A survey of the combinatorial theory of orthogonal polynomials and continued fractions

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Orthogonal polynomials

classical analysis

trigonometric
hypergeometric
Bessel, elliptic
) functions

special functions

numerical analysis

interpolation
mechanical quadrature
differential and integral equations

Probalities
theory

quantum
statistical
mechanics

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind

$$\int_{-1}^{+1} U_m(x) U_n(x) (1-x^2)^{1/2} = \frac{\pi}{2} \delta_{m,n}$$

$$\cos(n\theta) = T_n(\cos\theta)$$

$T_n(x)$ Tchebychef polynomial 1^{st} kind

$$\{P_n(x)\}_{n\geq 0}$$

sequence of polynomials

$$P_n(x) \in \mathbb{R}[x]$$

$$deg(P_n(x)) = n$$

degree

$$f(P(x)Q(x)) = \int_{\mathbb{R}} P(x)Q(x)\, d\mu(x)$$

measure μ on $\mathbb{R}$

origin: continued fractions

Euler

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§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}.$$

DIVERGENTIBVS. 225

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + \cfrac{4x}{1 + \cfrac{4x}{1 + \cfrac{5x}{1 + \cfrac{5x}{1 + \cfrac{6x}{1 + \cfrac{6x}{1 + \cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

AVCTORE
Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.4.6.6.8.8.10.10}{3.3.5.5.7.7.9.9.11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

continued fractions

Stieltjes

$$\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cfrac{\cdots\cdots}{1-\cfrac{\lambda_k t}{\cdots}}}}}$$

$\underbrace{\qquad\qquad\qquad}_{S(t;\lambda)}$

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cdots}}}}}}$$

Euler

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots}{1-b_kt-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}}$$

$$J(t;b,\lambda)$$

Jacobi continued fraction

$$b=\{b_k\}_{k\geq 0} \qquad \lambda=\{\lambda_k\}_{k\geq 1}$$

equivalence

orthogonal polynomials ⟷ continued fractions

classical theory

continued fractions

J-fraction

$$J(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

orthogonal polynomials

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

classical theory

continued fractions

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t-\cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments

generating function

orthogonal polynomials

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

classical theory

continued fractions | **orthogonal polynomials**

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_k t \cdots}}}$$

moments generating function

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

late 70's, early 80's

combinatorial interpretations

of classical orthogonal polynomials

Hermite, Laguerre, Jacobi

combinatorial interpretations

of linearization coefficients

$$P_k(x)\, P_\ell(x) = \sum_n a_{k\ell}^{n}\, P_n(x)$$

positivity

Combinatorial interpretation of some orthogonal polynomials

Hermite polynomials

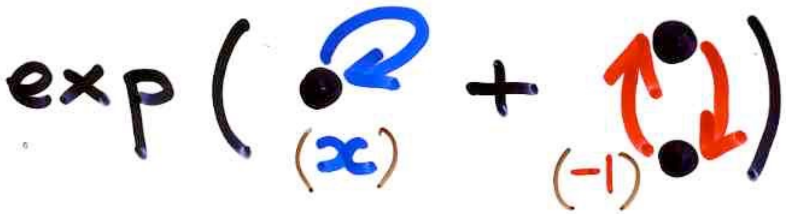
Hermite polynomial

$$H_n(x)$$

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2} dx = \sqrt{\pi}\, 2^n n!\, \delta_{nm}$$

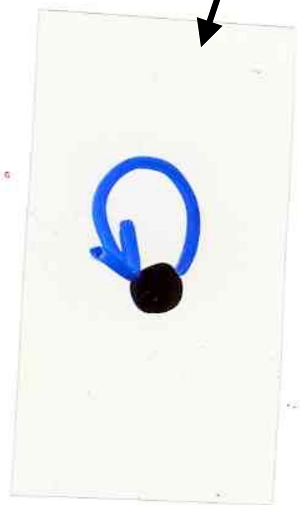
"physicists" Hermite polynomial $H_n(x)$

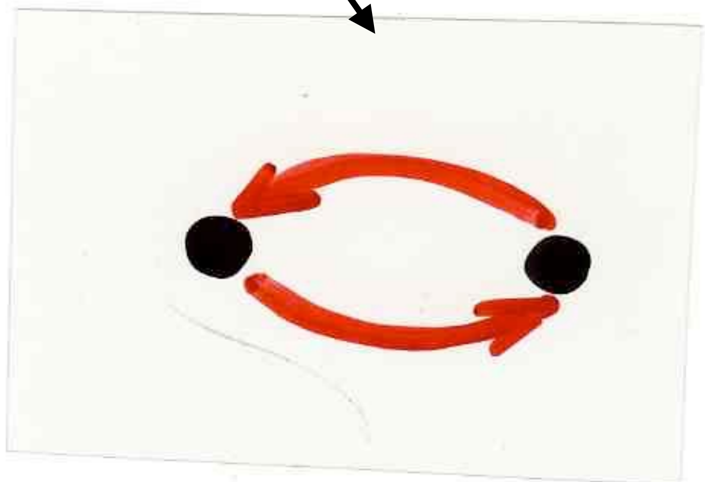
$$\sum_{n\geqslant 0} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

