Proofs without words:

the example of Ramanujan continued fraction

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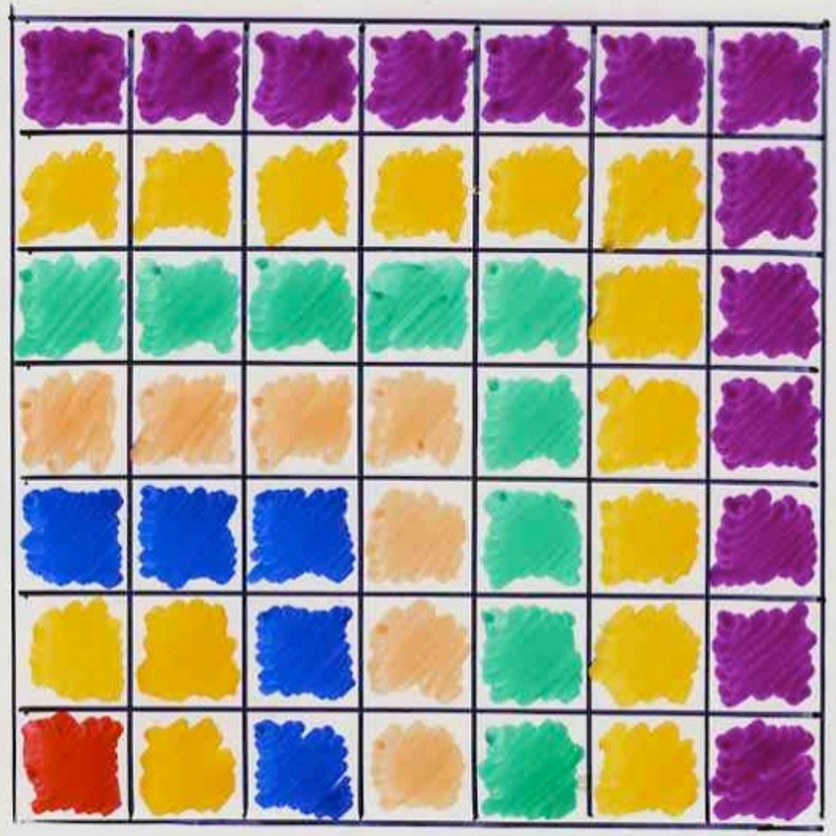
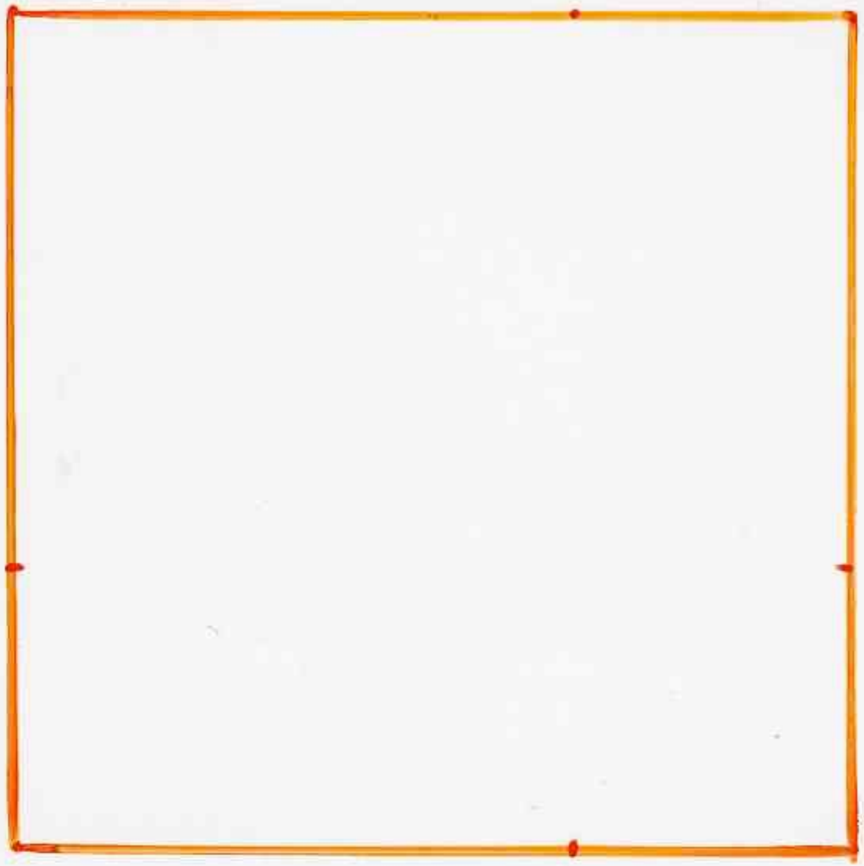
bijective proof of an identity

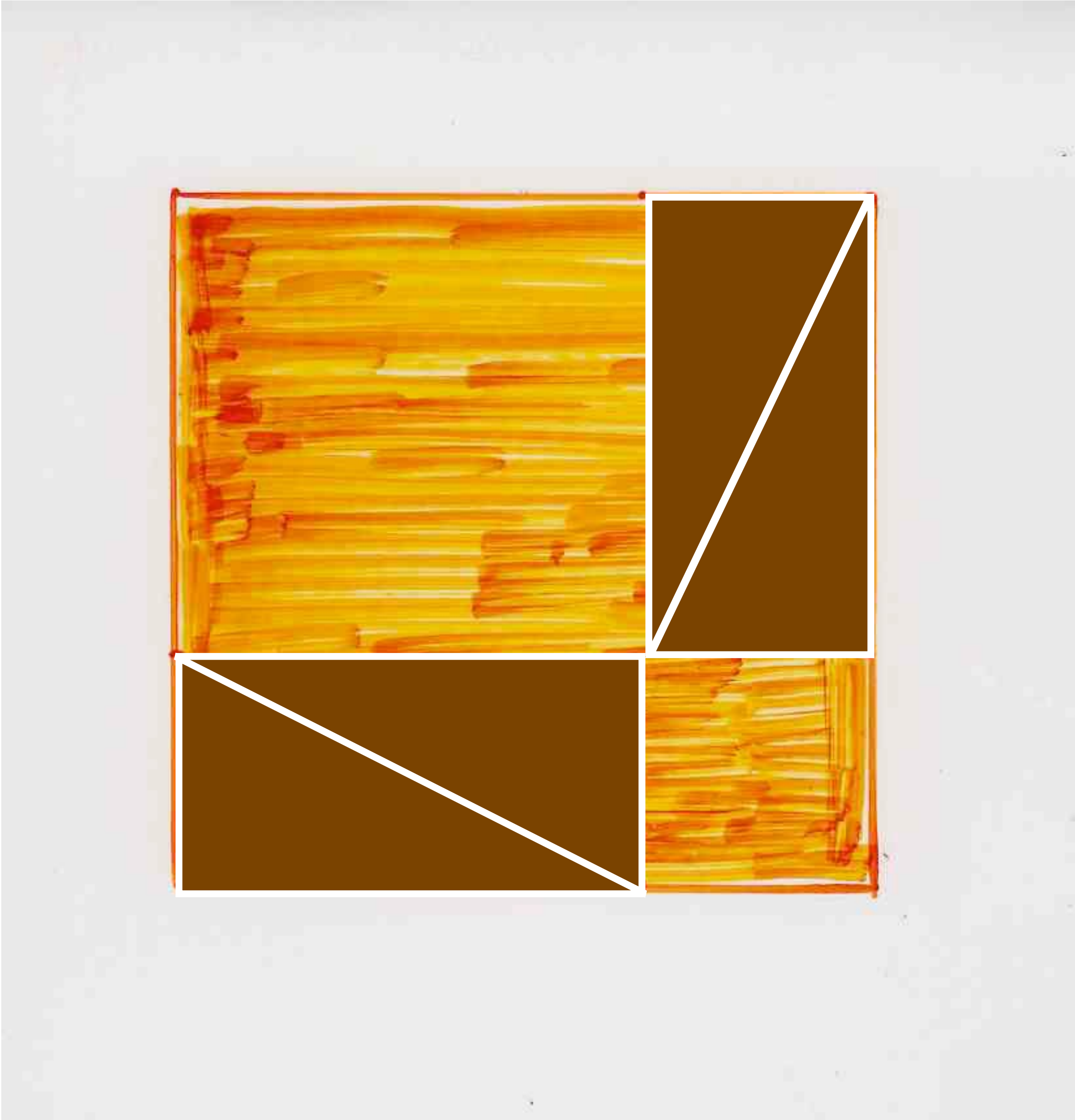


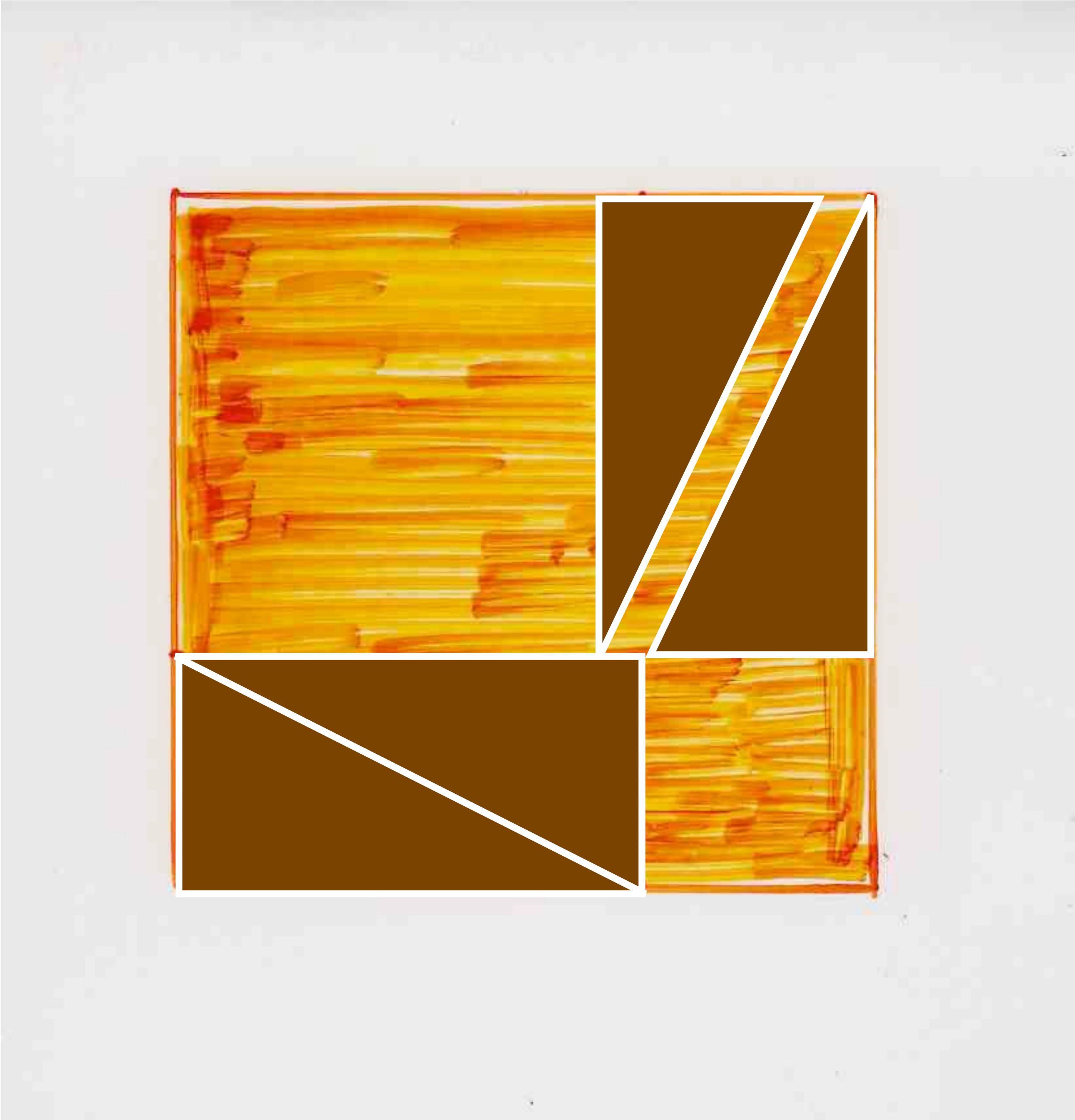
$$n^2 = 1 + 3 + \ldots + (2n-1)$$

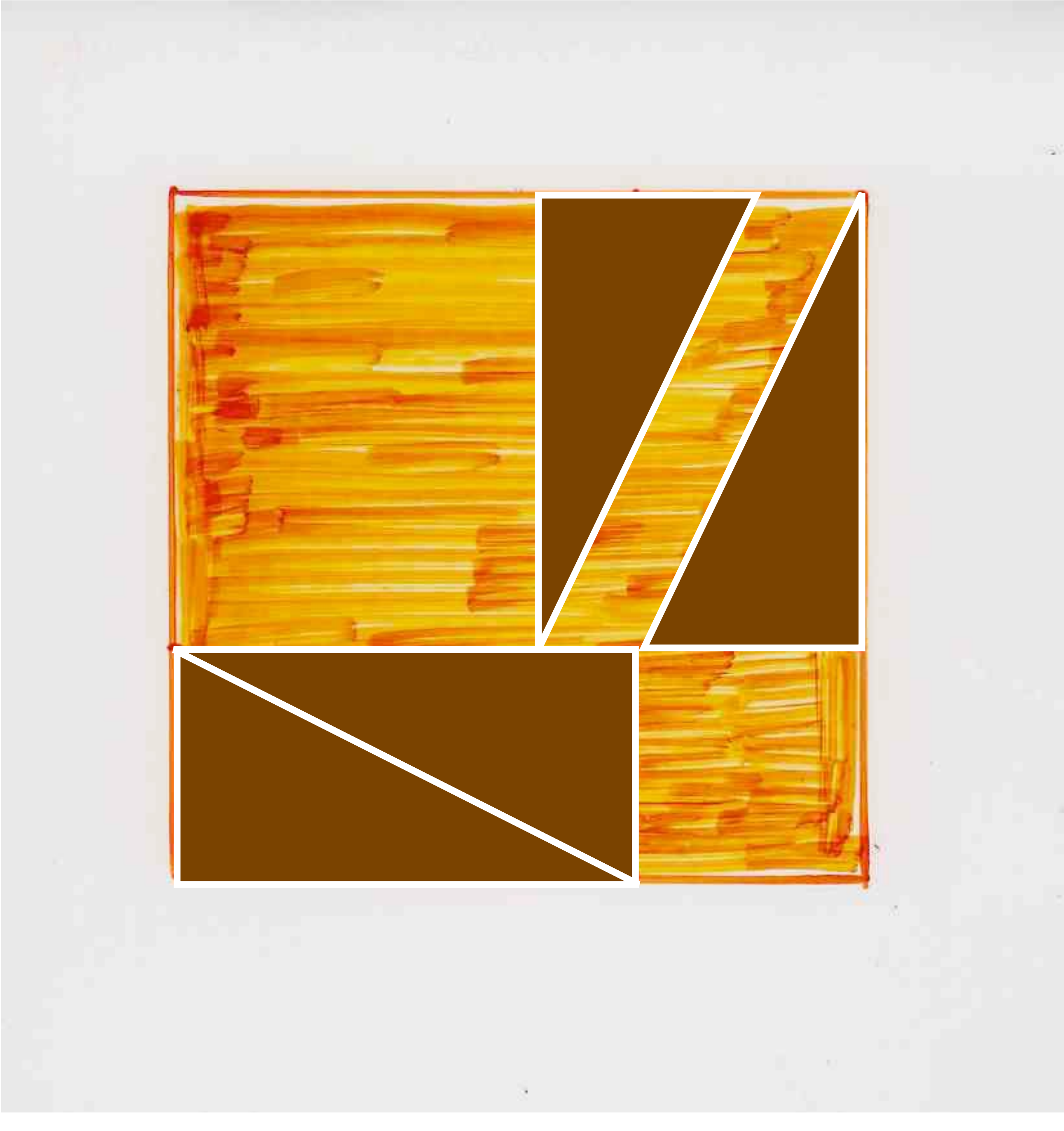
« visual proof »

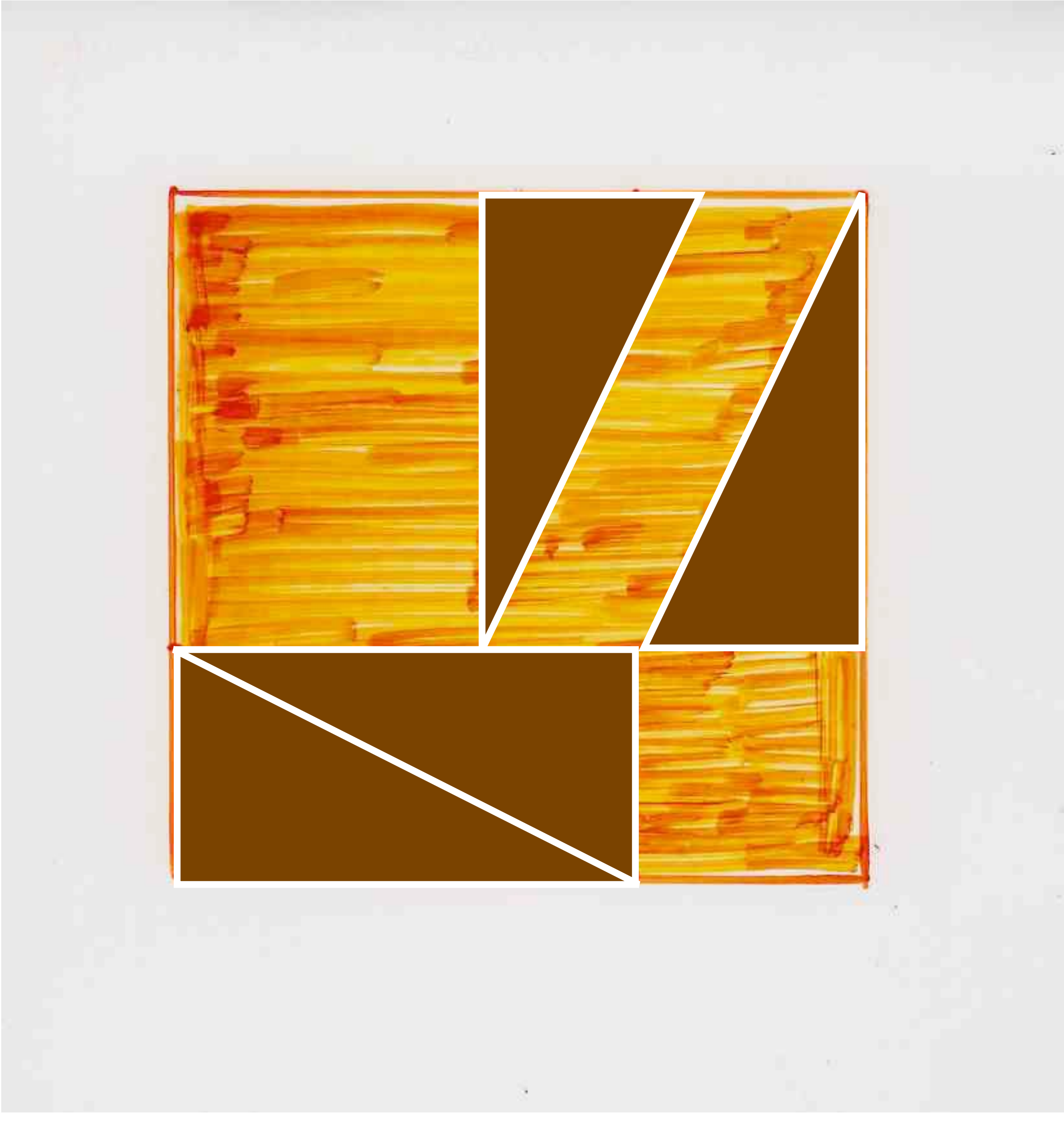
Pythagoras ....

