Laguerre heaps of segments for the PASEP

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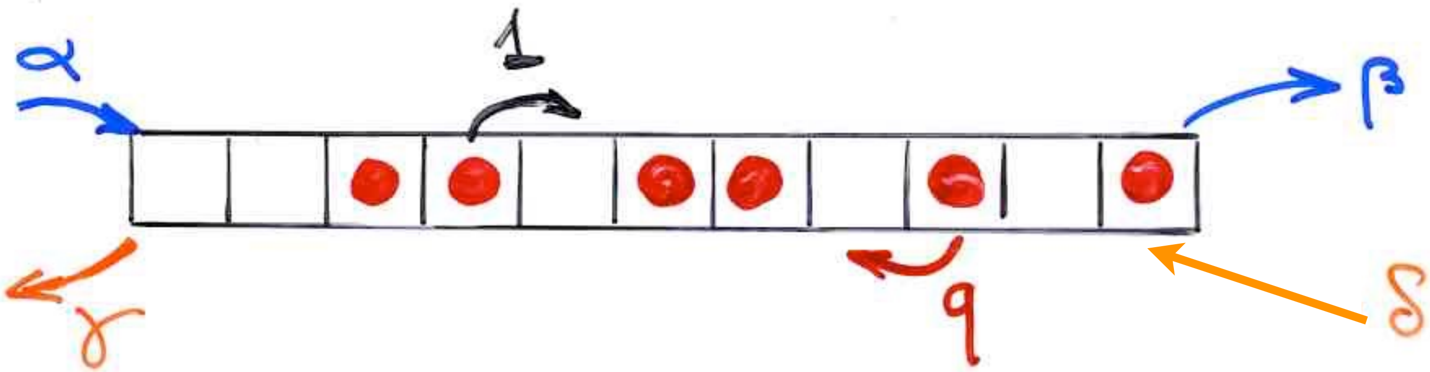
toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

$$\begin{cases} DE = qED + E + D \\ \langle W| (\alpha E - \gamma D) = \langle W| \\ (\beta D - \delta E)|V\rangle = |V\rangle \end{cases}$$

colum vector V

row vector W

• Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

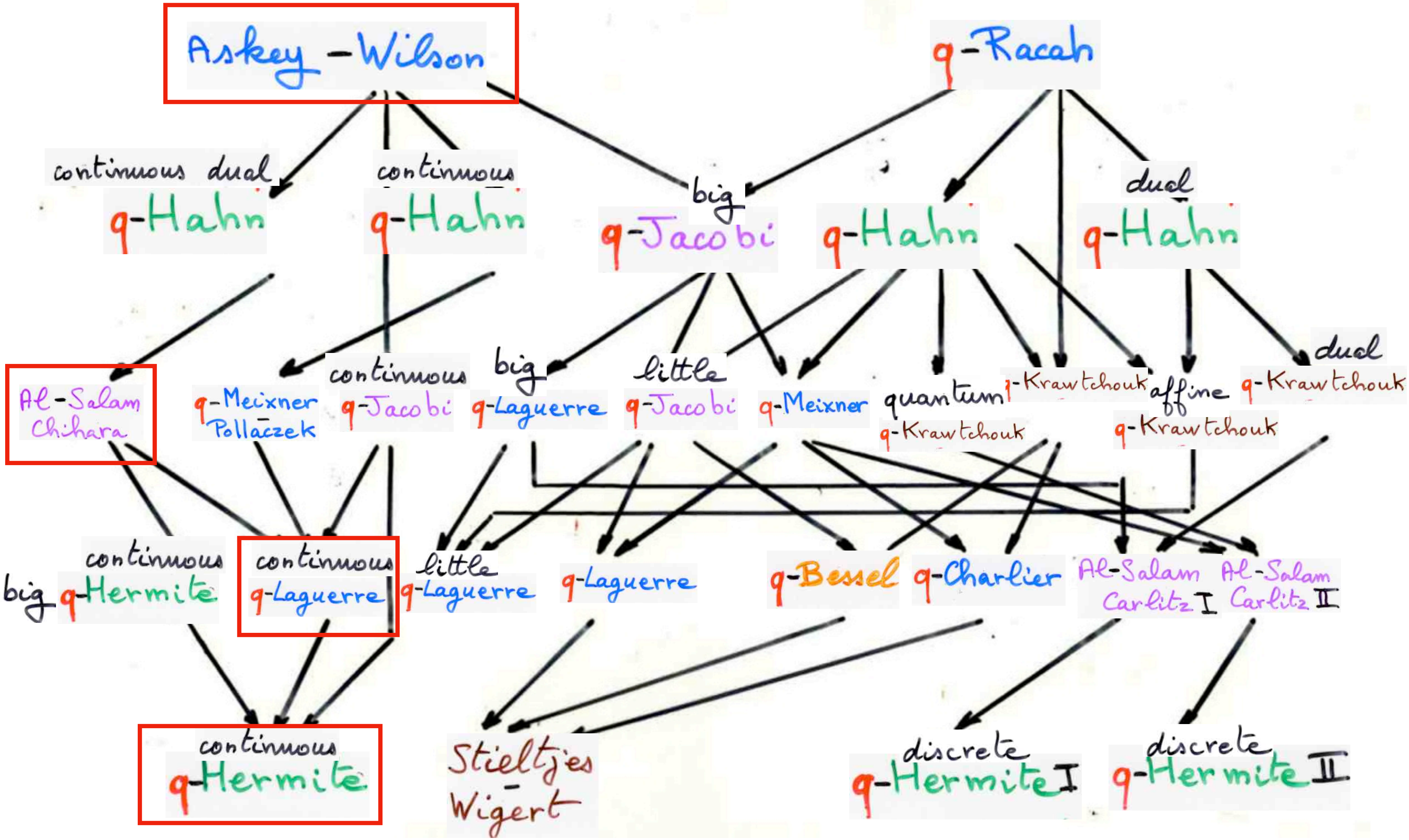
$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

scheme of basic hypergeometric orthogonal polynomials



Combinatorics of the PASEP

(1982) Shapiro, Zeilberger

(2004) Brak, Essam

(2005) Duchi, Schaeffer

(2006) Corteel Burstein

Brak, Corteel, Essam, Parviainen, Rechnitzer

Corteel, Williams

(2007) Corteel, Nadeau Corteel, Williams

Steingrimsson, Williams X.V.

(2008) X.V.

(2009) Corteel, Josuat-Vergès, Prellberg, Rubey

Josuat-Vergès Nadeau

(2010) Corteel, Williams

(2011) Josuat-Vergès Corteel, Kim
Corteel, Dasse-Hartaut
Corteel, Josuat-Vergès, Williams
Aval, Boussicault, Nadeau

(2012) Corteel, Stanley, Stanton, Williams

(2013) Aval, Boussicault, Bouvel, Silimbani
Aval, Boussicault, Nadeau
Aval, Boussicault, Dasse-Hartaut

(2014) E. Jin

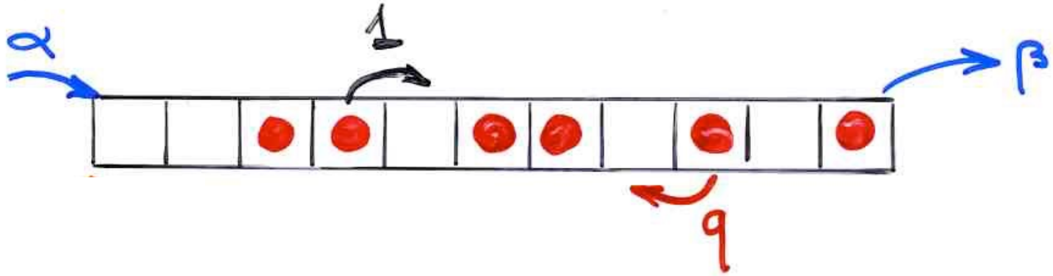
(2016) Aval, Boussicault, Delcroix-Oger, Hivert, Laborde-Zubieta
Corteel, Kim, Stanton Mandelshtam, X.V.

(2017) Corteel, Williams Mandelshtam, X.V.
Corteel, Mandelshtam, Williams
Corteel, Nunge
Laborde-Zubieta

alternative tableaux

PASEP with 3 parameters

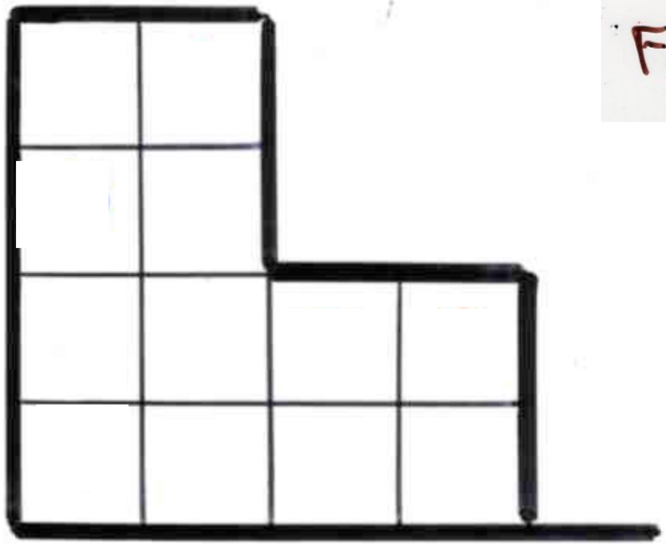
$\gamma = \delta = 0$ $\quad$ q, α, β



PASEP

alternative tableau

Definition



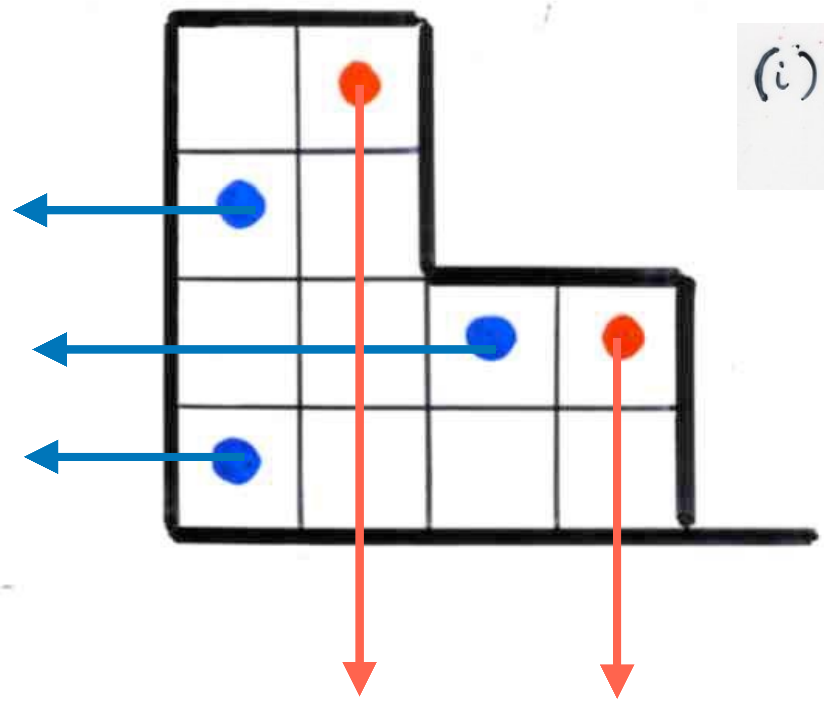
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

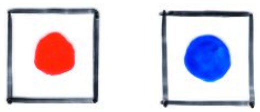
$\mathbf{n} = (\text{number of rows}) + (\text{number of columns})$

alternative tableau



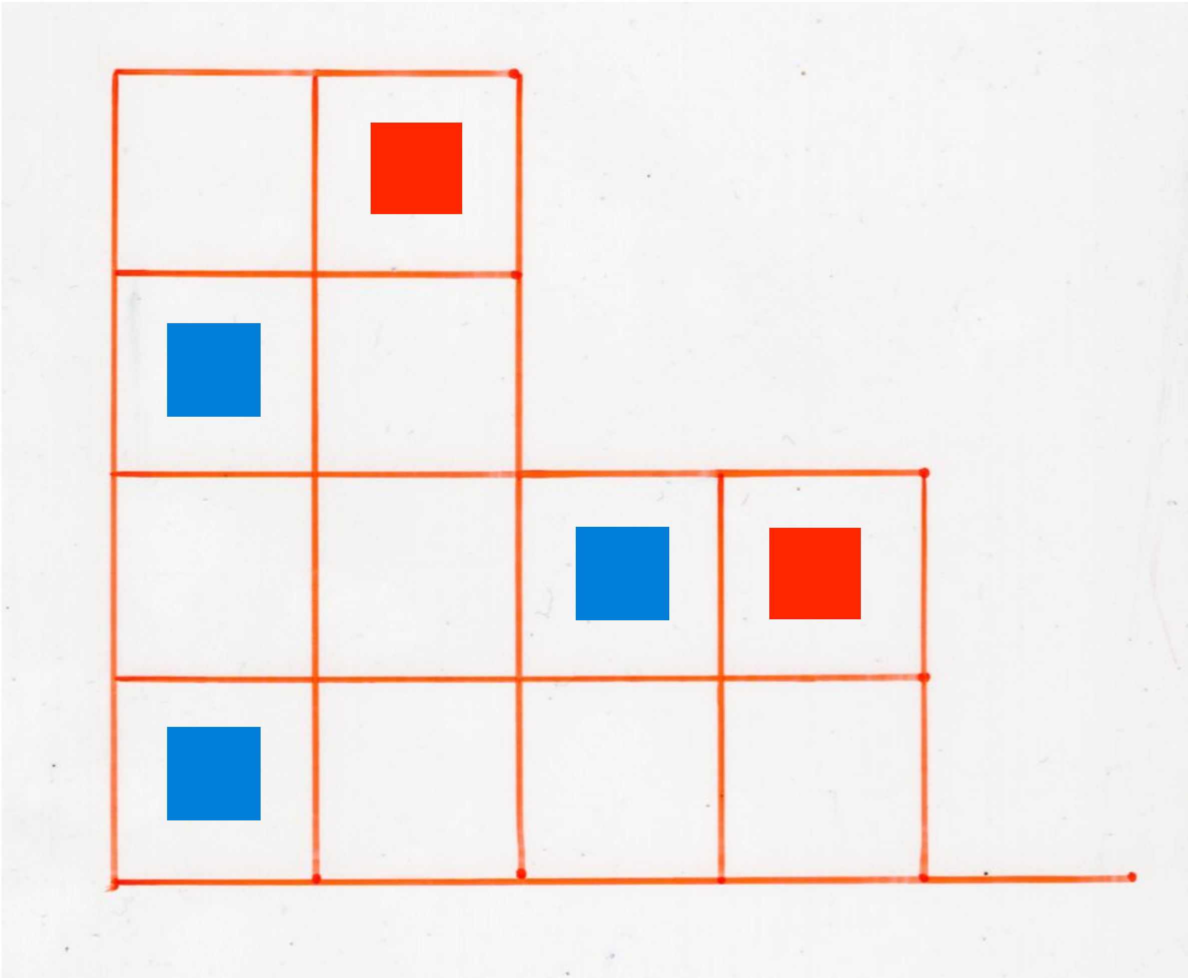
Definition

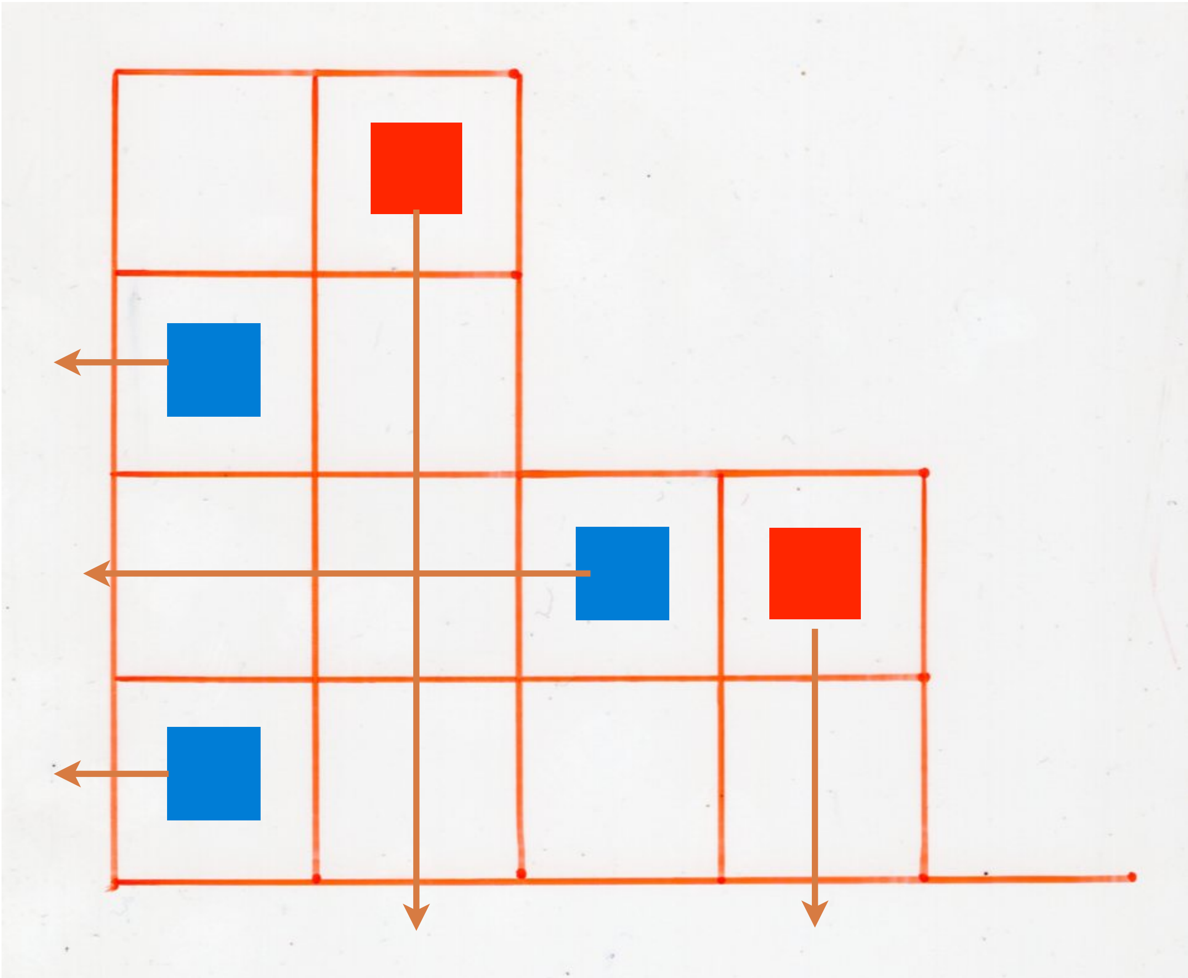
(i) some cells are coloured red or blue

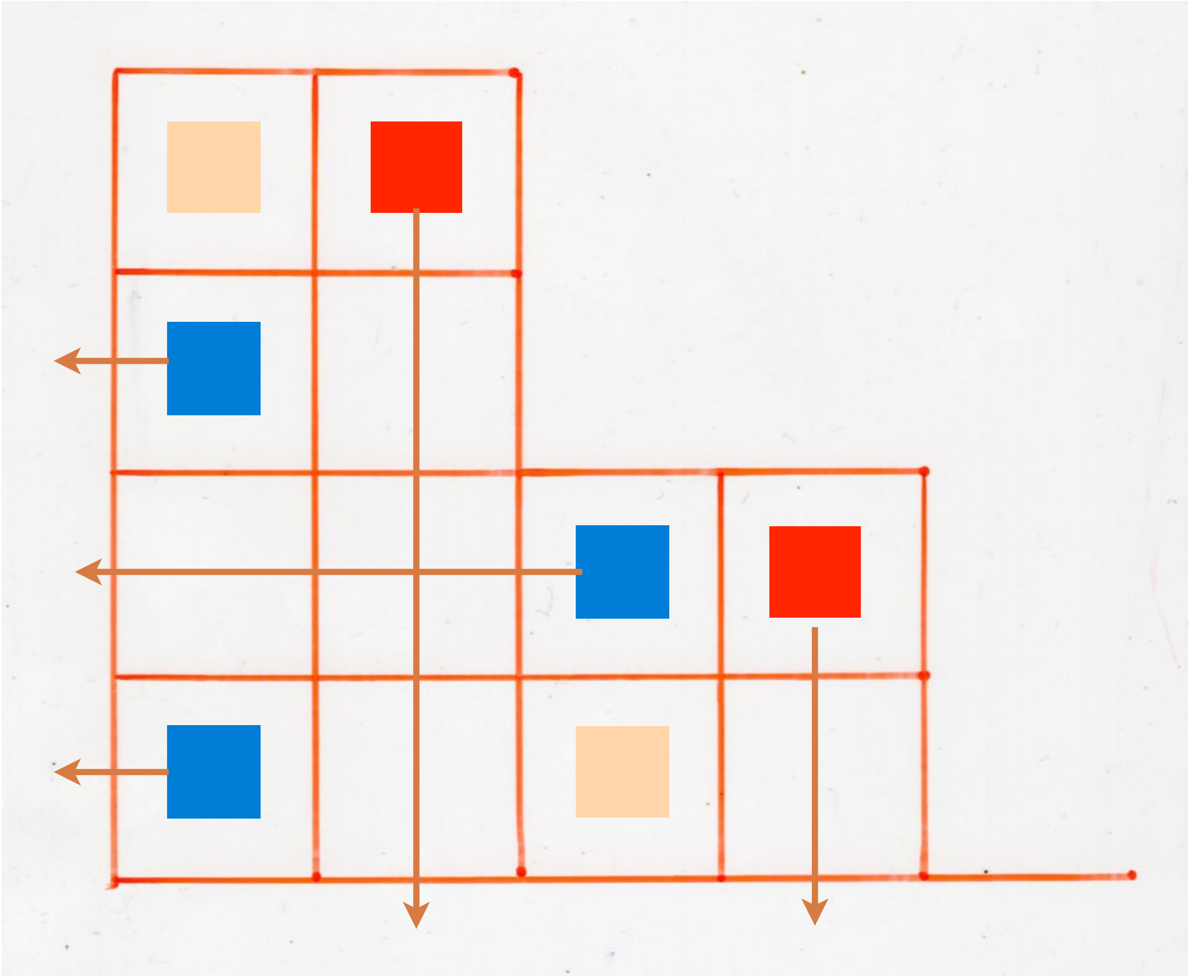


(ii) • no coloured cell at the left of a blue cell

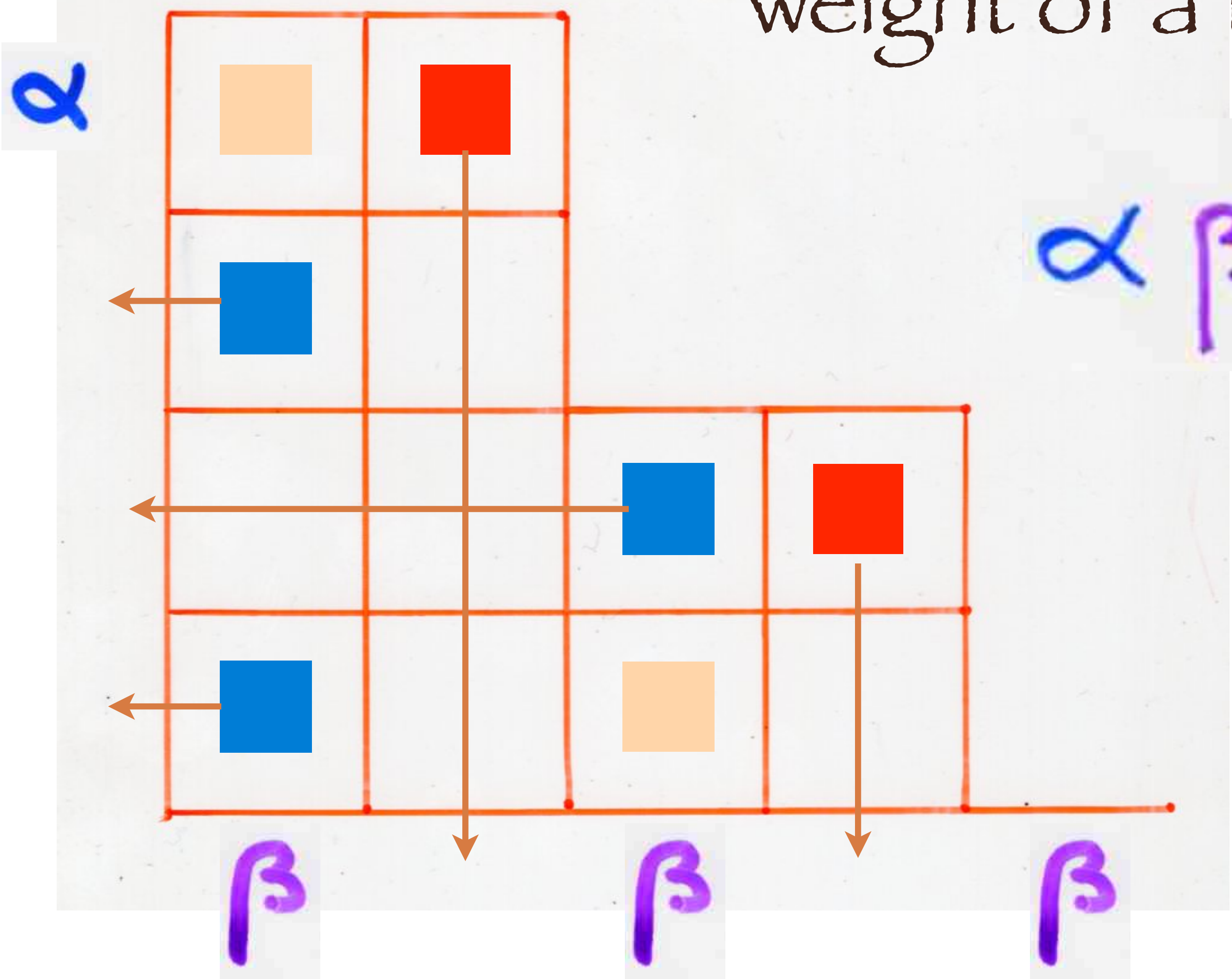
• no coloured cell below a red cell







weight of a tableau



$\alpha \beta^3 q^2$

q $k(T)$ = nb of cells

α $i(T)$ = nb of rows without

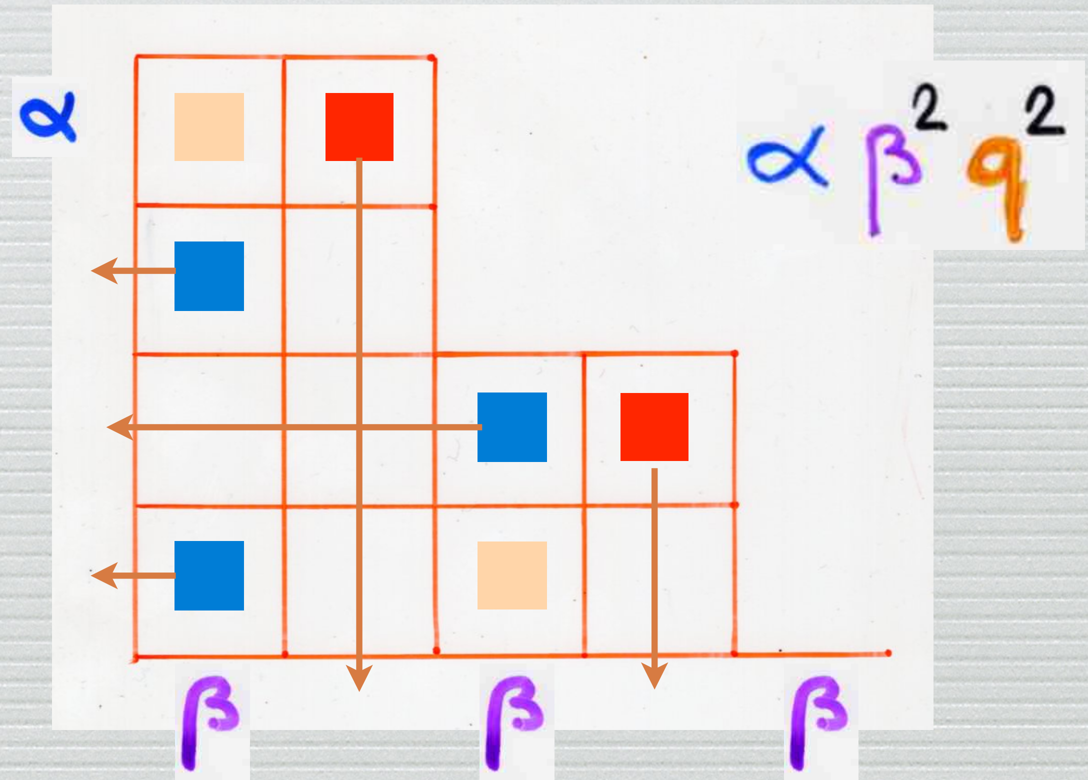
β $j(T)$ = nb of columns without

Partition function

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

$$Z_n$$

Sum of the weight of all tableaux of size n



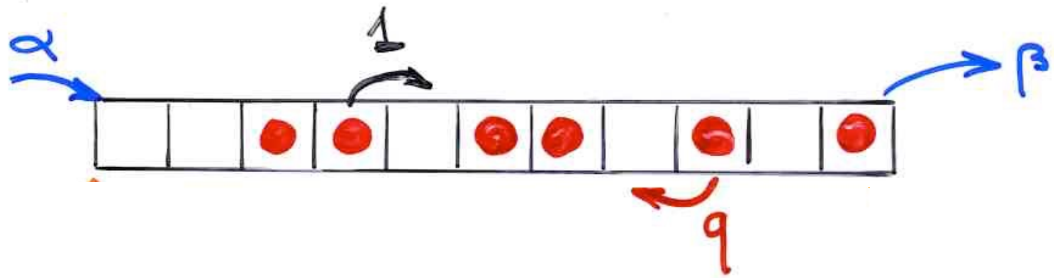
q $k(T)$ = nb of cells

α $i(T)$ = nb of rows without

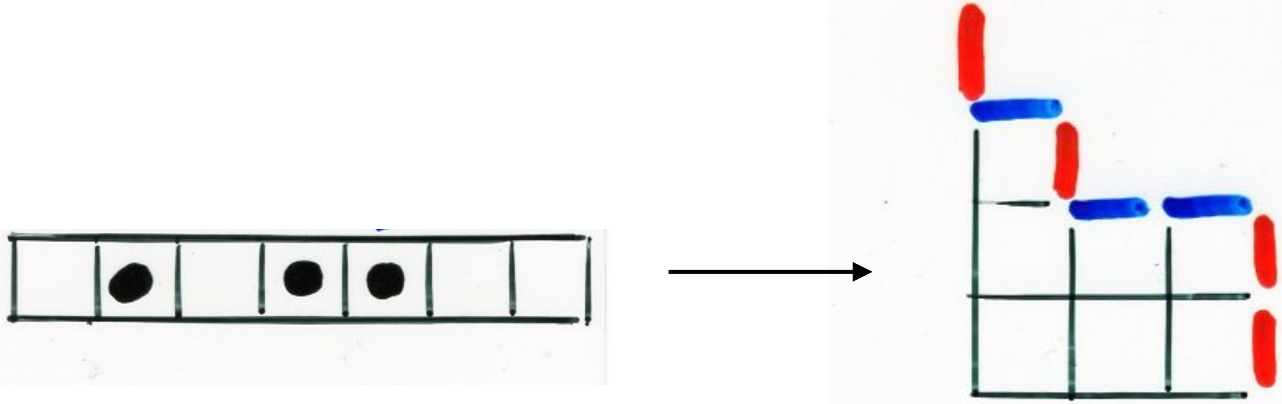
β $j(T)$ = nb of columns without

computation of the
"stationary probabilities"

PASEP with 3 parameters q, α, β



PASEP



<u>Corollary.</u> The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

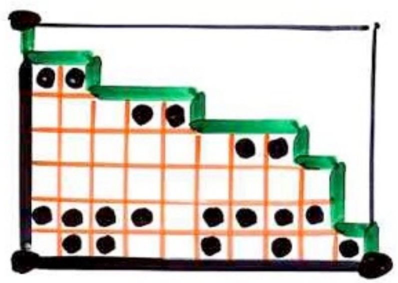
alternative tableau
X.V. (2008)

permutation tableau

S. Corteel, L. Williams
(2007, 2008, 2009)

permutation tableaux

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column: at least one 1

(ii) $\begin{array}{cc} 1 & 0 \\ & 1 \end{array}$ forbidden

□ = 0   ⊡ = 1

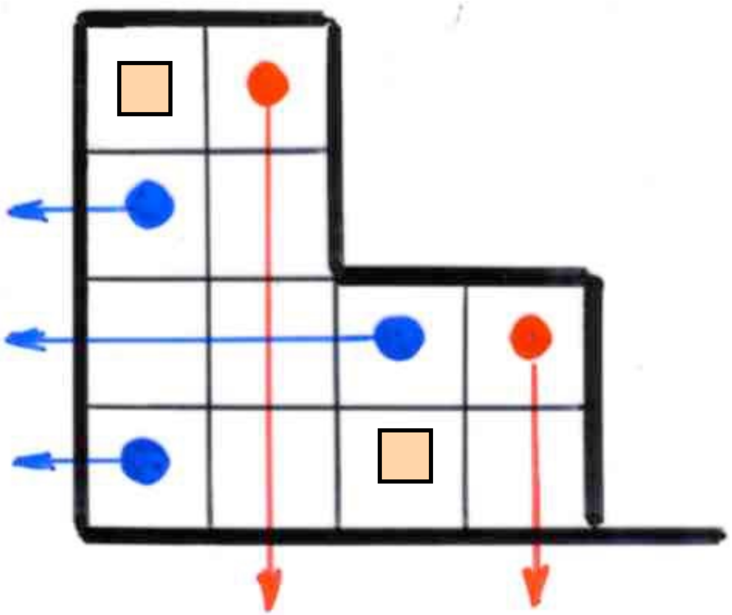
permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

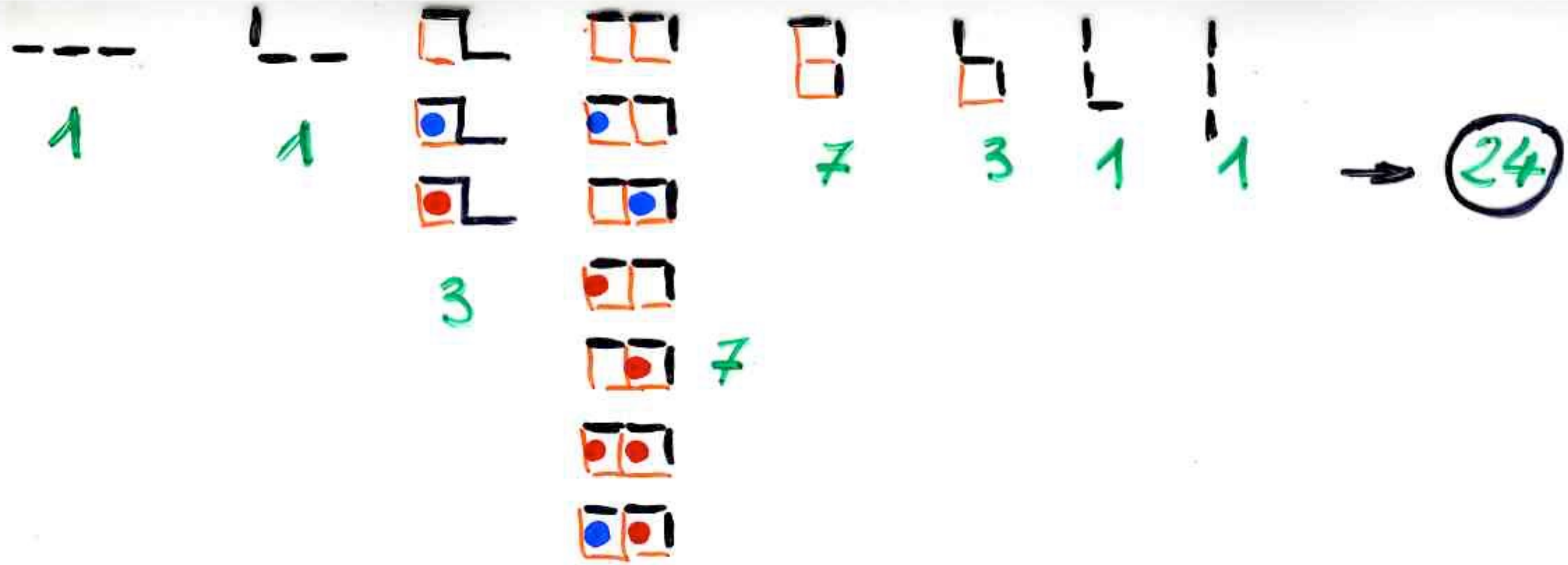
E. Steingrímsson, L. Williams (2005)

Enumeration of alternative tableaux



<u>Prop.</u> The number of alternative tableaux of size n is $(n+1)!$

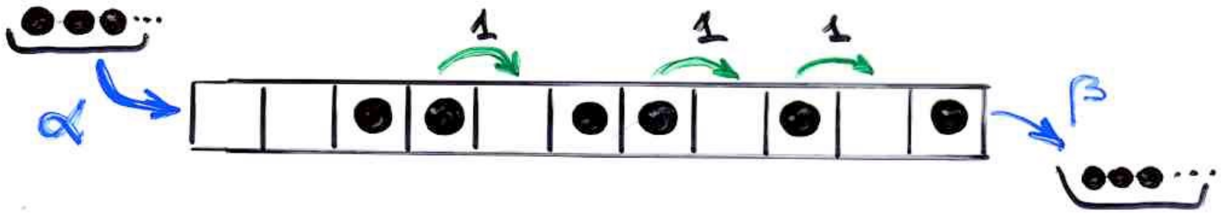
ex: n=2



$q = 0$ TASEP (α, β)

TASEP

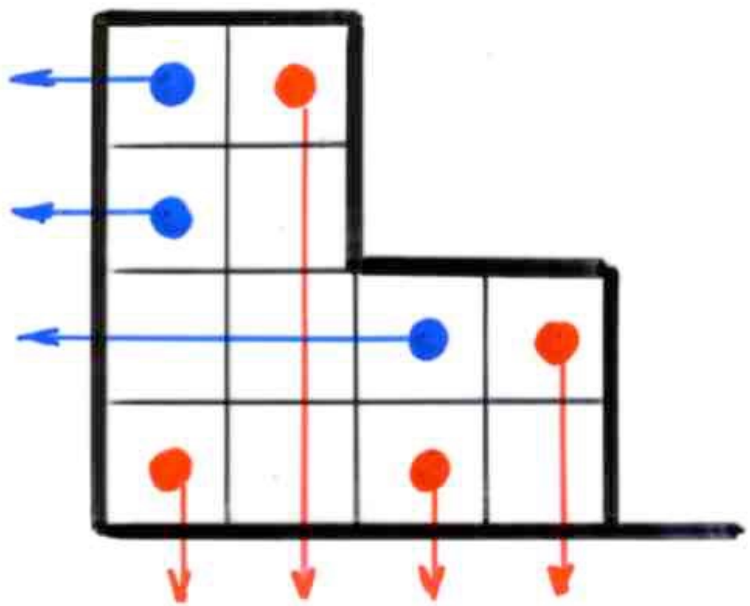
"totally asymmetric exclusion process"



Definition Catalan alternative tableau

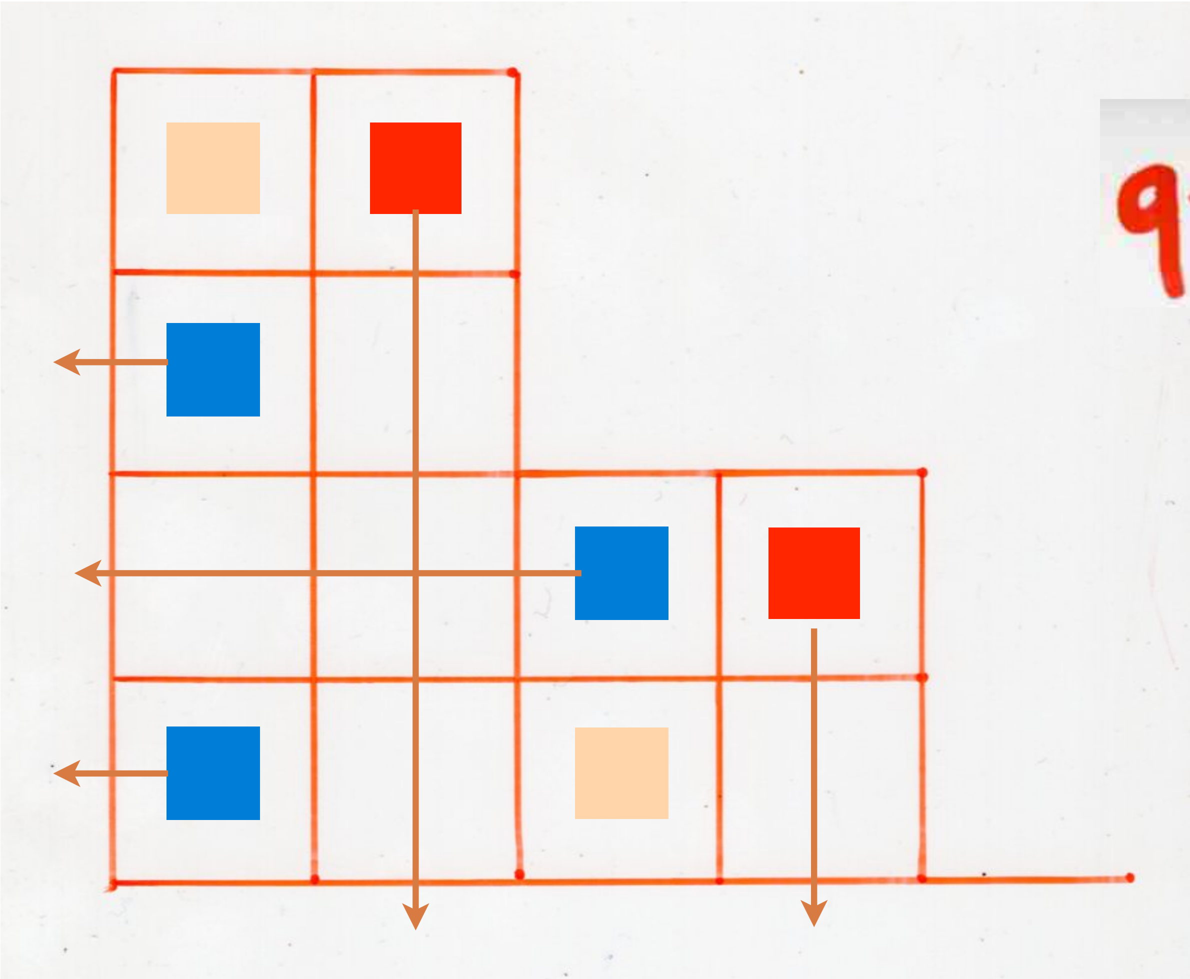
alternative tableau T without cells $\square$

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



q- analog of n!

?

q-analog of $n!$

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

Inv

number of inversions

Maj

Major index

Interpretation of the 3-parameters Partition function

q, α, β

Josuat-Vergès (2011)

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

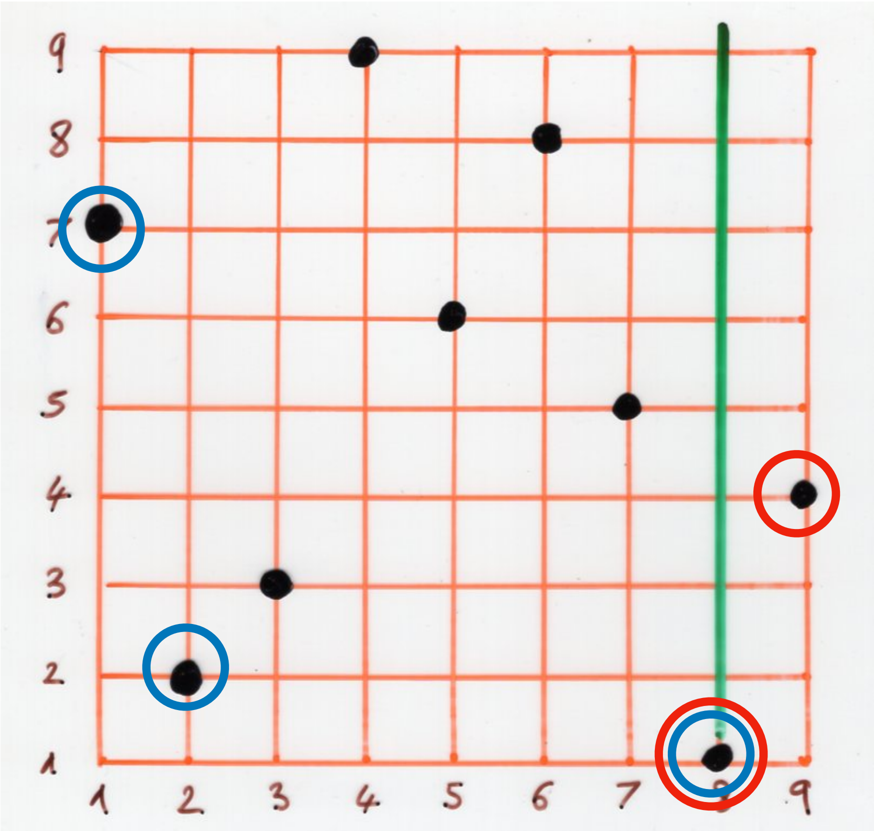
$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

$31\text{-}2(\sigma)$ = number of patterns 31-2



σ = 7 2 3 9 6 8 5 1 4

permutation

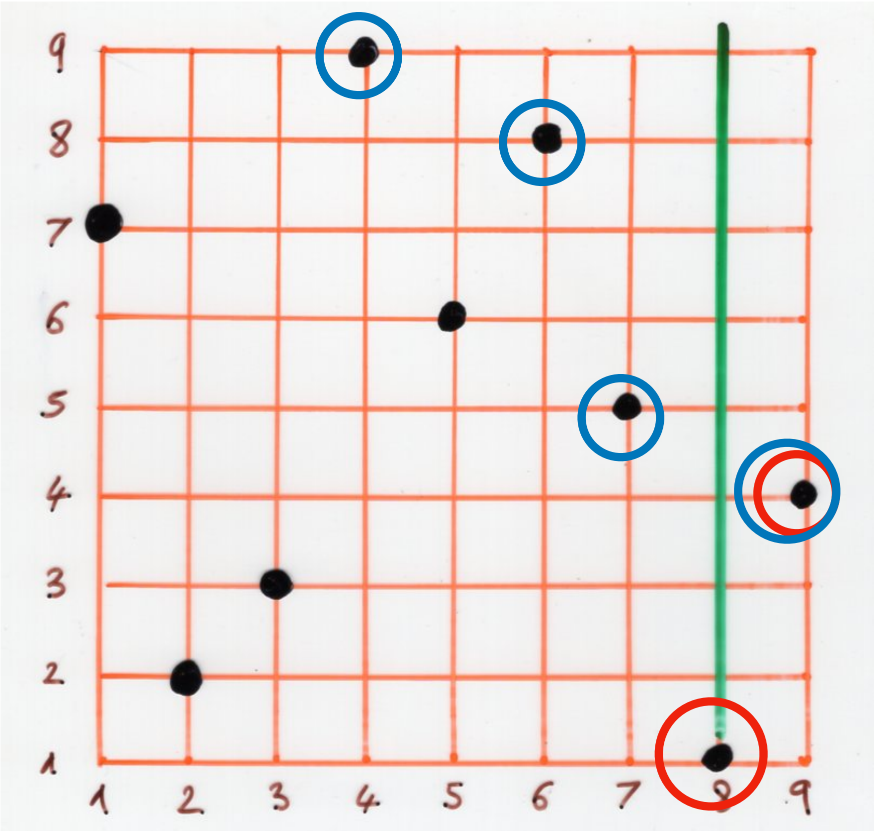
word

left-to-right

right-to-left

minimum

elements



$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ 1\ 4$

permutation word

$s(\sigma) =$ number right-to-left maxima

$t(\sigma) =$ number right-to-left minima

$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$

Josuat-Vergès (2011)

Proposition

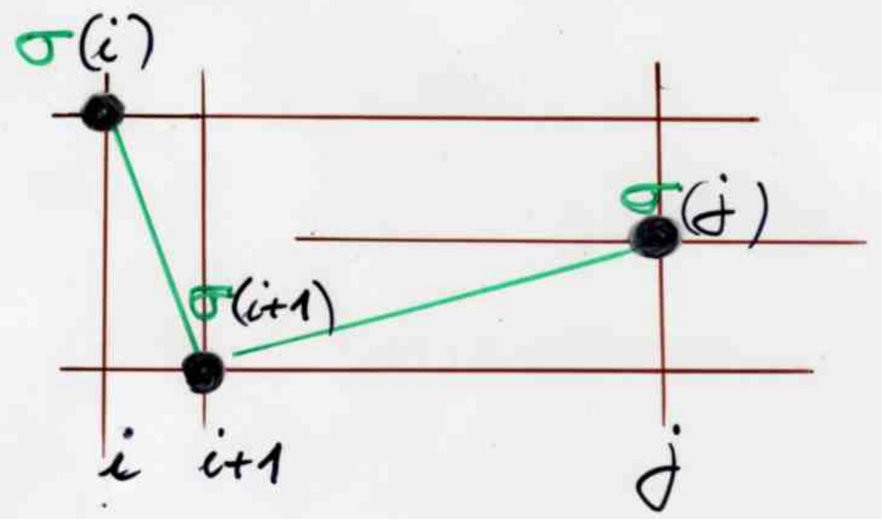
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

31-2



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Josuat-Vergès (2011)

<u>Proposition</u>

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- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Al-Salam-Chihara polynomials

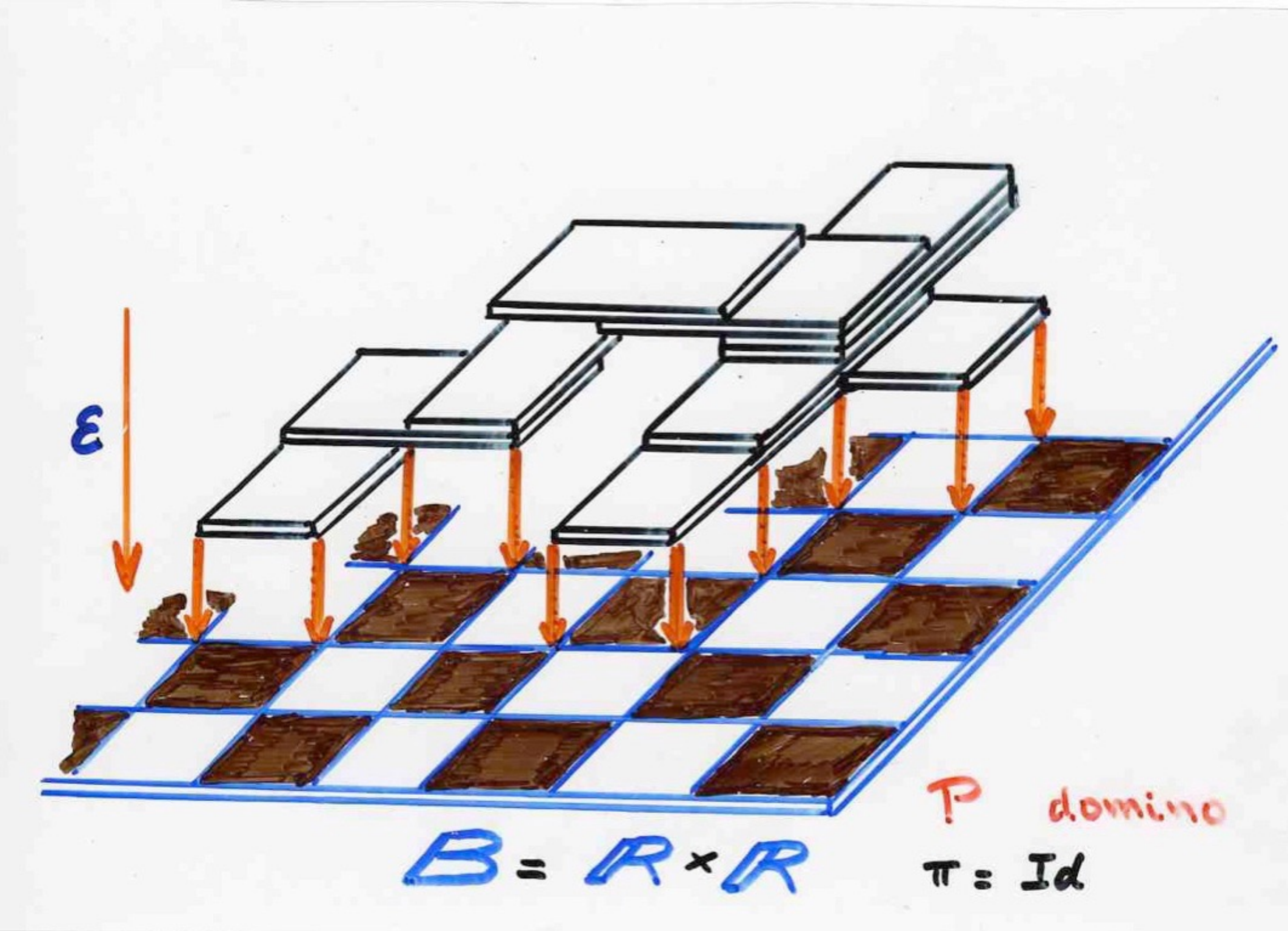
$$2x\, Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})\, Q_{n-1}(x)$$

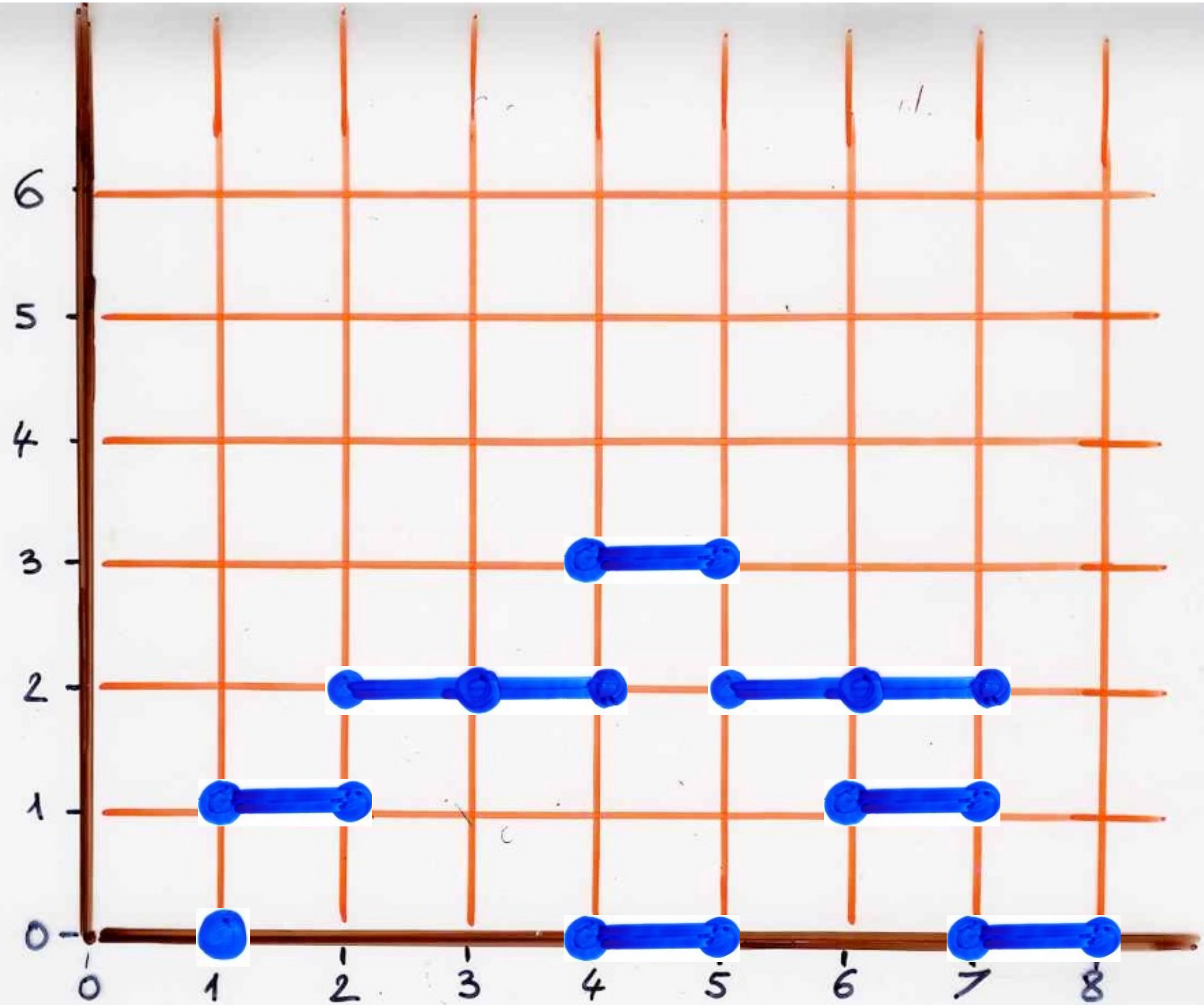
Laguerre heaps of segments

heaps of pieces

Introduction

Heaps




6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

heap definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P { symmetric, reflexive } (dependency relation)
- heap E, finite set of pairs (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

α: projection, i: level

(i)

(ii)

heap   definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases}\text{symmetric}\\\text{reflexive}\end{cases}$

  (**dependency** relation)

- heap E, finite set of pairs

  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  projection   level

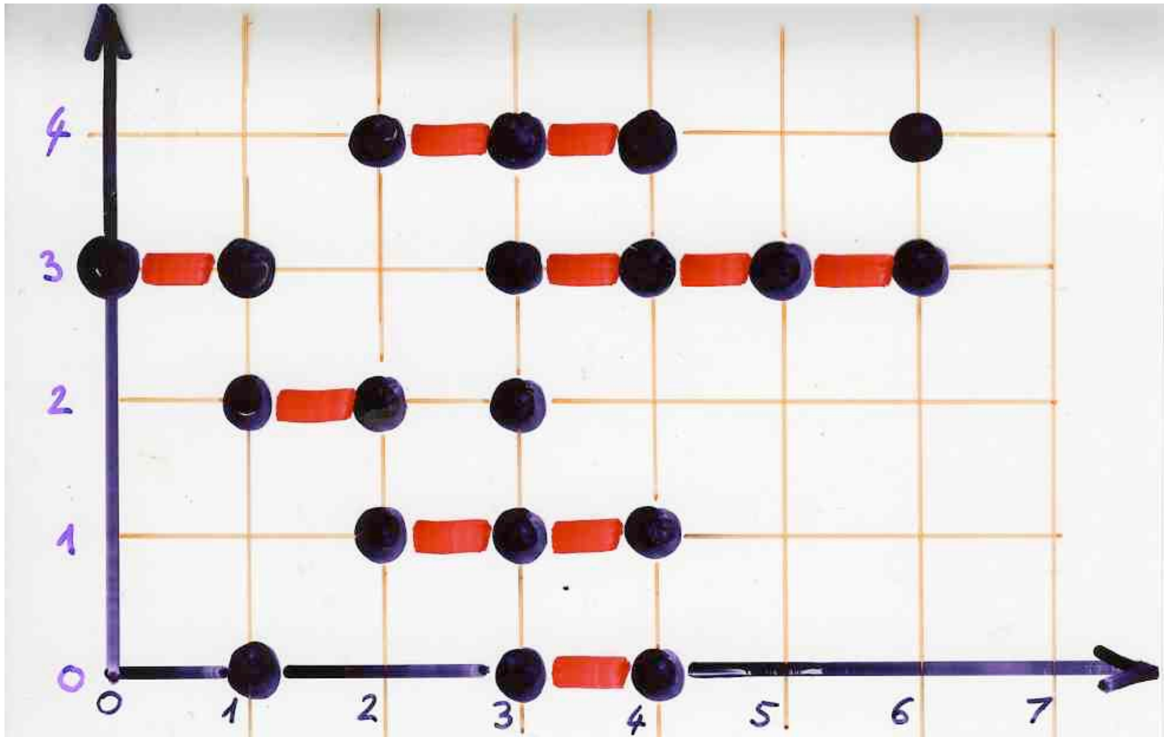
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \,\mathcal{C}\, \beta \implies i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \implies \exists \beta \in P$, $\alpha \,\mathcal{C}\, \beta$, $(\beta, i-1) \in E$

ex: heap of segments over $\mathbb{N}$

$$P = \{ [a,b] = \{a, a+1, \dots, b\} \ , 0 \leq a \leq b \}$$

$$\mathcal{C} \qquad [a,b] \,\mathcal{C}\, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$