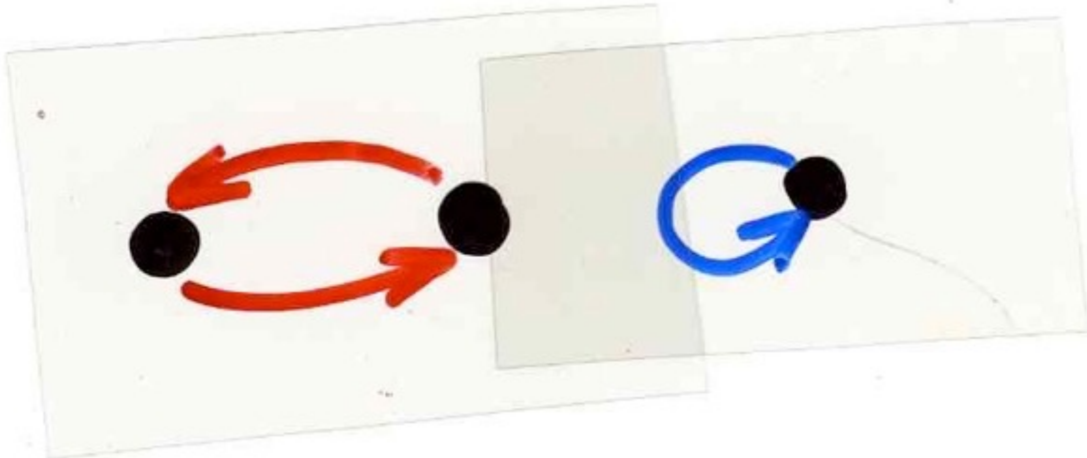


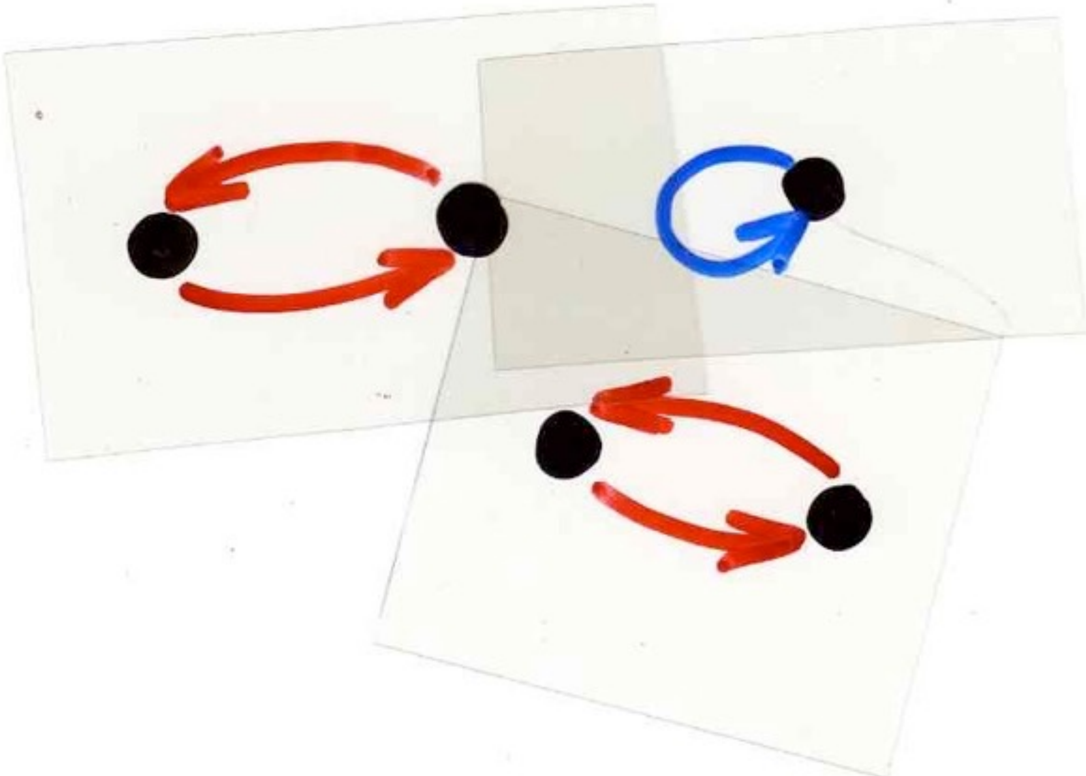
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