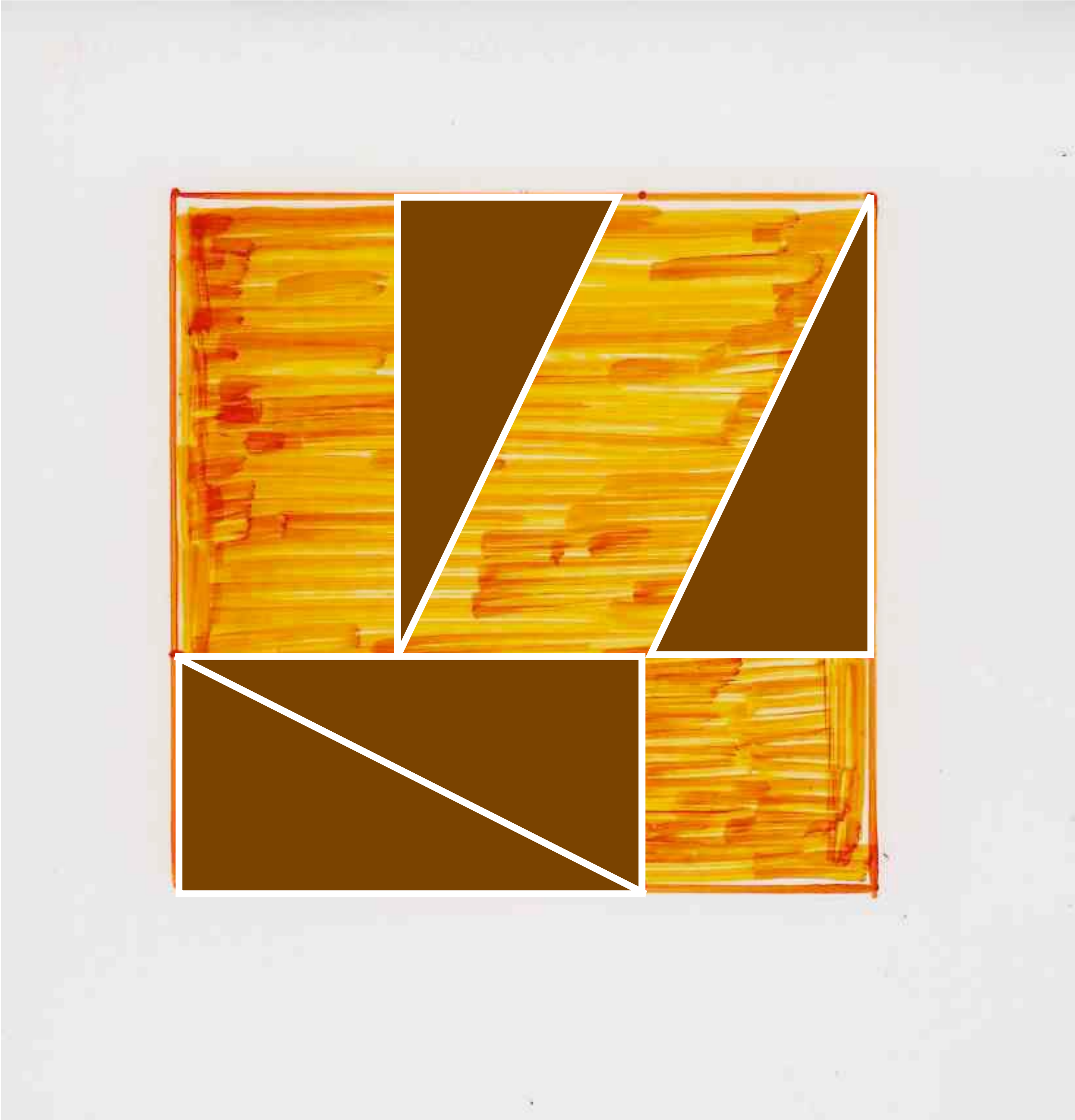


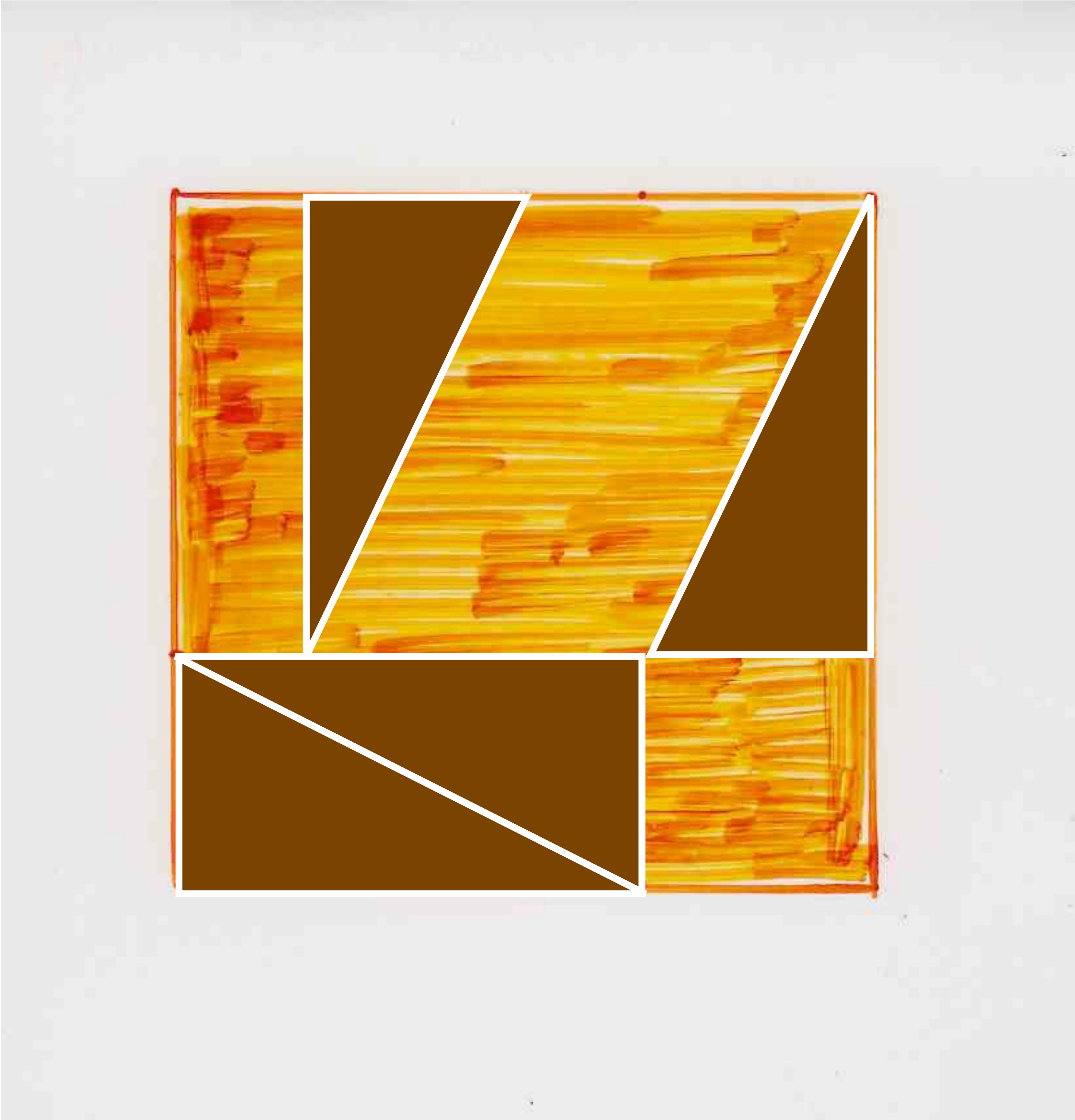


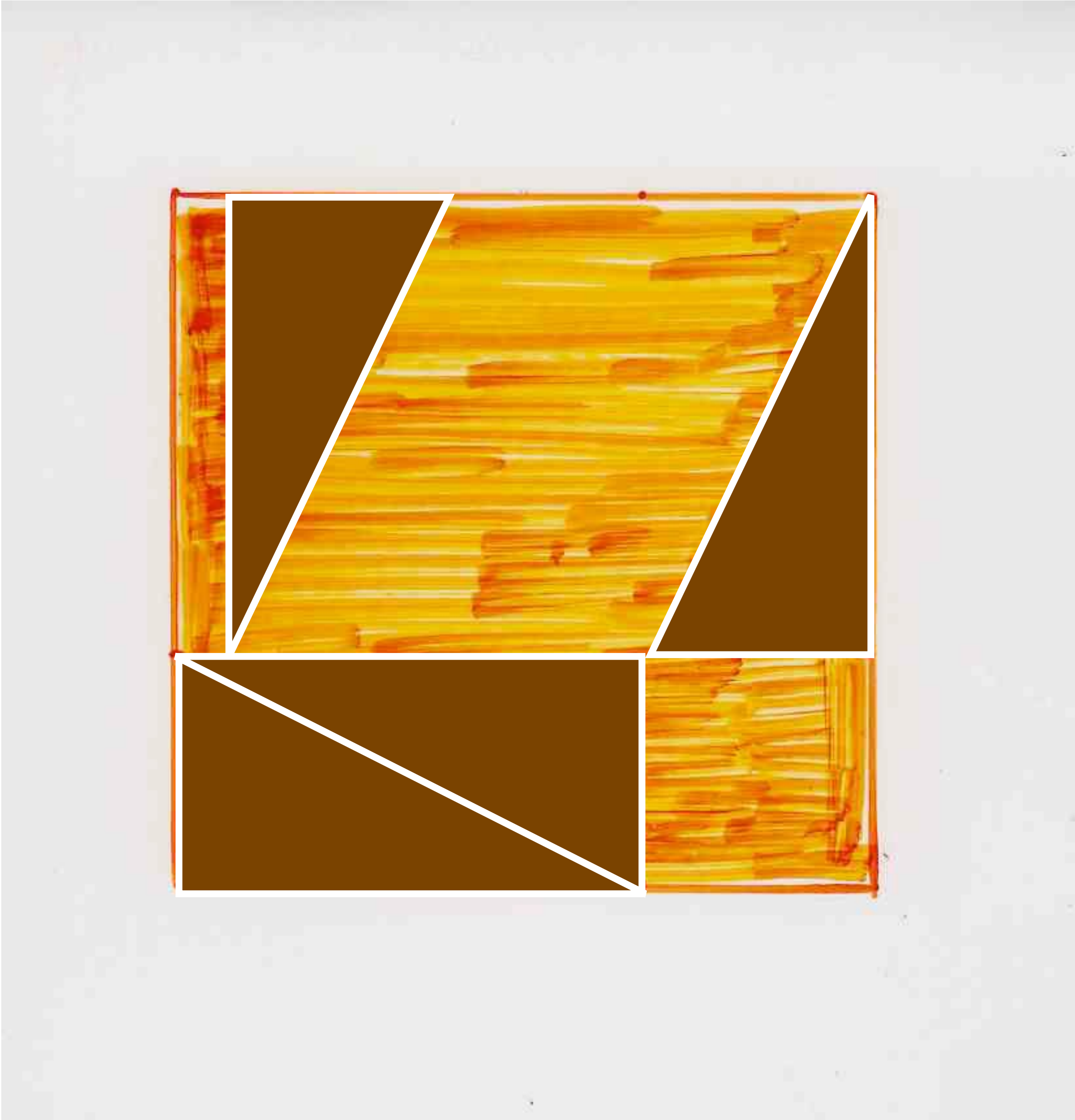


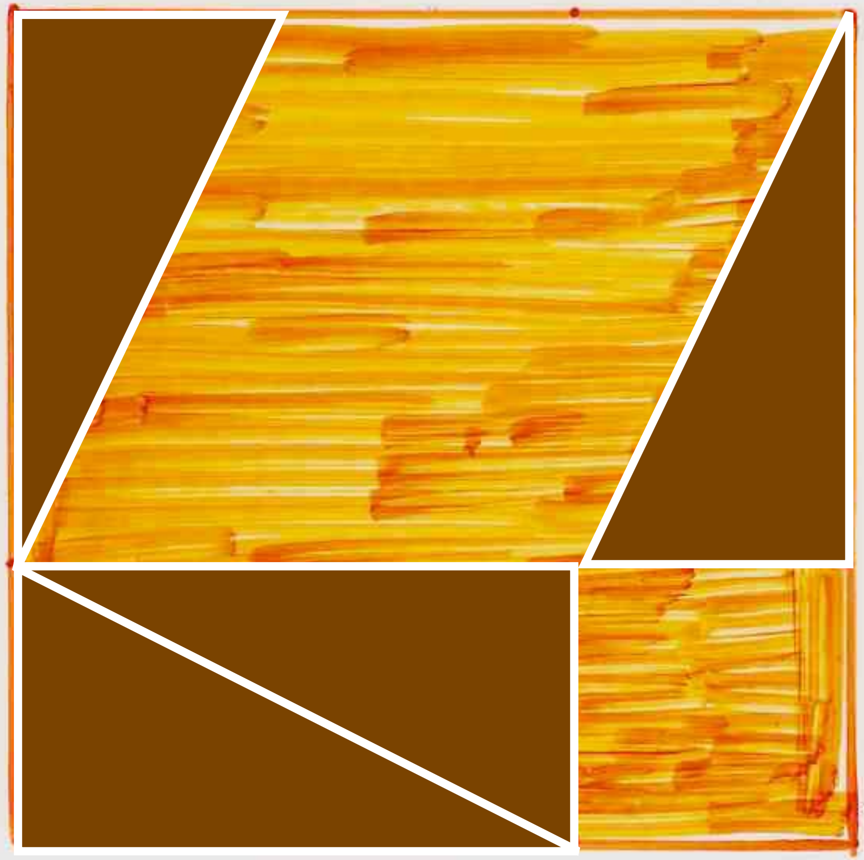


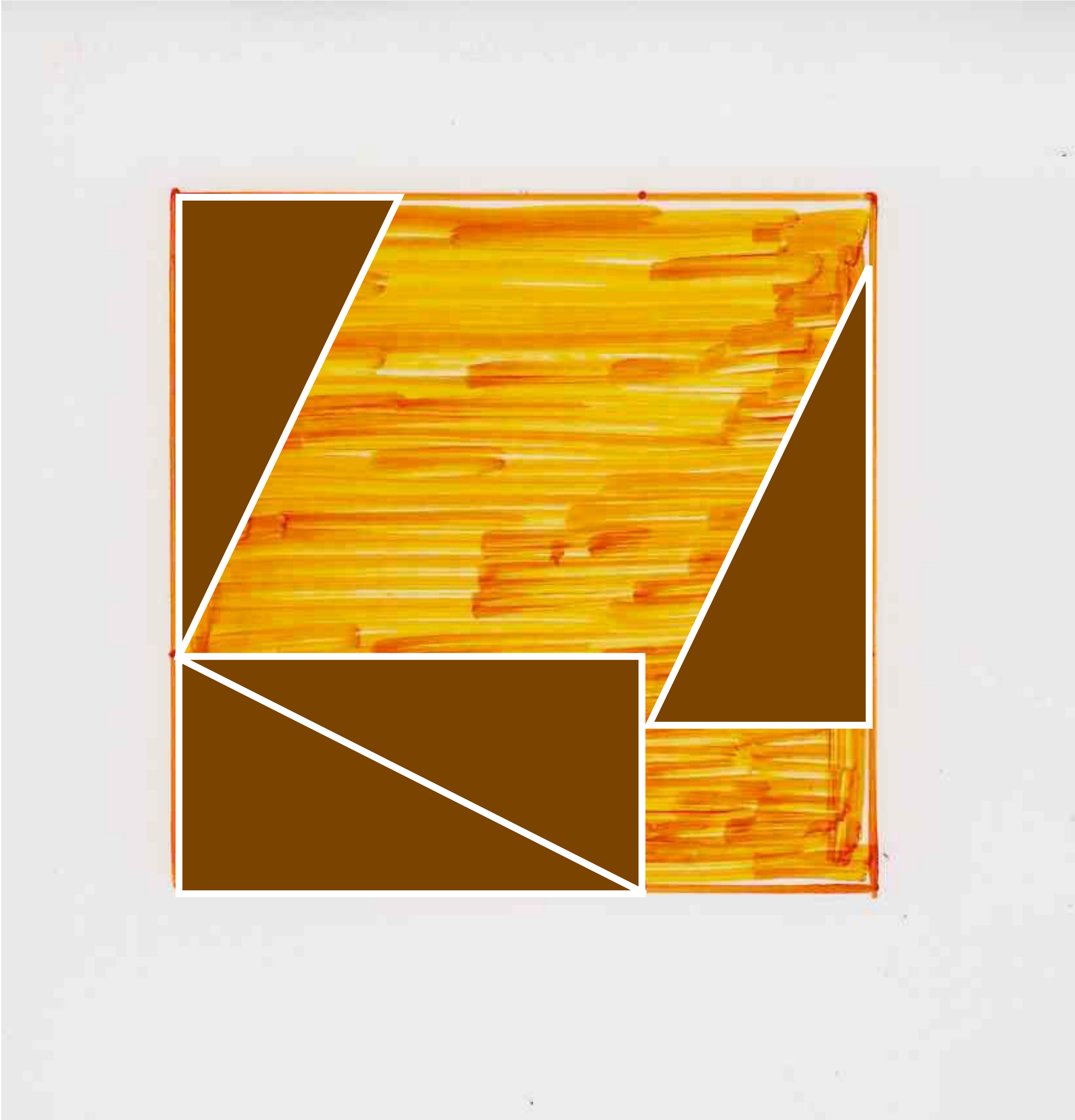


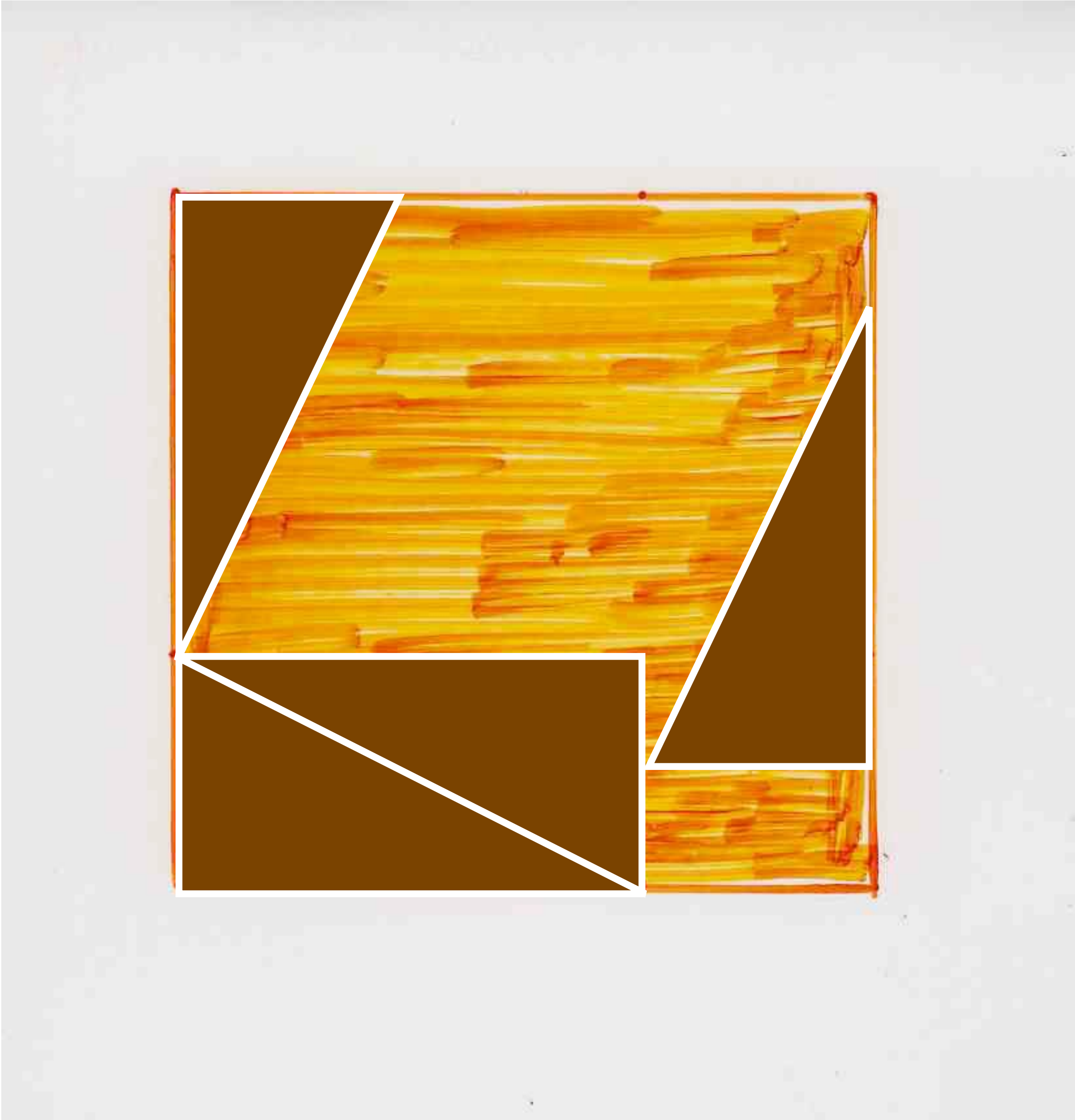


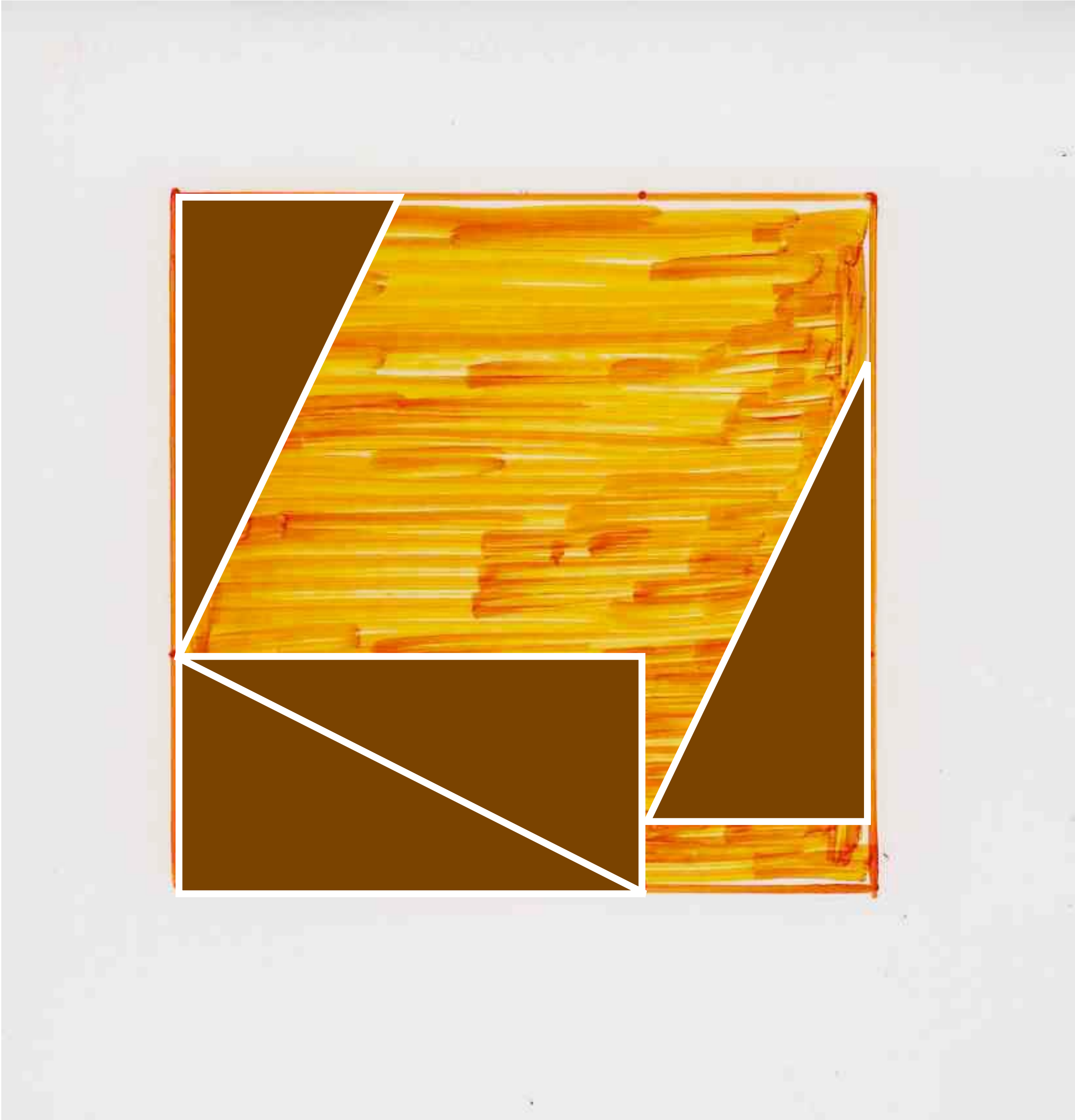


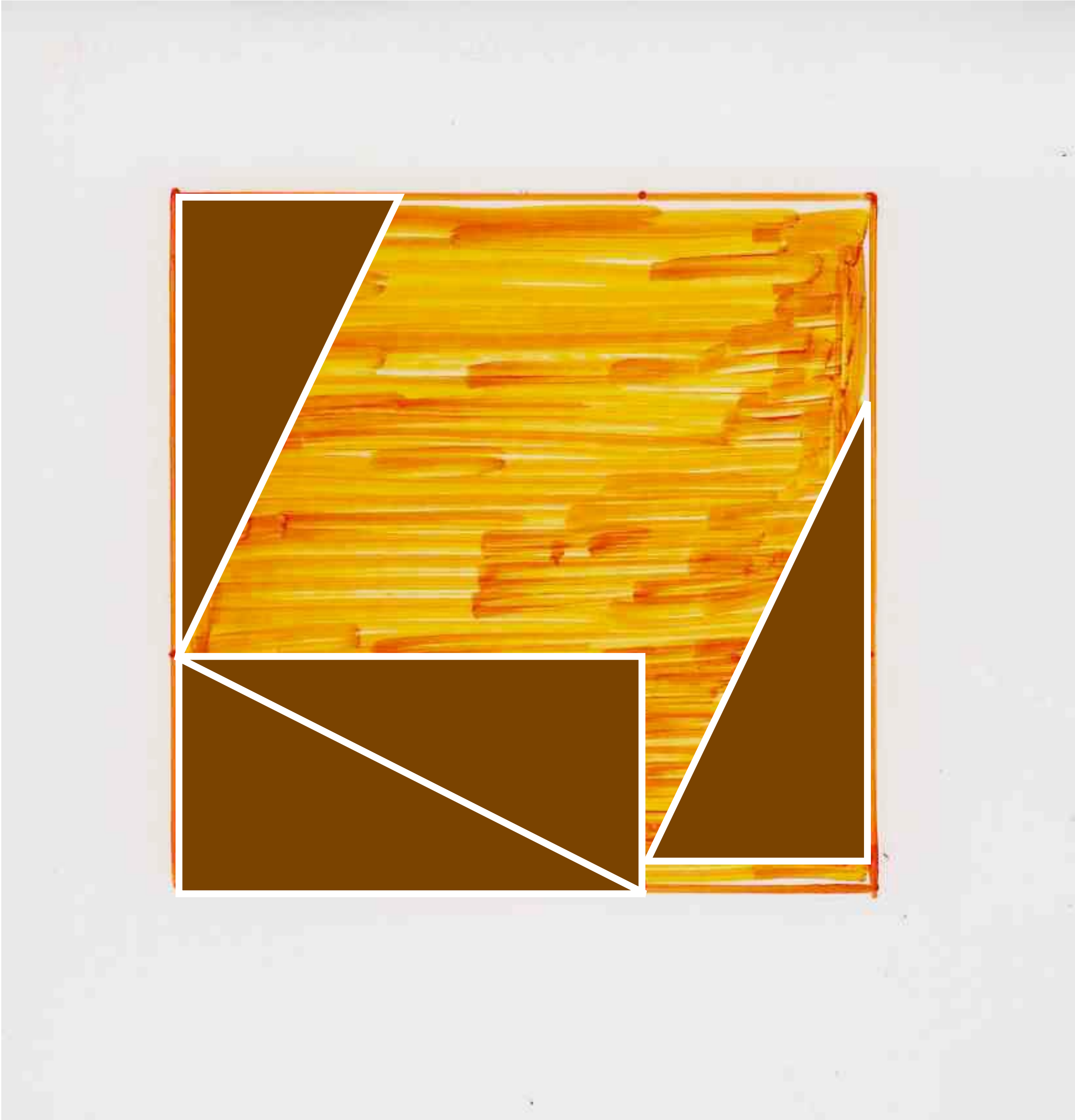


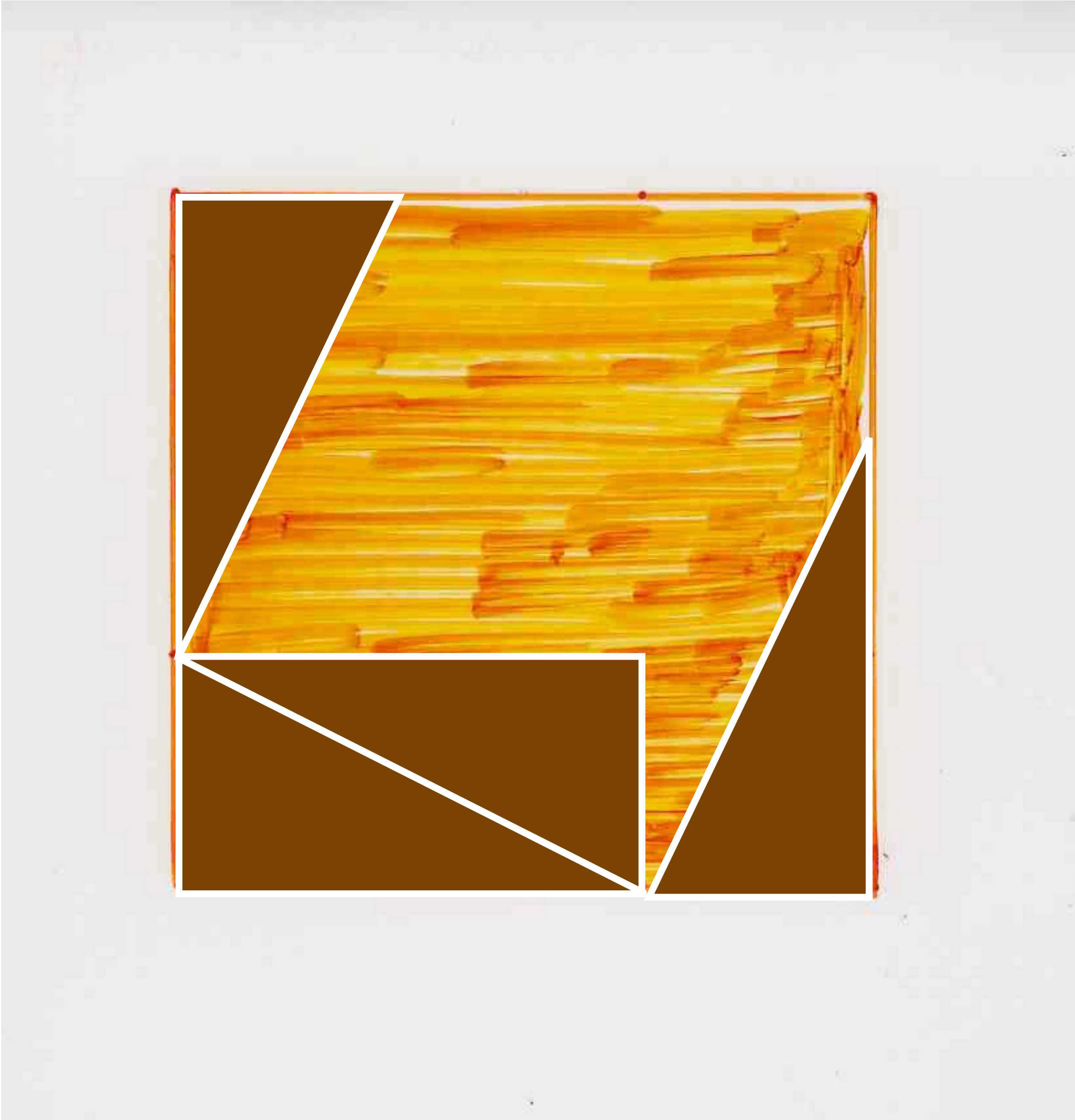


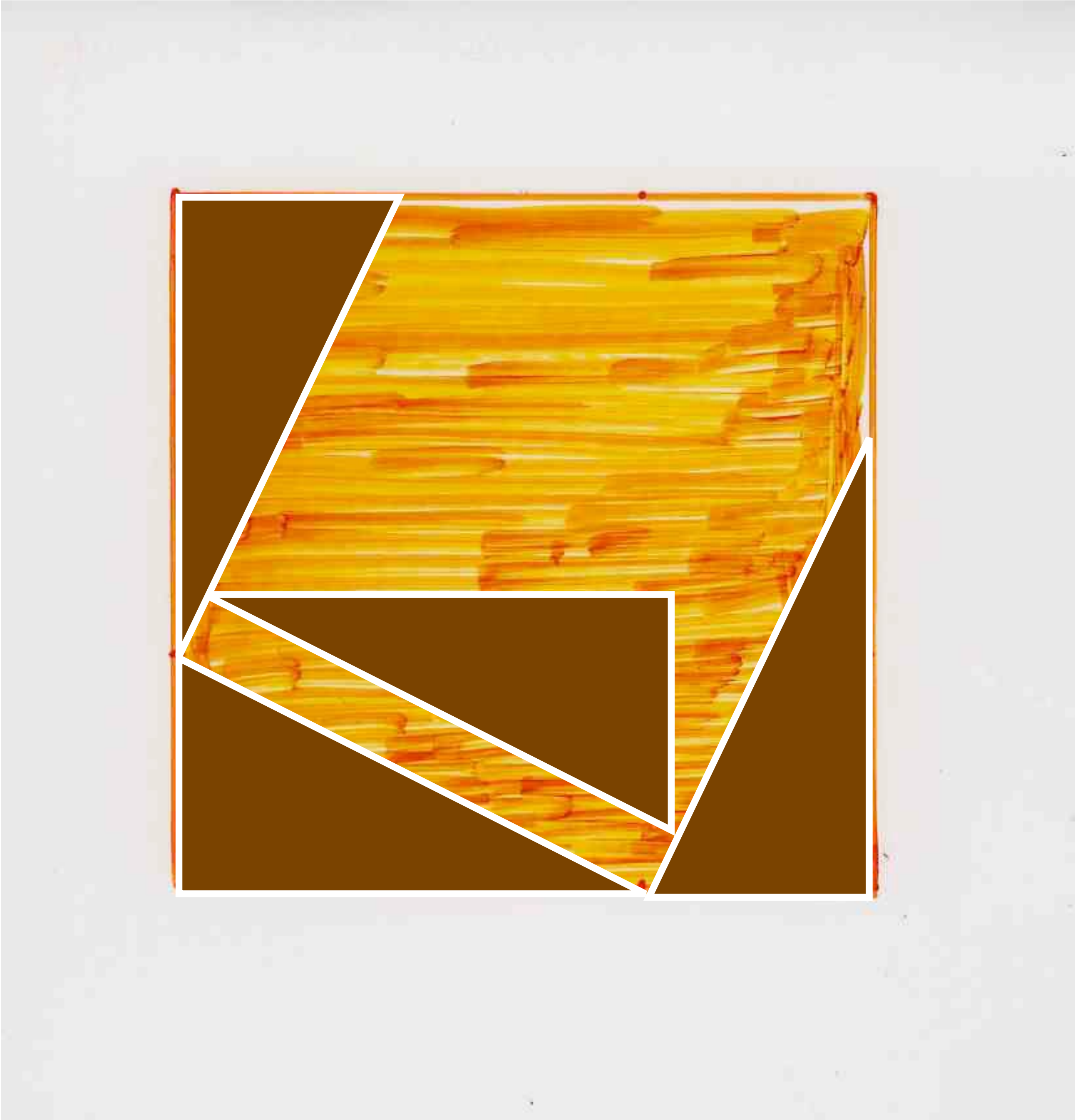


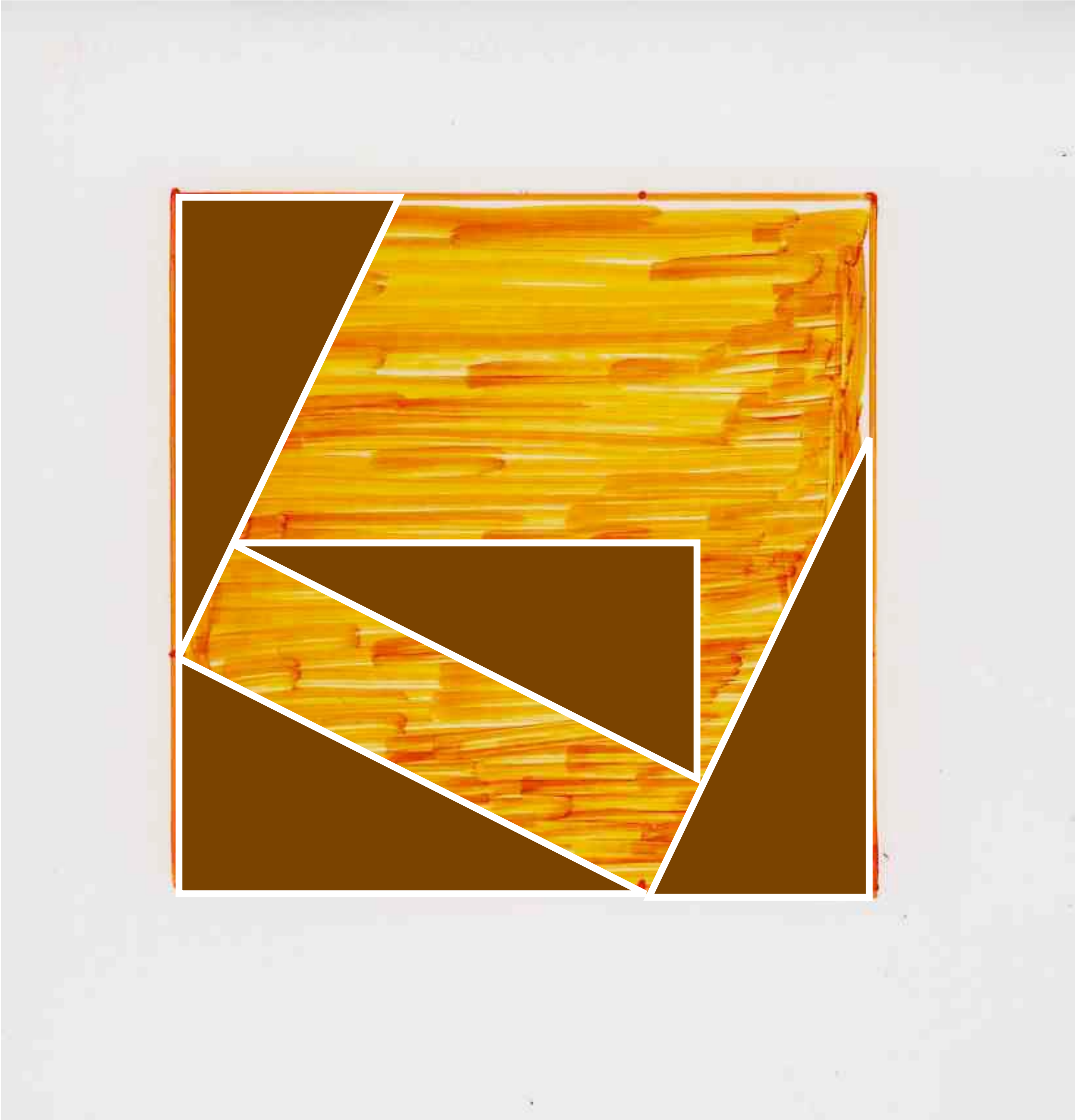


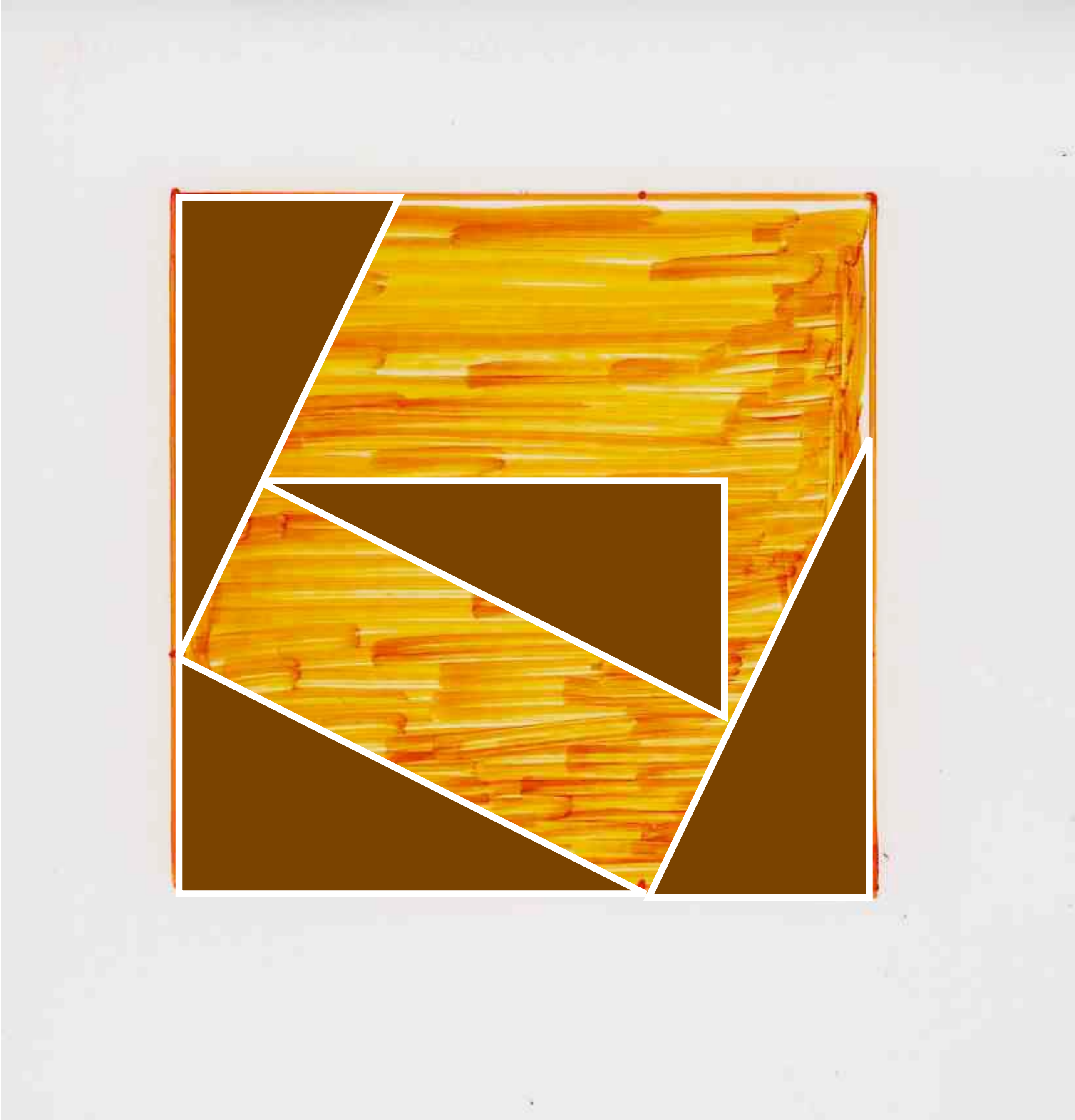


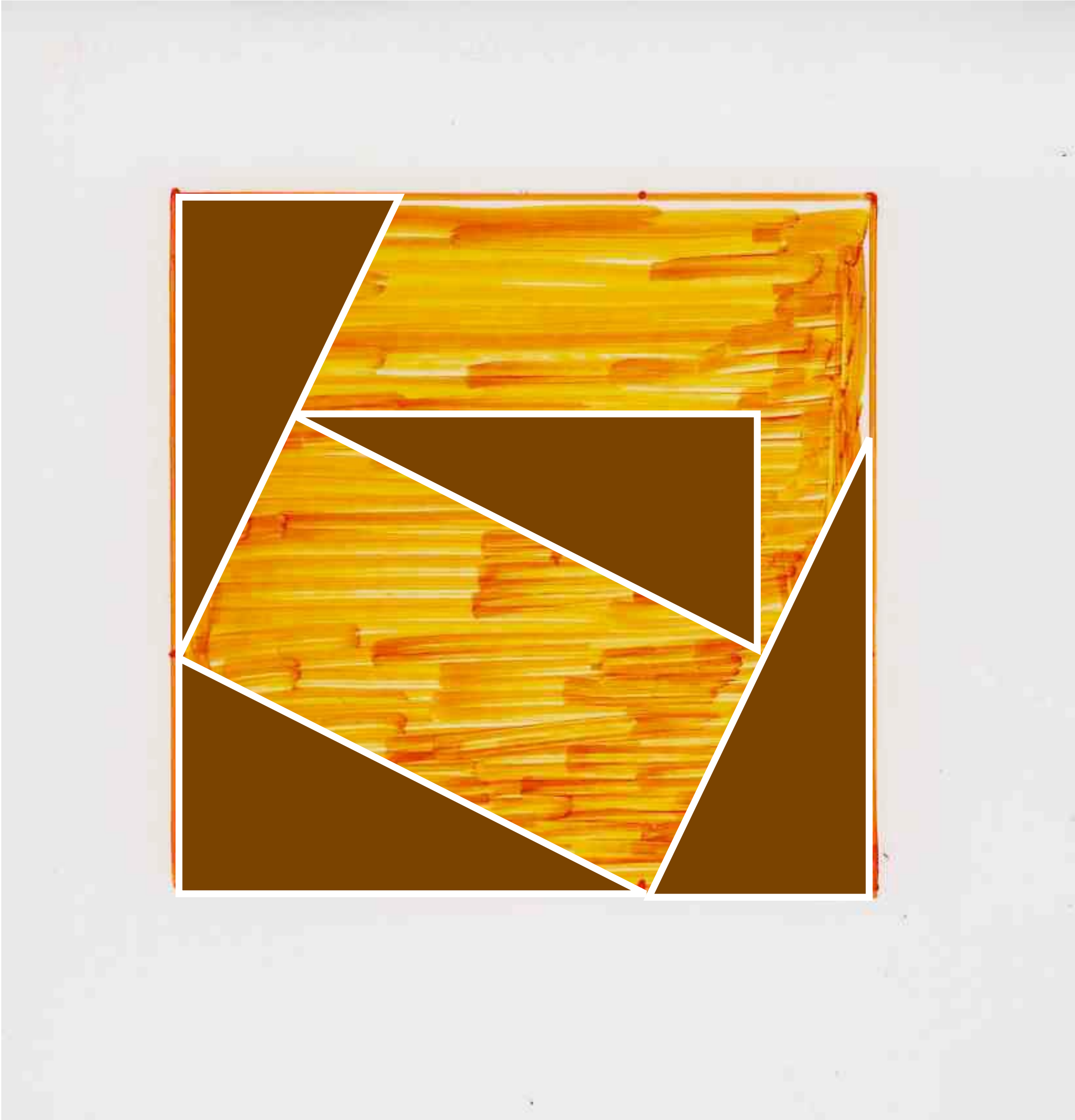


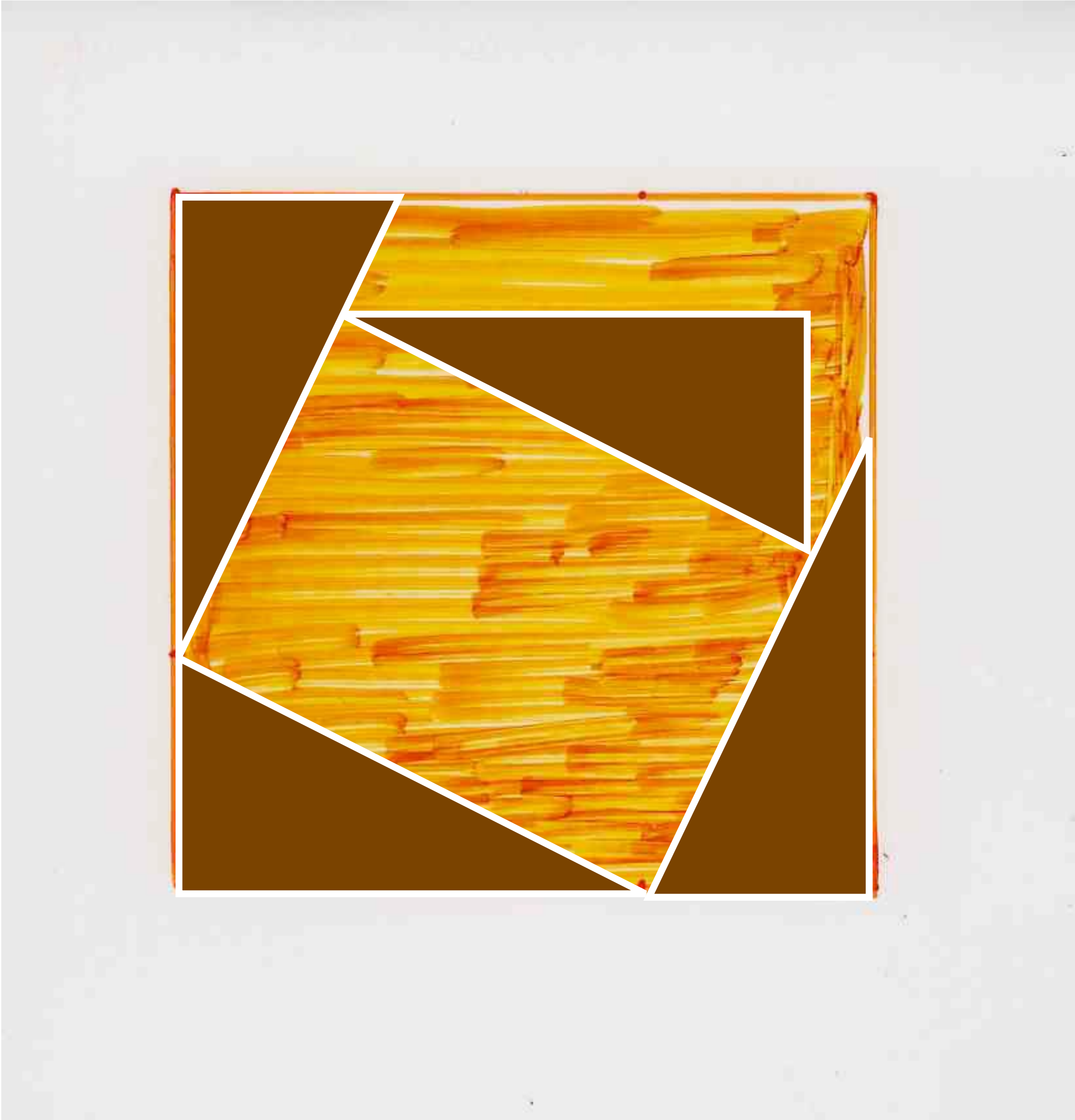


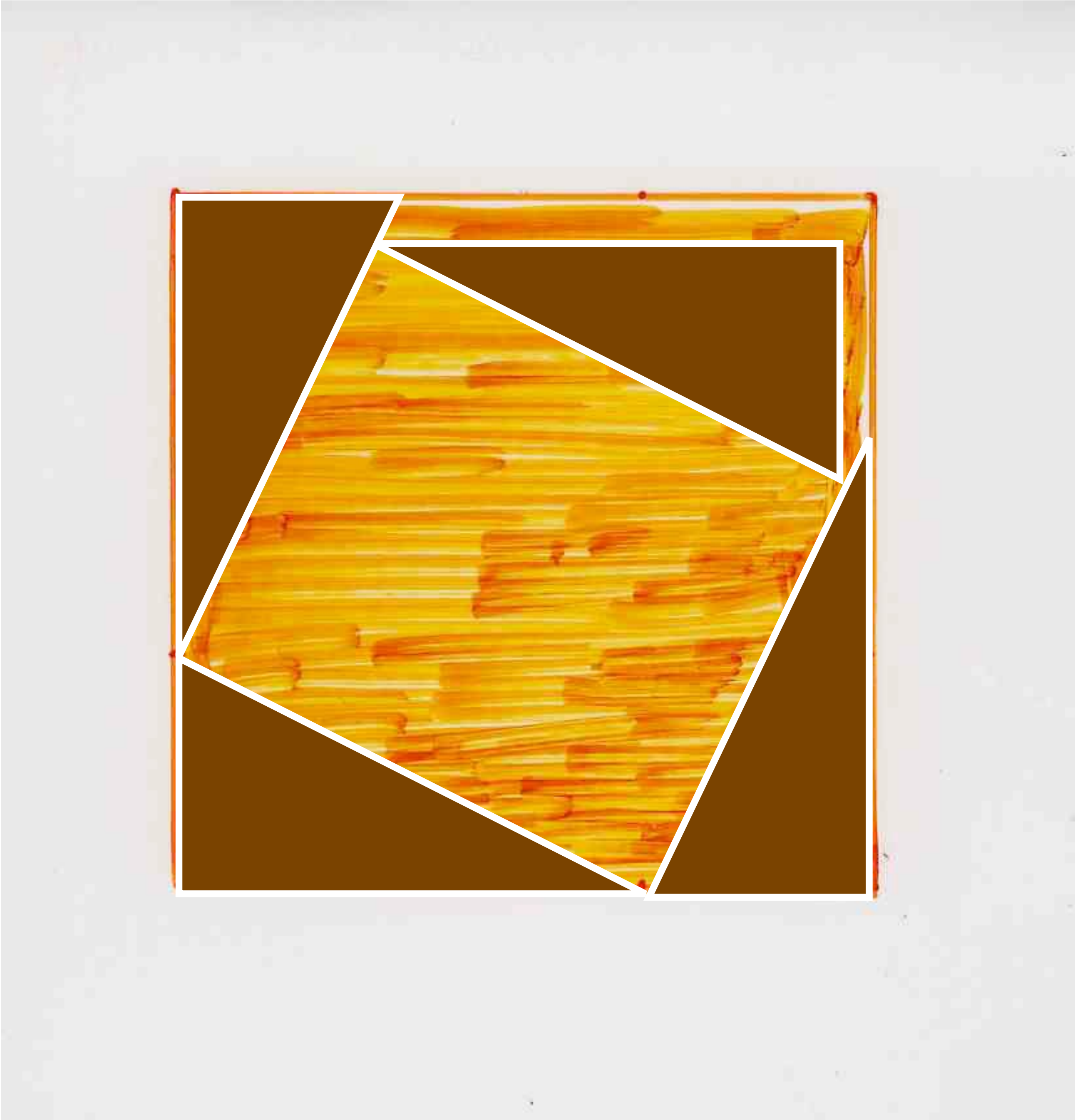


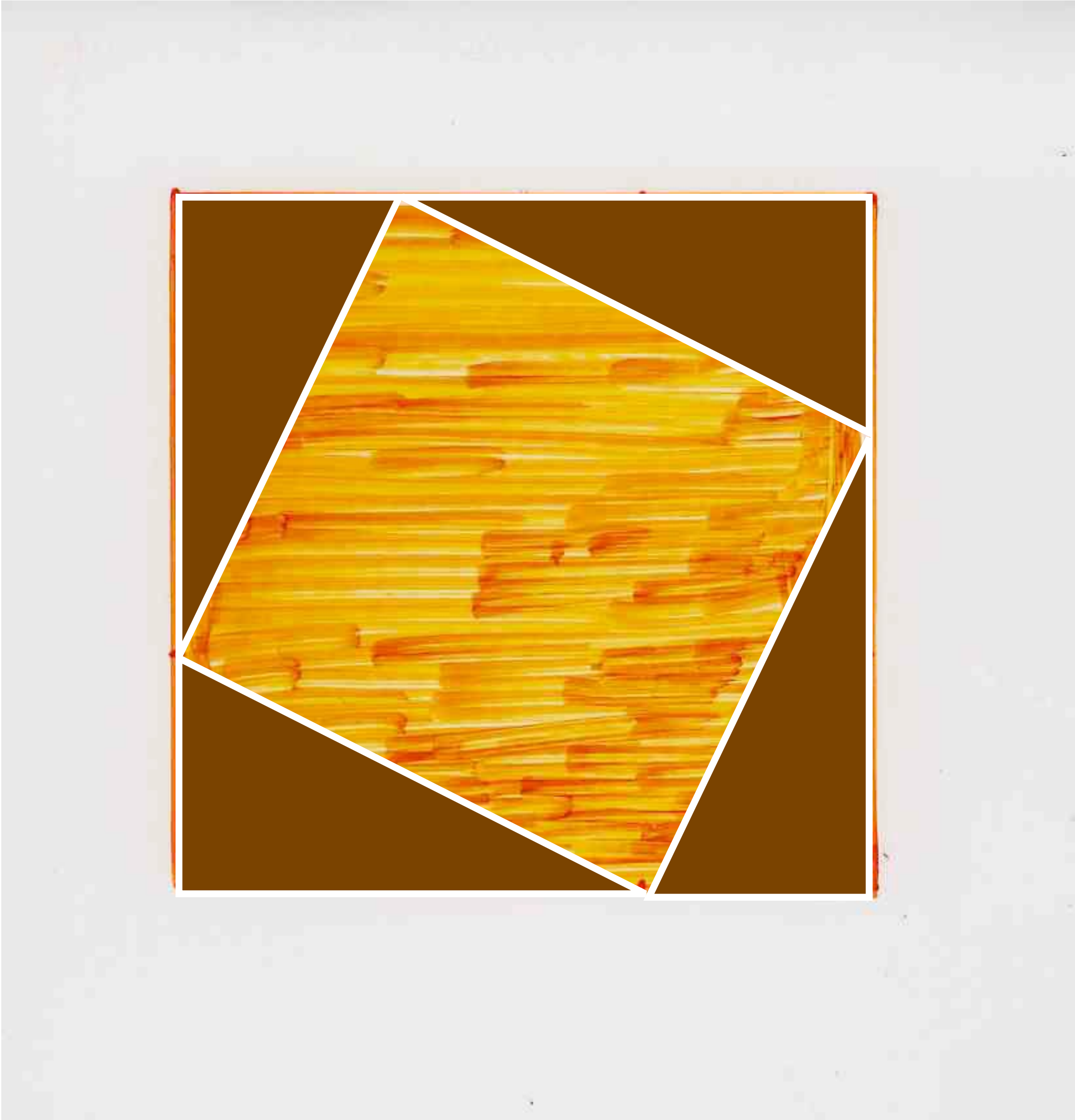


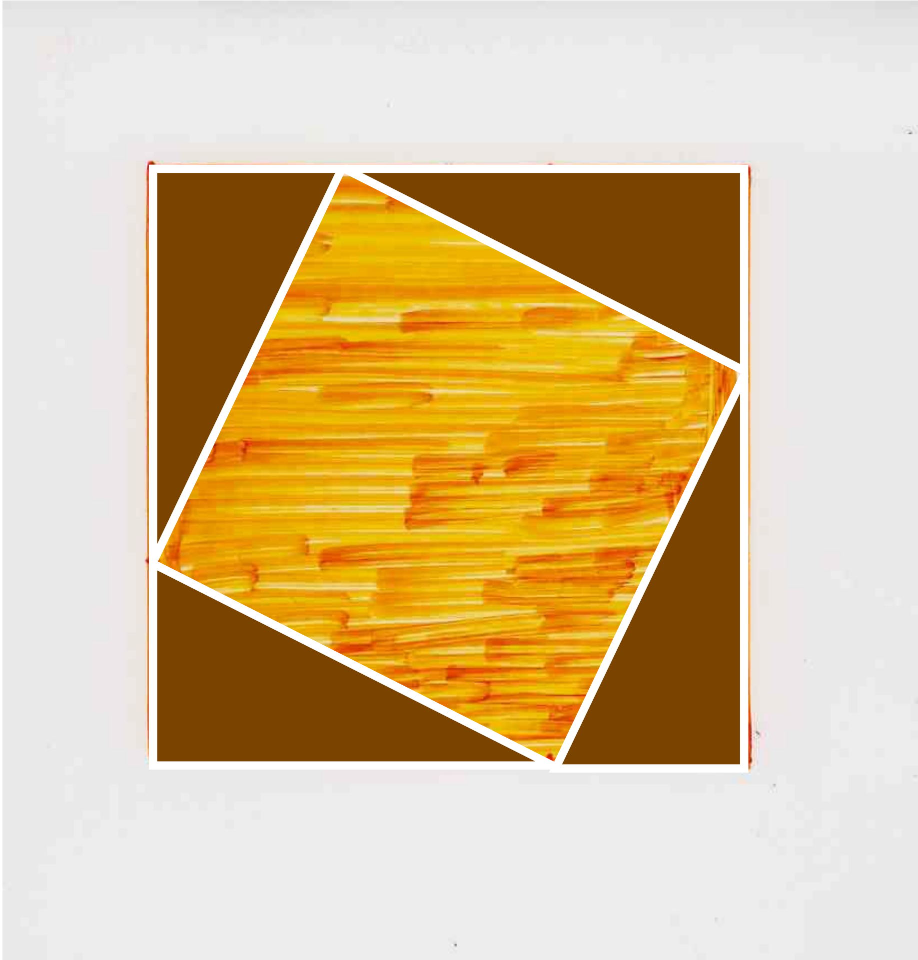




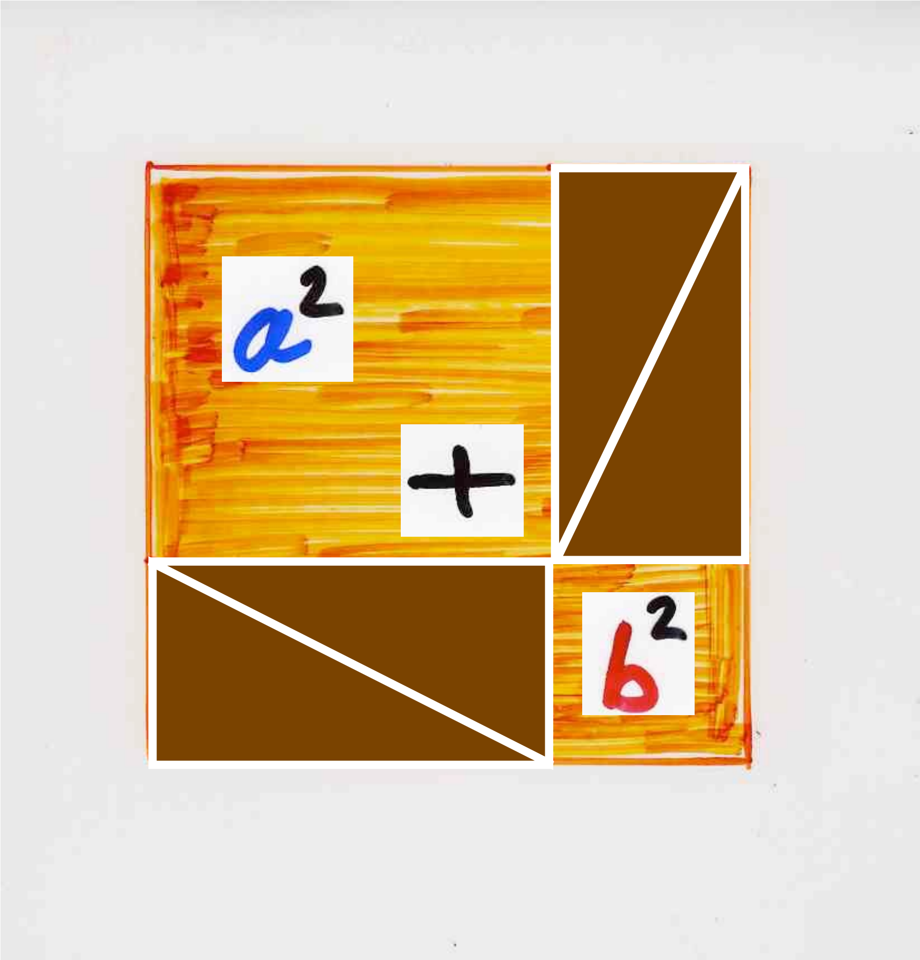


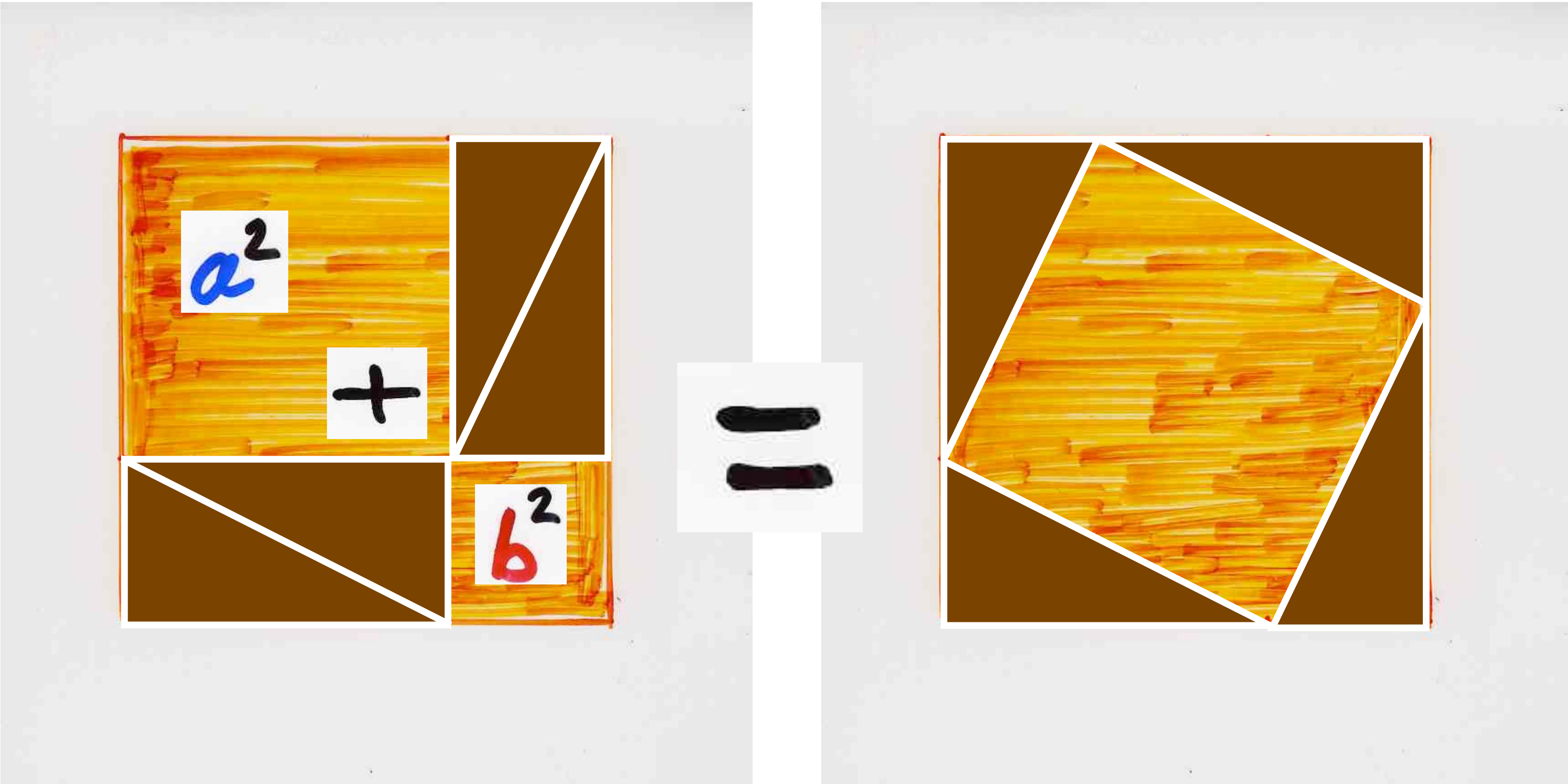


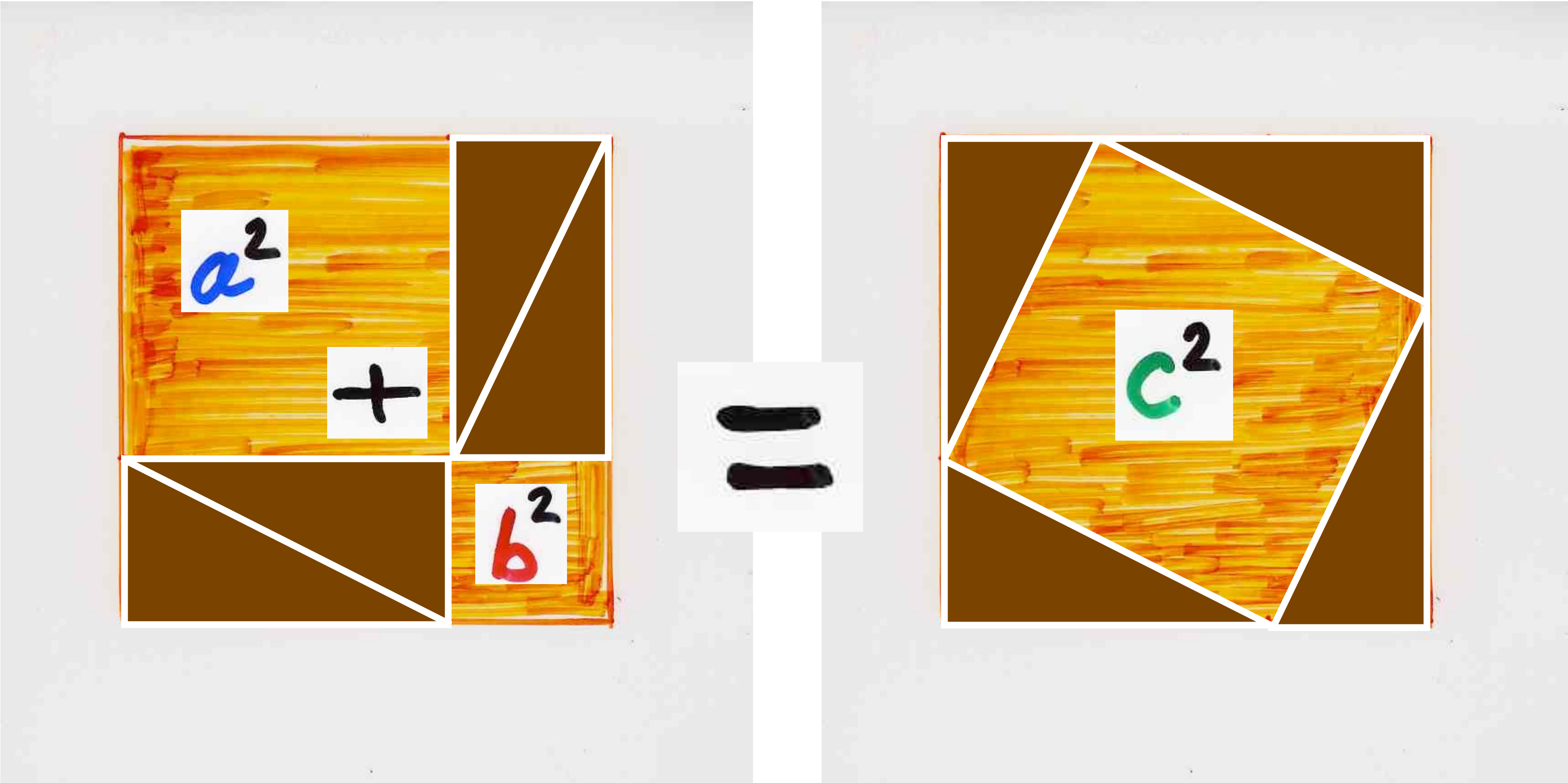



a2


a^2
+
b^2


a^2
+
b^2
=



a^2
+
b^2
=
c^2

Ramanujan continued fraction

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cdots\cdots \cfrac{}{1+\cfrac{q^k}{\cdots\cdots}}}}}$$

Ramanujan

continued fraction

Ramanujan's home
Sarangapani Street
Kumbakonam

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots}{1+\cfrac{q^k}{\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

Rogers-Ramanujan identities

Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

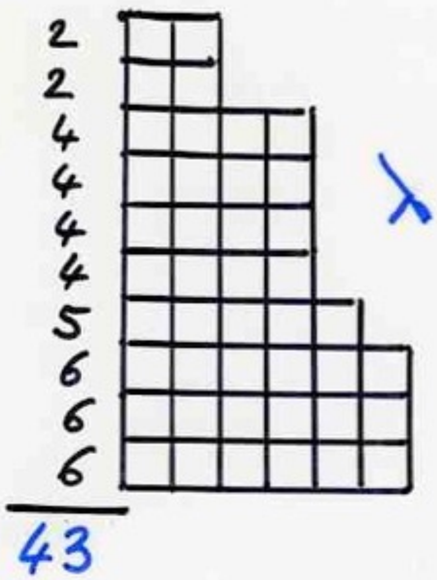
$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

formal power series
and
generating function

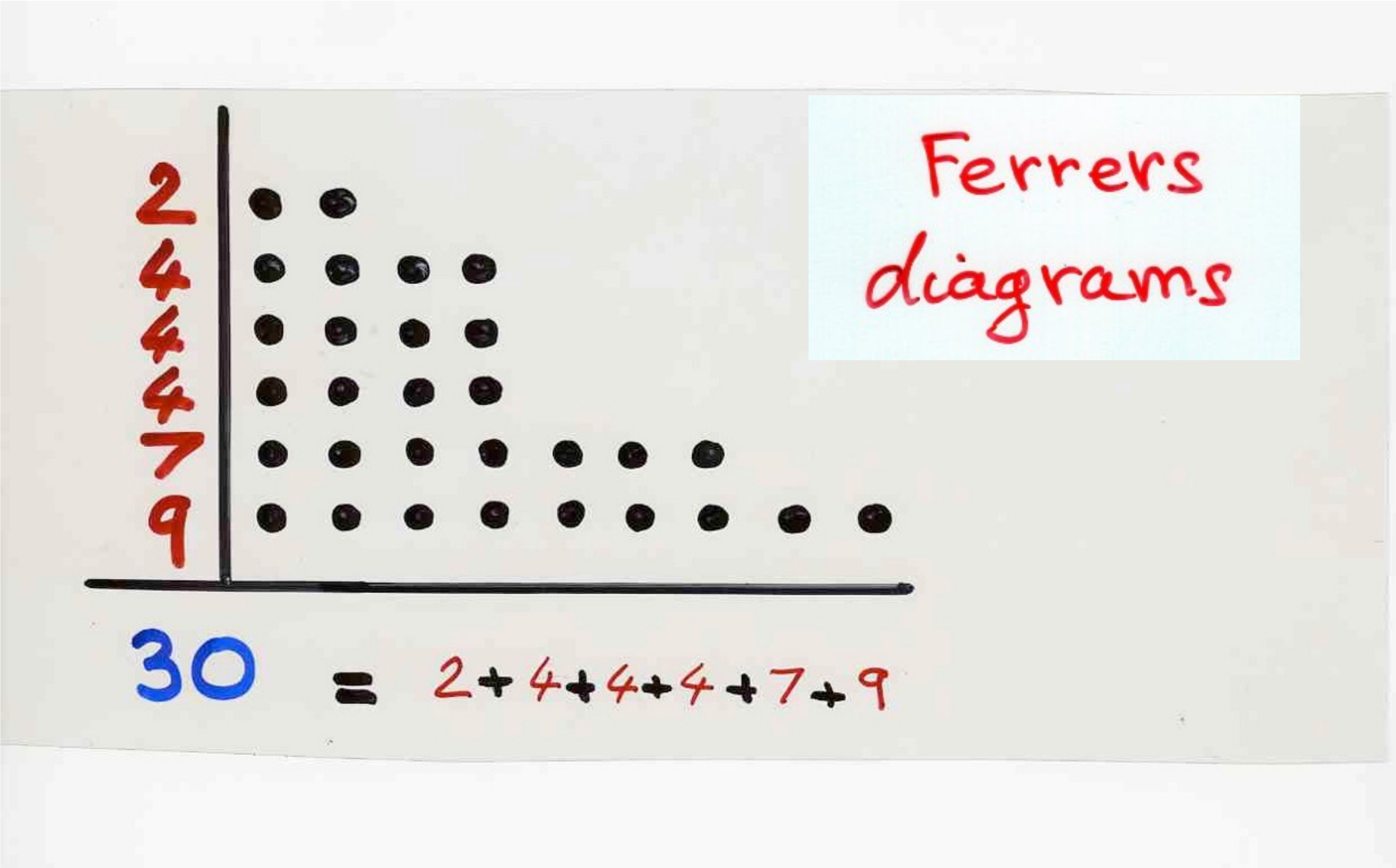
partition of an integer n

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram


Ferrers diagrams
2
4
4
4
7
9
30 = 2+4+4+4+7+9

(1)	1+1	1+1+1	1+1+1+1	1+1+1+1+1
	(2)	2+1	2+1+1	2+1+1+1
		(3)	3+1	2+2+1
			2+2	3+1+1
			(4)	3+2
				4+1
				(5)

$$1 \; , \; 2 \; , \; 3 \; , \; 5 \; , \; 7$$

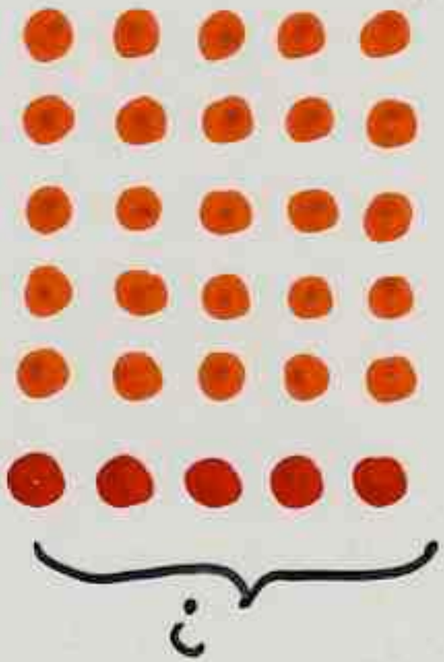
