The arguments for the sums S, T assume that the quantities S, T are well defined. The formal notion of convergence of a series was not defined until lathe 18th century. Most of the mathematicians of 17th and 18th century worked with infinite series for functions without worrying about their convergence, they manipulated such series with intuition as their guide, as we will see later.

We won't treat here in detail the various notions of convergence, except the notion of **absolute convergence**: a series $\sum_k a_k$ with positive summands is said to be absolutely convergent if the sum $\sum_k |a_k|$ converges (i.e., $\lim_{N\to\infty} \sum_{k<N} |a_k|$ exists); a series with both positive and negative summands is said to be absolute convergent if its positive and negative parts are absolutely convergent; a similar extension can be defined when the summands are complex numbers (that is, the real and imaginary parts are bounded in absolute value). Once we have verified that a series is absolutely convergent, we can apply our machinery of finite calculus, as the following example illustrates:

$$\begin{aligned}\sum_{k\geq 0} \frac{1}{(k+1)(k+2)} &= \sum_{k\geq 0} k^{\underline{-2}} \\ &= \lim_{N\to\infty} \sum_{0\leq k<N} k^{\underline{-2}} \\ &= \lim_{N\to\infty} -k^{\underline{-1}}\Big|_0^n \\ &= 1.\end{aligned}$$

But what about the sum $H := \sum_{k>0} 1/k$? Clearly, $H > R + 1$. But its terms are tending to zero? So it's not apparent whether H is bounded or unbounded. Let's define $H_n := \sum_{0<k\leq n} 1/k$, the familiar Harmonic series. The behaviour of H is the same as the limit $\lim_{n\to\infty} H_n$; but we know that $H_n \sim \ln n + \gamma$, where γ is the Euler-Mascheroni constant; thus as $n \to \infty$, the sum H tends to infinity as well. However, this argument depends upon a tight estimate of the nth harmonic function. We can salvage the approach by using calculus based estimate to show that

$$\ln n + \frac{1}{n} \leq H_n < \ln n + 1.$$

What about a direct argument? James Bernoulli (also known as Jakob or Jacques), one of the great Bernoullis, was an early pioneer in infinite series [4] He argued that the ordinary harmonic series is divergent. The proof is based upon the observation that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > 1 - \frac{1}{n};$$

each term is greater than n^{-2} and there are $n^2 - n$ terms. Thus

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > 1,$$

so, he said, we can collect such group of terms each contributing a term greater than one. This result was contrary to his belief, and the belief of many eighteenth century mathematicians such as Lagrange, that a series whose terms vanish cannot be infinite. He consequently showed closed forms for other infinite series, such as the following:

- inverse sum of the triangular numbers $k(k+1)/2$ is two (also shown by Huygens and Leibniz);
- the sum of terms $(a + kc)/(bd^k)$; and
- two beautiful sums $\sum_{k\geq 1} k^2/2^k = 6$ and $\sum_{k\geq 1} k^3/2^k = 26$.

Fueled by these successes he attempted the series

$$\zeta(s) := \sum_{k\geq 1} \frac{1}{k^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \quad (1)$$

The harmonic series is the sum $\zeta(1)$. So the next challenge was $s = 2$.

[4] There was some great work done by an Indian named Madhavan from the Kerala school of astronomy in the series expansion of trigonometric functions predating Newton by at least two centuries.

2 The Basel Problem

The Basel problem was to derive a "nice" form for the sum $\zeta(2)$. It was posed by Pietro Mengoli (1625-1686), an Italian, when he was nineteen years old. Though the problem became well known in 1689 when James Bernoulli wrote about it (a coeval-problem, though not to rise to fame till a later date, was Fermat's problem). He said in his *Tractatus*

> If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude.

The name "Basel" was associated with the problem because of the various attempt by the Bernoullis and the eventual solution by one of their disciples Euler.

James had made some progress in showing that, unlike the harmonic series case, the sum $\zeta(2)$ was bounded. This was based upon an easily verifiable observation that

$$\frac{1}{k^2} \leq \frac{2}{k(k+1)}.$$

The sum of the latter terms is

$$\begin{aligned}\sum_{k>0} \frac{2}{k(k+1)} &= 2\sum_{k>0}\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 2\left[(1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots\right] \\ &= 2,\end{aligned}$$

because of the "telescoping affect"; this result was already known to Leibniz, as we had mentioned earlier.[5]

Thus $\zeta(2)$, and hence $\zeta(s)$, $s > 2$, were all bounded from above.