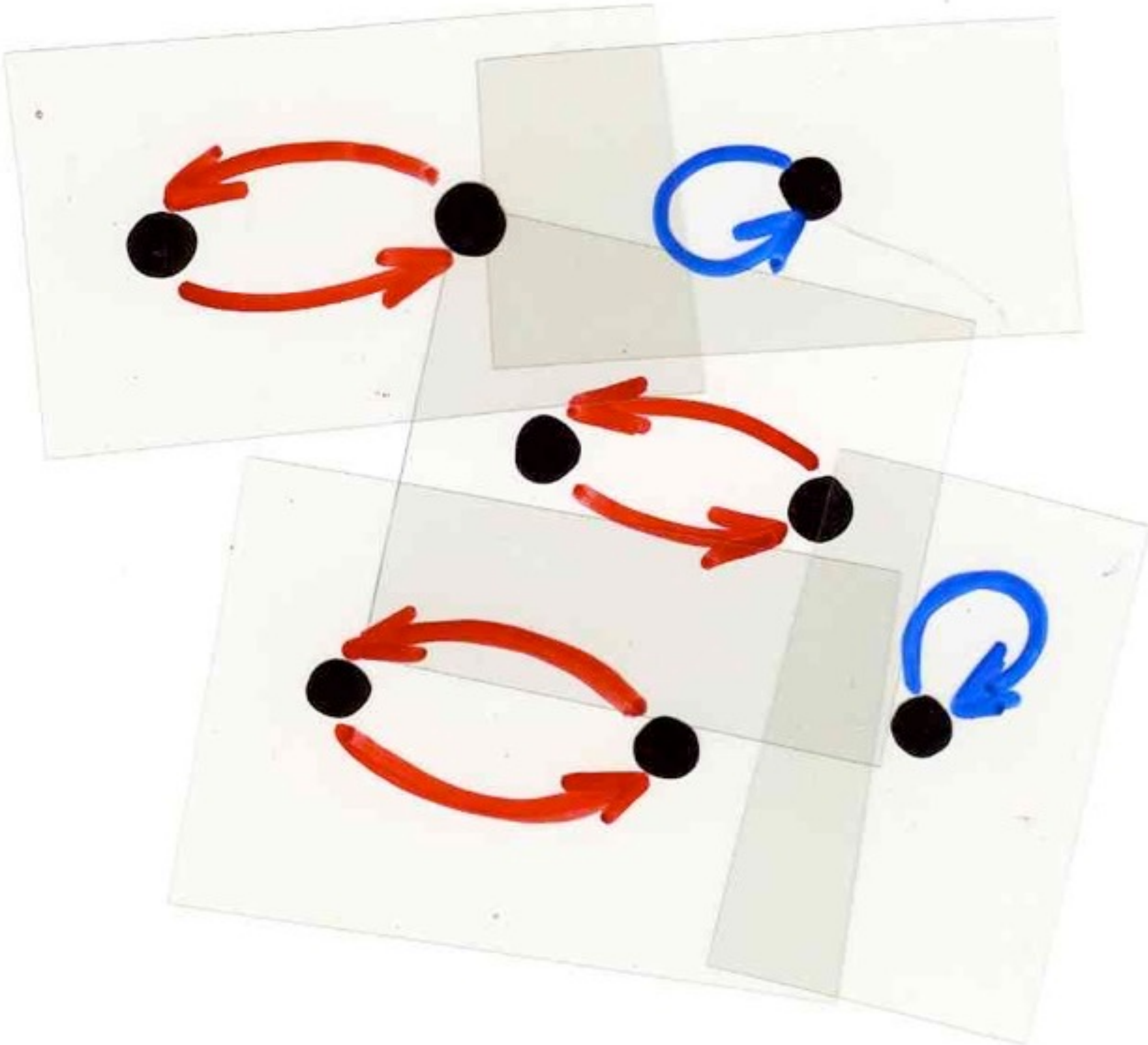
(combinatorial)
Hermite polynomials

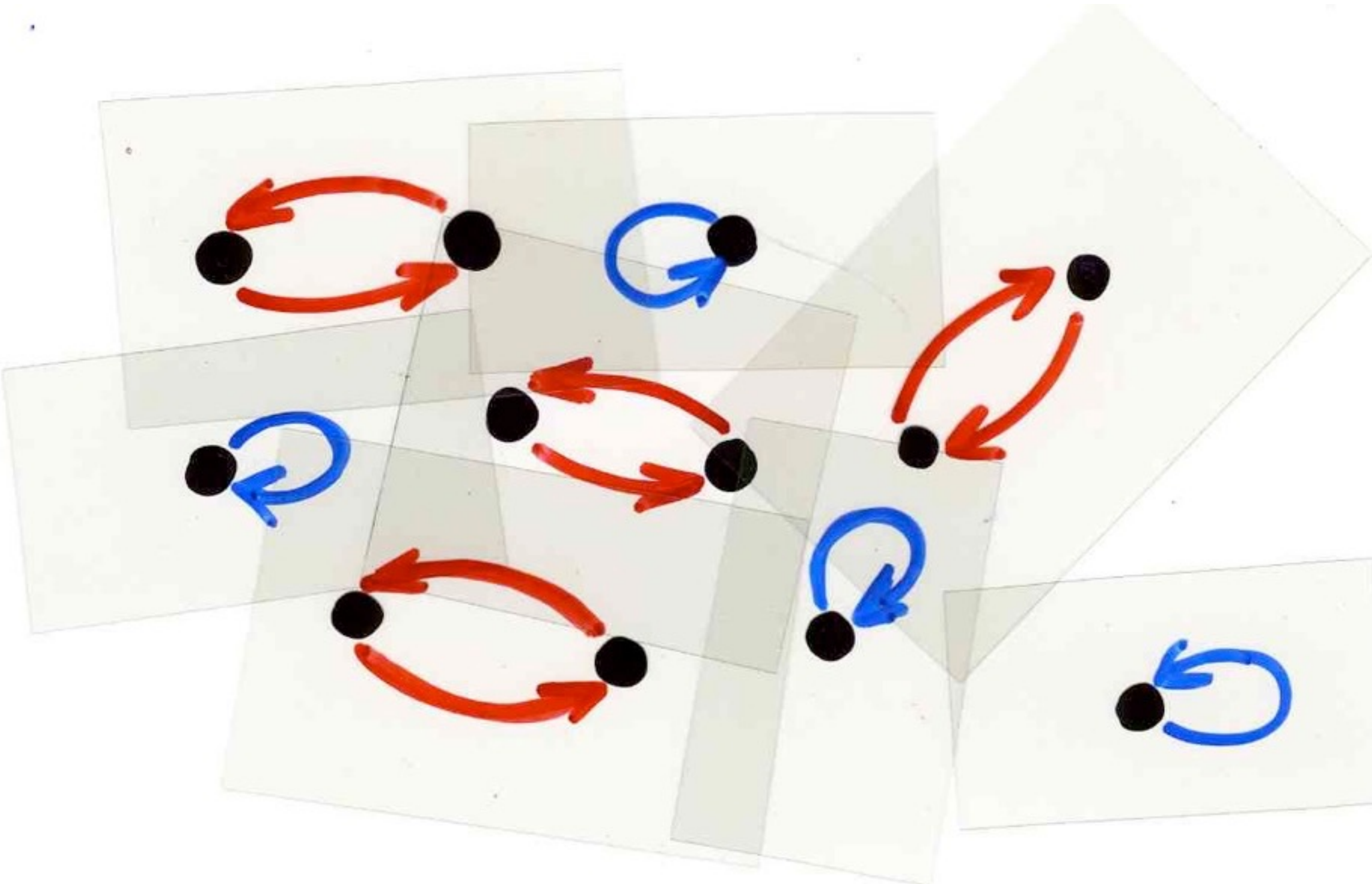


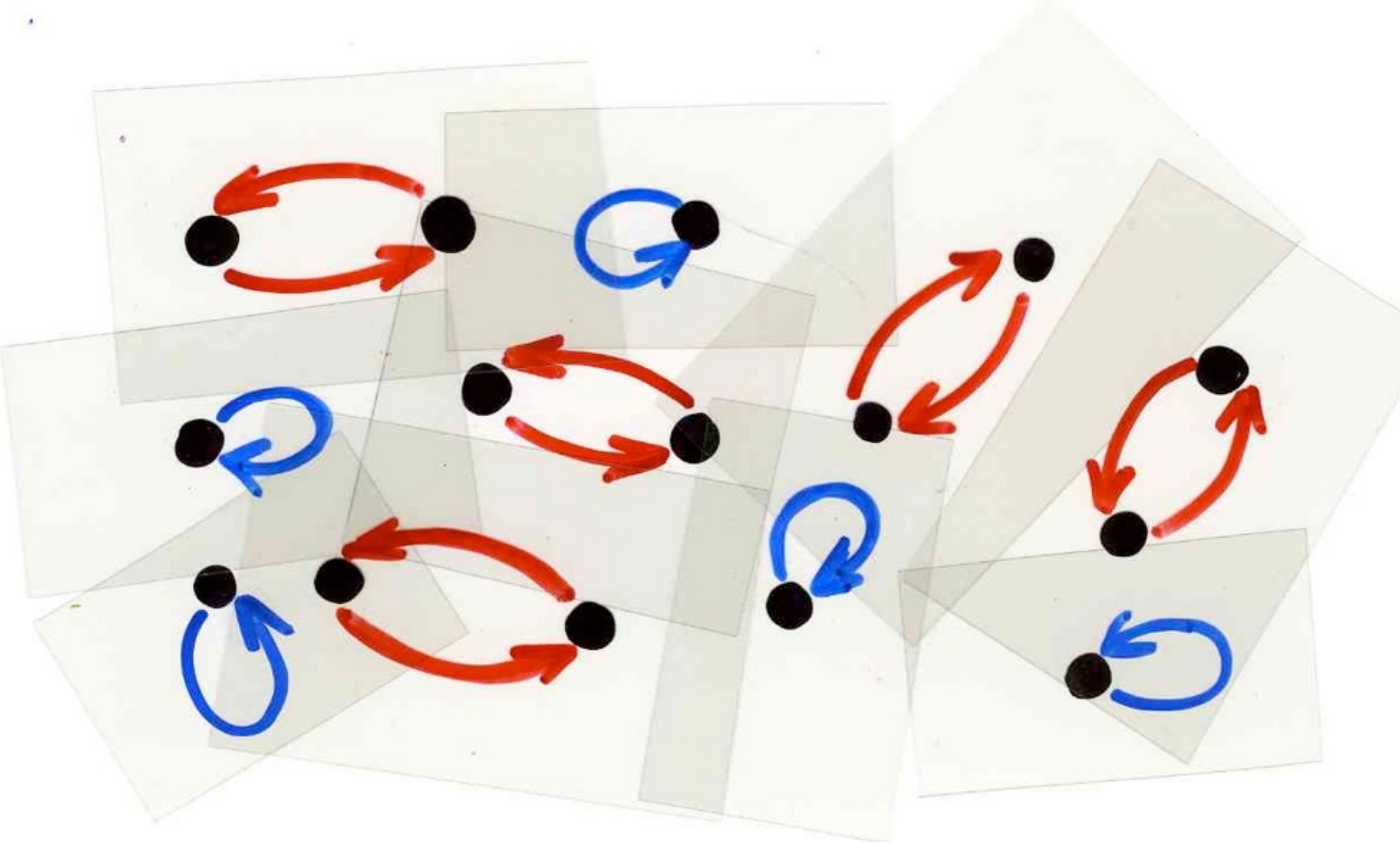




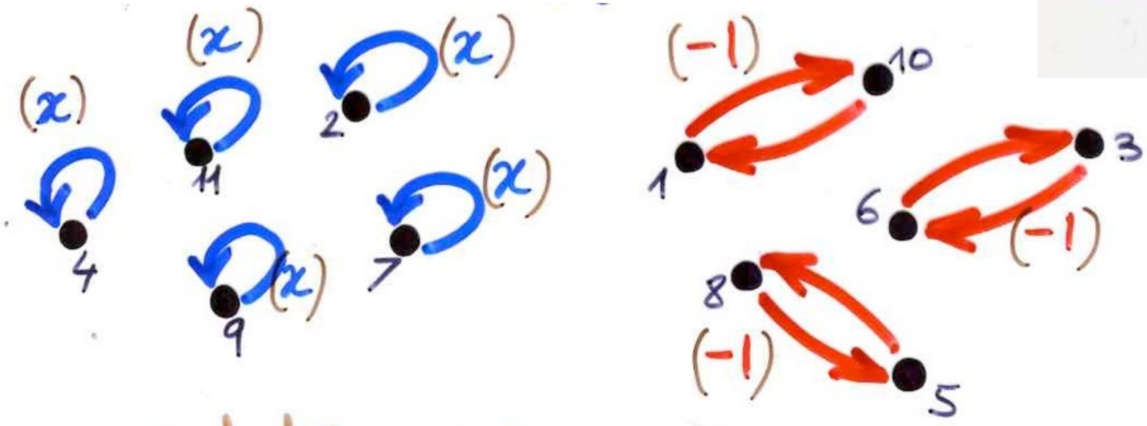








Hermite configuration



$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{fix(\sigma)}$$

(combinatorial)
Hermite polynomials

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 2 & 6 & 4 & 8 & 3 & 7 & 5 & 9 & 1 & 11 \end{pmatrix}$$

$$H_n(x) = \sum_{0 \le 2k \le n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!}\, x^{n-2k}$$

(combinatorial) Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)}\, x^{fix(\sigma)}$$

