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A Combinatorial Interpretation of the Quotient-Difference Algorithm

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1 Introduction

During the last thirty years, a growing interest for *Padé approximants* appeared in many theoretical and applied fields, such as numerical analysis, theoretical physics, chemistry, electronics, $\cdots$ as shown in the books Baker [1], Baker, Graves-Morris [2], Brezinski [3], Gilewicz [11]. Padé approximants are strongly connected with *continued fractions* (see for example Henrici [16], Jones, Thron [17], Wall [25]) and *orthogonal polynomials* (see for example Brezinski [4, 5], Draux [7], Van Rossum [22], Wynn [26]). The so-called quotient-difference algorithm, or *qd-algorithm*, plays an important role in these theories. It was originated in Steifel [21] and studied by Rutishauser [19], Henrici [16, 15]. (See also Brezinski [5], Gragg [12]).

The general theory of continued fractions and orthogonal polynomials has been lifted at the combinatorial level by Flajolet [8] and Viennot [23, 24]. The basic structures are the so-called *weighted Motzkin paths.* This paper follows the ideas of [23]. We show that the qd-algorithm can easily be derived from the geometry of the paths, without involving the usual determinant manipulations. Moreover, the Gessel-Viennot [9, 10] methodology, interpretating determinants as configurations of non-crossing paths, gives without calculus the classical expression of the coefficients of the qd-table in terms of Hankel determinants.

A combinatorial theory of Padé approximants, extending the combinatorial theory of orthogonal polynomials exposed in [8] and [23], has been done by E. Roblet in [18][1].

The qd-algorithm has been used for the enumeration of certain Young tableaux with bounded height (see Desainte-Catherine, Viennot [6]). These Young tableaux are encoded by certain configurations of non-crossing paths. These configurations can be "compressed" into a unique configuration with fractional weights for the paths. These rational numbers are given by the qd-algorithm and this "compression" of paths is at the basis of the present paper. The product of these fractional weights gives a "hook-length" type formula for the number of such Young tableaux.

[1] A preliminary version of this paper has been written as a technical report from LaBRI, Bordeaux University. A short version is exposed as Annexe B of Roblet's thesis [18].

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Very recently, this formula, with some extensions, reappeared in some considerations in statistical physics about directed polymers, vicious walkers and watermelons (see Guttmann, Owczarek, Viennot [14] and Guttmann, Krattenthaler, Viennot [13]).

The qd-algorithm has also been used many times in theoretical physics. See for example Sogo [20] for an application of the qd-algorithm to the solution of the Toda molecule equation.

2 Weighted Motzkin and Dyck paths

We briefly recall a few basic definitions and propositions of [8] and [23].

A *Motzkin path* is a path $w = (s_0, \cdots, s_n)$ (i.e. a sequence of points) of $\mathbb{N} \times \mathbb{N}$ such that the starting point is $s_0 = (0,0)$, the ending point is $s_n = (n, 0)$ and each "elementary step" (s_i, s_{i+1}) is North-East, East or South-East (see figure 1). The length of w is n and denoted by $|w|$. A Motzkin path having only North-East or South-East steps is called *Dyck path*.

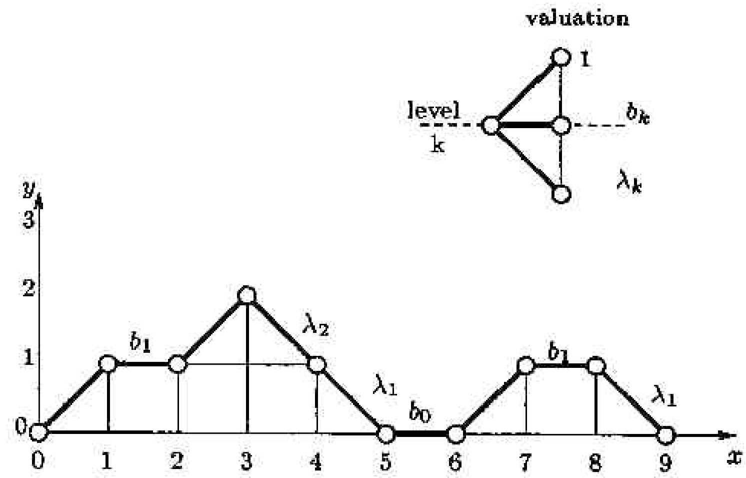

Fig. 1. A weighted Motzkin path, $v(w) = b_0 b_1^2 \lambda_1^2 \lambda_2$.