$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

generating function for (integer) partitions

$$\sum_{n \geqslant 0} a_n q^n$$

9^i



$$\frac{1}{1-q^{i}}$$

symbolic method

Philippe Flajolet (1948 – 2011)

(with Robert Sedgewick)

Analytic Combinatorics

(Cambridge Univ. Press, 2008)

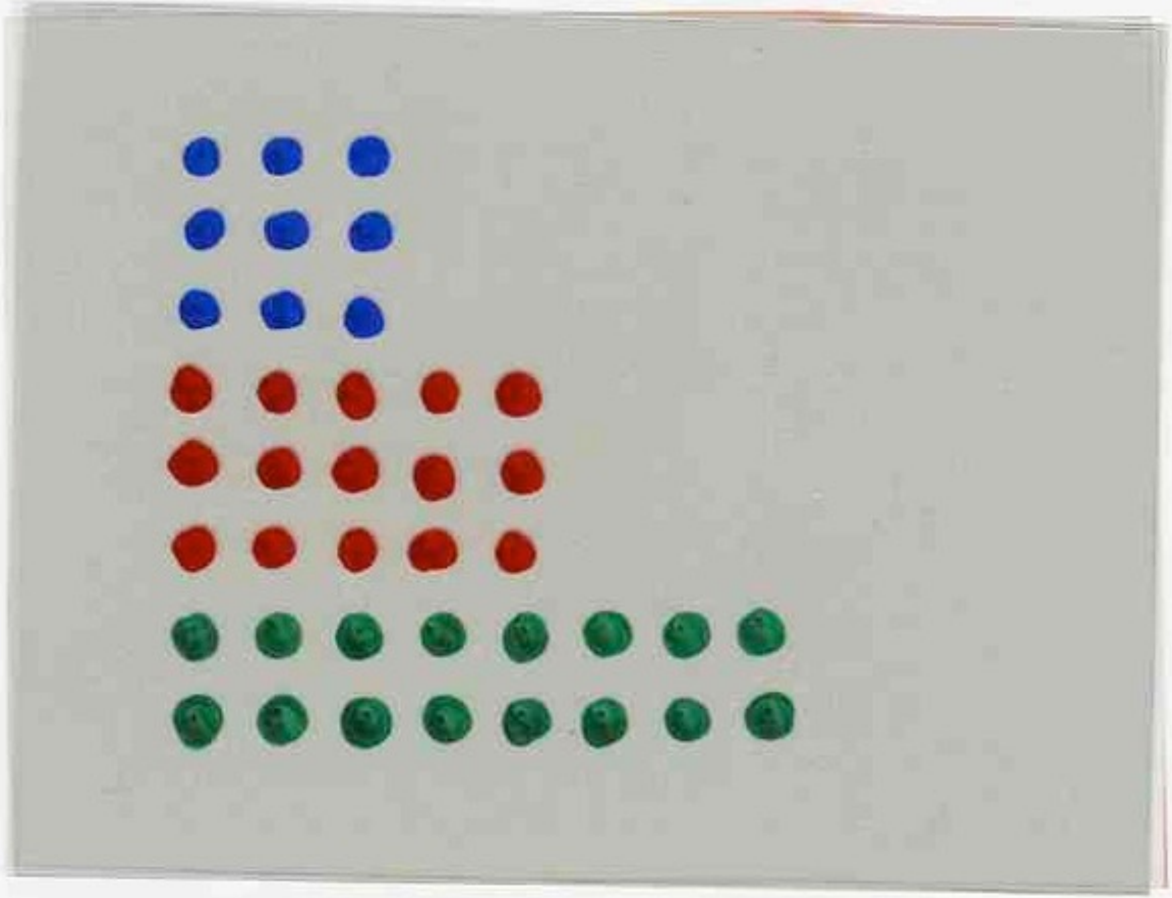
• sequence

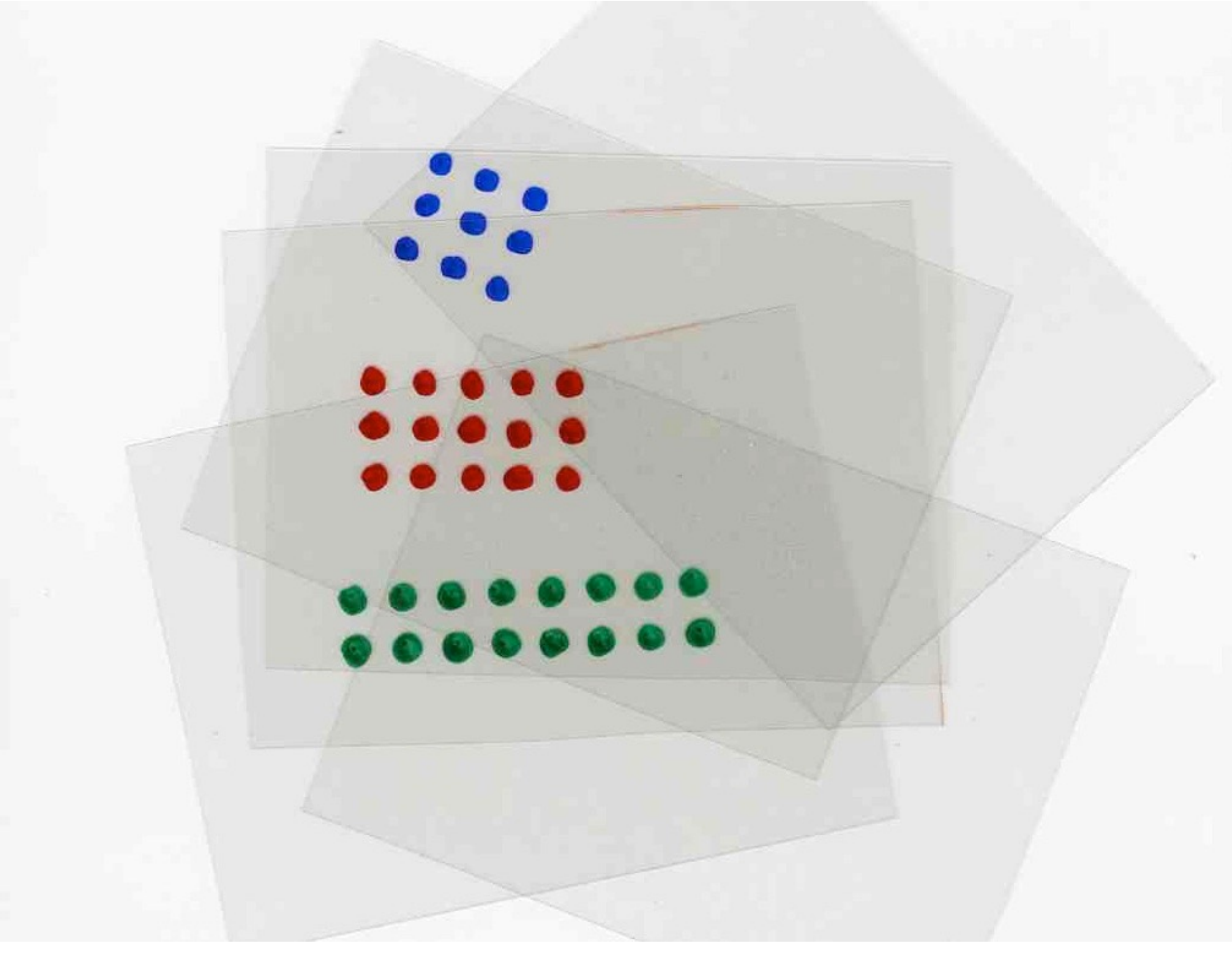
$\mathcal{A} = (A, v_A)$ $\qquad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^n + \ldots$$

$$= \mathcal{A}^*$$

Lemma $f_{\mathcal{A}^*} = \dfrac{1}{1 - f_{\mathcal{A}}}$





$$\frac{1}{(1-q)(1-q^2)\cdot\ldots\cdot(1-q^m)}$$

- product $\mathcal{A}\cdot\mathcal{B} = \mathcal{C}$

$= (C, v_C)$

- $C = A \times B$
- $(\alpha, \beta) \in C$ $\quad v_C(\alpha,\beta) = v_A(\alpha)\, v_B(\beta)$

ex: "size" $|(\alpha,\beta)| = |\alpha| + |\beta|$

ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

generating function
for the number of
partitions of an integer n

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

partitions

parts $\equiv 1, 4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \qquad \begin{cases} 1+\dots+1 \end{cases}$$

D- partition $\lambda_i - \lambda_{i+1} \geq 2$

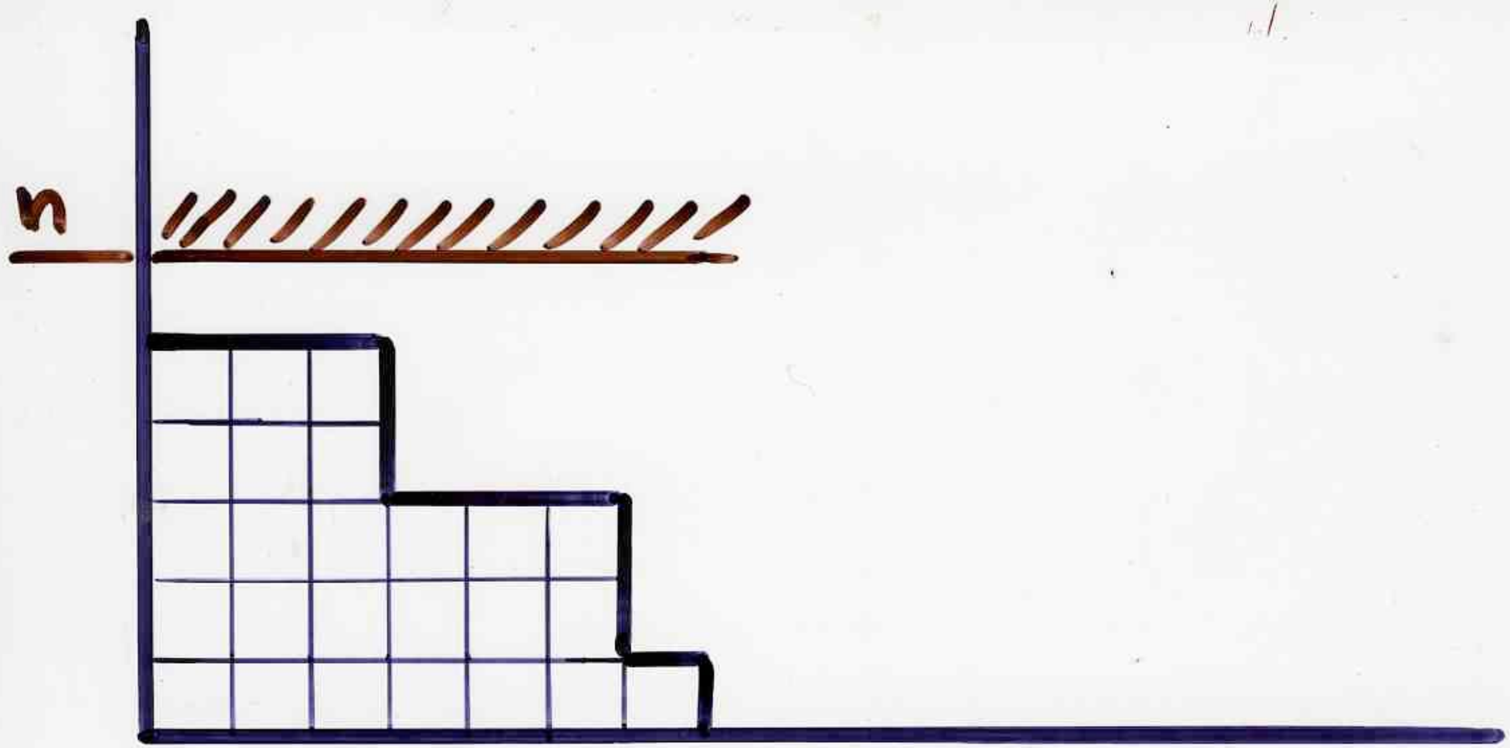
$\lambda = (\lambda_1, \ldots, \lambda_k)$ $(1 \leq i < k)$

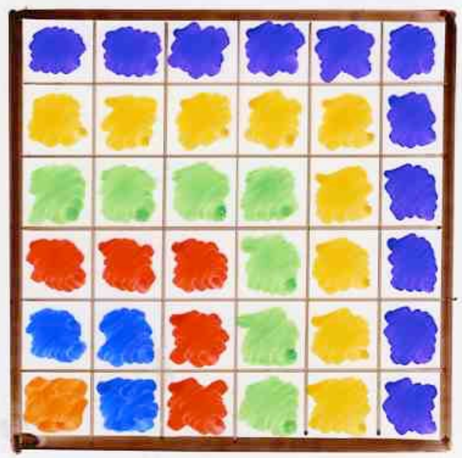
generating function for D- partitions

$$\sum_{m \geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

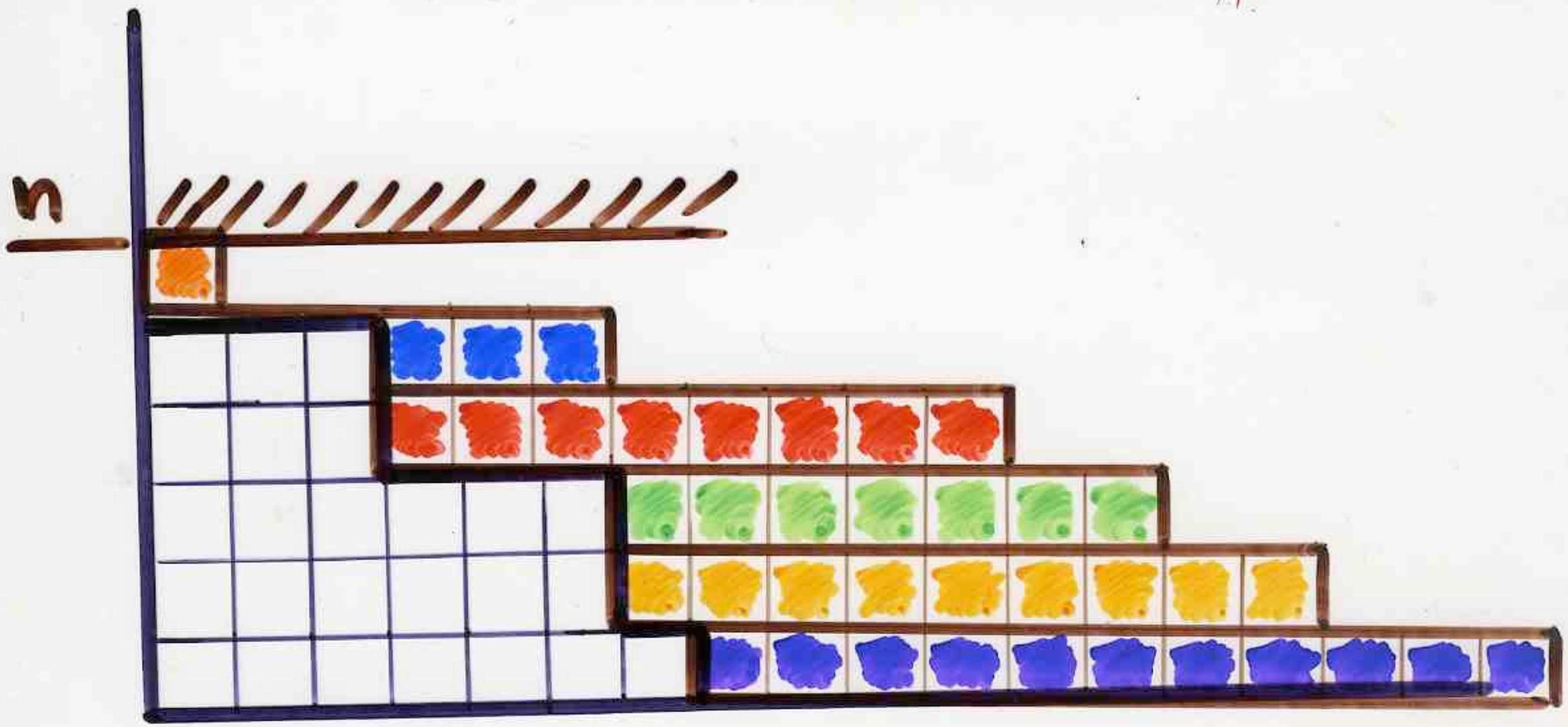
Partition ayant au plus n parts $\Longleftrightarrow$ D-partition ayant exactement n parts

$$0 \leq (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n) \qquad (1+\lambda_1, 3+\lambda_2, \ldots, (2n-1)+\lambda_n)$$


n



$$n^2 = 1 + 3 + .. + (2n - 1)$$


n

Rogers - Ramanujan identities

$$R_I \qquad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

D-partitions

- 9
- 8+1
- 7+2
- 6+3
- 5+3+1

?