Mehler identity
for Hermite polynomials

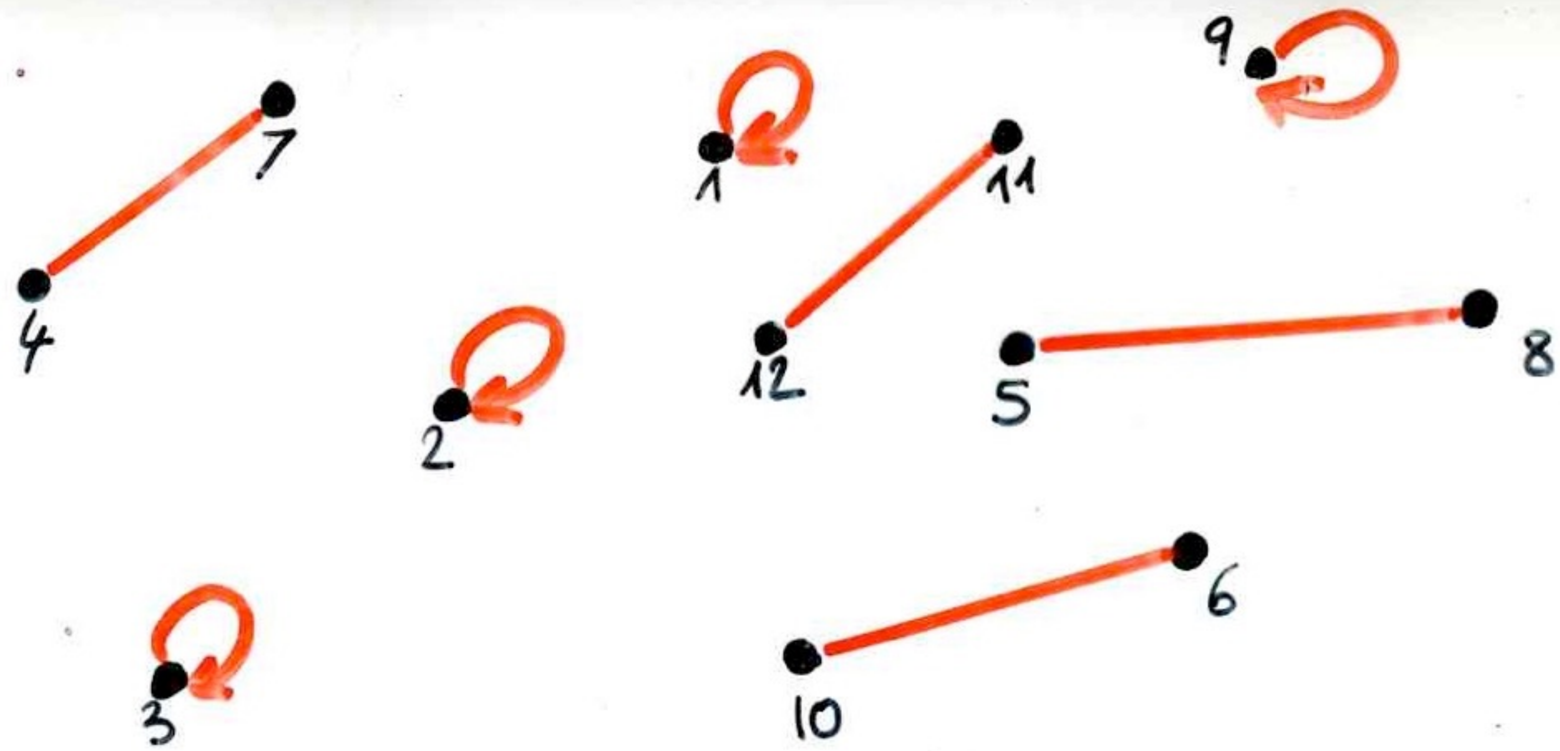
Foata (1978)