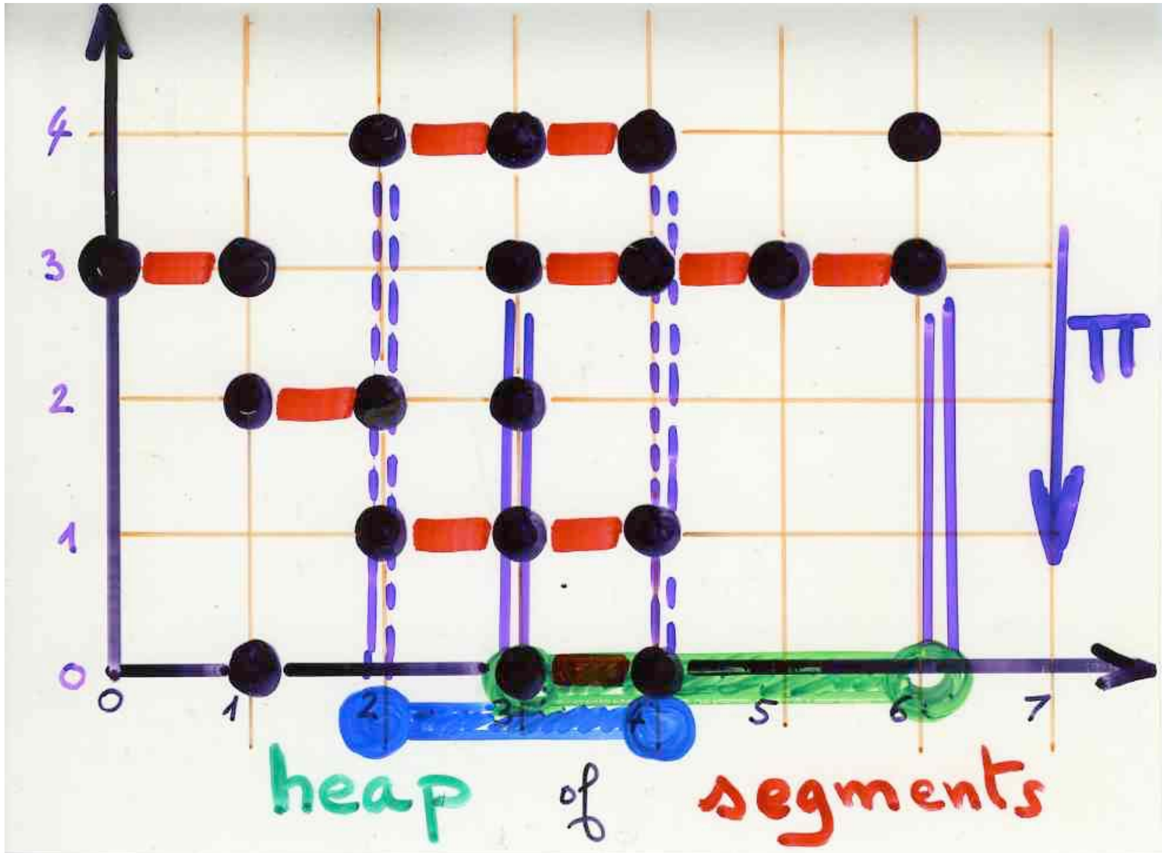


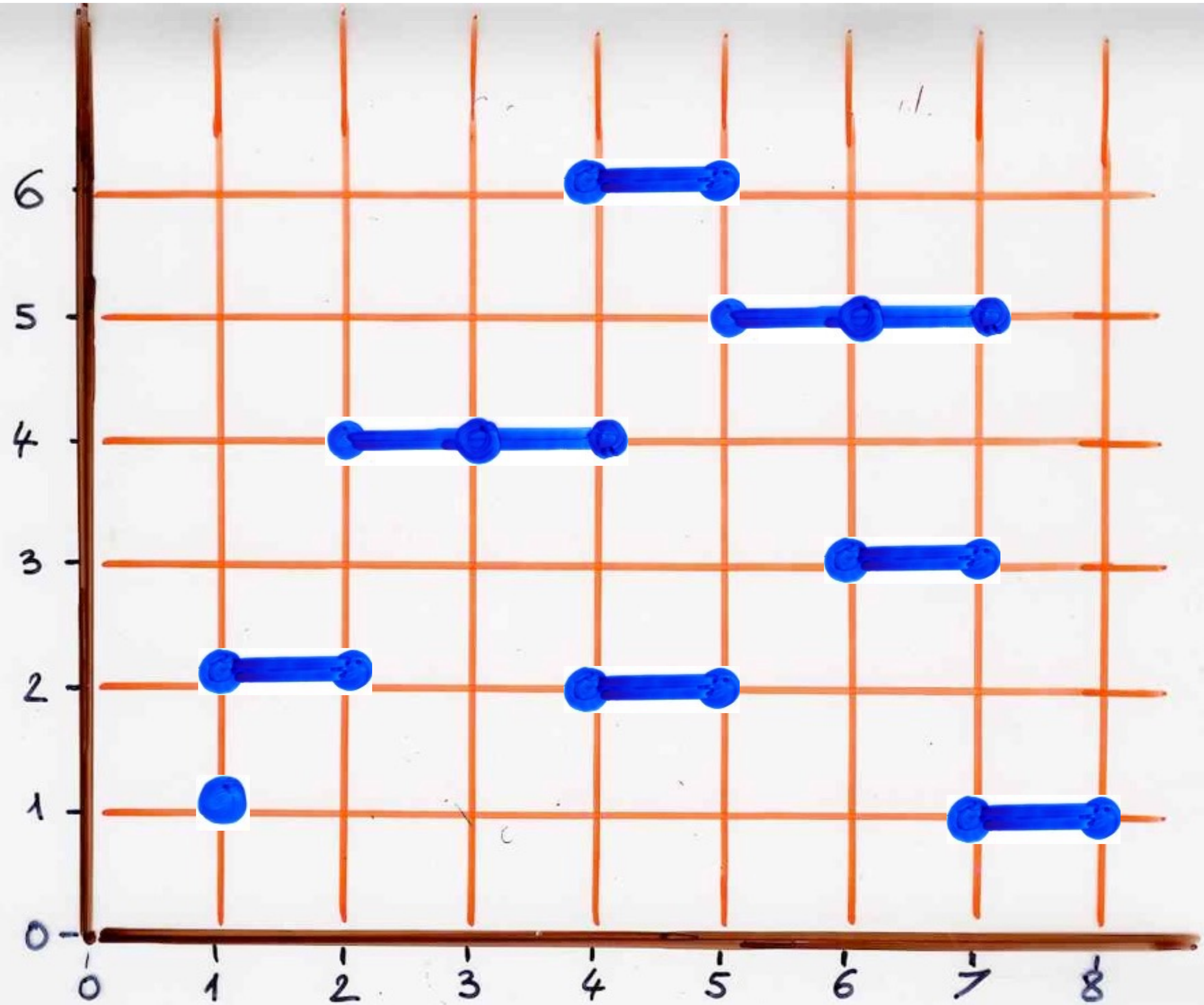
heap of segments

ex: heap of segments over $\mathbb{N}$

$$\mathcal{P} = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \le a \le b \}$$

$$\mathcal{C} \qquad [a,b] \,\mathcal{C}\, [c,d] \Leftrightarrow [a,b] \cap [c,d] \neq \emptyset$$

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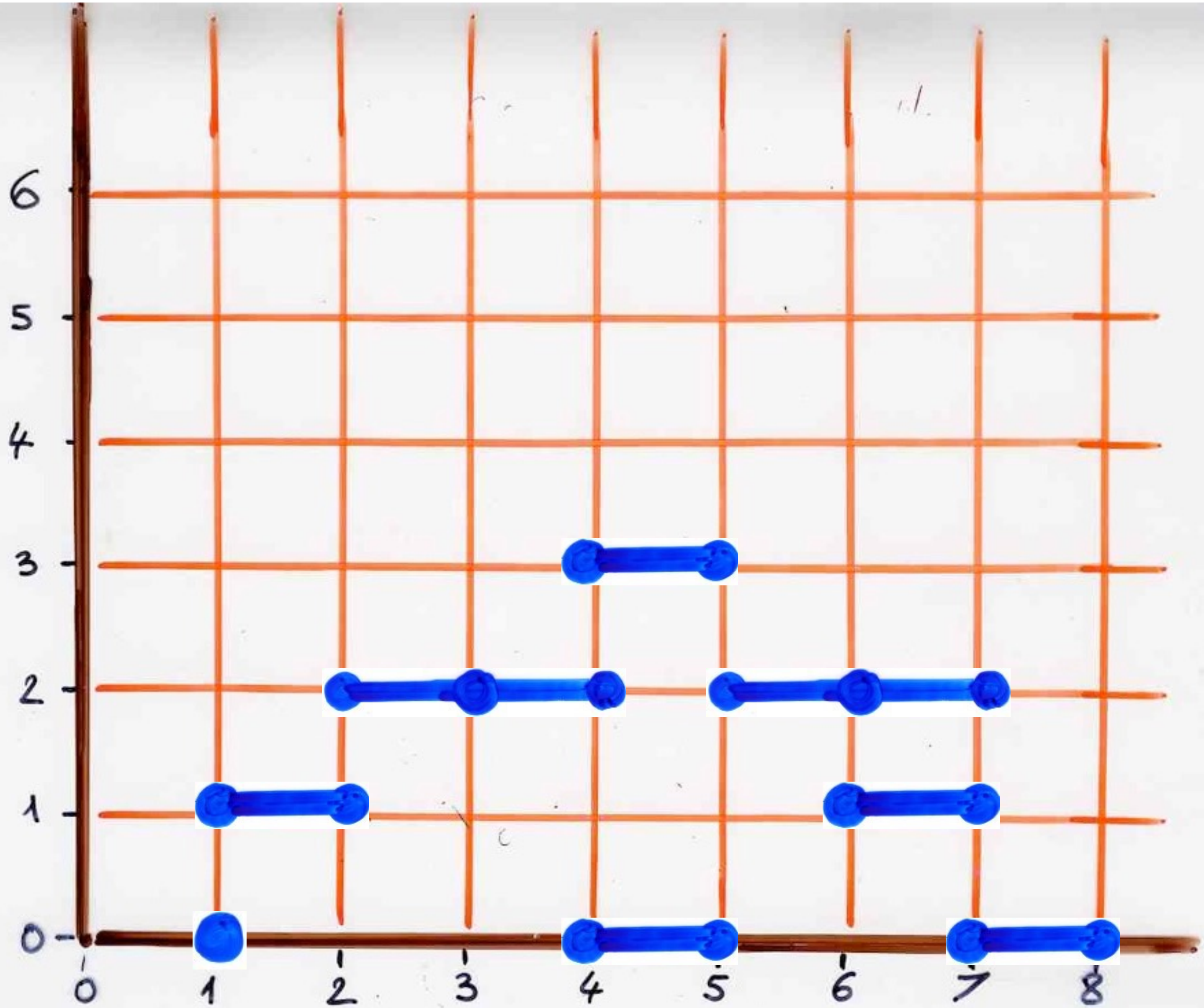

6
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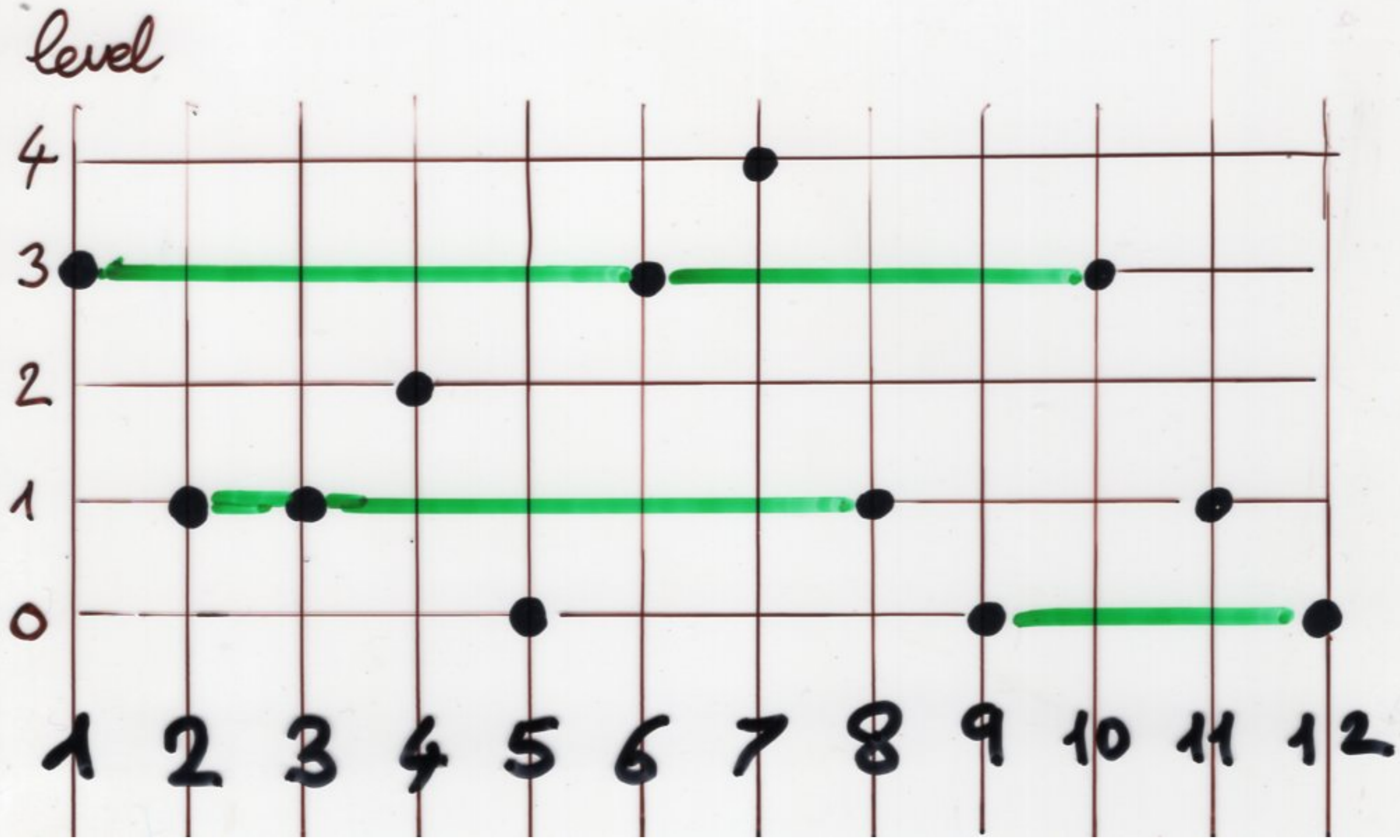
Definition

Laguerre heap on [1, n]



Definition

Laguerre heap on $[1, n]$

- basic piece: pointed segments

  segment $[a, b] = \{a, a+1, \dots, b\}$

  $0 \leq a \leq b$

- pointed: choice of points $a \leq j \leq b$

  including a and b

- dependency relation

  $$[a, b] \cap [c, d] \neq \emptyset$$

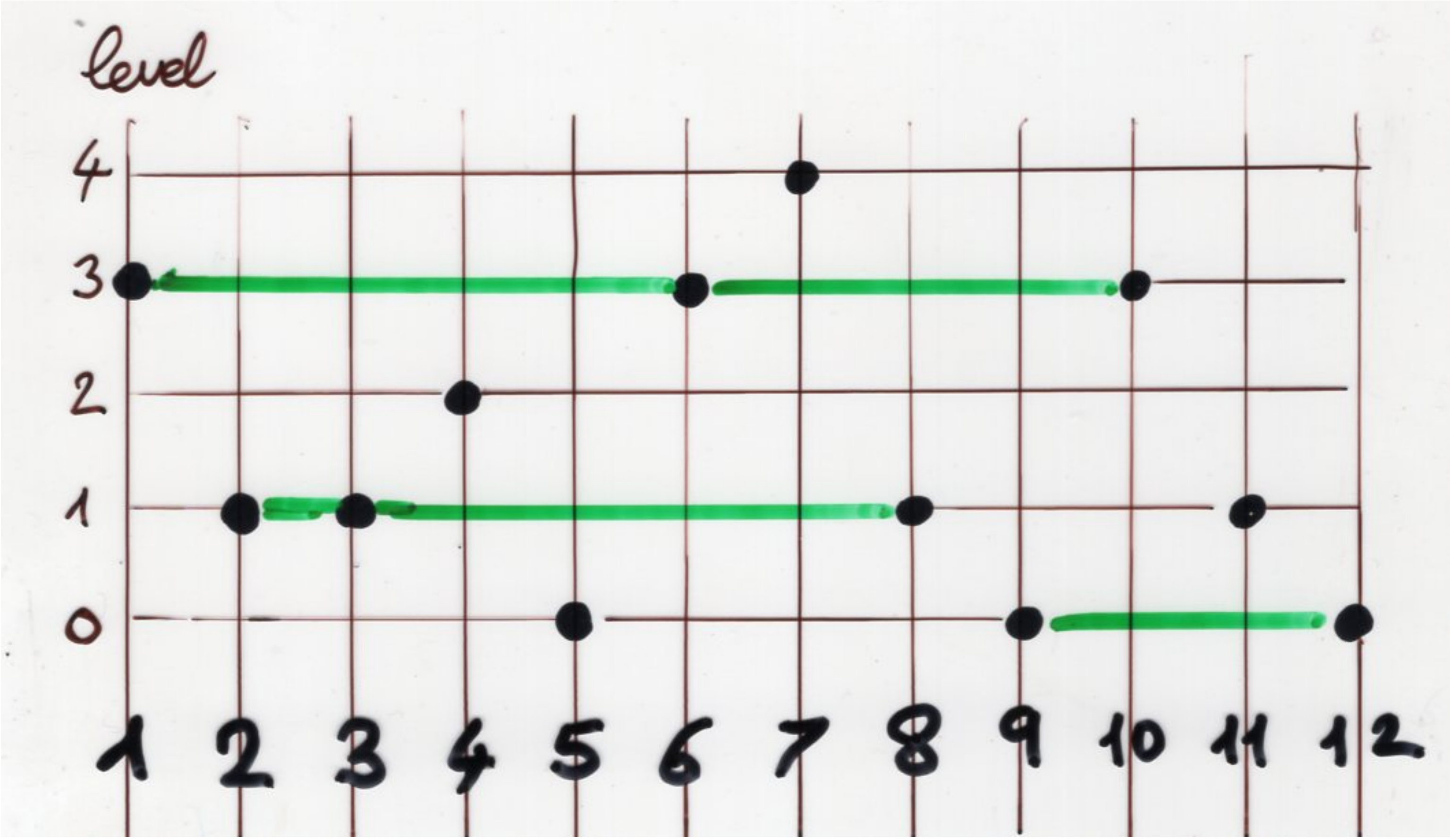
  (same as for segments)

Definition

Laguerre heap on $[1, n]$

- multilinear :

for each j, $1 \leq j \leq n$, there exist one and only one pointed segment of the heap such that j is one of the pointed element of that segment.


level
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3
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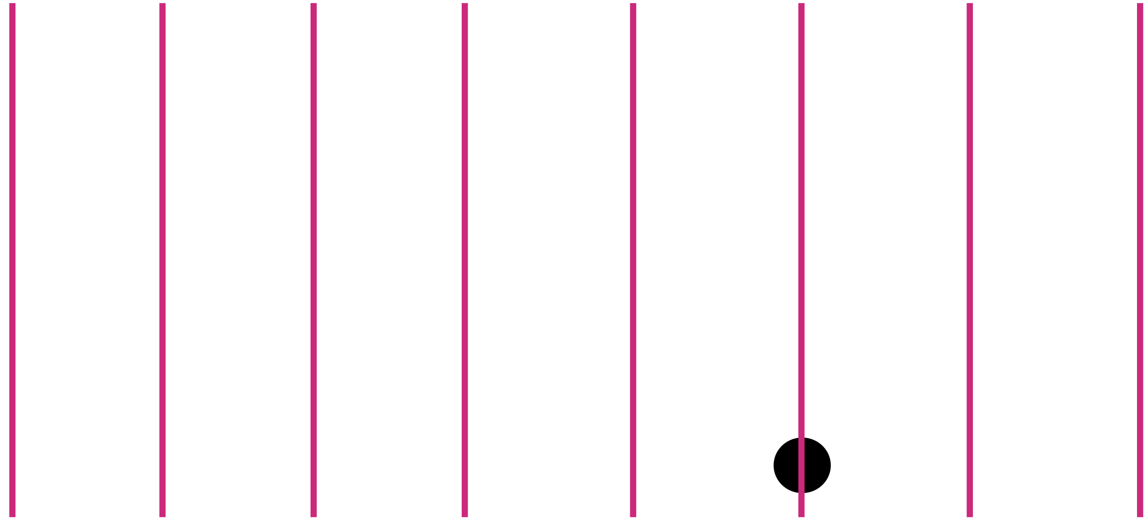
9
10
11
12

Bijection

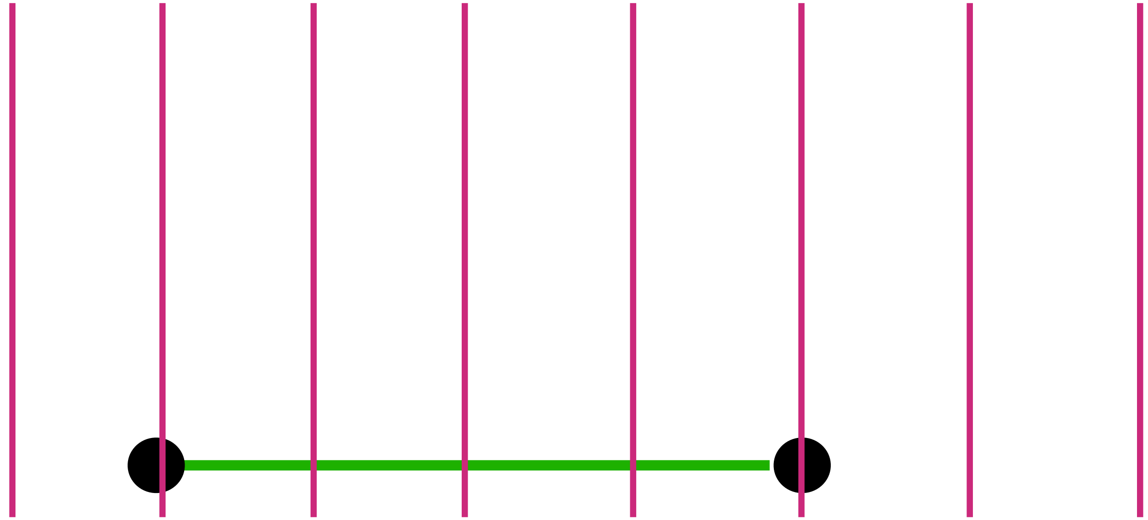
permutations

Laguerre heaps of segments



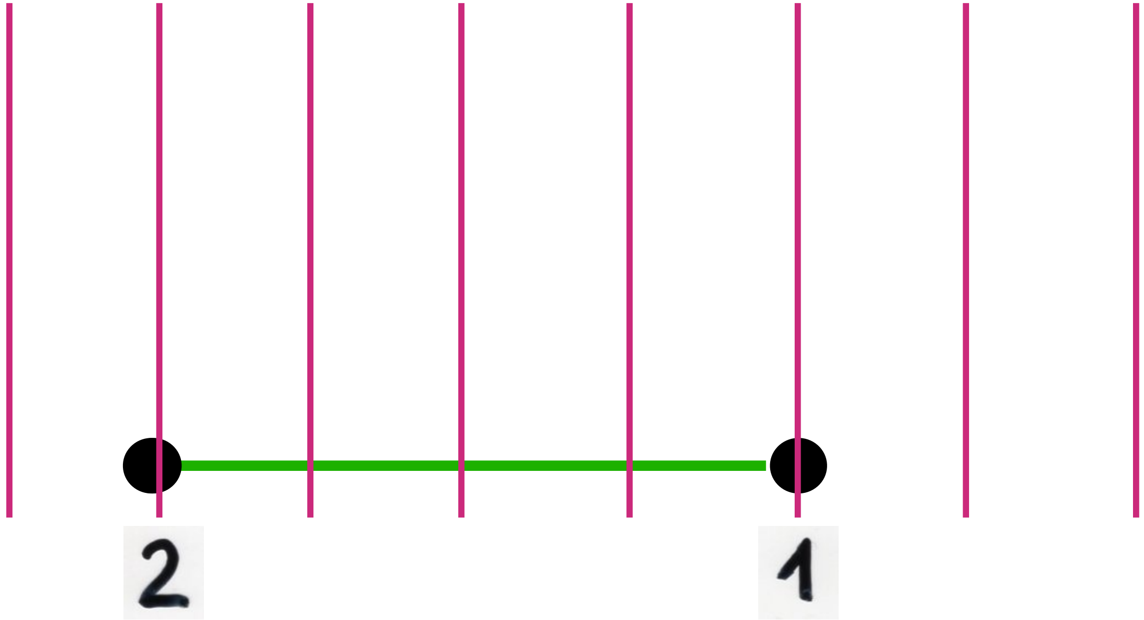
1

6 2 7 3 5 1 8 4

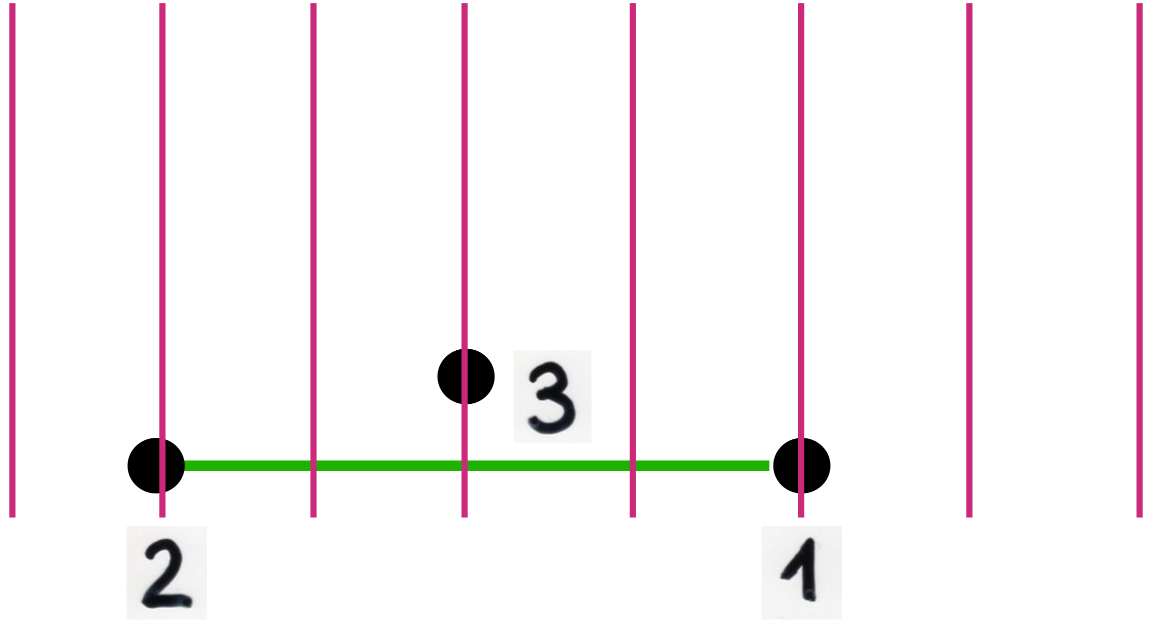


2 1

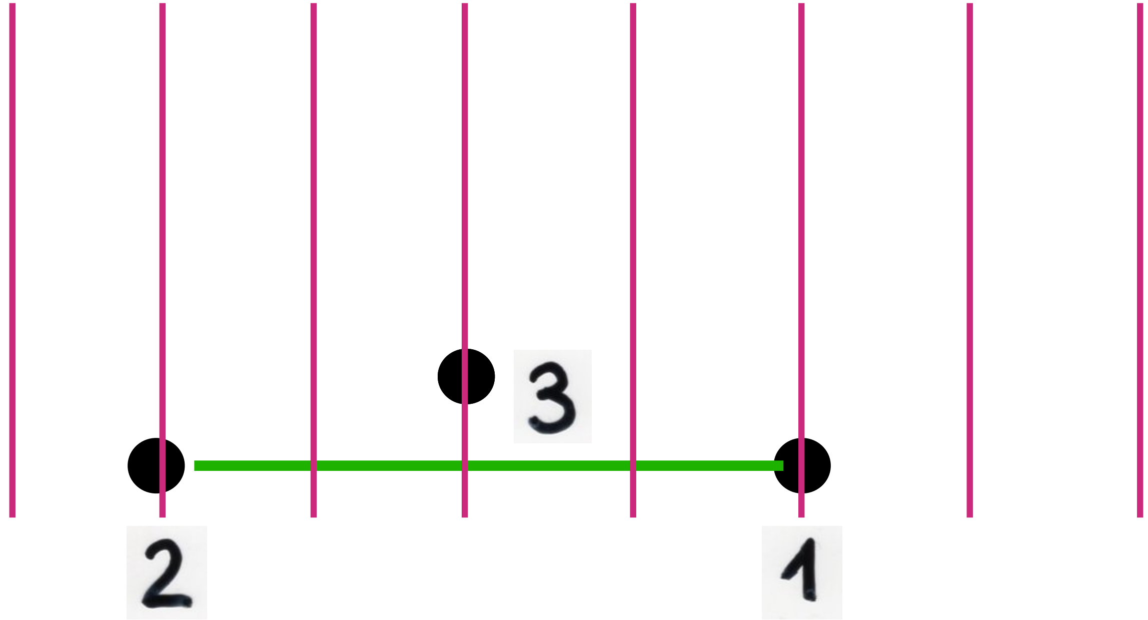
6 2 7 3 5 1 8 4



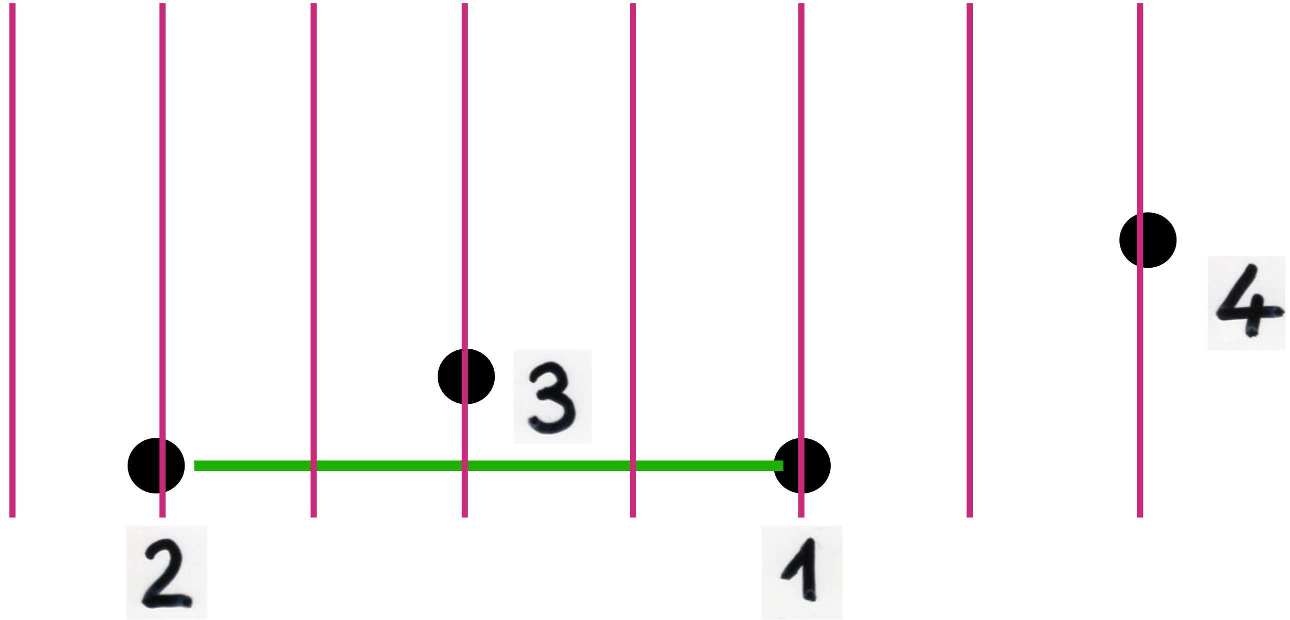
6 2 7 3 5 1 8 4



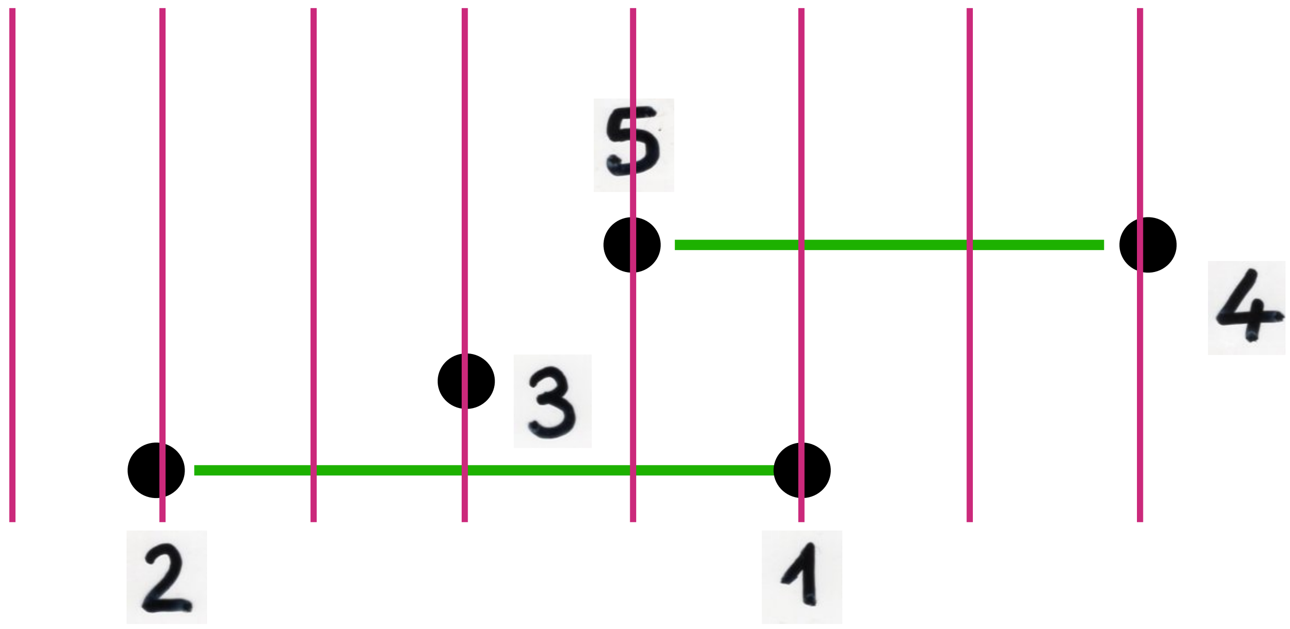
6 2 7 3 5 1 8 4



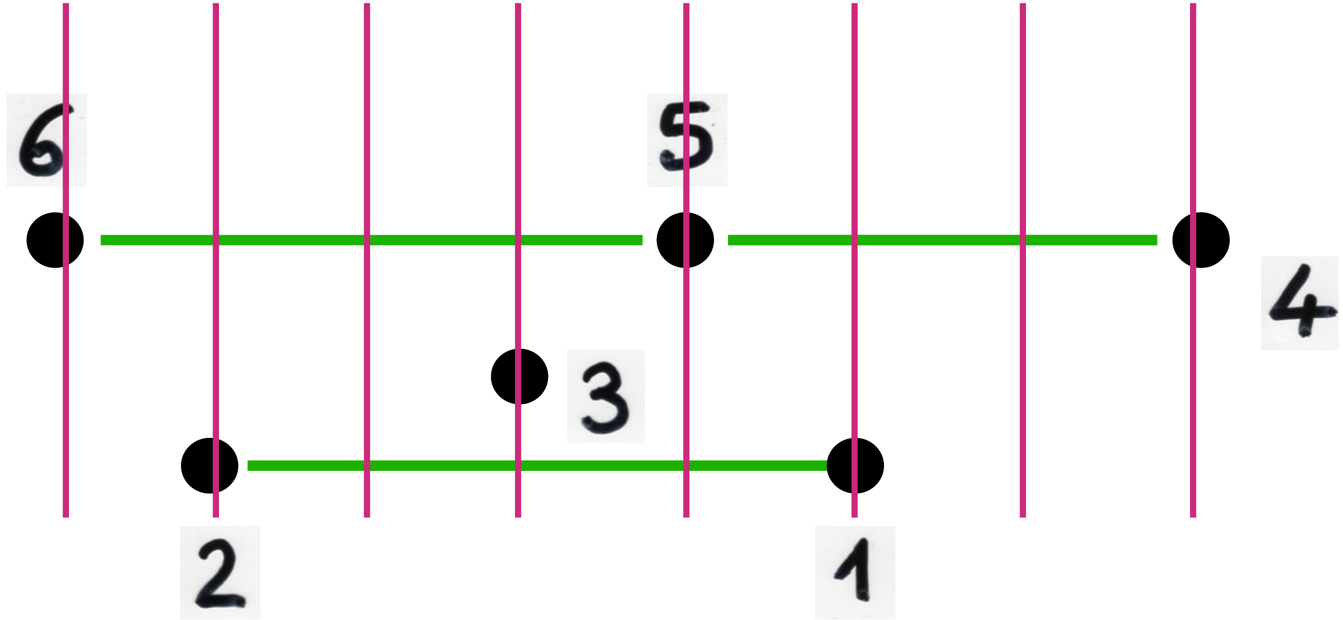
6 2 7 3 5 1 8 4



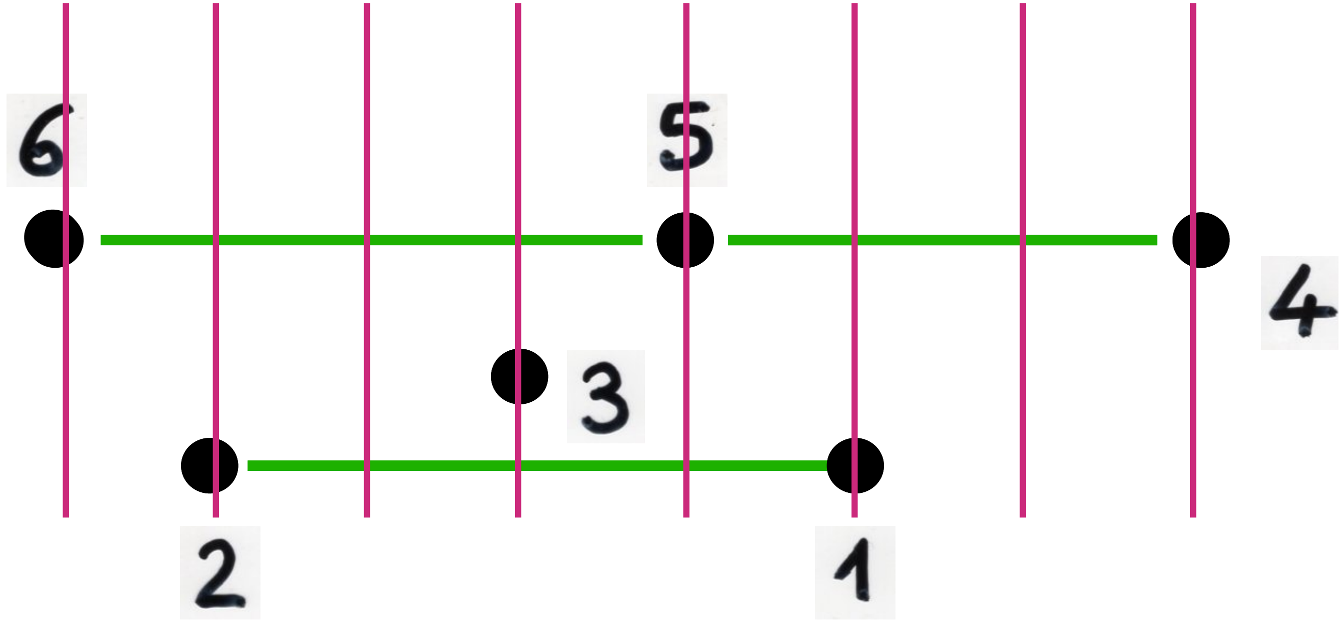
6 2 7 3 5 1 8 4



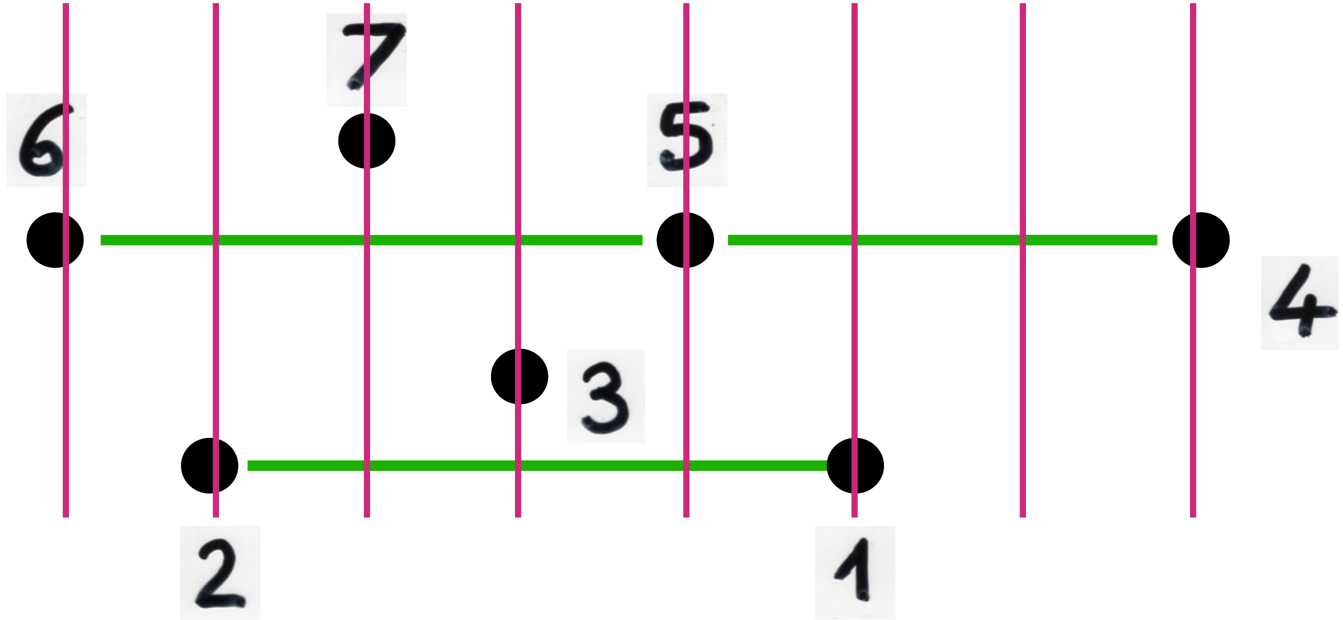
6 2 7 3 5 1 8 4



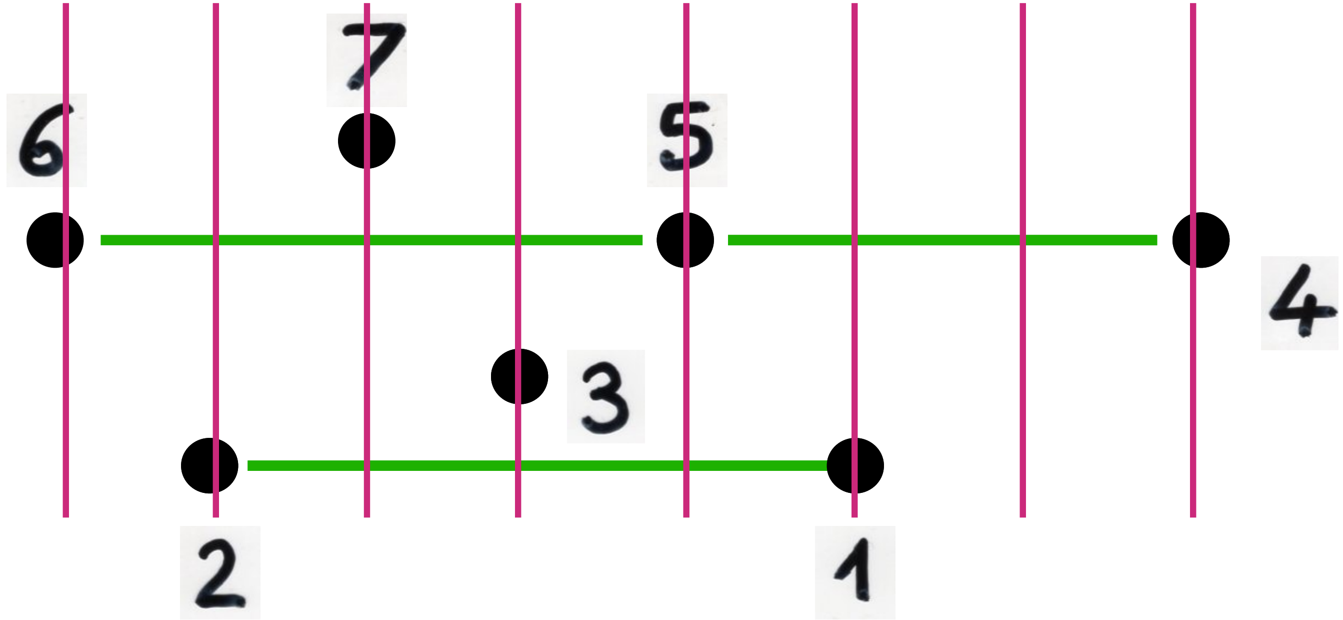
6 2 7 3 5 1 8 4



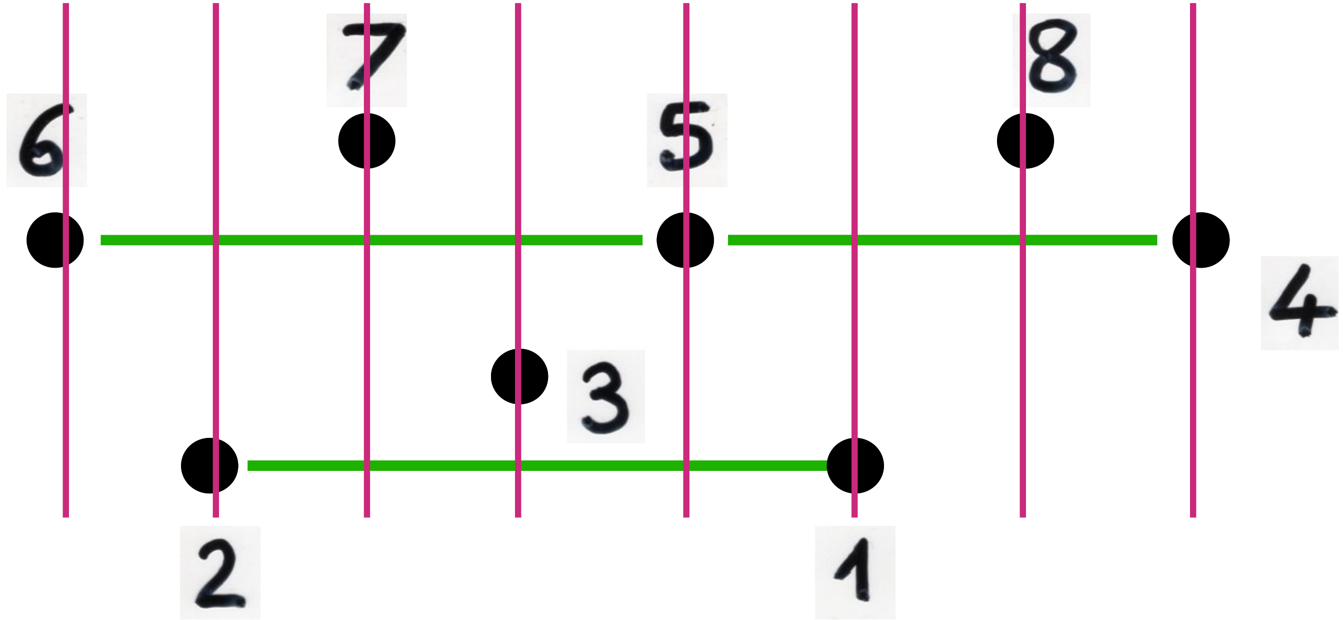
6 2 7 3 5 1 8 4



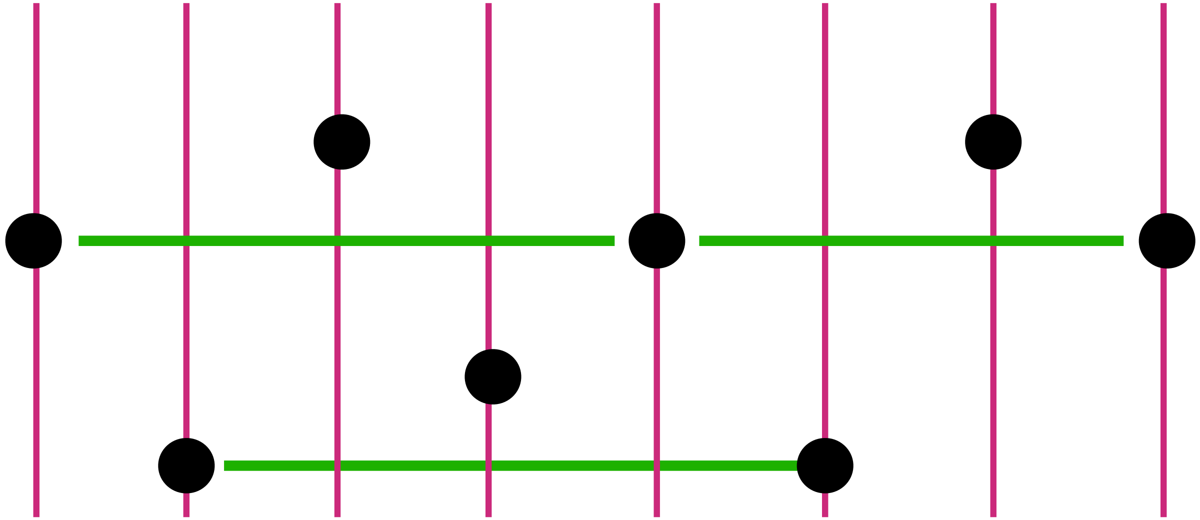
6 2 7 3 5 1 8 4



6 2 7 3 5 1 8 4



6 2 7 3 5 1 8 4

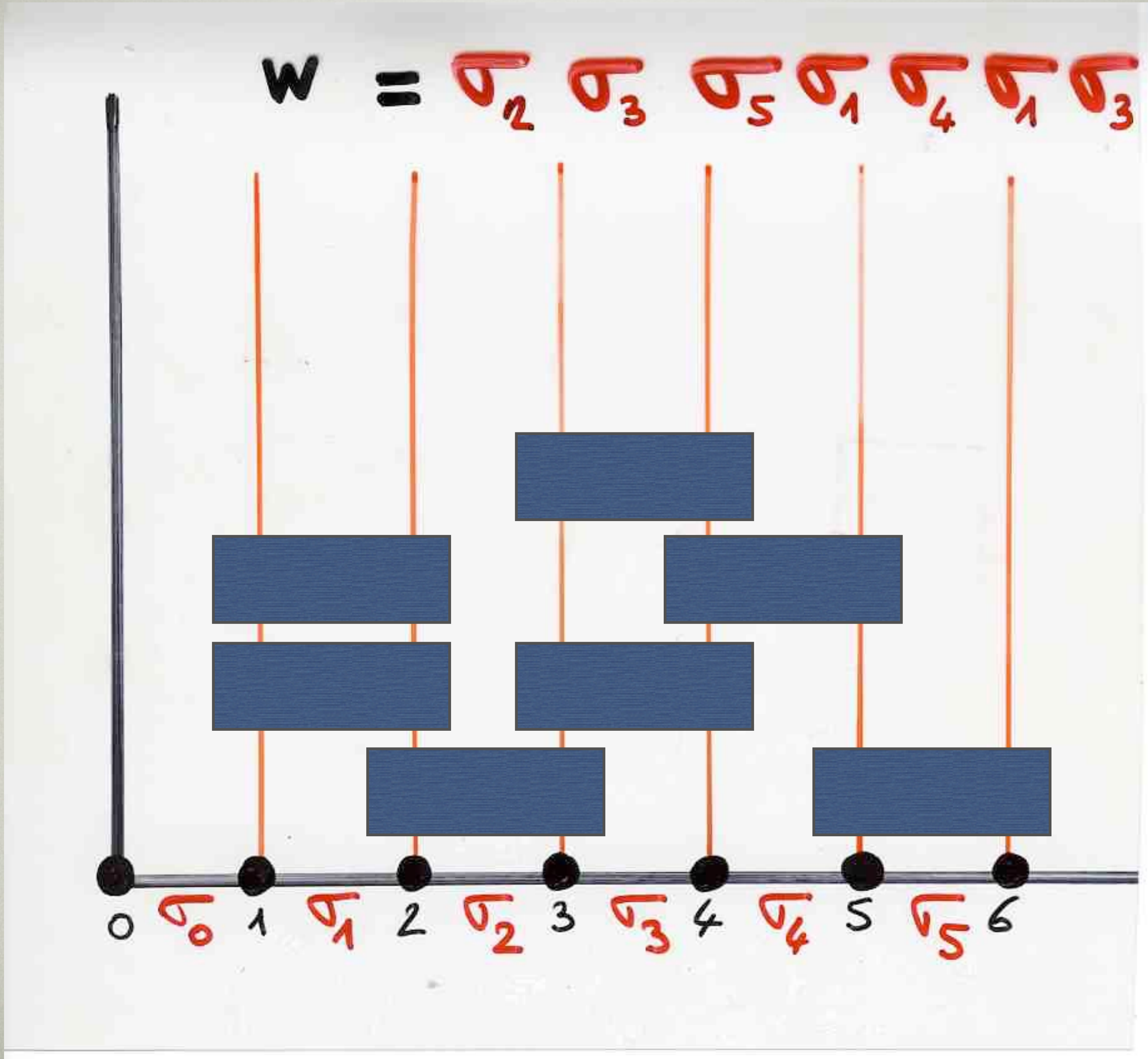


6 2 7 3 5 1 8 4

Reverse bijection

Lexicographic normal form of a heap

From ABJC 2, Ch 1b, p44


W = σ2 σ3 σ5 σ1 σ4 σ1 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

ex: heaps of dimers on $\mathbb{N}$

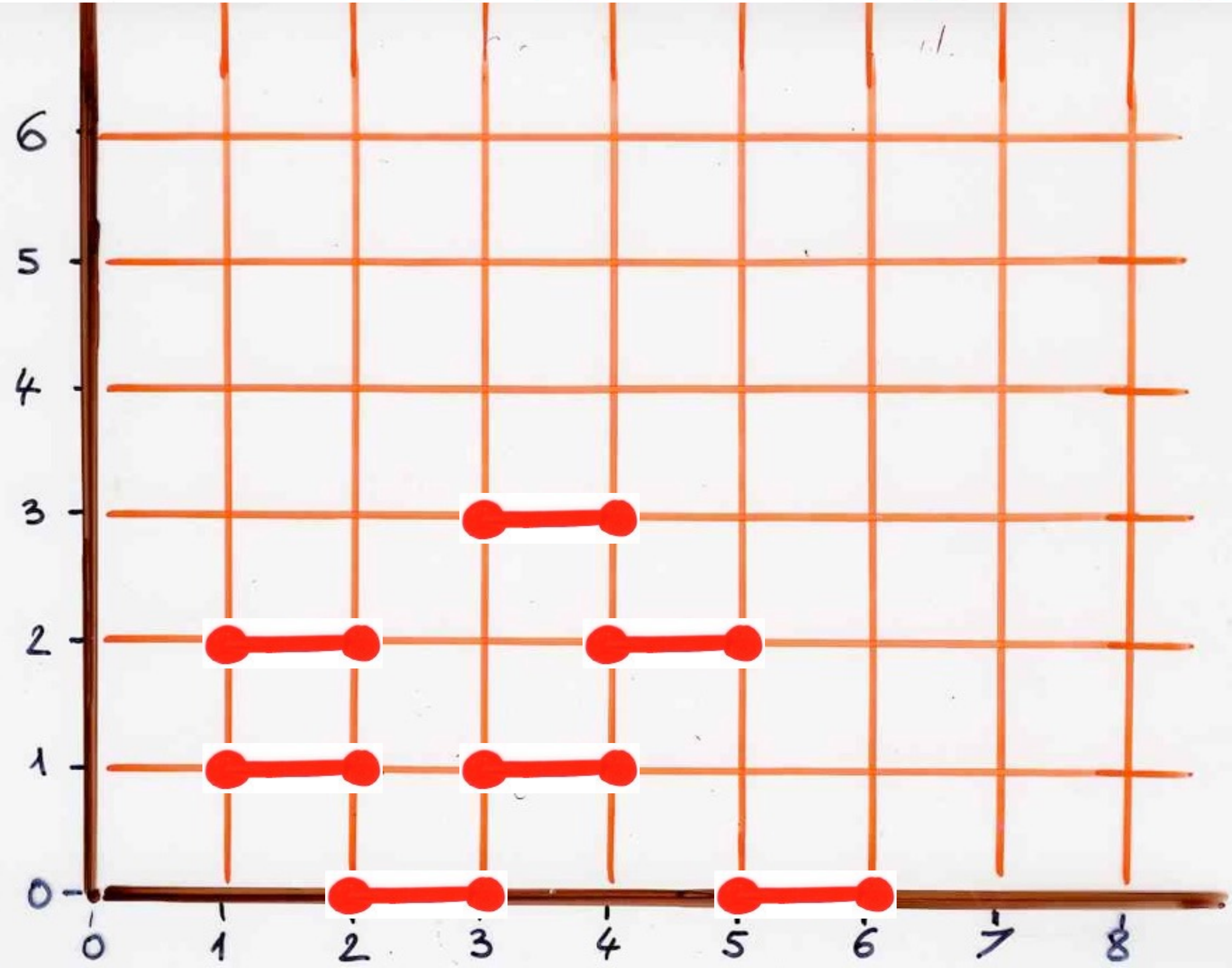
$$P = \{ [i, i+1] = \sigma_i \ , \ i \geq 0 \}$$

$\mathcal{C}$

C commutations

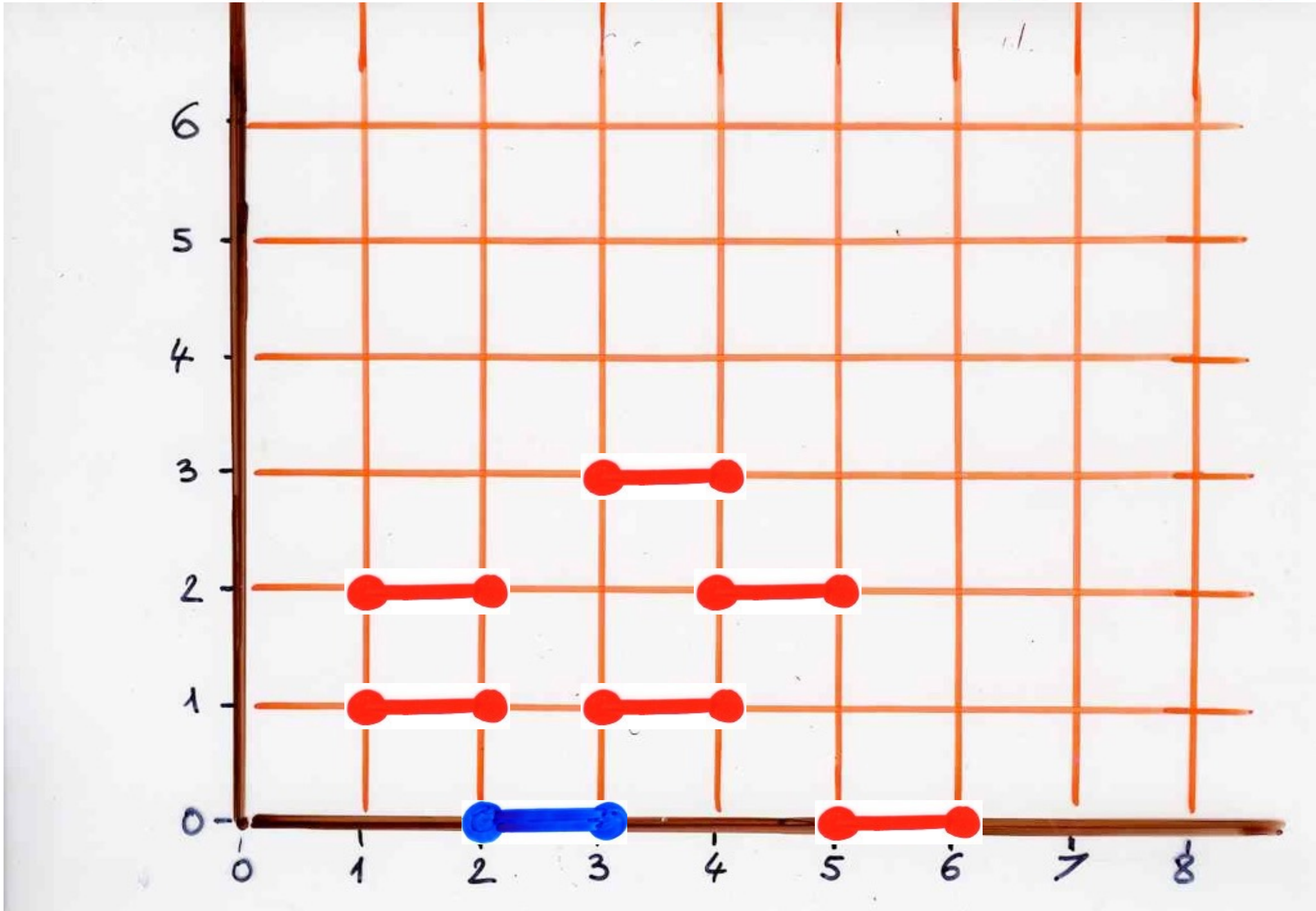
$$\sigma_i \ \sigma_j = \sigma_j \ \sigma_i \quad \text{iff} \ |i-j| \geq 2$$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



