Zeta function on graphs revisited with the theory of heaps of pieces

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zeta function of a graph

Ihara-Selberg zeta function of a graph

$Z_G(t)$

Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

prime numbers decomposition

$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Euler identity

$$\zeta(s) = \prod_{p} \left(\frac{1}{1 - p^{-s}} \right)$$

prime number

$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1 - t^{|C|})}$$

some "prime" over the graph G

Ihara-Selberg zeta function of a graph

$$Z_G(t) = \prod_{[C]} \frac{1}{(1 - t^{|C|})}$$

some "prime" over the graph G

equivalence class of a circuit C

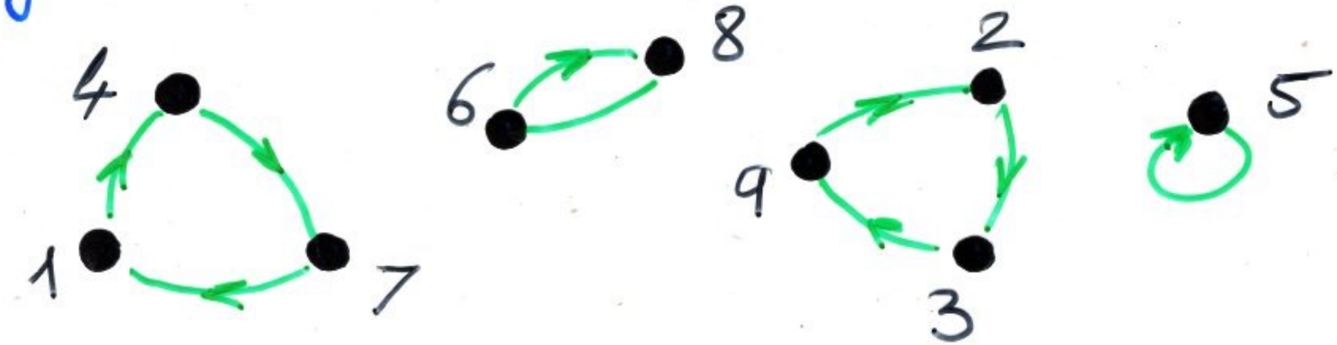
prime circuit

path on X

permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

cycles notation



path on X

$$\omega = (s_0, \dots, s_i, s_{i+1}, \dots, s_n)$$

$s_i \in X \quad i=0,\dots,n$

ω goes from s_0 to s_n

path on a graph G (oriented or not)

(s_i, s_{i+1}) edge of G

notation $\quad s_0 \overset{\omega}{\leadsto} s_n$

s_0 starting vertex

s_n ending vertex

(s_i, s_{i+1}) elementary step

length $|\omega| = n$
(number of elementary steps)
$n+1$ vertices

circuit

= path $\underset{u \rightsquigarrow u}{\omega}$

$\underset{u \rightsquigarrow s}{\omega'} \cdot \underset{s \rightsquigarrow t}{\omega''}$

product of two paths circuit

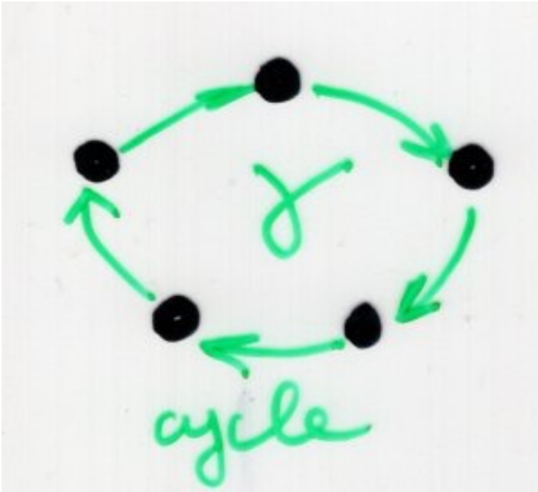
prime circuit

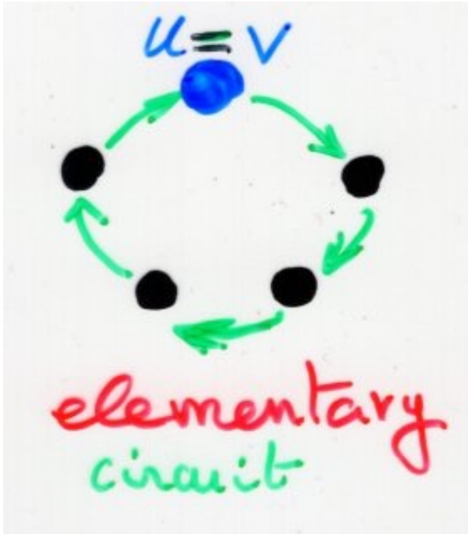
$C = \omega^p \Rightarrow p = 1$

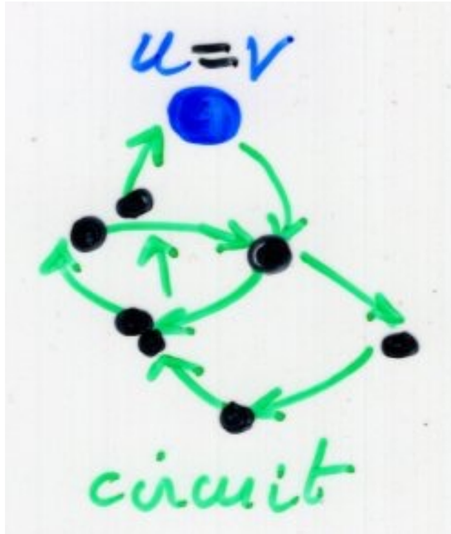
equivalence class of a circuit C

$[C]$

C as a word of edges up to a circular permutation







cycle = [C]

Ihara-Selberg zeta function of a graph

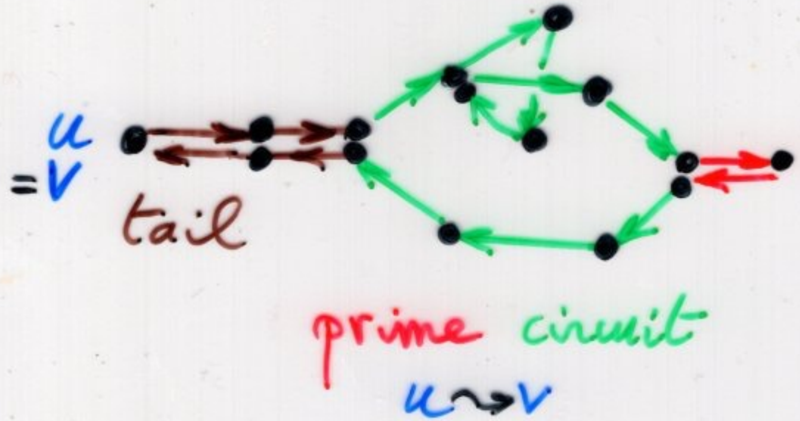
Ihara (1966)

$$(i) \quad Z_G(t) = \prod_{[C]} \frac{1}{(1 - t^{|C|})}$$

equivalence class

prime circuit

no backtracking



backtracking

- no tail
- no backtracking

Ihara-Selberg zeta function of a graph

(i) $$\zeta_G(t) = \prod_{[C]} \frac{1}{(1 - t^{|C|})}$$

(ii) $$\zeta_G(t) = \frac{1}{\det(1 - Ht)}$$

(iii) $$\zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det(I - tA + t^2(D - I))}$$

$$t \frac{d}{dt} \log \zeta_G(t)$$

Bass formula

Bass (1992) Hashimoto (1989)
Venkov, Nikitin (1994)
Sunada (1986, 88)

Stark, Terras (1996, 2000)
book

Northshield (1999)
Foata, Zeilberger (1999)
bijective proof

Bartholdi (1999)

Mizuno, Sato (2000, ..., 2009)

....
many others

→ quantum
walks

Giscard, Rochet (2016)
extending number theory
to paths on Graphs

(i) $\prod_{[c]} \frac{1}{(1-t^{|c|})}$

$$\log \zeta_G(t) = \sum_{[c]} \sum_{p\geq 1} \frac{1}{p} t^{p|c|}$$

$$t\frac{d}{dt} \log \zeta_G(t) = \sum_{[c]} \sum_{p\geq 1} |c|\, t^{p|c|}$$

equivalence class
prime circuit
no backtracking

$$= \sum_{[c]} |c|\, t^{|\omega|}$$

equivalence class
circuit
no backtracking

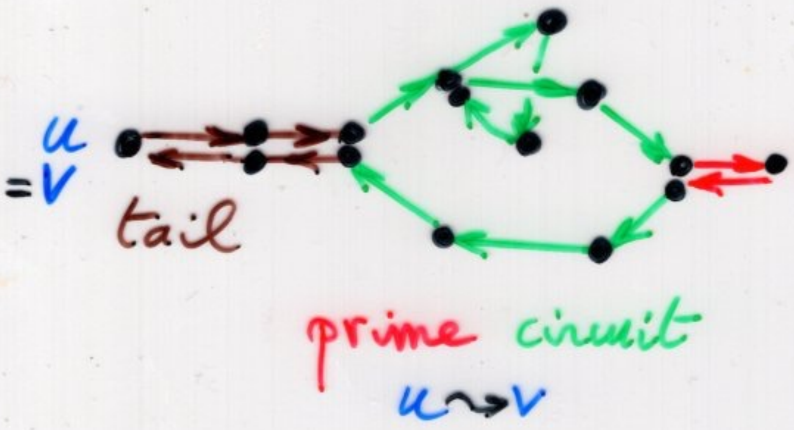
ω circuit $u \to u$ $= c^p$

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

(- no tail
- no back tracking

$$t\frac{d}{dt}\log \zeta_G(t) = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

(- no tail
- no back tracking



backtracking

$$\text{(ii)} \quad \zeta_G(t) = \frac{1}{\det(1 - Ht)}$$

$$\text{(iii)} \quad \zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I - tA + t^2(D - I)\right)}$$

Ramanujan graphs

spectrum of a graph G

multiset of eigenvalues of A of adjacency matrix

spectrum of A related to the poles of the zeta function $Z_G(t)$

G is Ramanujan $\iff$ the poles of $Z_G(q^{-s})$ occur only on $\{\mathcal{R}(s) = 1/2\}$ and at $\{0,1\}$

Ramanujan graph

G $(q+1)$-regular $\Rightarrow$ $(q+1)$ is an eigenvalue

G bipartite $\Rightarrow$ $-(q+1)$ is also an eigenvalue

G connected, these are the unique largest eigenvalue

(Alon - Boppana) as $|V|$ increases,
vertices
all the other eigenvalues tend to be constrained in $[-2\sqrt{q}, 2\sqrt{q}]$.

G $(q+1)$-regular graph is called Ramanuján if all its eigenvalues lie in the interval $[-2\sqrt{q}, 2\sqrt{q}]$, except possibly $\pm(q+1)$

first infinite family (Lubotsky, Phillips, Sarnak) was constructed using Ramanujan conjecture on coefficients of the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

$$q = e^{2\pi i \tau}$$

Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

Dedekind eta function

$$q = e^{2\pi i \tau}$$

$$\Delta(\tau) = (2\pi)^{12} \left(\eta(\tau)\right)^{24}.$$

modular discriminant

$$\frac{1}{(2\pi)^{12}} \Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

$$= \sum_{n=1}^{\infty} \tau(n) q^n$$

Weil's conjectures $\Rightarrow$ Ramanujan's conjecture

$$|\tau(n)| \leq 2\, n^{11/2}$$

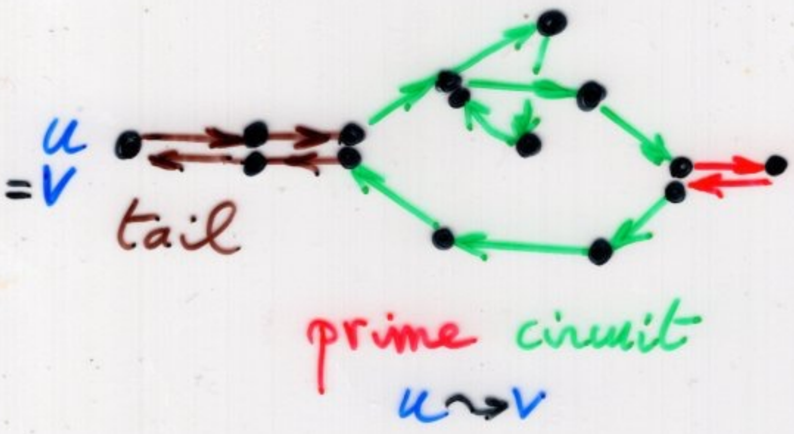
(i) $\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$

equivalence class
prime circuit

no backtracking

$t \frac{d}{dt} \log \zeta_G(t) = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$

- no tail
- no backtracking



backtracking

Paths in graphs
and
linear algebra

Lemma $X = \{1, 2, \dots, k\}$

$A = (a_{ij})_{1 \le i,j \le k}$ matrix

$$(I - A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

$$(I - A)^{-1} = I + A + \dots + A^n + \dots$$

$$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{\mathrm{cof}_{j,i}(I_k - A)}{\det(I_k - A)}$$

Ihara-Selberg zeta function of a graph

Ihara (1966)

$$(i) \quad \zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class

prime

circuit

no backtracking

$$(ii) \quad \zeta_G(t) = \frac{1}{\det(1-Ht)}$$

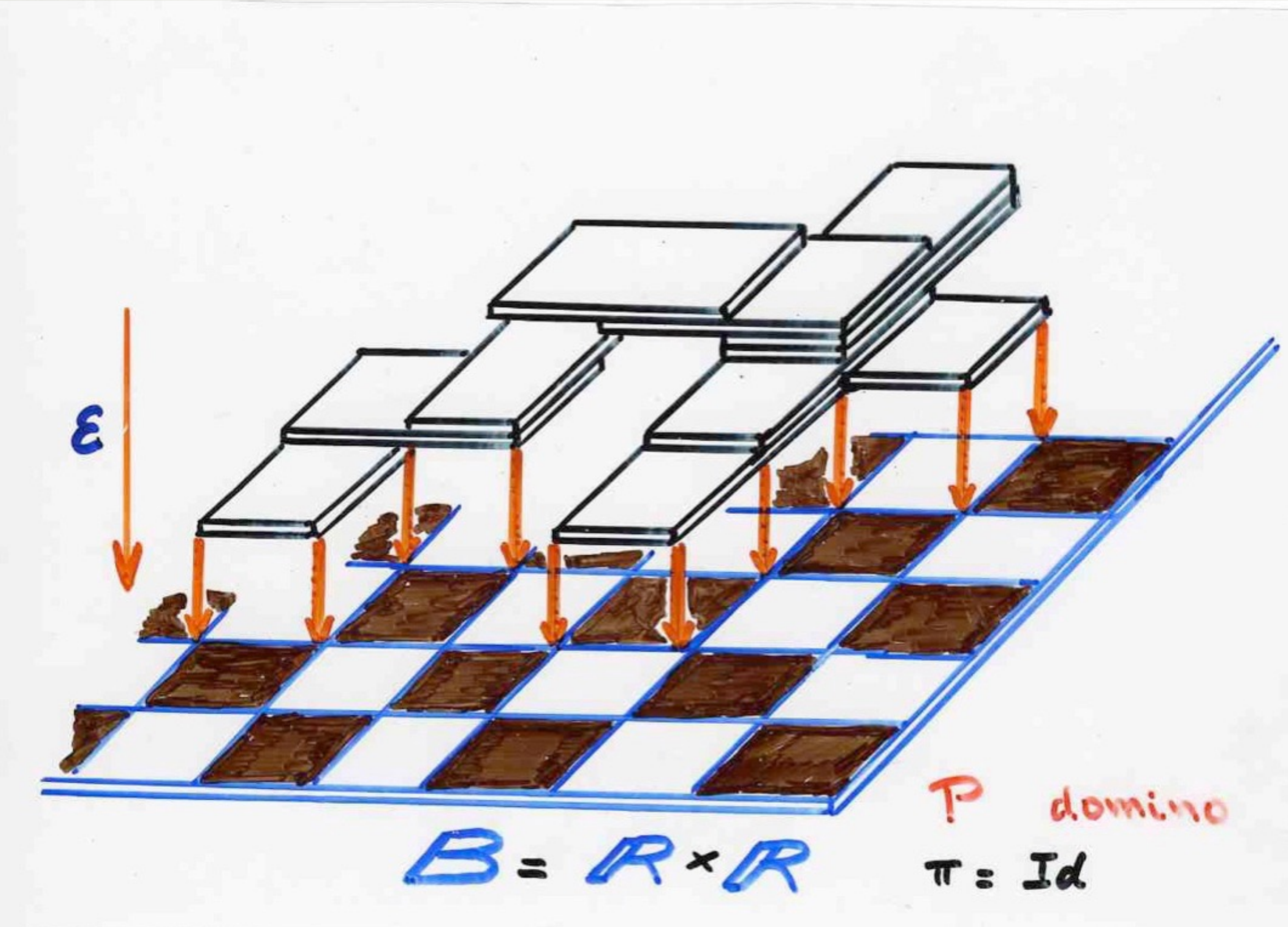
$$(iii) \quad \zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det(I-tA+t^2(D-I))}$$

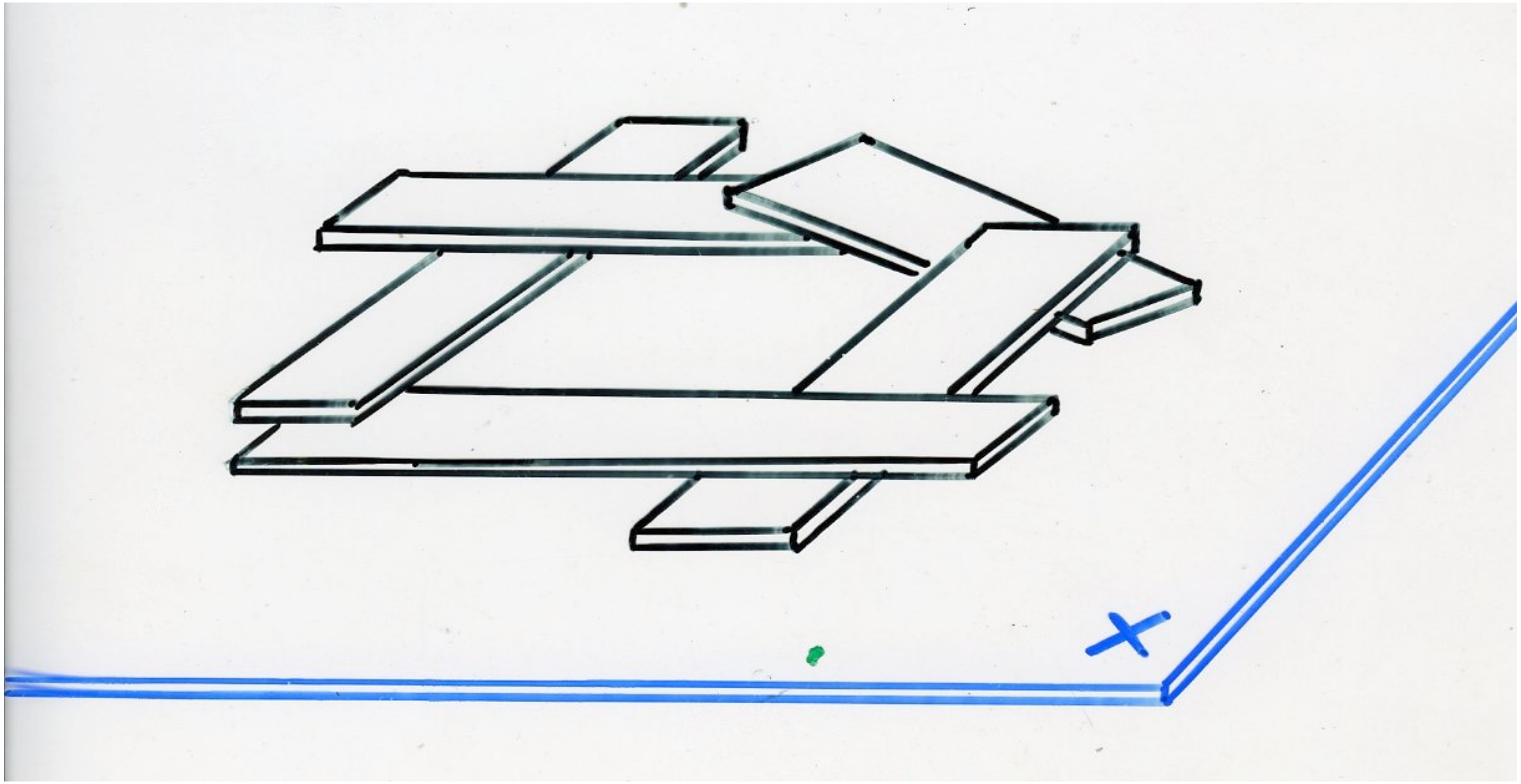
$$\frac{1}{\det(I-A)} \quad ?$$

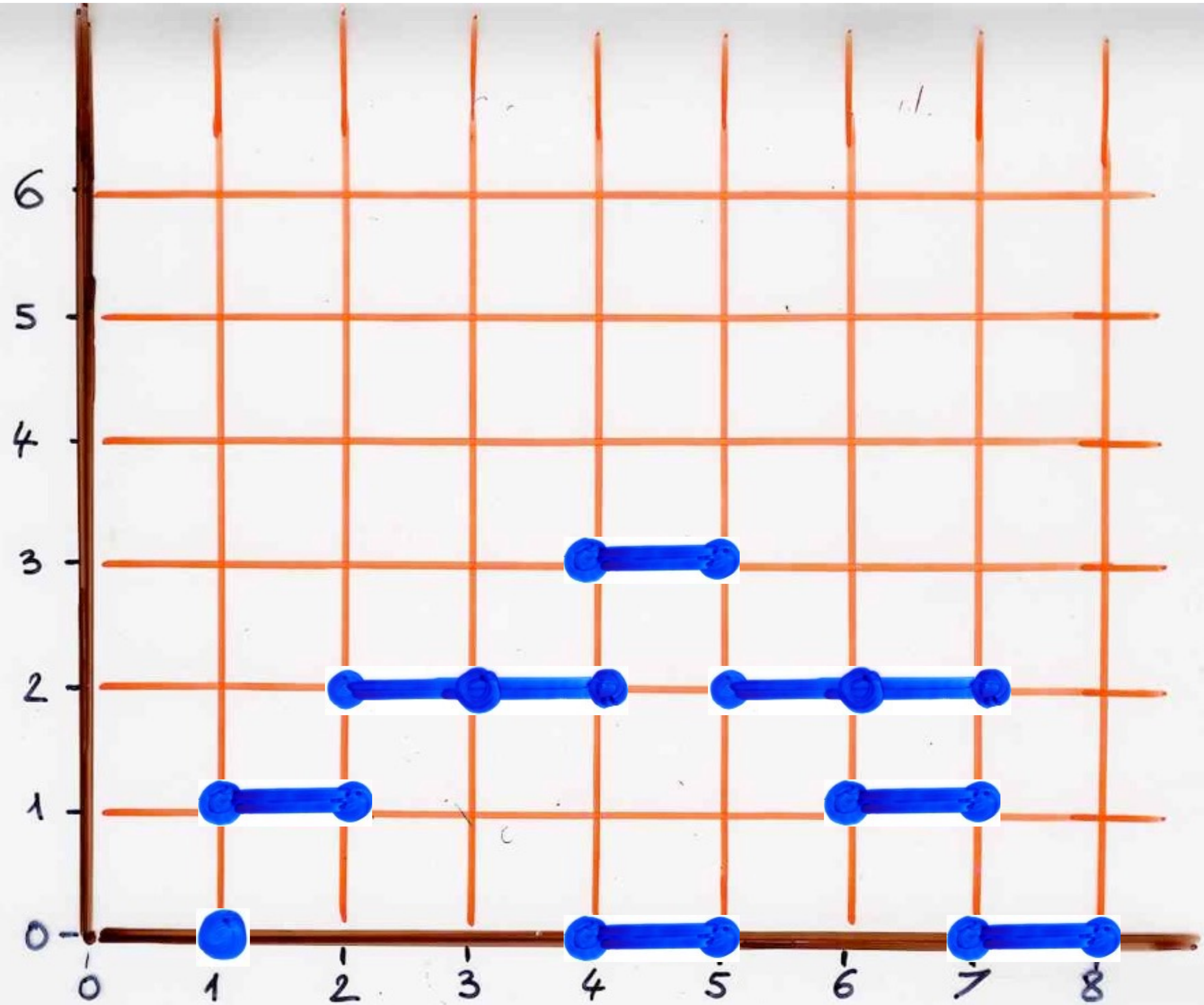
$$t\frac{d}{dt}\log \zeta_G(t) \quad ?$$

Heaps of pieces: definition with levels

Introduction
Heaps







6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

heap definition

- $\mathcal{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathcal{P}$ $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (dependency relation)
- heap E, finite set of pairs

  (α, i) $\alpha \in \mathcal{P}$, $i \in \mathbb{N}$ (called pieces)

  projection level

(i)

(ii)

heap definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P $\begin{cases}\text{symmetric}\\ \text{reflexive}\end{cases}$

  (dependency relation)
- heap E, finite set of pairs (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  α: projection, i: level

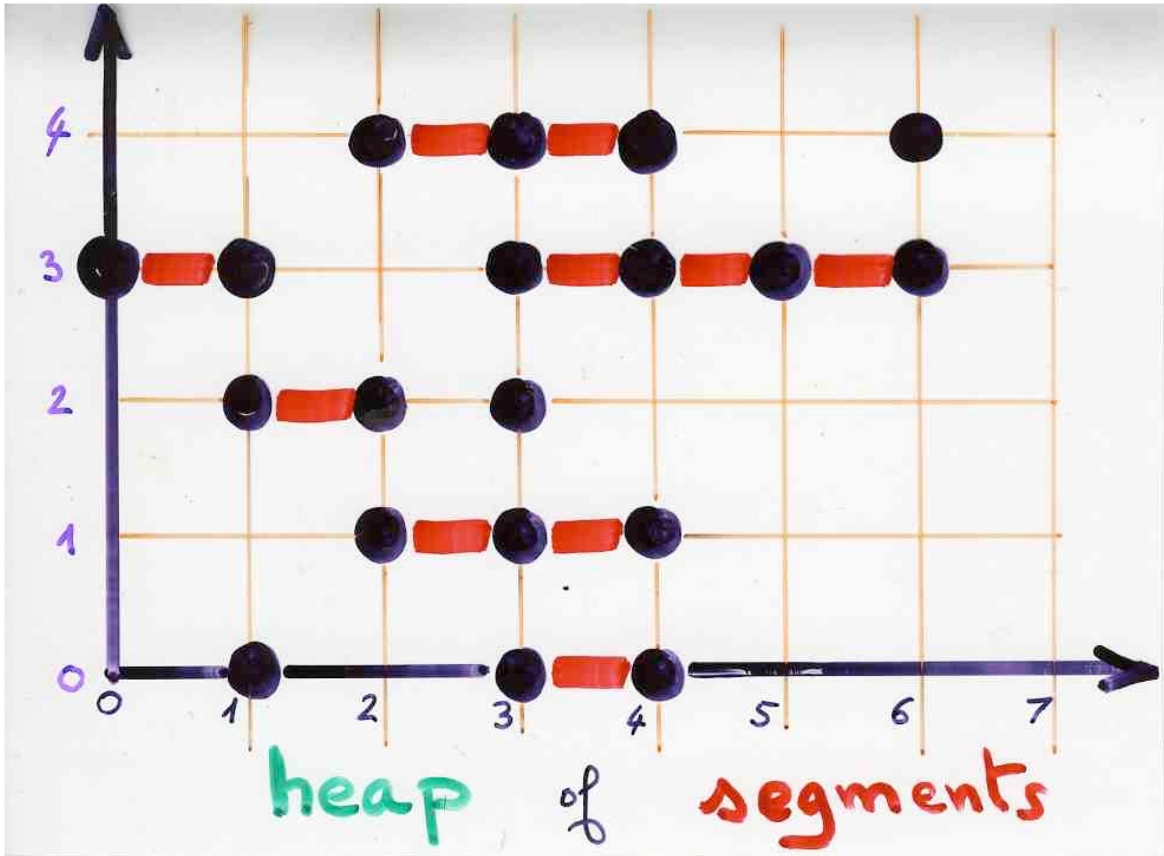
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \,\mathcal{C}\, \beta \Rightarrow i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \Rightarrow \exists \beta \in P$, $\alpha \,\mathcal{C}\, \beta$, $(\beta, i-1) \in E$

ex: heap of segments over $\mathbb{N}$

$P = \{ [a,b] = \{a, a+1, \dots, b\}, 0 \leq a \leq b \}$

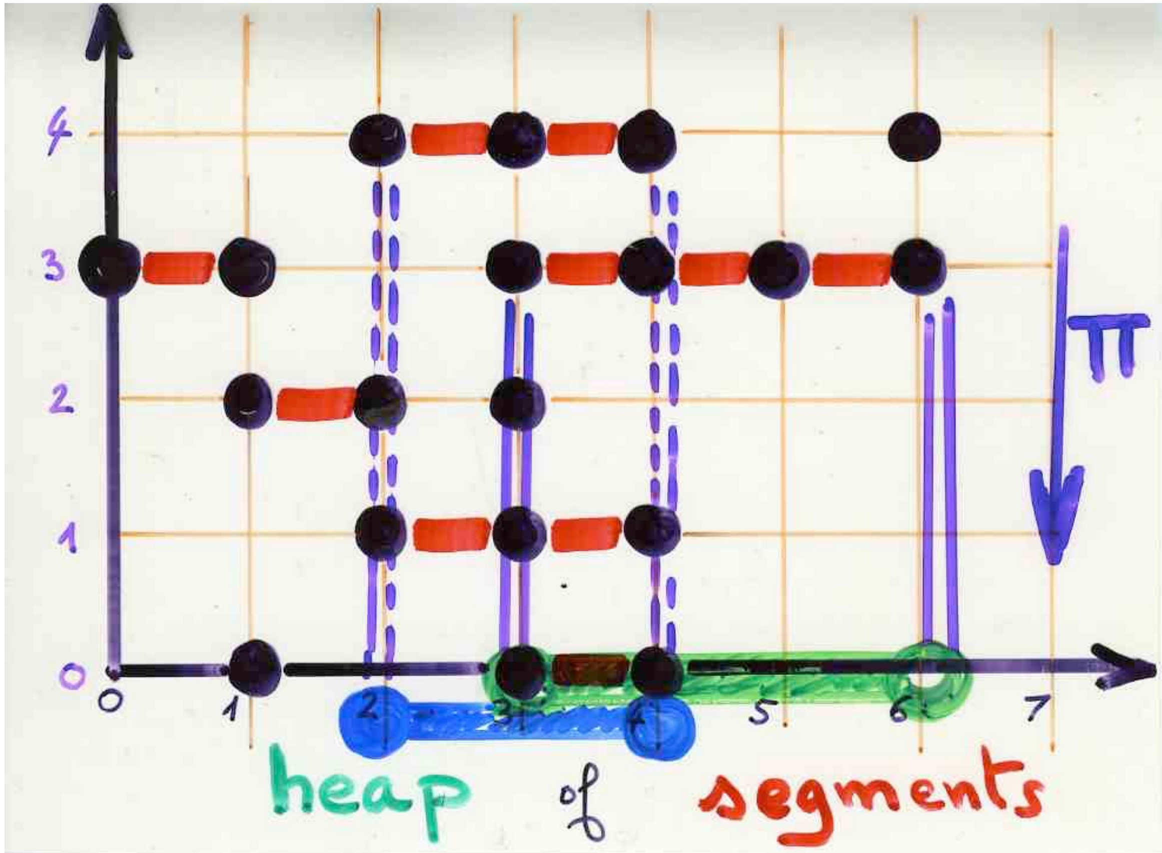
$\mathcal{C} \quad [a,b] \,\mathcal{C}\, [c,d] \Leftrightarrow [a,b] \cap [c,d] \neq \emptyset$



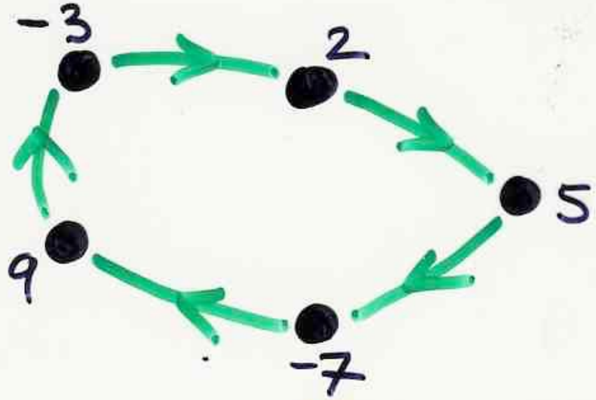
ex: heap of segments over $\mathbb{N}$

$$\mathcal{P} = \{ [a,b] = \{a, a+1, \dots, b\} \, , 0 \le a \le b \}$$

$$\mathcal{C} \qquad [a,b] \, \mathcal{C} \, [c,d] \Leftrightarrow [a,b] \cap [c,d] \neq \emptyset$$



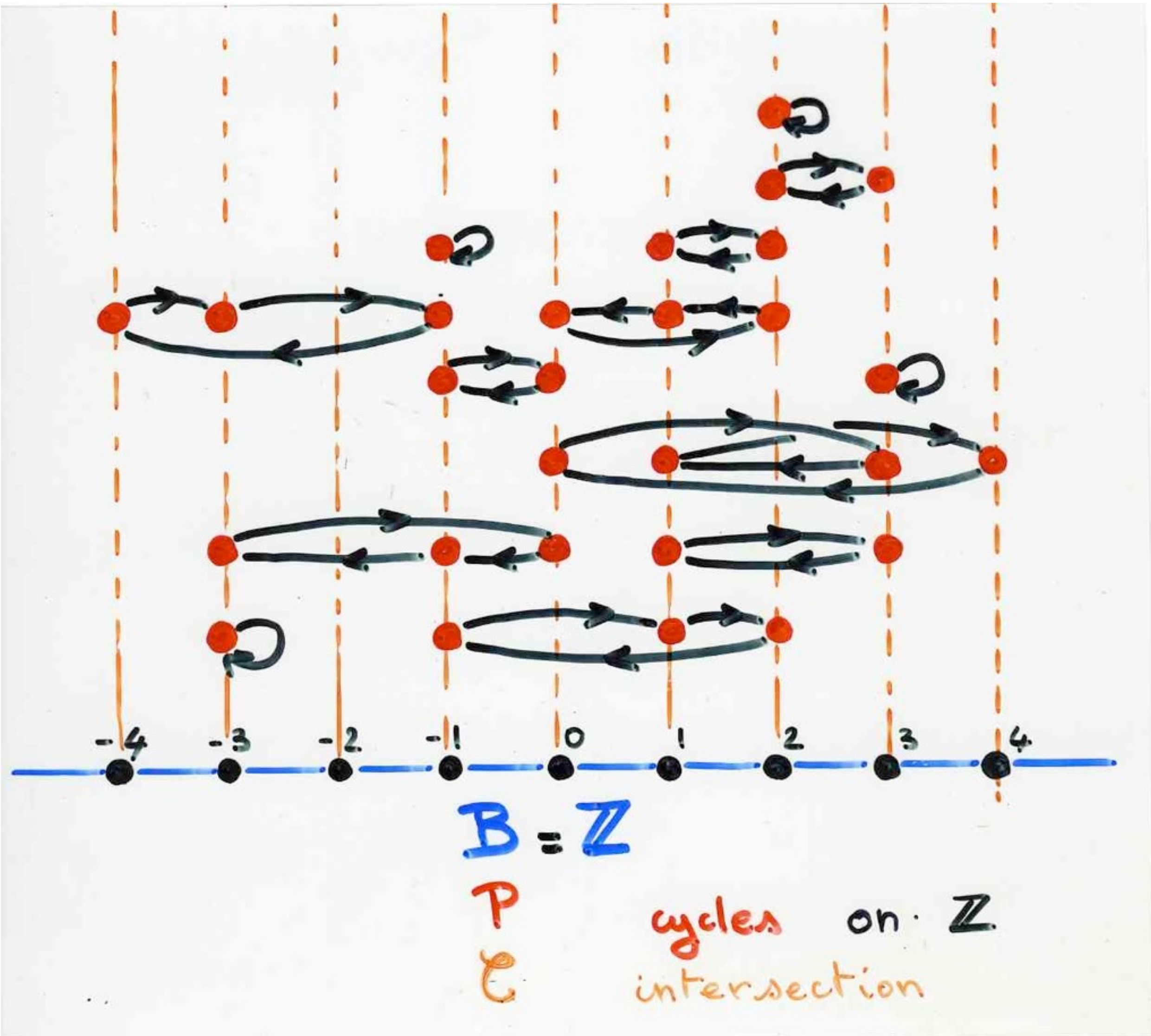
basic pieces $P = \{$ cycles on $\mathbb{Z}$ $\}$

$\gamma =$ 

Supp(γ) $= \{-7, -3, 2, 5, 9\}$

Support

$\mathcal{C}$ dependency relation $\quad \gamma \, \mathcal{C} \, \delta \iff \mathrm{Supp}(\gamma) \cap \mathrm{Supp}(\delta) \neq \emptyset$