The problem remained immune to attacks from various mathematicians for almost a century until Leonhard Euler (a student of Johann Bernoulli, brother of James) focussed his energies on it, and after six years finally solved it in 1735, at the age of 28.

Like many before him, Euler started with numerical approximations to the series. However, since the series convergence slowly such an approach was not illuminating; e.g. taking the sum to the first thousand terms gives us only two correct digits 1.64393. Euler's first break through in 1831 was to devise a representation of the sum that yielded much better numerical approximations. He showed that

$$\zeta(2) = \sum_{k\geq 1} \frac{1}{k2^{k-1}} + (\ln 2)^2. \tag{2}$$

Why is this a breakthrough? Because the series on the RHS is exponentially converging, we can derive a very good estimate for the LHS. So Euler showed that if we just use the first fourteen terms on the RHS then we can get an approximation good to six digits for $\zeta(2)$. This was much better than what was known earlier, and already gave a hint to what the sum could be. But first let's see how did Euler derive (2).

He started with the integral

$$\int_0^{\frac{1}{2}} -\frac{\ln(1-t)}{t} dt \tag{3}$$

[5] Applying our straightforward calculus based approach and see that $\int_1^\infty dx/x^2 < \zeta(2) < 1 + \int_1^\infty dx/x^2$. The second inequality gives us the desired upper bound.

and expressed it in two different ways (similar to the perturbation method). First, he replace $\ln(1-t)$ by its series expansion to get

$$\begin{aligned}\int_0^{\frac{1}{2}} -\frac{\ln(1-t)}{t}dt &= \int_0^{\frac{1}{2}} -\frac{-t-t^2/2-t^3/3-\cdots}{t}dt \\ &= \int_0^{\frac{1}{2}} \left(1+\frac{t}{2}+\frac{t^2}{3}+\cdots\right)dt \\ &= t+\frac{t^2}{4}+\frac{t^3}{9}+\cdots\Big|_0^{\frac{1}{2}}. \\ &= \sum_{k\geq 1}\frac{1}{k2^k}.\end{aligned}$$

Substituting $t=1$ in the penultimate step, gives us $\zeta(2)$, but the half was not an error as we will see next. Another way to derive the same integral was to replace $1-t$ with z, to obtain

$$\begin{aligned}I &= \int_1^{\frac{1}{2}} \frac{\ln z}{1-z} \\ &= \int_1^{\frac{1}{2}} \ln z(1+z+z^2+\cdots) \\ &= \int_1^{\frac{1}{2}} \ln z + \int_1^{\frac{1}{2}} z\ln z + \int_1^{\frac{1}{2}} z^2\ln z+\cdots.\end{aligned}$$

Choosing $du := z^k$ and $v := \ln z$, and applying the rule that $uv = \int udv + \int vdu$ we obtain that

$$\int_1^{\frac{1}{2}} z^k\ln z = \frac{z^{k+1}}{k+1}\ln z - \int_1^{\frac{1}{2}}\frac{z^k}{k+1} = \frac{z^{k+1}}{k+1}\ln z - \frac{z^{k+1}}{(k+1)^2}\Big|_1^{\frac{1}{2}}.$$

Substituting this in the equation for I we obtain that

$$\begin{aligned}I &= (z\ln z - z) + (\frac{z^2}{2}\ln z - \frac{z^2}{4}) + (\frac{z^3}{3}\ln z - \frac{z^3}{9}) + \cdots\Big|_1^{\frac{1}{2}} \\ &= \ln z(z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots) - (z+\frac{z^2}{4}+\frac{z^3}{9}+\cdots)\Big|_1^{\frac{1}{2}} \\ &= \ln z(-\ln(1-z)) - (z+\frac{z^2}{4}+\frac{z^3}{9}+\cdots)\Big|_1^{\frac{1}{2}} \\ &= -(\ln\frac{1}{2})^2 - \sum_{k\geq 1}\frac{1}{k2^k} + \zeta(2).\end{aligned}$$

Equating the two forms for I we get that

$$\zeta(2) = 2\sum_{k\geq 1}\frac{1}{k2^k} + (\ln 2)^2$$

the equation we wanted in (2).

Though this was significant progress, what Bernoulli had asked for was an *exact* formula. Four years later in 1835 Euler came up with an insightful proof based upon analogy, and said that

> Now, however, against all expectation I have found an elegant expression for the sum of the series $1+\frac{1}{4}+\frac{1}{9}+$ etc., which depends on the quadrature of the circle... I have found that six times the sum of this series is equal to the square of the circumference of a circle whose diameter is 1.