Combinatorial proof
of formulae

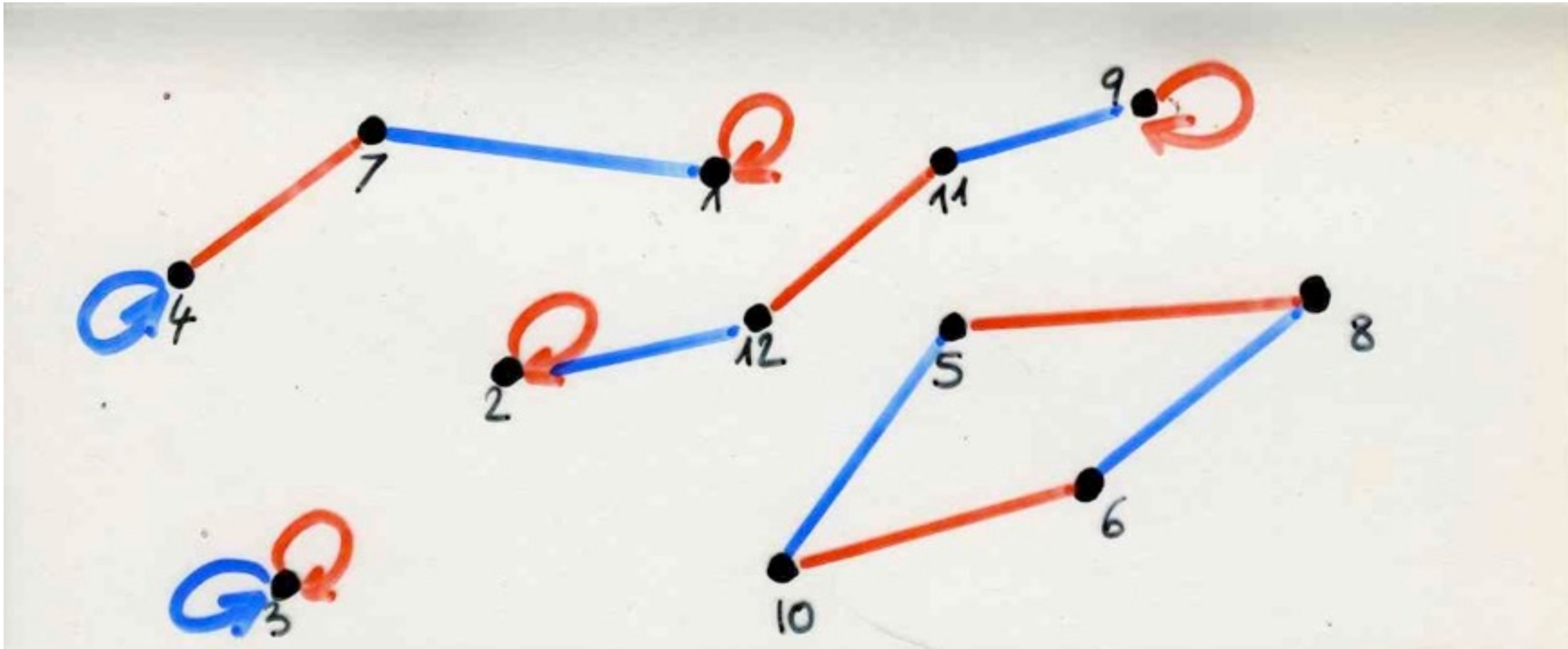
Mehler identity

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$



$$\sum_{n\geq 0} \mathbf{H}_n(x) \qquad \frac{t^n}{n!}$$



$$\sum_{n \geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

$$(1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

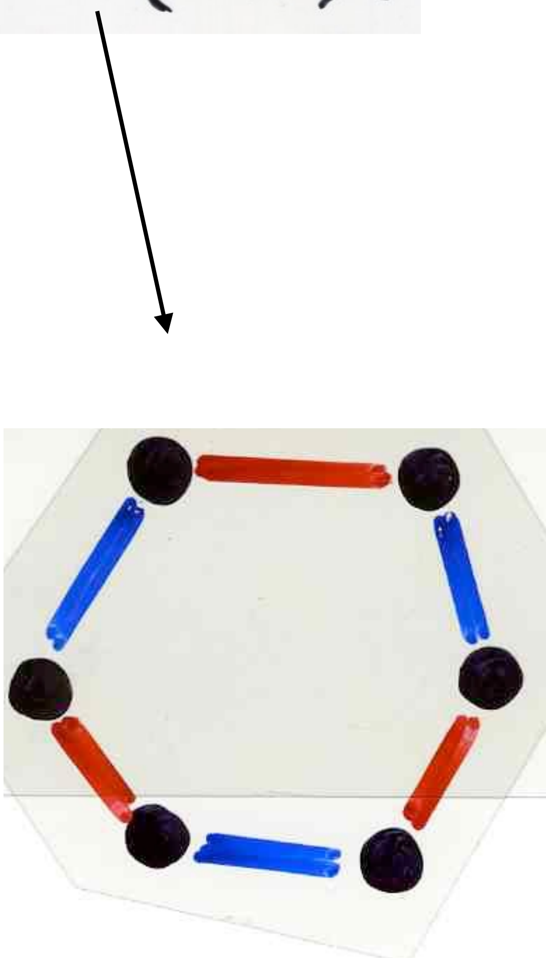
$$\frac{4xyt}{(1-4t^2)}$$

$$\frac{-4x^2t^2}{(1-4t^2)}$$

$$\frac{-4y^2t^2}{(1-4t^2)}$$

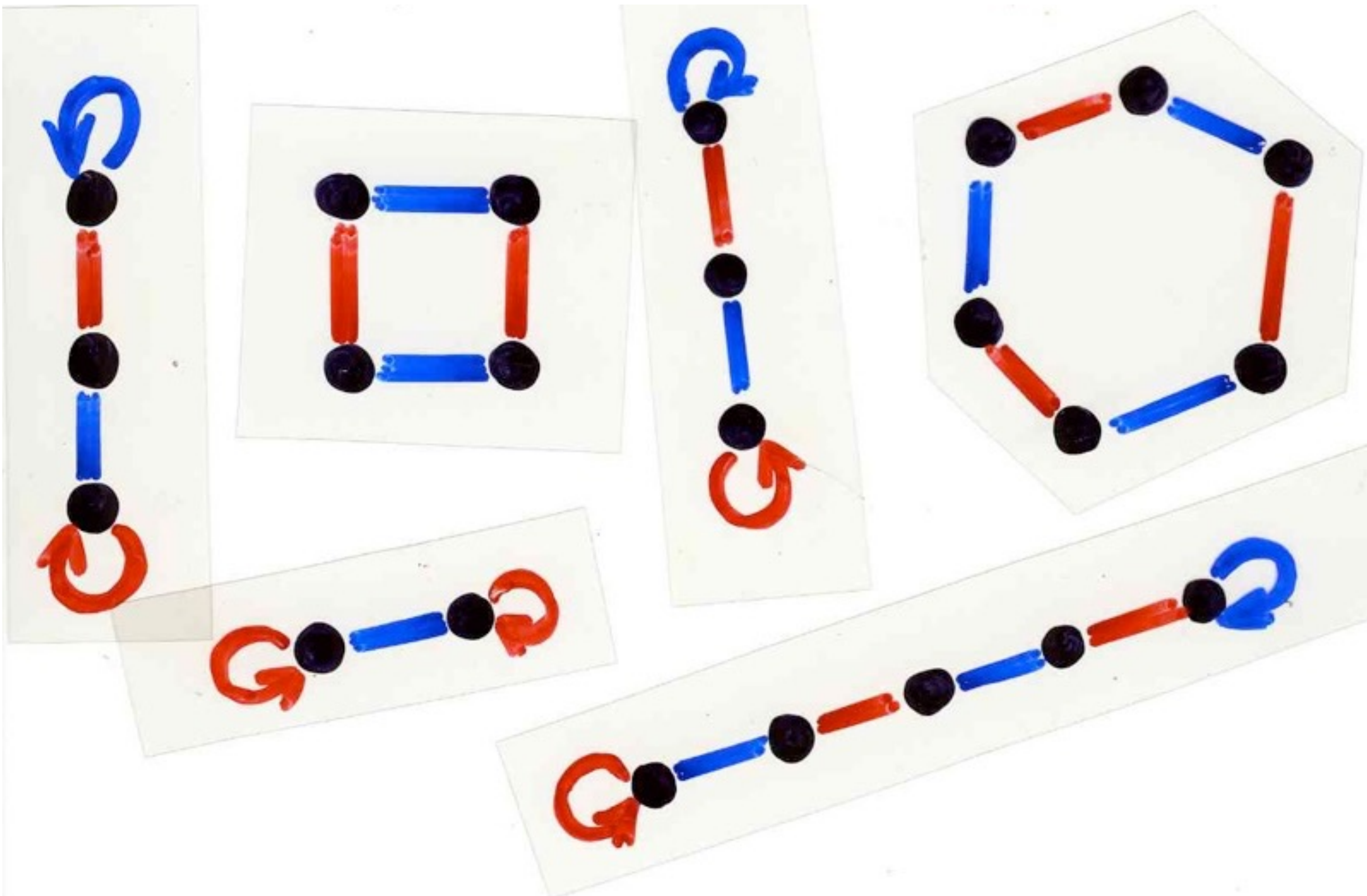
$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}\right]$$



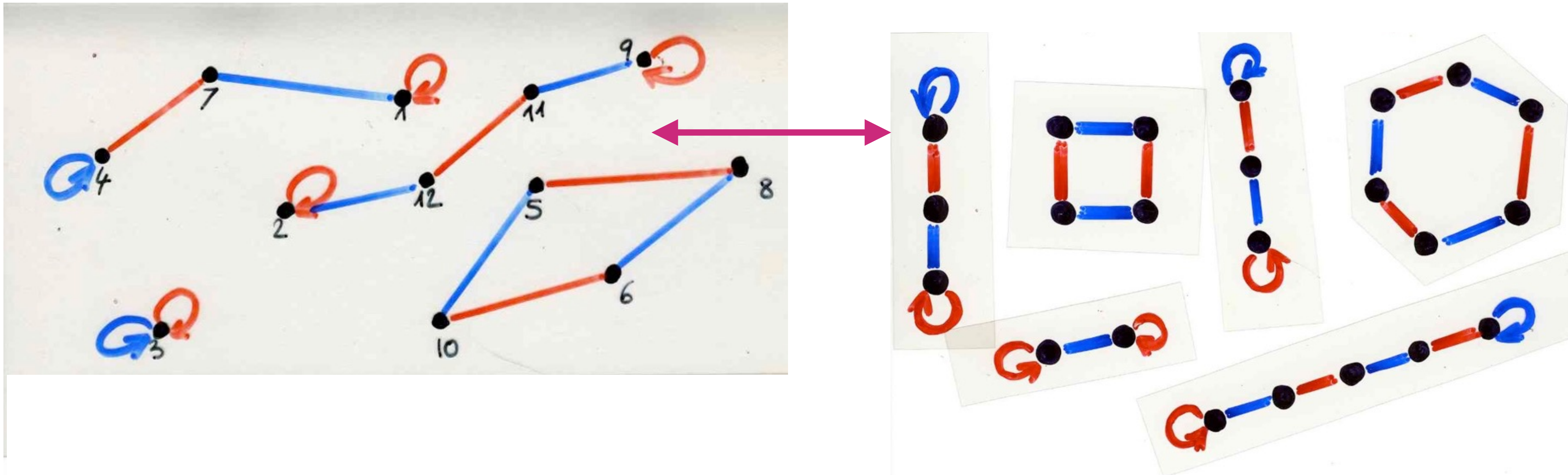
$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}\right]$$

