Growth diagrams, local rules and beyond

(part I)

21st Ramanujan Symposium:
National Conference on algebra and its applications

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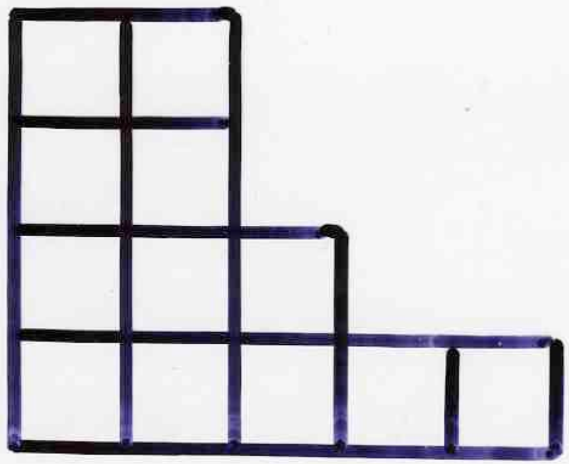
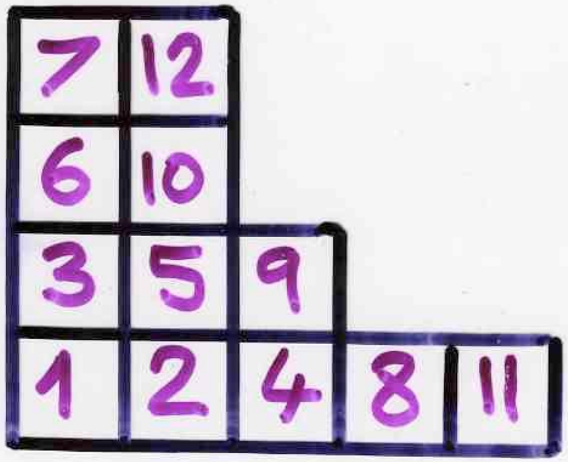
University of Madras
28 February 2018

RS
The Robinson-Schensted correspondence





Young tableau

shape λ

f_λ = number of Young tableaux with shape λ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$\updownarrow$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

Representation theory of groups

Case of the group $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) $=$ f_λ number of Young tableaux with shape λ

finite group G

$$|G| = \sum_{R} (\deg R)^2$$

R irreducible representation

for the symmetric group $\mathfrak{S}_n$ (permutations)

$$n! = \sum_{\lambda} (f_\lambda)^2$$

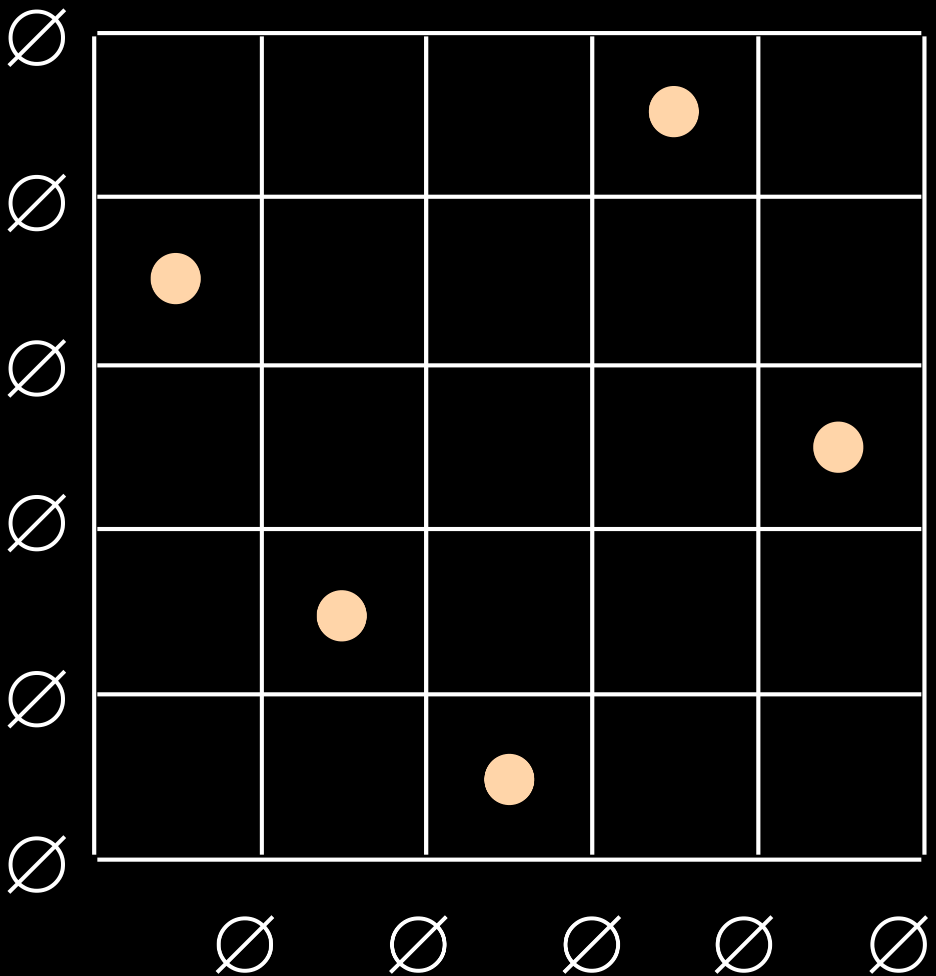
λ partition of n

“local” algorithm on a grid or “growth diagrams”

S. Fomin, 1986, 1994

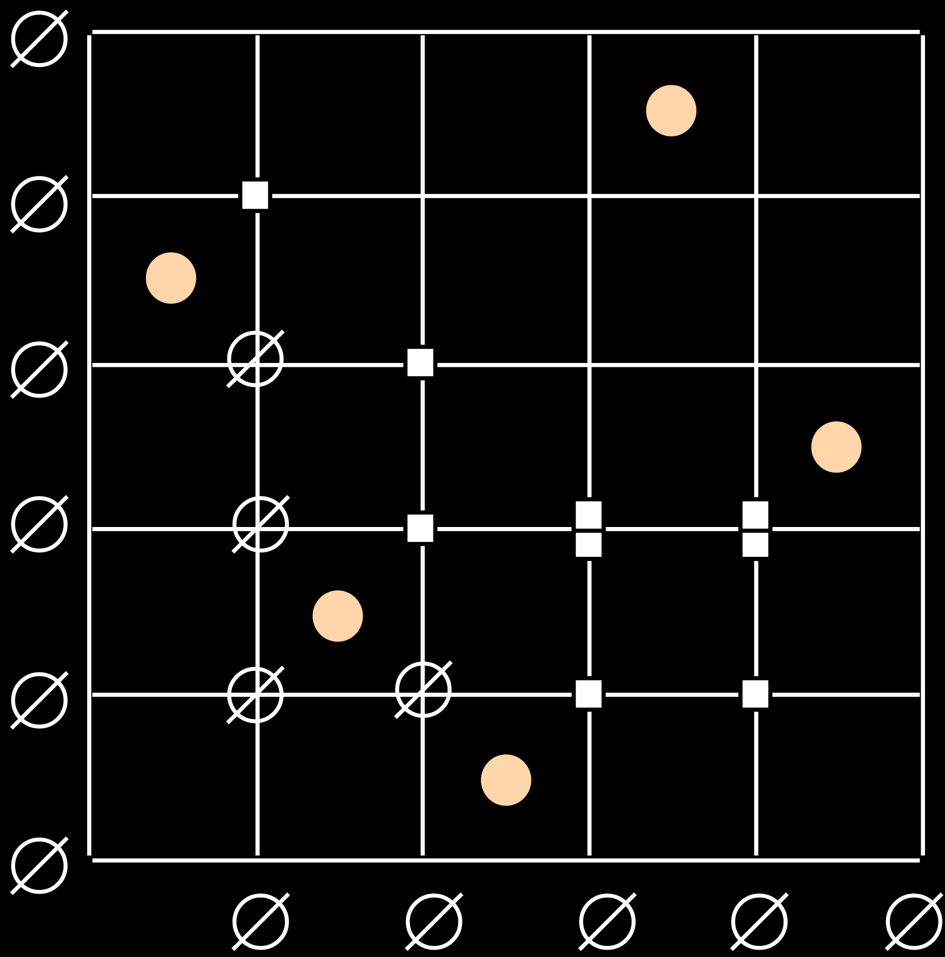
C.Krattenthaler

initial state



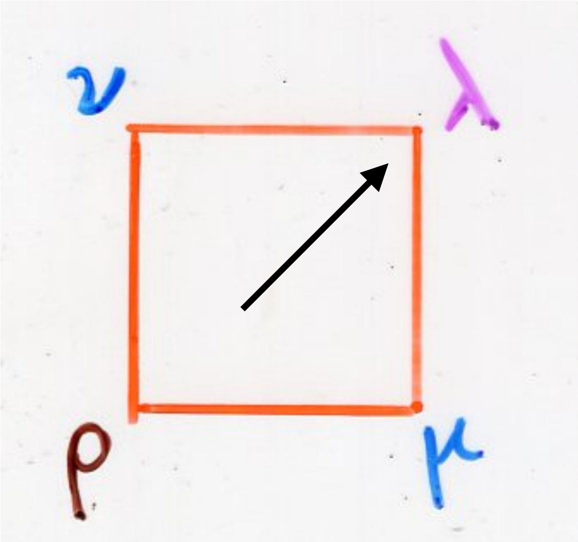
$\sigma = 4, 2, 1, 5, 3$

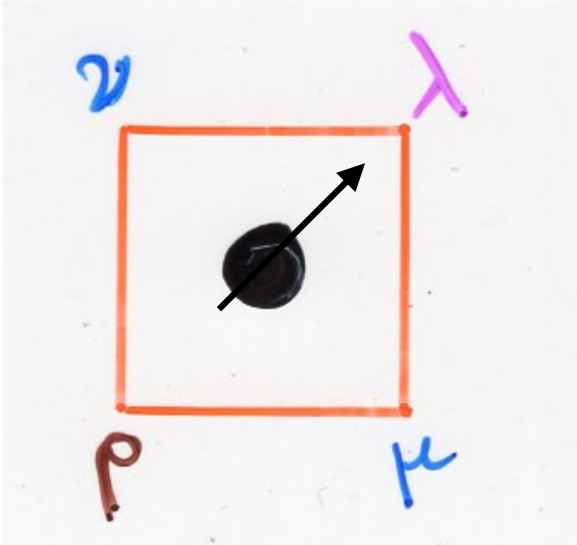
during the labeling process



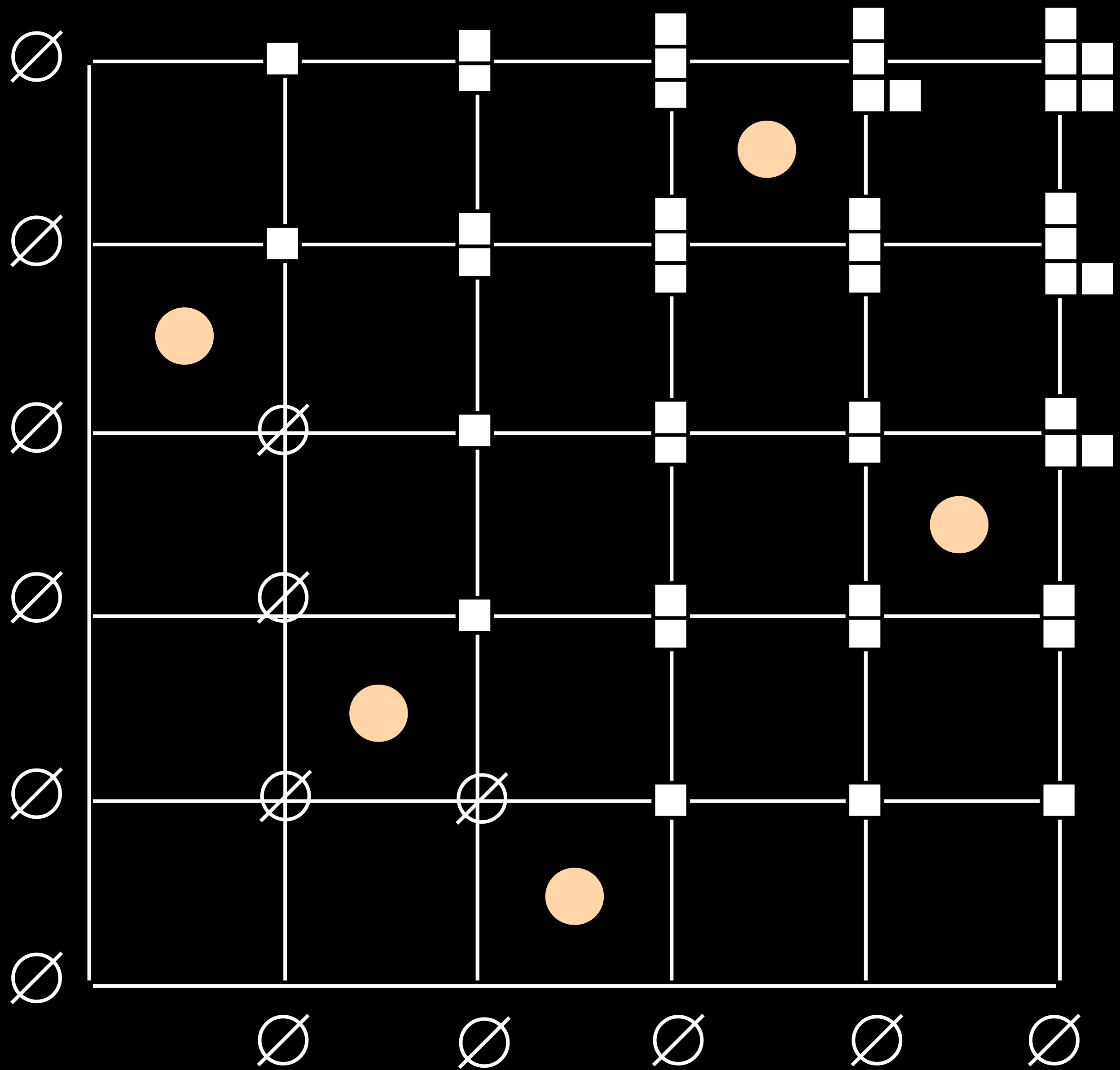
"growth diagrams"

"local rules"



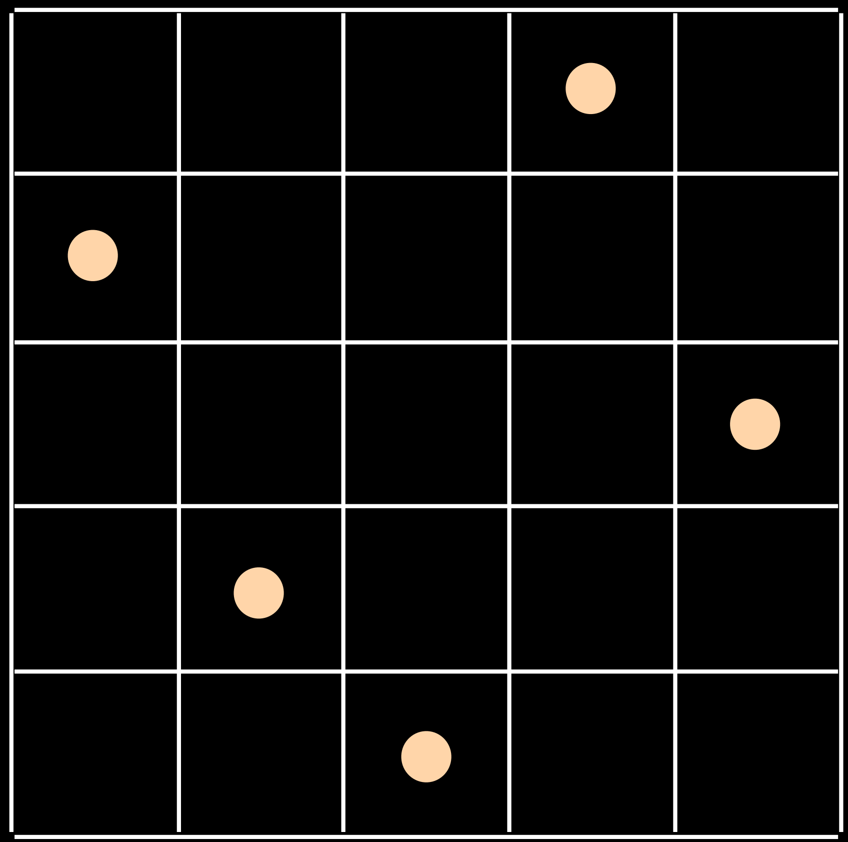


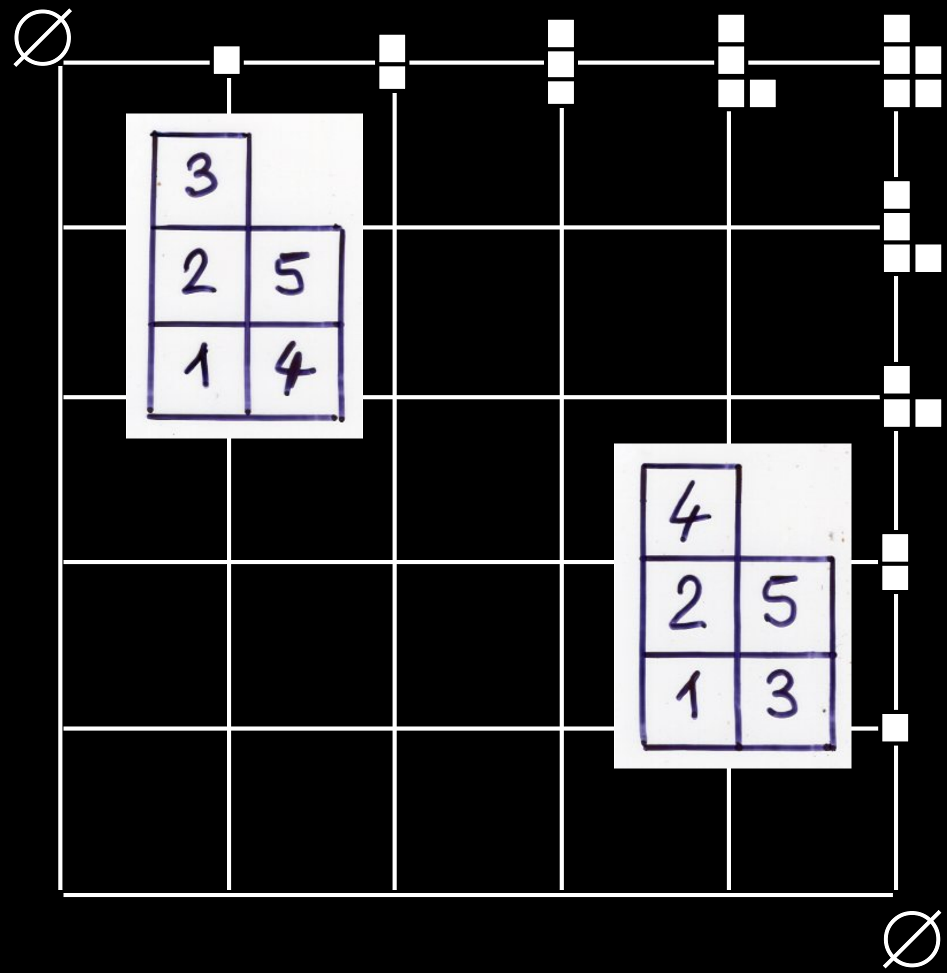
final state



"growth diagrams"

"local rules"

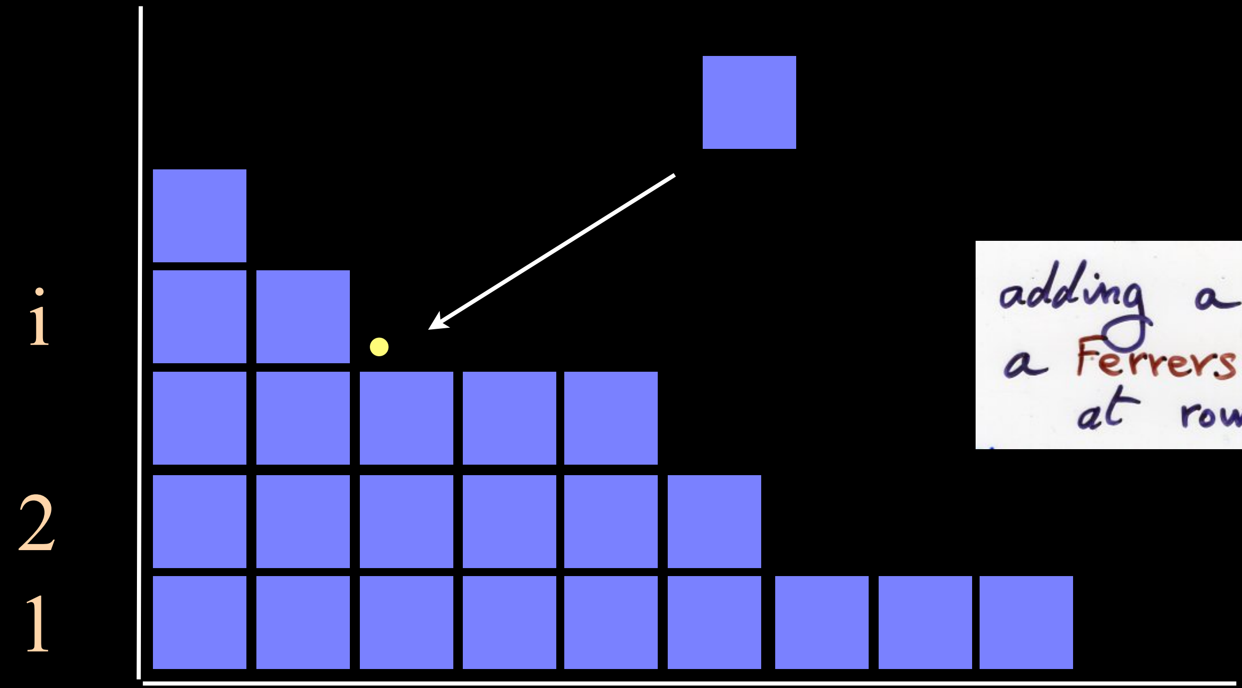




$\sigma = 4, 2, 1, 5, 3$

- this bijection is the same as the Robinson-Schensted correspondence notations

operator U_i



adding a cell in a Ferrers diagram ρ at row i

$$U_i(\rho) = \rho + (i)$$

"local rules"

(i) $\rho = \mu = \nu$ and $\begin{tikzcd} \nu \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[d, dash] \\ \rho \arrow[r, dash] & \mu \end{tikzcd}$ then $\lambda = \rho$

(ii) $\rho = \mu \neq \nu$, then $\lambda = \nu$

(iii) $\rho = \nu \neq \mu$, then $\lambda = \mu$

(iv) ρ, μ, ν pairwise $\neq$, then $\lambda = \mu \cup \nu$

(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$
given that $\mu = \nu$ and ρ differ in the i-th row
[in fact $\mu = \nu = \rho + (i)$]

(vi) $\rho = \mu = \nu$ and $\begin{tikzcd} \nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\bullet"] & \cdot \arrow[d, dash] \\ \rho \arrow[r, dash] & \mu \end{tikzcd}$, then $\lambda = \mu + (1)$

"local rules"

(ii) $\rho = \mu \neq \nu$, then $\lambda = \nu$

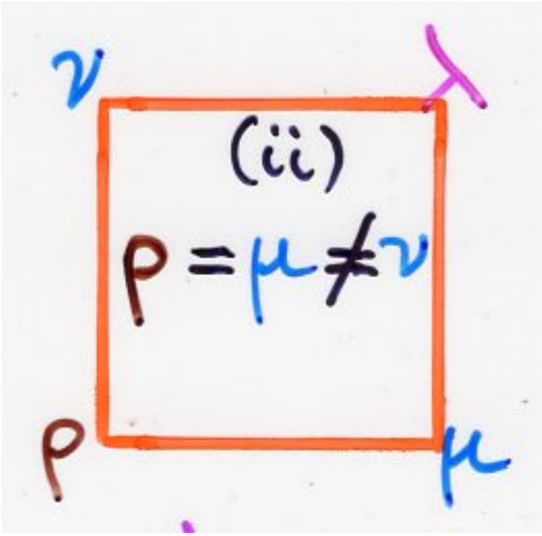
(iii) $\rho = \nu \neq \mu$, then $\lambda = \mu$

(iv) ρ , μ , ν pairwise $\neq$, then $\lambda = \mu \cup \nu$

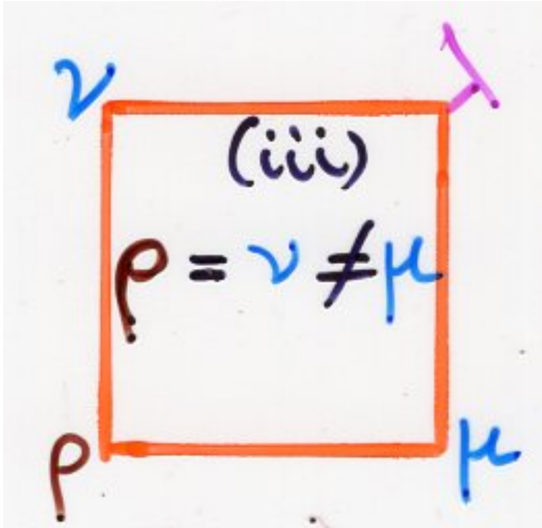
$\mu \neq \nu$

"local rules"

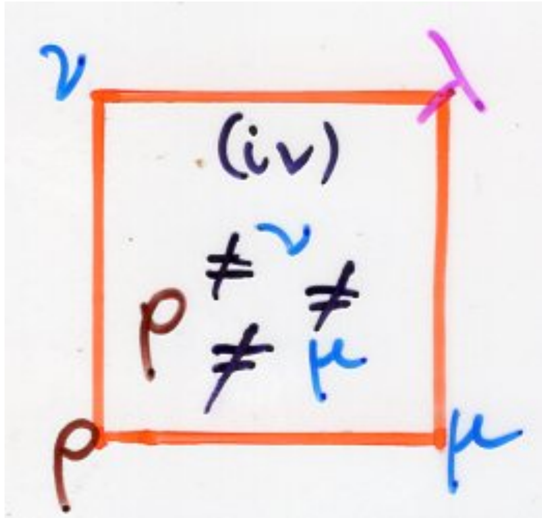
$\mu \neq \nu$



$\lambda = \nu$

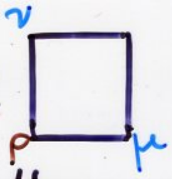


$\lambda = \mu$



$\lambda = \mu \cup \nu$

"local rules"

(i) $\rho = \mu = \nu$ and  then $\lambda = \rho$

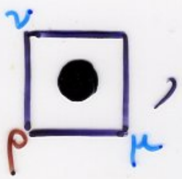
$\mu = \nu$

(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$

given that $\mu = \nu$ and ρ differ in the i-th row

[in fact $\mu = \nu = \rho + (i)$]

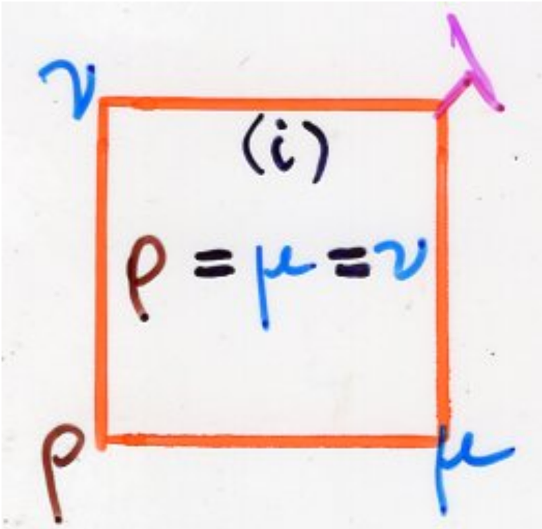
$\mu = \nu$

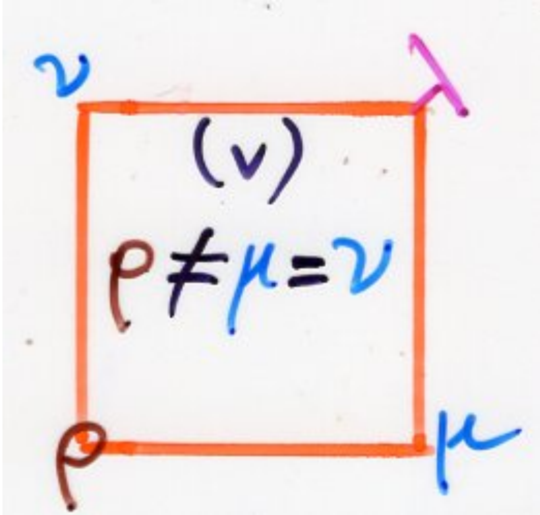
(vi) $\rho = \mu = \nu$ and , then $\lambda = \mu + (1)$

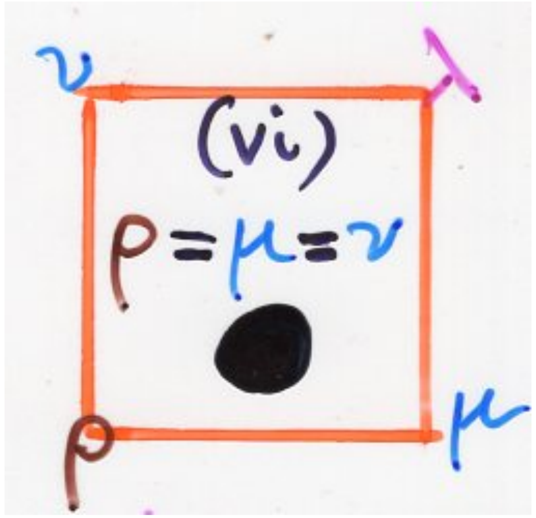
$\mu = \nu$

"local rules"

$$\mu = \nu$$



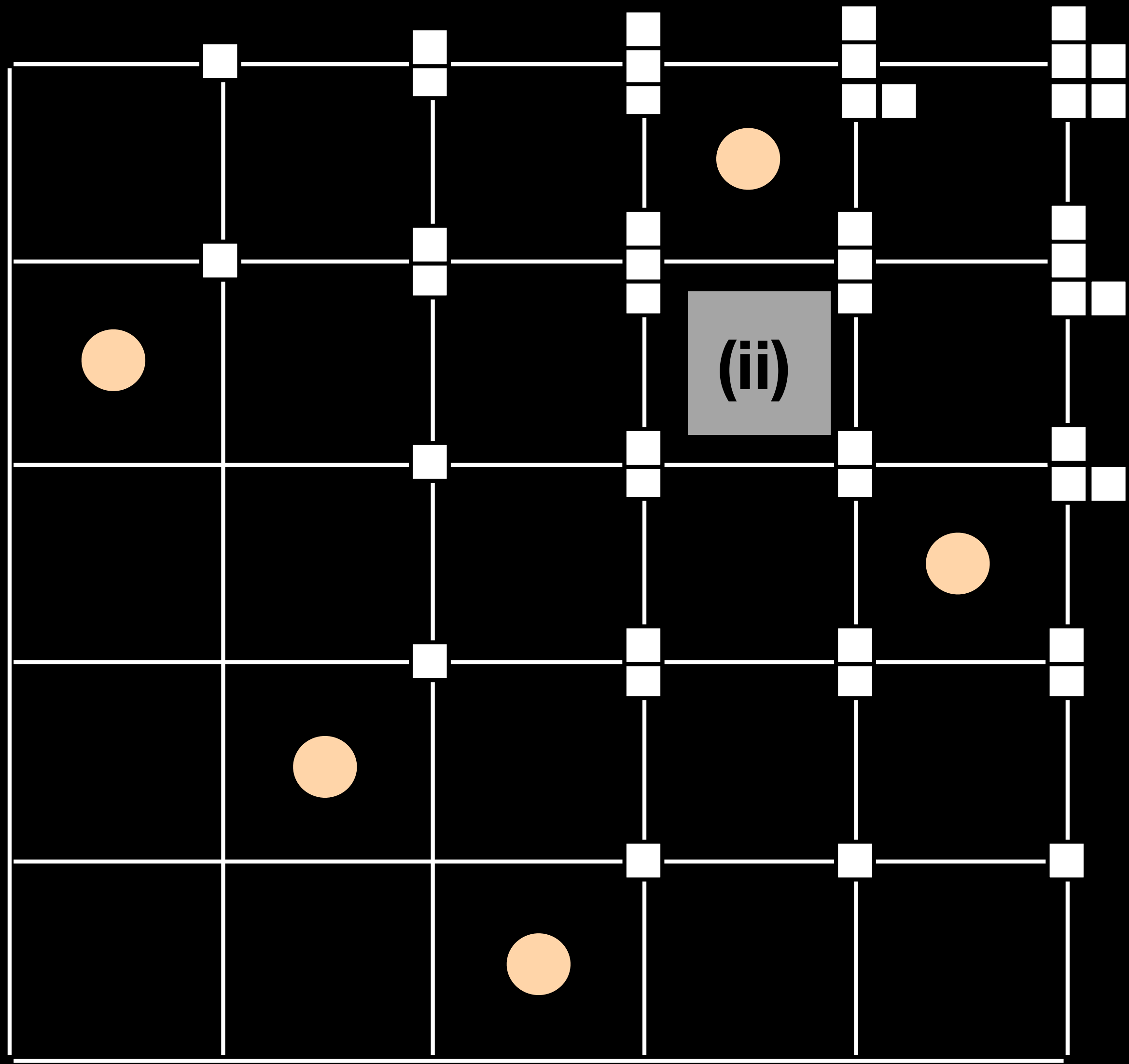


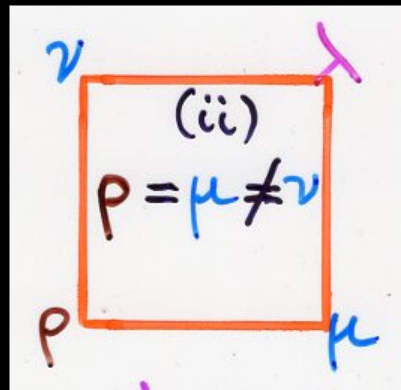


$$\lambda = \rho$$

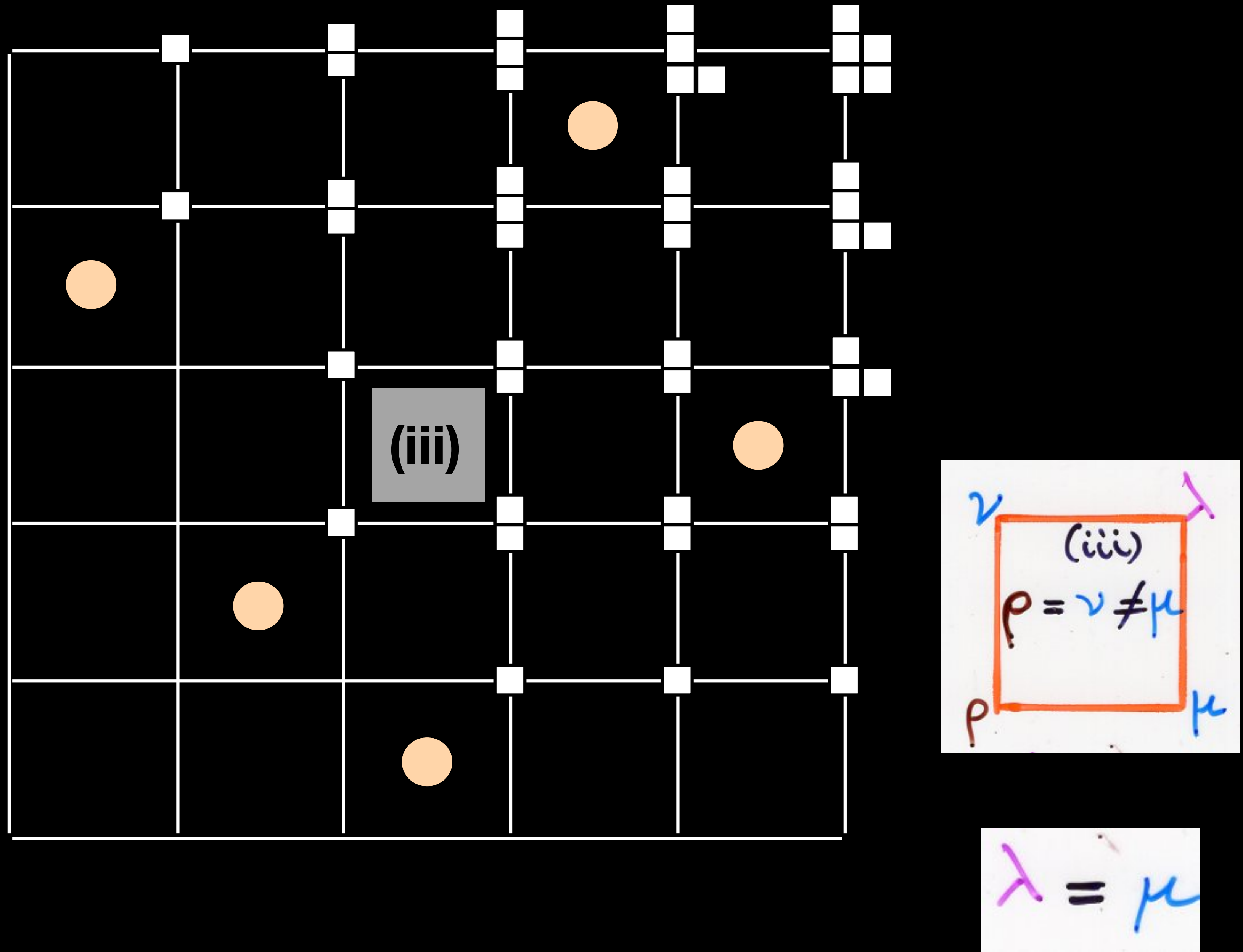
$$\lambda = \left\{\begin{matrix}\mu \\ \nu\end{matrix}\right. + (i+1)$$