Let K be a field, and let $b = \{b_k\}_{k\geq 0}$, $\lambda = \{\lambda_k\}_{k\geq 1}$ be two sequences of K. Each elementary step (s_i, s_{i+1}) is weighted by b_k (resp. λ_k, resp. 1) iff (s_i, s_{i+1}) is an East (resp. South-East, resp. North-East) step at level k (i.e. $s_i = (i, k)$). The *weight* (or *valuation*) $v(w)$ of the path w is the product of the valuations of elementary steps. We define

$$\mu_n = \sum_{|w|=n} v(w), \qquad (1)$$

where the summation is over all Motzkin paths of length n. We introduce the generating function of weighted Motzkin paths (according to the valuations b and λ).

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$$J(t;b,\lambda) = \sum_{n\geq 0} \mu_n t^n. \tag{2}$$

In the case of Dyck paths (i.e. $b_k = 0$ for every $k \geq 0$) we will use the notation

$$S(t;\lambda) = \sum_{w} v(w) t^{|w|/2}. \tag{3}$$

where the summation is over weighted Dyck paths. Note that

$$J(t;0,\lambda) = S(t^2;\lambda).$$

From direct paths consideration, one can easily see that there exist at most one pair (b,λ) of sequences (resp. a sequence λ) such that $J(t;b,\lambda)$ (resp. $S(t;\lambda)$) is a given generating function $\sum_{n\geq 0} \mu_n t^n$. The coefficients b_k, λ_k can be computed from the paths as soon as $\lambda_1, \cdots, \lambda_{k-1}$ are $\neq 0$ (see Viennot [23], Chapitre IV).

The notation J (resp. S) comes from the fact that the corresponding generating function has the following expansion into Jacobi (resp. Stieltjes) type continued fraction (see [8, 23]).

$$J(t;b,\lambda) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - b_2 t - \cfrac{\lambda_3 t^2}{\ddots}}}} \tag{4}$$

$$S(t;\lambda) = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{1 - \cfrac{\lambda_3 t}{\ddots}}}}$$

Let $\phi : K[X] \to K$ be the linear functional defined by $\phi(x^n) = \mu_n$, $n \geq 0$. The monic polynomials defined by the recurrence

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x), \; P_0 = 1, \; P_1 = x - b_0, \tag{5}$$

are orthogonal with respect to the scalar product $\phi(PQ)$.

These relations between weighted Motzkin paths and continued fractions and orthogonal polynomials will not be used in this paper. The qd-algorithm can be viewed as an algorithm for computing the expansion into Stieltjes continued fraction (4) of a given generating function $\sum_{n\geq 0} \mu_n t^n = S(t;\lambda)$. For our purpose, it is convenient to introduce a functional defined for almost all sequences. We propose the name "qd-transform".

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3 The qd-transform

Let $\gamma = \{\gamma_k\}_{k\geq 1}$ be a sequence of elements of the field K. The *qd-transform* $\gamma' = \{\gamma'_k\}_{k\geq 1}$ is the unique sequence (if such one exists) such that

$$S(t;\gamma) = 1 + \gamma_1 t S(t,\gamma'); \tag{6}$$

In general, such a sequence will exist and will be denoted by $\gamma' = qd(\gamma)$.

Example 1. For $\gamma_k = k$, then γ' is the sequence: $\gamma'_k = k$ if k is even, $\gamma'_k = k+2$ if k is odd.

Example 2. For $\gamma_k = 1$, then $\gamma'_{2k} = k/(k+1)$, $(k \geq 1)$, $\gamma'_{2k-1} = (k+1)/k$, $(k \geq 1)$, that is $\gamma' = (2, 1/2, 3/2, 2/3, 4/3, 3/4, \cdots)$.