$$\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



$$\sum_{n\geqslant 0} \mathbf{H}_n(x)\,\mathbf{H}_n(y)\,\frac{t^n}{n!}$$

$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$



Laguerre polynomials

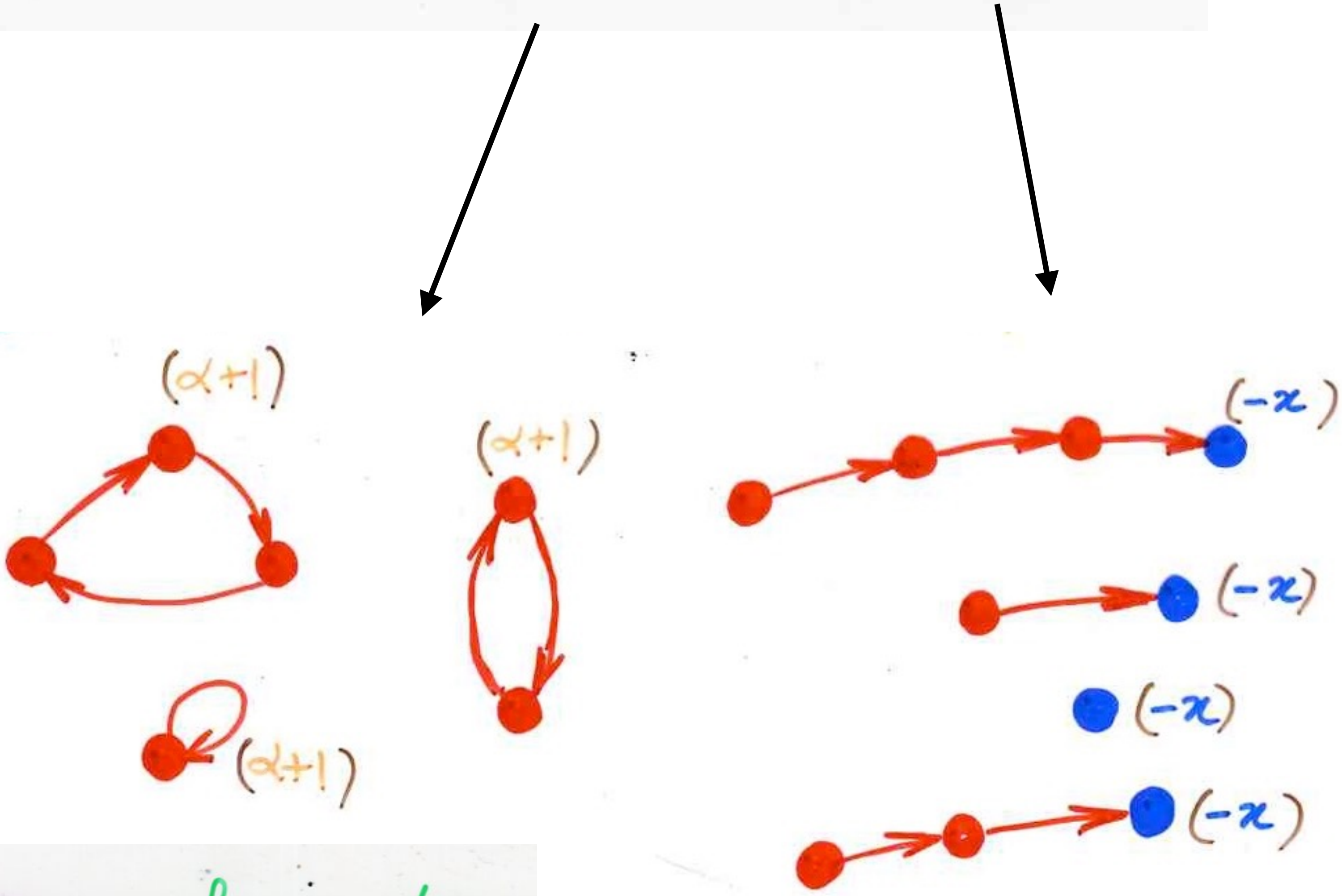
valued combinatorial objects

weight function

Laguerre polynomials

$$\int_0^\infty L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, e^{-x} x^{\alpha}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\, \delta_{mn}$$

$$\sum_{n \geq 0} \tilde{L}_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$