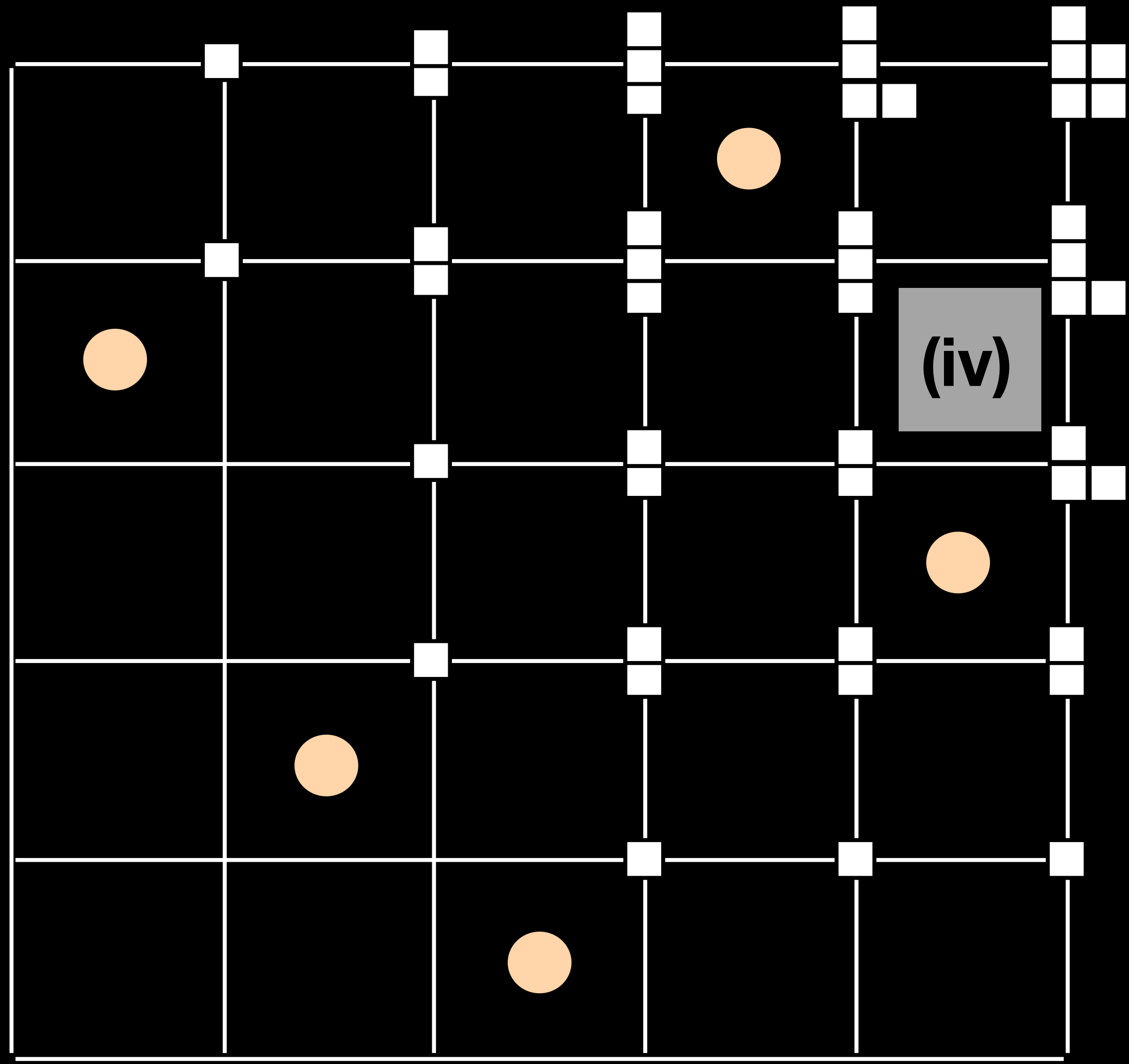
$$\lambda = \left\{\begin{matrix}\rho \\ \mu \\ \nu\end{matrix}\right. + (1)$$

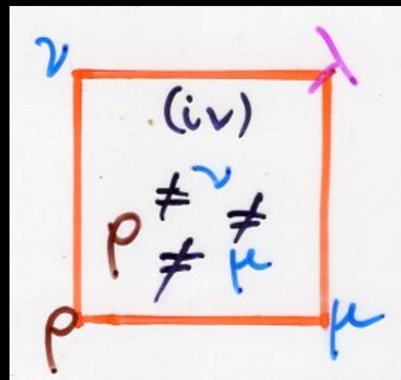




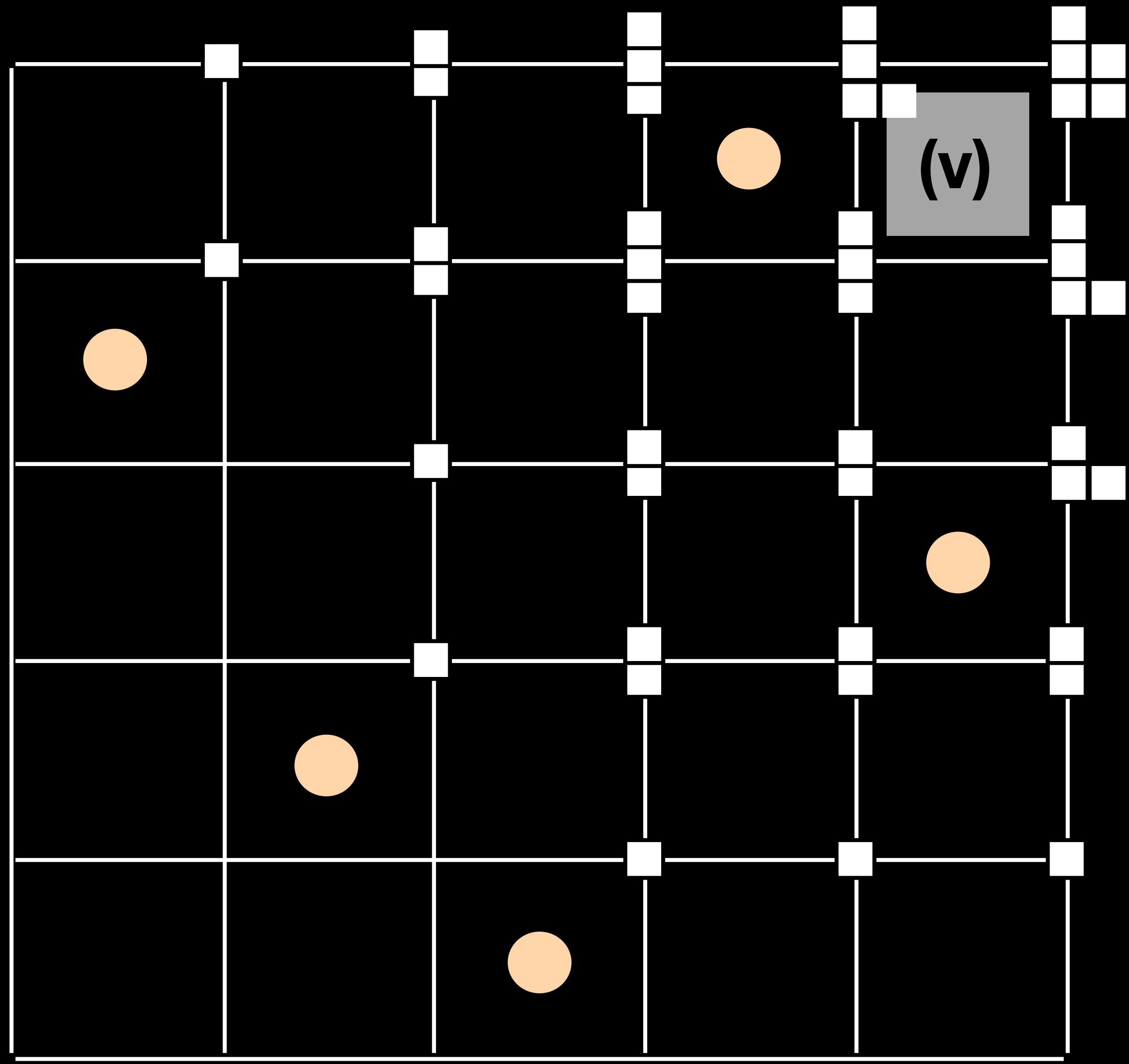
$$\lambda = \nu$$

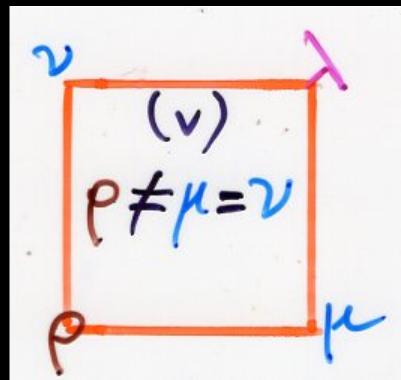

(iii)
ν
λ
(iii)
ρ = ν ≠ μ
ρ
μ
λ = μ



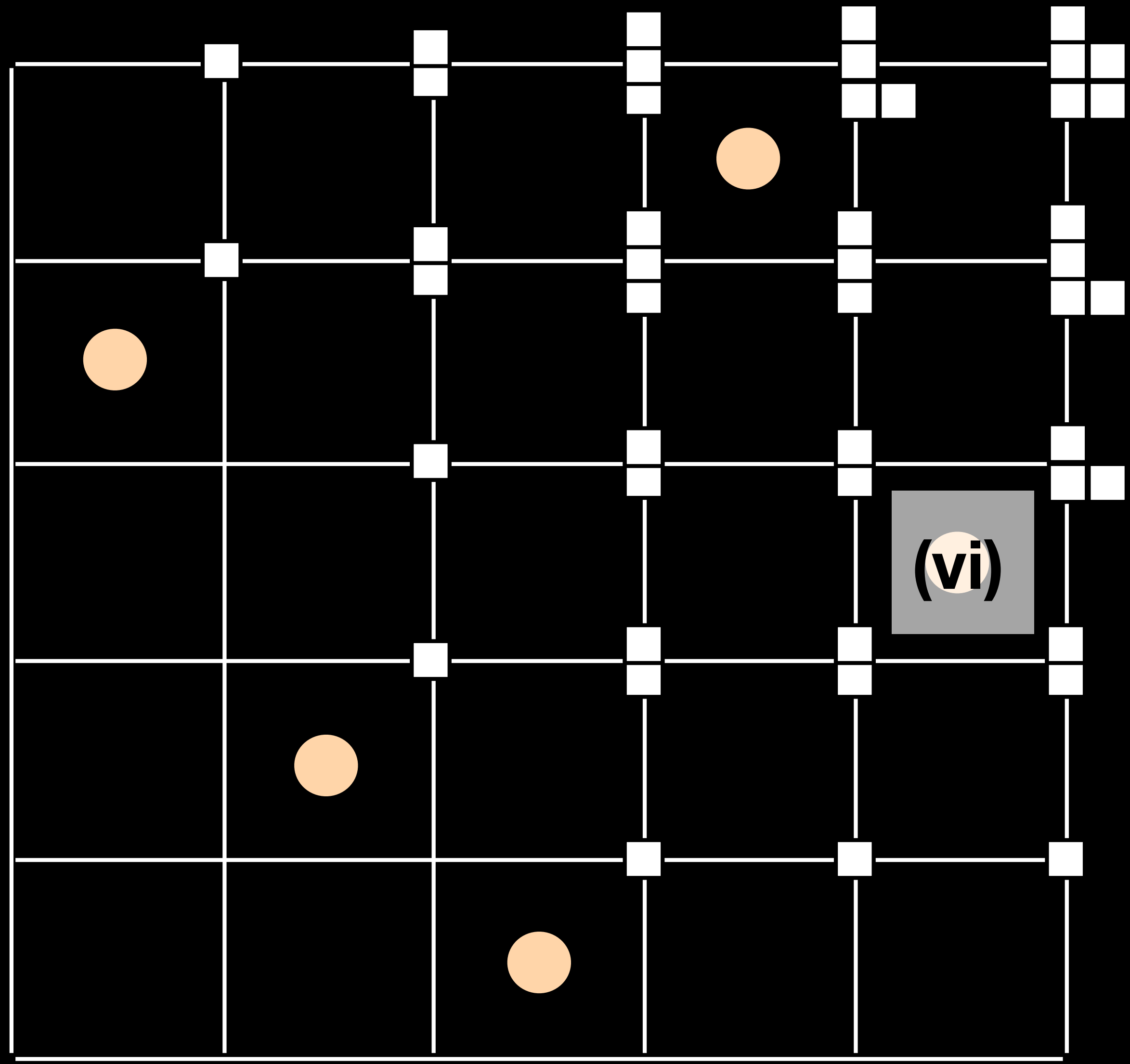


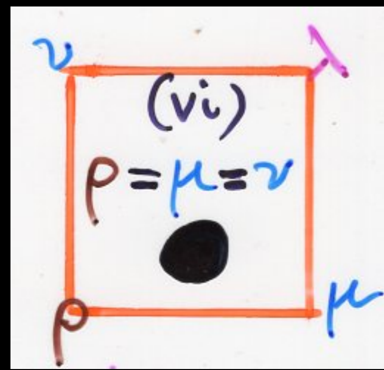
$$\lambda = \mu \cup \nu$$



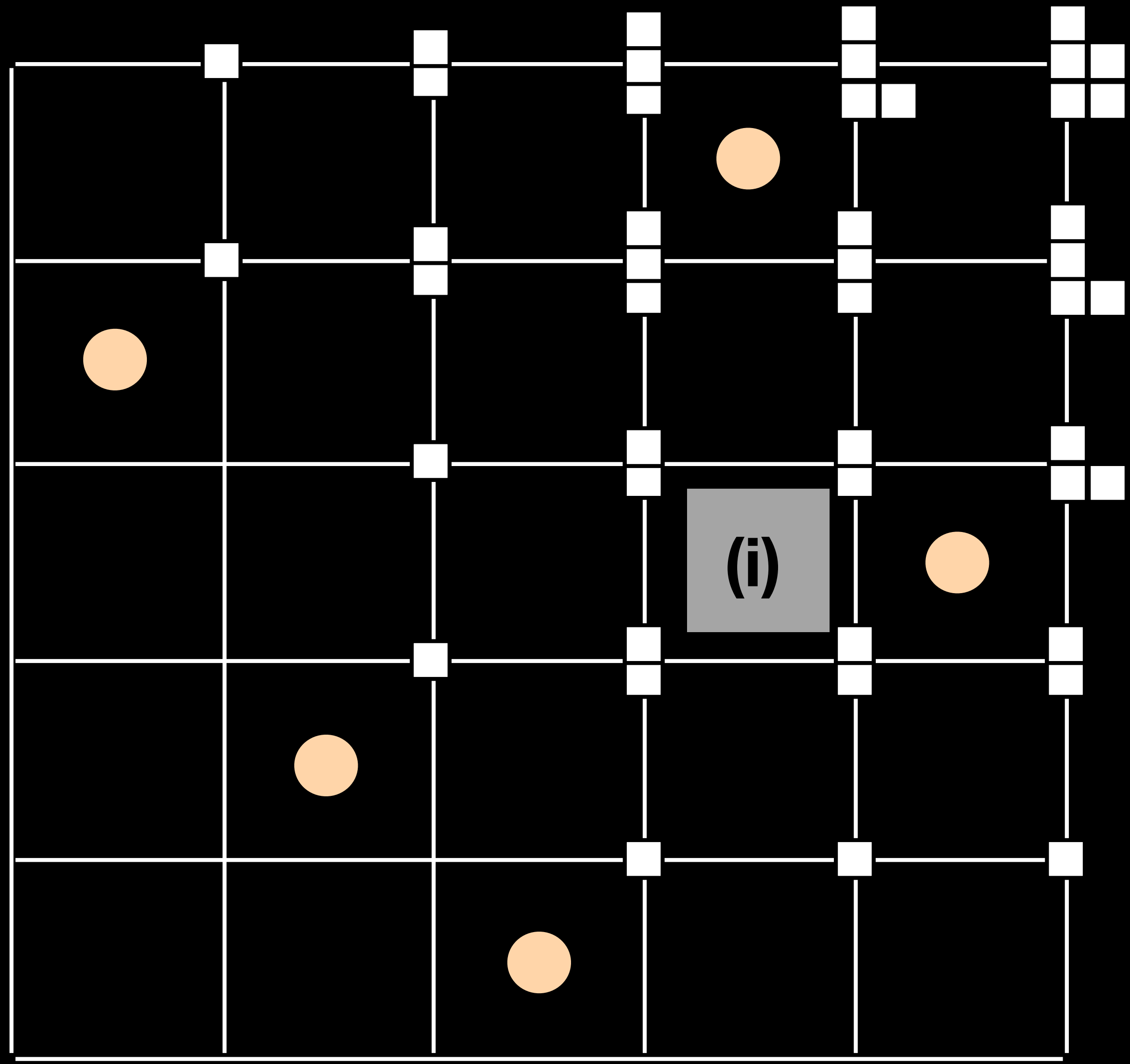


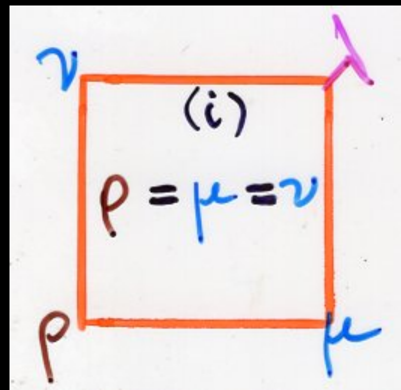
$$\lambda = \left\{ \begin{matrix} \mu \\ \nu \end{matrix} \right. + (i+1)$$





$$\lambda = \begin{cases} \rho \\ \mu \\ \nu \end{cases} + (1)$$





$\lambda = \rho$

« local rules on vertices »

Marc A. A. van Leeuwen (1996)

The Robinson-Schensted and Schützenberger algorithms, an elementary approach

C.Krattentheler, (2006).

GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

M.Rubey. (2007)

Increasing and Decreasing Sequences in Fillings of Moon Polyominoes

C.Krattentheler, (2016).

Proof of a conjecture of Burrill on oscillating tableaux

Combinatorial representation of

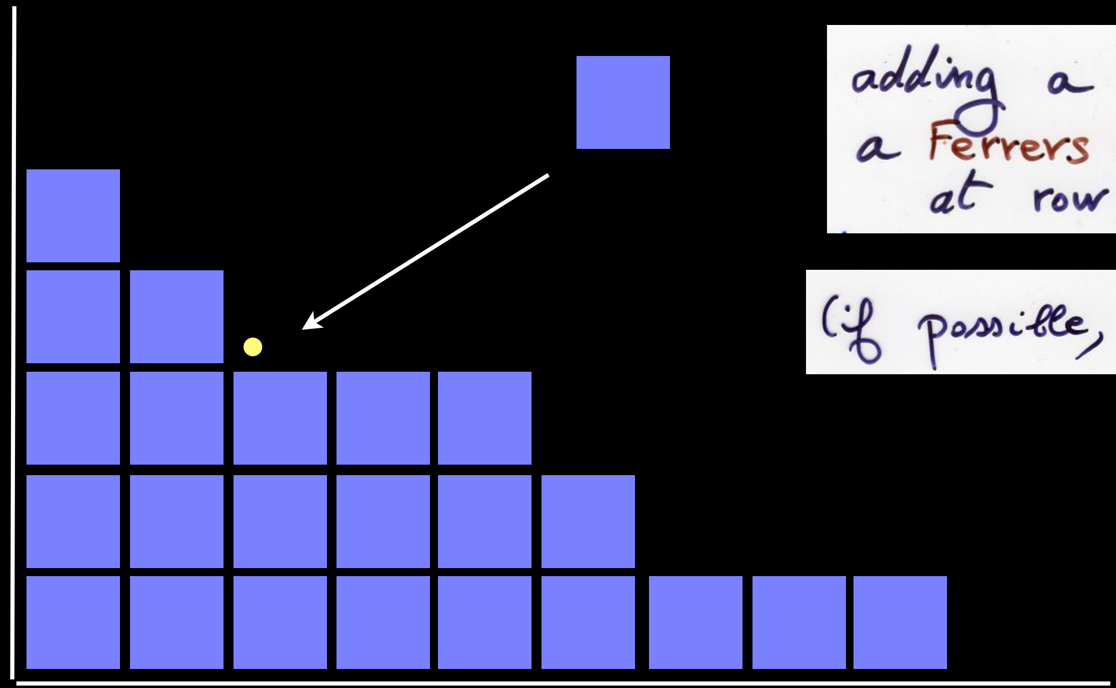
the algebra

$$UD = DU + Id$$

notations

operator U_i

i

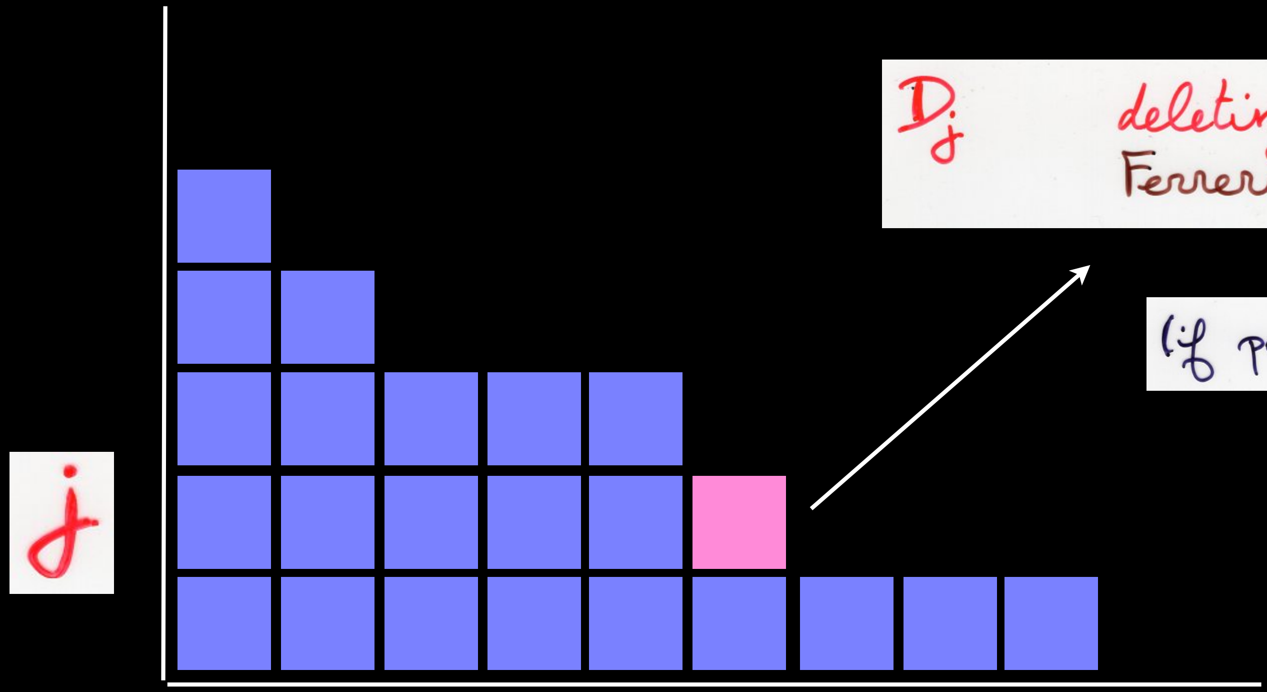


adding a cell in a Ferrers diagram ρ at row i

(if possible, else $U_i(\rho)=0$)

$$U_i(\rho) = \rho + (i)$$

$$D_j(\rho) = \rho - (j)$$

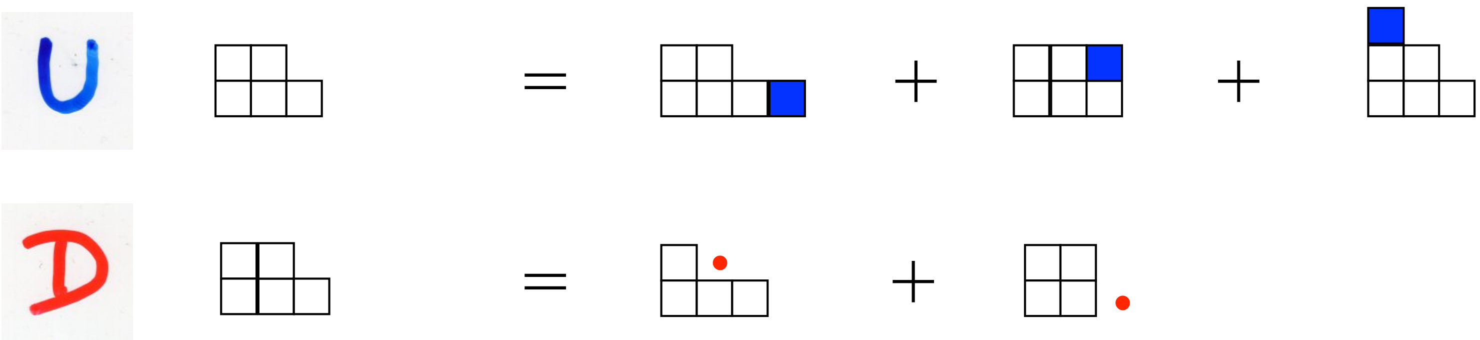


D_j deleting a cell in a Ferrers diagram ρ at row j

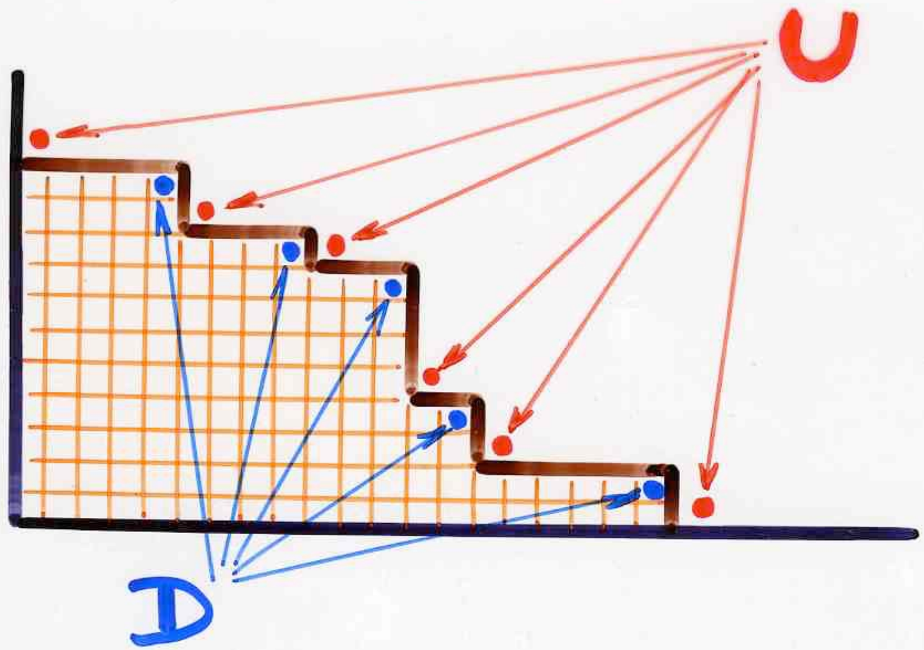
(if possible, else $D_j(\rho) = 0$)

$$U = \sum_{i \geqslant 1} U_i \qquad\qquad D = \sum_{i \geqslant 1} D_i$$

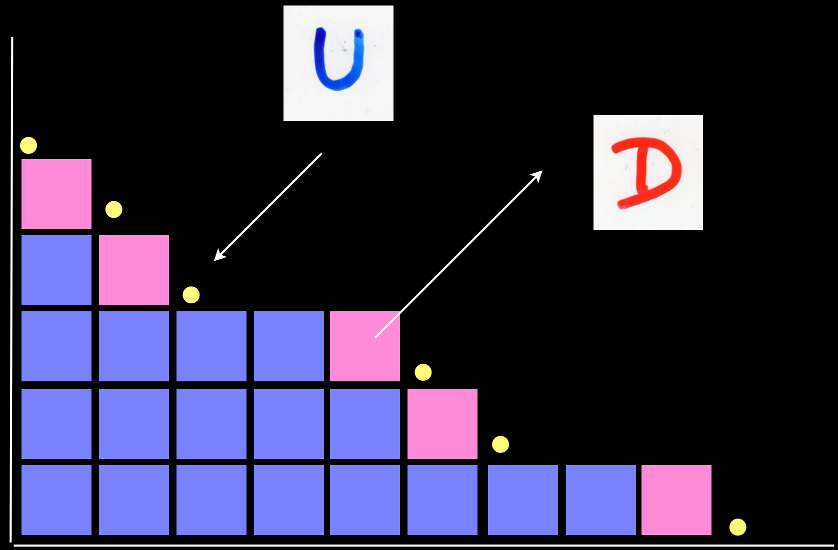
U and D are operators acting on the vector space generated by Ferrers diagrams



$$UD = DU + I$$

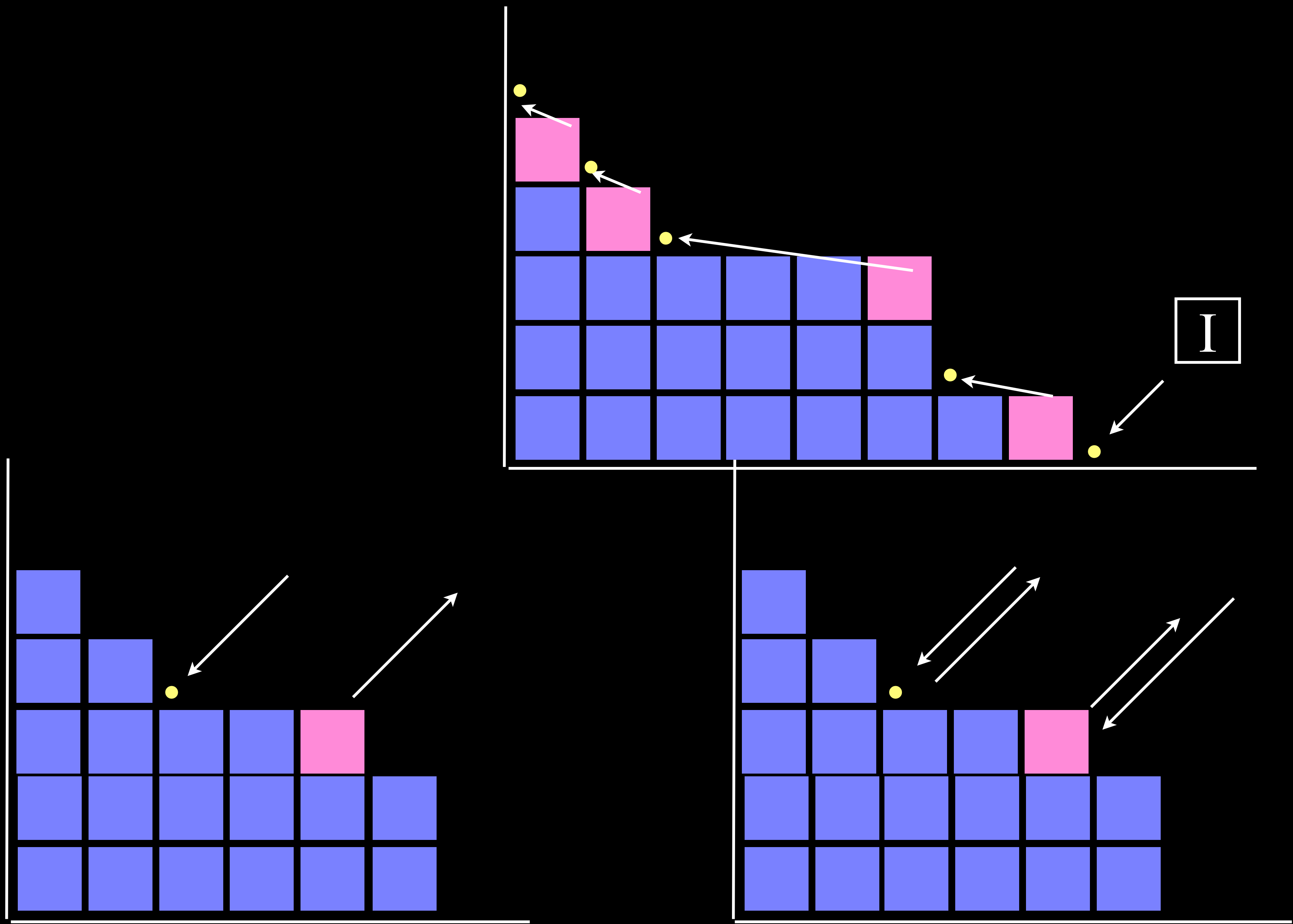


operators U and D



Young lattice

U adding a cell in a Ferrers diagram
D deleting a cell in a Ferrers diagram


I

$$UD = DU + I_v I_h$$

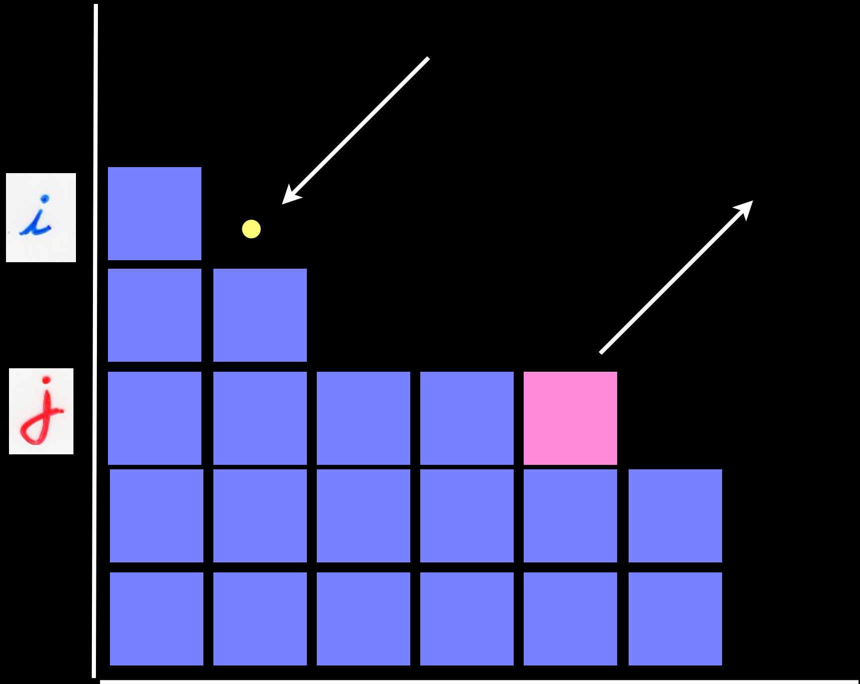
"commutation diagrams"

$$\begin{tikzcd} \nu \arrow[d, "I_v"'] \arrow[dr, phantom, "\bullet"] & \\ \rho \arrow[r, "I_h"'] & \mu \end{tikzcd} \qquad \begin{tikzcd} (\nu, p) \arrow[d, "D"'] & \\ (\rho, m) \arrow[r, "U"'] & \mu \end{tikzcd} \quad \xleftrightarrow{\text{bijection}} \quad \begin{tikzcd} (\nu, p') \arrow[r, "U"] & (\lambda, m') \arrow[d, "D"] \\ & \mu \end{tikzcd}$$

p, m, p', m' are "positions"

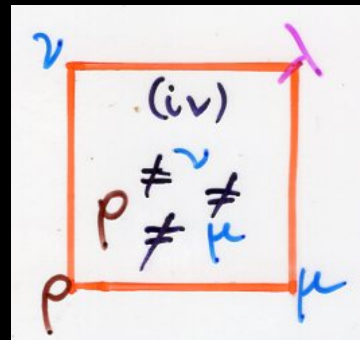
in ν, ρ, ν, λ respectively

$$(\nu,p) \xrightarrow{D} (\rho,m) \xrightarrow{U} \mu \quad \longleftrightarrow \quad (\nu,p') \xrightarrow{U} (\lambda,m') \xrightarrow{D} \mu$$



$p = j$
$m = i$

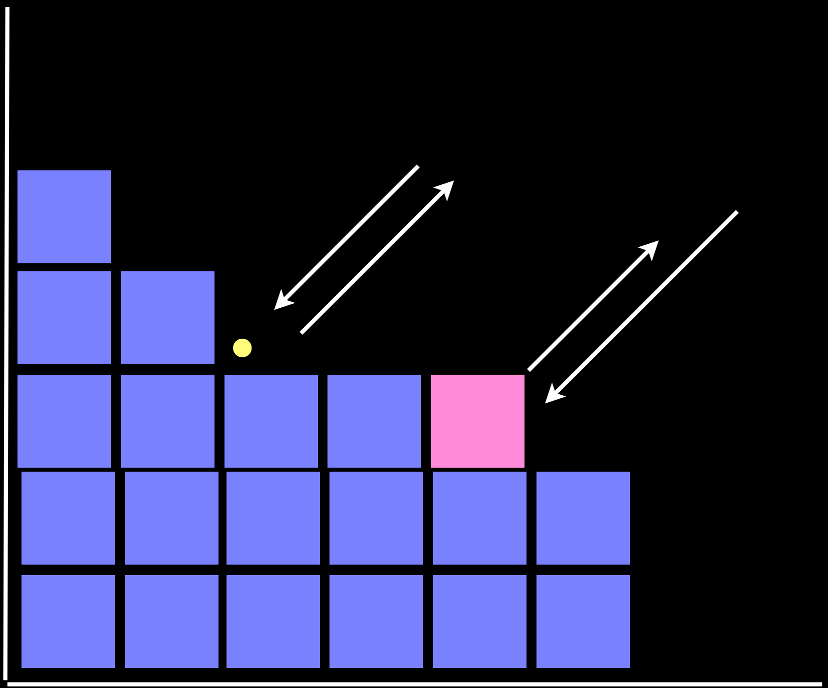
$p' = i$
$m' = j$



$\nu = \rho + (j)$
$\mu = \rho + (i)$
$\lambda = \rho + (i) + (j)$

$$
\begin{tikzcd}
 & (\nu, p') \arrow[r, "U"] & (\lambda, m') \arrow[d, "D"] \\
(\nu, p) \arrow[d, "D"'] \arrow[rr, leftrightarrow] & & \mu \\
(\rho, m) \arrow[r, "U"'] & \mu &
\end{tikzcd}
$$

i

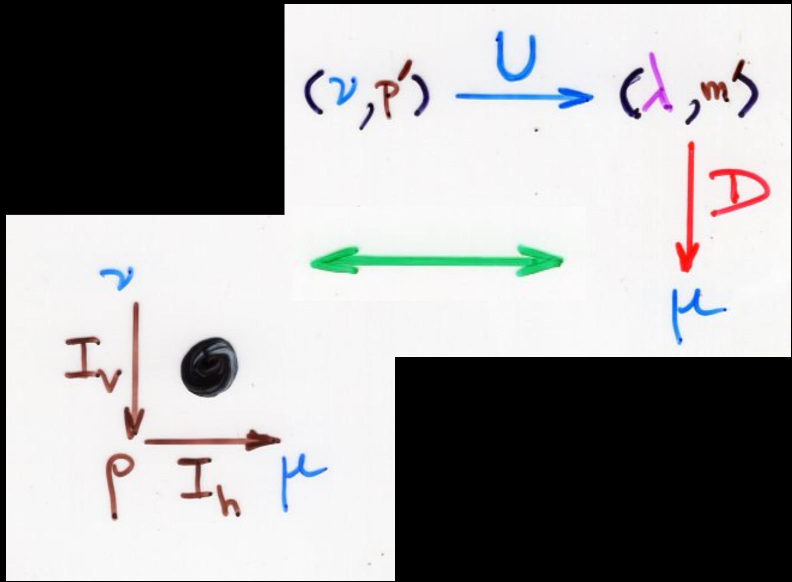


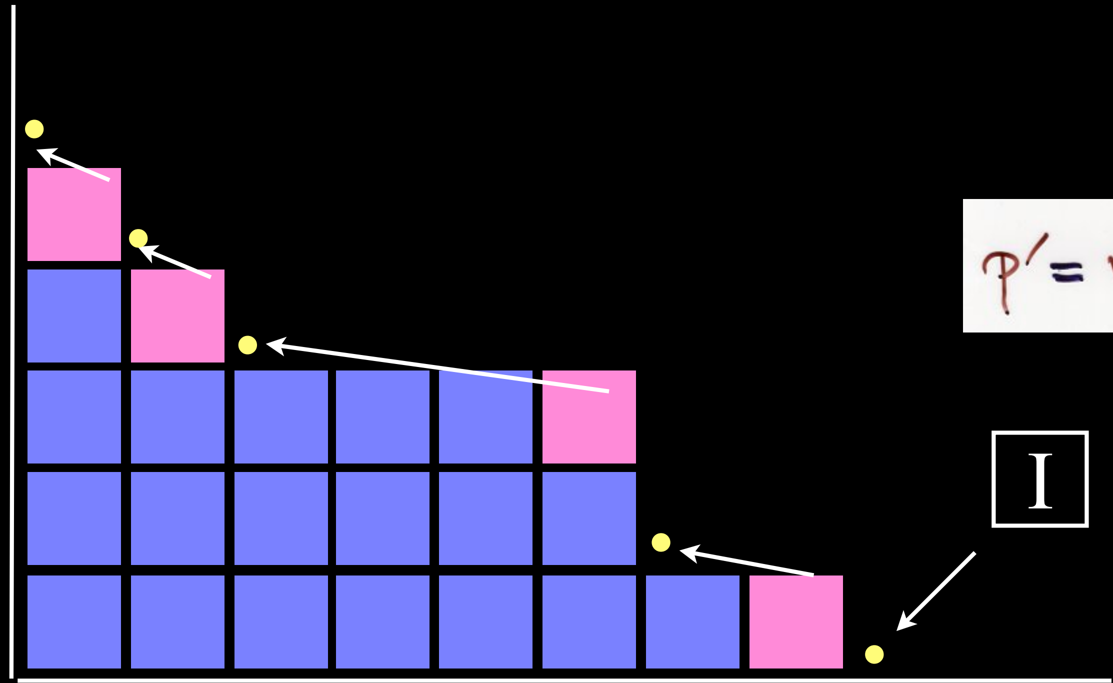
$p = i$
$m = i$

$p' = i+1$
$m' = i+1$

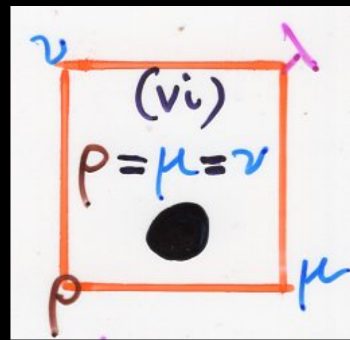
$$
\begin{tikzcd}
\nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "{(v)\ \rho \neq \mu = \nu}"] & \lambda \arrow[d, dash] \\
\rho \arrow[r, dash] & \mu
\end{tikzcd}
$$