Example 3. Let $\gamma_k = \lceil k/2 \rceil$ (smallest integer $\geq k/2$). Then $\gamma'_k = k/2$ if k is even, $\gamma'_k = \lceil k/2 \rceil + 1$ if k is odd, that is $\gamma = (1,1,2,2,3,3,\cdots)$ and $\gamma' = (2,1,3,2,4,3,\cdots)$

Example 4. Let $\gamma = (1,2,1,2,\cdots)$ that is $\gamma_k = 1$ if k is odd, $\gamma_k = 2$ if k is even. Then $\gamma' = (3, 2/3, 7/3, 6/7, 15/7, 14/15, 31/15, \cdots)$ that is $\gamma'_{2k} = \frac{2^{k+1}-2}{2^{k+1}-1}$, $\gamma'_{2k-1} = \frac{2^{k+1}-1}{2^k-1}$ $(k \geq 1)$.

These four examples follow immediately from the

Proposition 5. *Let $\gamma = (\gamma_k)_{k\geq 1}$ and $\gamma' = (\gamma'_k)_{k\geq 1}$ be two sequences of K. Then $\gamma' = qd(\gamma)$ iff we have the following relations for every $k \geq 0$:*

$$\gamma_{2k+1} + \gamma_{2k+2} = \gamma'_{2k} + \gamma'_{2k+1}, \; \gamma_{2k}\gamma_{2k+1} = \gamma'_{k-1}\gamma'_{2k}. \tag{7}$$

Using this relation, one can compute by recurrence the coefficients of γ' from the sequence γ as soon as $\gamma'_k \neq 0$ for $k \geq 1$. The proof of proposition 5 is based upon contraction of paths.

4 Contraction of paths

The idea, inspired from the renormalisation group in physics, is to "change the scale" of Dyck paths. We follow such paths w by successive jumps of two consecutive elementary steps. We have two types of such contraction, depending upon starting at the first (contraction T) or second (contraction T^+) vertex of the path w.

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a) Contraction T

If w is a Dyck path of length $2n$, $T(w)$ is a Motzkin path of length n, as shown on figure 2. From the valuation $\gamma = (\gamma_k)_{k\geq 1}$, we define the two valuations b and λ by

$$b_k = \gamma_{2k} + \gamma_{2k+1}, \quad \lambda_k = \gamma_{2k-1}\gamma_{2k}. \tag{8}$$

Let η be a Motzkin path weighted by the valuations b and λ. We have the following relation (just look at figure 2!)

$$v(\eta) = \sum_w v(w), \tag{9}$$

where the summation is over all weighted (according to γ) Dyck paths w such that $T(w) = \eta$. Note that the number of such paths is 2^m where m is the number of East steps of the path η.

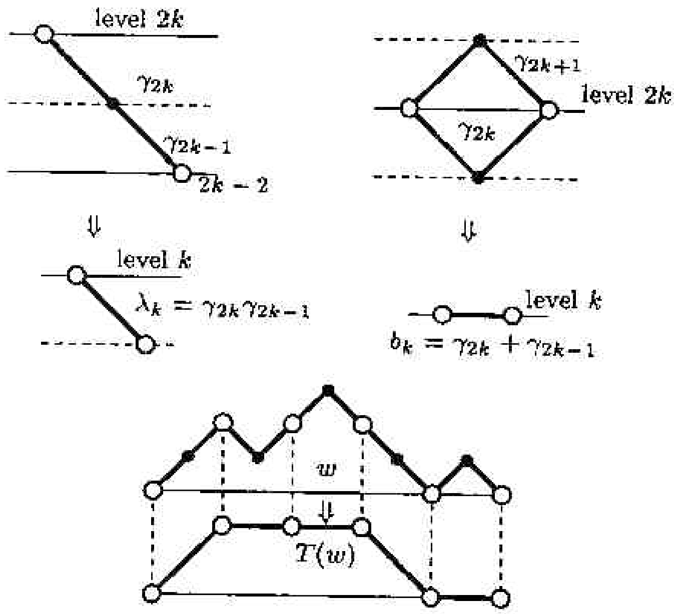

Fig. 2. The contraction T.