Laguerre configuration

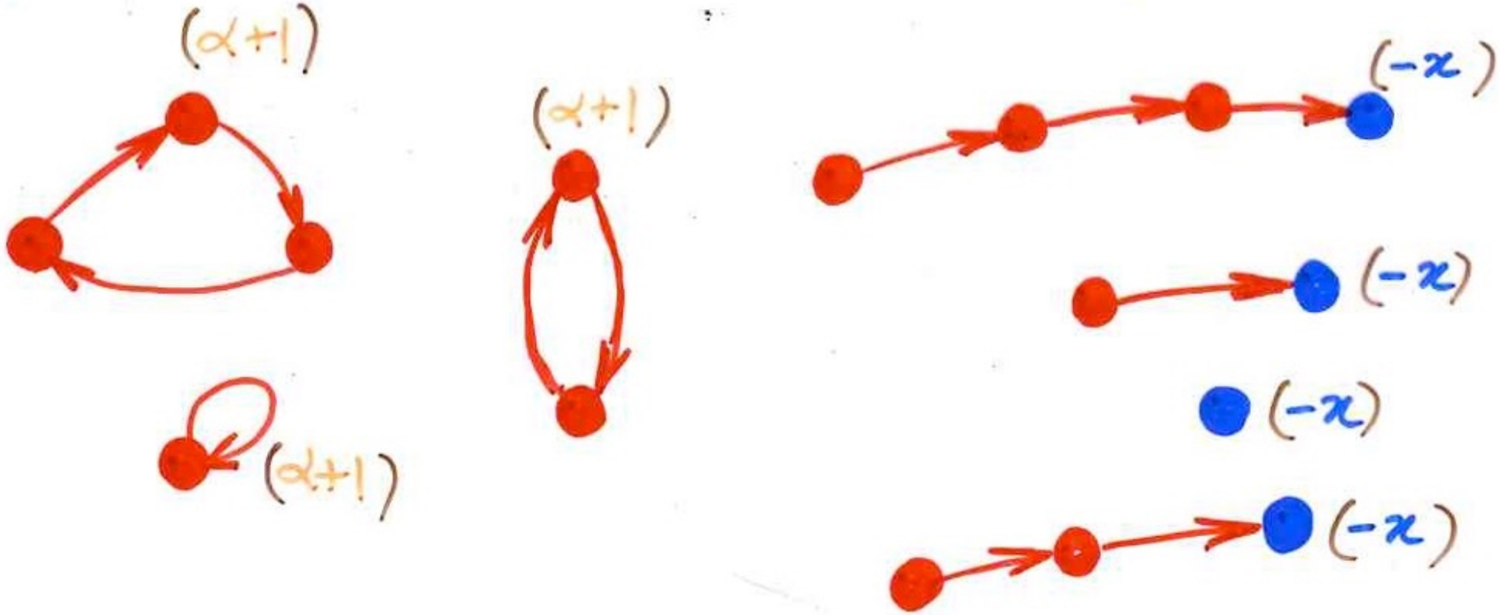
$$L_n^{\alpha}(x) = \sum_{LC} v(LC)$$

LC: Laguerre configurations on $[1,n]$

$$v(LC) = (\alpha+1)^{i}(-x)^{j}$$

i = number cycles 

j = number of chains 

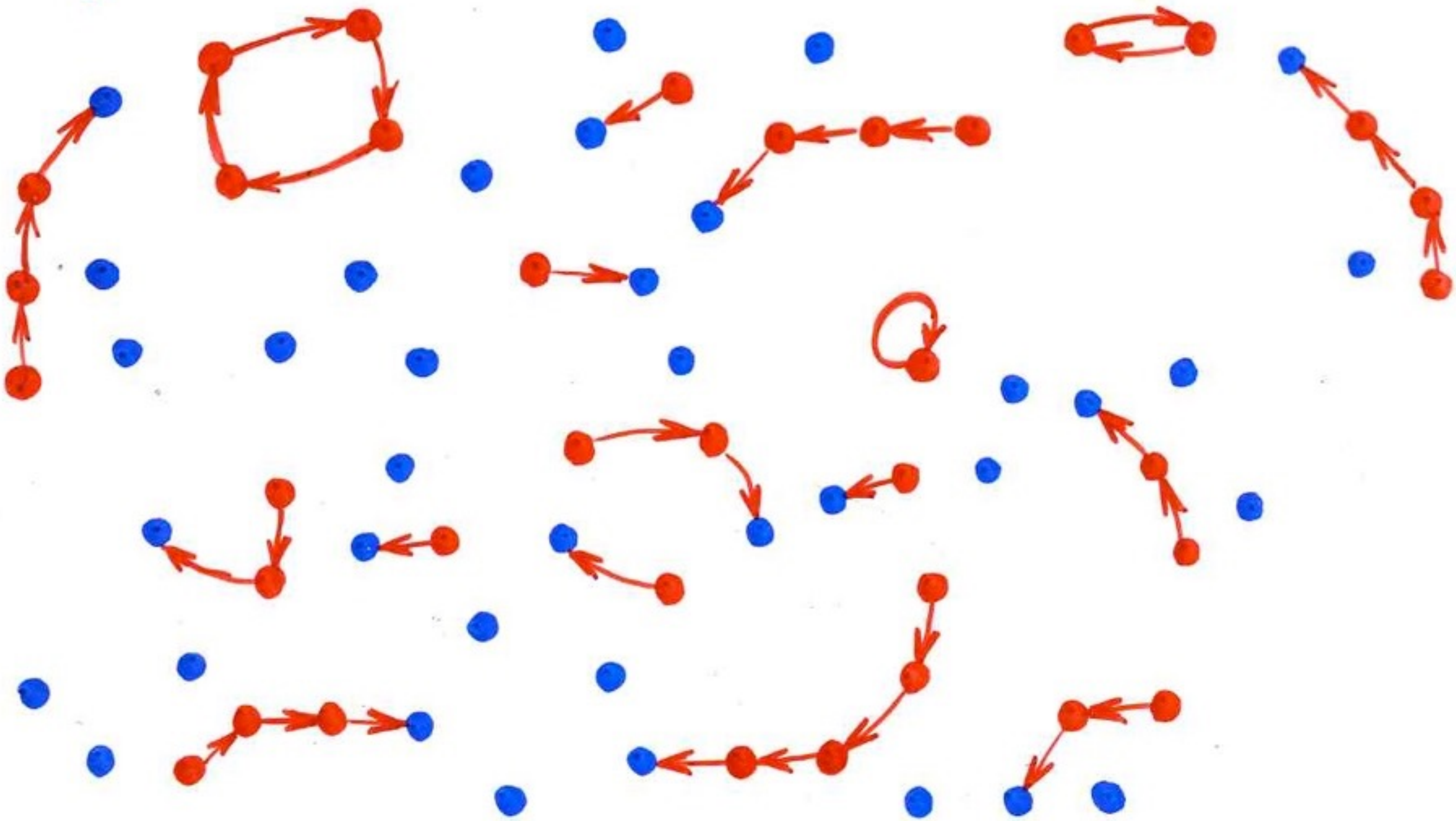


Jacobi polynomials

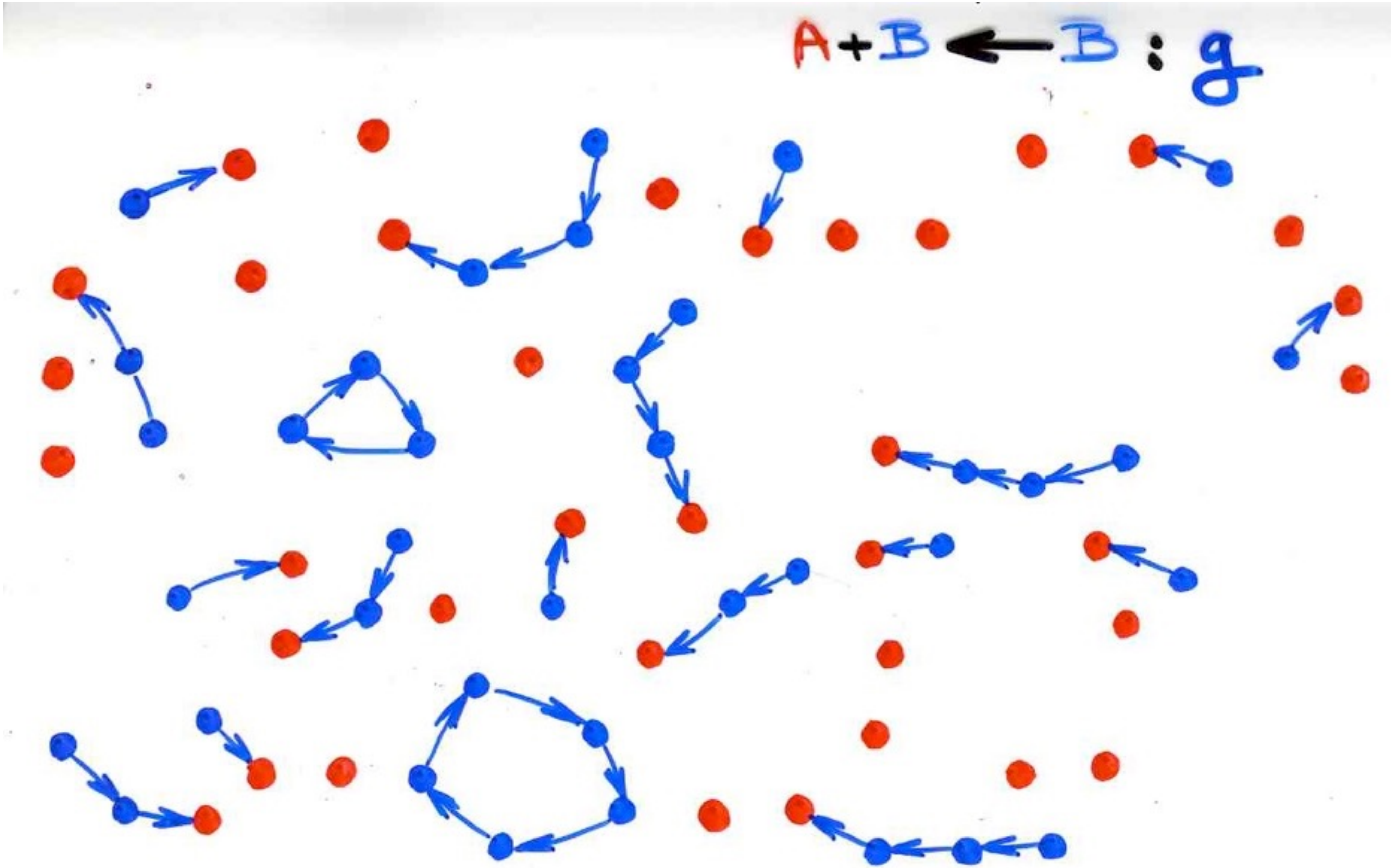
Foata, Leroux (1983)

(A, B)