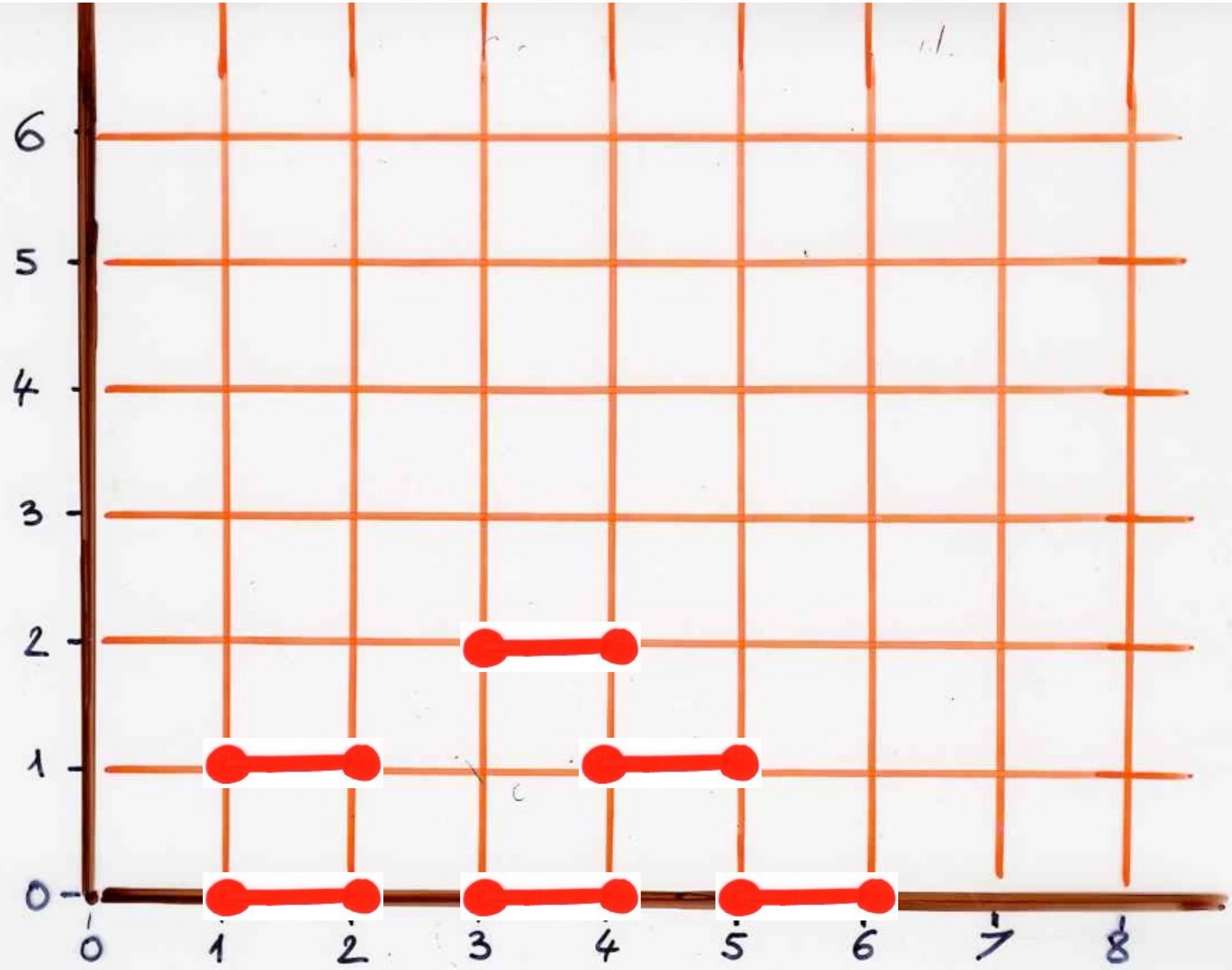
$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_5$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



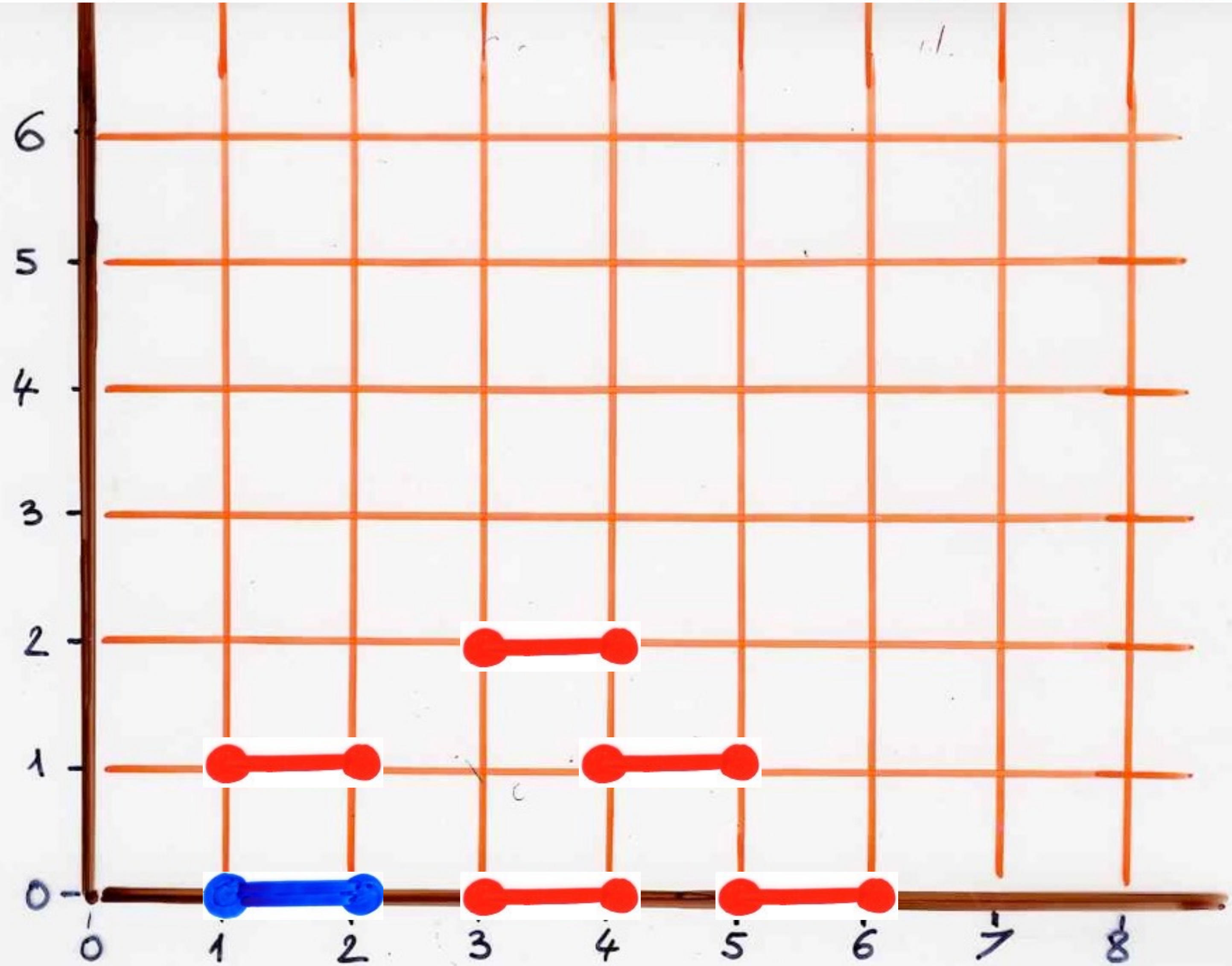
σ_2

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



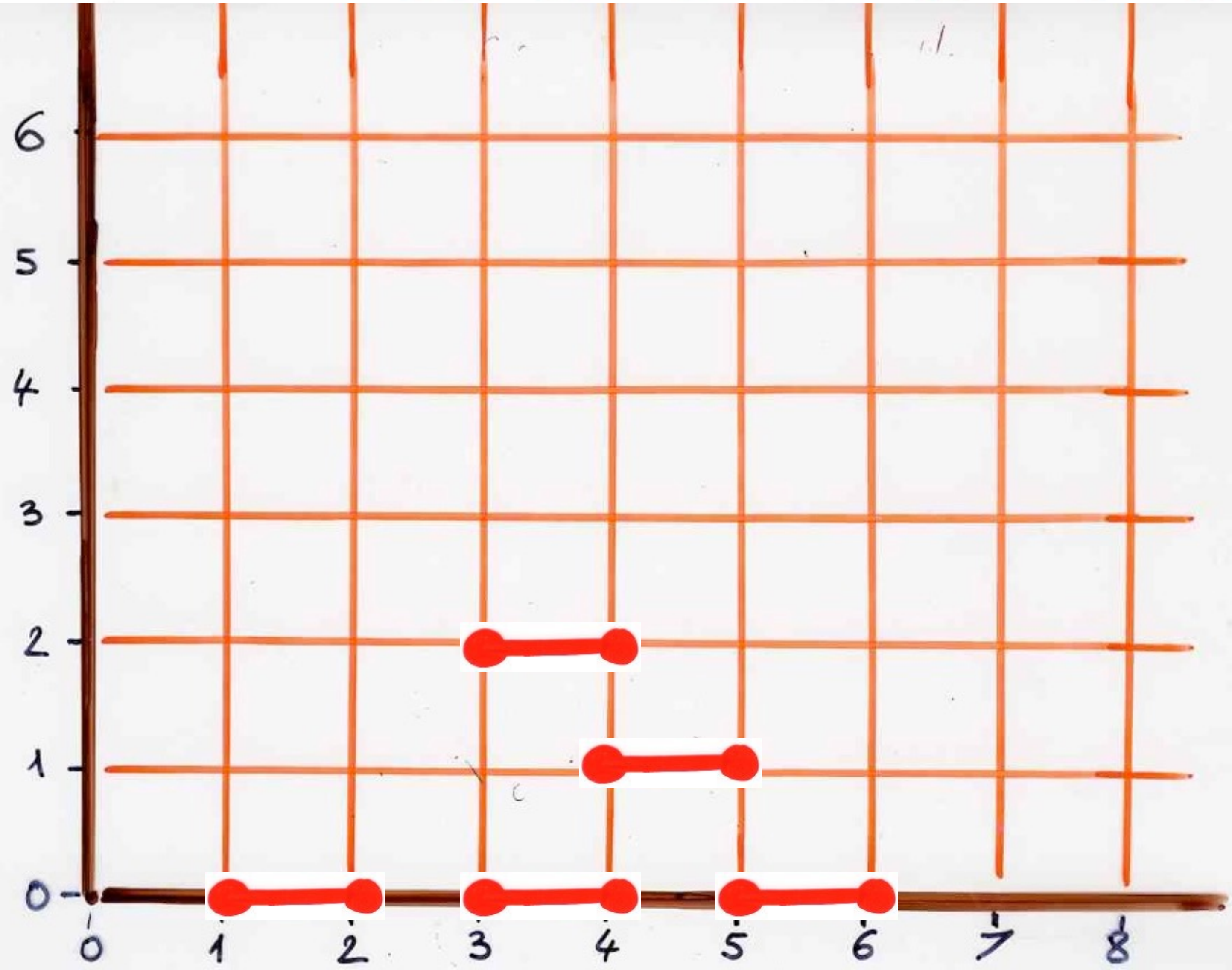
σ_2

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



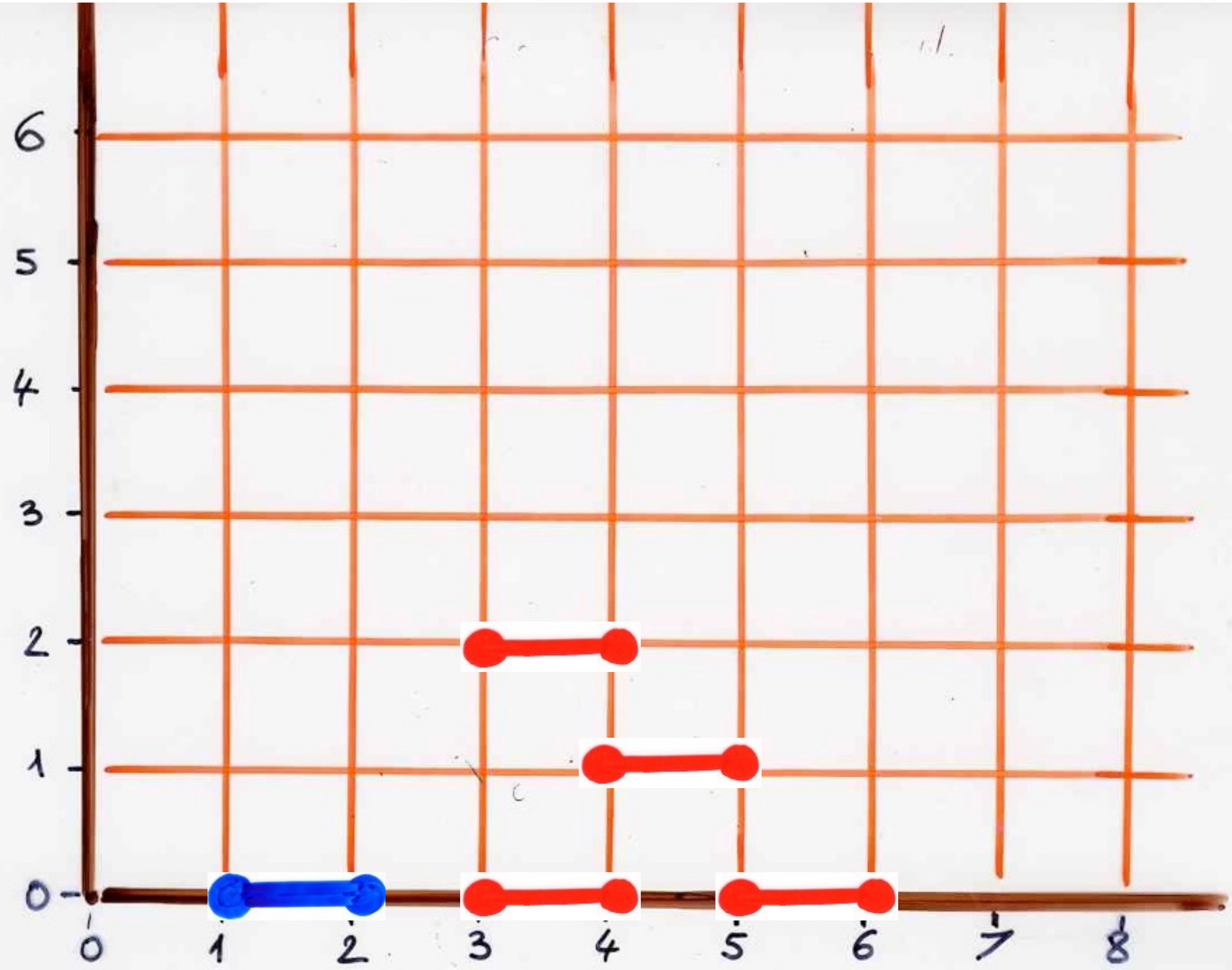
$\sigma_2 \sigma_1$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



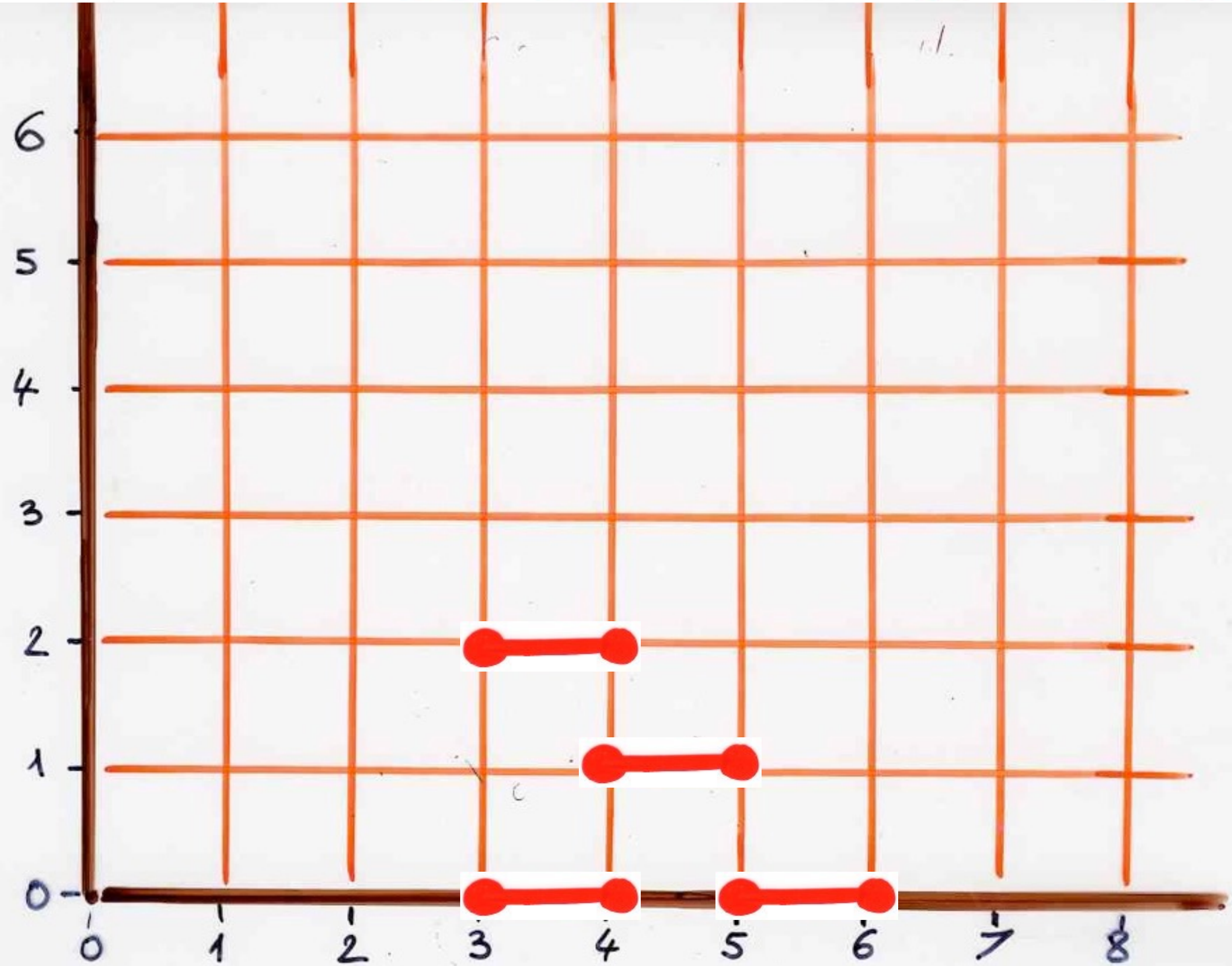
$\sigma_2\ \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



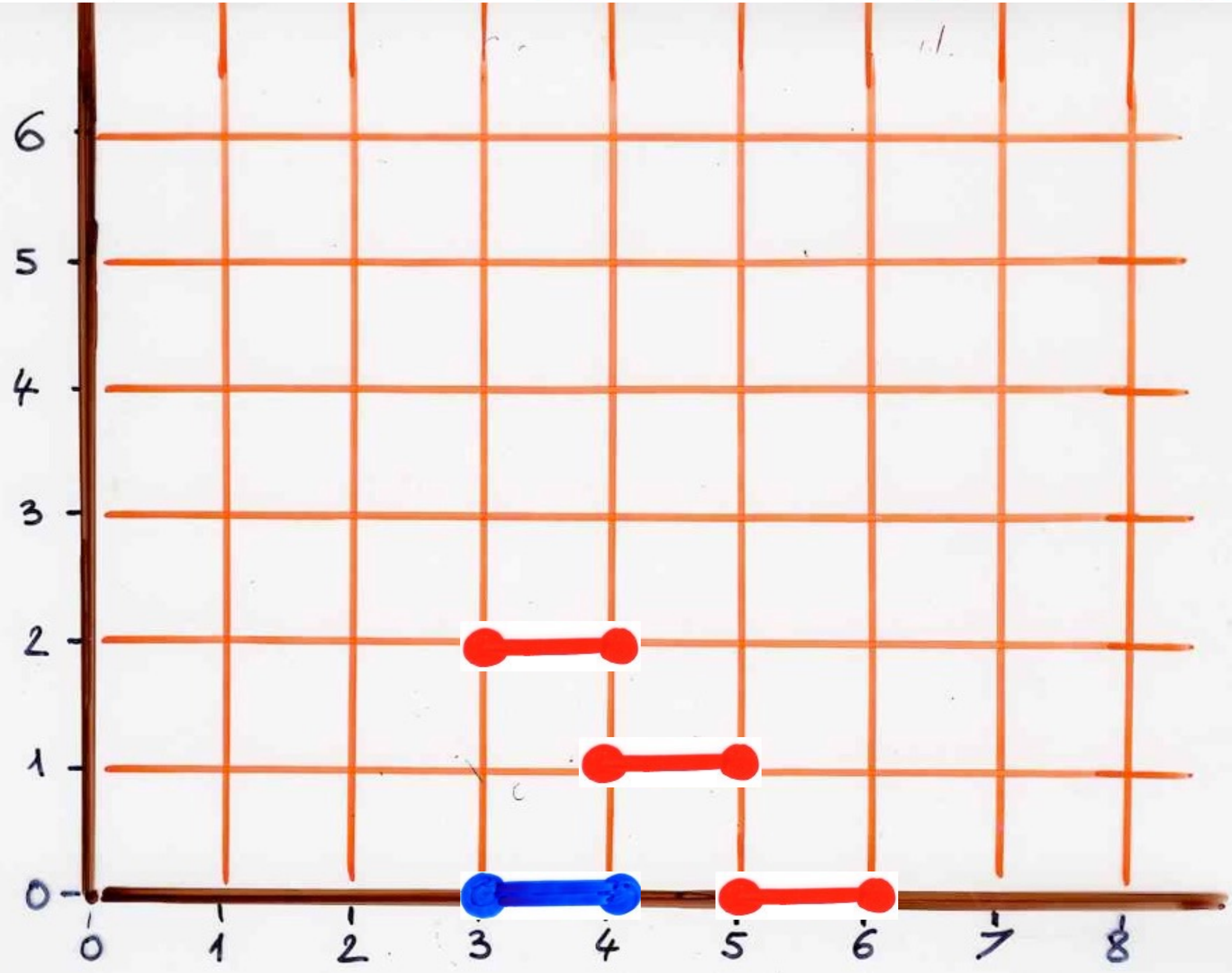
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



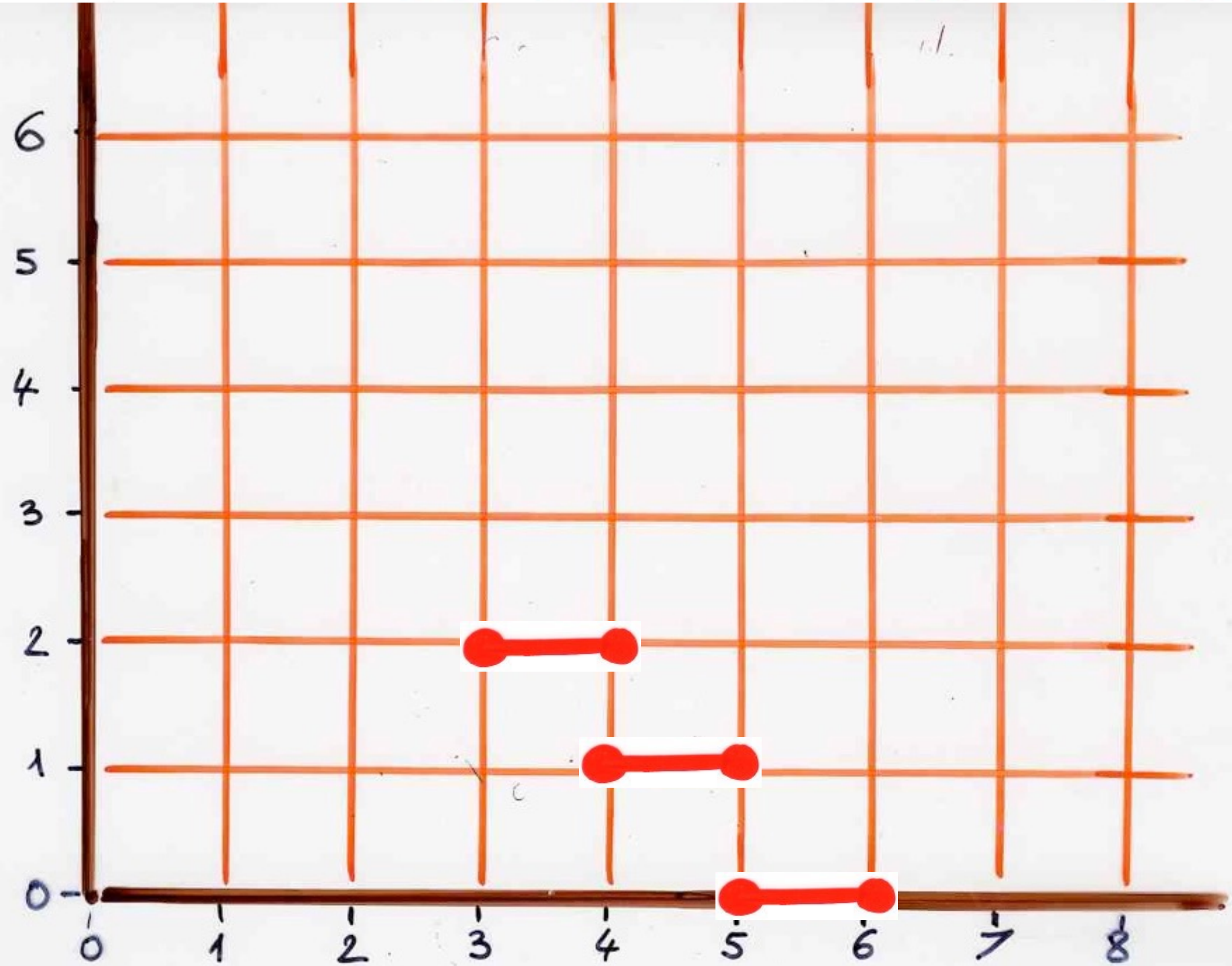
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



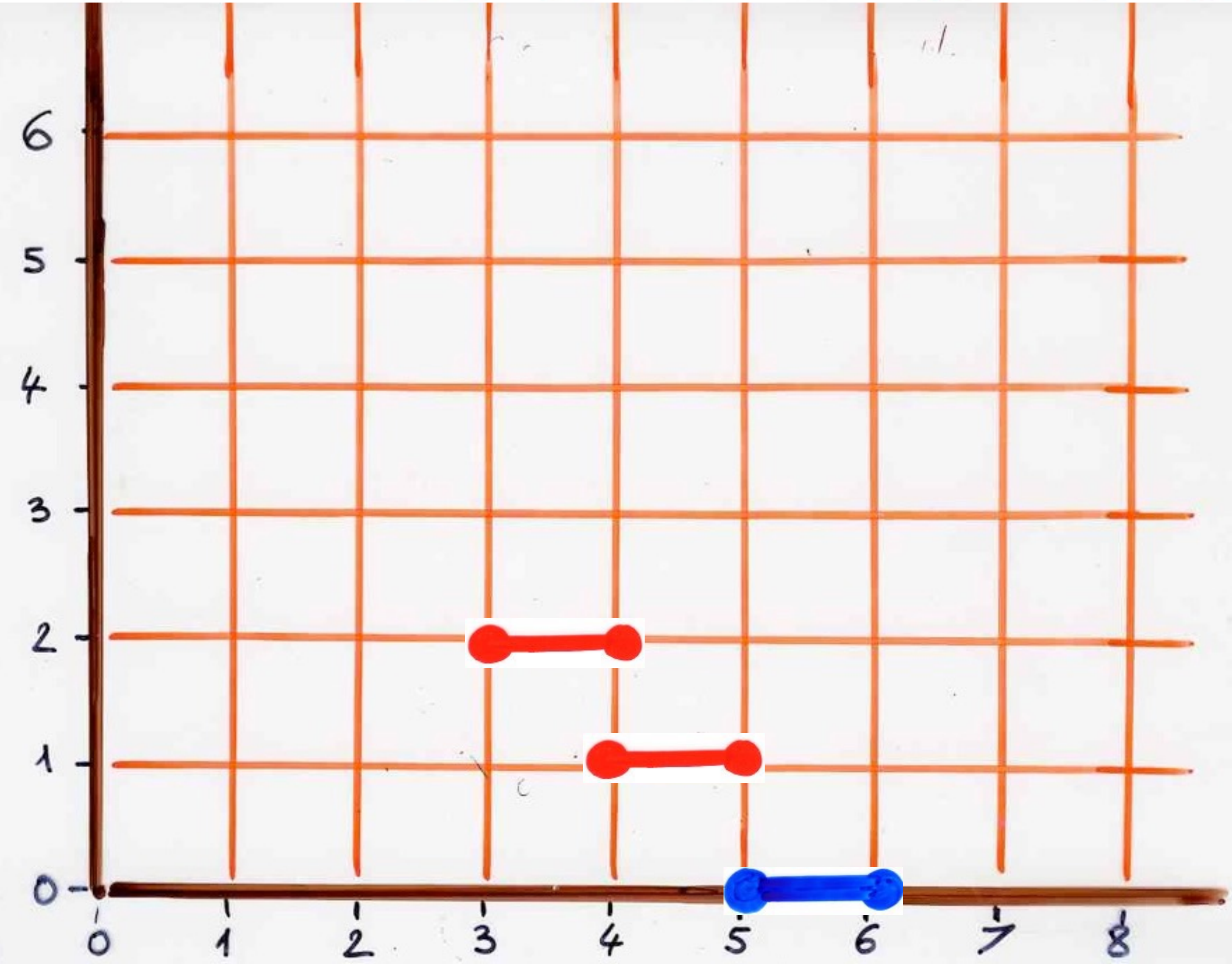
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



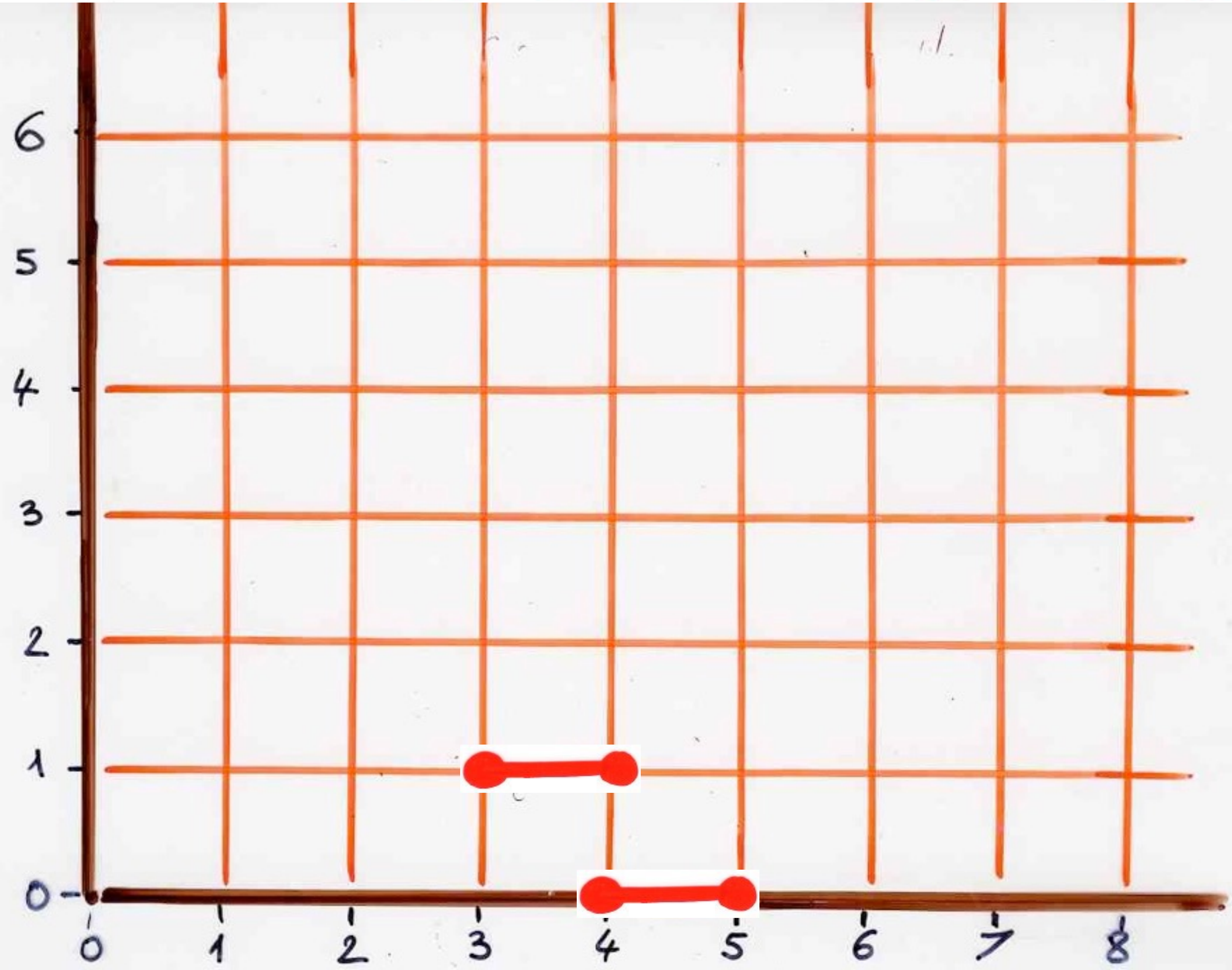
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



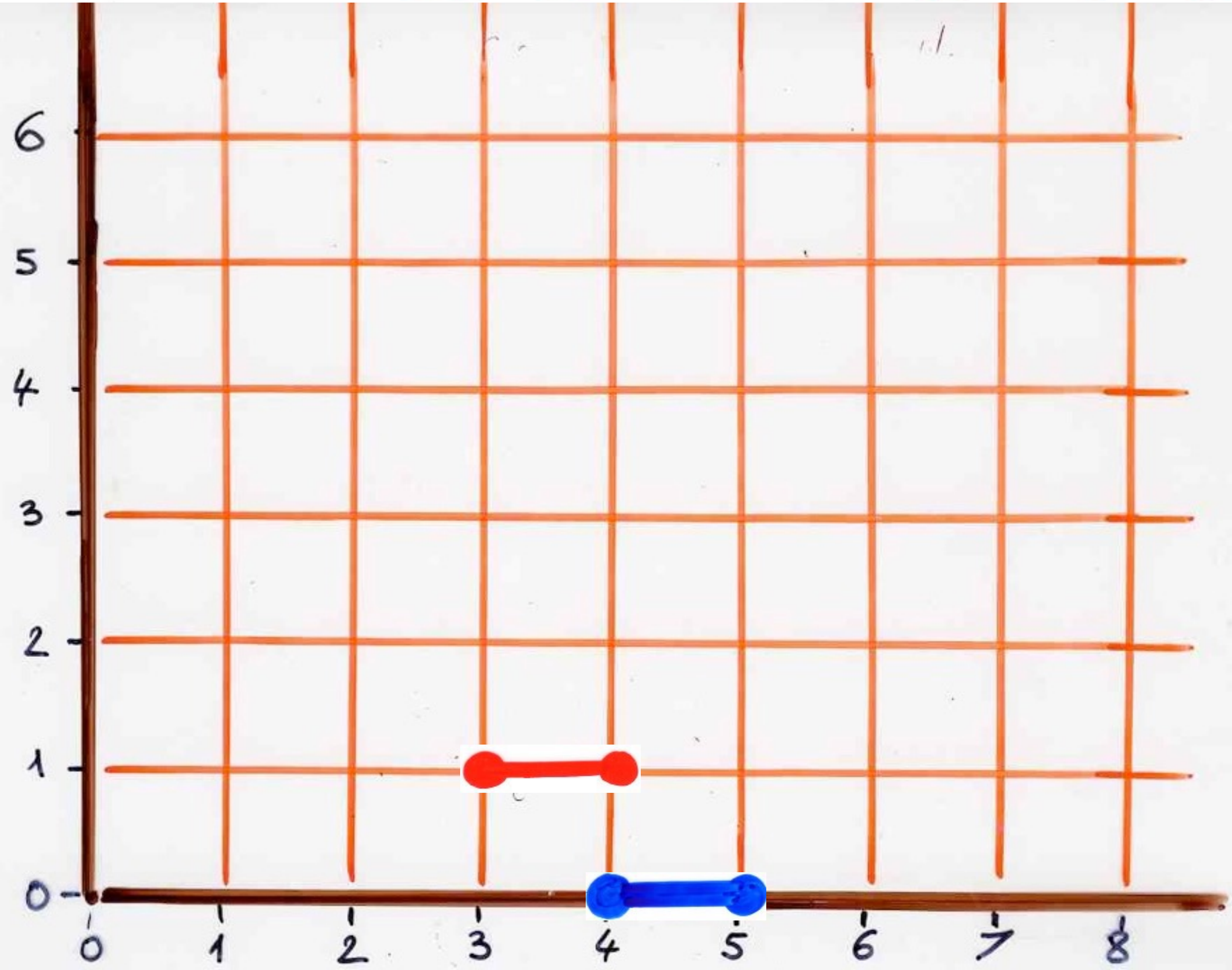
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



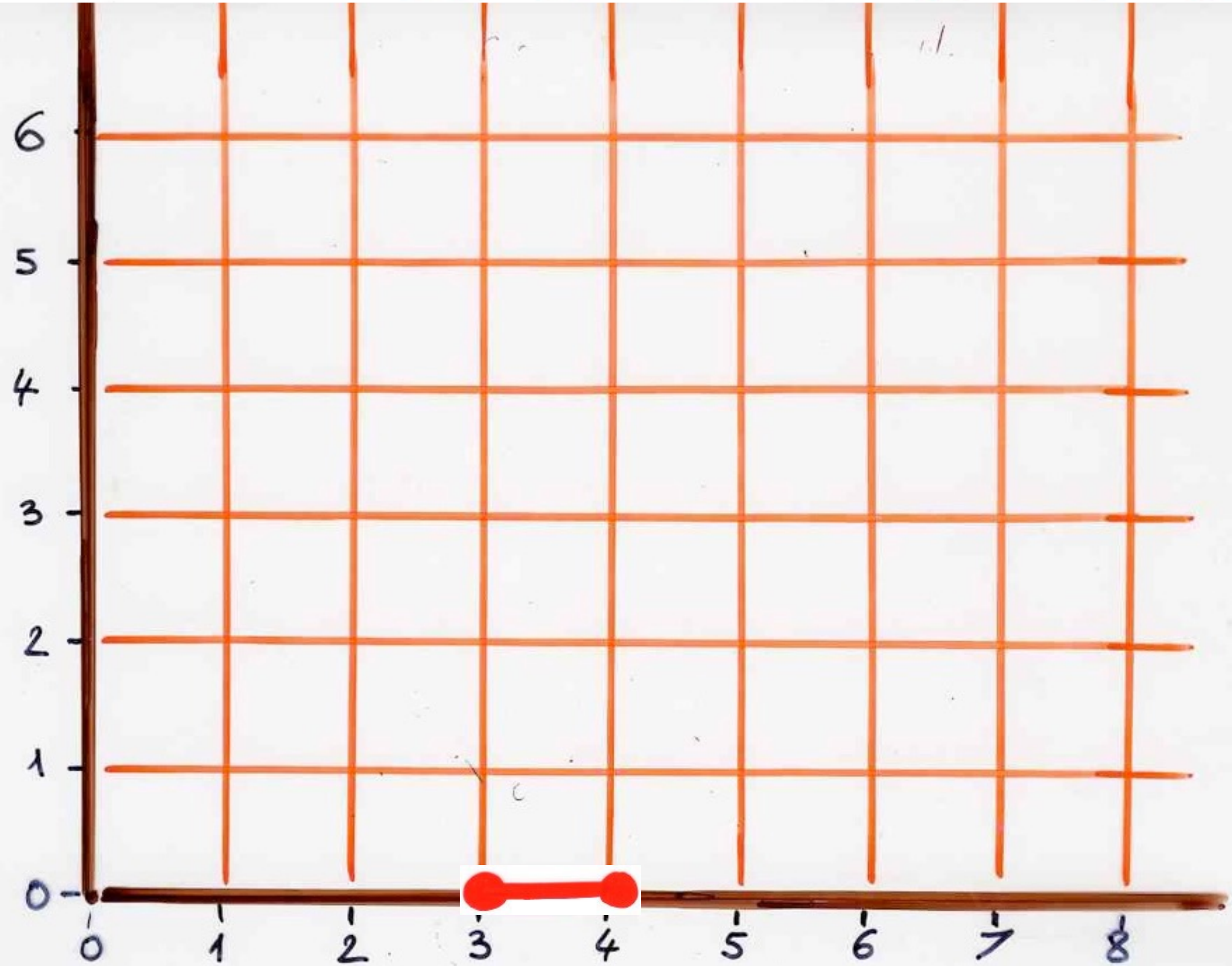
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



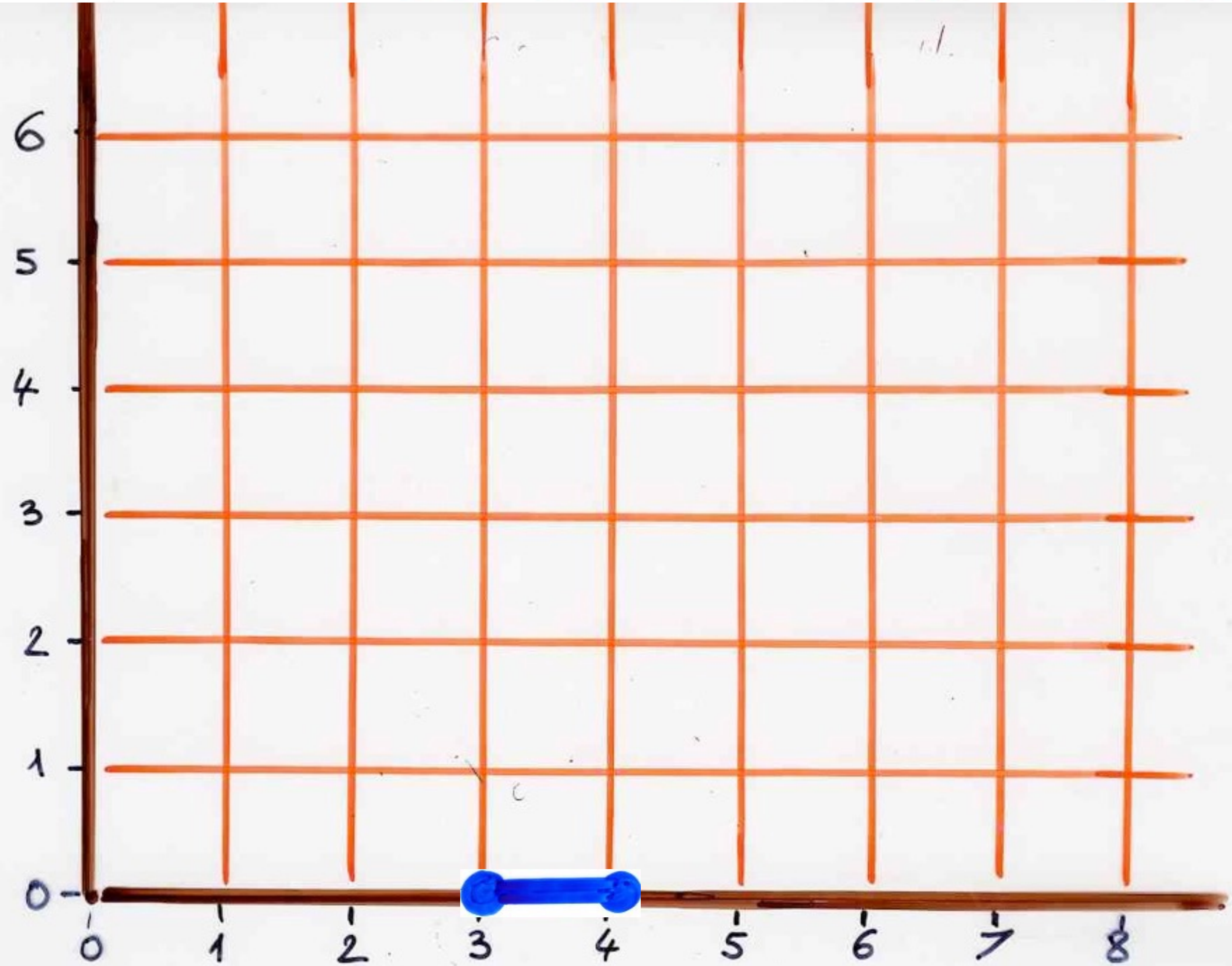
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4$

example $w = \sigma_2\, \sigma_3\, \sigma_5\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$



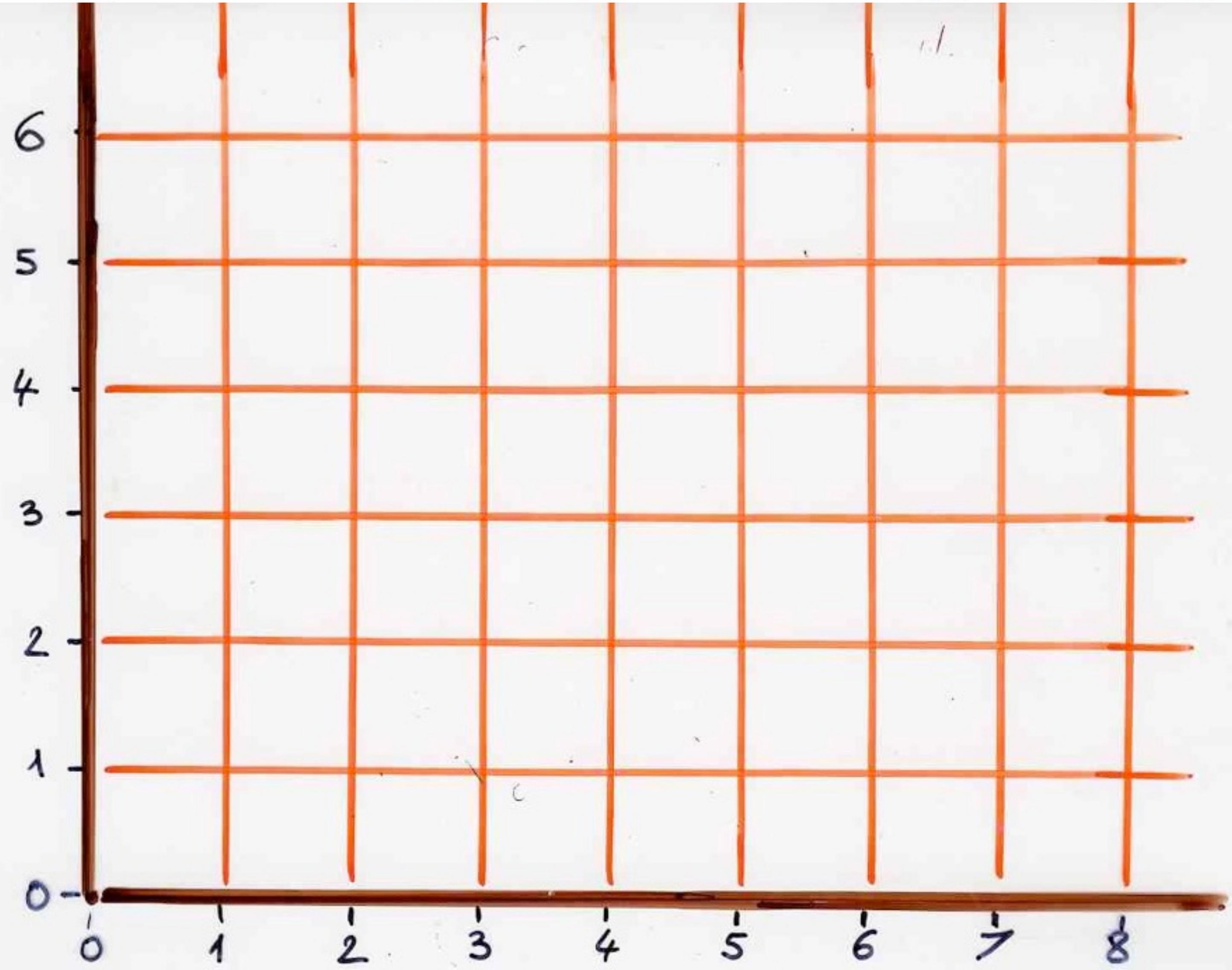
$\sigma_2\, \sigma_1\, \sigma_1\, \sigma_3\, \sigma_5\, \sigma_4$

<u>example</u> $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



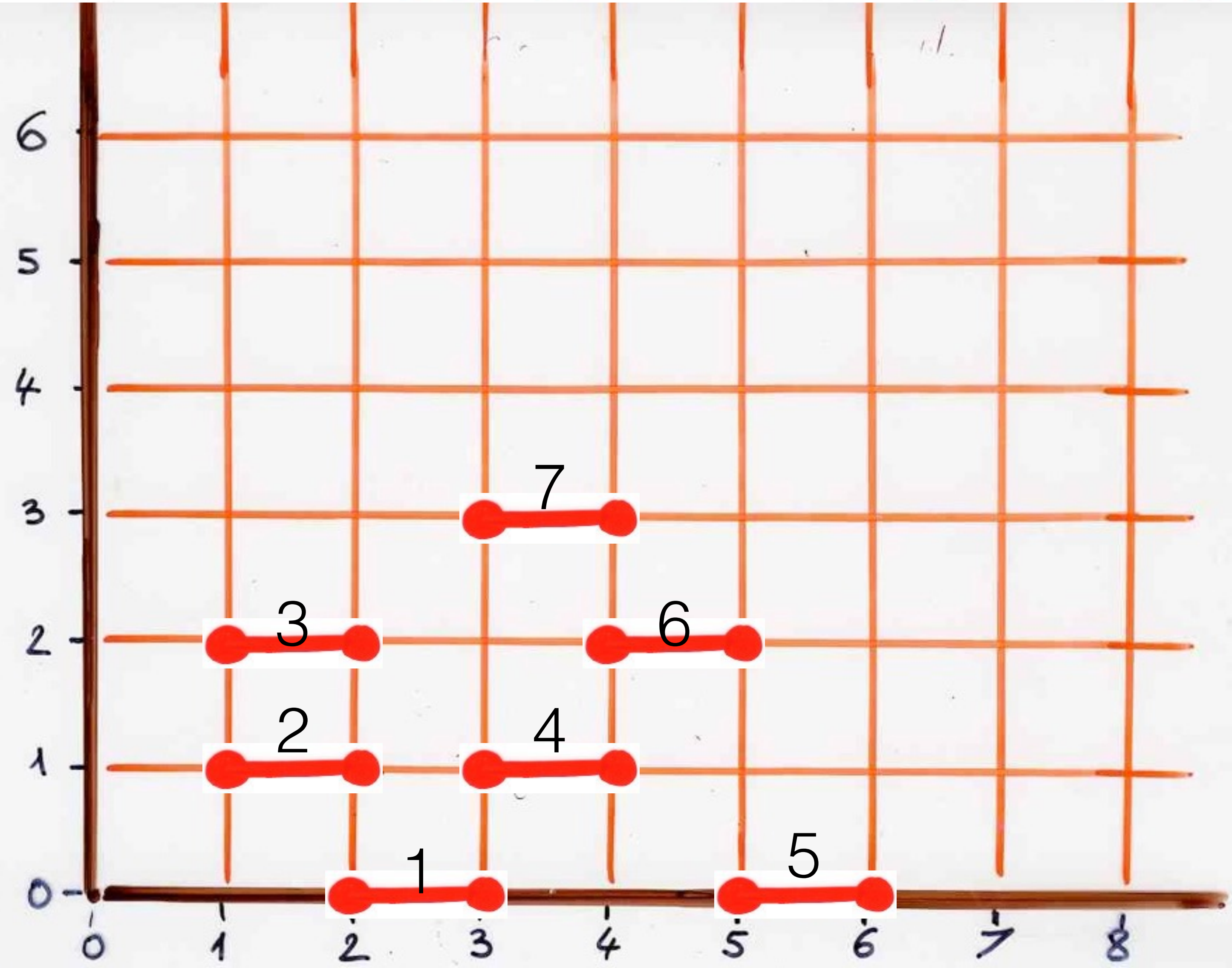
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$

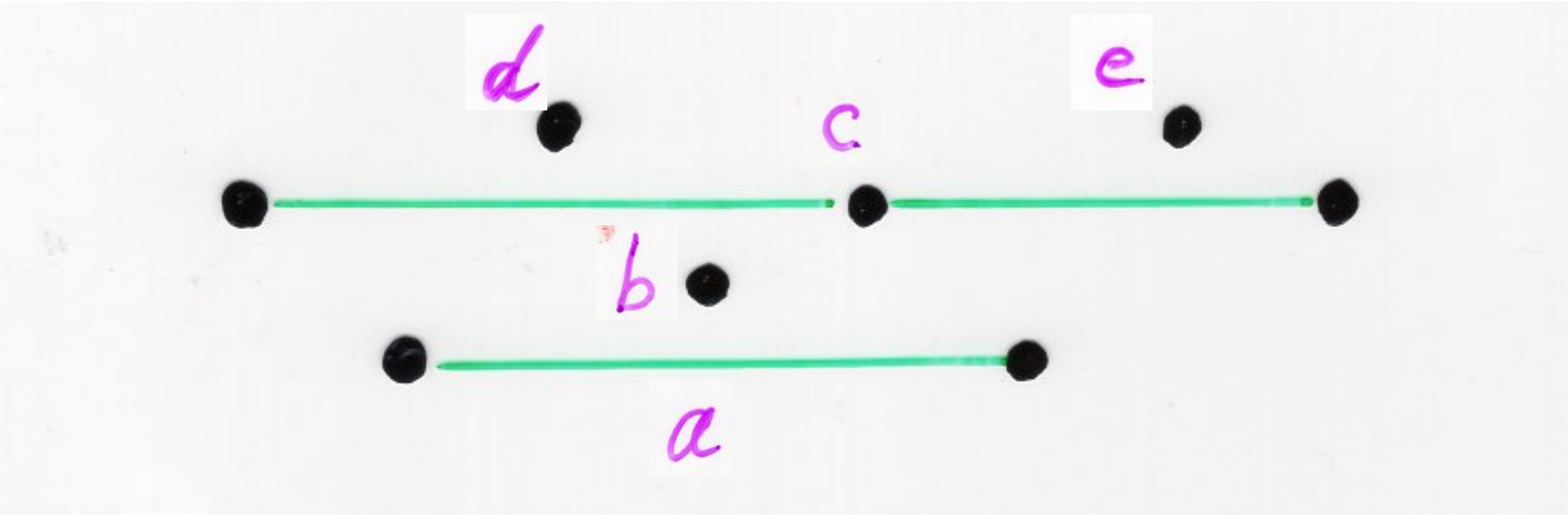


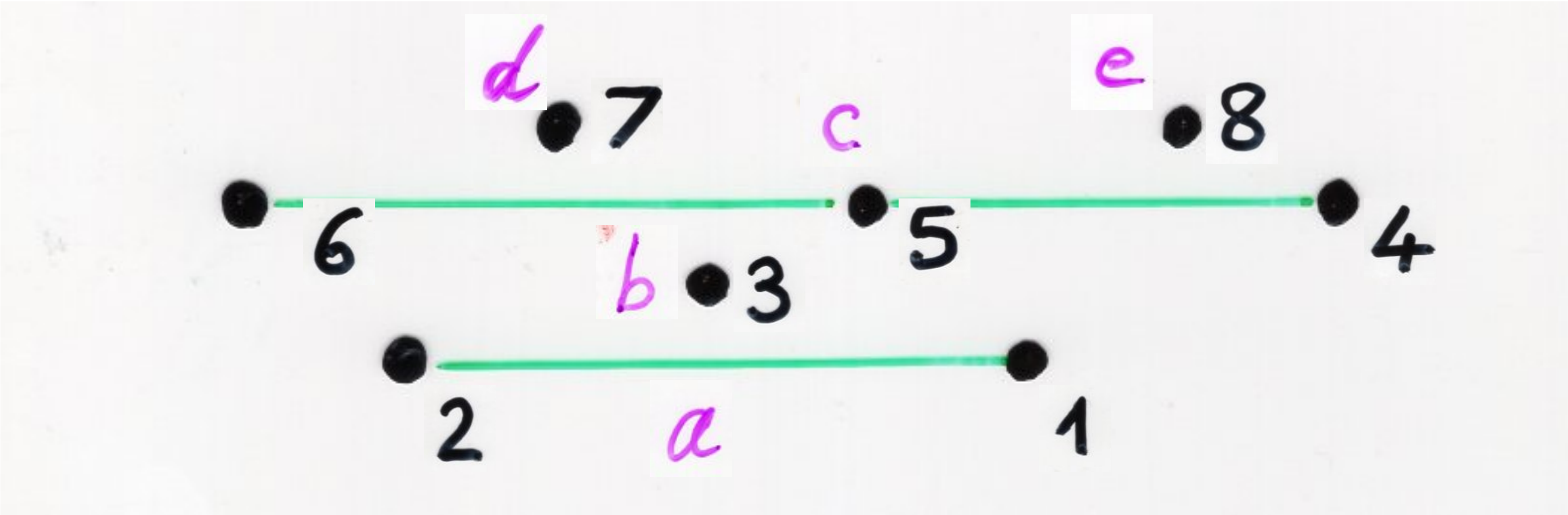
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4 \sigma_3$

Reverse bijection

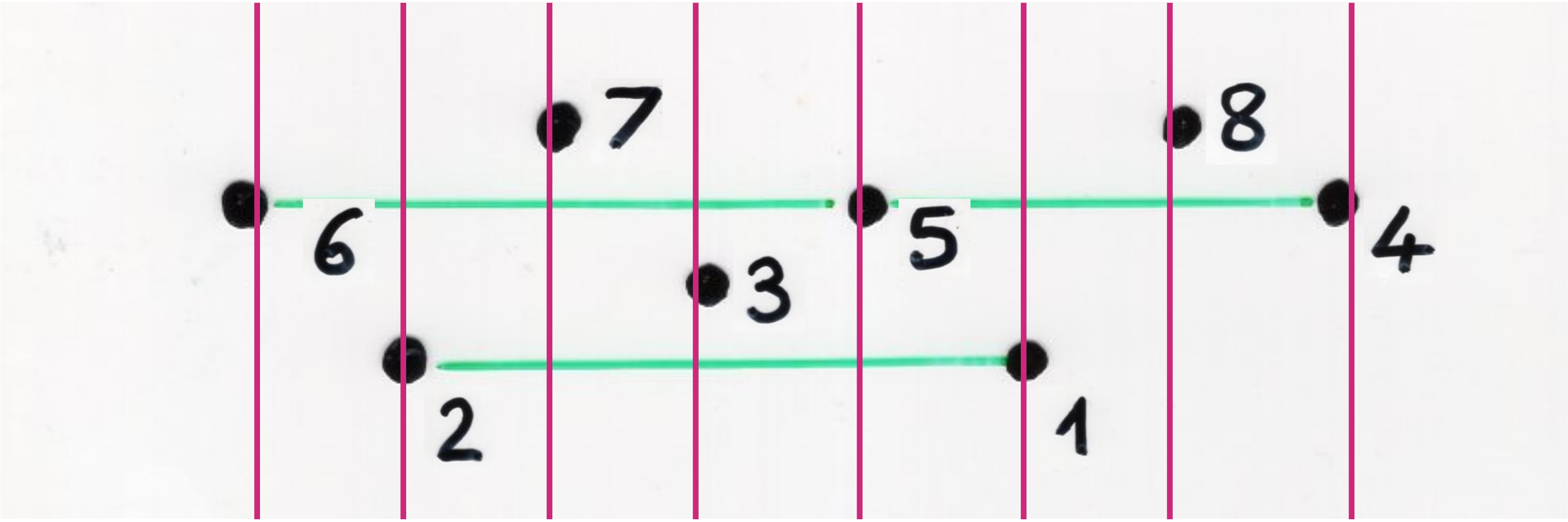
permutations

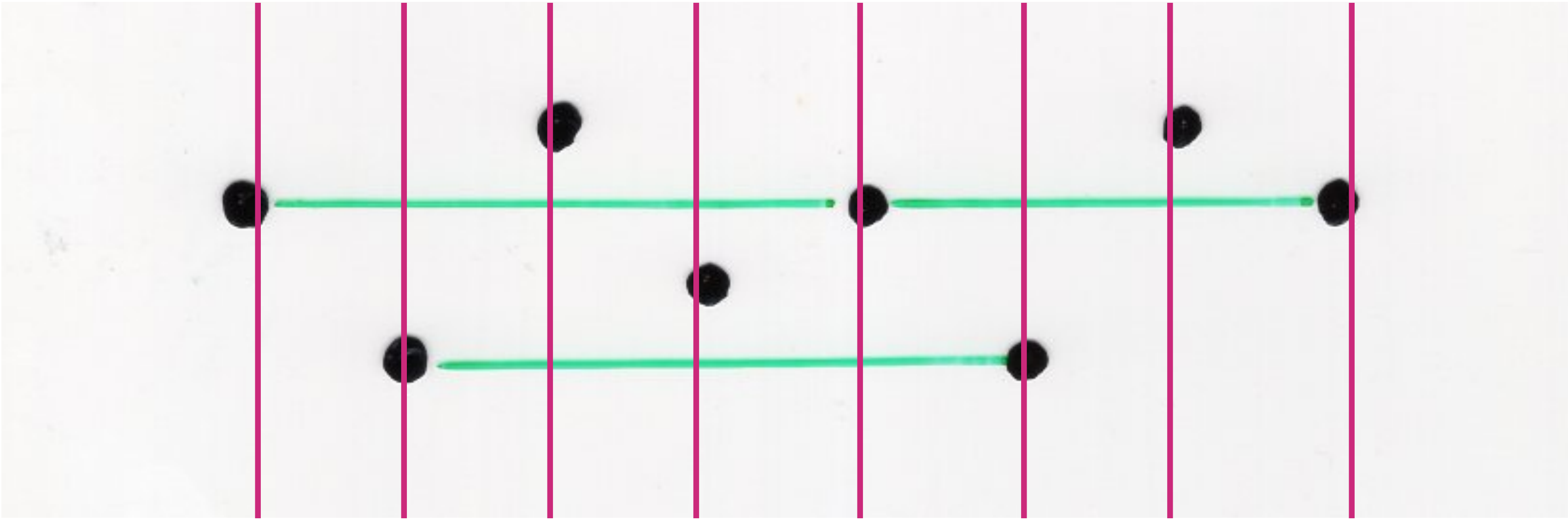
Laguerre heaps of segments


d
e
c
b
a



d
7
c
e
8
6
5
4
b
3
2
a
1


7
8
6
5
4
3
2
1



6 2 7 3 5 1 8 4

(formal) orthogonal polynomials

$$f\big(P(x)\,Q(x)\big) \quad = \int_{\mathbb{R}} P(x)\,Q(x)\,d\mu(x)$$

measure μ
on $\mathbb{R}$

$$f(x^n) = \int_{\mathbb{R}} x^n\,d\mu(x)$$

moments
problem

$$f(x^n) = \underbrace{\mu_n}_{\text{moments}}$$

$\mathbb{K}$ ring

field $\mathbb{R}$, $\mathbb{C}$
or $\mathbb{Q}[\alpha, \beta, \ldots]$

$\mathbb{K}[x]$
polynomials in x

$\{P_n(x)\}_{n\geq 0}$
sequence of polynomials

$P_n(x) \in \mathbb{K}[x]$.