From (9) we deduce

$$S(t;\gamma) = J(t;b,\lambda). \tag{10}$$

b) Contraction T^+

Starting from the second vertex of the Dyck path w of length $2n$, we define the Motzkin path $T^+(w)$ of length $n-1$, as shown in figure 3.

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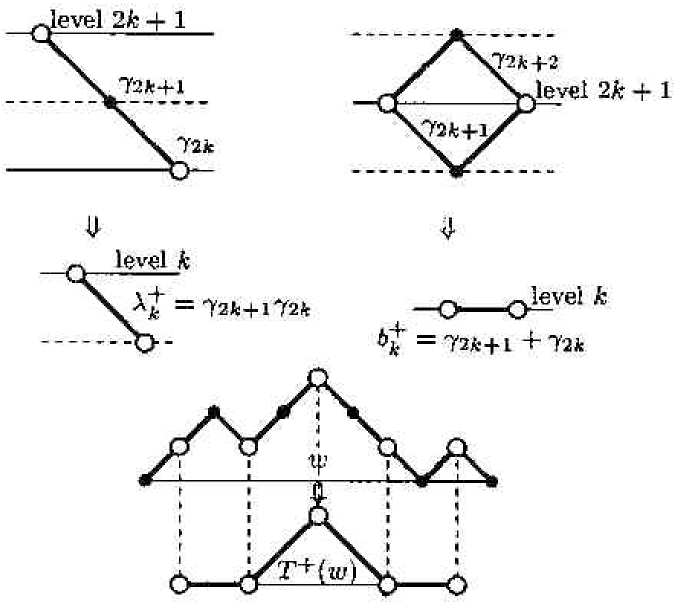

Fig. 3. The contraction T^+.

Let b^+ and λ^+ be the valuations defined by

$$b_k^+ = \gamma_{2k+1} + \gamma_{2k+2}, \ \lambda_k^+ = \gamma_{2k}\gamma_{2k+1}. \tag{11}$$

With the analogue summation of (9), we have

$$\gamma_1 v(\eta) = \sum_{w/T^+(w)=\eta} v(w), \tag{12}$$

which implies

$$S(t;\gamma) = 1 + \gamma_1 t J(t; b^+, \lambda^+). \tag{13}$$

Proof (Proof of Proposition 5). Combining (6), (10) and (13) we deduce that the sequences b', λ' associated to γ' by the relation (8) is the same as the sequences b^+, λ^+ associated to γ' by (11). We get (7). □

5 The qd-algorithm

The "qd-algorithm" is obtained by applying recursively the qd-transform to a sequence γ.

Let $\gamma = \gamma^{\{0\}} = \{\gamma_k\}_{k\geq 1}$ and $S(t;\gamma) = \sum_{n\geq 0} \mu_n t^n$ be the generating function defined by (3). Denoting by $\gamma^{\{m\}} = \{\gamma_k^{\{m\}}\}_{k\geq 1}$ the sequence $\gamma^{(m)} = qd^{(m)}(\gamma)$, we have successively

$$\begin{aligned} S(t;\gamma) &= 1 + \gamma_1^{(0)} t S(t;\gamma^{(1)}), \\ S(t;\gamma) &= 1 + \gamma_1^{(0)} t + \gamma_1^{(0)}\gamma_1^{(1)} t^2 S(t;\gamma^{(2)}), \\ &\dots, \\ S(t;\gamma) &= 1 + \gamma_1^{(0)} t + \cdots + \gamma_1^{(0)}\gamma_1^{(1)}\cdots\gamma_1^{(n-1)} t^n S(t;\gamma^{(n)}). \end{aligned} \tag{14}$$

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Thus $\mu_n = \gamma_1^{(0)}\gamma_1^{(1)}\cdots\gamma_1^{(n-1)}$ and

$$\gamma_1^{(n)} = \mu_{n+1}/\mu_n, \ n \geq 0. \tag{15}$$

From (7), the coefficients $\gamma_k^{(n)}$ are related by the relations

$$\gamma_{2k+1}^{(n)} + \gamma_{2k+2}^{(n)} = \gamma_{2k}^{(n+1)} + \gamma_{2k+1}^{(n+1)}, \ \gamma_{2k}^{(n)}\gamma_{2k+1}^{(n)} = \gamma_{2k-1}^{(n+1)}\gamma_{2k}^{(n+1)}. \tag{16}$$

If the moments μ_n are given, we can construct recursively the whole table from the initial condition (15) (in the case where no division by 0 occurs). This is the classical form of the qd-algorithm. The rules (16) are called the *Rhombus rules*.

