Growth diagrams, local rules and beyond

(part II)

21st Ramanujan Symposium:
National Conference on algebra and its applications

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28 February 2018

mirror website
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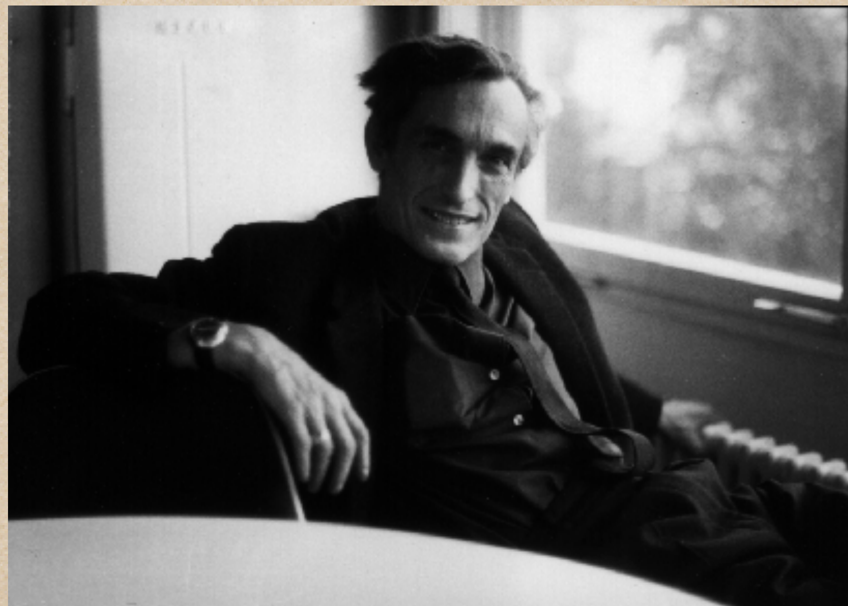
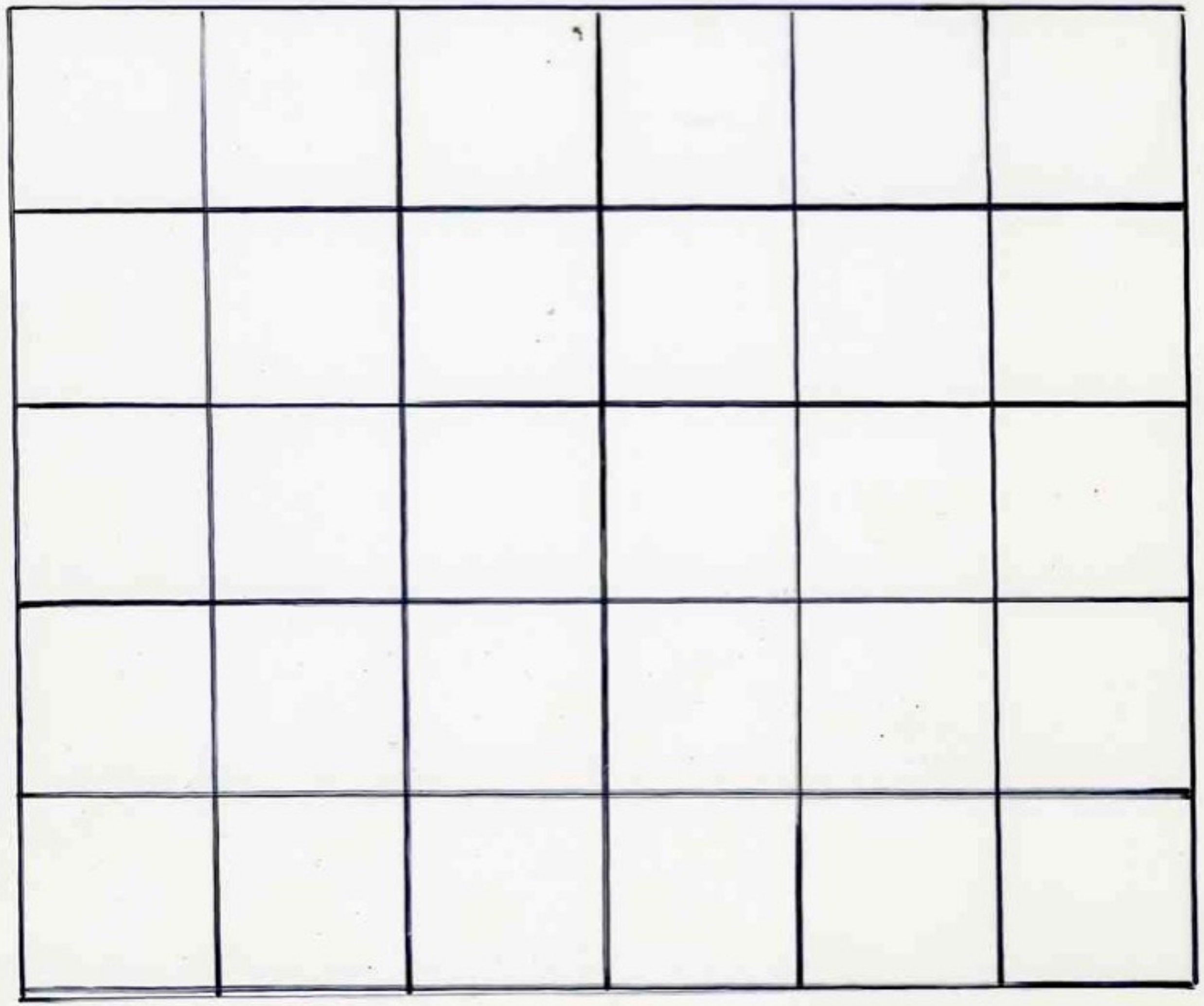
Jeu de taquin

M.P. Schützenberger

(1976)