Definition $\{P_n(x)\}_{n\geq 0}$ sequence of polynomials orthogonal iff $\exists$ $f : \mathbb{K}[x] \longrightarrow \mathbb{K}$ linear functional

(i) $\deg(P_n) = n$, for $n \geq 0$

degree

(ii) $f(P_k P_l) = 0$, for $k \neq l \geq 0$

(iii) $f(P_k^2) \neq 0$, for $k \geq 0$

$$f(x^n) = \mu_n$$

moments

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

moments of (Tchebychev) 1 st kind

2 nd kind

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan number

$$\frac{2}{\pi}\int_{-1}^{+1} x^{2n} (1-x^2)^{1/2}\, dx = \frac{1}{4^n} C_n$$

Catalan

E_{2n}

secant number

Meixner – Pollaczek $\rightarrow$ Laguerre

Jacobi $\rightarrow$ Laguerre

Meixner $\rightarrow$ Charlier

number of ordered partitions

Laguerre $\rightarrow$ Hermite

Charlier $\rightarrow$ Hermite

$\mu_n = n!$

$(\alpha+1)(\alpha+2)\cdots(\alpha+n)$

B_n Bell number

number of partitions

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

Combinatorial theory
of orthogonal polynomials

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

(formal) Favard's Theorem

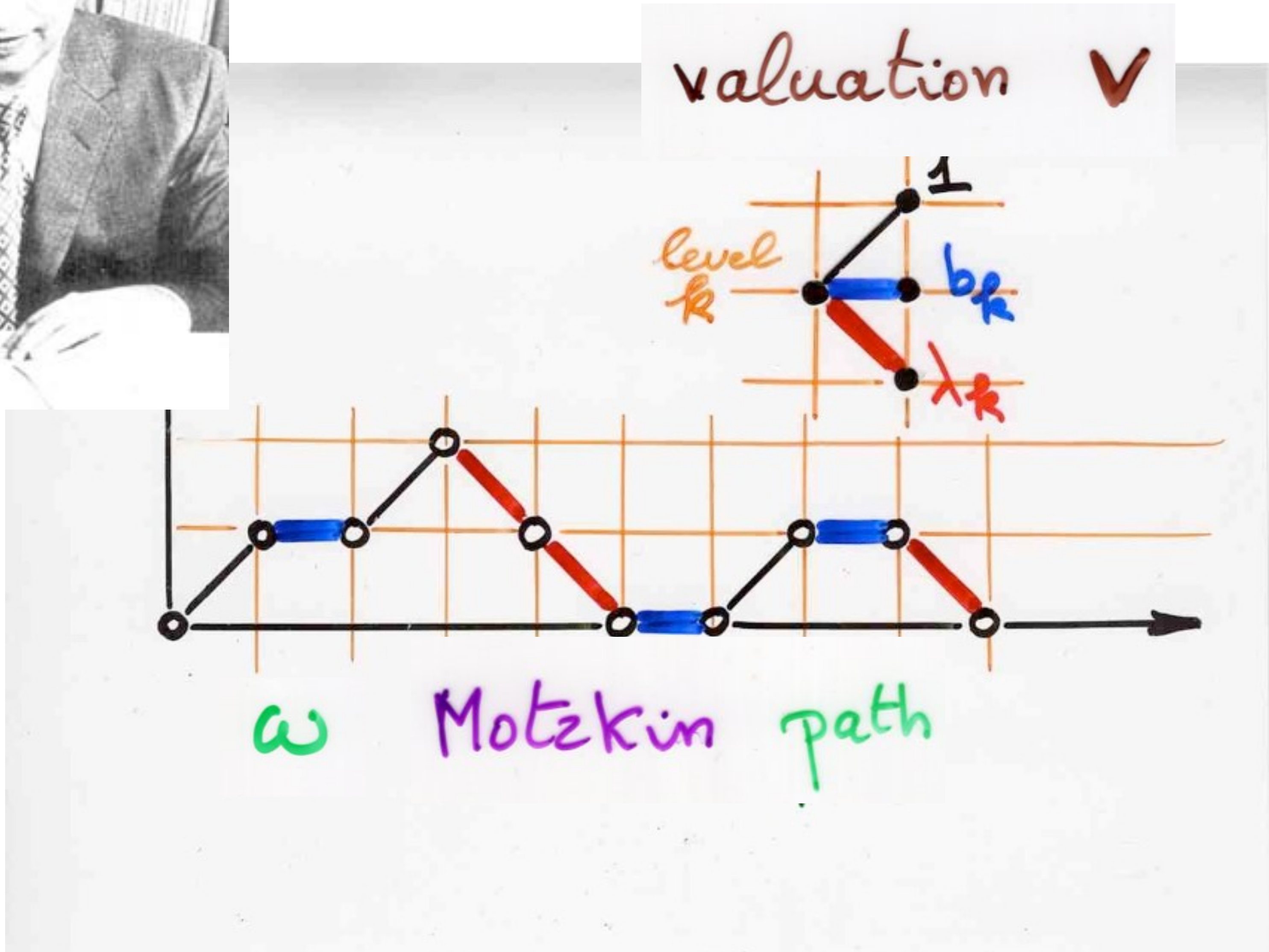
3-terms linear recurrence relation

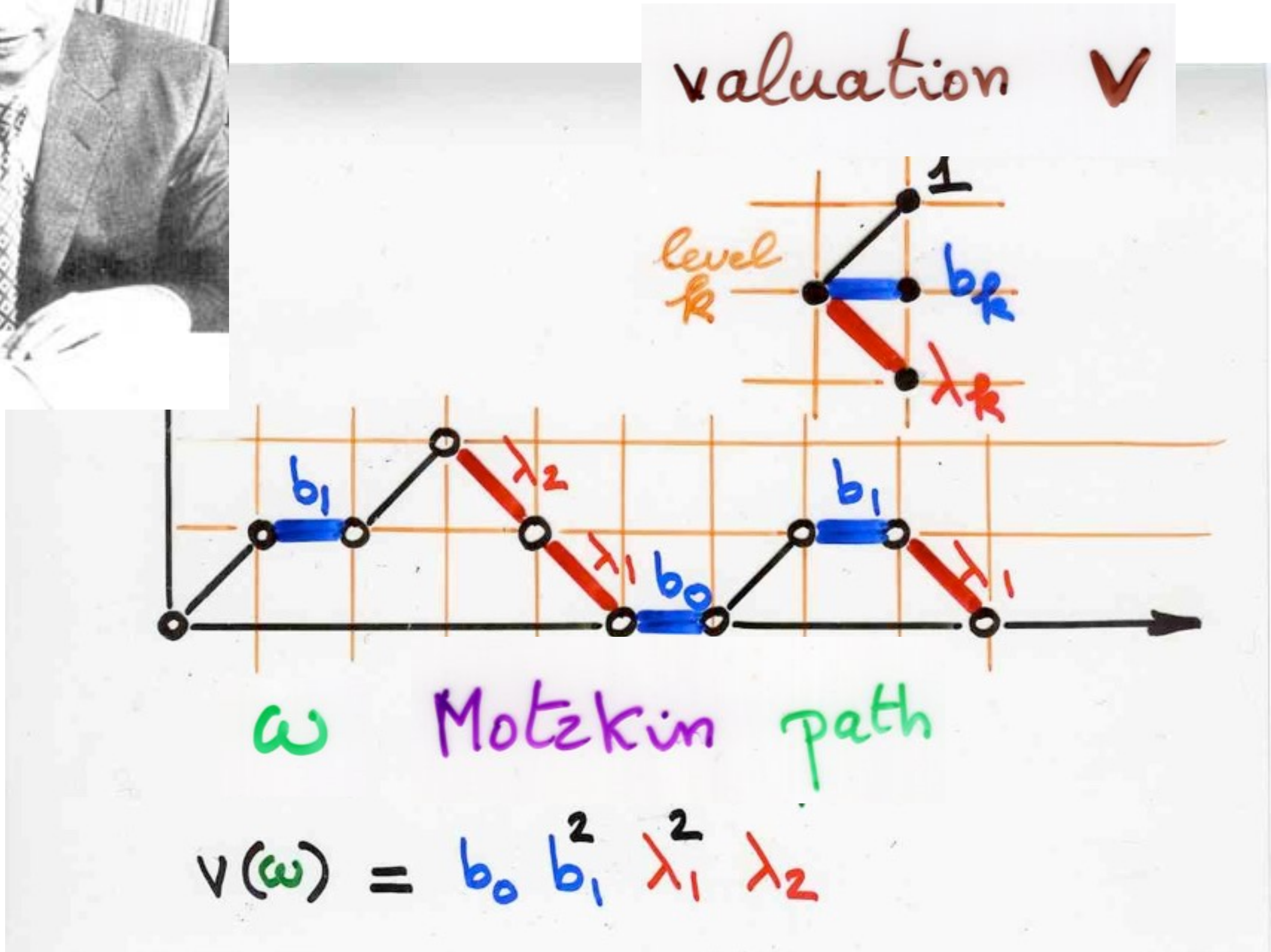
$\Rightarrow$ orthogonality

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \underset{\text{ring}}{\mathbb{K}}$

μ_n ?


valuation v
1
level k
b_k
λ_k
ω Motzkin path


valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

$$f(x^n) = \mu_n$$

length

Laguerre histories

The FV bijection

Laguerre polynomials

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$

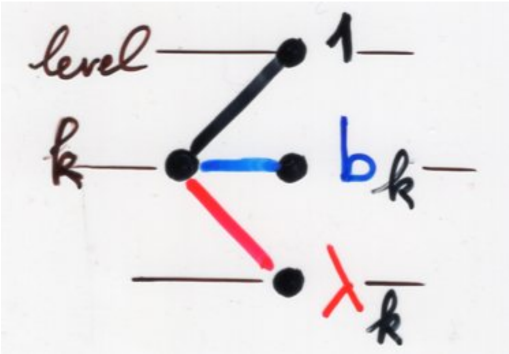
Laguerre polynomials

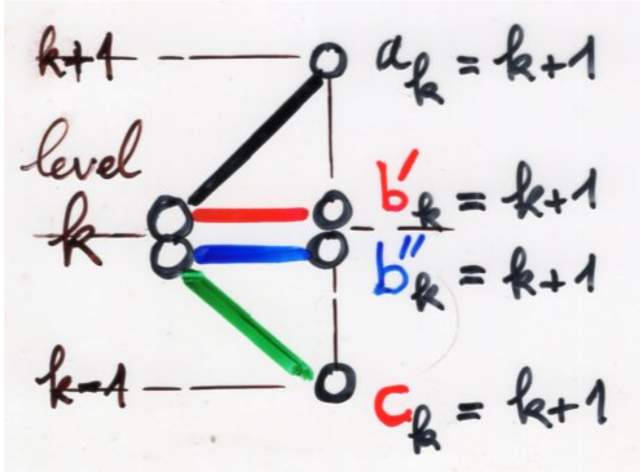
$L_n^{(1)}(x)$

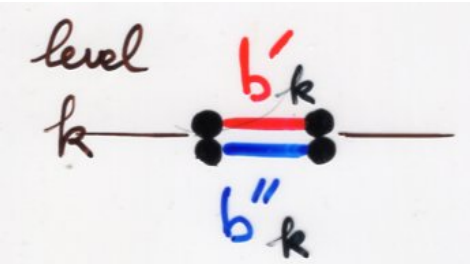
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

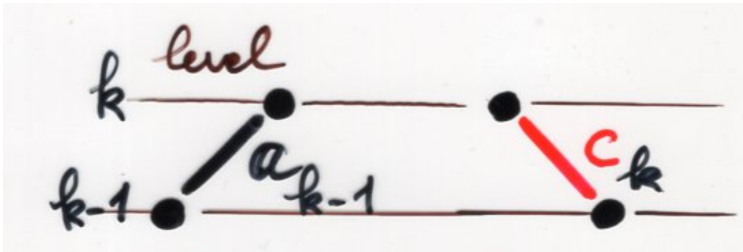
moments

$\mu_n = (n+1)!$



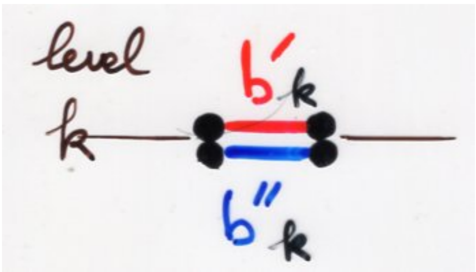




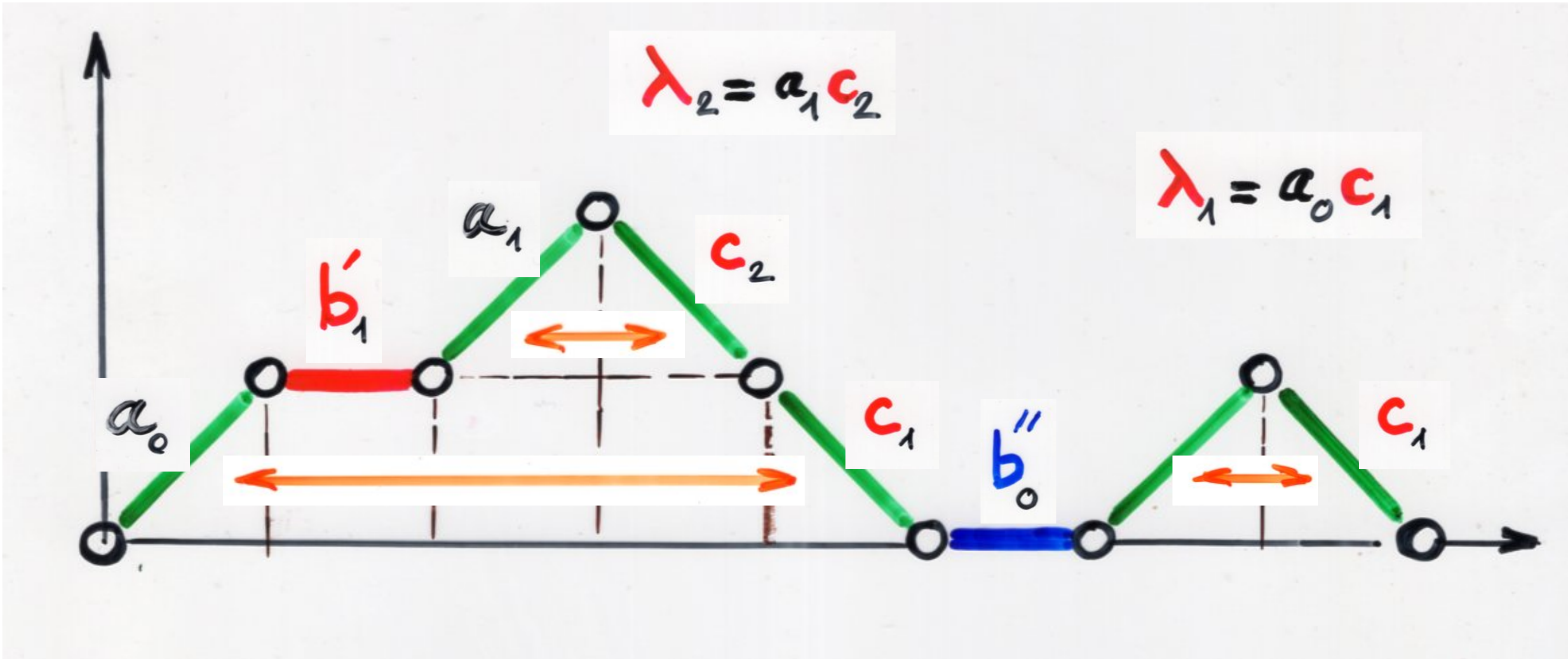


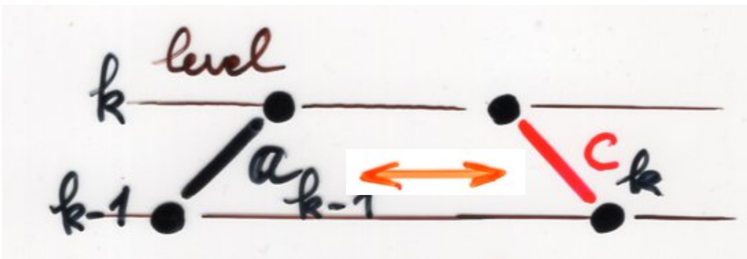
$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$



$$b_k = b'_k + b''_k$$



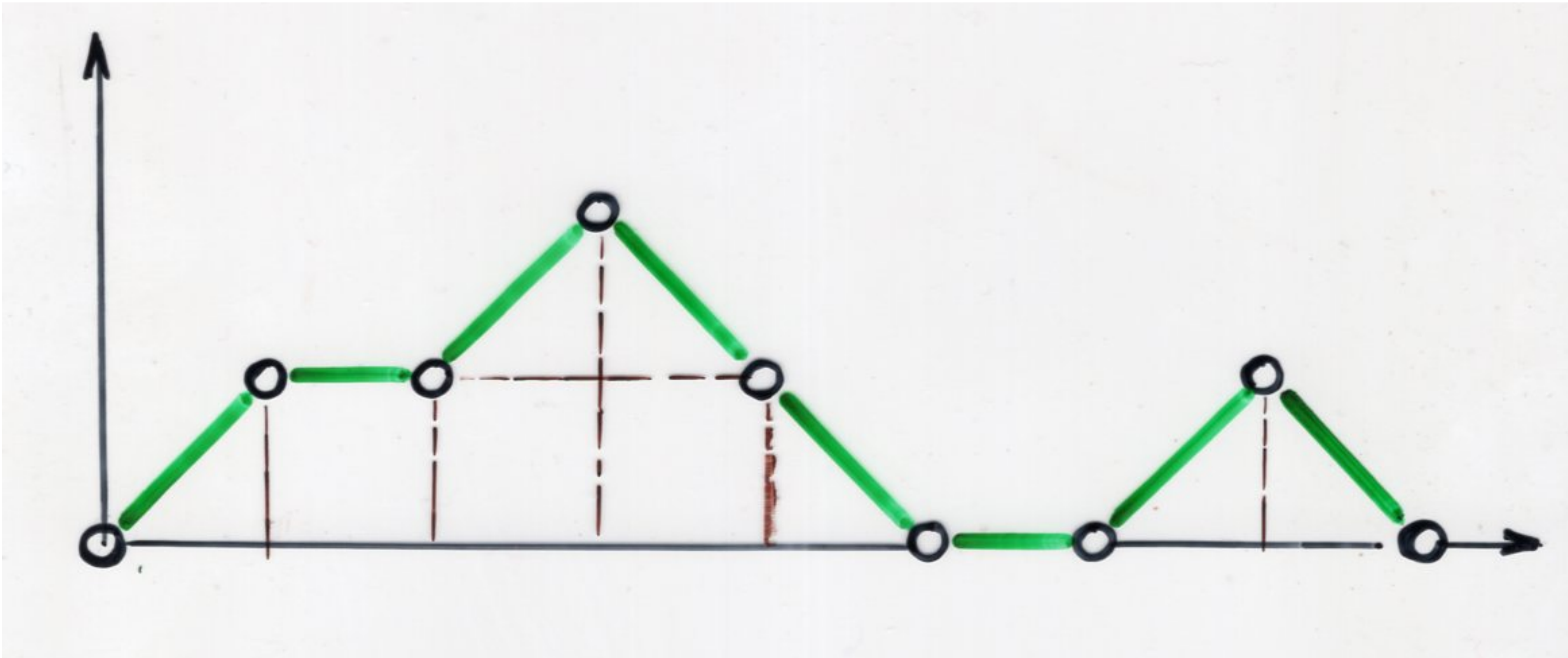


$$a_{k-1}\, c_k = \lambda_k$$

Laguerre history

$h = (\omega_c, p)$

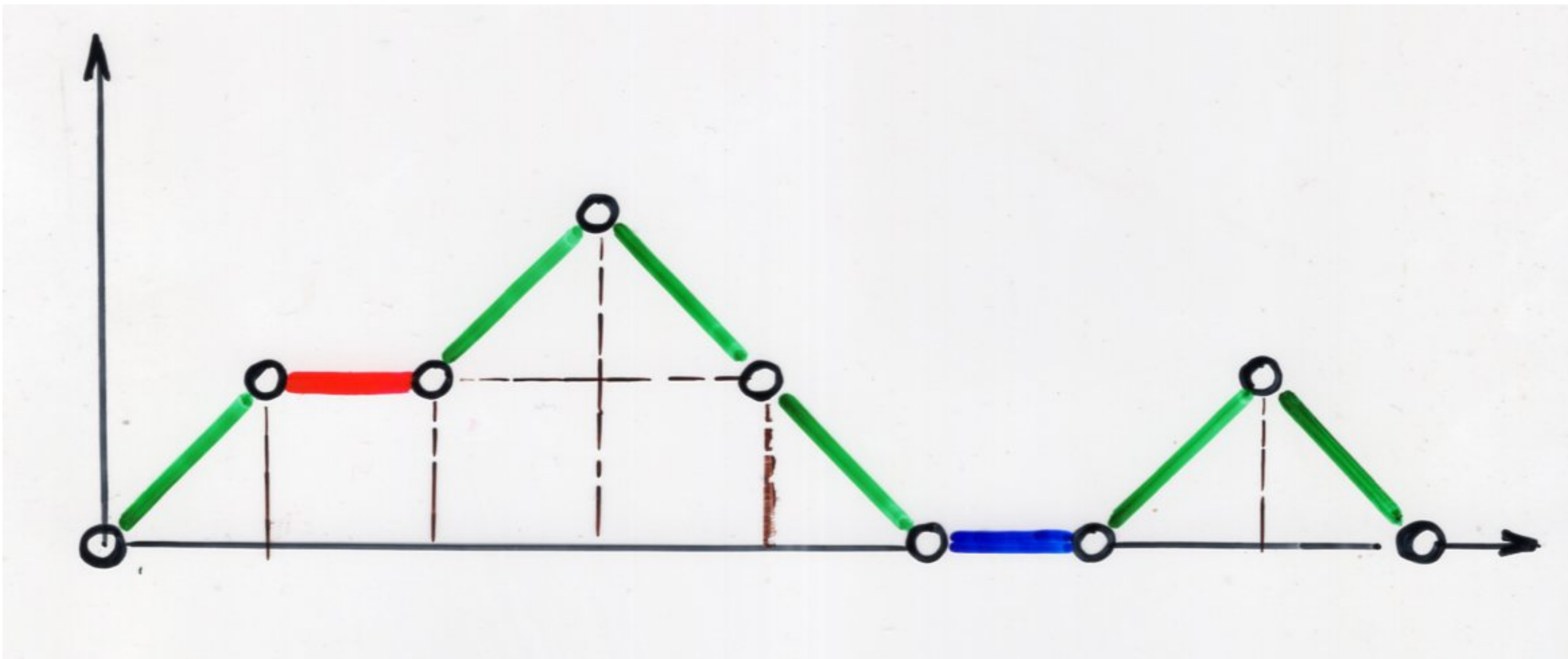
Motzkin path



Laguerre history $h = (\omega_c, p)$

Motzkin path

2 colors East steps

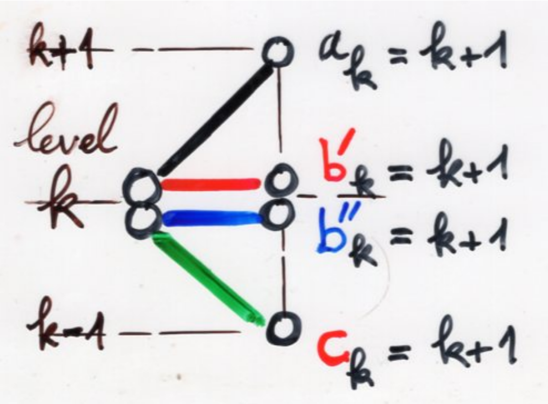


Laguerre history

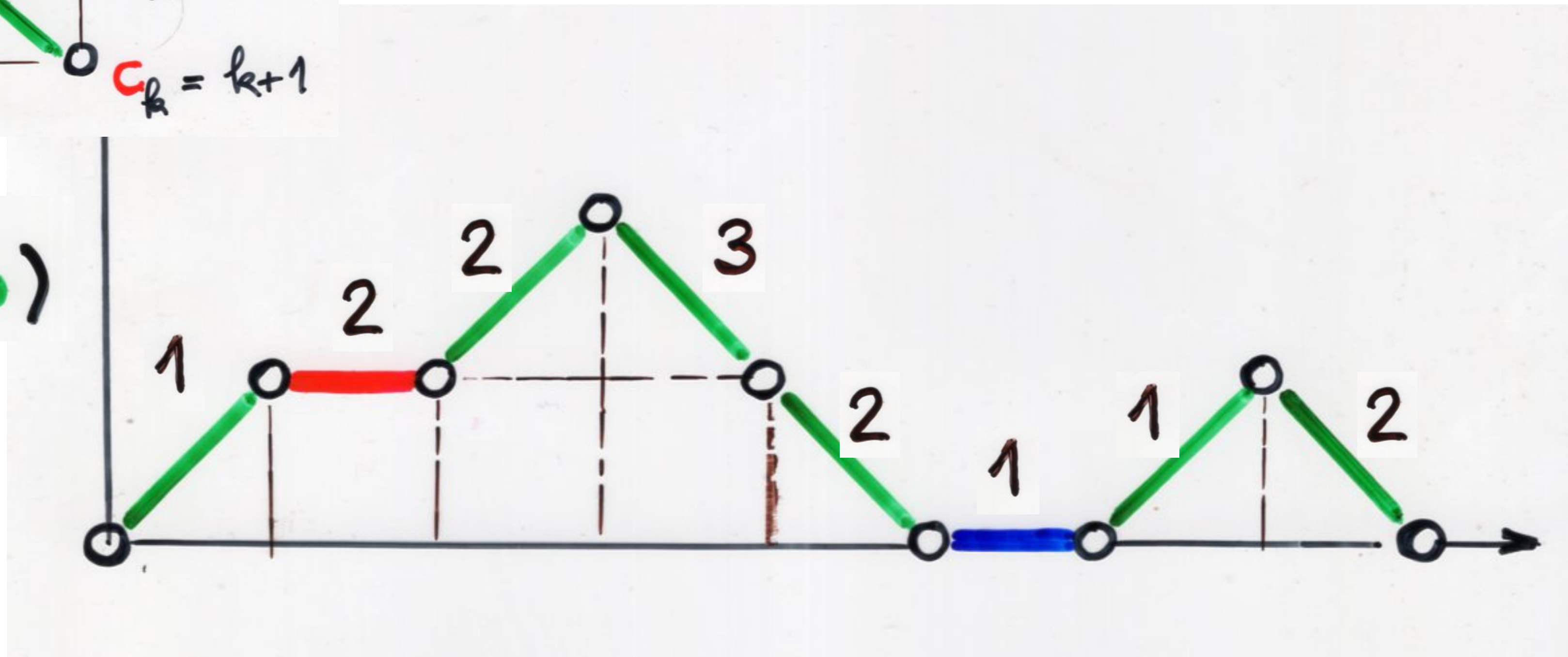
$h = (\omega_c , p)$

Motzkin path

2 colors East steps



$v^*(\omega)$



Laguerre history

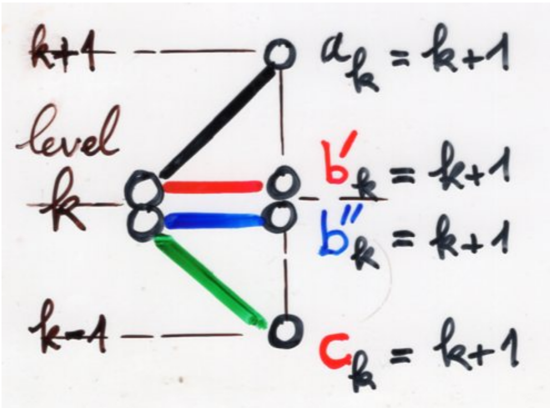
$h = (\omega_c, P)$

Motzkin path

2 colors East steps

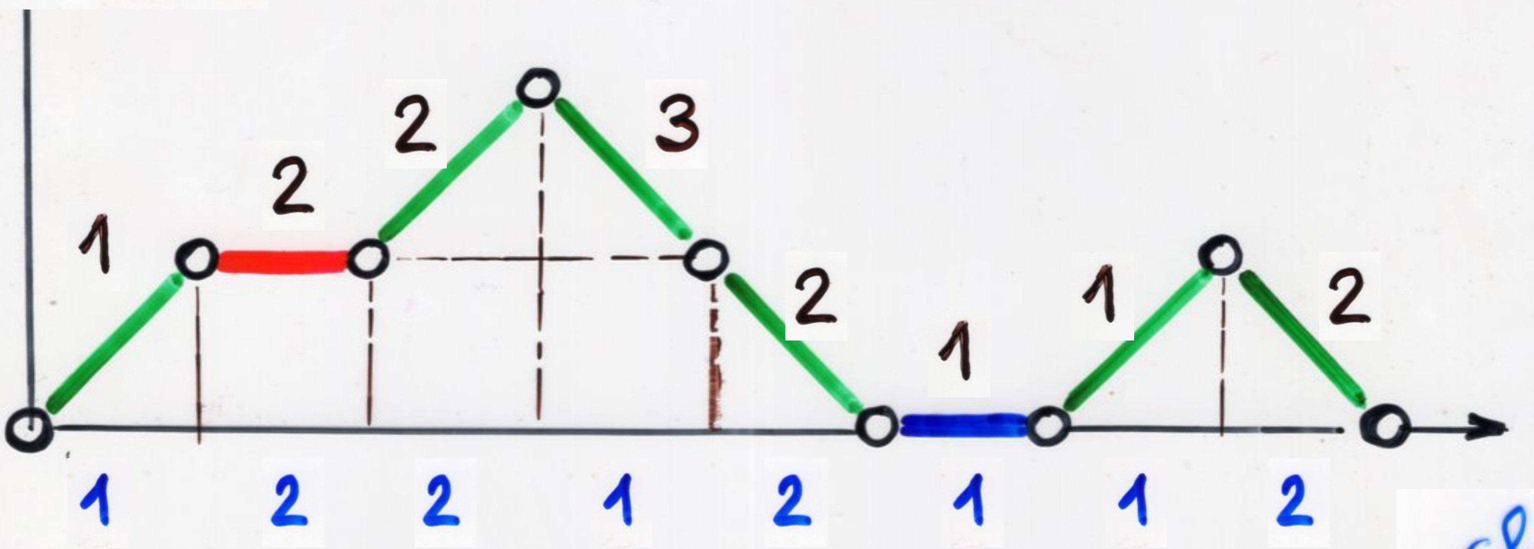
$P = (p_1, \dots, p_n)$

$1 \leq p_i \leq v(\omega_i)$



$\omega = (\omega_1 \cdots \omega_n)$

$v^*(\omega)$



choice function

bijection

$$h = (\omega_c ; \underbrace{(p_1, \dots, p_n)}_{p}) \longleftrightarrow \text{permutations } \sigma \in \mathfrak{S}_{n+1}$$

$|\omega| = n$

Laguerre histories

$(n+1)!$

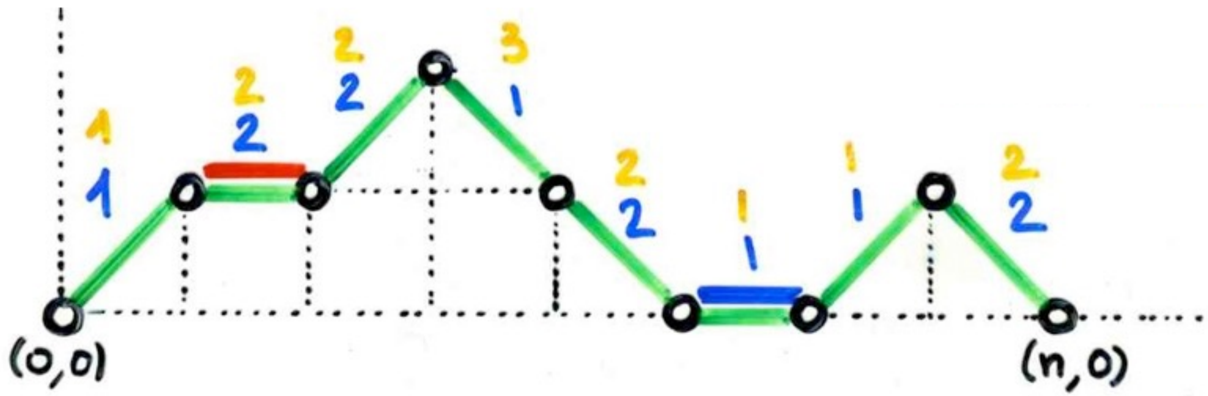
$|h| = |\omega|$

length of the history

J. Françon, X.V. (1979)

Laguerre history

1	1	1
2	2	2
3	2	2
4	3	1
5	2	2
6	1	1
7	1	1
8	2	2



$$\langle k|\, A = (k+1)\,\langle (k+1)|$$

$$\langle k|\, K = (k+1)\,\langle k|$$

$$\langle k|\, J = (k+1)\,\langle k|$$

$$\langle k|\, S = (k+1)\,\langle (k-1)|$$

1	1	1
2	2	2
3	2	2
4	3	1
5	2	2
6	1	1
7	1	1
8	2	2

␣

A

␣ 1 ␣

K

␣ 1 ␣ 2

A

␣ 1 ␣ 3 ␣ 2

S

4 1 ␣ 3 ␣ 2

S

4 1 ␣ 3 5 2

J

4 1 6 ␣ 3 5 2

A

4 1 6 ␣ 7 ␣ 3 5 2

S

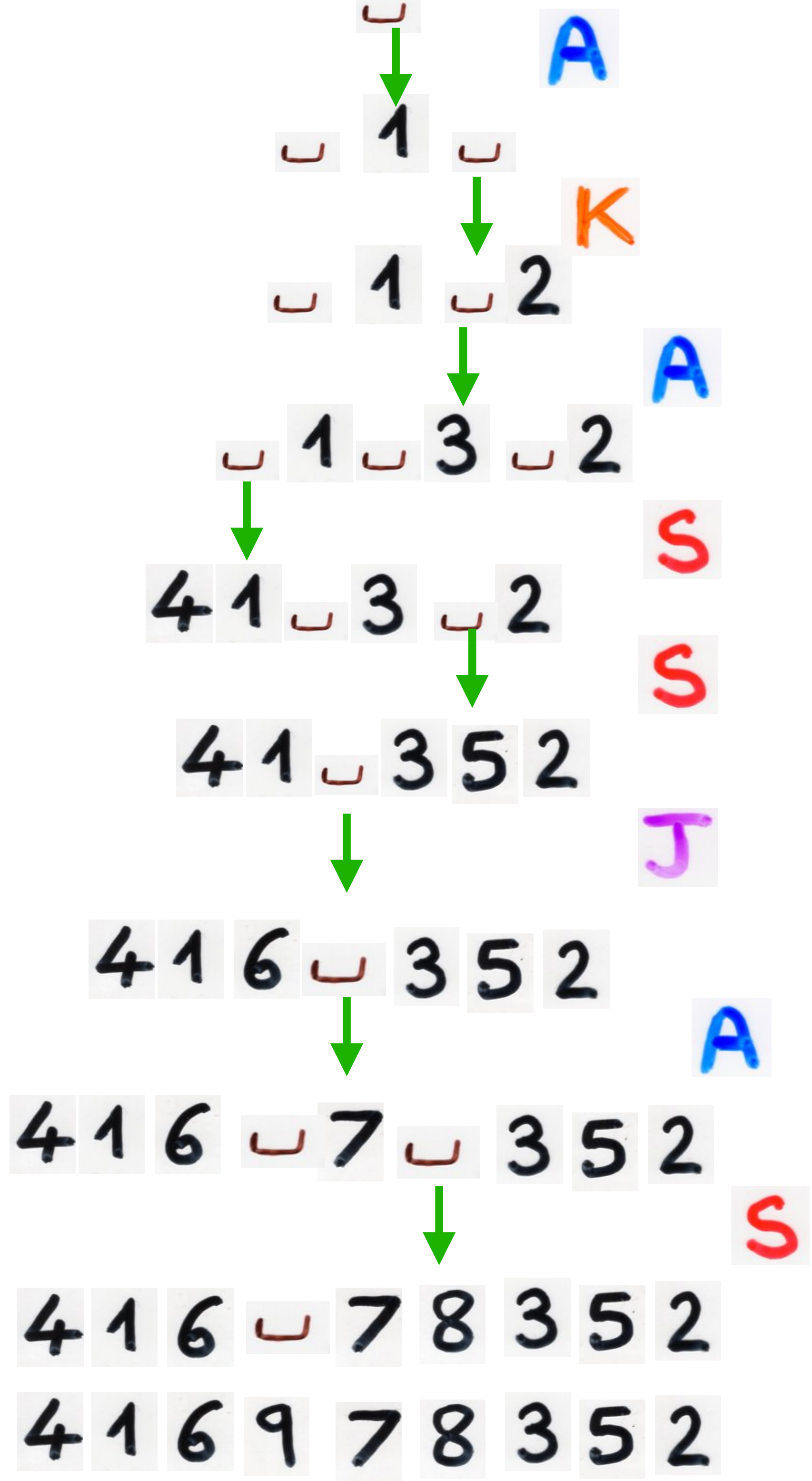
4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2

q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$



"q-analogue" of Laguerre histories

choice function

$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories q^4

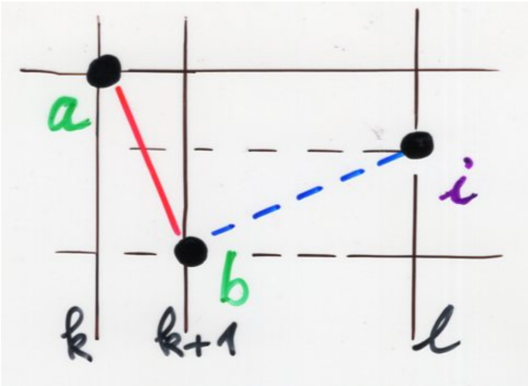
Lemma

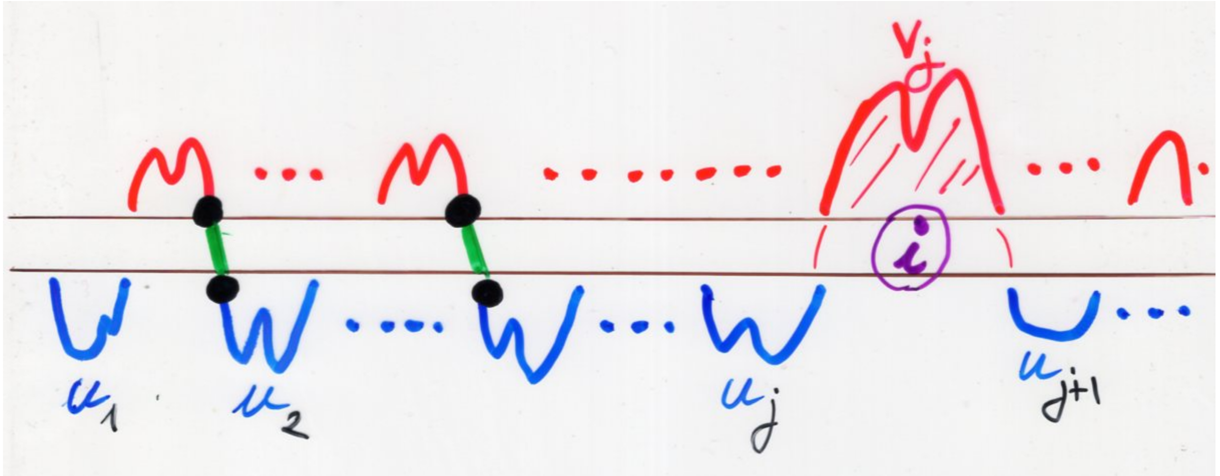
$P_i = j$ is also defined by:

$j = 1 +$ number of triples (a, b, i) having the patter (31-2), that is:

$a = \sigma(k)$, $b = \sigma(k+1)$, $i = \sigma(\ell)$

with $k < k+1 < \ell$ and $b < i < a$

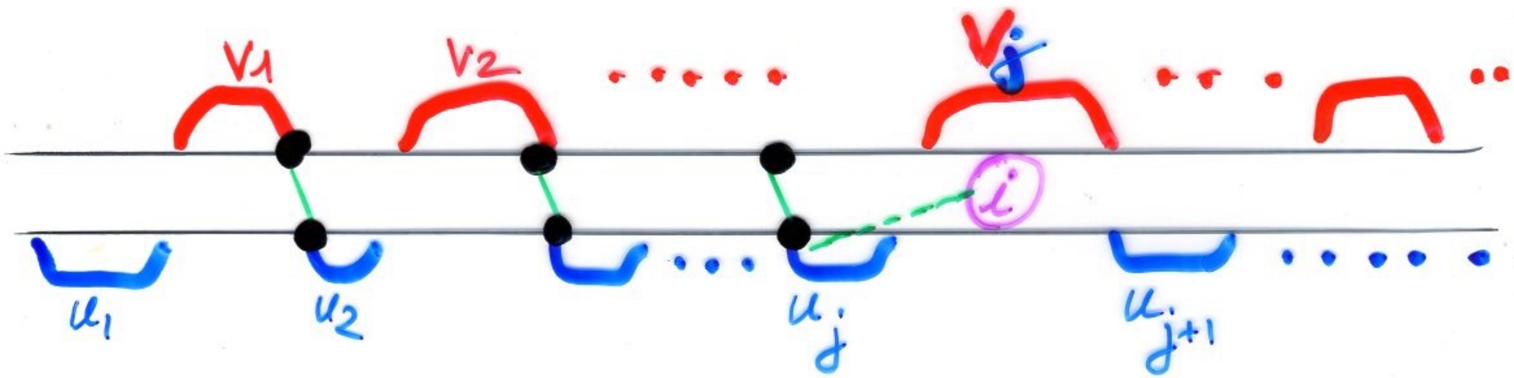




weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n} (p_i - 1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern (31-2)



q- Laguerre I

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\mu_n = (n+1)!$$

q- Laguerre
restricted
histories

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\mu_n = n!$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008)^y

q - Laguerre II

if $\mu_n = [n!]_q$

then
$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

Bijection (restricted) Laguerre histories

and Laguerre heaps of segments

Laguerre histories

$$\langle k|\, A = (k+1)\, \langle (k+1)|$$
$$\langle k|\, K = (k+1)\, \langle k|$$
$$\langle k|\, J = (k+1)\, \langle k|$$
$$\langle k|\, S = (k+1)\, \langle (k-1)|$$

$$\mu_n = (n+1)!$$

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

restricted Laguerre histories

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$K\,|k\rangle = (k+1)\,|k\rangle$$
$$J^b\,|k\rangle = k\,|k\rangle$$
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$

$$\mu_n = n!$$

$$b_k = (2k+1)$$
$$\lambda_k = k^2$$

$$A|k\rangle = (k+1)\,|(k+1)\rangle$$

$$J^b|k\rangle = k\,|k\rangle$$

$$K|k\rangle = (k+1)\,|k\rangle$$

$$S^b|k\rangle = k\,|(k-1)\rangle$$

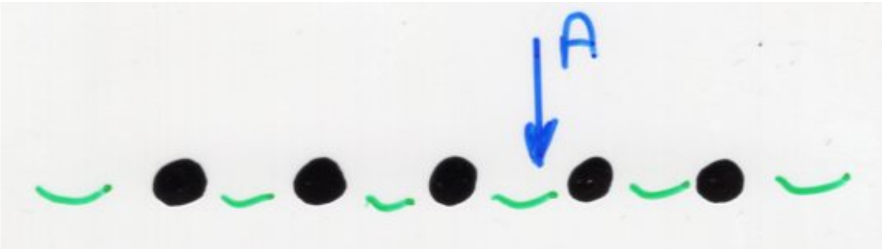
dictionnary data structure

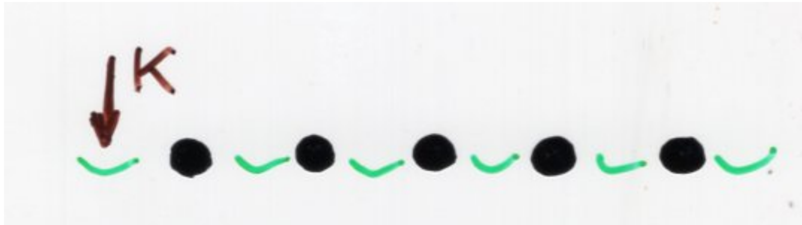
add or delete any element

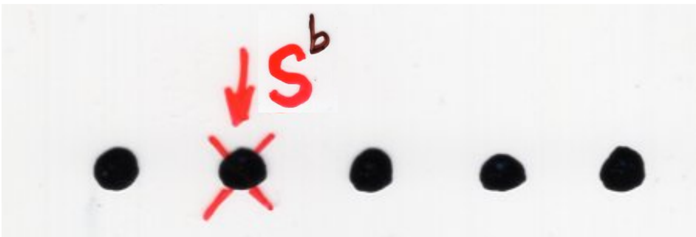
ask questions

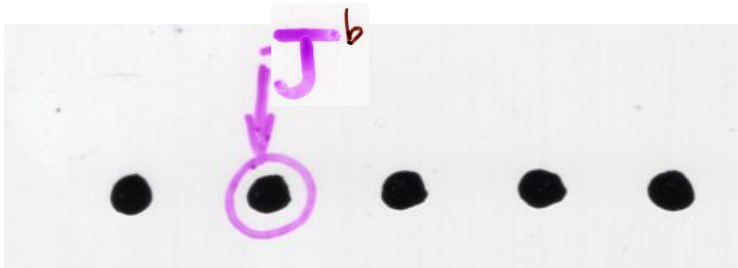
J^b positive

K negative





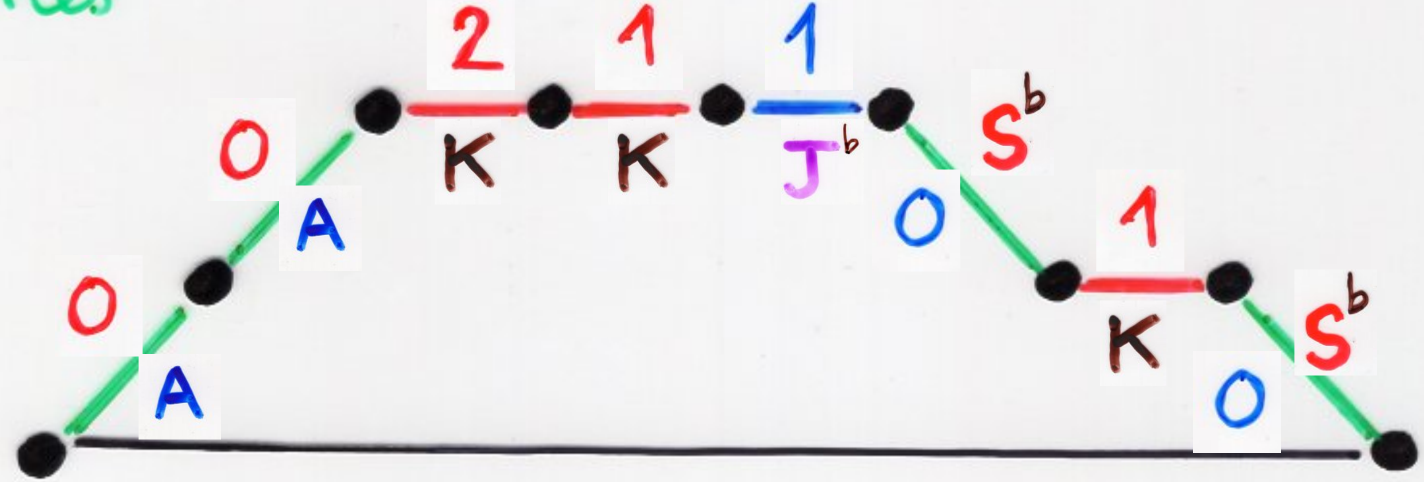


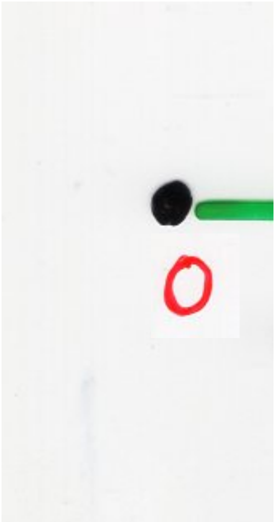


$$A|k\rangle = (k+1)\,|(k+1)\rangle \qquad J^b|k\rangle = k\,|k\rangle$$

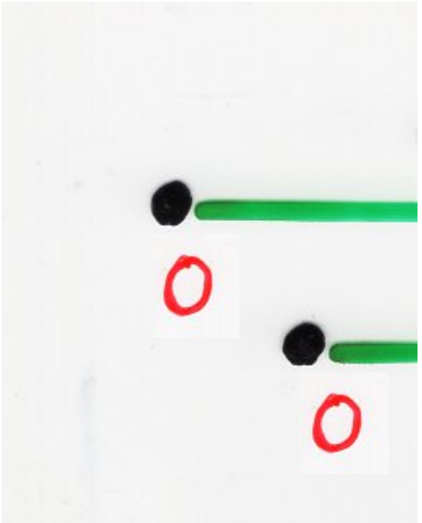
$$K|k\rangle = (k+1)\,|k\rangle \qquad S^b|k\rangle = k\,|(k-1)\rangle$$