Usually the following more convenient notations are used:

$$\gamma_{2k}^{(n)} = e_k^{(n)}, \ \gamma_{2k+1}^{(n)} = q_{k+1}^{(n)}. \tag{17}$$

The Rhombus rule (16) becomes

$$q_{k+1}^{(n)} + e_{k+1}^{(n)} = q_{k+1}^{(n+1)} + e_k^{(n+1)}, \ e_k^{(n)}q_{k+1}^{(n)} = e_k^{(n+1)}q_k^{(n+1)}, \tag{18}$$

with initial conditions

$$e_0^{(n)} = 0, \ q_1^{(n)} = \frac{\mu_{n+1}}{\mu_n} \text{ for every } n \geq 0. \tag{19}$$

This is visualized on figure 4, called the qd-table

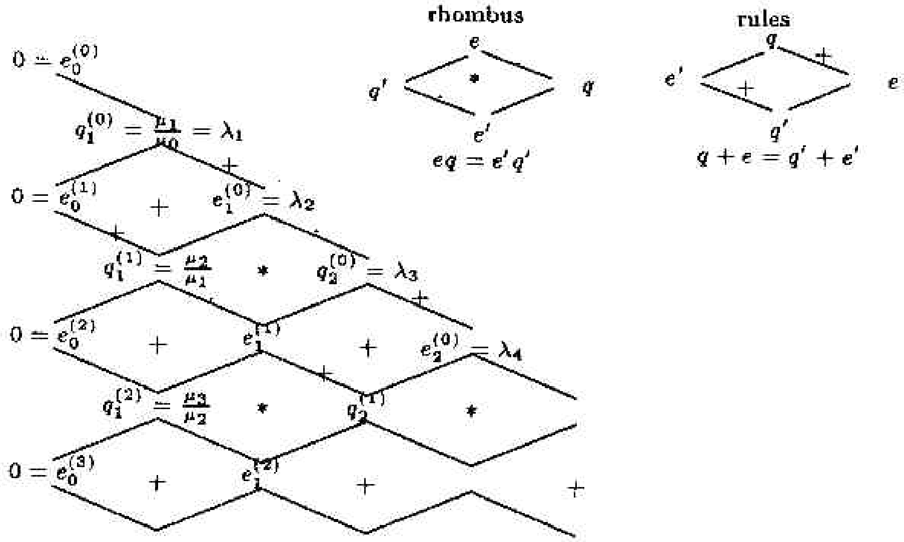

Fig. 4. The qd-table.

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Example 6.