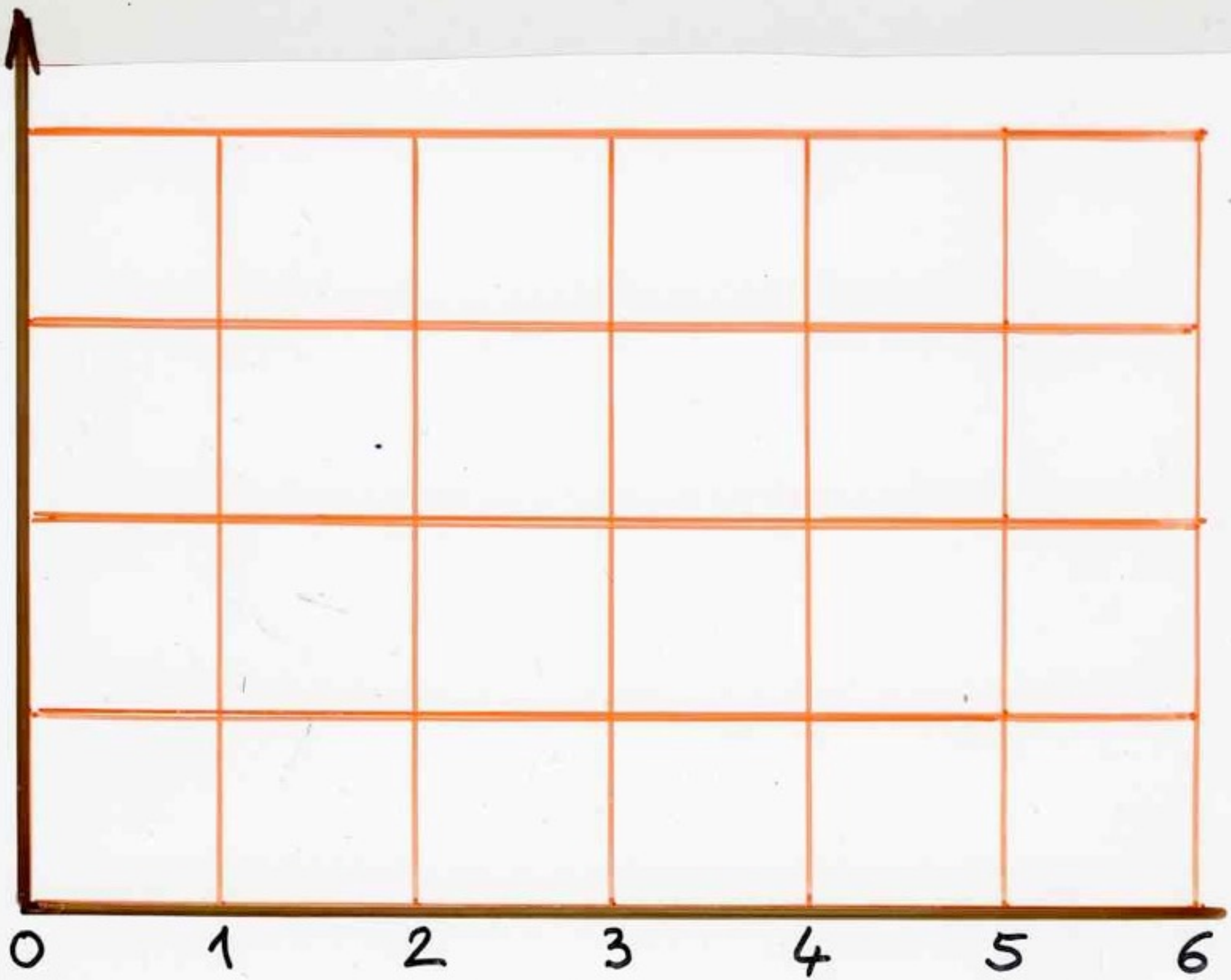

-4
-3
-2
-1
0
1
2
3
4
B = ℤ
P cycles on ℤ
C intersection

Heaps of pieces
and
commutation monoids

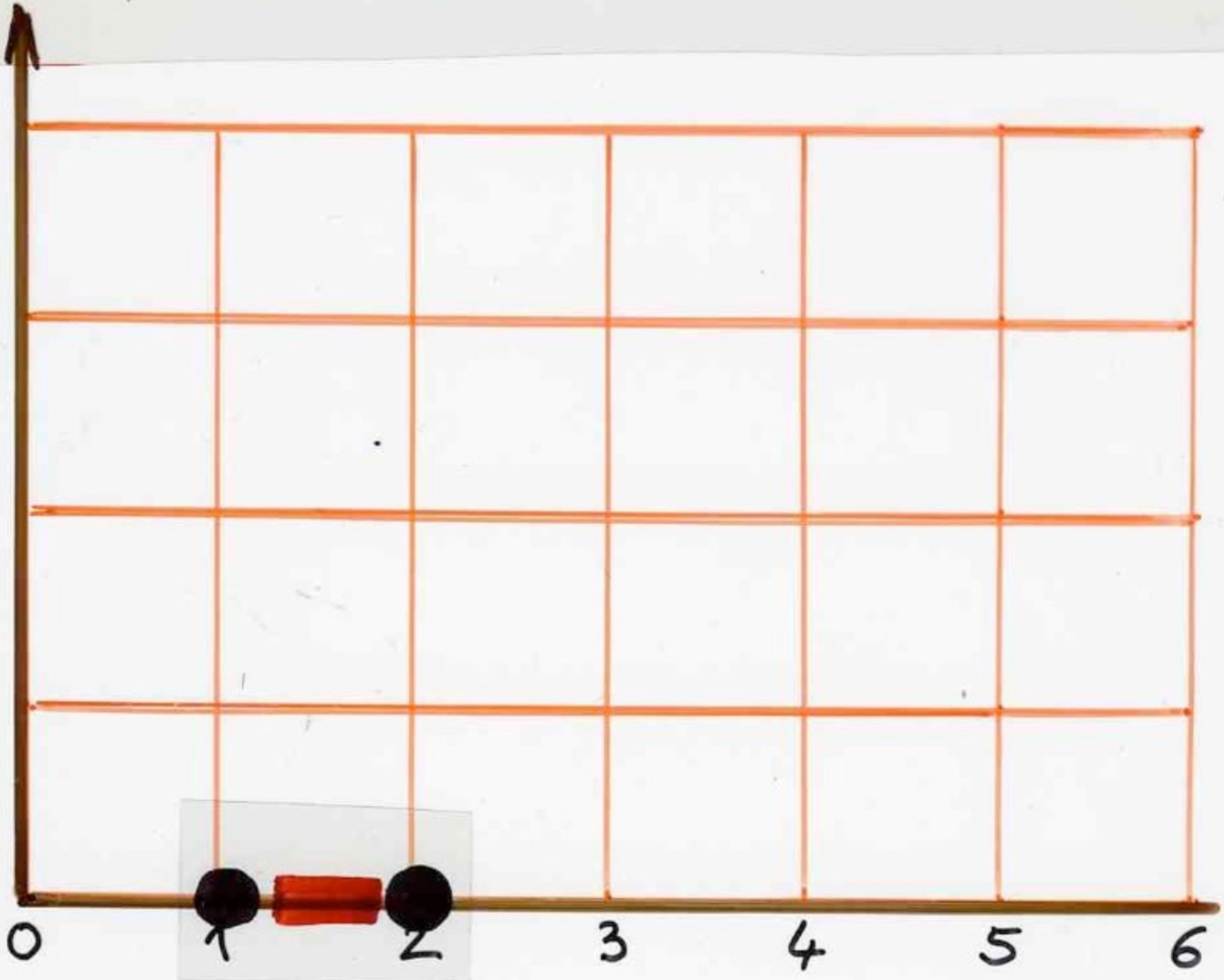
<u>ex</u> : heaps of dimers on $\mathbb{N}$

$$P = \{ [i, i+1] = \sigma_i \, , \, i \geq 0 \}$$

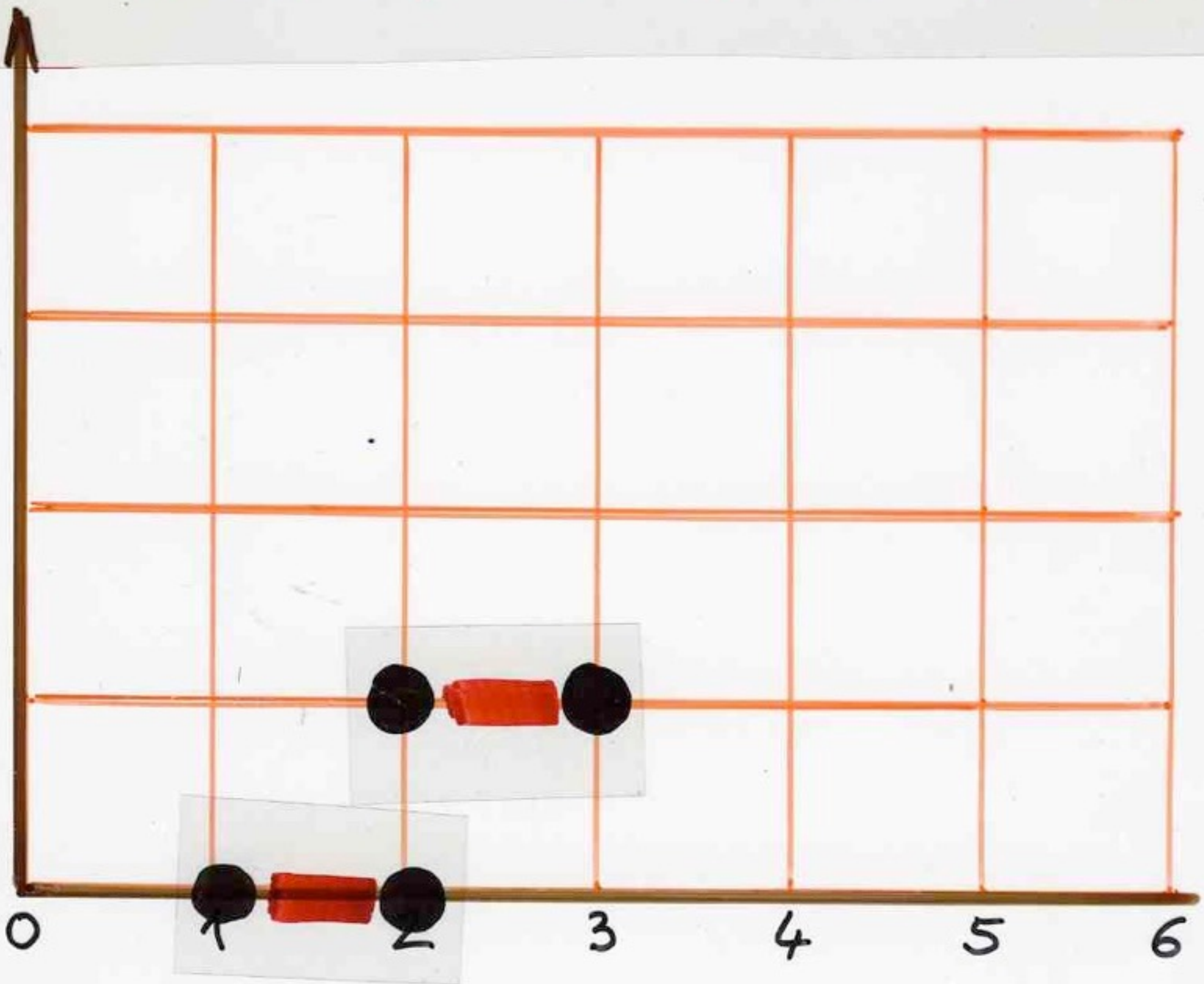
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



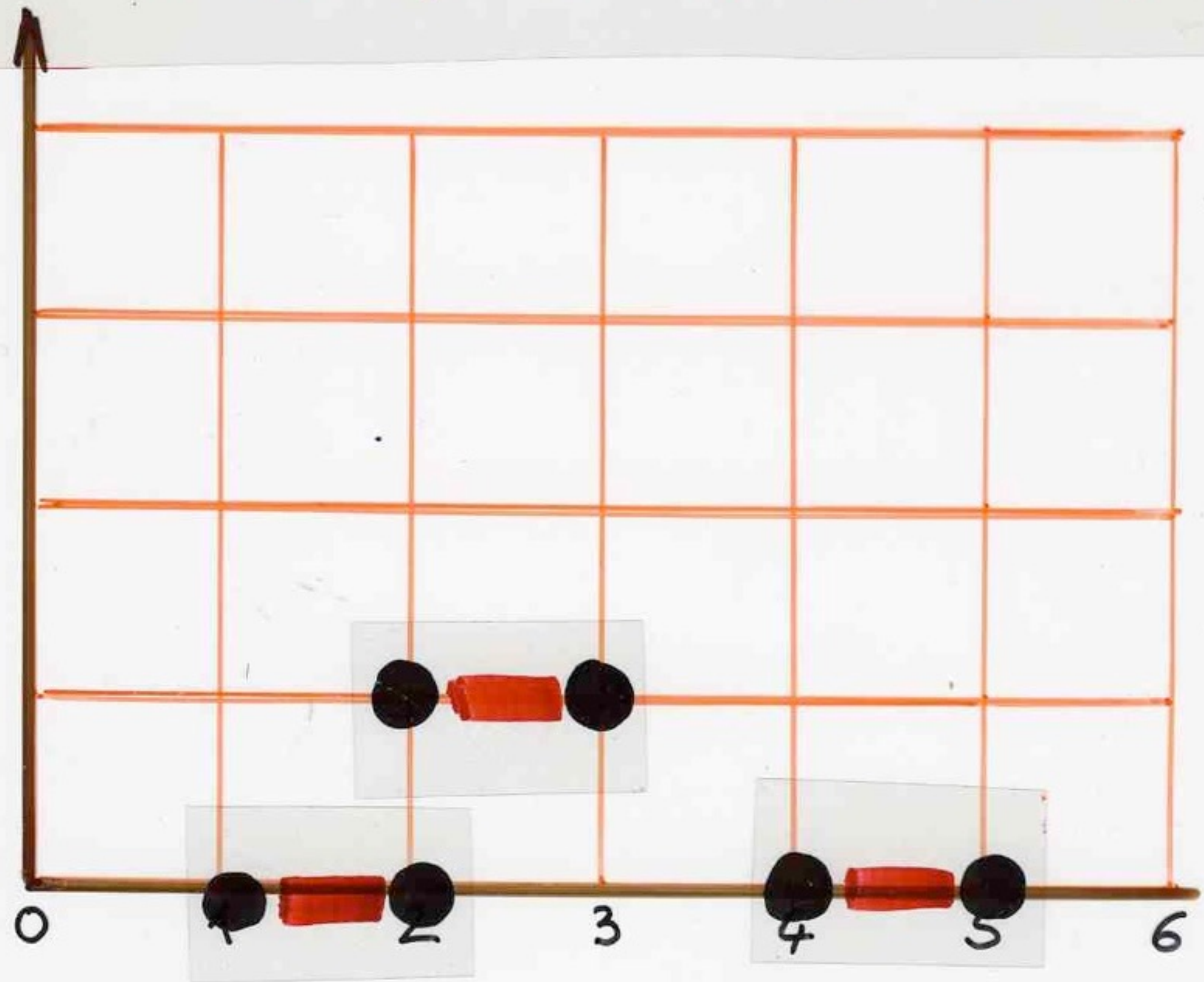
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



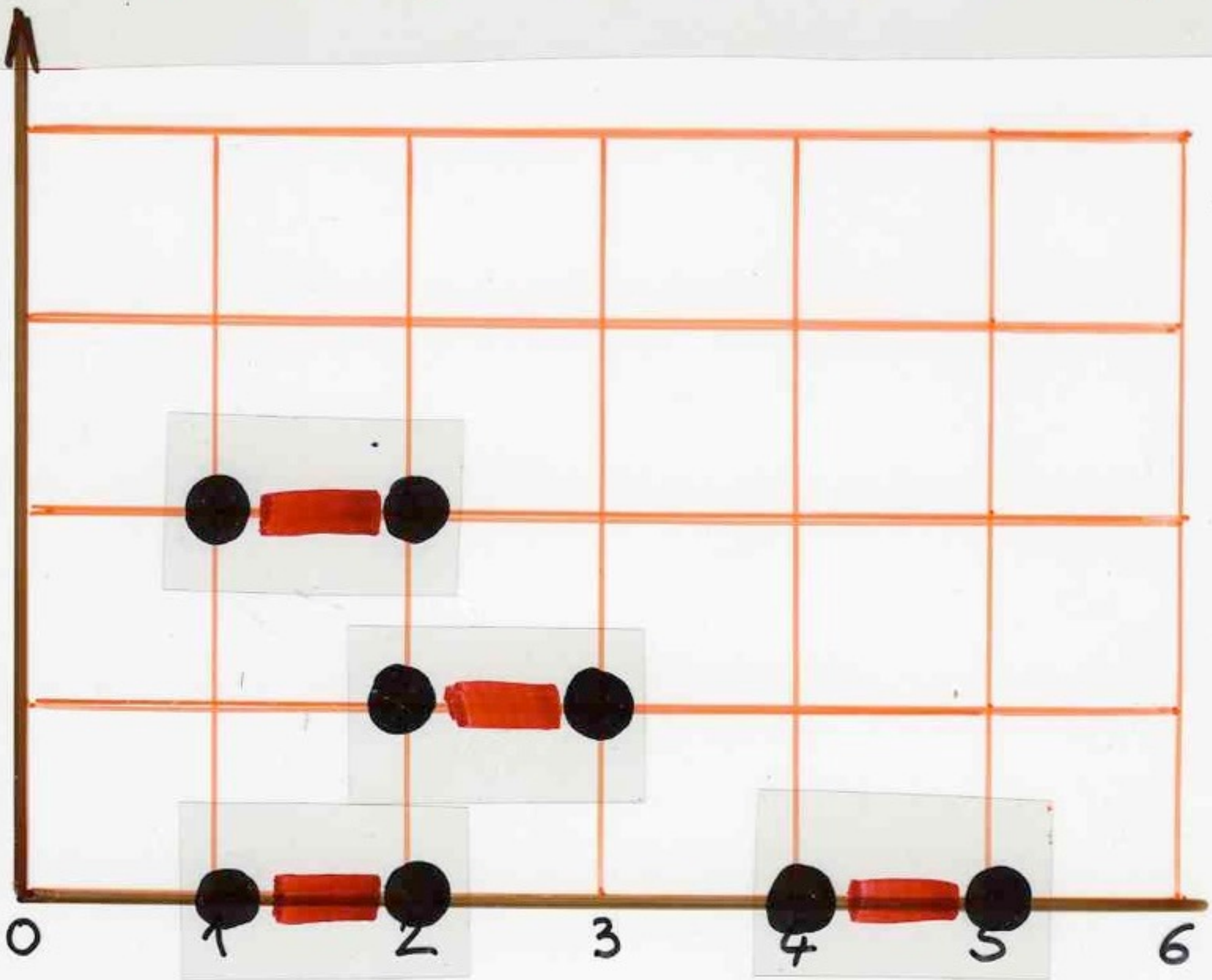
$$w = \sigma_1\,\sigma_2\,\sigma_4\,\sigma_1\,\sigma_4\,\sigma_3\,\sigma_0\,\sigma_4$$



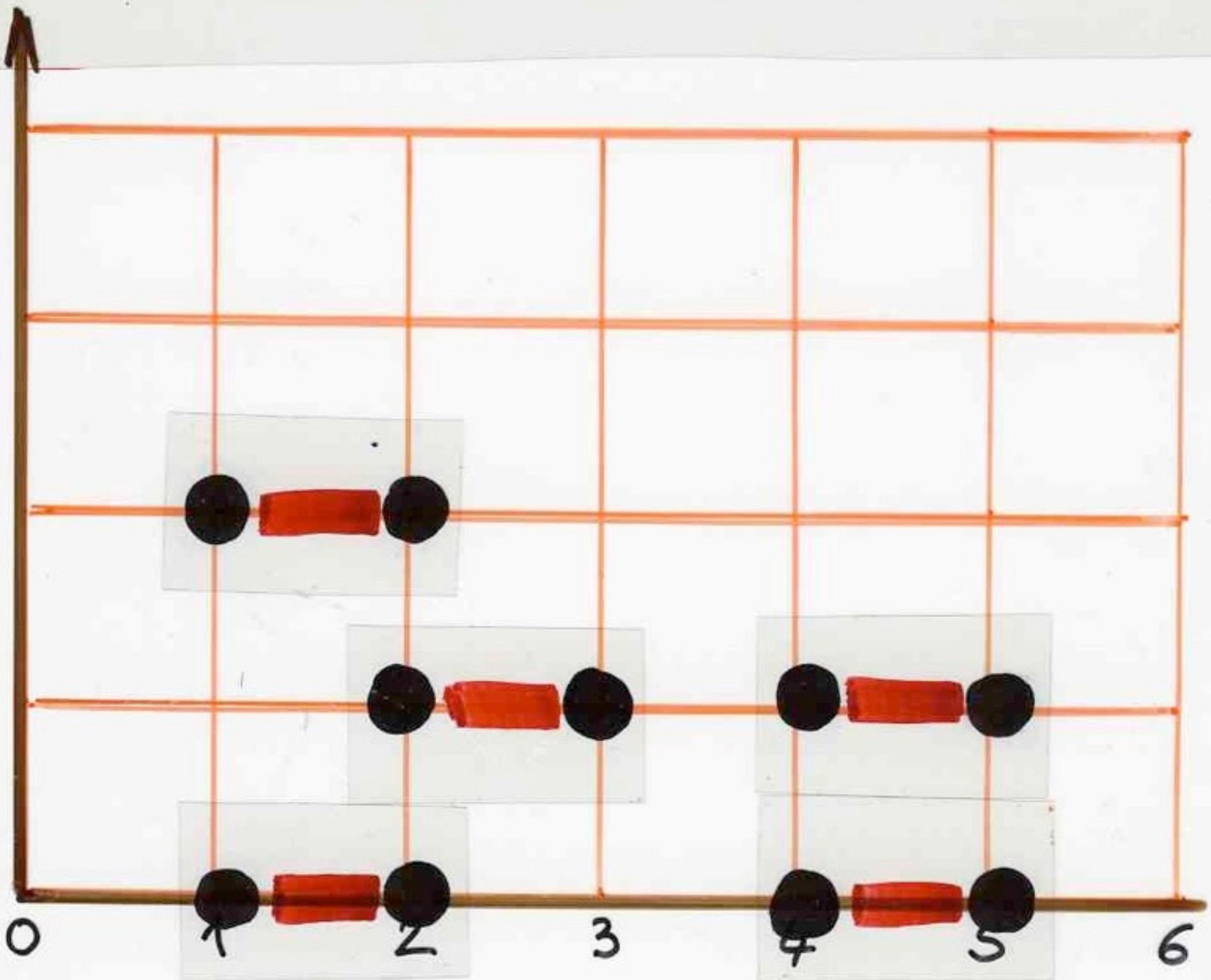
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



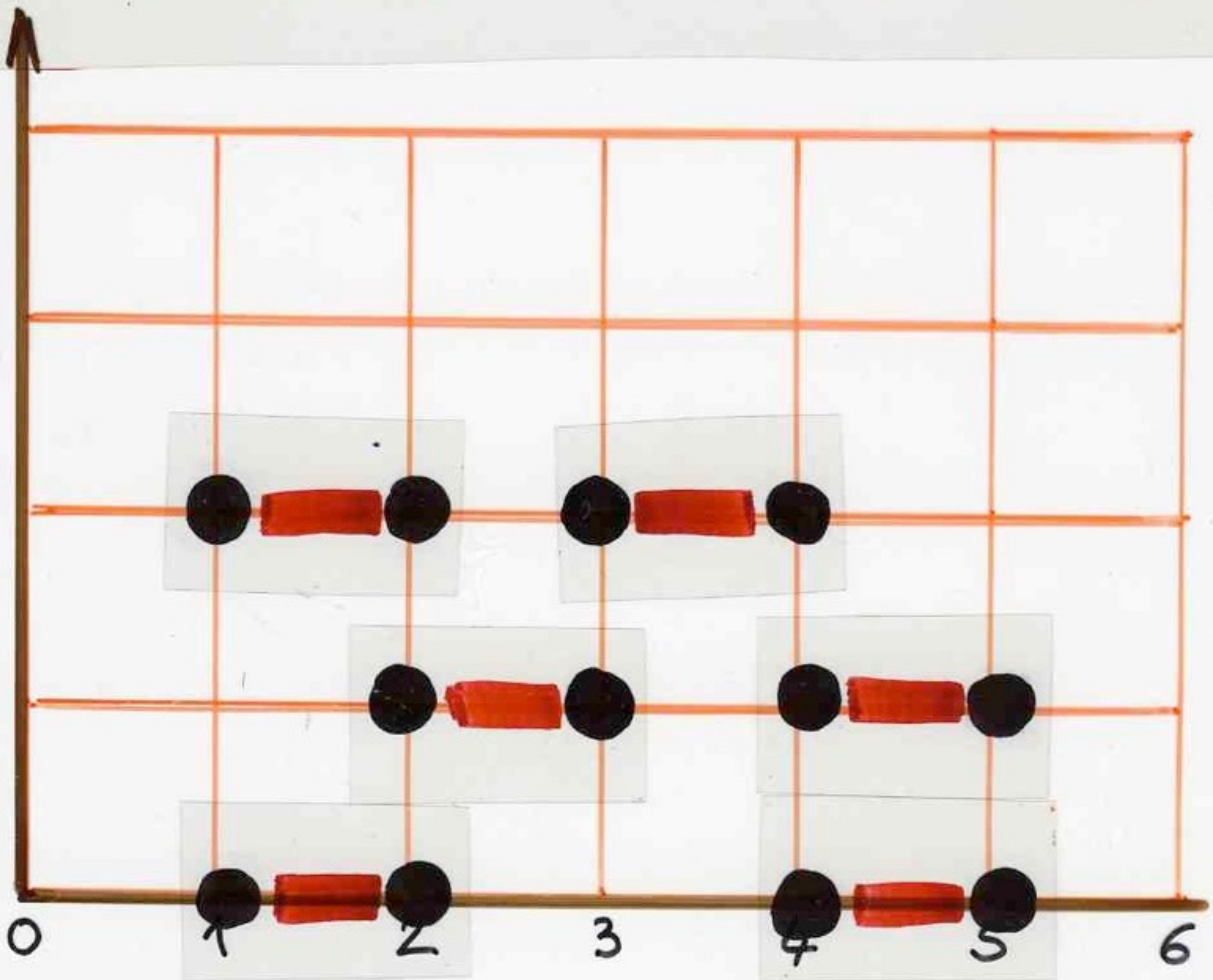
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



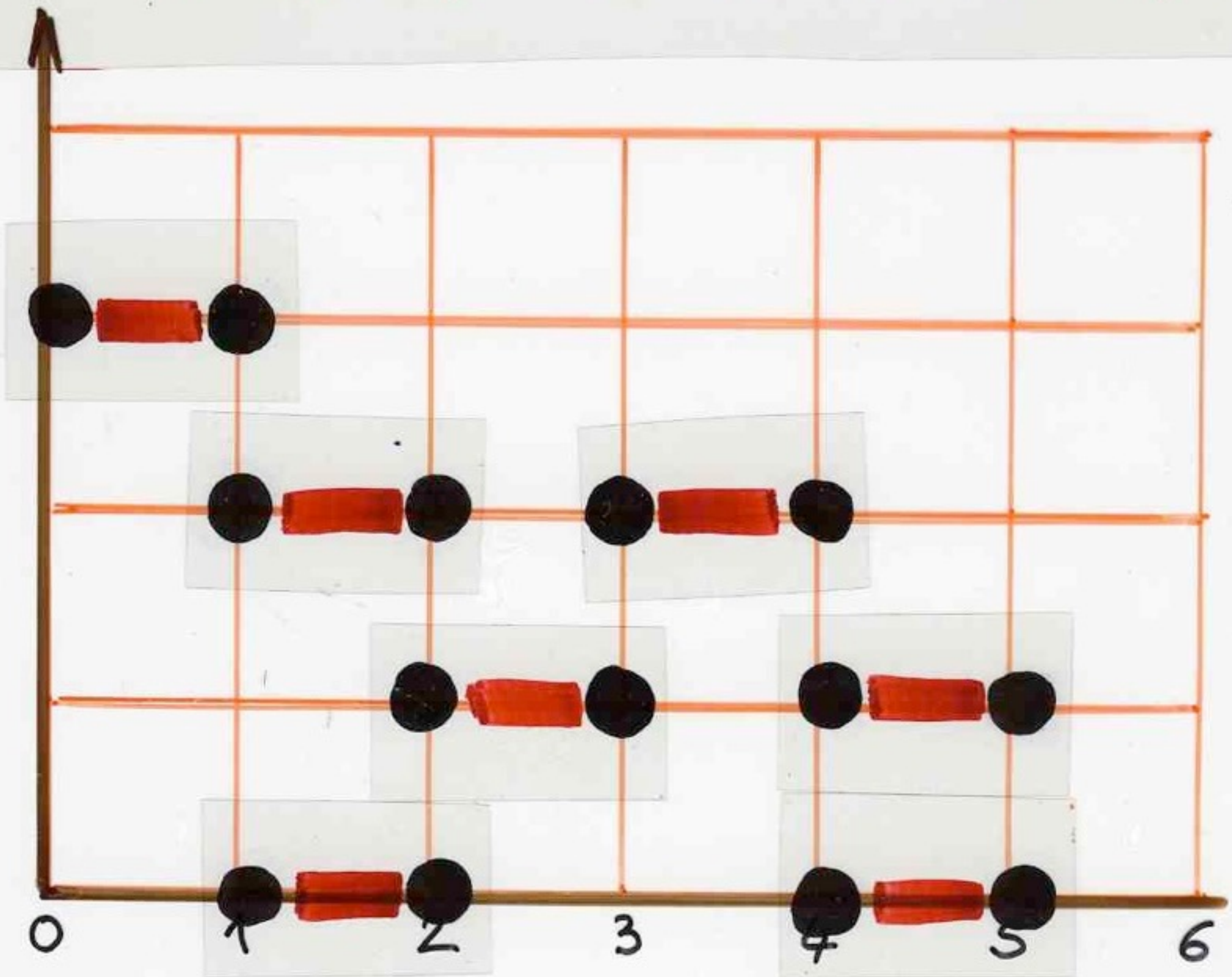
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



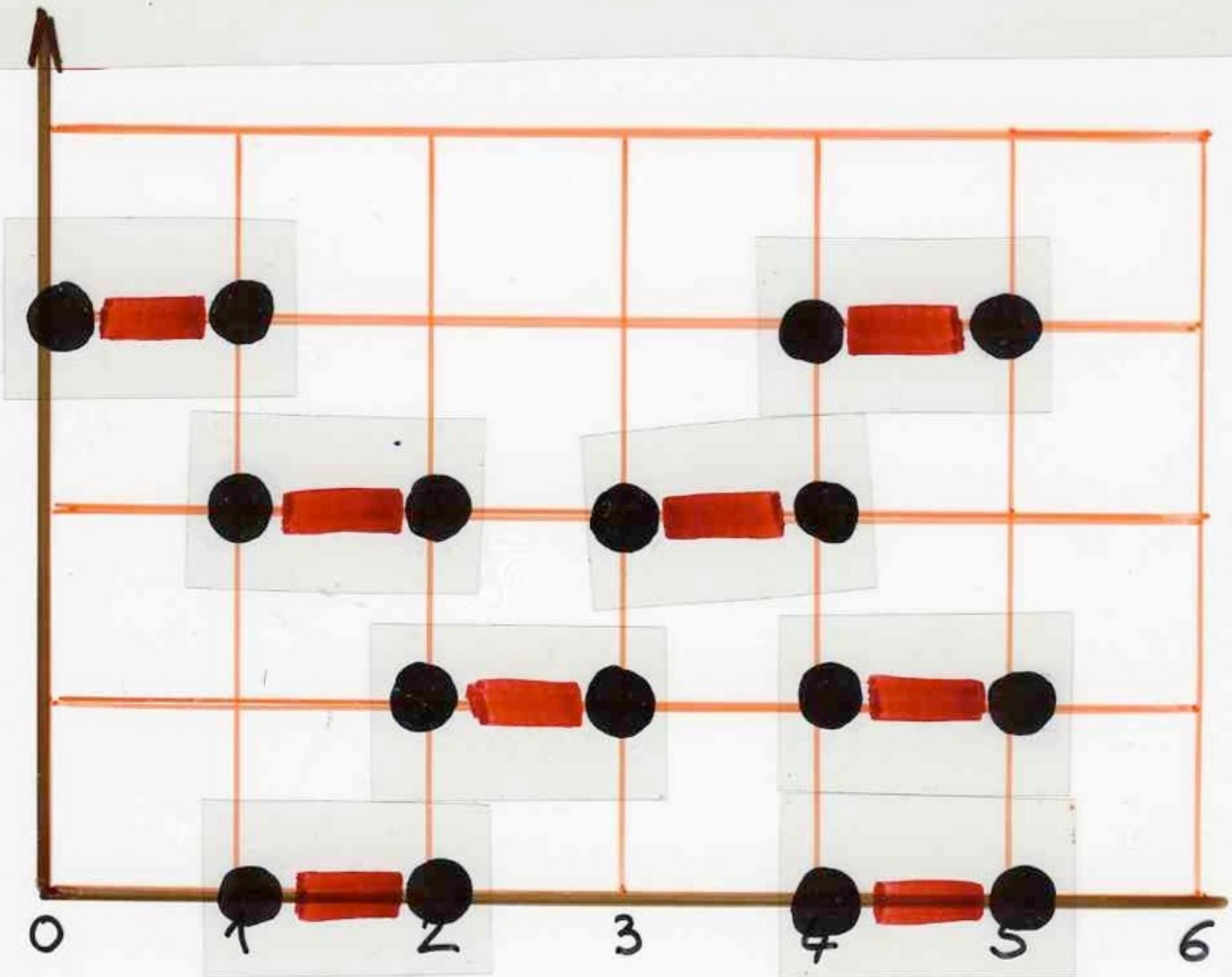
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

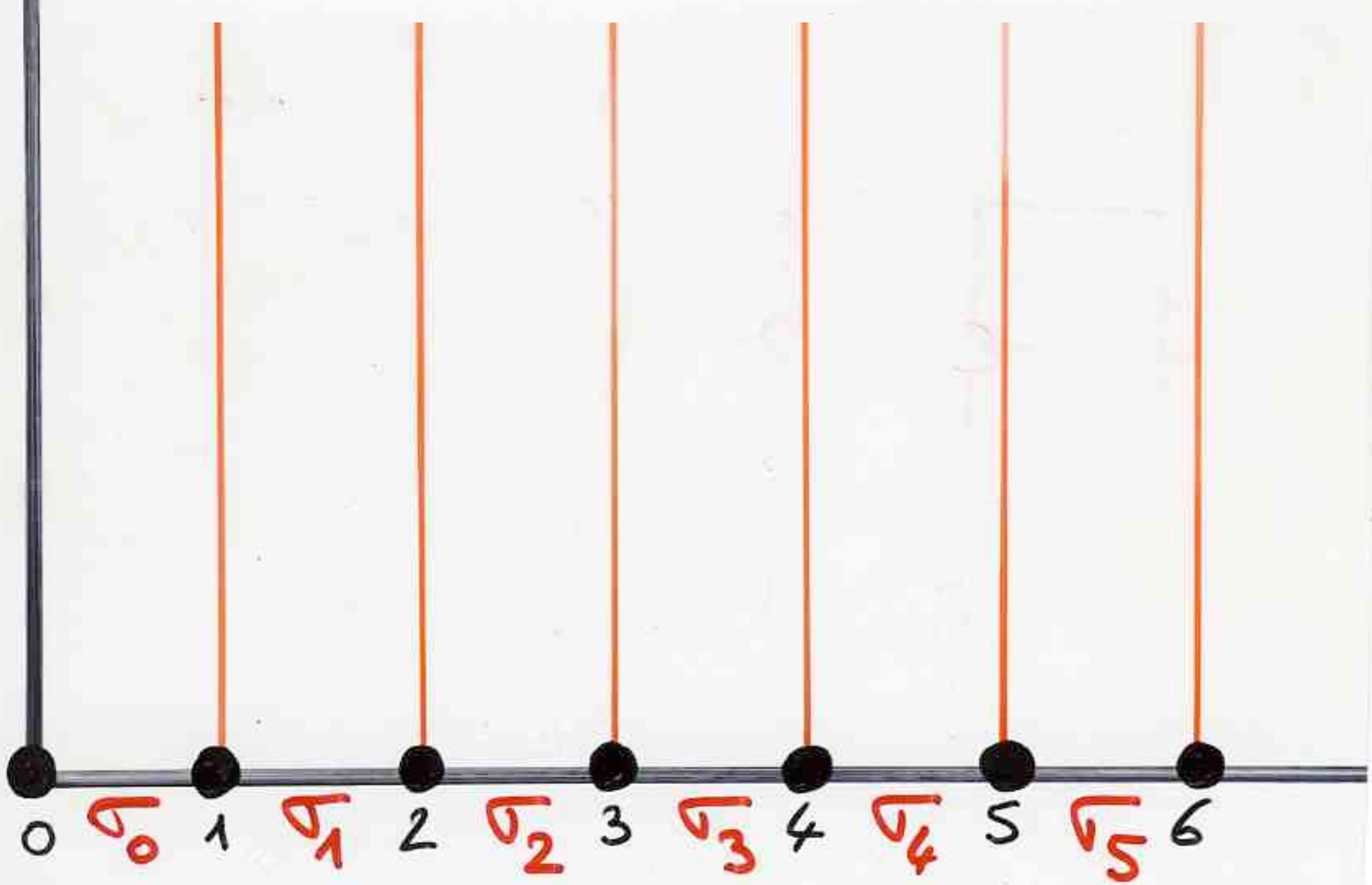


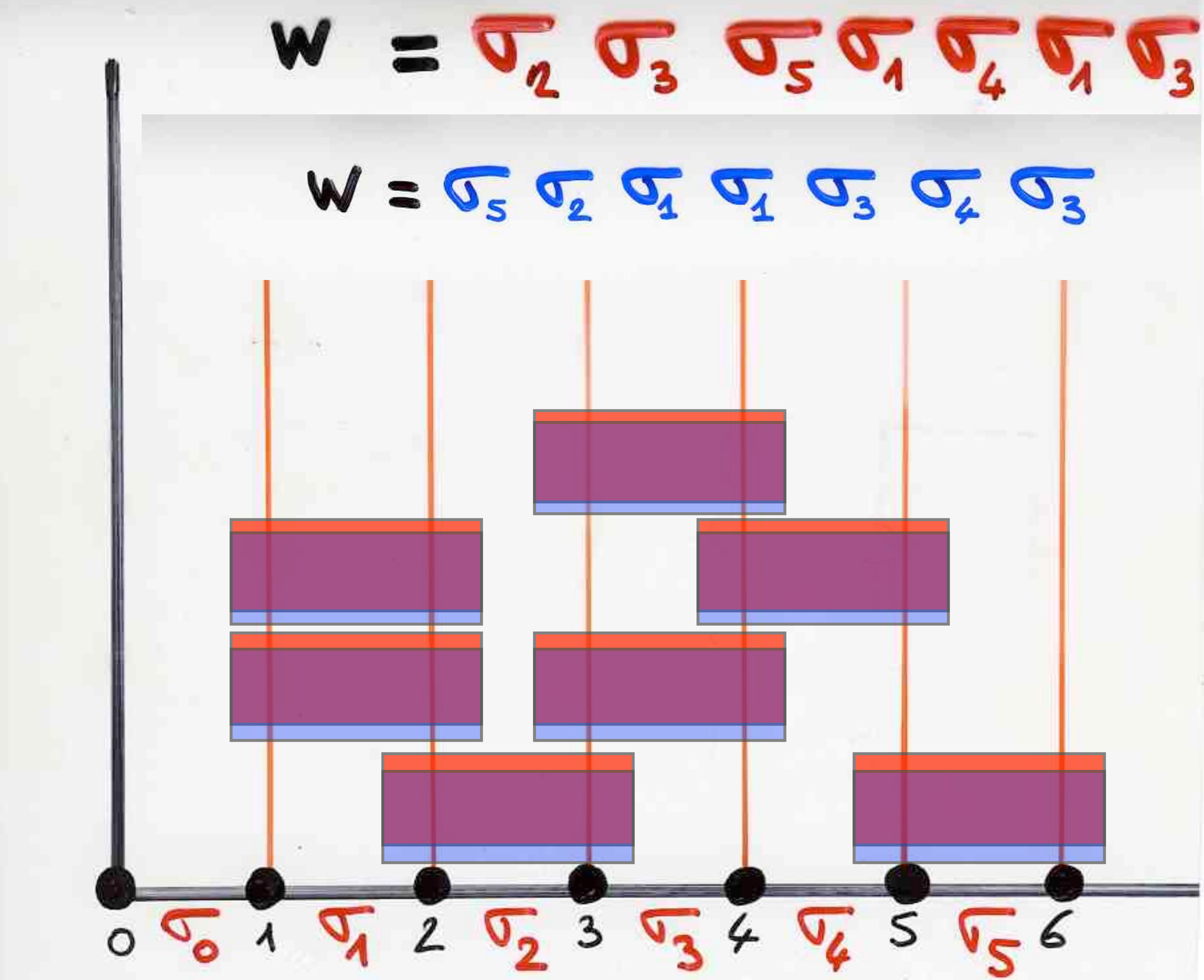
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$W = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$$

$$W = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$$





$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
$w = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6