$\mu = \nu = \rho + (i)$
$\lambda = \mu + (i+1)$

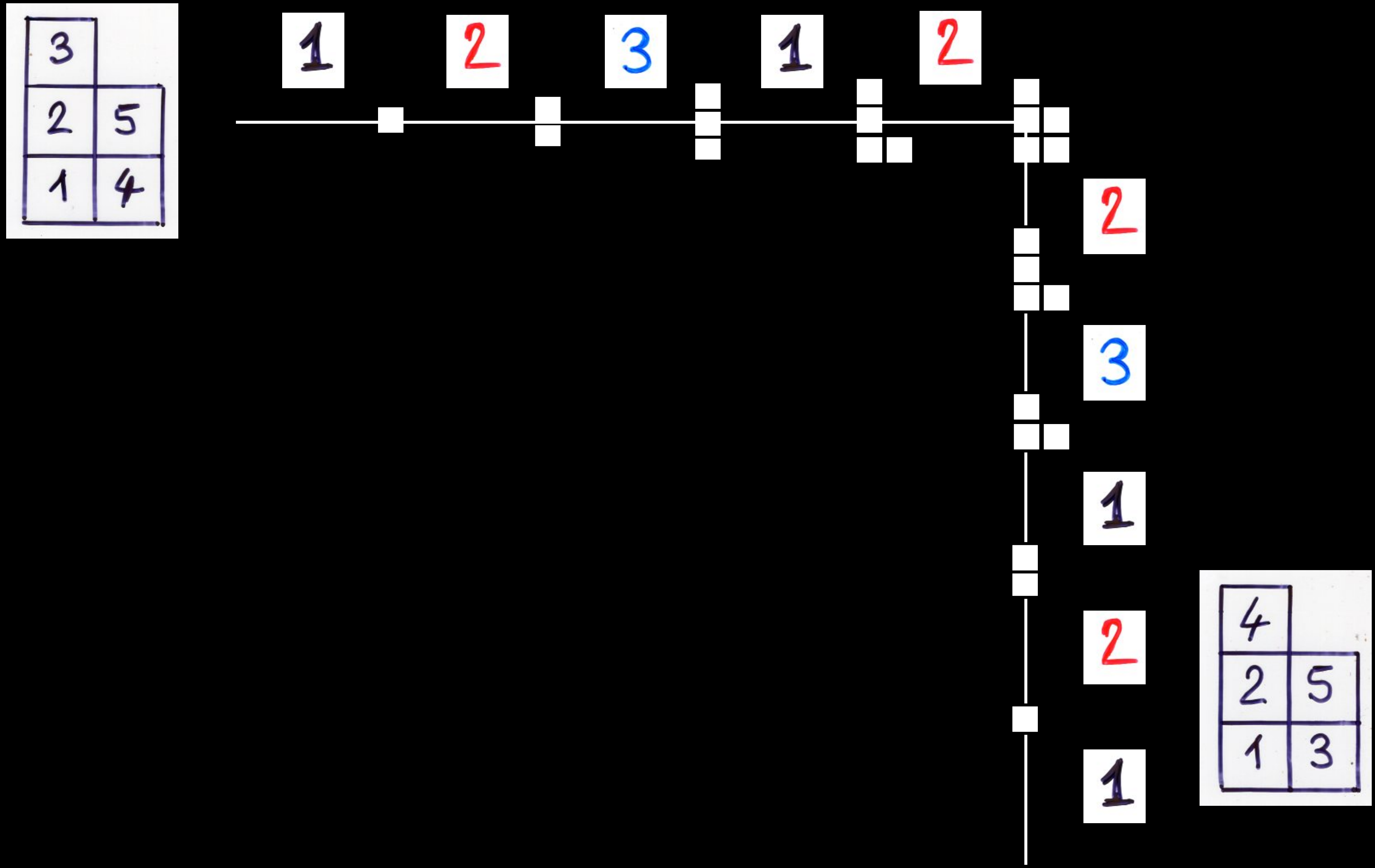




$p' = m' = 1$



$\lambda = \left\{\begin{matrix}\rho\\ \mu\\ \nu\end{matrix}\right. + (1)$


3
2 5
1 4
1
2
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1
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1
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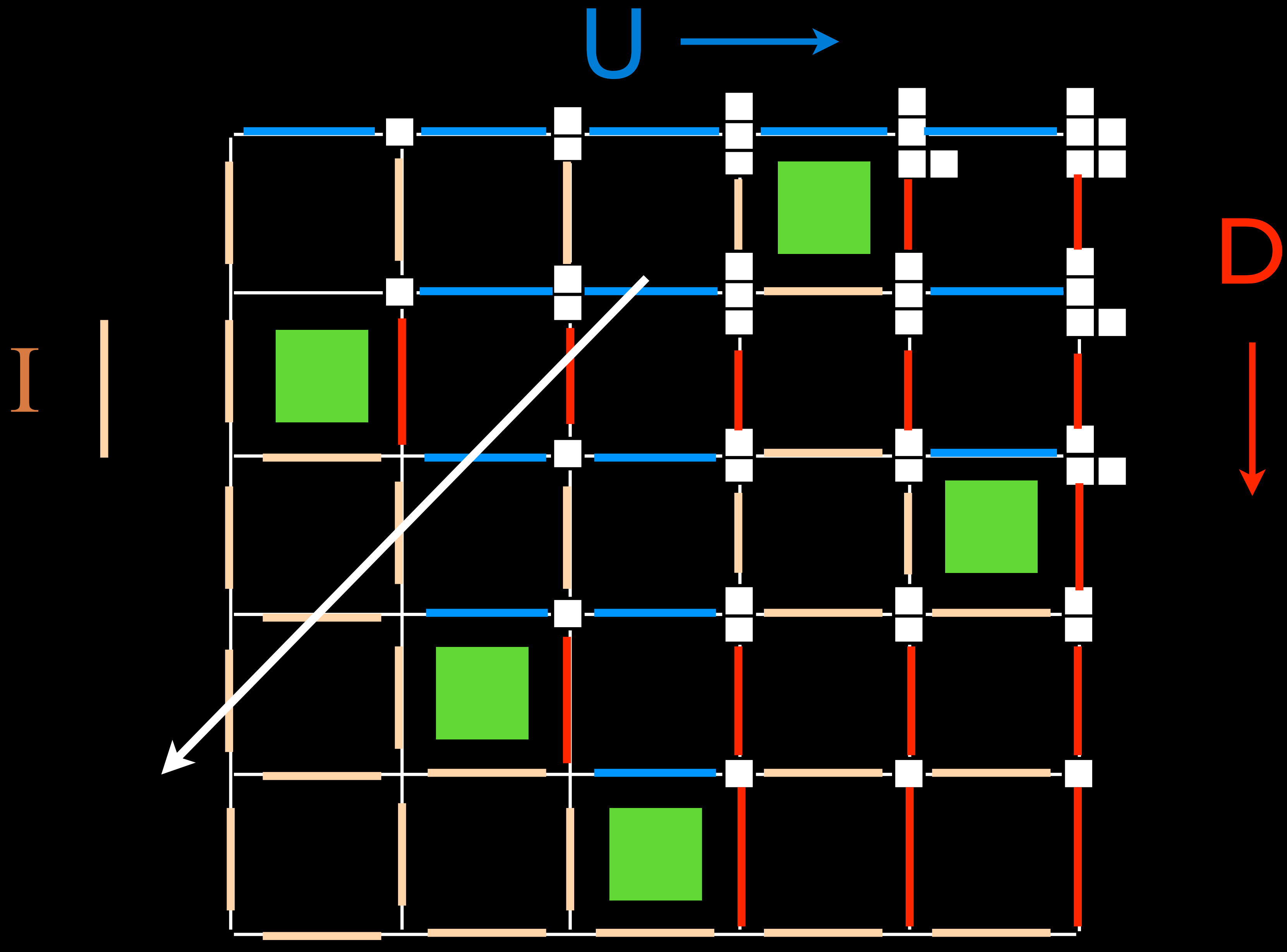
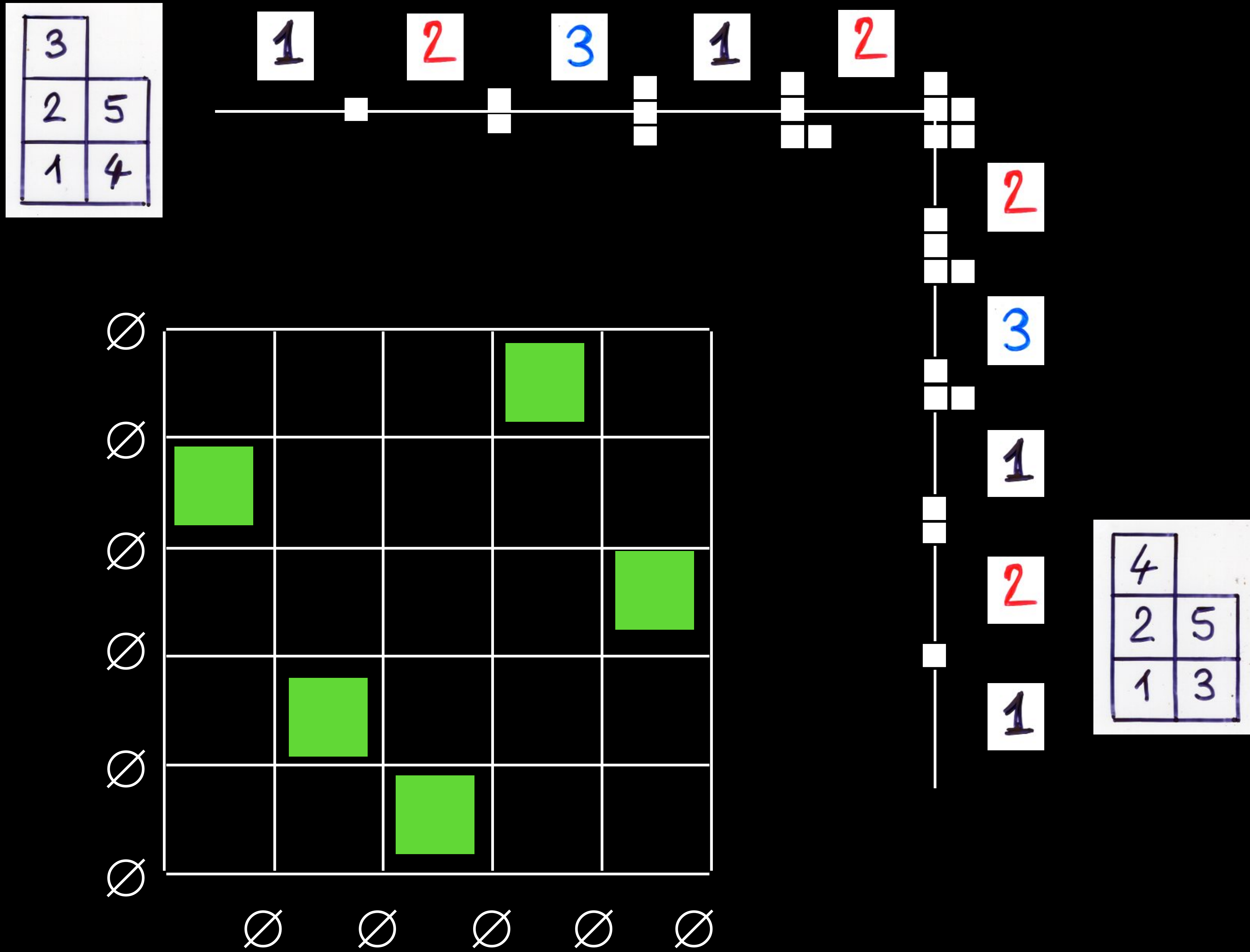
1
4
2 5
1 3



This "propagation" algorithm is exactly the reverse of Fomin's "growth diagrams"

I


3
2 5
1 4
1 2 3 1 2
2 3 1 2 1
4
2 5
1 3
∅ ∅ ∅ ∅ ∅ ∅
∅ ∅ ∅ ∅ ∅

"The cellular ansatz"

quadratic algebra Q

$UD = DU + \mathrm{Id}$

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

bijections

RSK

representation of Q by combinatorial operators

pairs of Young tableaux

(i) first step

(ii) second step

quadratic algebra Q

Q-tableaux

$UD = DU + Id$

permutations

Planarization of the rewriting rules

$UD = DU + Id$ commutations

$UD \to DU \qquad UD \to Id$ rewriting rules

$UUDD$

$$\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2UD \\
&= (DUDU + DU) + 2(DU + Id) \\
&= (DDUU + 2DU) + 2(DU + Id) \\
&= DDUU + 4DU + 2Id
\end{aligned}$$

$UD = DU + Id$

commutations

<u>Lemma</u> Every word w with letters U and D can be written in a unique way

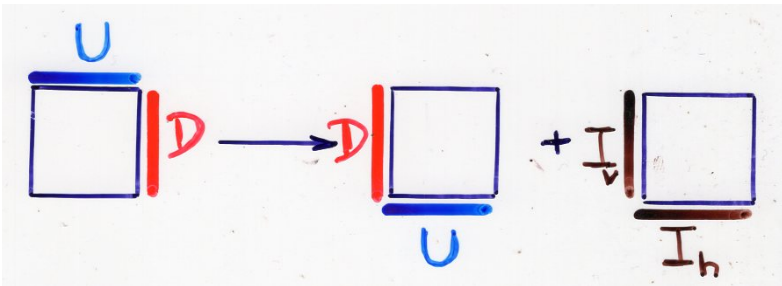
$$w = \sum_{i,j \geq 0} c_{ij}(w)\, D^i U^j$$

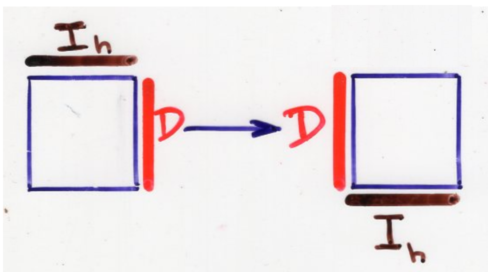
normal ordering in physics

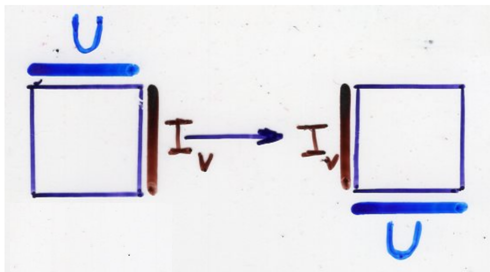
$UD = DU + Id$ commutations

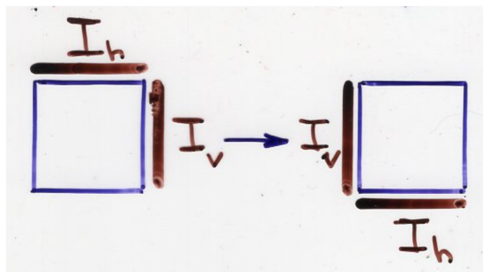
$UD \to DU$ $UD \to Id$ rewriting rules

planarization of the rewriting rules







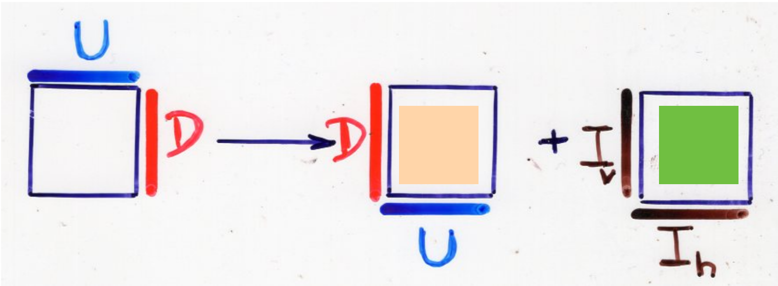


homogenization
of the system
of commutations
relations

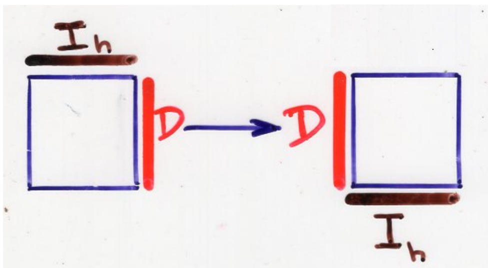
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

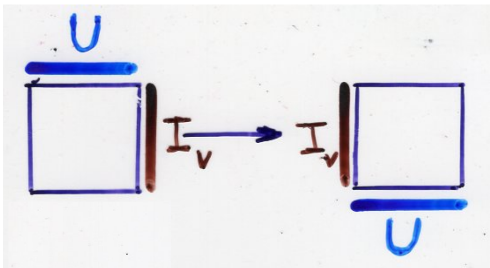
$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

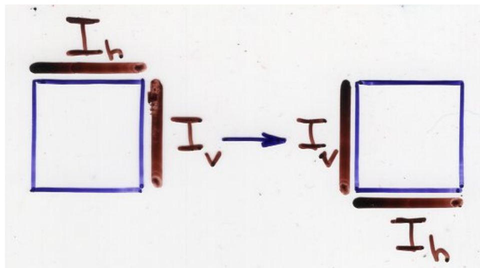
rewriting rules

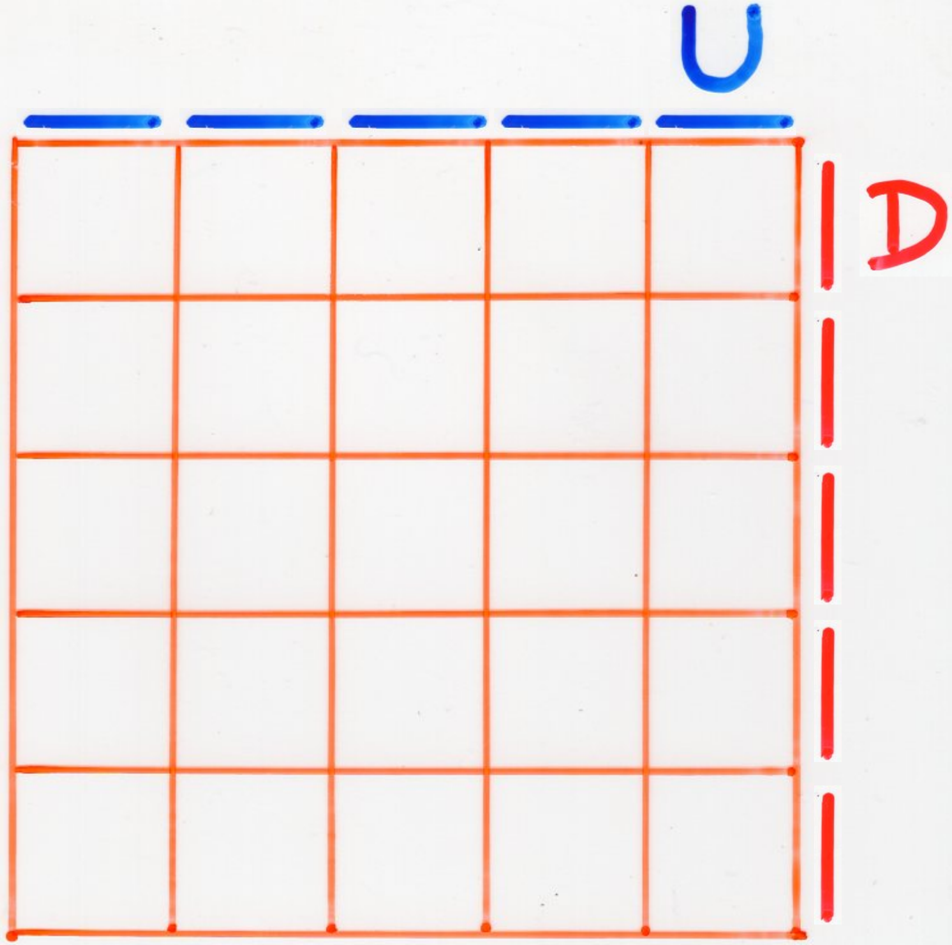


"planarization" of the "rewriting rules"





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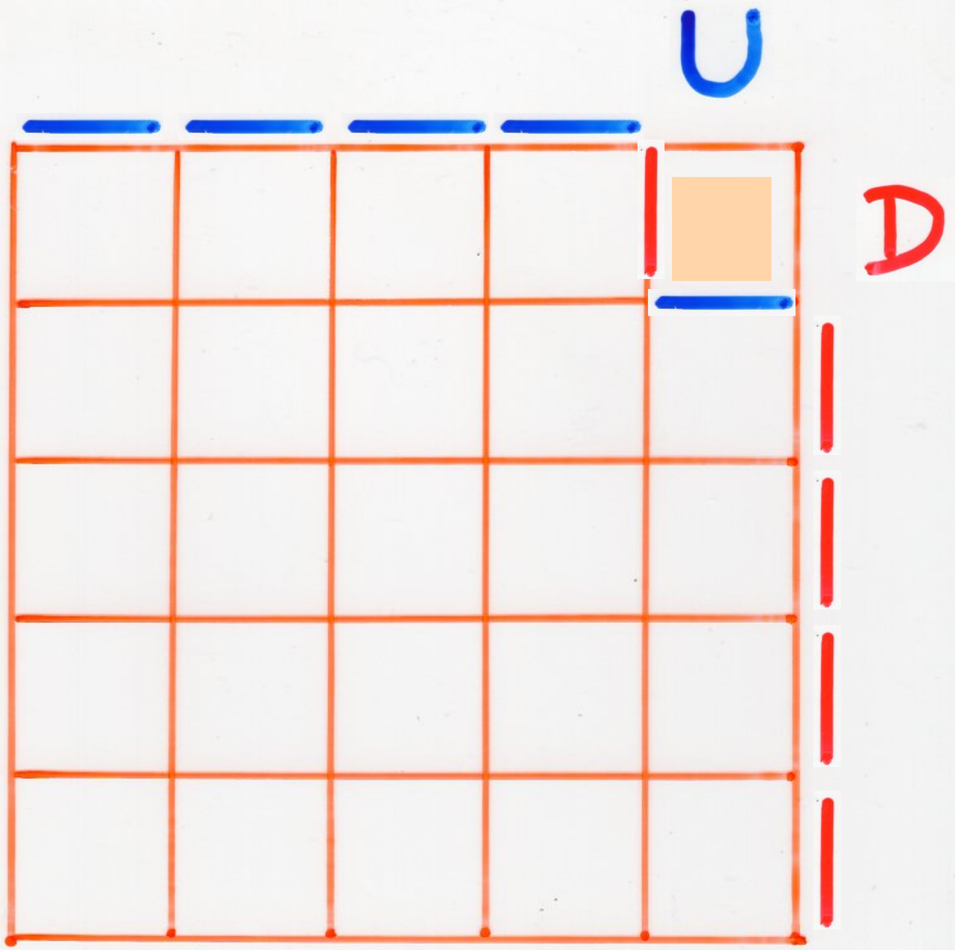
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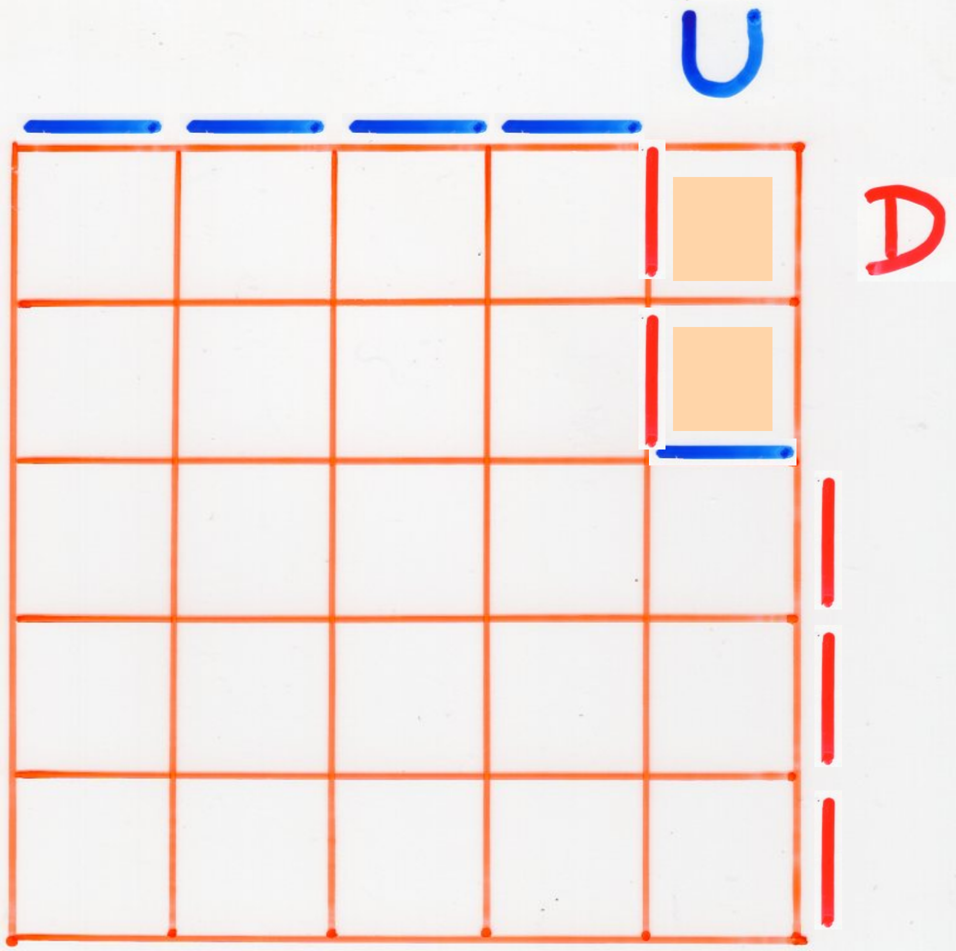
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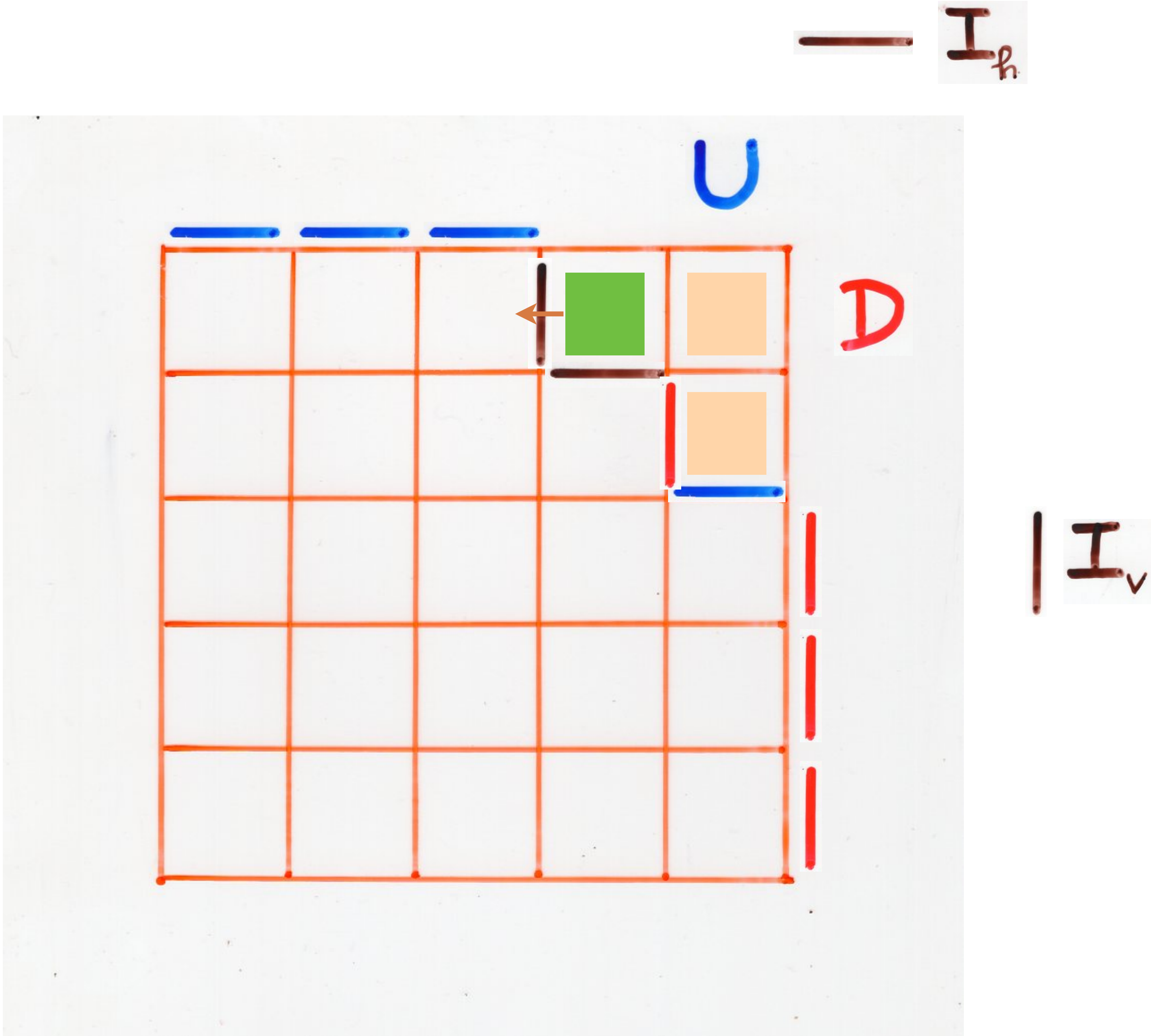