$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$



$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

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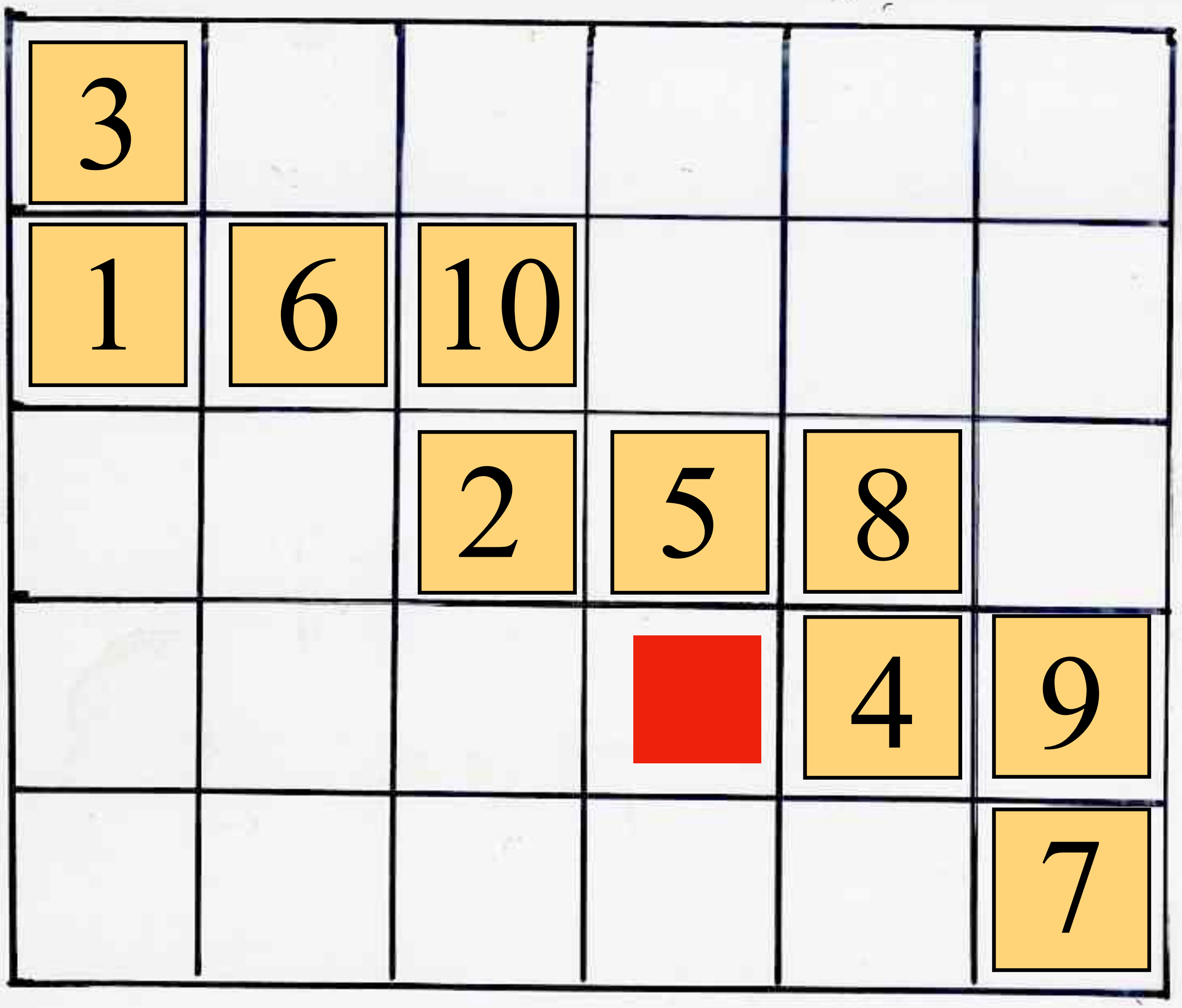
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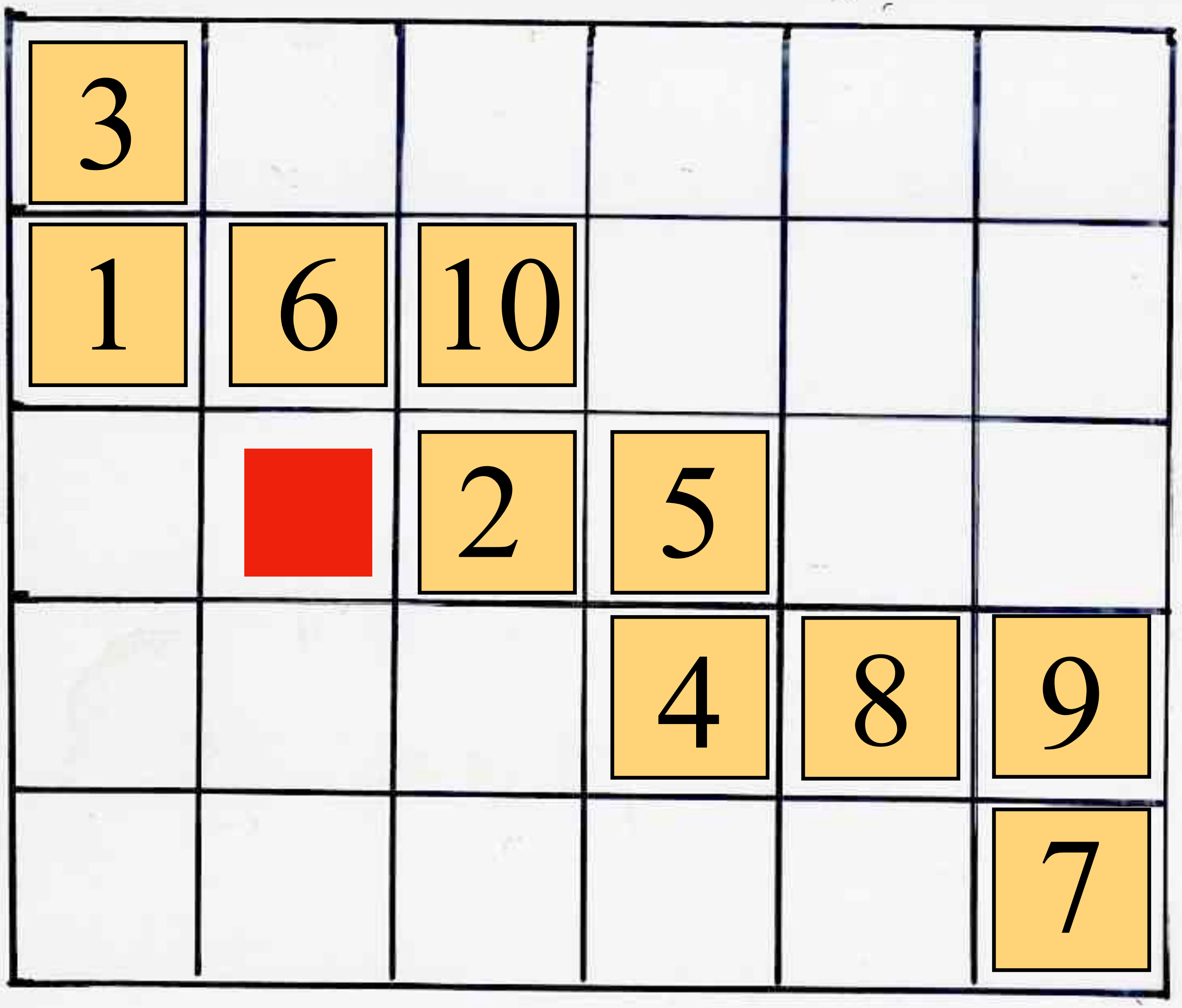
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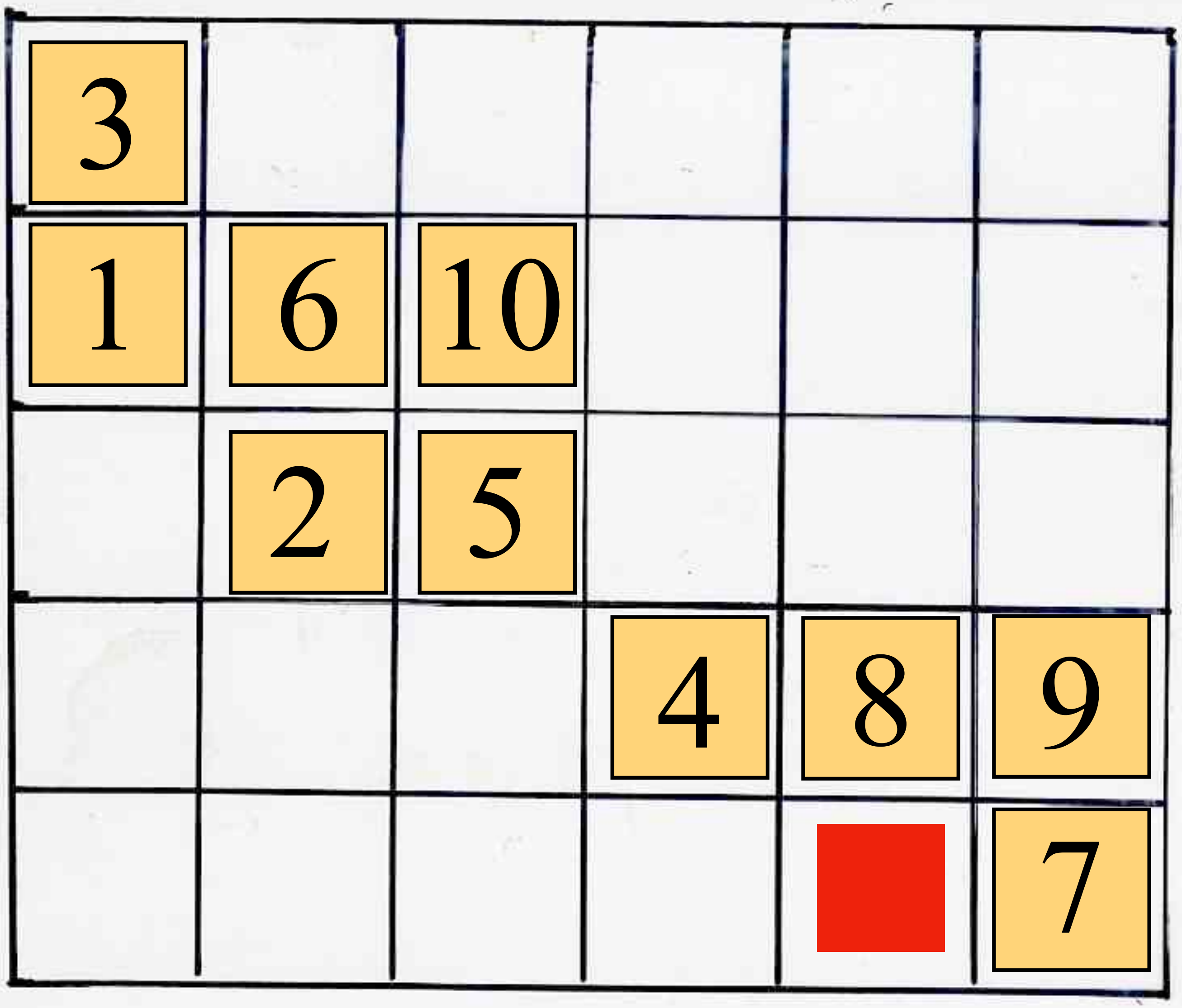
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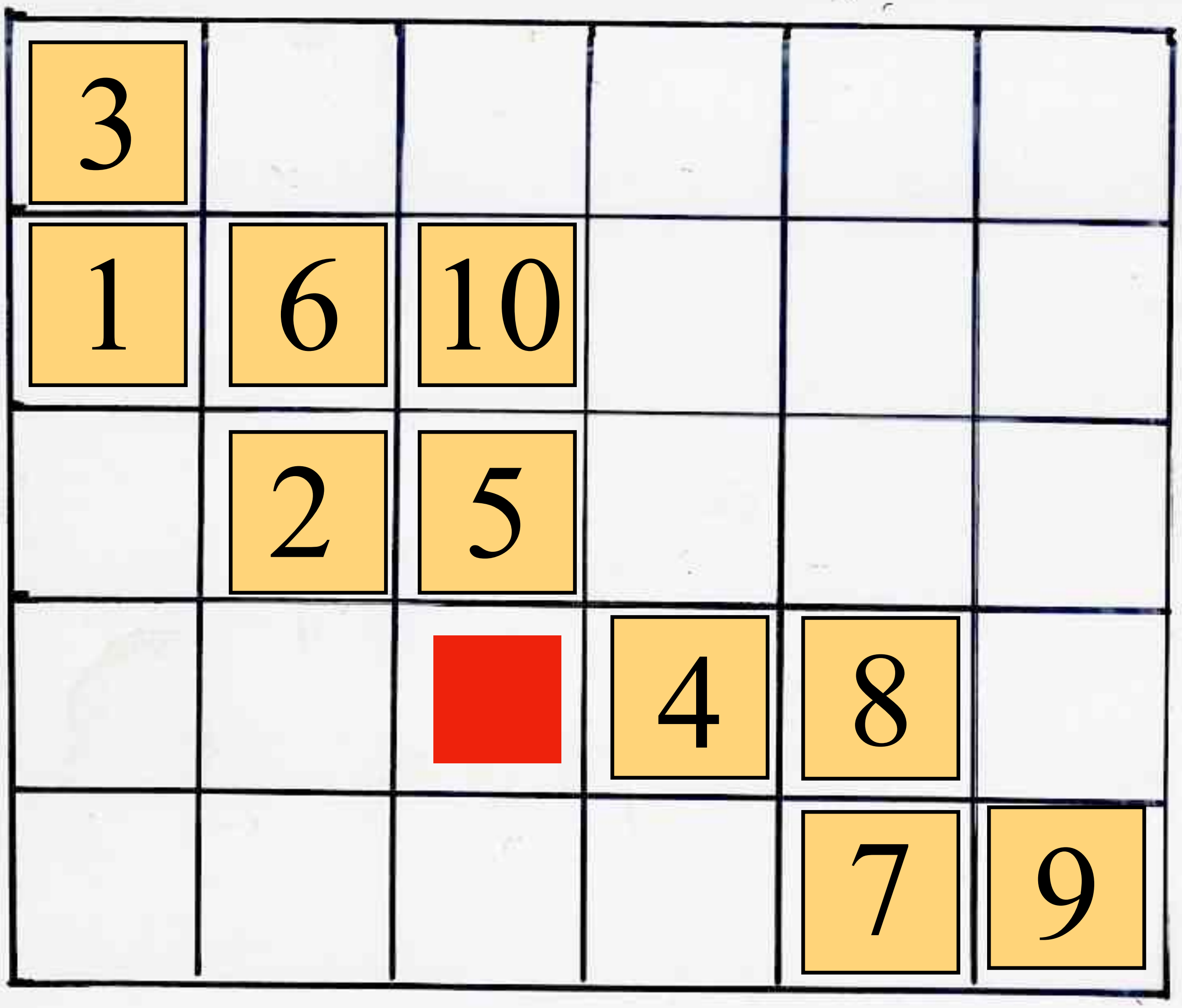
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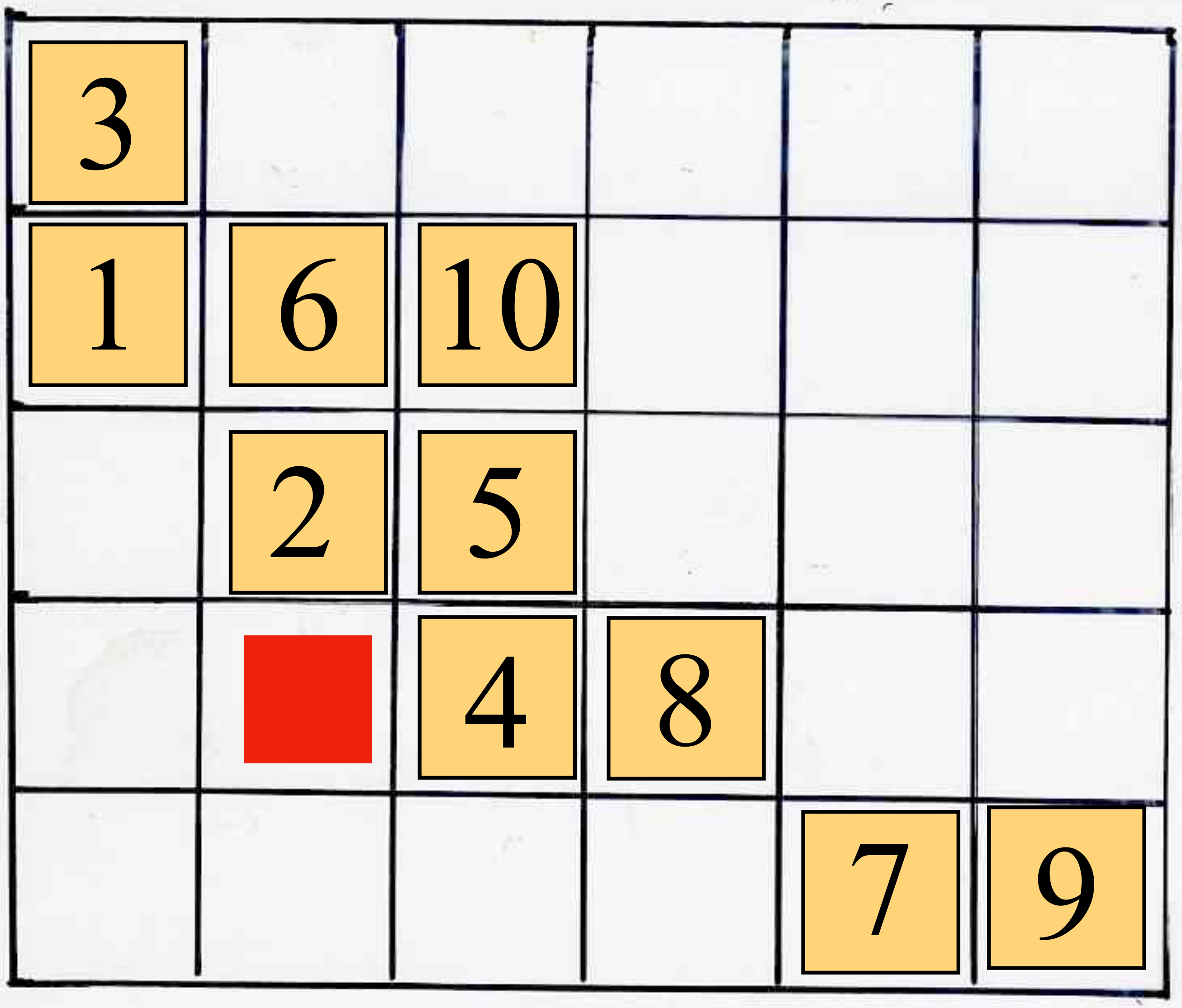
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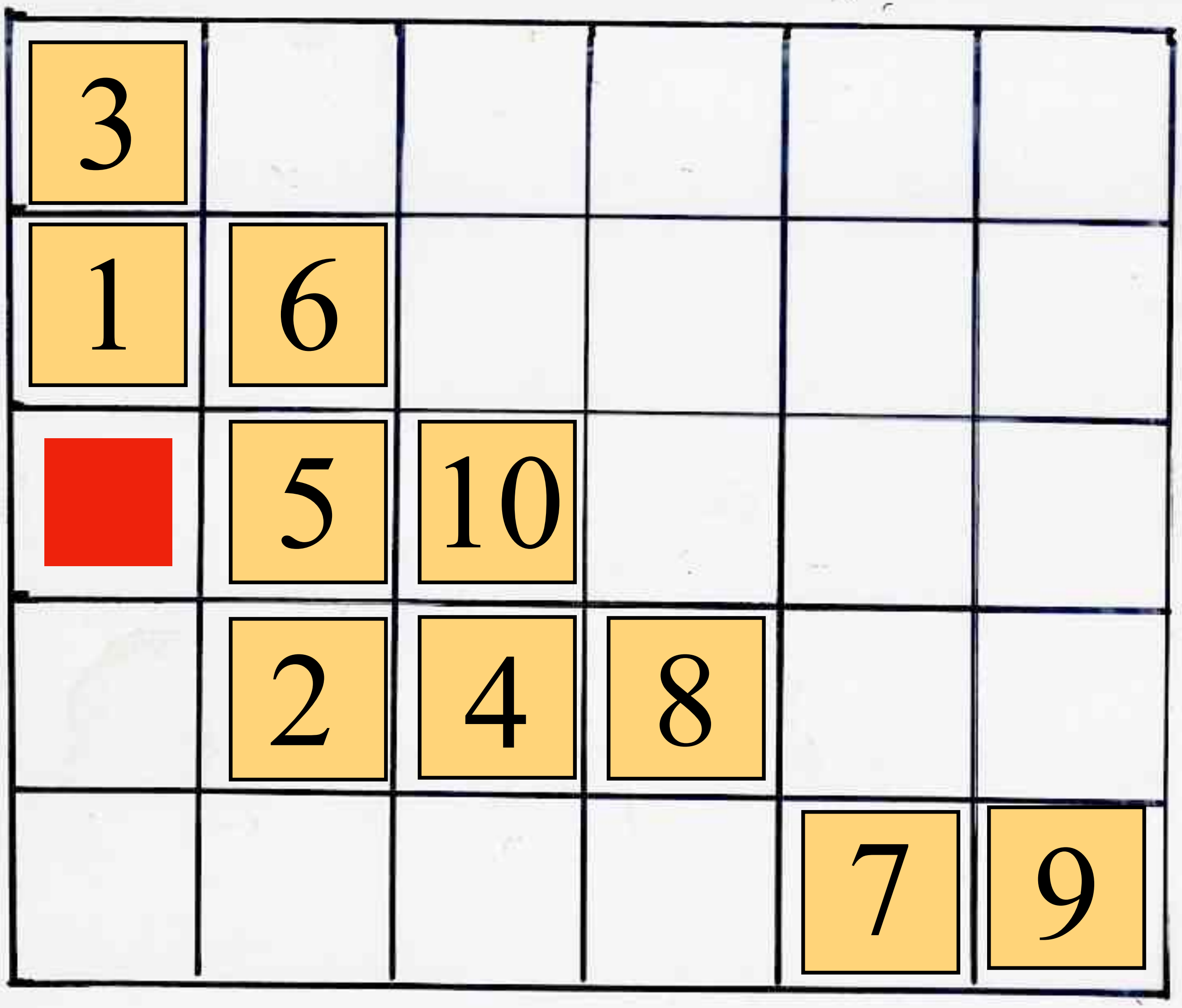
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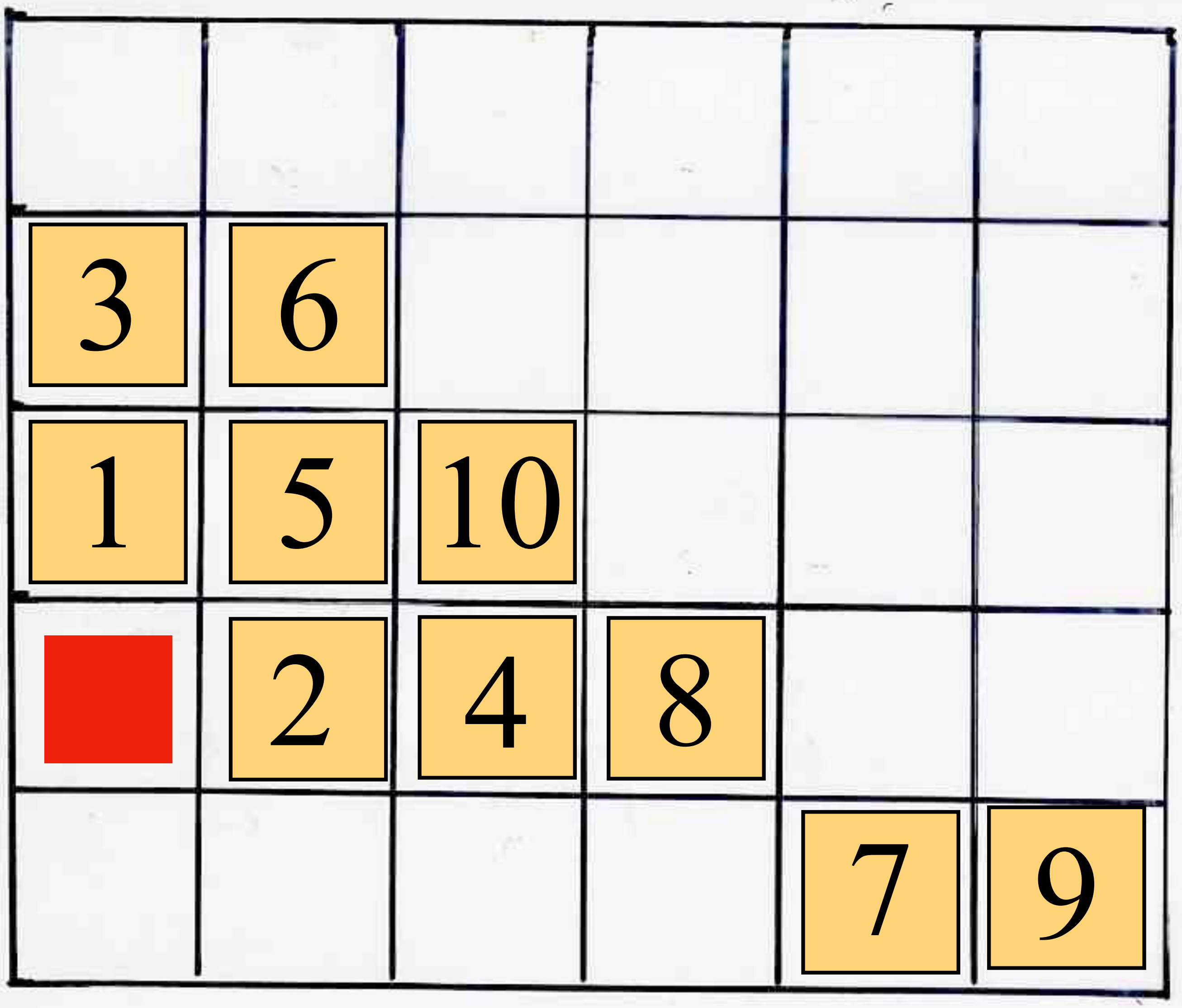
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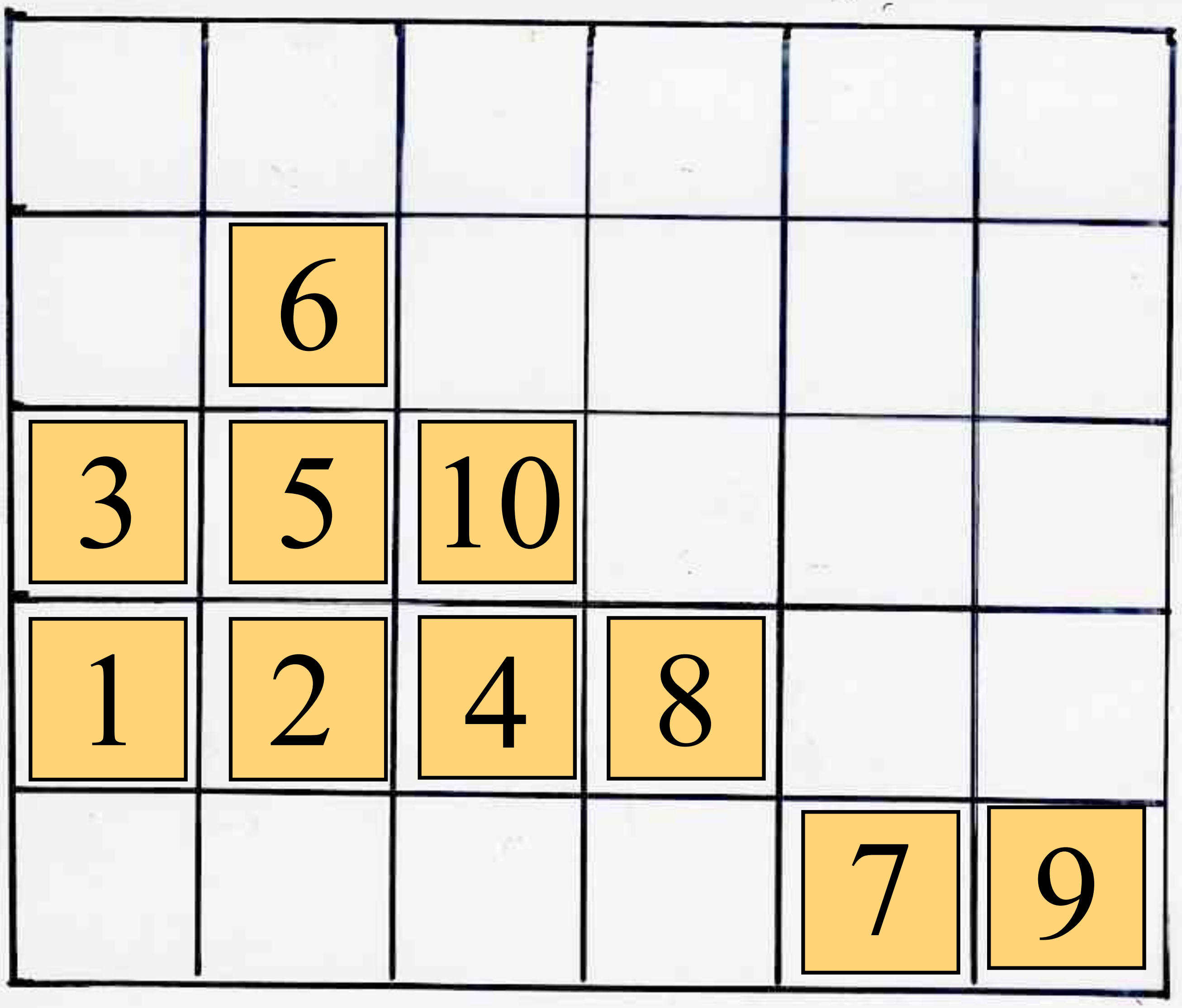
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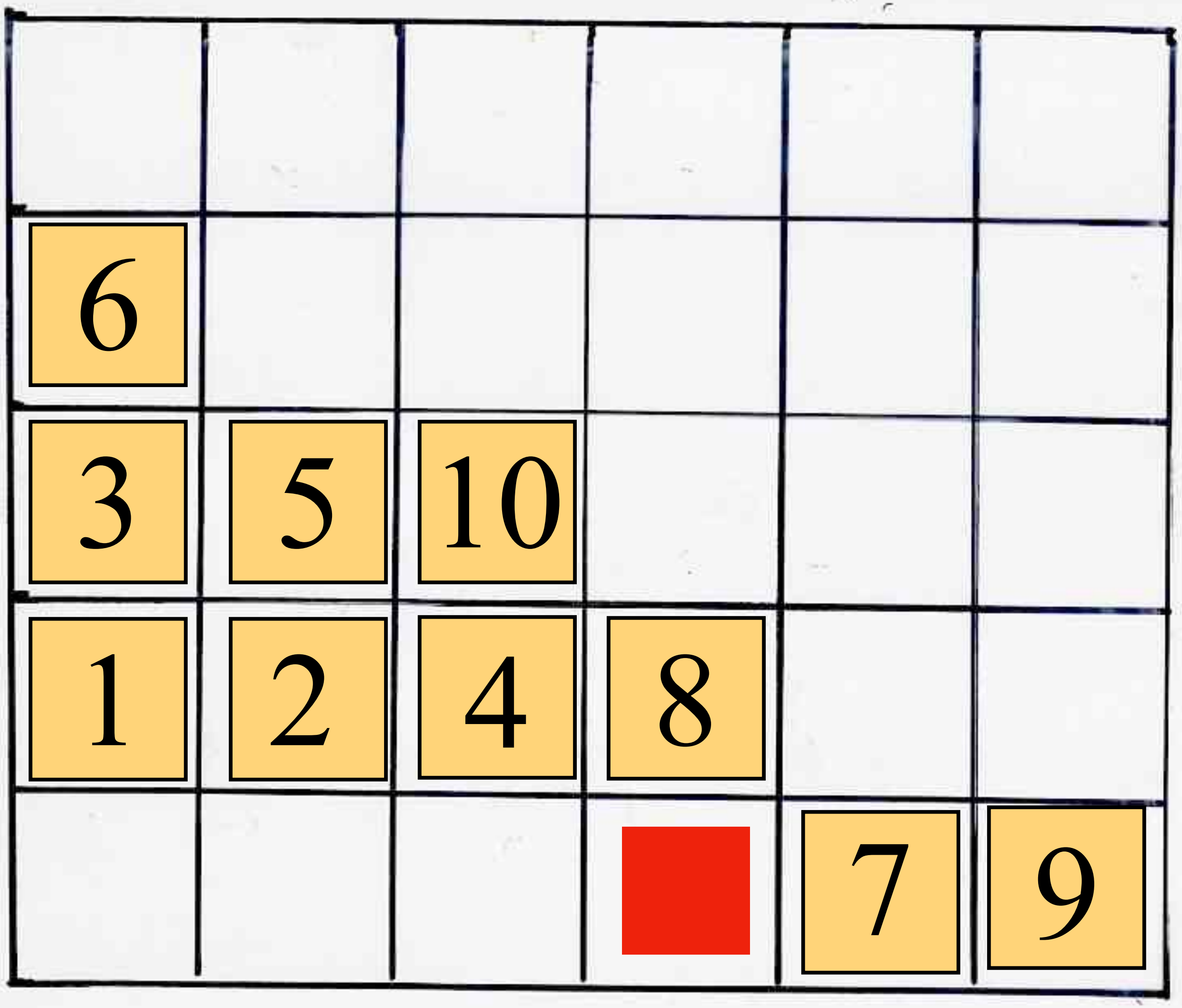
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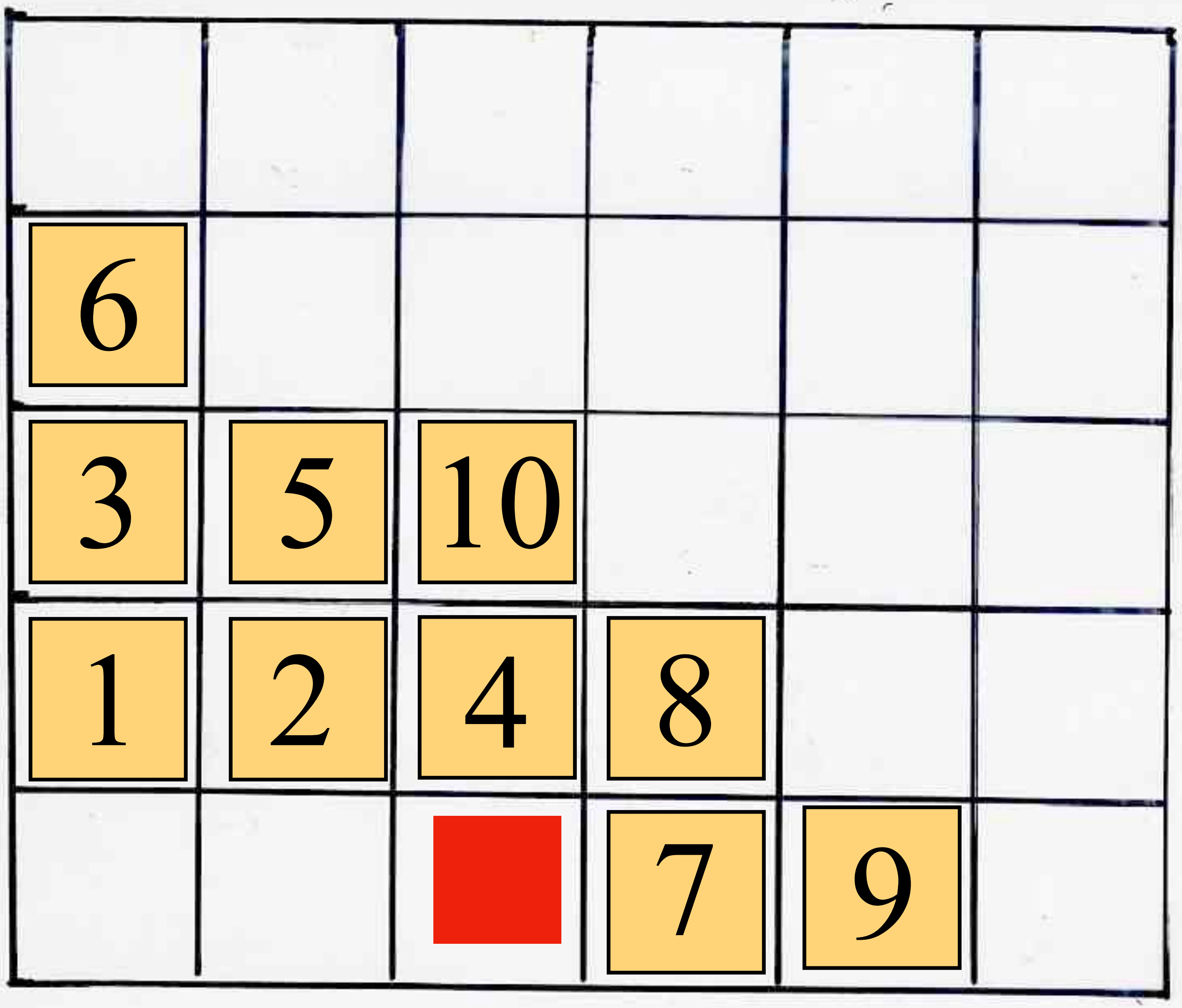
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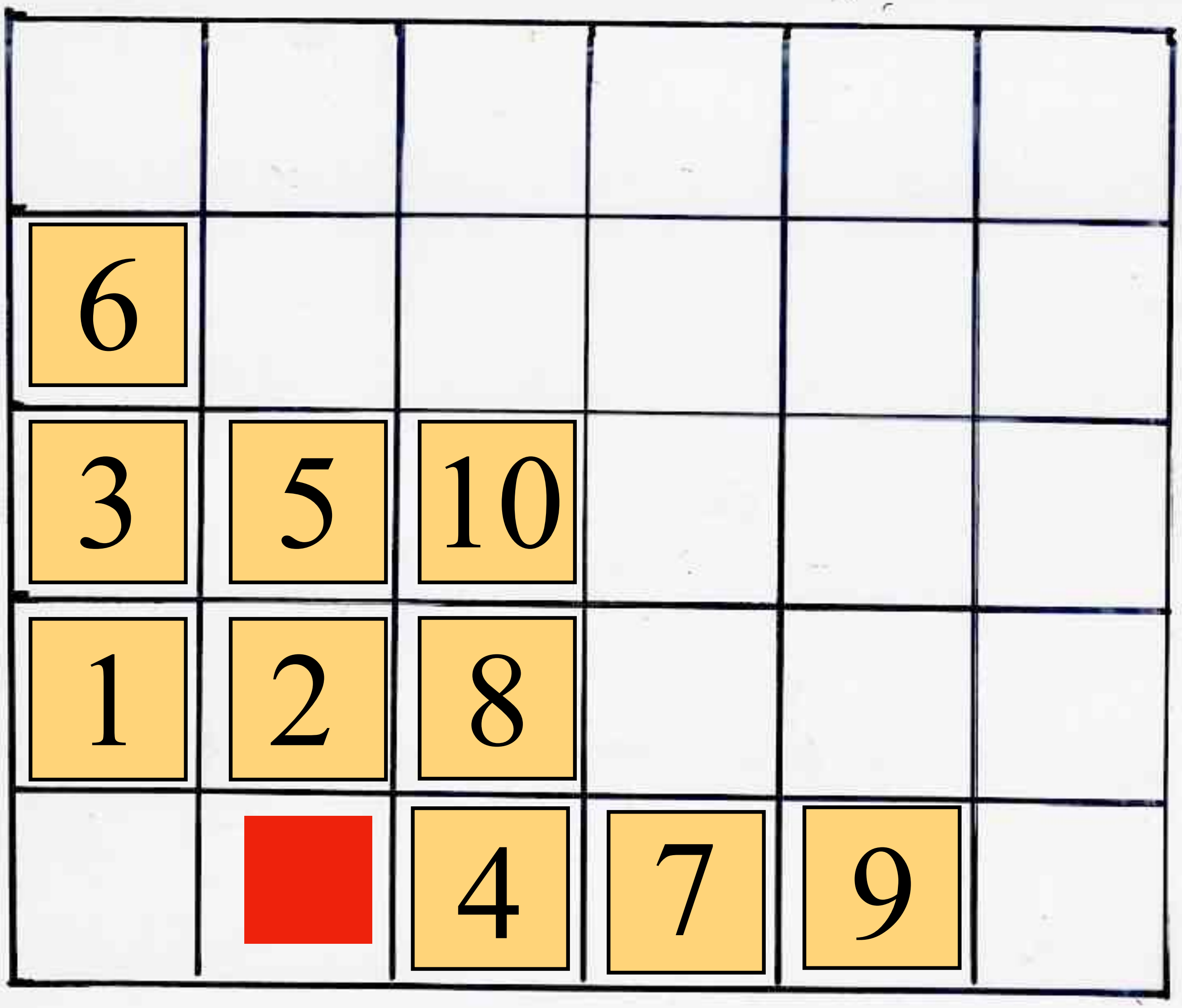
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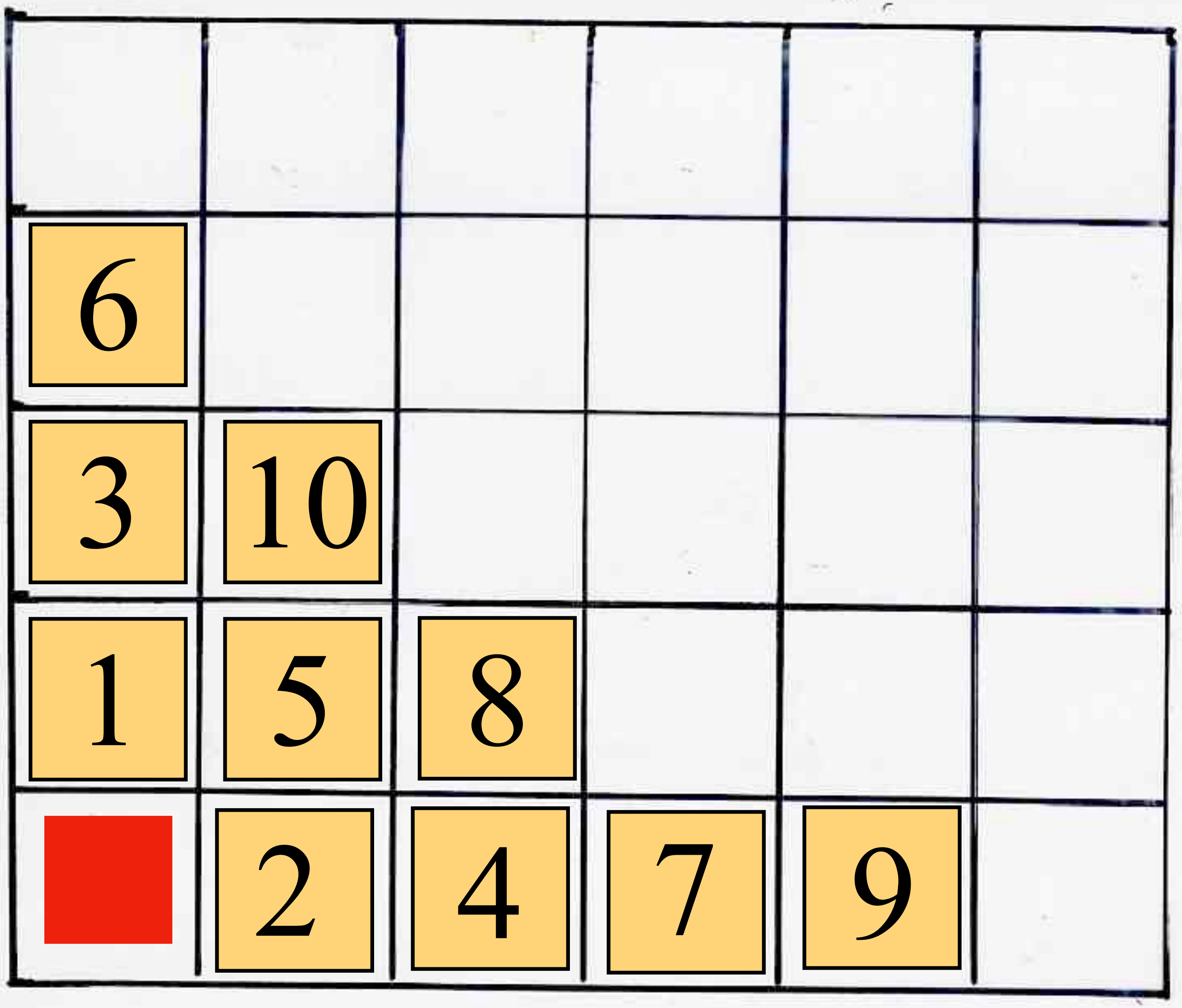
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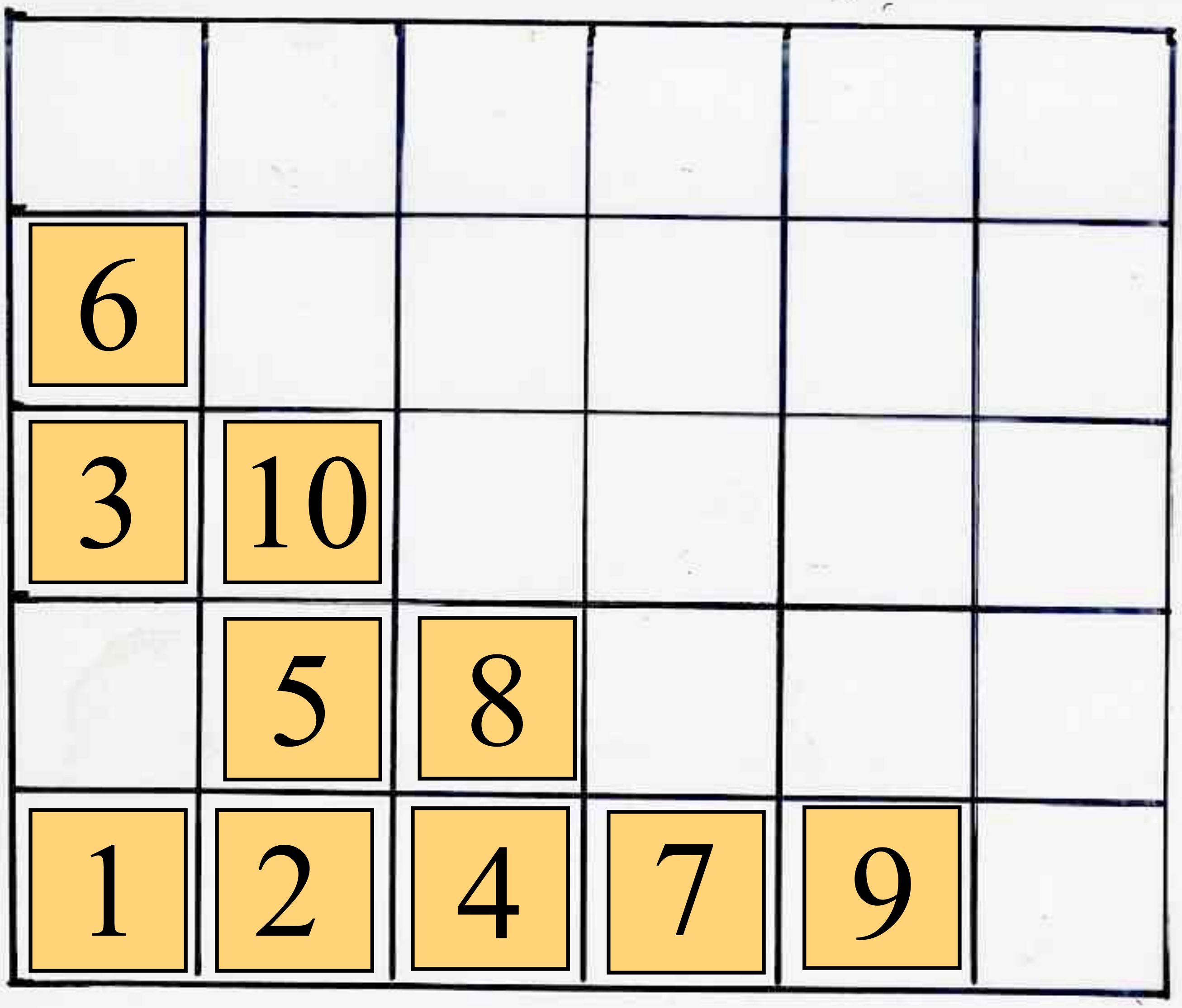
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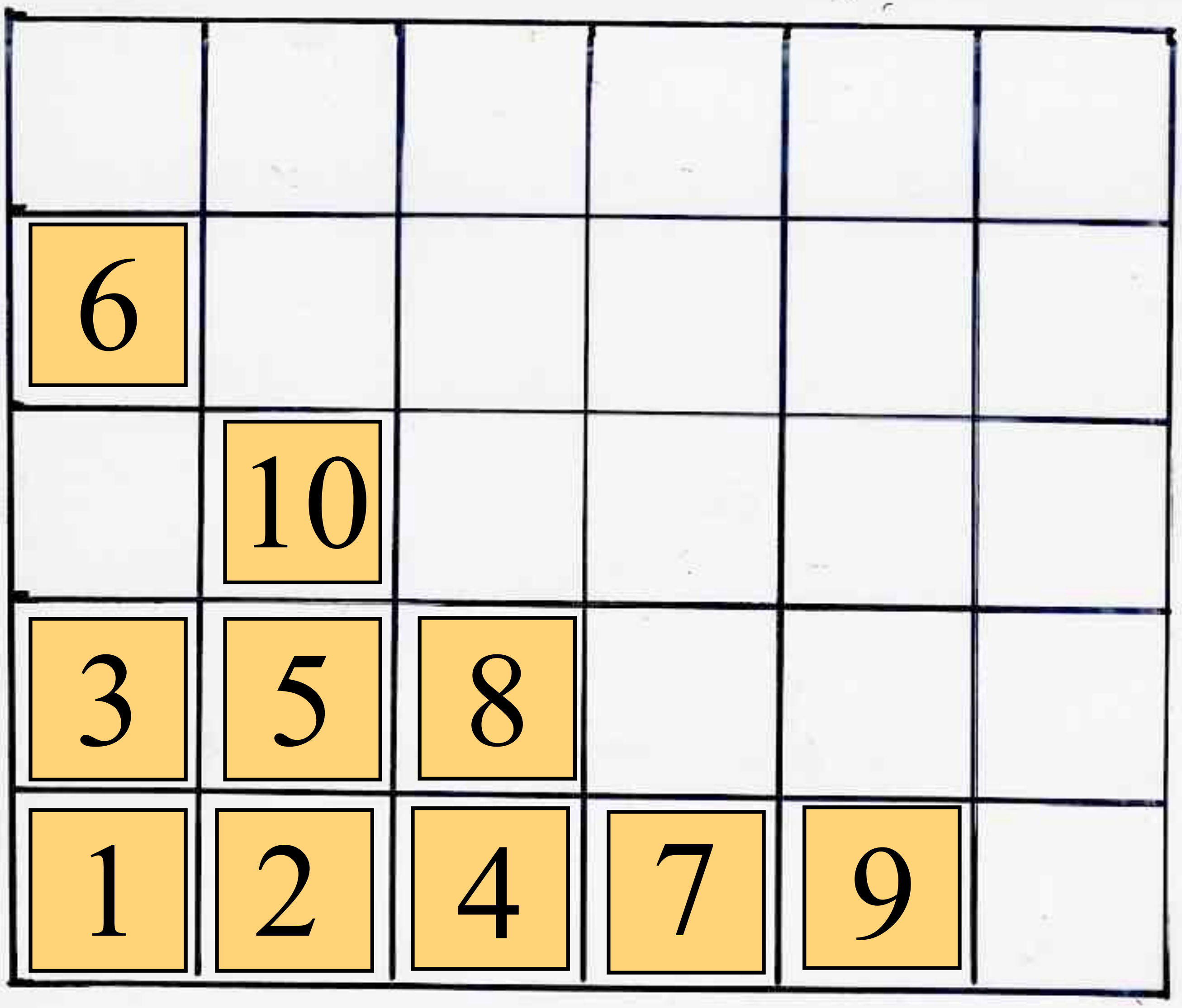
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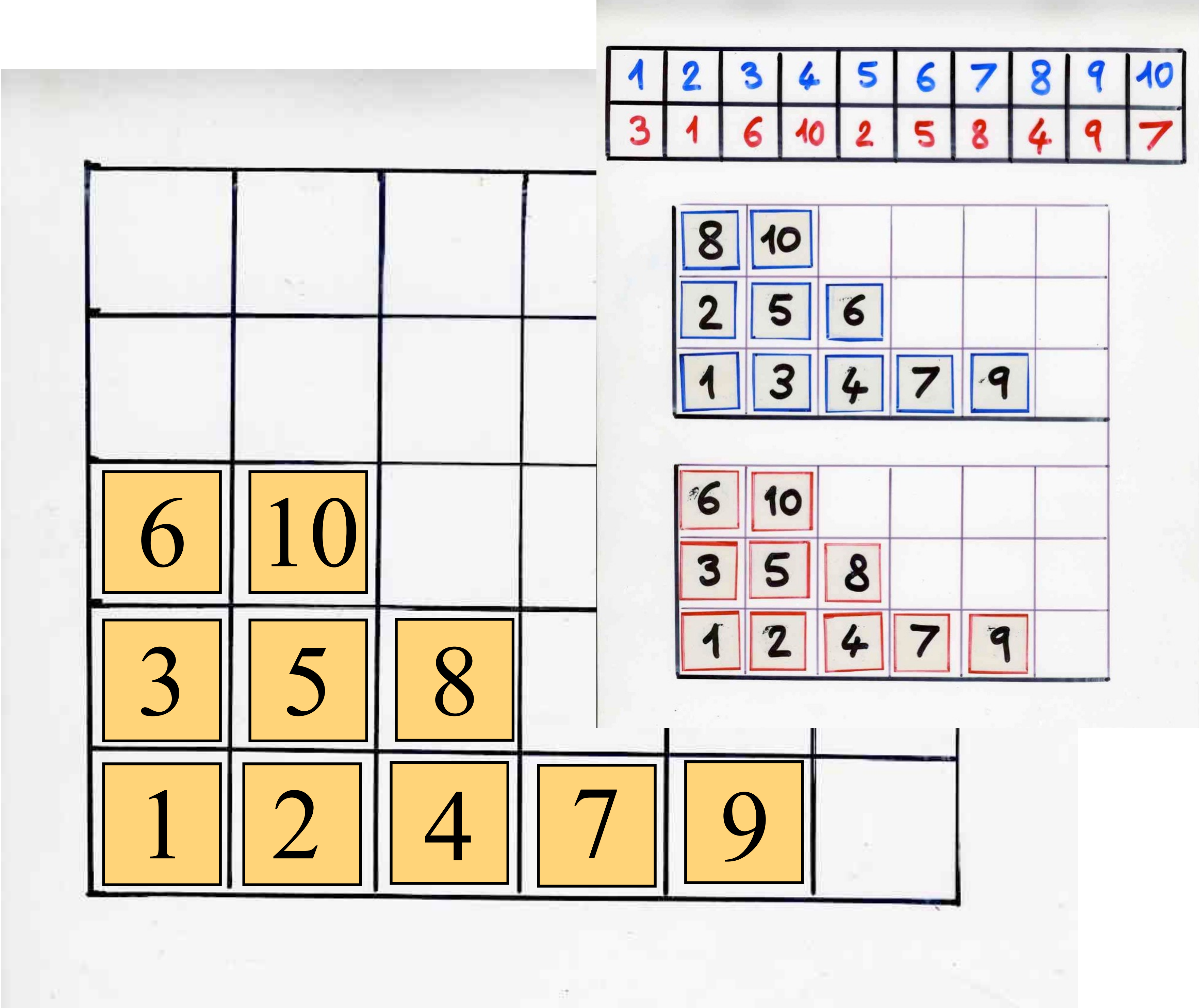