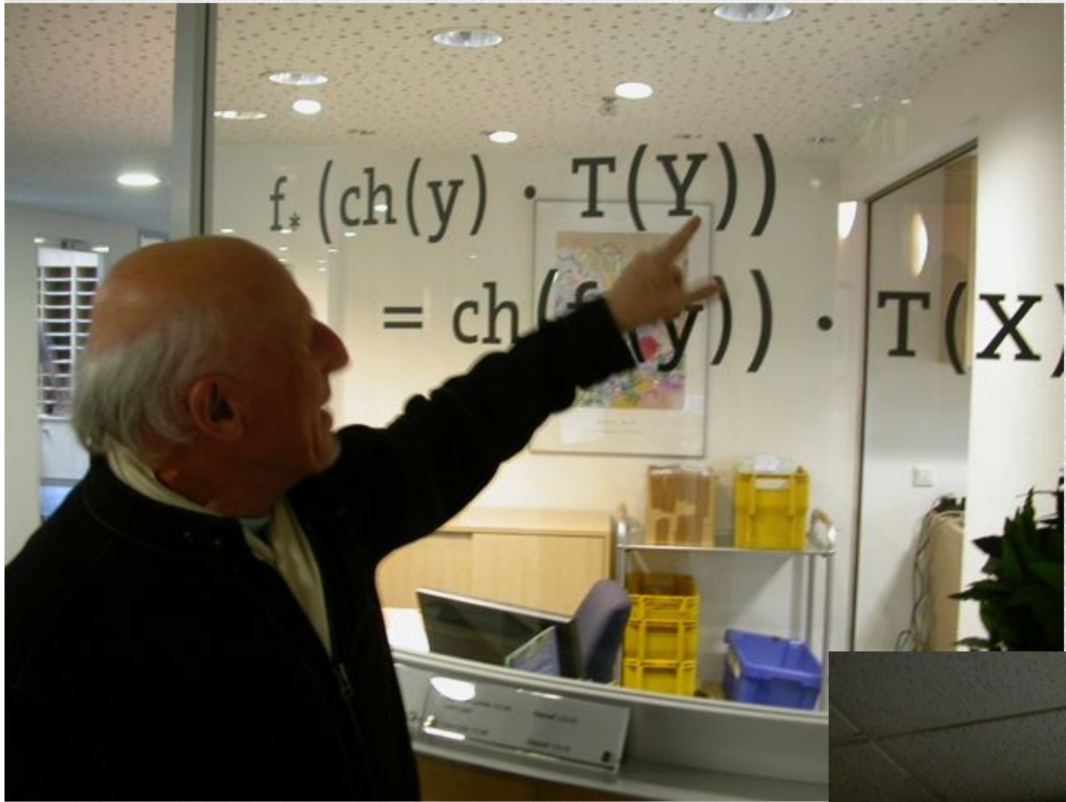
ex: heaps of dimers on $\mathbb{N}$

$$P = \{ [i, i+1] = \sigma_i , i \geq 0 \}$$

$\mathcal{C}$

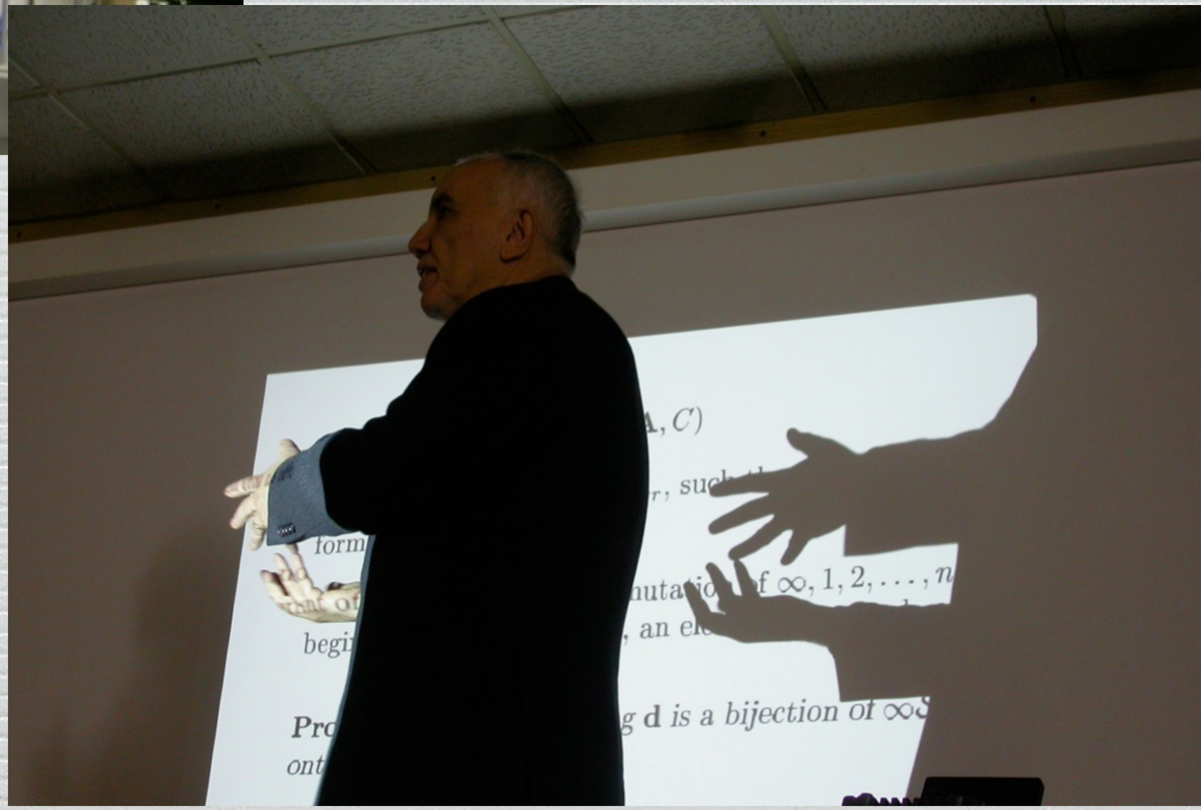
C commutations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ iff } |i-j| \geq 2$$



Cartier- Foata
commutation Monoids
(1969)

Heaps of pieces
(X.V. 1985)



Trace monoids

Computer Science

model for parallelism

concurrency access to data structures

Trace

Mazurkiewicz (1977)

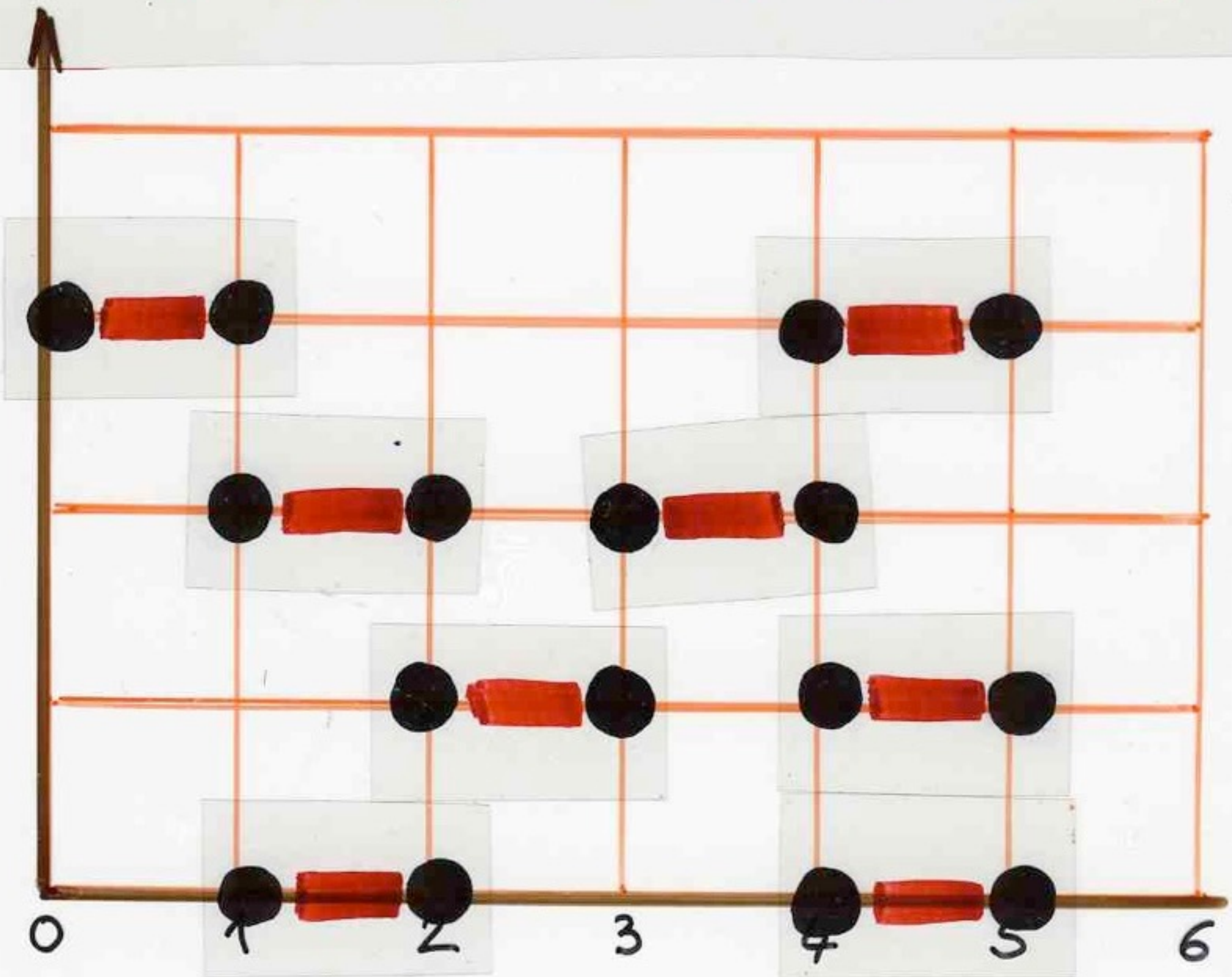
model of the logical behavior of safe Petri nets

Dieckert, Rosenberg ed. (1995)

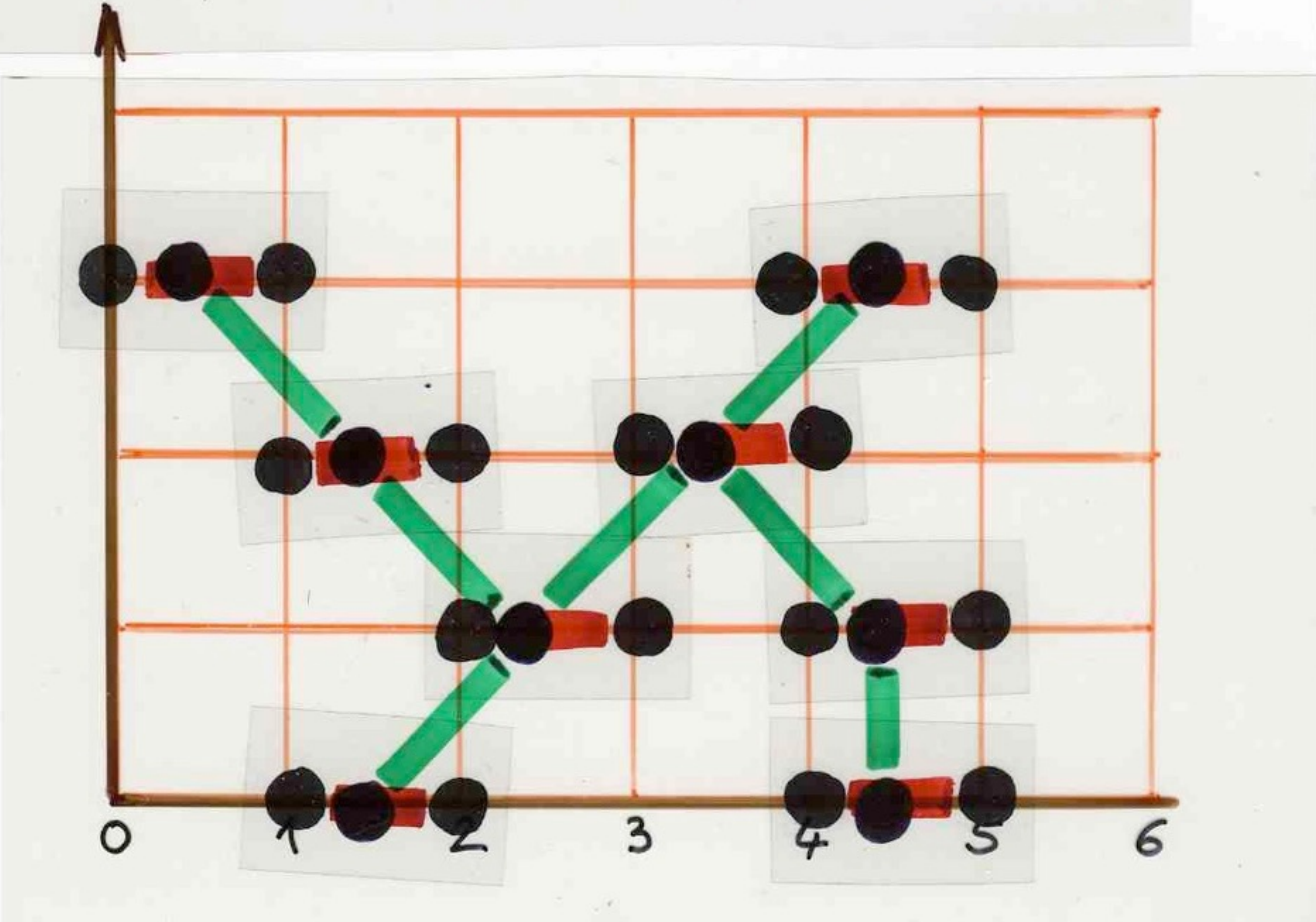
The book of traces

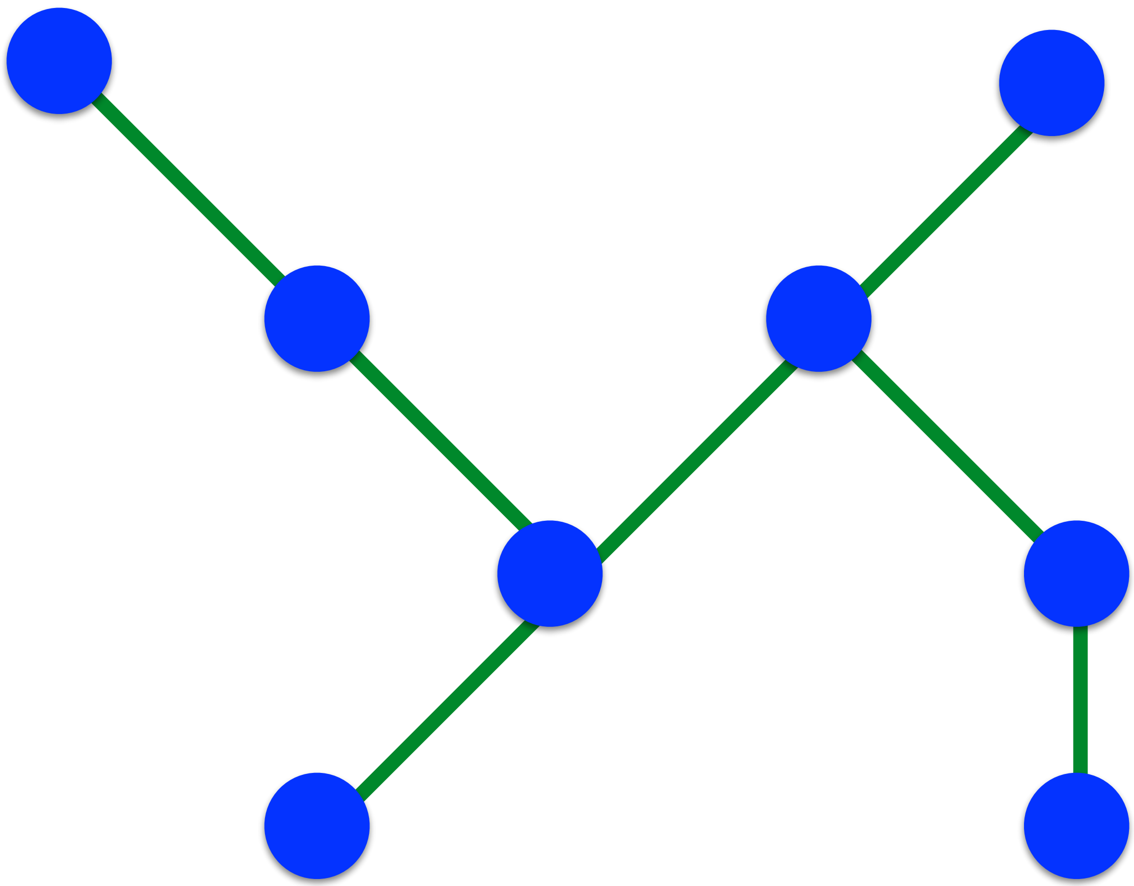
Heaps of pieces as a poset

$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$w = \sigma_1 \, \sigma_2 \, \sigma_4 \, \sigma_1 \, \sigma_4 \, \sigma_3 \, \sigma_0 \, \sigma_4$$





Second definition:

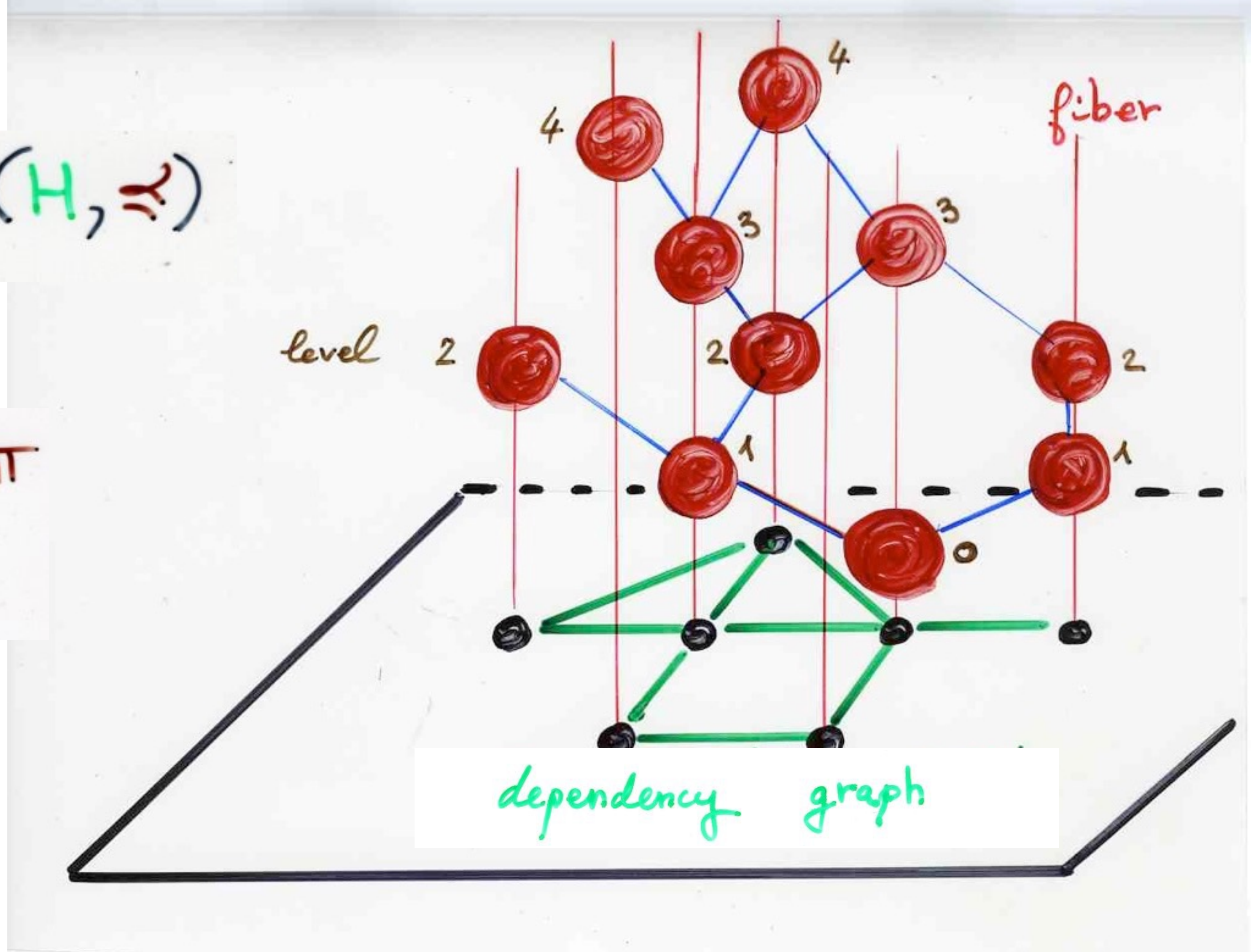
heaps as a poset on a graph

$\Gamma = (S, E)$ Γ is the dependency graph

finite poset $(H, \preceq)$

labeling map π

$H \xrightarrow[\pi]{} \Gamma$

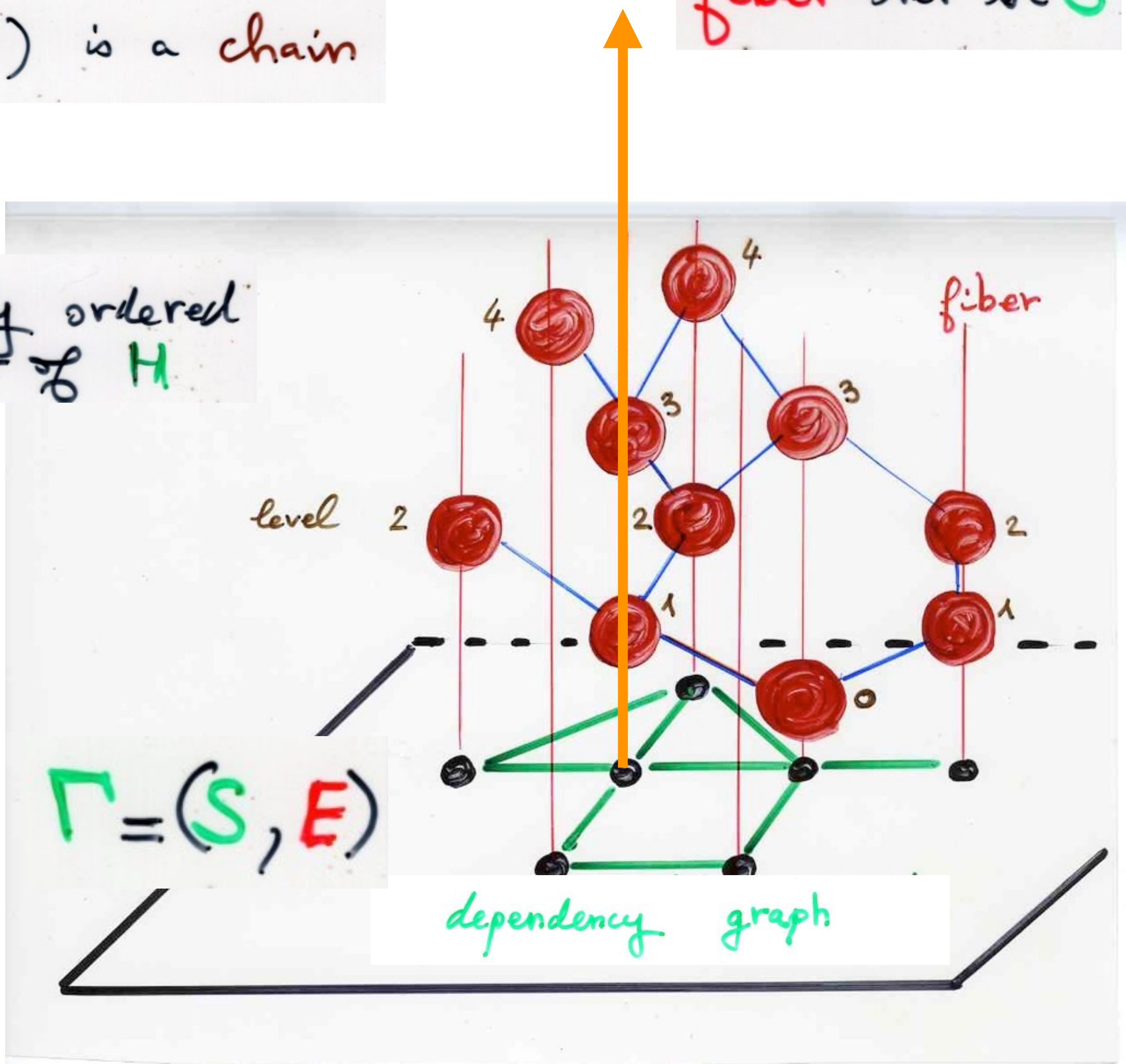


for every vertex $s \in S$

$H_s = \pi^{-1}(\{s\})$ is a chain

fiber over $s \in S$

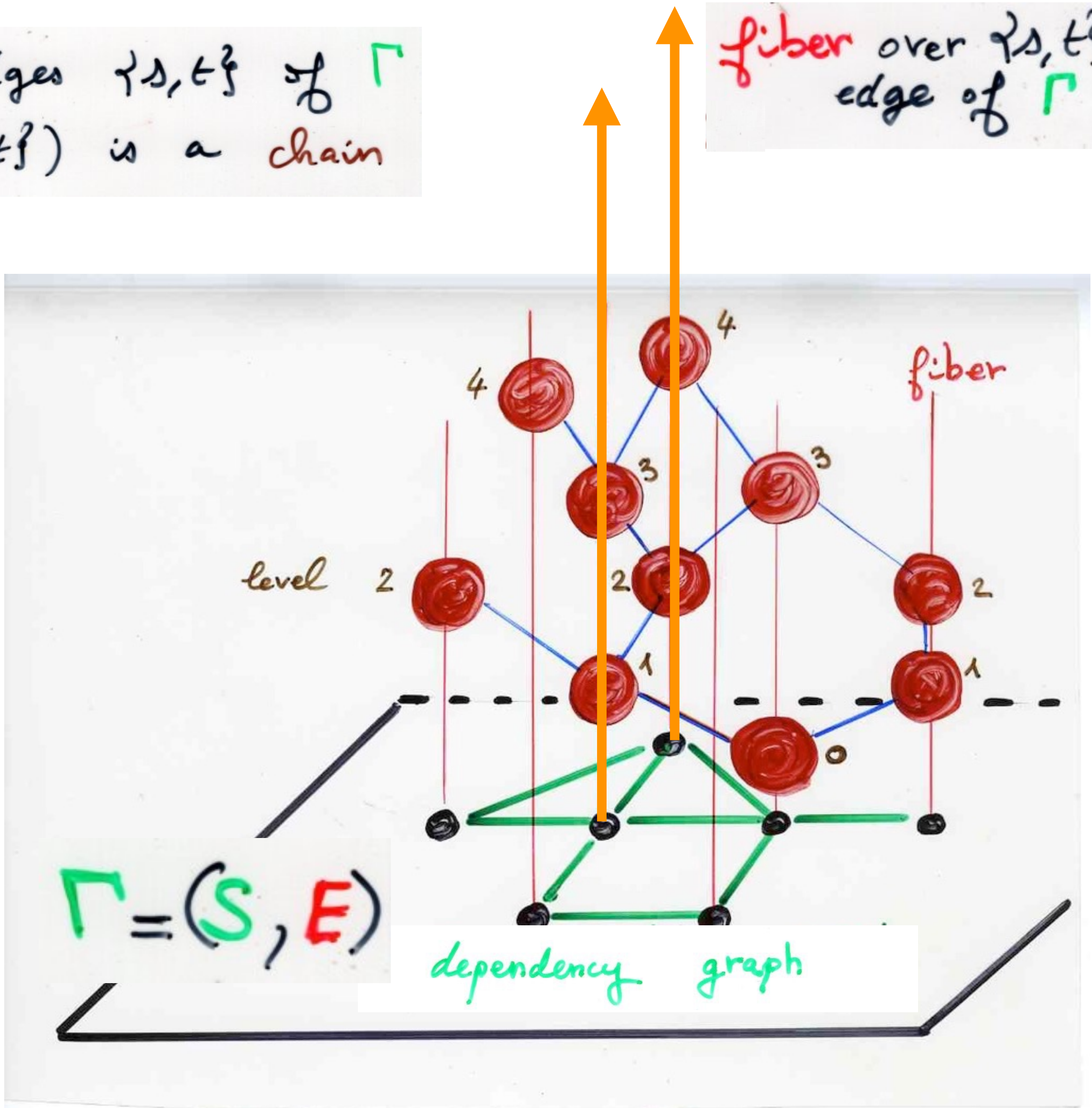
chain = totally ordered subset of H



for any edges $\{s,t\}$ of Γ

$H_{s,t} = \pi^{-1}(\{s,t\})$ is a chain

fiber over $\{s,t\}$ edge of Γ



The order relation $\preccurlyeq$ is the transitive closure of the relations given by all chains of $^{(i)'}$

$$H_s \qquad H_{s,t}$$

(i.e. the smallest partial ordering containing these chains)

Algebraic graph theory revisited with heaps of pieces

an example:

chromatic polynomial

and

acyclic orientations of a graph

graph $G = (V, E)$

$\gamma_G(\lambda)$ chromatic polynomial

number of (proper) coloring of the graph G with λ colors

$a(G)$ number of acyclic orientations of G

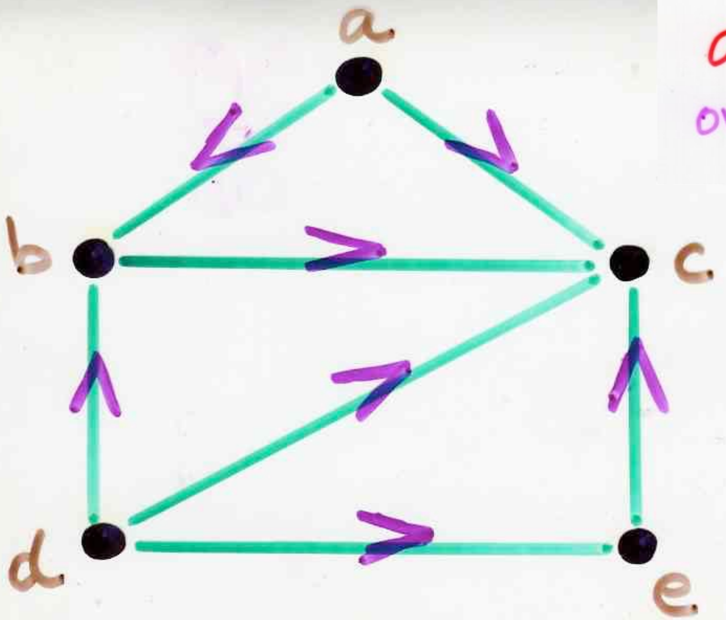
$n(G) = |V|$ number of vertices

Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$

Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$



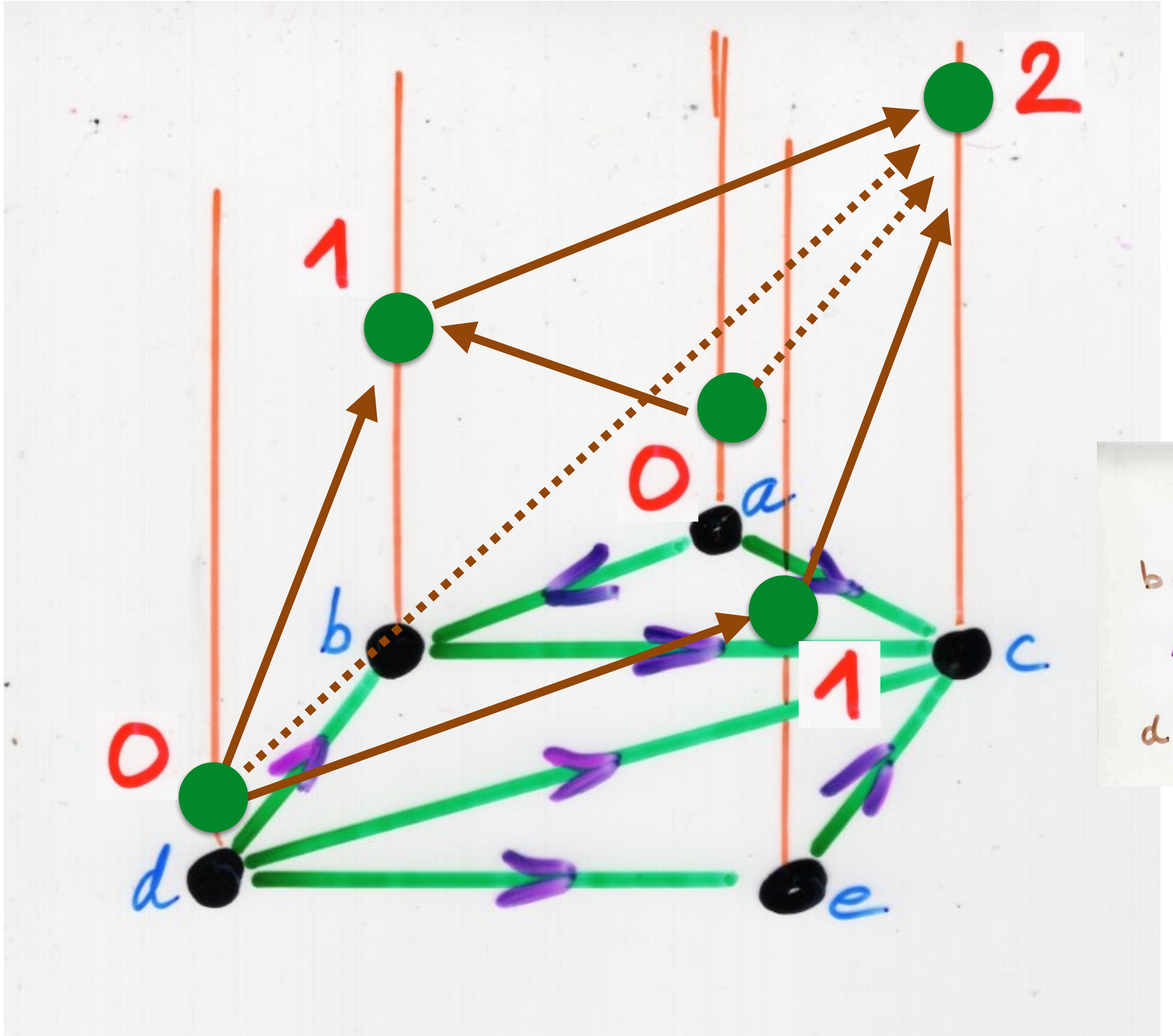
acyclic
orientations

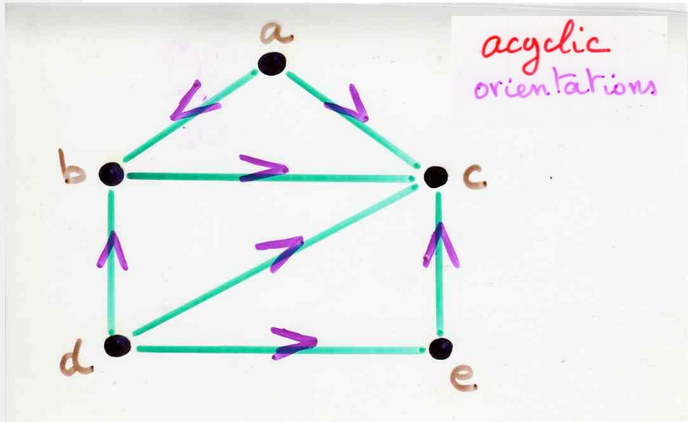
proof using
commutation
(Cartier-Foata)
monoid

from Gessel
(1985)?

Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G

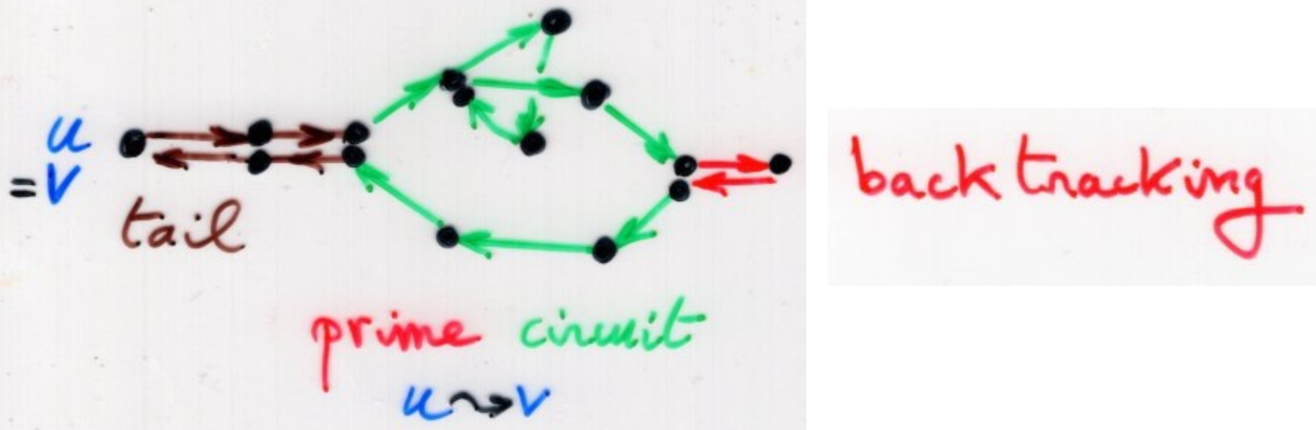




3 basic lemma on heaps

$$t \frac{d}{dt} \log \zeta_G(t) = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

(- no tail
- no back tracking



$$\text{(ii)} \quad \zeta_G(t) = \frac{1}{\det(1 - Ht)}$$

$$\text{(iii)} \quad \zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I - tA + t^2(D - I)\right)}$$

3 basic lemma:

- Inversion lemma
- Logarithmic lemma
- circuit = heap of cycles

First basic lemma on heaps:
the inversion lemma

1/D

the inversion lemma

$$(\text{Heaps}) = \frac{1}{(\text{Trivial heaps})}$$

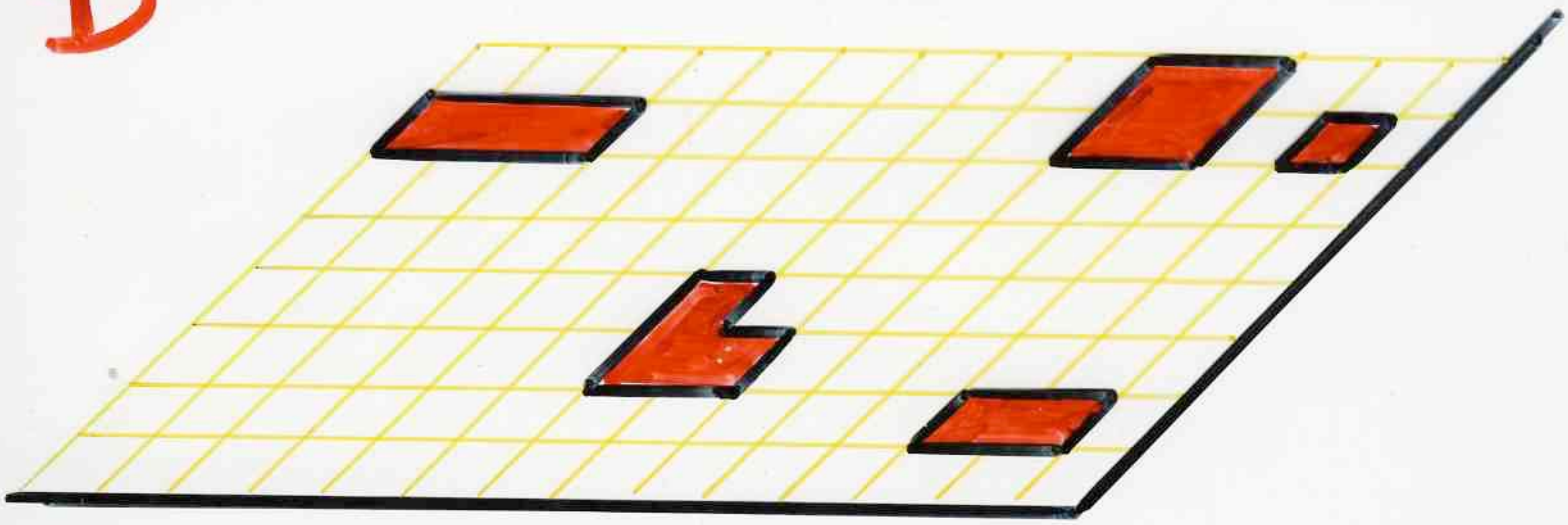
all pieces (α, i)
at level 0

trivial heap

F

all pieces (α, i) at level 0

D



weight
valuation

$v(E)$

- $v : \mathbf{P} \longrightarrow \mathbb{K}[x, y, \dots]$

  basic piece

- $v(\alpha, i) = v(\alpha)$

  piece

- $v(E) = \prod_{(\alpha, i) \in E} v(\alpha, i)$

  heap

the inversion lemma

$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right)}$$

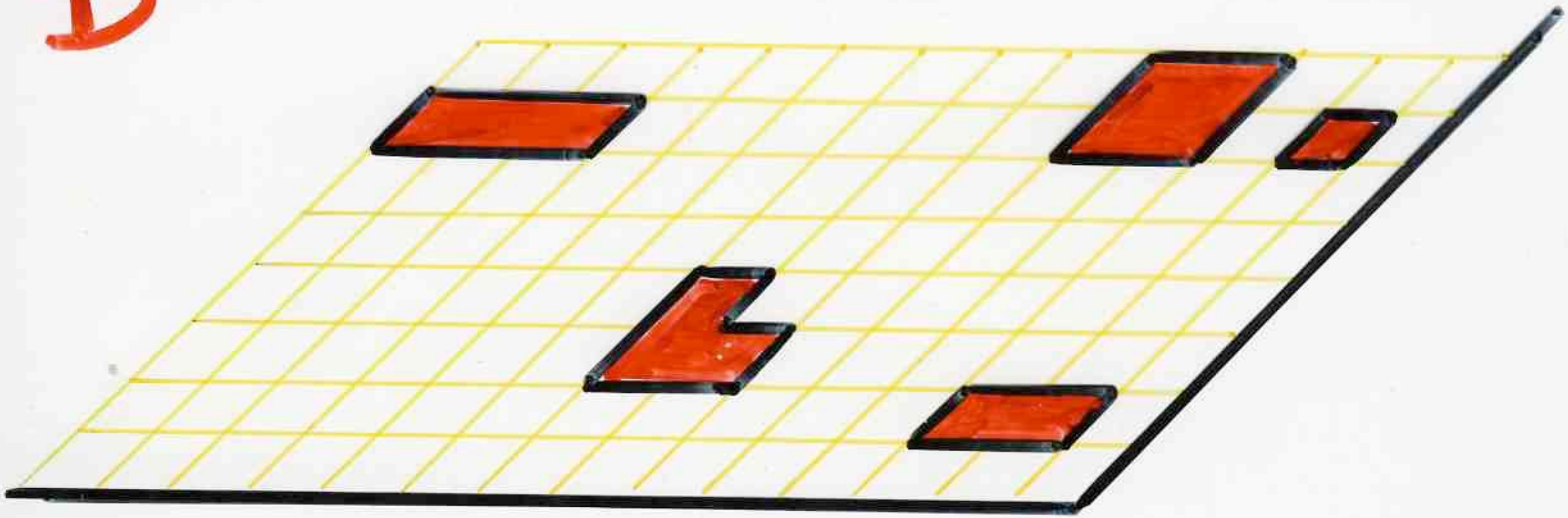
the inversion lemma

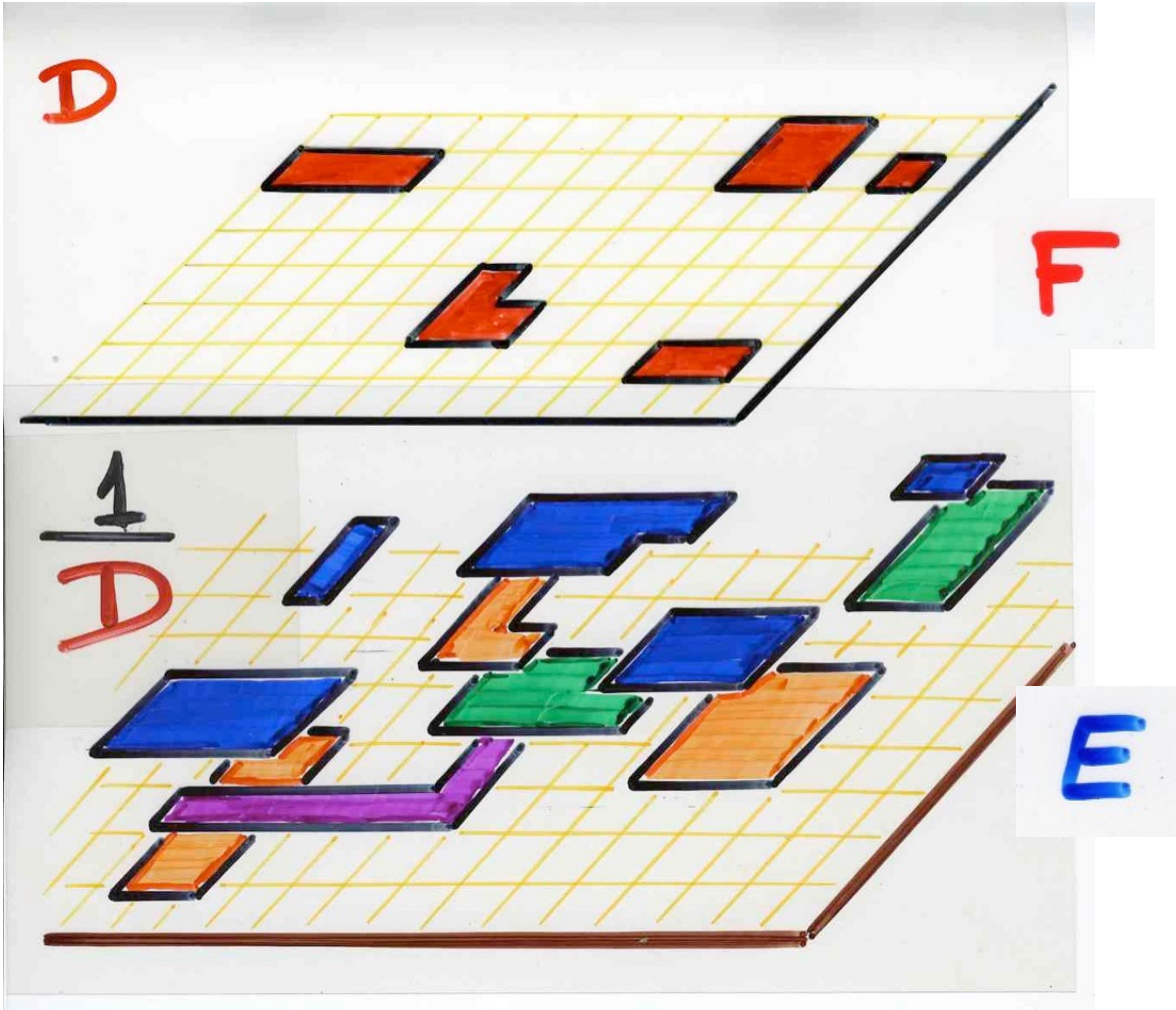
$$\left(\sum_{\substack{E\\ \text{heaps}}} v(E)\right) = \frac{1}{\boxed{\left(\sum_{\substack{F\\ \text{trivial}\\ \text{heaps}}} (-1)^{|F|} v(F)\right)}}$$

D

F

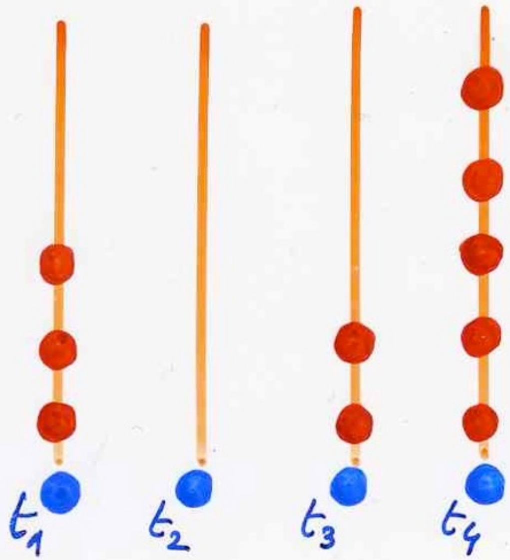
D





D
F
1
D
E

$$\frac{1}{1-t} = 1 + t + t^2 + \dots + t^n \dots$$



$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} = \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \geq 0} t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} t_4^{\alpha_4}$$

the logarithmic lemma

weight
valuation

$v(E)$

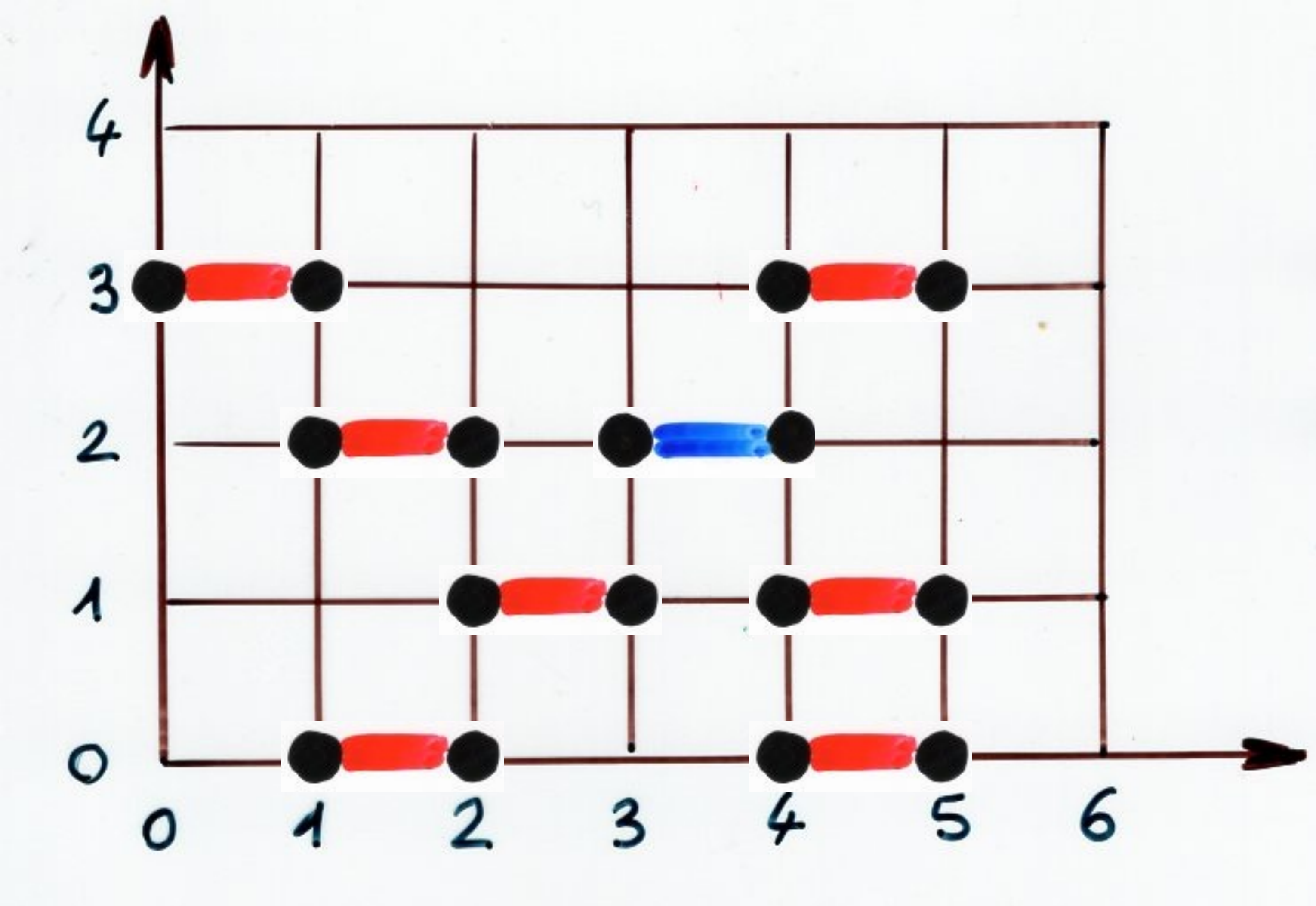
- $v : \underset{\text{basic piece}}{P} \longrightarrow \mathbb{K}[x, y, \ldots]$

- $v(\underset{\text{piece}}{\alpha, i}) = v(\alpha)\, t$

- $v(\underset{\text{heap}}{E}) = \prod_{(\alpha, i) \in E} v(\alpha\ i)\; t^{|E|}$

The logarithmic Lemma

$$t \frac{d}{dt} \log \left(\sum_{\substack{E \\ \text{heap}}} v(E)\, t^{|E|} \right) = \sum_{\substack{P \\ \text{pyramid}}} v(P)\, t^{|P|}$$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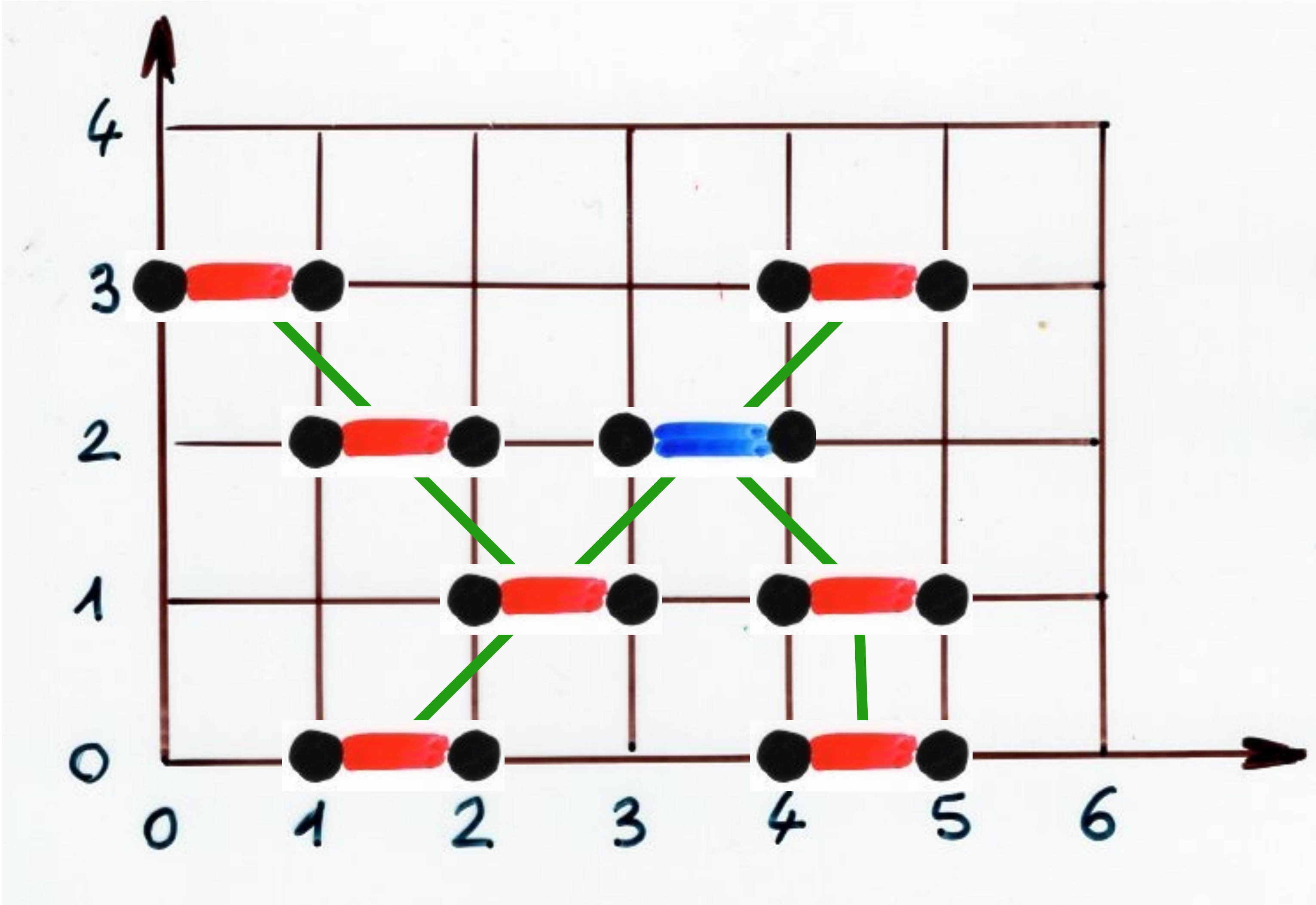
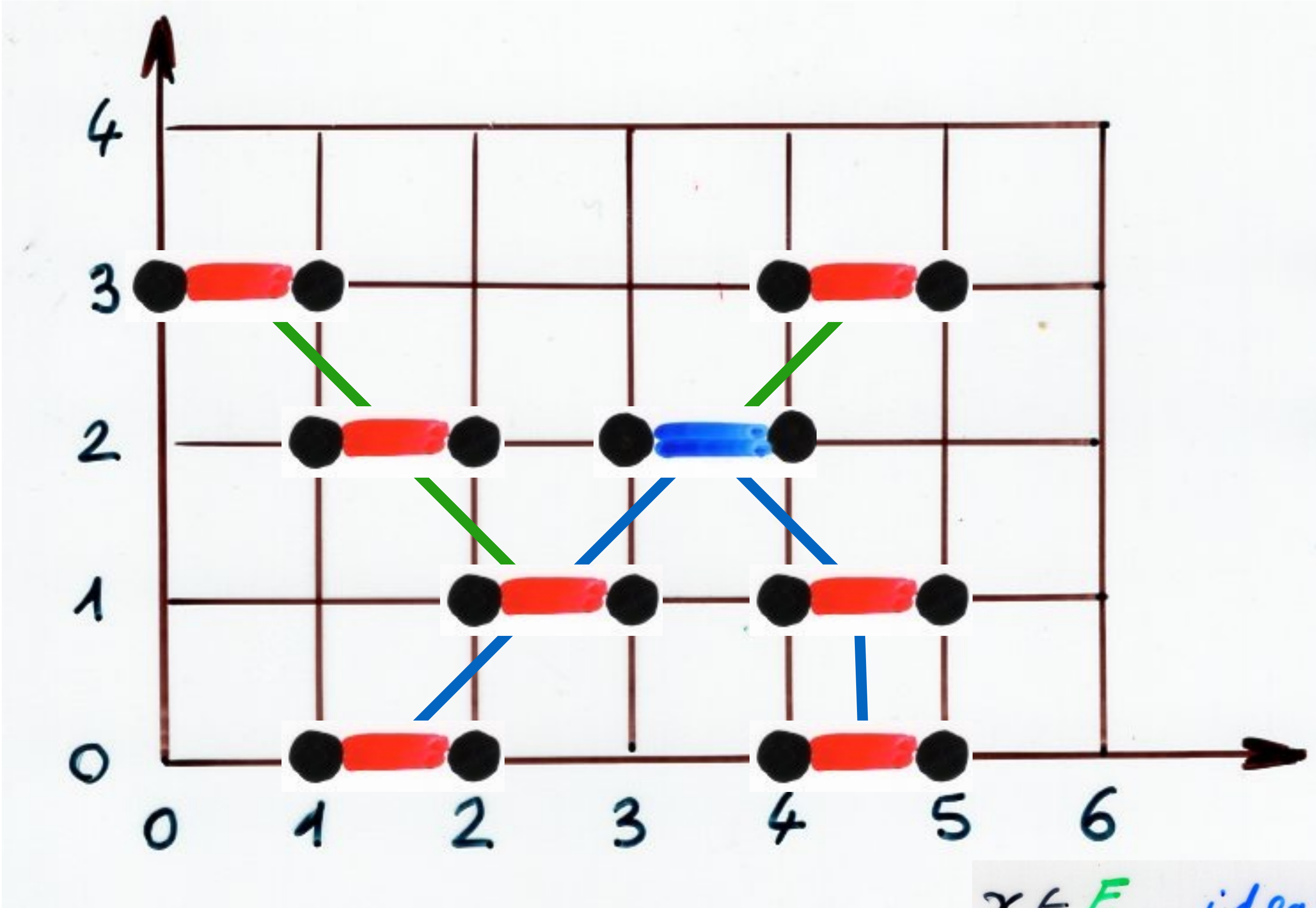


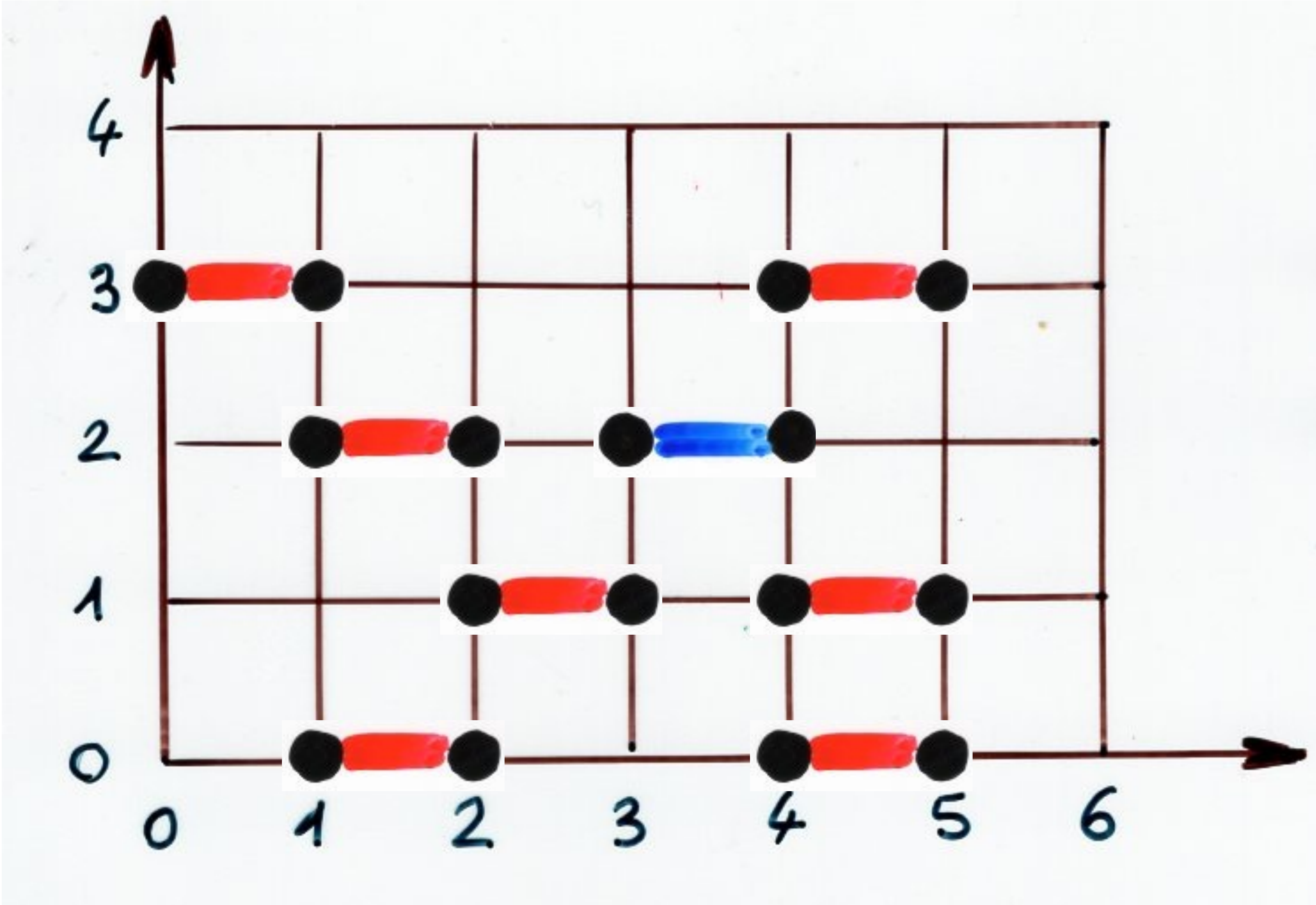
poset $\leqslant$ underlying the heap E



$x \in E$ ideal generated by x

pyramid

$P_x = \{ y \in E,\ y \preccurlyeq x \}$



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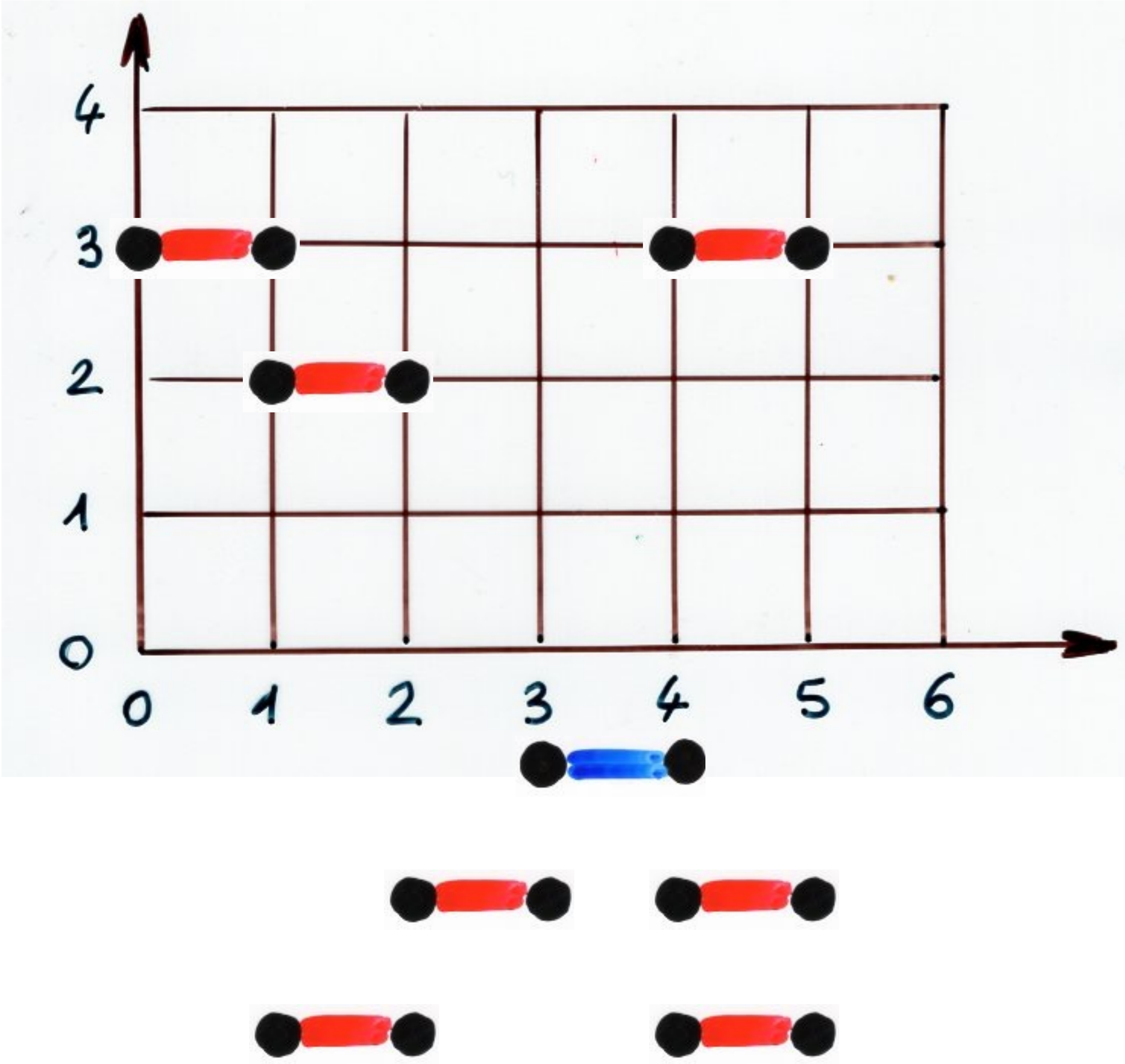