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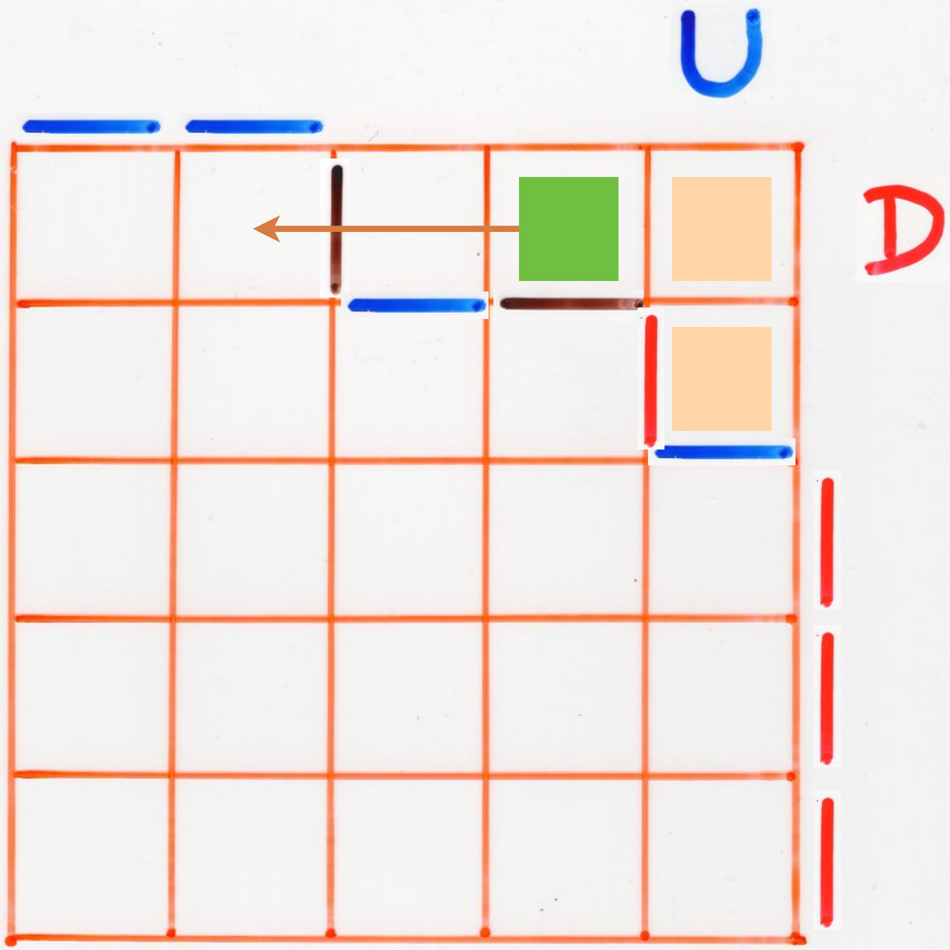
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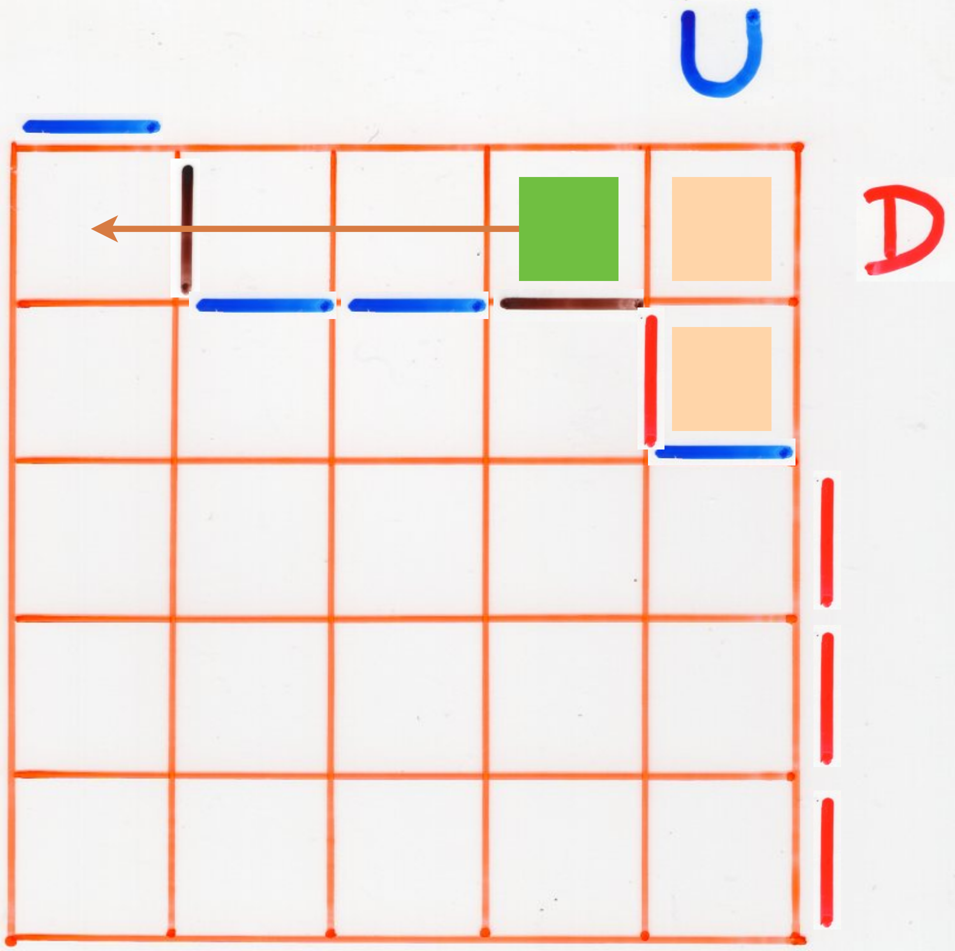
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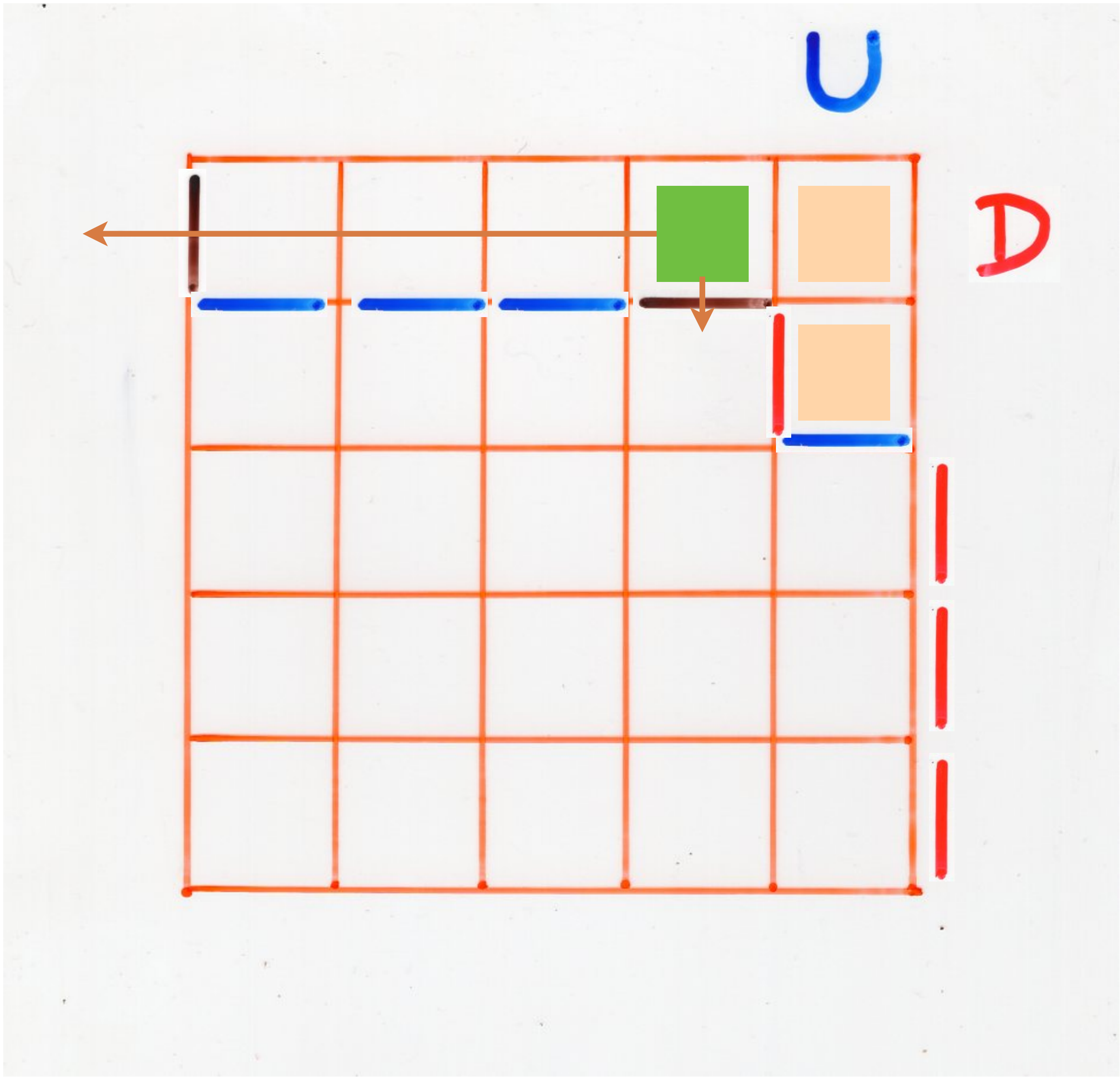
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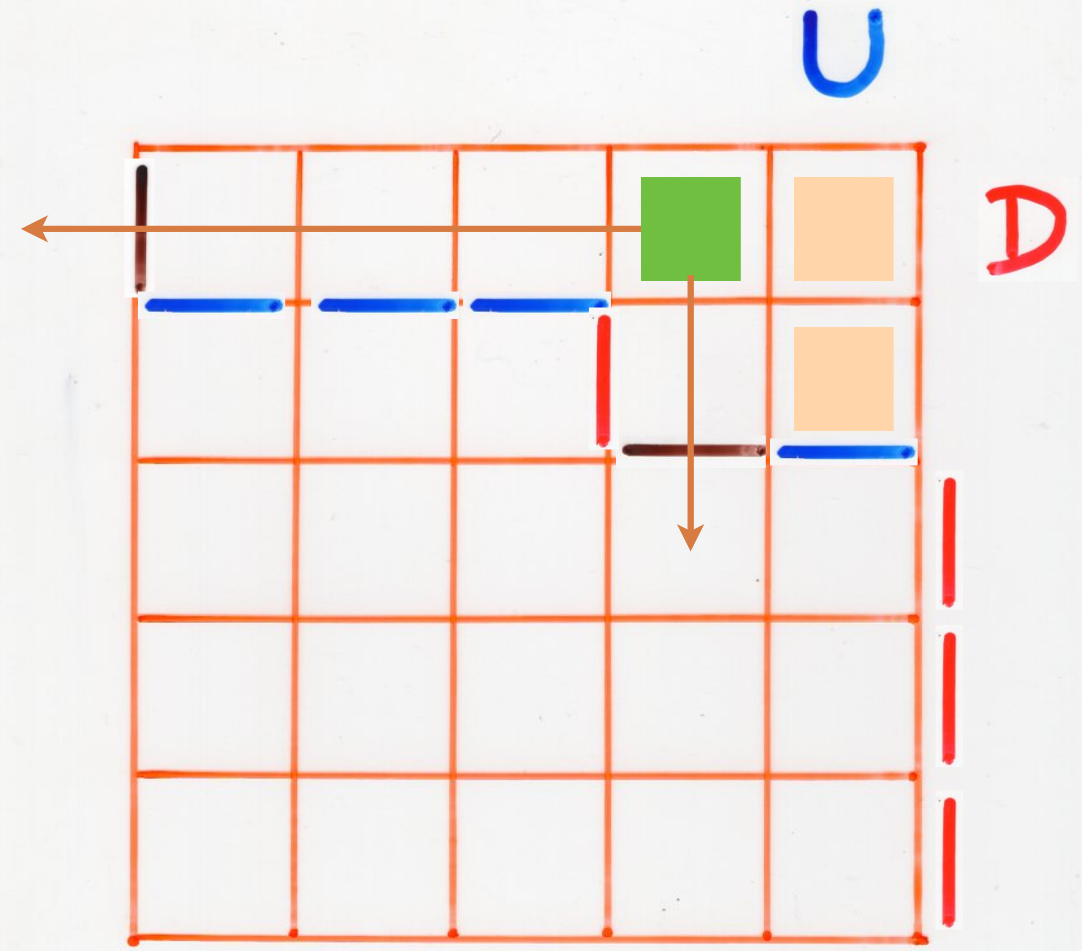
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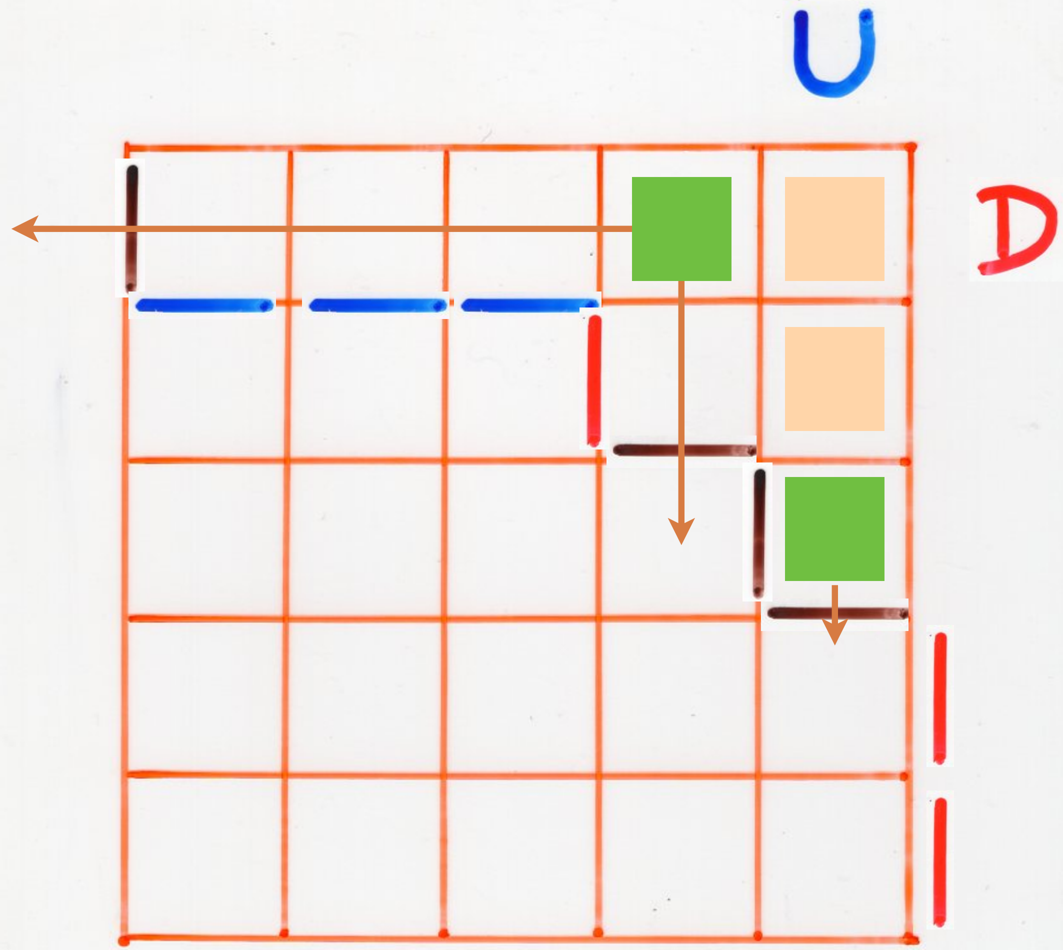
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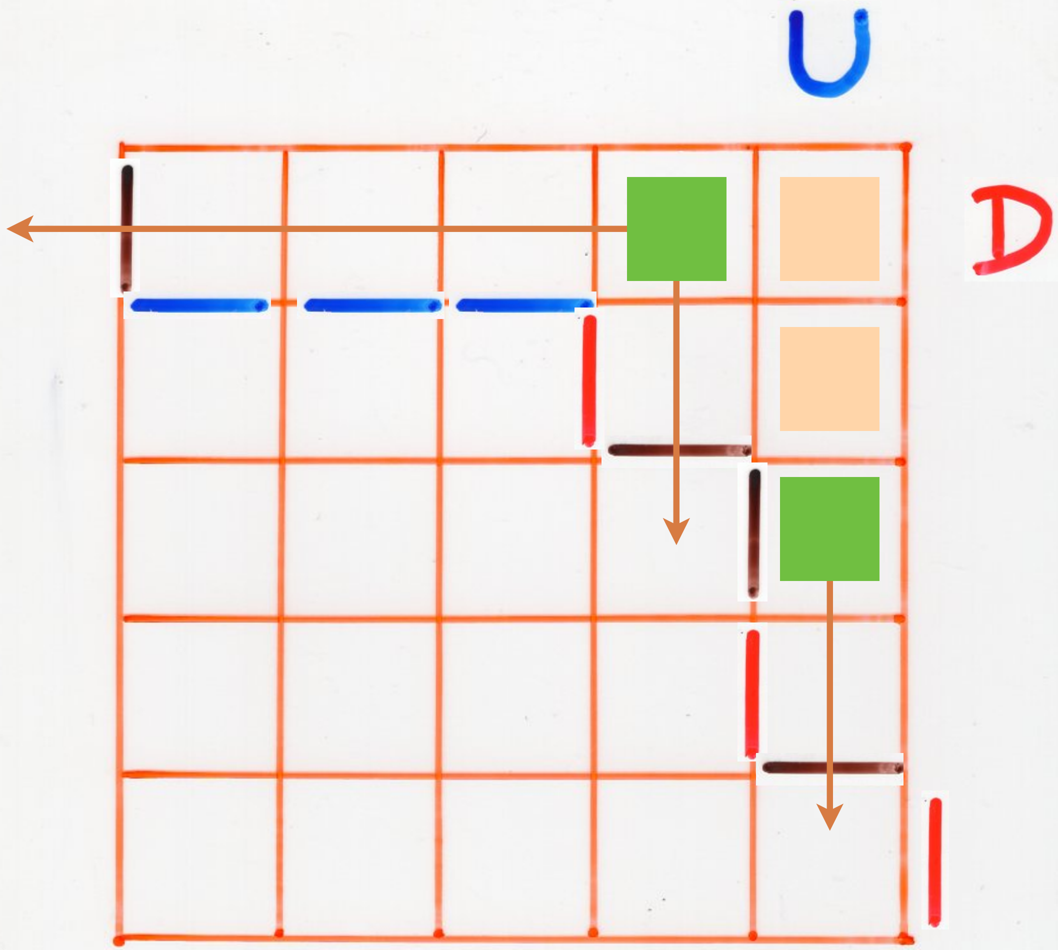
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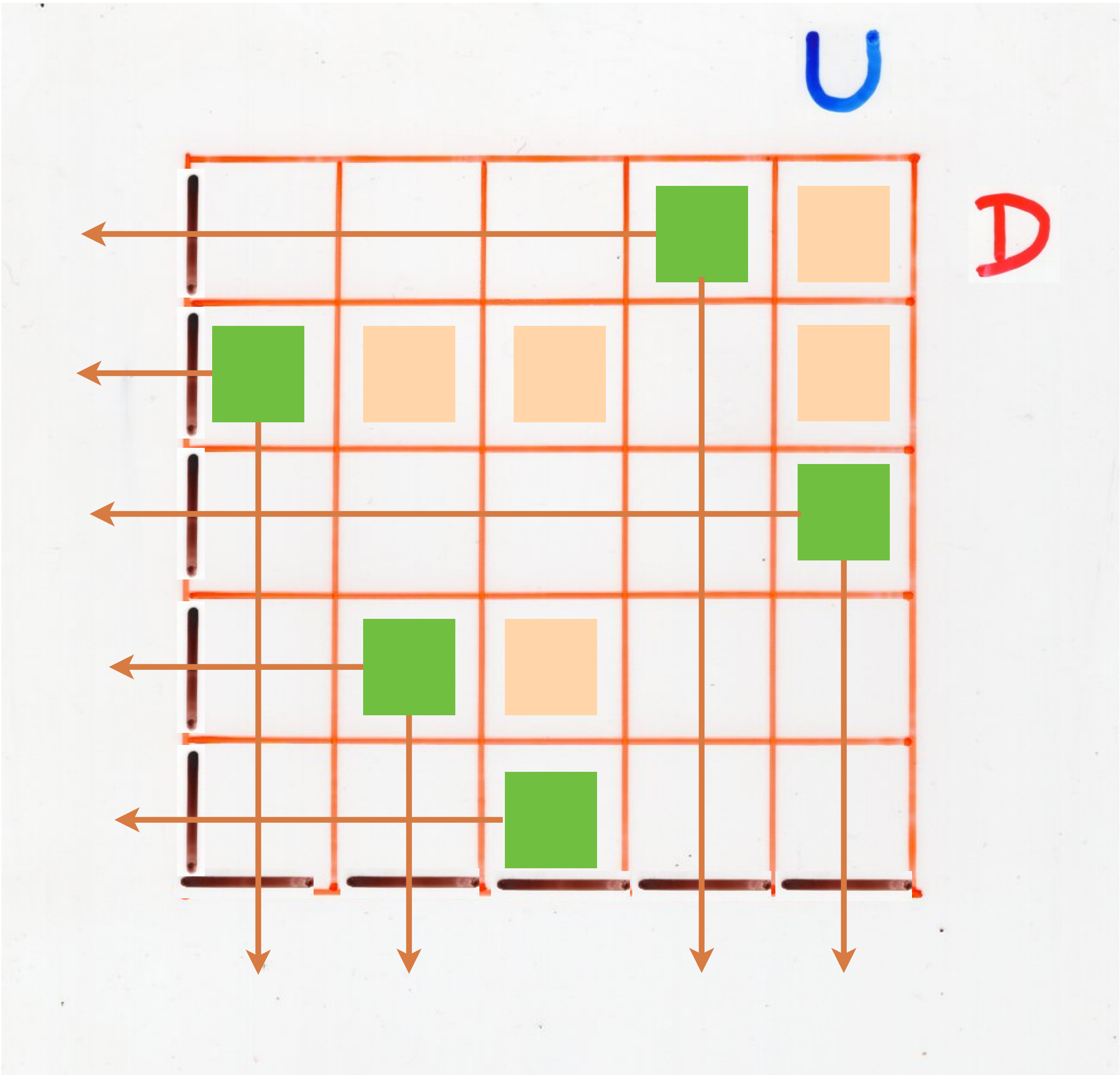
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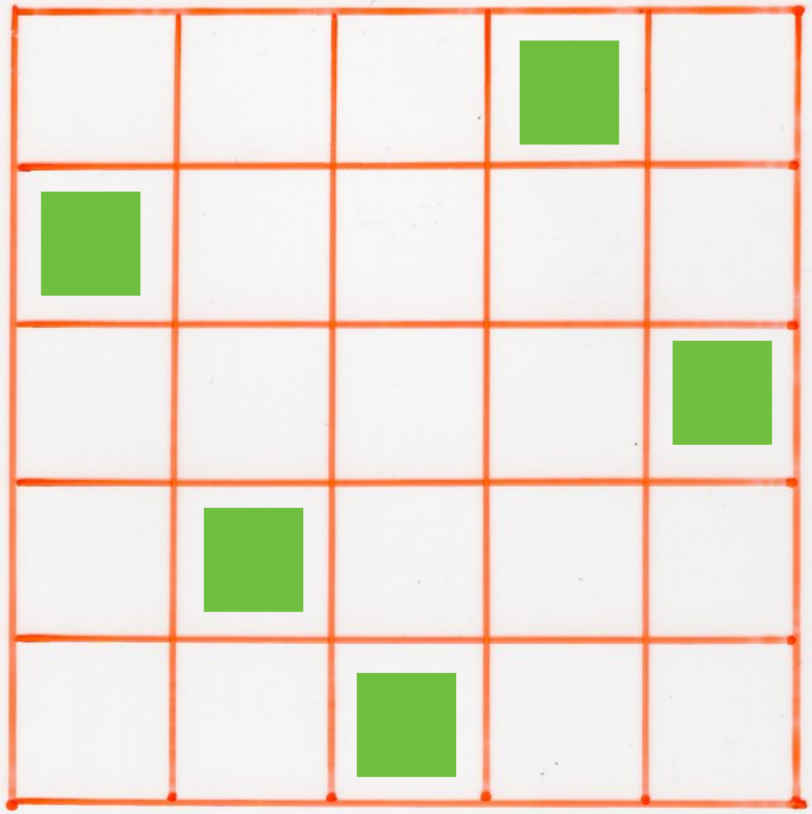


quadratic algebra **Q**

$UD = DU + Id$

permutation as a **Q**-tableau

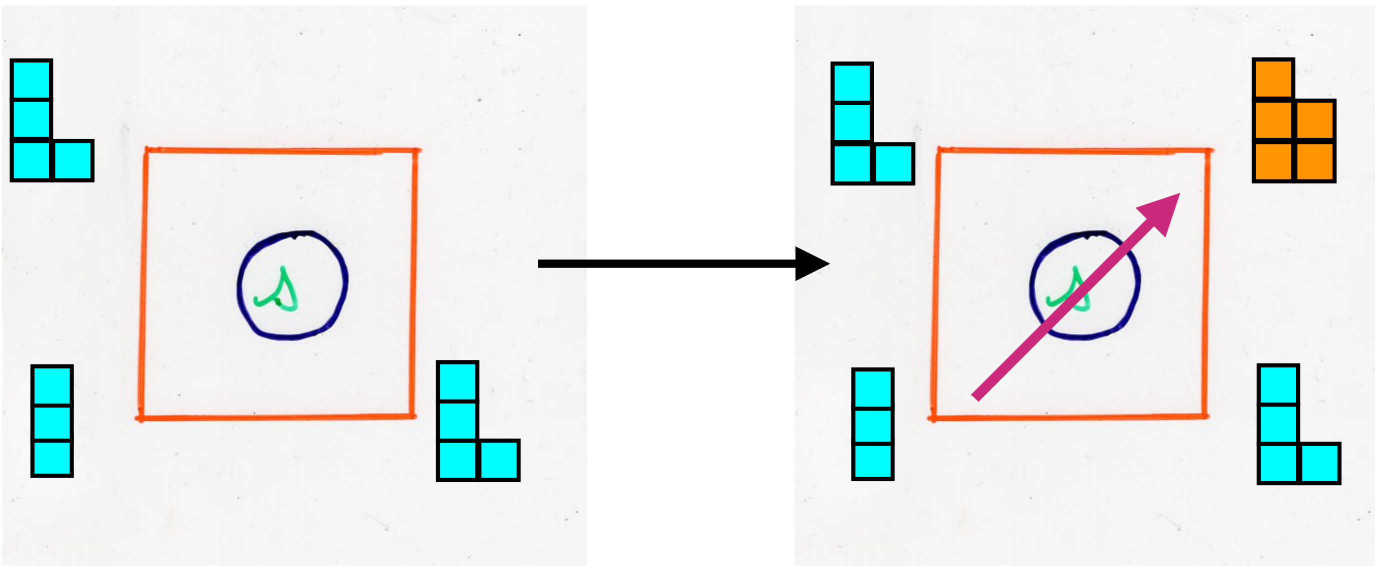
From « local rules » on vertices

to

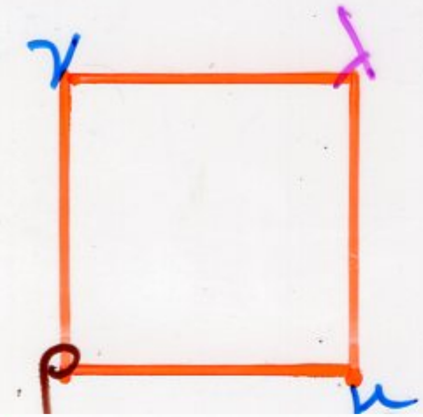
« local rules » on edges

Fomin's "local rules"

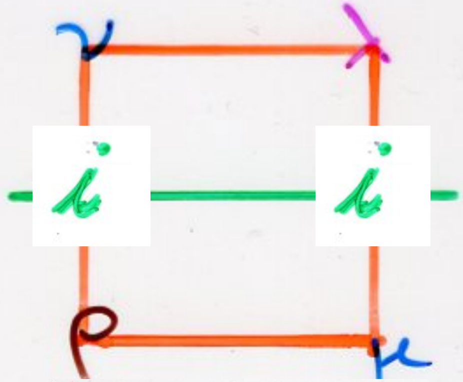
"growth diagrams"



"local rules" on the vertices

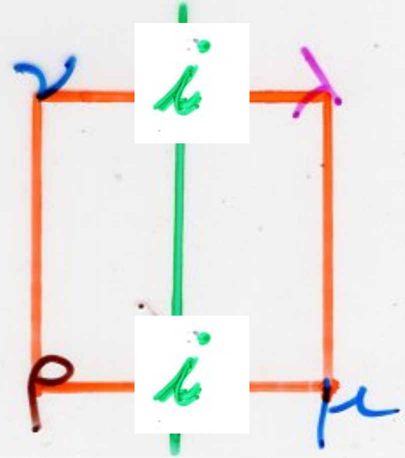


$\lambda = \rho = \mu = \nu$



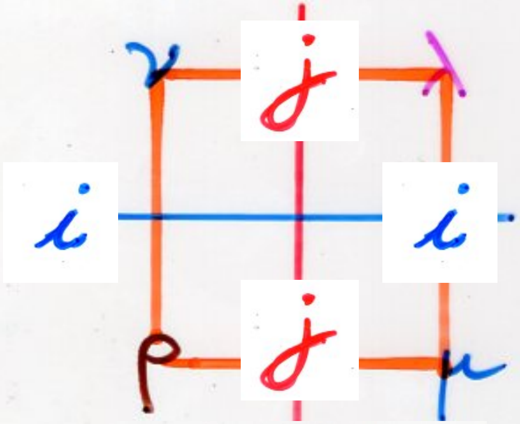
$\rho = \mu$

$\lambda = \nu = \rho + (i)$



$\rho = \nu$

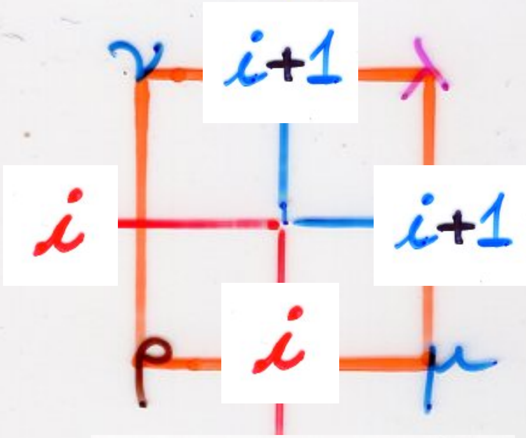
$\lambda = \mu = \rho + (j)$



$\nu = \rho + (i)$

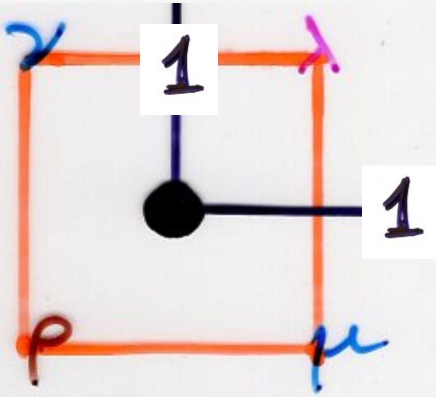
$\mu = \rho + (j)$

$\lambda = \rho + (i) + (j)$



$\mu = \nu = \rho + (i)$

$\lambda = \mu + (i+1)$



$\lambda = \left\{\begin{matrix}\rho\\ \mu\\ \nu\end{matrix}\right. + (1)$

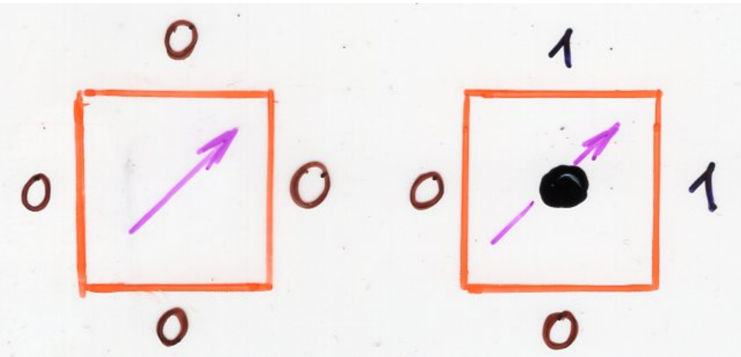
"local rules" on the edges

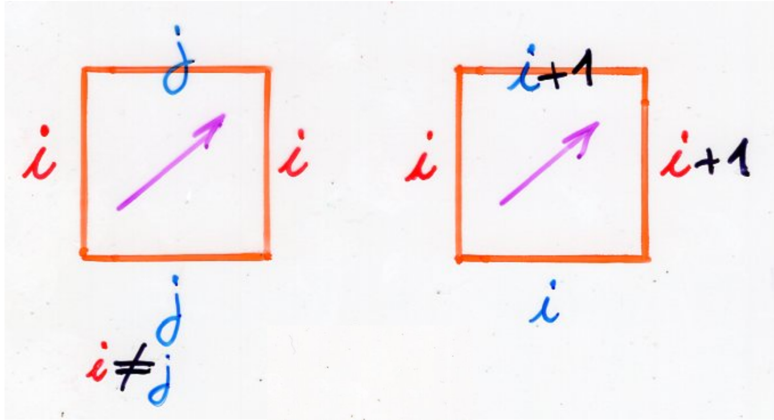
state {0, 1, 2, ...}

state {0, 1, 2, ...}

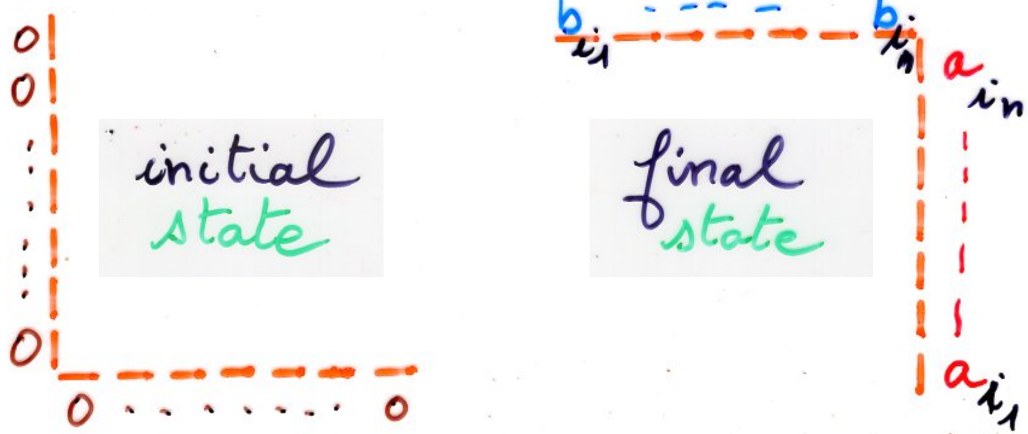
set of labels

$L = \{\square, \boxdot\}$

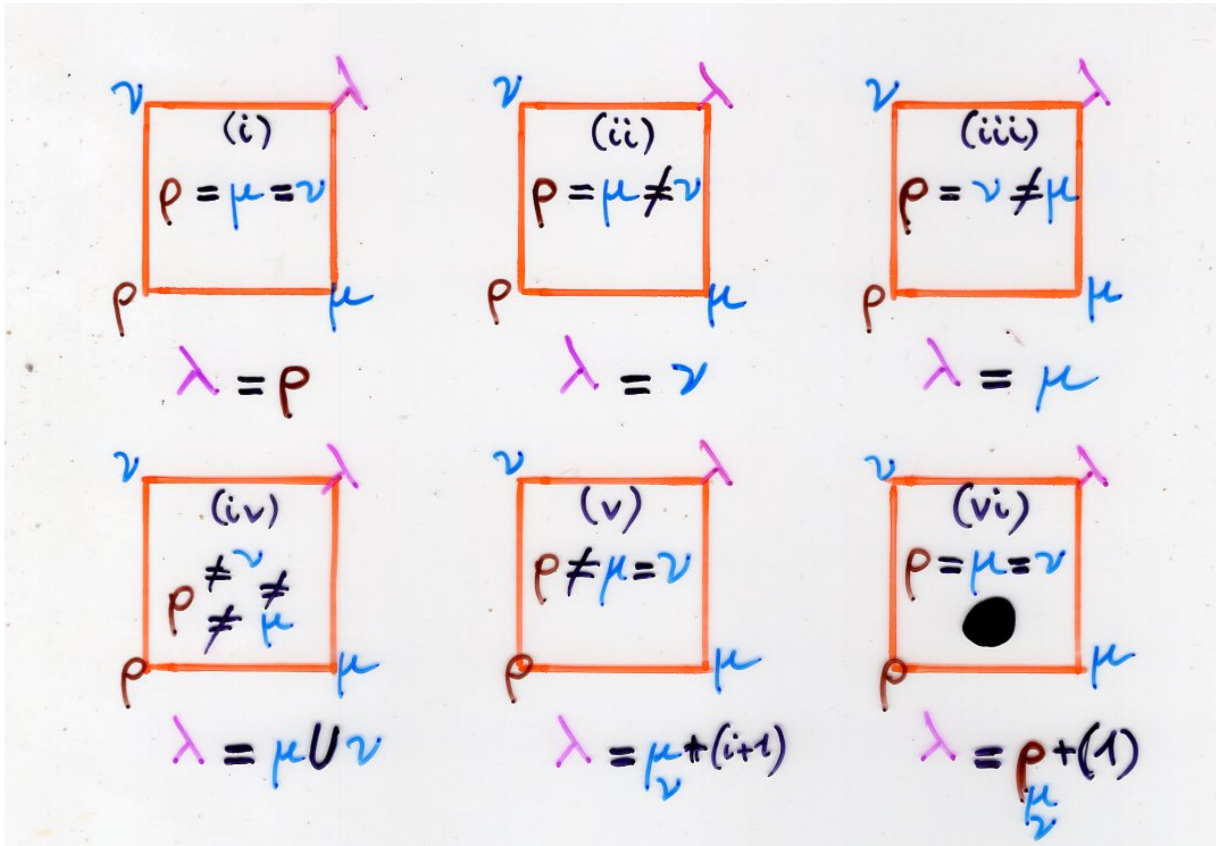




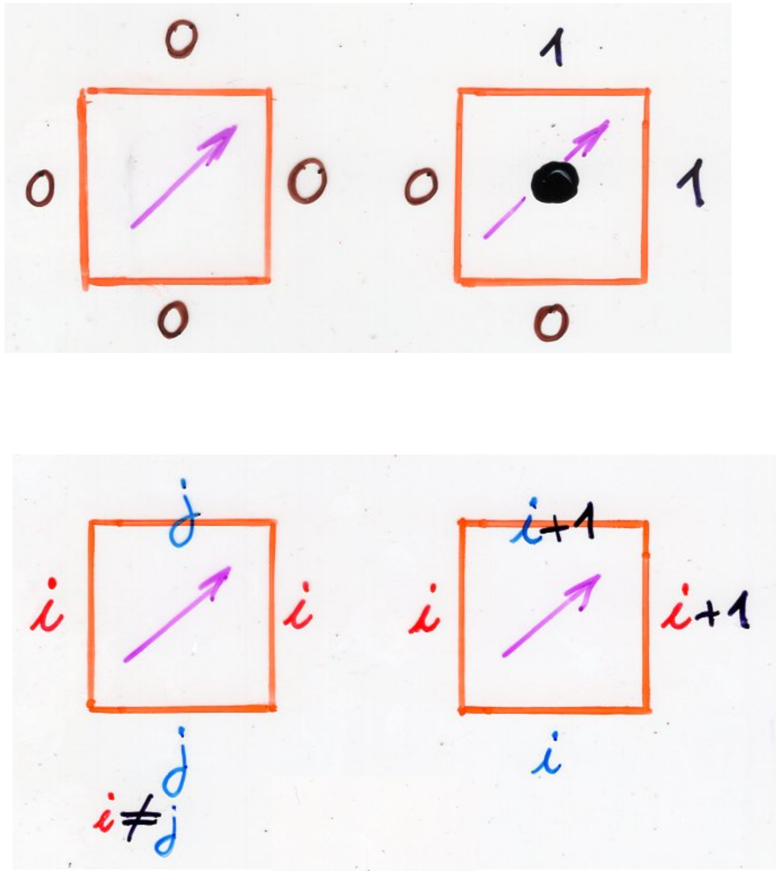
"planar rewriting"