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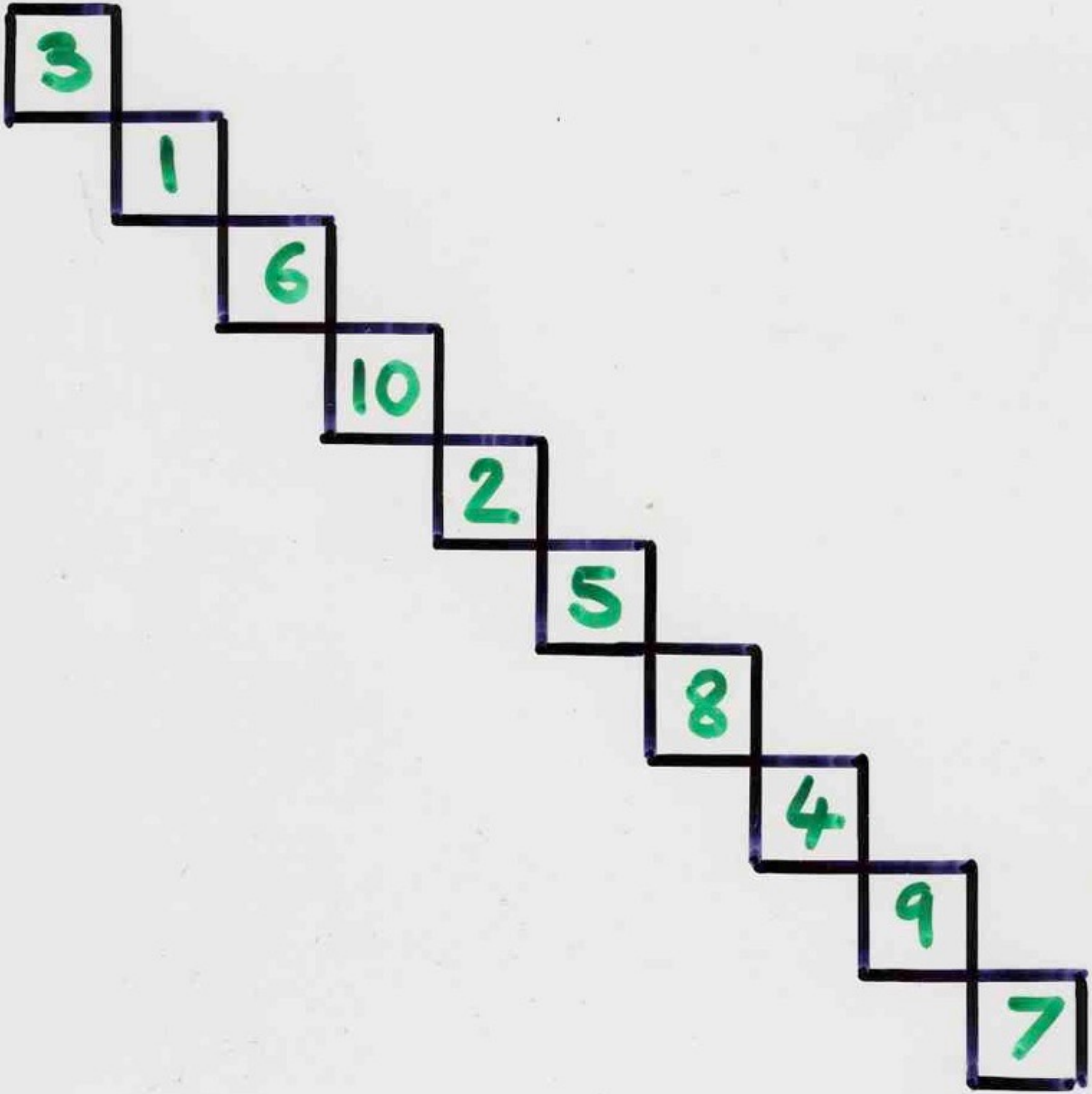
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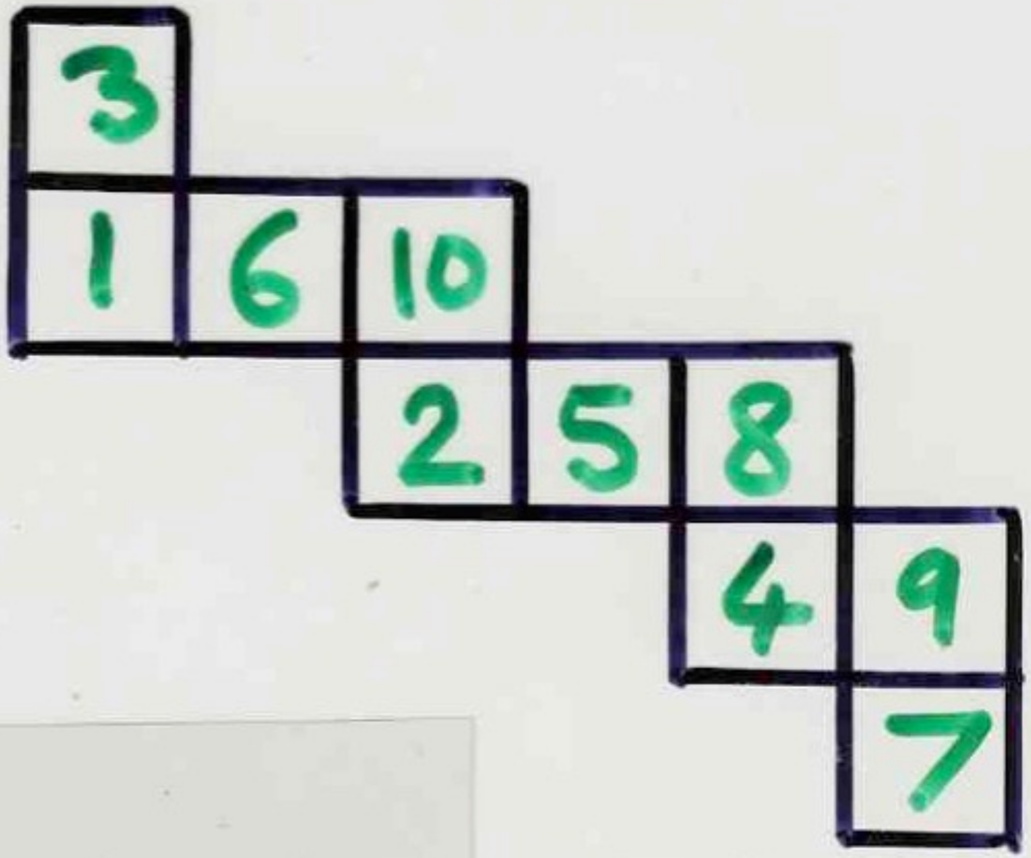
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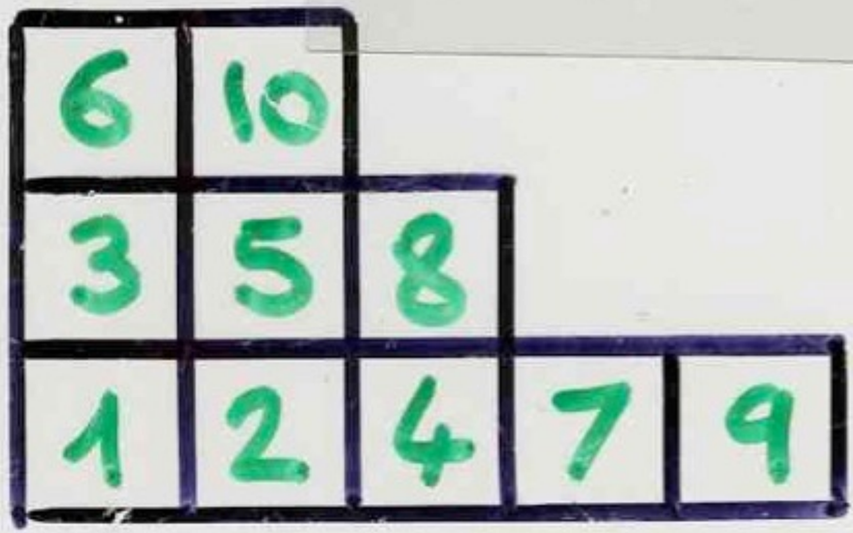
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Schur functions

and

jeu de taquin

Schur Functions

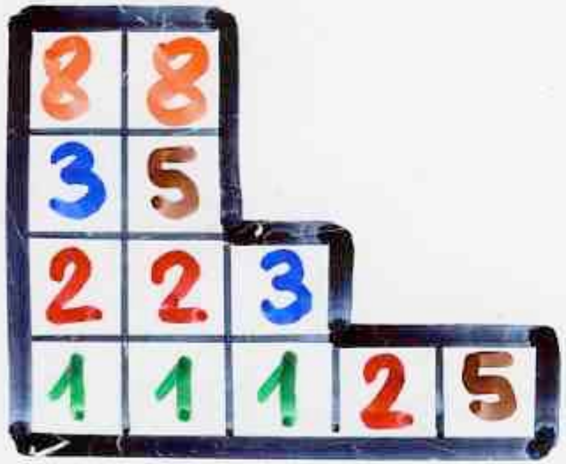
$$S_\lambda(x_1, x_2, \ldots, x_m) = \sum_T v(T)$$

T Young tableau
shape λ
entries $1, 2, \ldots, m$

Jacobi (1841)
Schur (1901)

Littlewood-Richardson (1934)

basis of **symmetric** functions

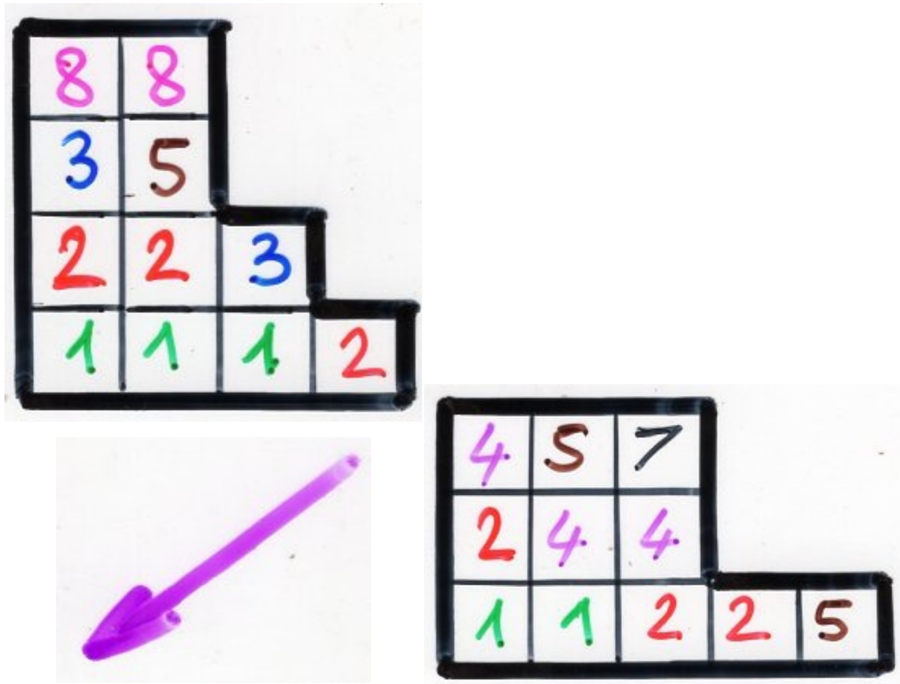


Schur functions

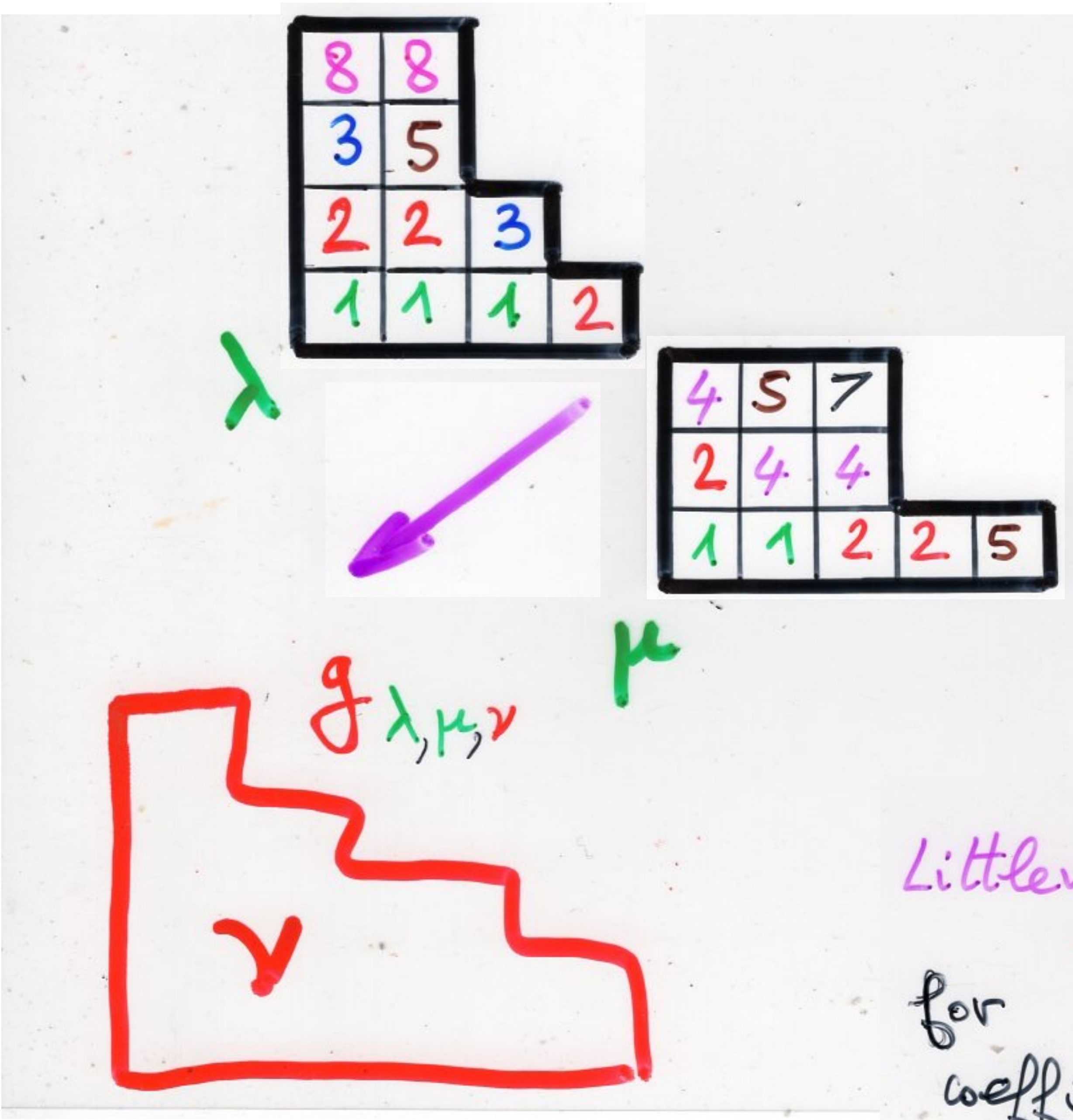
$$s_\lambda s_\mu = \sum_\nu g_{\lambda,\mu,\nu} s_\nu$$

$s_\lambda(x_1, \ldots, x_m)$

Littlewood-Richardson



Jeu de taquin



Jeu de taquin

Littlewood-Richardson rule (1934) for computing the coefficients $g_{\lambda,\mu,\nu}$

jeu de taquin in recent research work

- algebraic combinatorics

  Pechenik, Yong (2015)

  analogue of Littlewood-Richardson coefficients in the "equivariant K-theory" of the Grassmannian

- bijective combinatorics

  Fang (2015)

  - bijective proof of a character identity
    (Frobenius, Murnaghan-Nakayama)

  Krattenthaler (2016)

  - bijection between oscillating tableaux
    (Burrill conjecture)

- probabilistic combinatorics

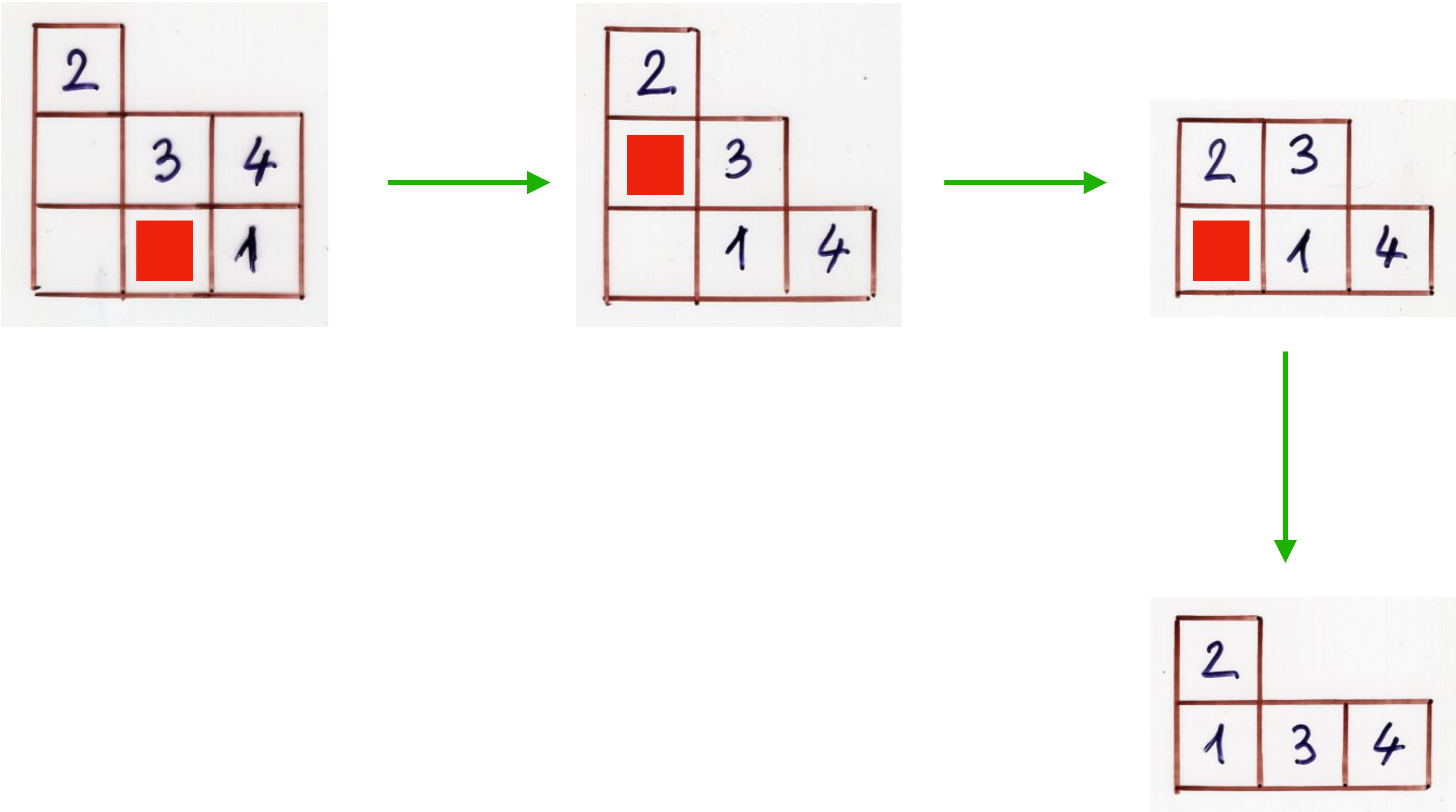
  Romik, Śniady (2015)

  random infinite tableaux

Jeu de taquin
with growth diagrams

S. Fomin, 1986, 1994

Сергей Владимирович Фомин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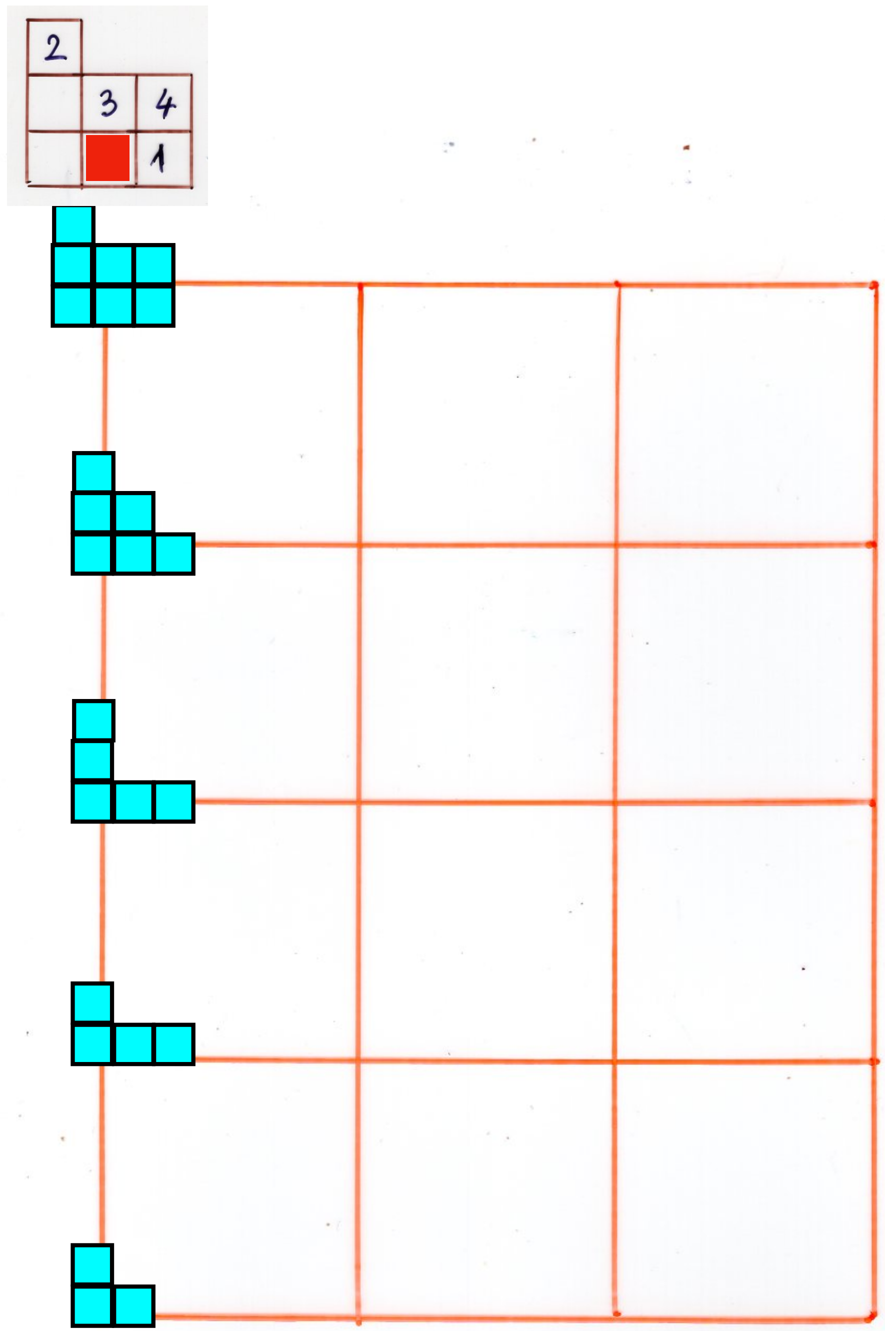
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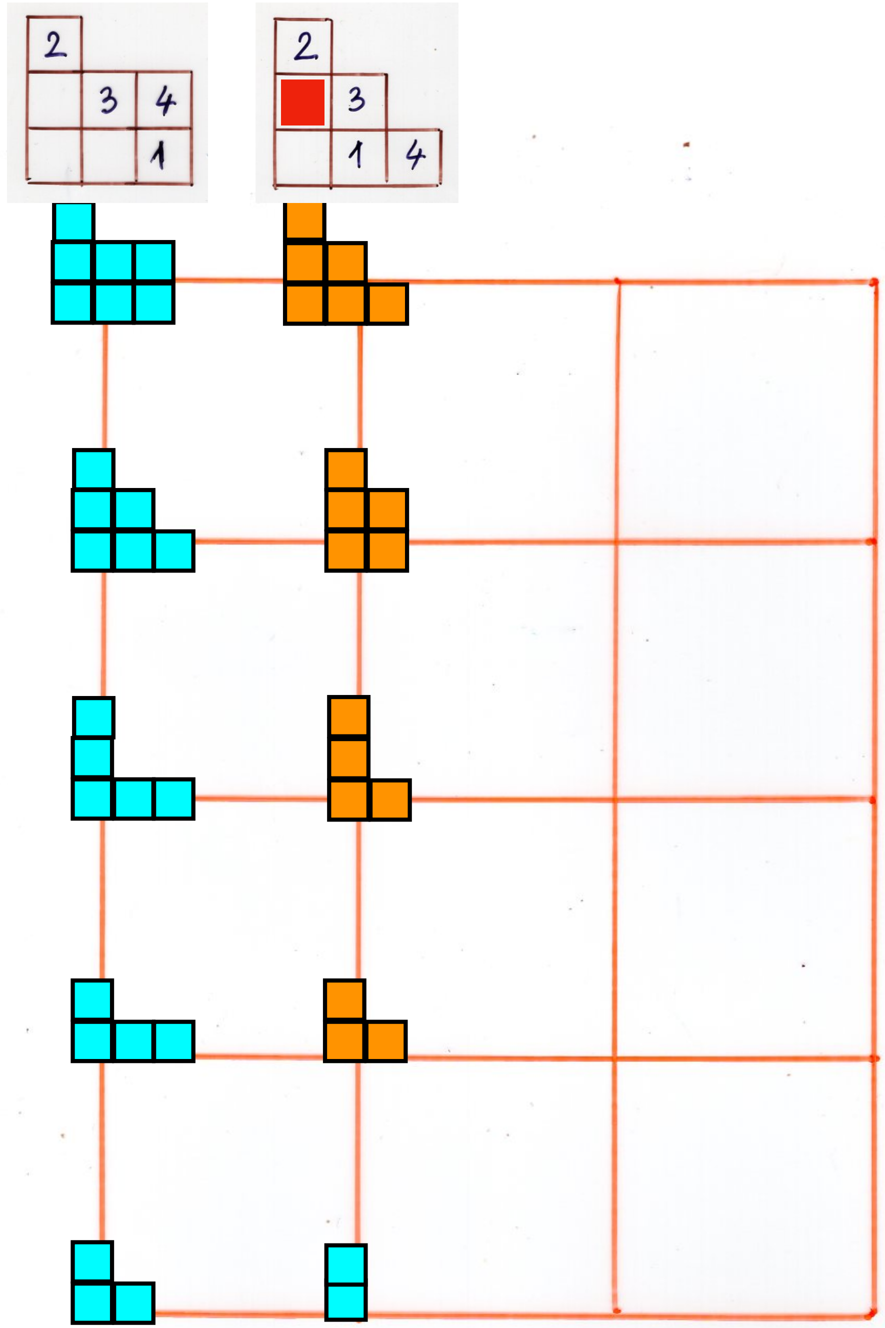
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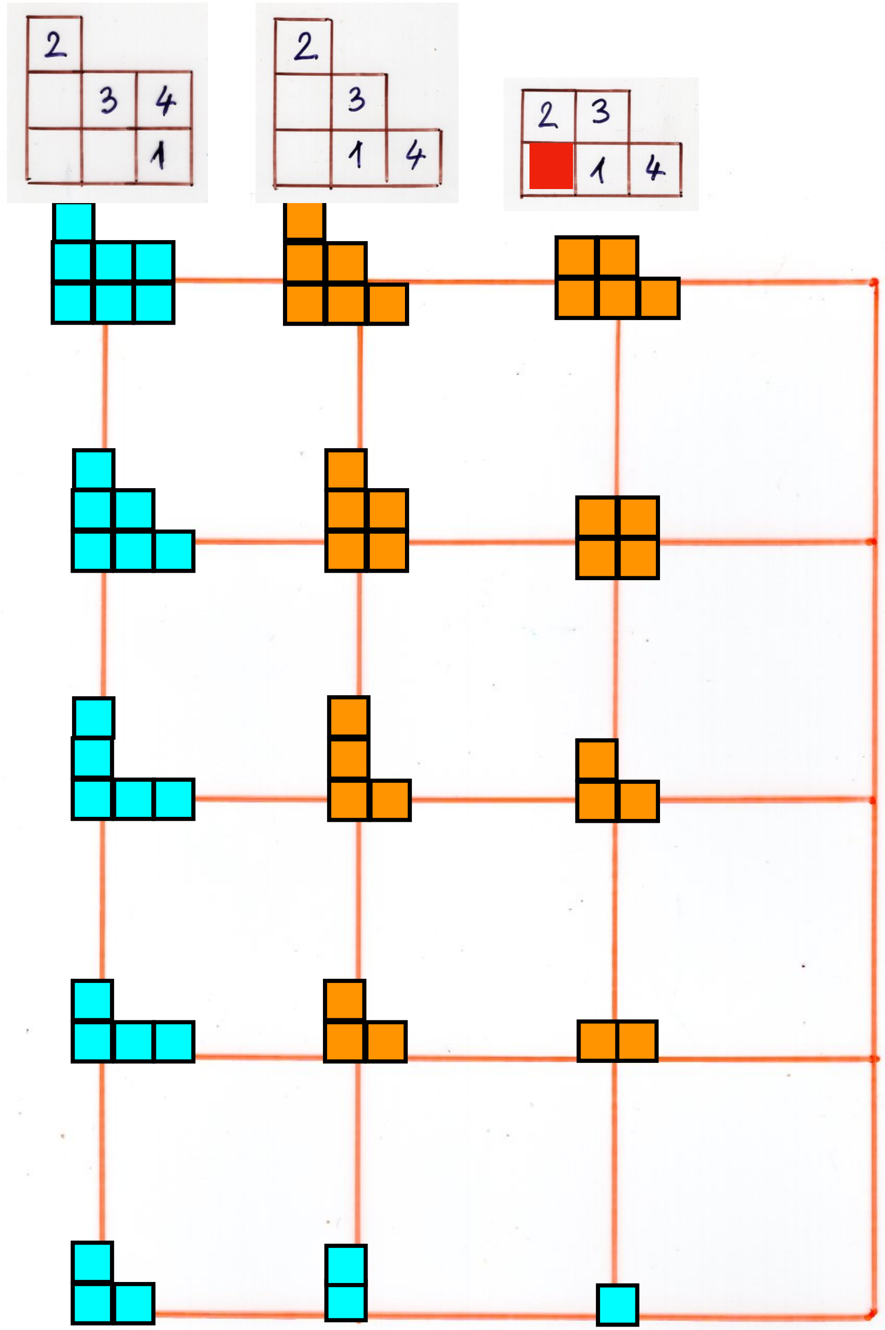

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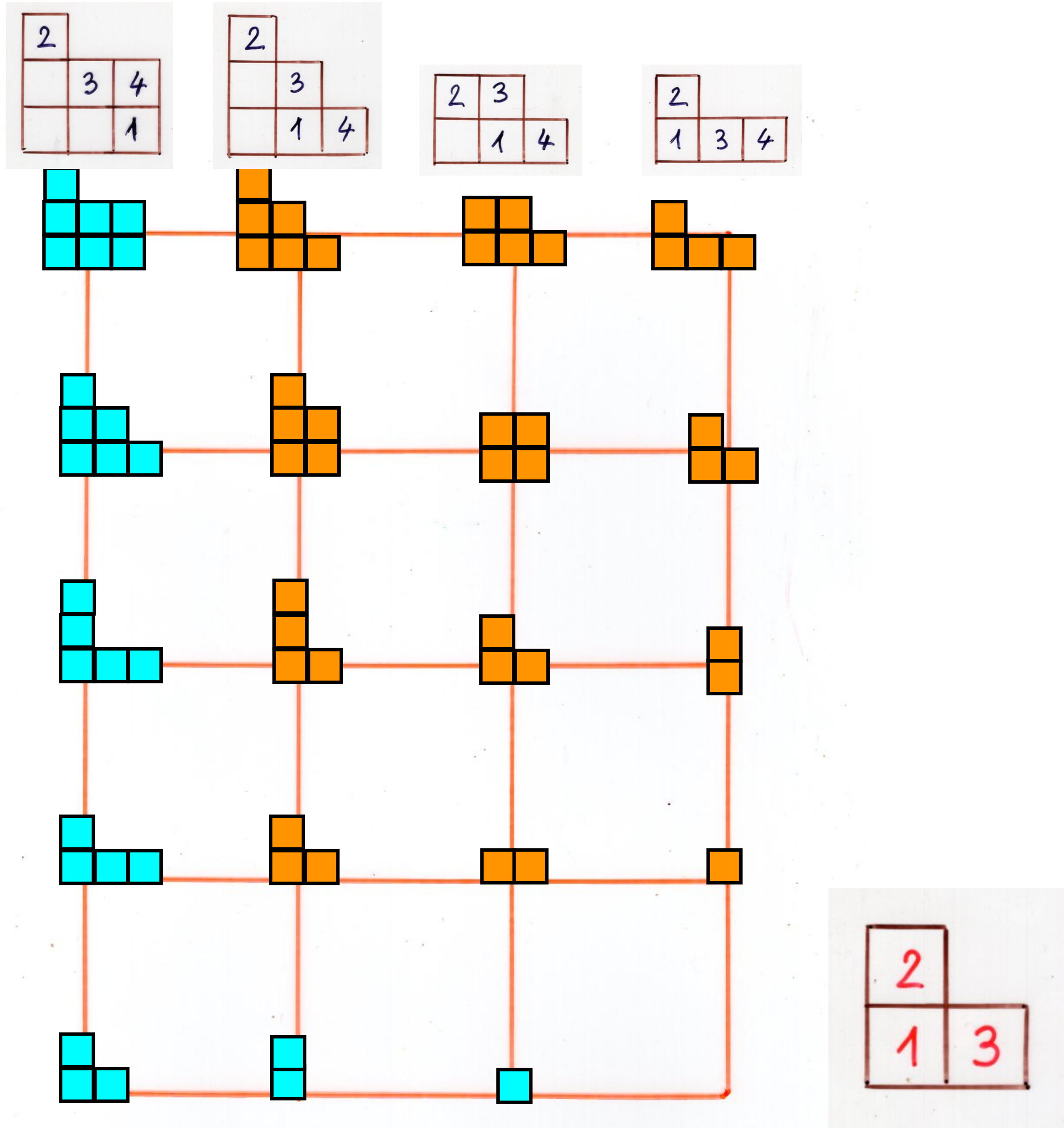