partitions

parts ≡ 1, 4 mod 5

- 9
- 4+4+1
- 6+1+1+1
- 4+1+1+1+1+1
- 1+...+1

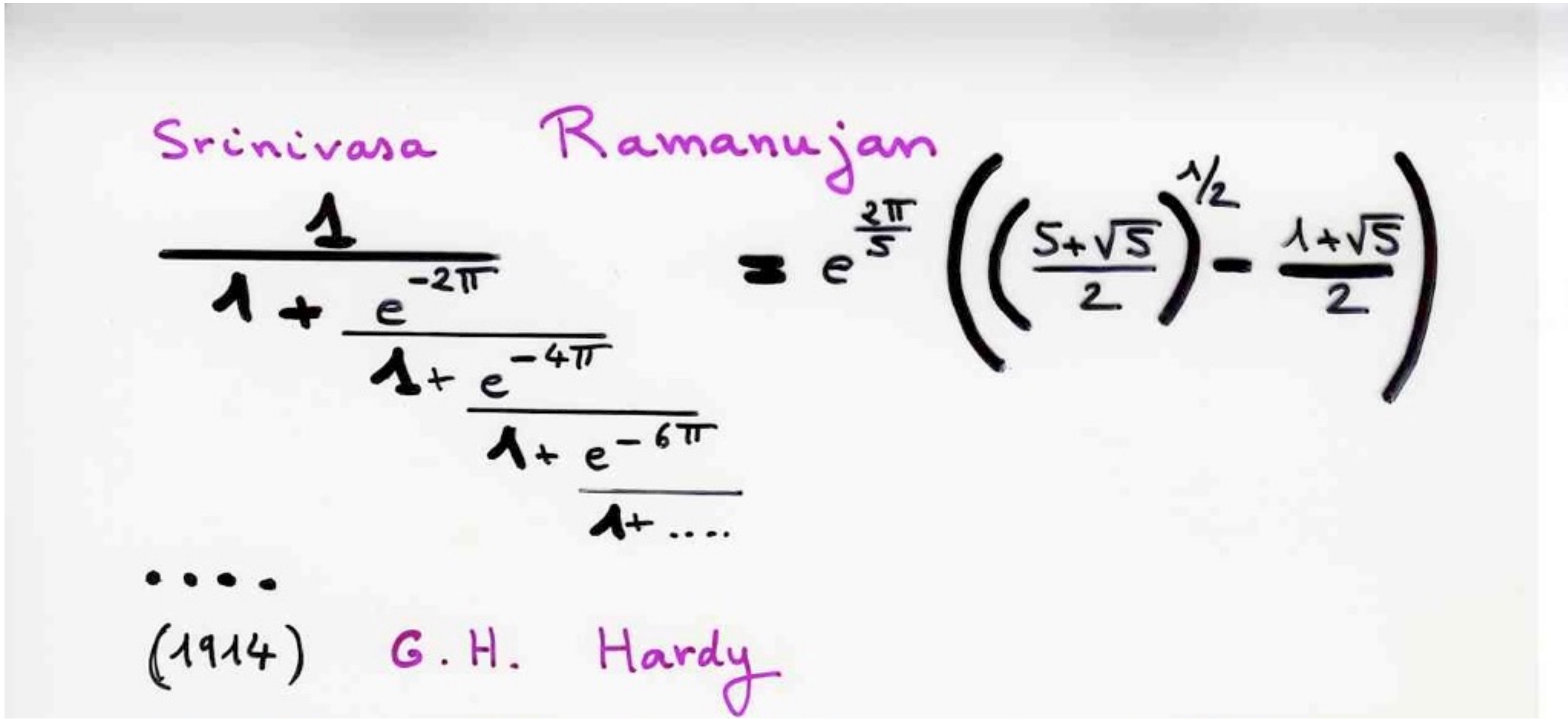
R_{II}

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

D-partitions		Partitions
parts $\neq 1$		parts $\equiv 2, 3$ mod 5
7 + 2	?	2 + 2 + 2 + 3
6 + 3		3 + 3 + 3
9		7 + 2

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\prod_{\substack{i\equiv 2,3 \\ \bmod 5}} (1-q^i)}{\displaystyle\prod_{\substack{i\equiv 1,4 \\ \bmod 5}} (1-q^i)}$$



«These identities defeated me completely; I had never seen anything like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. »

Srinivasa Ramanujan
1887 - 1920

Godfrey Harold Hardy
1877 - 1947

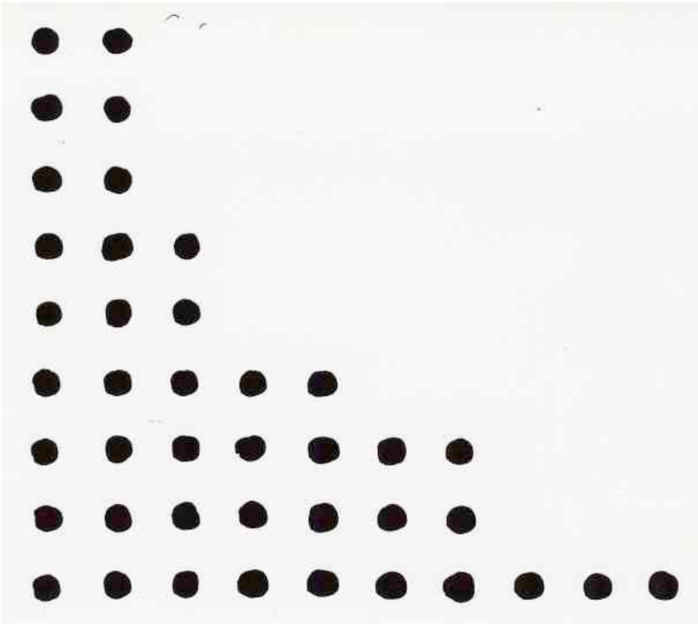
bijective proof of an identity

The "bijective paradigm"

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1}\frac{1}{(1-q^i)}$$

right handside

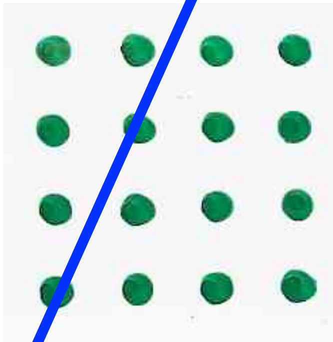


$$=\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

Ferrers diagram (= partition of an integer) left handside

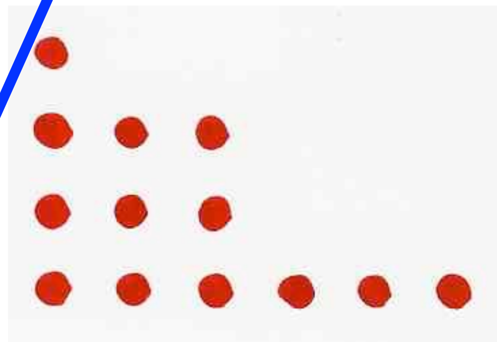
$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1}\frac{1}{(1-q^i)}$$

$$q^{m^2}$$



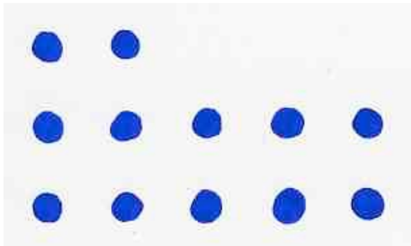
m × m square

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$



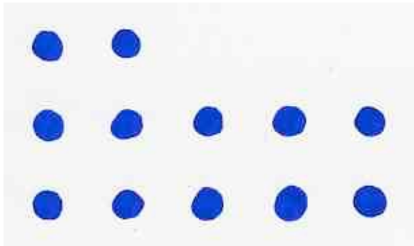
} at most m rows

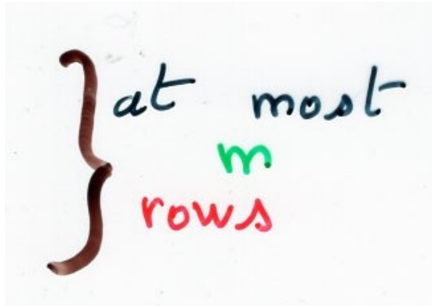
$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

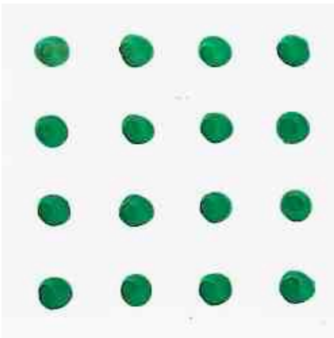


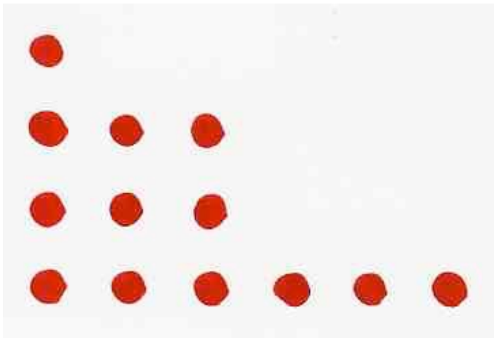
} at most m rows

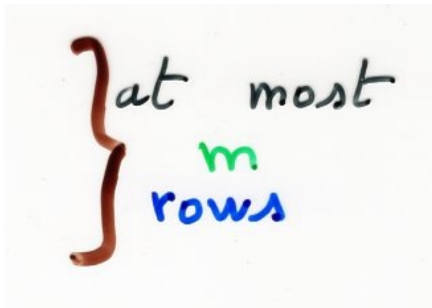
$m \times m$ square

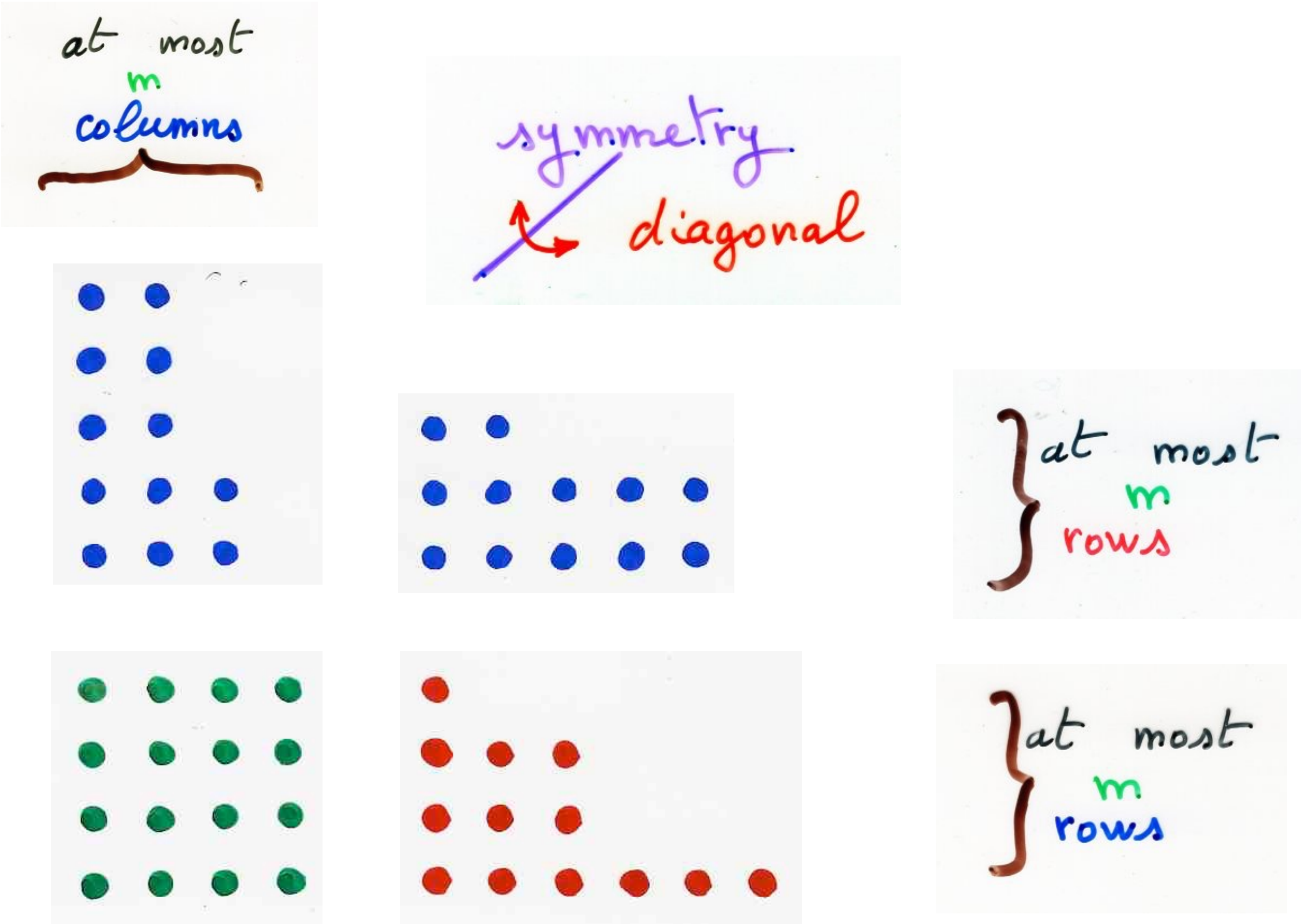


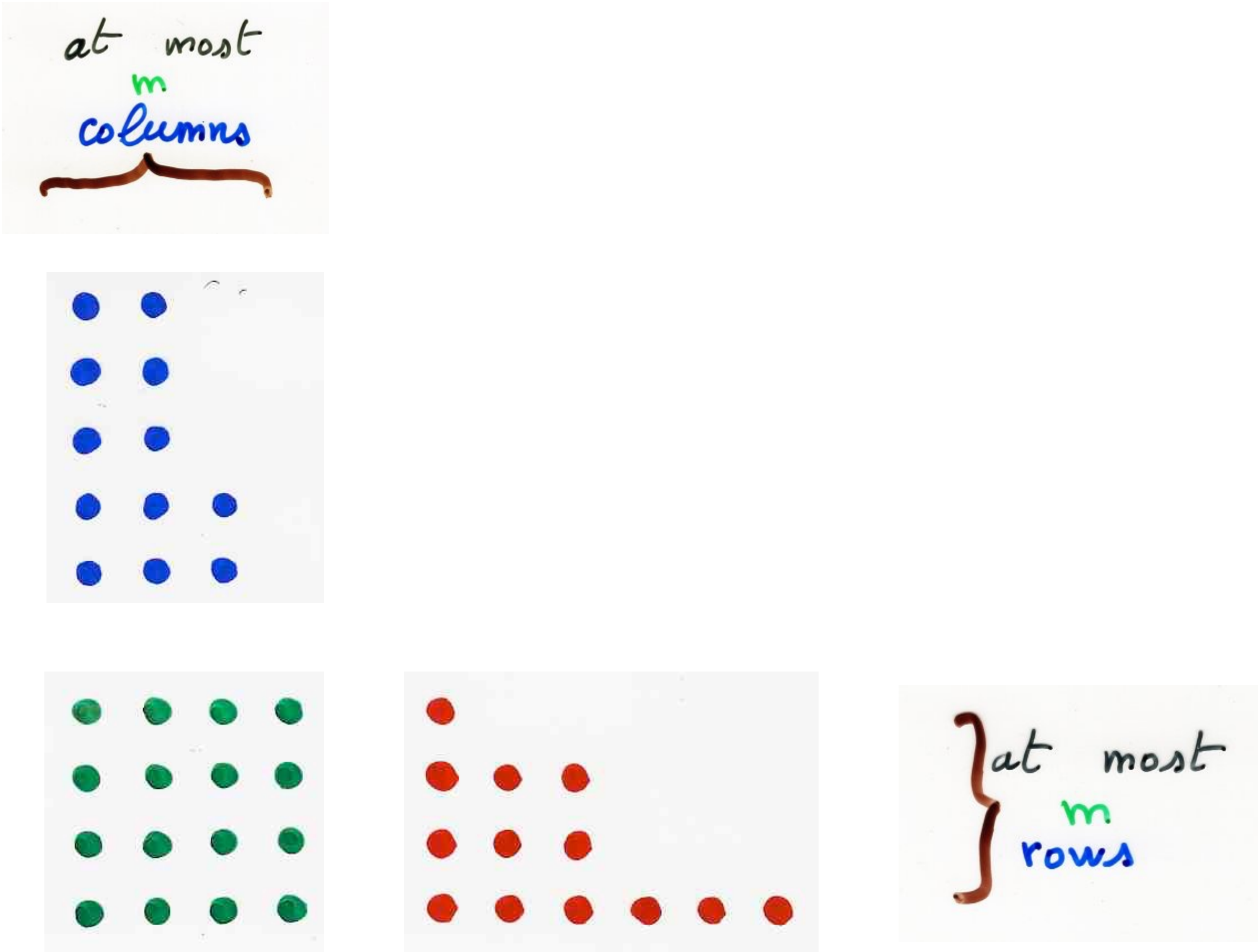






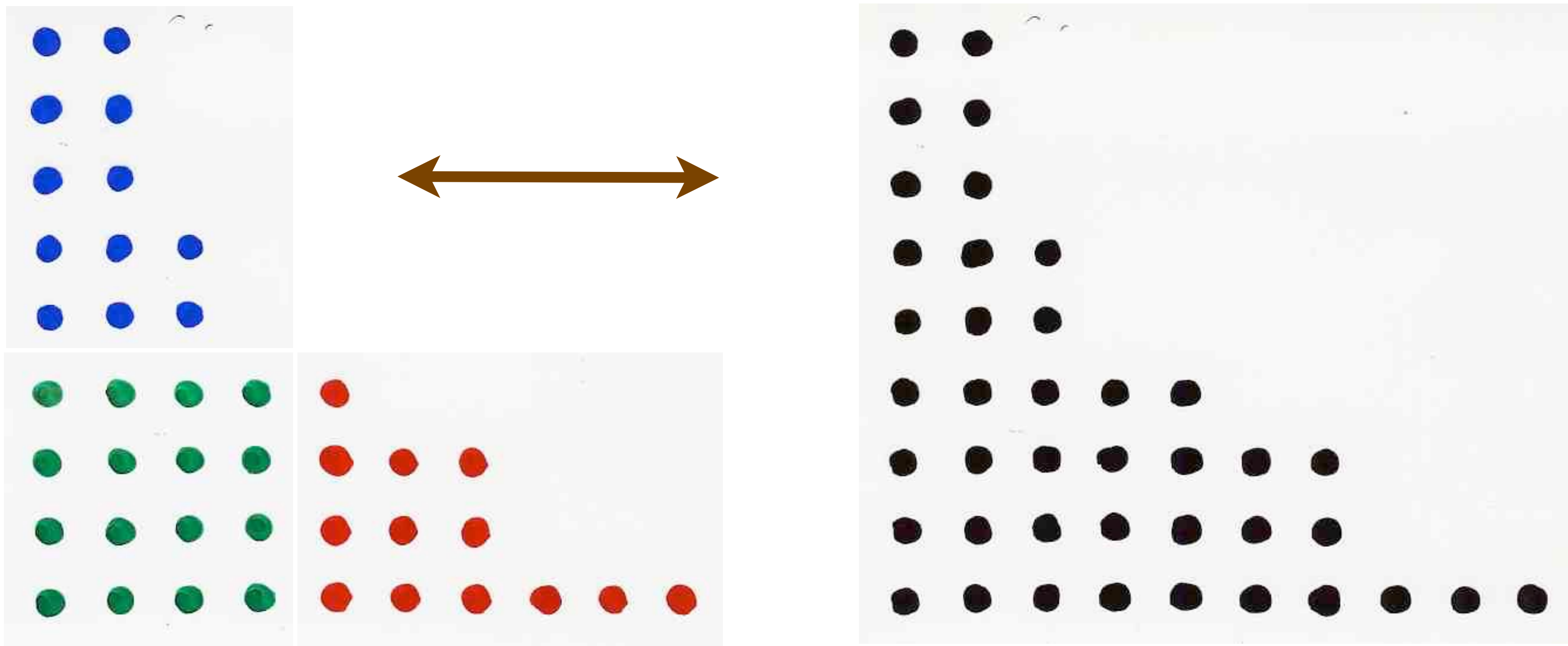



at most
m
columns
symmetry
diagonal
at most
m
rows
at most
m
rows

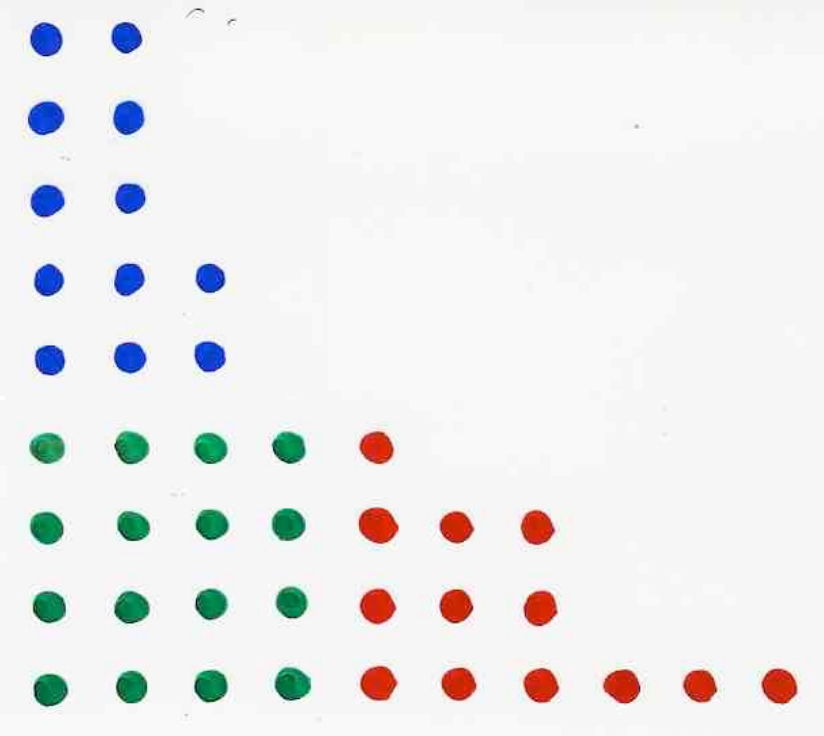

at most m columns
at most m rows

left handside

right handside



The identity means:



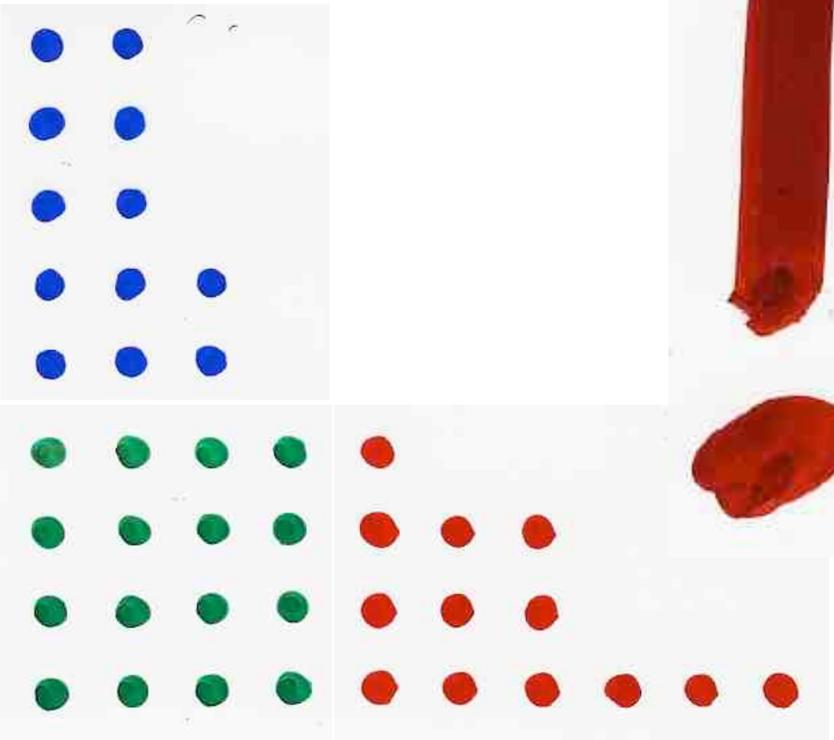
extract the biggest square ⊆ Ferrers diagram

What remains

- diagram having at most m rows
- diagram having at most m columns

m size of the square

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1}\frac{1}{(1-q^i)}$$



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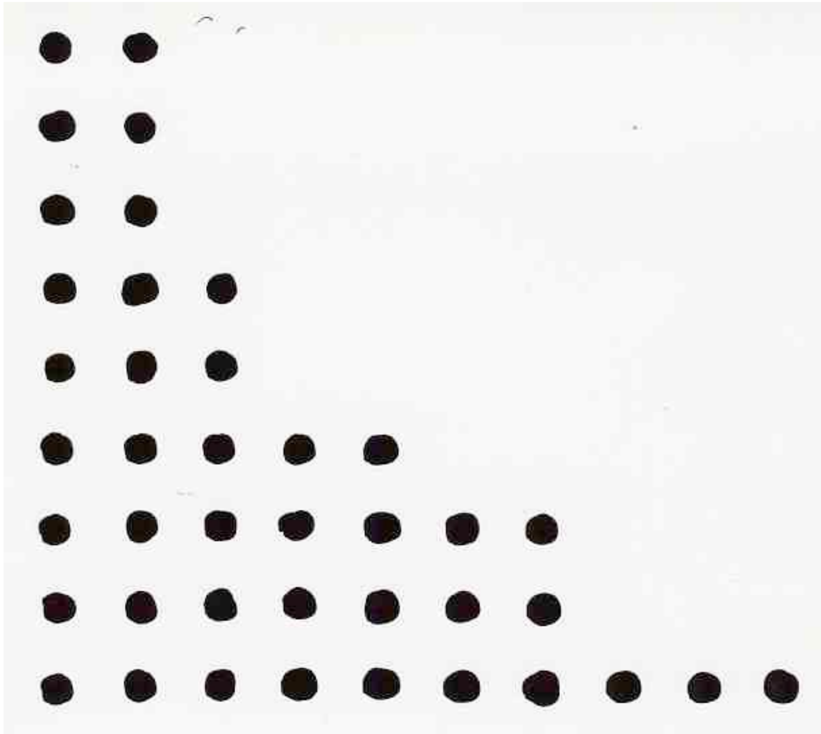


"drawing" calculus . . .

computing with "drawings"

(figures)

=

better
understanding

↔

combinatorial interpretation

of

Ramanujan continued fraction

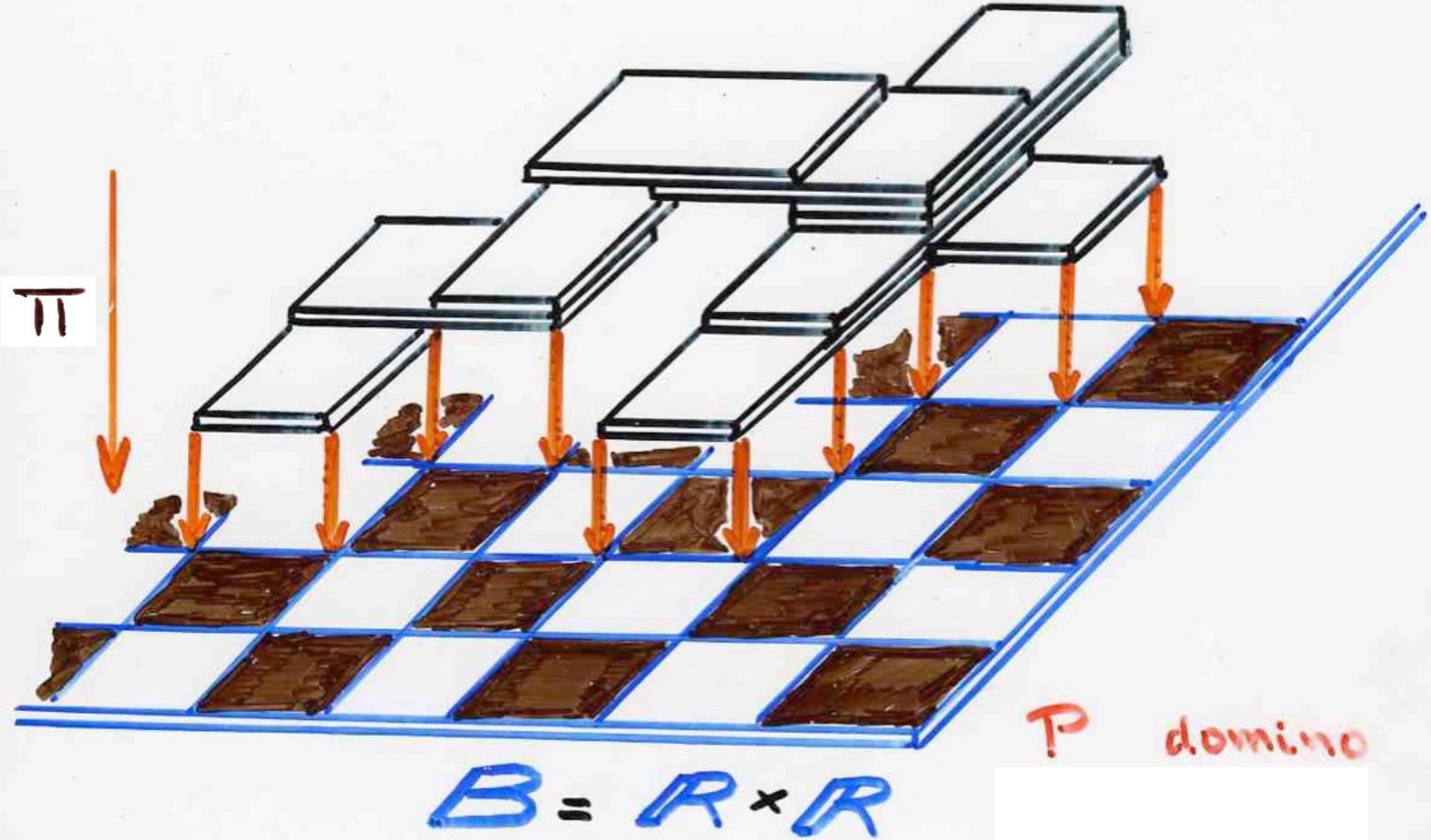
with heaps of pieces

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cdots\cdots\cfrac{}{1+\cfrac{q^k}{\cdots\cdots}}}}}$$

Ramanujan

continued fraction

Introduction
Heaps



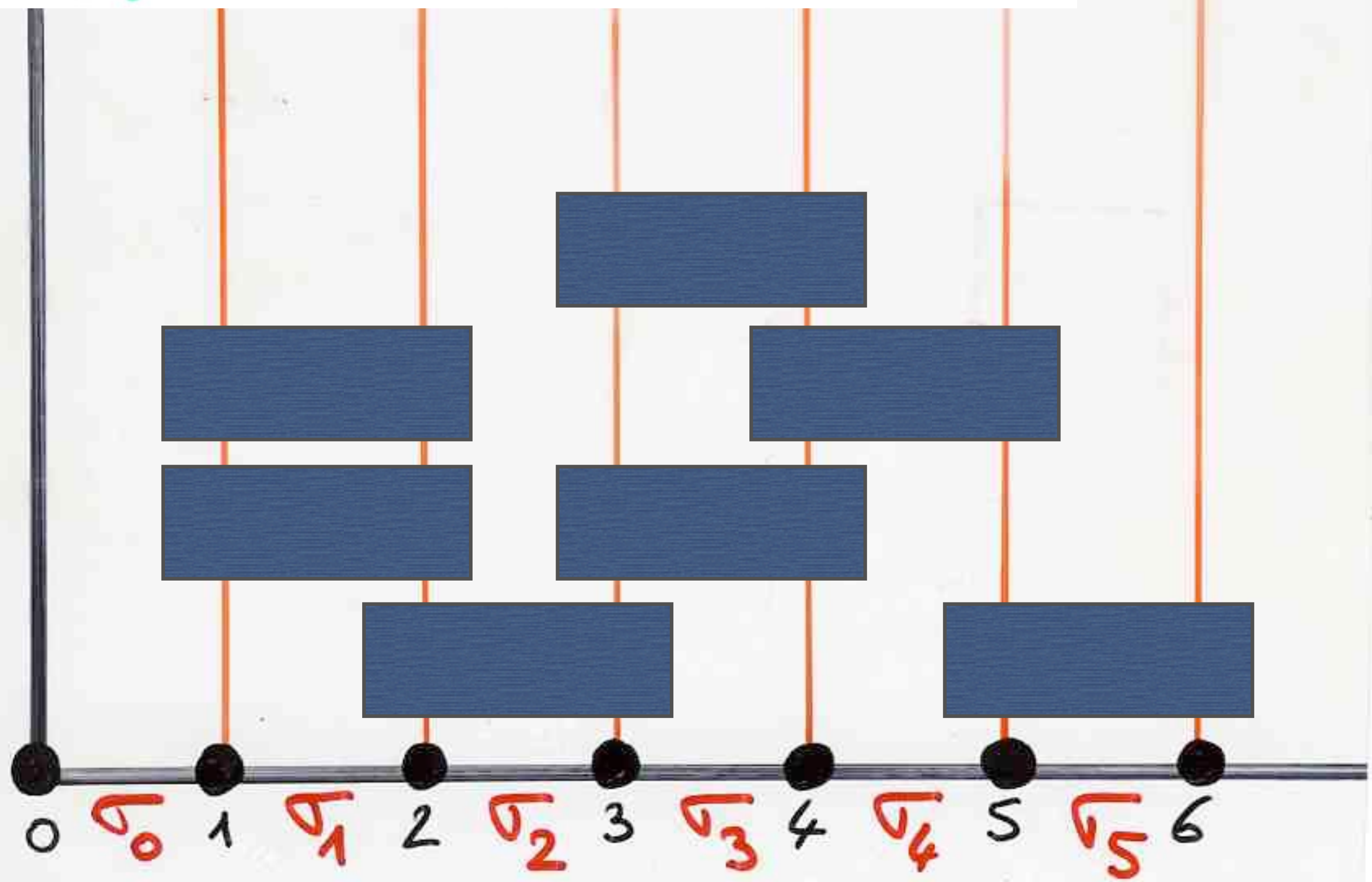
heaps of dimers on $\mathbb{N}$

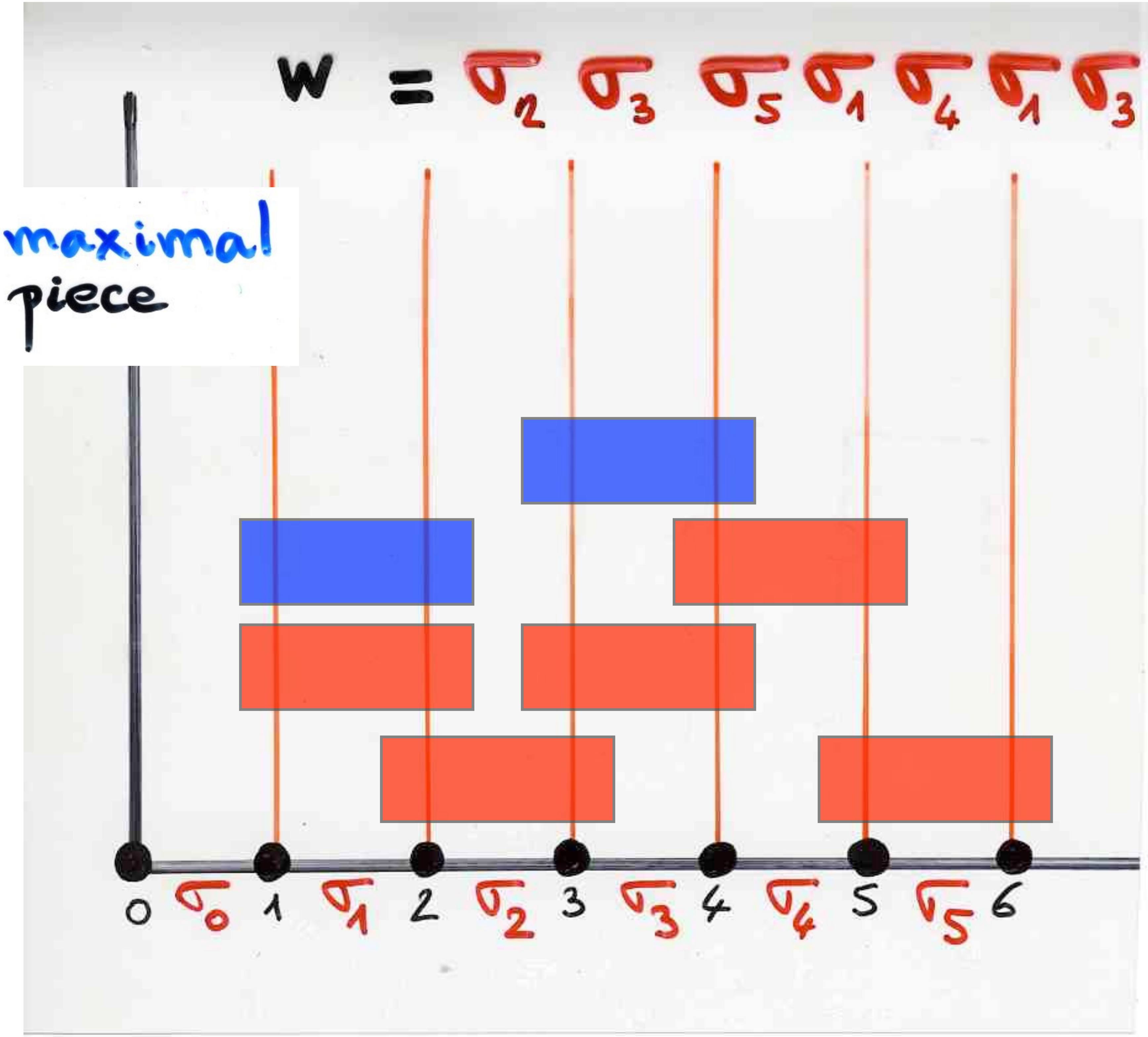
$$\mathcal{P} = \{ [i, i+1] = \sigma_i \,,\, i \geq 0 \}$$

basic pieces

heaps of dimers on $\mathbb{N}$

$$w = \sigma_2\, \sigma_3\, \sigma_5\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$$