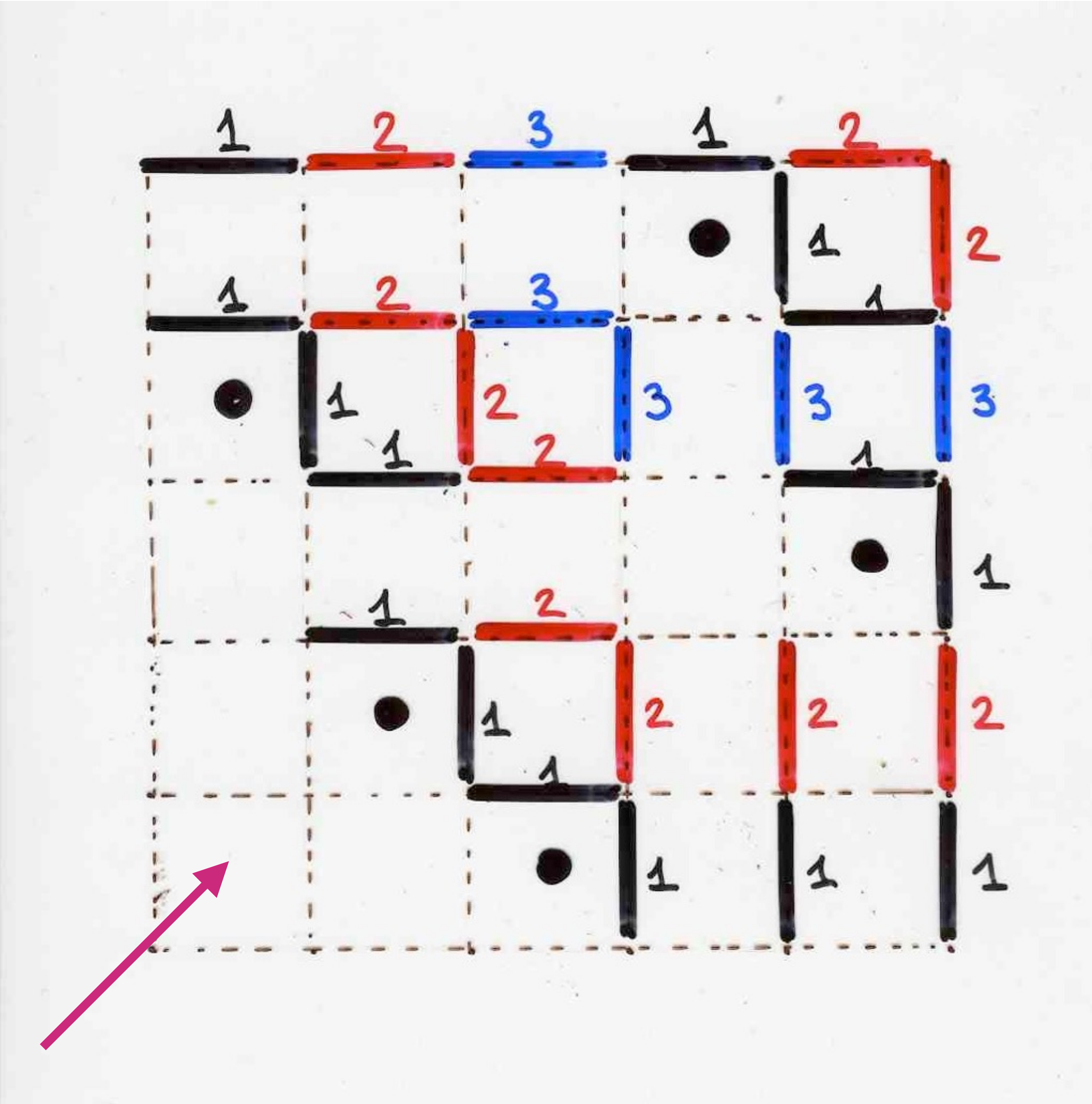


"local rules" on the vertices

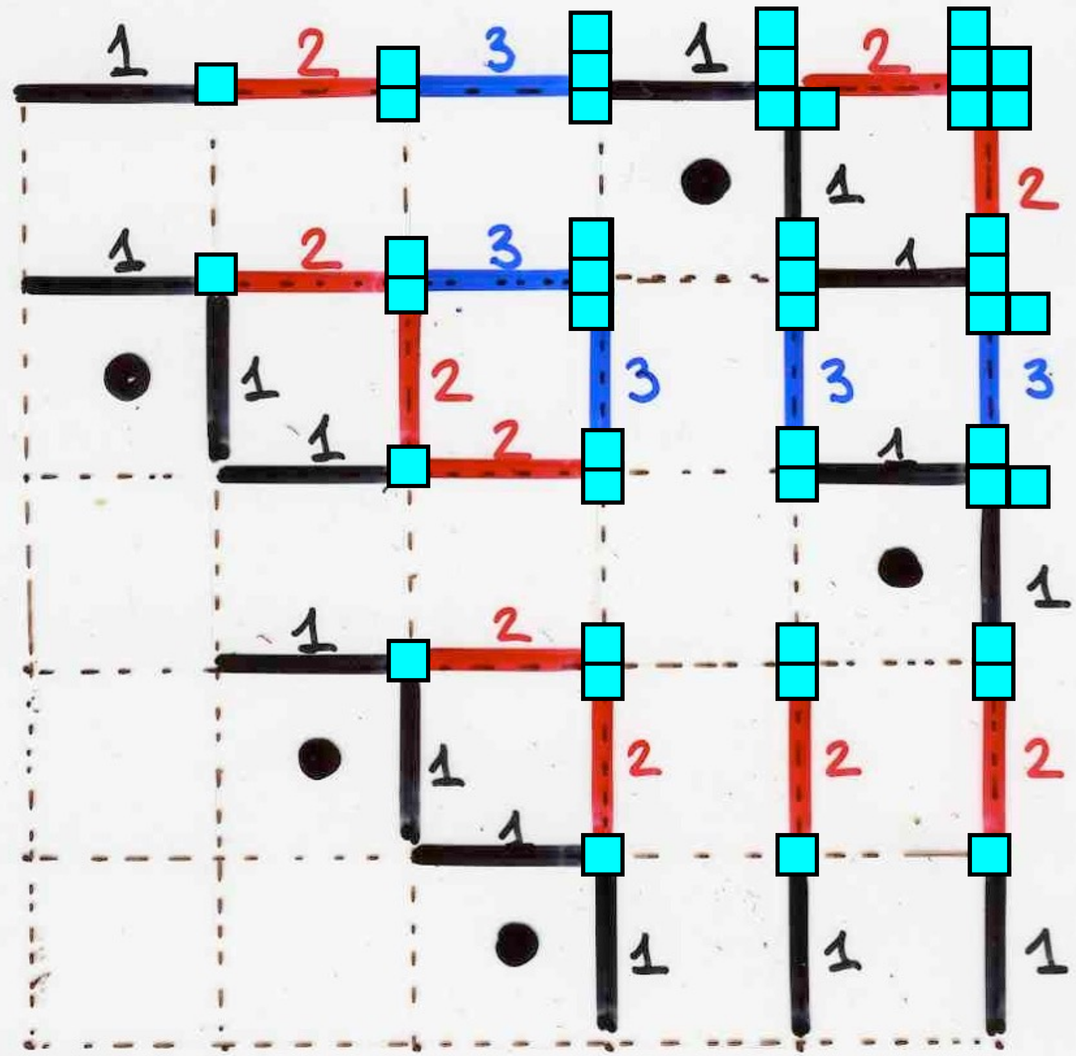


"local rules" on the edges





1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1



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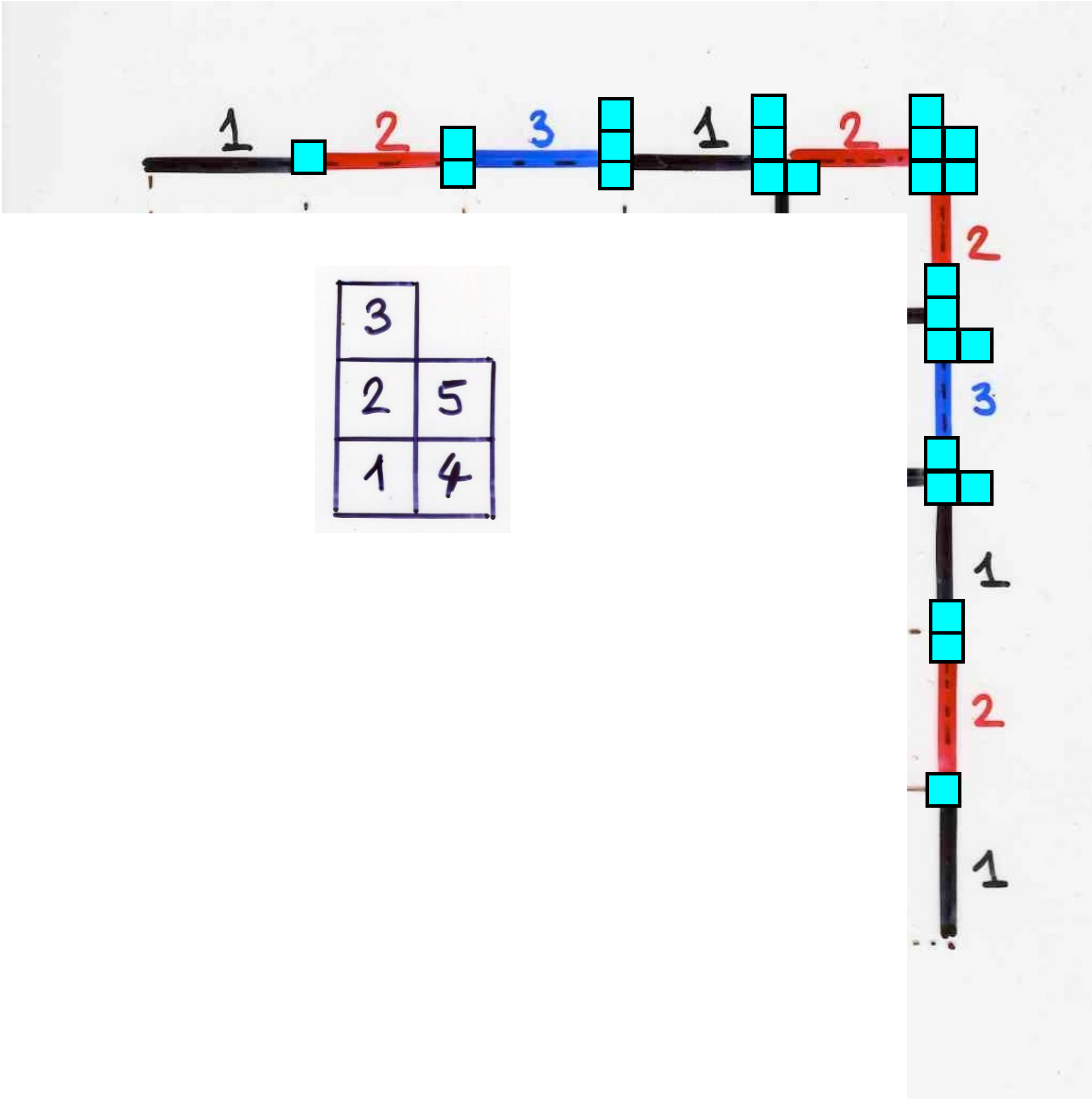
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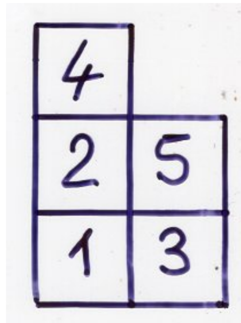
4
2
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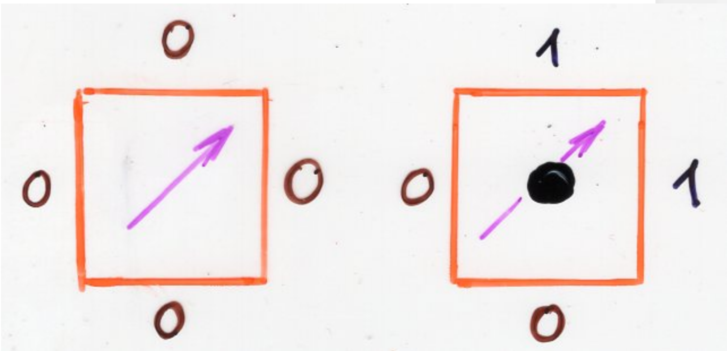
1
3

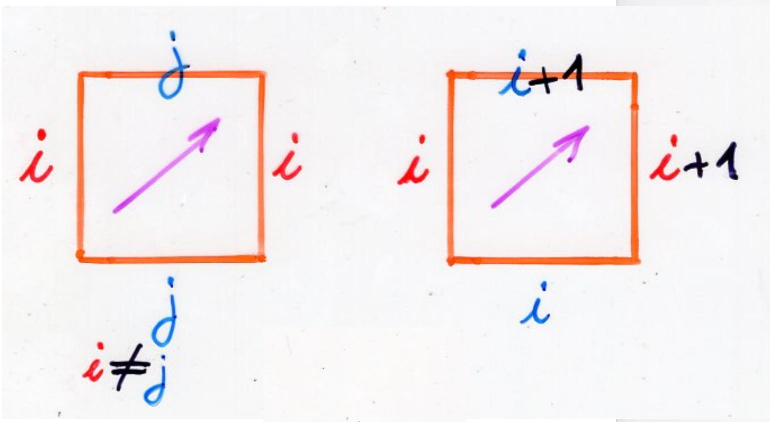
I claim that much attention should be given to the « local rules on edges » rather that « local rules on vertices ».

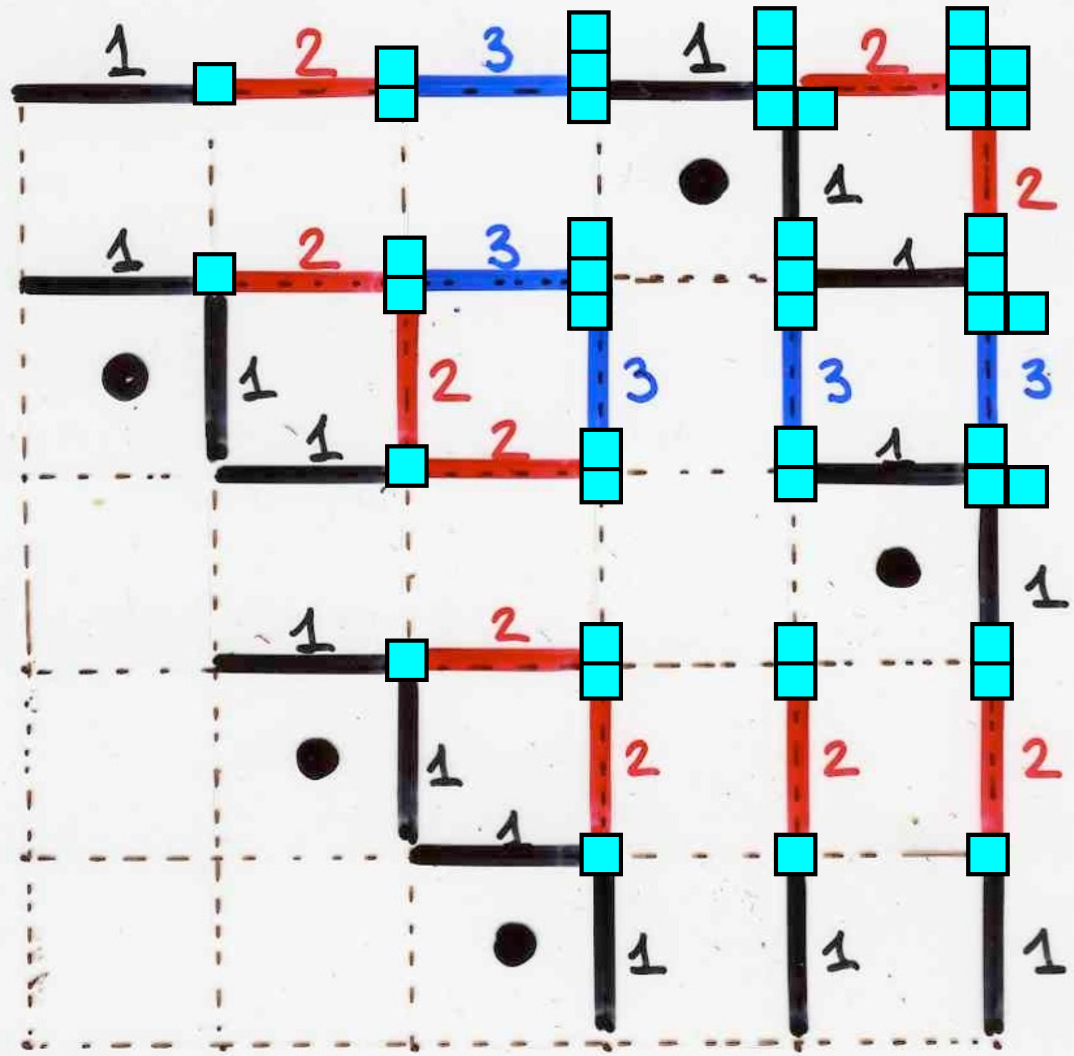
This is part of the philosophy of the « cellular ansatz »

The bilateral RSK planar automaton

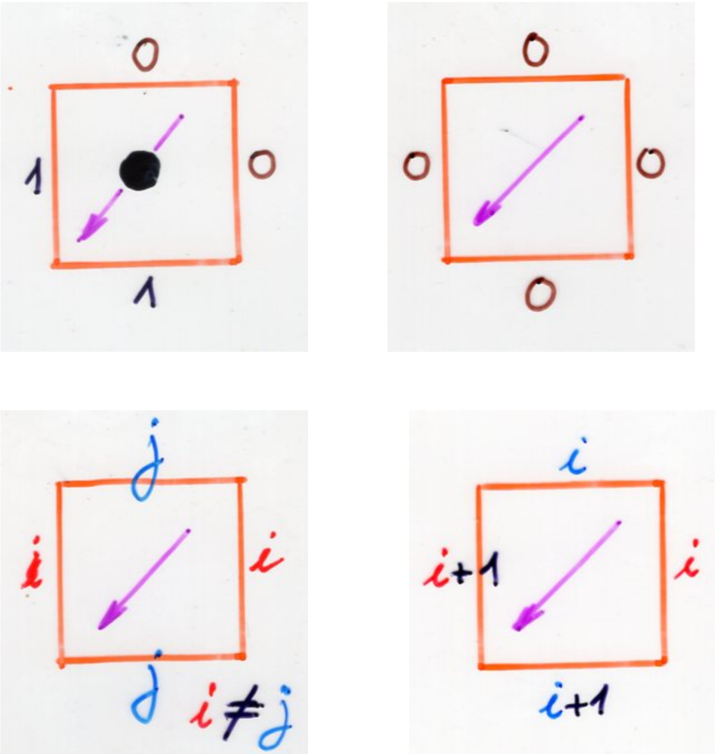
"local rules" on the edges

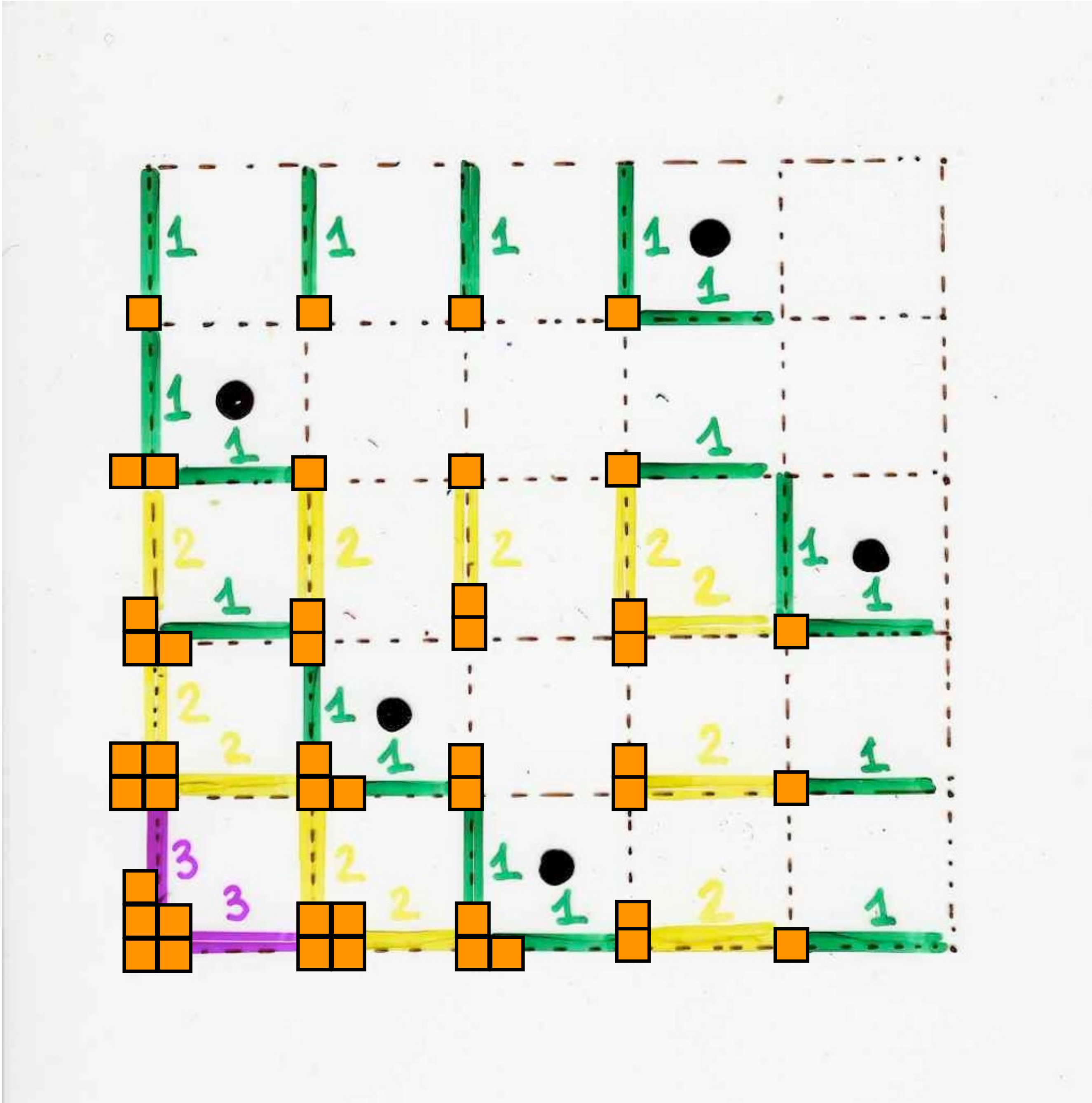




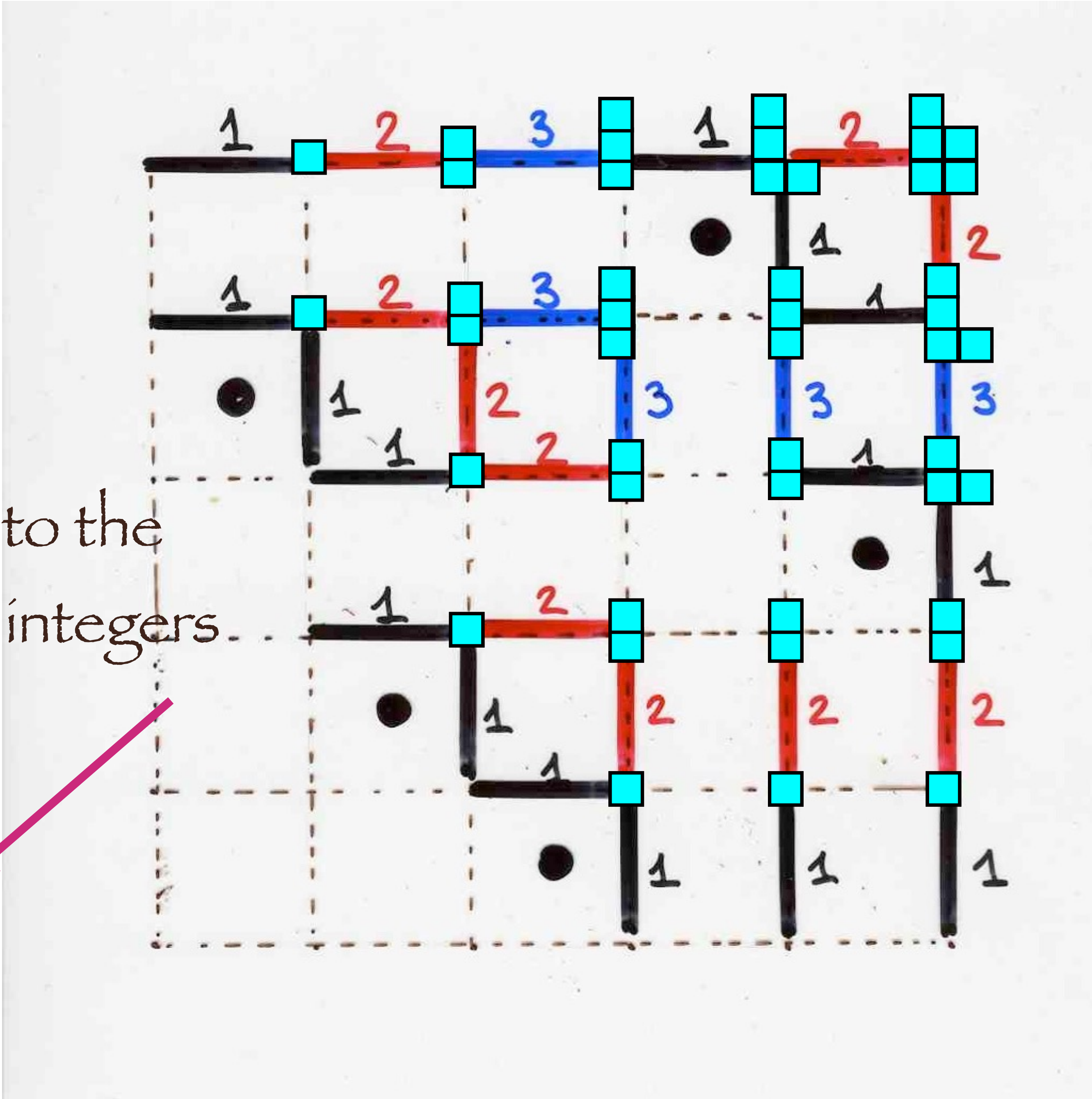


"local rules" on the edges



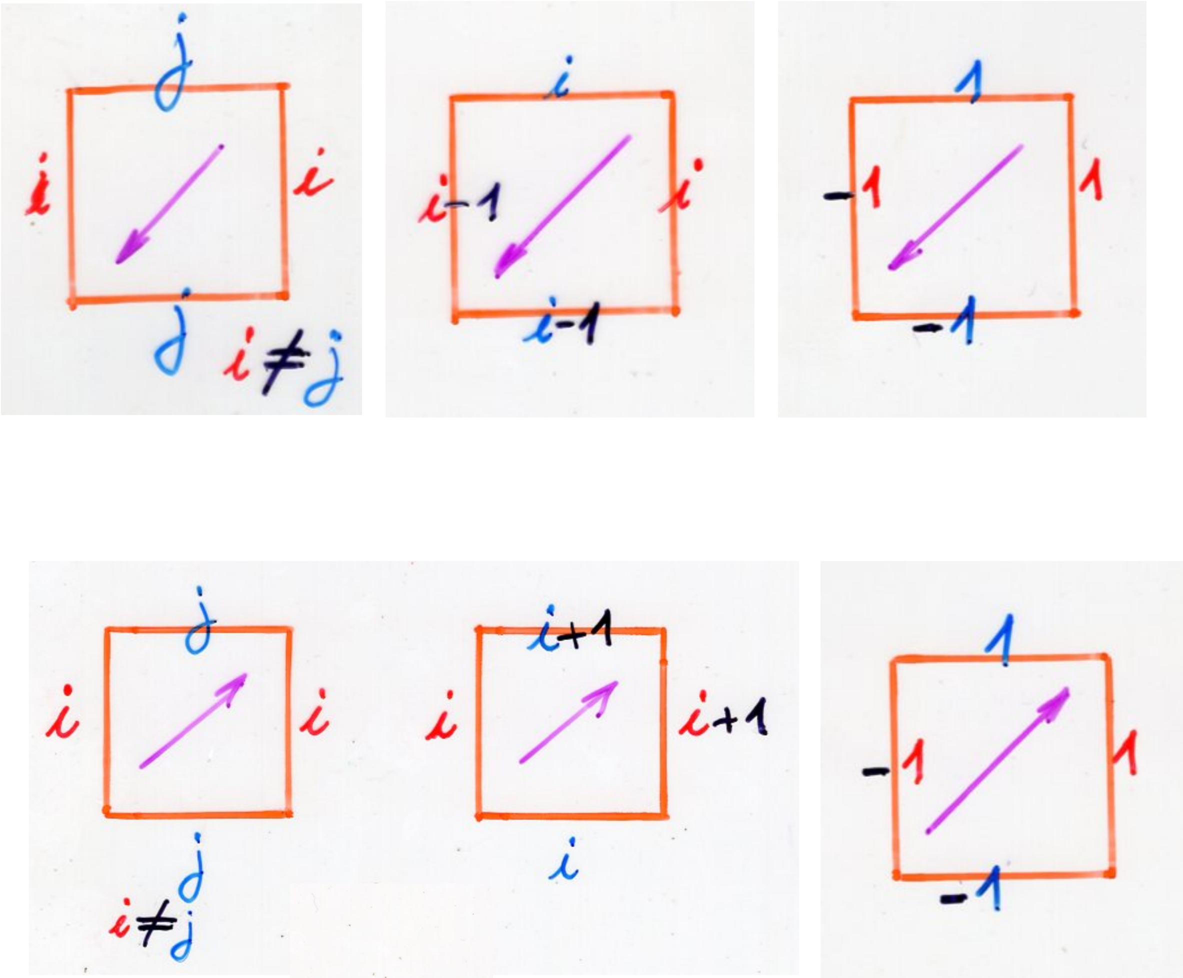


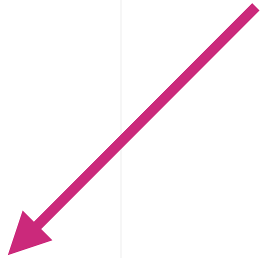
Going to the negative integers



bilateral

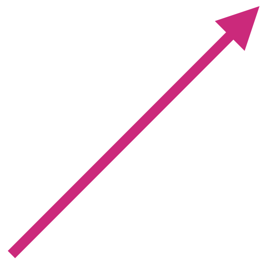
planar automaton RSK

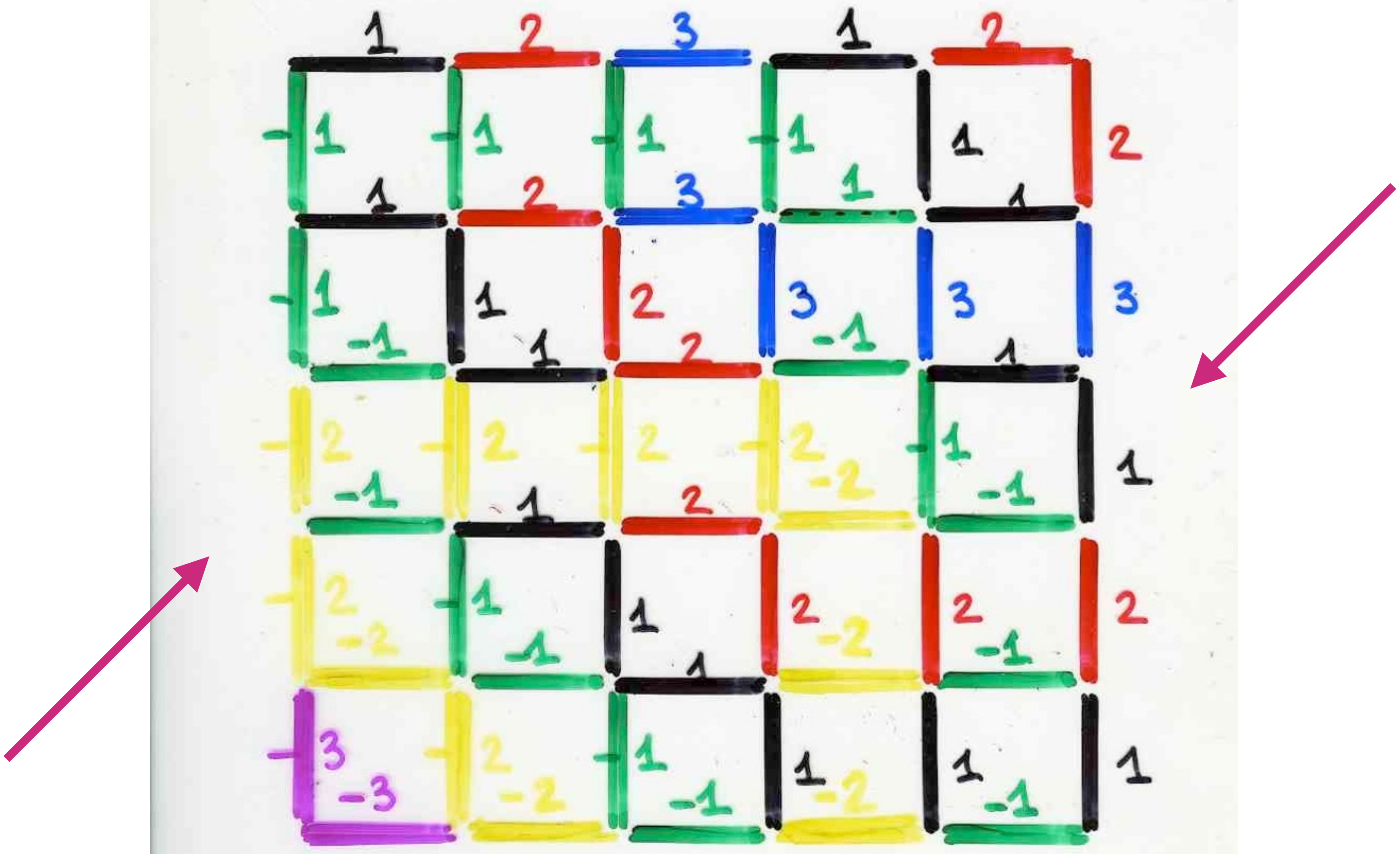


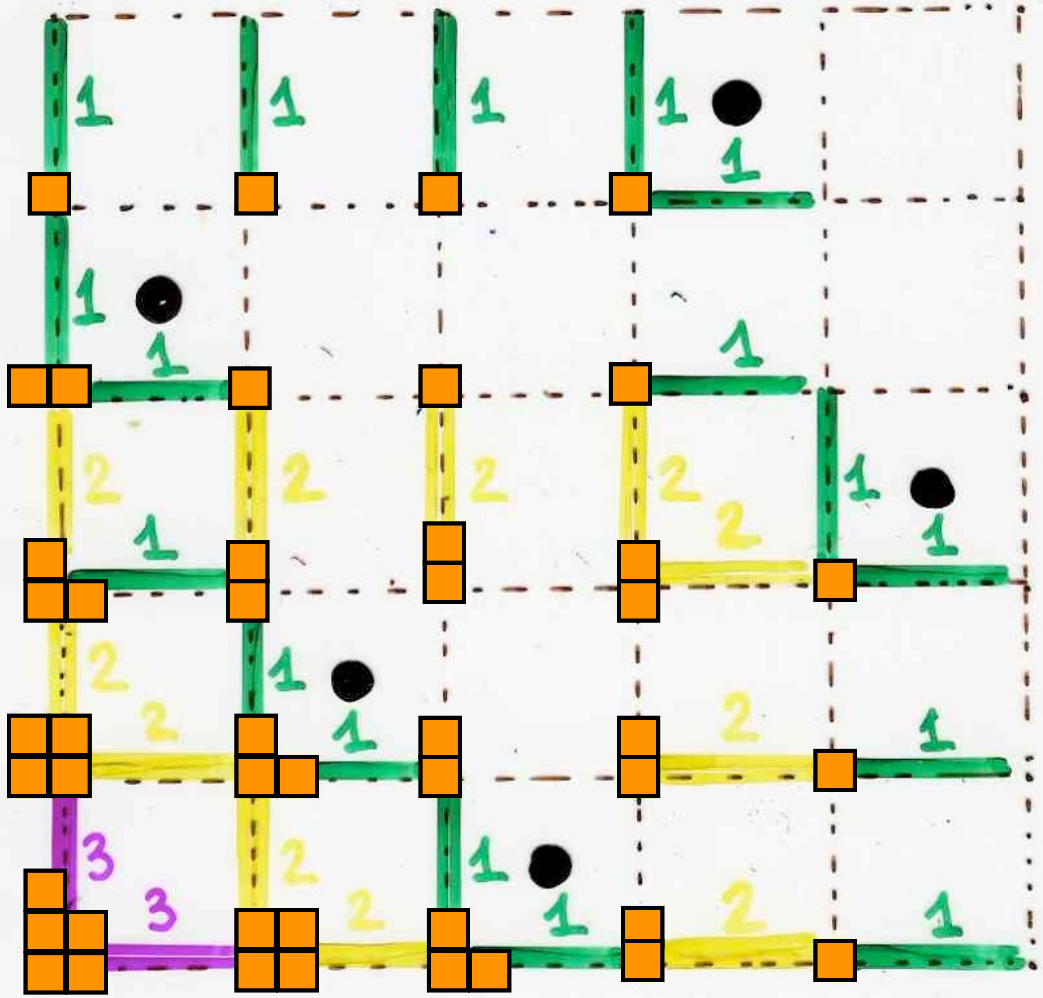


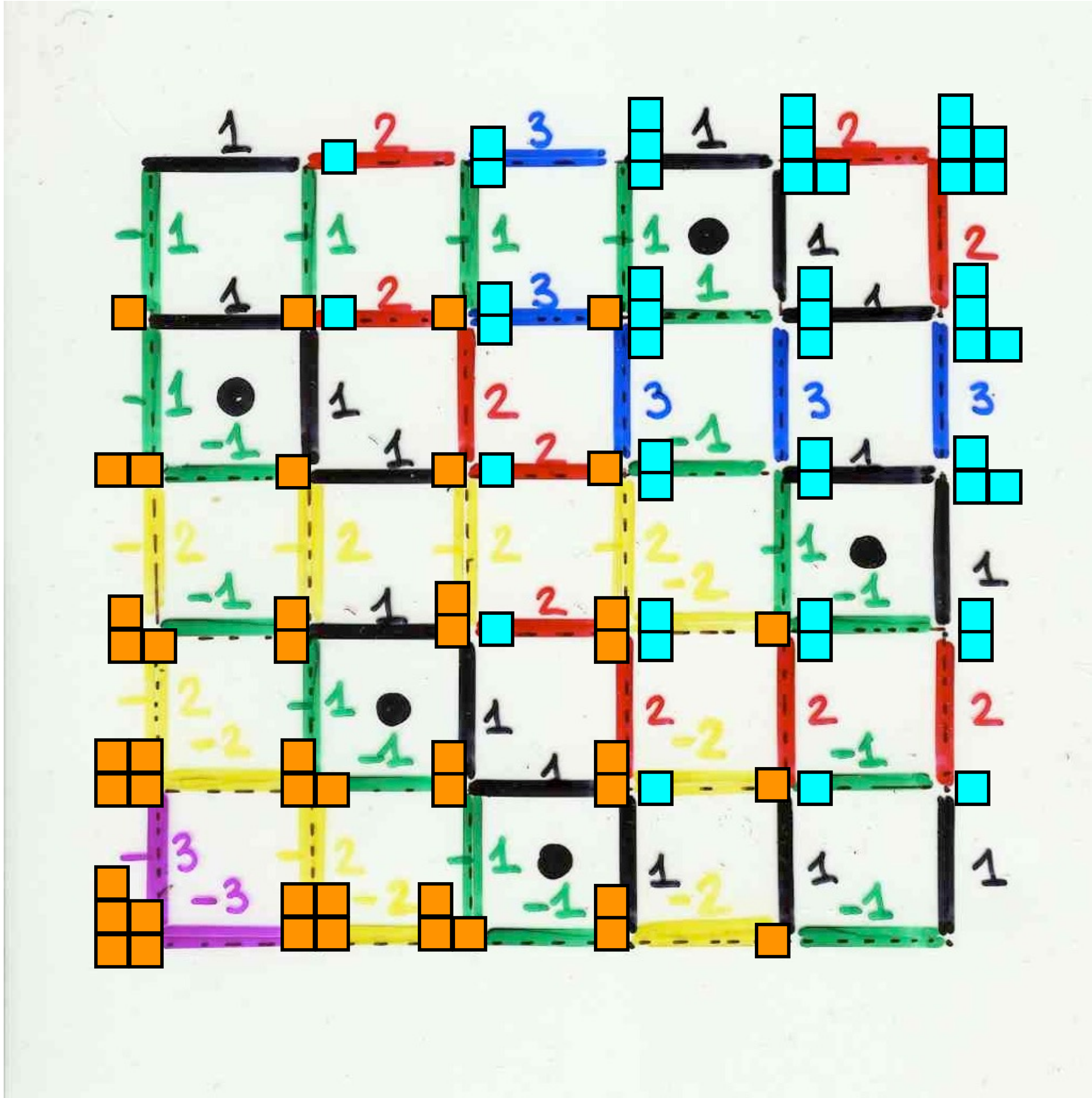
$i \in \mathbb{Z} - \{0\}$

$j \in \mathbb{Z} - \{0\}$



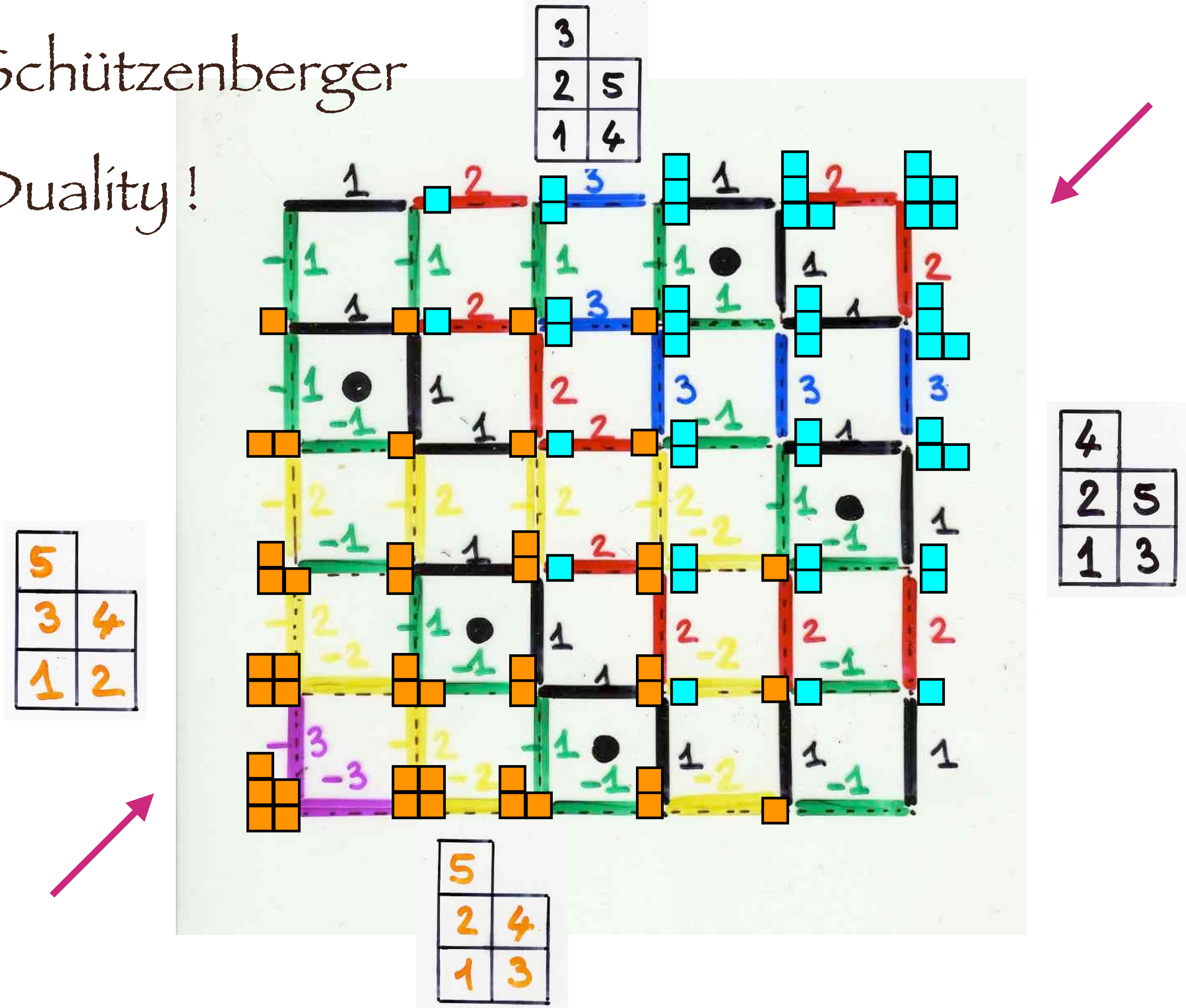






Schützenberger

Duality !