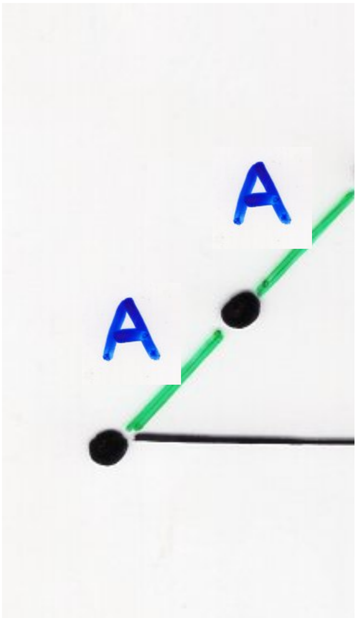
restricted Laguerre histories

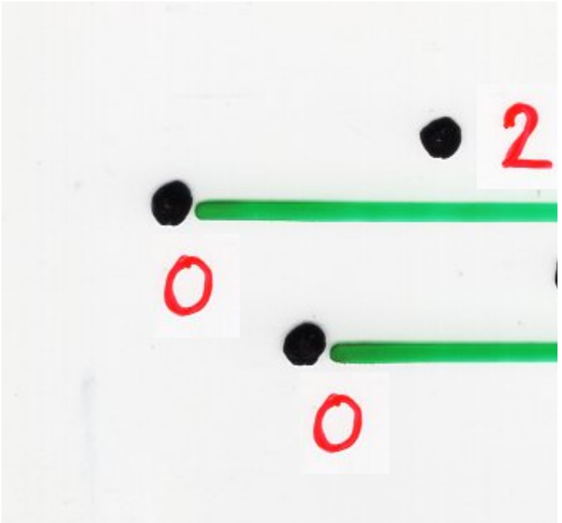


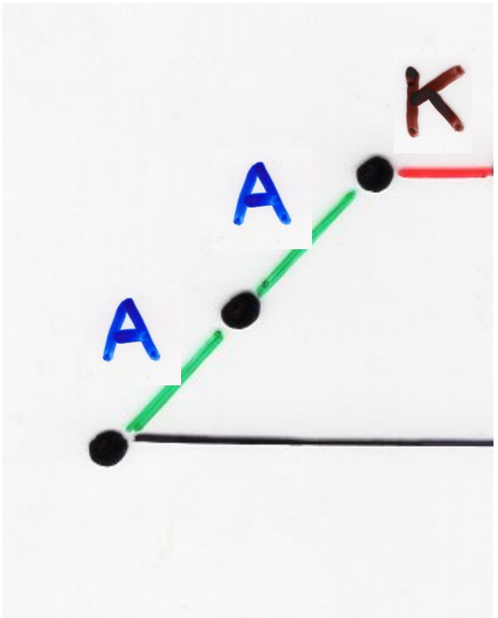

O

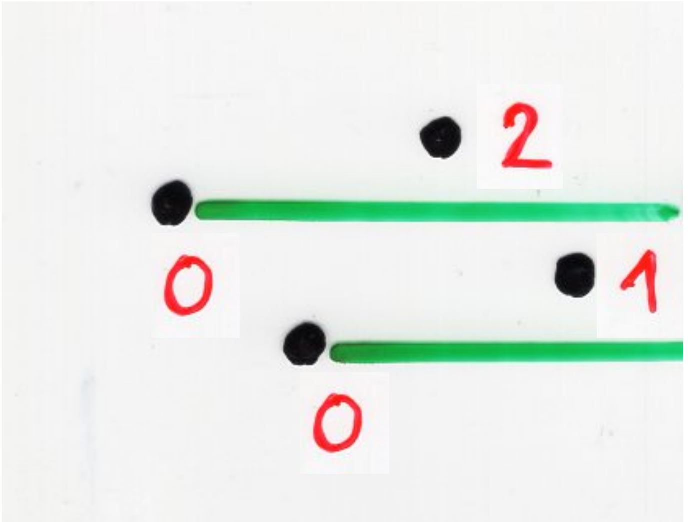

A

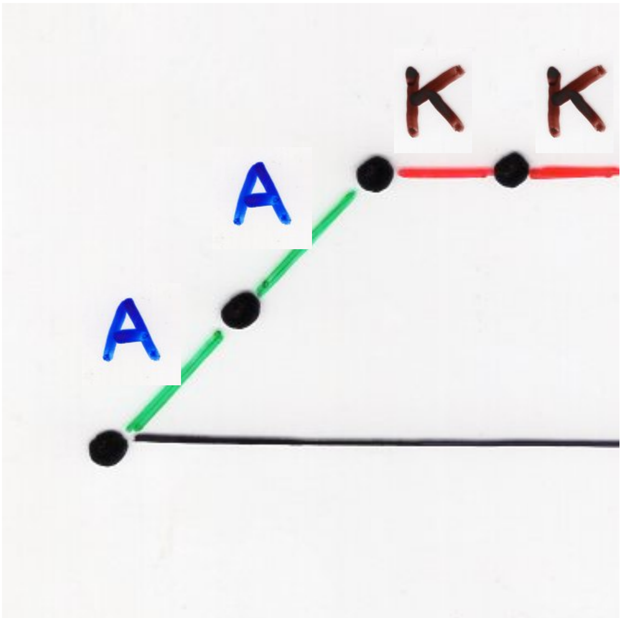

O
O

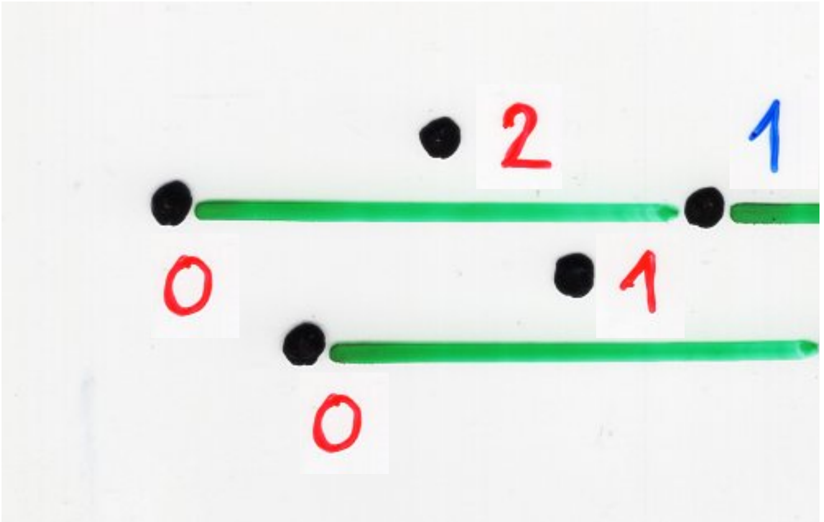

A
A

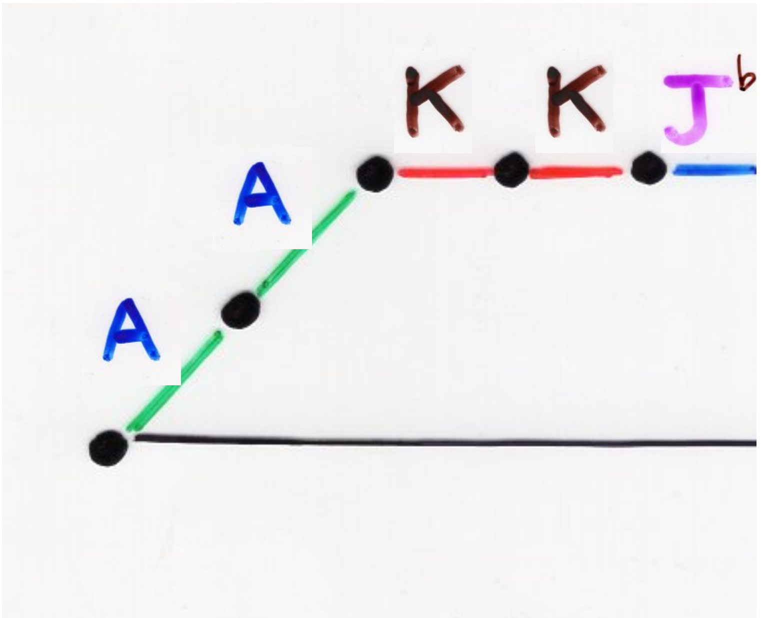

2
0
0

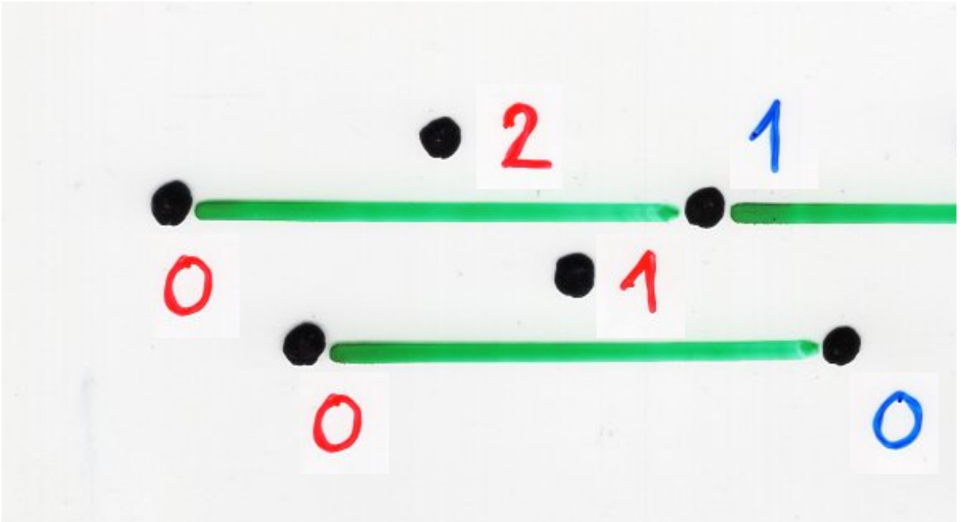

K
A
A


2
0
1
0

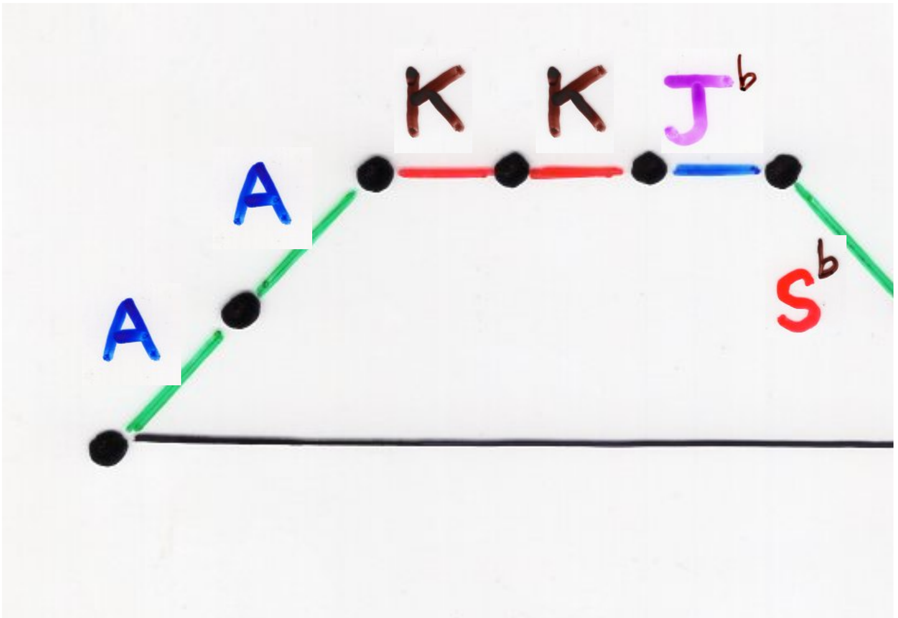

K
K
A
A


2
1
0
1
0

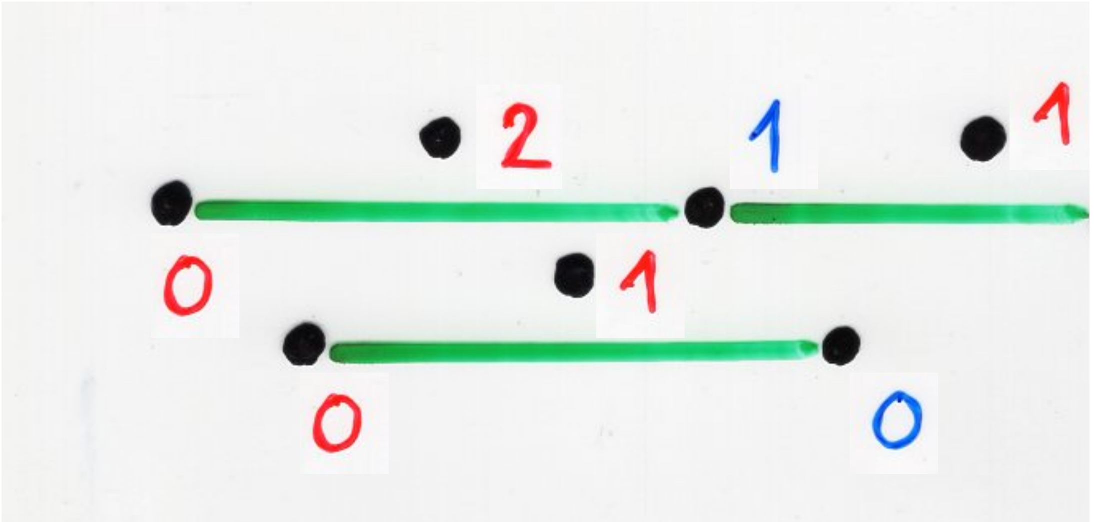

K
K
J
b
A
A

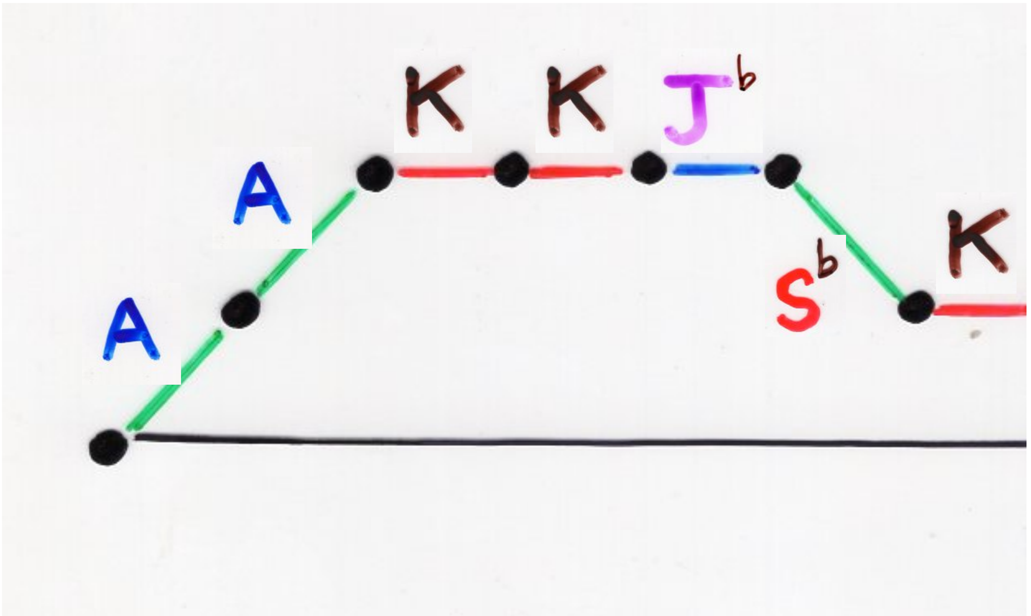


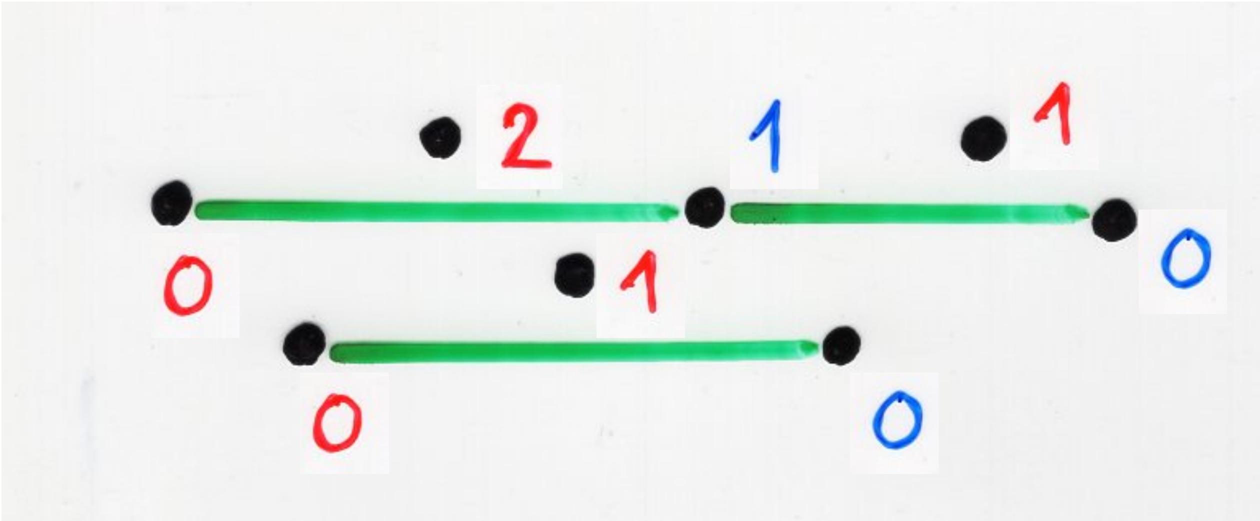
2
1
0
1
0
0

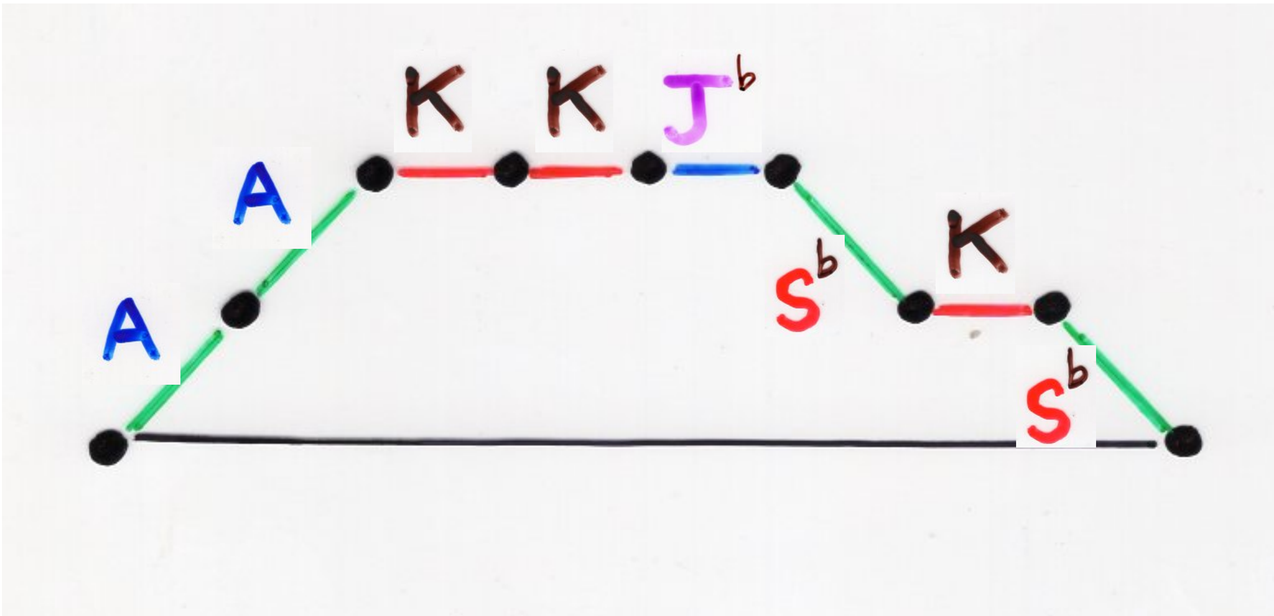


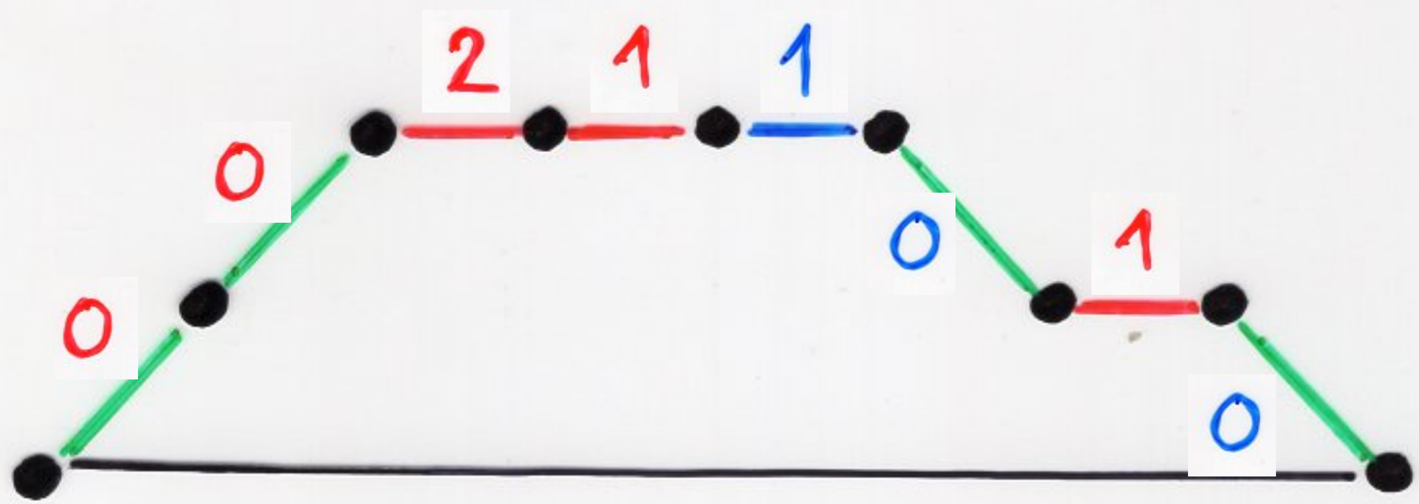
K
K
J^b
A
S^b
A


2
1
1
0
1
0
0

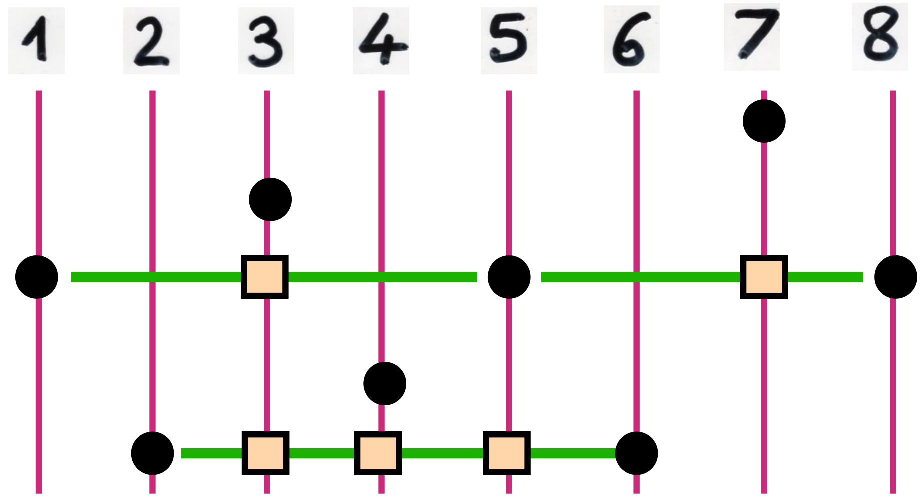

K
K
J^b
A
A
S^b
K

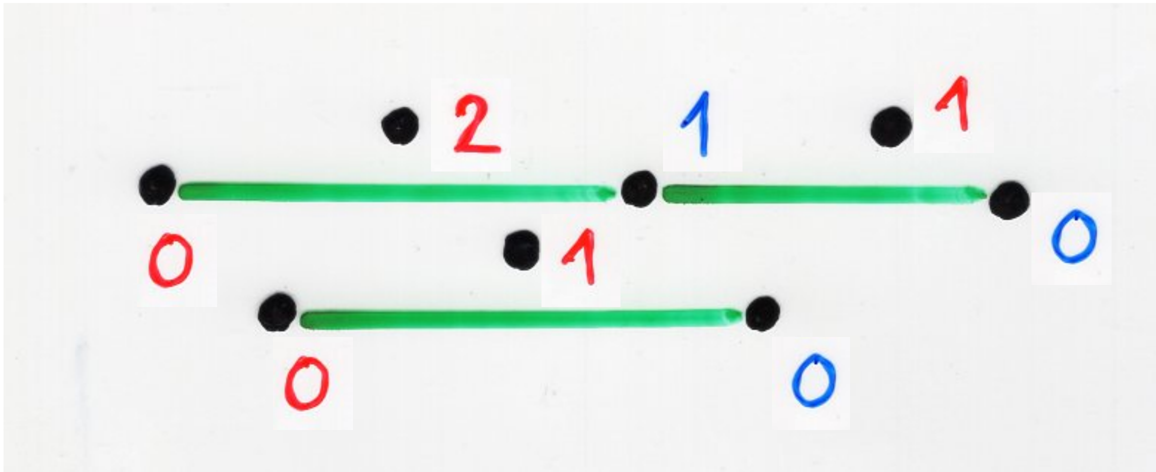

2
1
1
0
0
1
0
0

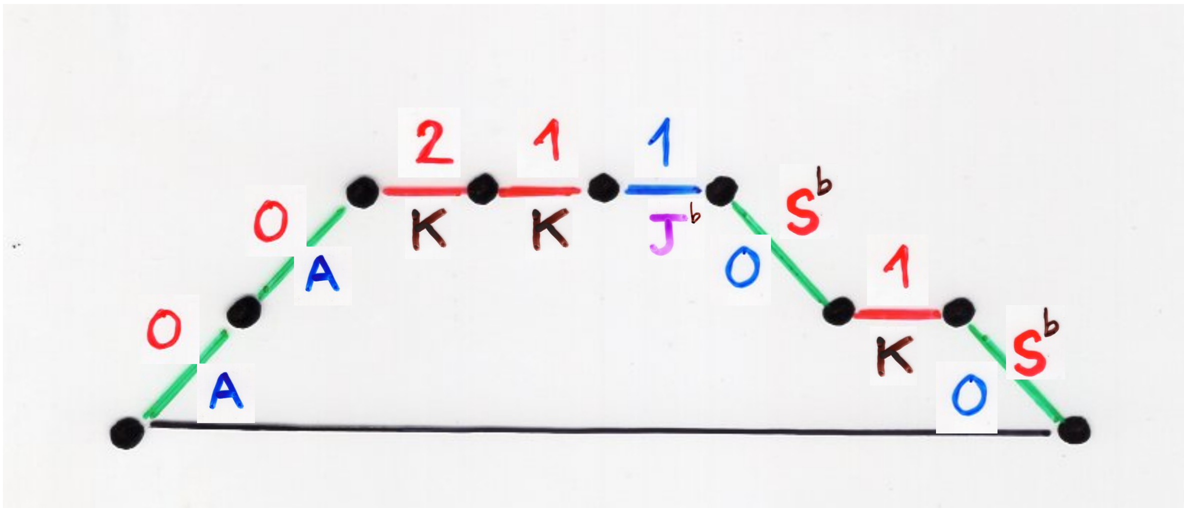

K
K
J^b
A
A
S^b
K
S^b



0
0
2
1
1
0
1
0


1 2 3 4 5 6 7 8


2
1
1
0
0
1
0
0


2
1
1
0
A
0
A
K
K
J^b
S^b
0
1
K
S^b
0

From (restricted) Laguerre histories

to

Permutations
(word notation)

$$\langle k|\, A = (k+1)\, \langle (k+1)|$$

$$\langle k|\, K = (k+1)\, \langle k|$$

$$\langle k|\, J = (k+1)\, \langle k|$$

$$\langle k|\, S = (k+1)\, \langle (k-1)|$$

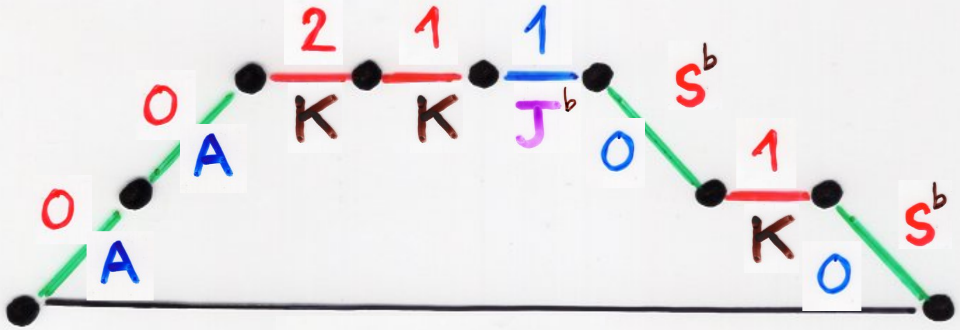
$$A\, |k\rangle = (k+1)\, |(k+1)\rangle$$

$$K\, |k\rangle = (k+1)\, |k\rangle$$

$$J^b\, |k\rangle = k\, |k\rangle$$

$$S^b\, |k\rangle = k\, |(k-1)\rangle$$

Laguerre history



restricted Laguerre histories

$\sqcup$

$A \quad \sqcup 1 \sqcup$

$A \quad \sqcup 2 \sqcup 1 \sqcup$

$K \quad \sqcup 2 \sqcup 1\ 3 \sqcup$

$K \quad \sqcup 2\ 4 \sqcup 1\ 3 \sqcup$

$J^b \quad \sqcup 2\ 4 \sqcup 5\ 1\ 3 \sqcup$

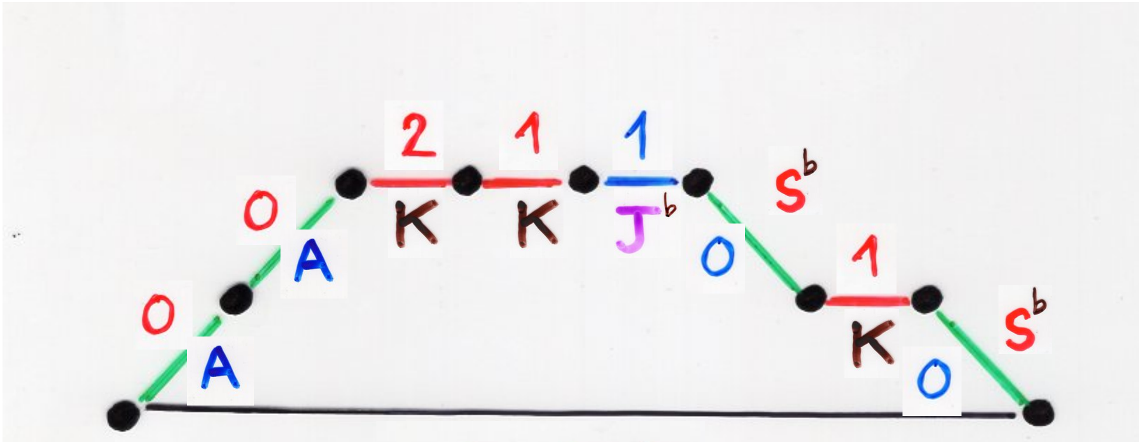
$S^b \quad 6\ 2\ 4 \sqcup 5\ 1\ 3 \sqcup$

$K \quad 6\ 2\ 4 \sqcup 5\ 1\ 3\ 7 \sqcup$

$S^b \quad 6\ 2\ 4\ 8\ 5\ 1\ 3\ 7 \sqcup$

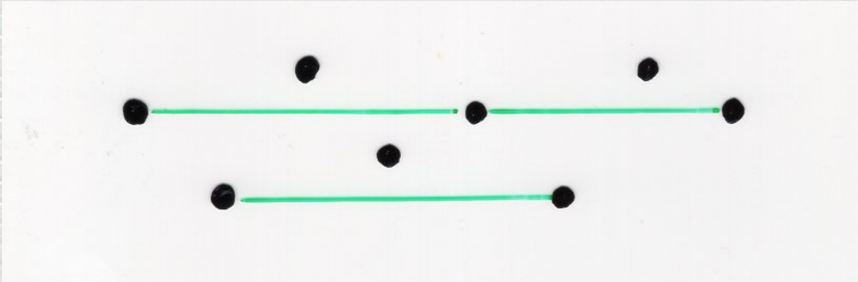
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$K\,|k\rangle = (k+1)\,|k\rangle$$

$$J^b\,|k\rangle = k\,|k\rangle$$
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$

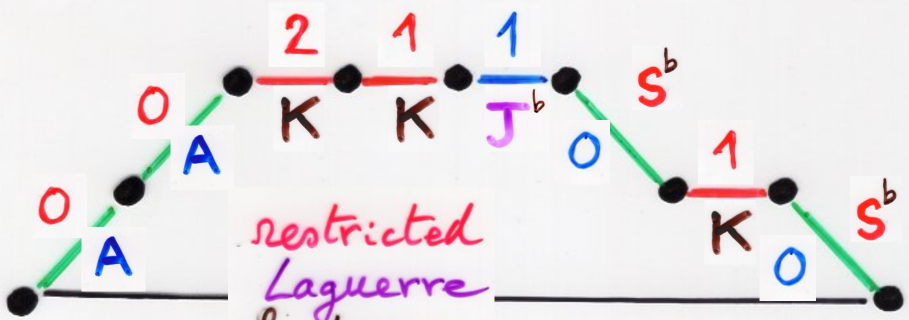


restricted
Laguerre
histories

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$



Laguerre heap of segment



restricted Laguerre histories

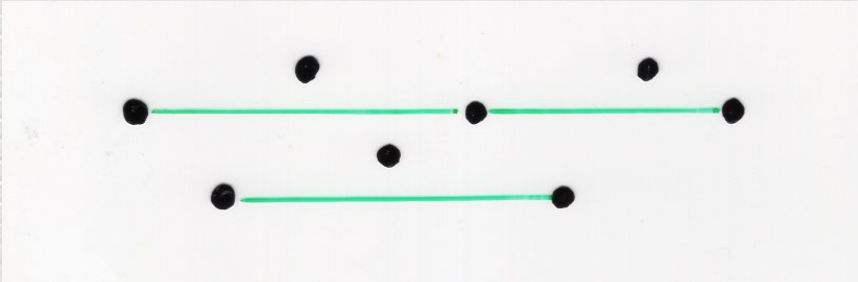
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 7 & 3 & 5 & 1 & 8 & 4 \end{pmatrix}$$

permutation

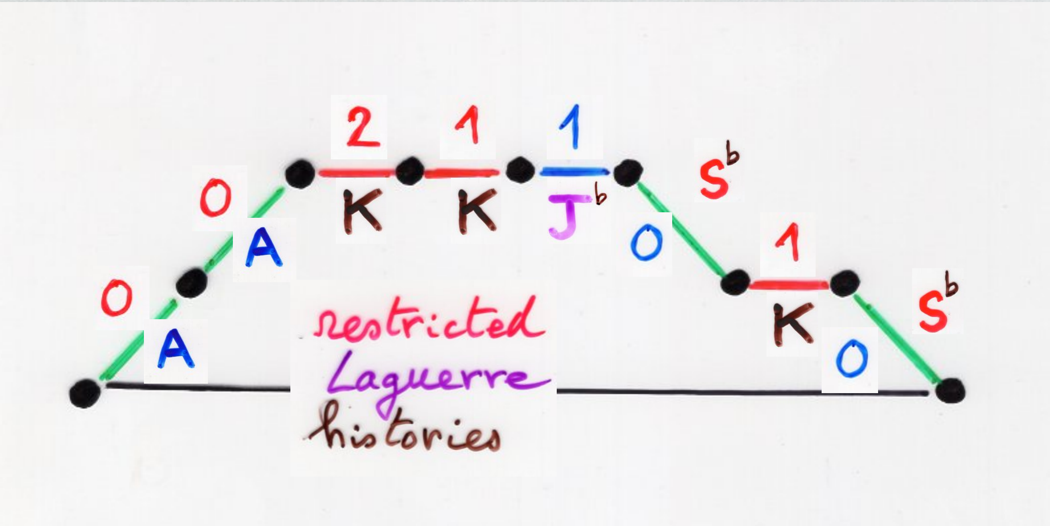
σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$

permutation



Laguerre heap of segment



σ^{-1}

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 7 & 3 & 5 & 1 & 8 & 4 \end{pmatrix}$$

permutation

σ

permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 4 & 8 & 5 & 1 & 3 & 7 \end{pmatrix}$$

Proof of Josuat-Vergès proposition

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

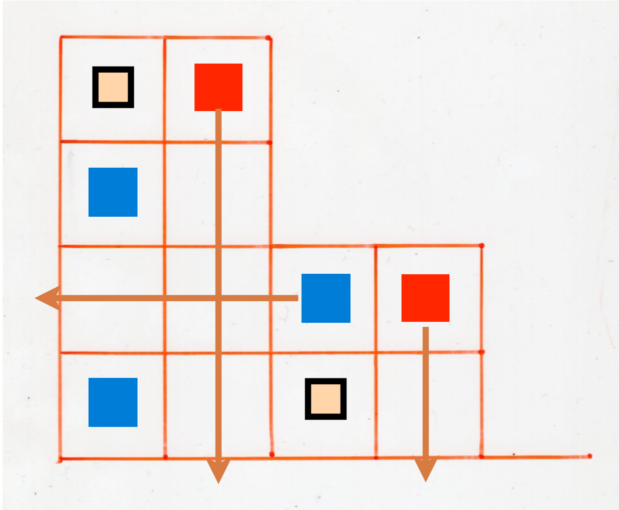
$31\text{-}2(\sigma)$ = number of patterns 31-2

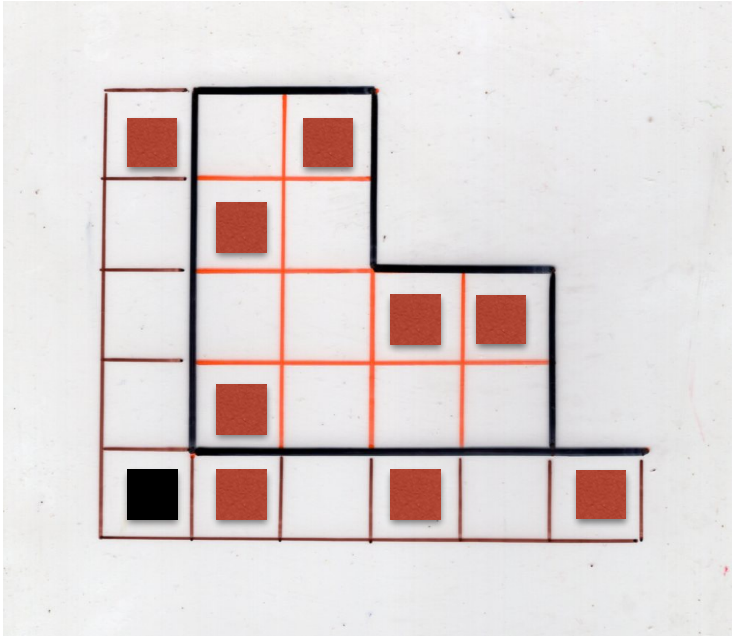
- direct bijection (with tree-like tableaux)

Aval, Boussicault, Nadeau (2011)

$$\begin{array}{c}\text{tableaux}\\ \text{size } (n+1)\end{array} \longleftrightarrow \left(\begin{array}{c}\text{tableaux}\\ \text{size } n\end{array}, 1 \leqslant i \leqslant n+1\right)$$

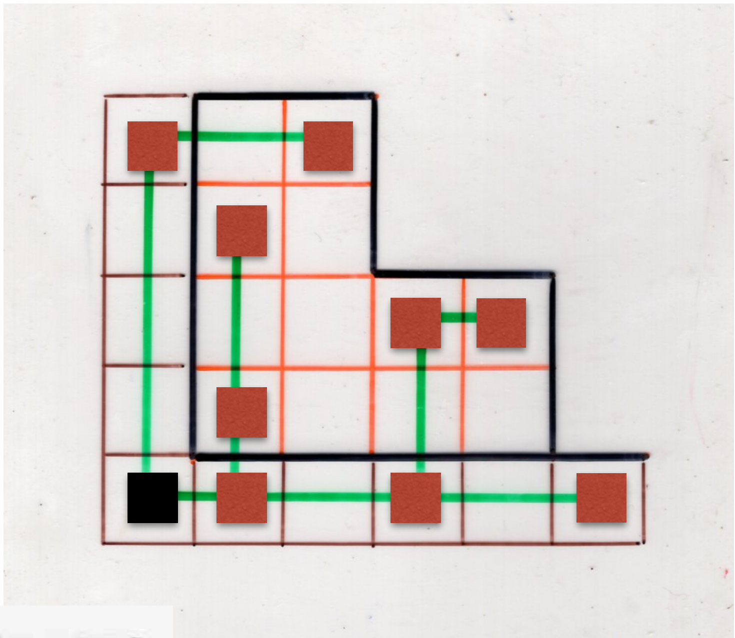
$$(n+1)!$$



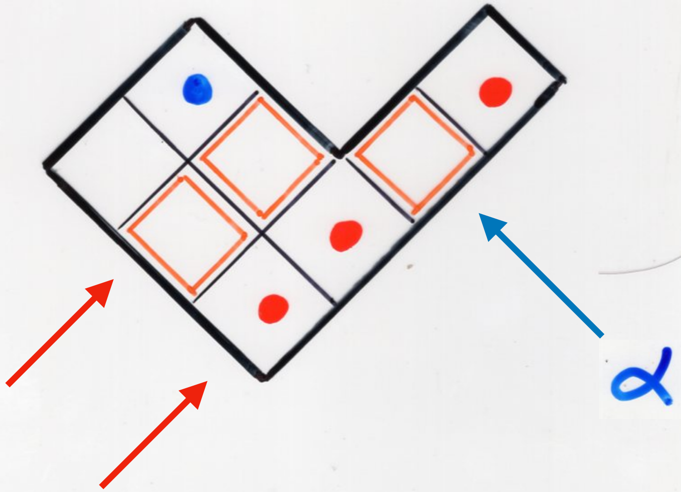


alternative tableaux

tree-like tableaux

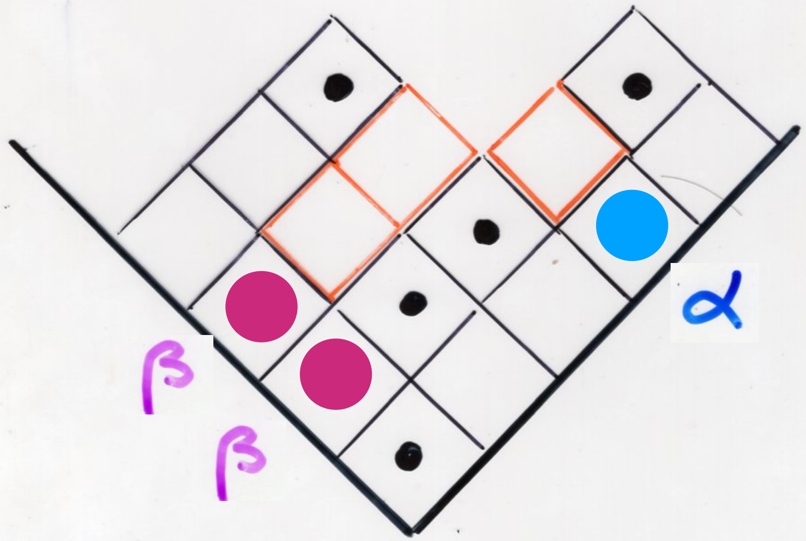


Aval, Boussicault, Nadeau (2013)



bijection

$f \longrightarrow T$



1

1

2 1

1

2 3 1

0

2 3 1 4

2

2 3 5 1 4

5

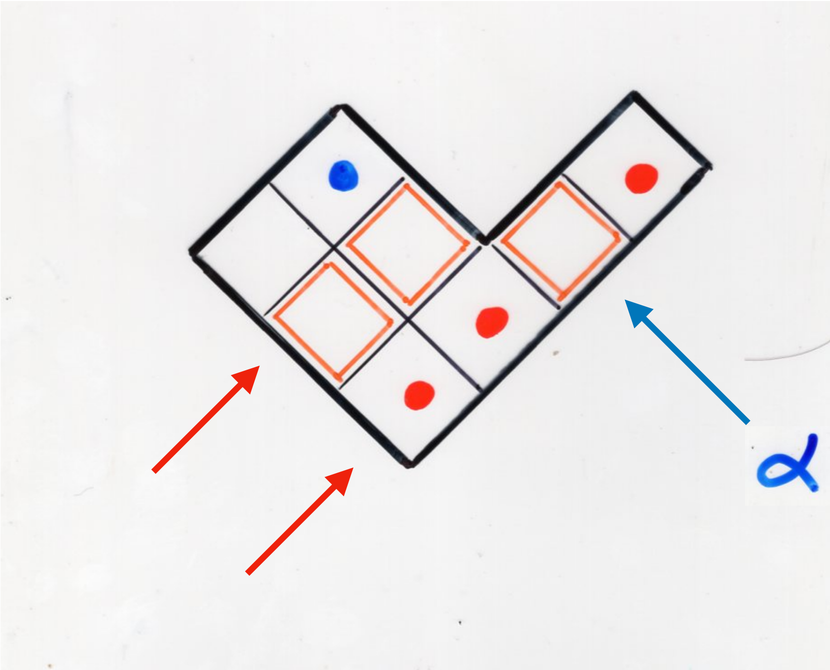
6 2 3 5 1 4

4

6 2 7 3 5 1 4

1

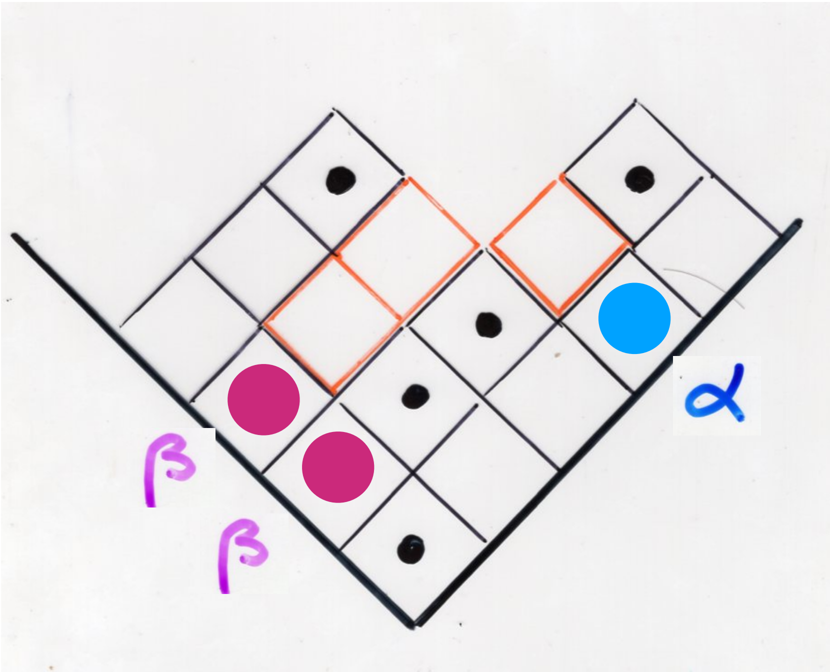
6 2 7 3 5 1 8 4

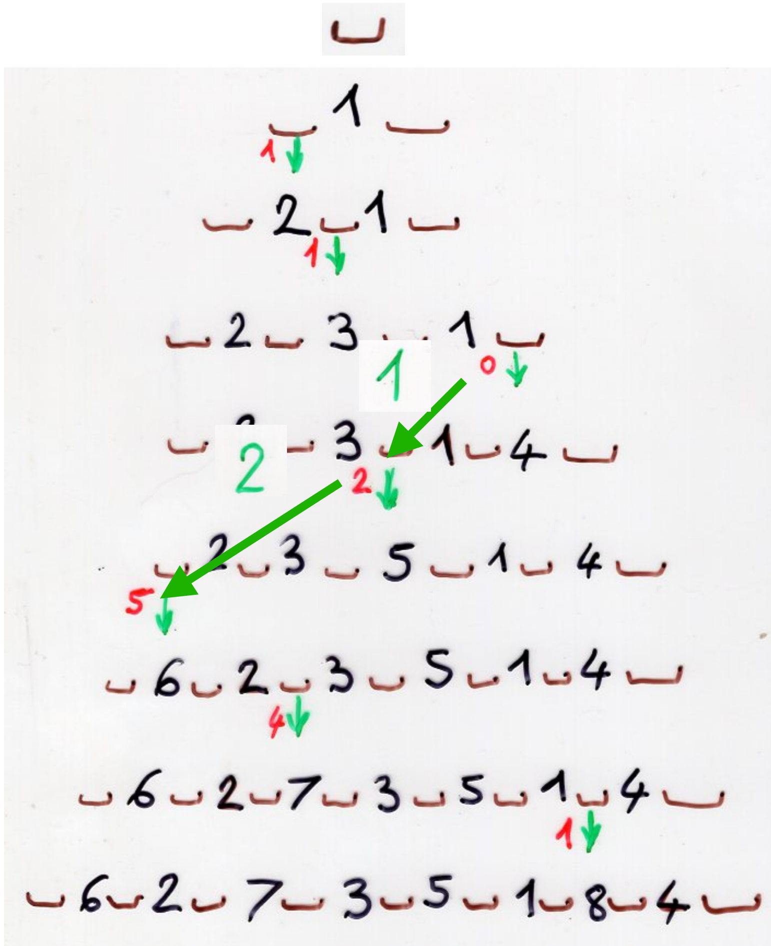


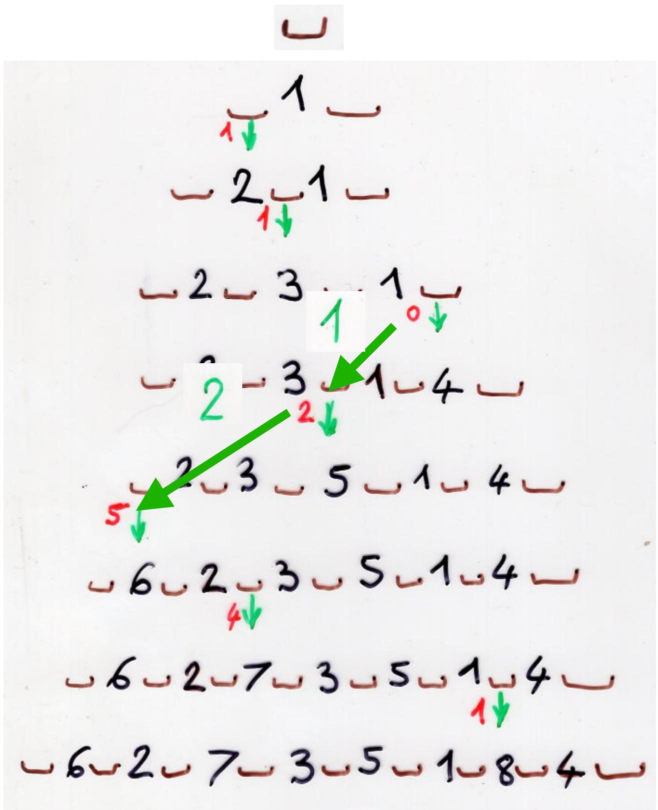
number of crossings

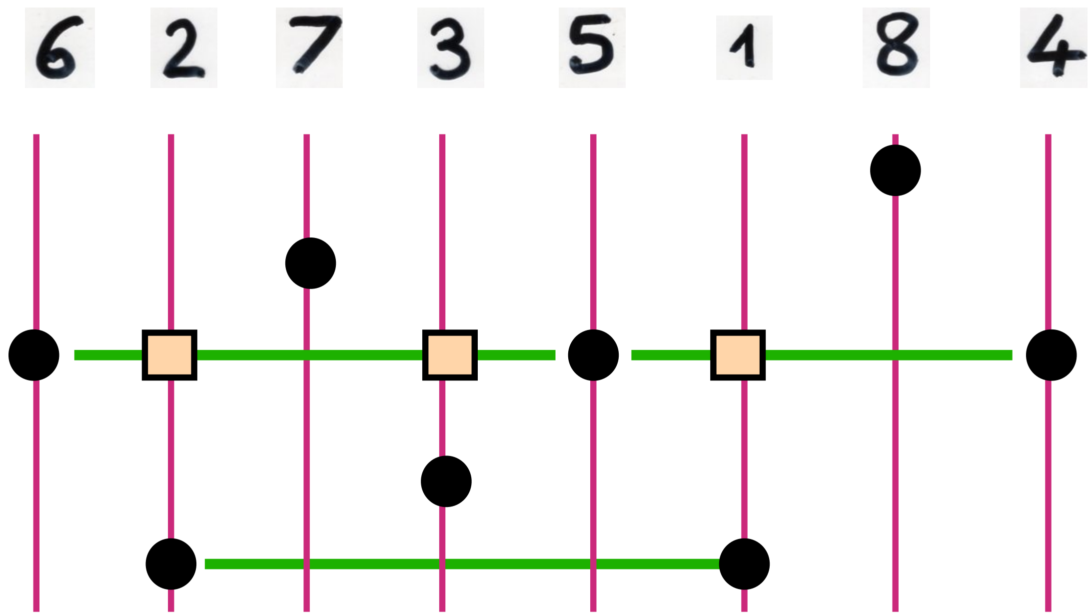
cr(T)

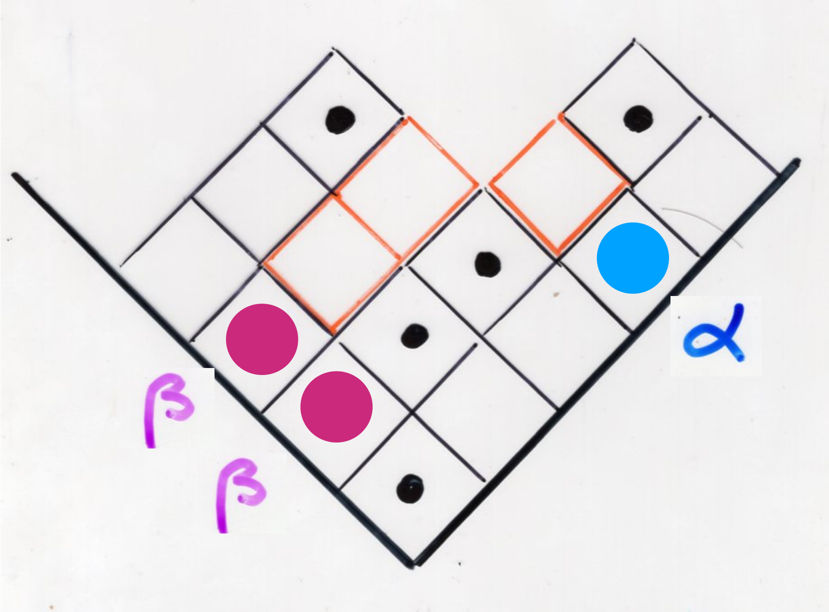
$$= \sum_{1 \leq i \leq (n-1)} \max\left[\left(f(i+1) - f(i) - 1\right), 0\right]$$

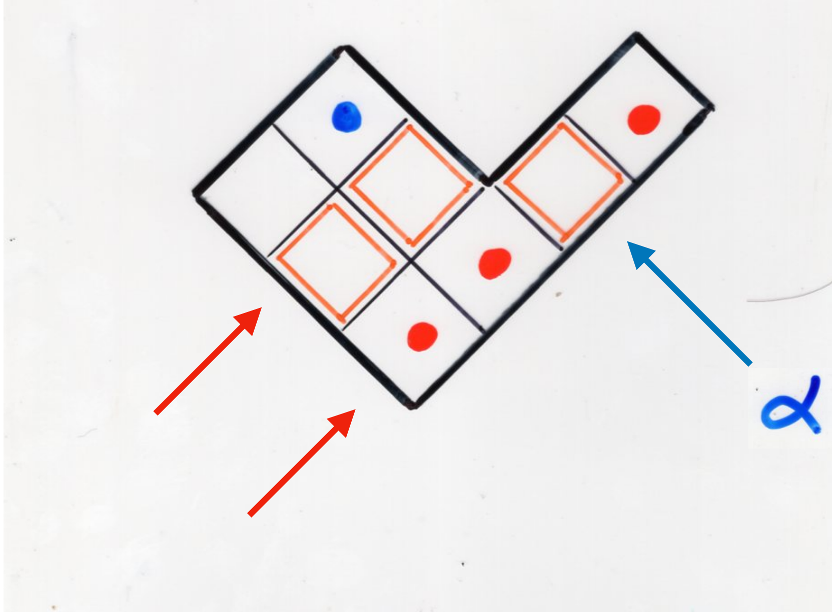


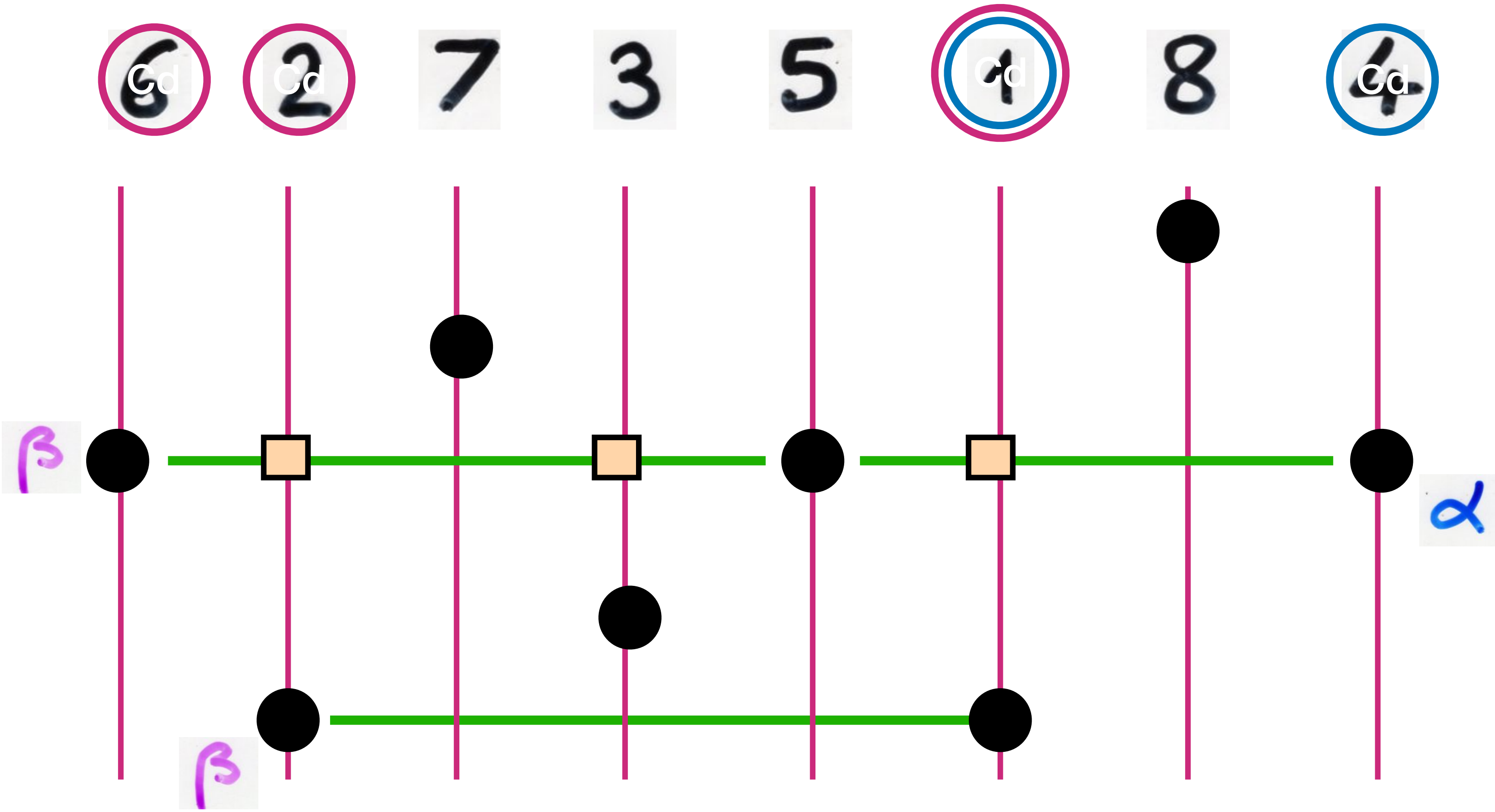


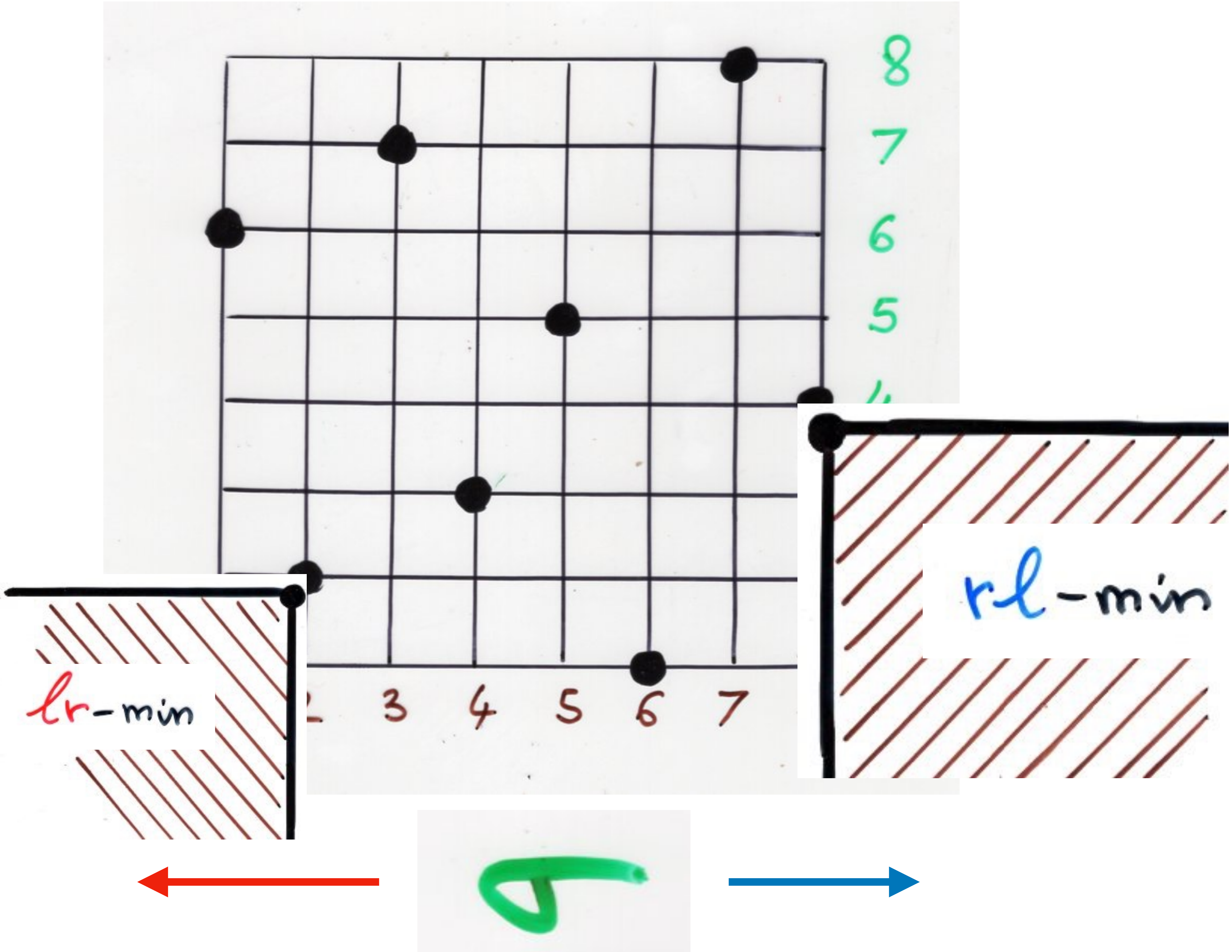



6 2 7 3 5 1 8 4

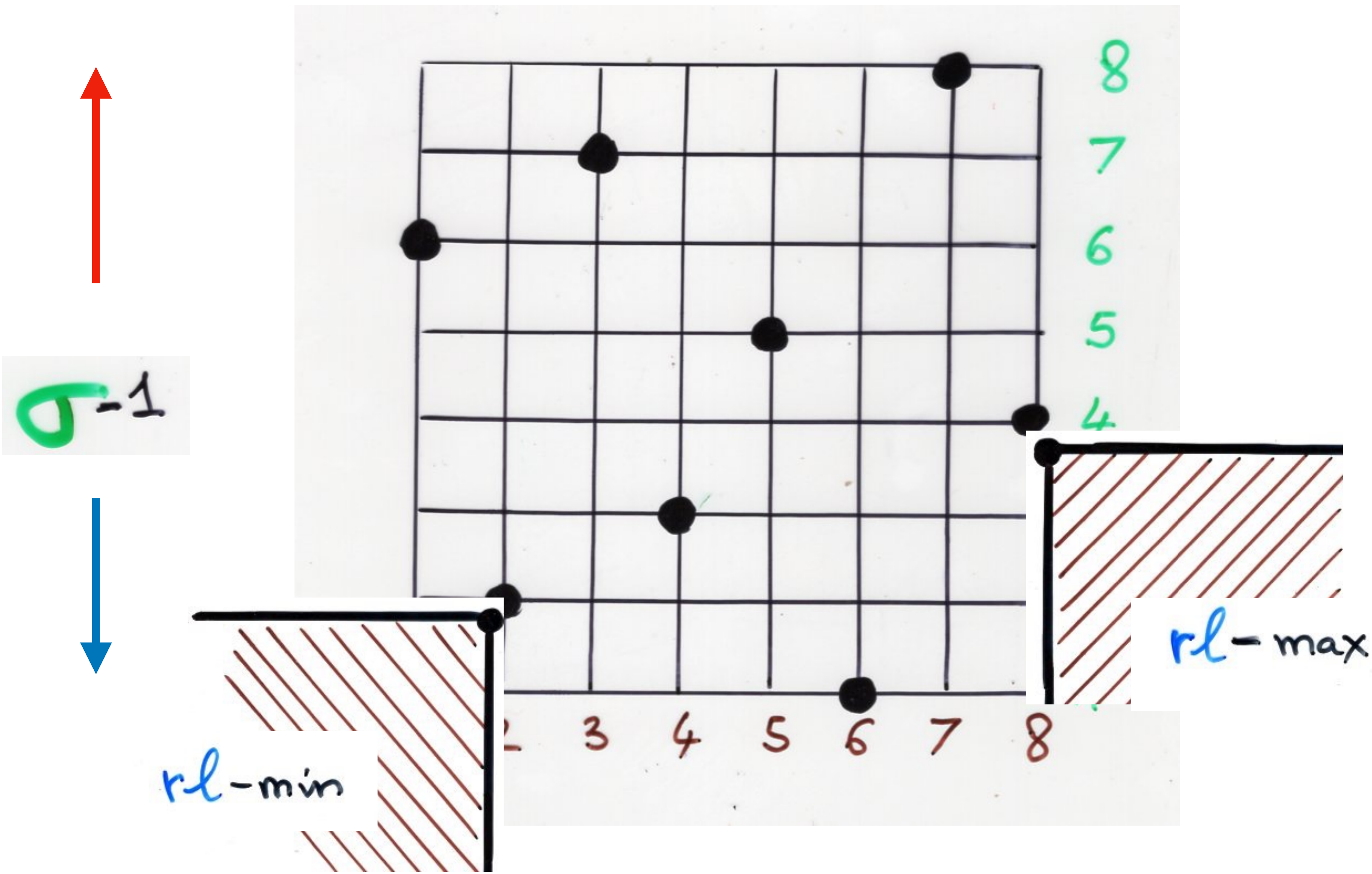

β
β
α


α


6 2 7 3 5 1 8 4
Cd
β
α
β



8
7
6
5
4
lr-min
2
3
4
5
6
7
rl-min

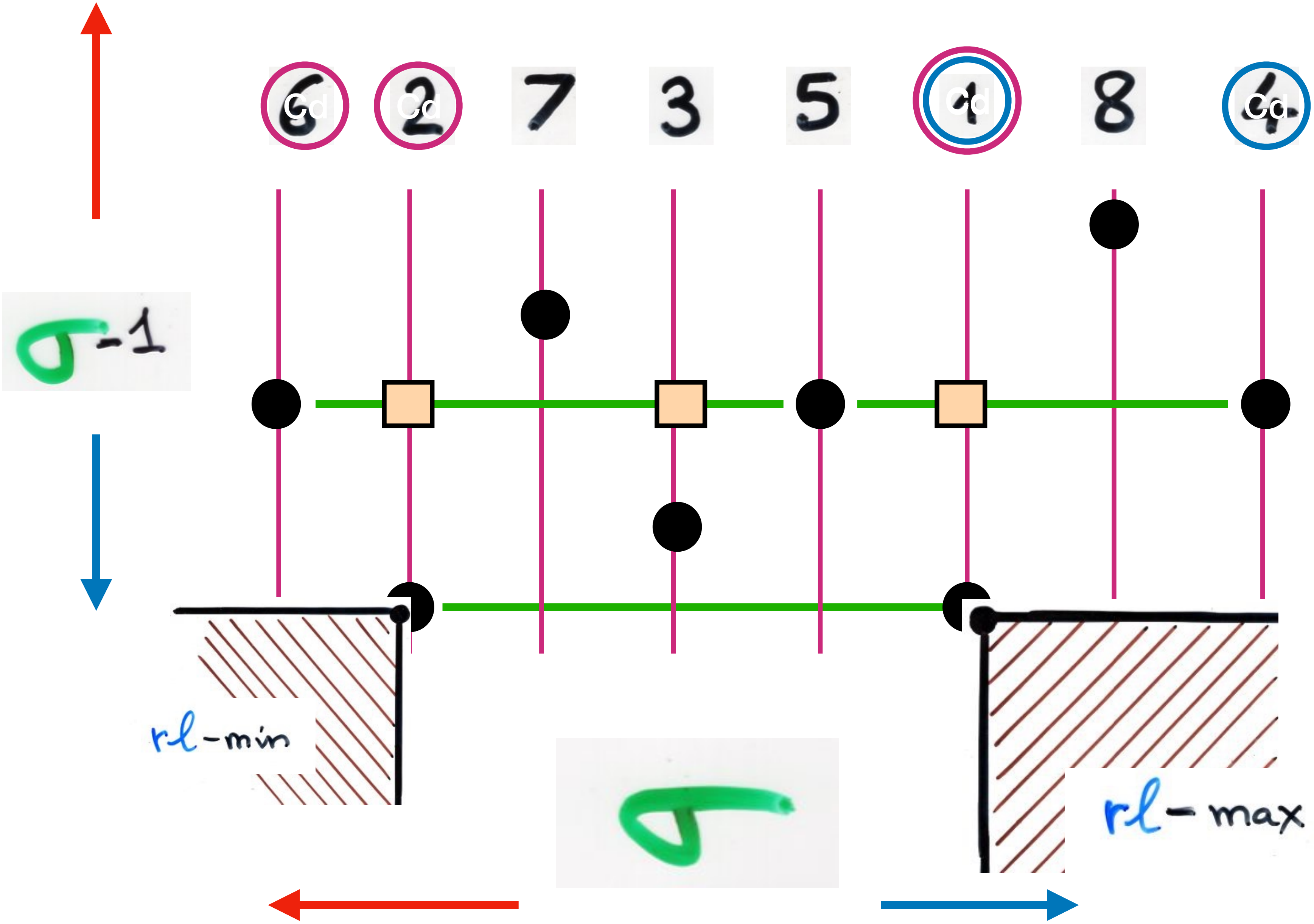


σ-1
8
7
6
5
4
2
3
4
5
6
7
8
rl-min
rl-max

Proposition

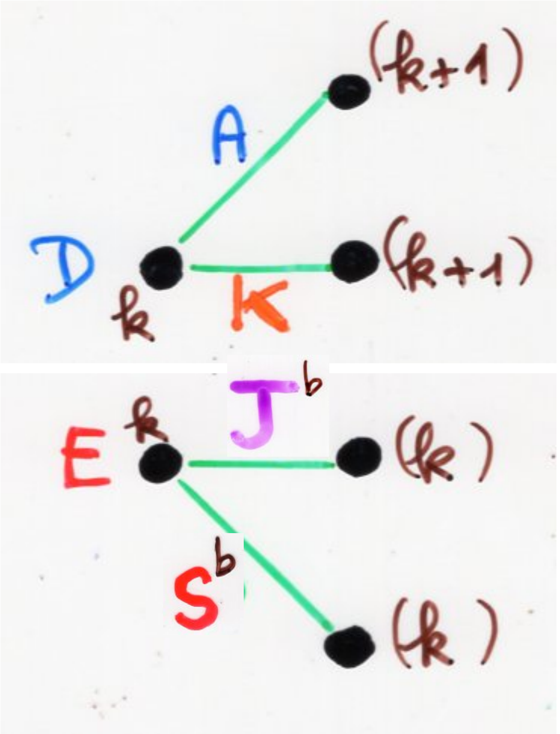
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