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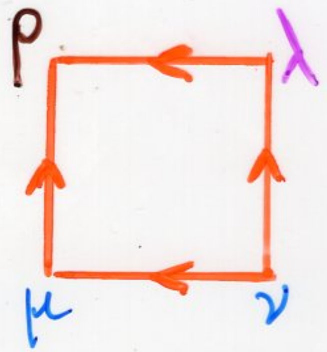
Proposition   jeu de taquin local rules   (Fomin)

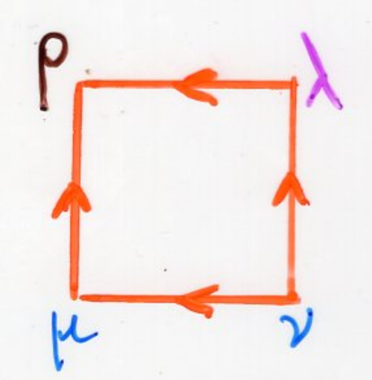


cell of the jeu de taquin growth diagram
(ρ covers μ and λ, μ and λ cover ν)

Then λ is uniquely determined from μ, ν, ρ by the following "local rule":

(i) • if μ is the only shape of its size that contains ν and is contained in ρ then $\lambda = \mu$

(ii) • otherwise there is a unique such shape different from μ, and this is λ



jeu de taquin
local rules

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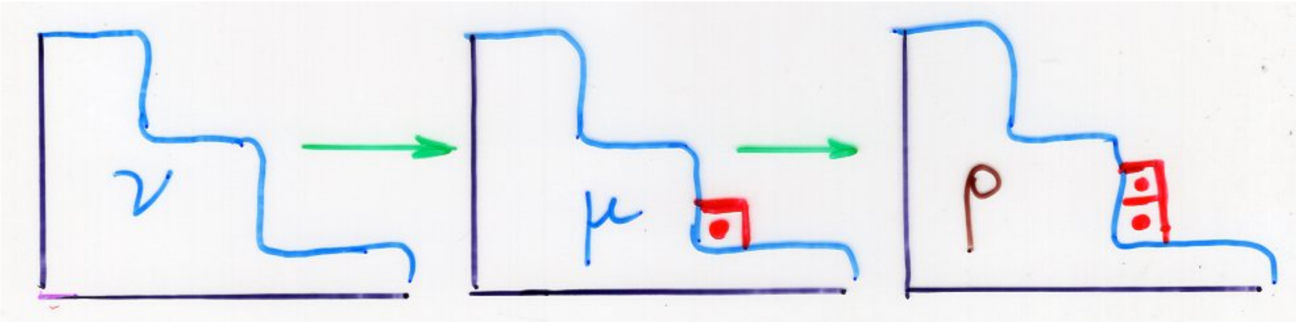
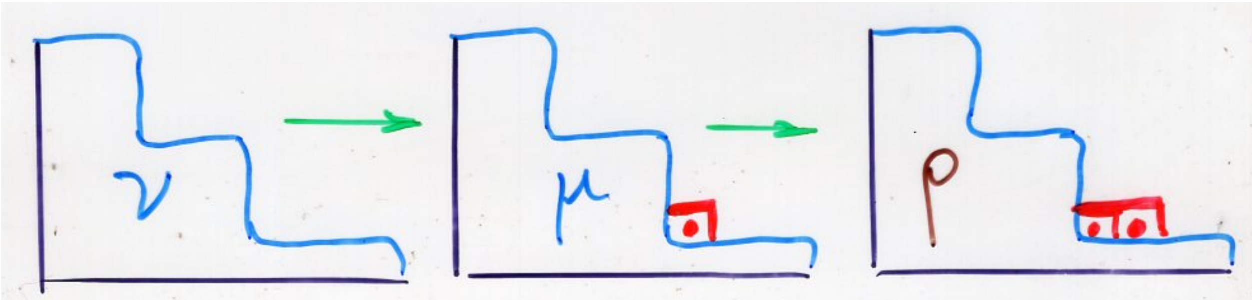
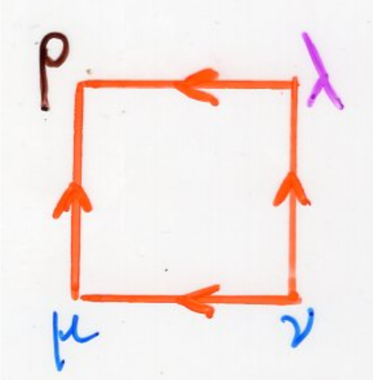




jeu de taquin local rules



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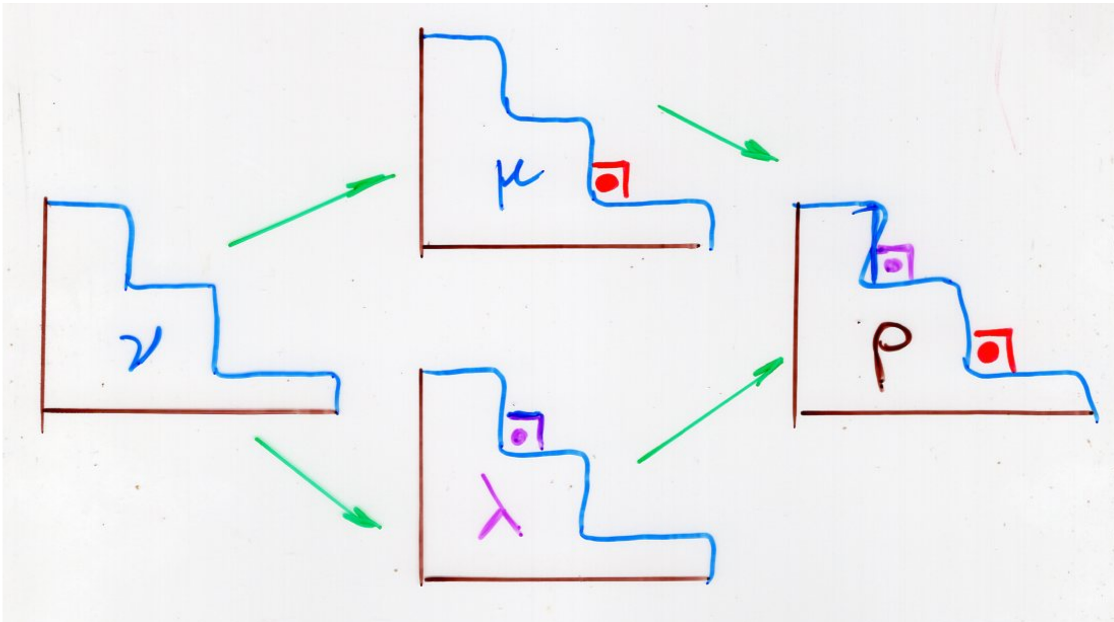
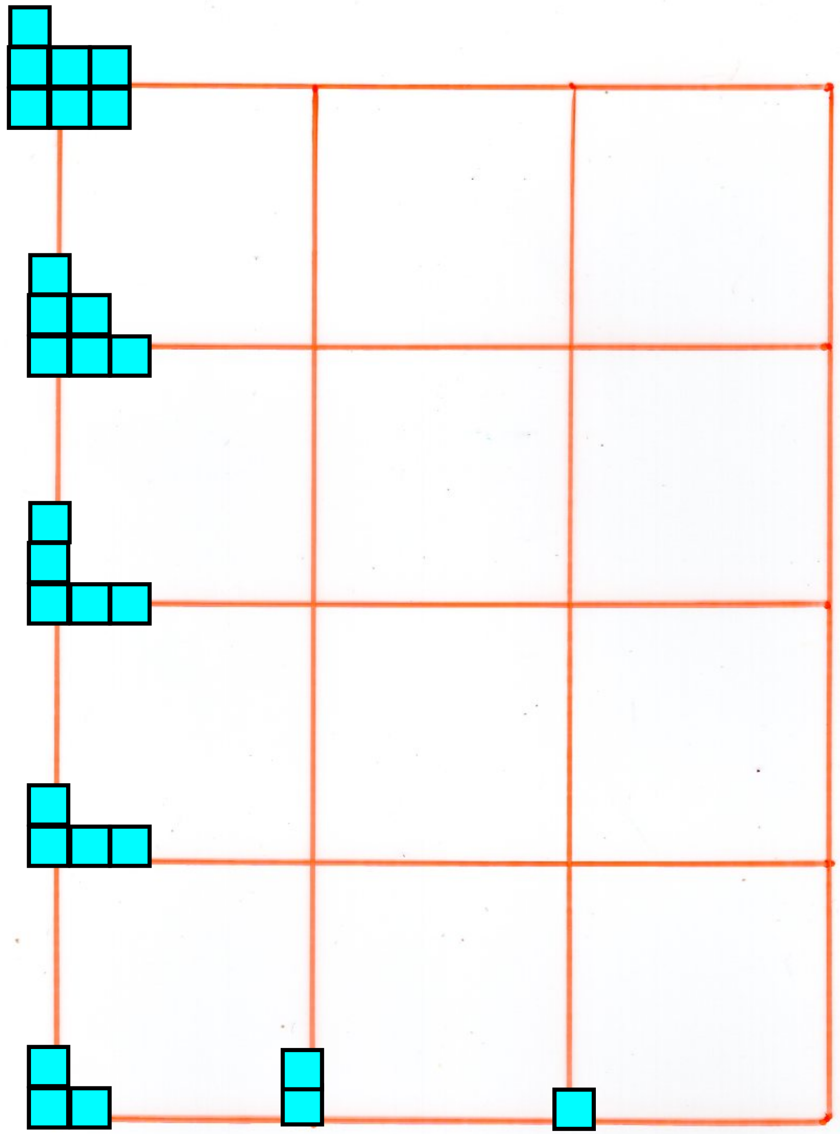
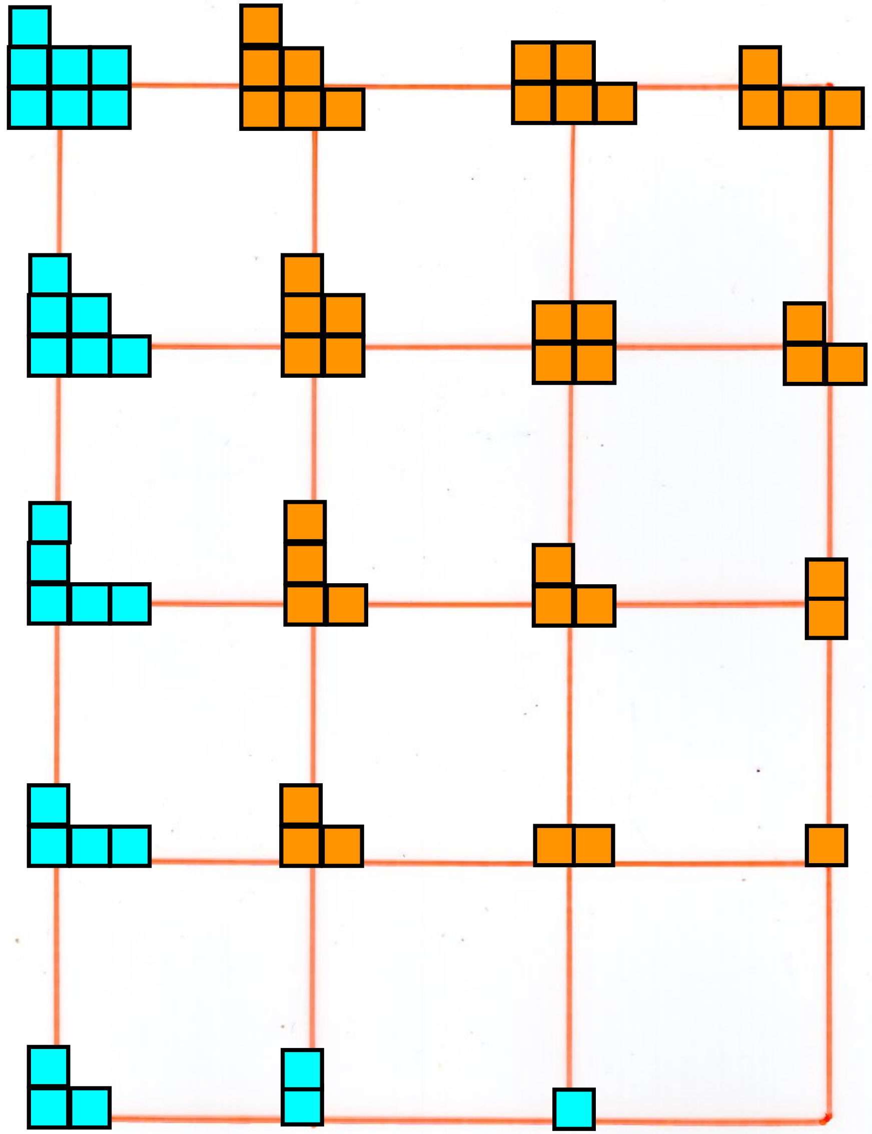
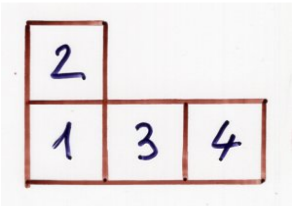


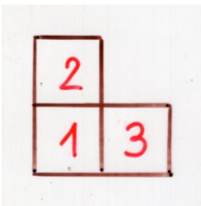


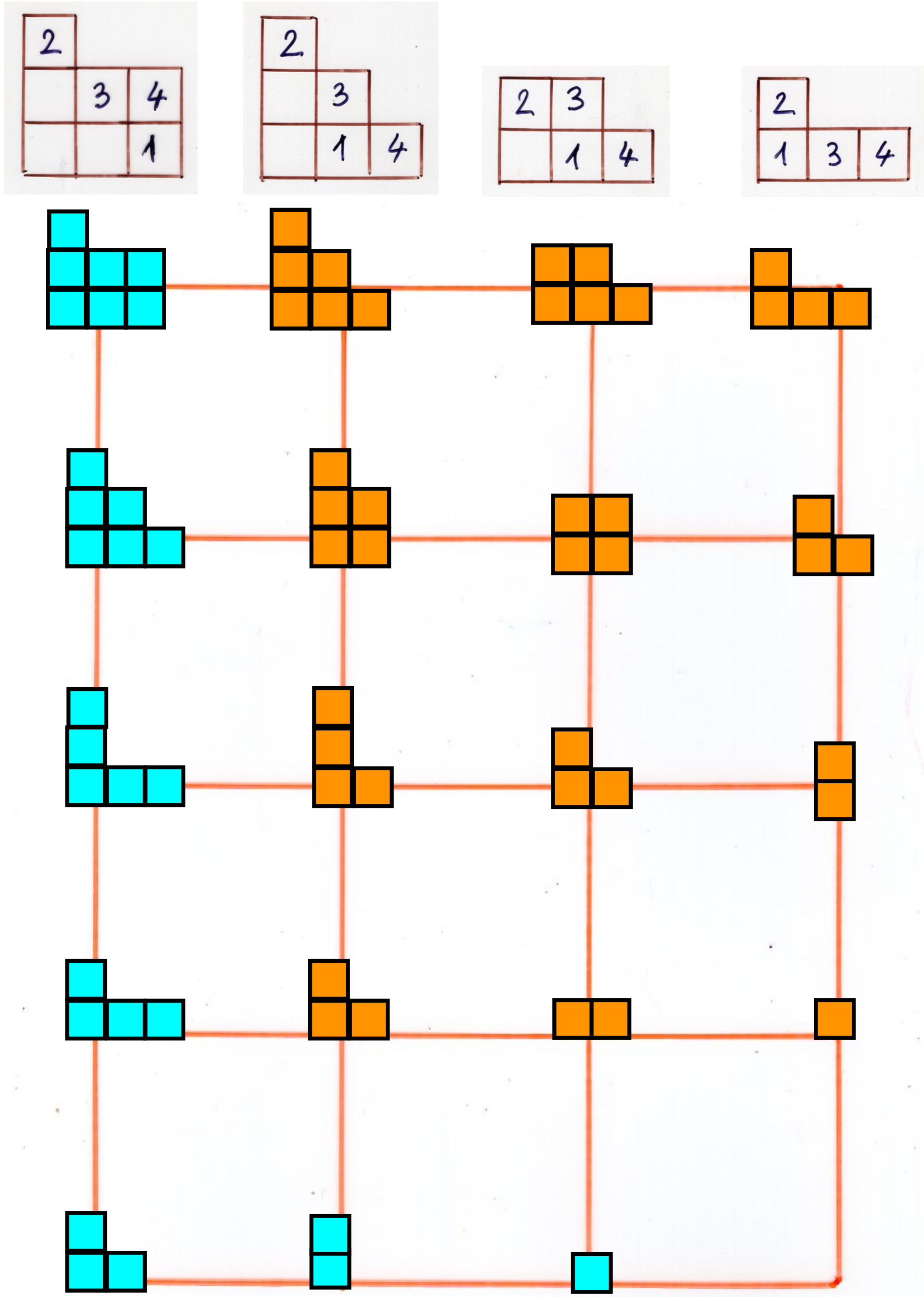


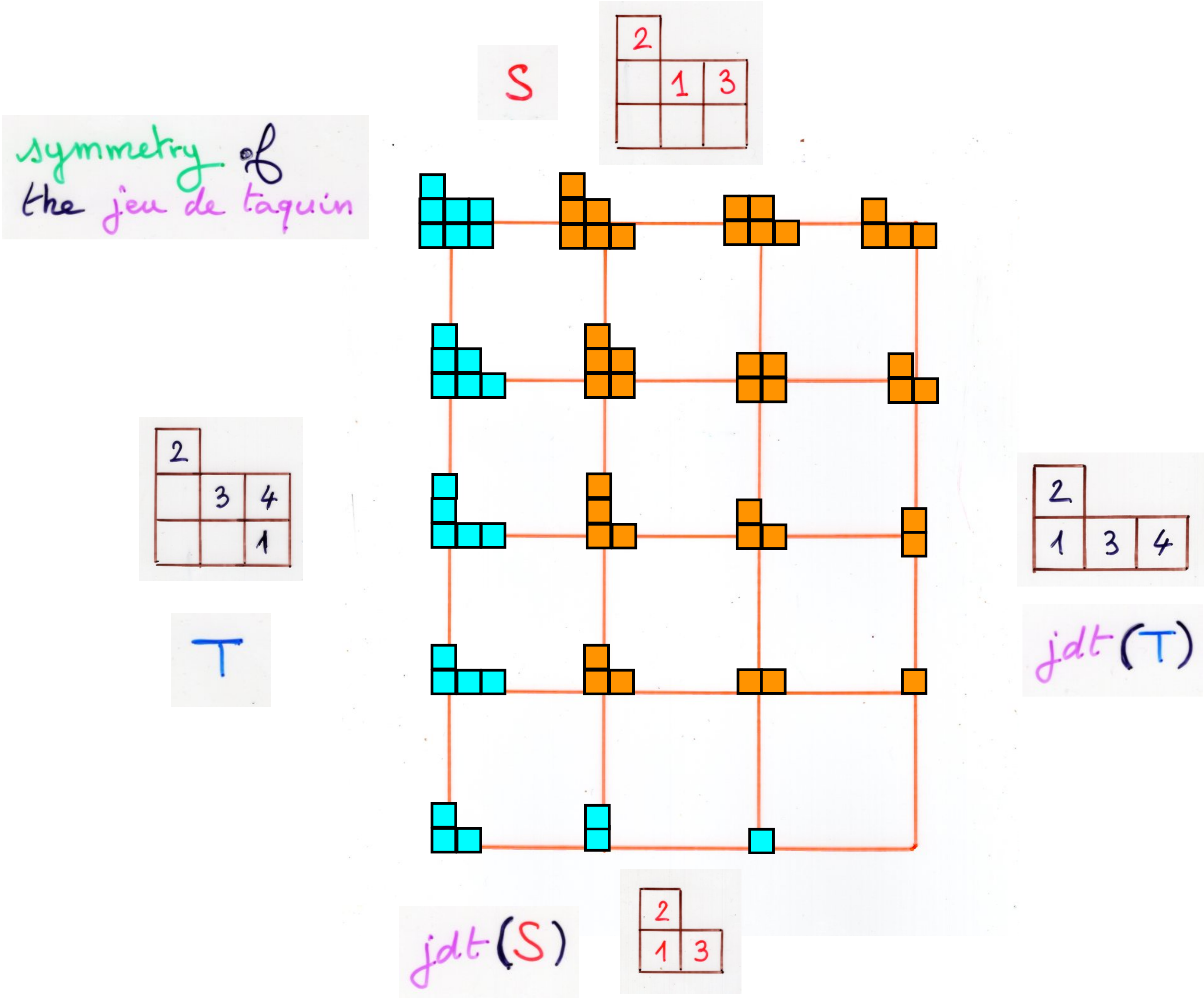
the tableau



is independant of the choice of the tableau






symmetry of the jeu de taquin
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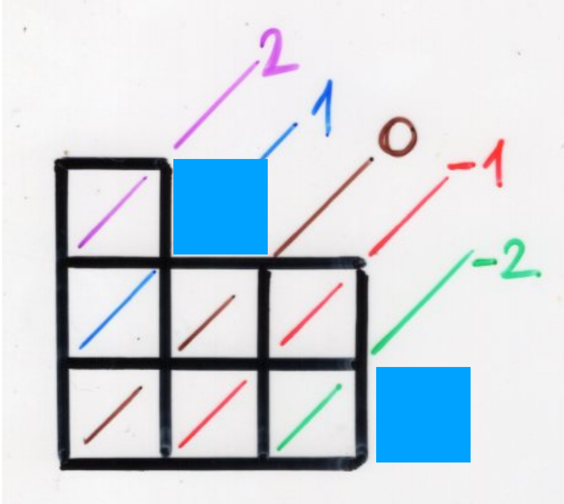
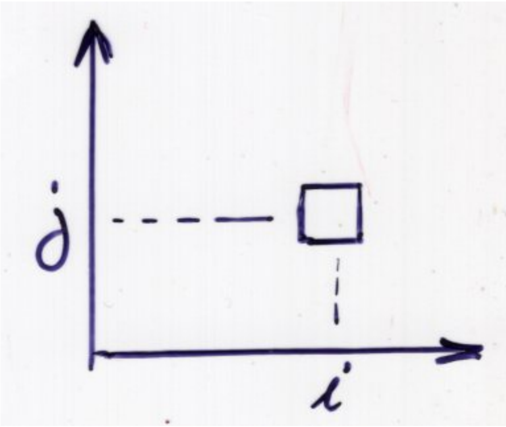
Jeu de taquin

with local rules on edges ?

diagonal operators

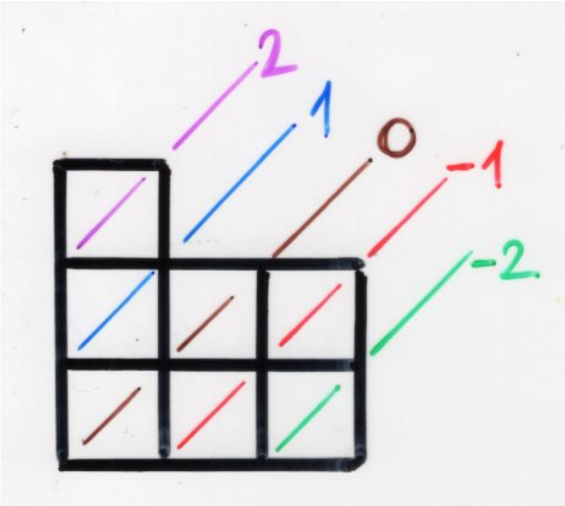
$\Delta_i \quad i \in \mathbb{Z}$





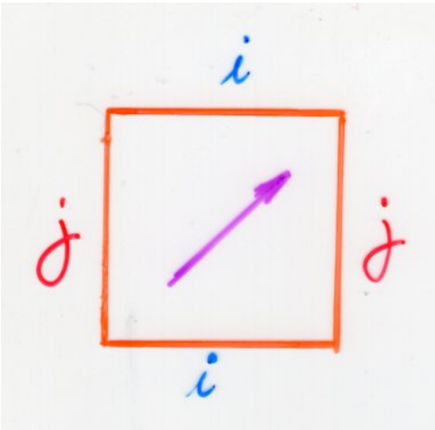
$(i, j) \longrightarrow j - i$

content

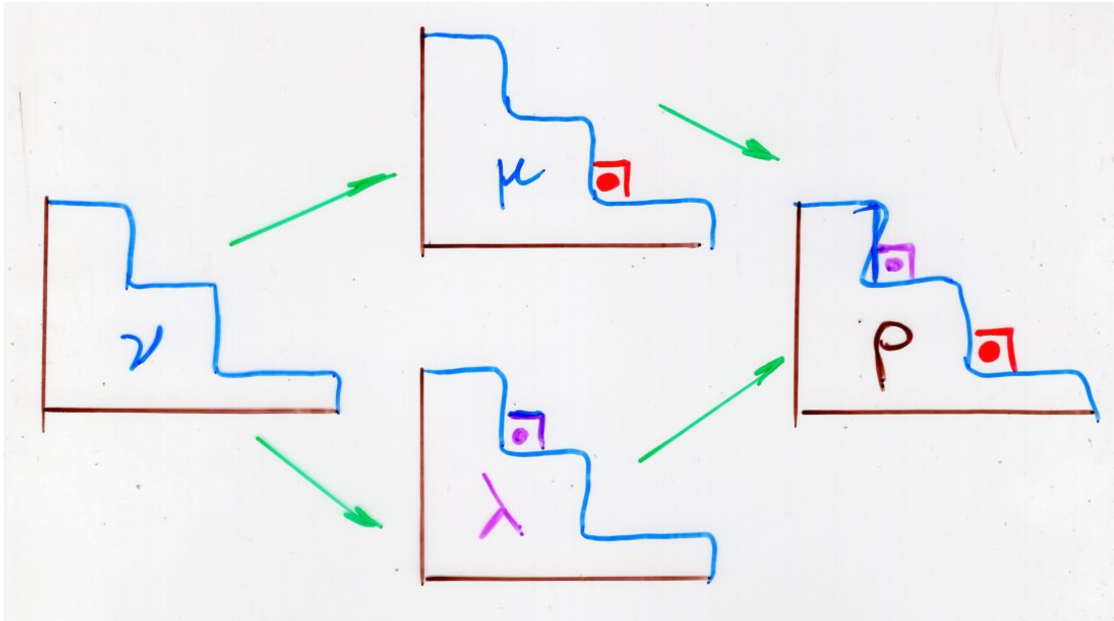


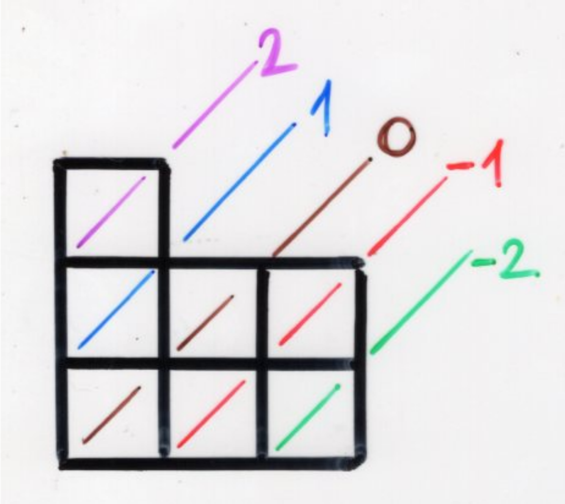
jeu de taquin

local rules on edges



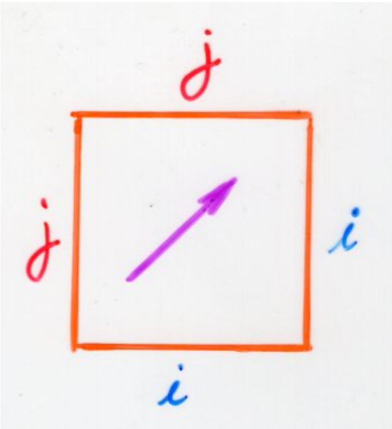
$$|i-j| \geq 2$$





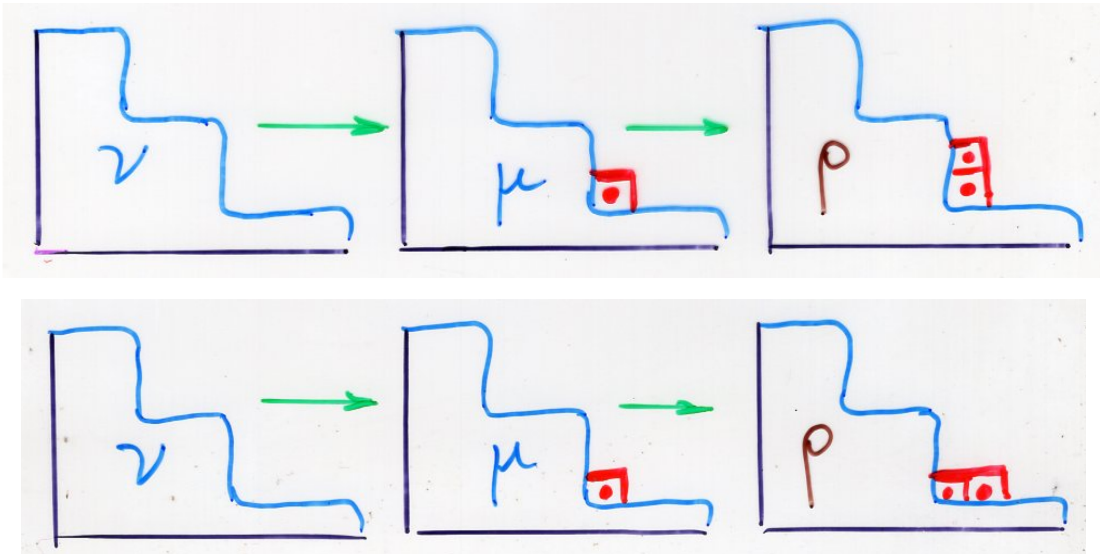
jeu de taquin

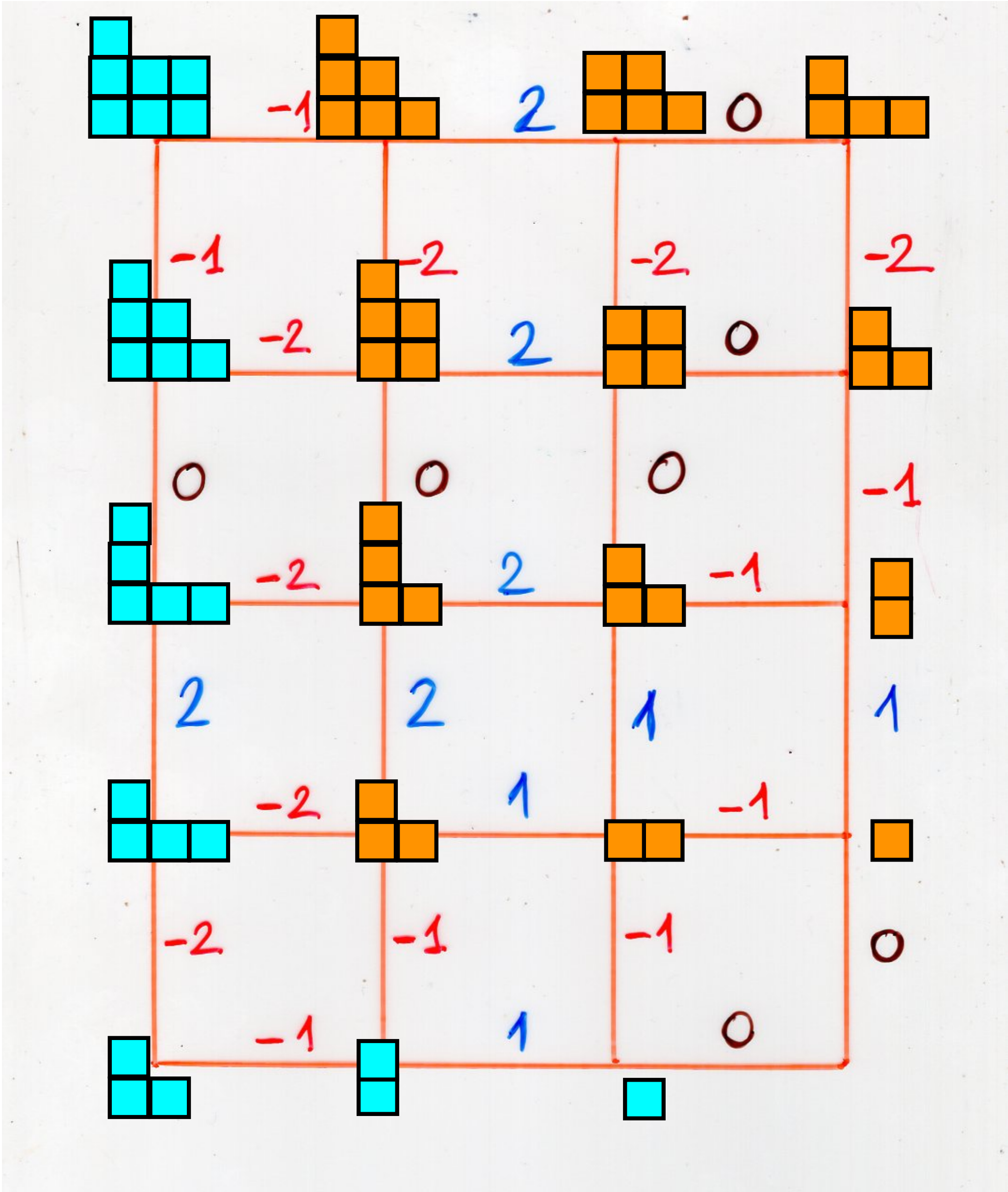
local rules on edges



$|i-j| \leq 1$

or





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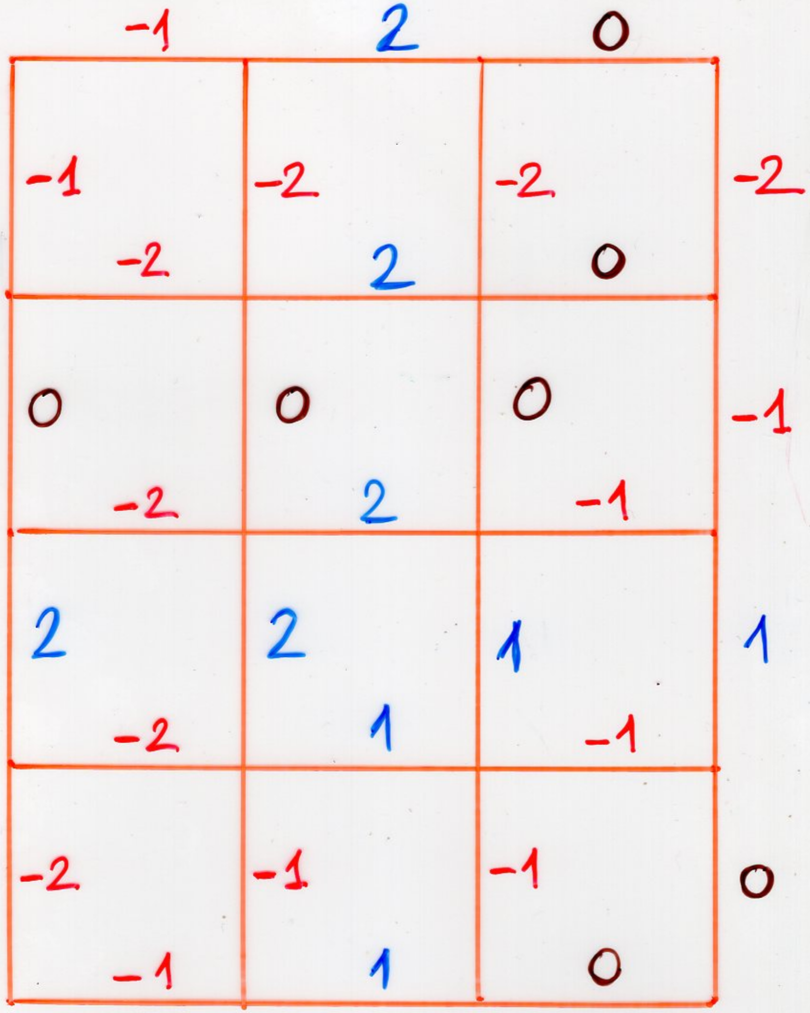