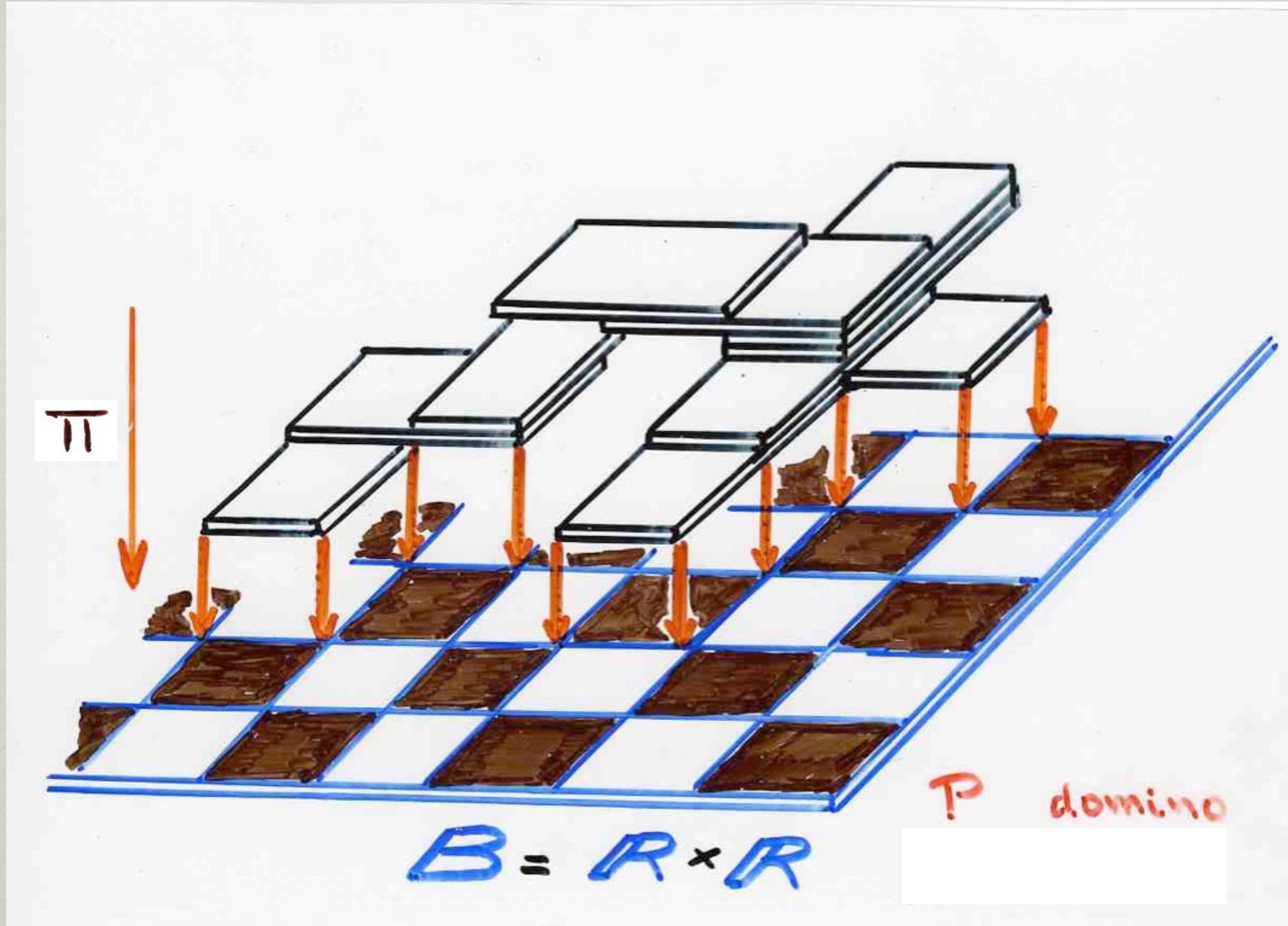




W = σ2 σ3 σ5 σ1 σ4 σ1 σ3
maximal piece
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

Pyramid

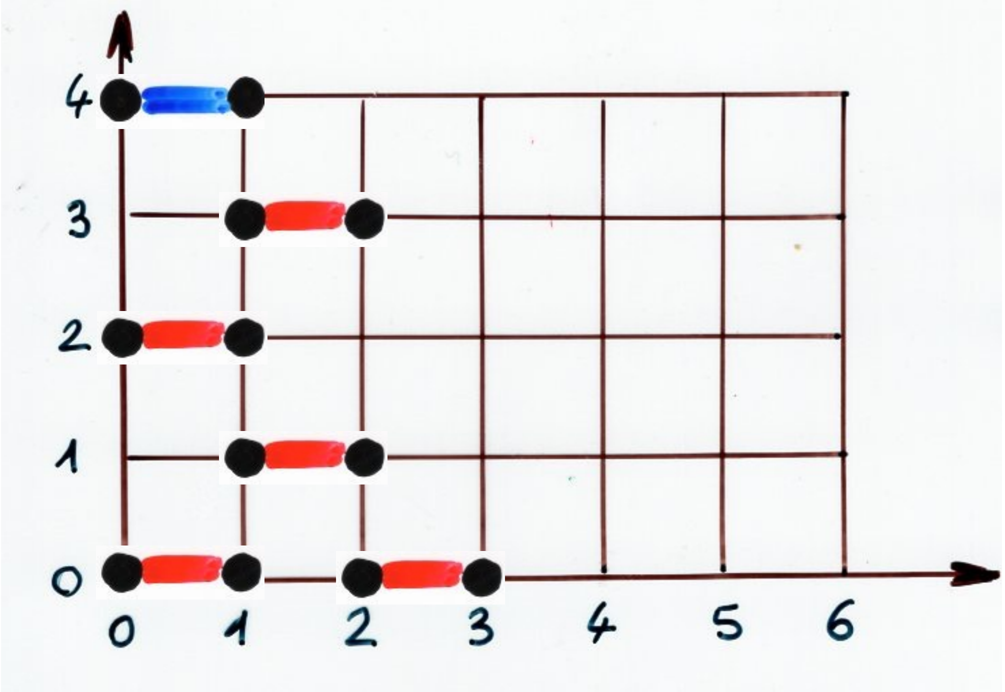
Def- Heap having only one maximal piece



π
P
domino
B = ℝ × ℝ

semi-pyramid of dimers on $\mathbb{N}$

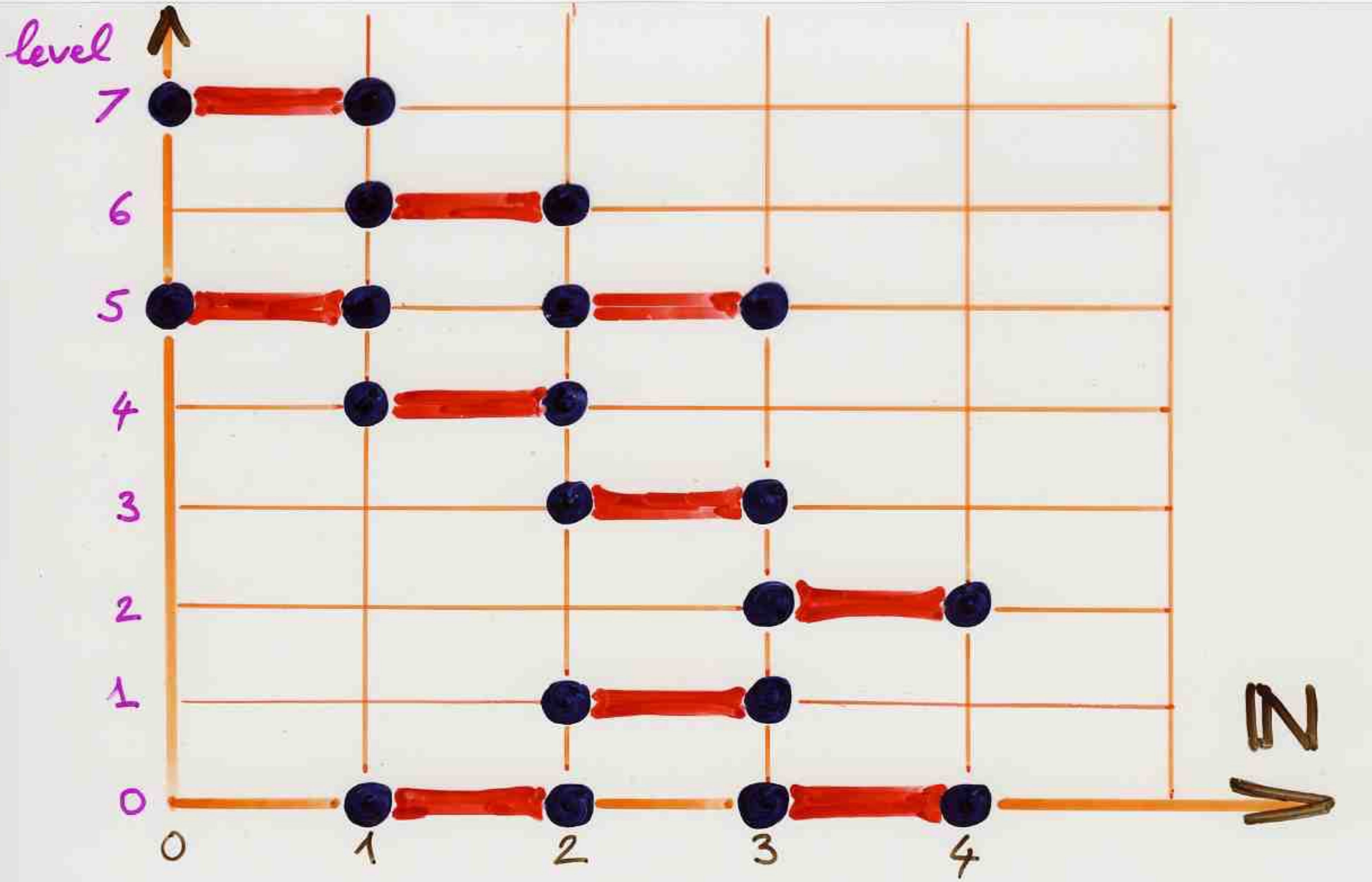
the unique maximal piece has projection $[0, 1]$



The number of semi-pyramids of dimers on $\mathbb{N}$ with n dimers is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$C_n = 1, 2, 5, 14, 42, \ldots$$


level
7
6
5
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3
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1
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0
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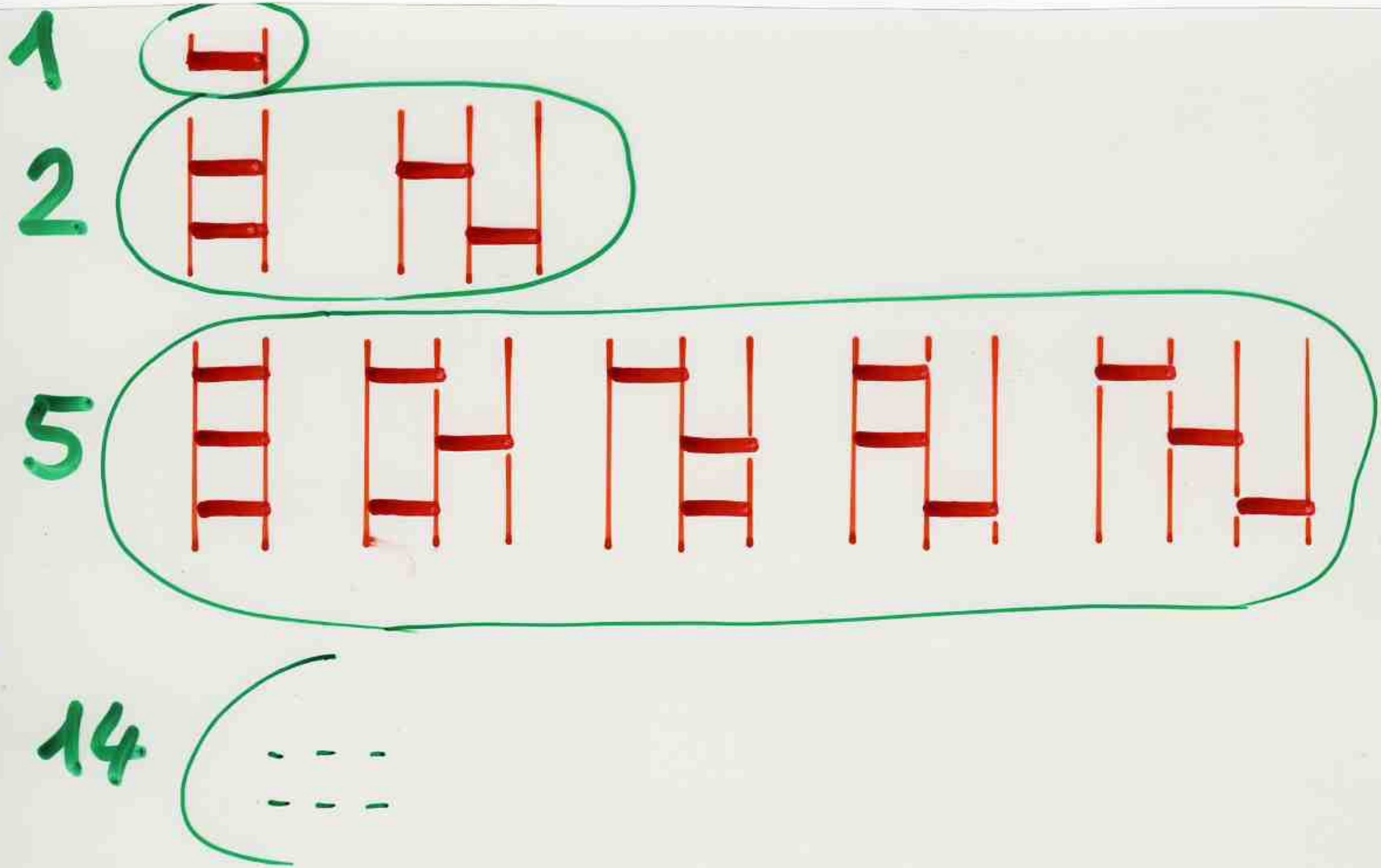
2
3
4
ℕ

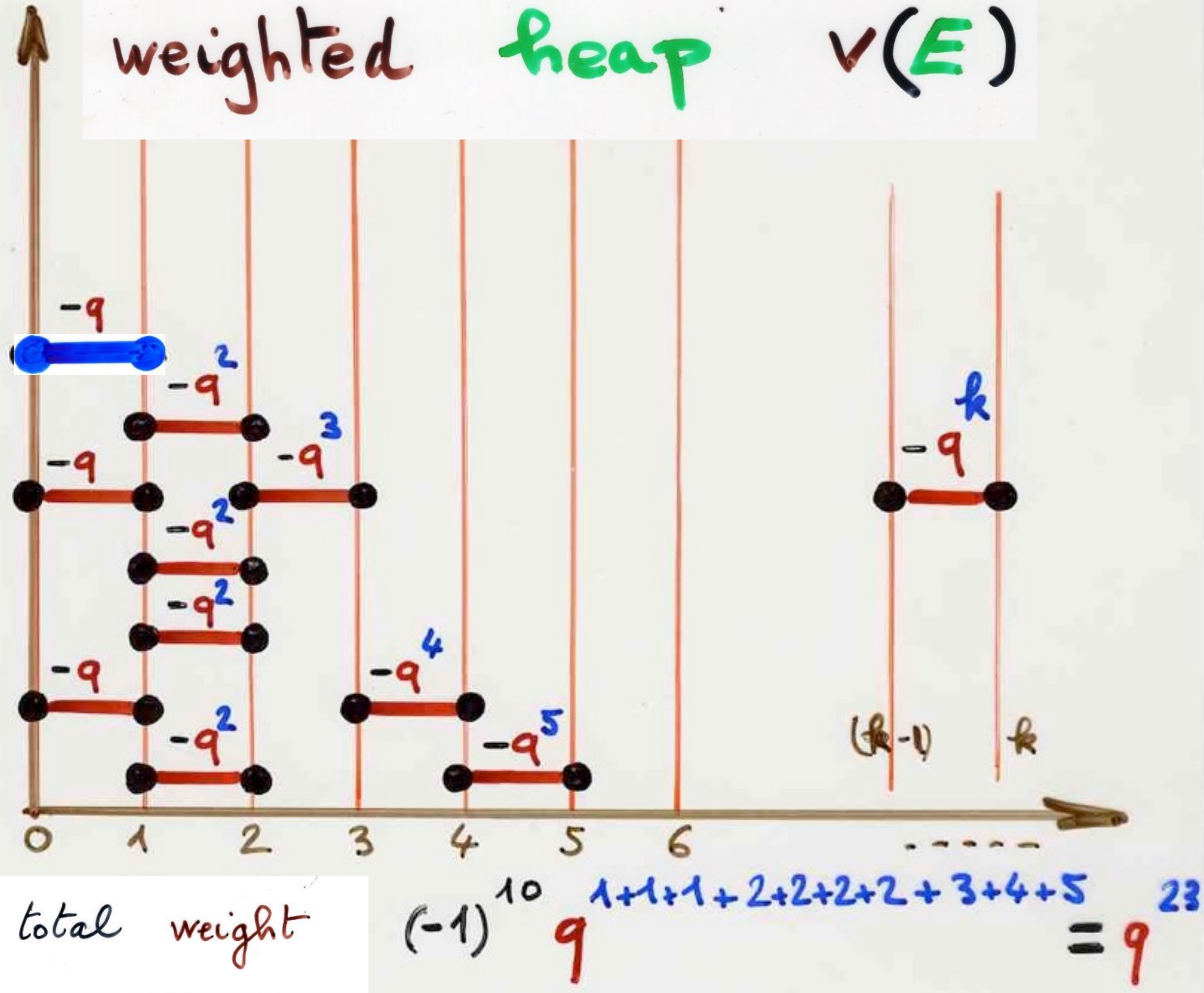

1
2
5
14

weighted heap $v(E)$

$$v(E) = \prod_{\alpha \in E} v(\alpha)$$

$$v(\alpha) = v(\pi(\alpha))$$ π "projection"

$$v([i-1, i]) = -q^{i}$$



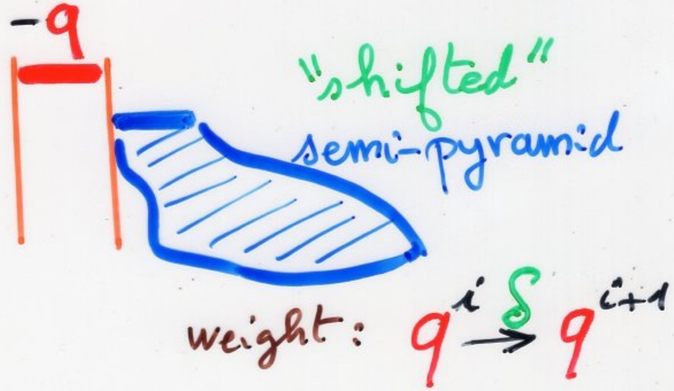
weighted heap v(E)
-q
-q2
-q3
-q
-q2
-q2
-q
-q4
-q2
-q5
-qk
(k-1)
k
0
1
2
3
4
5
6
total weight
(-1)10 q1+1+1+2+2+2+2+3+4+5
= q23

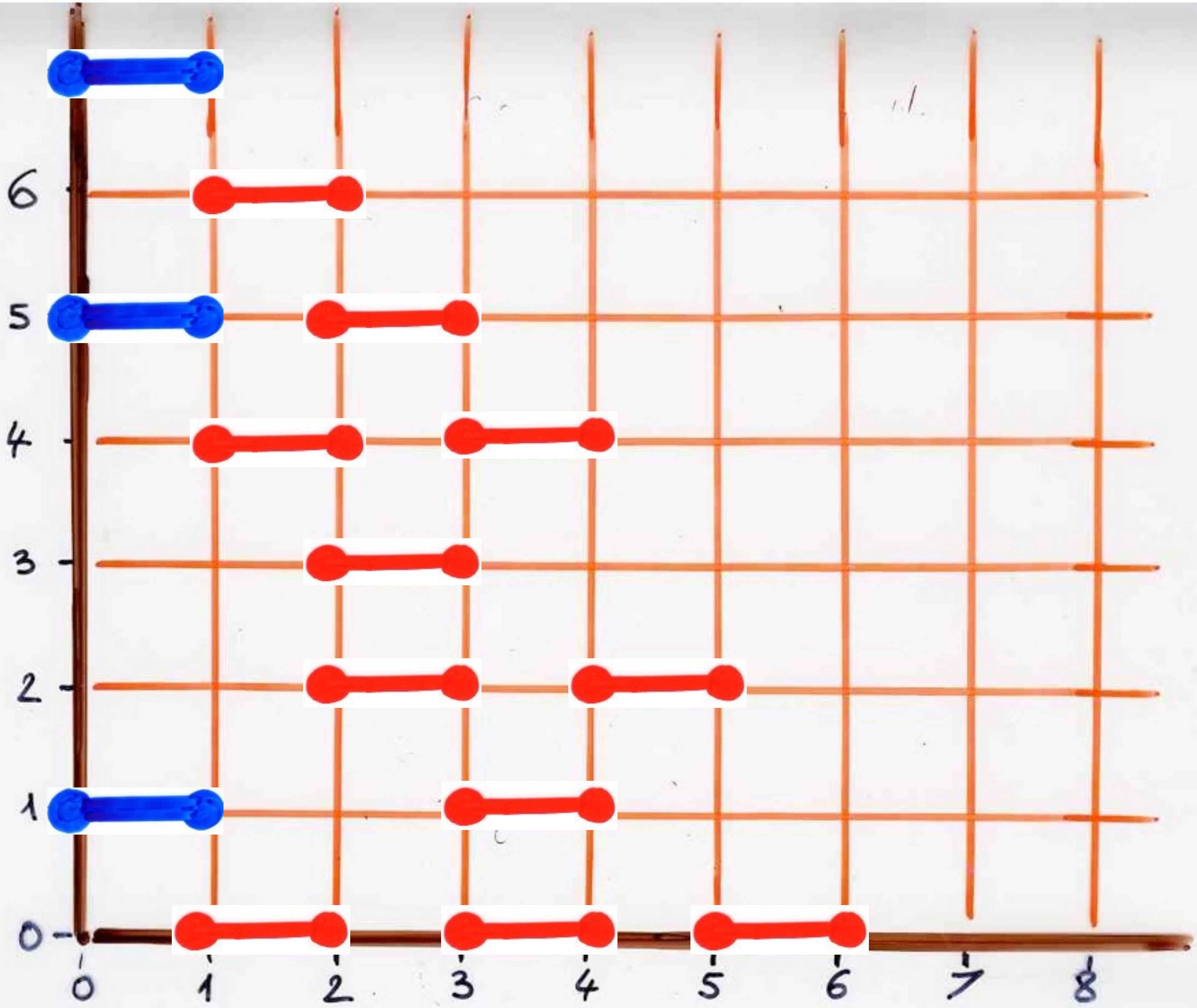
$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$

Semi-pyramid

= sequence of "primitive" semi-pyramids

"Primitive" semi-pyramid =





0
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5
6
7
8
0
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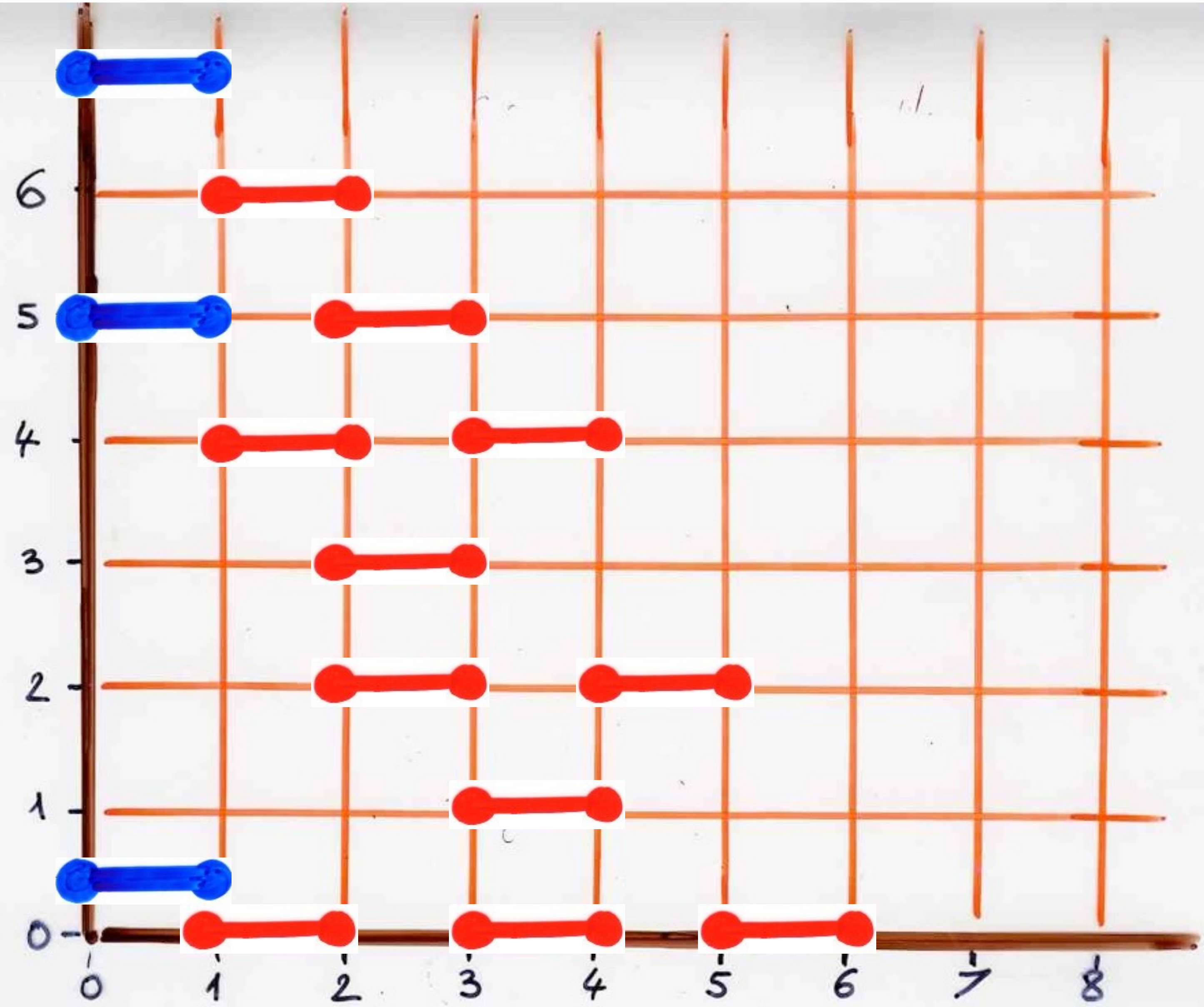
4
5
6



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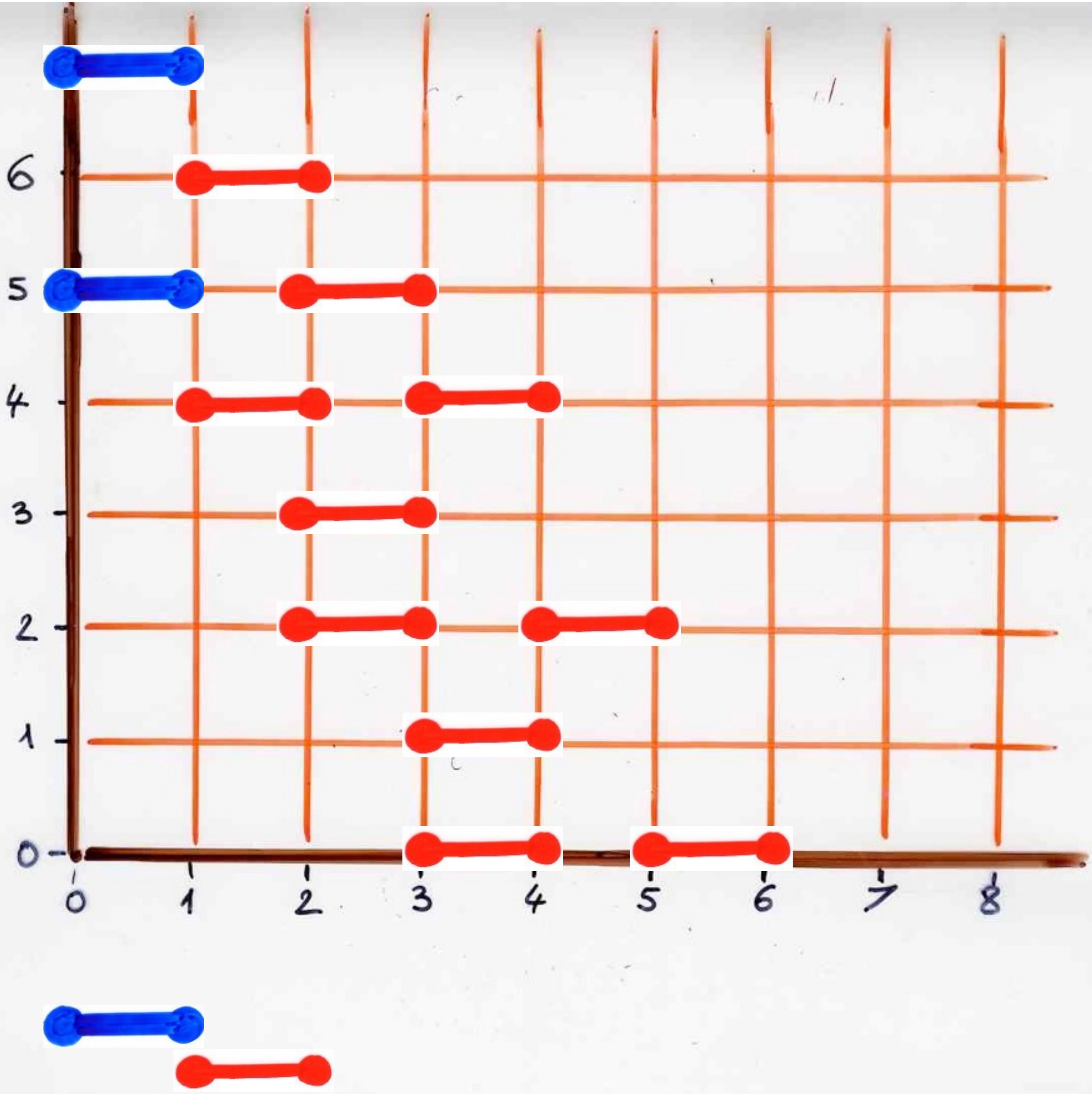

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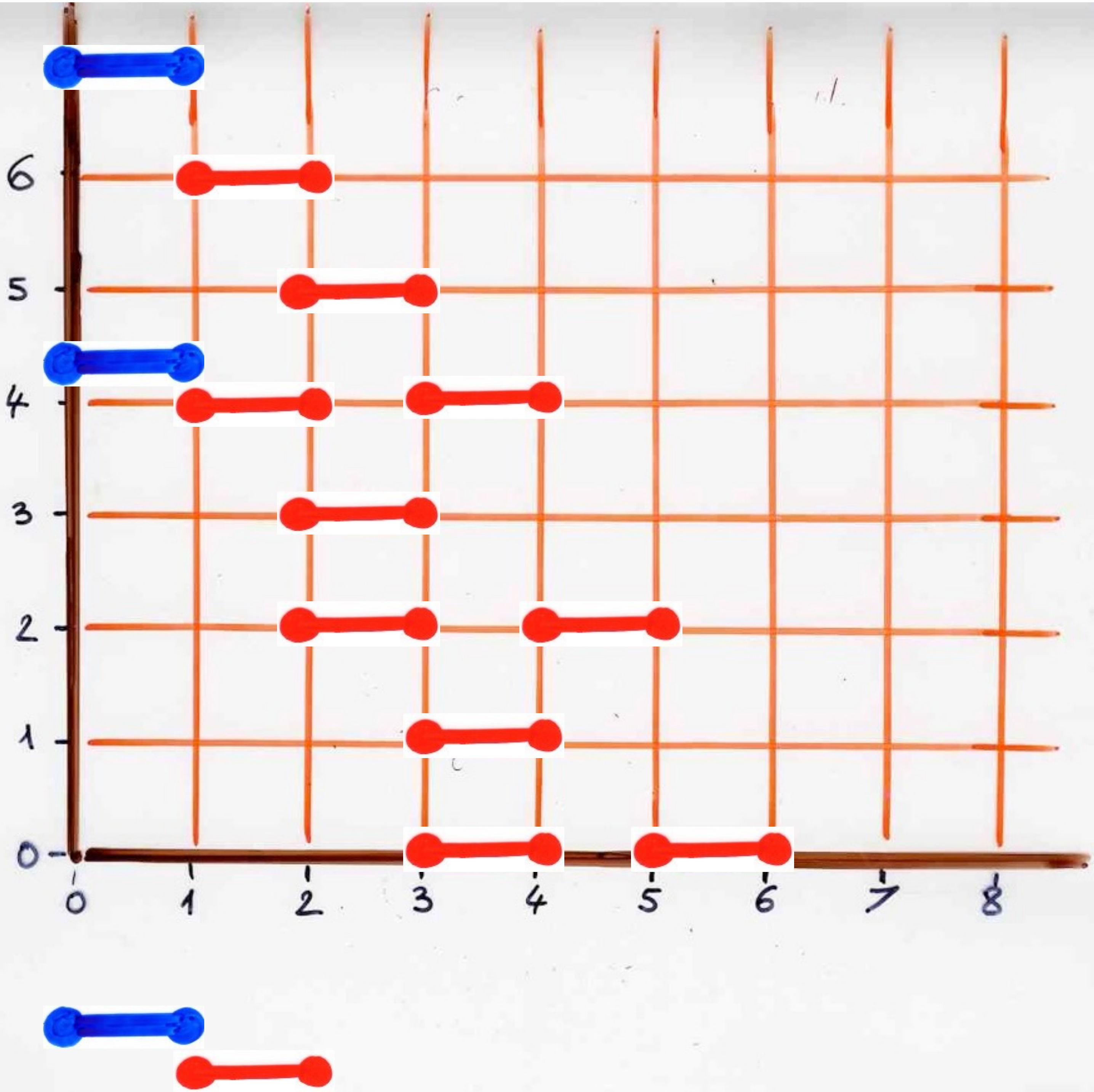

6
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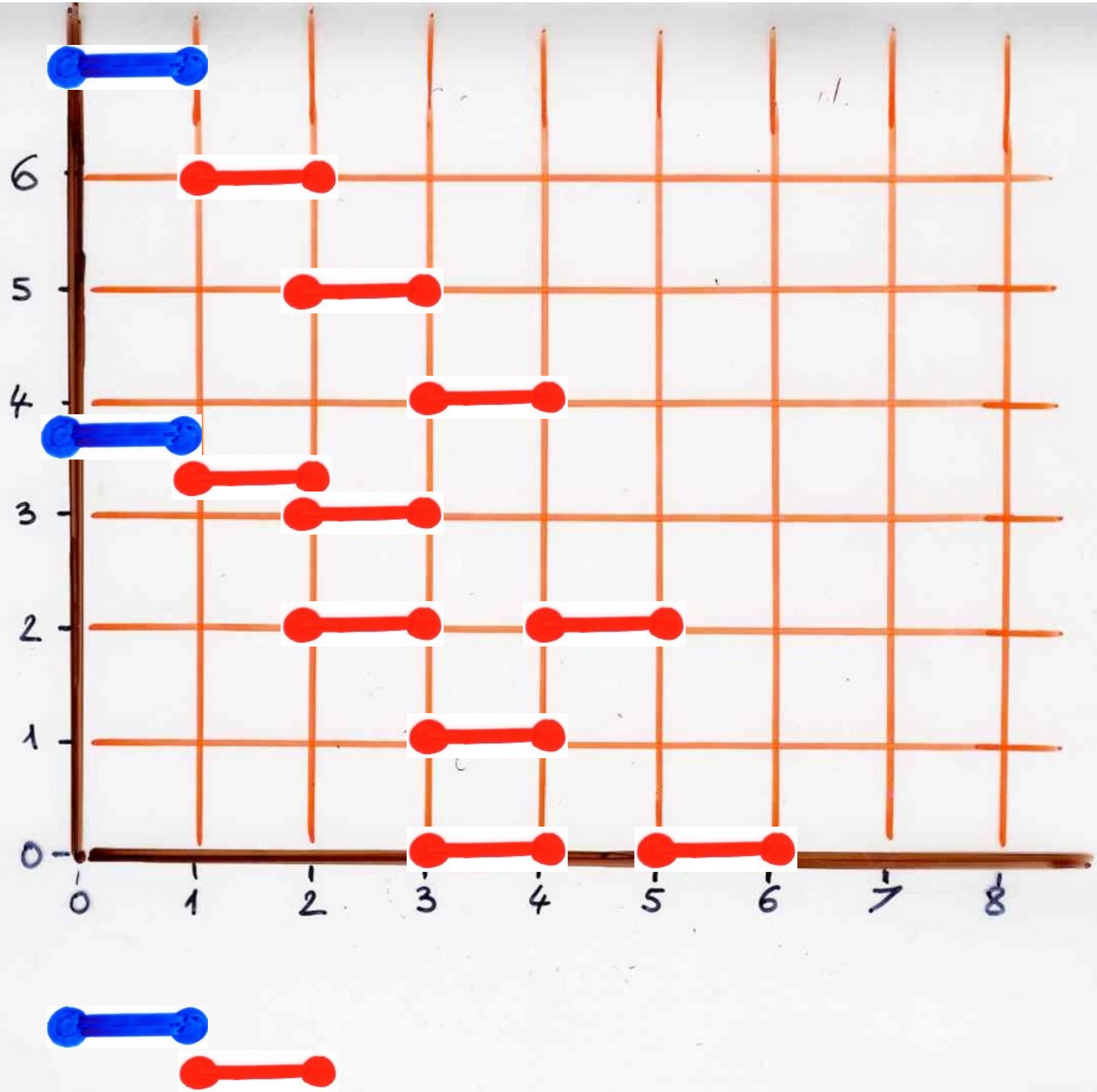