Josuat-Vergès (2011)



Bijection

Laguerre heaps — alternative tableaux



D, E "restricted"

$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

with the « cellular ansatz »:

bijection Laguerre heaps — alternative tableaux

« The cellular ansatz »

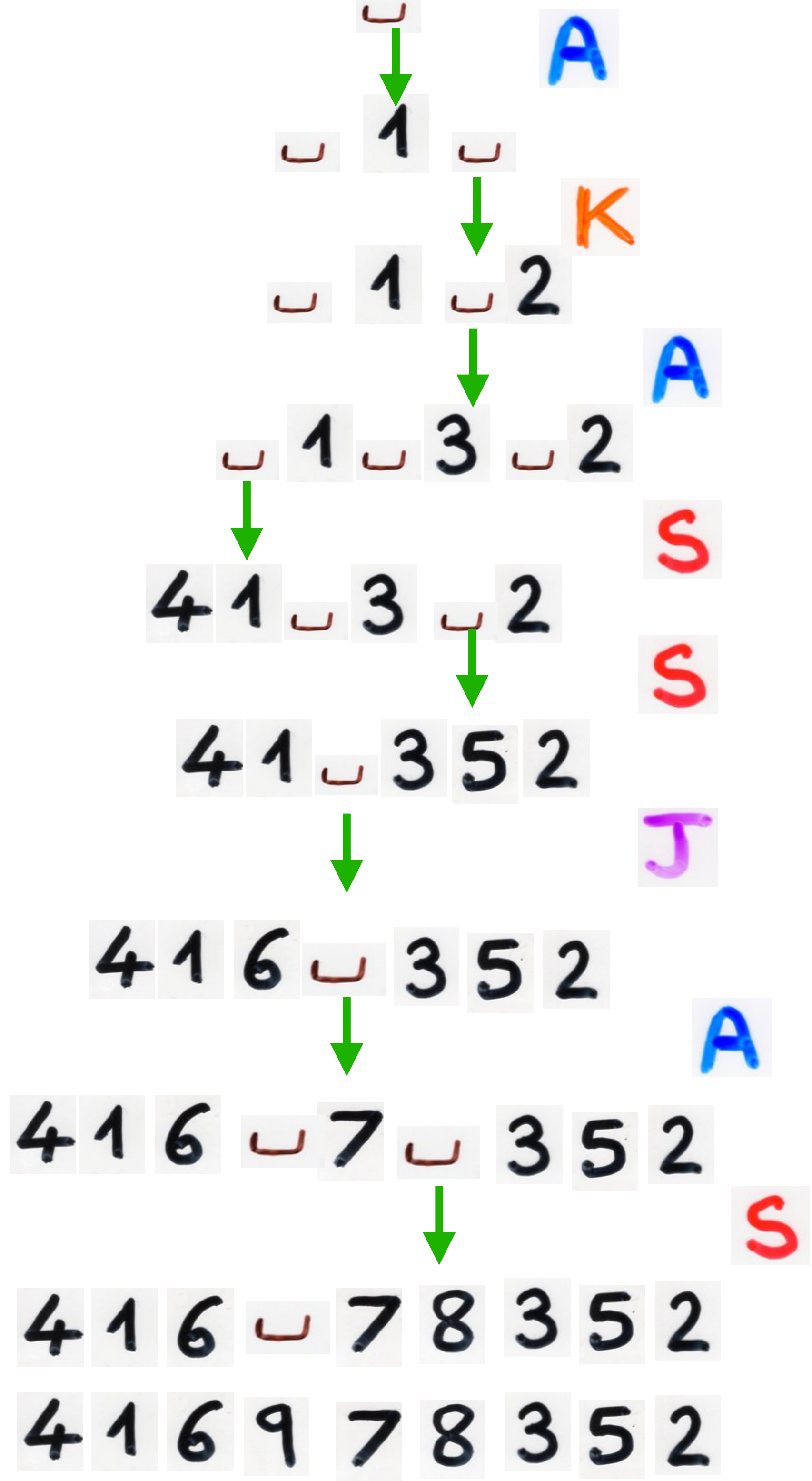
From a combinatorial representation
of the PASEP algebra

$$DE = ED + E + D$$

Bijection

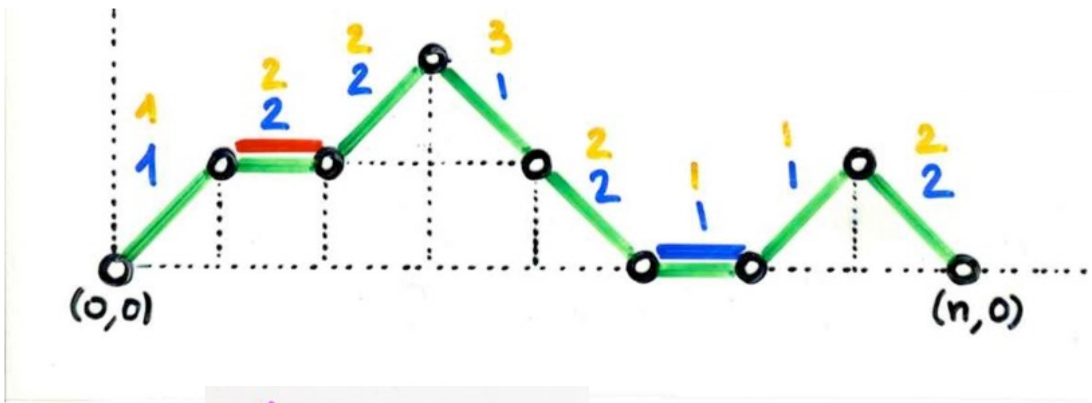
alternative tableaux — permutations

with « local rules »



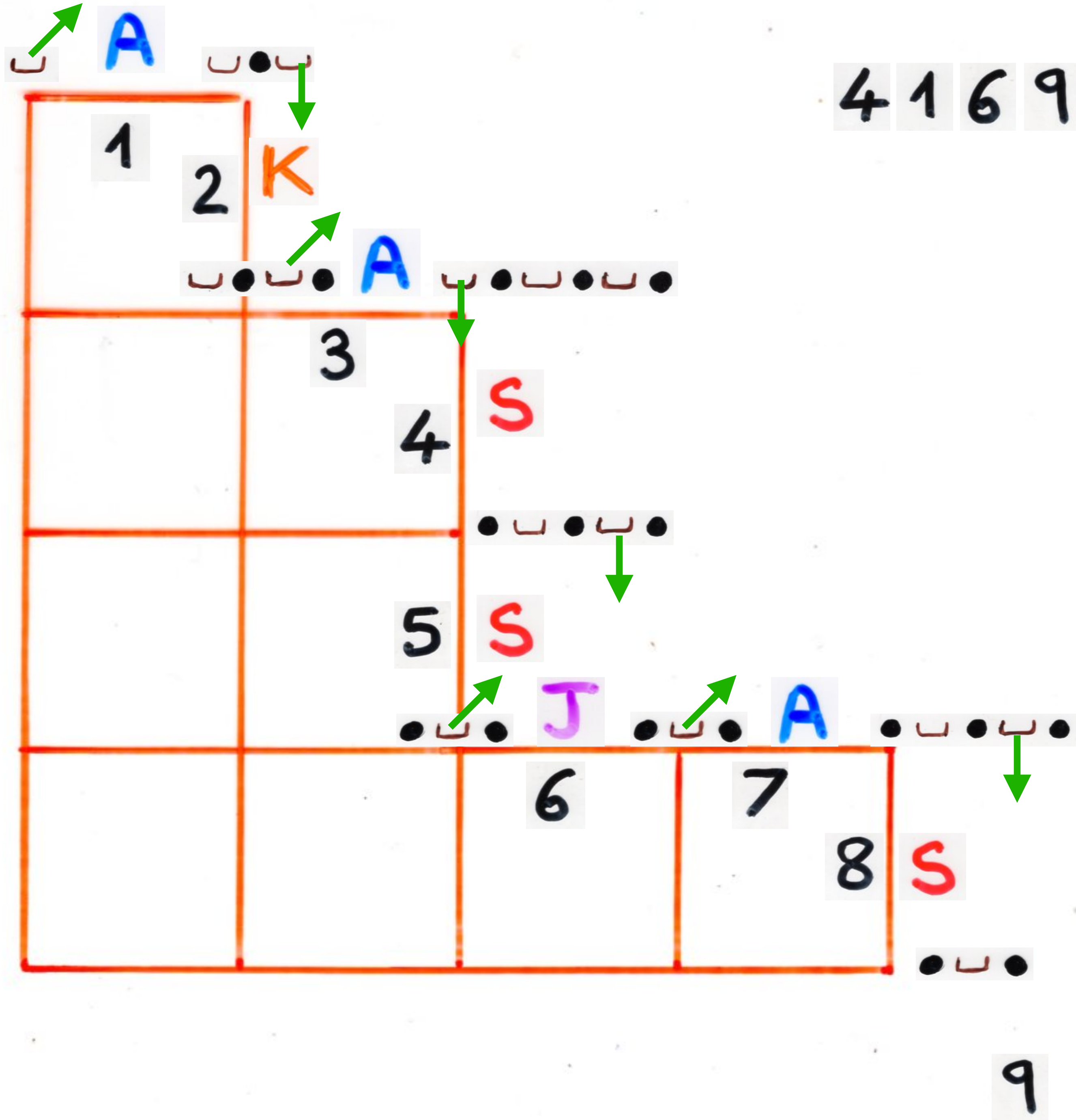
$$D = A + K$$
$$E = S + J$$

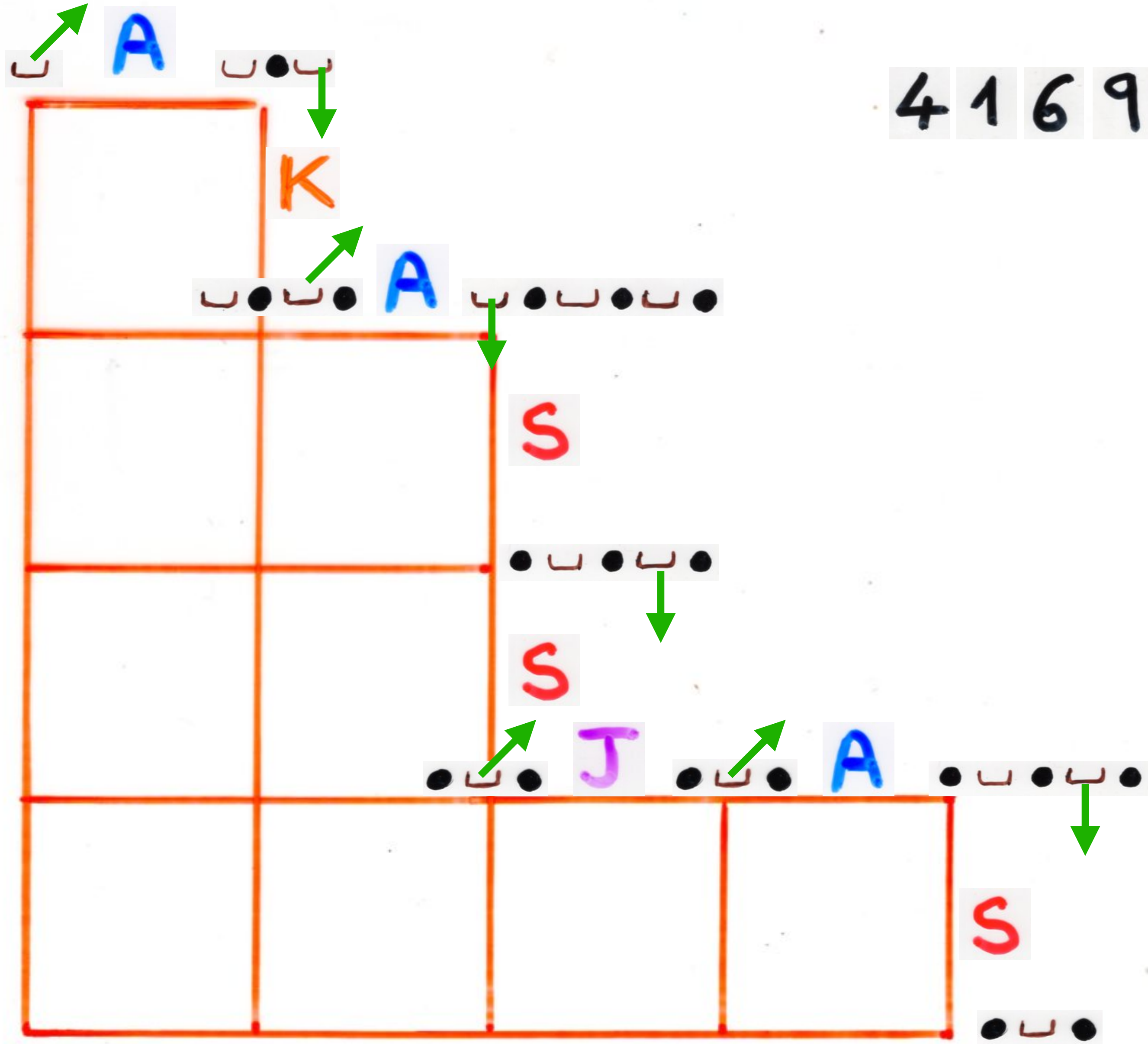
$$DE = ED + E + D$$



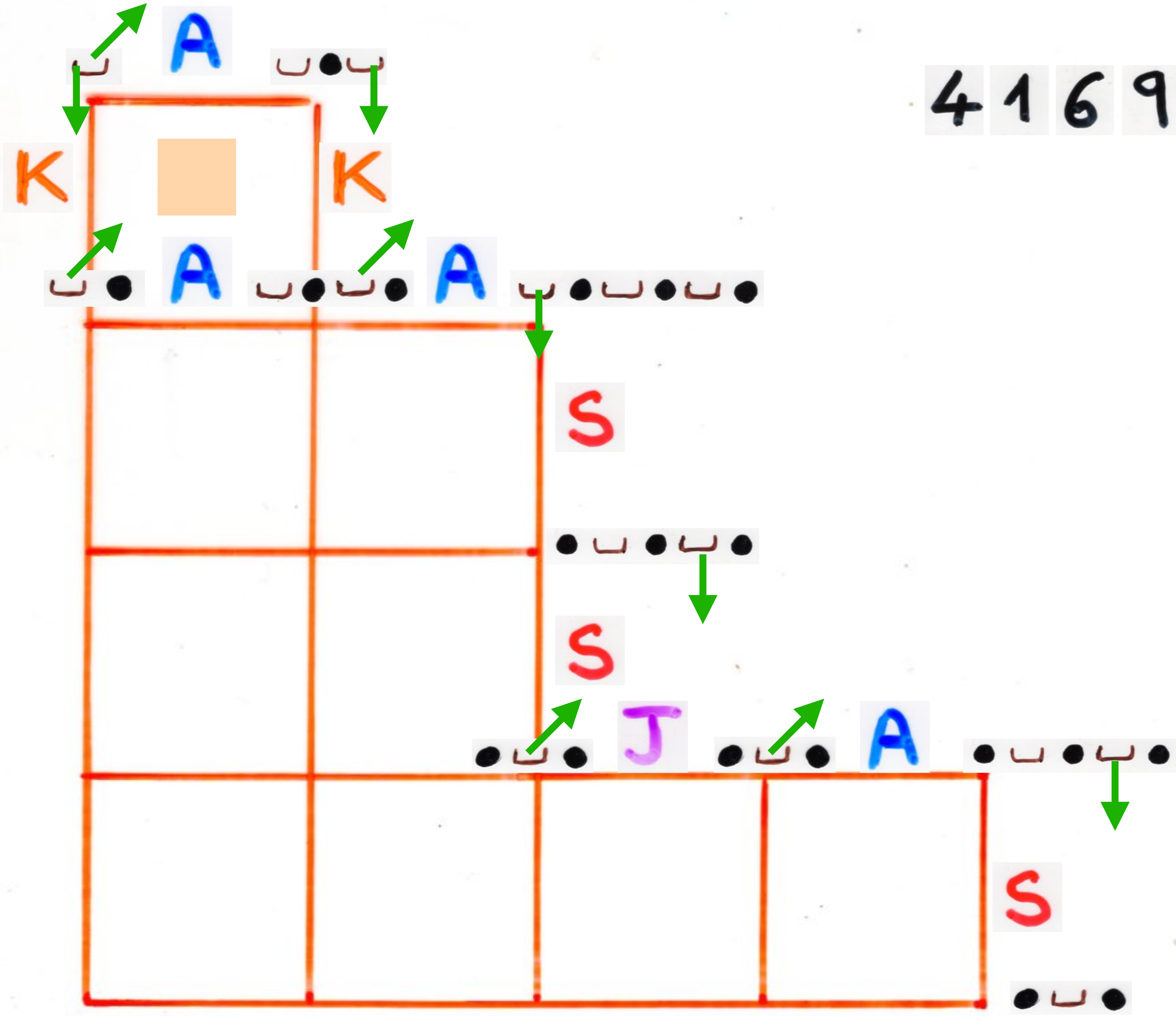
Laguerre histories

4 1 6 9 7 8 3 5 2



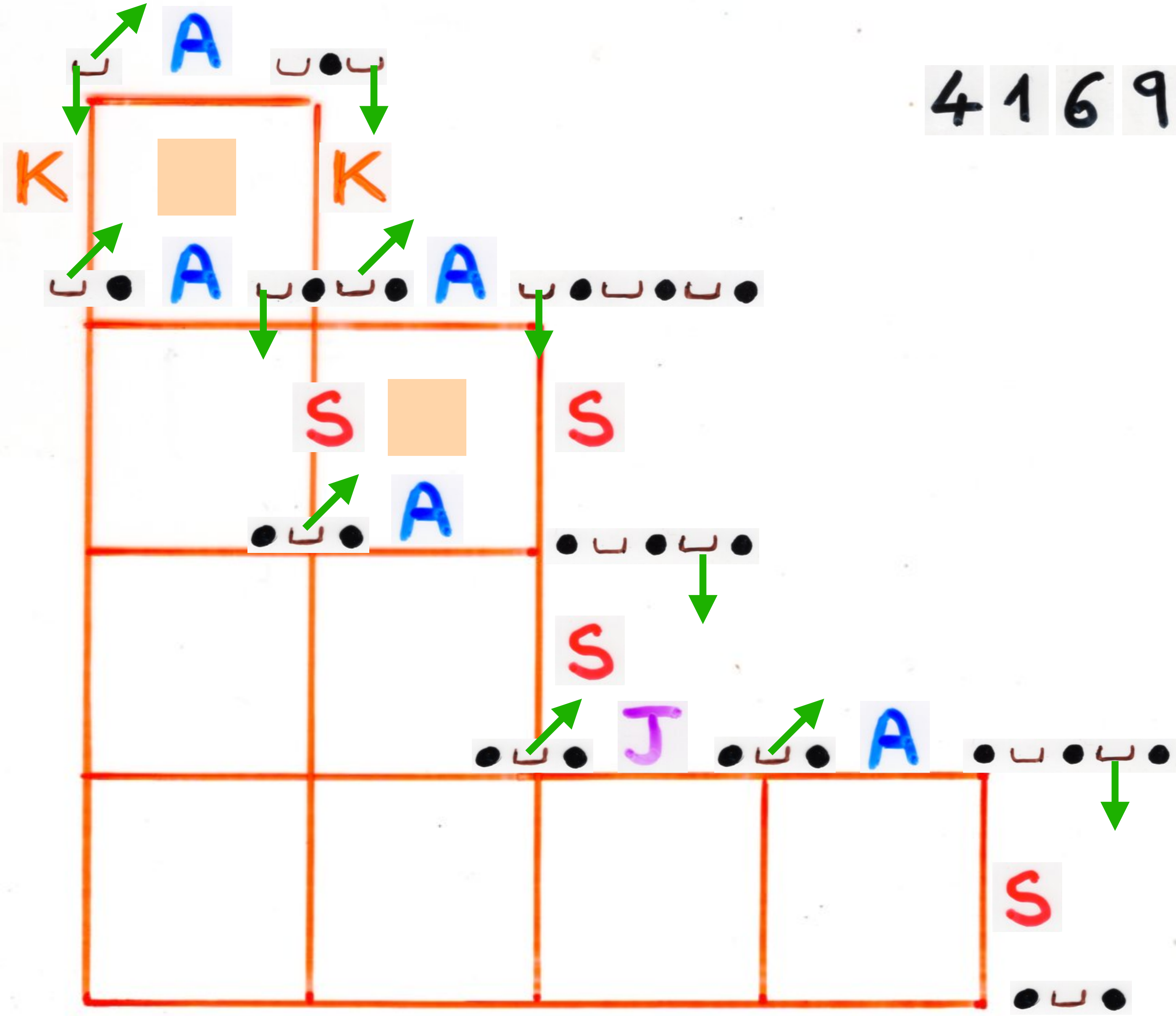


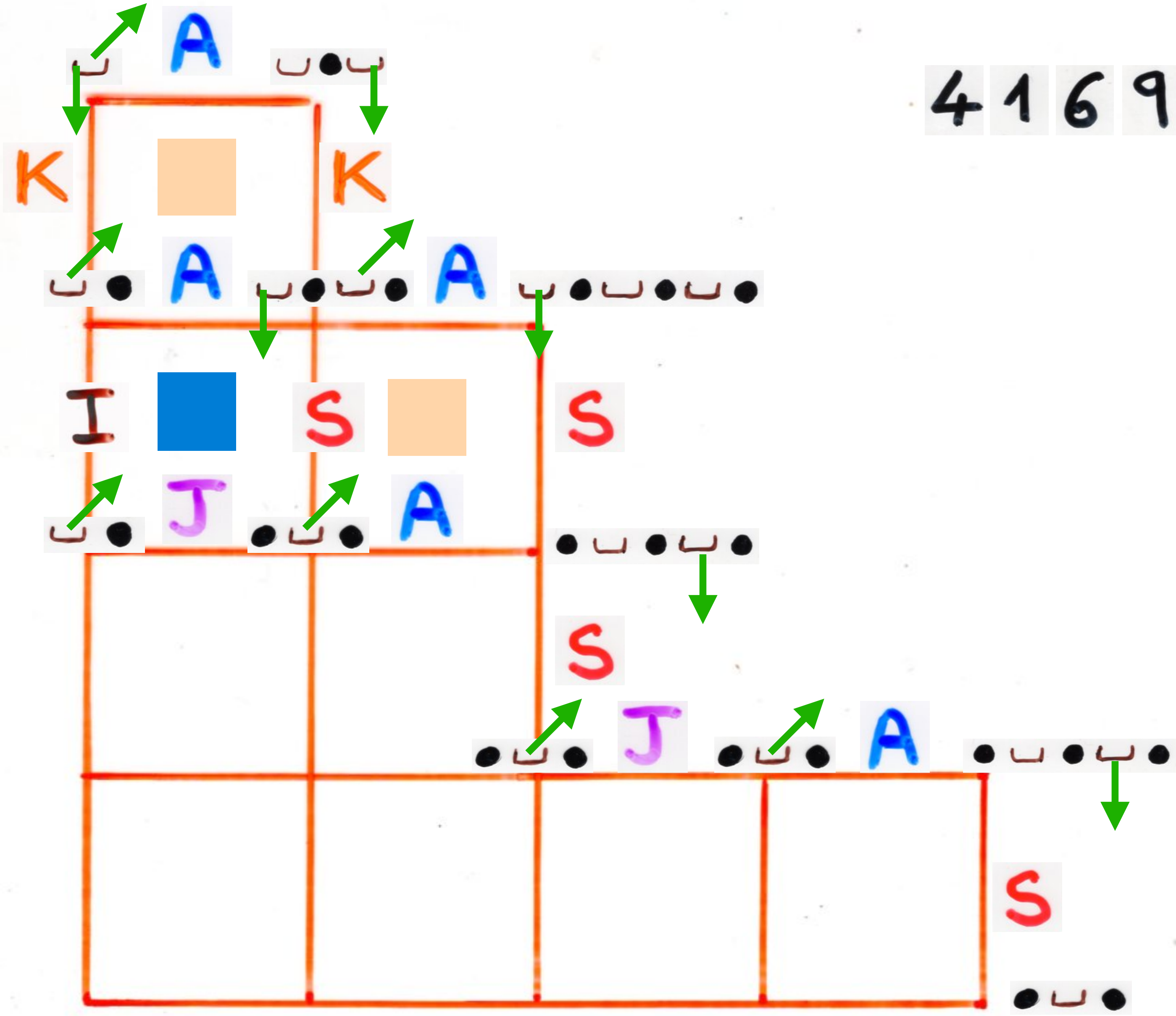
4 1 6 9 7 8 3 5 2



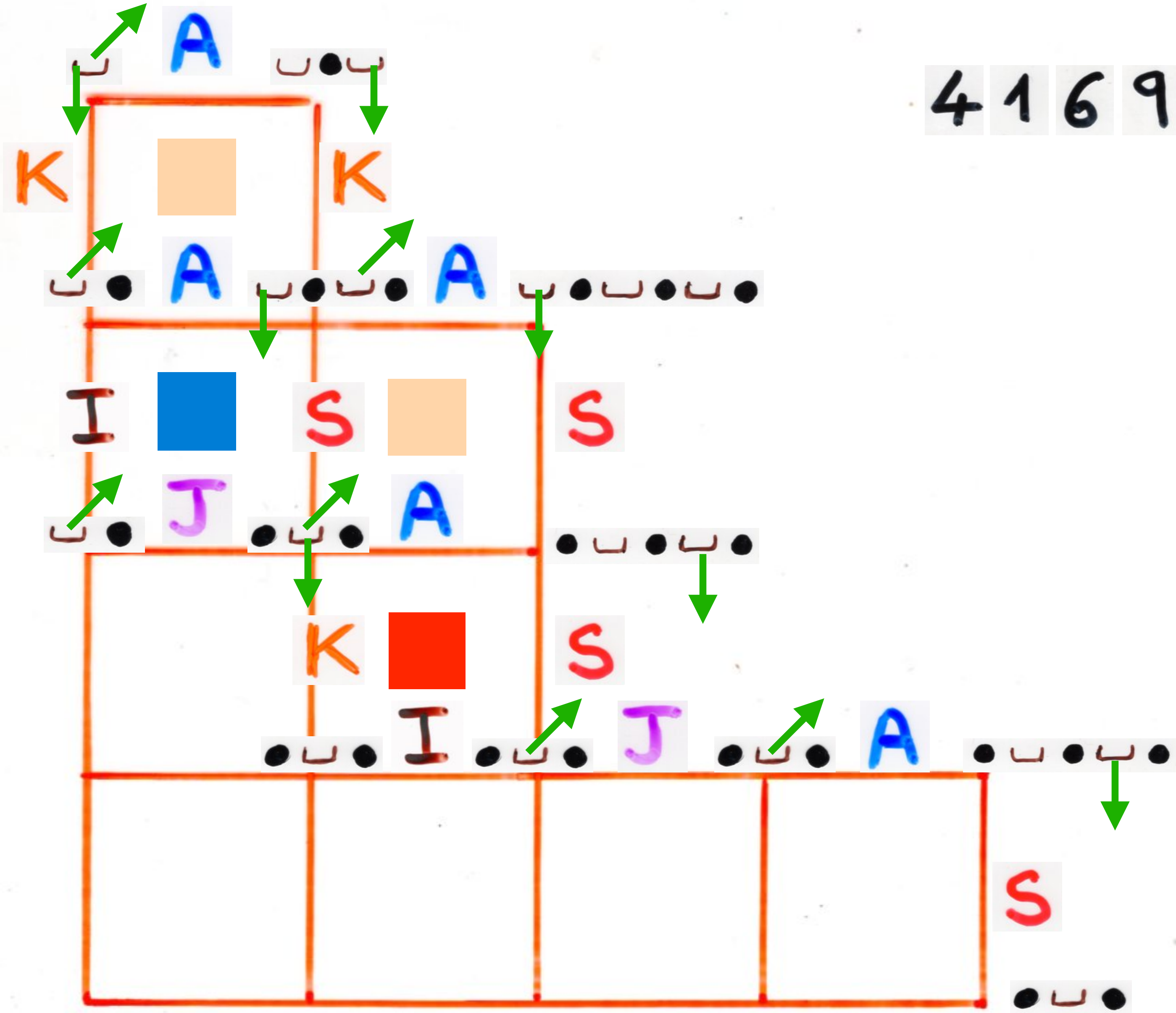
4 1 6 9 7 8 3 5 2

4 1 6 9 7 8 3 5 2

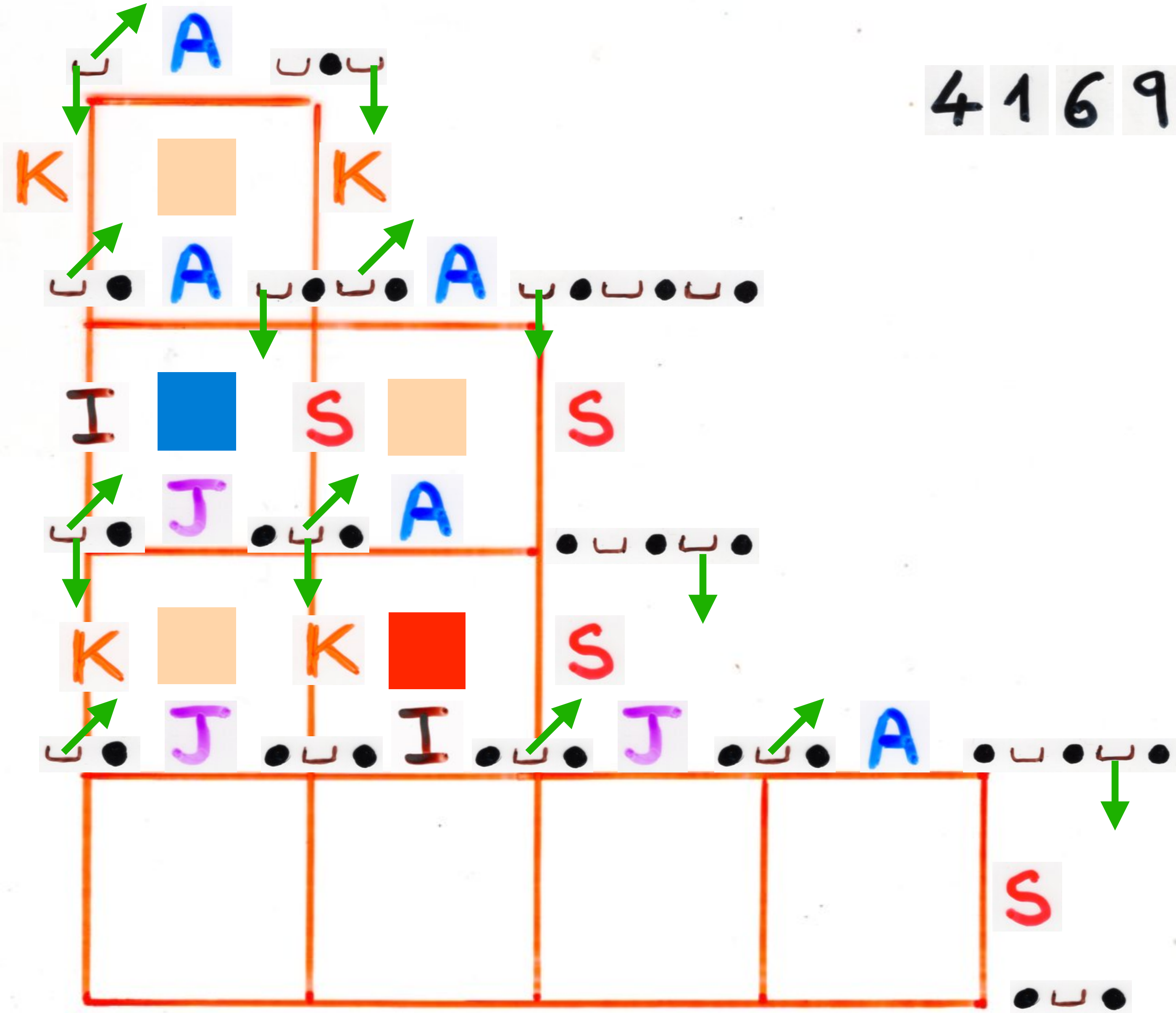




4 1 6 9 7 8 3 5 2

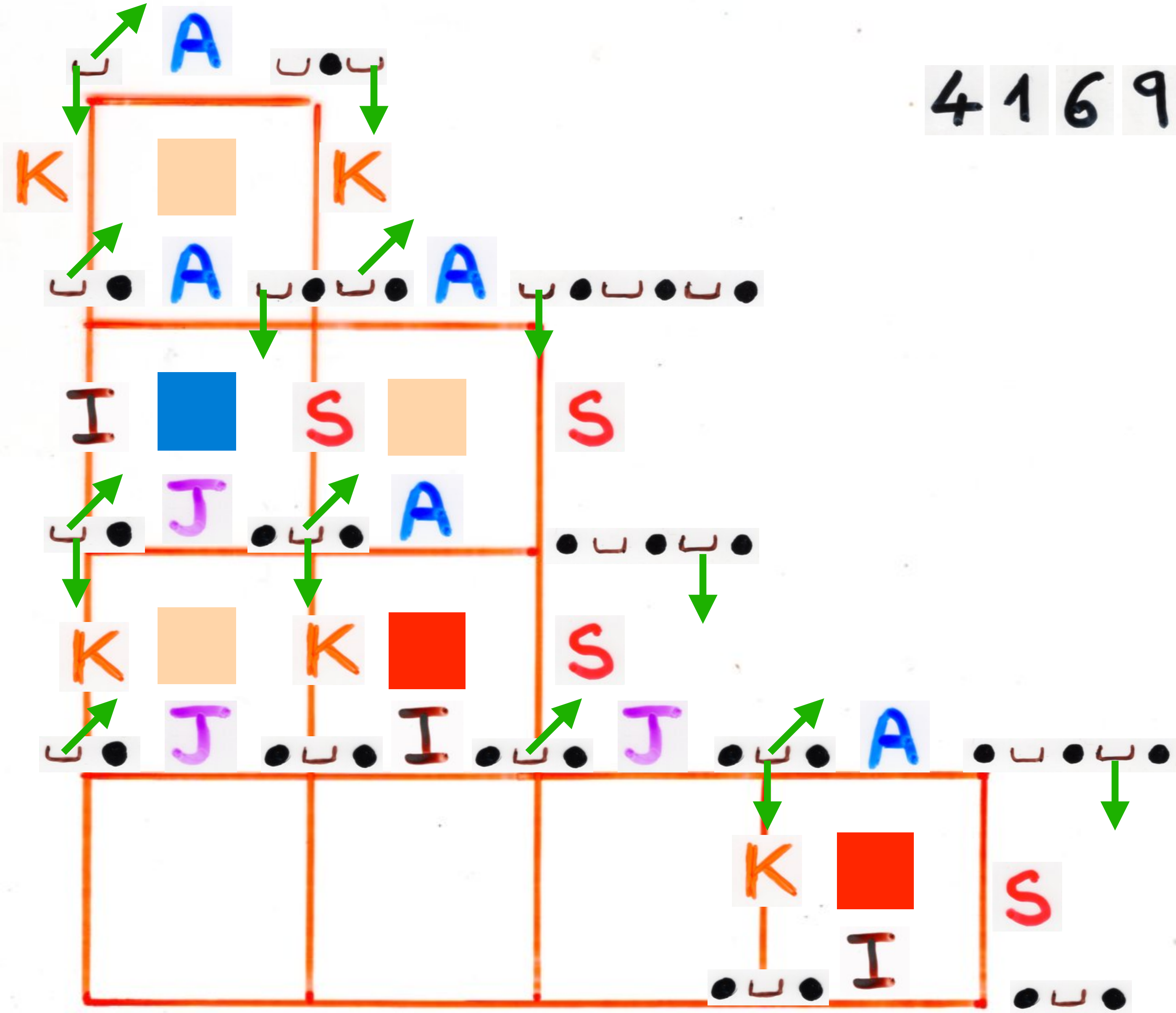


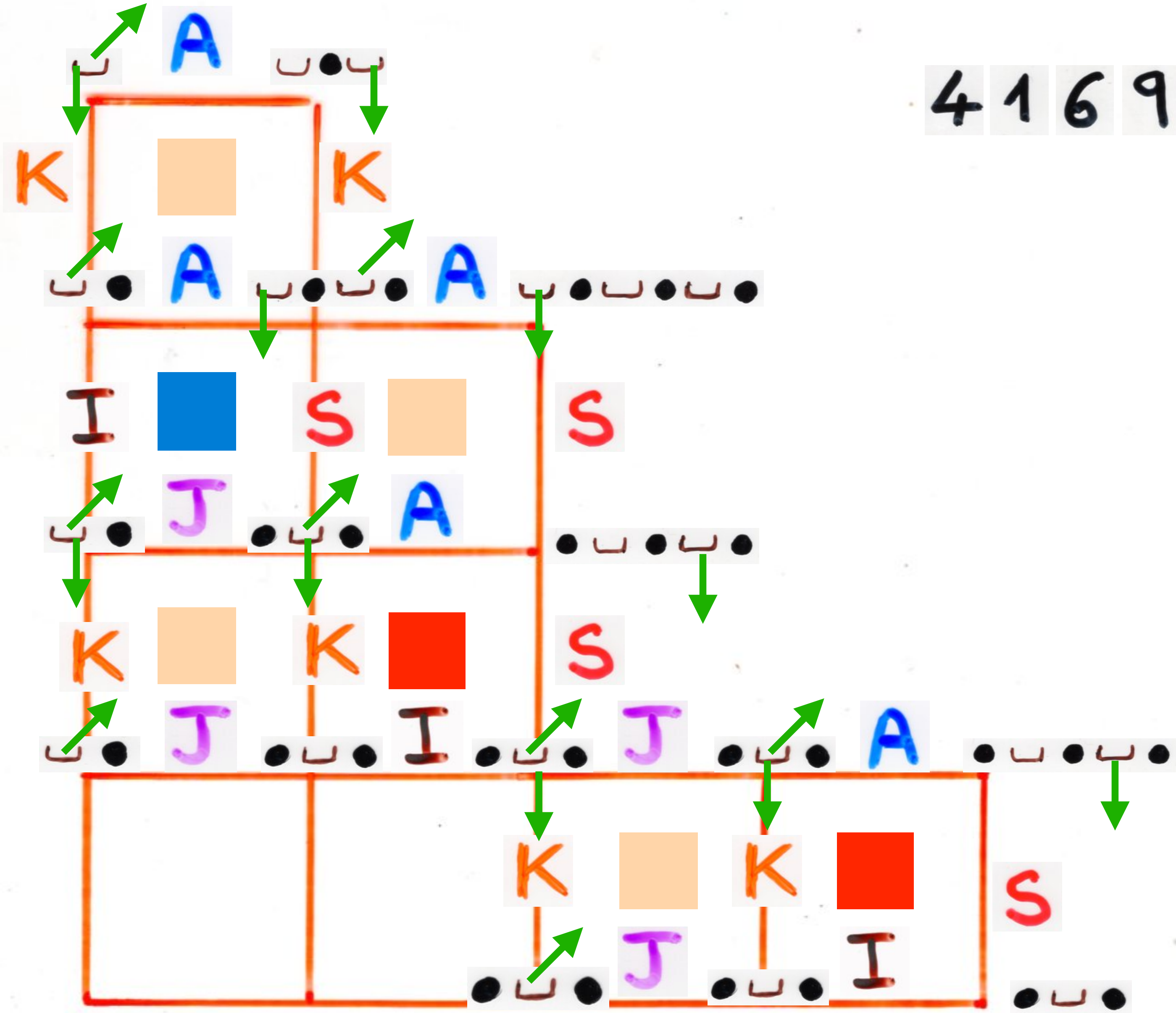
4 1 6 9 7 8 3 5 2



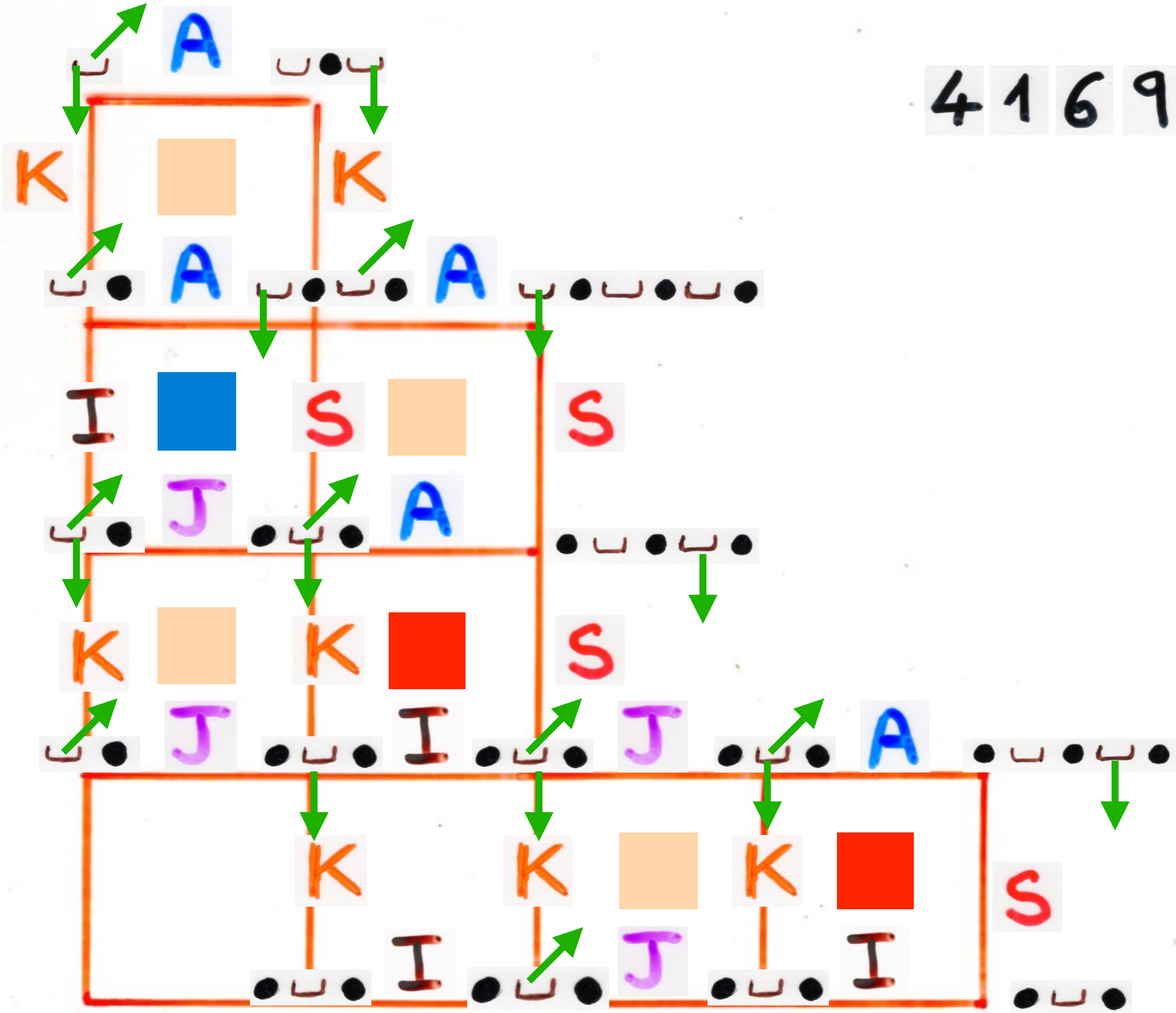
4 1 6 9 7 8 3 5 2

4 1 6 9 7 8 3 5 2

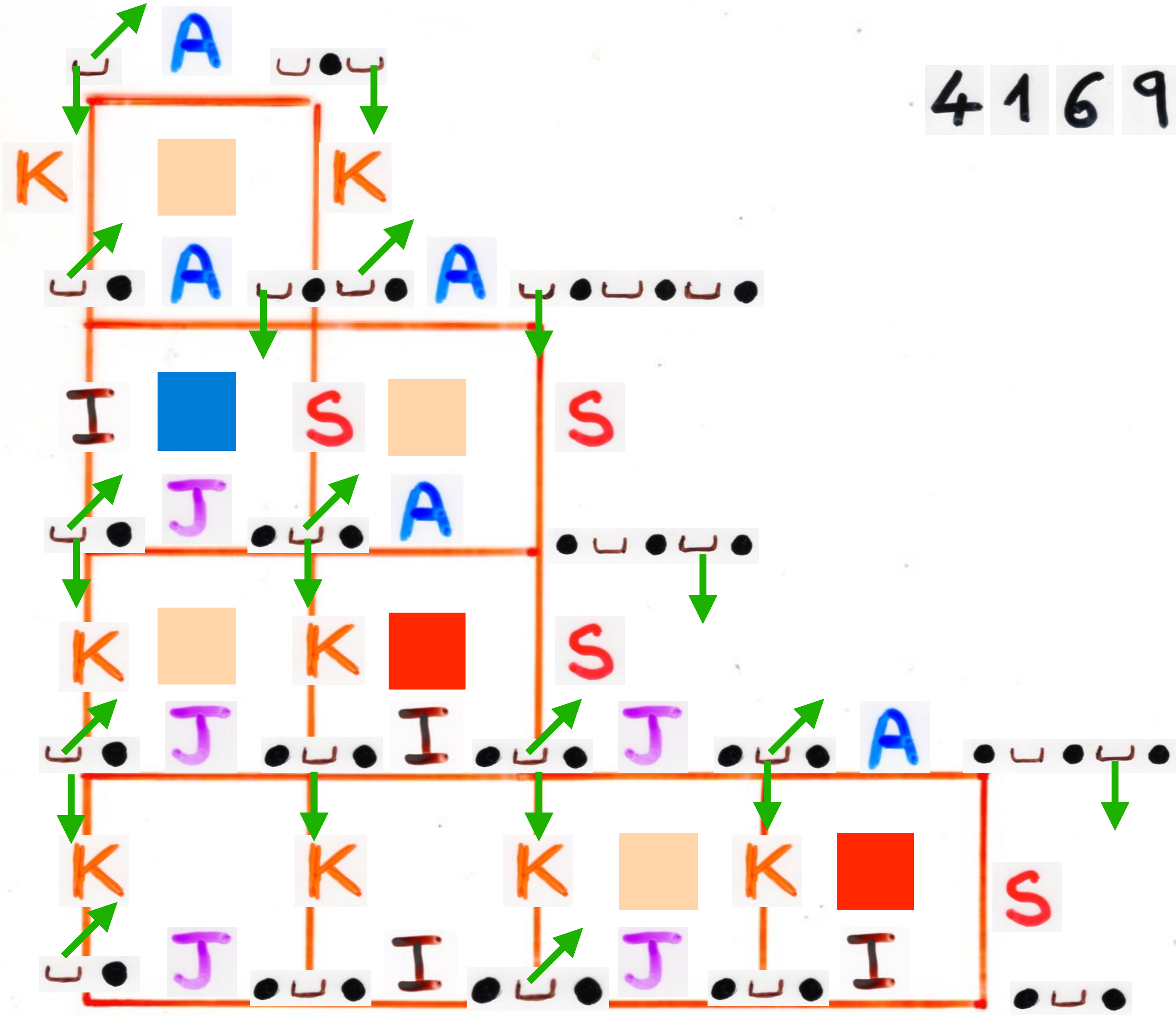




416978352



4 1 6 9 7 8 3 5 2



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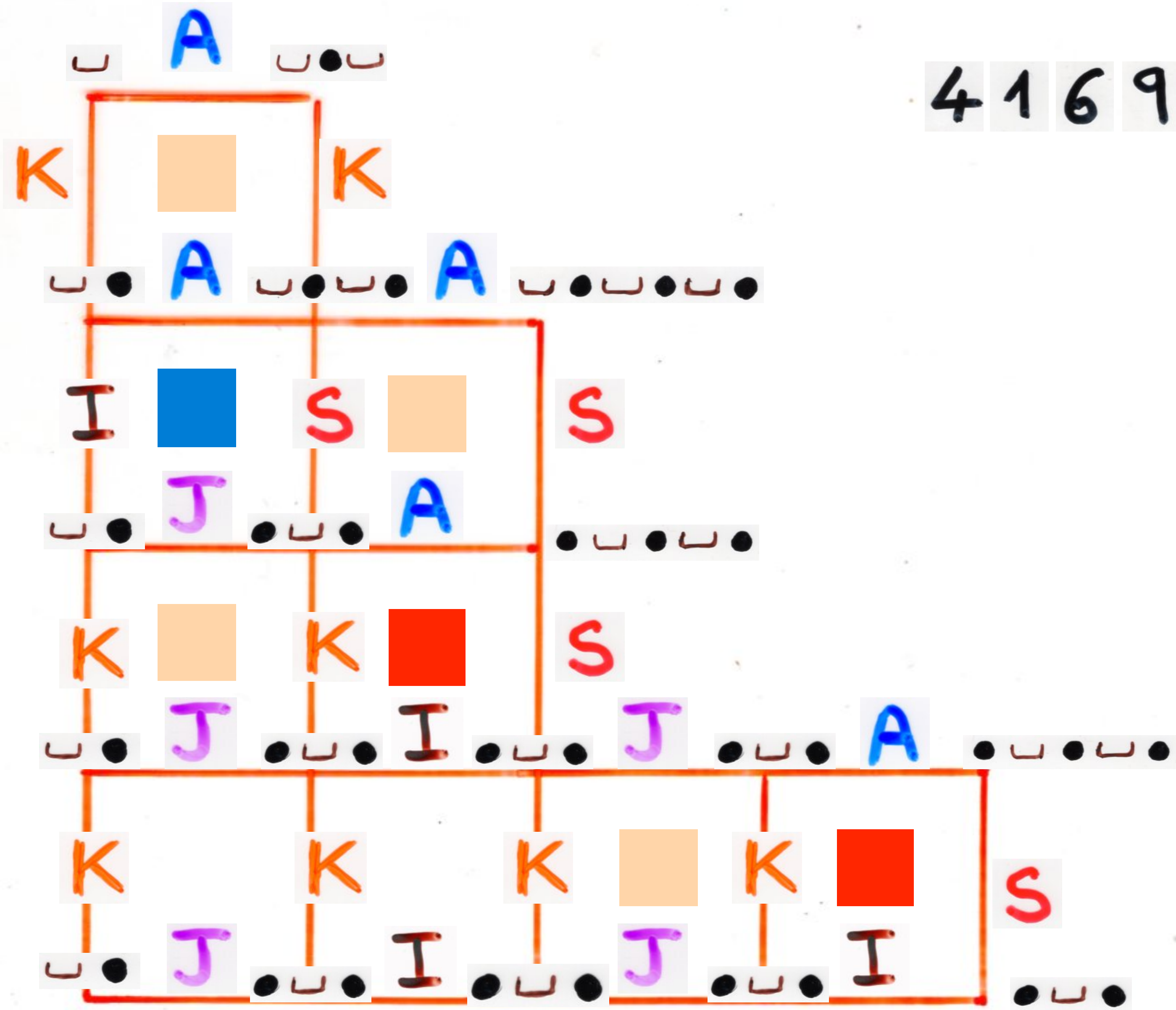
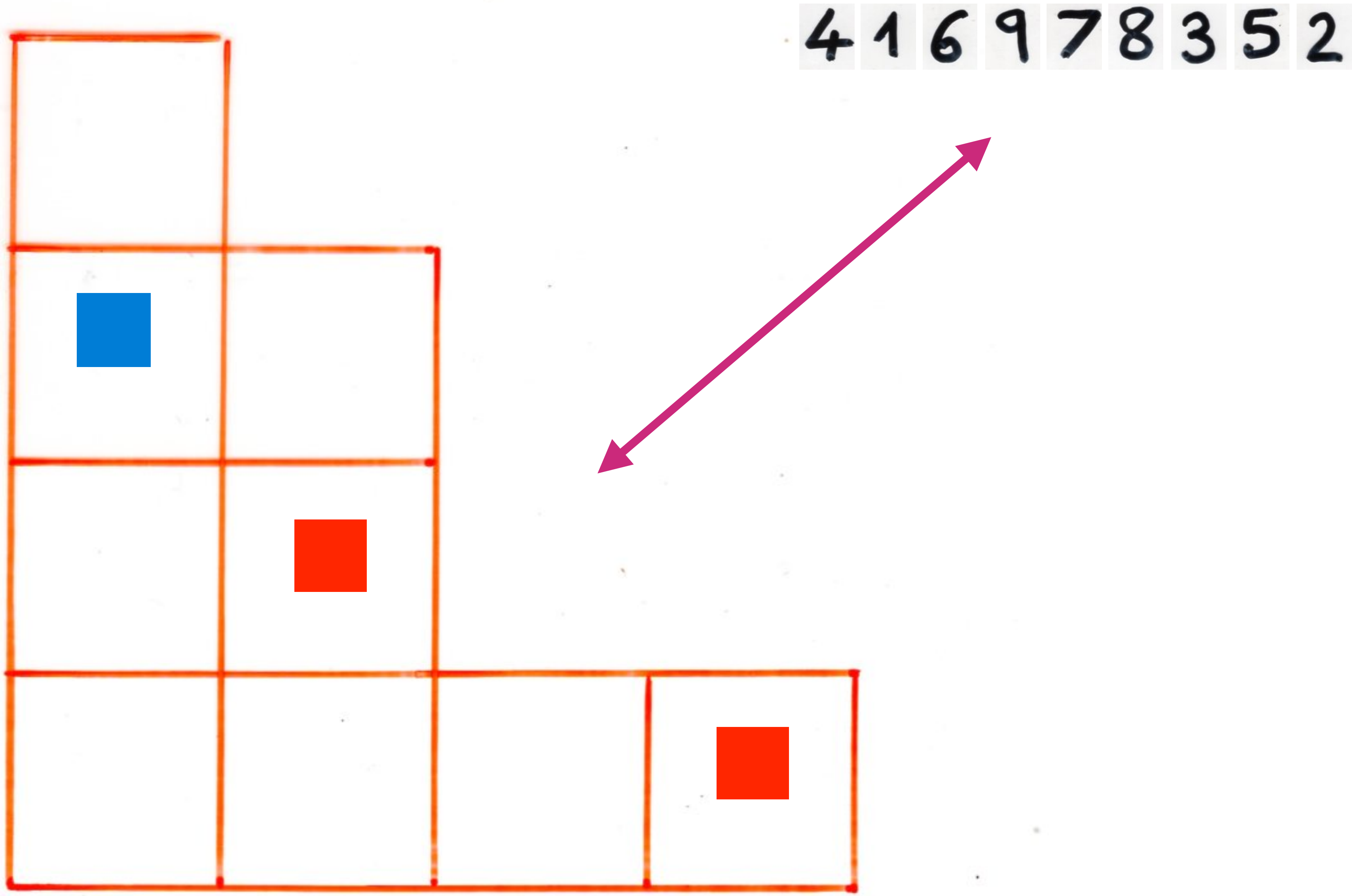
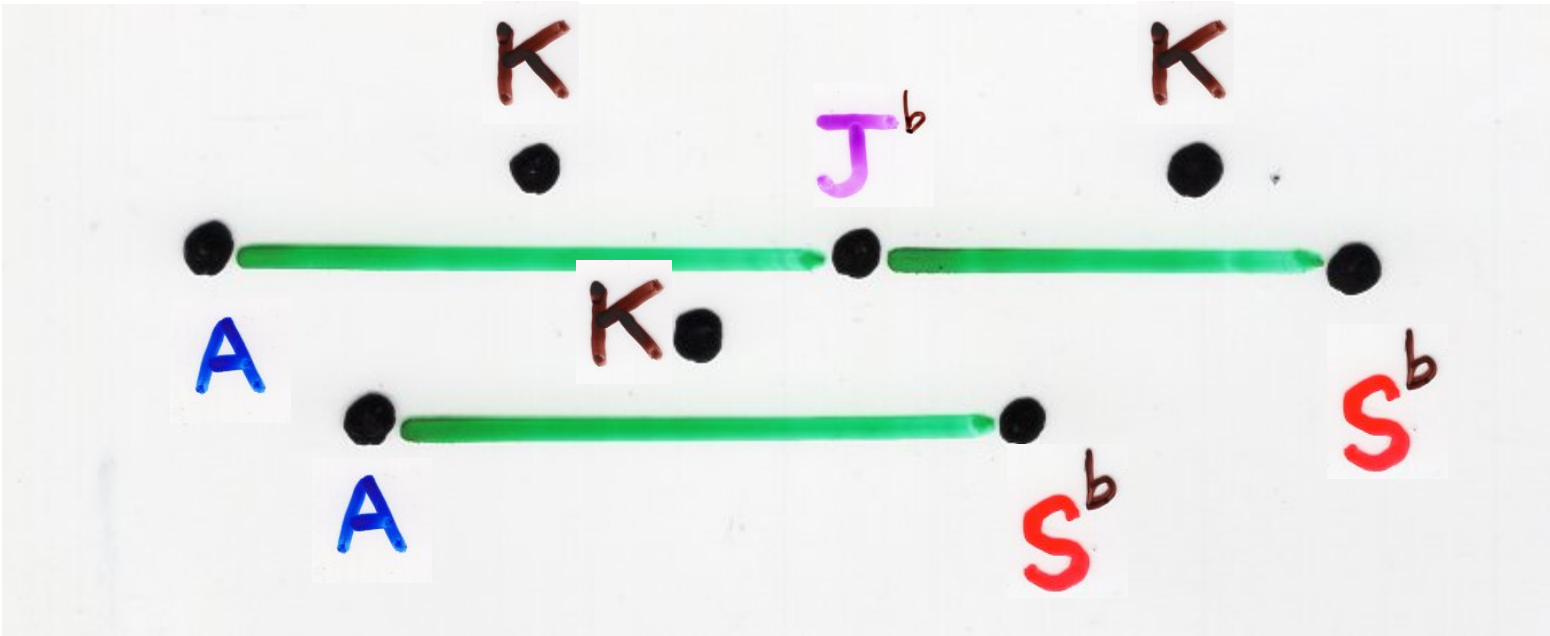



4 1 6 9 7 8 3 5 2

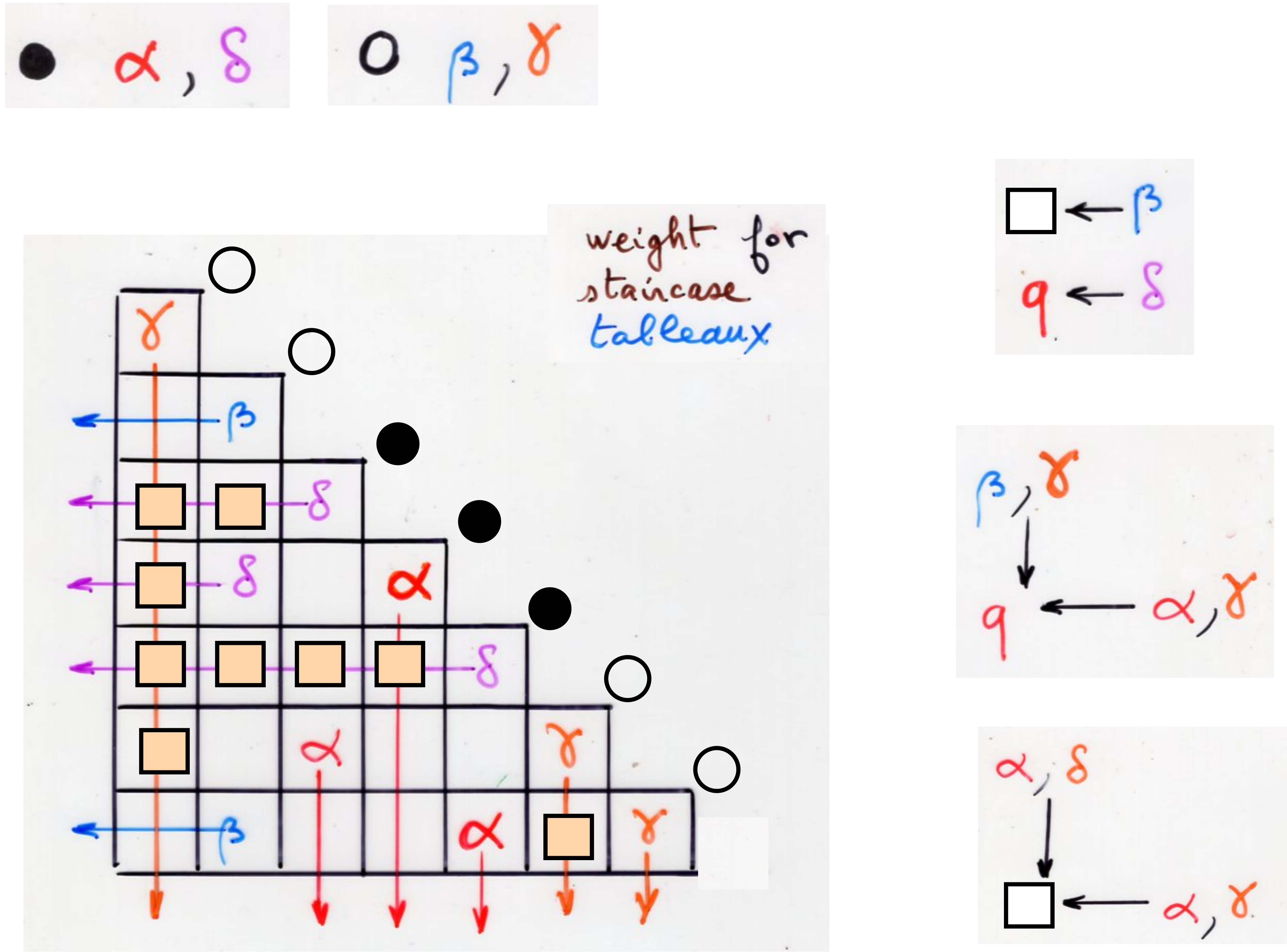
with the « cellular ansatz »:
bijection Laguerre heaps — alternative tableaux



$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

Why to insist on the 3 parameters model ?