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-2 -1 -1 0
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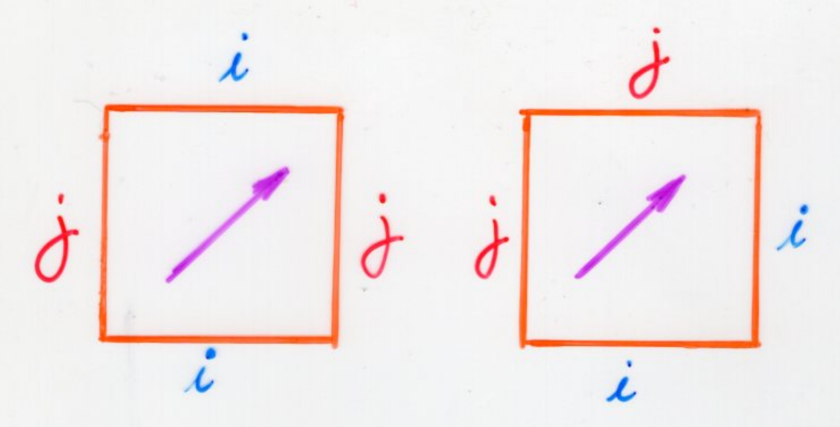
jeu de taquin
local rules on edges

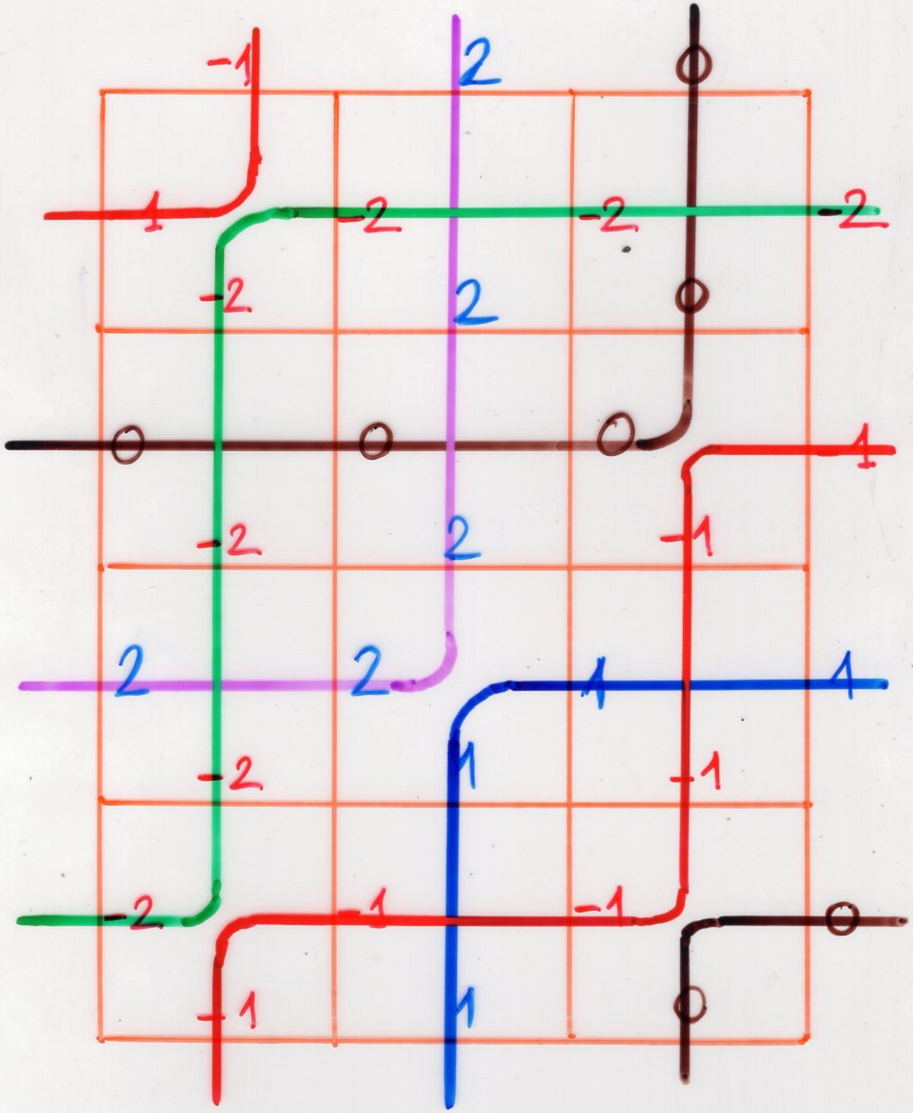


$i, j \in \mathbb{Z}$

$|i-j| \geqslant 2$ $|i-j| \leqslant 1$

in fact here $i = j$ impossible

nil-Temperley-Lieb
planar automaton



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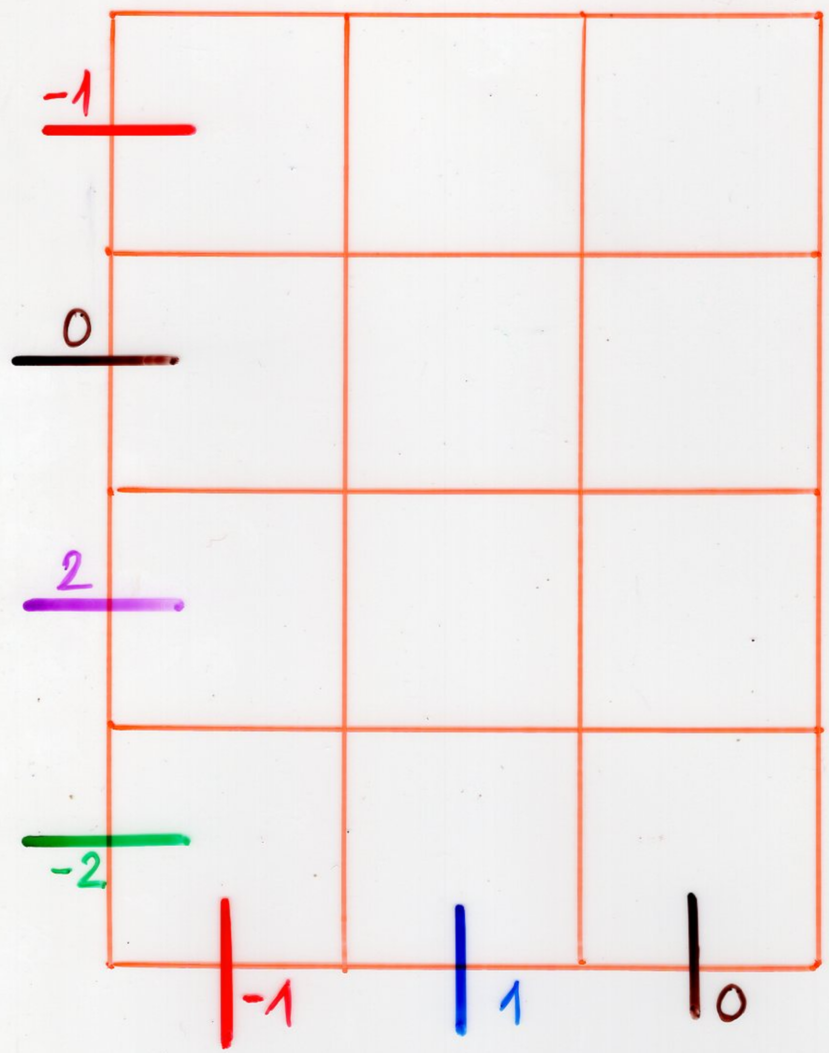

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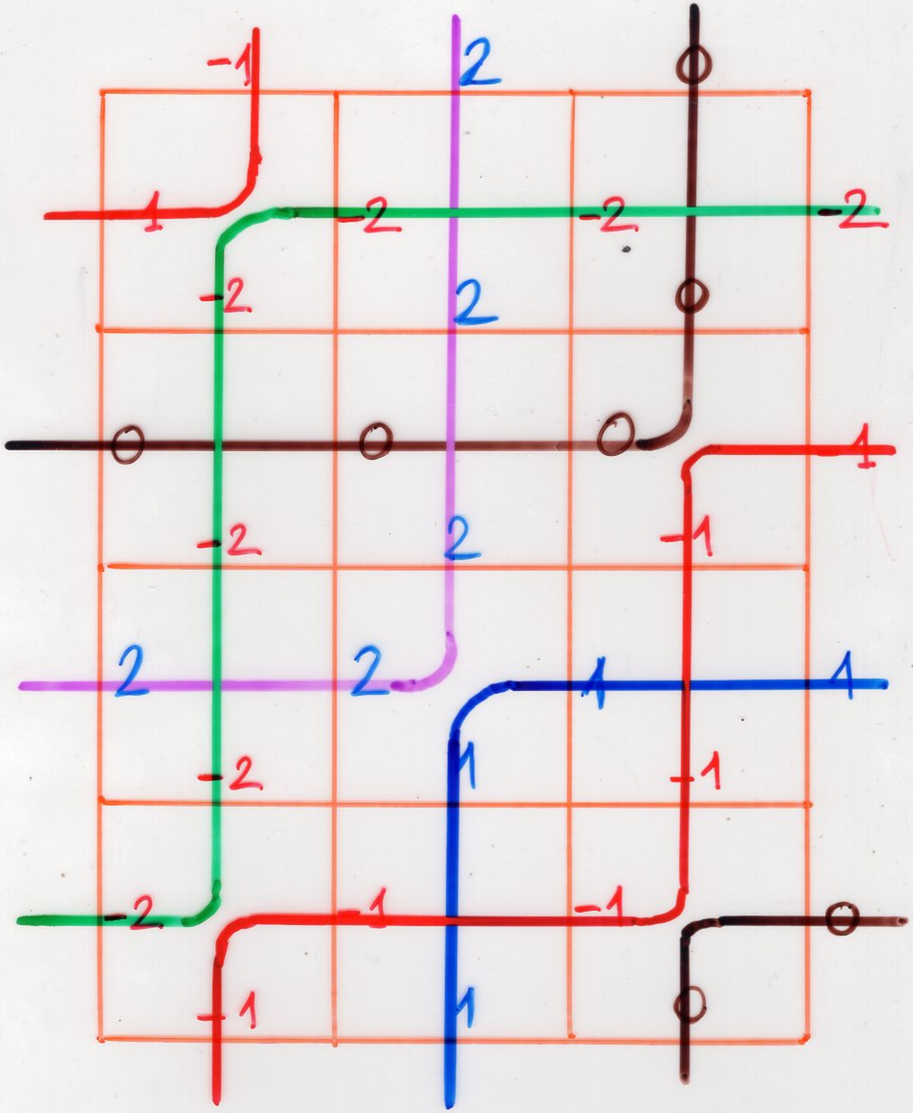

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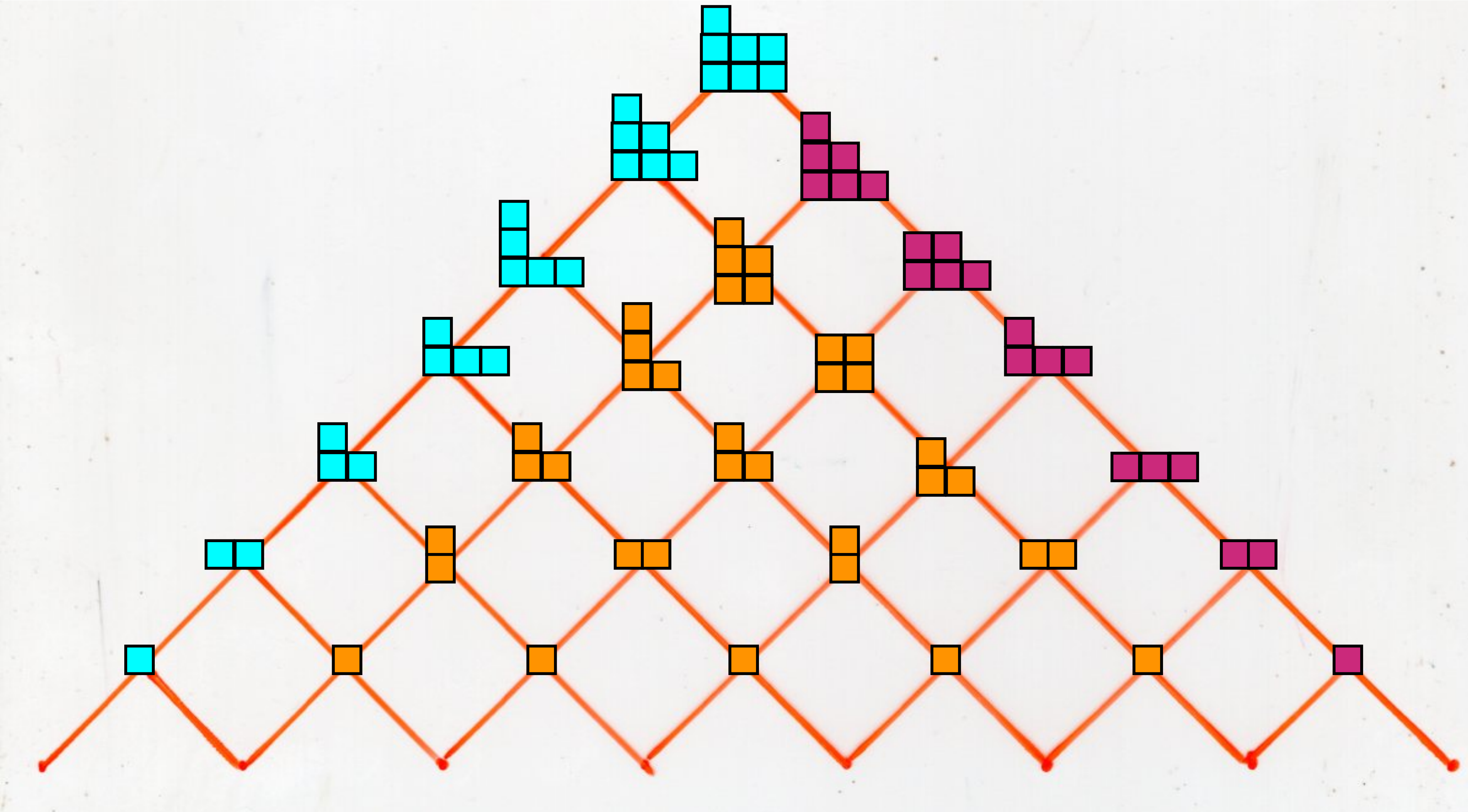
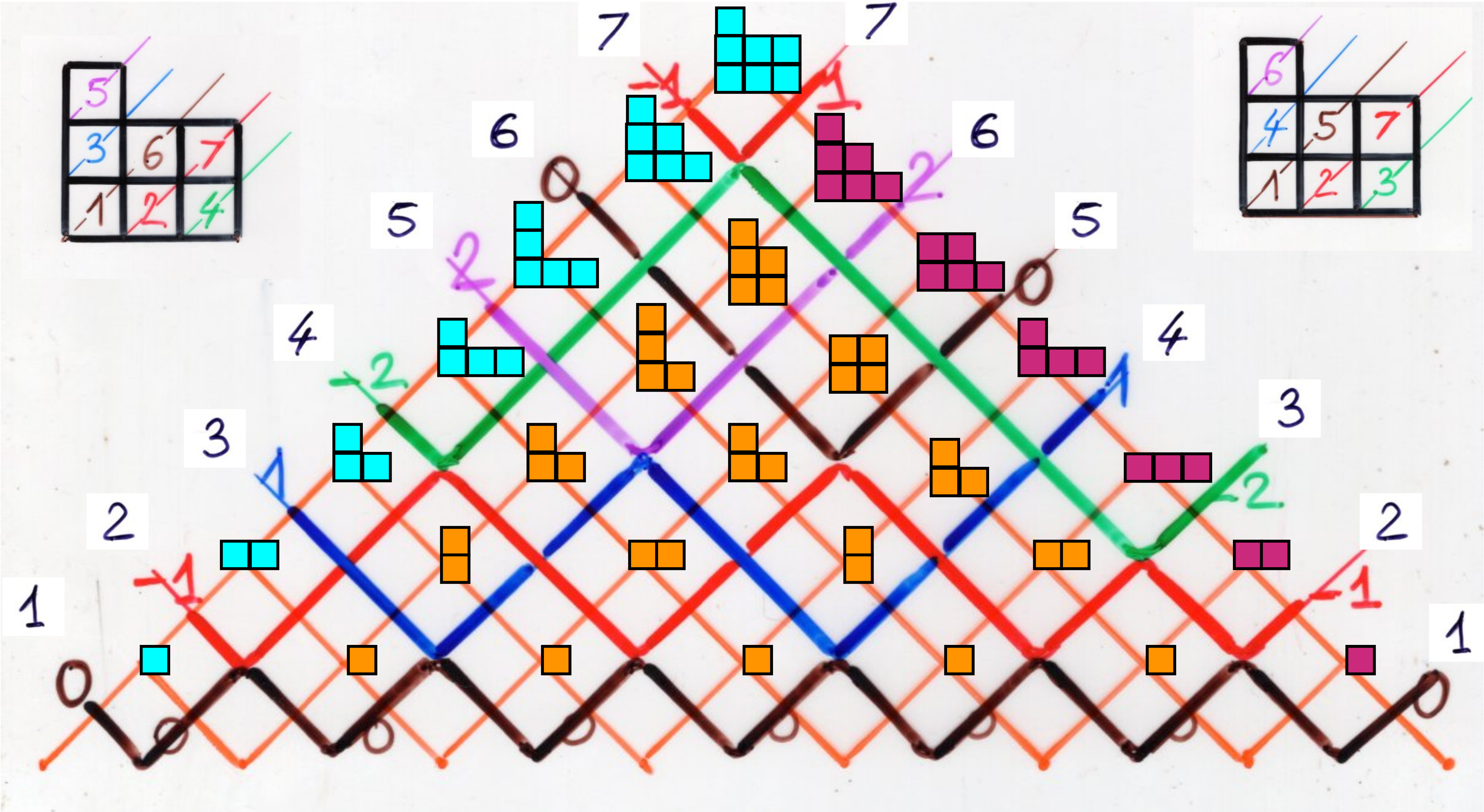
dual of a tableau



Schützenberger involution

dual of a tableau



Schützenberger involution

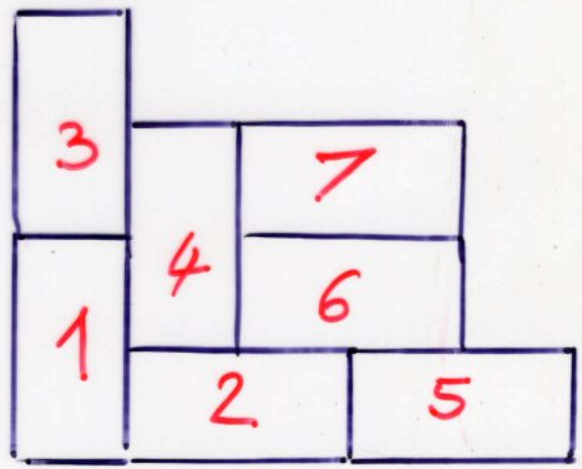
Proposition The map $T \longrightarrow T^*$ is an involution $(T^*)^* = T$

T Young tableau
T^* dual tableau

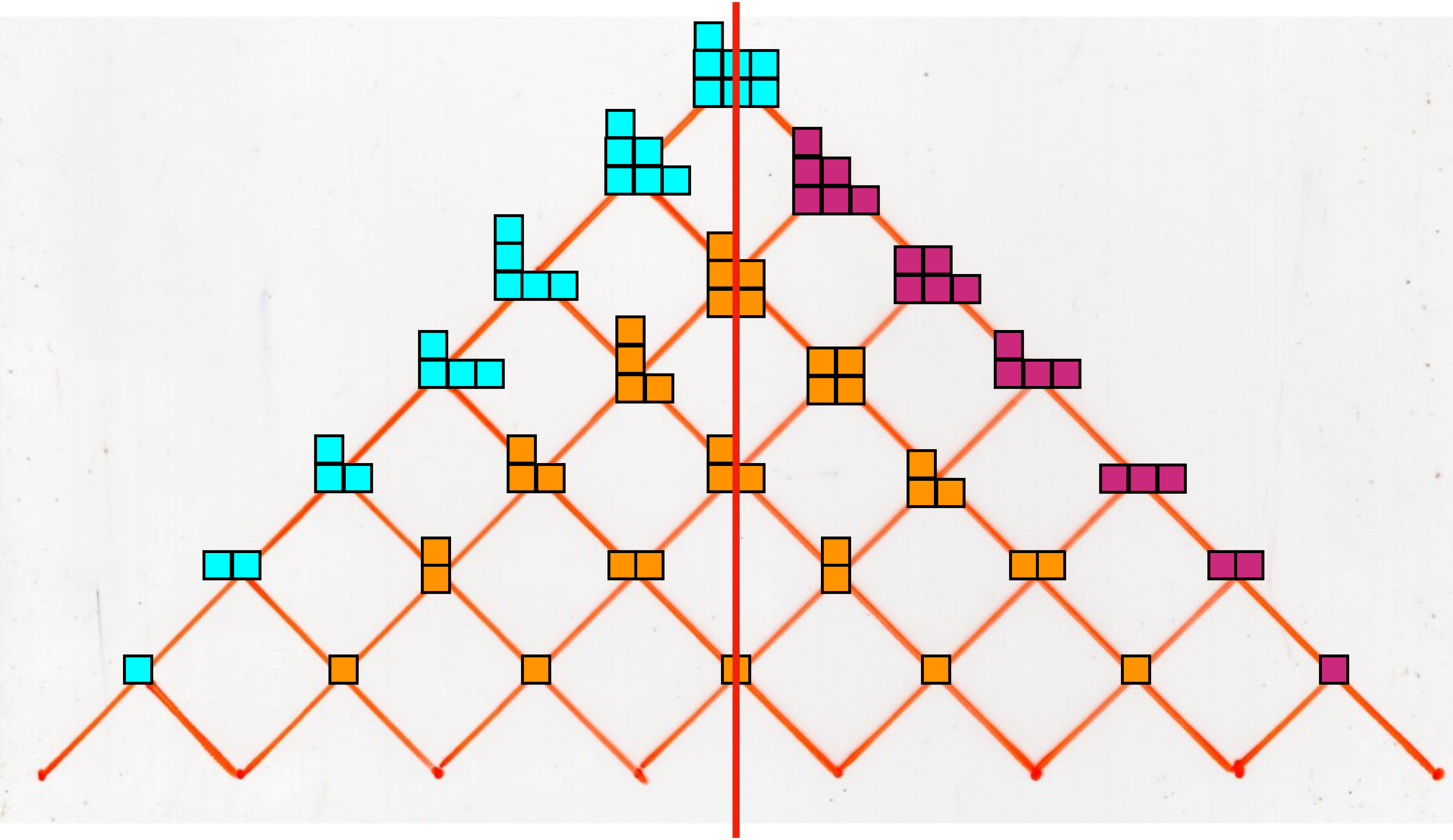
evac(T)
other notation

Proposition

tableaux such that $T = T^*$ are in bijection with domino tableaux



dual of a tableau



Schützenberger involution

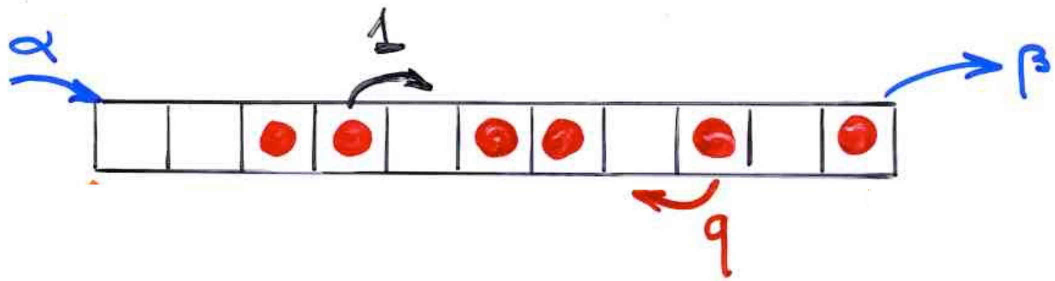
The PASEP algebra

$$DE = qED + E + D$$

alternative tableaux

toy model in the physics of dynamical systems far from equilibrium

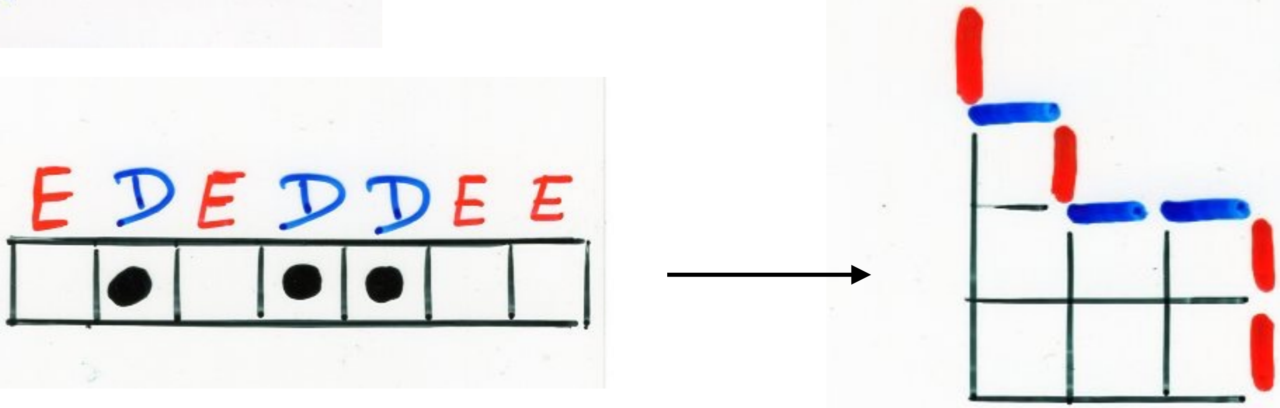
ASEP
TASEP
PASEP



computation of the "stationary probabilities"

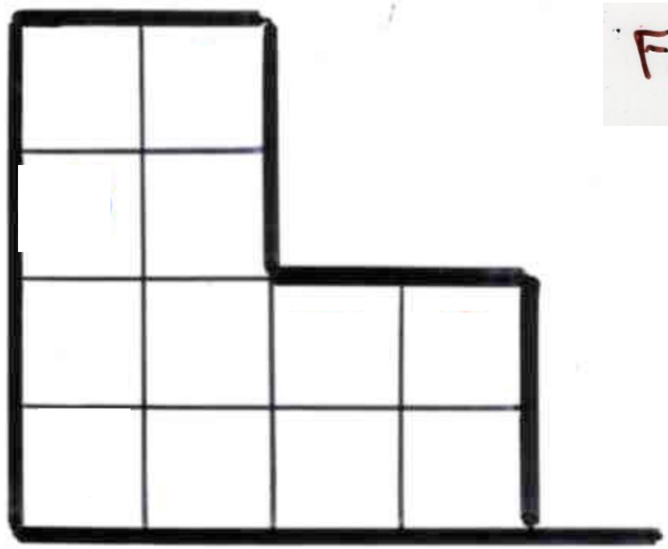
$$DE = qED + E + D$$

The PASEP algebra



alternative tableau

Definition



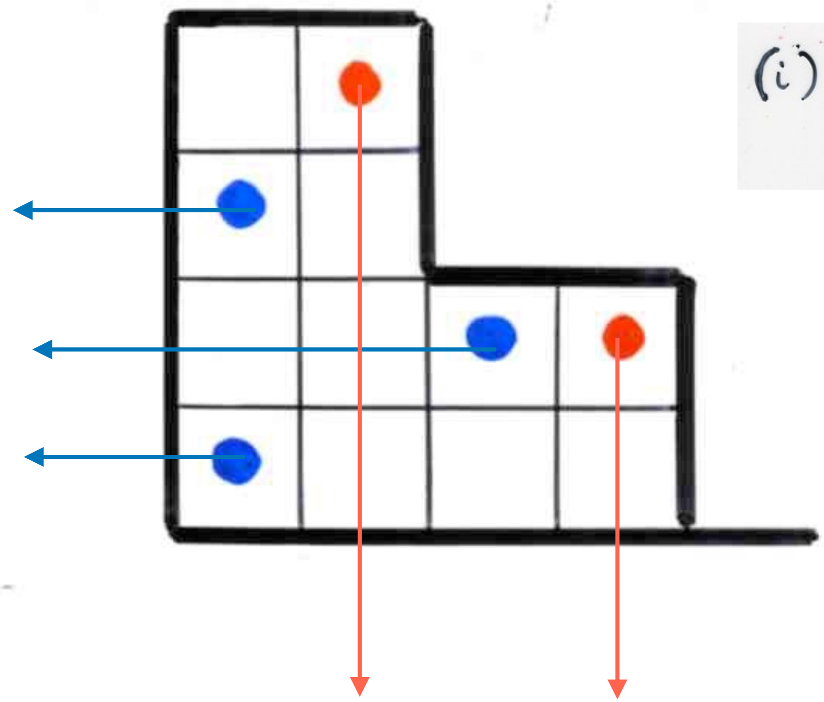
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