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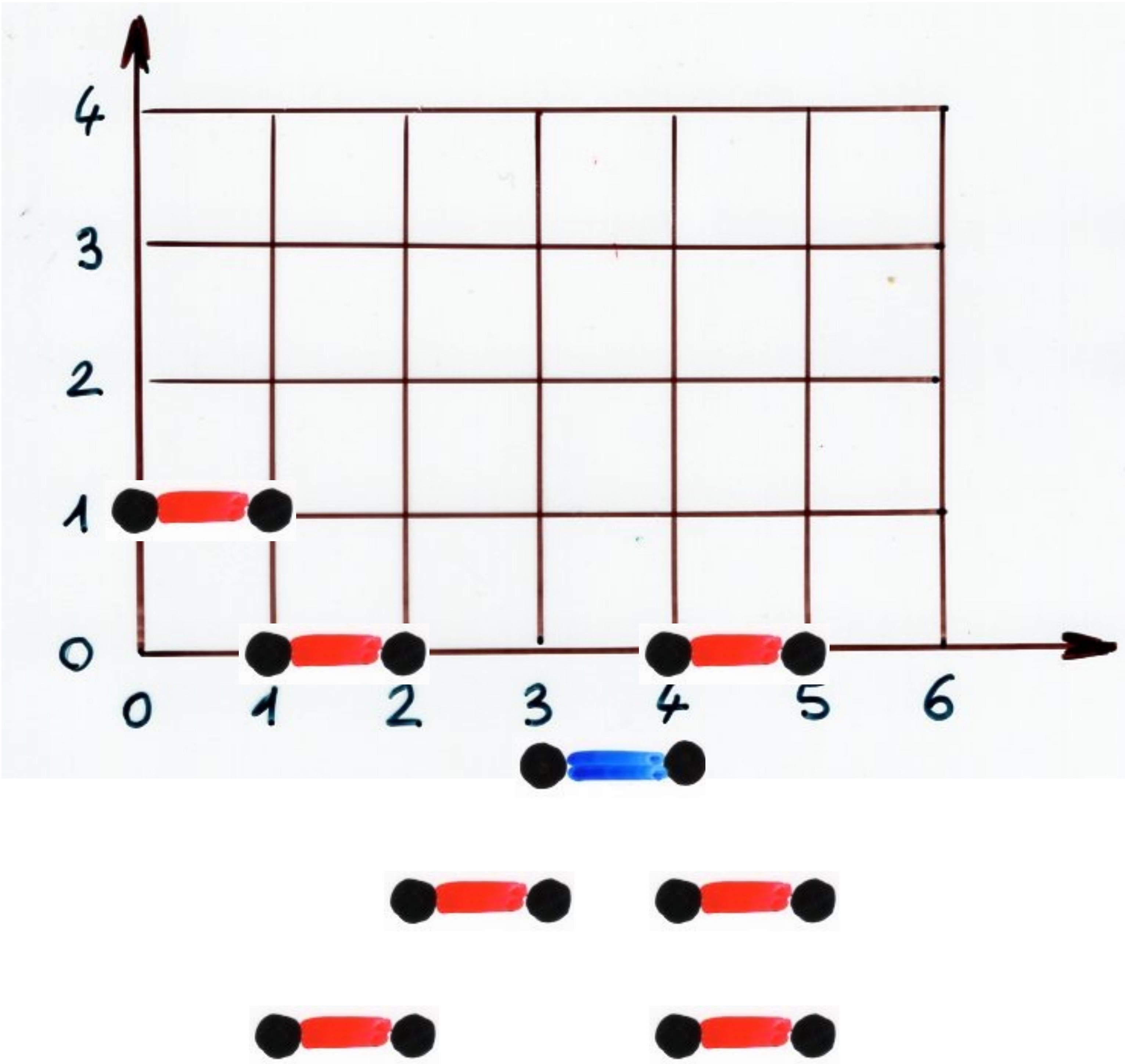
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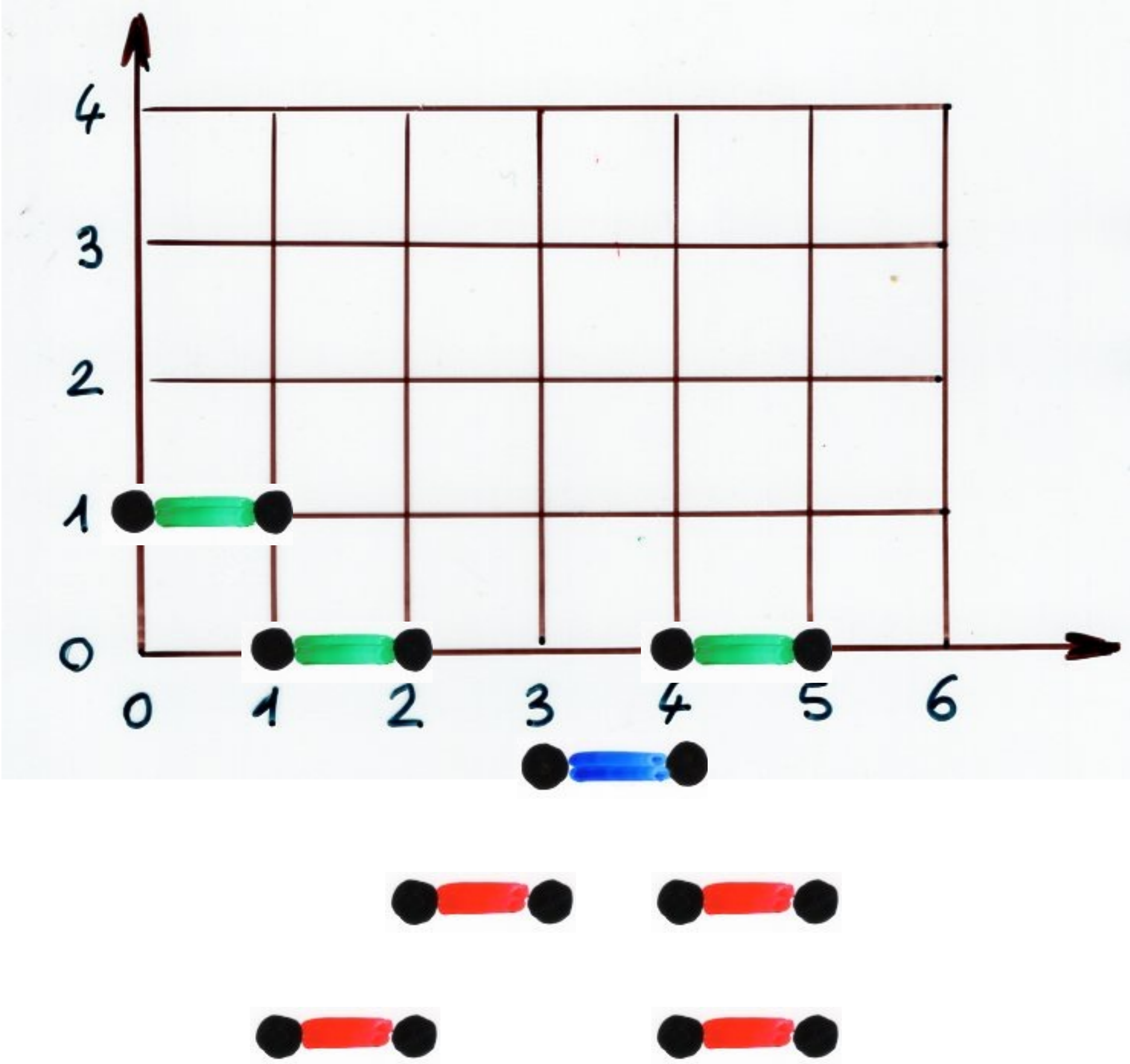
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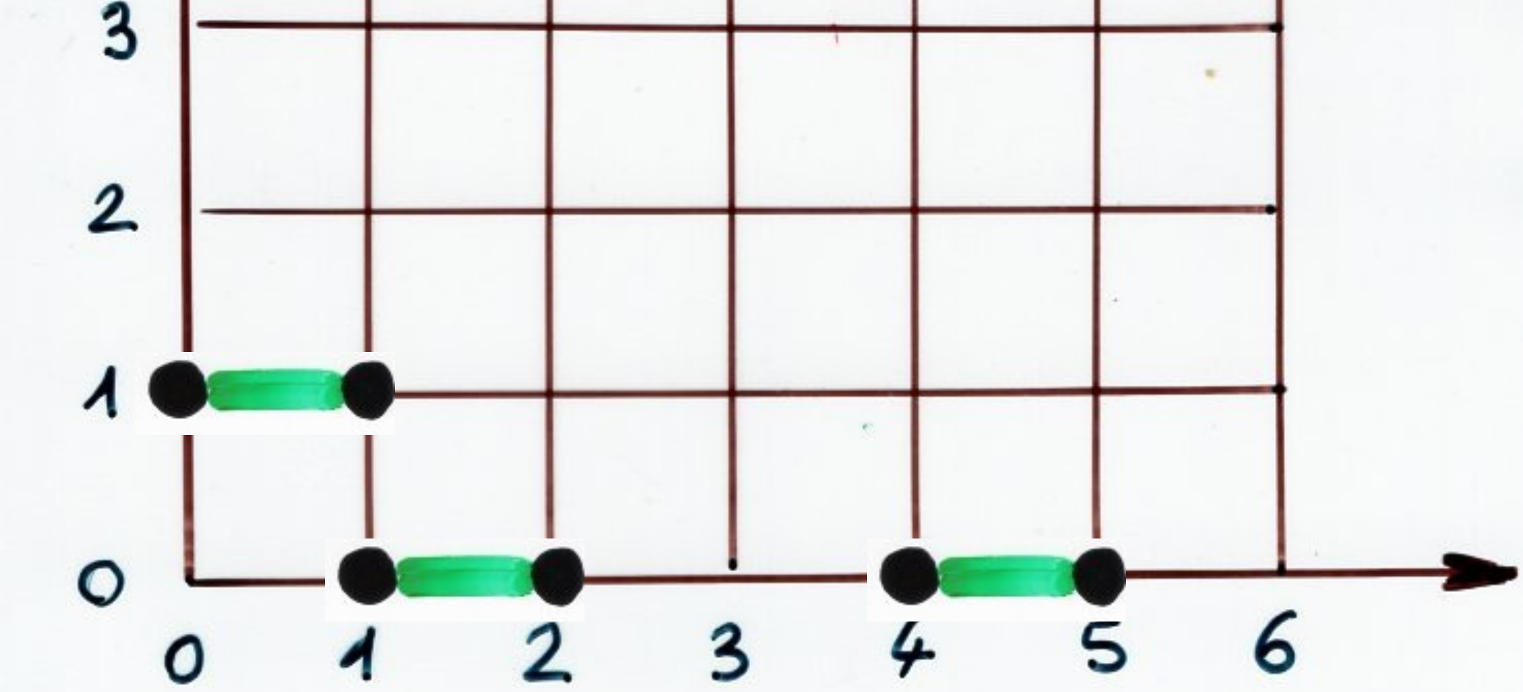
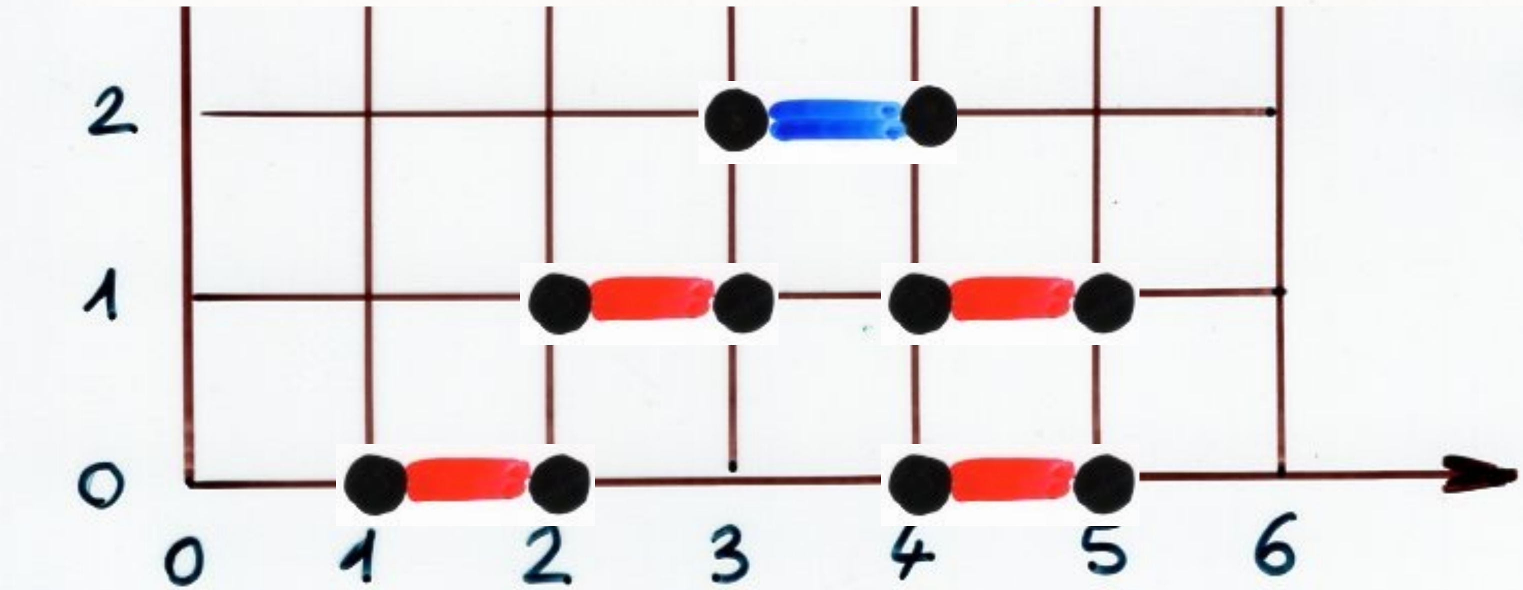


Pointed heap = Pyramid × heap



$$t\,y' = z\,y$$

Pointed heap = Pyramid × heap

$$z = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

$$\frac{t\,y'}{y} = z$$

$|E|$ number of elements

$$t\frac{d}{dt}\log\Big(\sum_{\substack{E \\ \text{heap}}} v(E)\,t^{|E|}\Big) = \sum_{\substack{P \\ \text{pyramid}}} v(P)\,t^{|P|}$$

The logarithmic Lemma

$\square$

The logarithmic Lemma

equivalent form

$$\log\Big(\sum_{\substack{E\\ \text{heap}}} v(E)\, t^{|E|}\Big) = \sum_{\text{pyramid}} v(P)\,\frac{t^{|P|}}{|P|}$$

also :

$$-\log\Big(\sum_{\substack{E\\ \text{trivial heap}}} v(E)\,(-t)^{|E|}\Big)$$

Interpretation of

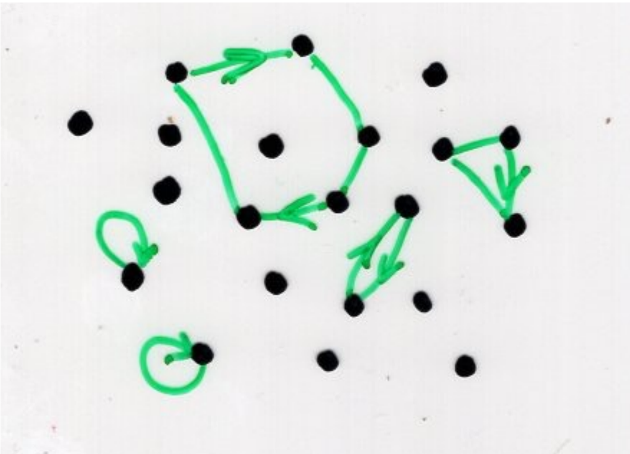
$$t \frac{d}{dt} \log \zeta_G(t)$$

$$\text{(ii)} \quad \zeta_G(t) = \frac{1}{\det(1-Ht)}$$

$$A = (a_{ij})_{1 \leq i,j \leq k}$$

$$\det(I - A) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{inv(\sigma)} a_{1\sigma(1)} \cdots a_{k\sigma(k)}$$

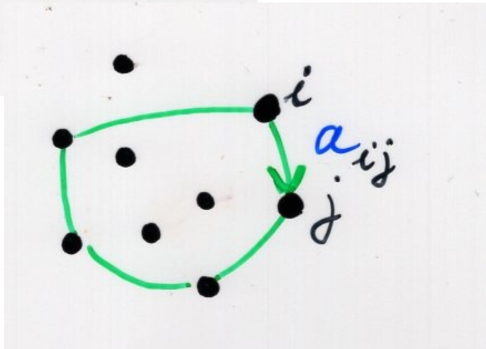
permutation



$$\sum_{\gamma_1, \ldots, \gamma_r} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$

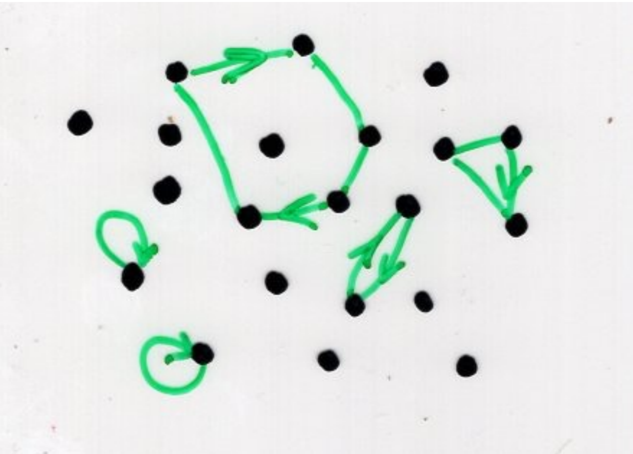
2 by 2 disjoint cycles

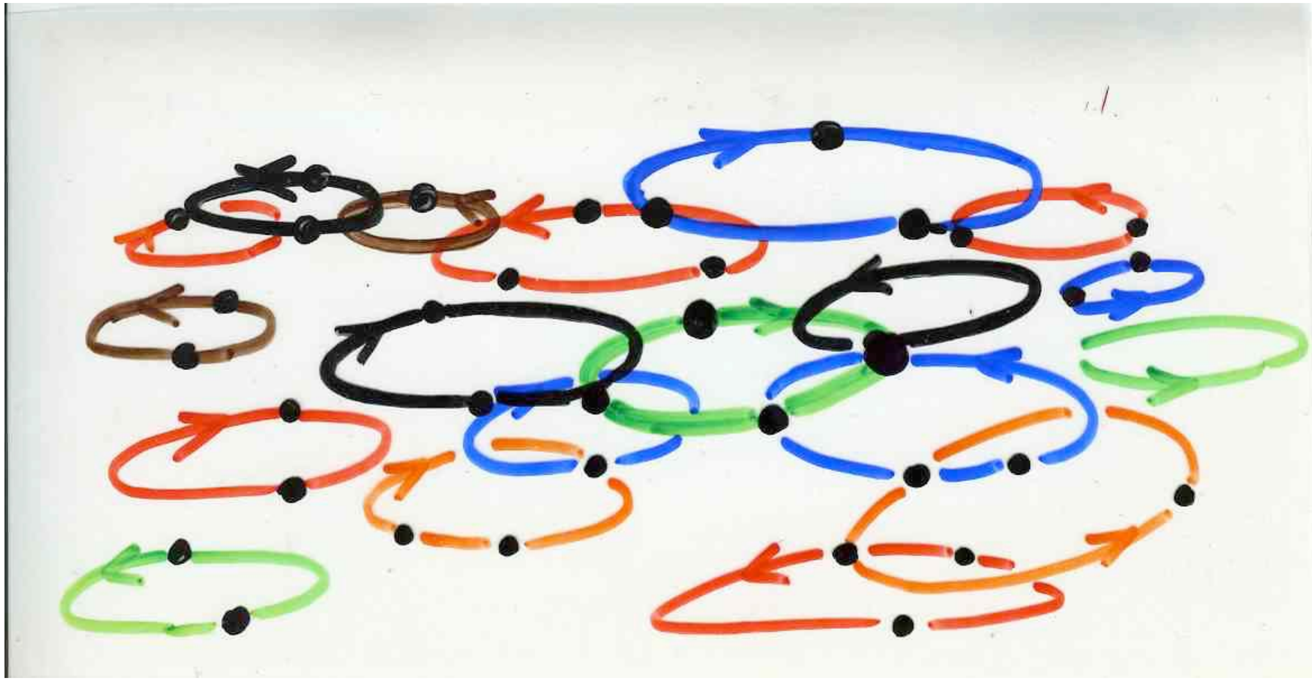
$$X = [1, k]$$



inversion lemma

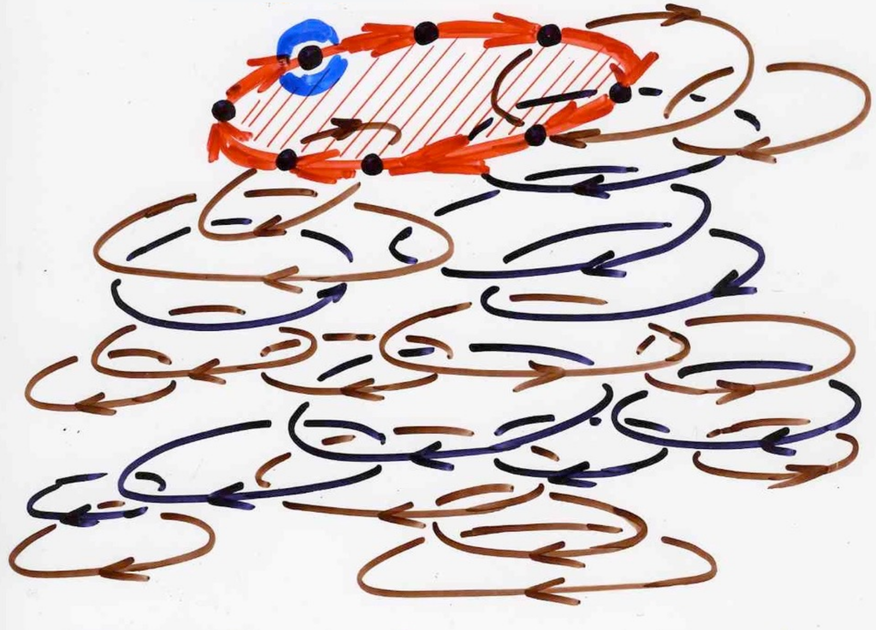
$$\frac{1}{\det(I-A)} = \sum_{\substack{E \\ \text{heap of cycles on } [1,k]}} v(E)$$





$$\log \frac{1}{\det(I-A)} = \sum_{\substack{P \\ \text{pointed} \\ \text{pyramid} \\ \text{of cycles} \\ (P,x)}} v(P) \frac{t^{\ell(P)}}{\ell(P)}$$

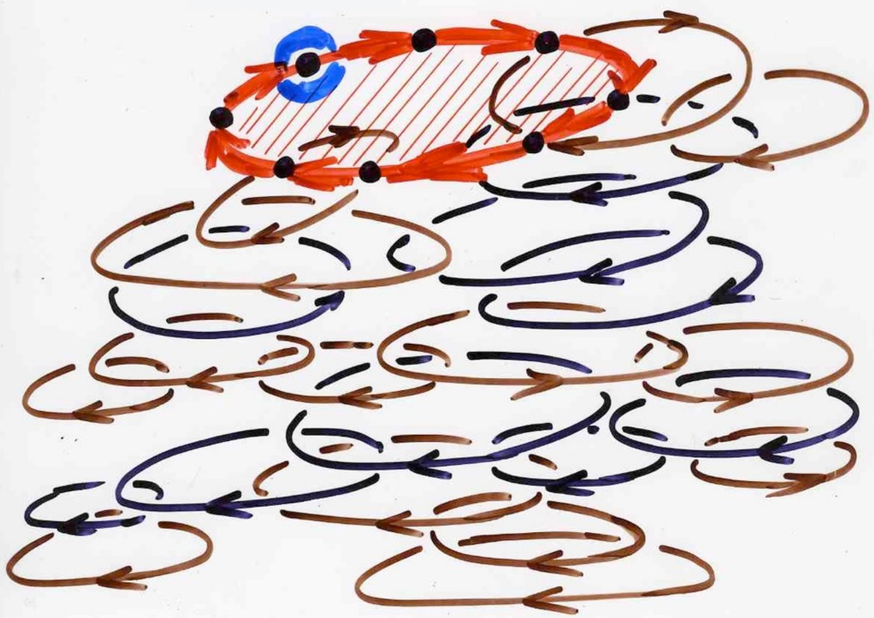
$x \in \gamma_{max}$



pointed pyramid of cycles

= the unique cycle maximal piece has a distinguished vertex (or edge)

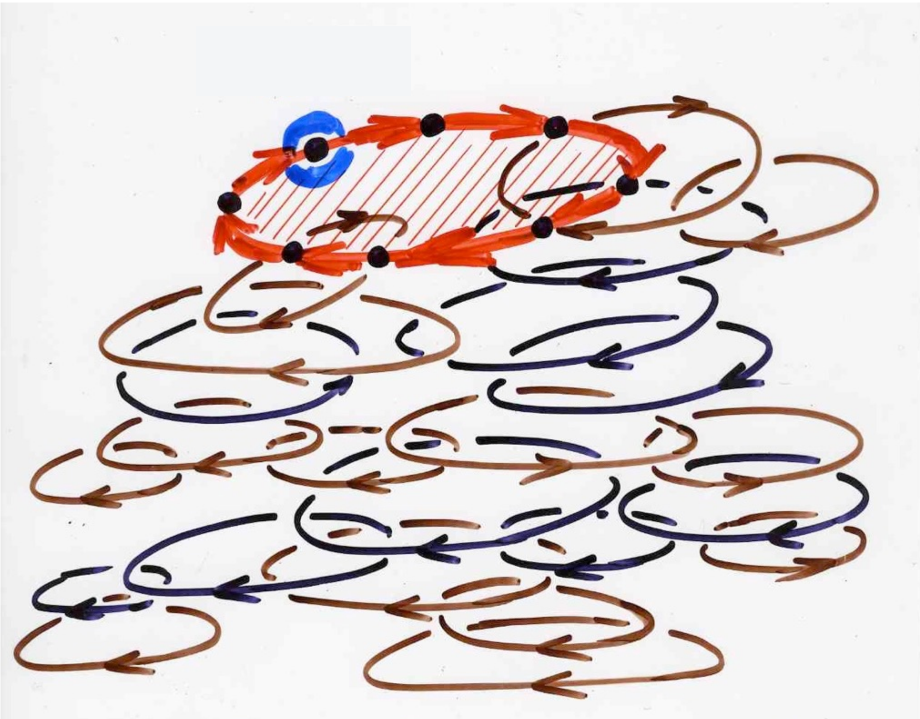
$$t \frac{d}{dt} \log \frac{1}{\det(I-A)} = \sum_{\substack{P \\ \text{pointed pyramid} \\ \text{of cycles} \\ (P,x) \\ x \in \gamma_{max}}} v(P)\, t^{\ell(P)}$$



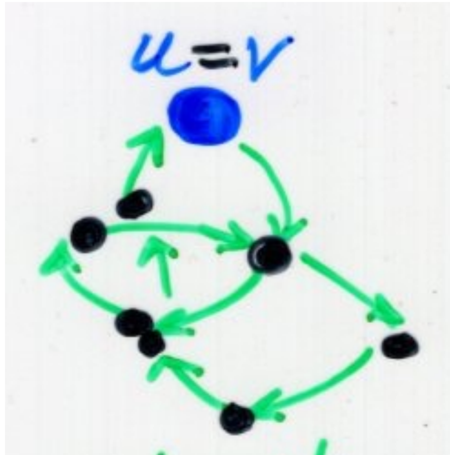
pointed pyramid of cycles

From the third lemma:

Circuit
path $\omega = (s_0, \dots, s_n)$ with $s_n = s_0$



pointed pyramid of cycles



circuit

are in bijection with
pointed pyramids of cycles

The third lemma:

Paths and heaps of cycles

Bijection

$u, v \in X$

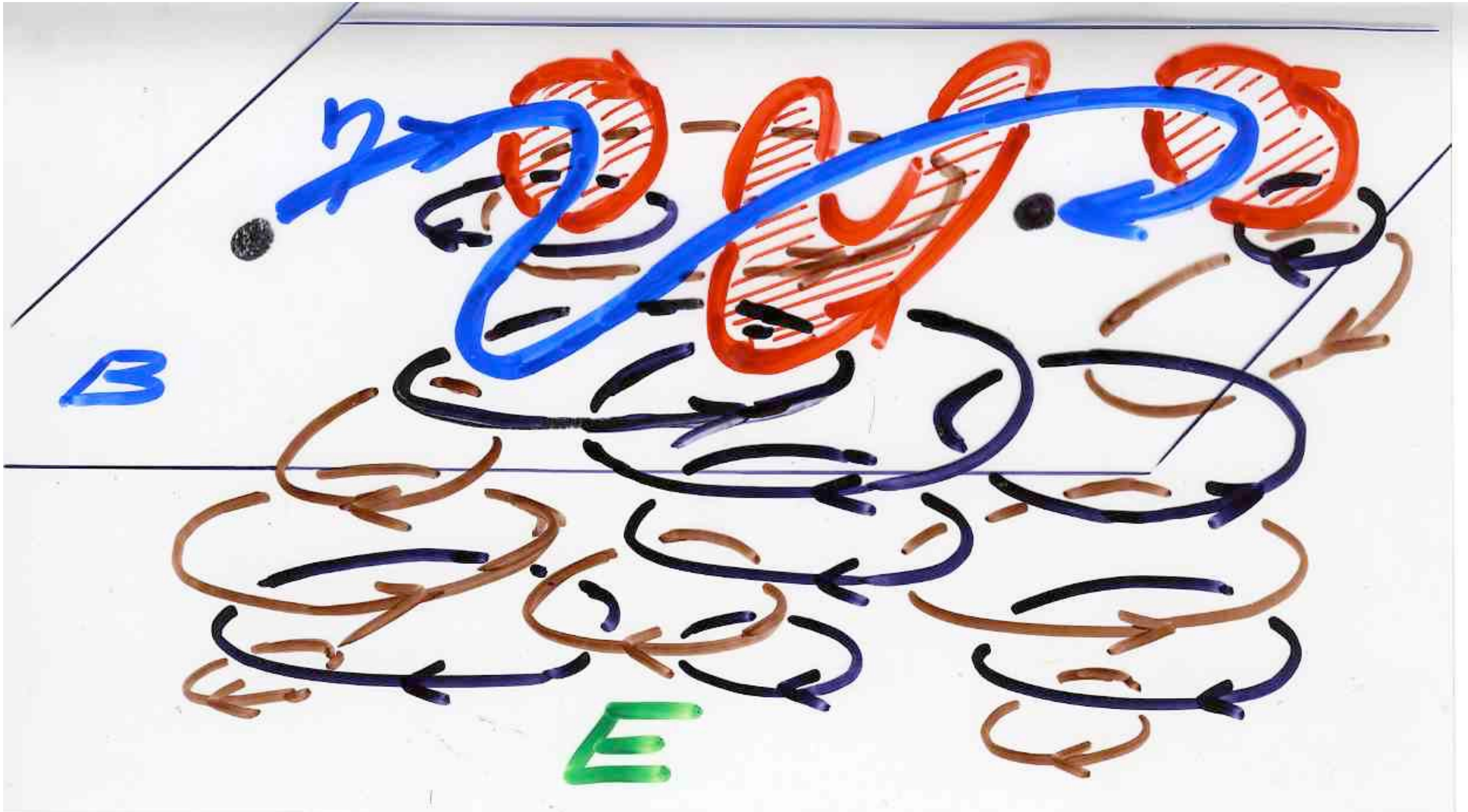
path ω on X going from u to v $\overset{\chi}{\longleftrightarrow}$ (η, E)

- η self-avoiding path going from u to v
- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η ($\underset{\text{cycle}}{\alpha}$ and $\underset{\text{path}}{\eta}$ has a common vertex)

for any $s, t \in X$
the numbers of occurrences of the edge (s, t) in ω and in (η, E) are the same.

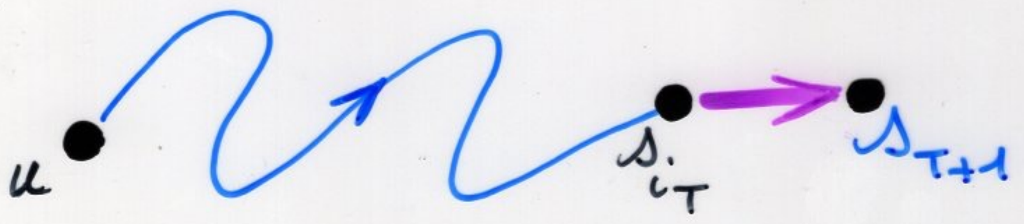
$\Rightarrow \quad v(\omega) = v(\eta)\, v(E)$

The bijection X



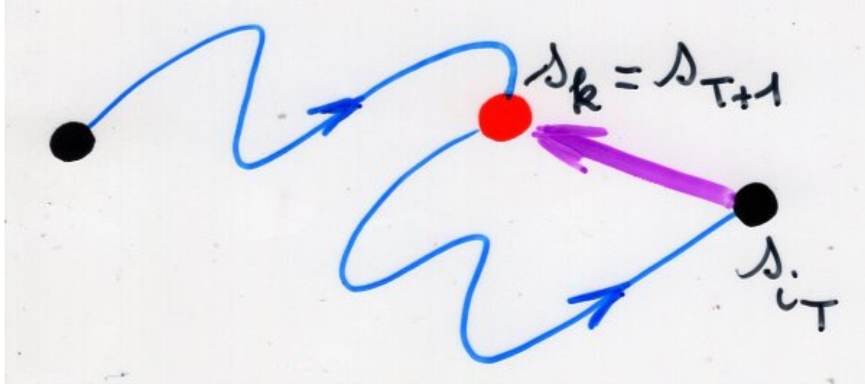
• suppose $\begin{cases} Cut_T(\omega) = (s_0 = u, \dots, s_{i_T}) \\ E_T(\omega) \text{ heap of cycles} \end{cases}$

(i) if $s_{T+1} \notin Cut_T(\omega)$



$$\begin{cases} Cut_{T+1}(\omega) = (s_0 = u, \dots, s_{i_T}, s_{T+1}) \\ E_{T+1}(\omega) = E_T(\omega) \end{cases}$$

(ii) if $s_{T+1} \in Cut_T(\omega)$, $s_{T+1} = s_k$



$$\begin{cases} Cut_{T+1}(\omega) = (s_0 = u, \dots, s_k) \\ E_{T+1}(\omega) = E_T(\omega) \odot \gamma \end{cases}$$

$$\dot{\gamma} = (s_k, \dots, s_{i_T}, s_{T+1} = s_k)$$

$$\omega \xrightarrow{\chi} (\eta, E)$$

$$\eta = \mathrm{Cut}_n(\omega)$$

$$E = E_n(\omega)$$

loop-erased process

LERW

Lawler, 1987

$$\omega \longrightarrow \left(\eta \, ; \, (\dot{\gamma}_1, \ldots, \dot{\gamma}_{r_n}) \right)$$

self-avoiding path $u \rightsquigarrow v$

sequence of pointed cycles

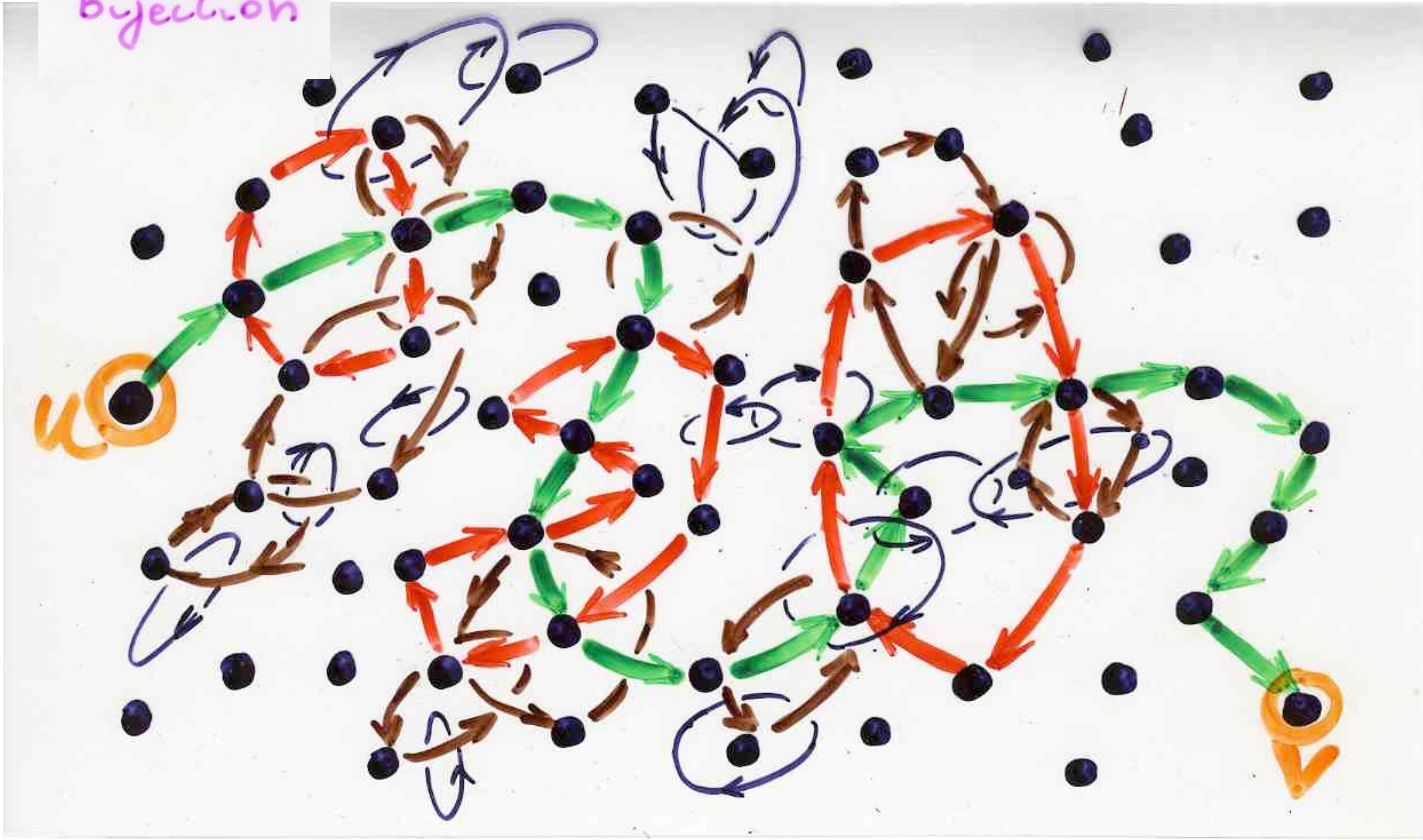
from the pair $(\eta \, ; (\dot{\gamma}_1, \ldots ; \dot{\gamma}_{r_n}))$
we can reconstruct the path ω