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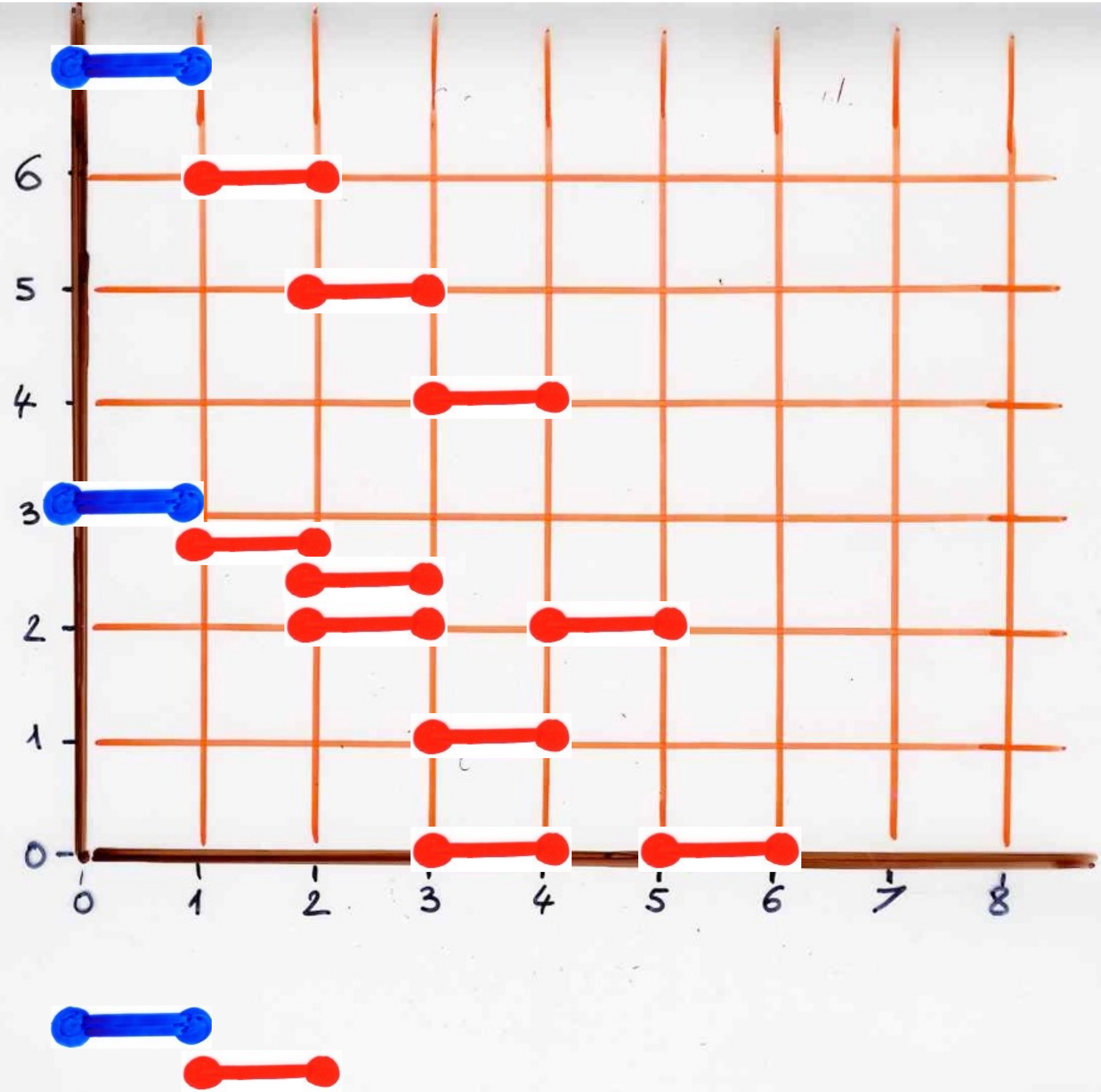
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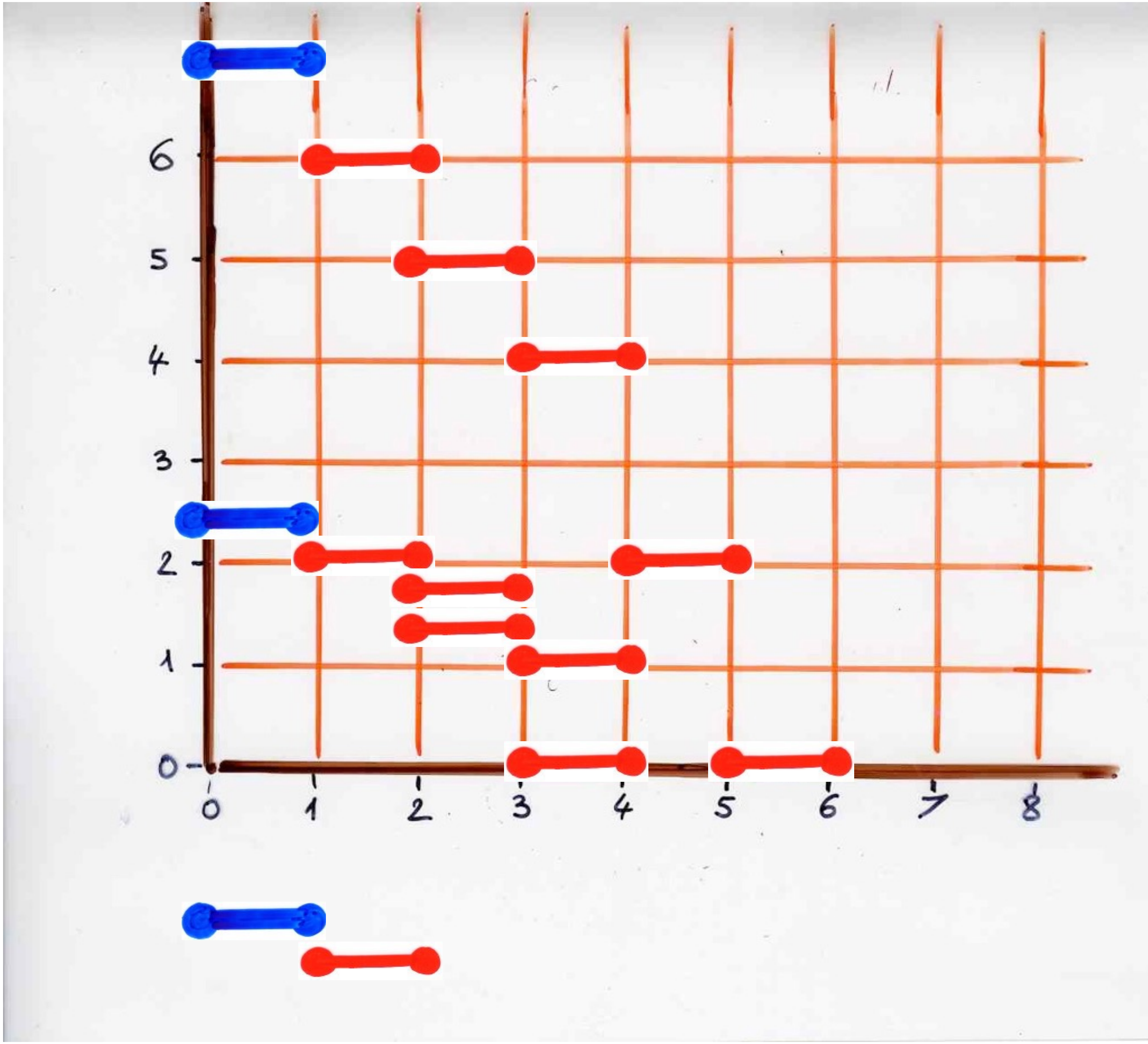

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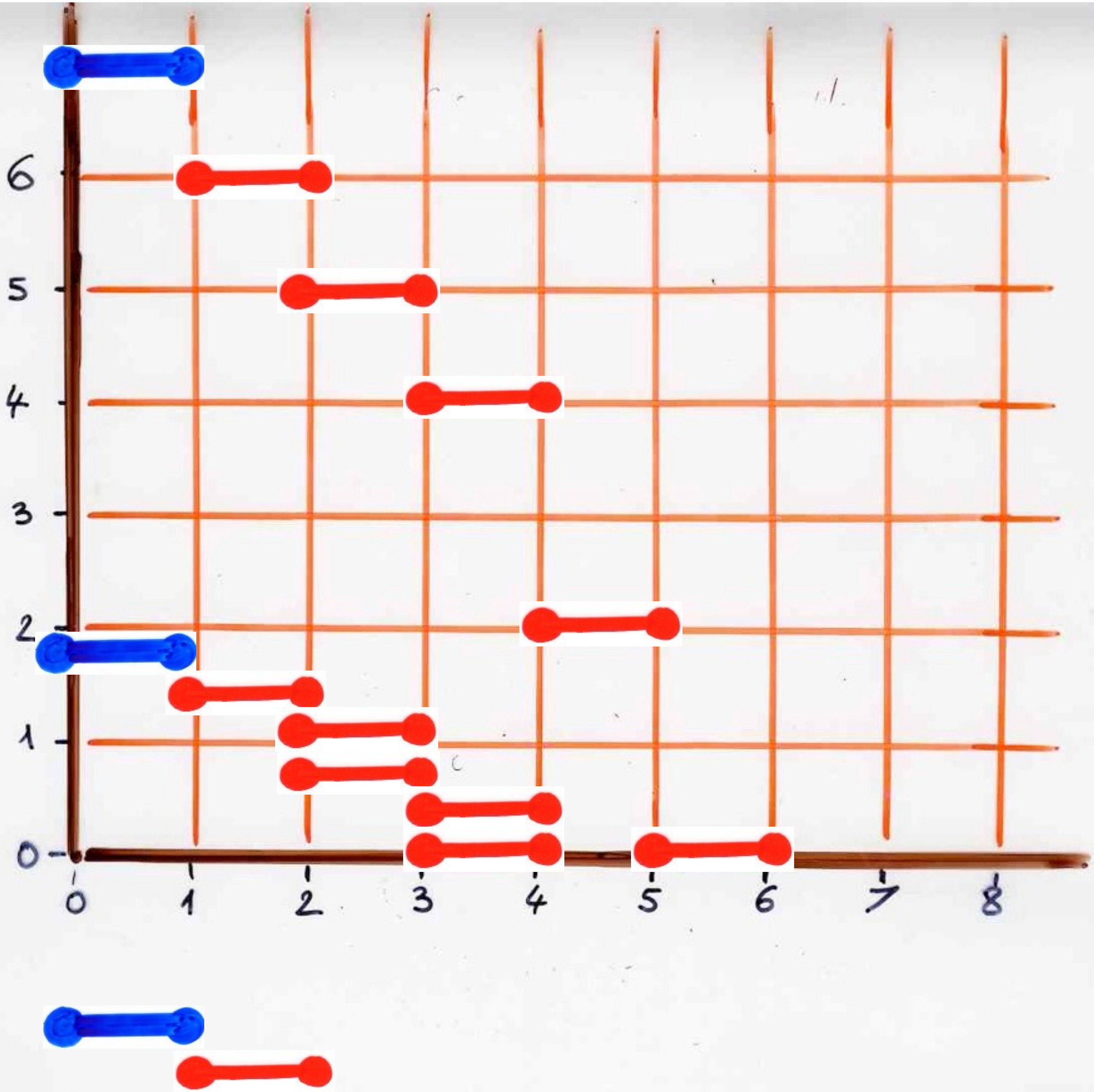

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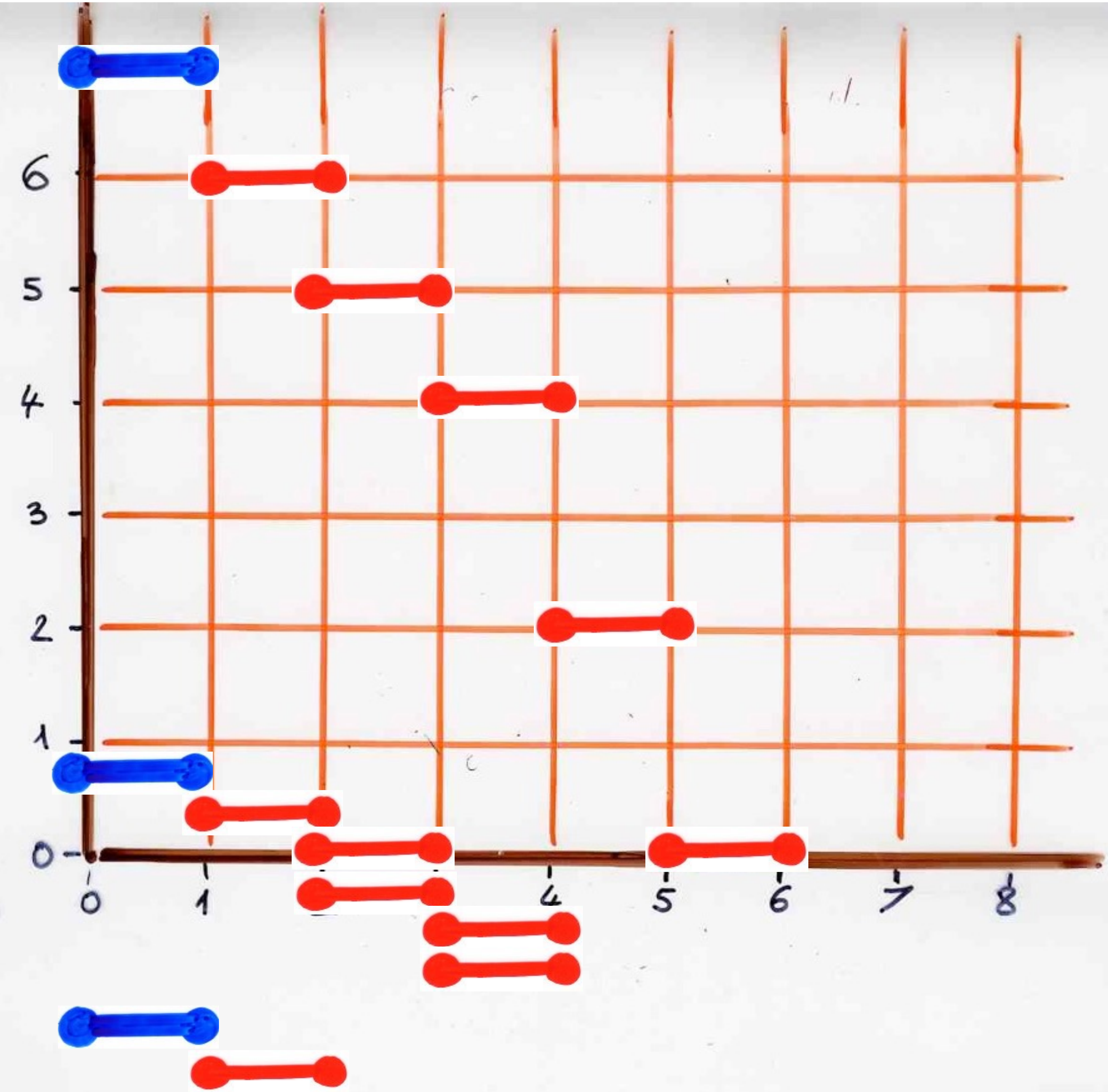

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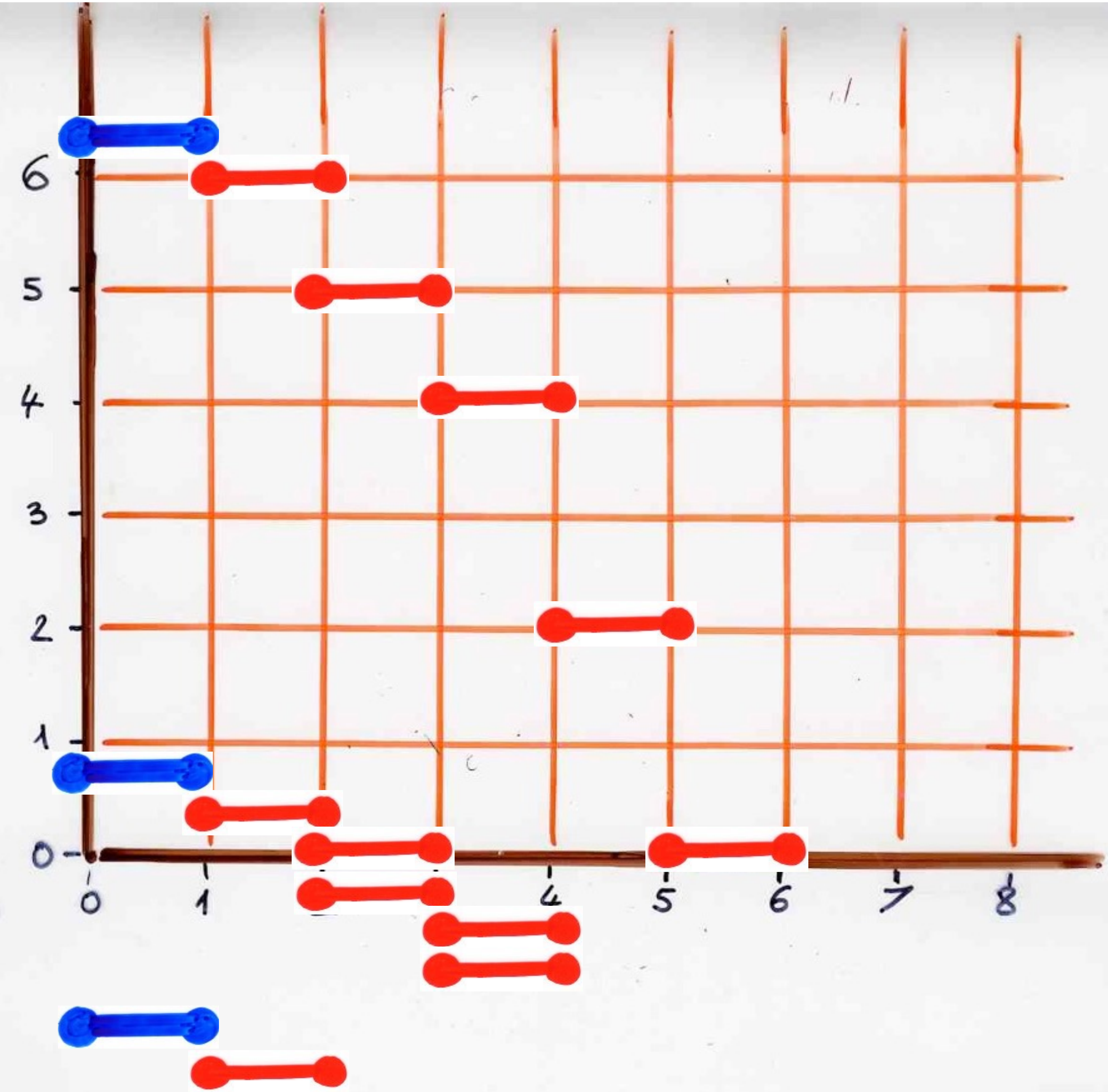
8



0
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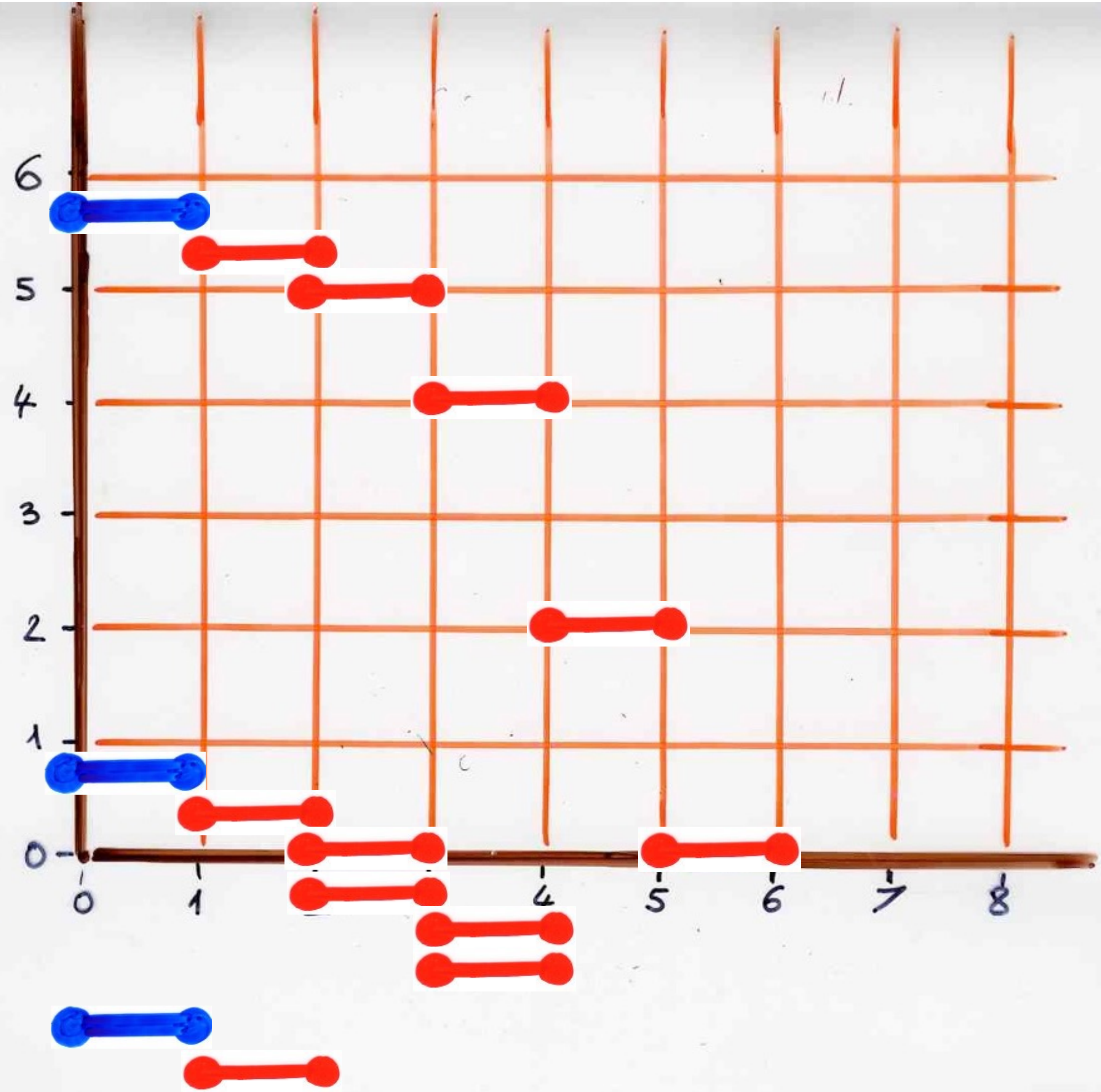

6
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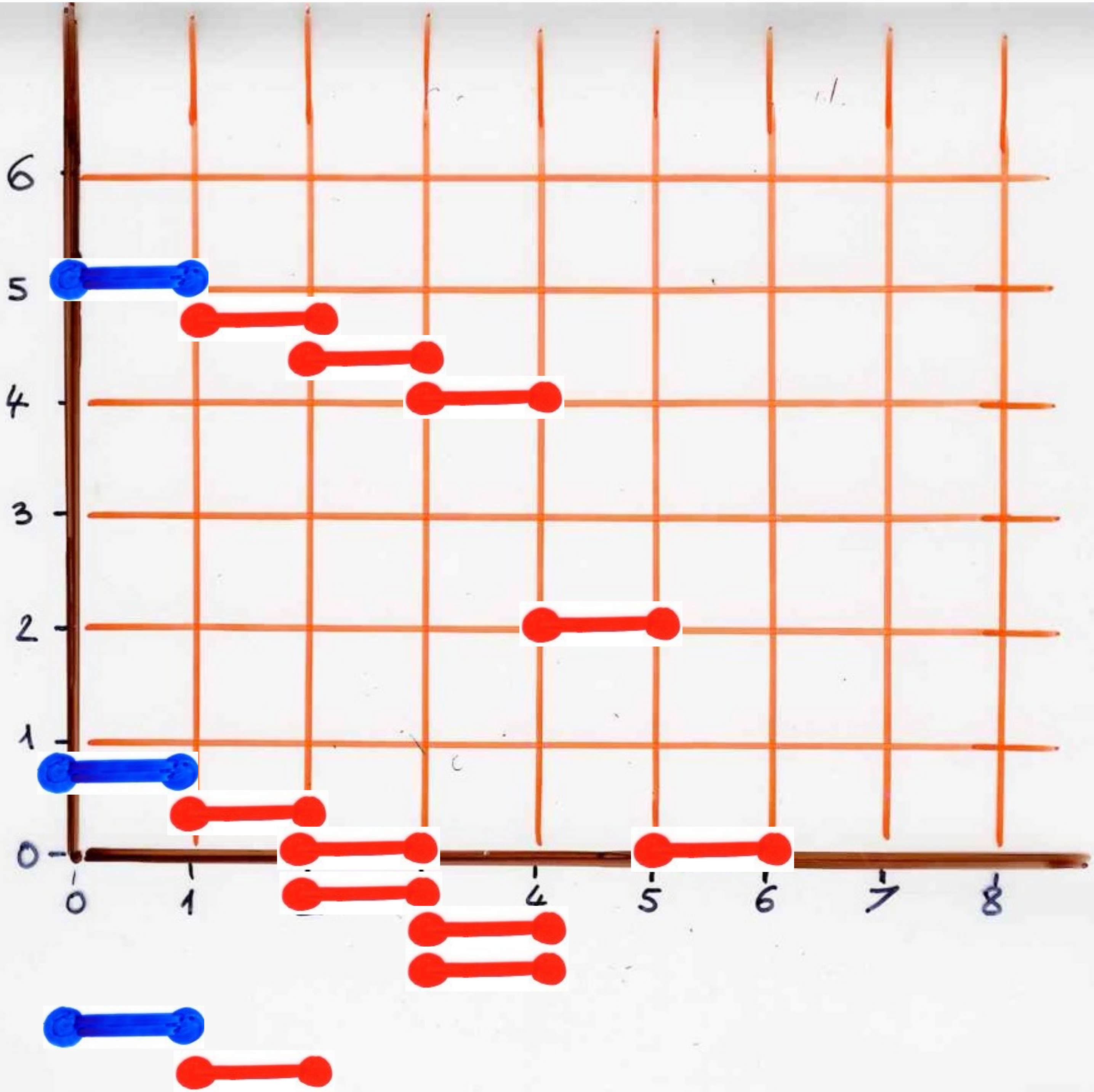
6
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4
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7
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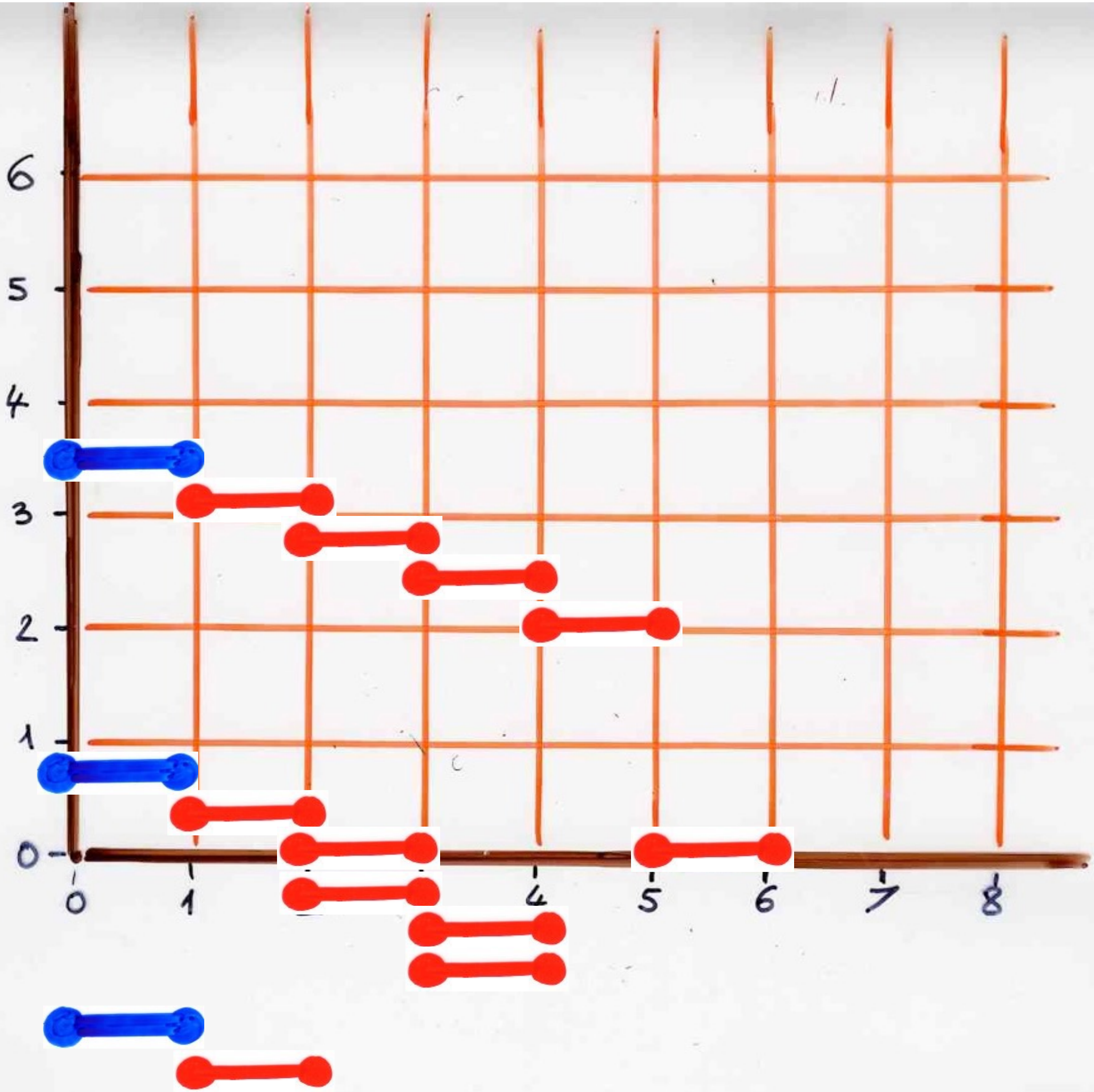
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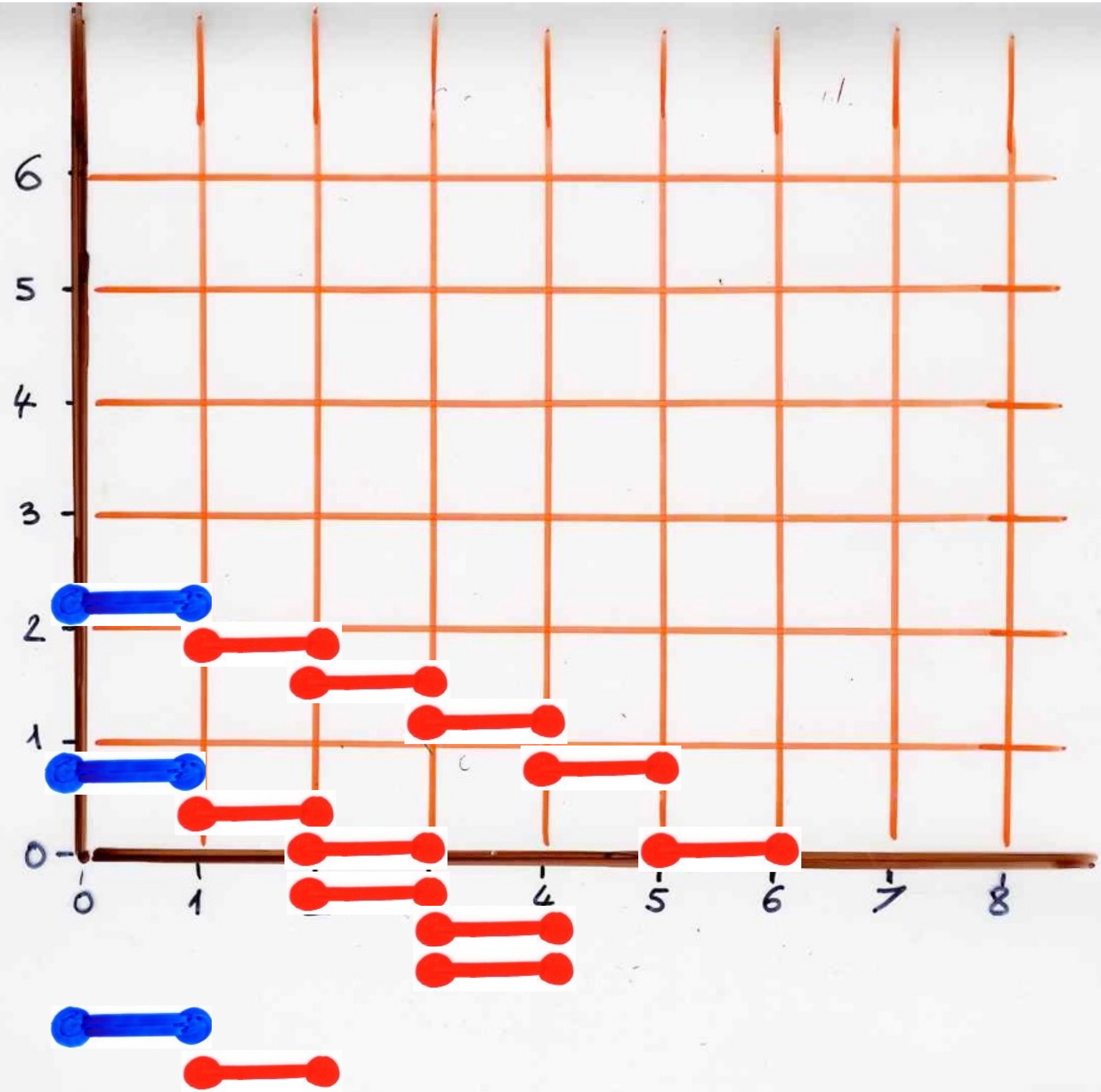
6
5
4
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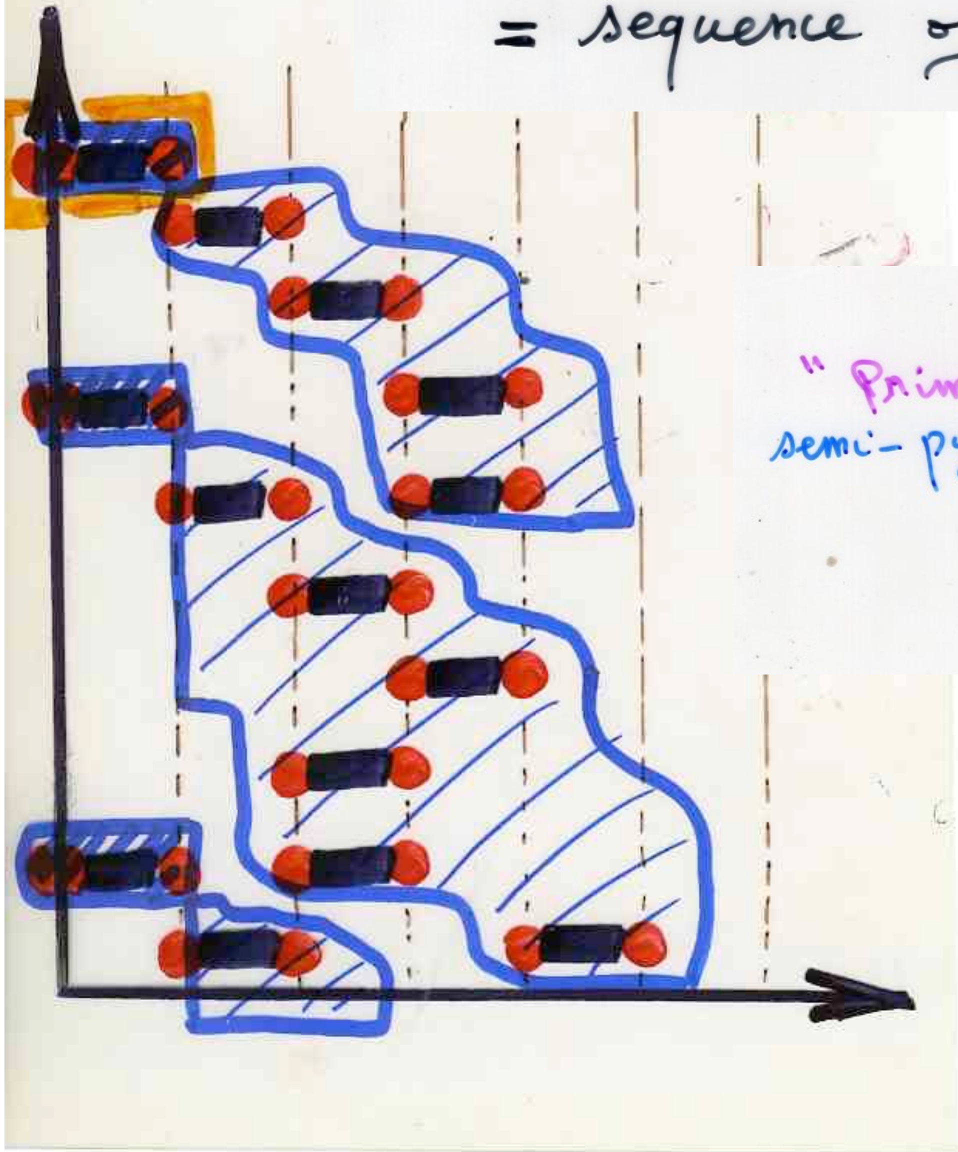
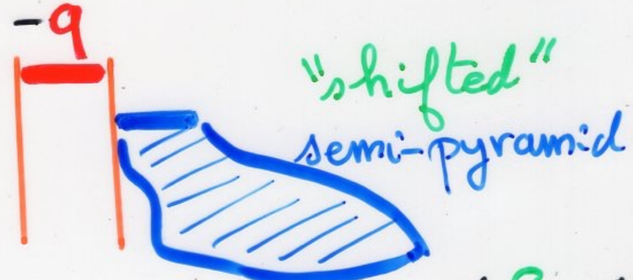
1
0
0
1
4
5
6
7
8

Semi-pyramid

= sequence of "primitive" semi-pyramids



"Primitive" semi-pyramid =



weight: $q^i \xrightarrow{\delta} q^{i+1}$

$$\sum_{\substack{E \\ \text{semi-pyramids}}} v(E) = \frac{1}{1-(-q)\sum\limits_{\substack{E \\ \text{semi-pyramids}}} \delta v(E)}$$

$$= \frac{1}{1+\dfrac{q}{1+q^2\sum\limits_{\substack{E \\ \text{semi-pyramids}}} \delta^2 v(E)}}$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$