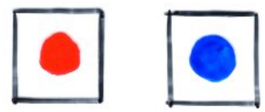
n = (number of rows) + (number of columns)

alternative tableau

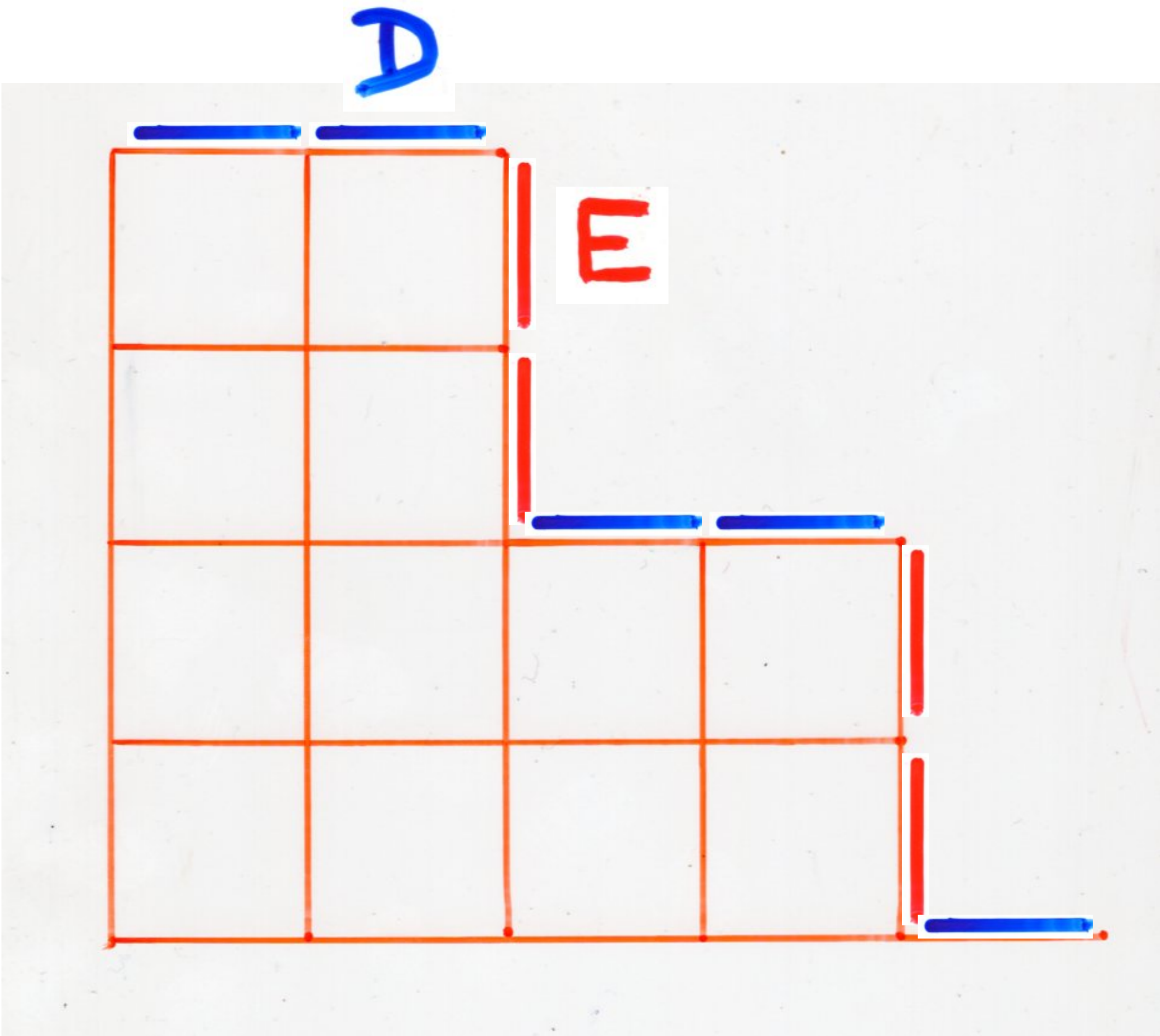


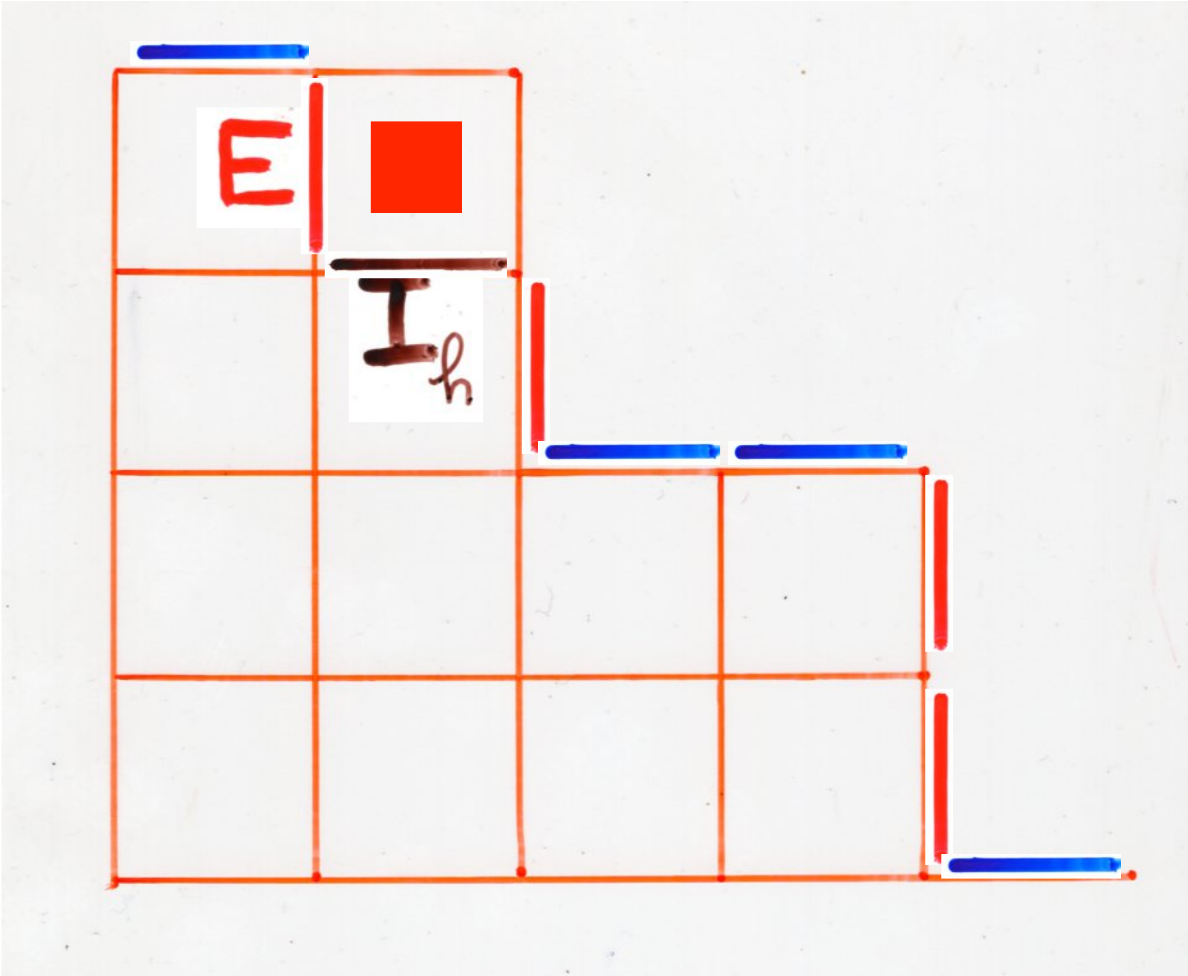
Definition

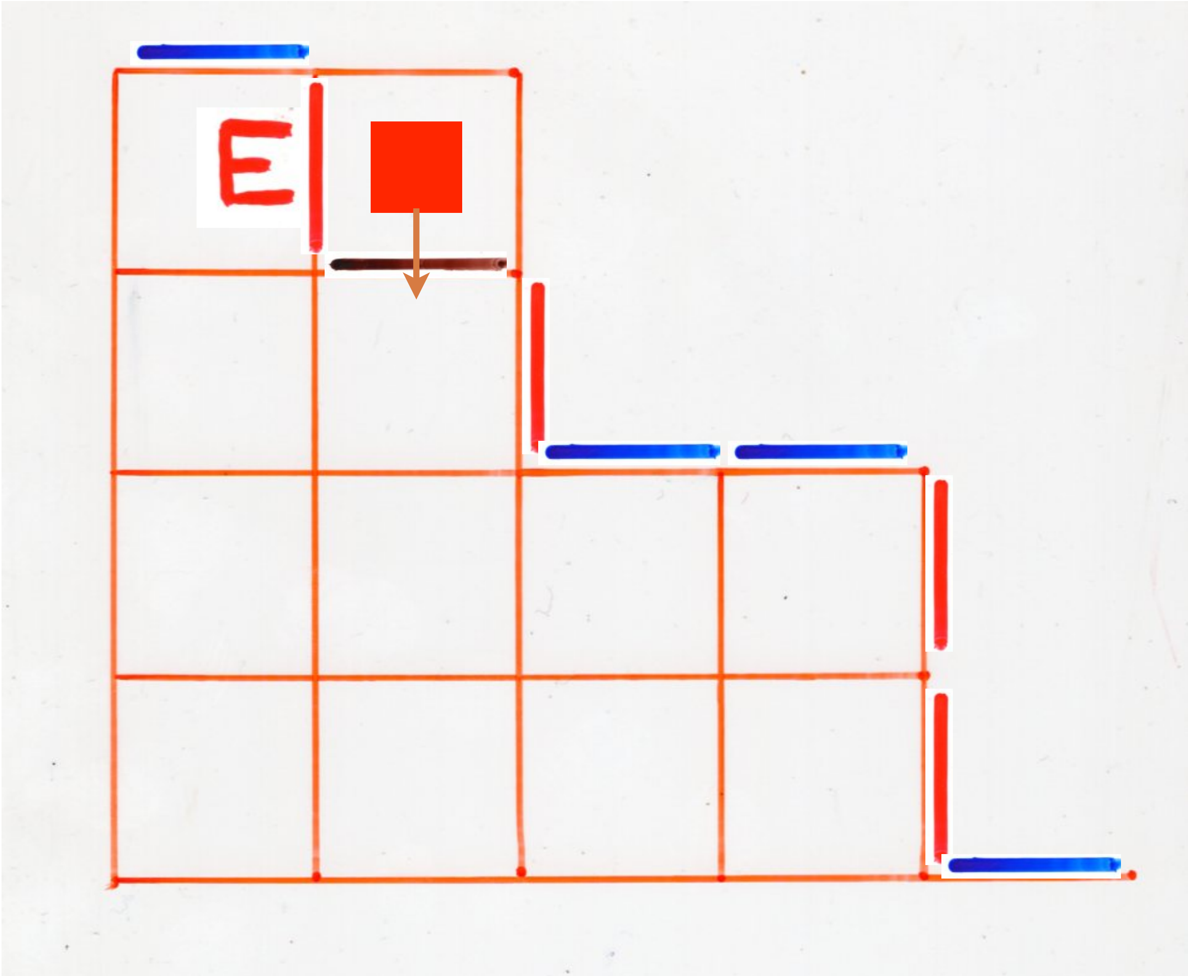
(i) some cells are coloured red or blue

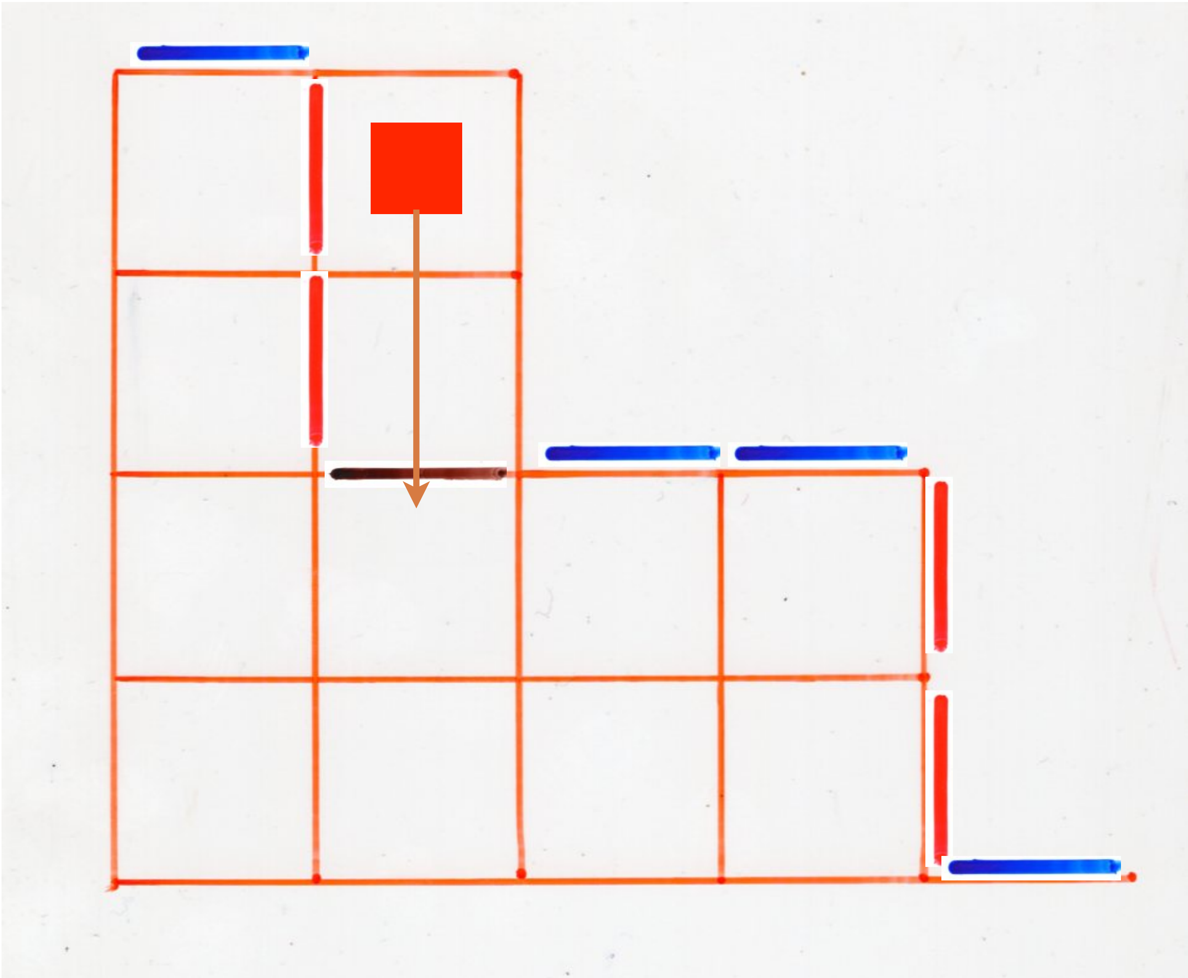


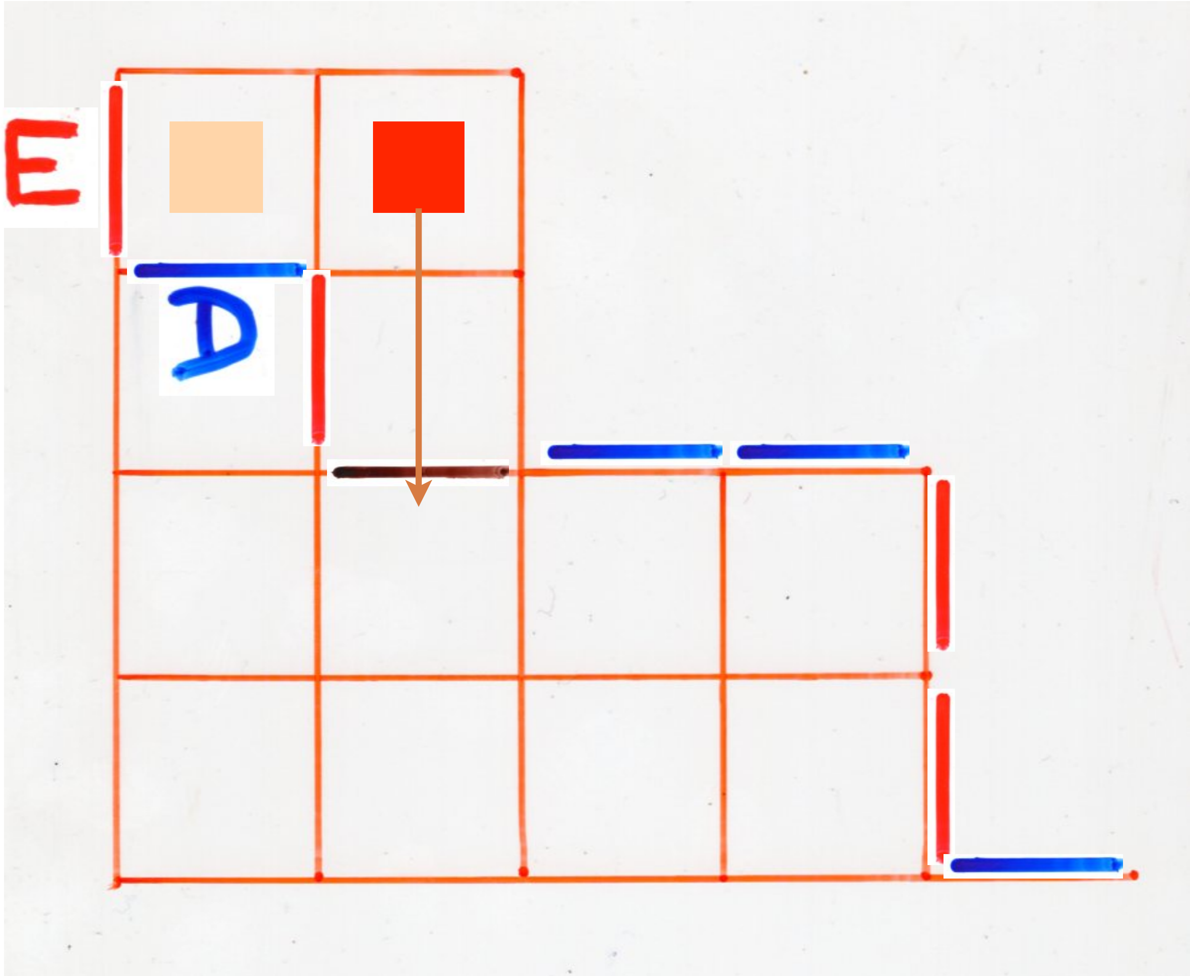
(ii)
- no coloured cell at the left of a blue cell
- no coloured cell below a red cell


D
E

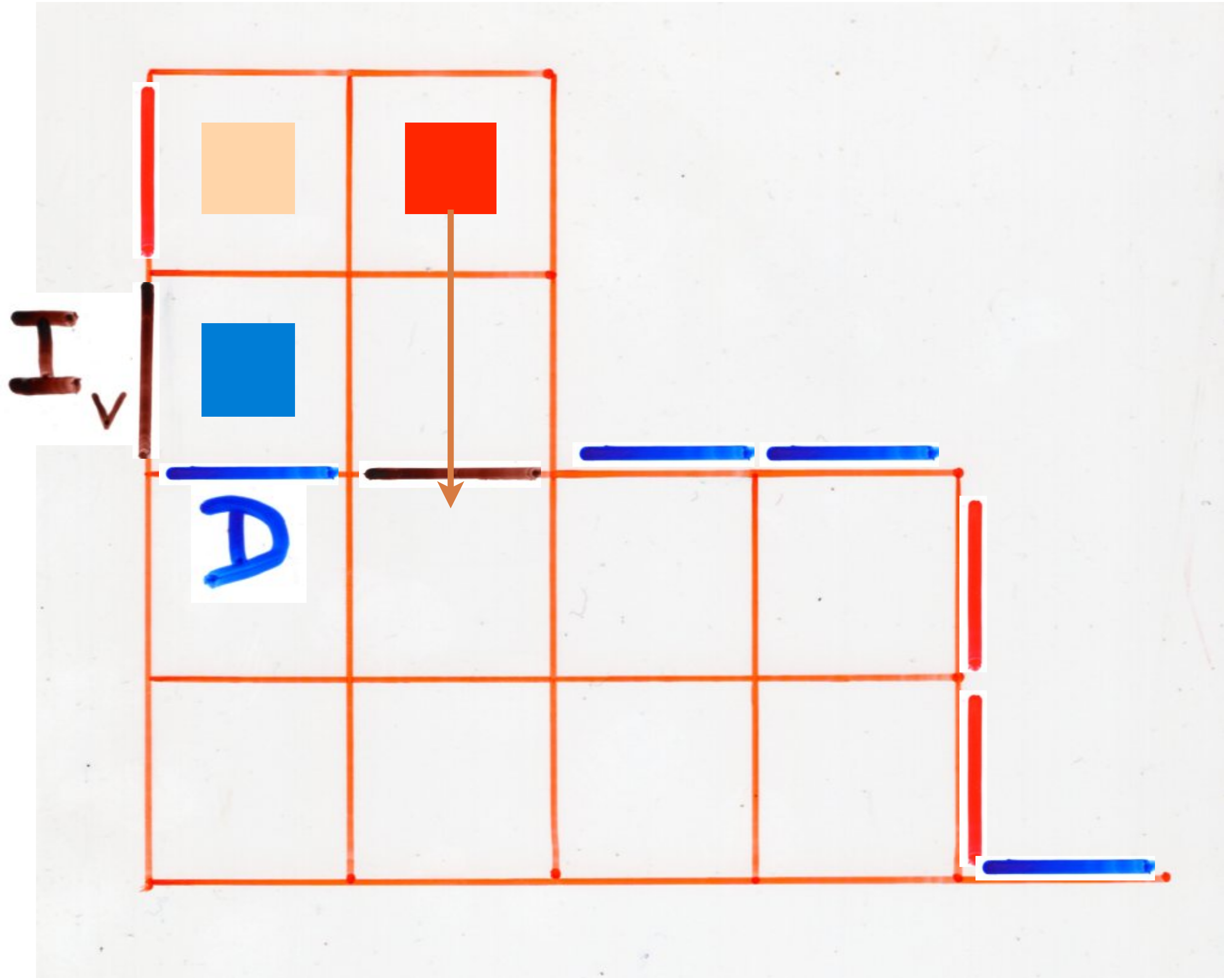

E
I_h


E

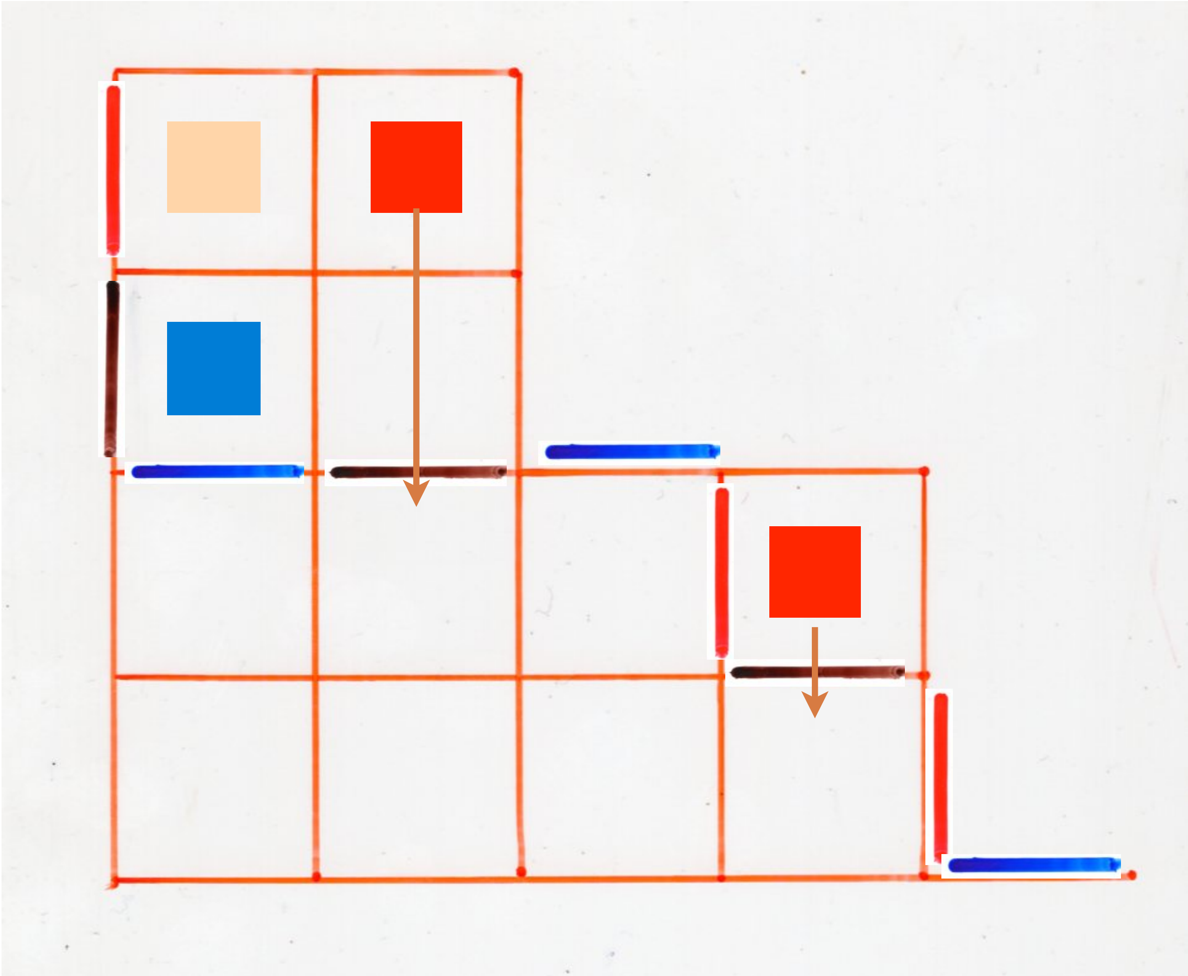


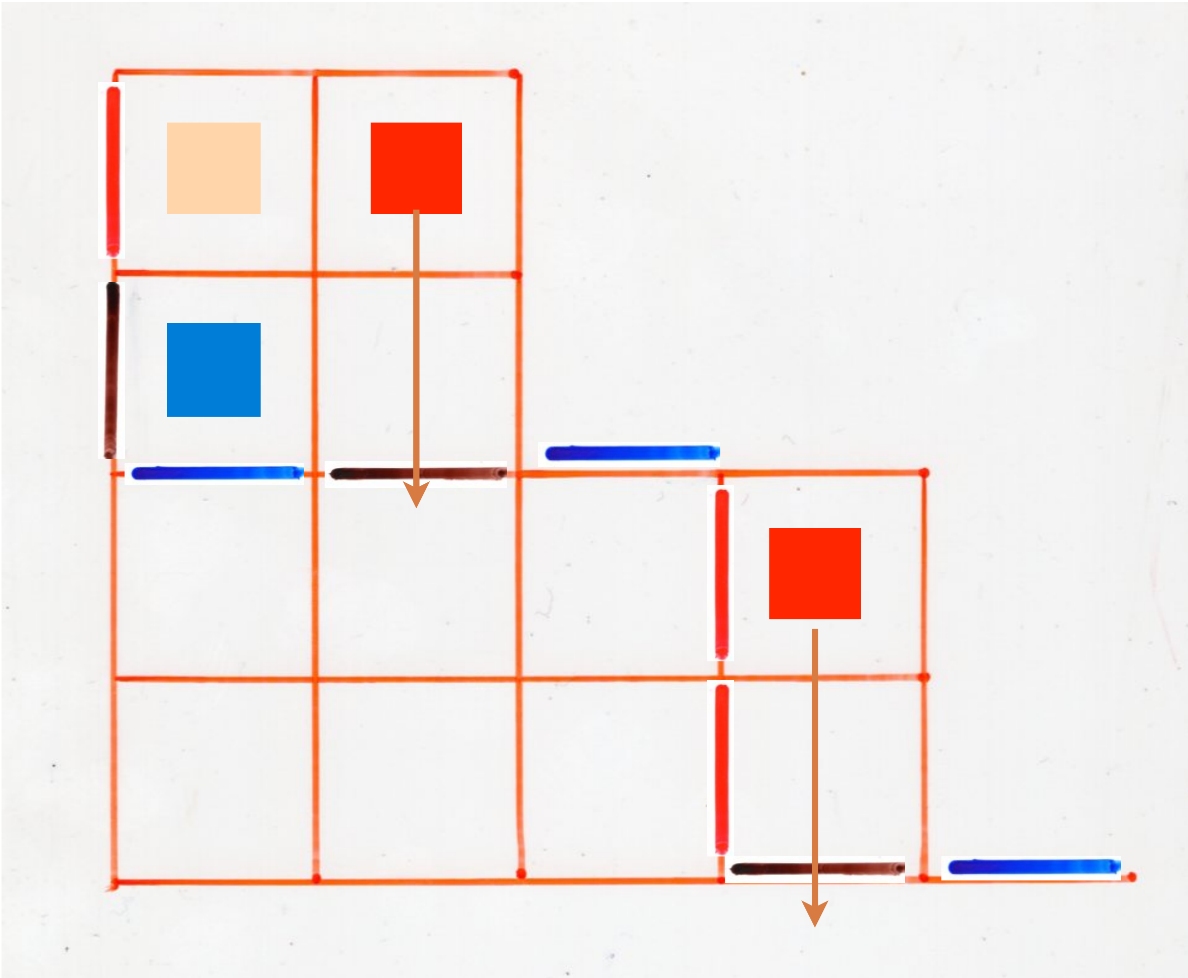


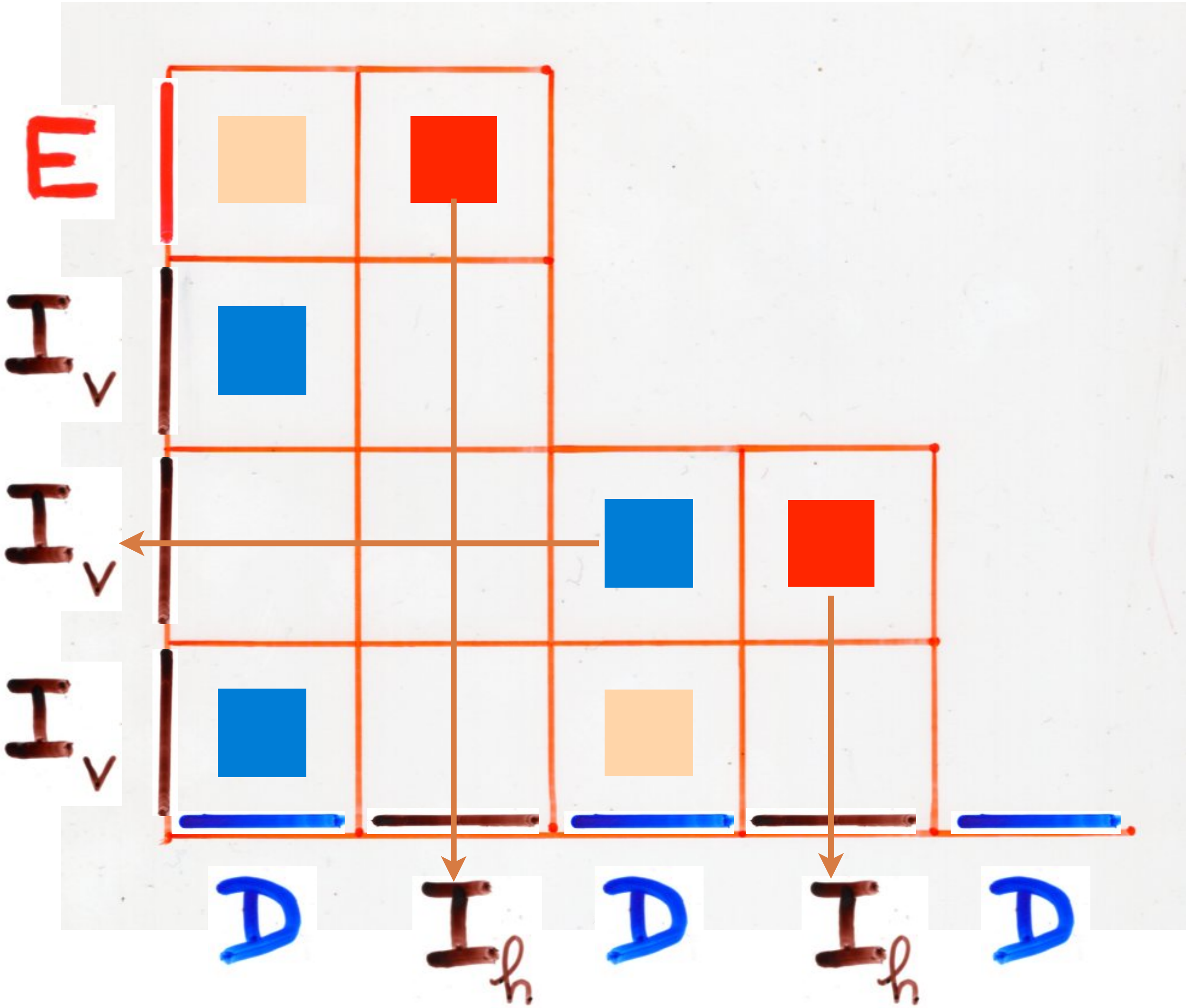
E
D



I_v
D






E
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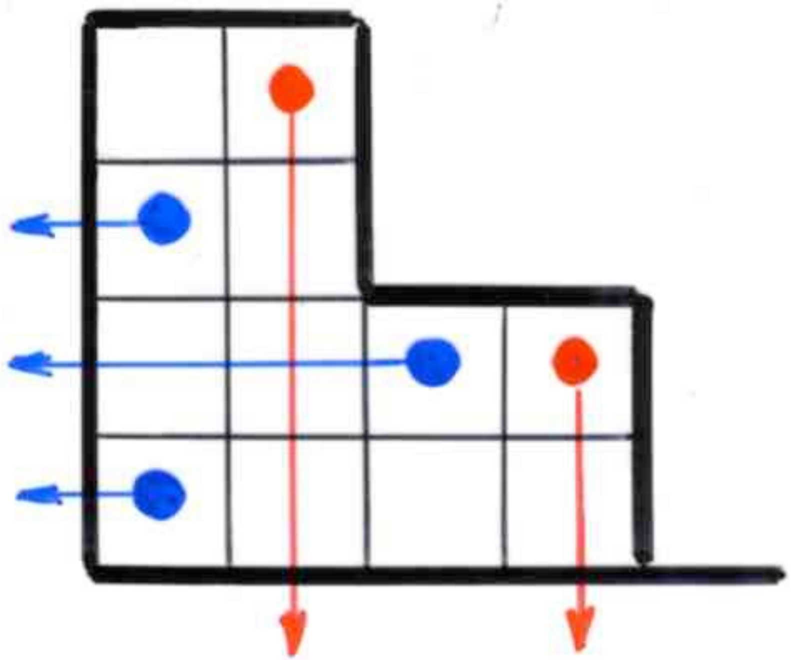
I_v
I_v
D
I_h
D
I_h
D



Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

"The cellular ansatz"

quadratic algebra Q

$UD = qDU + Id$

Physics

$DE = qED + E + D$

commutations

rewriting rules

planarization

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

alternative tableaux

representation of Q by combinatorial operators

bijections

RSK ⟷

pairs of Young tableaux

EXF ⟷

"Laguerre histories"

permutations

orthogonal polynomials

the bijection

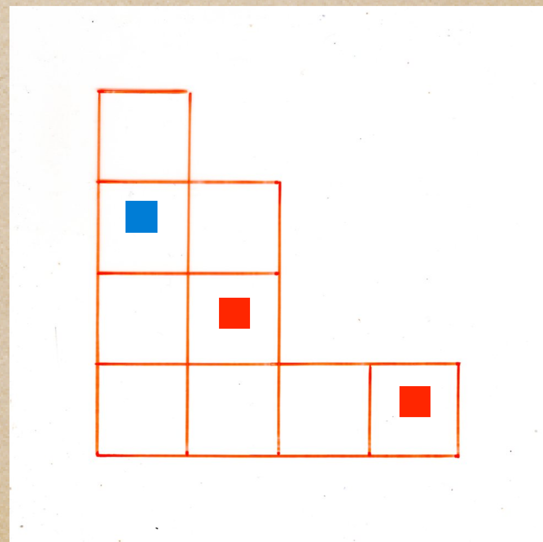
permutations — alternative tableaux

(Laguerre histories)

with local rules

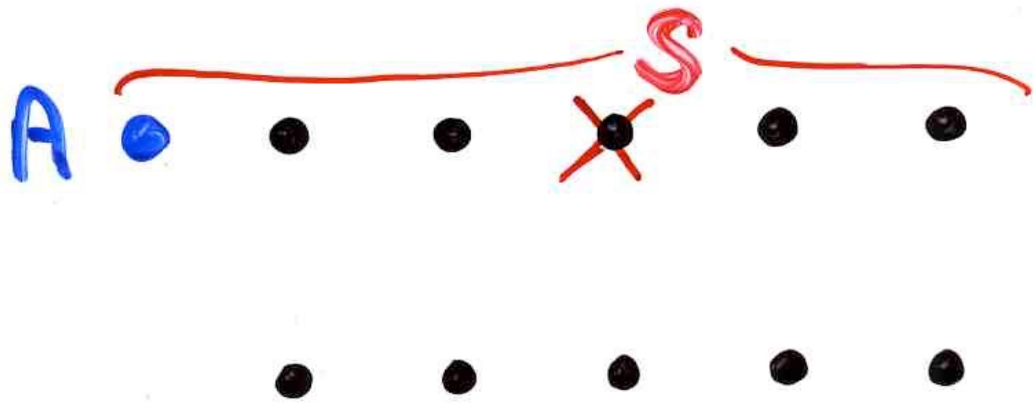
(commutation diagrams)

4 1 6 9 7 8 3 5 2 $\longrightarrow$



Polya urn

$$A\,|k\rangle = 1\,|(k+1)\rangle$$
$$S\,|k\rangle = k\,|(k-1)\rangle$$

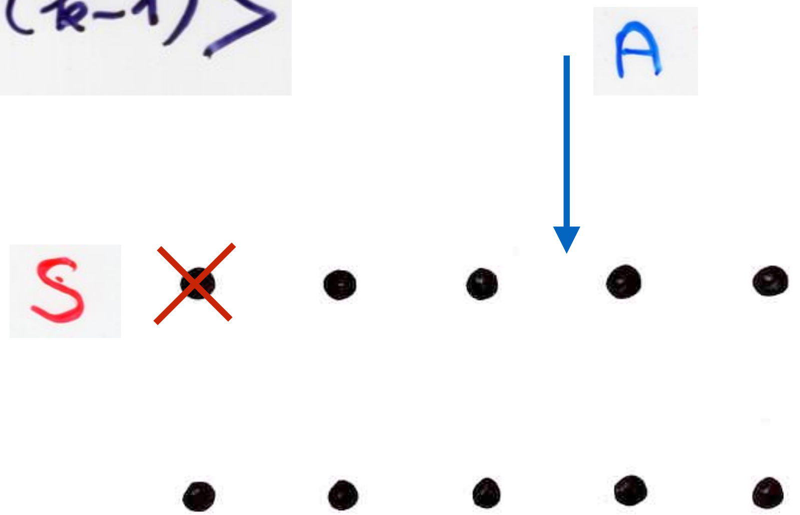


$$AS - SA = I$$

Priority queue

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

$$S\,|k\rangle = 1\,|(k-1)\rangle$$



$$AS - SA = I$$

data structures

Computer Science

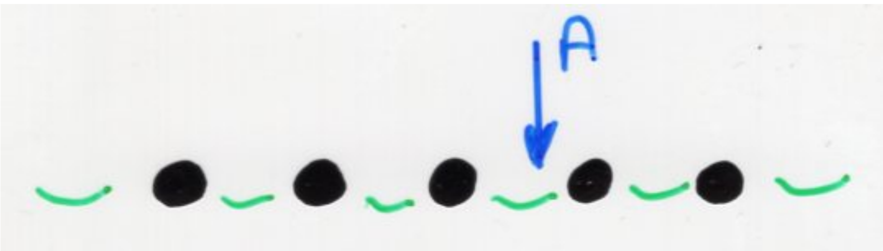
dictionnary data structure

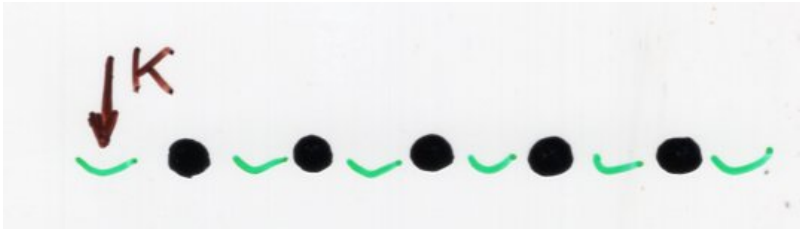
add or delete any element

ask questions

J positive

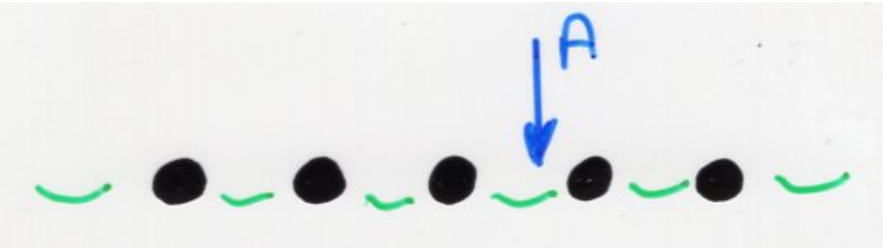
K negative

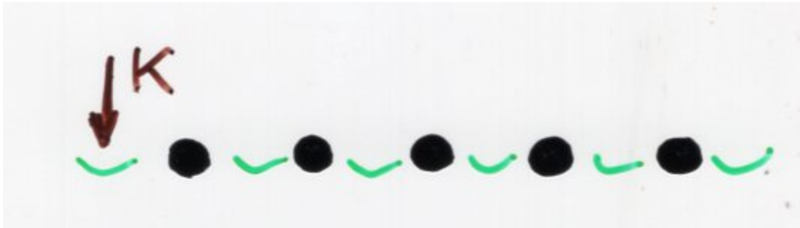


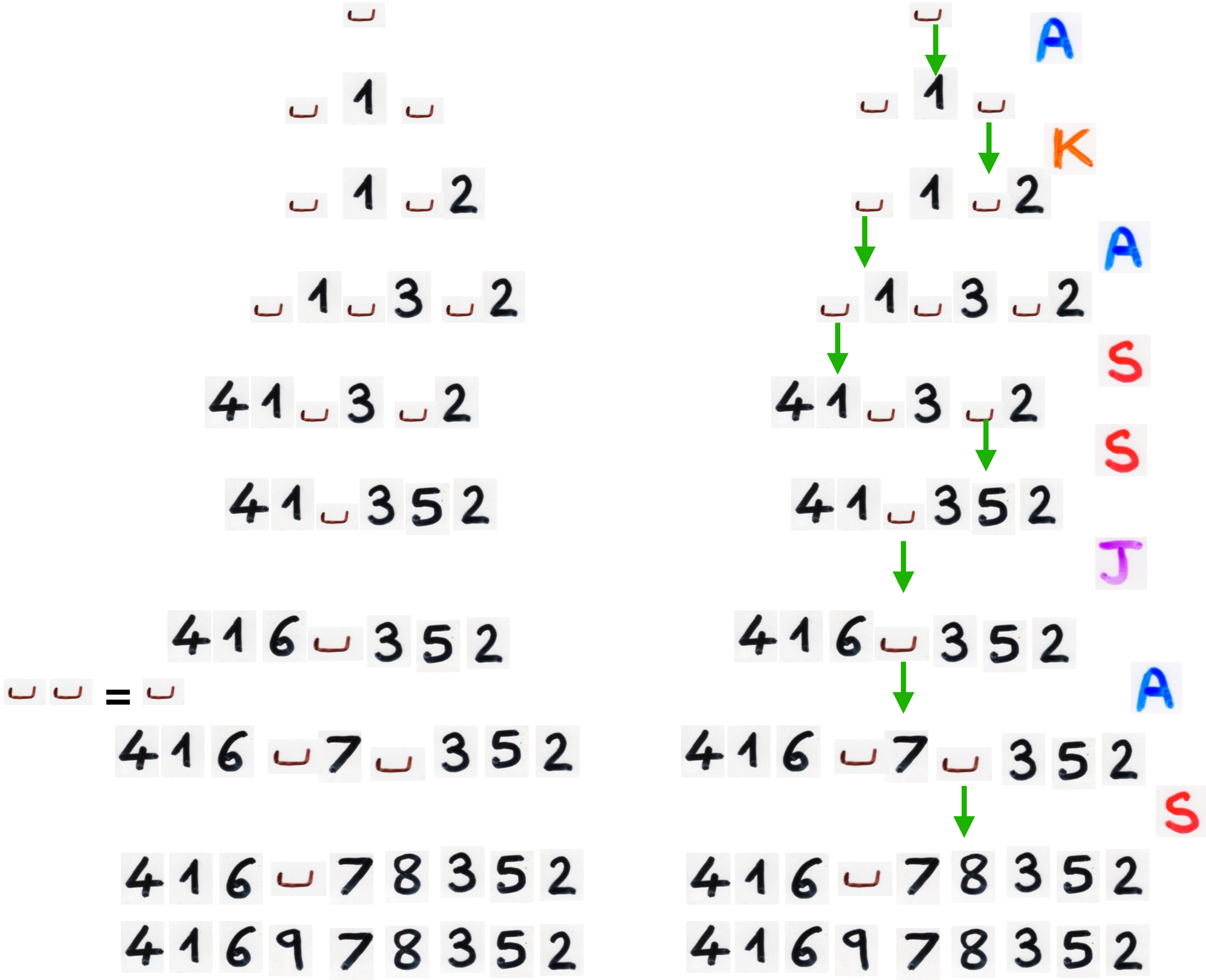


$$\begin{cases} D = A + K \\ E = S + J \end{cases}$$

$$DE = ED + E + D$$







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4 1 6 3 5 2
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4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2
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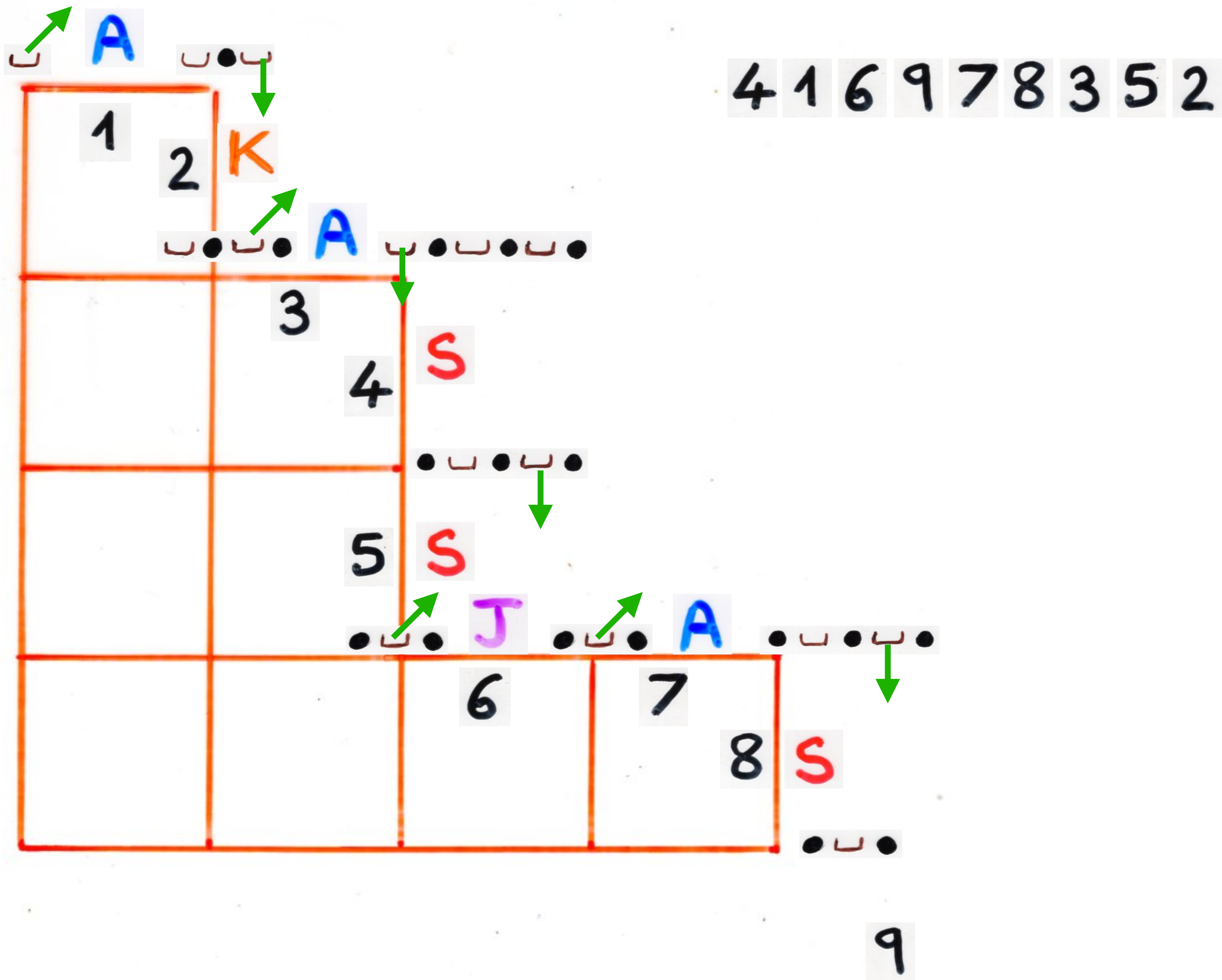
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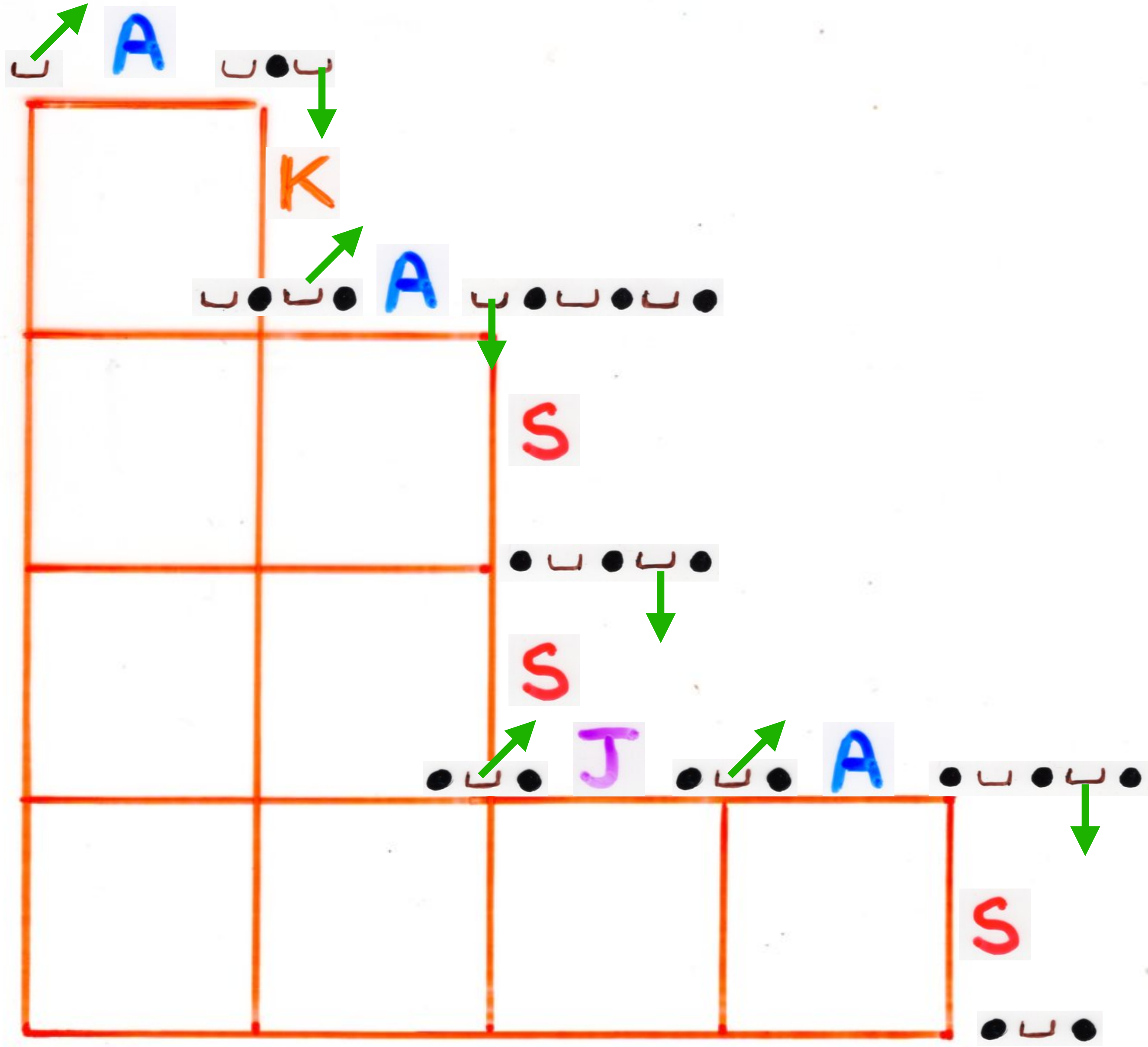