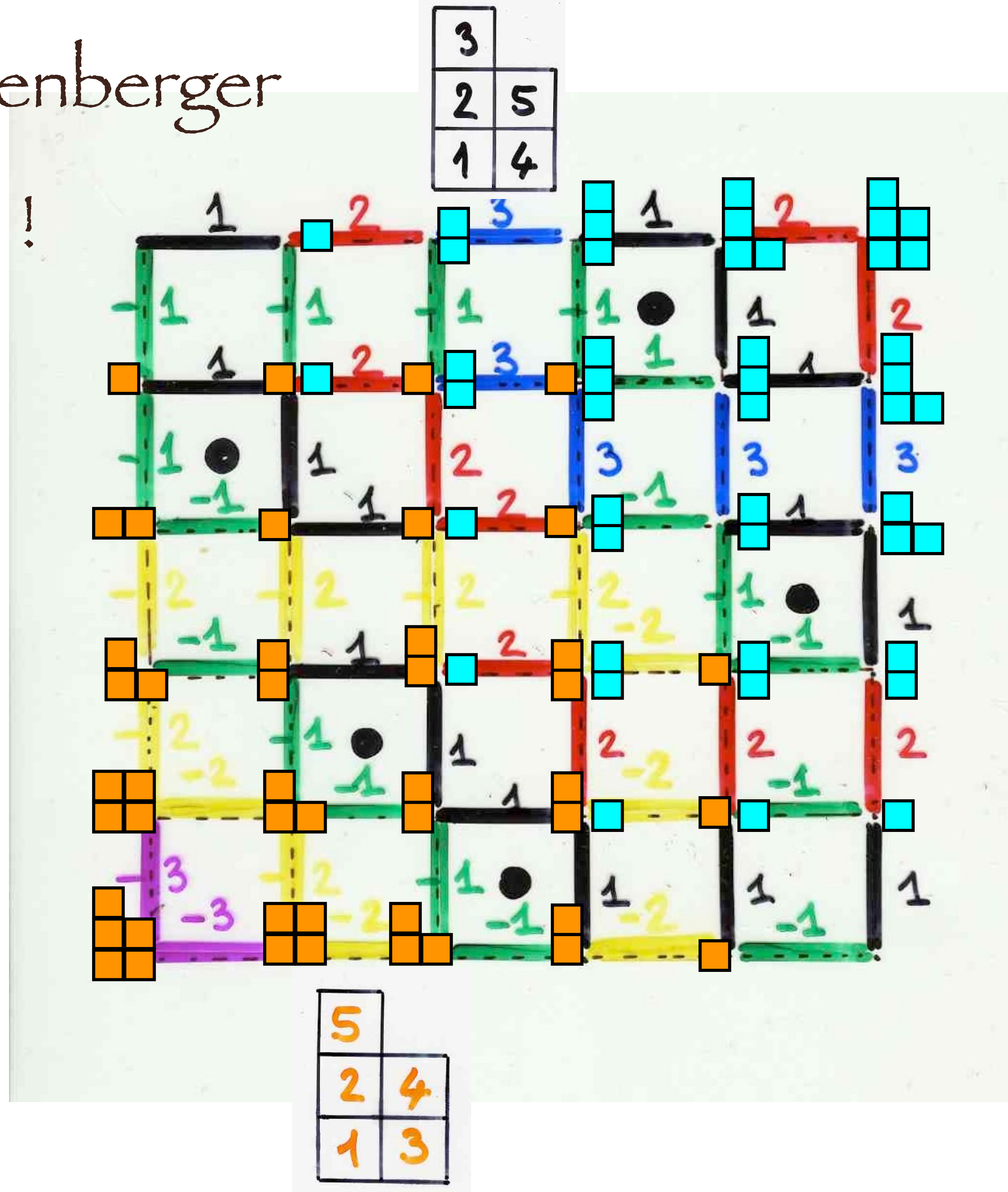


Schützenberger

Duality !