The inversion lemma

1/D

weighted heap $v(E)$

$$v(E) = \prod_{\alpha \in E} v(\alpha)$$

$$v(\alpha) = v(\pi(\alpha))$$ π "projection"

$$v([i-1, i]) = -q^{i}$$

Inversion lemma

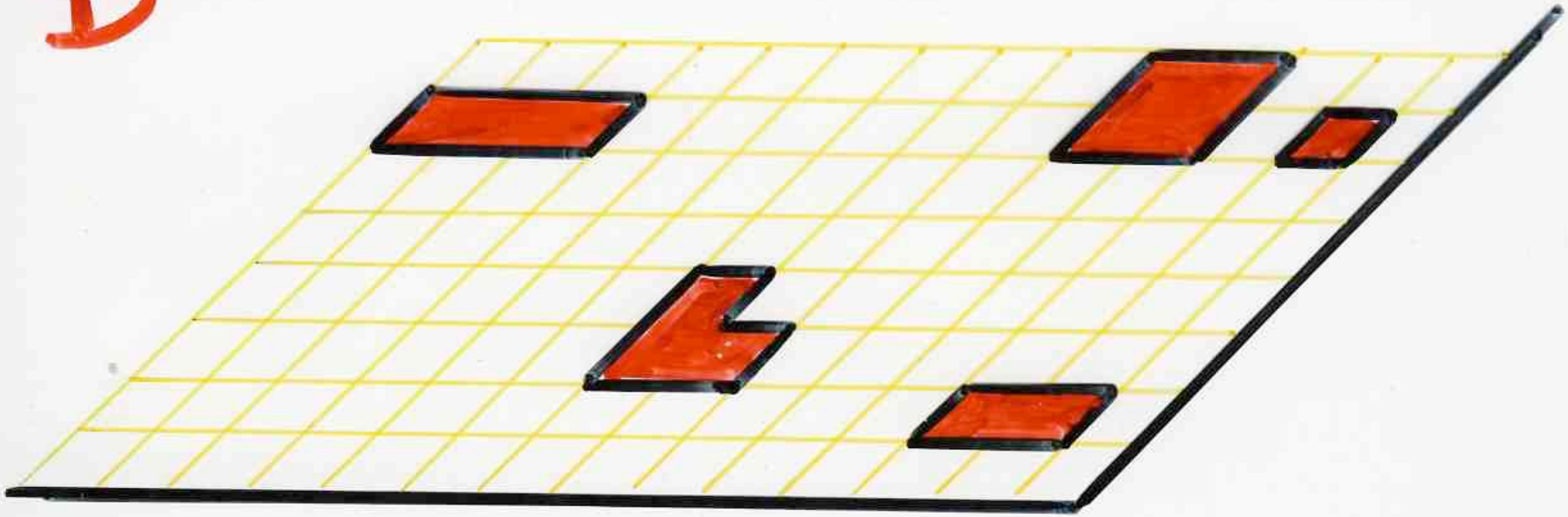
$$\sum_{\substack{E \\ \text{heaps}}} v(E) = \frac{1}{\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)}$$

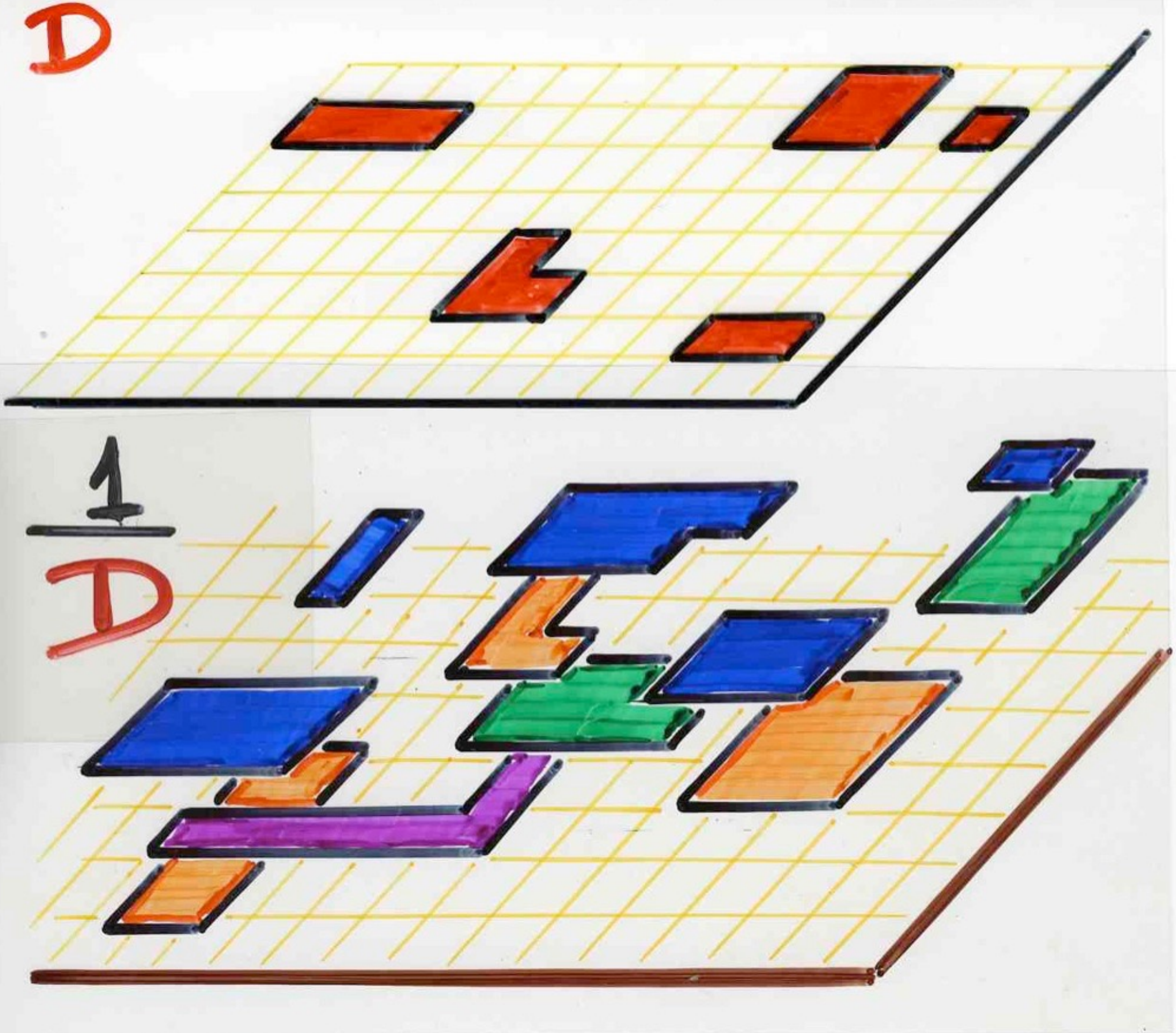
Inversion lemma

$$\sum_{\substack{E \\ \text{heaps}}} v(E) = \frac{1}{\sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)}$$

D

D




D
1/D

Extension of the inversion lemma

N/D

extension of the inversion lemma

$M \subseteq P$

$$\sum_{E} v(E) = \frac{N}{D}$$

π(maximal pieces) $\in M$

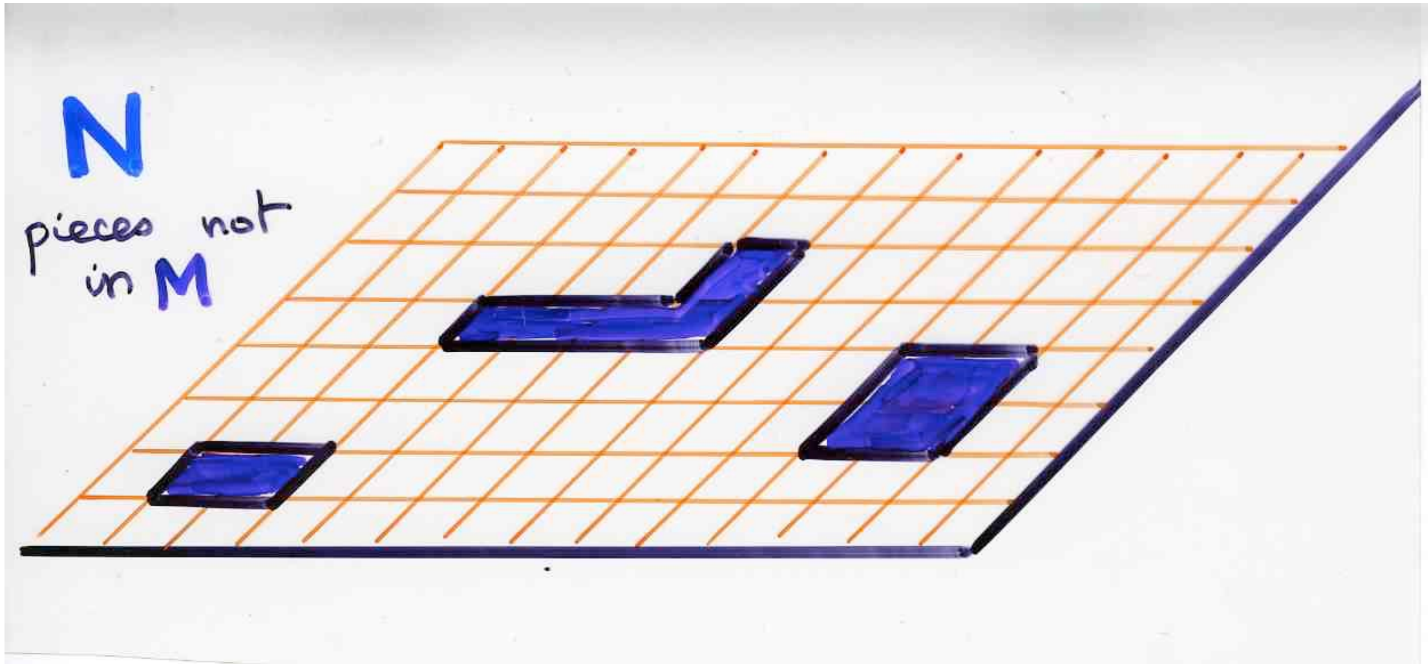
$$D = \sum_{F} (-1)^{|F|} v(F)$$

trivial heaps

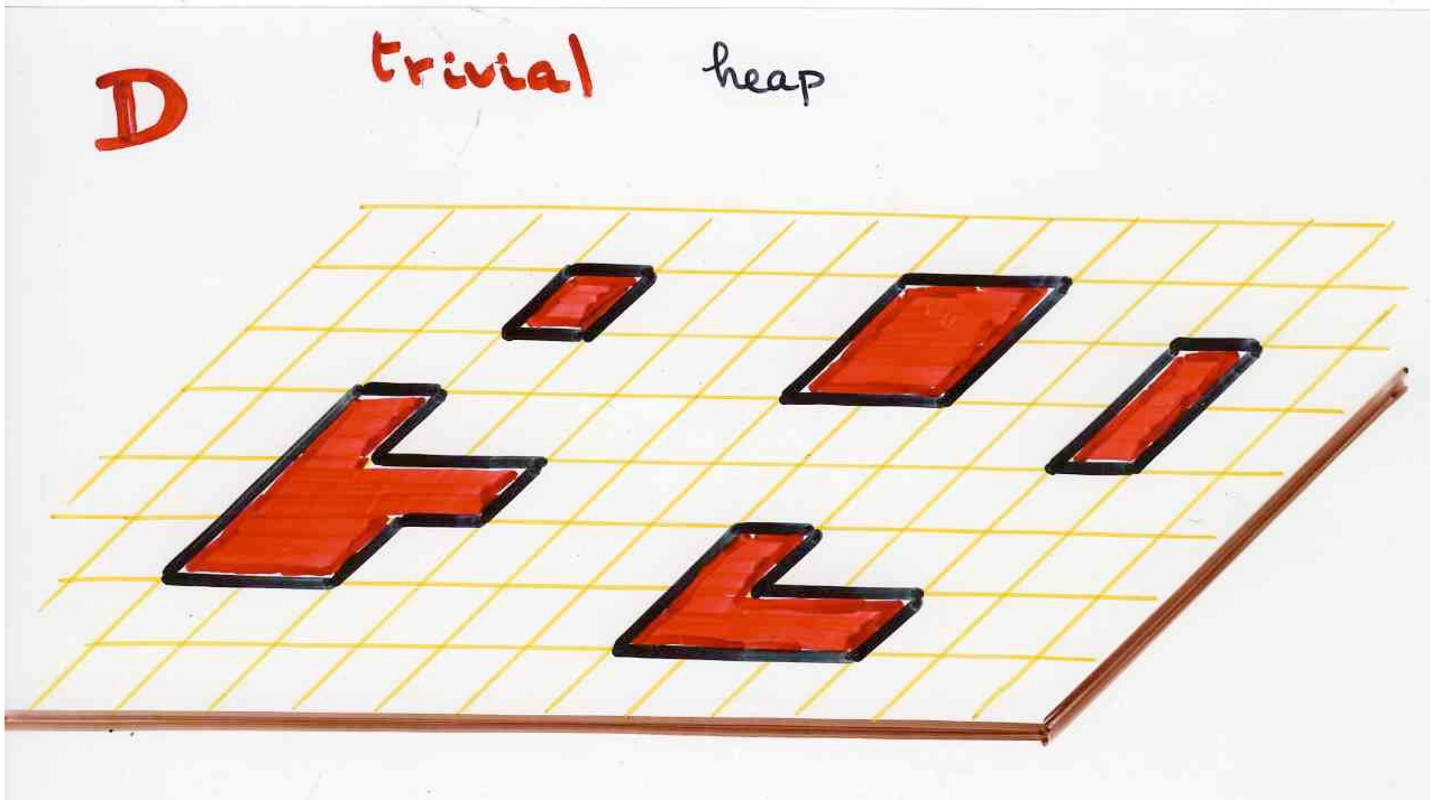
$$N = \sum_{F} (-1)^{|F|} v(F)$$

trivial heaps

pieces $\notin M$



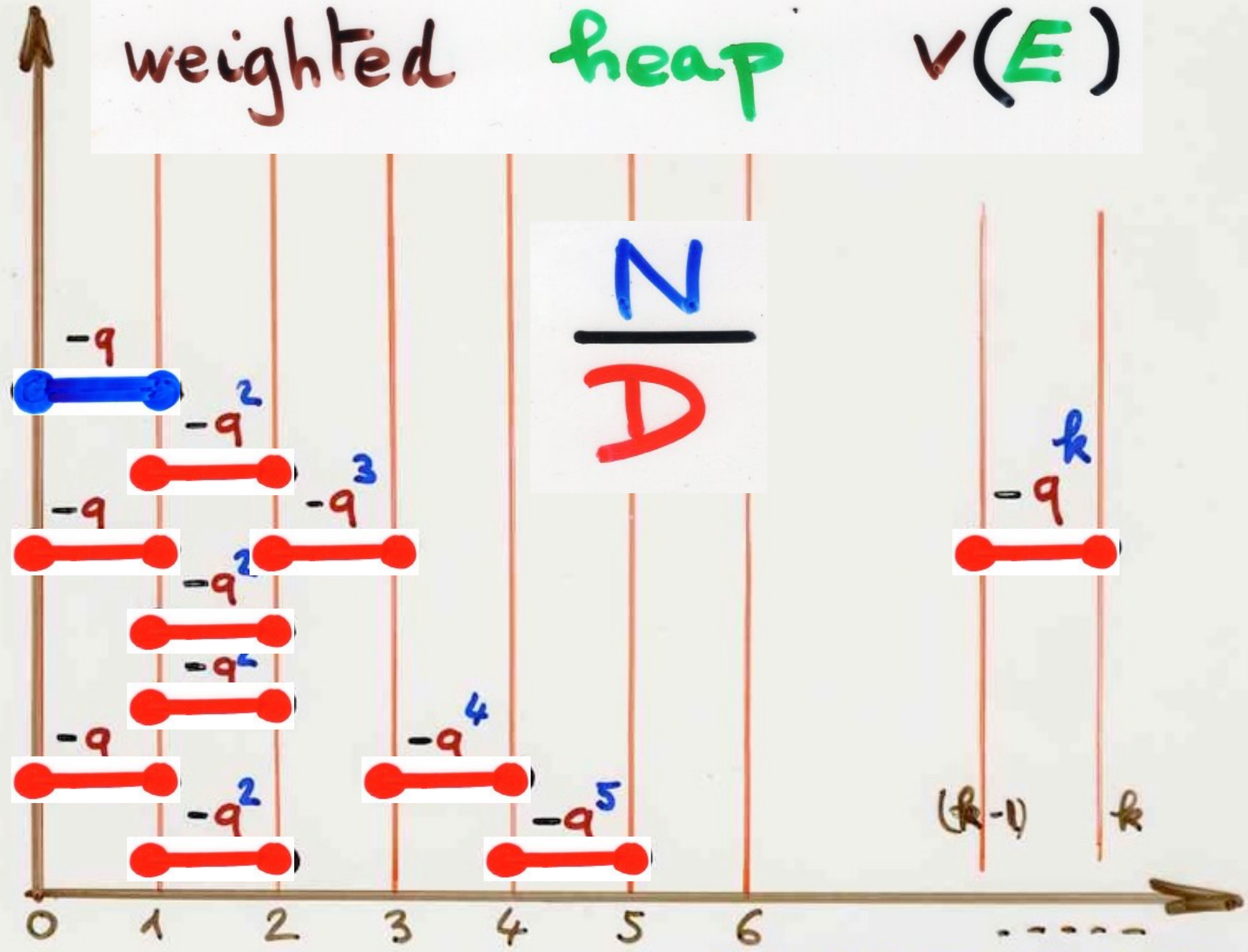
N
pieces not in M



D
trivial heap

back to
Ramanujan continued fraction

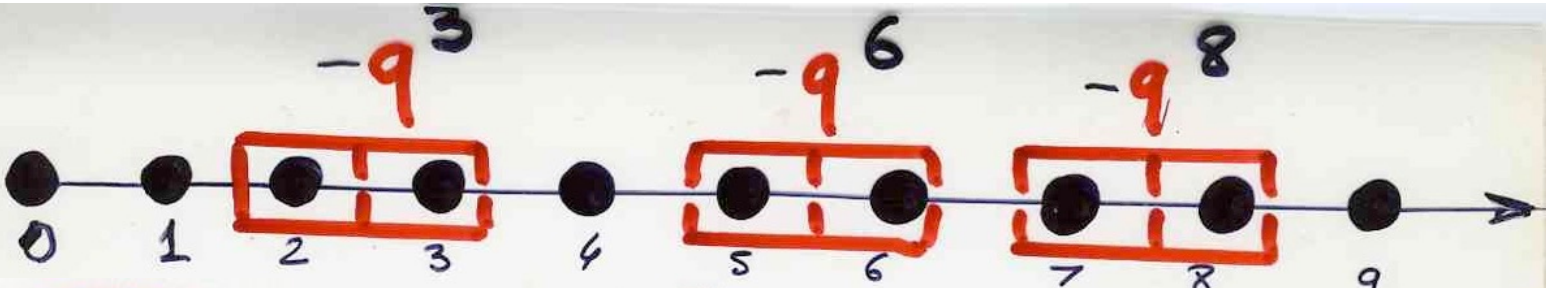
weighted heap $v(E)$



total weight $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

Rogers-Ramanujan 1st identity

$$D = \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$



$$D = \sum_{E} (-1)^{|E|} v(E)$$

E: trivial heaps of dimers on $\mathbb{N}$

$$v([k-1, k]) = -q^k$$

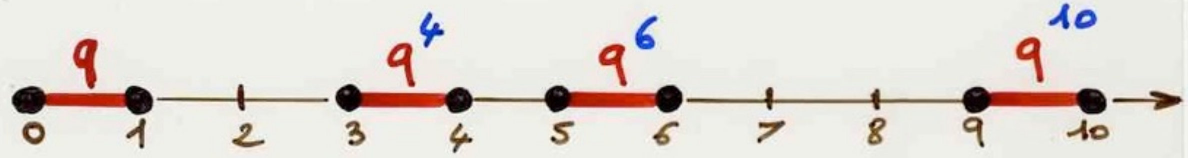
Rogers-Ramanujan

1st identity

$$D = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

poids total

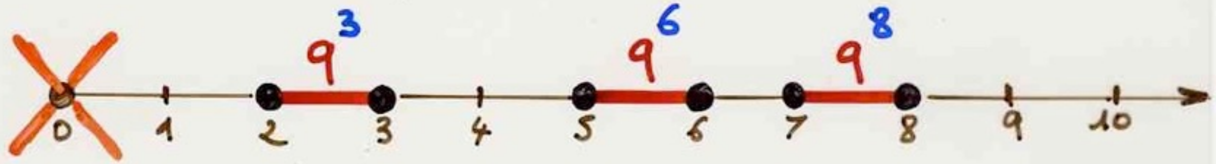
q^{21}



D-partition $\lambda = (10, 6, 4, 1)$

$21 = 10 + 6 + 4 + 1$

q^{17}



$\lambda = (8, 6, 3)$

$17 = 8 + 6 + 3$

$$N = \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \delta D$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

convergents for $q=1$

for $q = 1$

$$\rightarrow \phi - 1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}$$

ϕ golden ratio $\dfrac{1+\sqrt{5}}{2}$

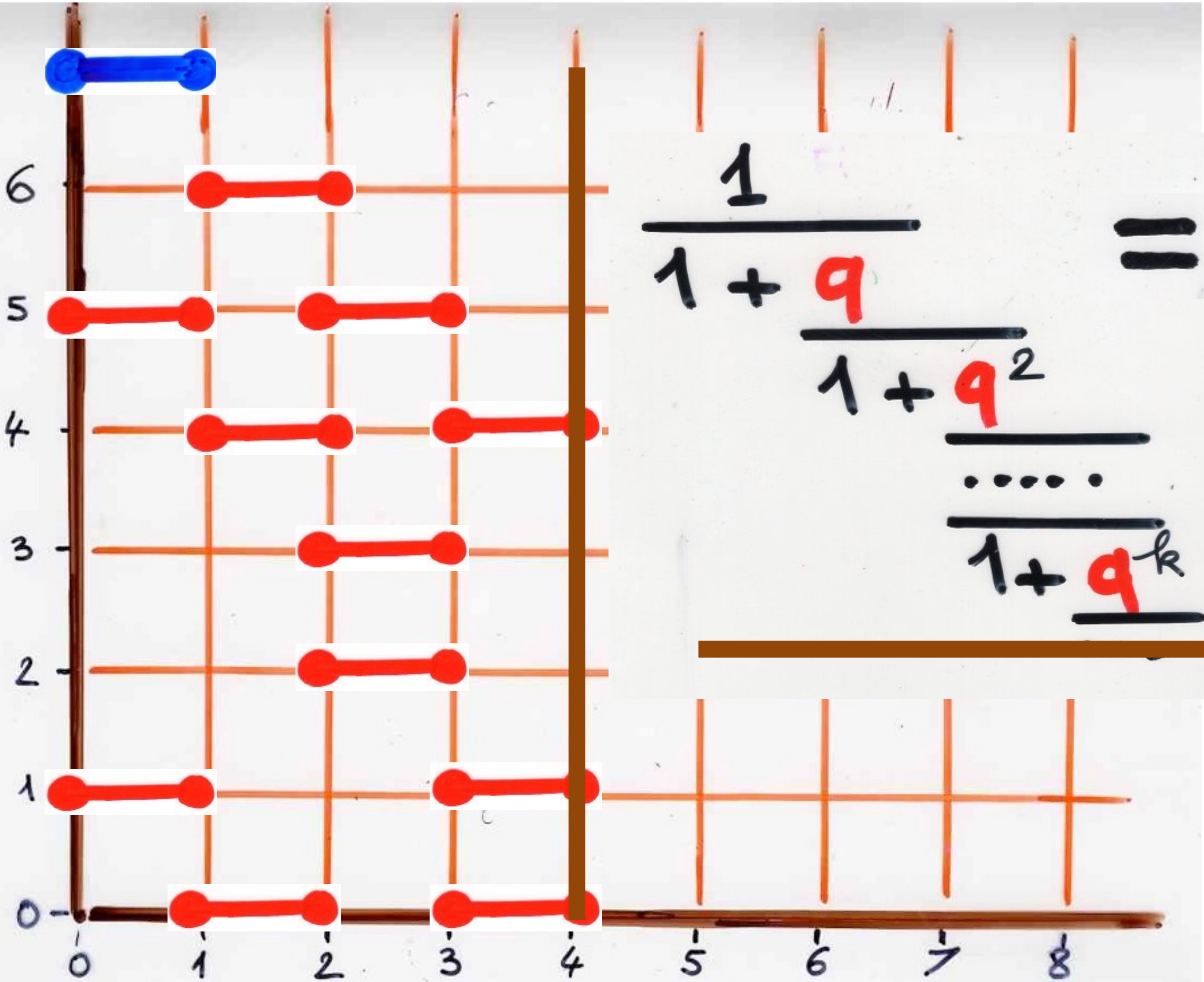
convergent

$$\left.\cfrac{1}{1 + \cfrac{1}{1 + 1}}\right\}2 = \frac{2}{3}$$

$$\left.\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}}\right\}3 = \frac{3}{5}$$

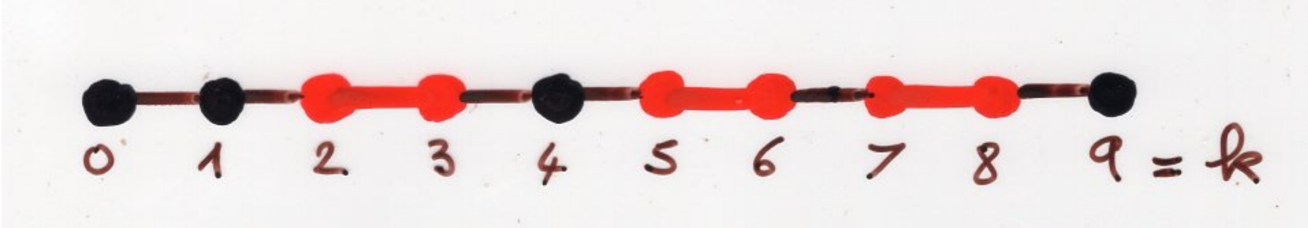
$1, 1, 2, 3, 5, 8, 13 \ldots$

$F_0 \quad F_1 \quad F_2 \; \ldots \; F_6 \; \ldots$



$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots}{1+q^k}}}} = \frac{N}{D}$$

D



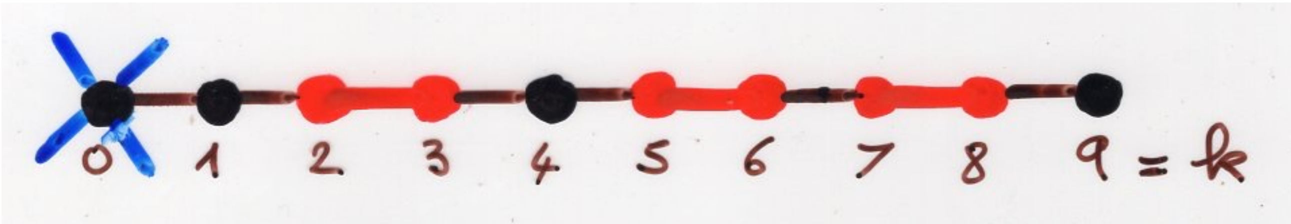
matching of [1, k]
= set of 2 by 2 disjoint edges (i, i+1)
(or dimers)

F_{k+1}

Fibonacci numbers

$$F_{k+1} = F_k + F_{k-1}$$

N



F_k

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13\ \cdots$$

$$F_0\quad F_1\quad F_2\quad \cdots\cdots\quad F_6\quad \cdots$$

Fibonacci numbers

k^{th} convergent

$$\underbrace{\cfrac{1}{1+\cfrac{1}{1+\cdots\cfrac{1}{1+1}}}}_{k} = \frac{F_k}{F_{k+1}}$$

Andrews theorem about the «reciprocal» of Ramanujan continued fraction

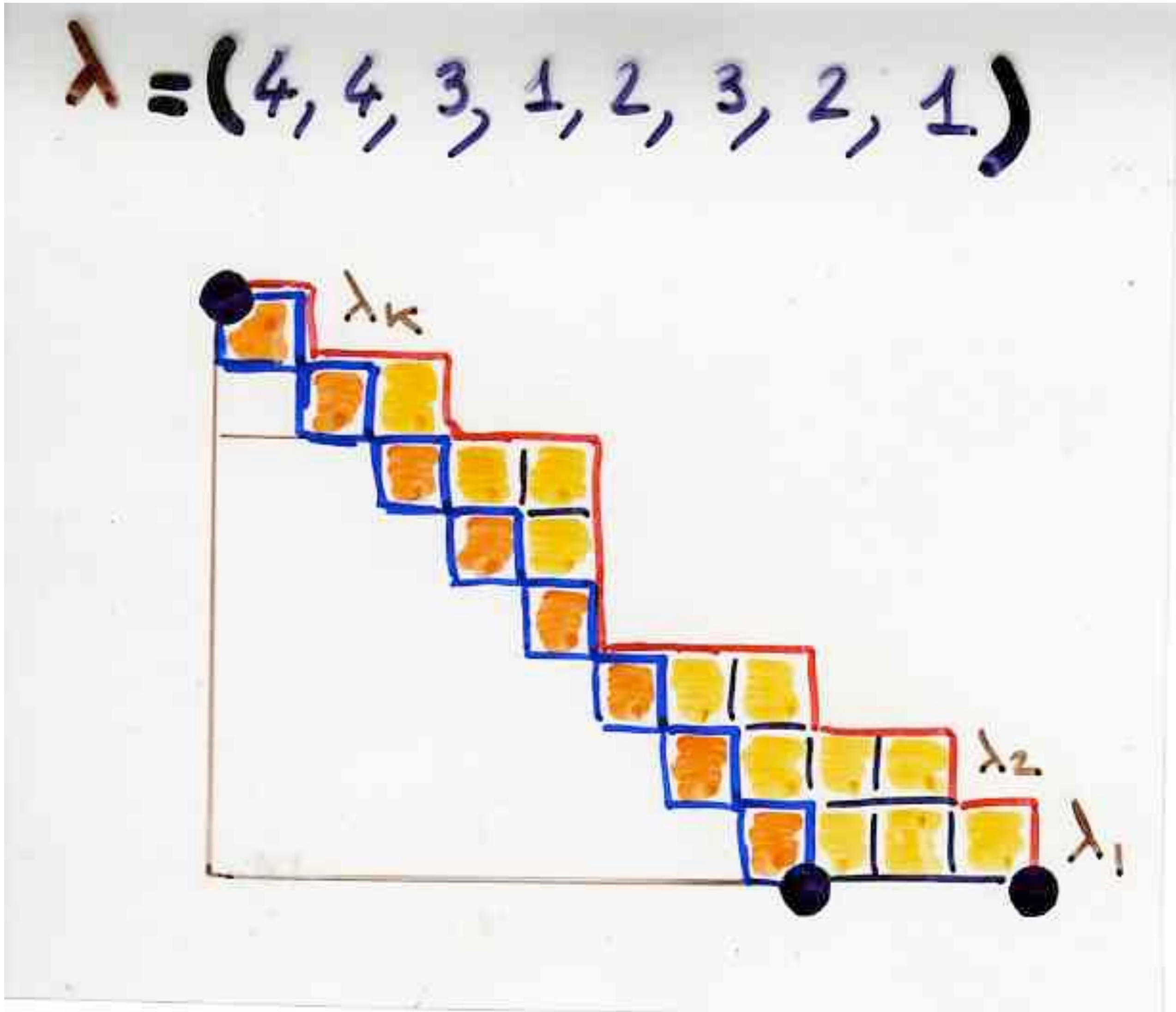
quasi-partitions of n

G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$1 + \lambda_i \geq \lambda_{i+1}$$

$i = 1, \ldots, k-1$

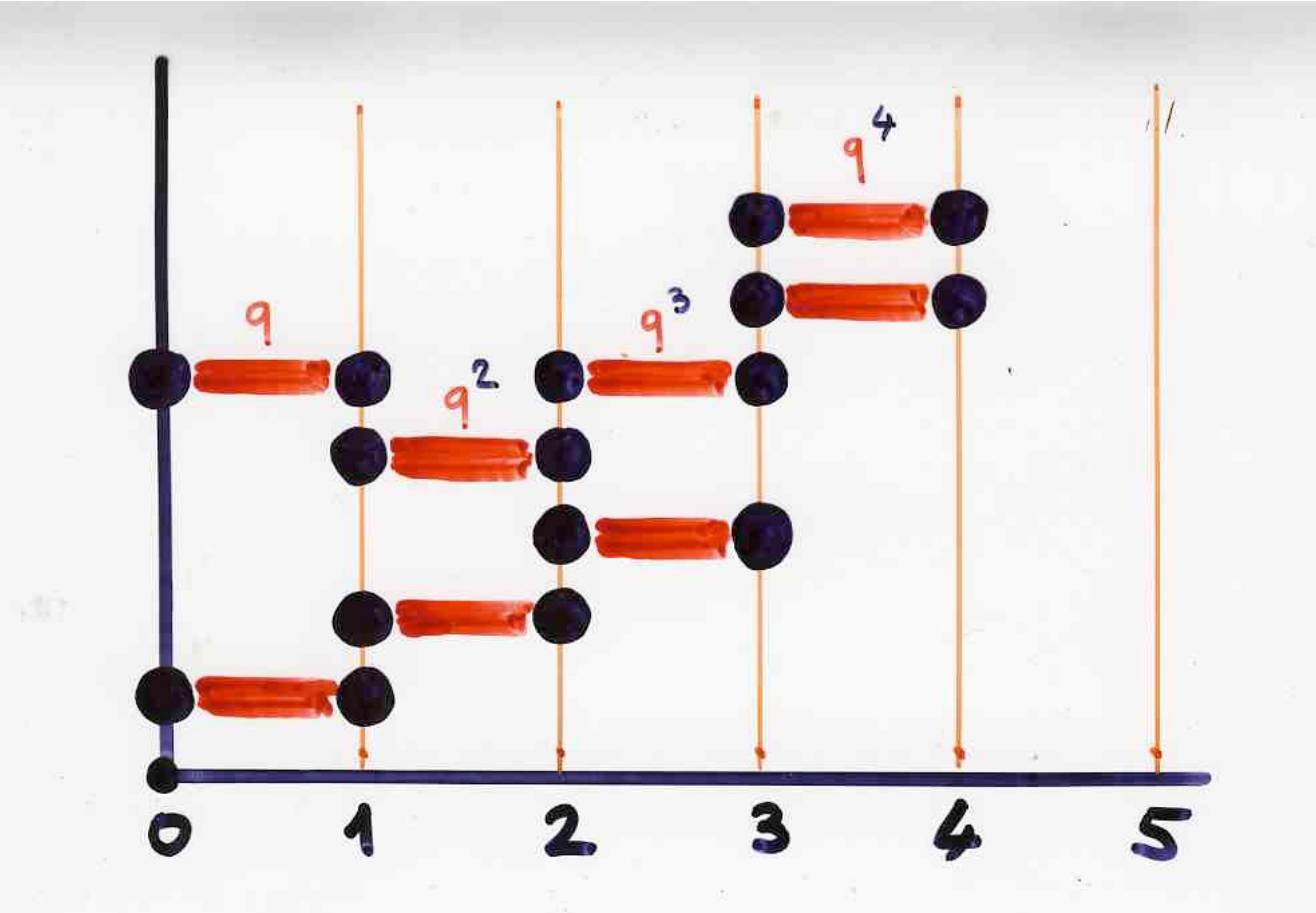


$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$
λ_k
λ_2
λ_1

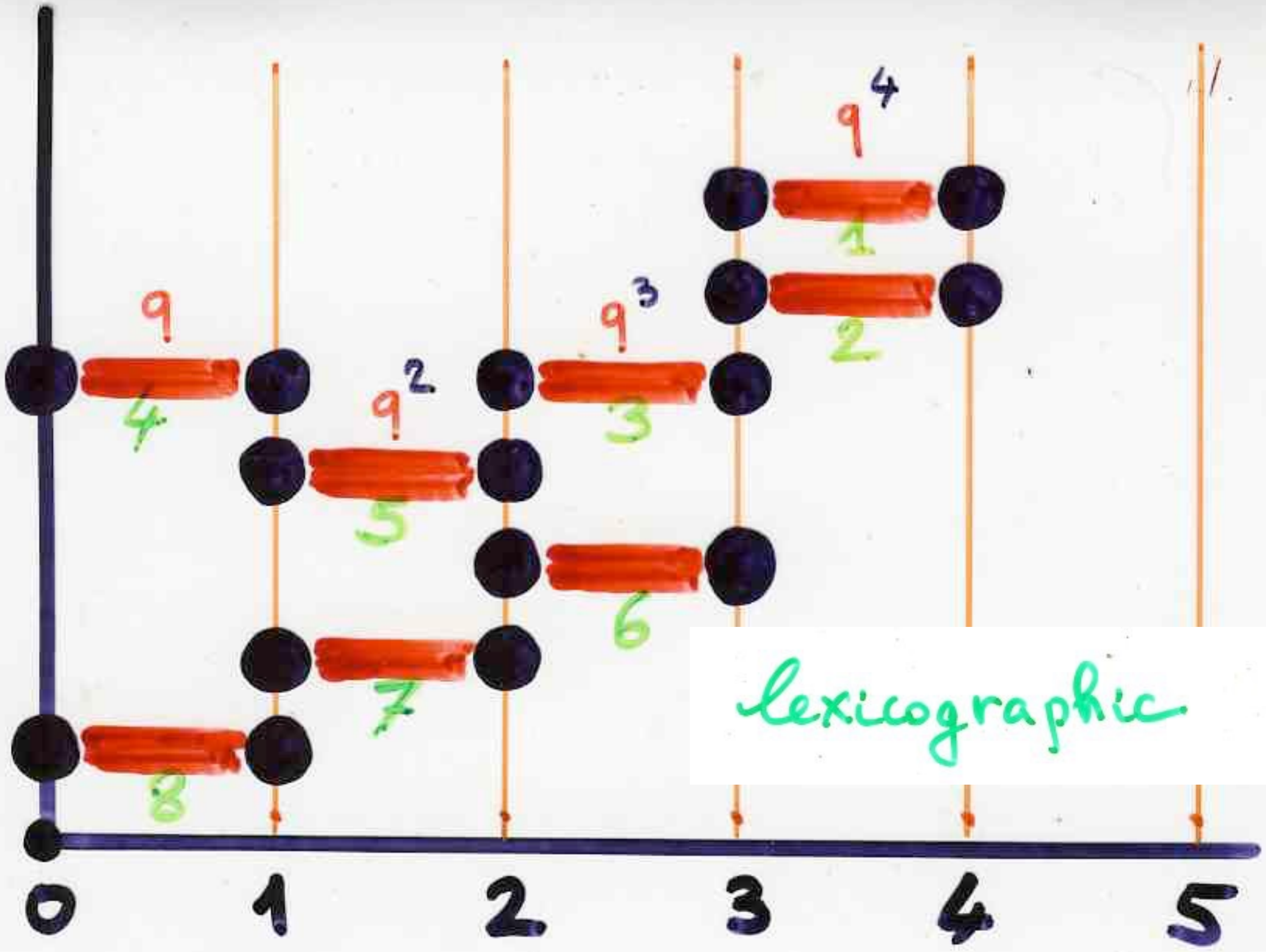
$$\frac{1}{R_I} = \sum_{\substack{\lambda \\ \text{quasi-} \\ \text{partitions}}} (-1)^{\ell(\lambda)} q^{|\lambda|}$$