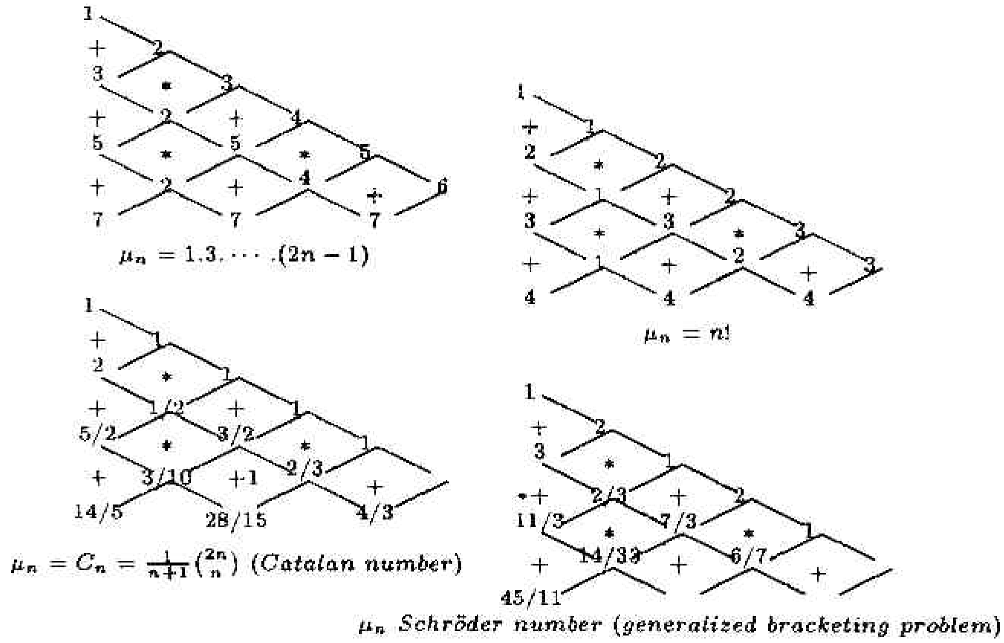

Fig. 5. (Table 1.)

For the first two examples, the reader will easily guess all the coefficients of the table (which are integers). A general formula for the coefficients corresponding to Catalan numbers will be given below.

Remark 7. The relation (14) can also be written in the following form:

$$\frac{\mu_{n+i}}{\mu_n} = \sum_{|w|=2i} v^{(n)}(w), \tag{20}$$

where the summation is over all Dyck paths of length $2i$, weighted by the valuation $\gamma^{(n)}$. This is the *"compression"* of paths refered to in the introduction: Dyck paths of length $2n + 2i$ are "compressed" into weighted paths of length $2i$.

6 Hankel determinants

Let $\mu = \{\mu_n\}_{n\geq 1}$ be a sequence of elements of the field K. For $0 \leq \alpha_1 < \cdots < \alpha_p$ and $0 \leq \beta_1 < \cdots < \beta_p$, the *Hankel determinant* $H\binom{\alpha_1,\cdots,\alpha_p}{\beta_1,\cdots,\beta_p}$ is the determinant of the $p \times p$ matrix with general term $\mu_{\alpha_i+\beta_j}$, for $1 \leq i, j \leq p$. From the general methodology of Gessel, Viennot [9, 10] one deduce the following interpretation (see also [23], Ch. IV).

$$H\begin{pmatrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_p\end{pmatrix} = \sum_{(w_1,\cdots,w_p)} v(w_1)\cdots v(w_p), \tag{21}$$

where the summation is over all configurations $\theta = (w_1, \cdots, w_p)$ of Dyck paths such that: (i) for any i, $1 \leq i \leq p$, the path w_i goes from $A_i = (-2\alpha_i, 0)$ to $B_i = (2\beta_i, 0)$ and (ii) the paths w_i are two by two disjoint (no common vertex).

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One of the configurations interpretating the Hankel determinant $H\binom{1,3,4}{0,1,3}$ is displayed in figure 6.

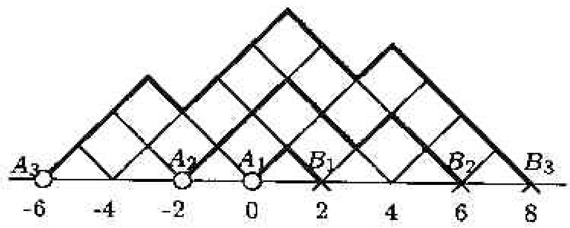

Fig. 6. The Hankel determinant $H\binom{1,3,4}{0,1,3}$.

Notation. $H_k^{(n)} = H\binom{n,n+1,\cdots,n+k-1}{n,n+1,\cdots,n+k-1}$.