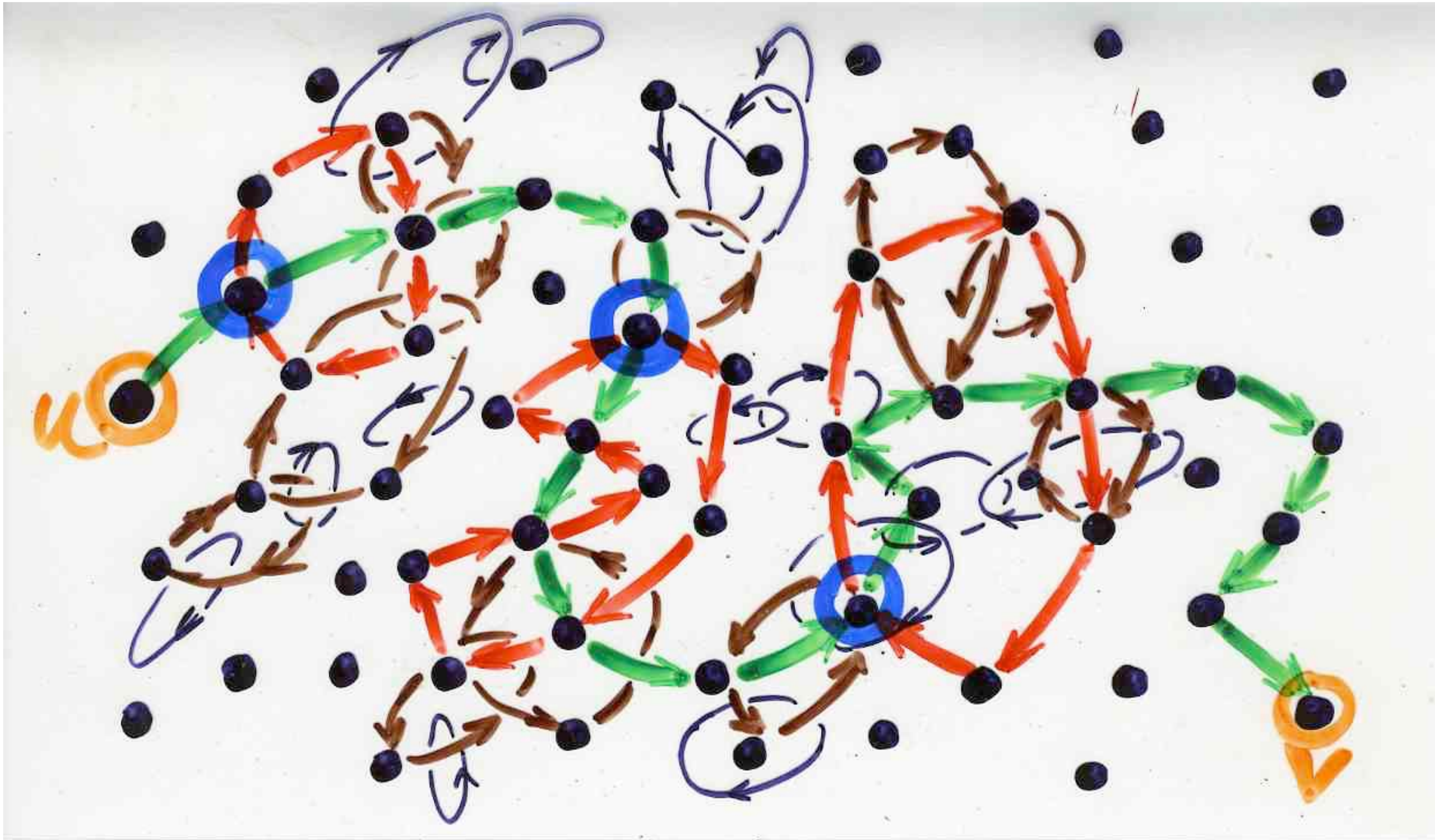
$$(\dot{\gamma}_1, \ldots, \dot{\gamma}_{r_n}) \longrightarrow E = \gamma_1 \odot \cdots \odot \gamma_{r_n}$$

$$\omega \longrightarrow (\eta \, , E)$$

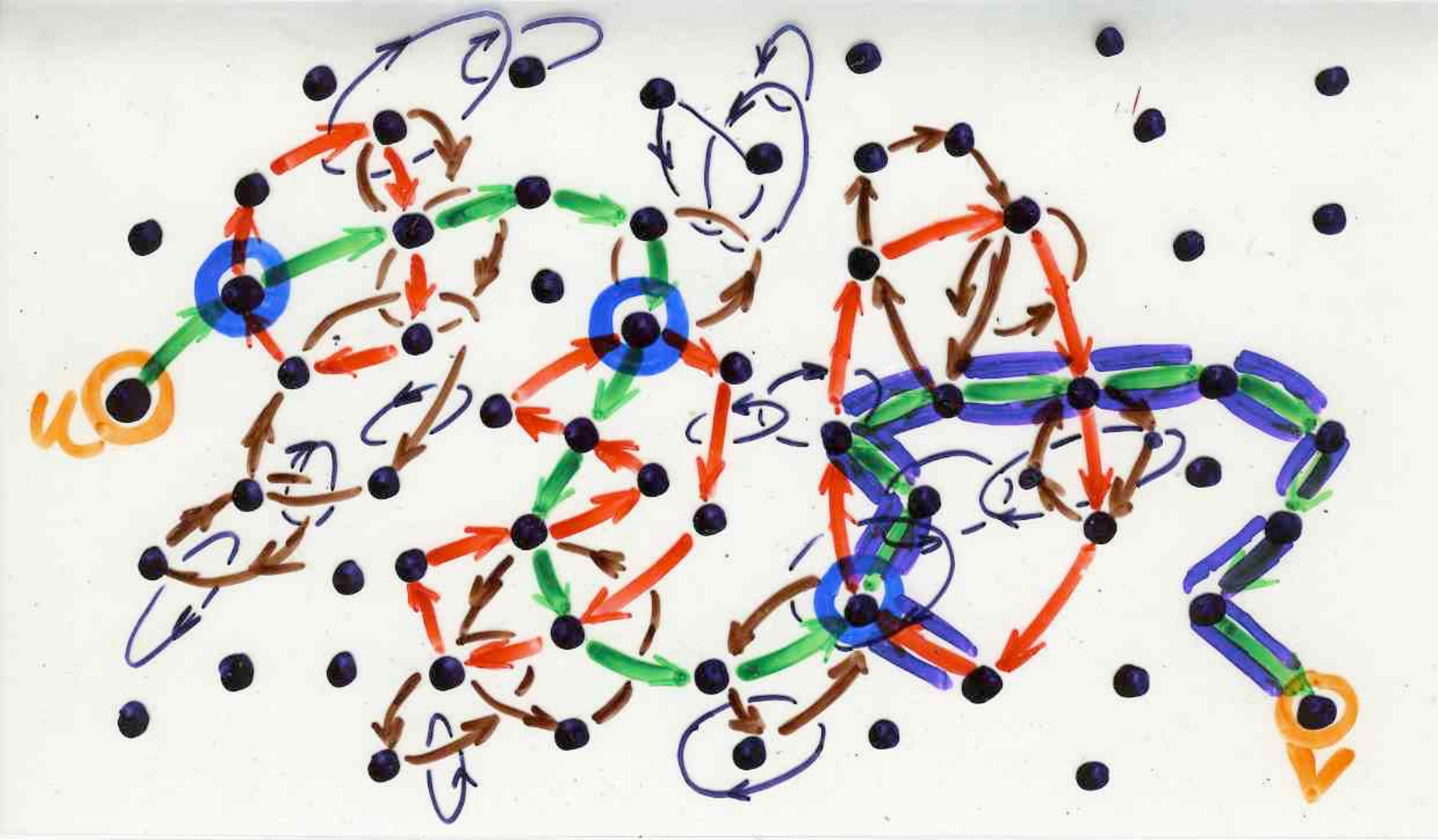
heaps of cycles on X
monoid

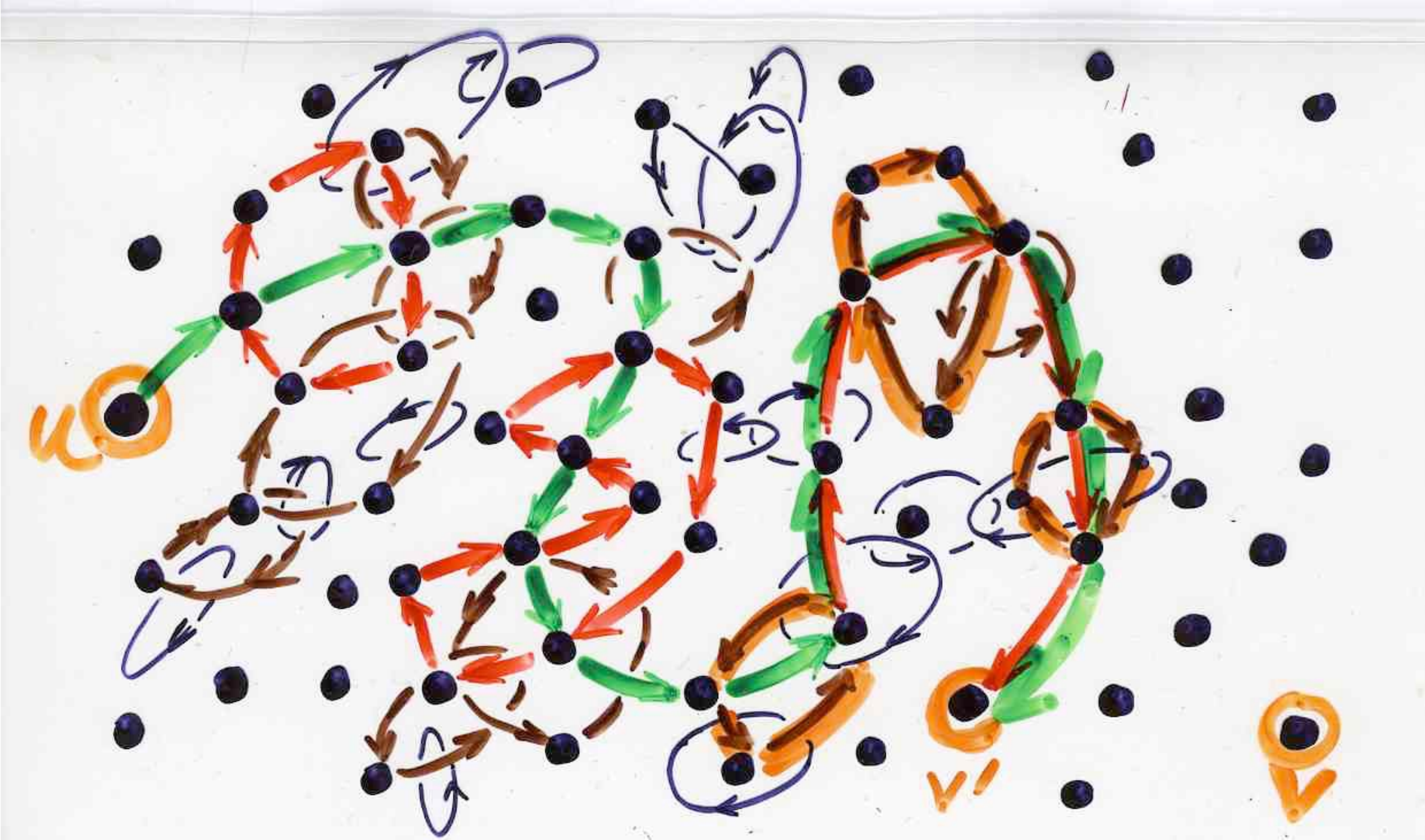
reverse bijection





u
v

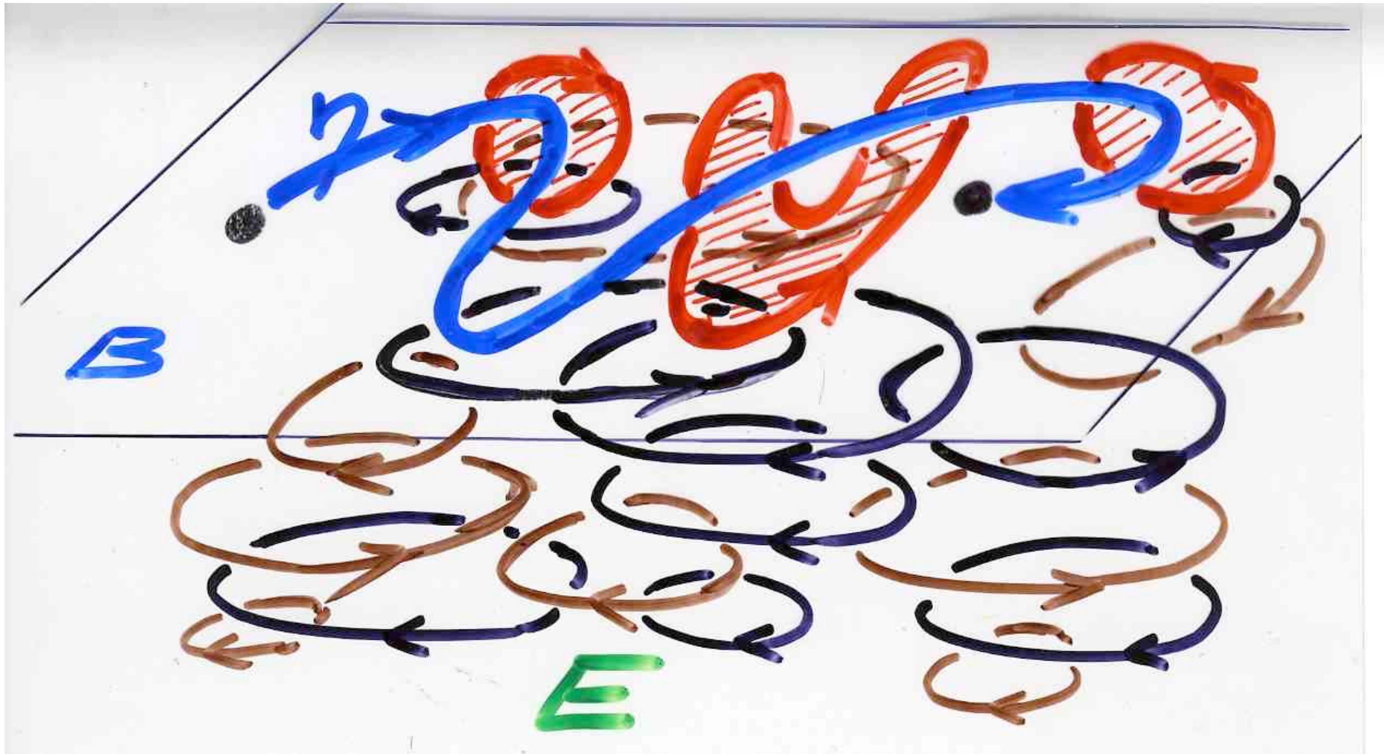

u
v


u
v'
v

The bijection χ

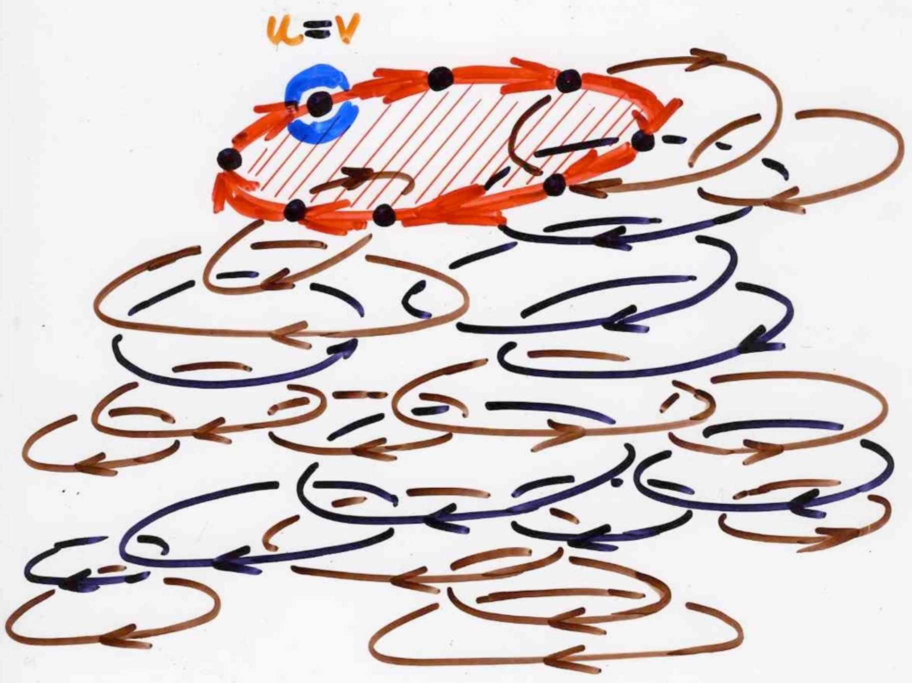
for circuits $u = v$

η is reduced to the vertex $u = v$

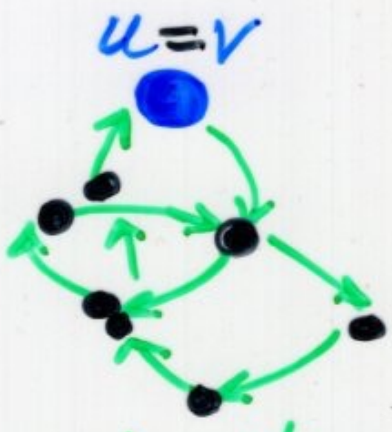


The bijection χ for circuits $u=v$

Corollary Circuits on X are in bijection with pointed pyramids of cycles

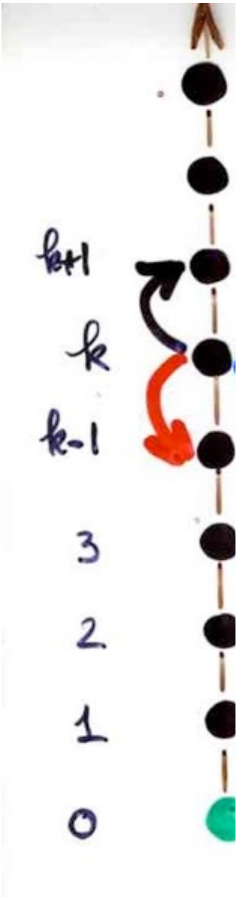


pointed cycles pyramid

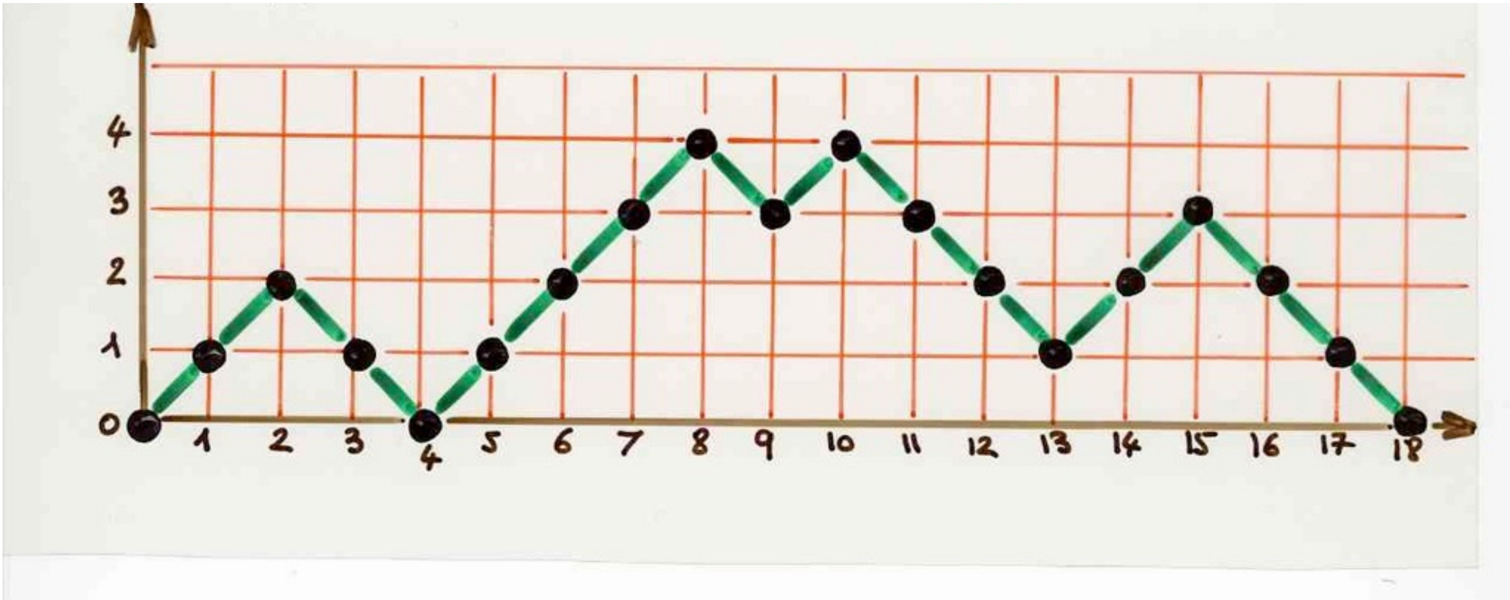


circuit

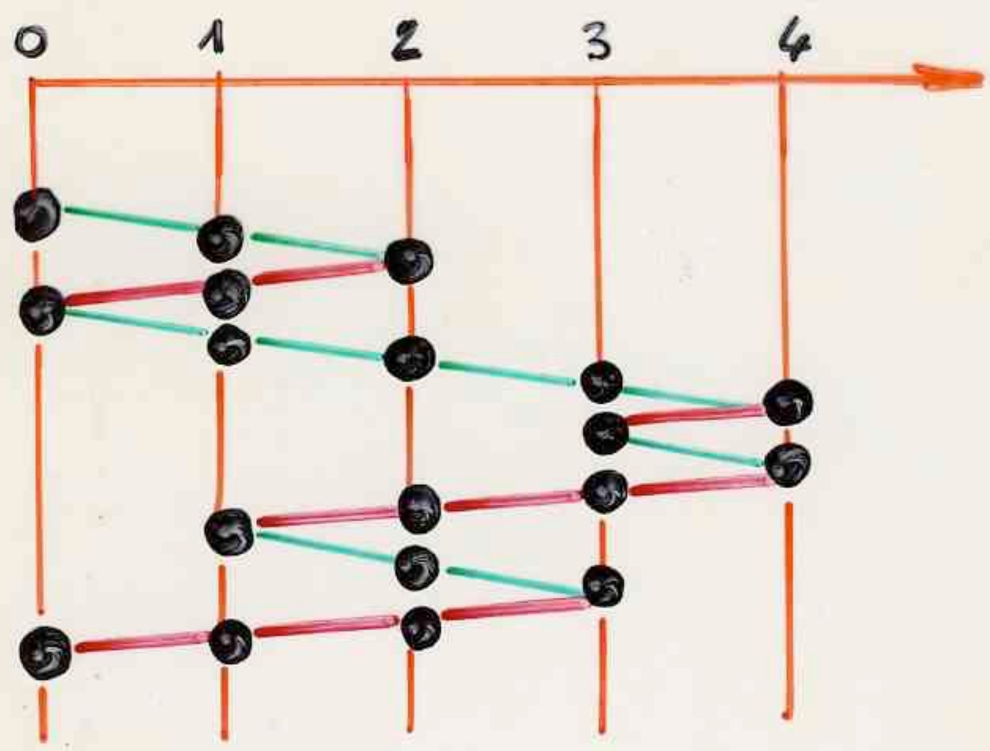
an example with Dyck paths



Dyck path



$$C_n = \frac{1}{(2n+1)} \binom{2n+1}{n}$$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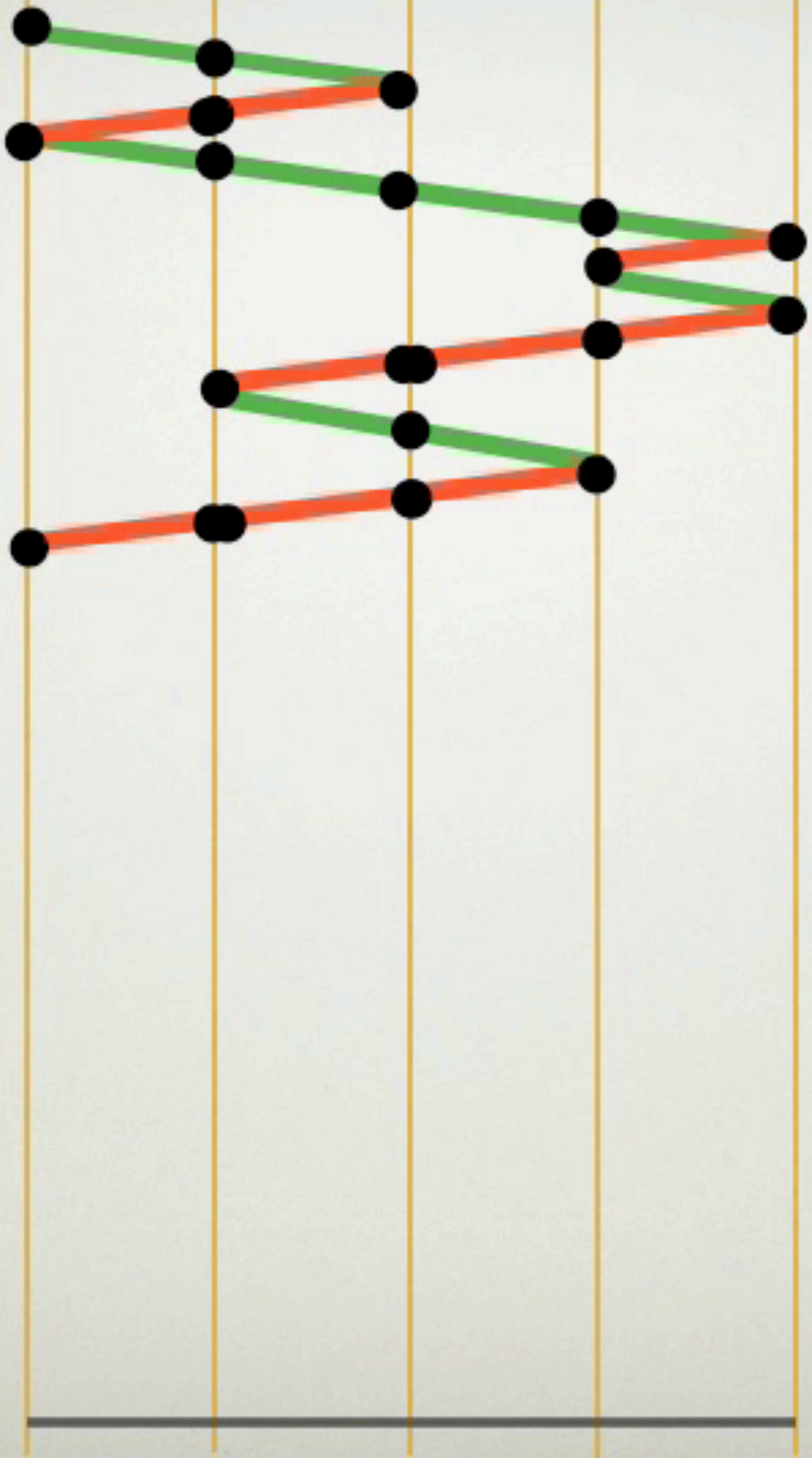
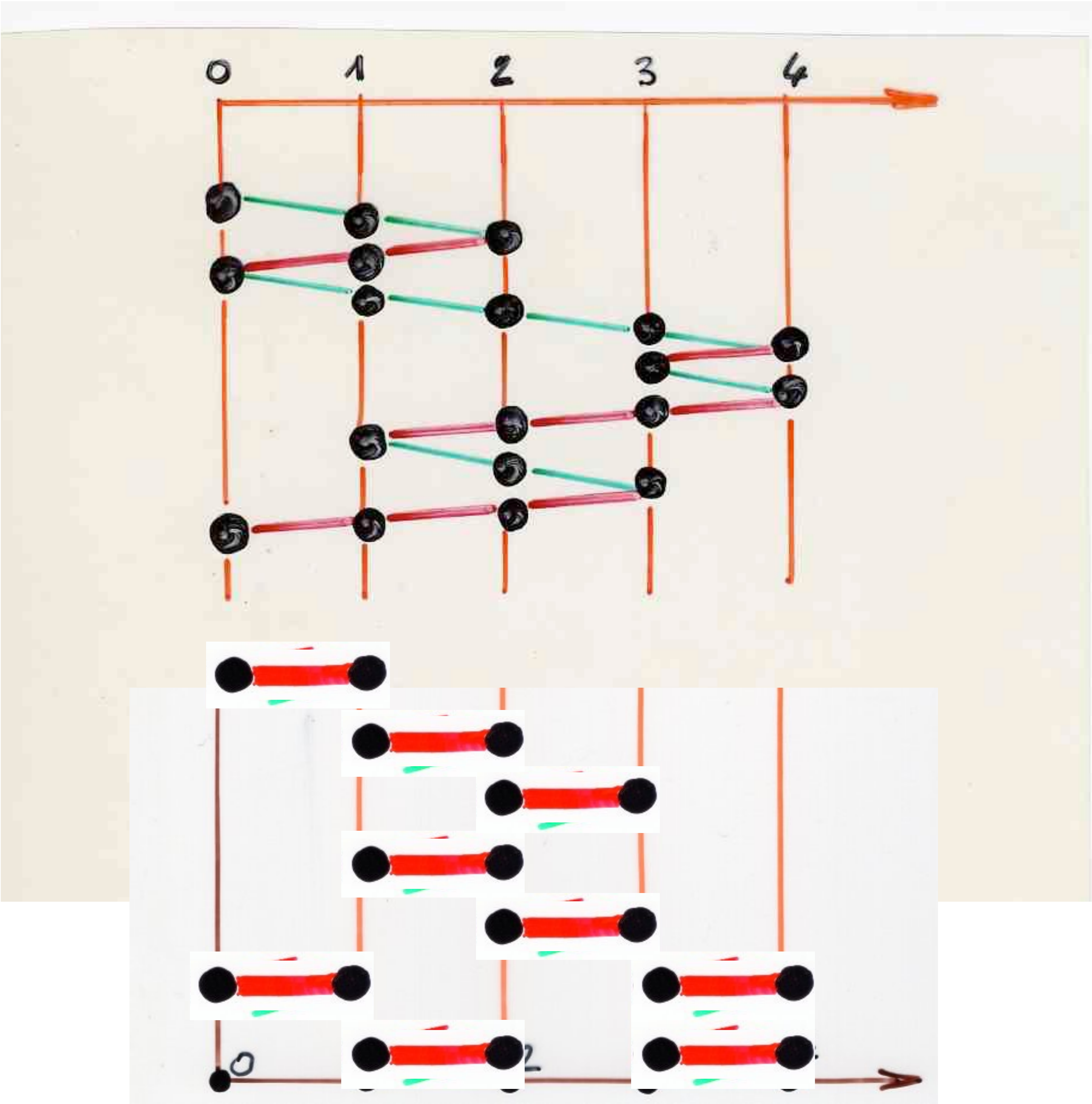


violin:
G. Duchamp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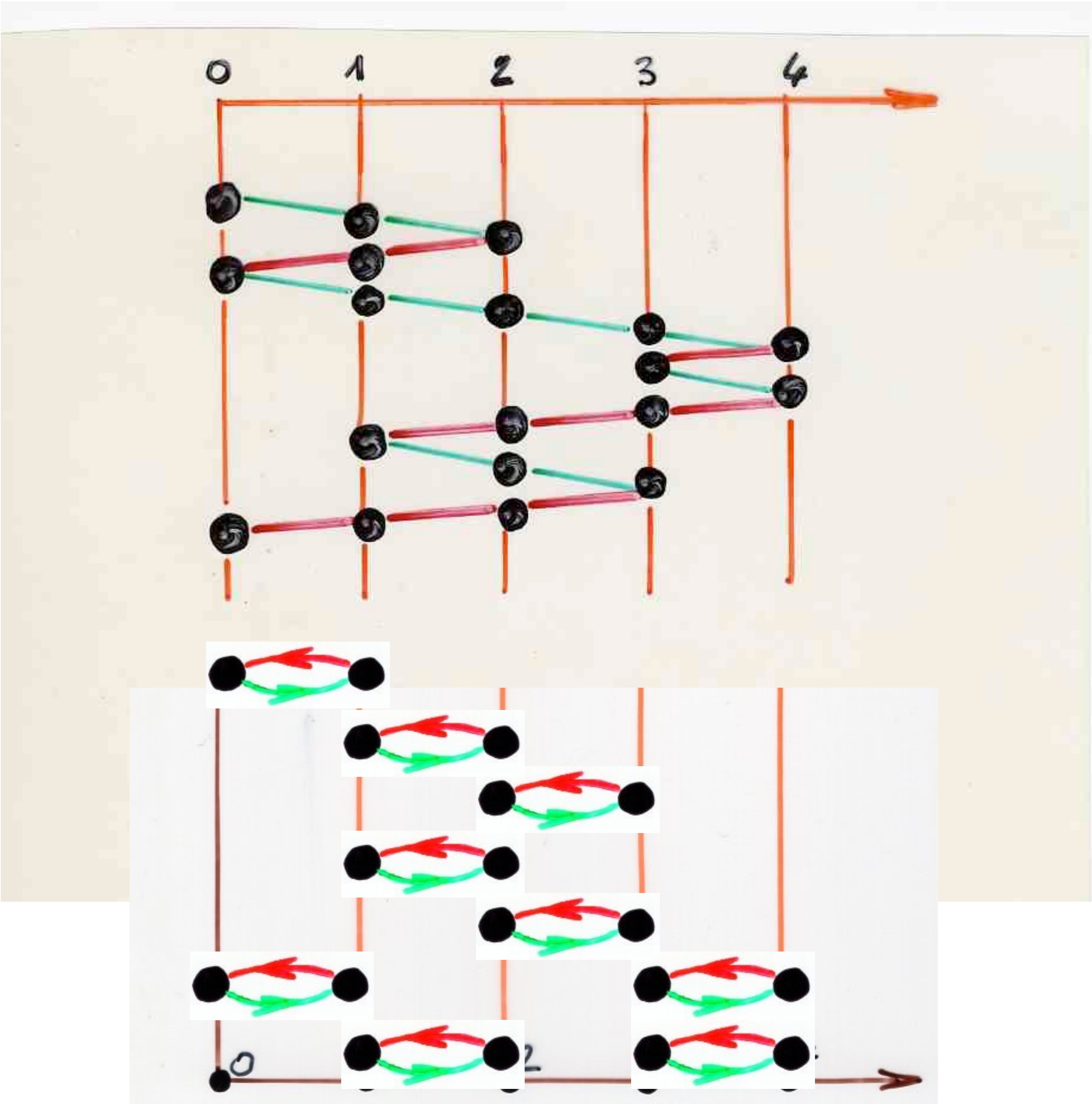
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paths on a tree

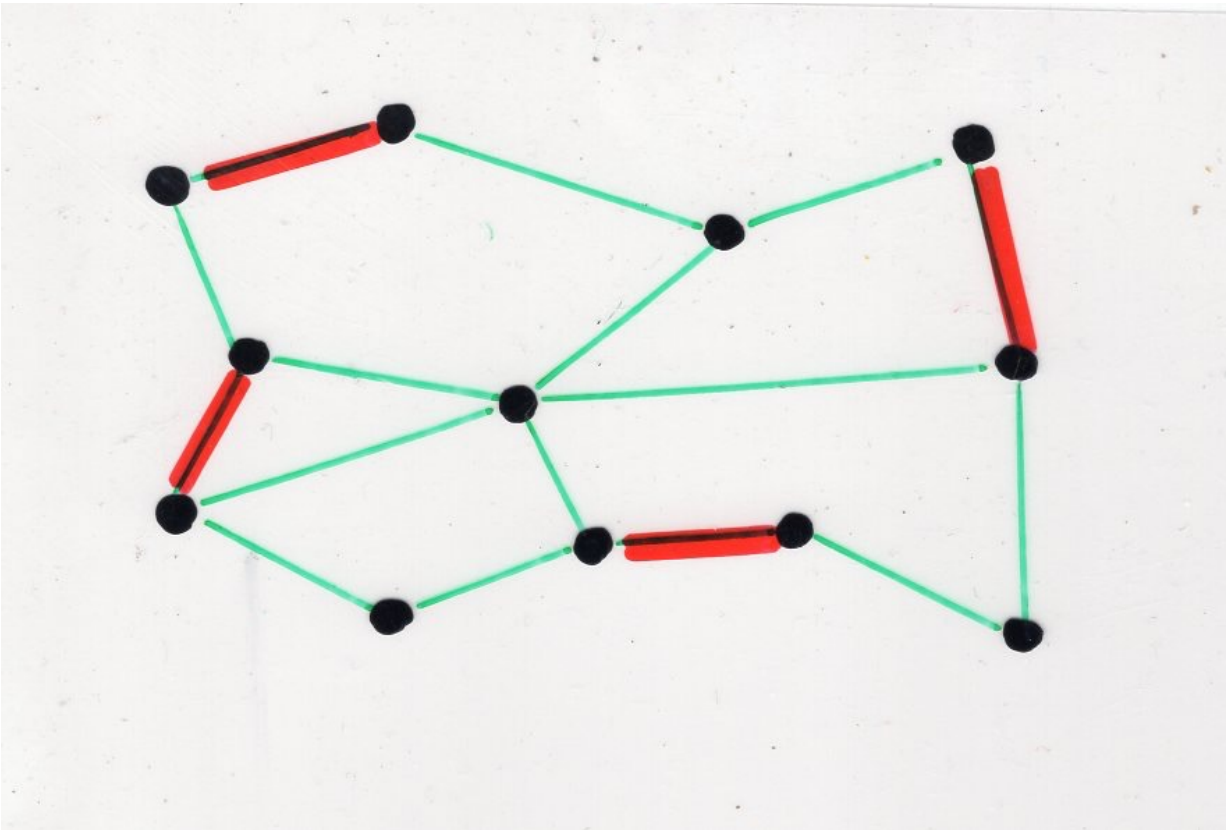
Godsil (1981)

tree-like paths

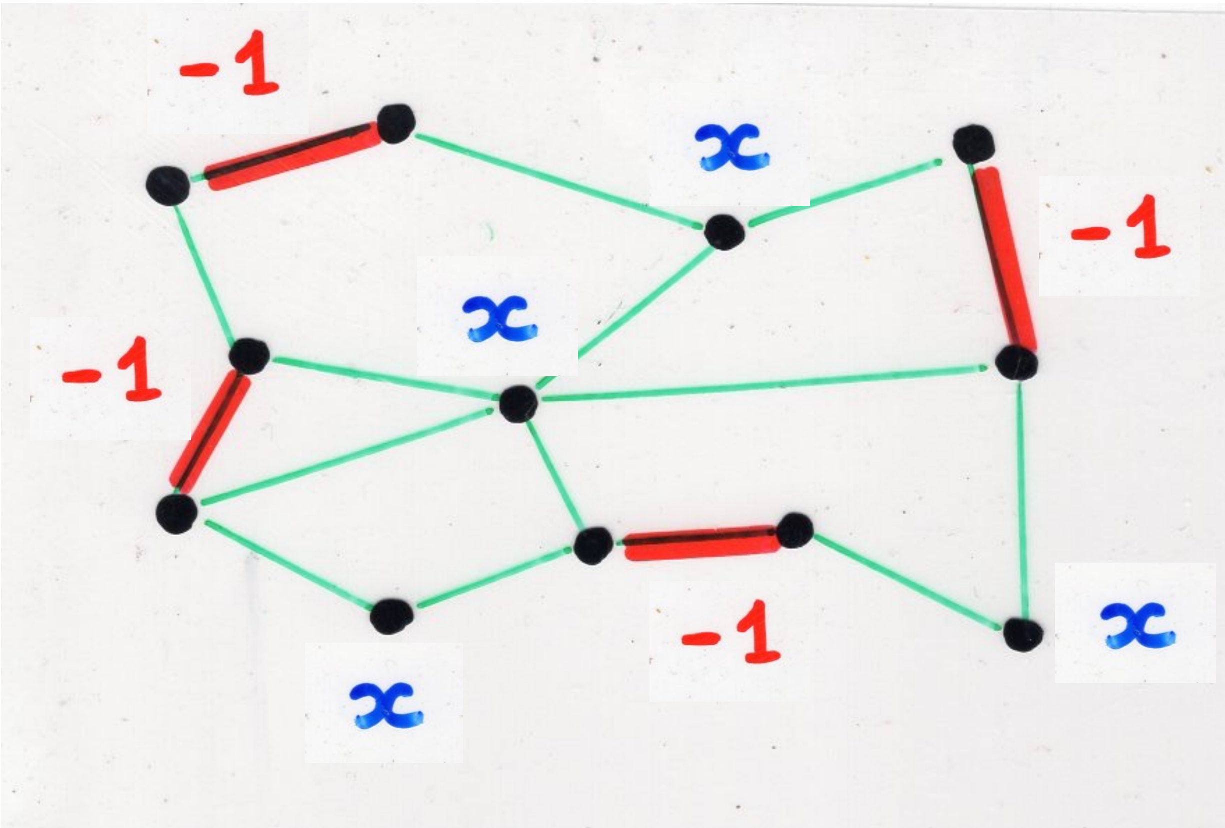
G graph, ~~χ~~

ω path on G with $\omega \longrightarrow (\eta, E)$.