Note that indirect reference to π; the notation that we use now was accepted only later, though some mathematicians had used it at that time; Euler himself started with p initially and in later years used π.

His argument requires two fundamental observations, and a vivid imagination. We know from the fundamental theorem of algebra that a polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ has n roots $\alpha_1, \ldots, \alpha_n$ and that

$$P(x) = a_n(x - \alpha_1) \ldots (x - \alpha_n). \tag{4}$$

Moreover, comparing the like powers of x on both sides, we can express the coefficients in terms of the roots. Two such relations are

$$a_{n-1} = -a_n(\alpha_1 + \cdots + \alpha_n).$$

and $a_0 = (-1)^n a_n \alpha_1 \ldots \alpha_n$. Another way to express (4) is

$$a_0 + a_1 x + \cdots a_n x^n = a_n \alpha_1 \ldots \alpha_n (\frac{x}{\alpha_1} - 1) \ldots (\frac{x}{\alpha_n} - 1) = a_0(1 - \frac{x}{\alpha_1}) \ldots (1 - \frac{x}{\alpha_n}) \tag{5}$$

which is valid if all the roots are not zero, or equivalently $a_0 \neq 0$. This equation gives us that

$$a_1 = -a_0 \left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_n} \right). \tag{6}$$

This equation can be used to derive a relation in the especial case when the polynomial has even degree $2n$, has the form

$$Q(x) := b_0 - b_1 x^2 + b_2 x^4 - b_3 x^6 + \cdots + b_n x^{2n},$$

and the $2n$ roots $\pm\beta_1, \pm\beta_2, \ldots, \pm\beta_n$. By plugging $y := x^2$, we can reduce the polynomial to the form in (4) where $\alpha_i := \beta_i^2$, $i = 1, \ldots, n$, and use (6) to obtain the relation

$$b_1 = b_0 \left(\frac{1}{\beta_1^2} + \cdots + \frac{1}{\beta_n^2} \right). \tag{7}$$

Suppose we had a "polynomial" of the form $Q(x)$ whose roots are $\pm\pi, \pm 2\pi, \ldots, \pm n\pi$ where n goes to infinity. Then (7) suggests that

$$\frac{b_1}{b_0} = \sum_{k=1}^{n} \frac{1}{(k\pi)^2}.$$

What could be such a "polynomial"? Letting our imagination run free for a moment we see that the "polynomial" is $\sin x$, except that 0 is also its root, so we divide it by x and claim

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \ldots. \tag{8}$$

Using the series expansion for $\sin x = \sum_{k \geq 1} (-1)^{k-1} x^k / (2k-1)!$ we get that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

Equating the coefficient of x^2 in the two representations for $\sin x / x$ we obtain the beautiful formula

$$\frac{\pi^2}{6} = \zeta(2).$$

The formula that James Bernoulli wanted, and that Euler derived with such a beautiful analogy. When Johann Bernoulli learned of the solution he wrote:

> Utinam Frater superstes effet! (If only my brother were alive!)

There was some concern with the way Euler proceeded in deriving an infinite product form for $\sin x$; e.g., the series $e^x \sin x$ has the same roots as $\sin x$, but its series expansion is not the same. Euler himself had doubts about his argument. He kept on verifying it numerically, and used the same approach to derive already known results, such as an alternate solution to Wallis's formula and Leibniz's result on π. Almost a decade later, he came up with a rigorous proof based upon calculus (perhaps not by modern standards). We will see next a rigorous proof.

3 A Rigorous Proof

This proof is from an article in the *Mathematical Intelligencer* by Apostol in 1983; the version here is from [AZ].

The main idea is similar to the one we used in the derivation of (2): we will represent $\zeta(2)$ as a double integral and give another representation of the integral that yields the desired equality.