α, δ
β, γ
weight for staircase tableaux
β
q
δ
β, γ
α, γ
α, δ

$\tau = (\tau_1, \dots, \tau_n)$

$$Z_\tau = \sum_{T} v(T)$$

T staircase tableaux size n

profile of T

S. Corteel, L. Williams (2009)

$$Z_n(\alpha, \beta, \gamma, \delta ; q) = \sum_{T} v(T)$$

partition function

T staircase tableaux size n

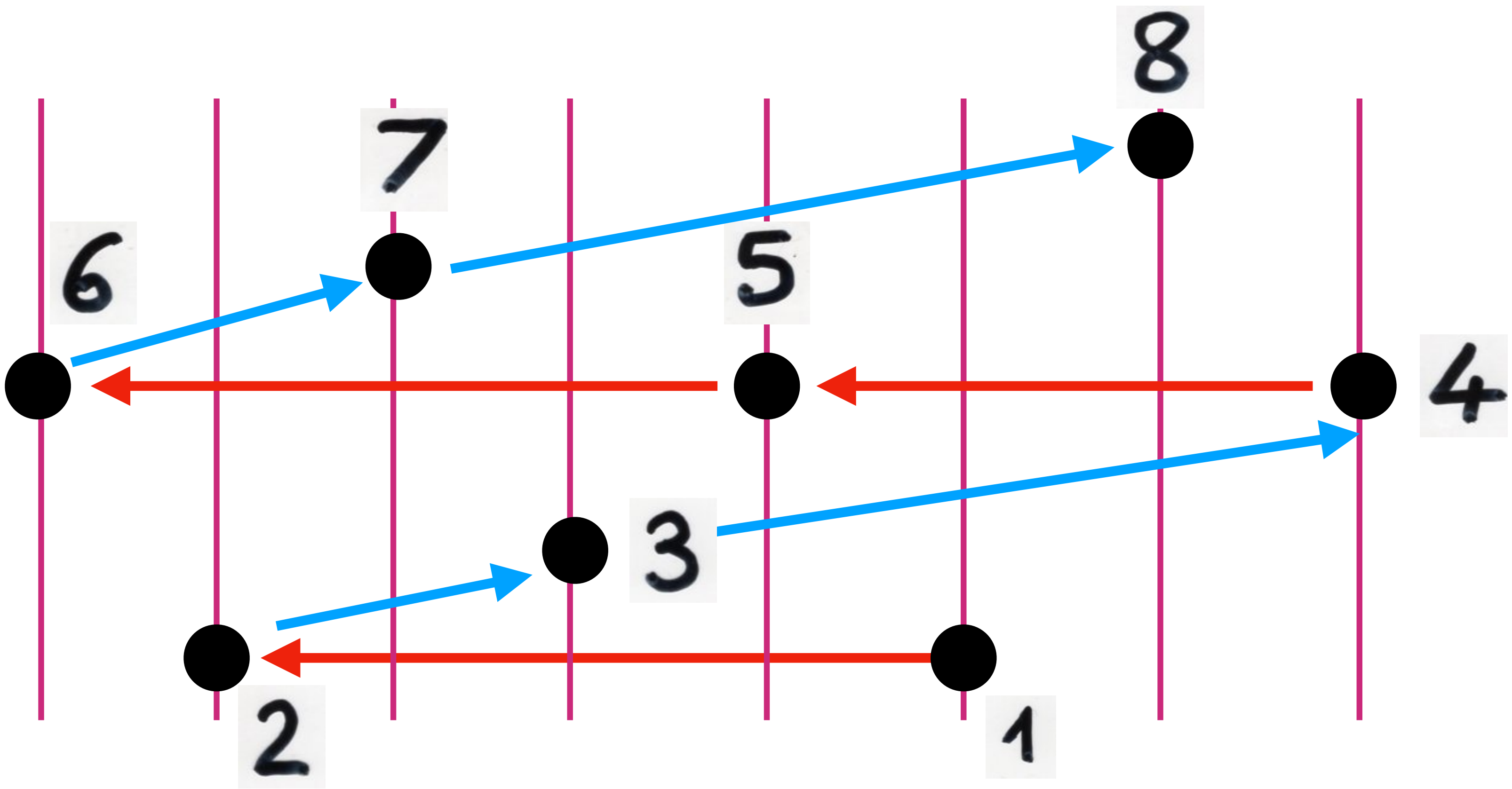
→ expression for the moments of the Askey-Wilson polynomials

S. Corteel, L. Williams, R. Stanley, D. Stanton (2010)

Why to insist
on the 3 parameters model ?

symmetries !

my dream ...



8
7
6
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4
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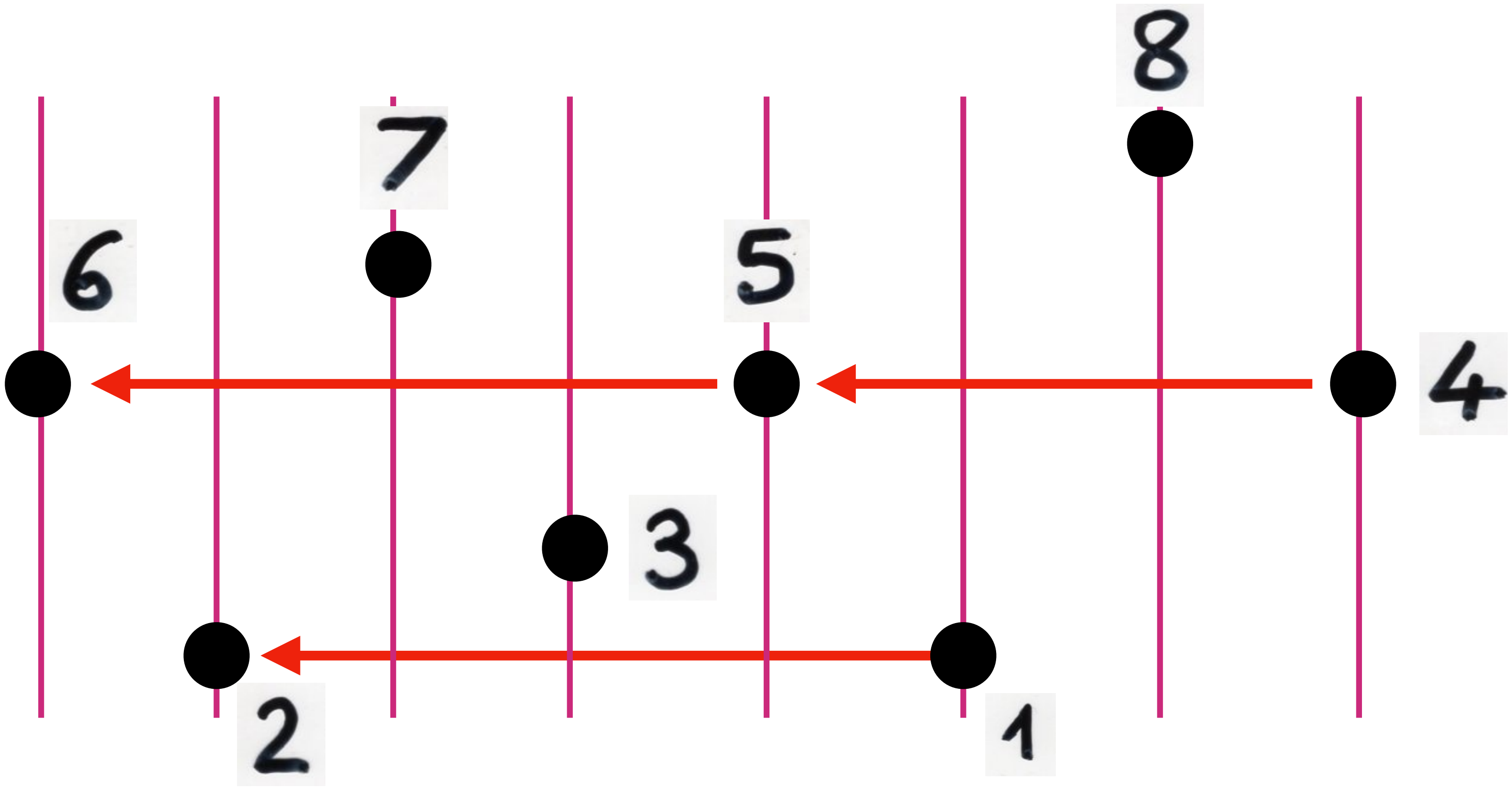
2
1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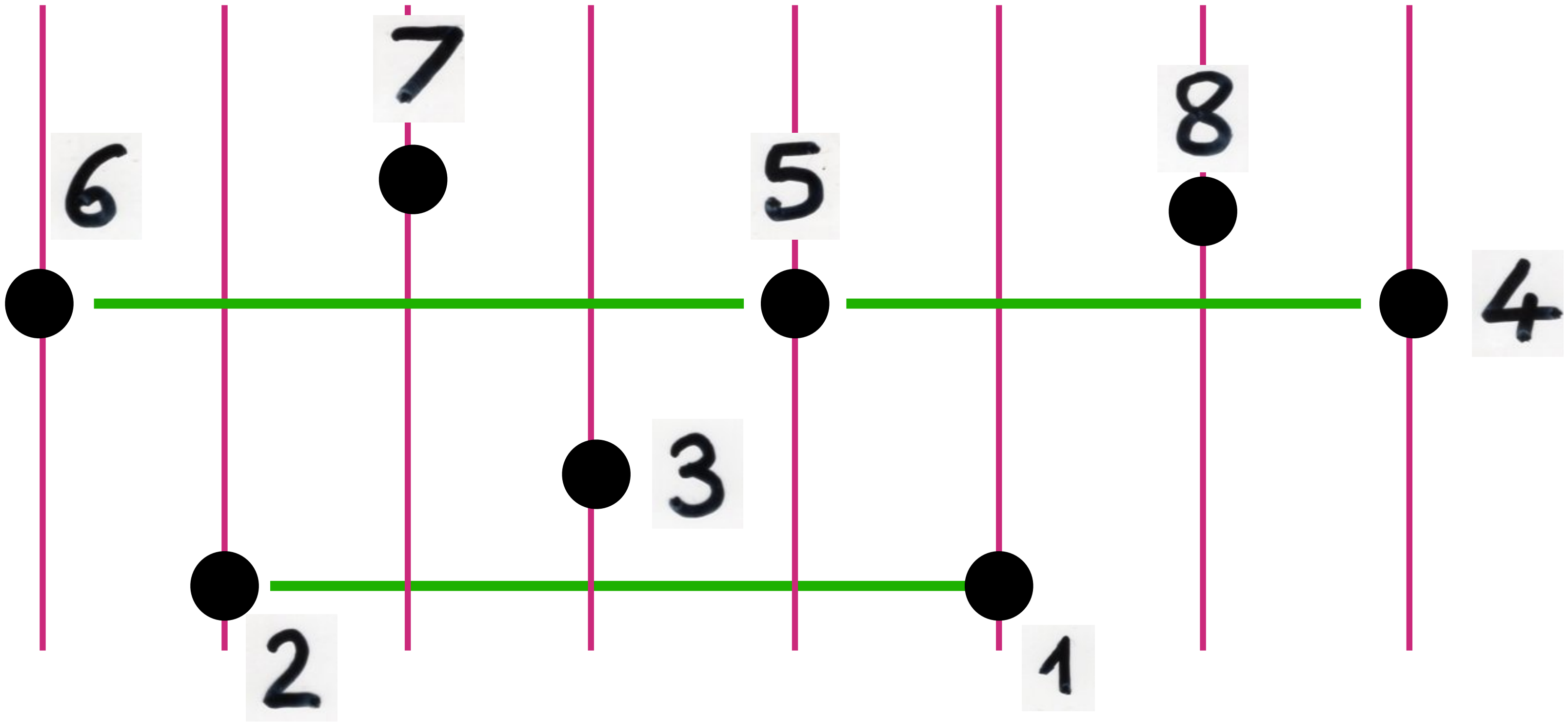

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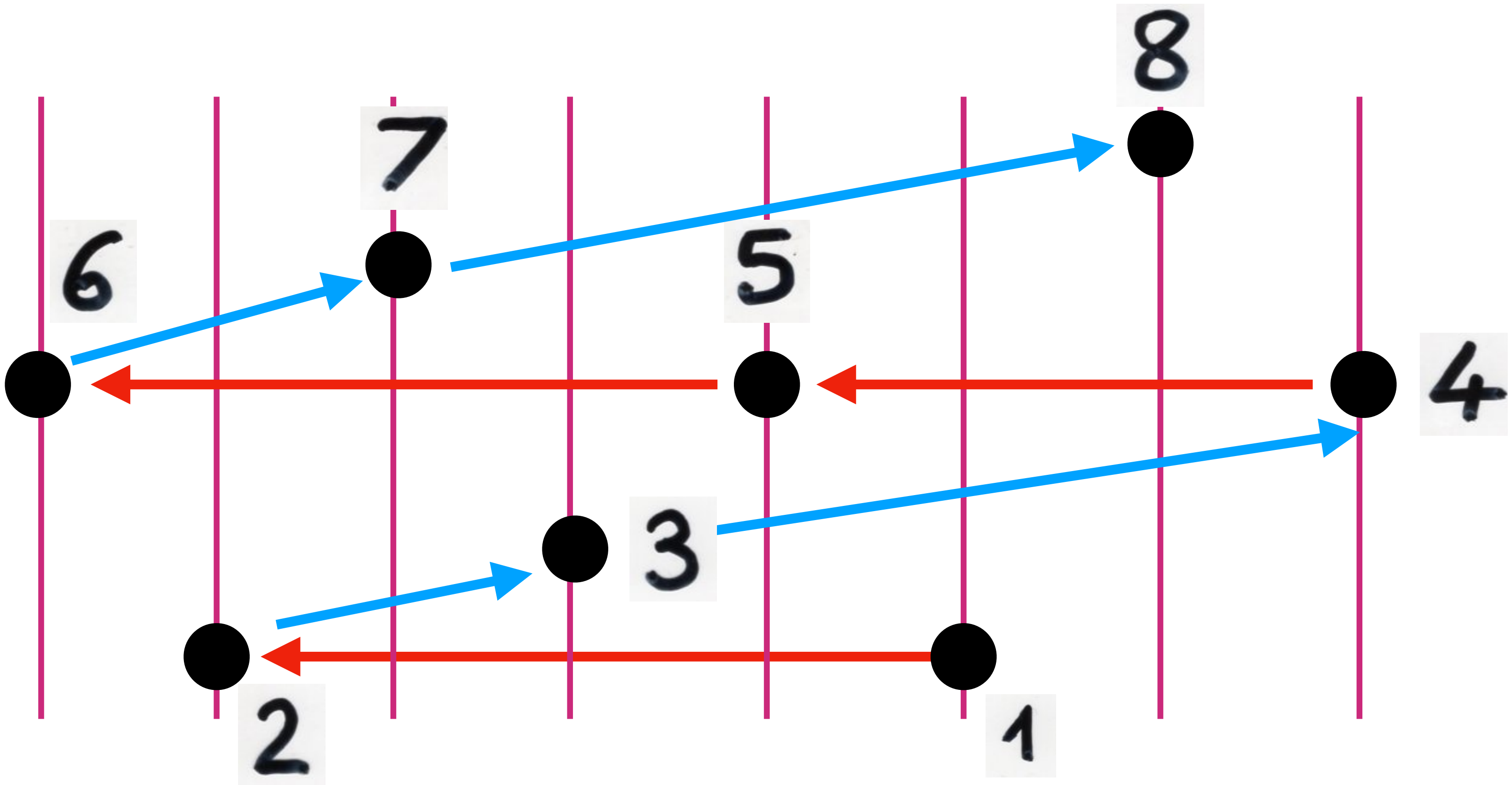
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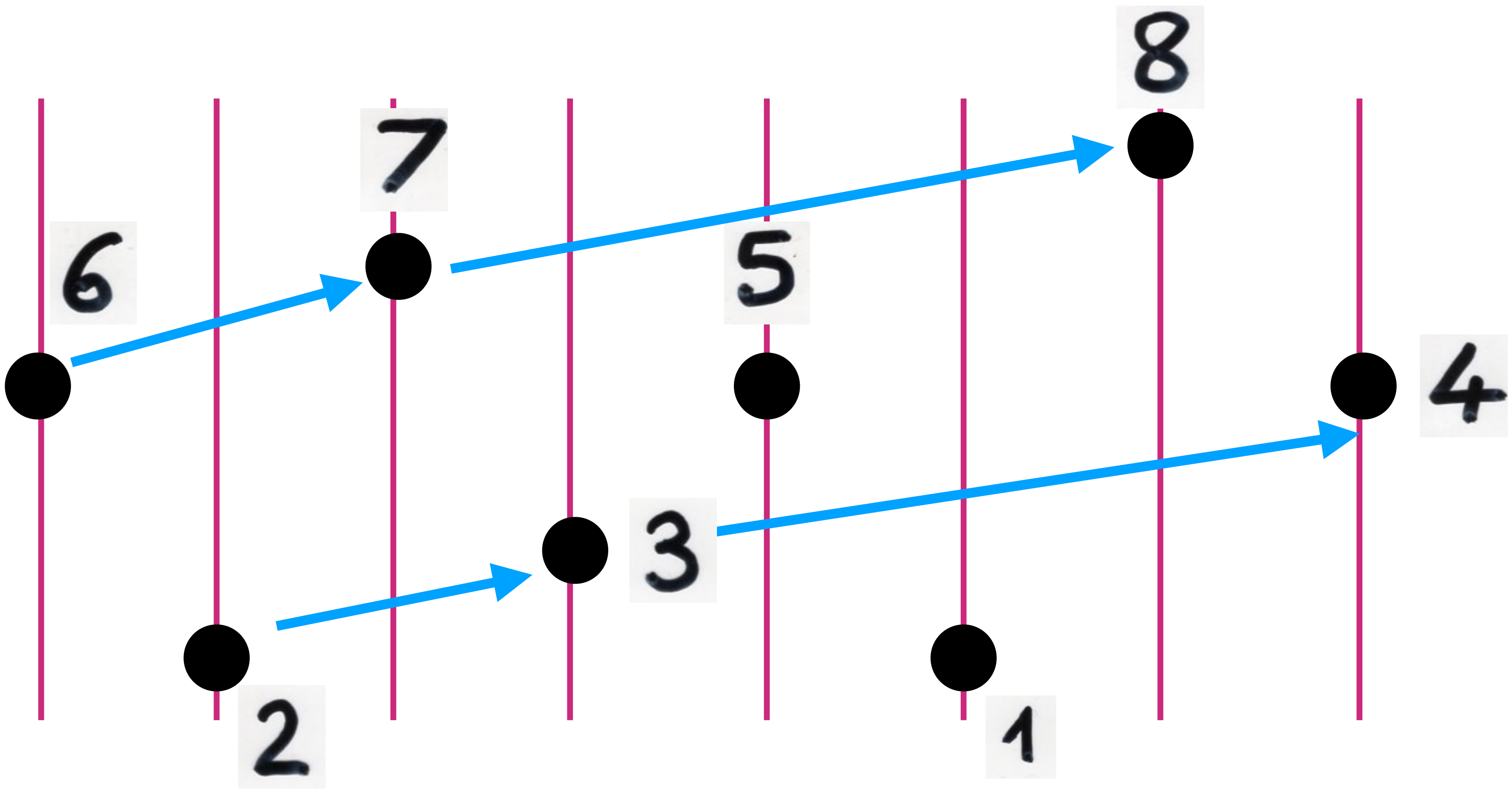
8
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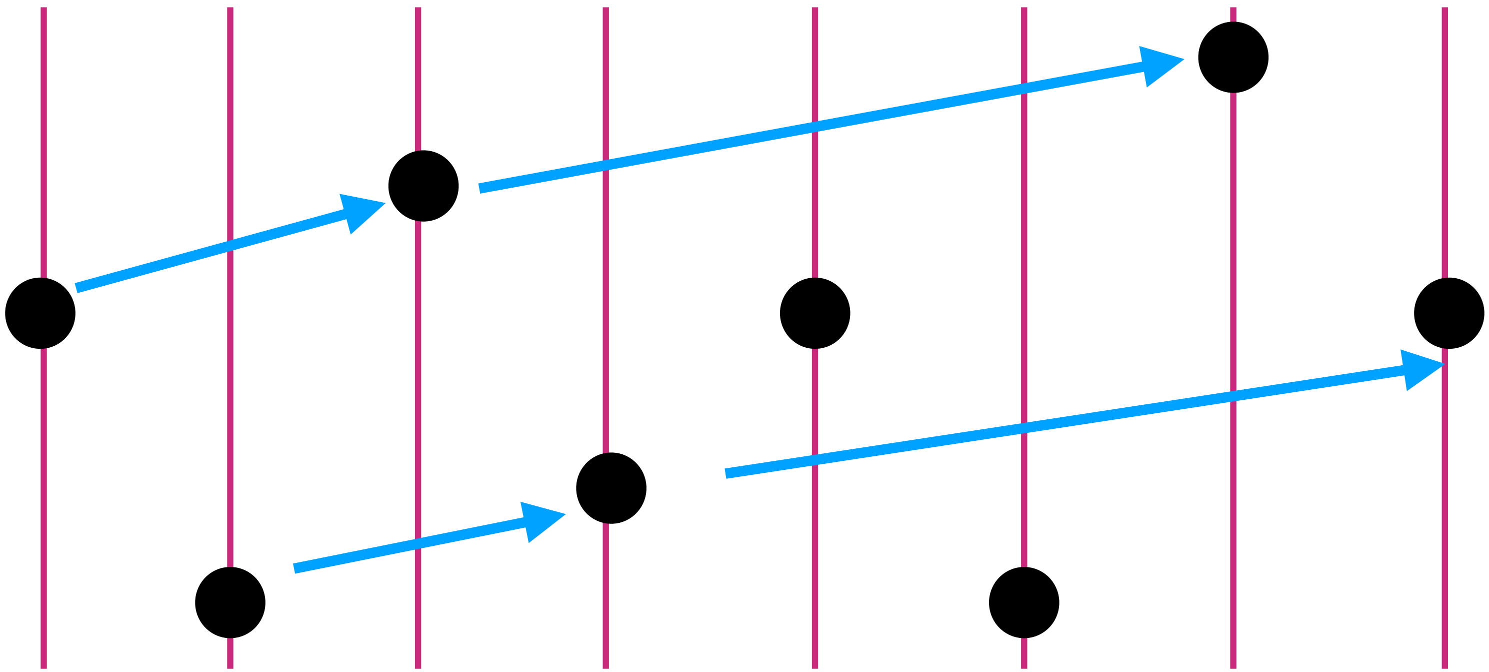
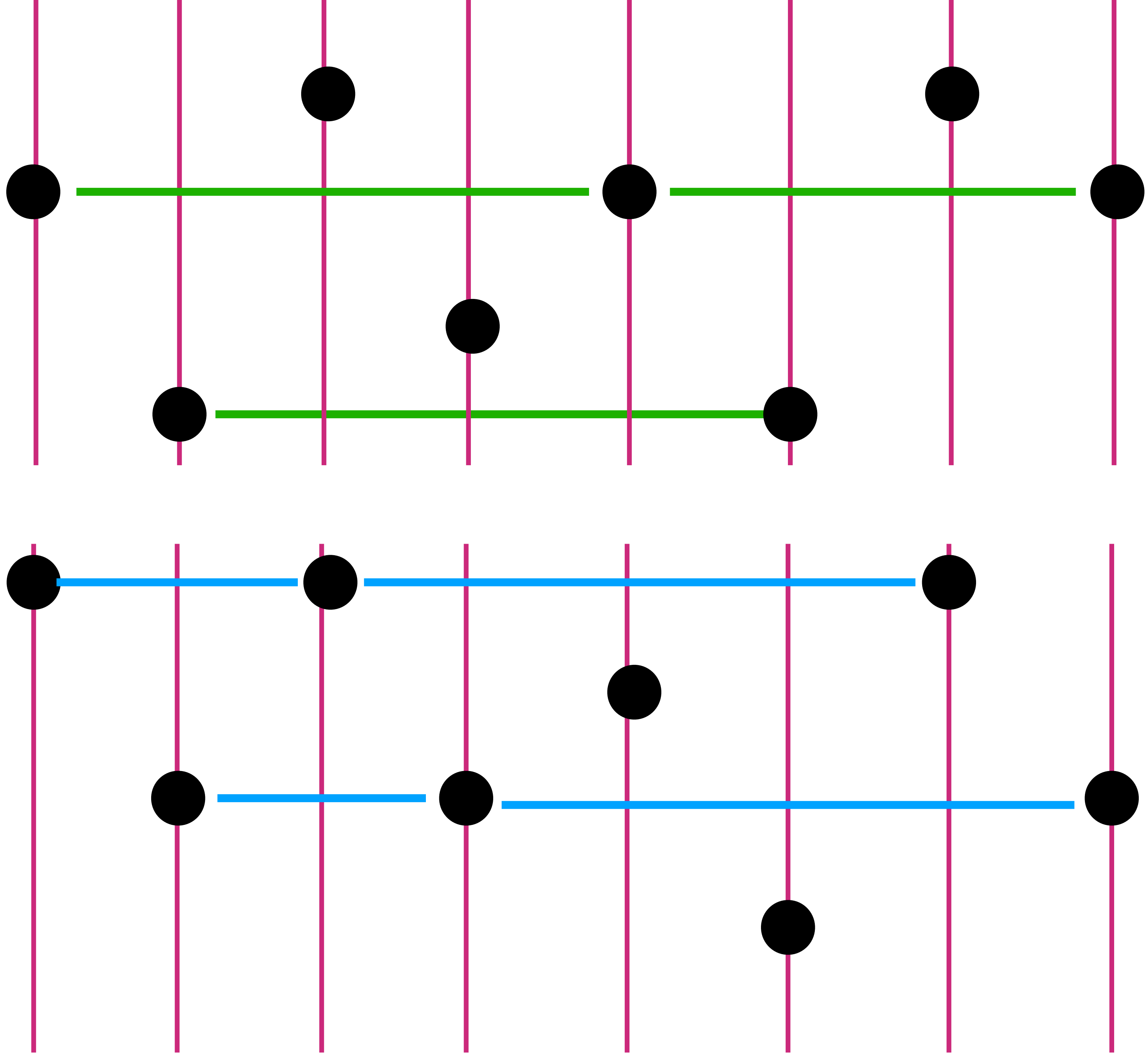


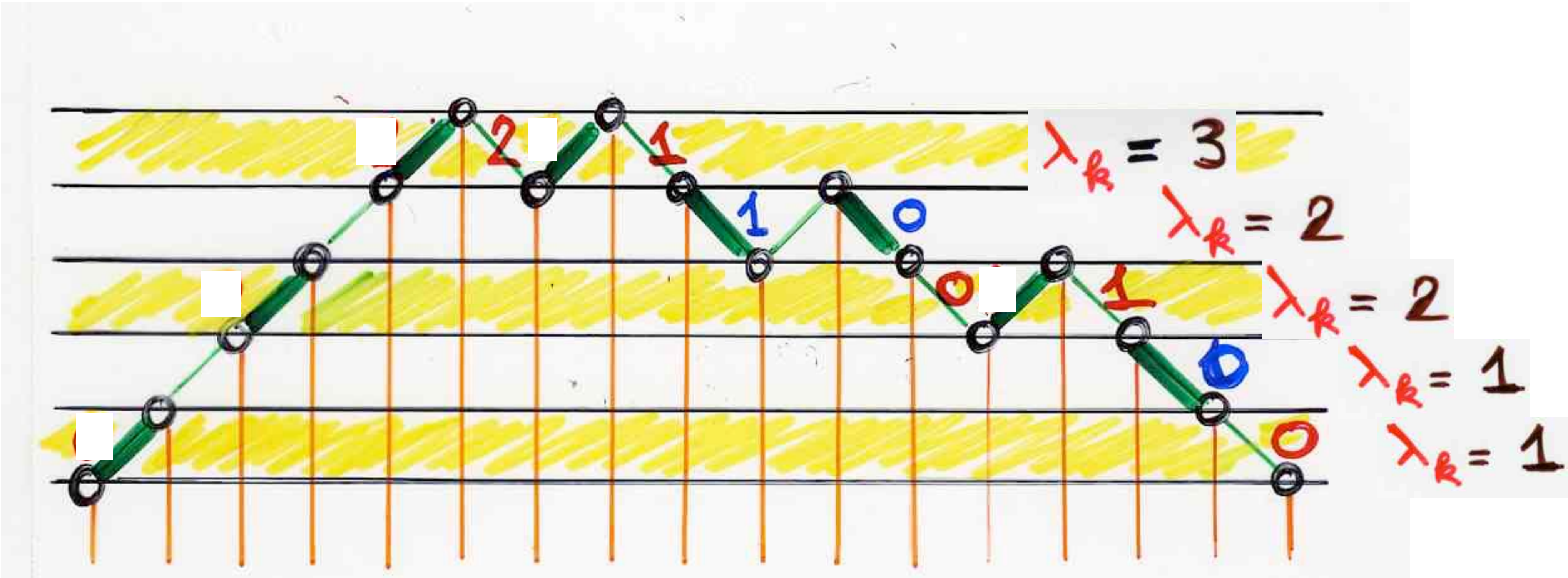


A festival of bijections

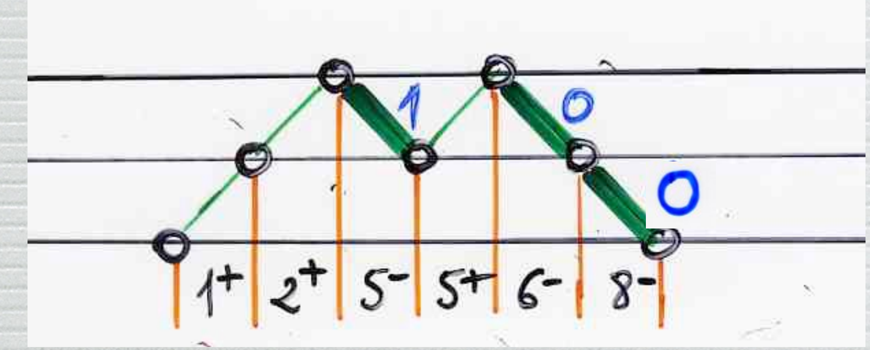
$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \dots}}}}}}$$

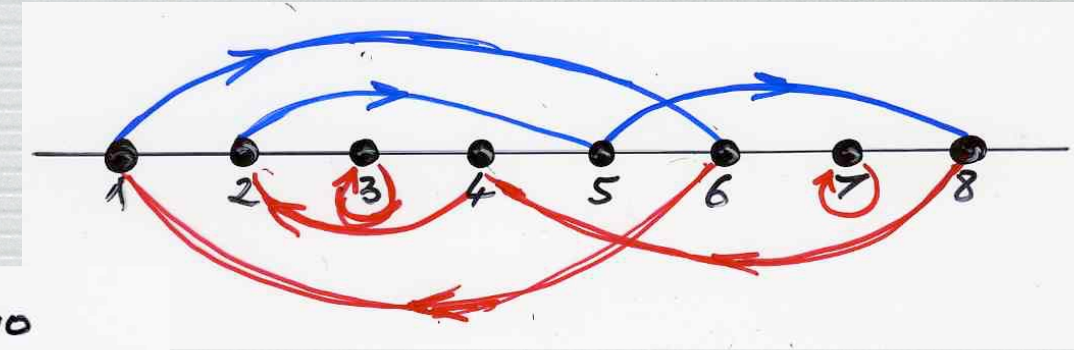
$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

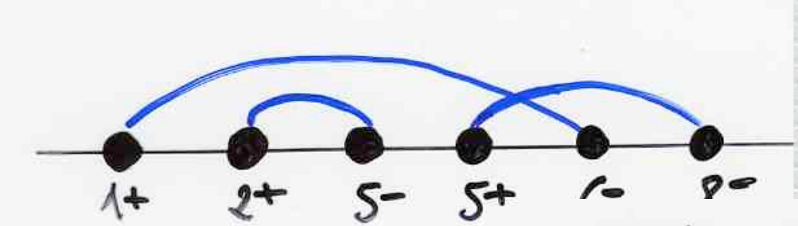


subdivided Laguerre history

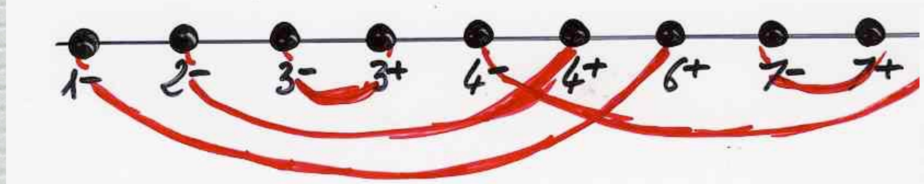


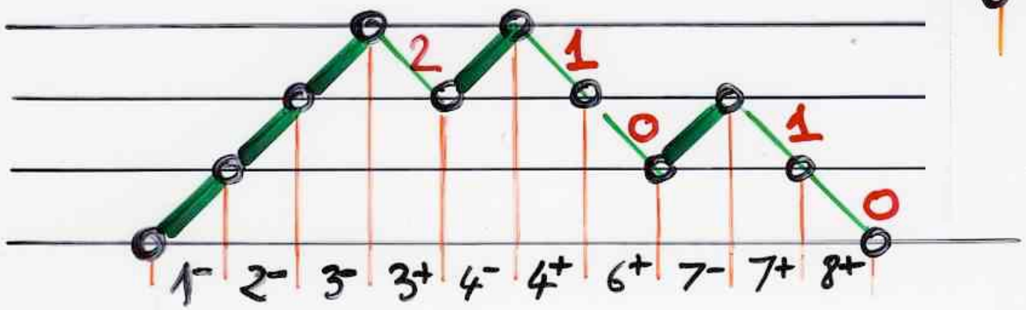
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

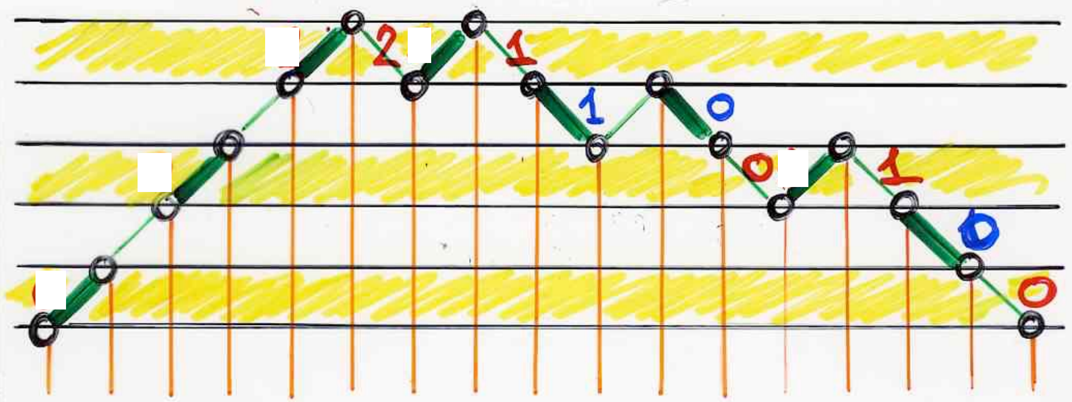




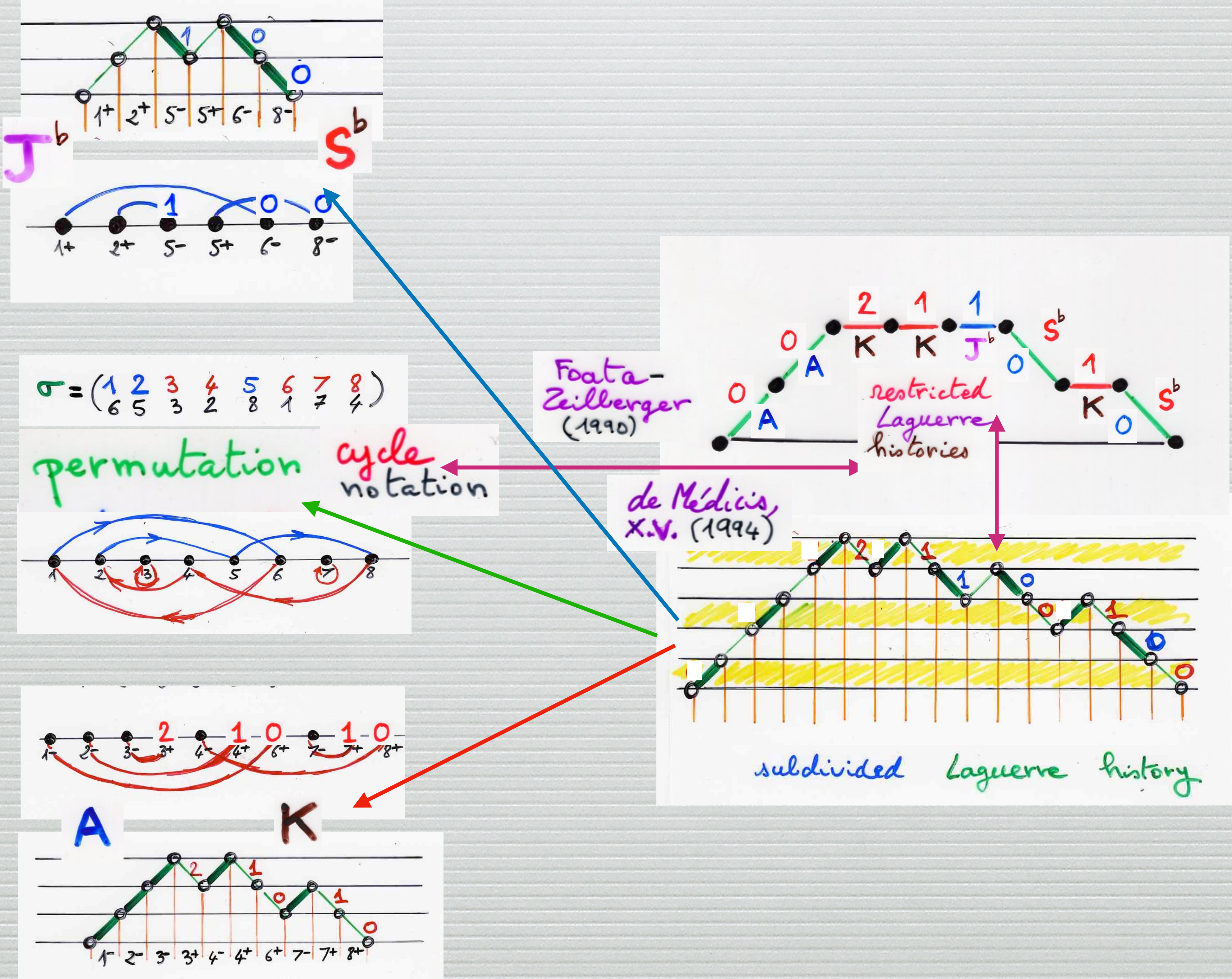
pair of two Hermite histories ("shuffle")



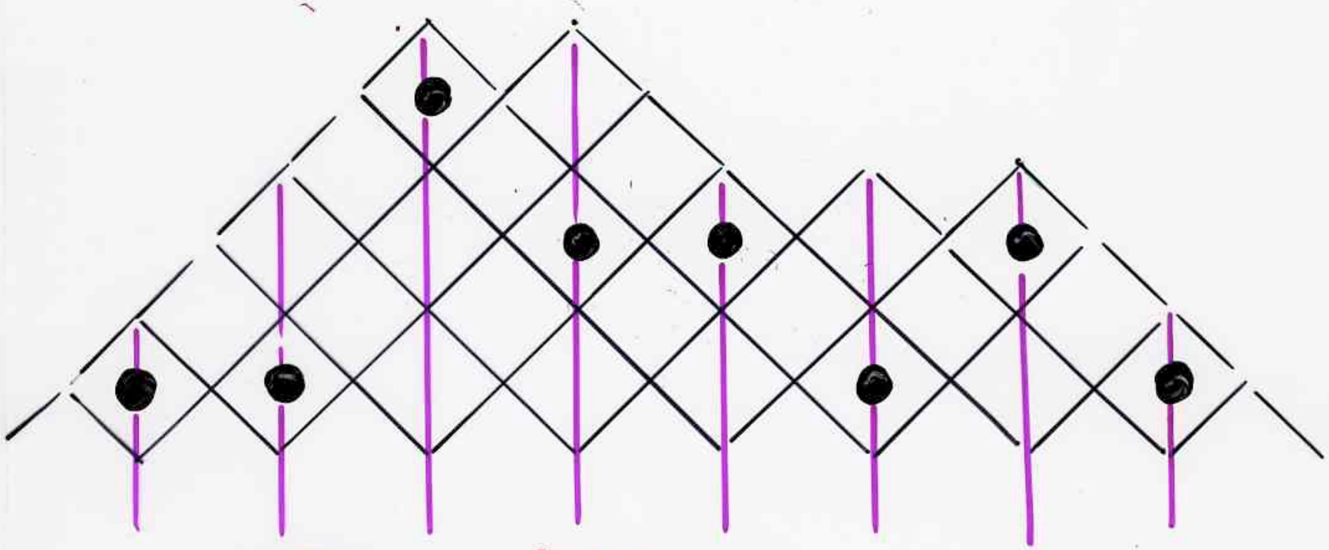




subdivided Laguerre history

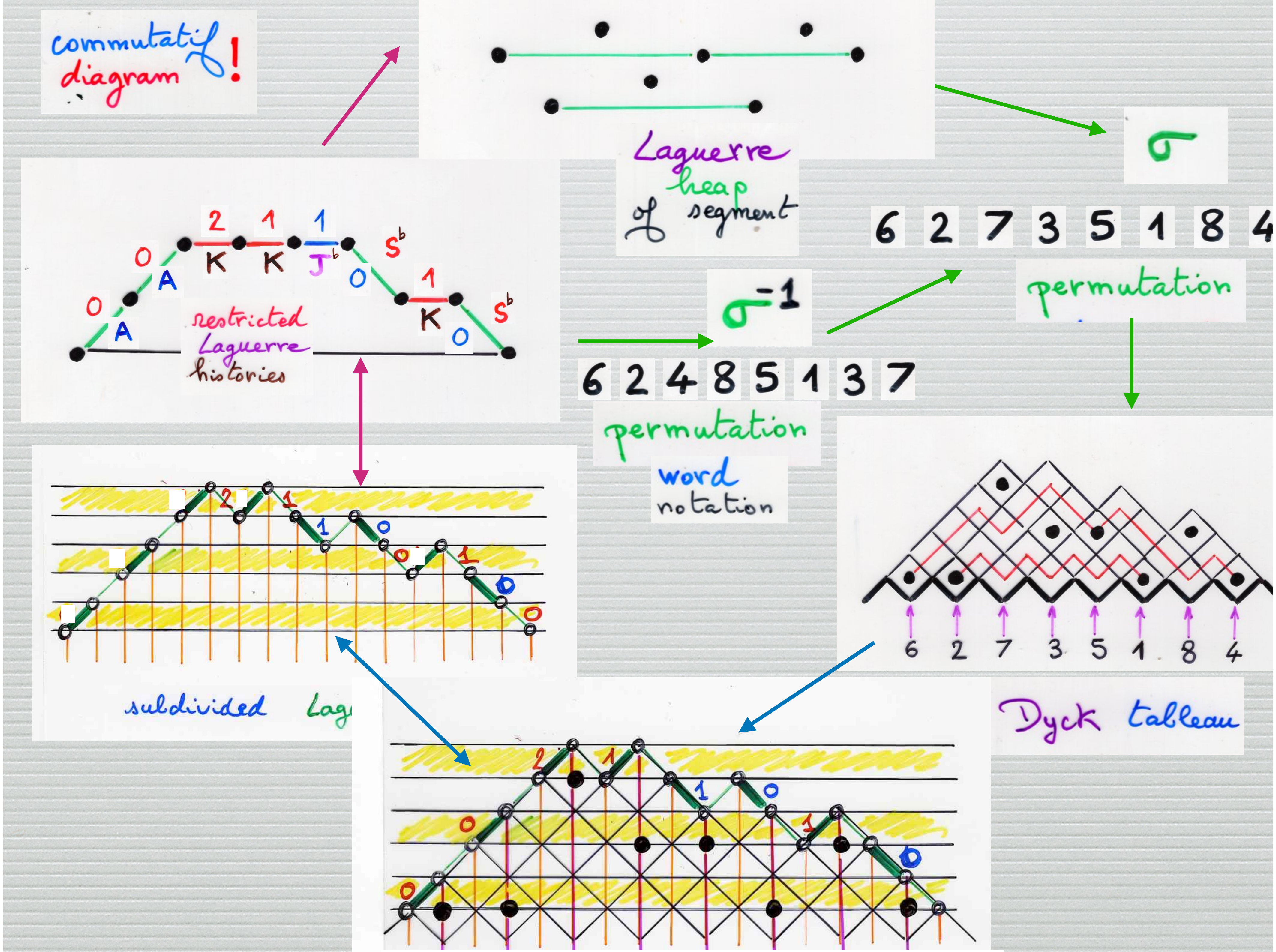


J^b
S^b
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$
permutation
cycle notation
Foata-Zeilberger (1990)
restricted Laguerre histories
de Médicis, X.V. (1994)
subdivided Laguerre history
A
K



Dyck tableau

J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)



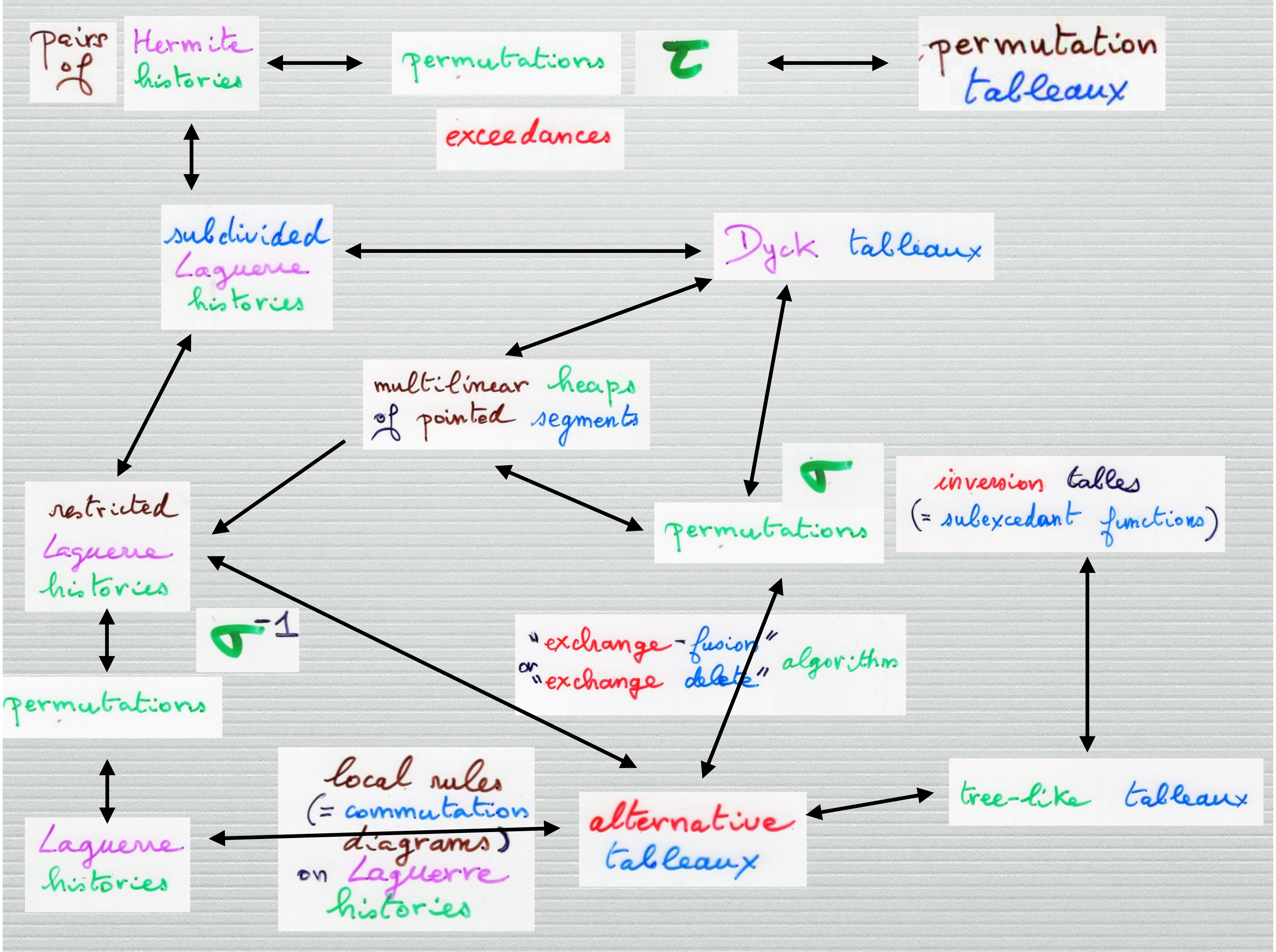
commutatif diagram !
Laguerre heap of segment
σ
6 2 7 3 5 1 8 4
permutation
σ^-1
6 2 4 8 5 1 3 7
permutation word notation
restricted Laguerre histories
2 1 1
O A O A K K J^b O S^b 1 K O S^b
subdivided
Dyck tableau
6 2 7 3 5 1 8 4