ω is tree-like iff the heap E contains only cycles of length 2.



matching of a graph G = set of 2 by 2 disjoint edges



matching polynomial of a graph G

Proposition The zeros of the characteristic polynomial of a graph G are real numbers

$$\chi(x) = \det(Ix - A)$$

Linear algebra
revisited with heaps of pieces

combinatorial (bijective) proofs
of classical theorem
in linear algebra

with the 3 basic lemma:

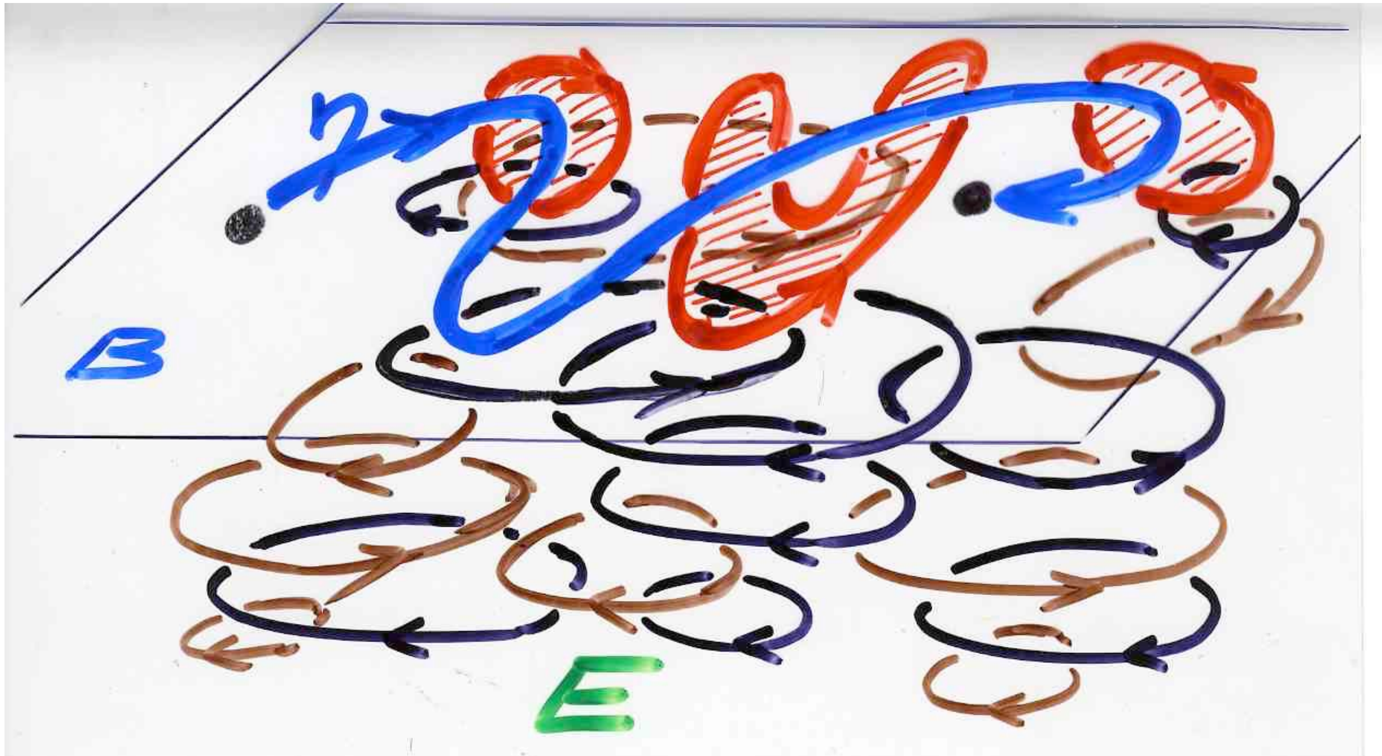
- Inversion lemma
- Logarithmic lemma
- circuit = heap of cycles

- MacMahon "master theorem"
  Cartier-Foata (1969)
- Matrix inversion
  Foata (1979)
- Jacobi identity (log det)
  Jackson (1977)
  Foata (1980)

- Cayley-Hamilton theorem
  Straubing (1983)
  Zeilberger (1985)
- Jacobi identity (duality)
  Lalonde (1990, 1996)
  Fomin (2001), Talaska (2012)

$$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{\mathrm{cof}_{ji}(I_k - A)}{\det(I_k - A)}$$

$$(1 - A)^{-1}$$

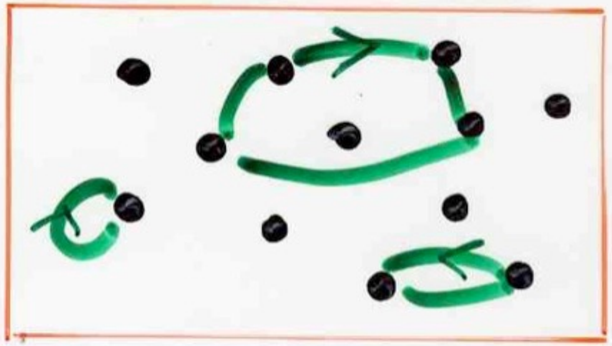


Prop. $$\sum_{\omega:\, i \rightsquigarrow j} v(\omega) = \frac{N_{ij}}{D} \qquad N_{ij} = \sum_{\eta} v(\eta) N_\eta$$

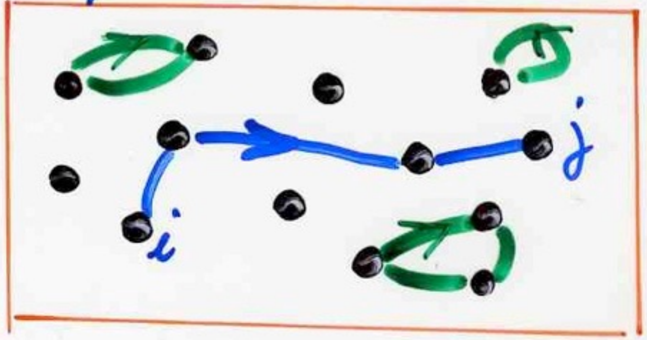
η self-avoiding path $i \rightsquigarrow j$

$$D = \sum_{\{\gamma_1, \dots, \gamma_r\}} (-1)^r v(\gamma_1) \dots v(\gamma_r)$$

$\{\gamma_1, \dots, \gamma_r\}$ 2 by 2 disjoint cycles



$$N_{ij} = \sum_{\{\eta;\, \gamma_1, \dots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \dots v(\gamma_r)$$



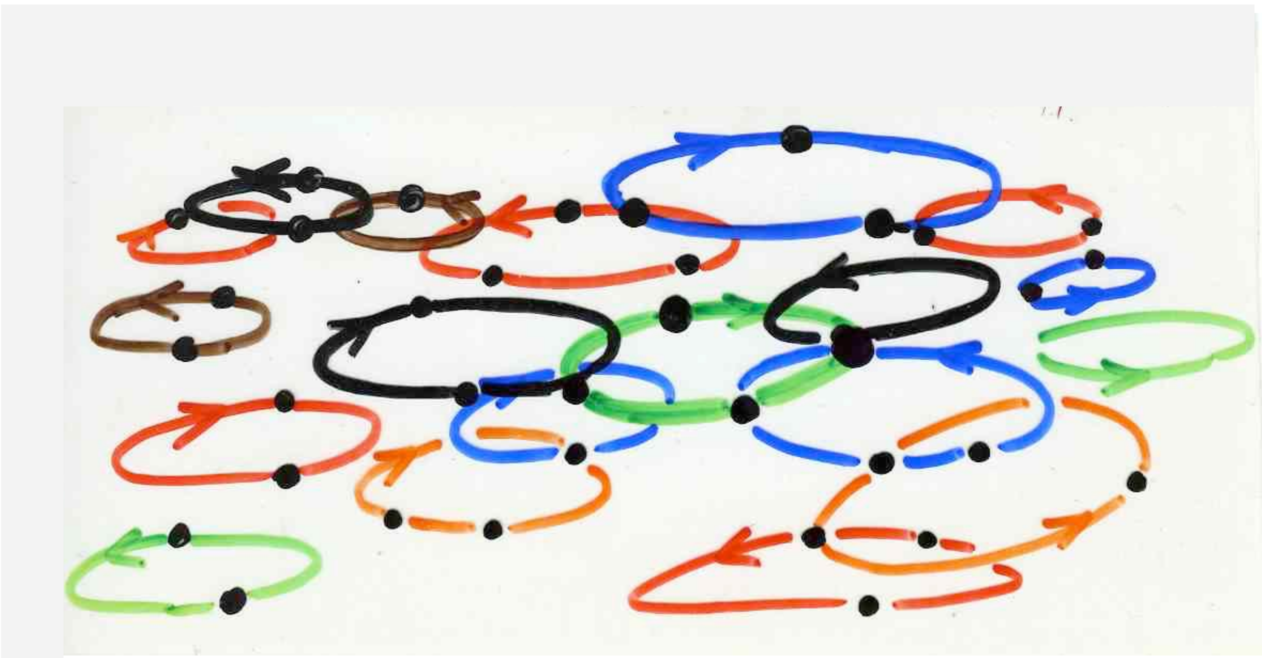
Jacobi identity

$$\log(\det(B)) = \mathrm{Tr}(\log(B))$$

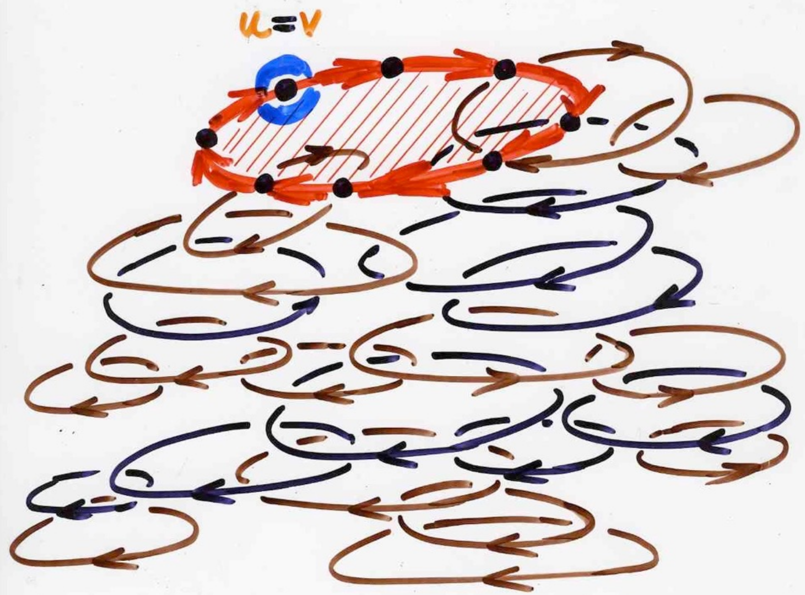
$$B = (1-A)^{-1}$$

$$\frac{1}{\det(I-A)} = \sum_{E} v(E)$$

E heap of cycles on $[1,k]$



Paths with no backtracking

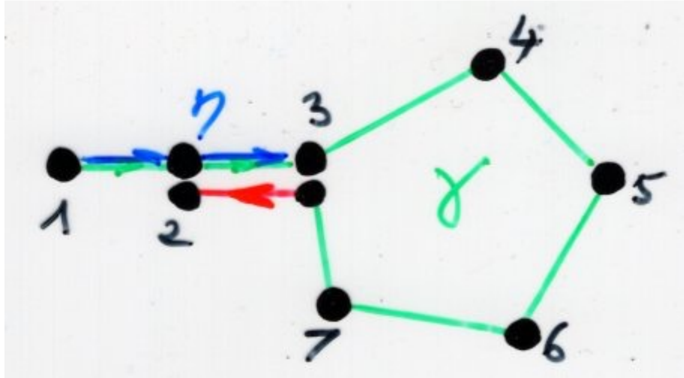


pointed cycles pyramid

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

- no tail
- no back tracking

no backtracking for ω $\Longleftrightarrow$ no cycle length 2 in E ?



$$\omega \xrightarrow{\chi} (\eta, E)$$

$$\omega \to (\underbrace{\bullet \to \bullet}_{\eta}, \underbrace{d \odot \gamma}_{E})$$

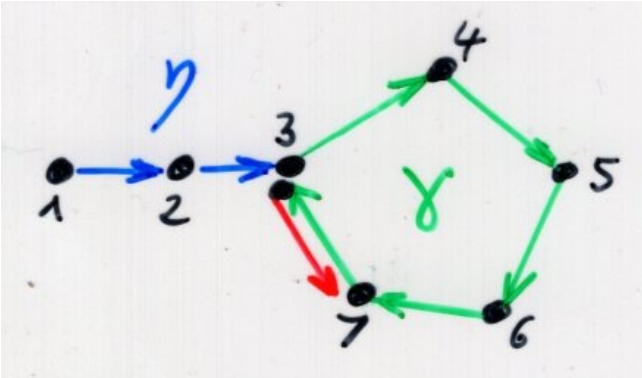
$$\omega = (1, 2, 3, 4, 5, 6, 7, 3, 2)$$

$$d = [2, 3, 2] \quad \text{cycle} \quad |d| = 2$$

$$\eta = (1, 2) \qquad \gamma = [3, 4, 5, 6, 7, 3]$$

no backtracking for ω

a cycle length **2** in E



$$\omega \xrightarrow{\cancel{\gamma}} (\eta, E)$$

$$\omega = (1, 2, 3, 4, 5, 6, 7, 3, 7)$$

$$E = (\gamma)$$

$$\eta = (1, 2, 3, 7)$$

$$\gamma = [3, 4, 5, 6, 7, 3]$$

a backtracking for ω

no cycle length **2** in E

second bijection

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

$\omega = (s_0, s_1, \ldots, s_i, \ldots, s_n)$

ω path of G

ω path on V

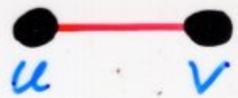
$\longrightarrow \vec{L}(\omega)$

$$\big((s_0, s_1), (s_1, s_2) \cdots, (s_i, s_{i+1}), \ldots, (s_{n-1}, s_n)\big)$$

path of $\vec{L}G$

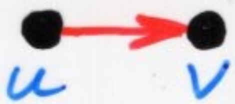
$G = (V, E)$

E

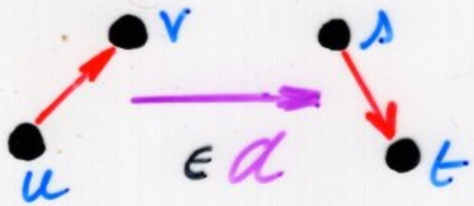


$\overrightarrow{LG} = (\vec{E}, \mathcal{A})$

$\vec{E}$



oriented line graph



$\iff v = s$

second bijection ψ

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

$\underset{u \rightsquigarrow v}{\eta}$ trail

trail = path having all oriented edges distinct

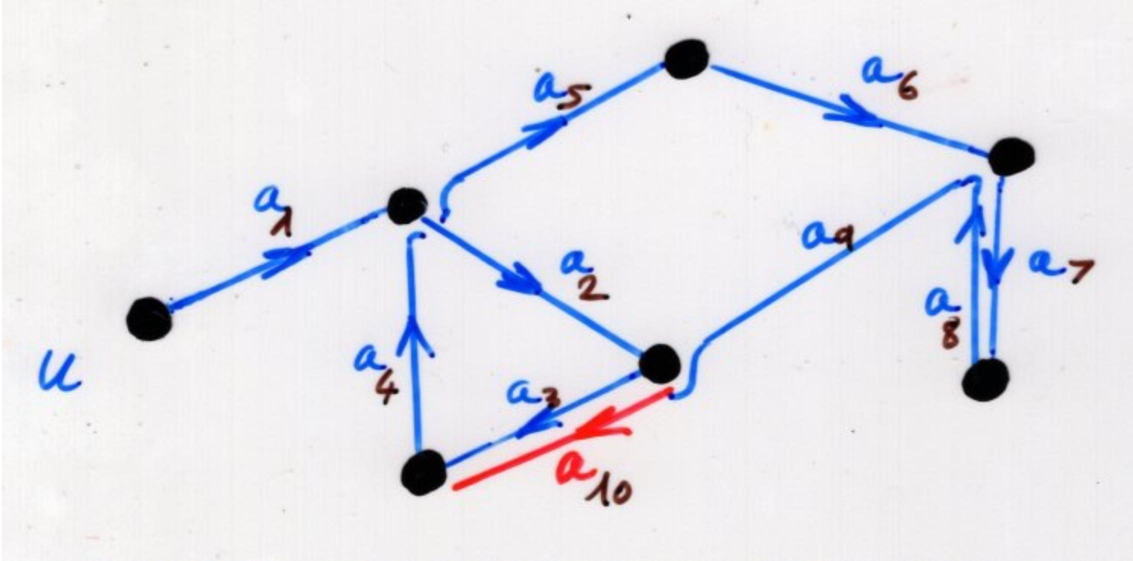
F heap of "oriented loops"

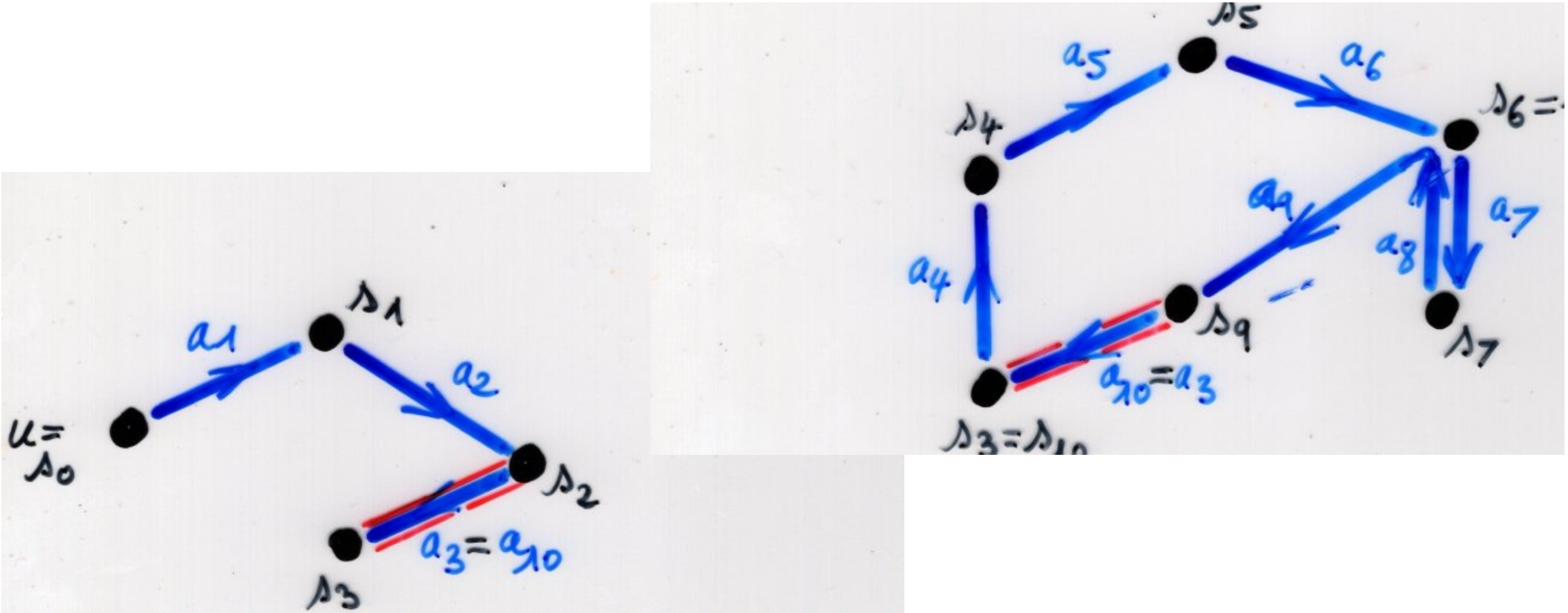
oriented loop

equivalence class of trail

trail $\underset{u \rightsquigarrow u}{\eta}$ up to a circular permutation of its edges

description of
the bijection φ





description of the bijection φ

ω path on V

$\omega = (s_0, \dots, s_n)$ $\quad u = s_0$, $v = s_n$

$\longrightarrow \vec{L}(\omega)$ $\quad \big((s_0, s_1), (s_1, s_2) \cdots, (s_i, s_{i+1}), \dots, (s_{n-1}, s_n)\big)$

$\vec{L}(\omega) = (e_1, \dots, e_n)$

$e_i = (s_{i-1}, s_i)$ oriented edges

at time $T=0$ - $\eta_0 = \emptyset$ $F = \emptyset$

- suppose $\omega_T = (s_0, \dots, s_T) \to (\eta_T, F_T)$

$\eta_T = (a_1, \dots, a_{i_T})$ trail going from s_0 to s_T

$a_1 = (u, s_{j_T})$

$F_T =$ heap of oriented loops $\pi(\max(\text{pieces}))$ intersect η

at time $T+1$, two cases

(i) (s_T, s_{T+1}) does not appear in η_T

then $\eta_{T+1} = \left(a_1, \dots, a_{i_T}, (s_T, s_{T+1}) \right)$

$F_{T+1} = F_T$

at time $T+1$, two cases

(ii) else $(s_T, s_{T+1}) = a_j$ edge of η_T

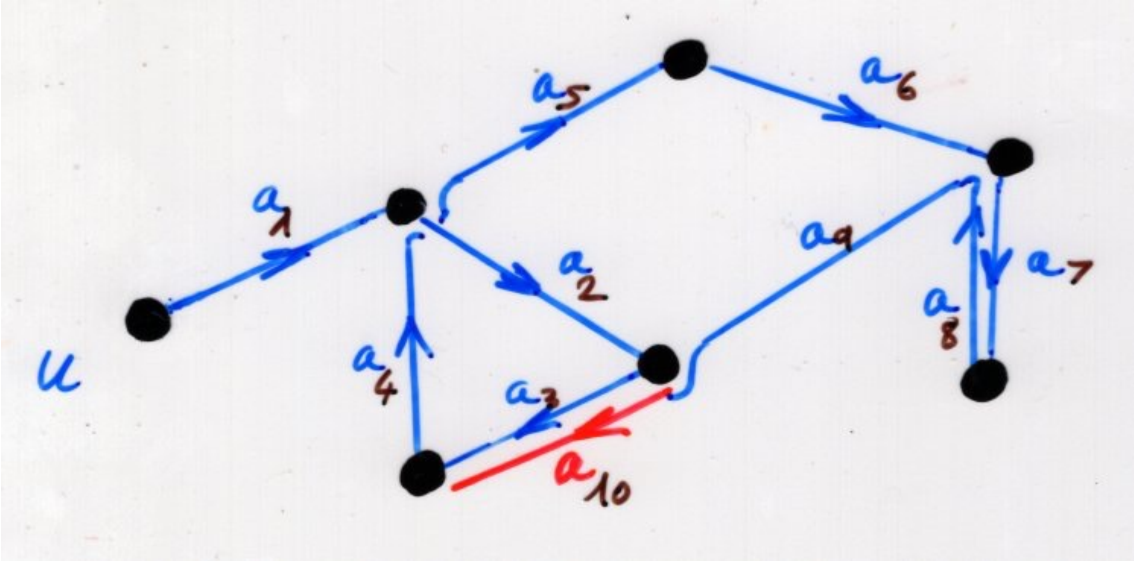
then $\eta_{T+1} = (a_1, \ldots, a_j)$

let $\Gamma_{T+1} = [\, a_j, \ldots, a_{i_T} \,]$

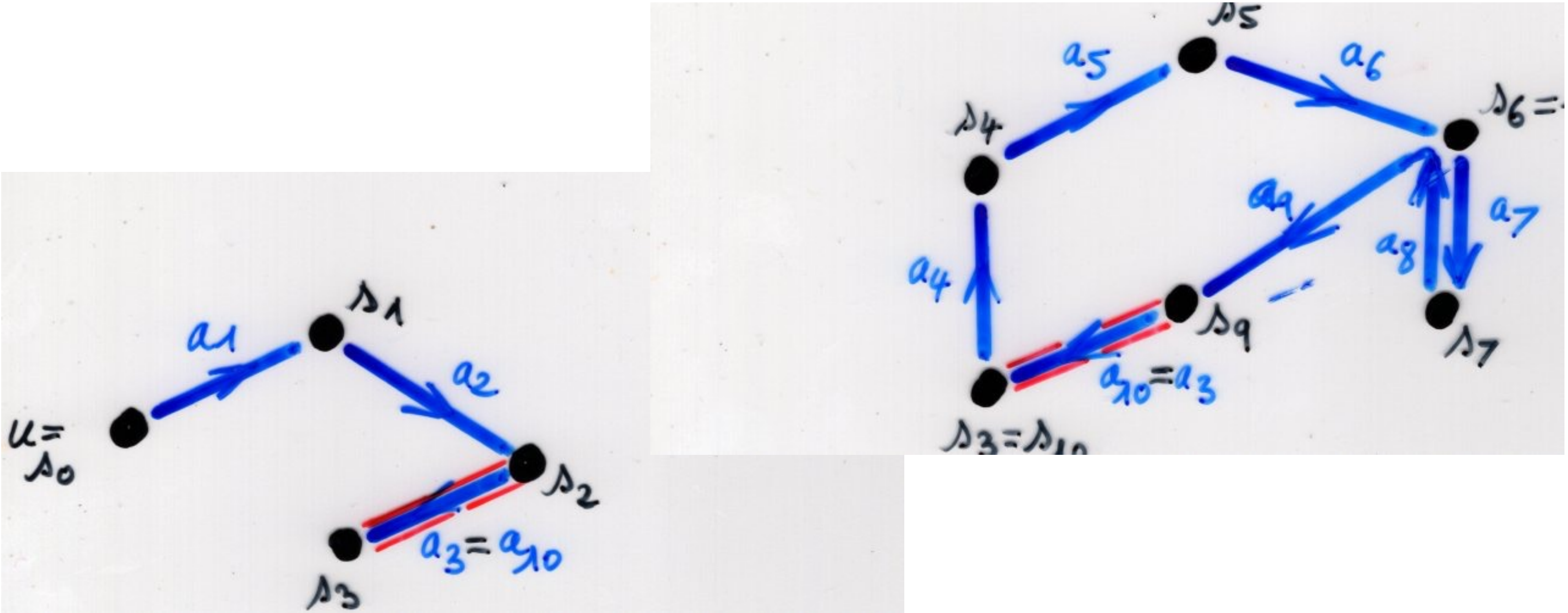
$F_{T+1} = \Gamma_{T+1} \odot F_T$

$\phi(\omega) = (\eta_n, F_n)$

$T = n$



a1
a2
a3
a4
a5
a6
a7
a8
a9
a10
u



u=s0
s1
s2
s3
a1
a2
a3=a10
s4
s5
s6=
s7
s9
s3=s10
a4
a5
a6
a7
a8
a9
a10=a3

Proposition

$$\underset{\text{path}}{\omega} \xrightarrow{\psi} (\eta, F)$$

ω (no tail
non backtracking

$\Longleftrightarrow$

- η is non backtracking, no tail

nd

- each oriented loops of F is non backtracking

Proof of the second formula for the zeta function of a graph

(ii) $\zeta_G(t) = \dfrac{1}{\det(1 - Ht)}$

$H = T - B$

T = adjacency matrix of the oriented line graph $\overrightarrow{LG} = (\vec{E}, a)$

$T = \left(t_{(i,j),(k,l)} \right)$

$t_{(i,j),(k,l)} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$

B submatrix of T

$B = \left(b_{(i,j)(k,l)} \right)$

$b_{(i,j),(j,i)} = \begin{cases} 1 \\ 0 & \text{else} \end{cases}$

backtracking

$$t \frac{d}{dt} \log \frac{1}{\det(1 - \mathbf{H}t)}$$

= generating function

(by number of edges)

pointed pyramids of non backtracking oriented loops

pointed = one of the edge of the maximal piece is pointed

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

- no tail
- no back tracking

$$t\frac{d}{dt}\log\frac{1}{\det(1-Ht)} = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

$$t\frac{d}{dt}\log Z_G(t)$$

- no tail
- no backtracking

The third formula for zeta

(iii)

$$\zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I - tA + t^2(D-I)\right)}$$

$$G = (V, E)$$

$m = |E|$ number of edges

$n = |V|$ number of vertices

$A = (a_{ij})$ incidence matrix of G

D diagonal matrix

$$D = (d_{ii})$$

$$d_{ii} = \deg v_i$$

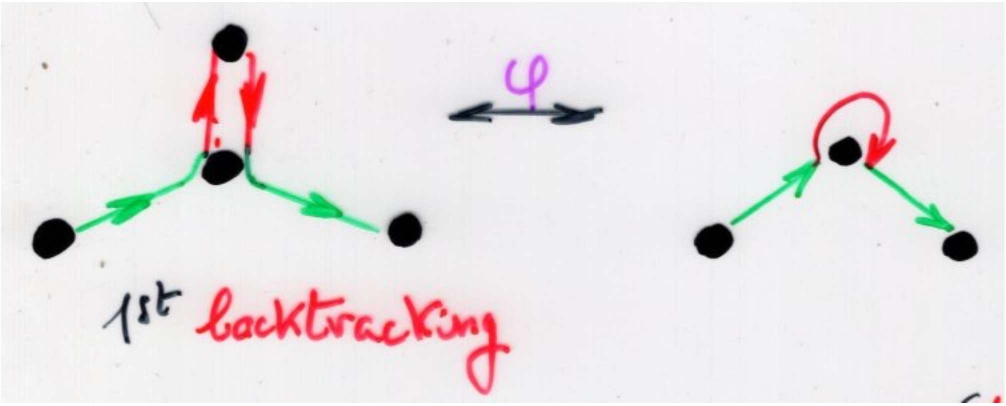
$$V = \{v_1, \ldots, v_n\}$$

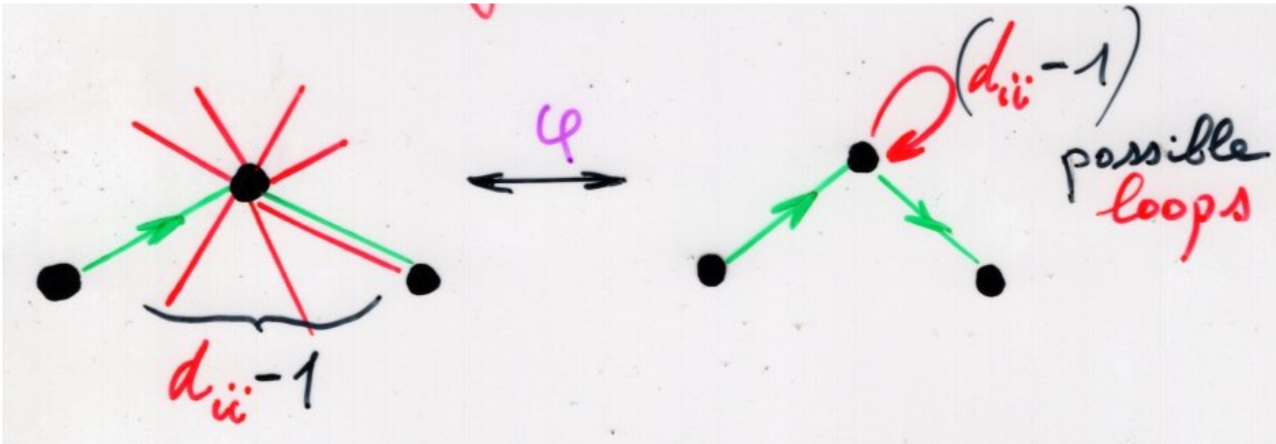
$$t \frac{d}{dt} \log \frac{1}{\det\left(I - tA + t^2(D - I)\right)} = \sum_{\substack{\omega \\ \text{circuit}}} v(\omega)$$

$$V = \{1, .., n\}$$

$$\begin{cases} v(i,j) = t \\ v(i,i) = -t^2\left((\deg i) - 1\right) \end{cases}$$

The idea ...