G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

$$\frac{1}{D} = \sum_{\substack{E \\ \text{heaps of dimers}}} v(E)$$



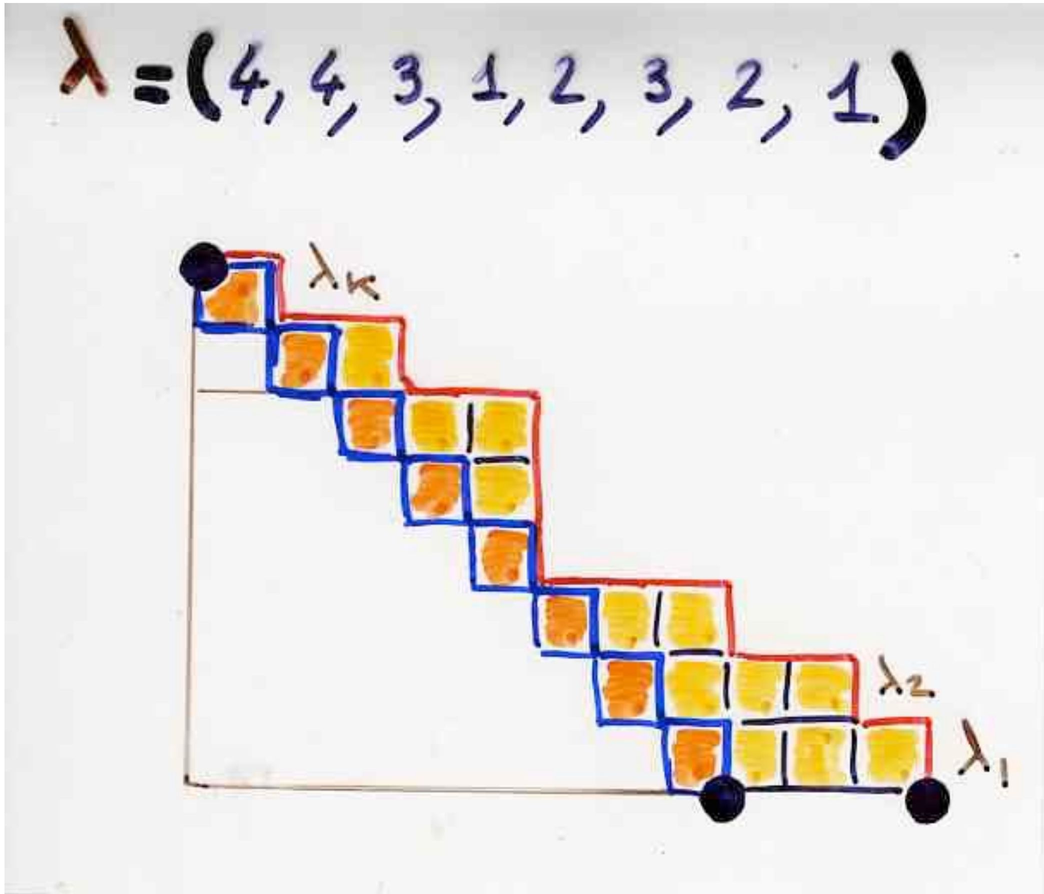
q
q^2
q^3
q^4
0
1
2
3
4
5



lexicographic normal form

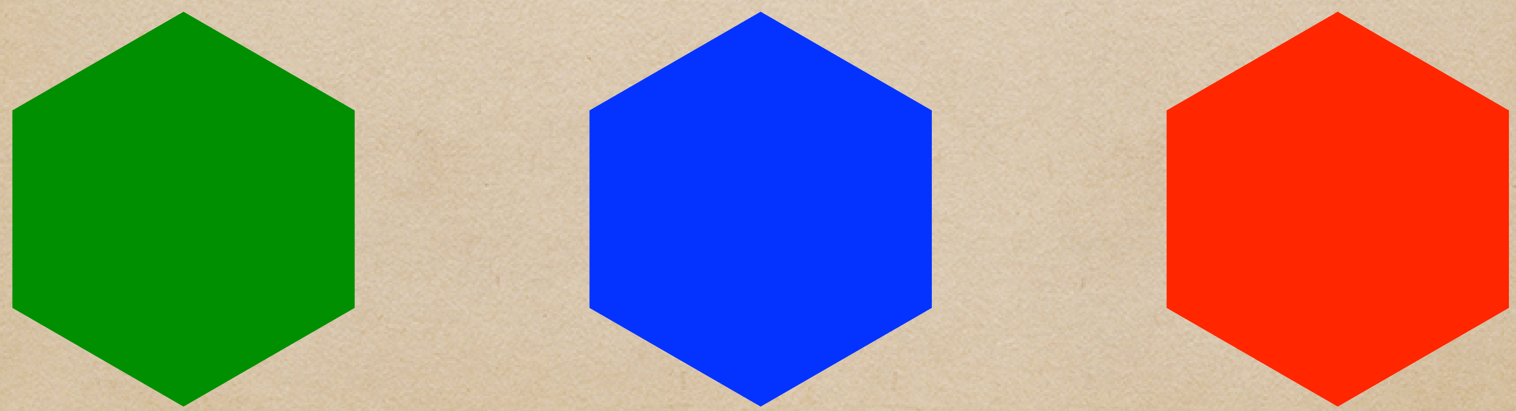
$$H \longrightarrow \lambda = (\underset{1}{4}, \underset{2}{4}, \underset{3}{3}, \underset{4}{1}, \underset{5}{2}, \underset{6}{3}, \underset{7}{2}, \underset{8}{1})$$

quasi-partition


λ = (4, 4, 3, 1, 2, 3, 2, 1)
λk
λ2
λ1

epilogue

Ramanujan
and the hard hexagons



phase transition
critical phenomena

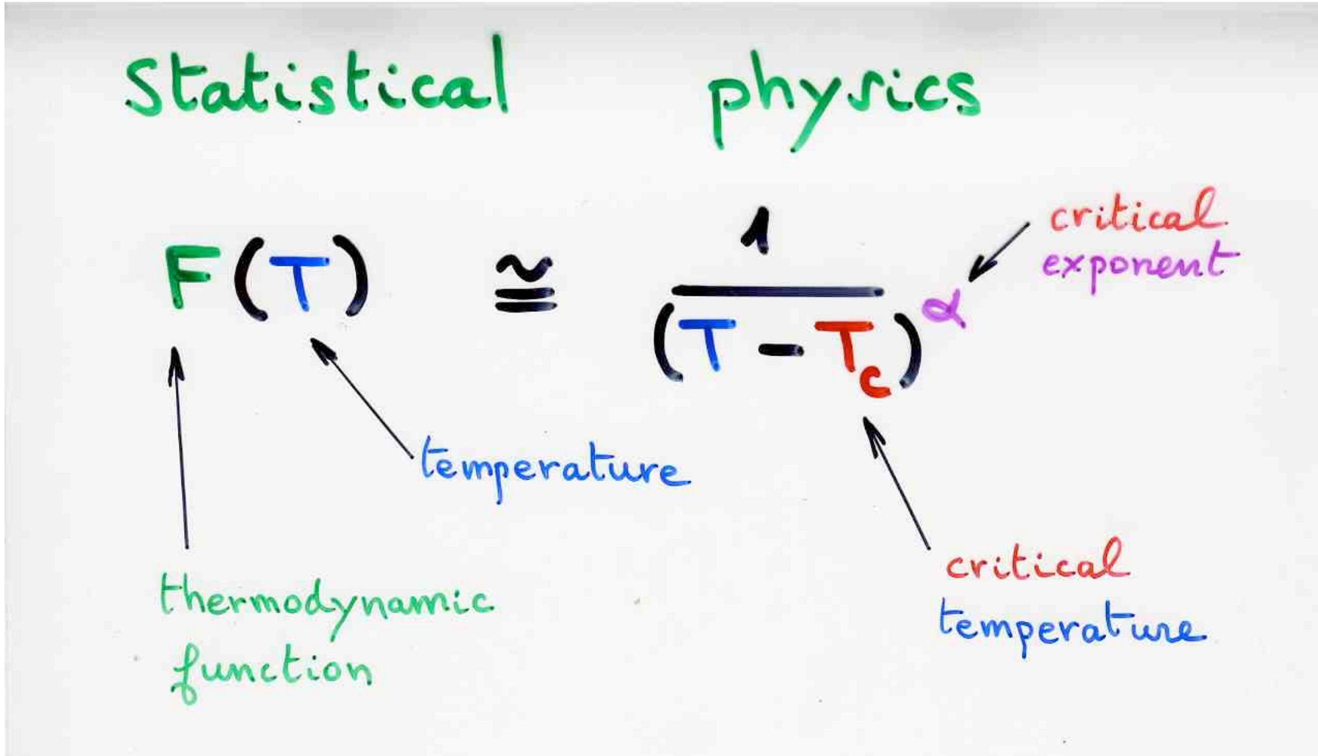
from local interactions
→ global behaviour

exactly
solved
models

Ising
model

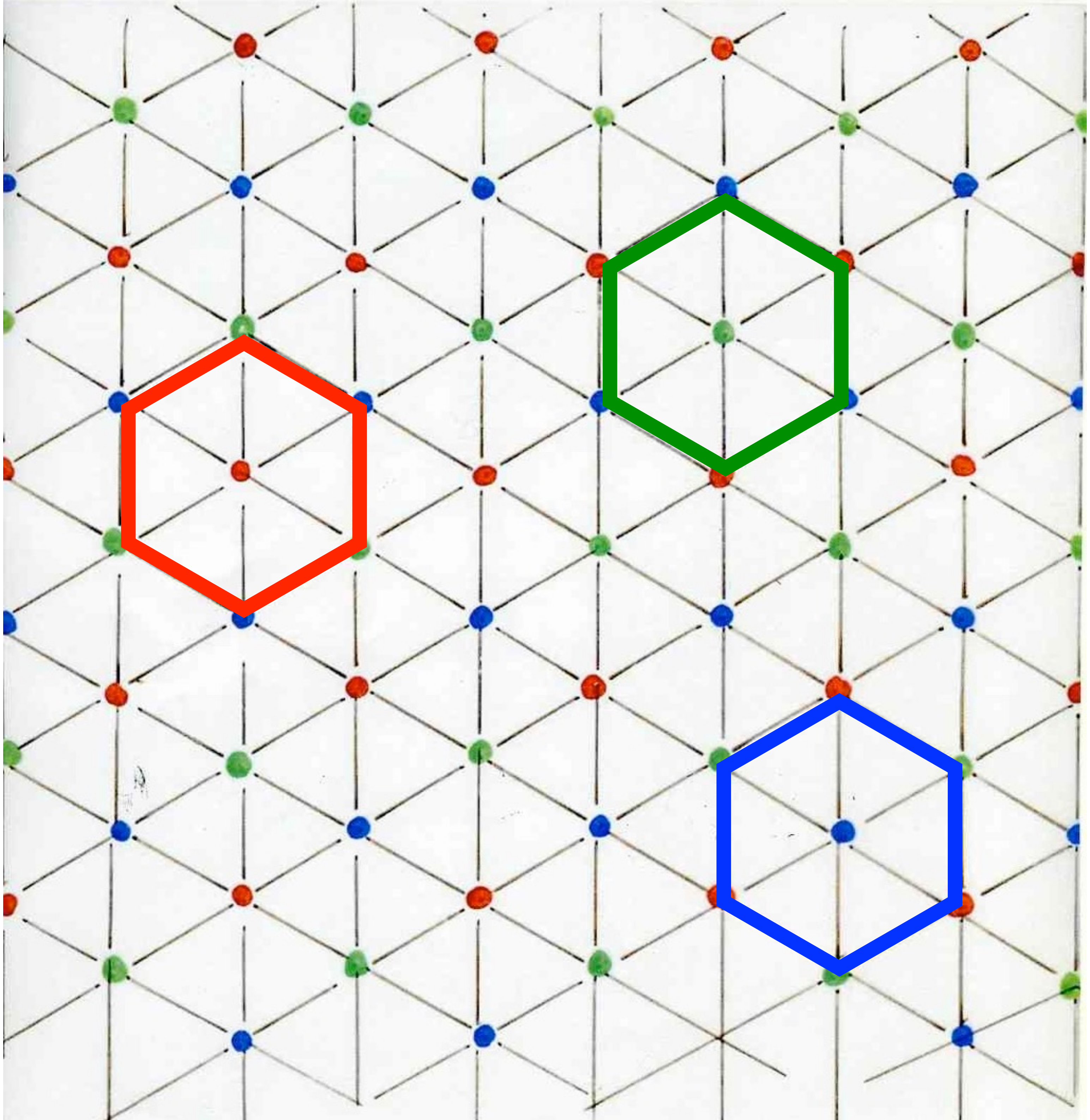
Baxter
book (1982)

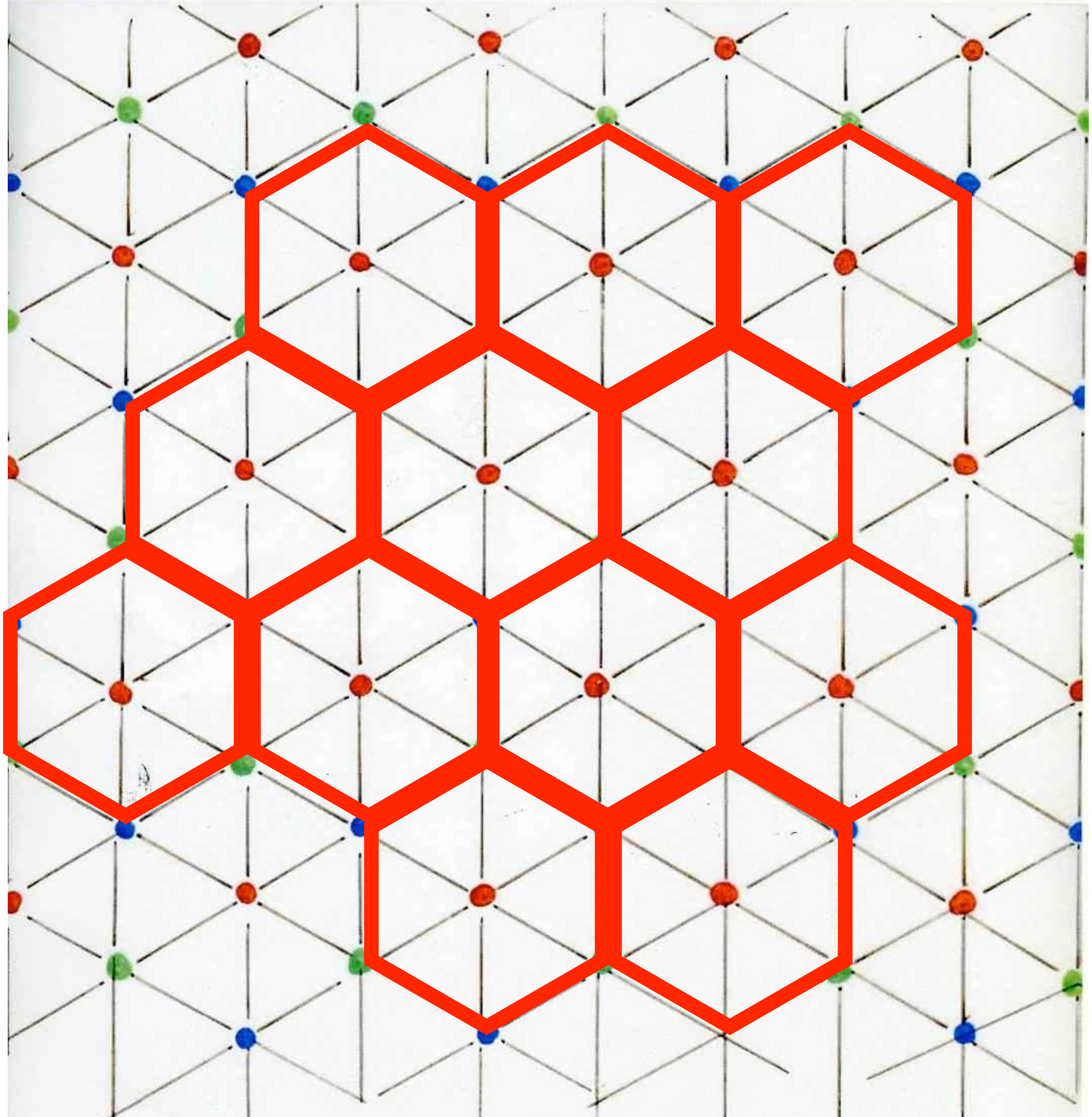
Onsager (1944)

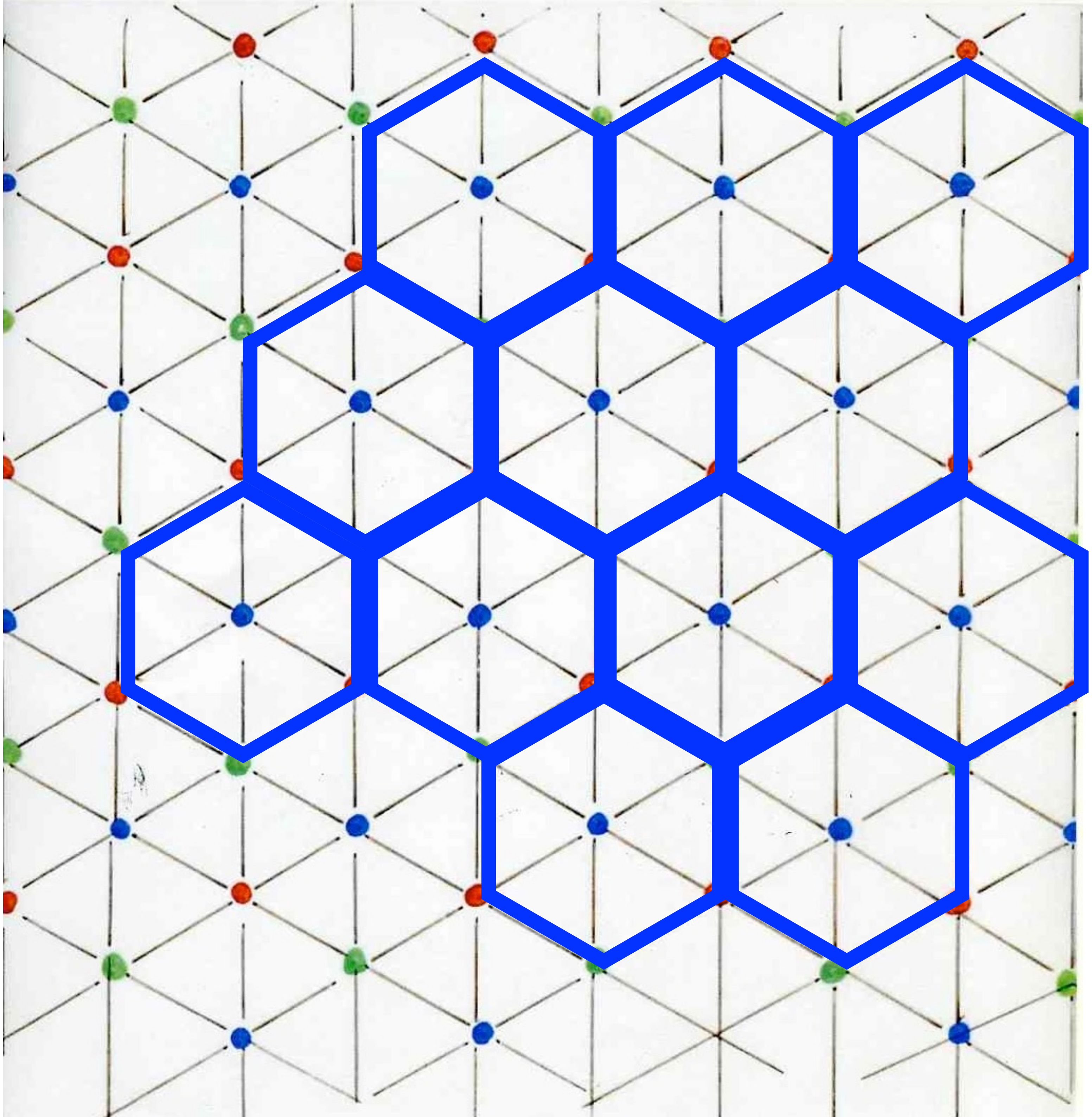

Statistical physics
$F(T) \cong \frac{1}{(T-T_c)^{\alpha}}$
critical exponent
temperature
thermodynamic function
critical temperature

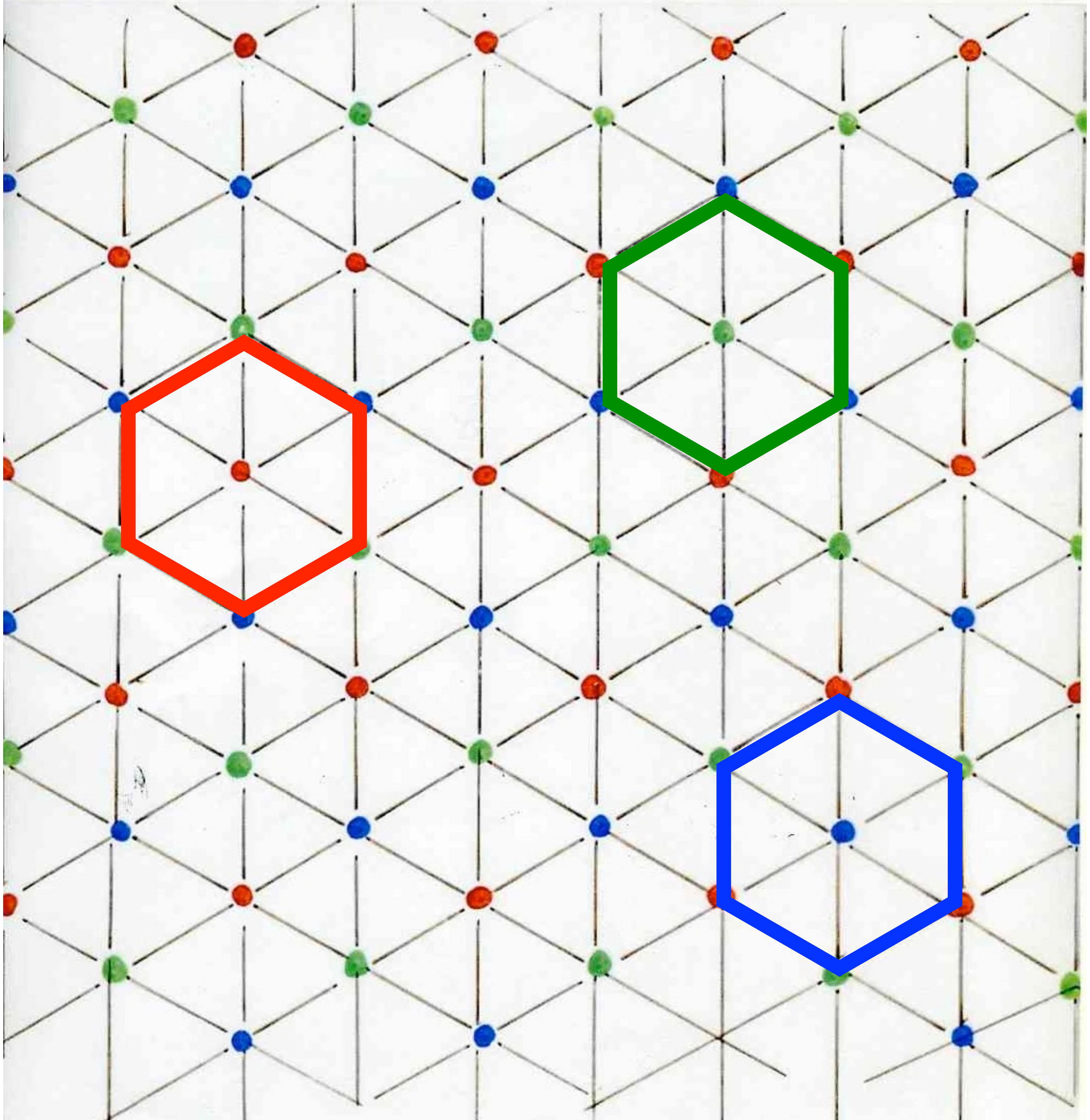
hard hexagons model

gas model



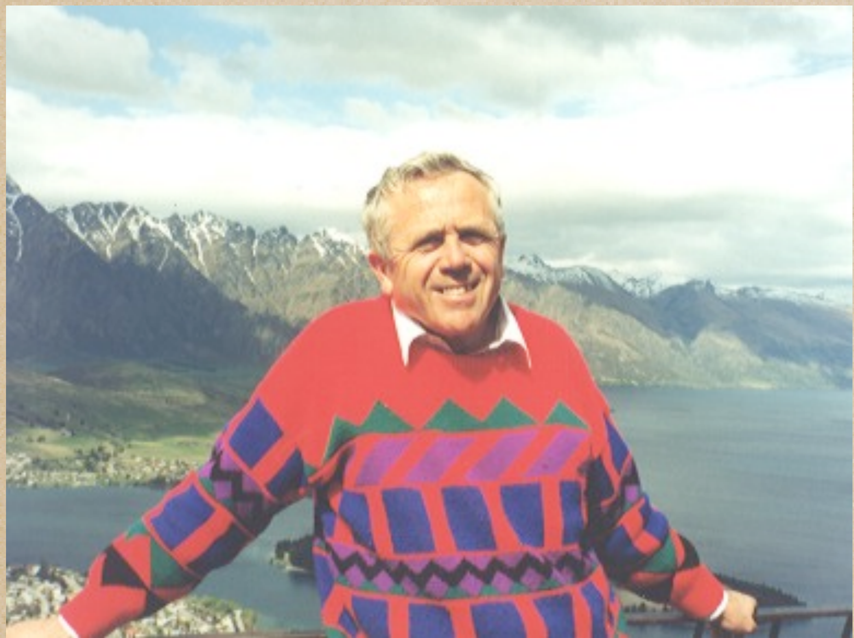






solution of the hard hexagons model

(R. Baxter, 1980)



Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

"La fraction continue" de Ramanujan

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

$$R(q) = \prod_{n \geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

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$$t = -q\left[R(q)\right]^5$$

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$$t = -q\left[R(q)\right]^5$$

$$Y(q) = \prod_{n \geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$$R(q) = \prod_{n\geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

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$$Z(t) = Y(q(t))$$

Z partition function

Température critique
pour $q = 1$

Fraction continue
de Ramanujan
pour $q = 1$

$$\rightarrow \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{1}{\phi}$$

nombre d'or

"Température critique"

$$T_c = (\phi)^5 = \frac{11 + 5\sqrt{5}}{2}$$

Baxter

(1980)

- critical temperature

$$T_c = \frac{11 + 5\sqrt{5}}{2}$$

- critical exponent $\frac{5}{6}$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^5$$

helium monolayer
absorbed onto
a graphite surface

(Riedel 1981)

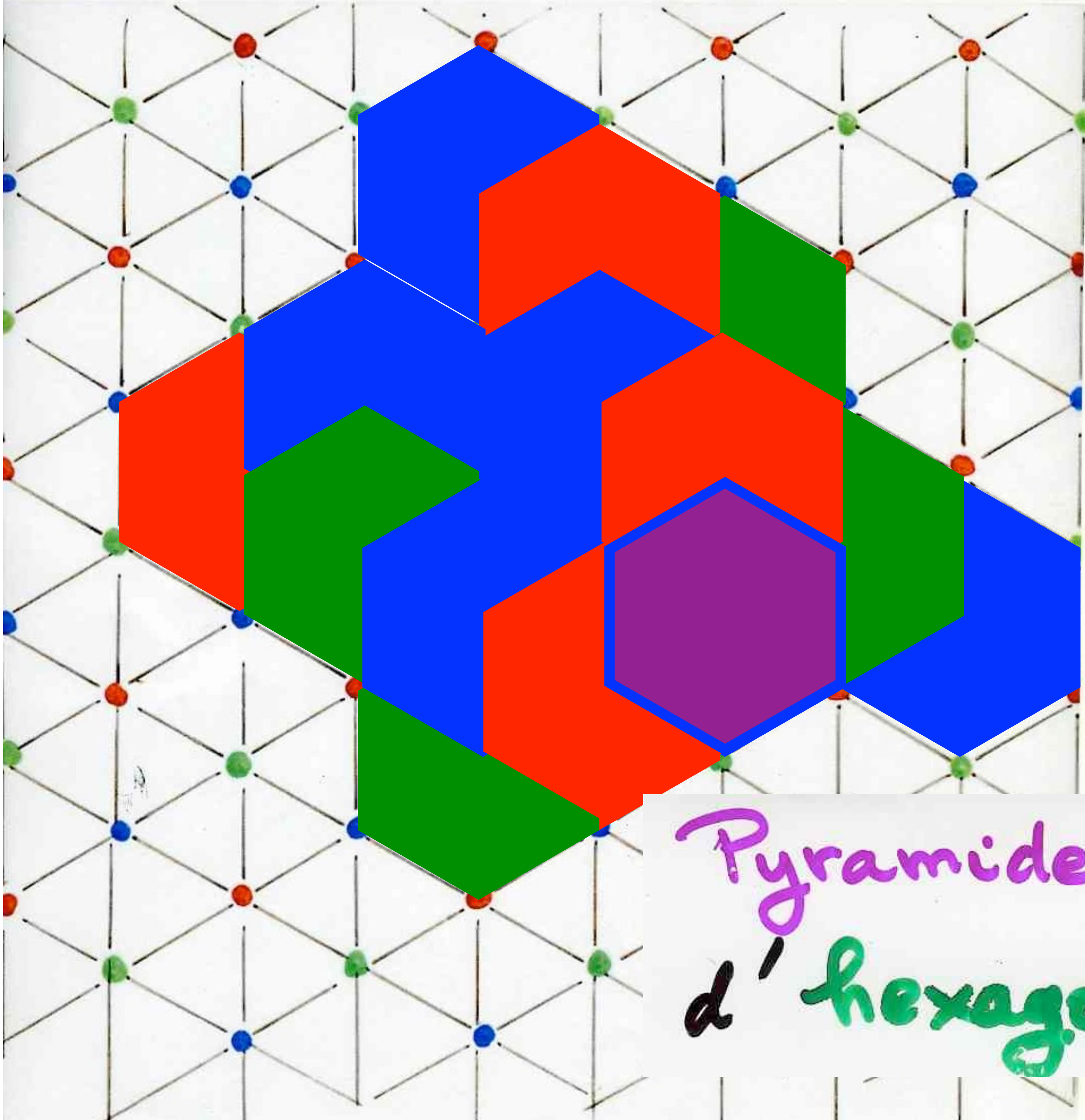
la densité du gaz

$$y = t \frac{d}{dt} \log Z(t)$$

Dans cette série

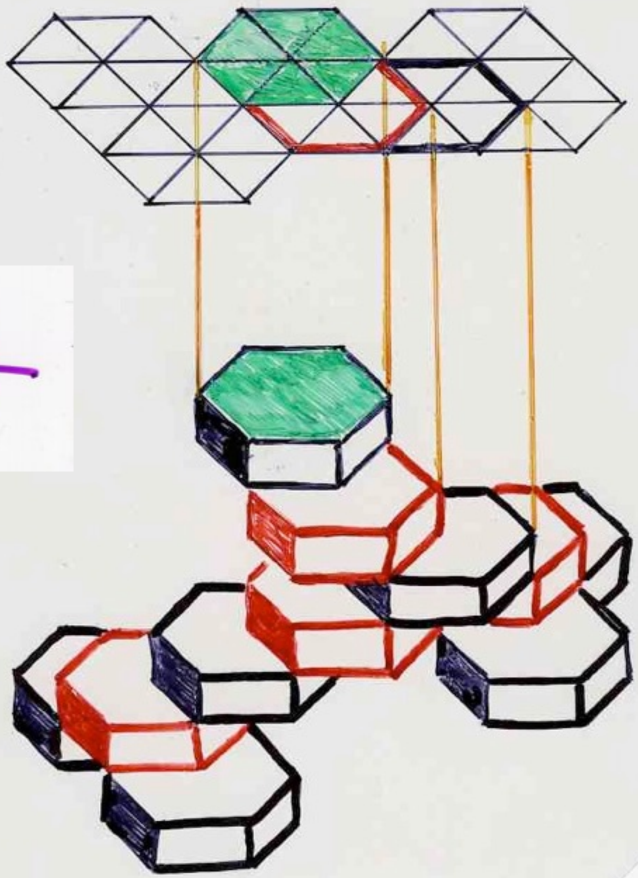
$$y = \sum_{n \geq 1} (-1)^n b_n t^n$$

le coefficient b_n est le nombre de pyramides d'hexagones formées de n hexagones



Pyramide d'hexagones

$$-\rho(-t) = y$$



ρ density

HC.

la densité du gaz

$$y = \frac{t\,d}{dt} \log Z(t)$$

vérifie l'équation algébrique suivante :

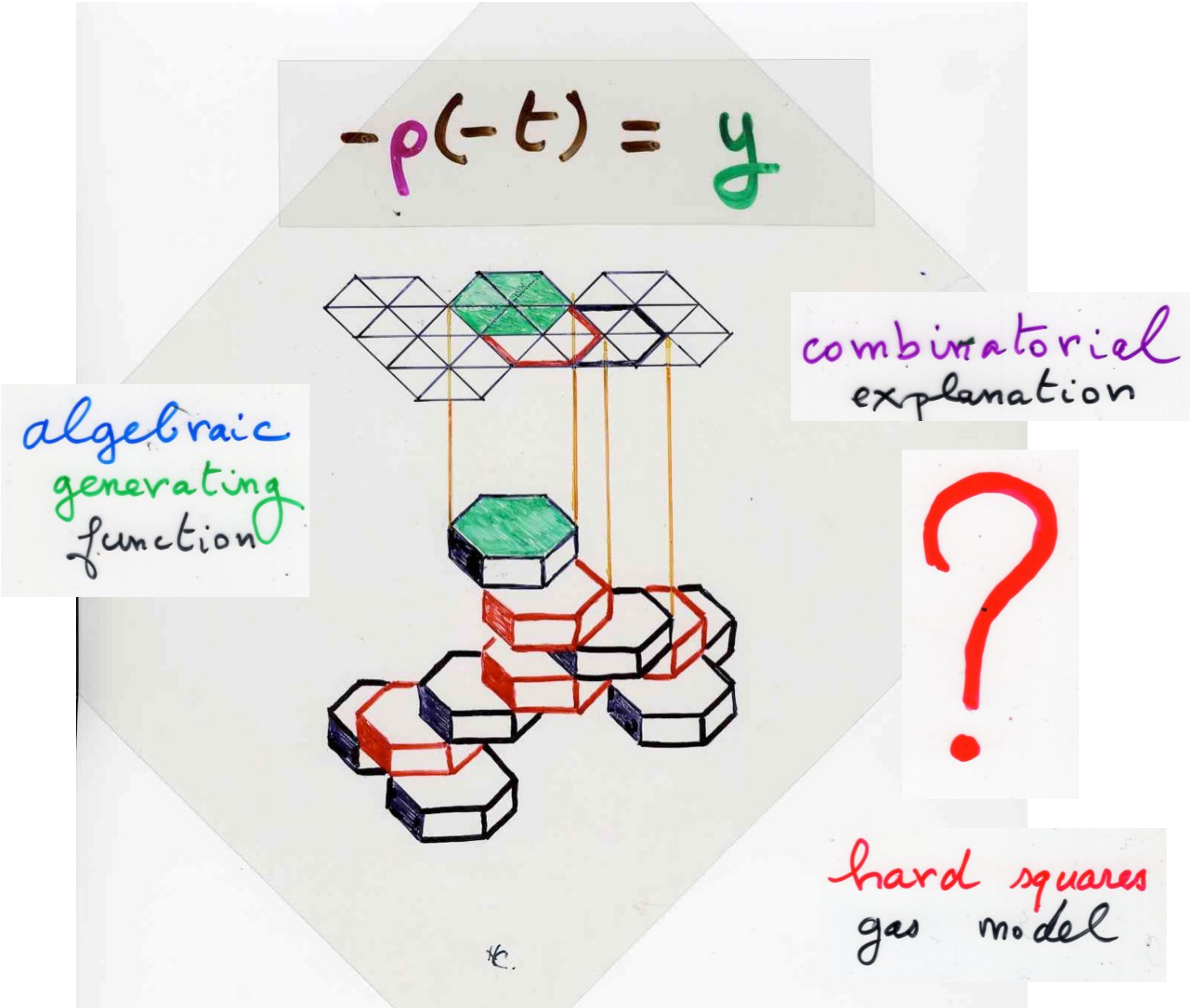
$$y\left(1+14t+97t^{2}+415t^{3}+1180t^{4}+2321t^{5}+3247t^{6}+3300t^{7}+2475t^{8}+1375t^{9}+550t^{10}+143t^{11}+18t^{12}\right)+$$

$$y^{2}\left(1+17t+83t^{2}+601t^{3}+1647t^{4}+4606t^{5}+7809t^{6}+710t^{7}+124t^{8}-608t^{9}-440t^{10}-92t^{11}-36t^{12}\right)+$$

$$y^{3}\left(3+50t+381t^{2}+1715t^{3}+5040t^{4}+10130t^{5}+14062t^{6}+13002t^{7}+6930t^{8}+715t^{9}-1595t^{10}-488t^{11}-198t^{12}\right)+$$

$$y^{4}\left(1+17t+131t^{2}+595t^{3}+1765t^{4}+3574t^{5}+4939t^{6}+4356t^{7}+1815t^{8}-605t^{9}-1210t^{10}-616t^{11}-126t^{12}\right)$$

$$=\left(t+11t^{2}+55t^{33}+165t^{4}+330t^{5}+462t^{6}+462t^{7}+330t^{8}+165t^{9}+55t^{10}+11t^{11}+t^{12}\right)$$



$-p(-t) = y$
algebraic
generating
function
combinatorial
explanation
?
hard squares
gas model

Thank you !