If some confusions are possible we will also write $H_k^{(n)}(\mu)$. For $\mu = \{\mu_n\}_{n\geq 0}$, we define $\mu^{(n)} = \{\mu_i^{(n)}\}_{i\geq 0}$ and $\overline{\mu}^{(n)} = \{\overline{\mu}_i^{(n)}\}_{i\geq 0}$ by $\mu_i^{(n)} = \mu_{n+i}$ and $\overline{\mu}_i^{(n)} = \mu_{n+i}/\mu_n$ for $i, n \geq 0$. With (20), we can state the following relations

$$H_k^{(n)}(\mu) = H_k^{(0)}(\mu^{(n)}),\ H_k^{(n+1)}(\mu) = H_k^{(1)}(\mu^{(n)}),$$

$$H_k^{(n)}(\mu) = (\mu_n)^k H_k^{(0)}(\overline{\mu}^{(n)}), \tag{22}$$

$$H_k^{(n+1)}(\mu) = (\mu_n)^k H_k^{(1)}(\overline{\mu}^{(n)}). \tag{23}$$

The Hankel determinant of the right-hand side of (22) (resp. (23)) is interpretated by a single configuration of non-crossing paths, as shown on figure 7 (resp. 8). We have thus defined a kind of "compression" of configurations of non-crossing Dyck paths.

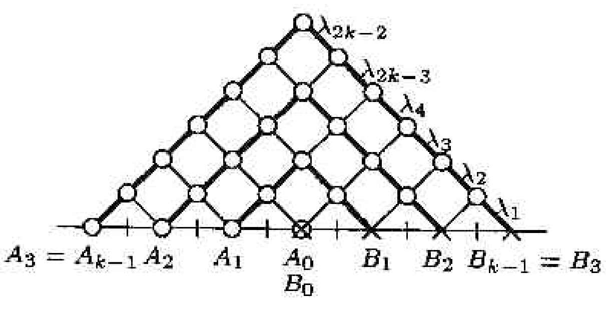

Fig. 7. The Hankel determinant $H_k^{(0)}(\lambda)$.

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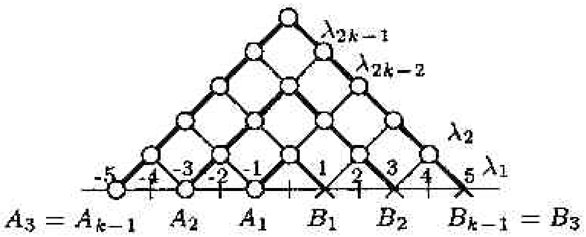

Fig. 8. The Hankel determinant $H_k^{(1)}(\lambda)$.

For any valuation λ, the geometry of the configurations displayed on figures 7 and 8 shows that the ratios $H_k^{(0)}(\lambda)/H_{k-1}^{(0)}(\lambda)$ and $H_k^{(1)}(\lambda)/H_{k-1}^{(1)}(\lambda)$ are respectively the weight of the longest path of each configuration.

With the notations of Section 5, we take $\lambda = \gamma^{(n)}$. Remark that (14) or (20) can also be written: $S(t;\gamma^{(n)}) = \sum_{i\geq 0} \overline{\mu}_i^{(n)} t^i$, for every $n \geq 0$.

With (22) and (23) we deduce

$$\frac{H_k^{(n)}(\mu)}{H_{k-1}^{(n)}(\mu)} = \mu_n \gamma_1^{(n)} \gamma_2^{(n)} \cdots \gamma_{2k-2}^{(n)}, \quad \frac{H_k^{(n+1)}(\mu)}{H_{k-1}^{(n+1)}(\mu)} = \mu_n \gamma_1^{(n)} \gamma_2^{(n)} \cdots \gamma_{2k-1}^{(n)}. \tag{24}$$

With the notations (17) we get back the classical formulae for the coefficients of the qd-table

$$q_k^{(n)} = \frac{H_k^{(n+1)} H_{k-1}^{(n)}}{H_{k-1}^{(n+1)} H_k^{(n)}}, \quad e_k^{(n)} = \frac{H_{k+1}^{(n)} H_{k-1}^{(n+1)}}{H_k^{(n)} H_k^{(n+1)}}, \tag{25}$$

We also have the two corollaries (where we suppose $\mu_n \neq 0$):

Corollary 8. *Starting from the sequence $\{\mu_n\}_{n\geq 0}$, the qd-algorithm can be performed iff $H_k^{(n)}(\mu) \neq 0$ for every $n, k \geq 0$.*

Corollary 9. *The qd-transform of the sequence γ exists iff $H_k^{(1)}(\mu)$ and $H_k^{(2)}(\mu)$ are $\neq 0$ for every $k \geq 0$ (with $\mu = (\mu_n)_{n\geq 0}$ defined by (1)).*

7 Application to enumeration

This section results from a joined work with M. Desainte-Catherine.