$P^{*} =$ dual

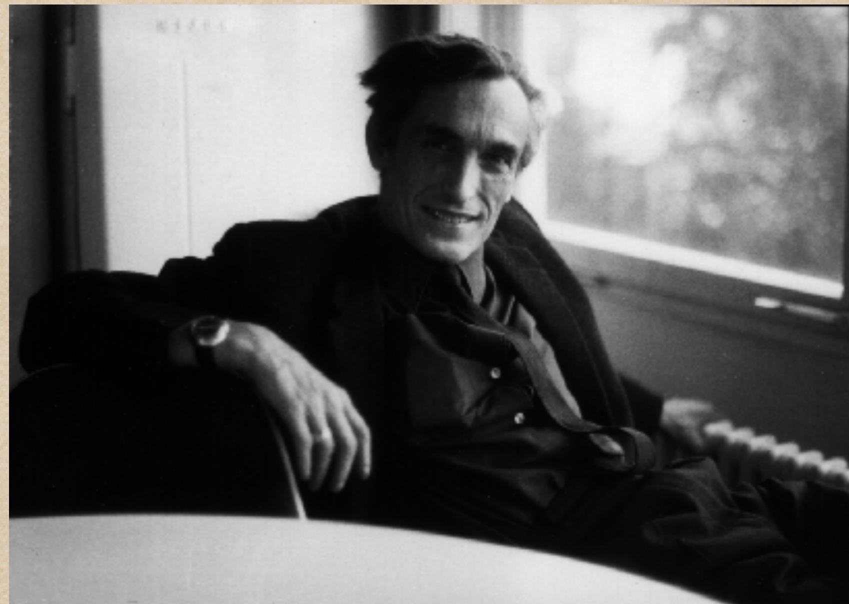
5	
3	4
1	2



$P =$

4	
2	5
1	3

dual of a Young tableau



M.P. Schützenberger

6	10			
3	5	8		
1	2	4	7	9

6	10			
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	2	4	7	9

6	10			
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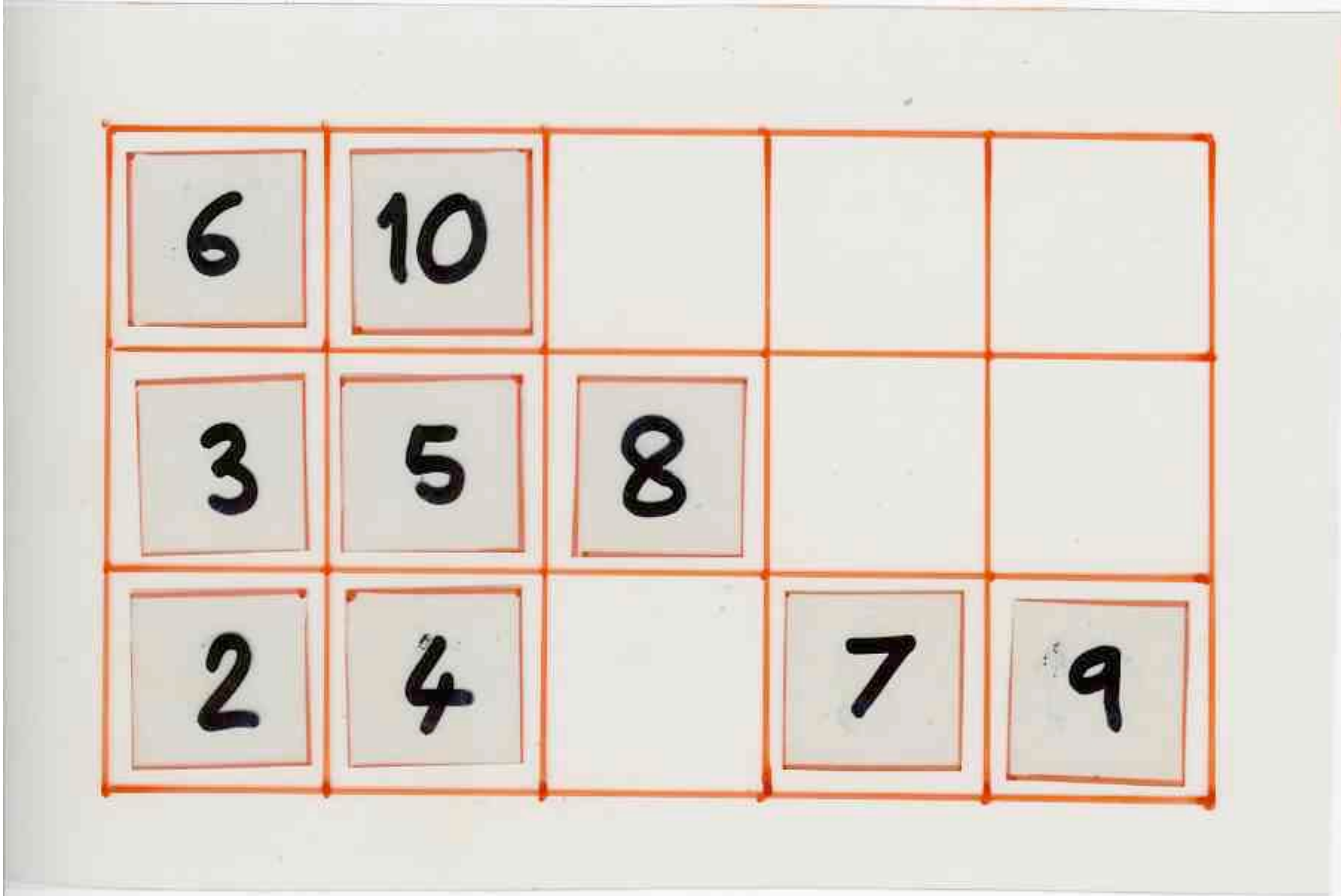
6 10
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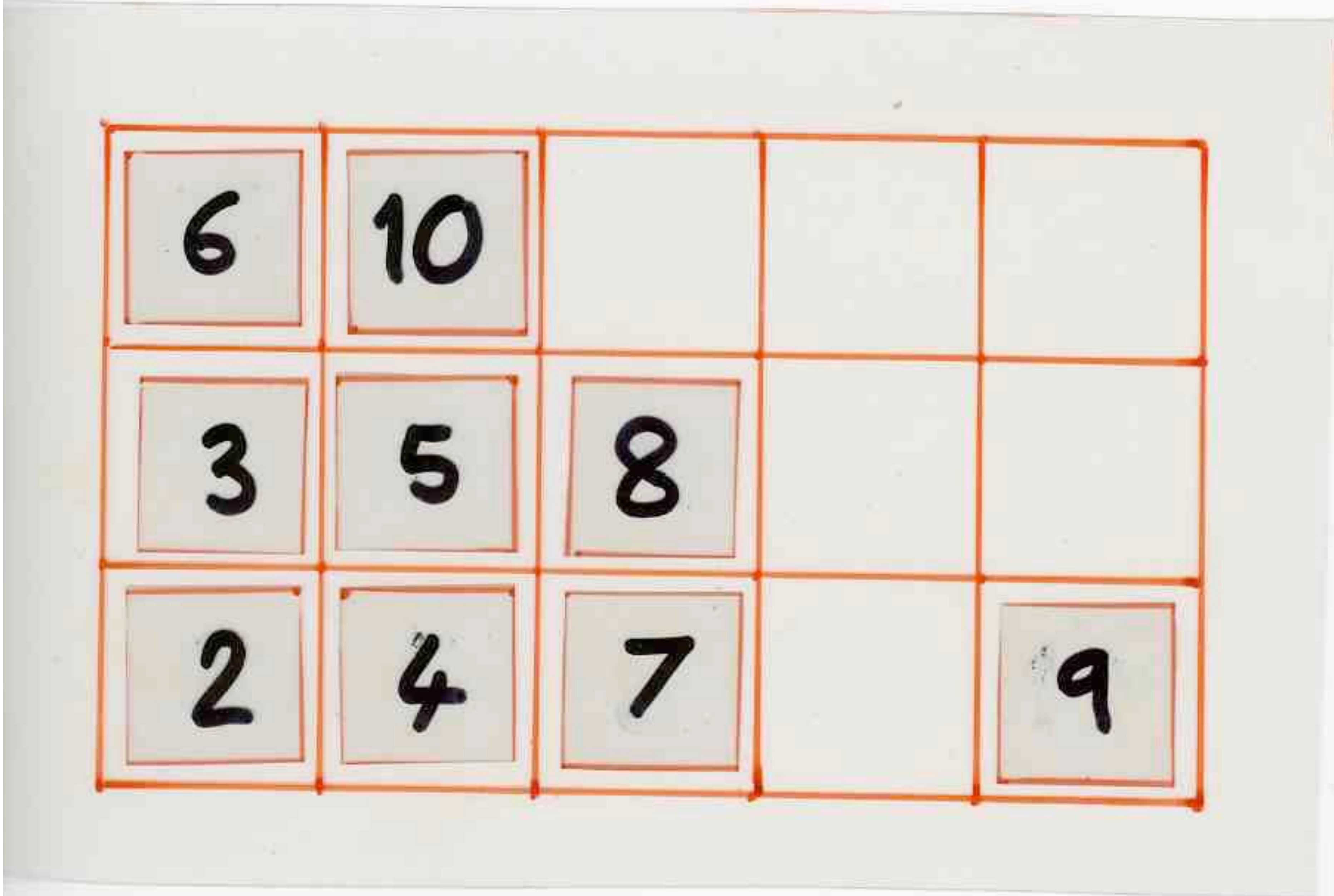
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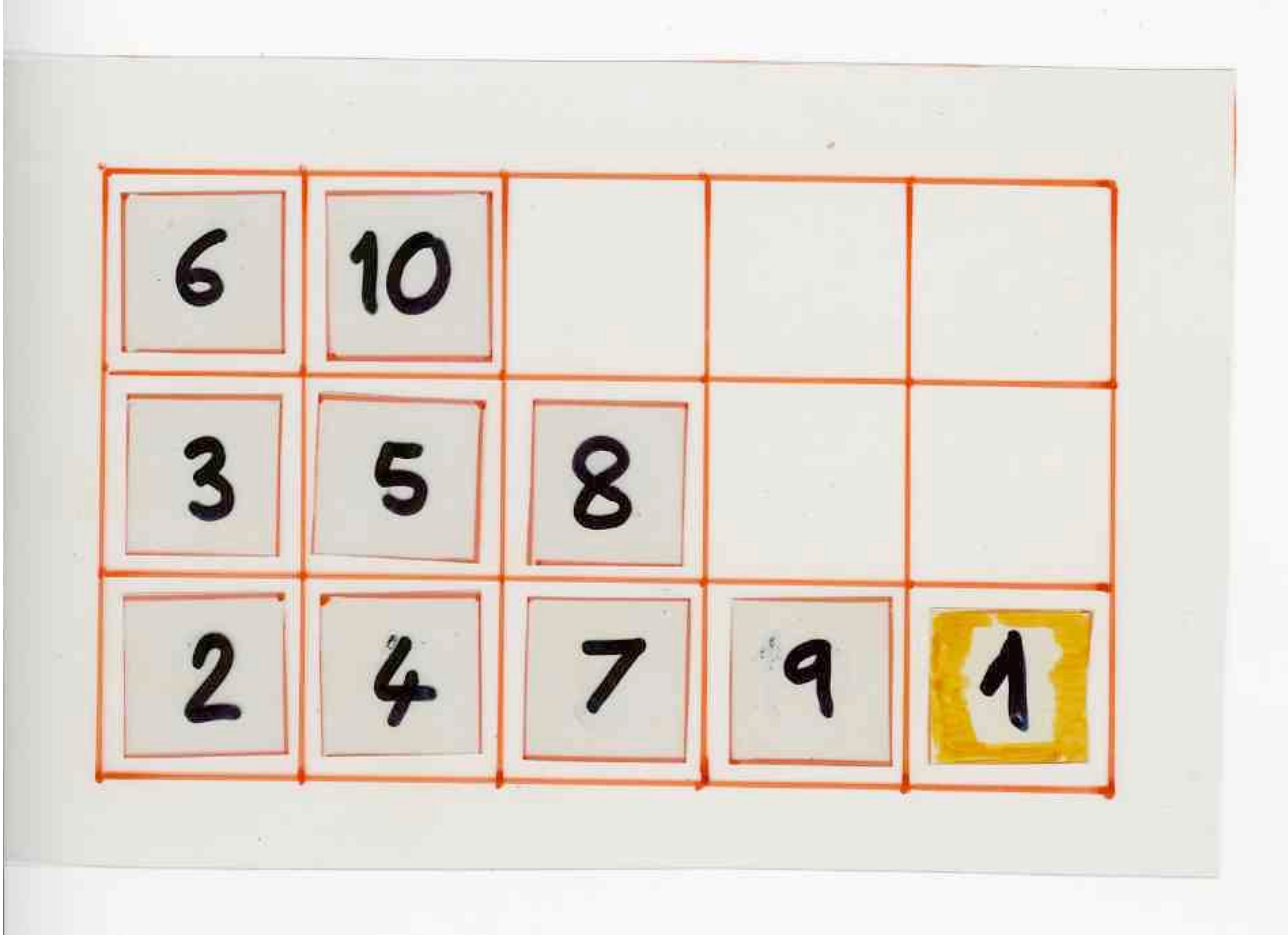
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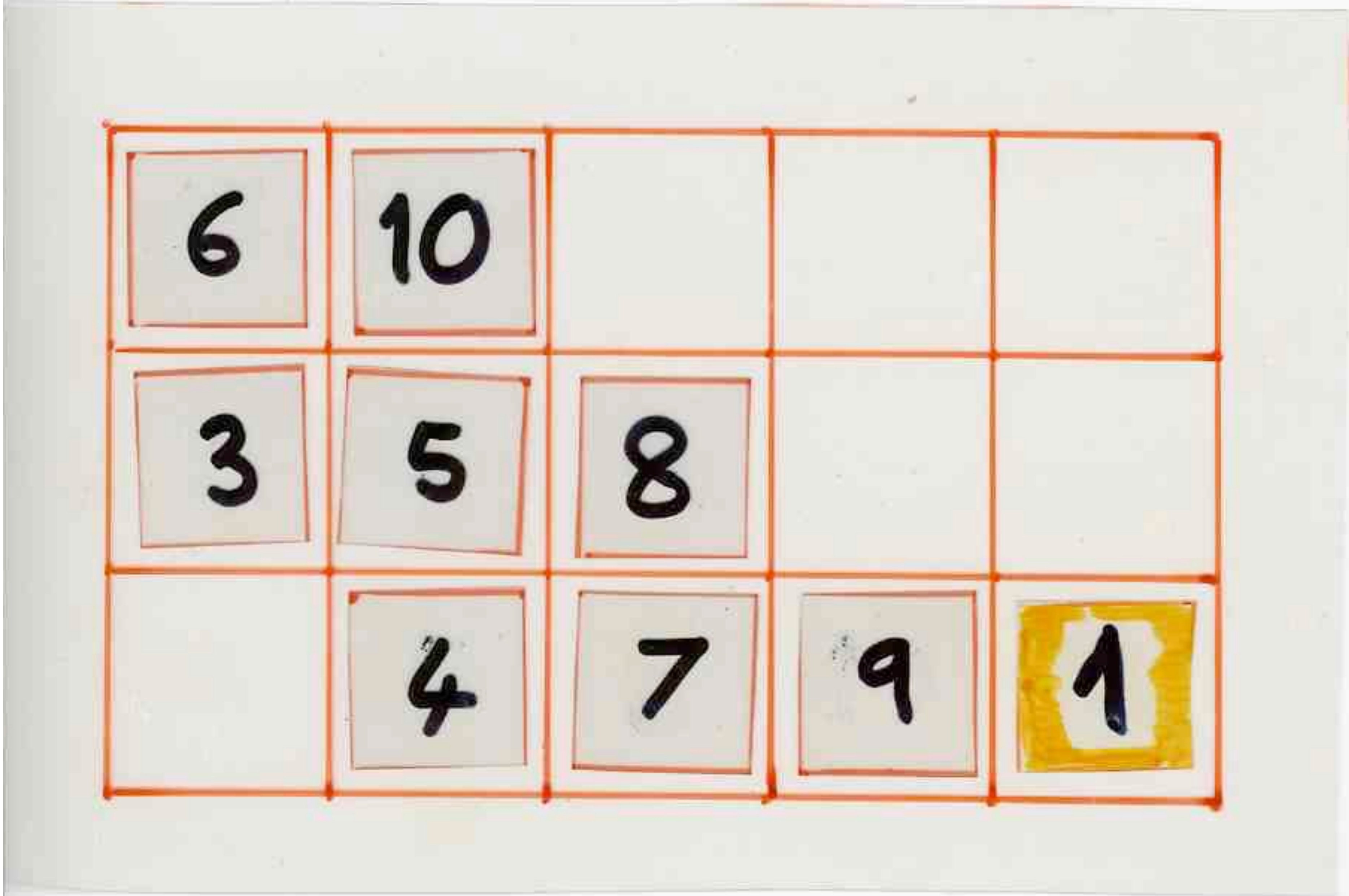
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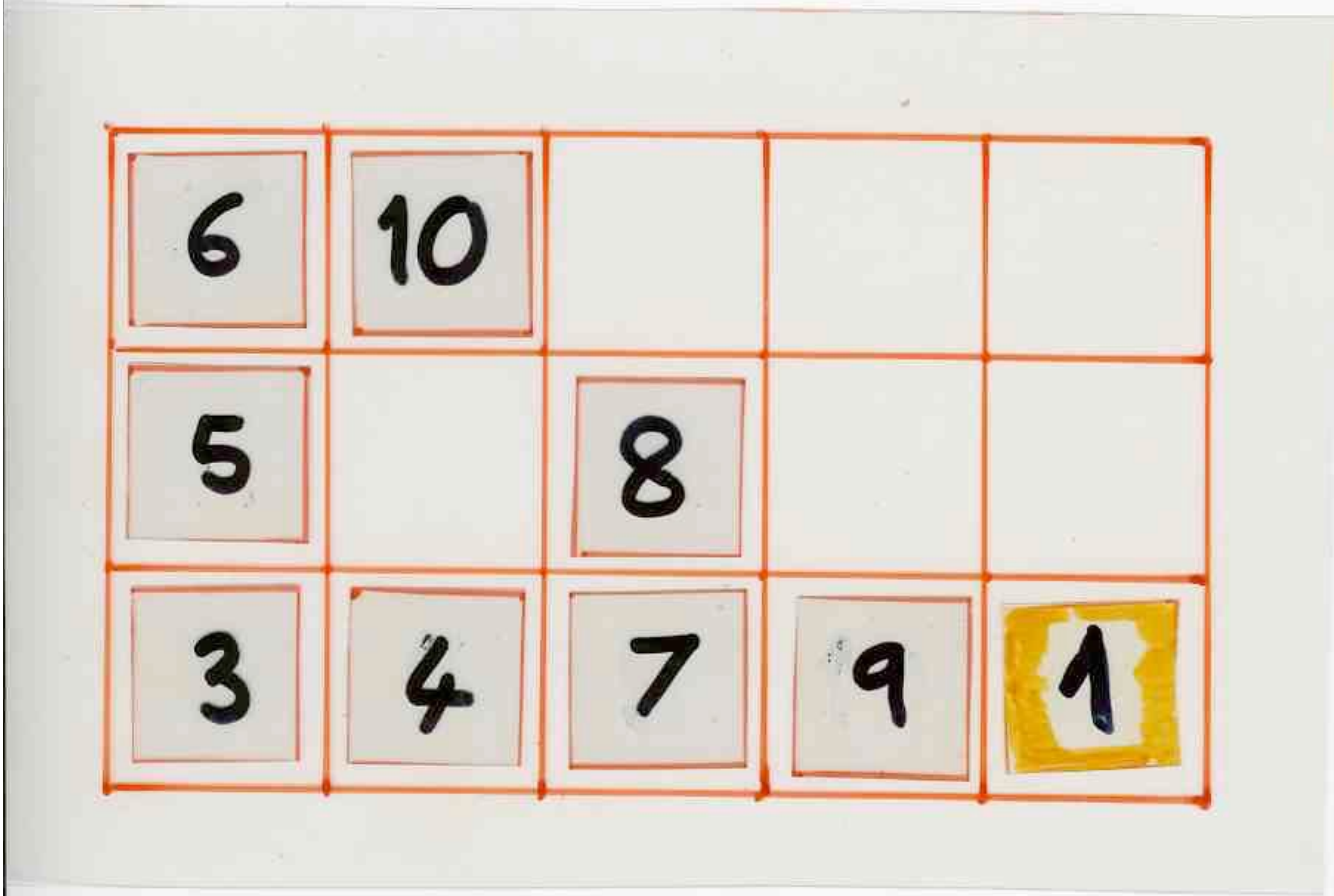
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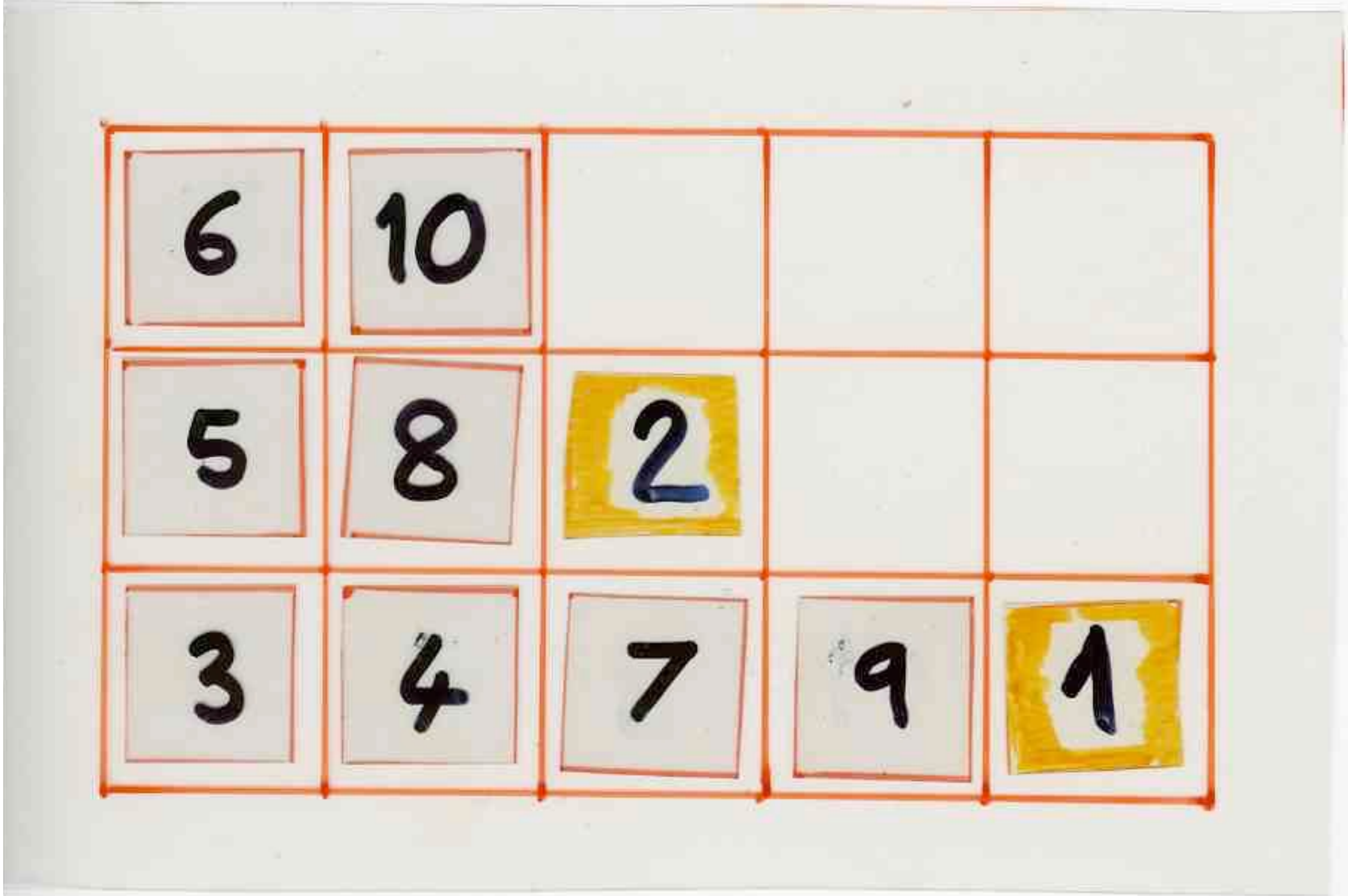
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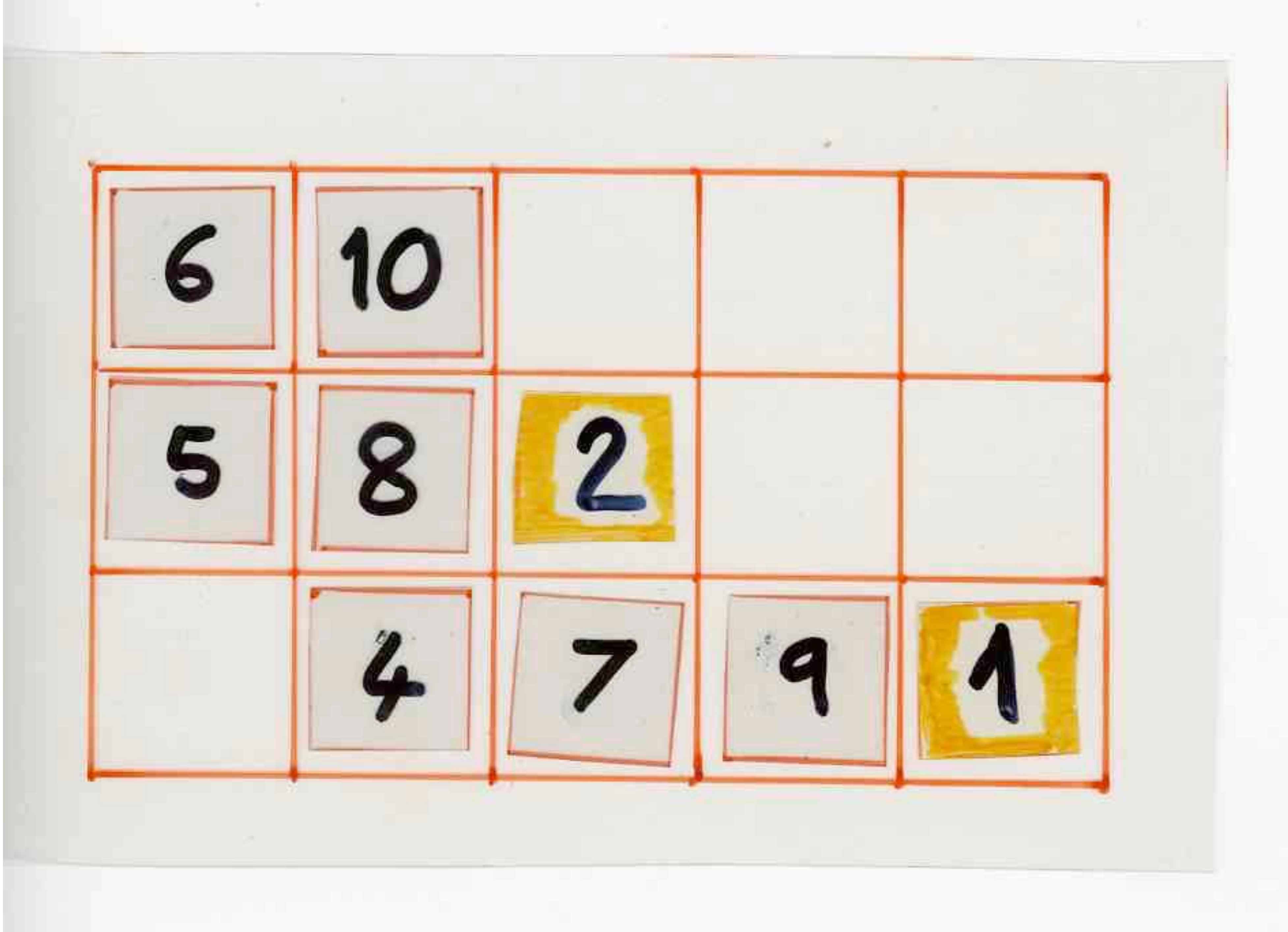
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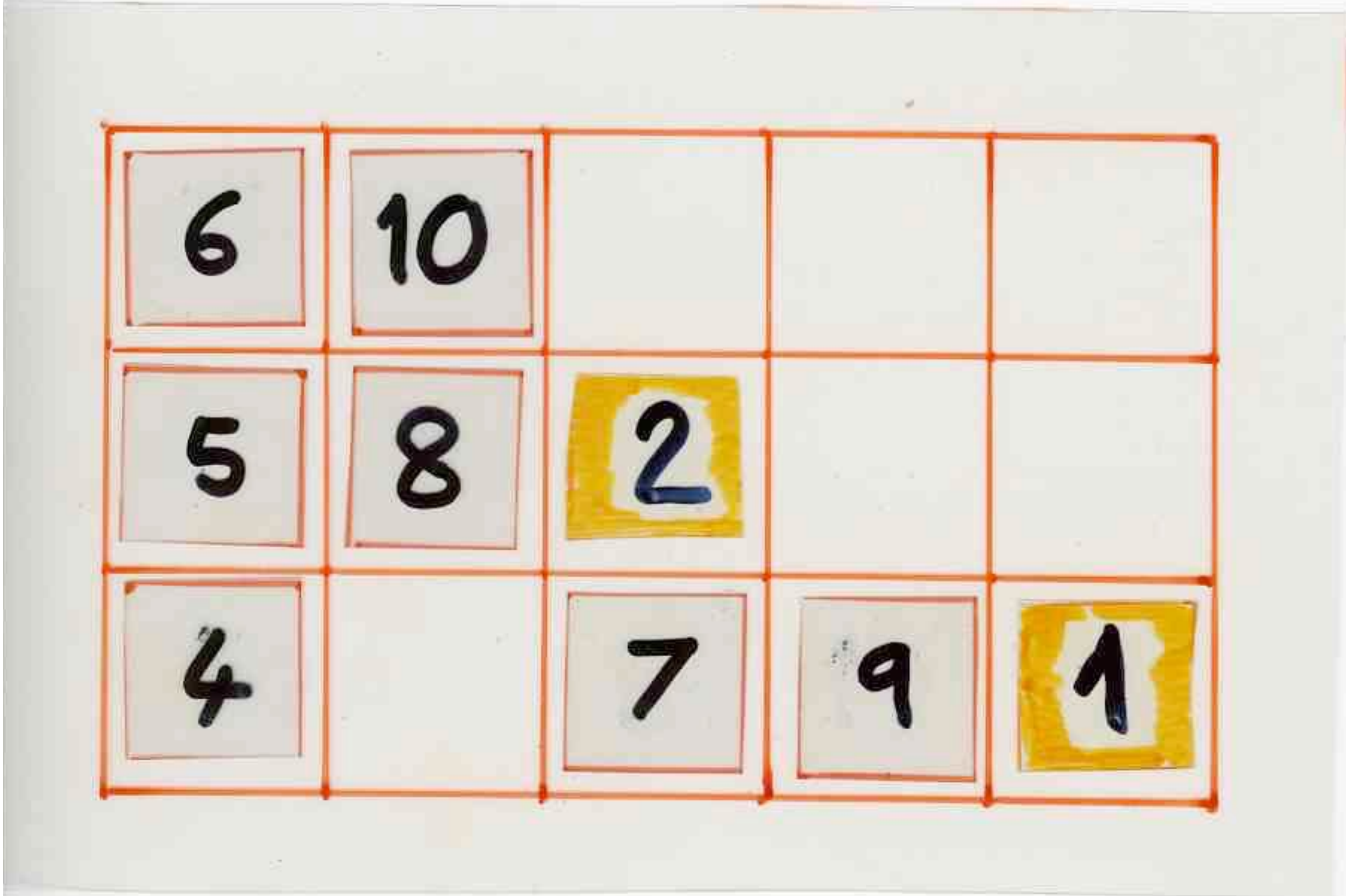
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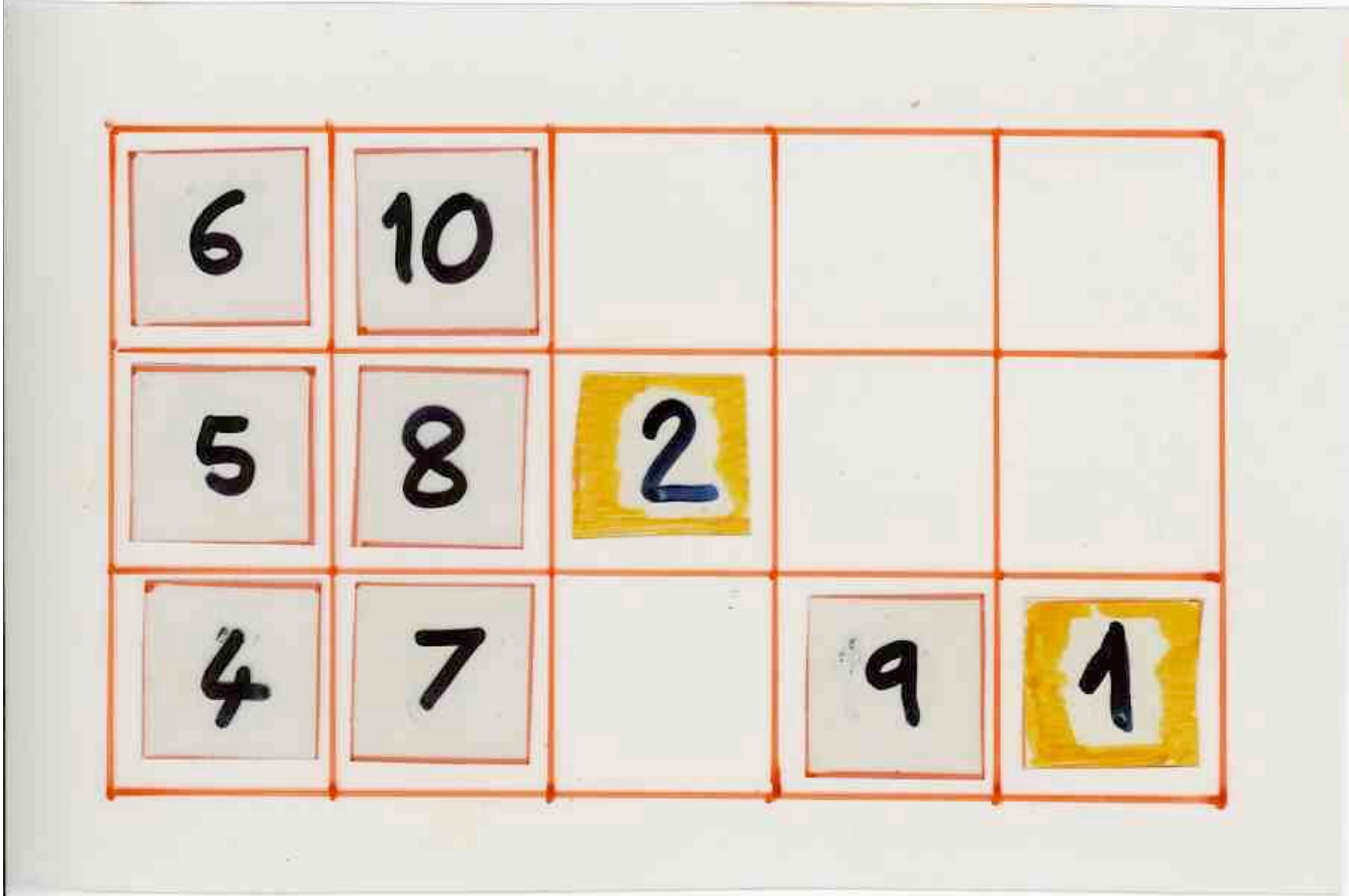
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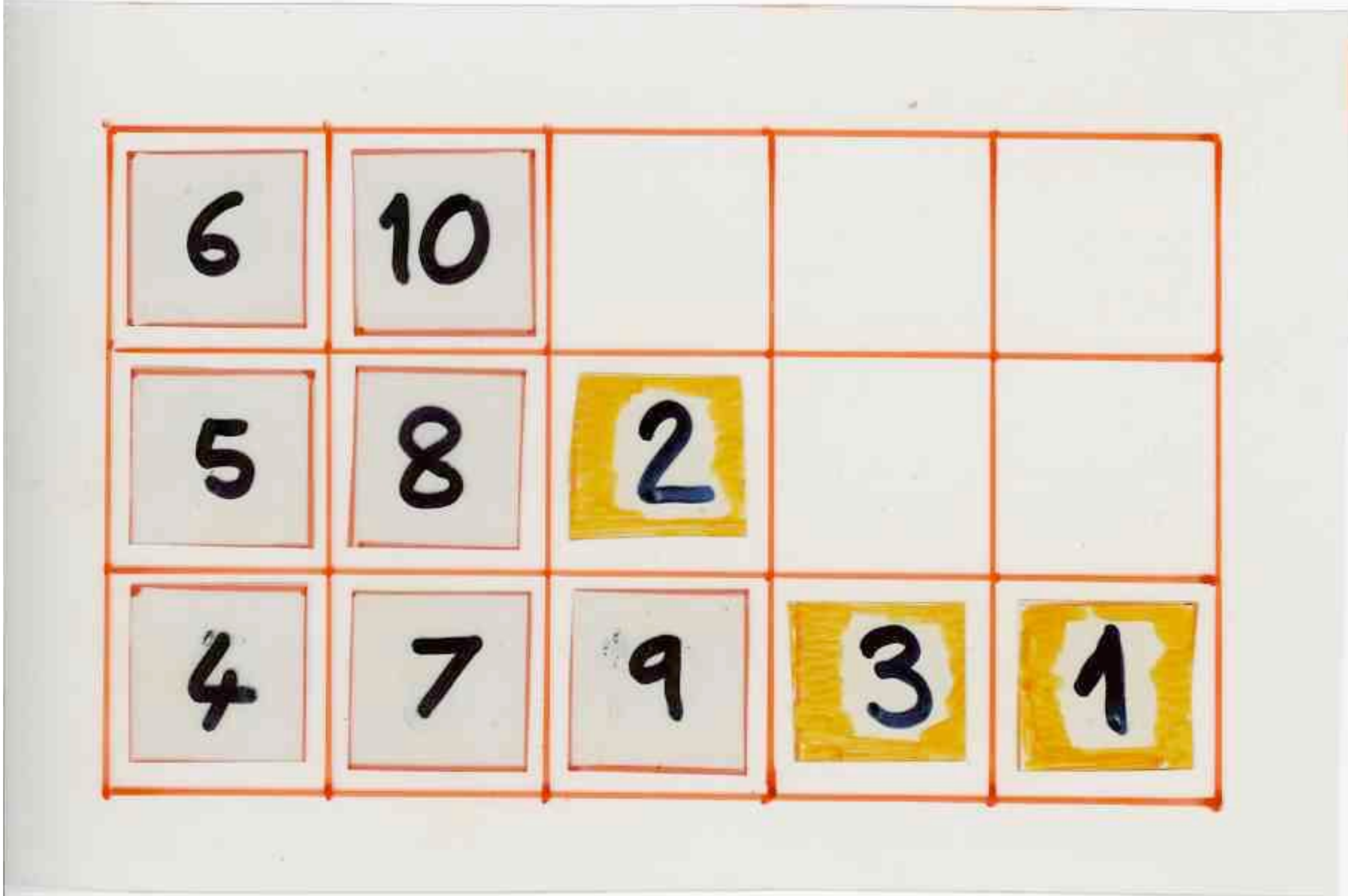
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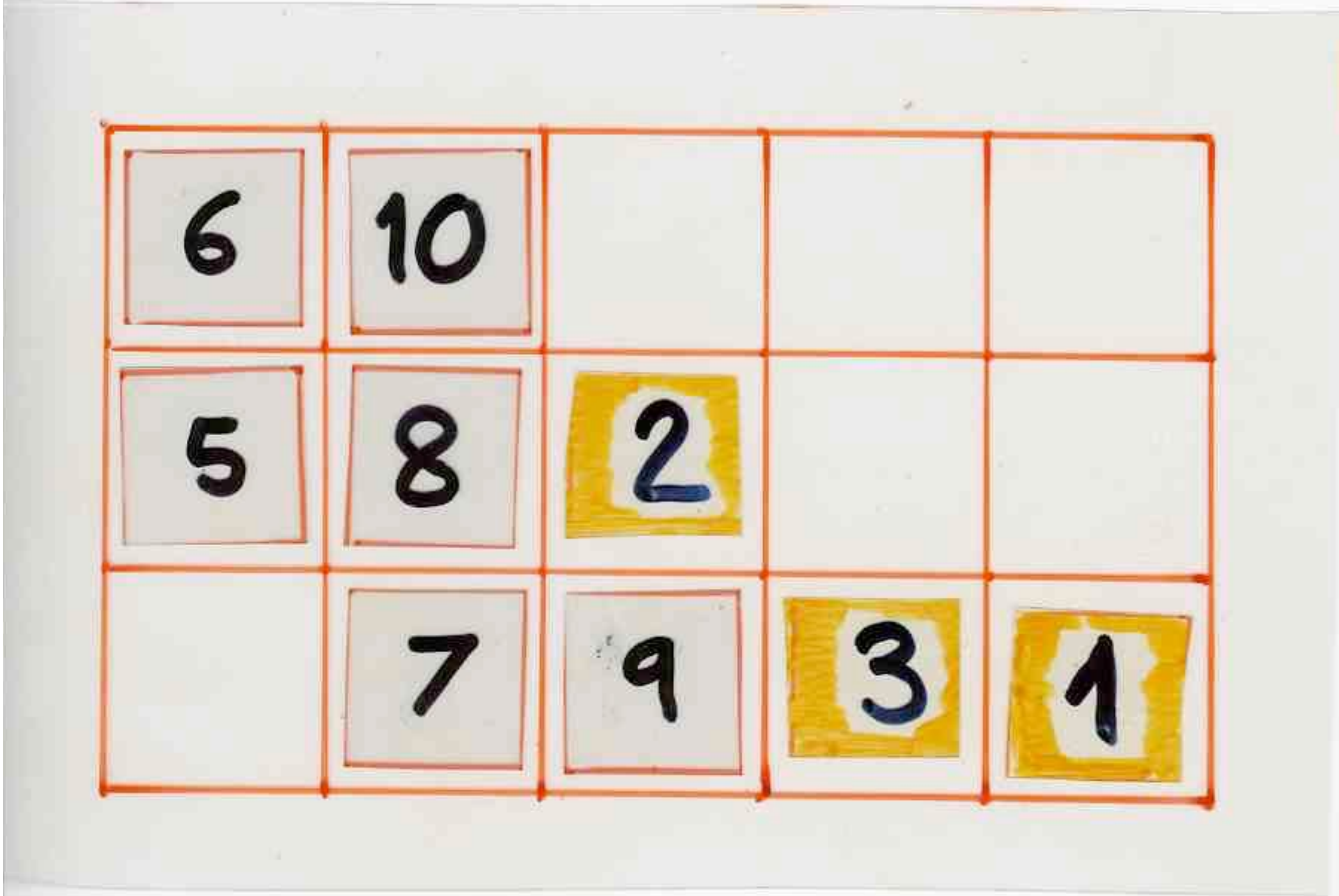
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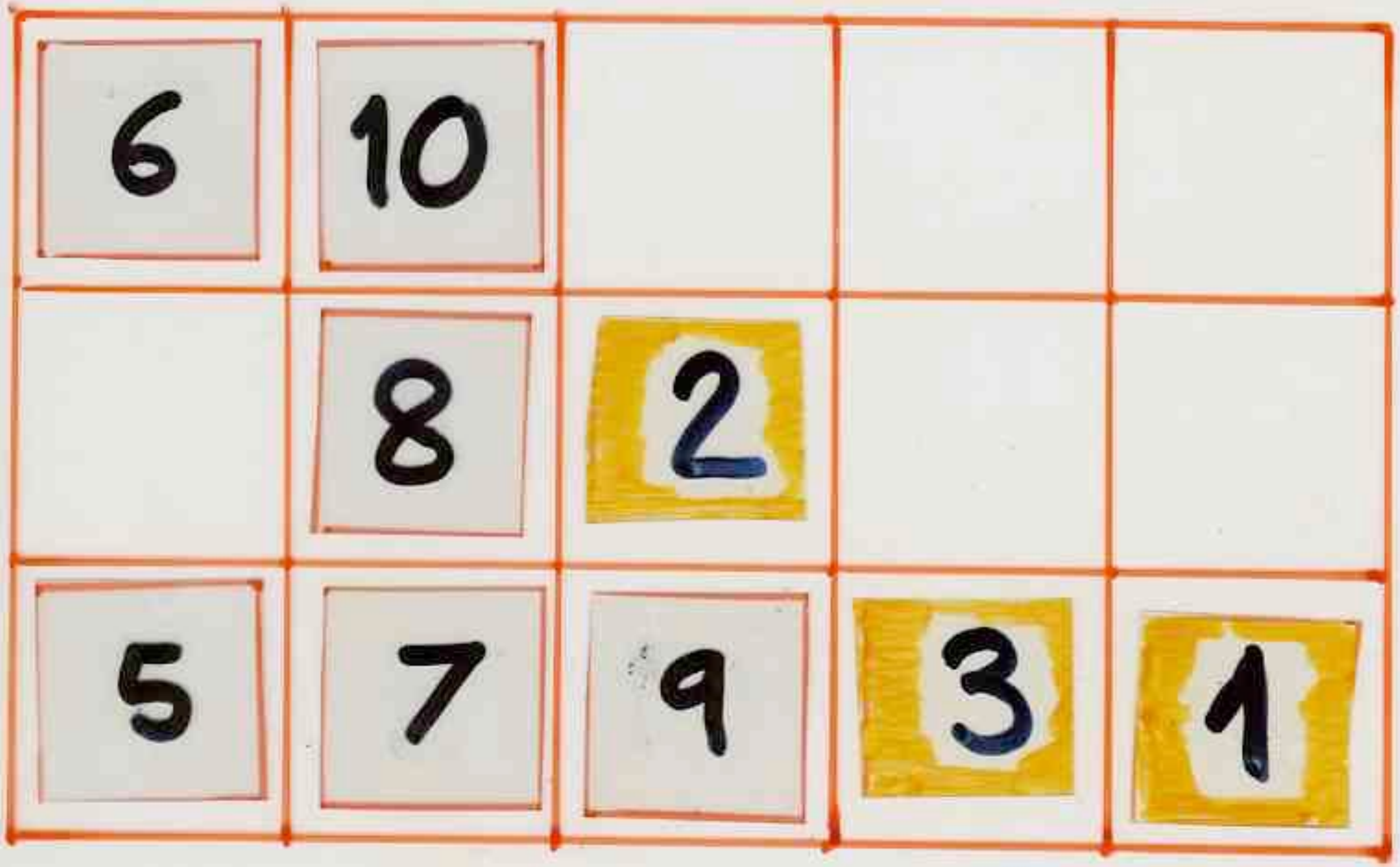
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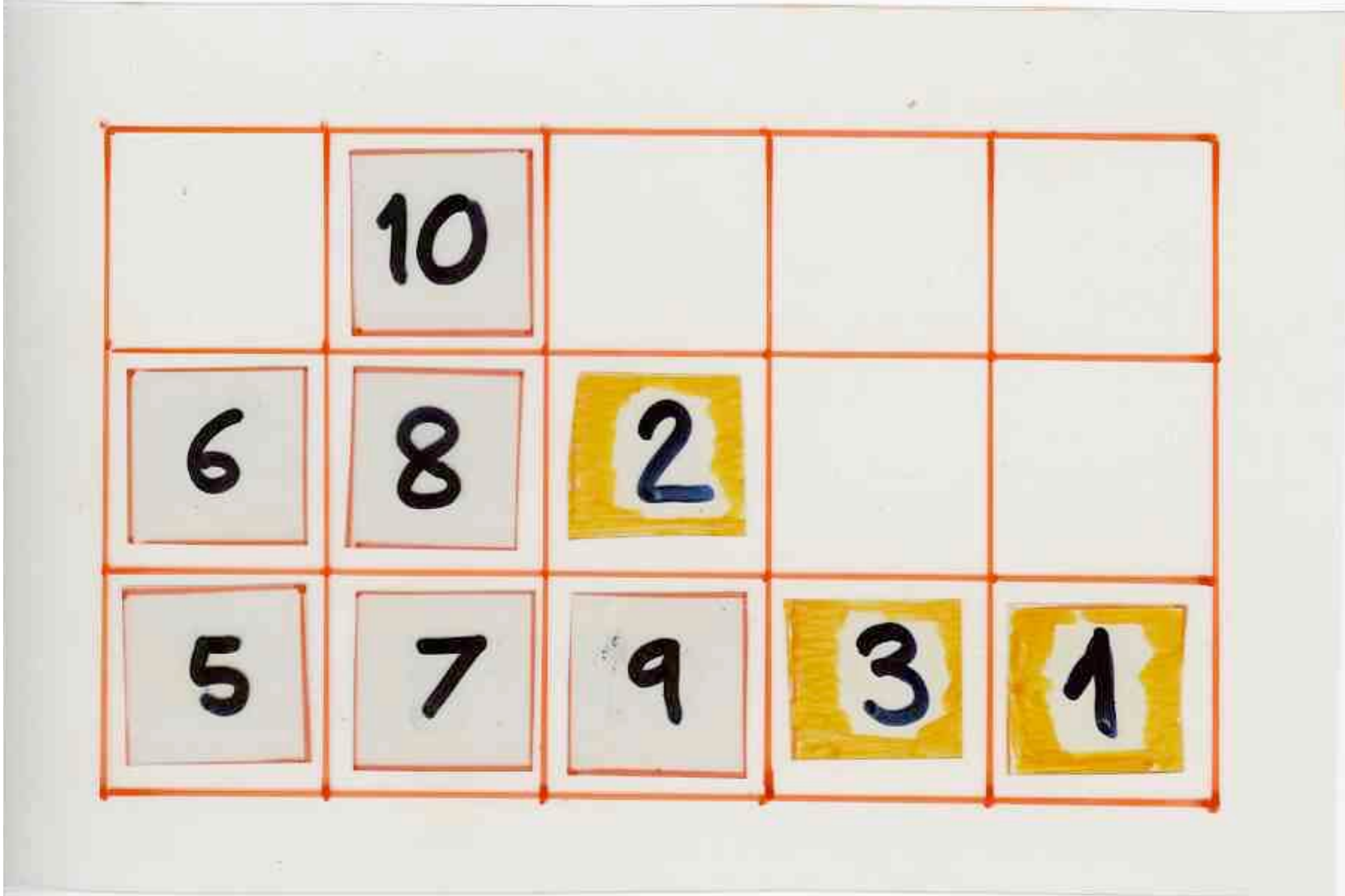
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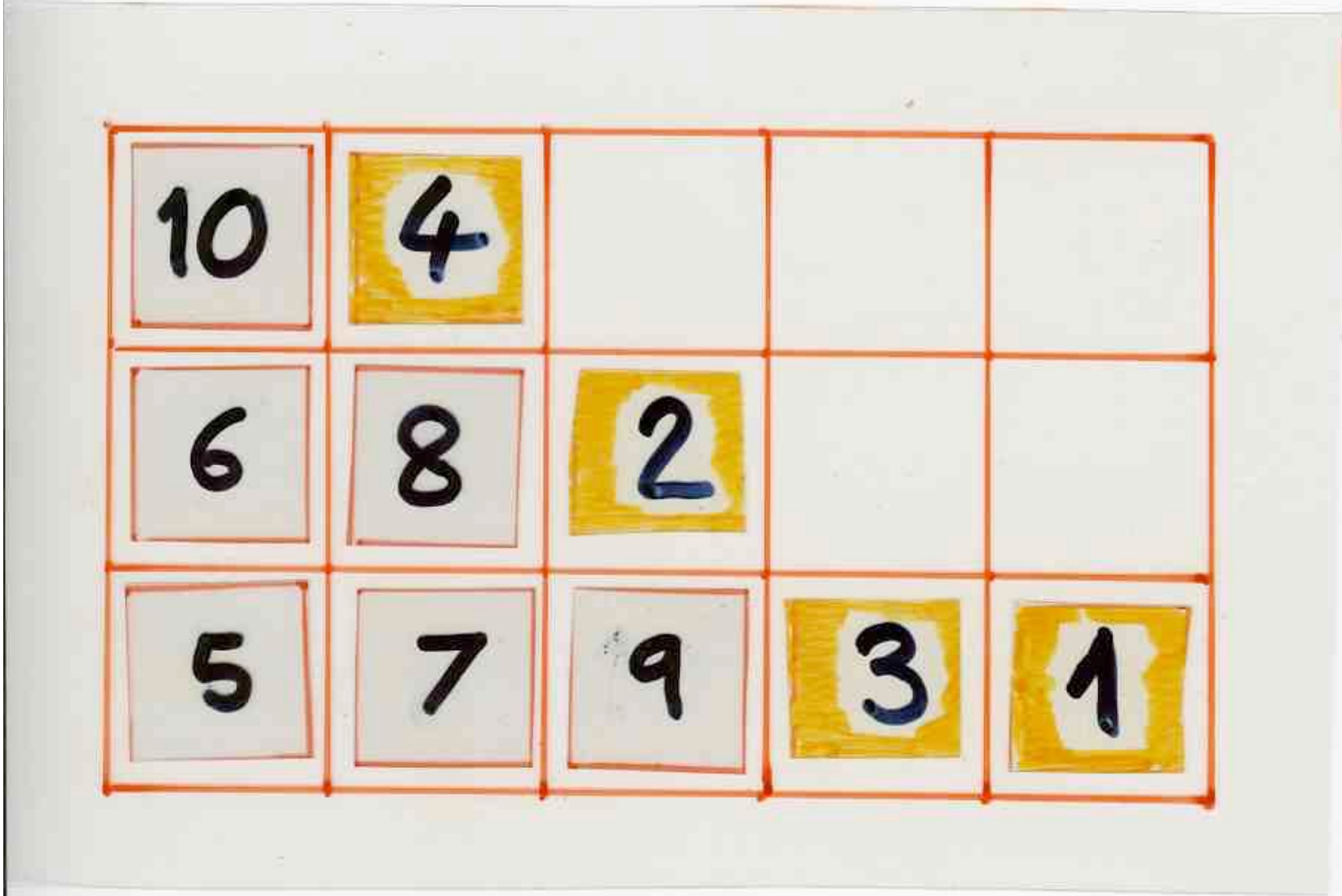
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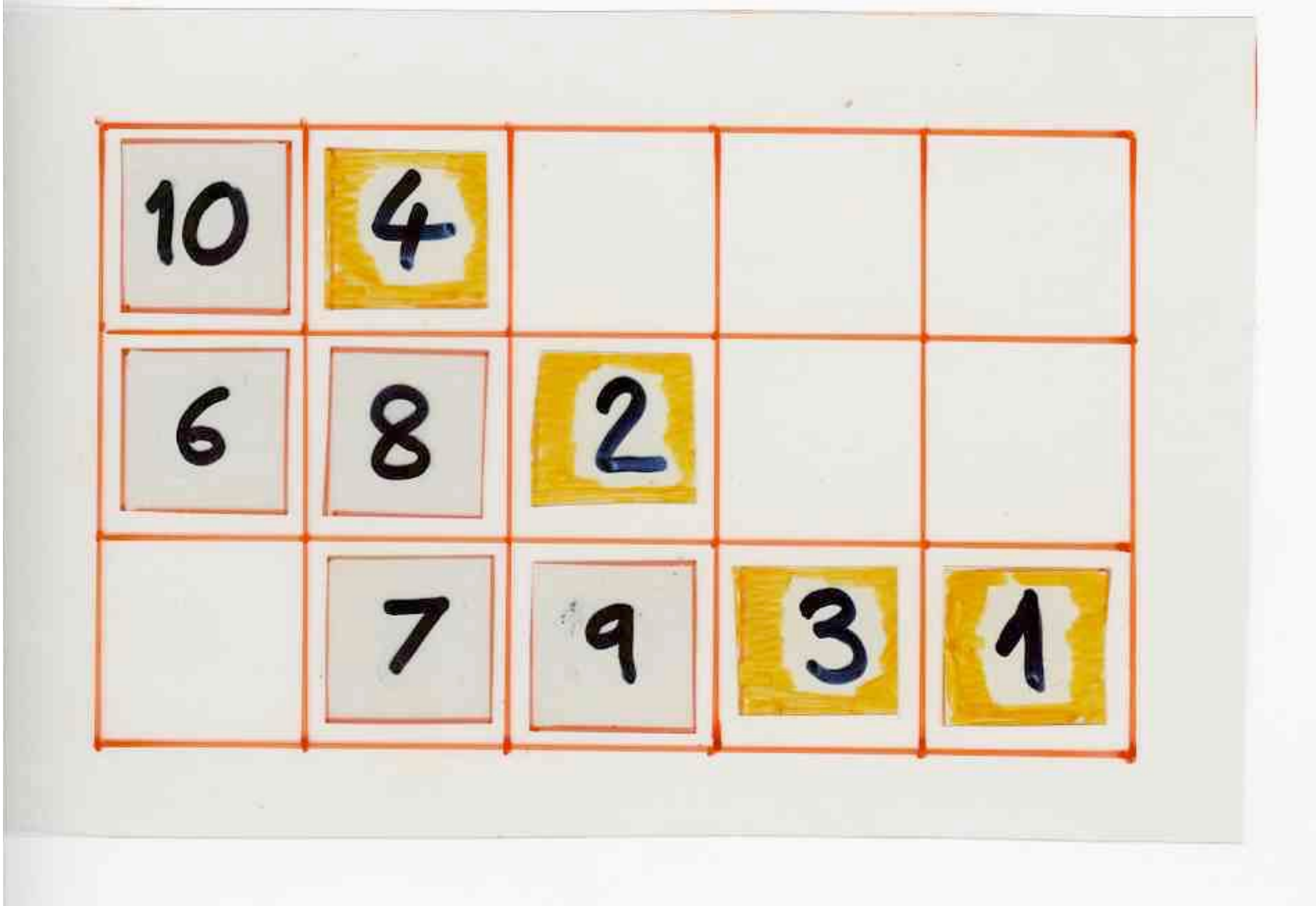
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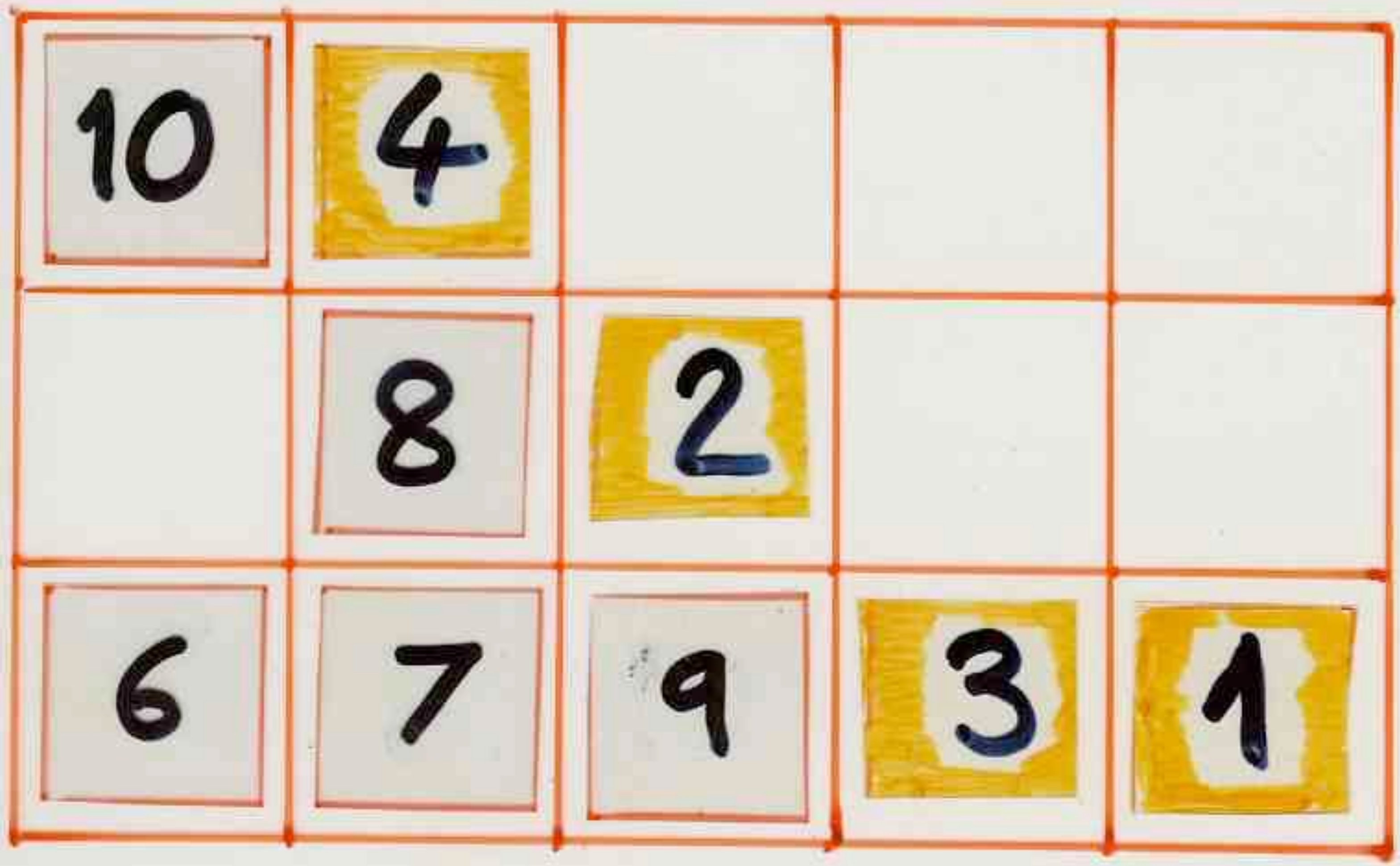
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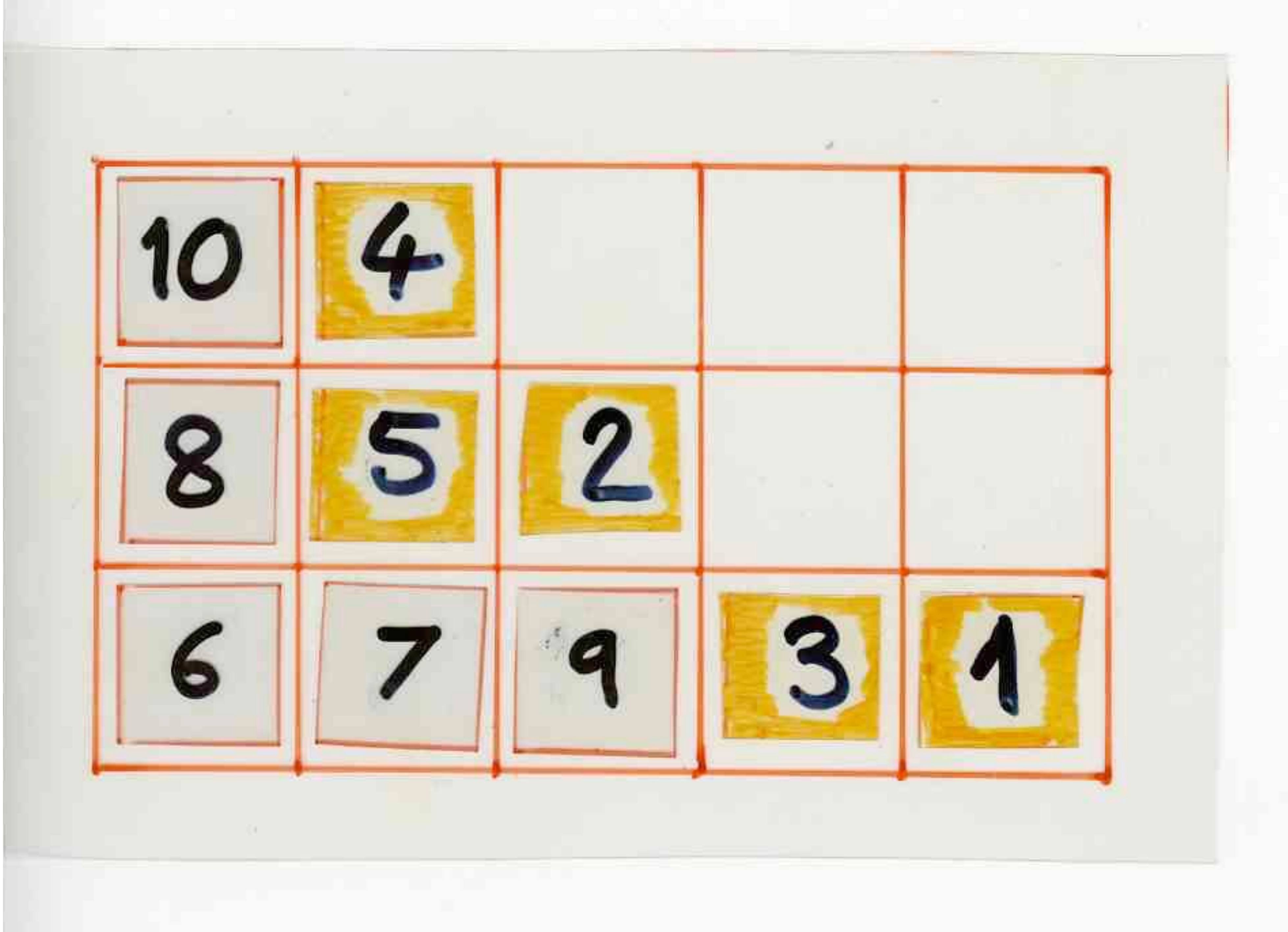
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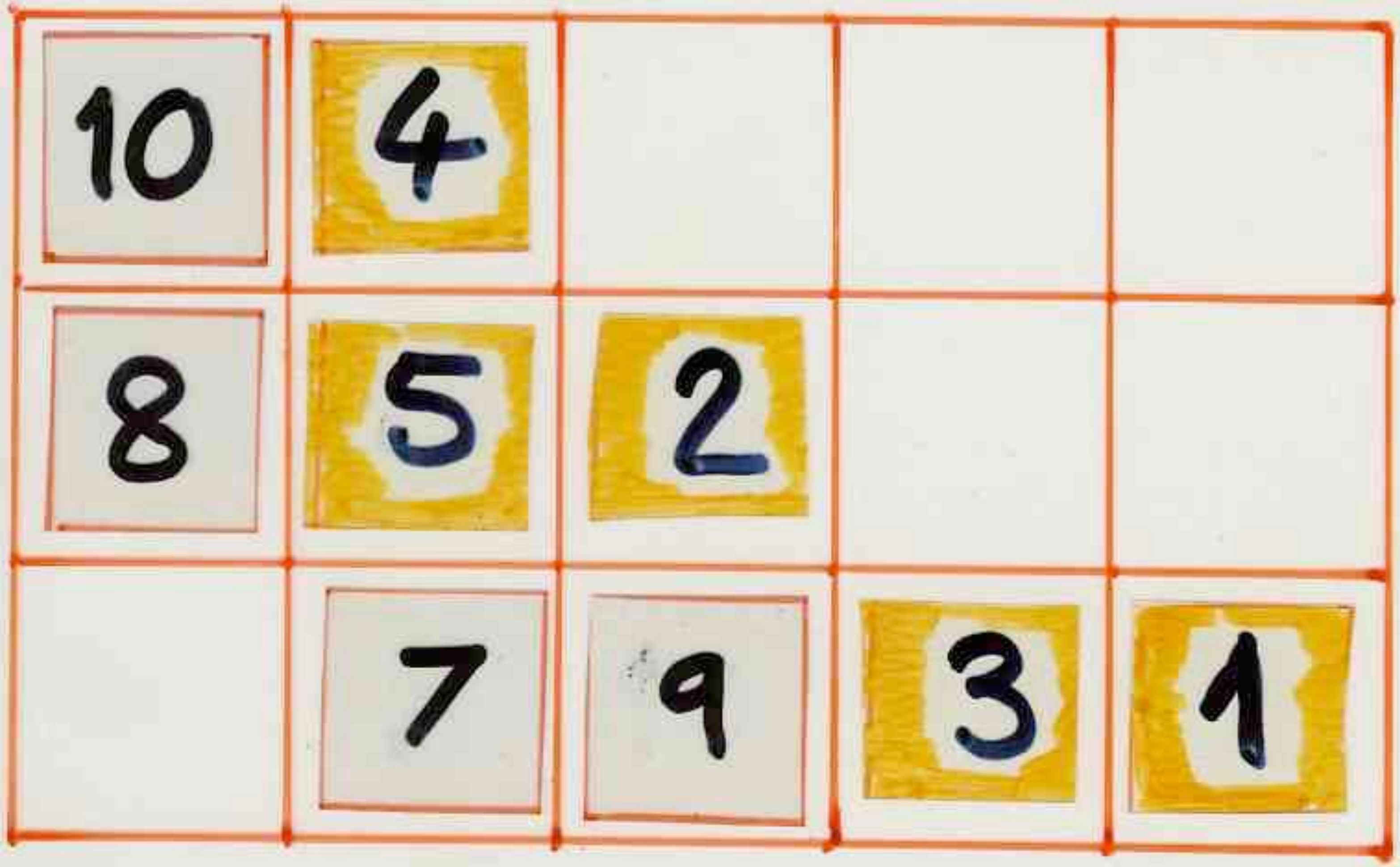
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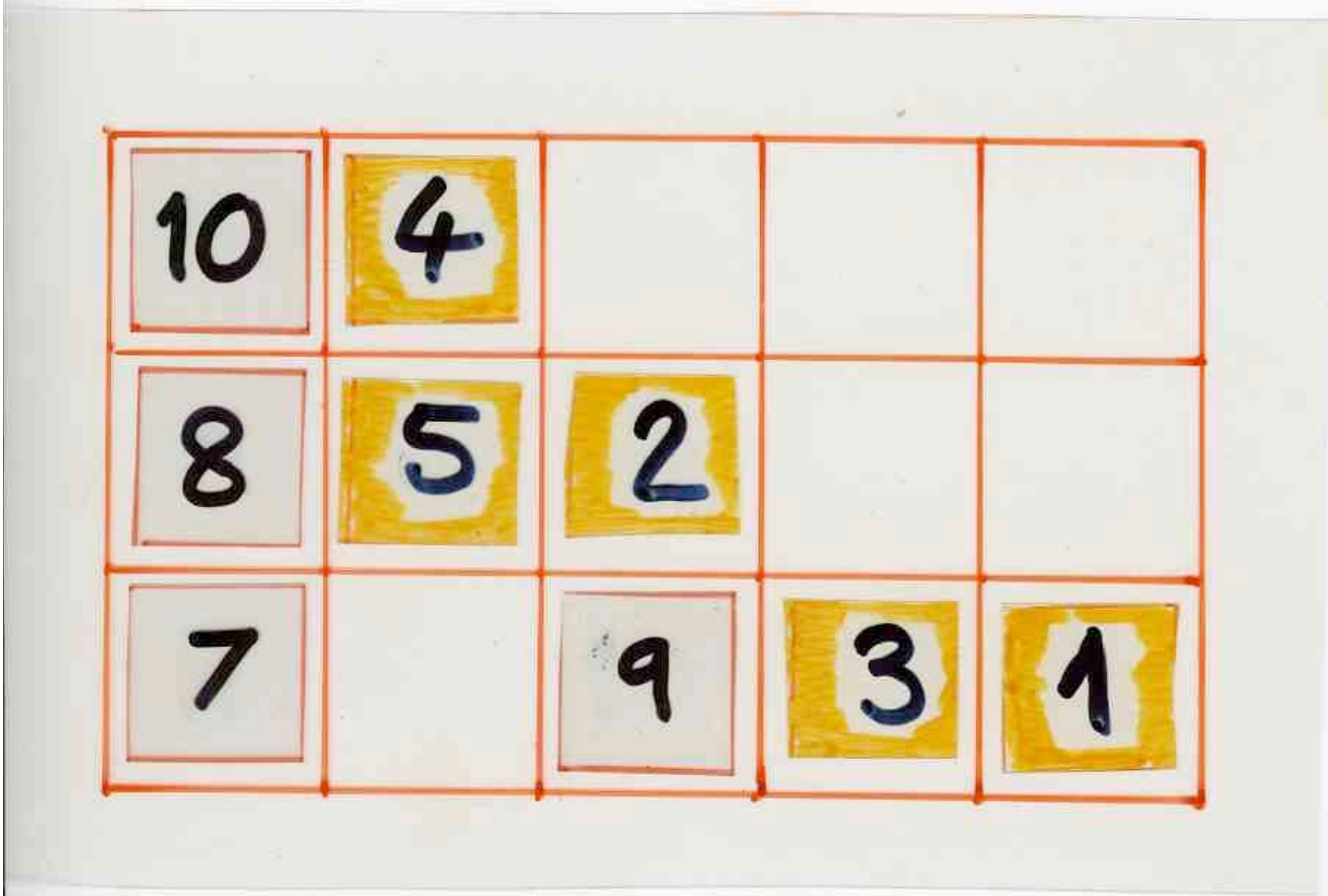