Our starting point is to represent $1/k^2$ as an integral. Let's start with $1/k$. It is easy to see that

$$\int_0^1 x^{k-1} dx = \frac{x^k}{k}\bigg|_0^1 = \frac{1}{k}.$$

So if we multiply to such integrals, over different variables we get

$$\int_0^1 x^{k-1} dx \int_0^1 y^{k-1} dy = \int_0^1 \int_0^1 (xy)^{k-1} dx\, dy = \frac{1}{k^2}.$$

Thus

$$\begin{aligned}\zeta(2) &= \sum_{k\geq 1} \frac{1}{k^2} \\ &= \sum_{k\geq 1} \int_0^1 \int_0^1 (xy)^{k-1} dx\, dy \\ &= \int_0^1 \int_0^1 \sum_{k\geq 1} (xy)^{k-1} dx\, dy \\ &= \int_0^1 \int_0^1 \frac{dx\, dy}{1-xy}.\end{aligned}$$

Let I be this new integral. We have done half the task. We now perform a coordinate transform

$$u := \frac{y+x}{2} \text{ and } v := \frac{y-x}{2}.$$

In the new coordinate system the domain of integration is a unit square with edge length $1/\sqrt{2}$ and rotated by $\pi/4$, as shown in Figure 1.

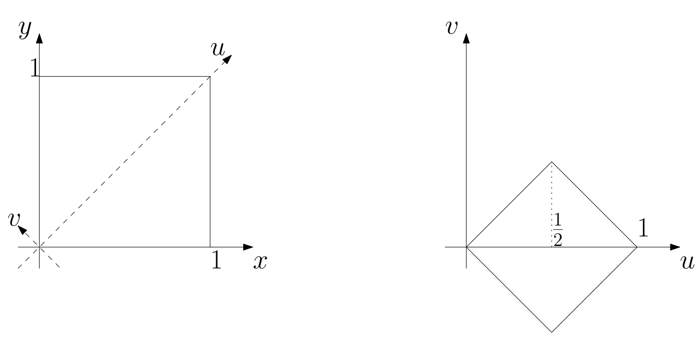

Figure 1: The domain of integration: the unit square in the xy-plane, and the transformed square in the uv-plane.

What is the new integral? Since $x = (u - v)$ and $y = (u + v)$,

$$\frac{1}{1-xy} = \frac{1}{1-u^2+v^2}$$

and since the transformation reduces the area of an axis aligned square by half, we have $dx\, dy = 2du\, dv$. Since the square in the uv-domain is symmetric around the u-axis, we just have to compute the integral in the upper-half region and double it. Moreover, while u goes from 0 to $1/2$, v goes from 0 to u, and when u goes from $1/2$ to 1, v goes from 0 to $1 - v$. Thus our original integral is equal

$$I = 4\int_0^{\frac{1}{2}}\int_0^u \frac{dv}{1-u^2+v^2}du + 4\int_{\frac{1}{2}}^1\int_0^{1-u}\frac{dv}{1-u^2+v^2}du.$$

Since

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + \text{ const.}$$

by taking $a := \sqrt{1-u^2}$ we obtain

$$I = 4\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-u^2}}\arctan\frac{u}{\sqrt{1-u^2}}du + 4\int_{\frac{1}{2}}^1\frac{1}{\sqrt{1-u^2}}\arctan\frac{1-u}{\sqrt{1-u^2}}du.$$

Let $g(u)$ be the $\arctan$ in the first integral and $h(u)$ be the arctan in the second integral. Then $g'(u) = 1/\sqrt{1-u^2}$ and $h'(u) = -1/2\sqrt{1-u^2}$. So we have

$$\begin{aligned}
I &= 4\int_0^{\frac{1}{2}} g'(u)g(u) - 8\int_{\frac{1}{2}}^1 h'(u)h(u)\\
&= 2g(u)^2\Big|_0^{\frac{1}{2}} - 4h(u)^2\Big|_{\frac{1}{2}}^1\\
&= 2(g(\frac{1}{2})^2 - g(0)^2) - 4(h(1)^2 - h(\frac{1}{2})^2)\\
&= 2(\frac{\pi}{6})^2 - 0 - 0 + 4(\frac{\pi}{6})^2\\
&= \frac{\pi^2}{6}.
\end{aligned}$$

4 Current Status

Euler had already given formulas for $\zeta(2n)$. What about $\zeta(3)$? The problem is still open. Roger Apery '78 has shown that it is irrational. We don't even have the equivalent of Leibniz's result for $\sum_k 1/(k^3-1)$. Given these obstacles, researchers have tried the weaker sums $\sum_{\mathbb{Z}} 1/(n^k-1)$ and have showed that they are transcendental for $k = 3, 4, 5$ and for all k if Schneider's theorem holds (see Weatherby –Transcendence of Infinite Series). For more than two centuries, we haven't come close to a solution that James Bernoulli would have wanted.