In the Young tableaux enumerative problem of Desainte-Catherine, Viennot [6], we need to compute the qd-table in the case of the Catalan numbers, that is $\mu_n = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proposition 10. *The coefficients of the qd-table corresponding to the Catalan numbers $\mu_n = C_n$ are given by*

$$q_k^{(n)} = \frac{(2n+2k-1)(2n+2k)}{(n+2k-1)(n+2k)}, \quad e_k^{(n)} = \frac{2k(2k+1)}{(n+2k)(n+2k+1)}. \tag{26}$$

Proof. We just have to check that these numbers satisfy the rhombus rules (18). We have successively

$$e_k^{(n)}q_{k+1}^{(n)} = \frac{2k(2k+1)(2n+2k+1)(2n+2k+2)}{(n+2k)(n+2k+1)(n+2k+1)(n+2k+2)} = e_k^{(n+1)}q_k^{(n+1)}.$$

$$\begin{aligned} q_{k+1}^{(n)} + e_{k+1}^{(n)} &= \frac{(2n+2k+1)(2n+2k+2)}{(n+2k+1)(n+2k+2)} + \frac{(2k+2)(2k+3)}{(n+2k+2)(n+2k+3)}, \\ &= \frac{4n^3 + n^2(16k+18) + n(24k^2+52k+26) + 2(2k+1)(2k+2)(2k+3)}{(n+2k+1)(n+2k+2)(n+2k+3)}, \\ &= \frac{(2n+2k+3)(2n+2k+4)}{(n+2k+2)(n+2k+3)} + \frac{2k(2k+1)}{(n+2k+1)(n+2k+2)}, \\ &= q_{k+1}^{(n+1)} + e_k^{(n+1)}. \square \end{aligned}$$

Corollary 11. *The number of non-crossing configurations of k Dyck paths $\eta = (w_1, w_2, \cdots, w_k)$ such that for i, $1 \le i \le k$, w_i goes from the point $(-2i+2, 0)$ to the point $(2n+2i-2, 0)$ is*

$$d_{n,k} = \prod_{1 \le i < j < n} \frac{i+j+2k}{i+j}.$$

Proof. From the above considerations, this number is the $k \times k$ Hankel determinant (for $\mu_n = C_n$)

$$H_k^{(n)} = (C_n)^k (q_1^{(n)}e_1^{(n)})^{k-1}(q_2^{(n)}e_2^{(n)})^{k-2} \cdots (q_{k-1}^{(n)}e_{k-1}^{(n)}), \tag{27}$$

with $q_k^{(n)}$ and $e_k^{(n)}$ defined by (26). We have successively

$$C_n q_1^{(n)} e_1^{(n)} q_2^{(n)} e_2^{(n)} \cdots q_{k-1}^{(n)} e_{k-1}^{(n)} = \frac{(2k-1)!(2n+2k-2)!}{(n+2k-1)!(n+2k-2)},$$

$$d_{n,k}/d_{n,k-1} = \frac{(2k+n)(2k+n+1)\cdots(2k+2n-2)}{2k(2k+1)\cdots(2k+n-2)},$$

$$d_{n,k}/d_{n,k-1} = H_k^{(n)}/H_{k-1}^{(n)},\ (n \ge 1, k \ge 2). \tag{28}$$

With $d_{n,1} = C_n = H_1^{(n)}$ $(n \ge 1)$. We deduce $H_k^{(n)} = d_{n,k}$. □

The formula of corollary 11 reappeared in Physics in the context of directed polymers with watermelons topoly in the presence of a wall. Extensions are given in Guttmann, Krattenthaler, Viennot [13]. This work follows Guttmann, Owczarek, Viennot [14].

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