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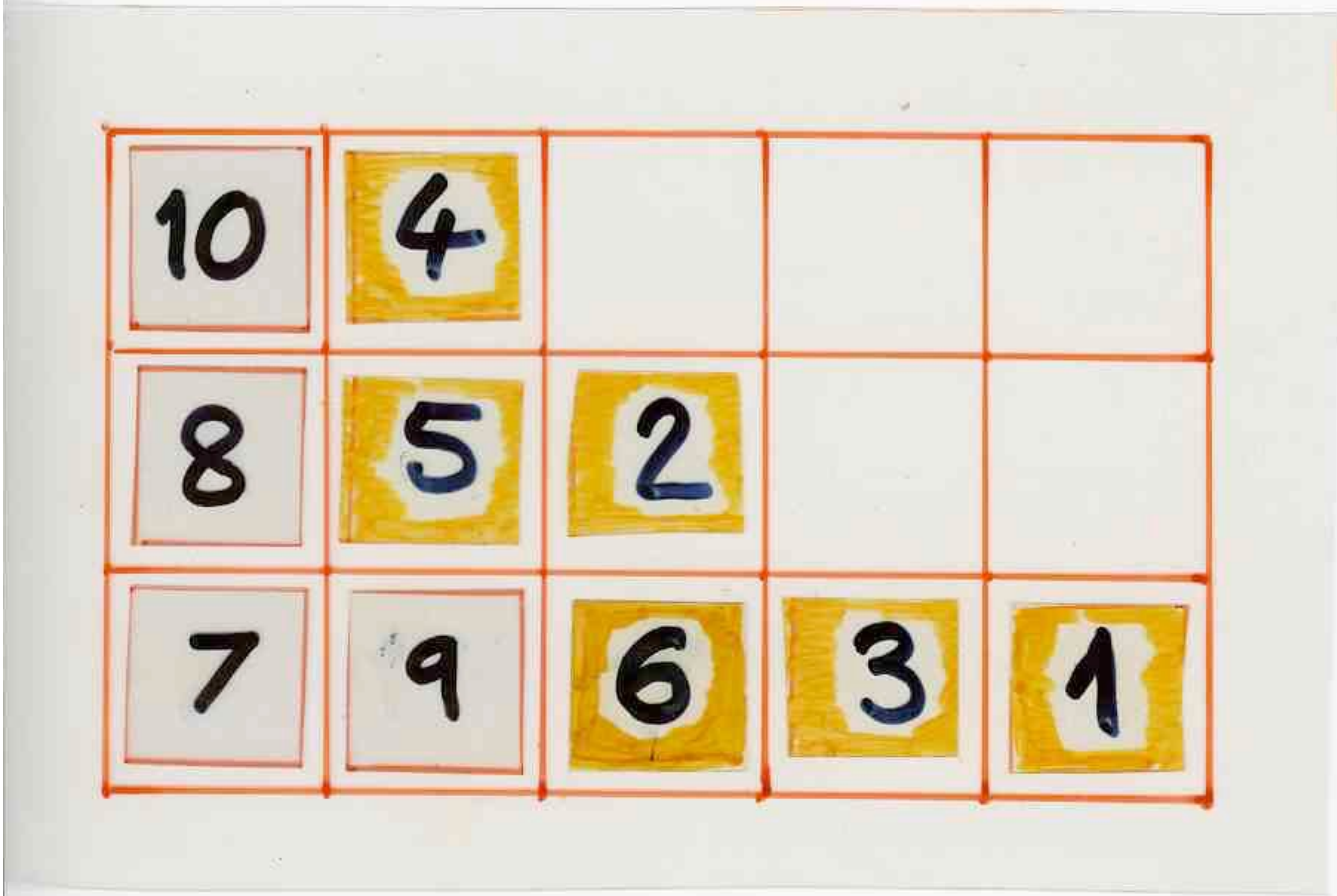
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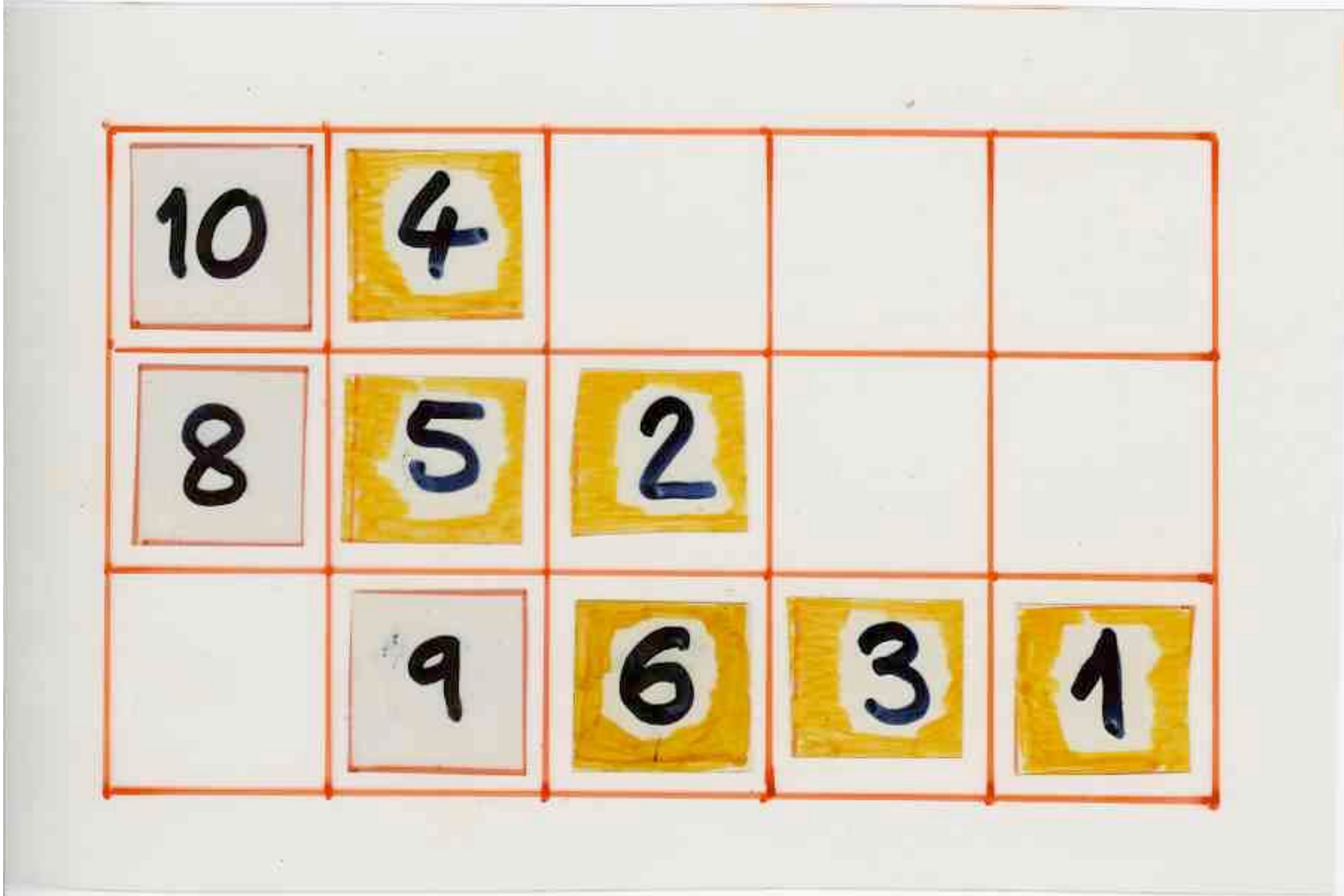
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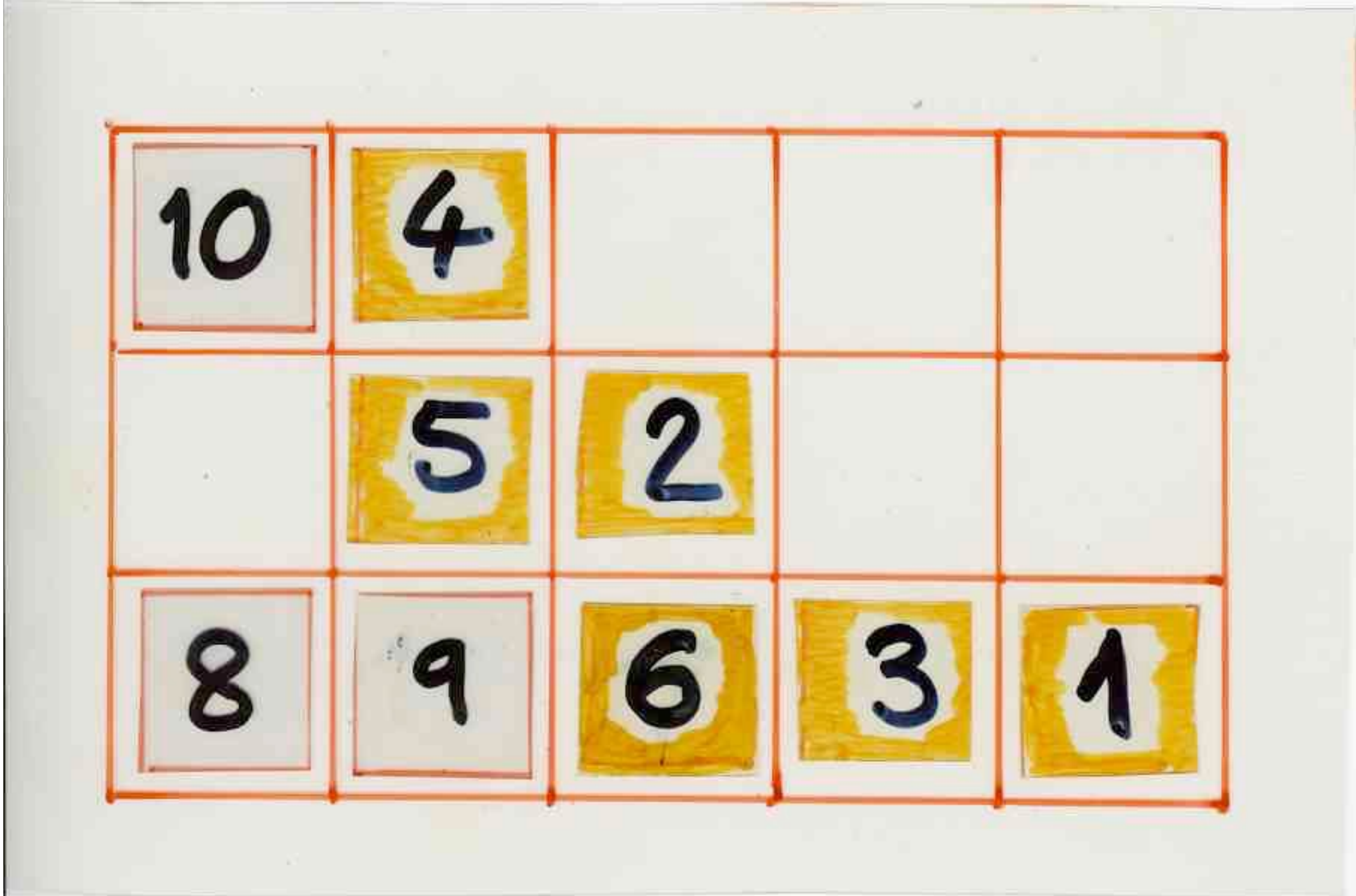
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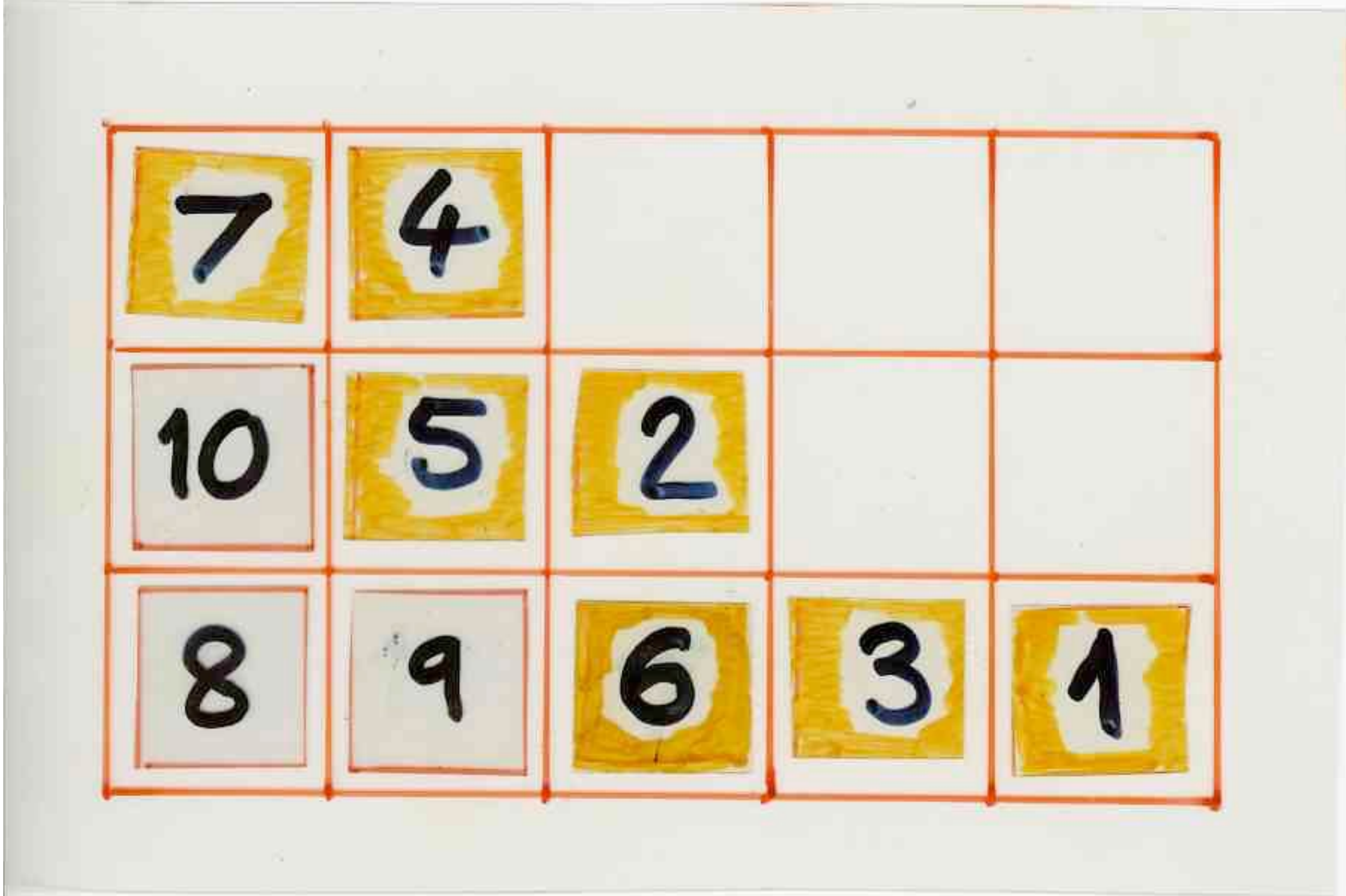
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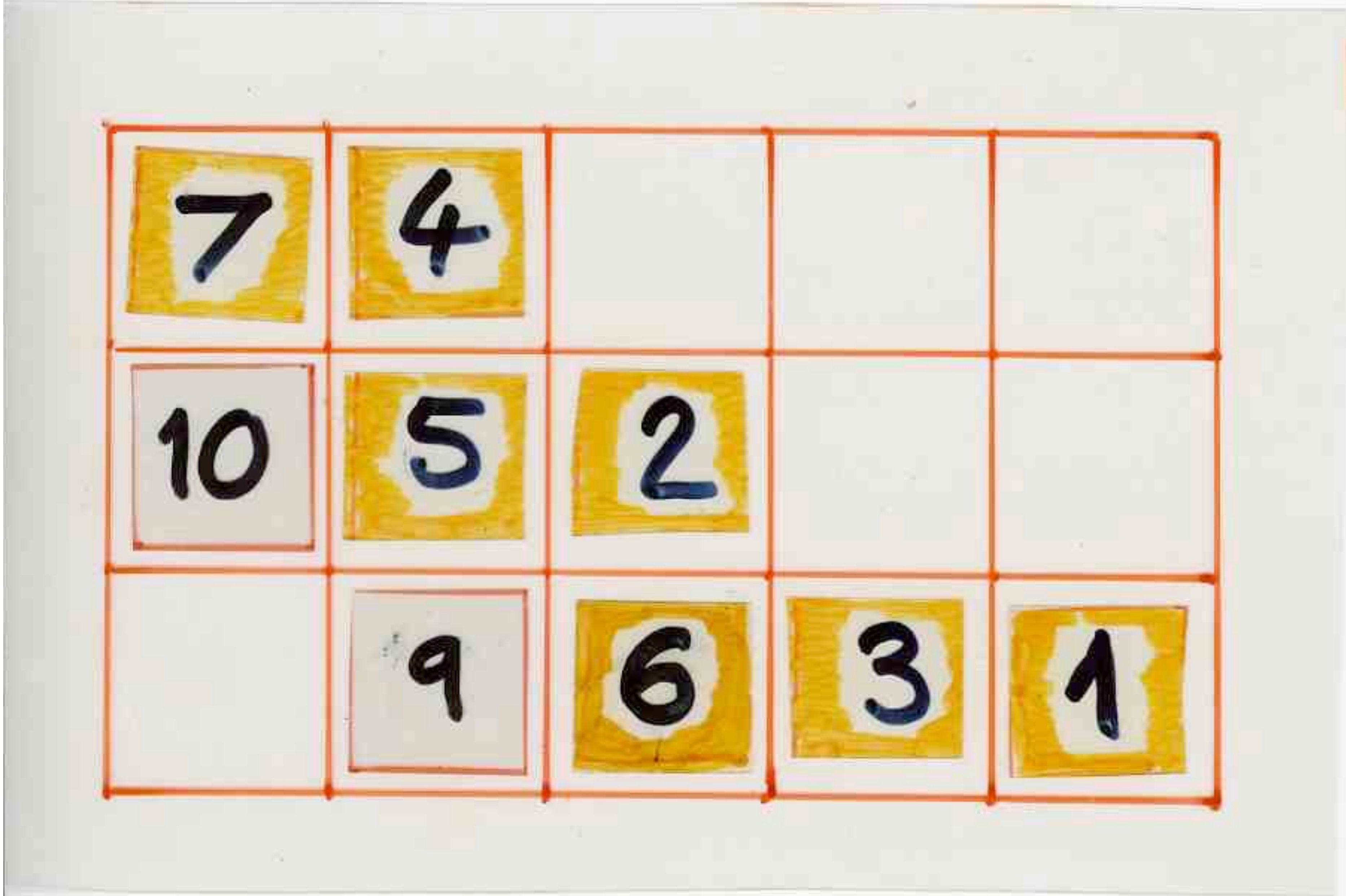
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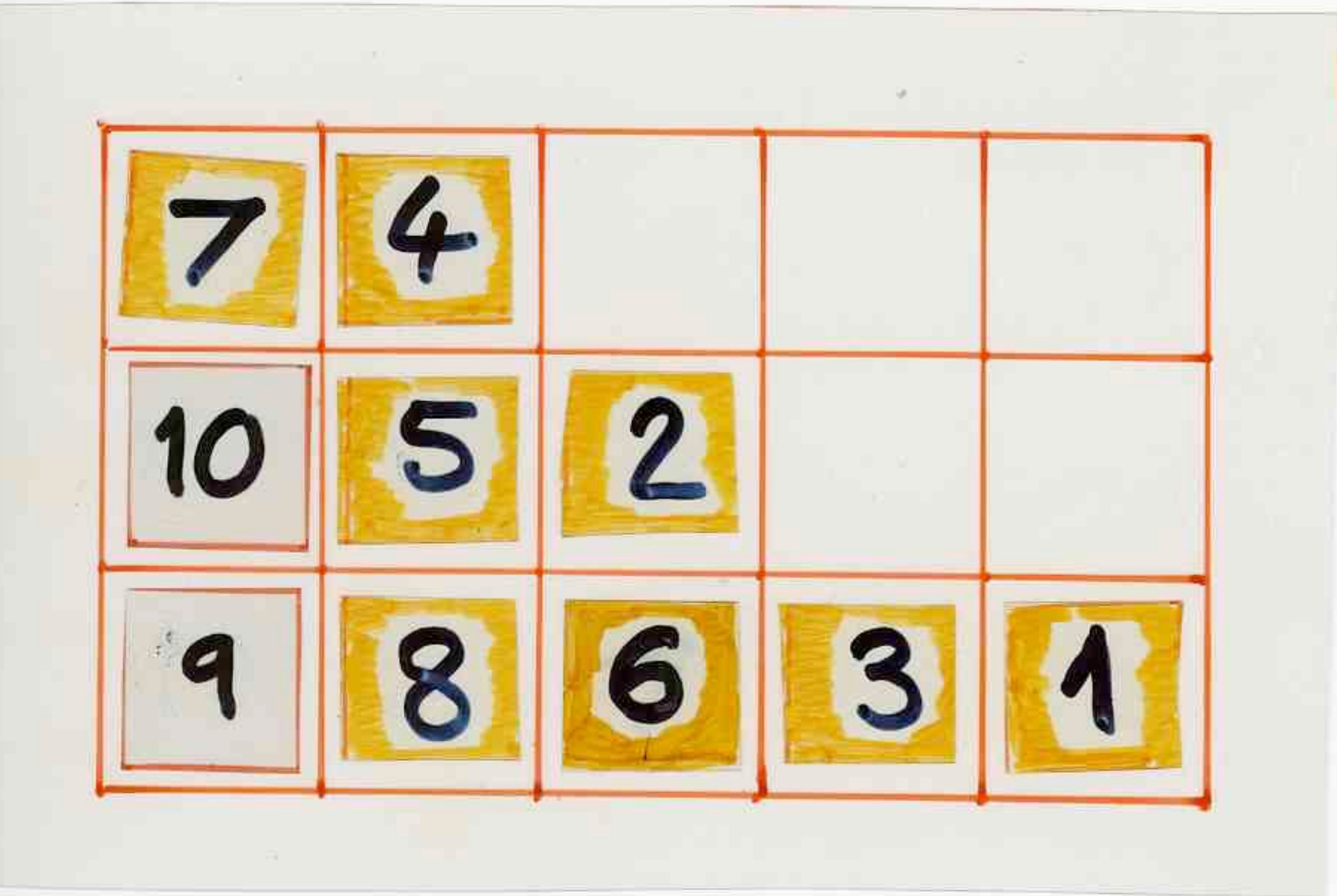
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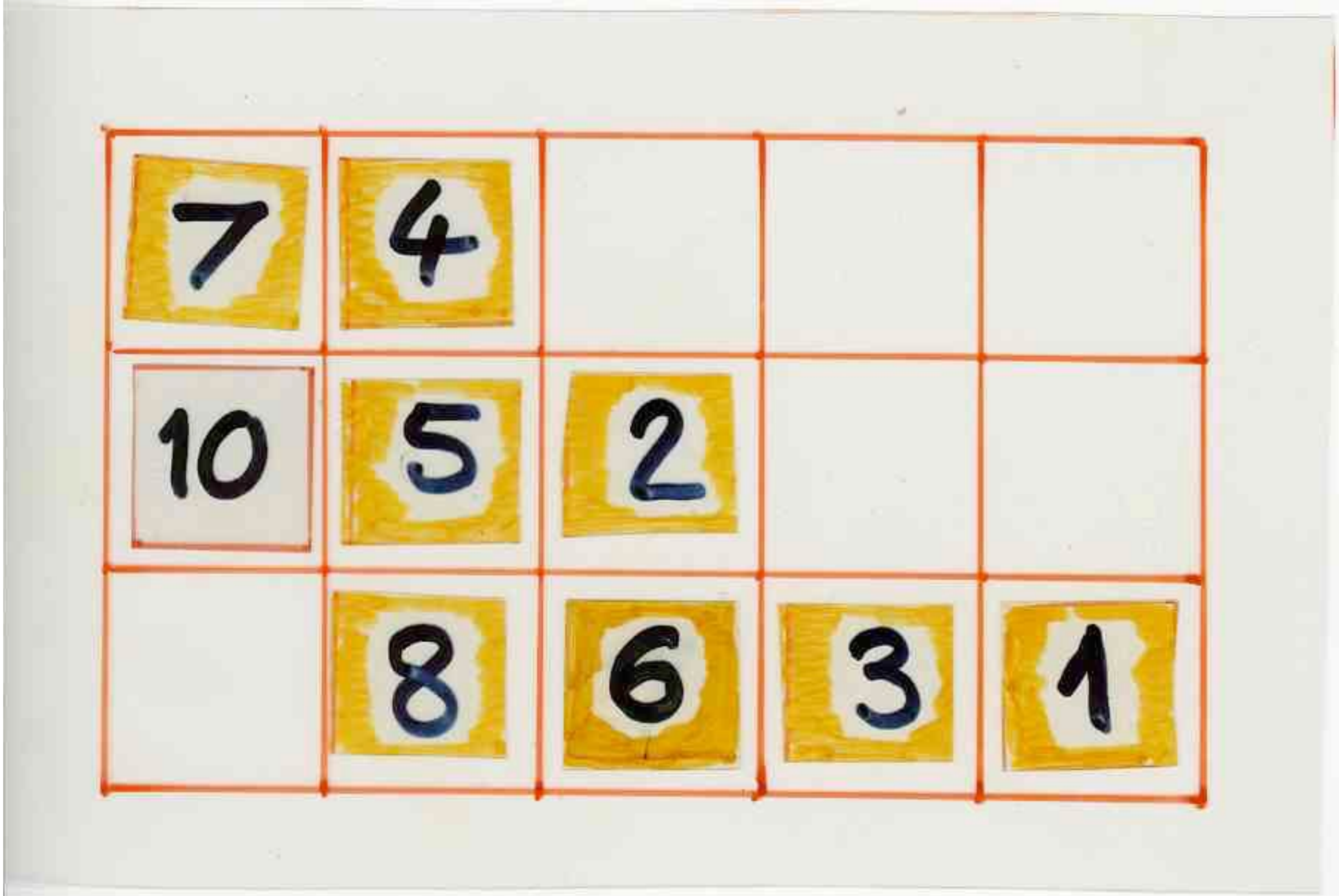
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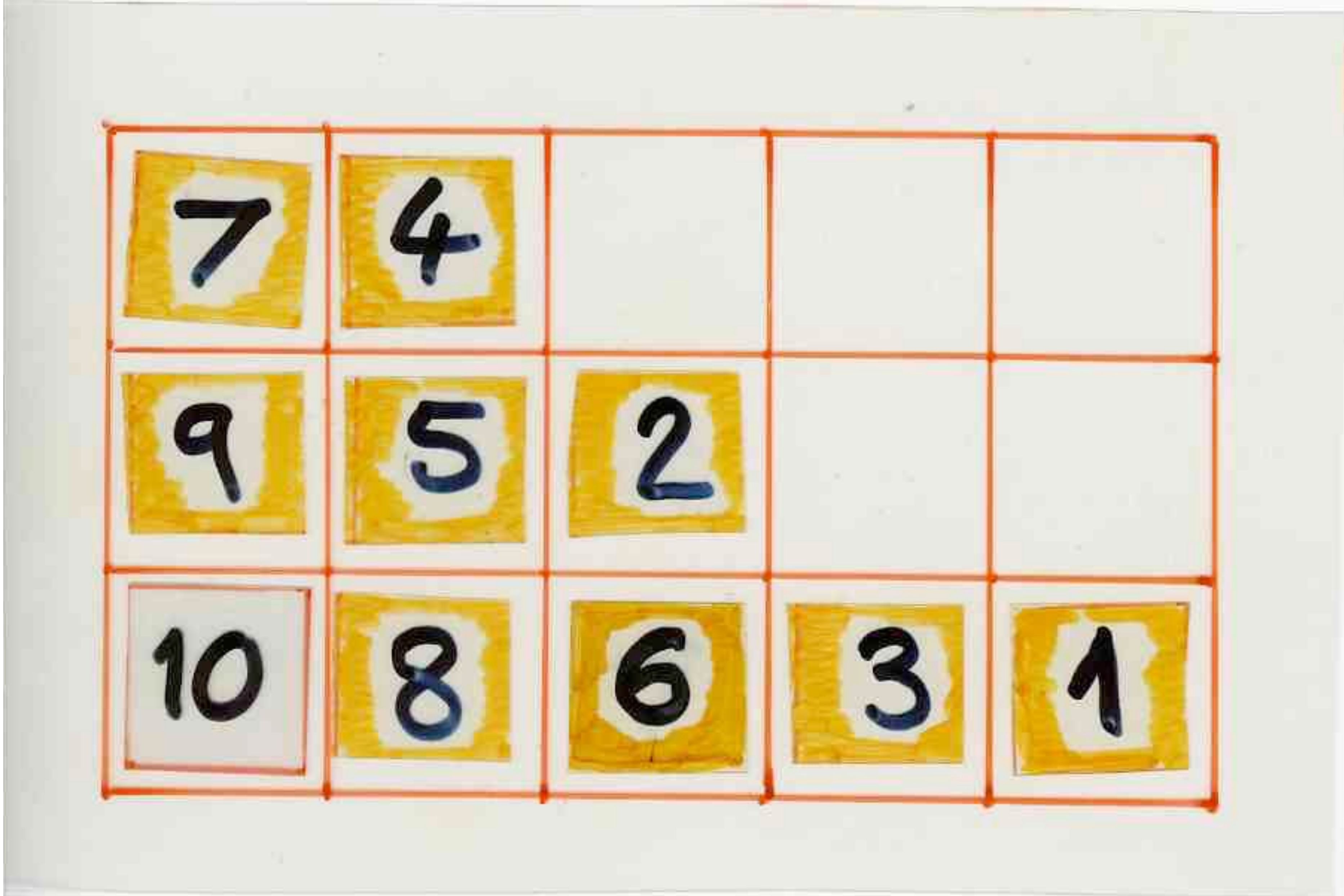
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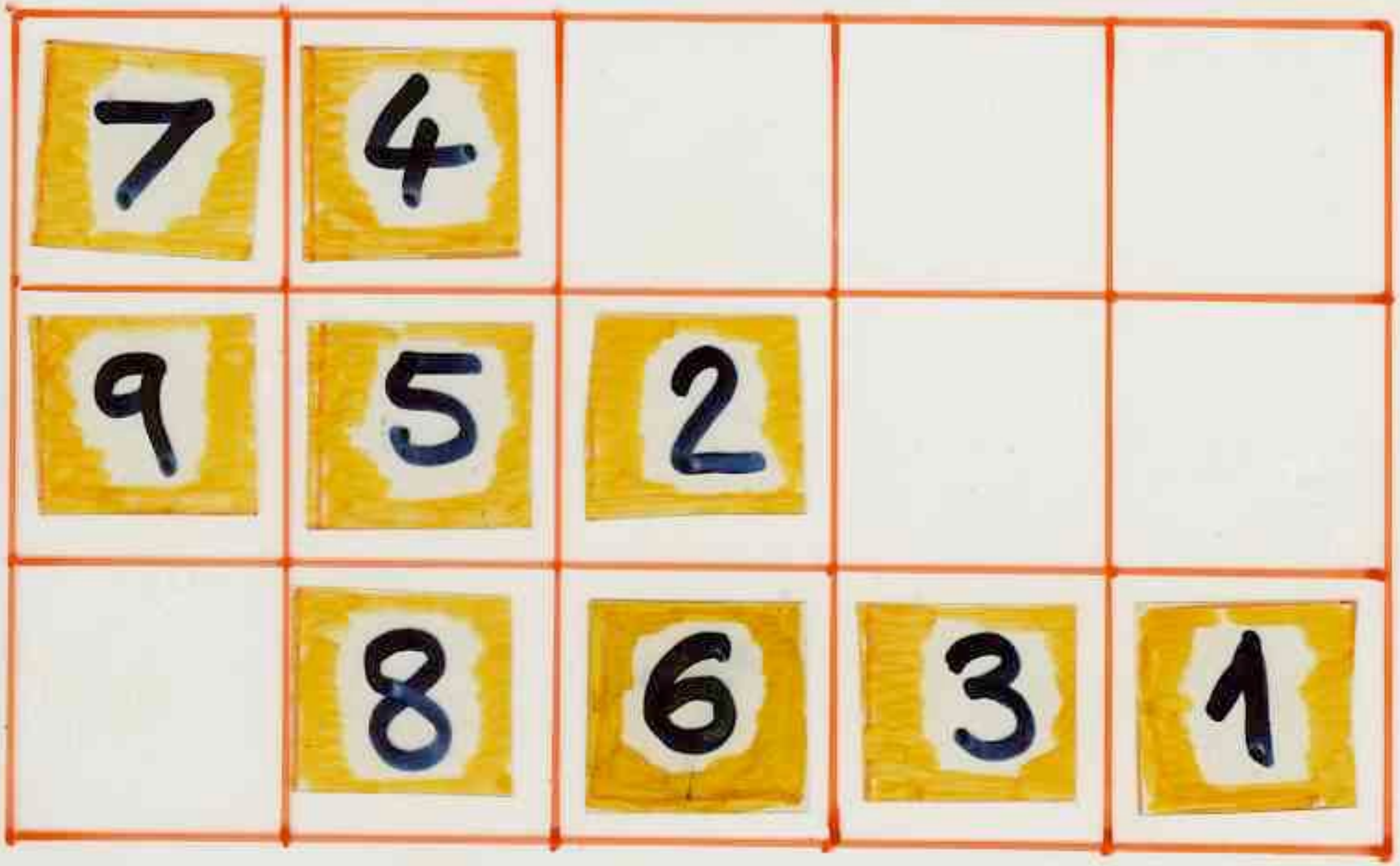
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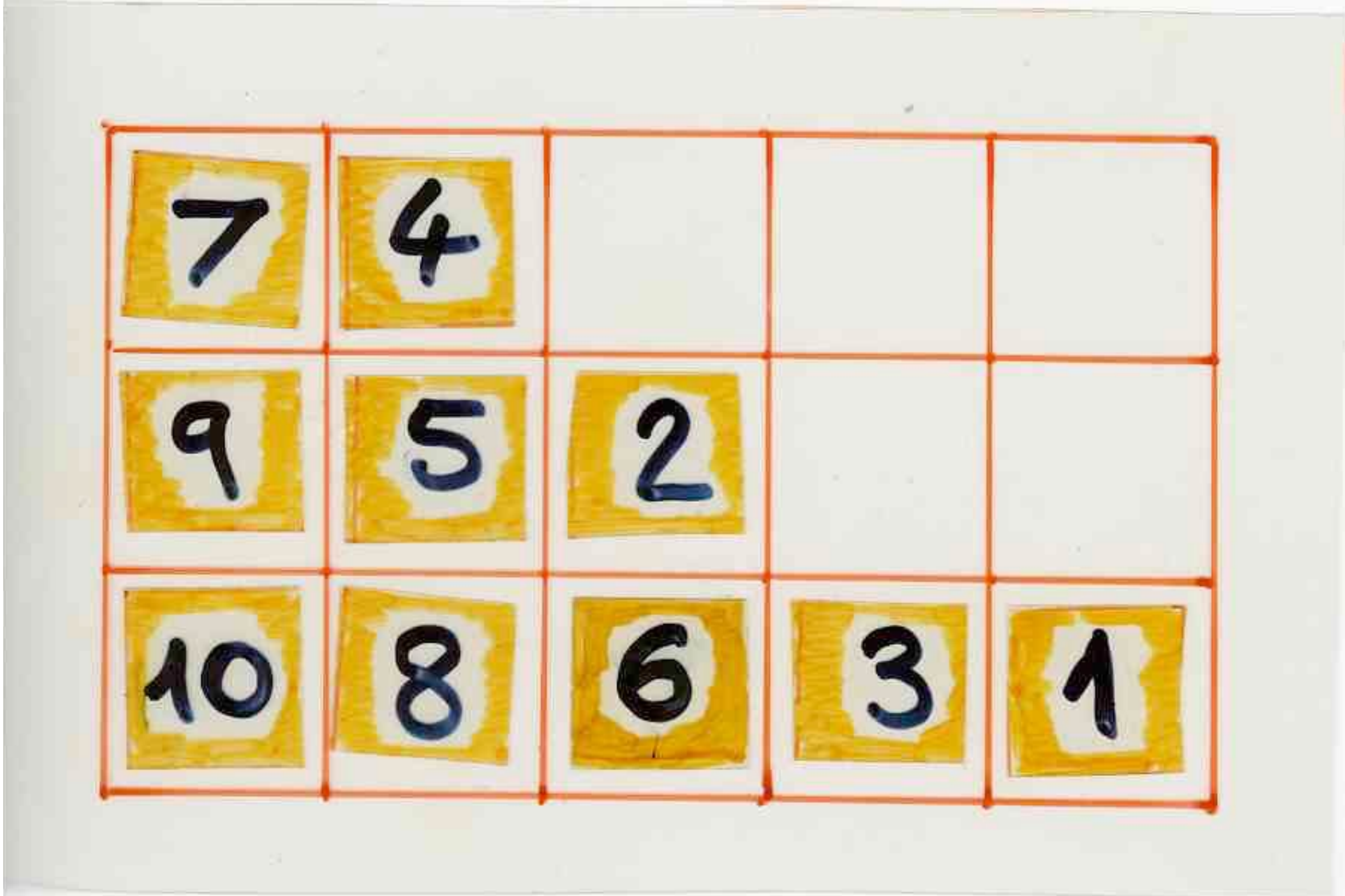
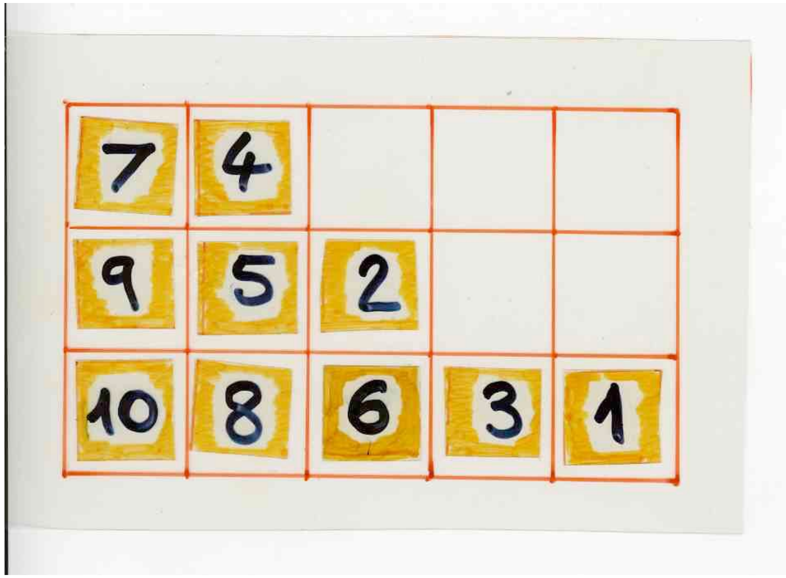

7 4
9 5 2
10 8 6 3 1



P^* =
dual

4	7			
2	6	9		
1	3	5	8	10

complement

$(i)^c = n+1-i$

P =

6	10			
3	5	8		
1	2	4	7	9

Schützenberger Duality !