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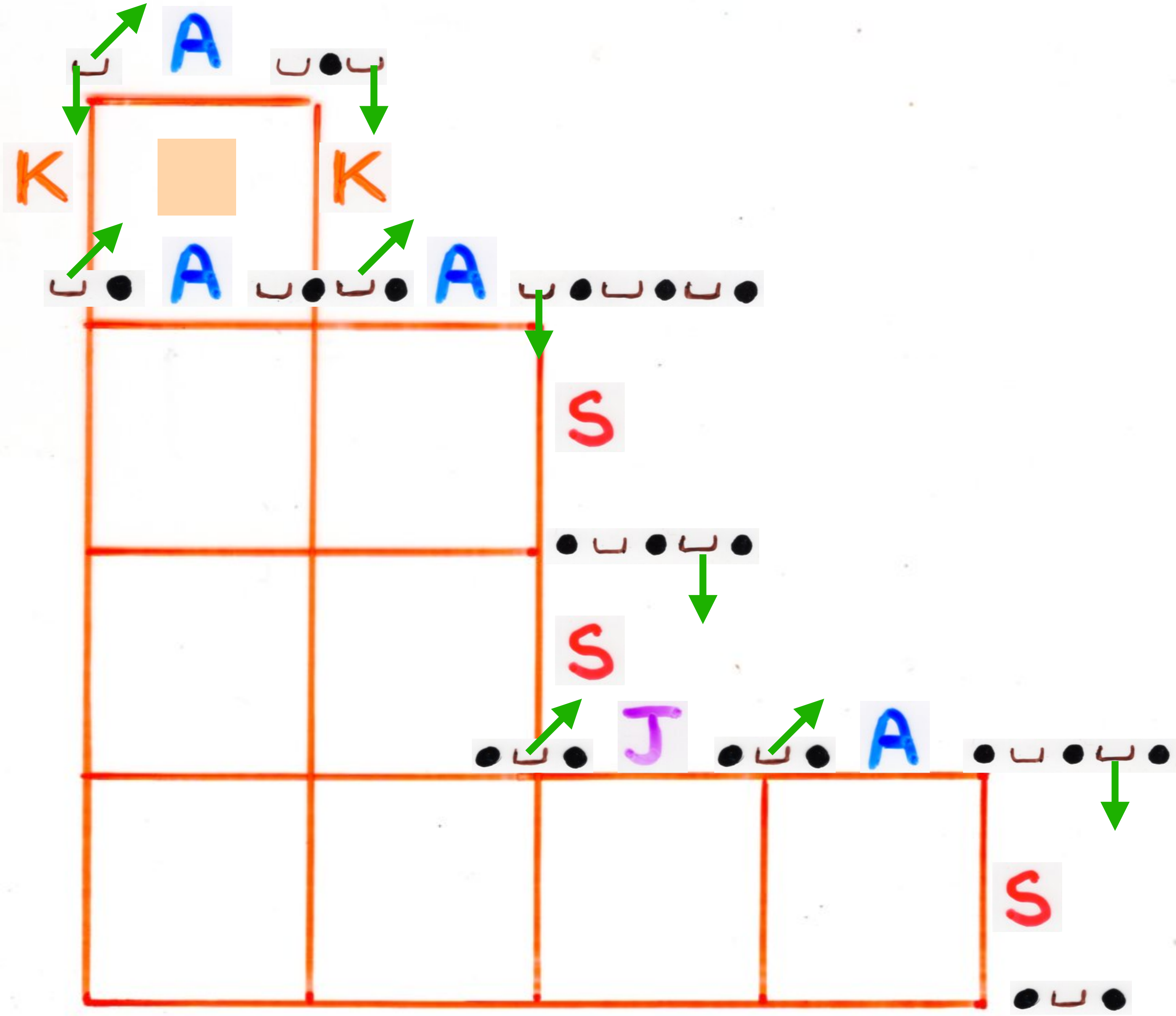
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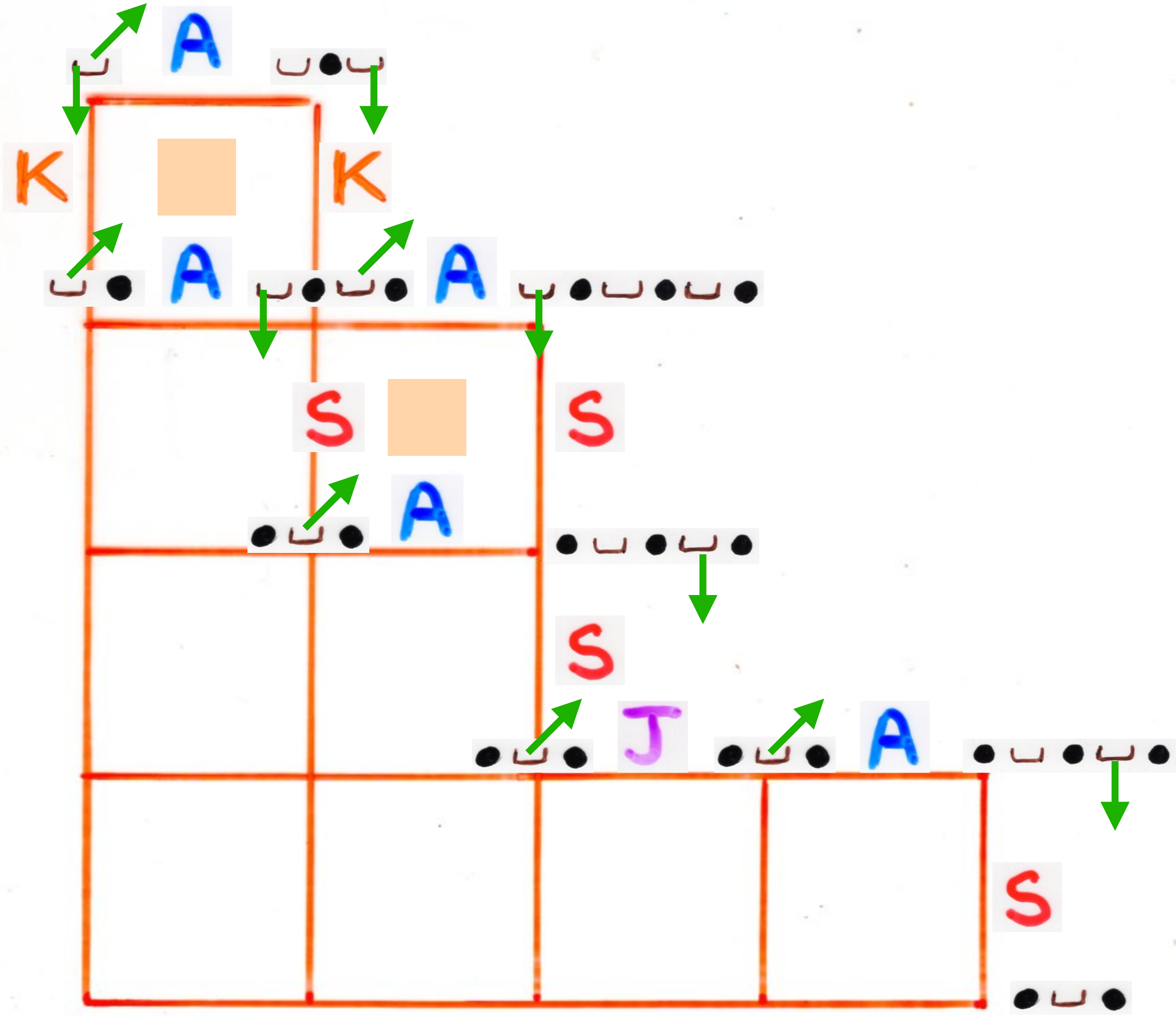
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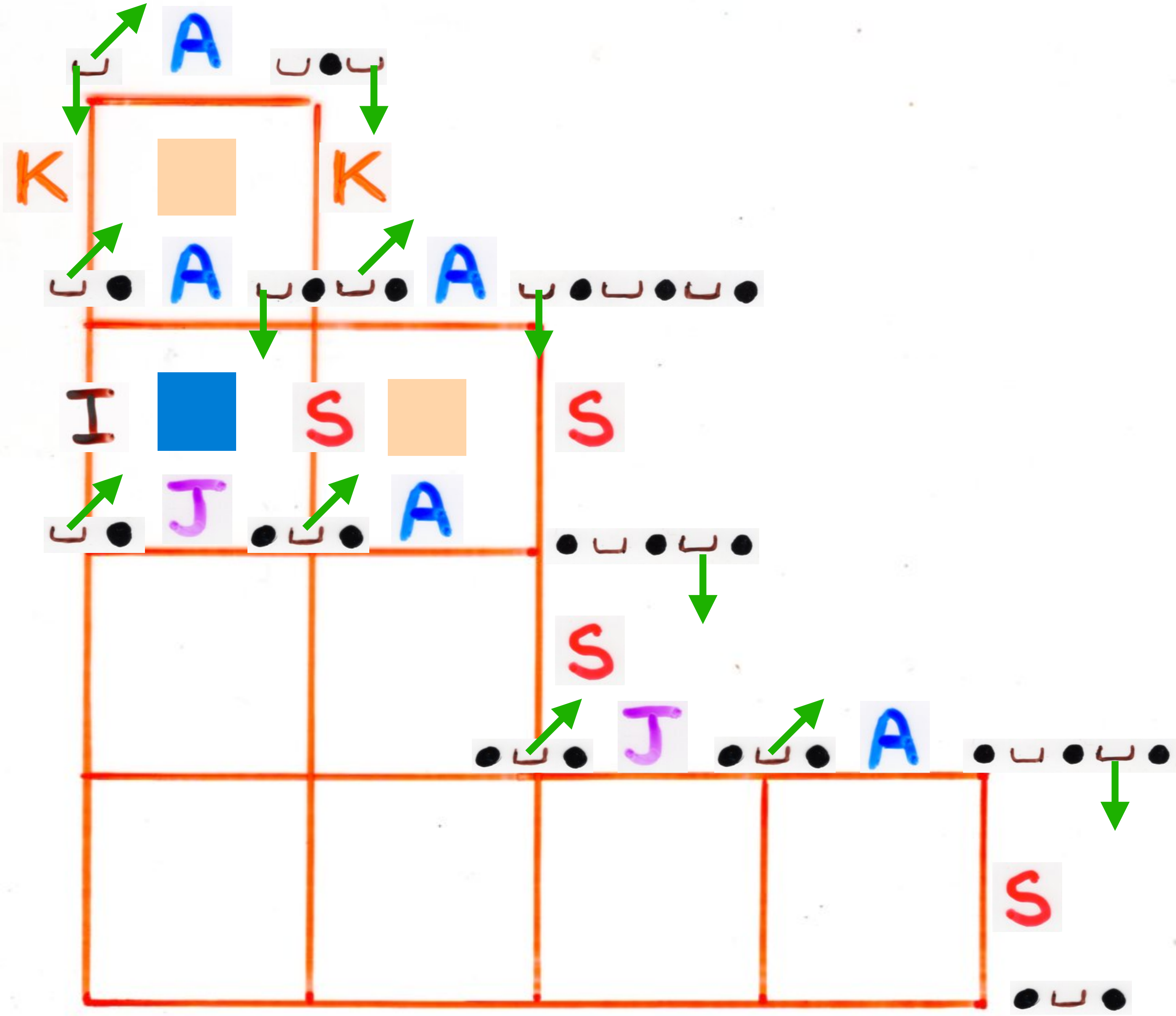
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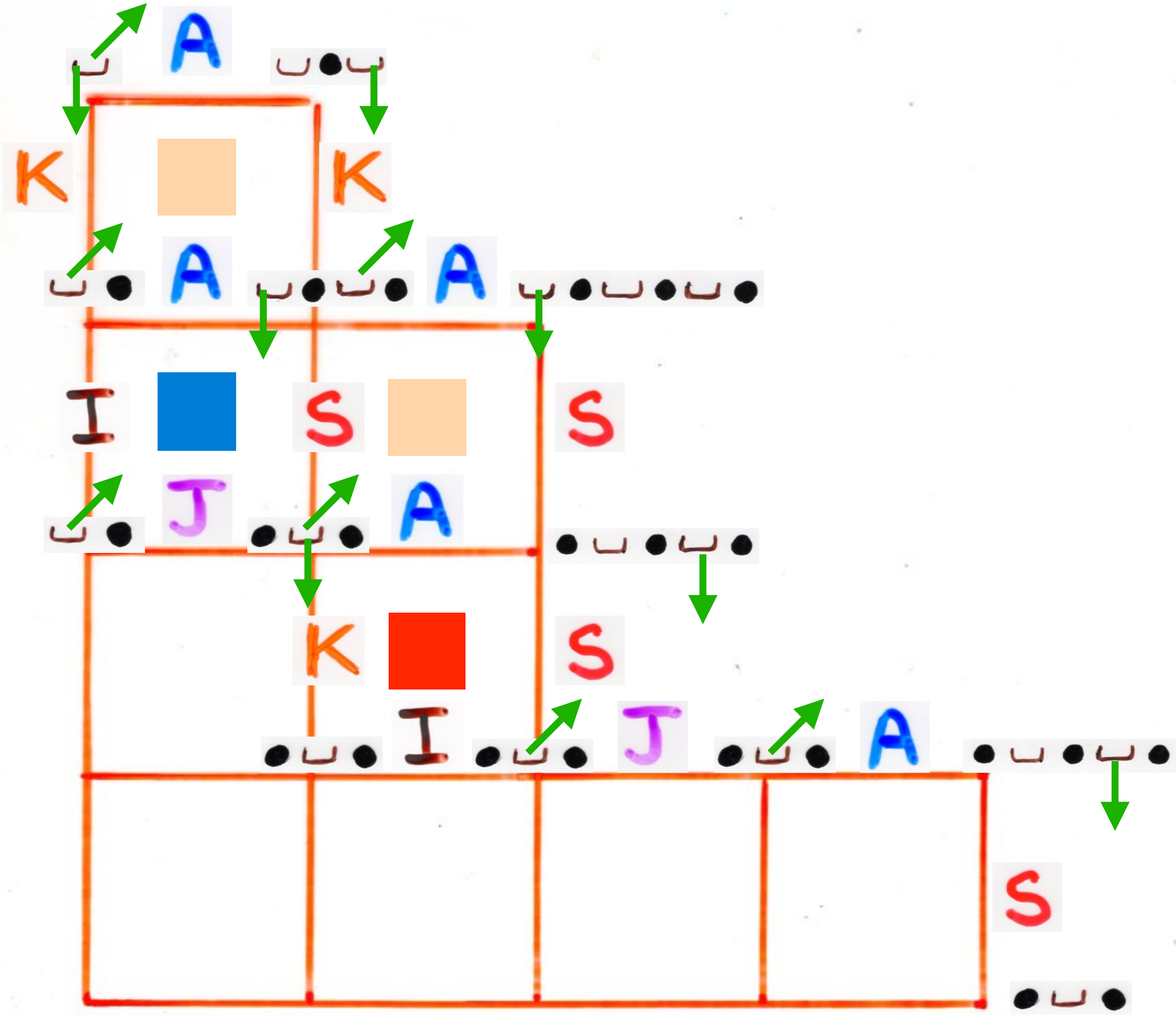
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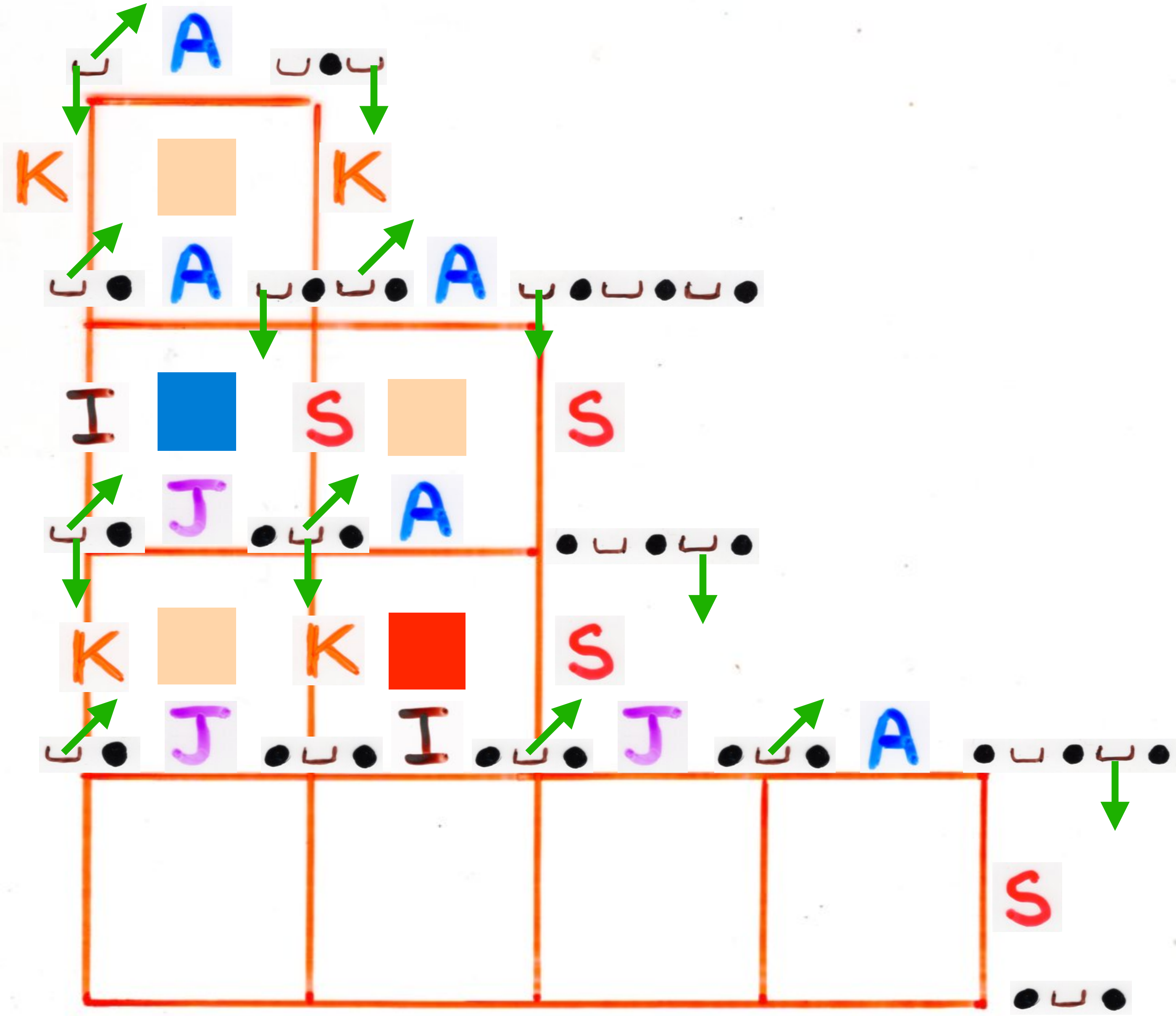
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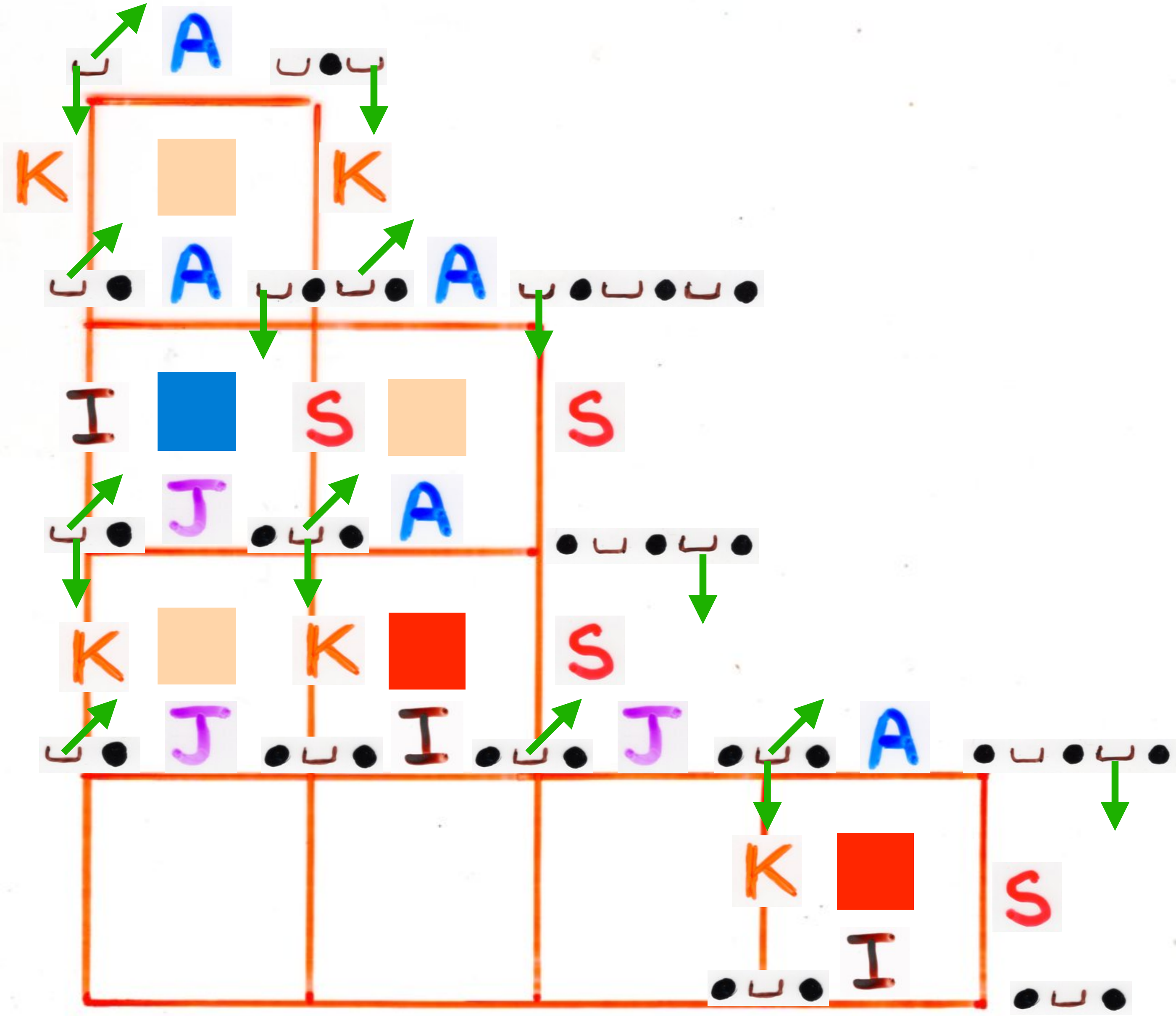
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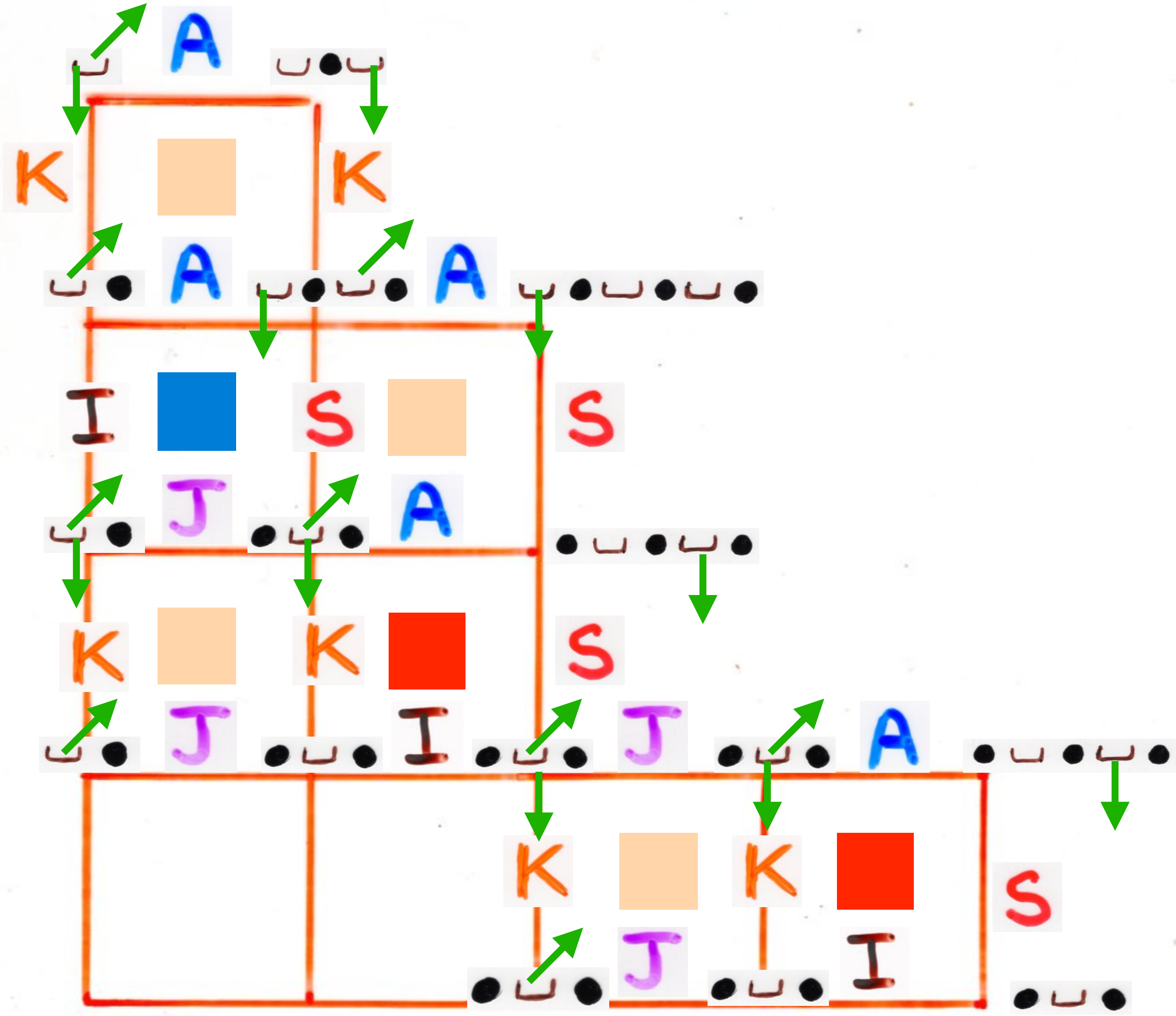

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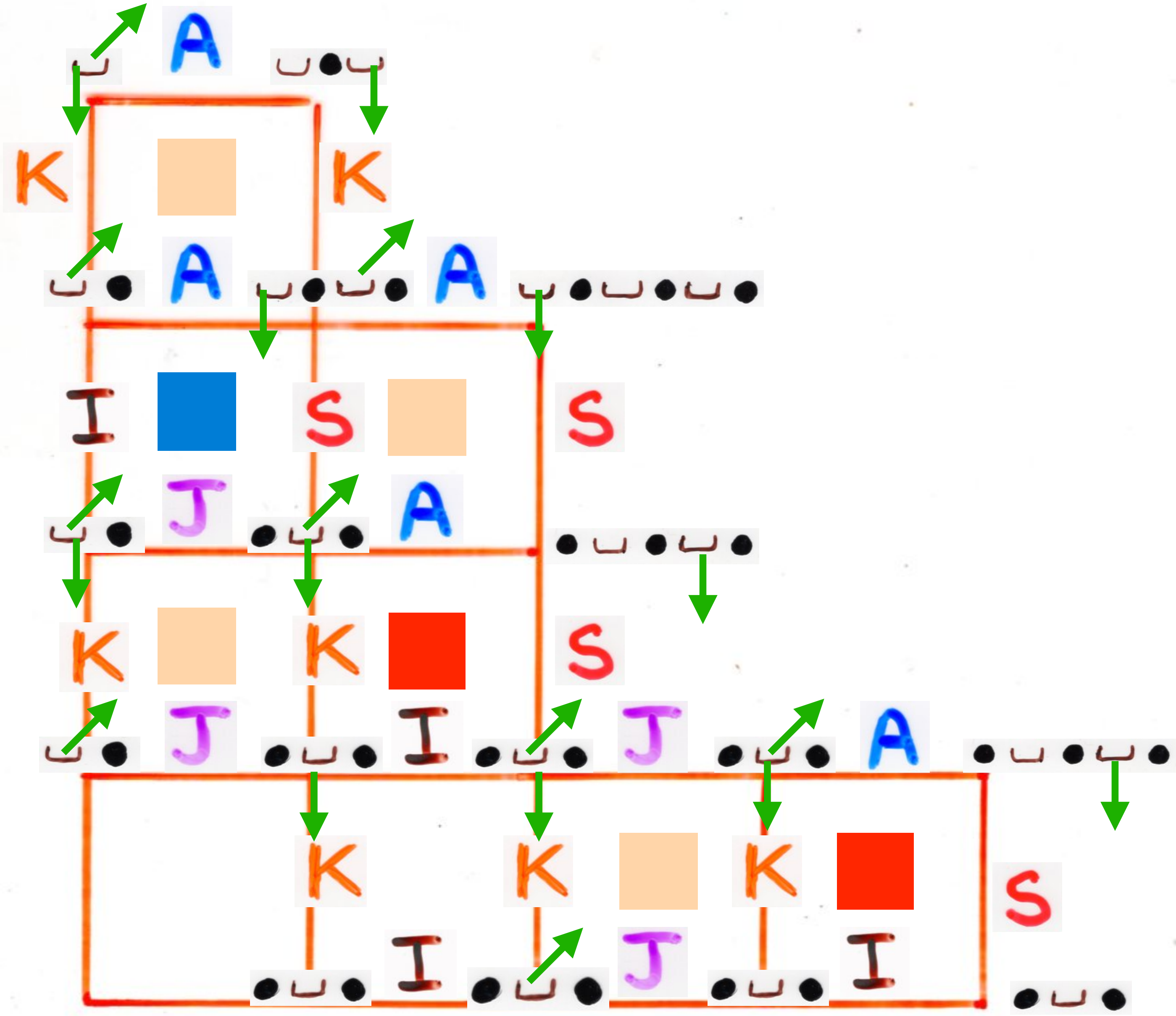
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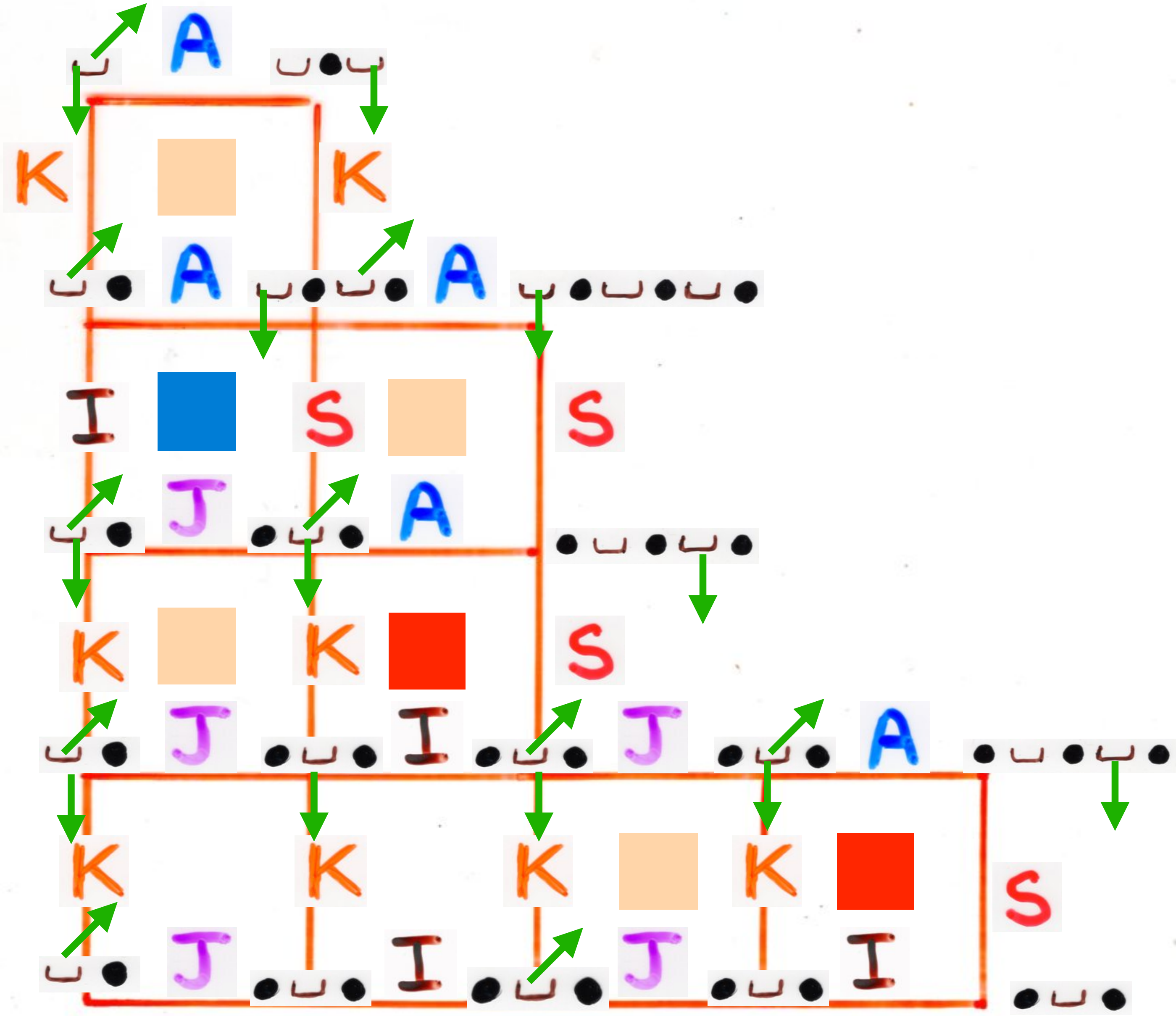
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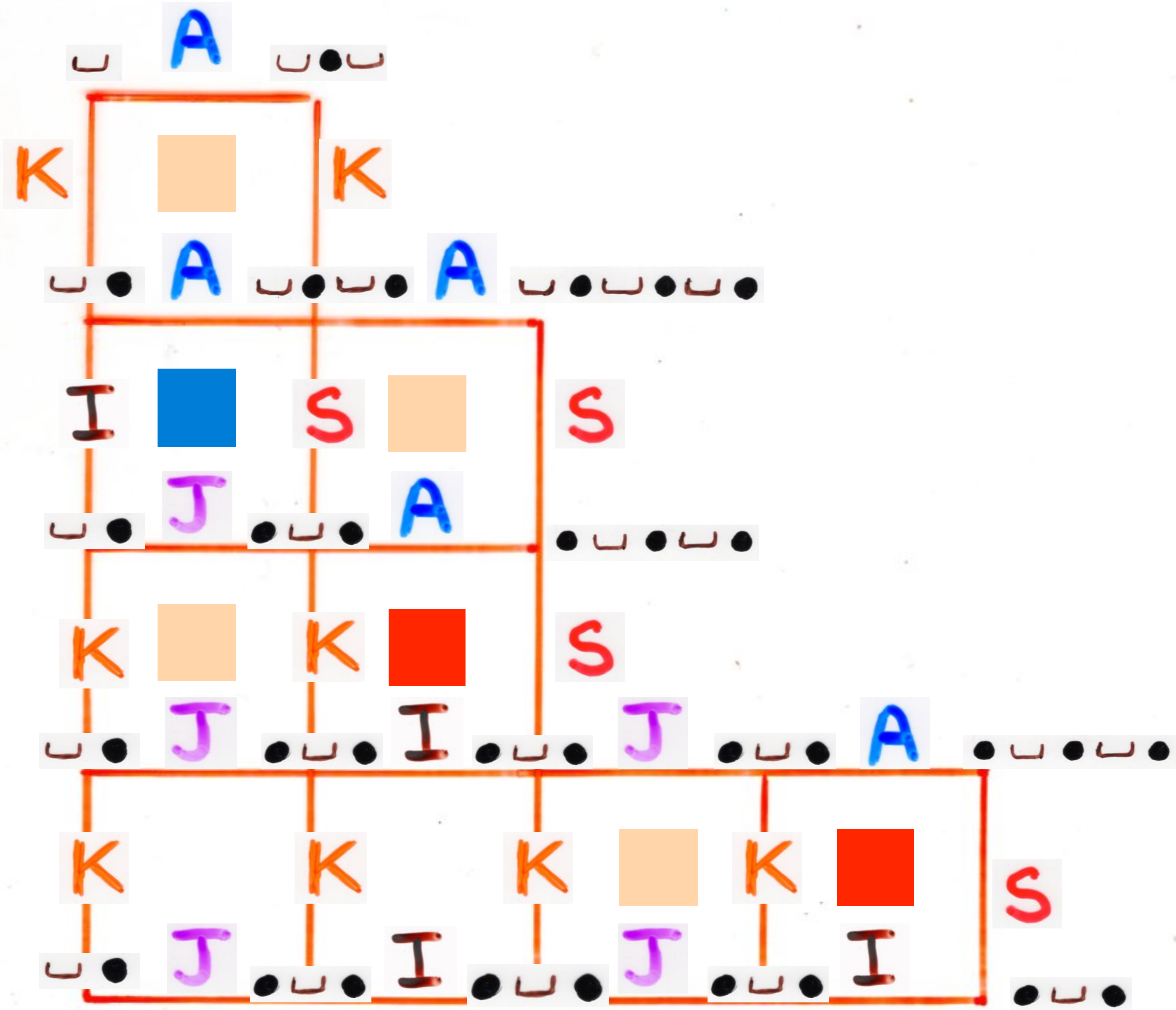

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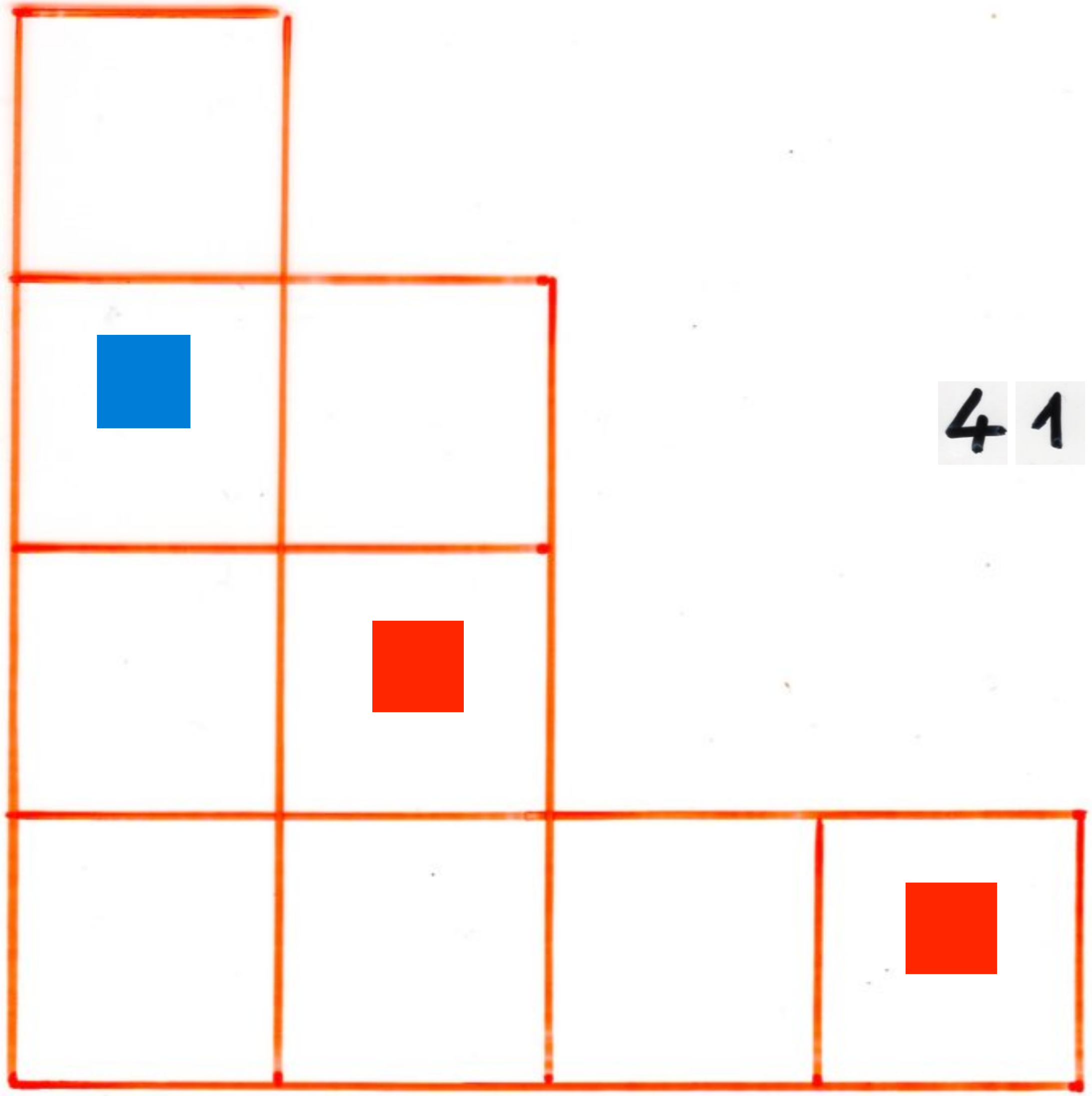

4 1 6 9 7 8 3 5 2

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

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permutations

towers placements

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"Laguerre histories"

permutations

data structures "histories"

orthogonal polynomials

commutations

rewriting rules

ASM alternating sign matrices

tilings

planarization

"planar automata"

non-crossing paths

8-vertex model

⟷ ?

more material in the course at IMSc

The bijective combinatorics course, Part III

The cellular ansatz:
bijective combinatorics and quadratic algebra

www.imsc.res.in/~viennot

Thank you !