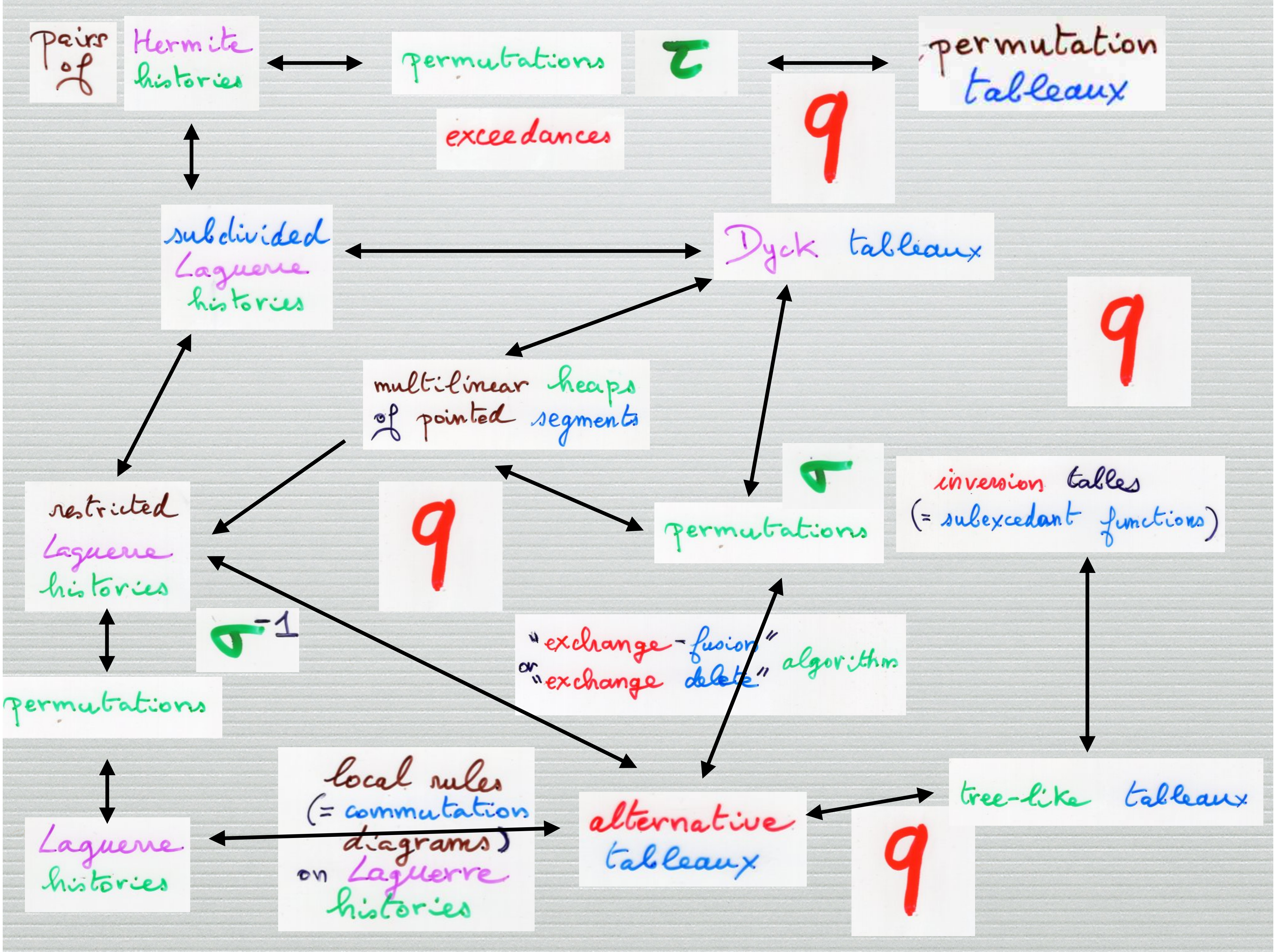
Epilogue

The « essence » of bijections …

SLC81, Krattenthalerfest

9-12 September, Strobl, Austria


pairs of Hermite histories
permutations τ
exceedances
permutation tableaux
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
inversion tables (= subexcedant functions)
restricted Laguerre histories
σ^{-1}
permutations
"exchange-fusion" or "exchange delete" algorithm
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories



pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
q
subdivided Laguerre histories
Dyck tableaux
q
multilinear heaps of pointed segments
restricted Laguerre histories
q
σ
permutations
inversion tables (= subexcedant functions)
σ^{-1}
"exchange-fusion" or "exchange delete" algorithms
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories
q

pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

Dyck tableaux

contraction of paths

multilinear heaps of pointed segments

σ

permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

"exchange-fusion" or "exchange delete" algorithm

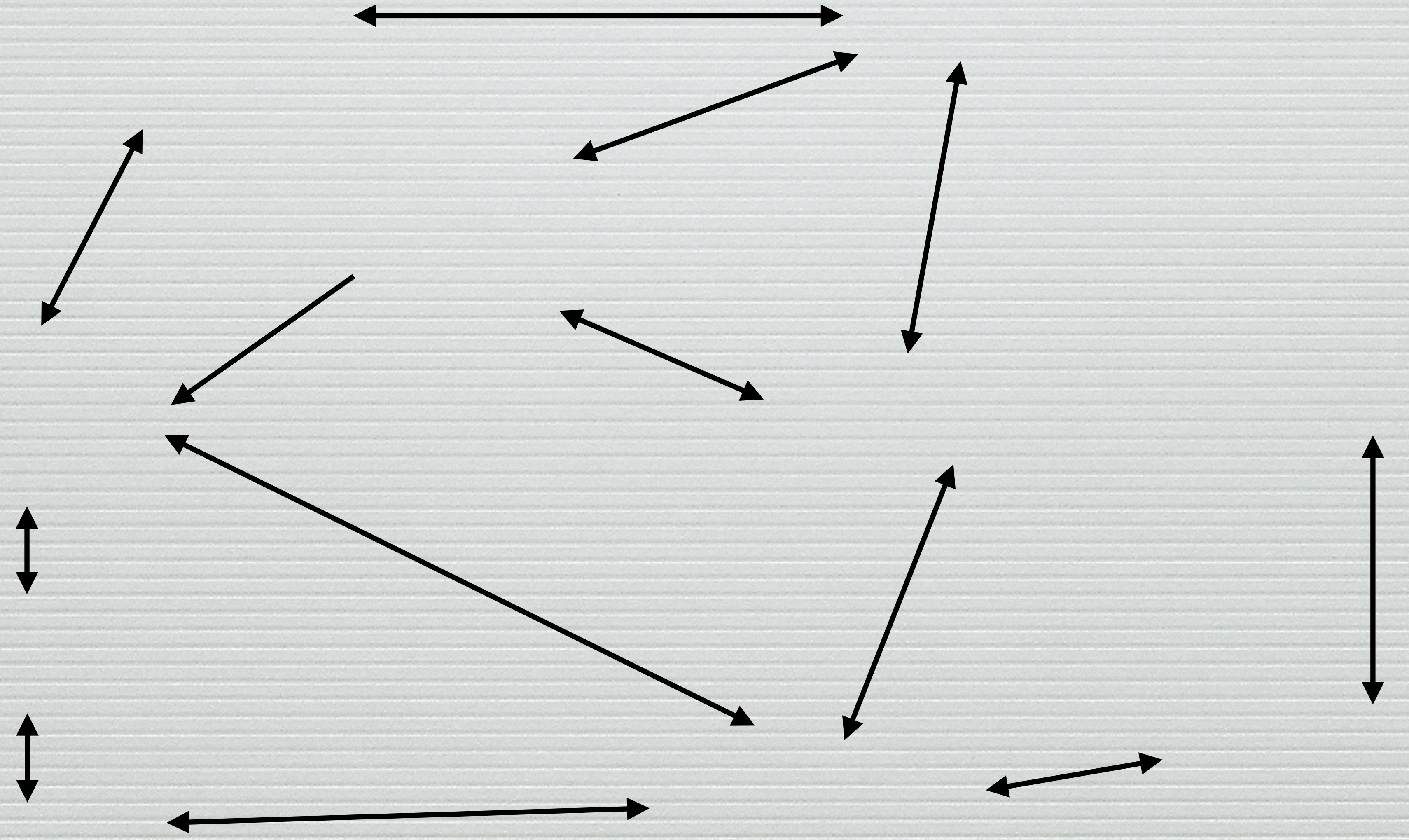
permutations

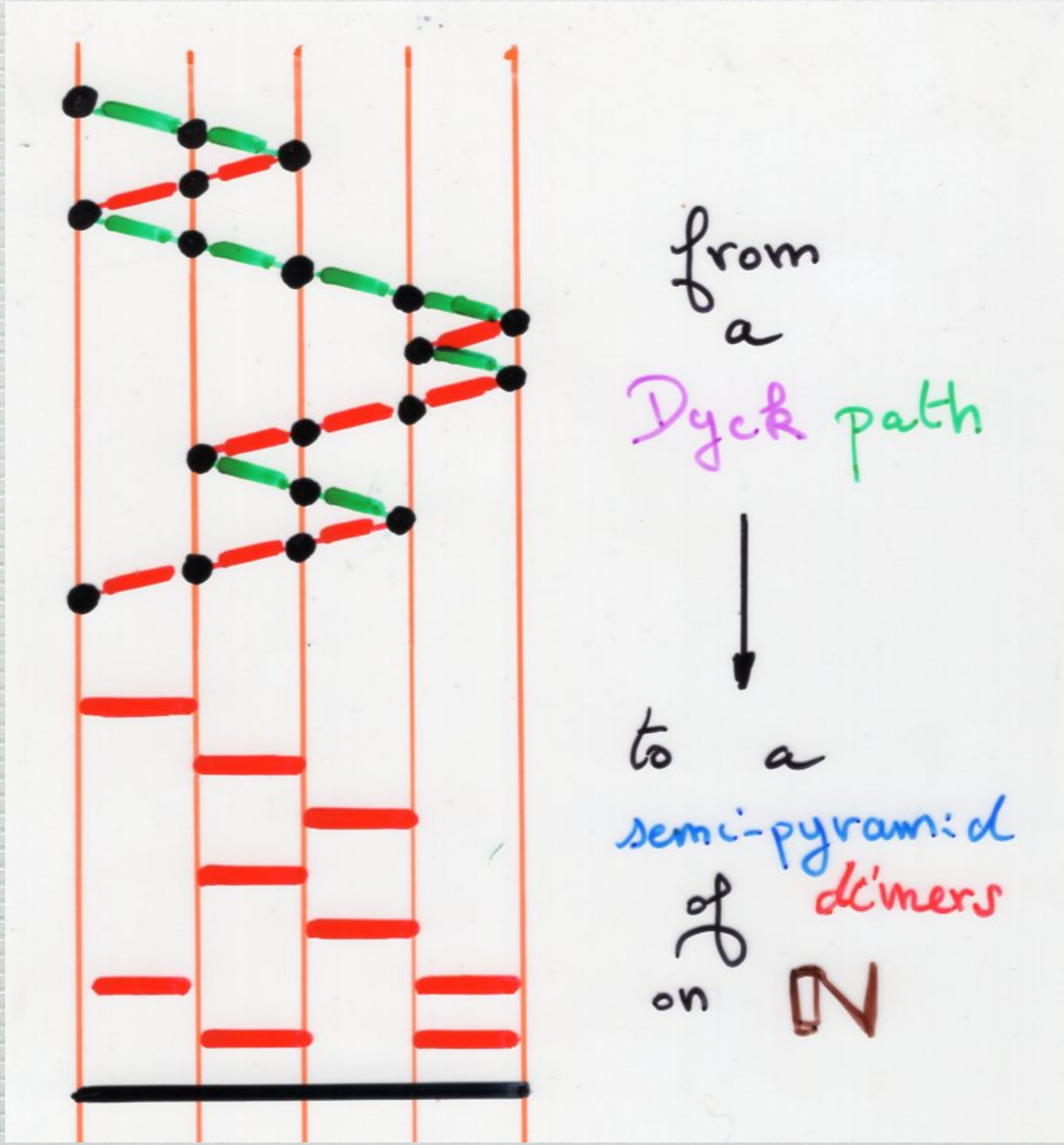
tree-like tableaux

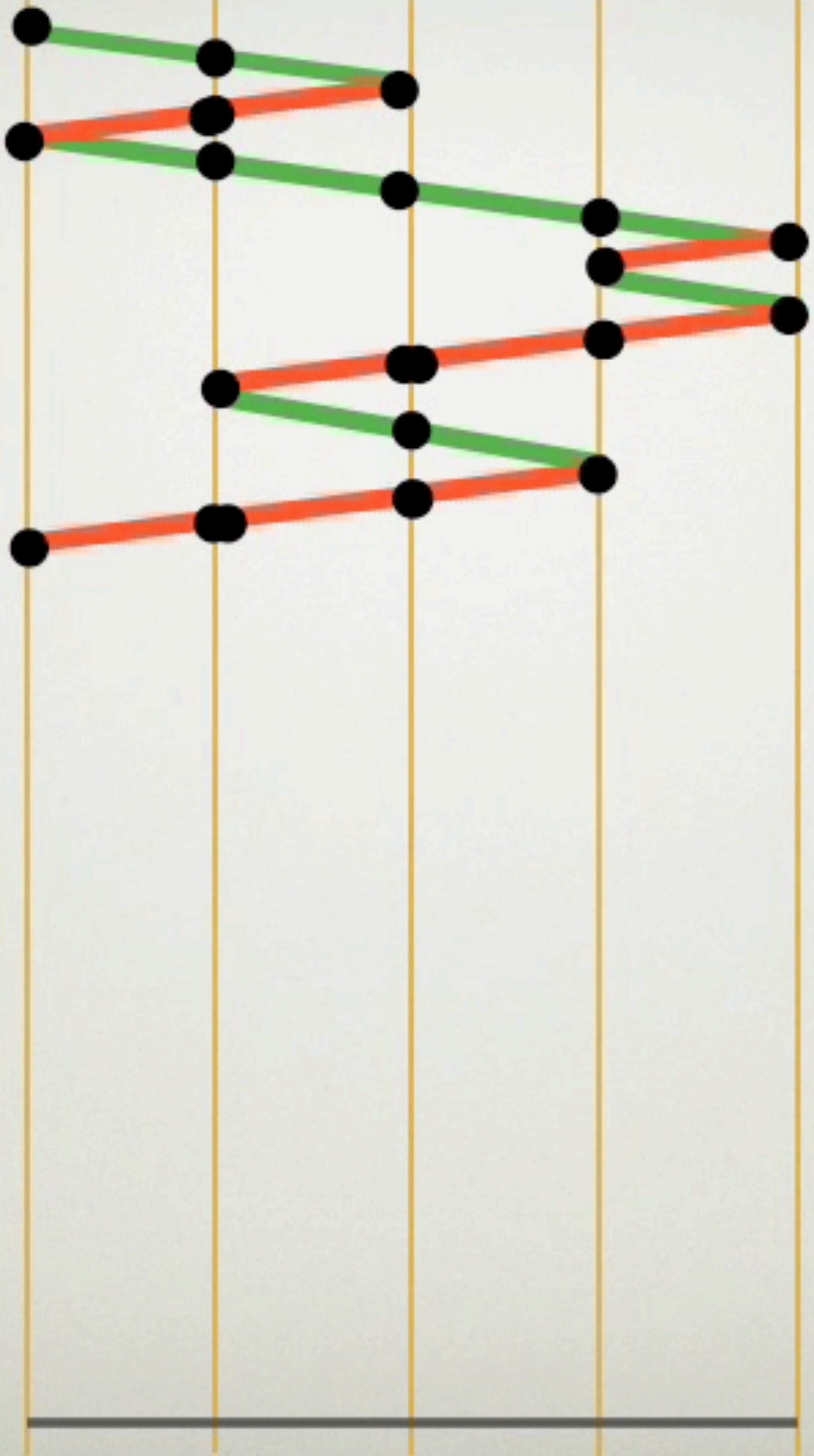
alternative tableaux

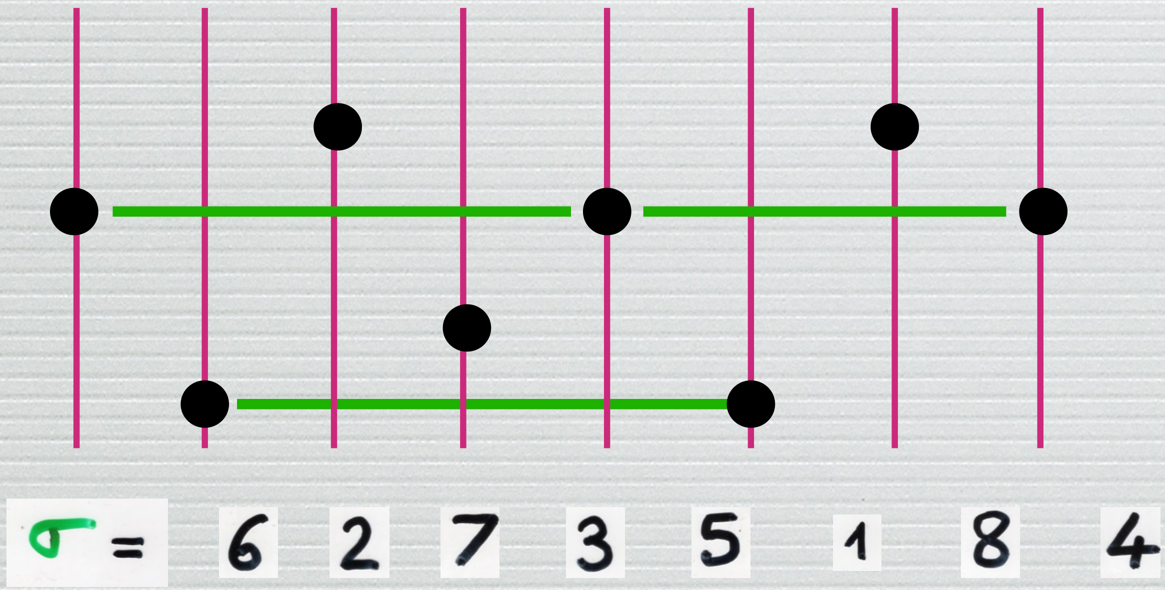
Laguerre histories

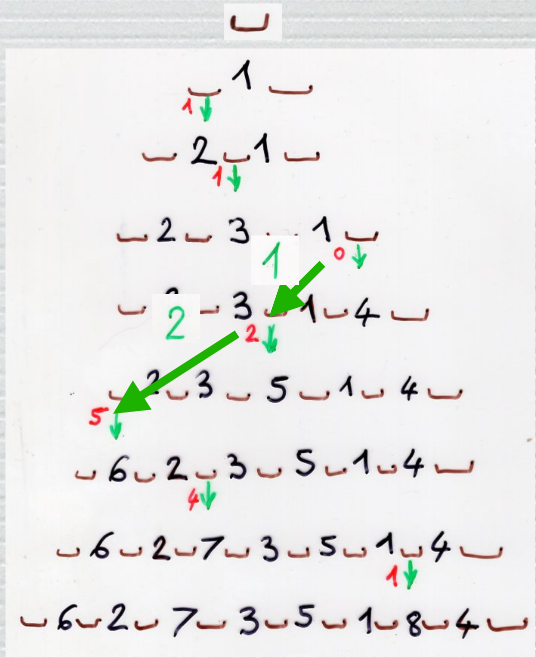
The «essence» of bijections …



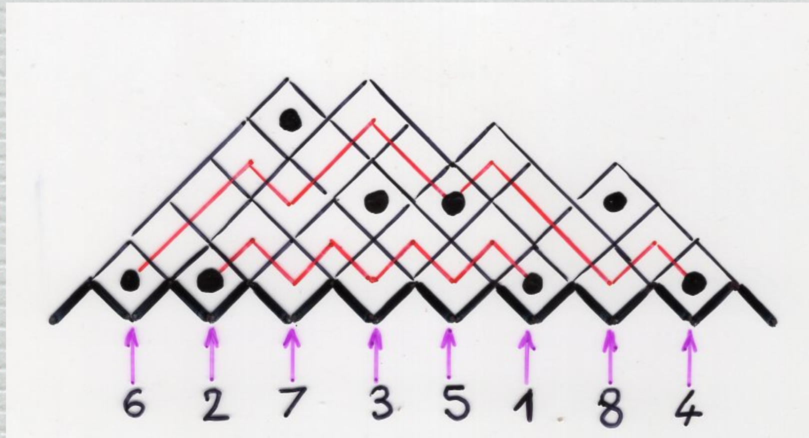

from
a
Dyck path
to a
semi-pyramid
of dimers
on N

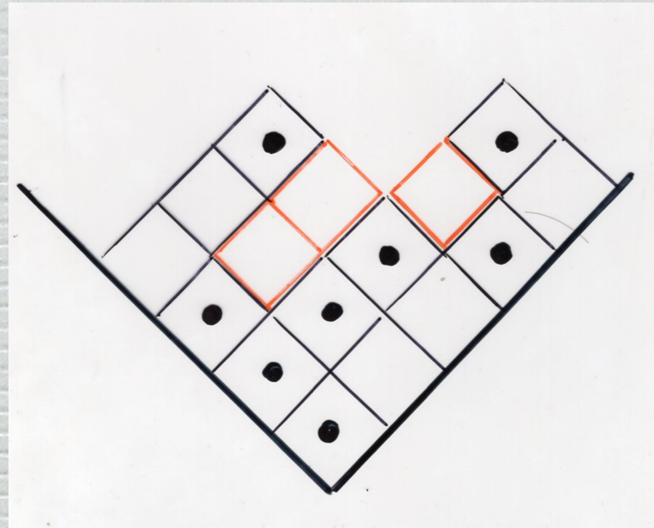


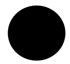


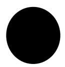


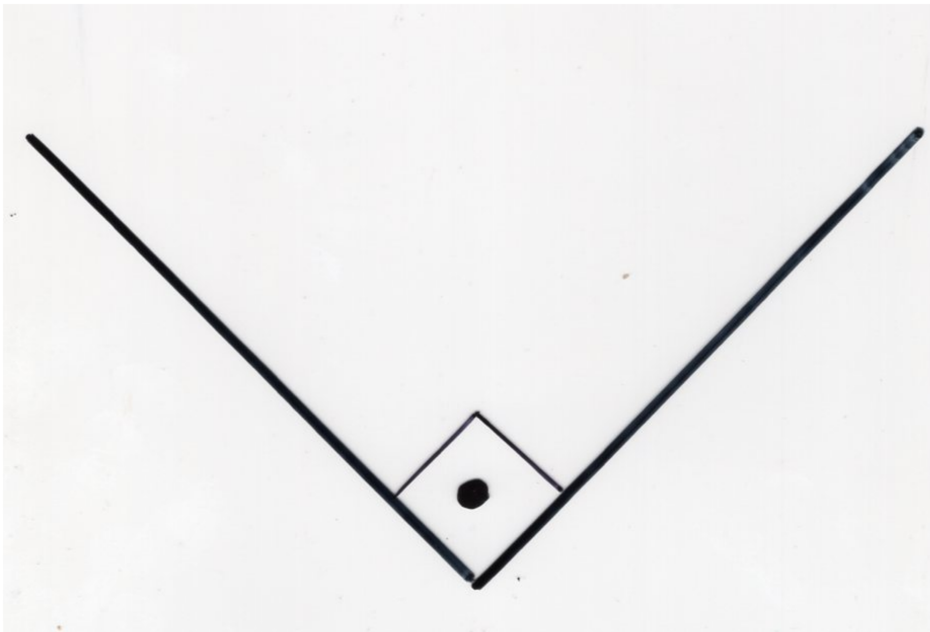
The « essence » of 3 bijections in parallel

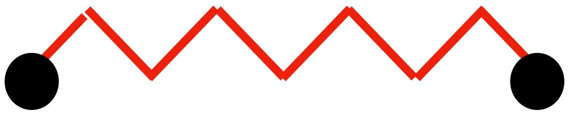


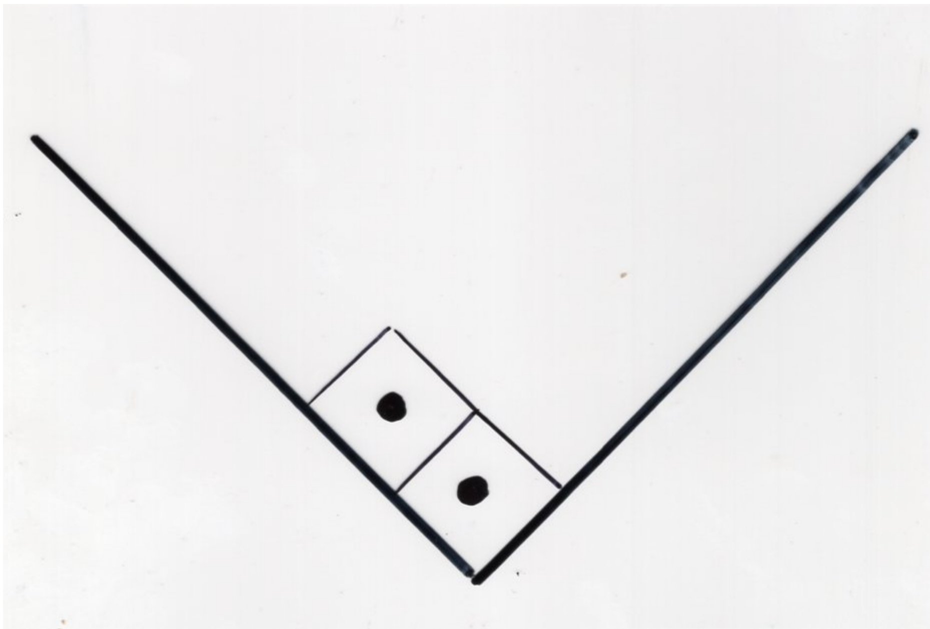


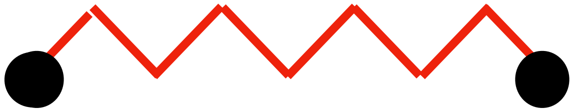


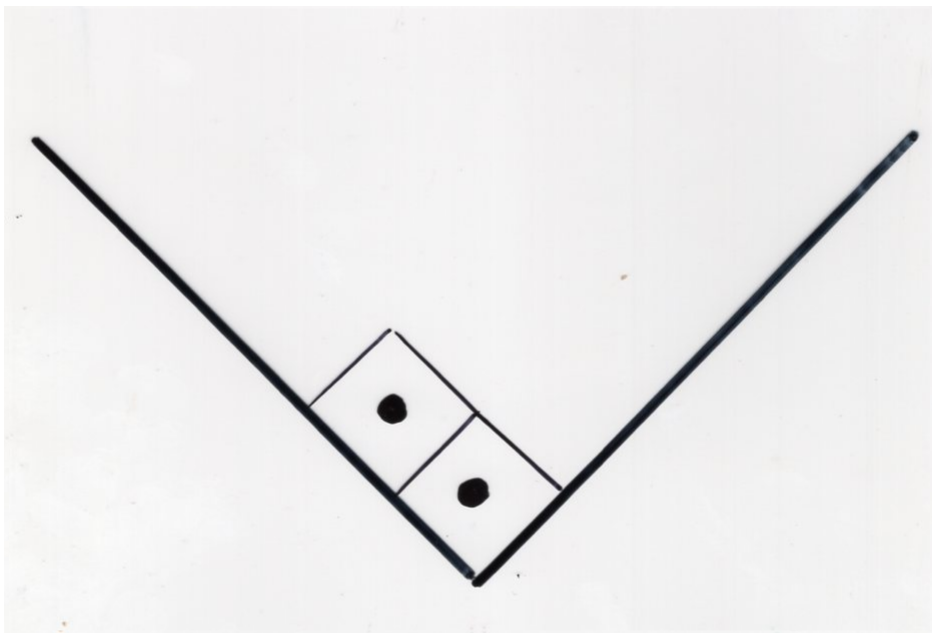


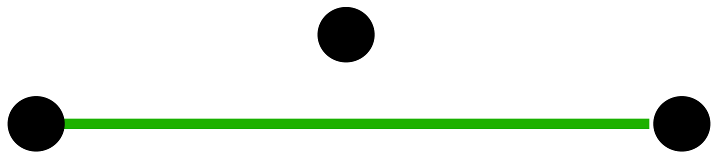


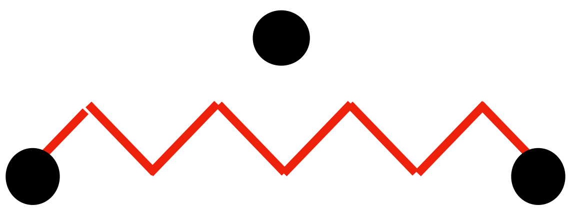


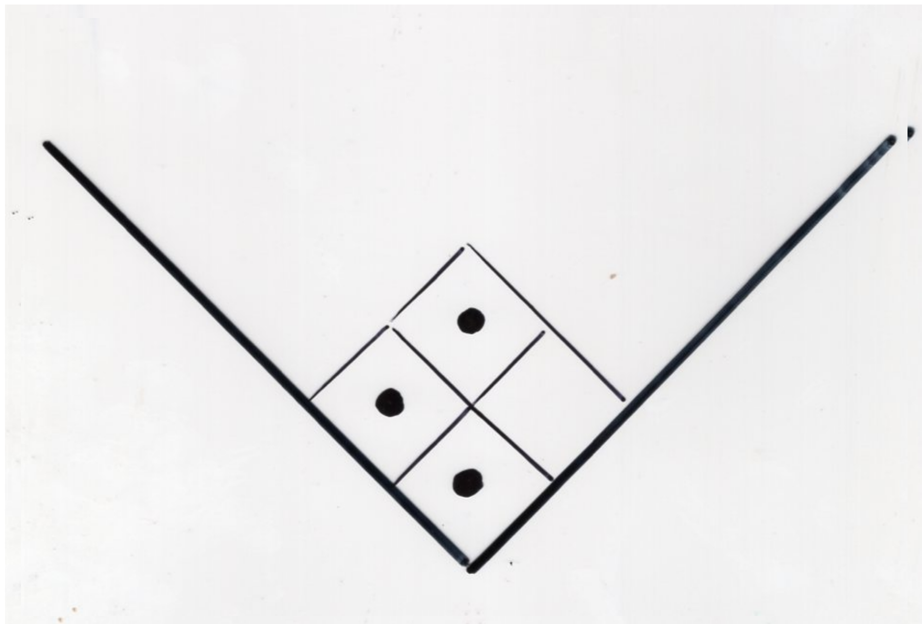


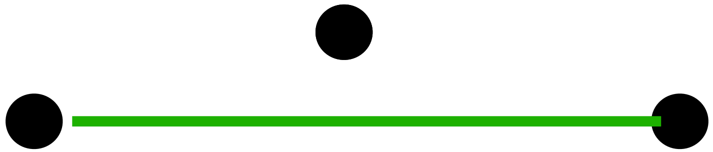


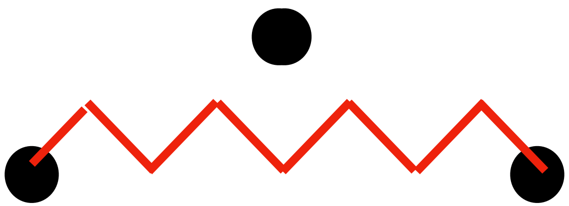


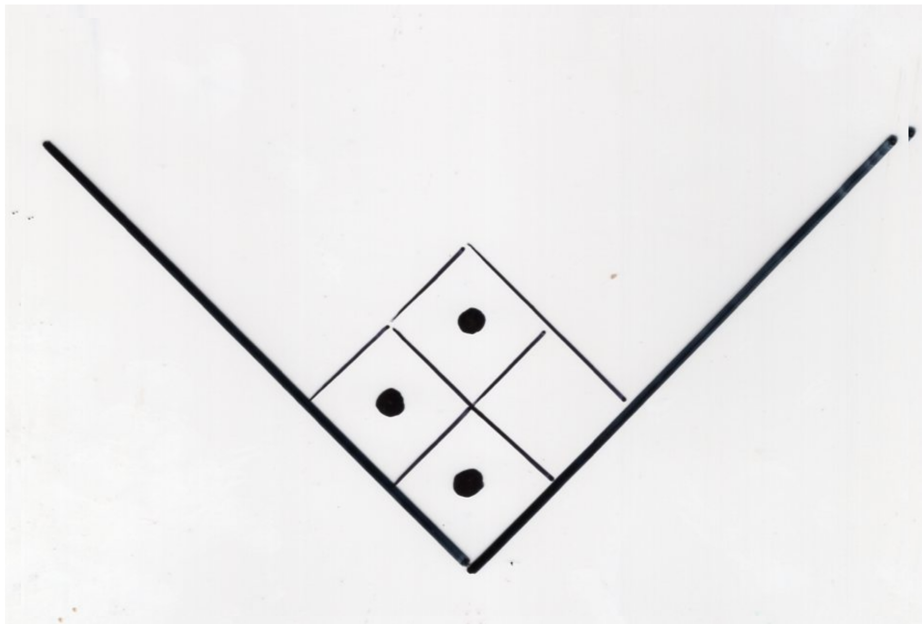


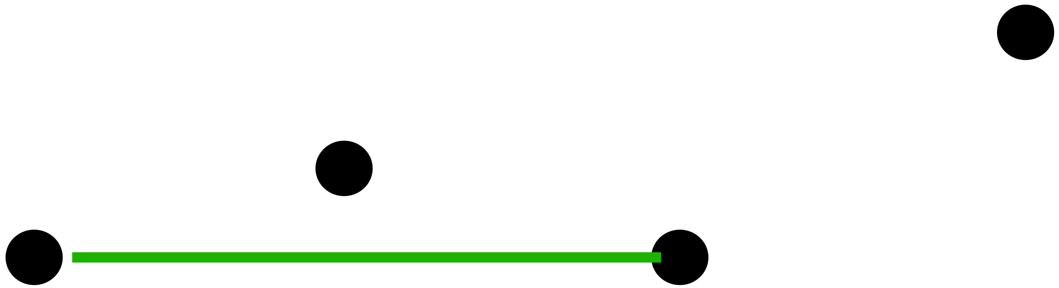


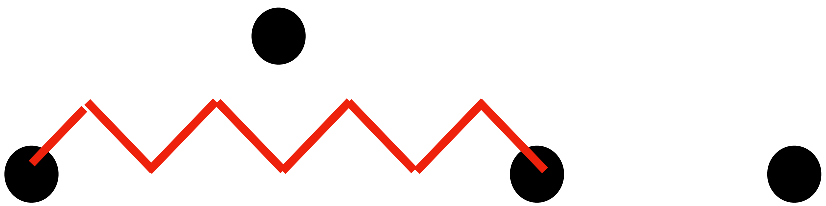


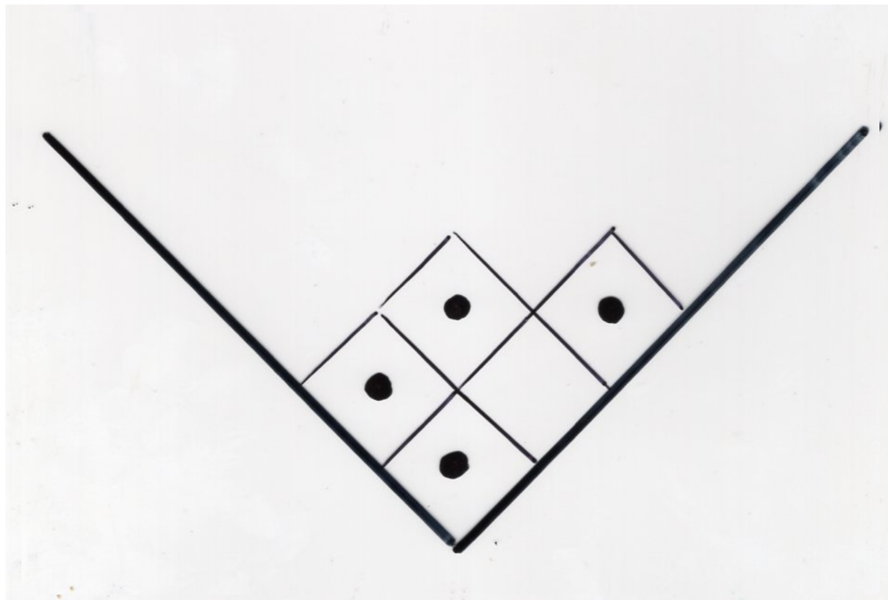


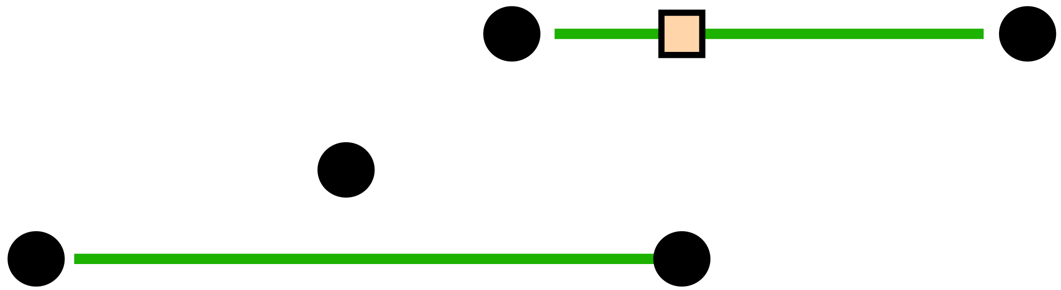


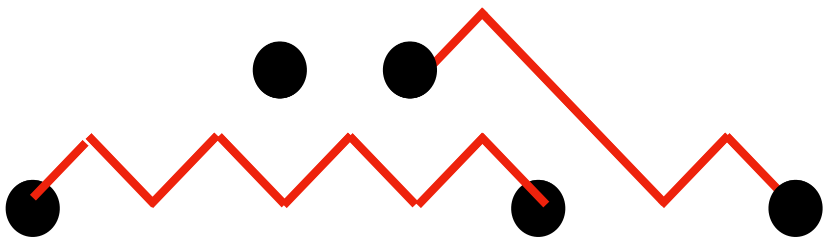


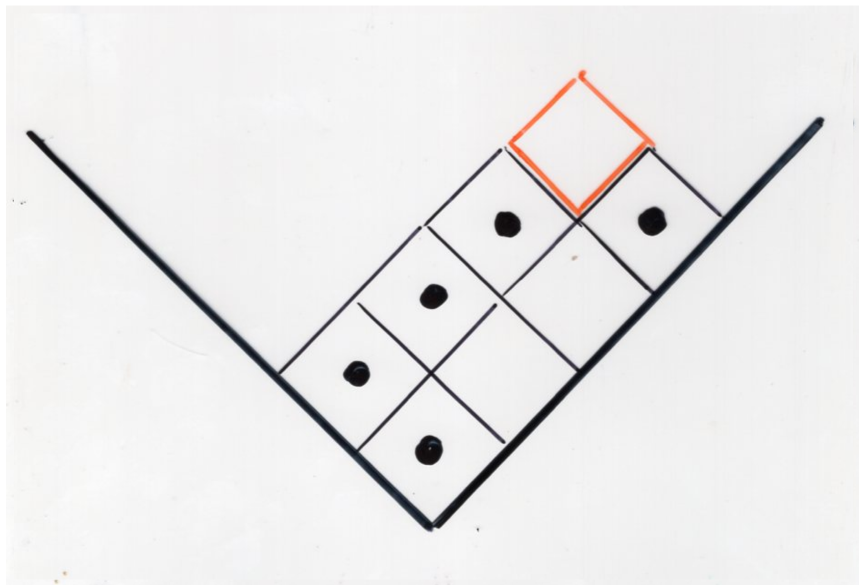


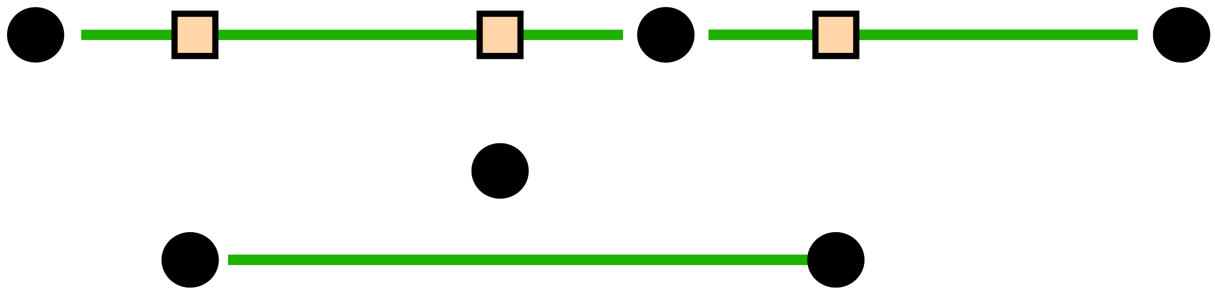


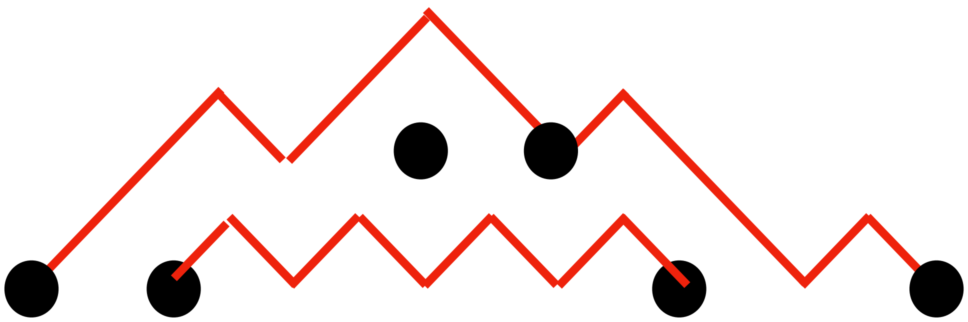


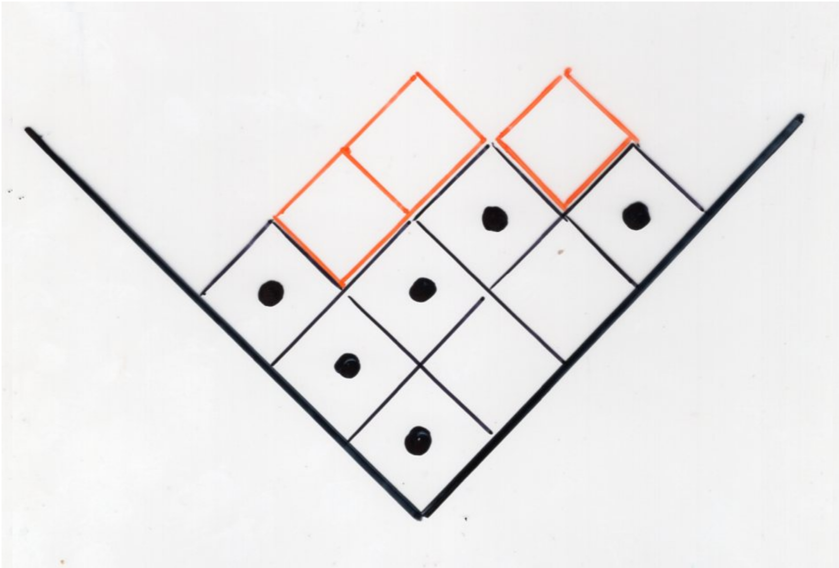


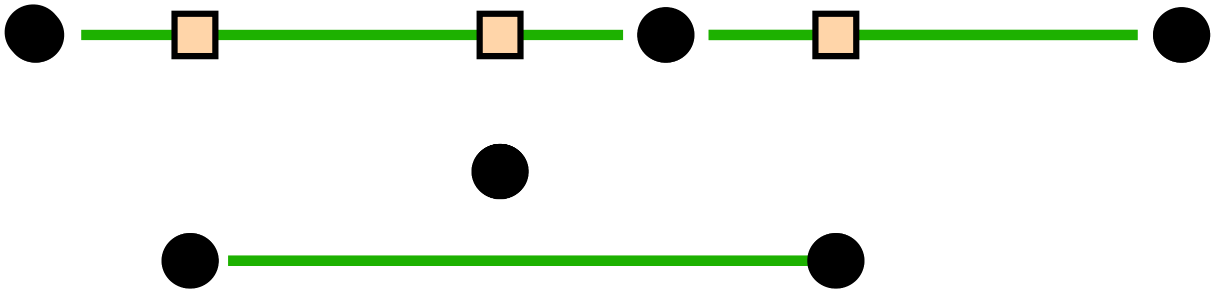


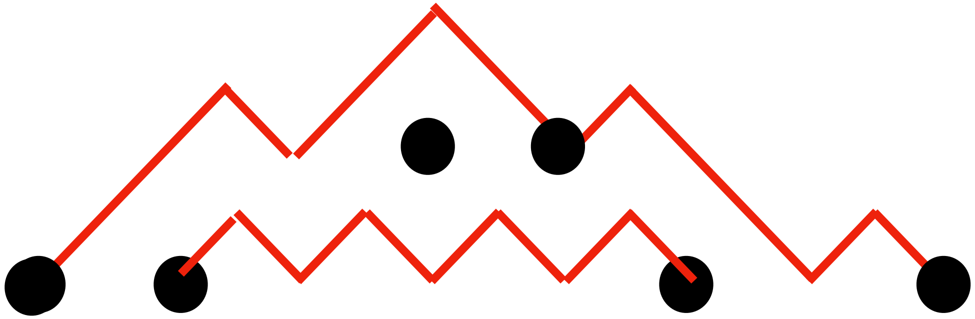


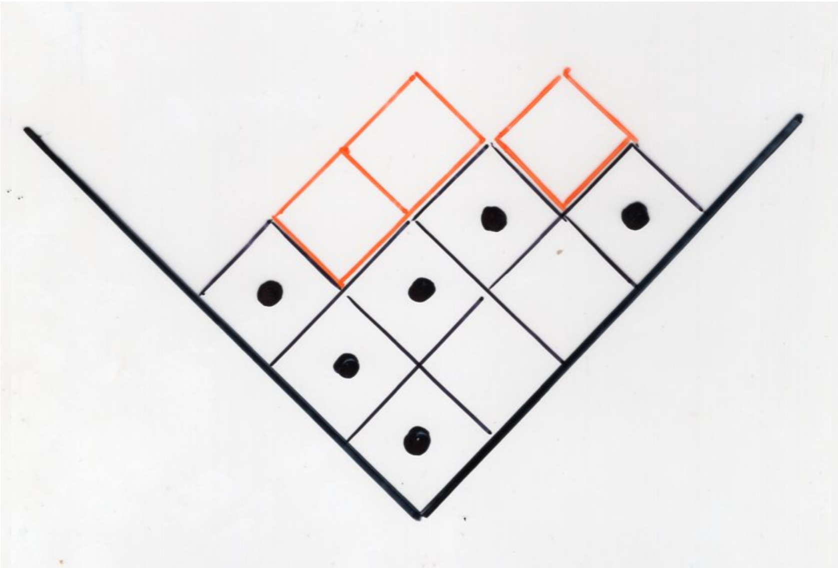


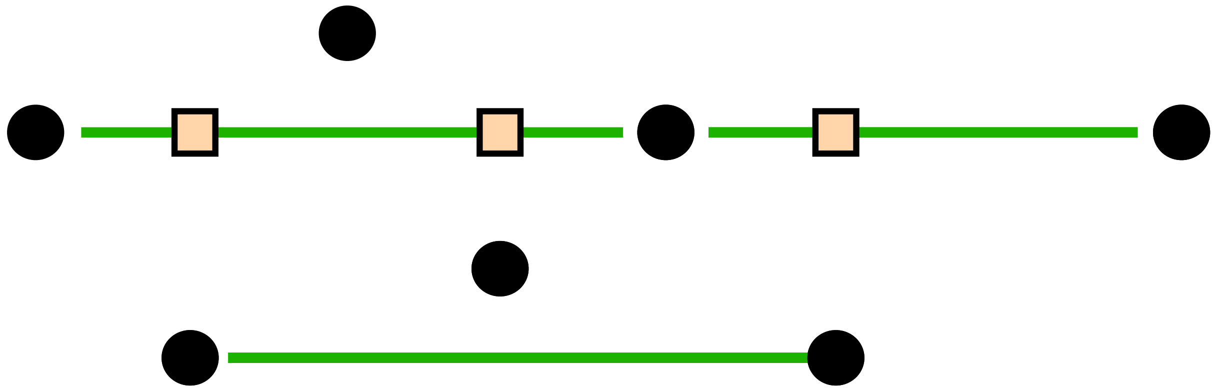


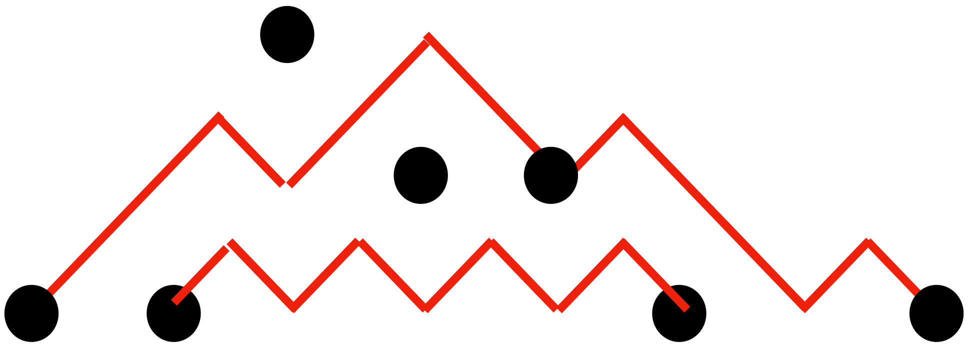


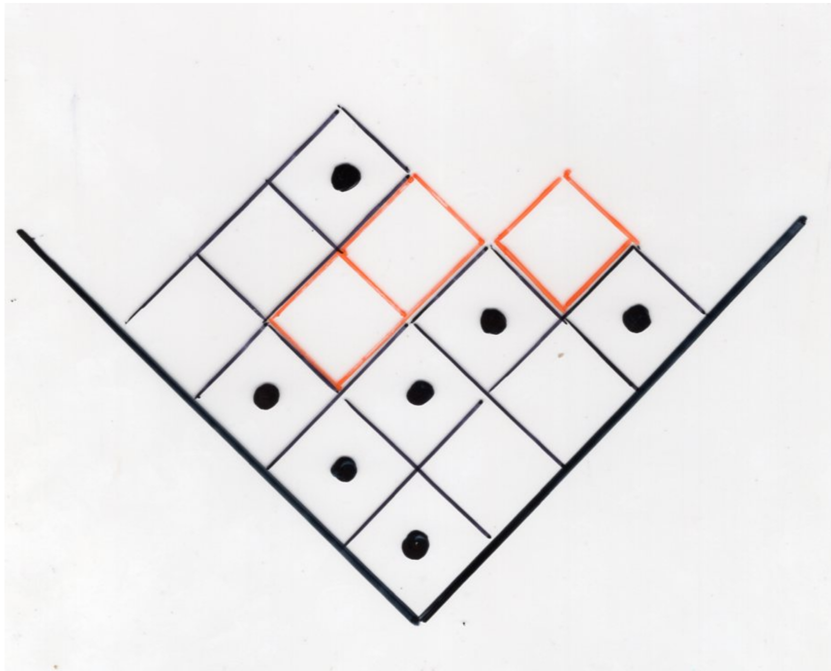


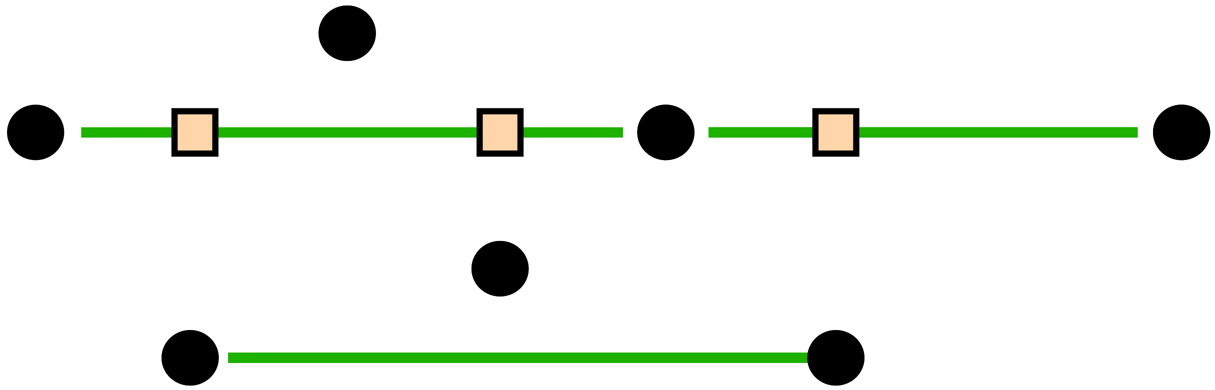


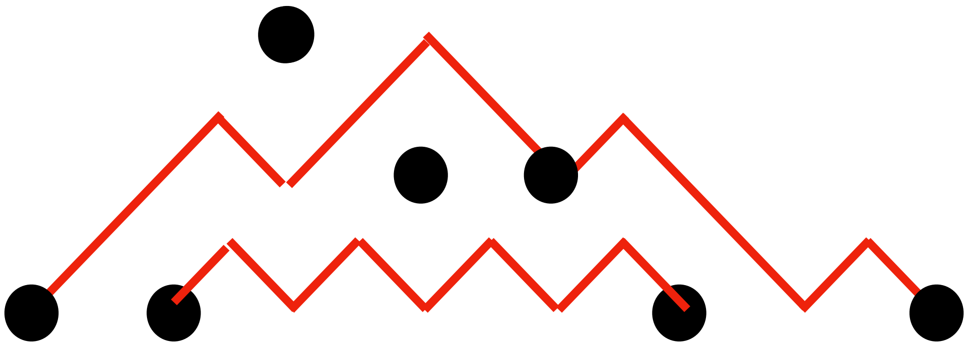


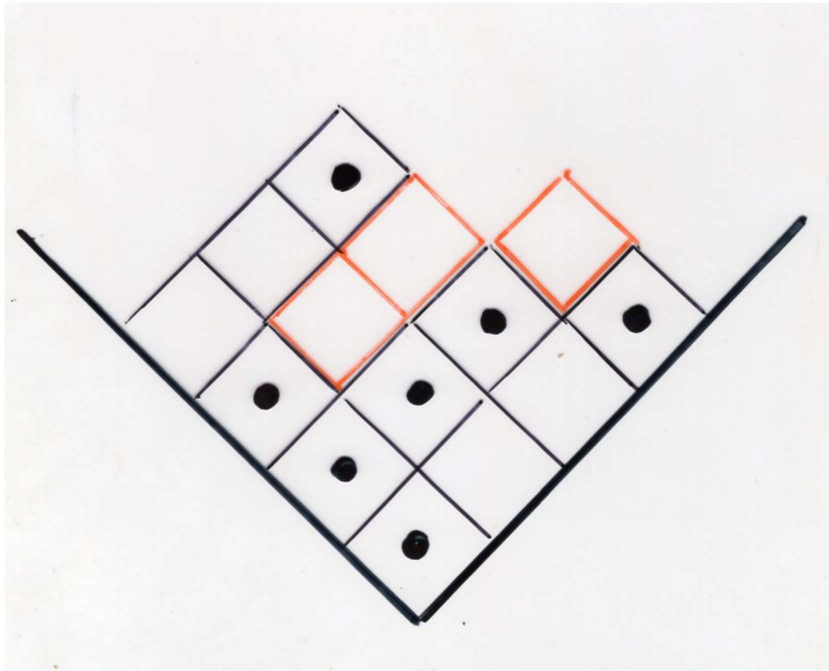


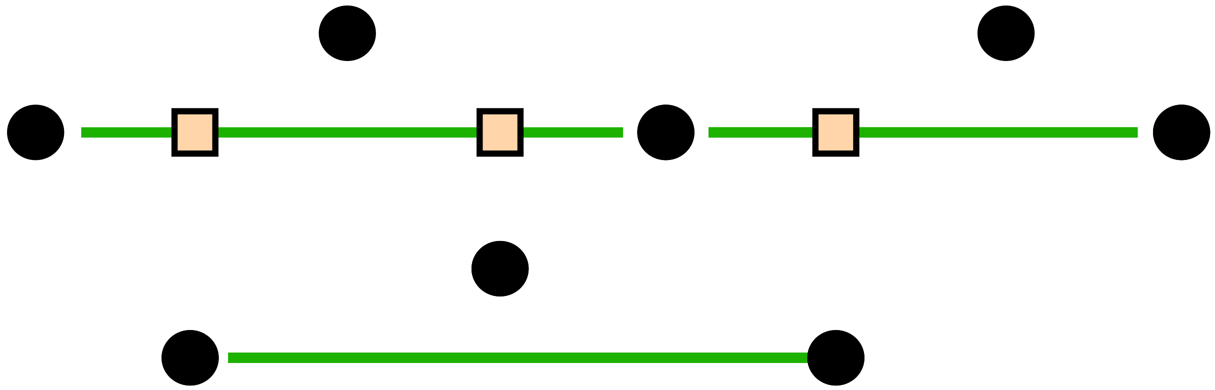


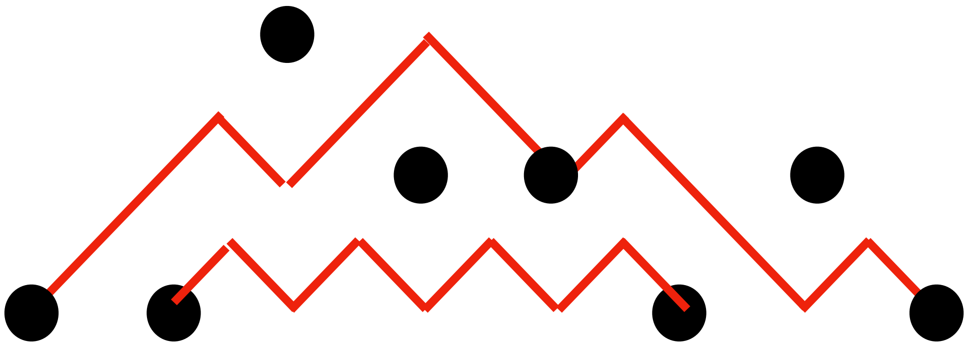


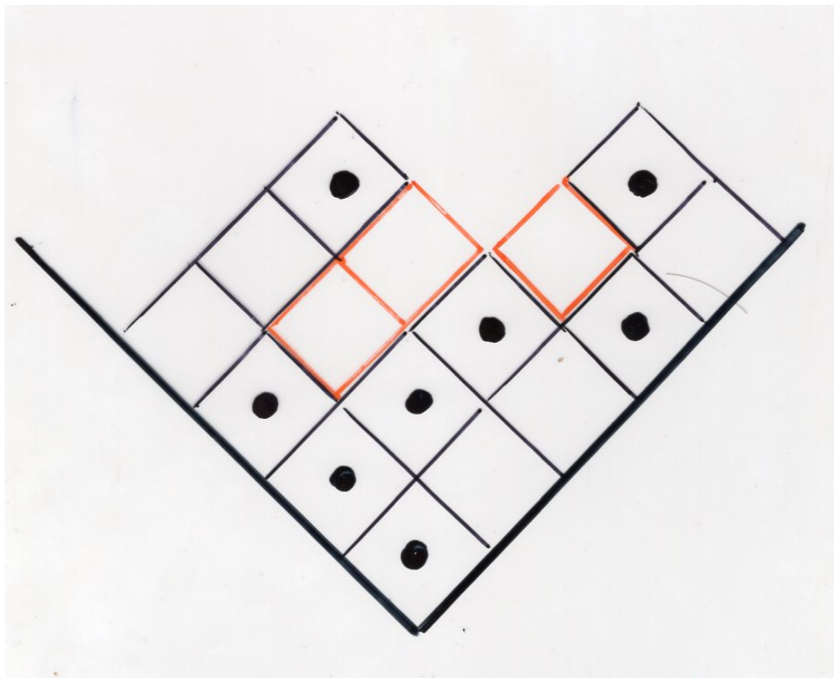


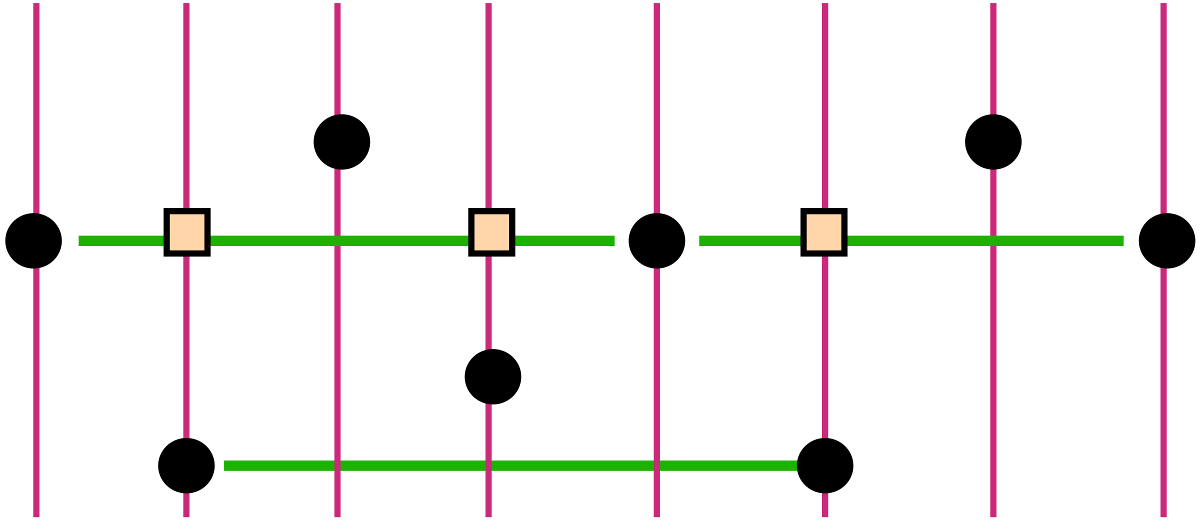


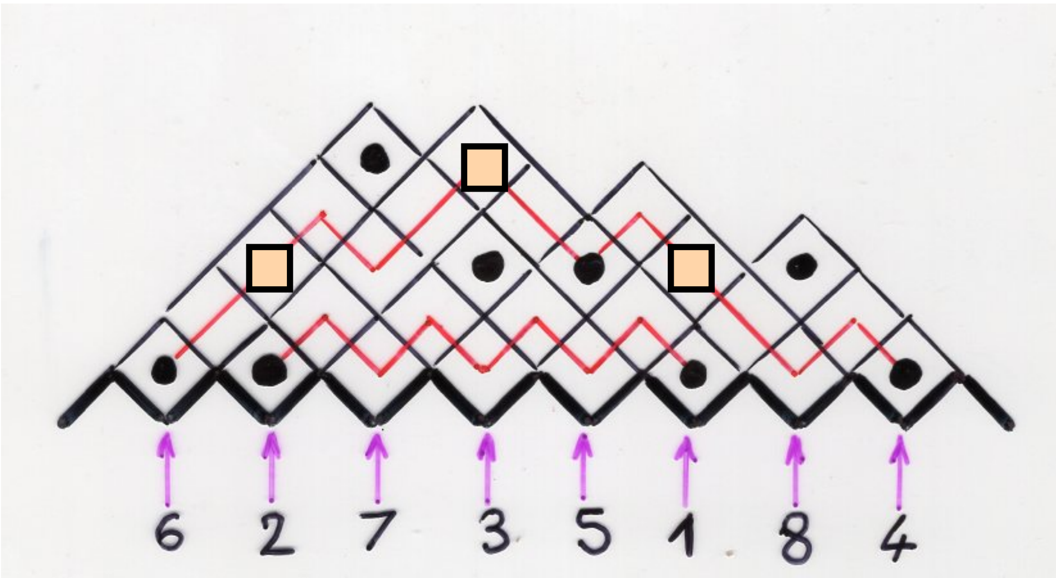








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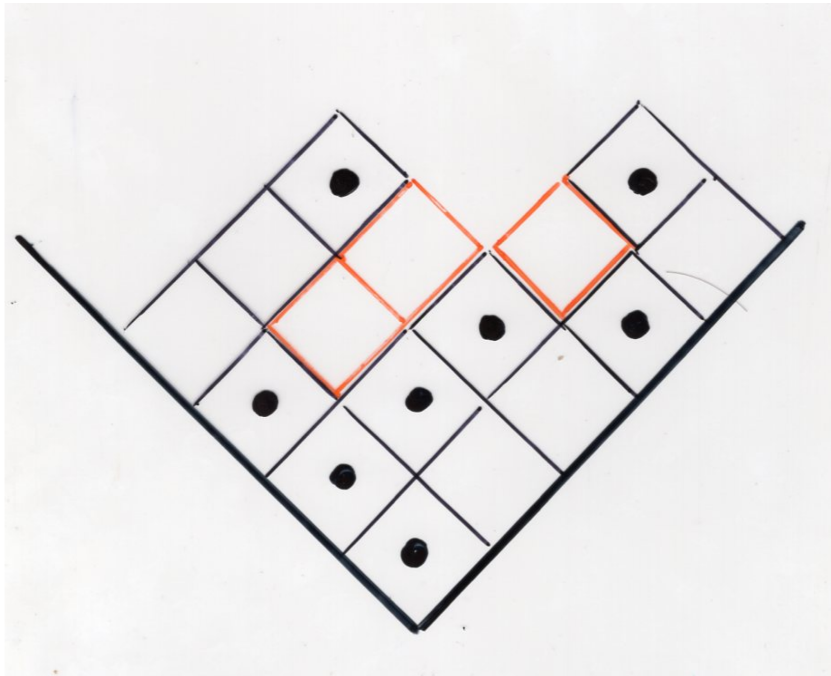

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