Back to number theory

$$\zeta(s) = \prod_{\substack{p \\ \text{prime} \\ \text{number}}} \left(\frac{1}{1-p^{-s}} \right)$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

some "prime" over the graph G

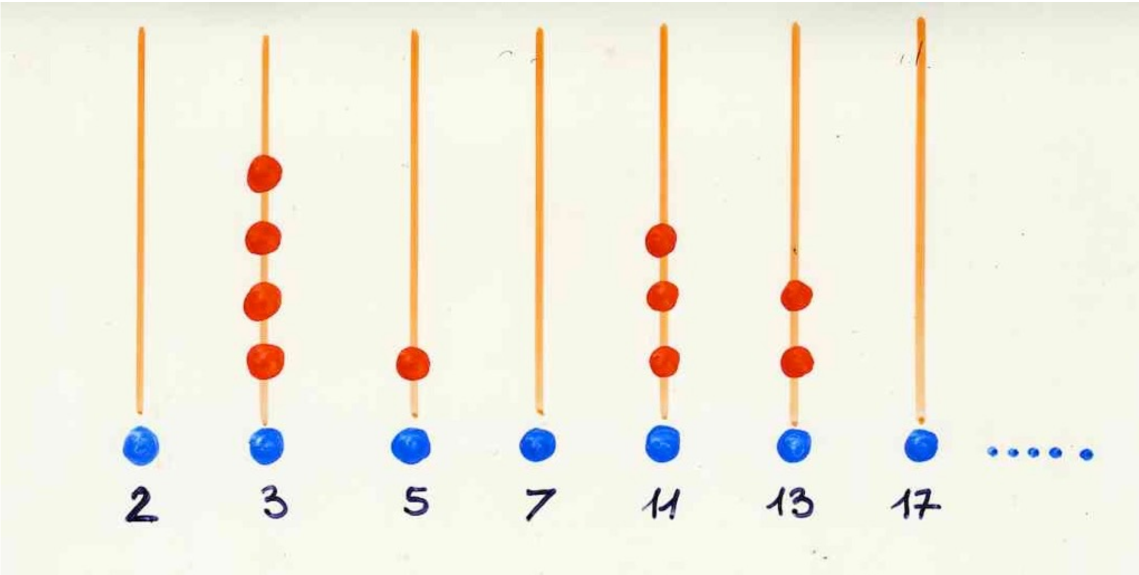
$$\zeta(s) = \prod_{\substack{p \\ \text{prime} \\ \text{number}}} \left(\frac{1}{1-p^{-s}} \right)$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class
prime circuit

no backtracking

$$\sum_{n\geqslant 1} n^{-s} = \left(\sum_{n\geqslant 1} \mu(n)\, n^{-s}\right)^{-1}$$

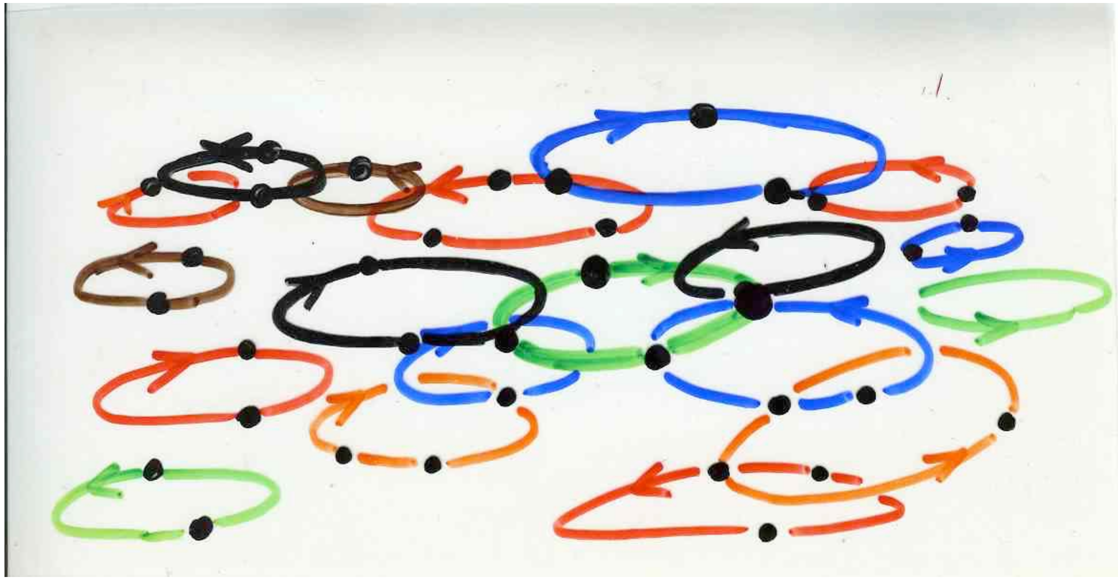


$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Euler identity

$$\zeta(s) = \prod_{\substack{p \\ \text{prime} \\ \text{number}}} \left(\frac{1}{1-p^{-s}}\right)$$

$$\sum_{n\geq 1} n^{-s} = \left(\sum_{n\geq 1} \mu(n)\, n^{-s}\right)^{-1}$$



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{\left(1 - t^{|C|}\right)}$$

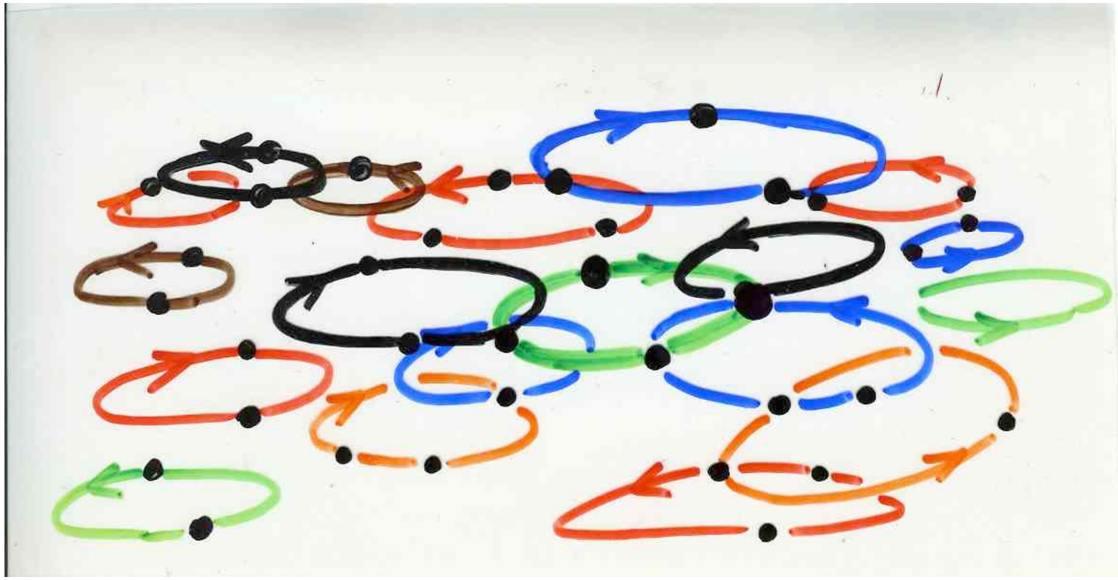
equivalence class

prime

circuit

$$\zeta_G(t) = \frac{1}{\det(I-A)}$$

Giscard, Rochet (2016)
extending number theory to paths on Graphs

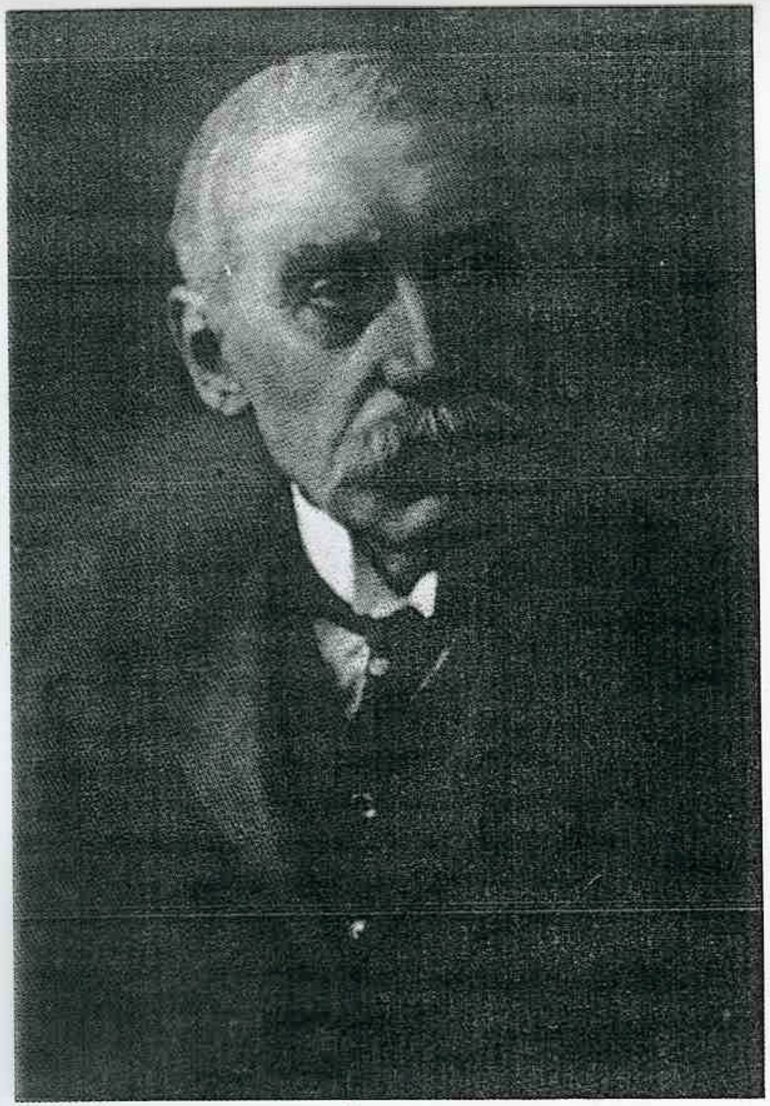


$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{\left(1-t^{|C|}\right)}$$

equivalence class
prime circuit

MacMahon Master theorem



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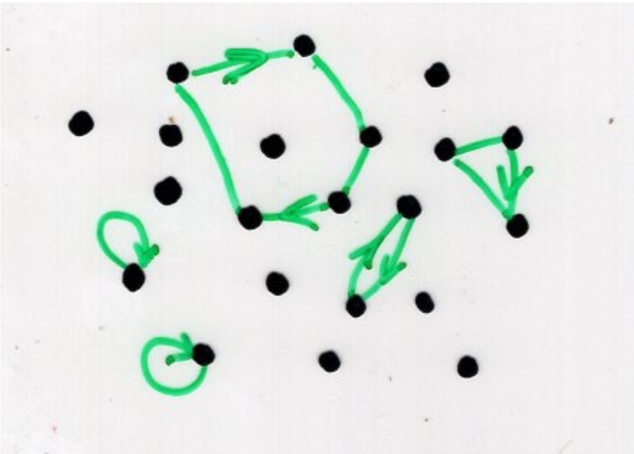
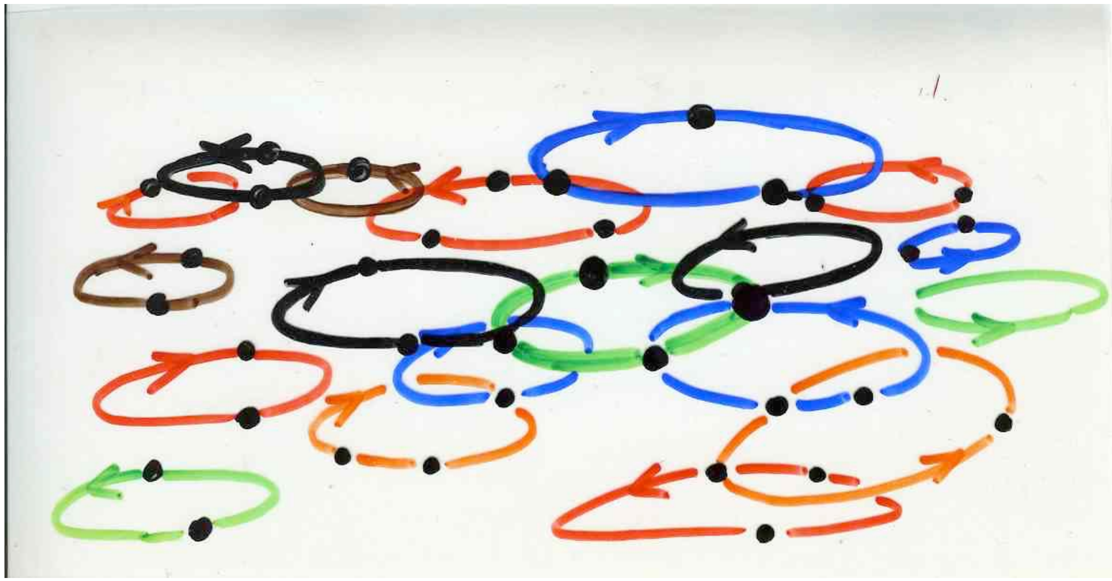
$$\frac{1}{\det(I-A)}$$

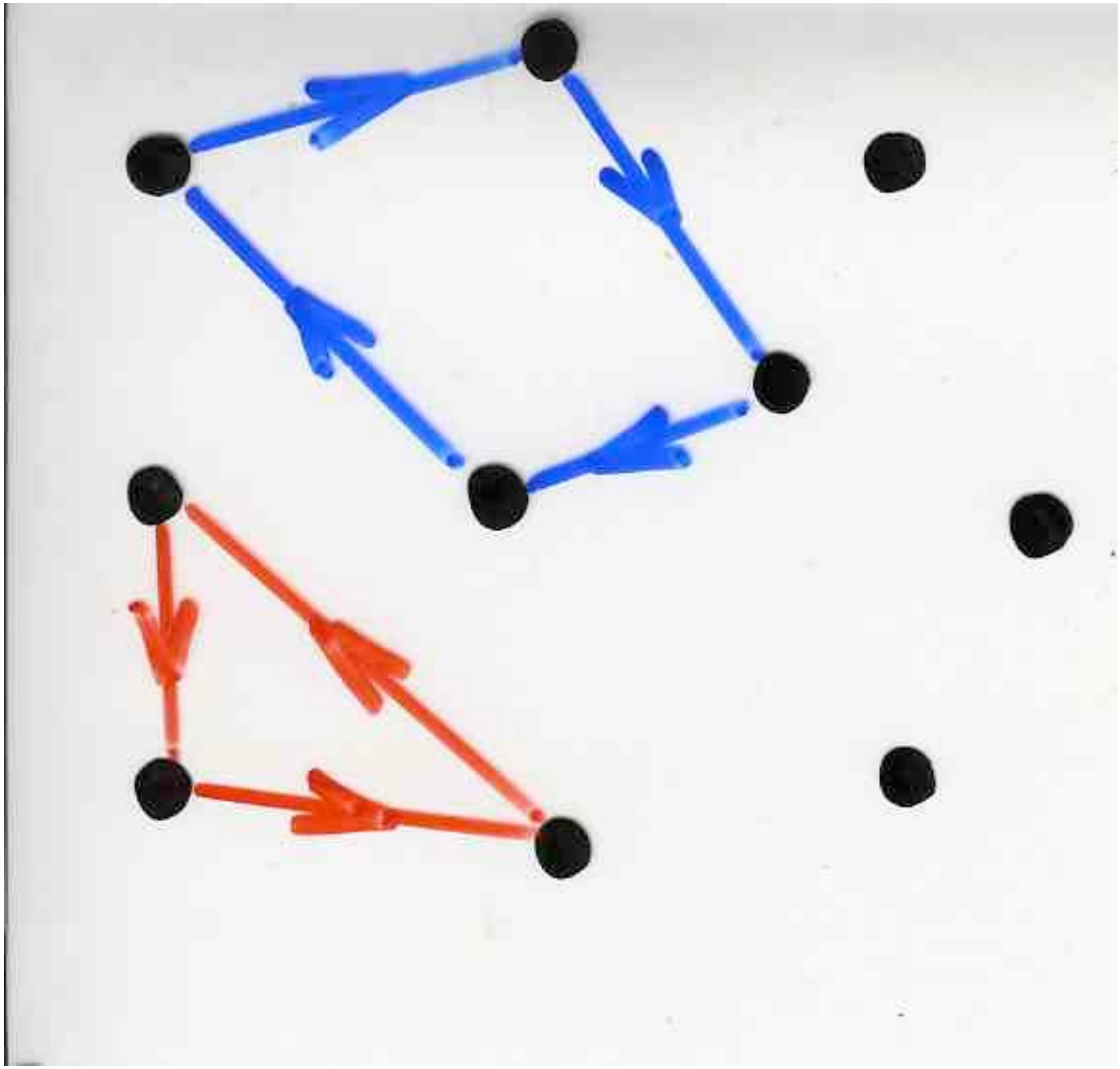
inversion lemma

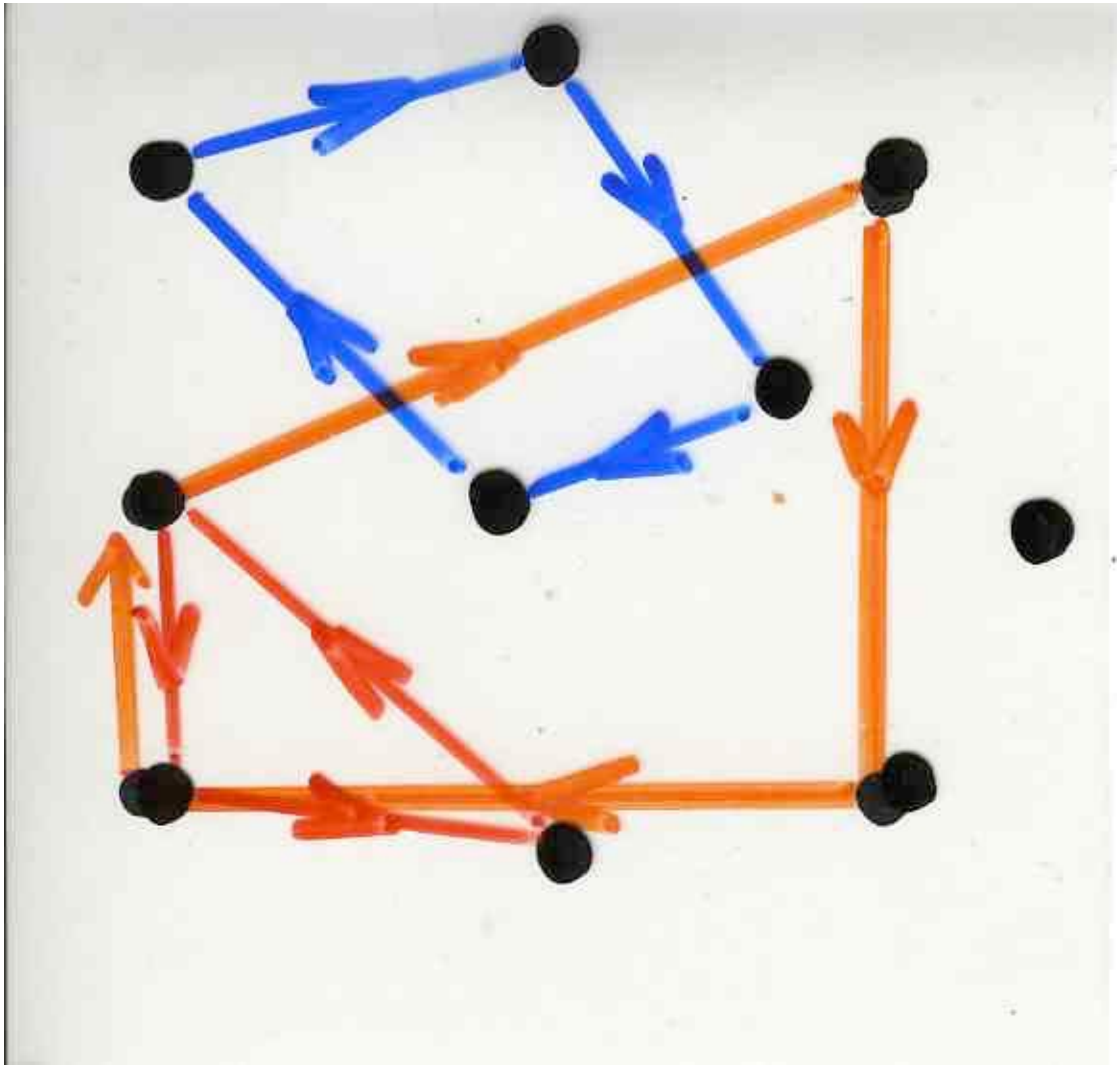
$$\frac{1}{\det(I-A)} = \sum_{E} v(E)$$

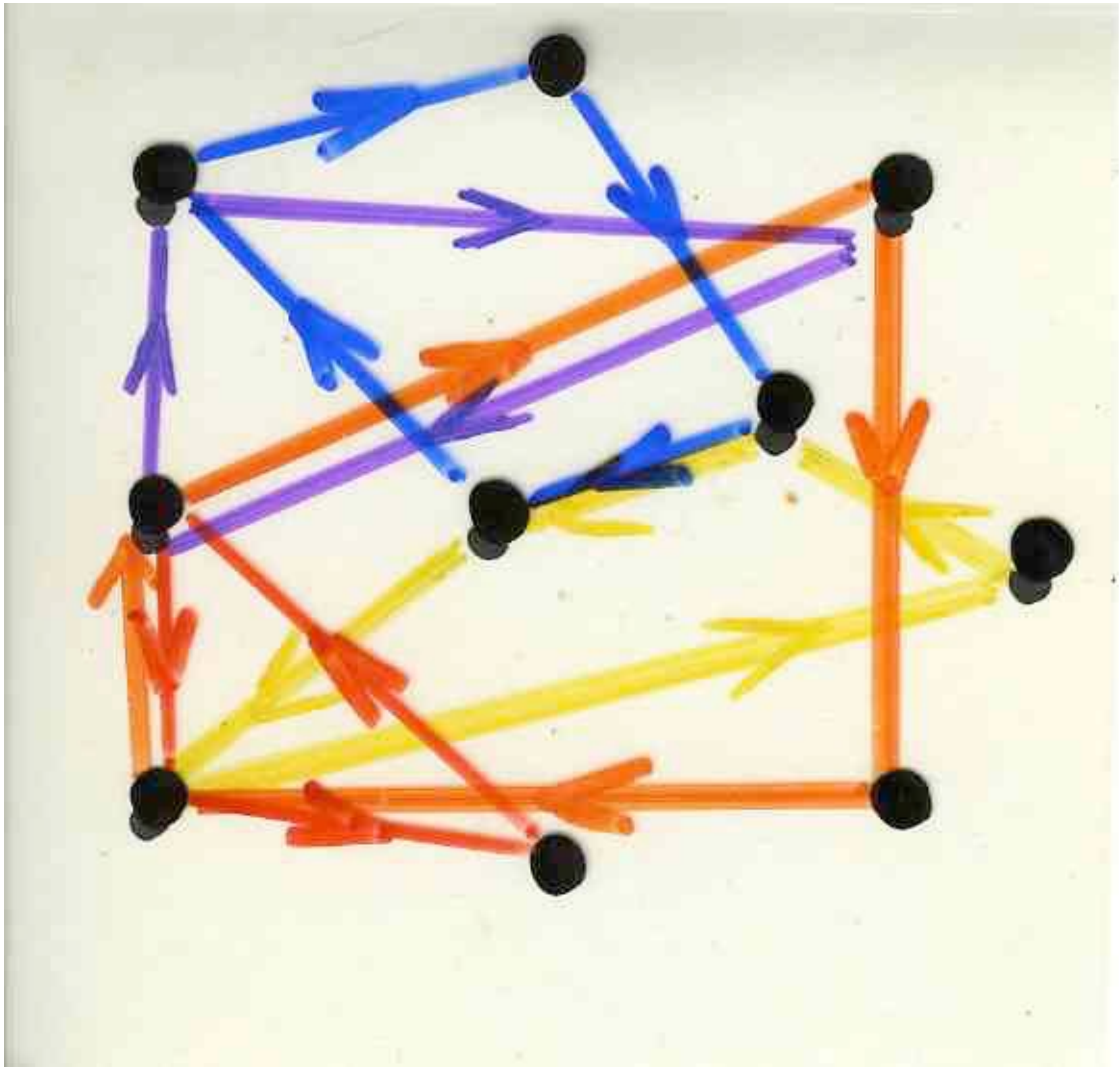
E heap of cycles on $[1,k]$

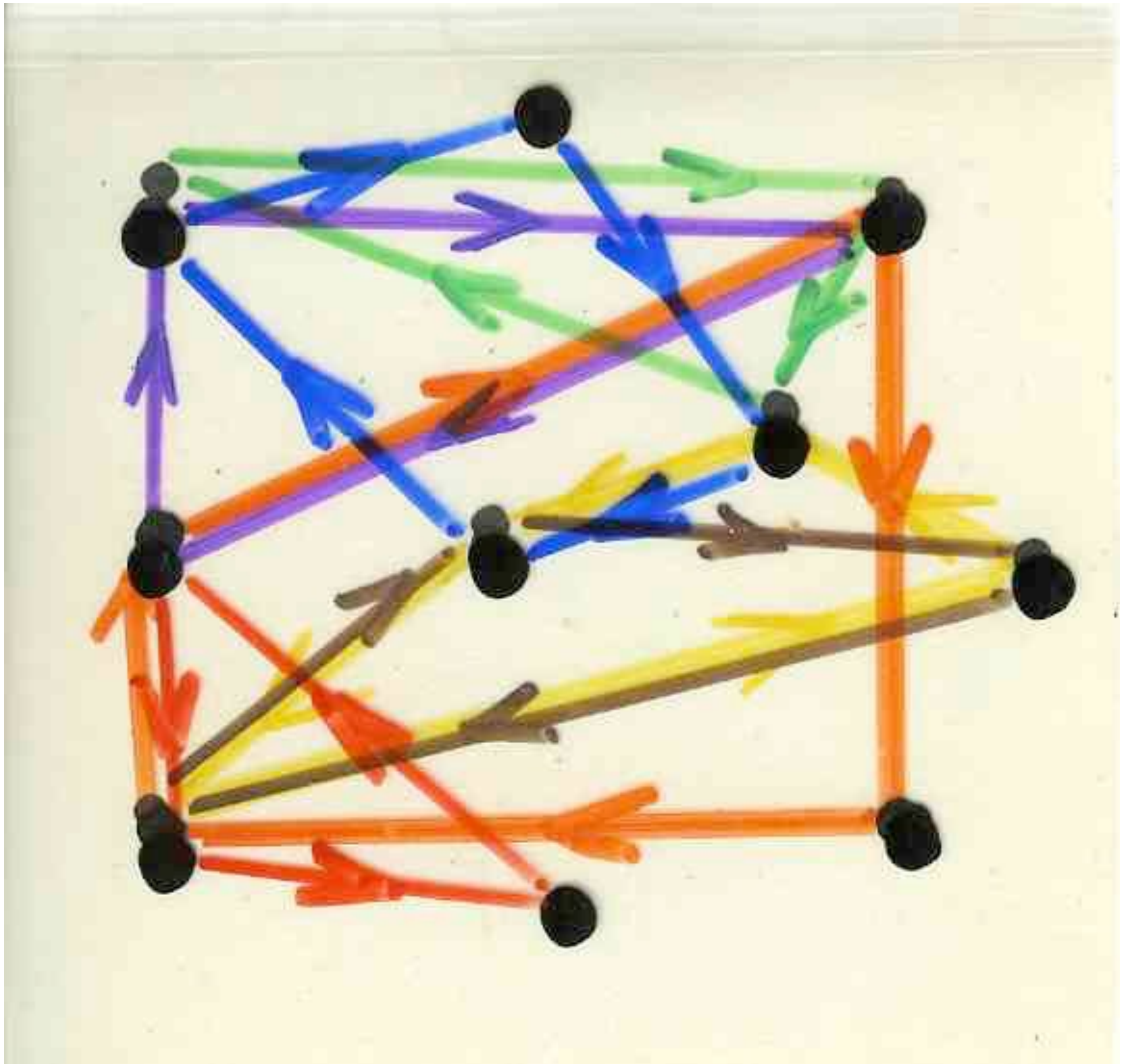


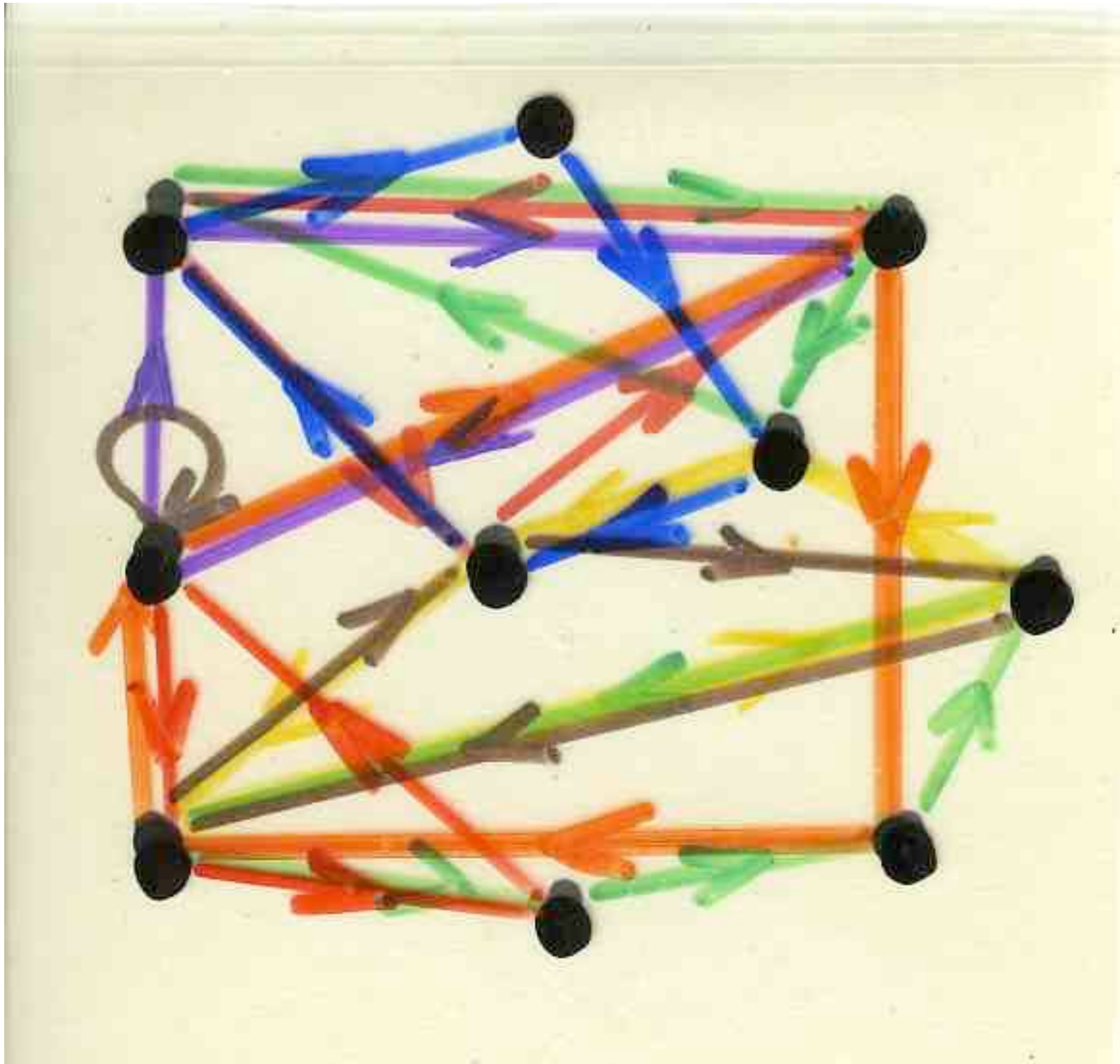






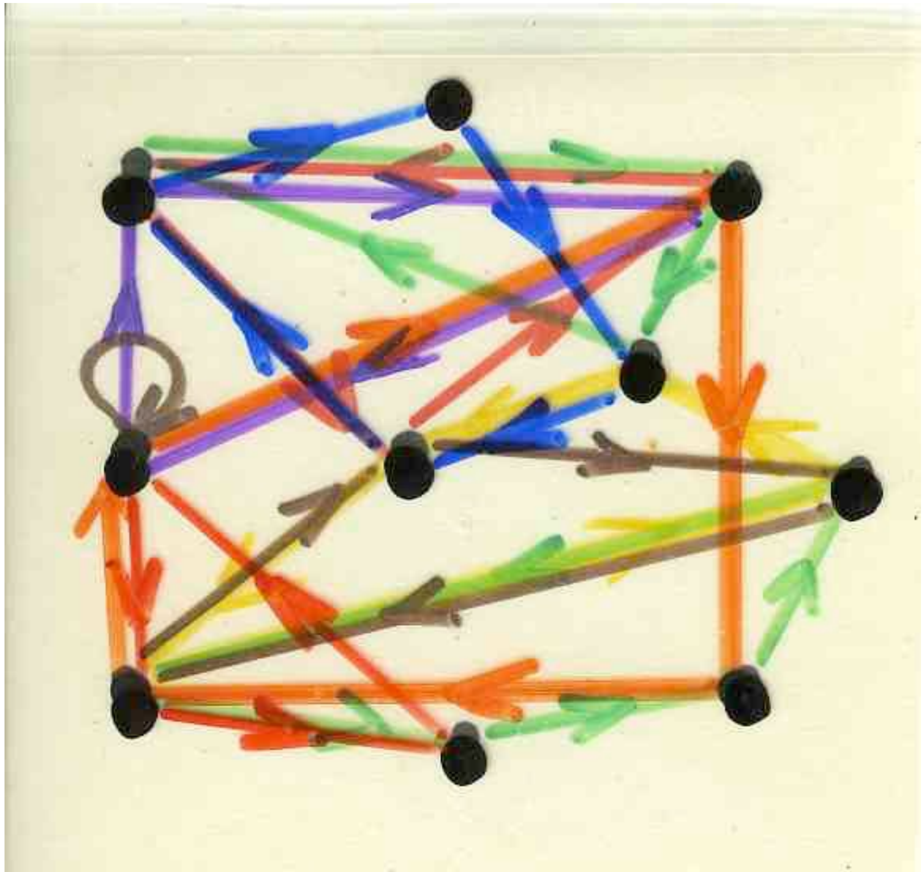






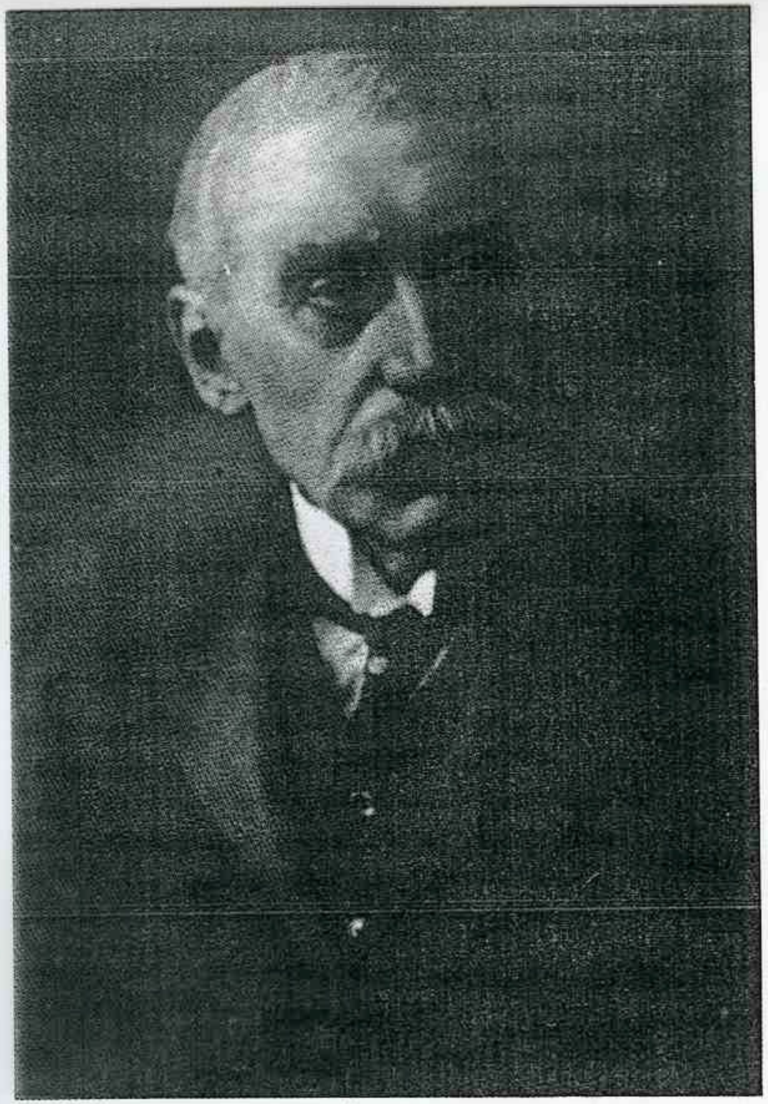
$$\frac{1}{\det(I-A)} = \sum_{E} v(E)$$

E heap of cycles on $[1,k]$



$$= \sum_{\Phi} v(\Phi)$$

Φ rearrangements on $[1,k]$



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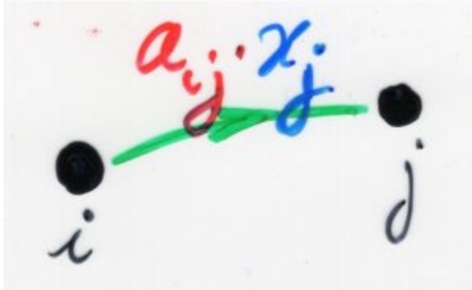
$$\frac{1}{\det(I-A)} = \sum_{\Phi} v(\Phi)$$

Φ rearrangements on $[1,k]$

Where is my MASTER THEOREM ?

MacMahon master theorem



$$A = (a_{ij})_{n\times n}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$AX = (a_{ij}\, x_j)_{n\times n}$$

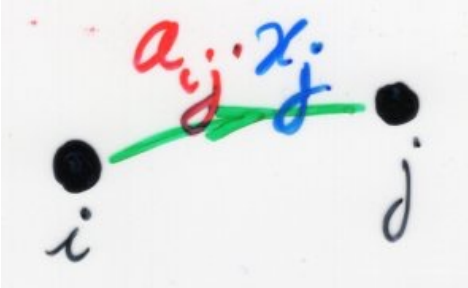
$$Y_i = \sum_{j=1}^{n} a_{ij}\, x_j$$

$$\frac{1}{\det(I - AX)}$$

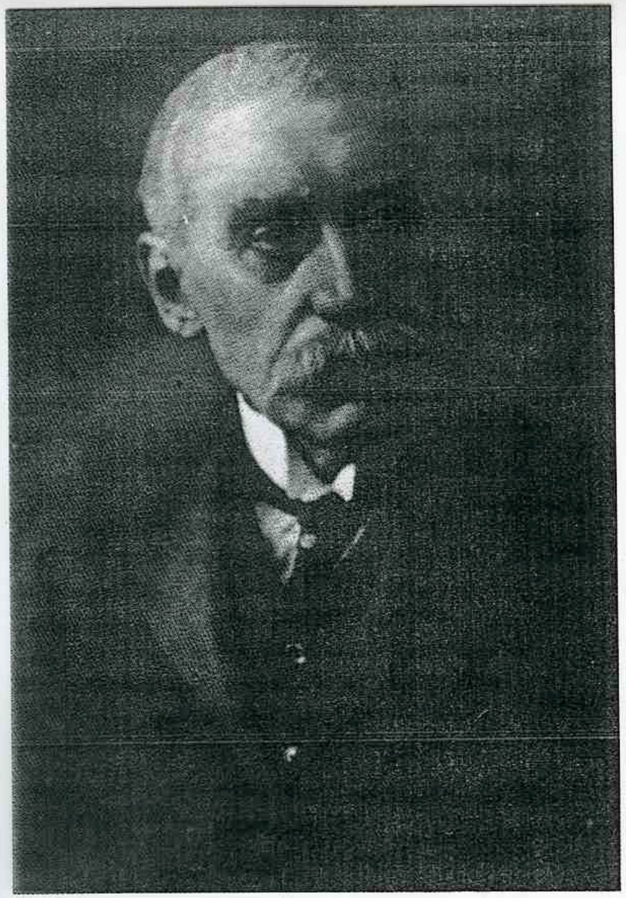
coeff. de $x_1^{\alpha_1} \cdots\cdots x_n^{\alpha_n}$

$$Y_1^{\alpha_1} \cdots\cdots Y_n^{\alpha_n}$$

MacMahon master theorem



The coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ in $\dfrac{1}{\det(I - AX)}$
is the same as
the coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$
in $Y_1^{\alpha_1} \dots Y_n^{\alpha_n}$



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ॐ सरस्वत्यै नमः।

The Art of Bijective Combinatorics

Part II, commutations and heaps of pieces

(video-book, course IMSc Chennai, 2017)

mirror website

www.viennot.org

www.imsc.res.in/~viennot

Chapter 4 Heaps and linear algebra

Chapter 5 Heaps and algebraic graph theory

Chapter 5b, zeta function of a graph

Chapter 6 Heaps and Coxeter groups