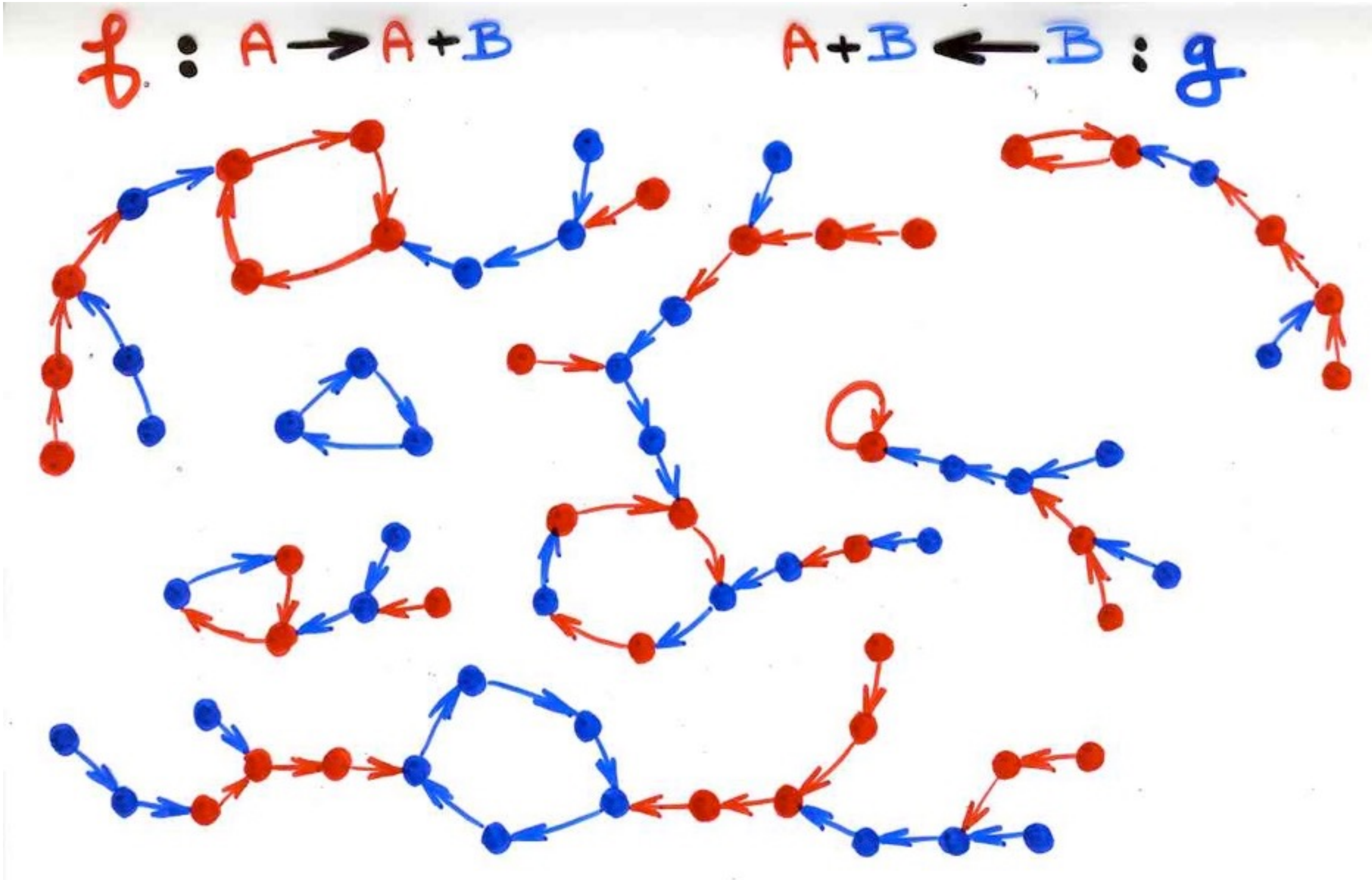
$f : A \to A + B$



(A, B)

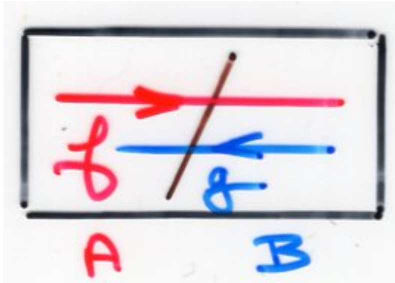


(A, B)

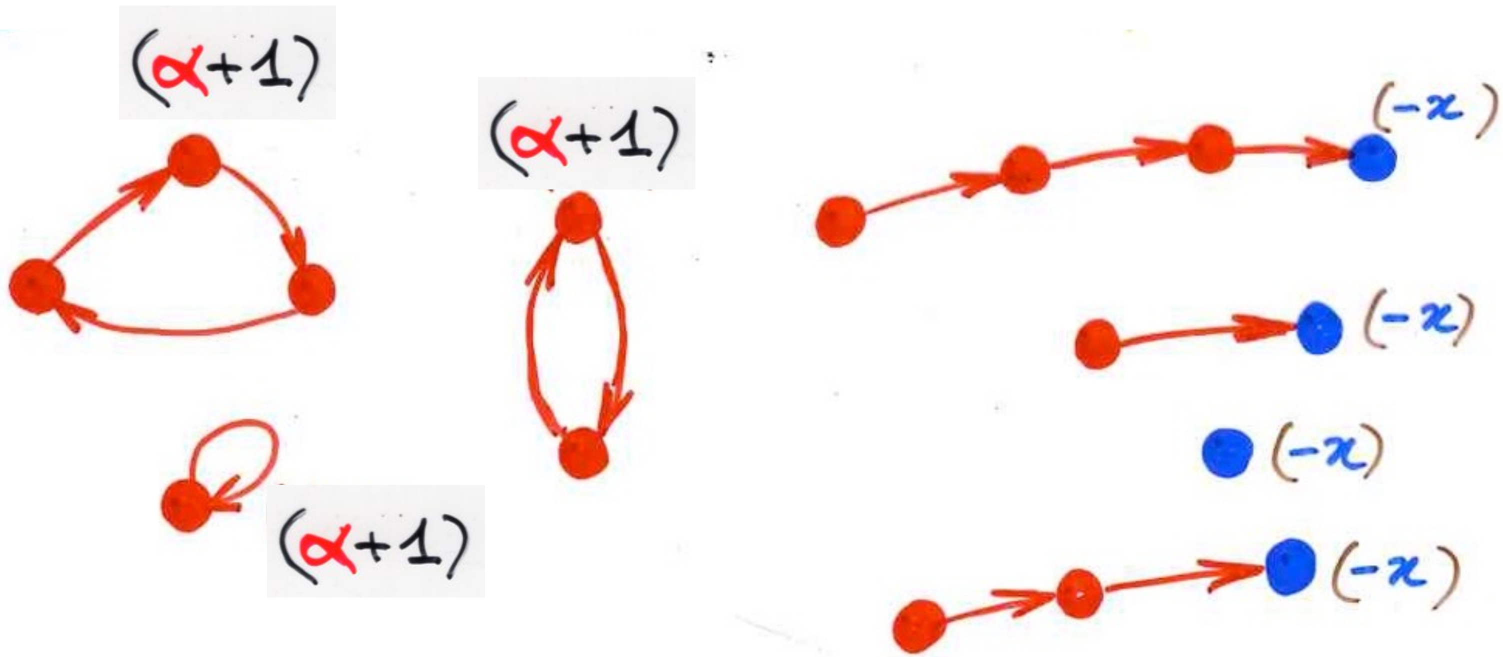


Jacobi configurations

$$\mathbf{J}[A,B] = \mathbf{L}[A,B] \times \mathbf{L}[B,A]$$

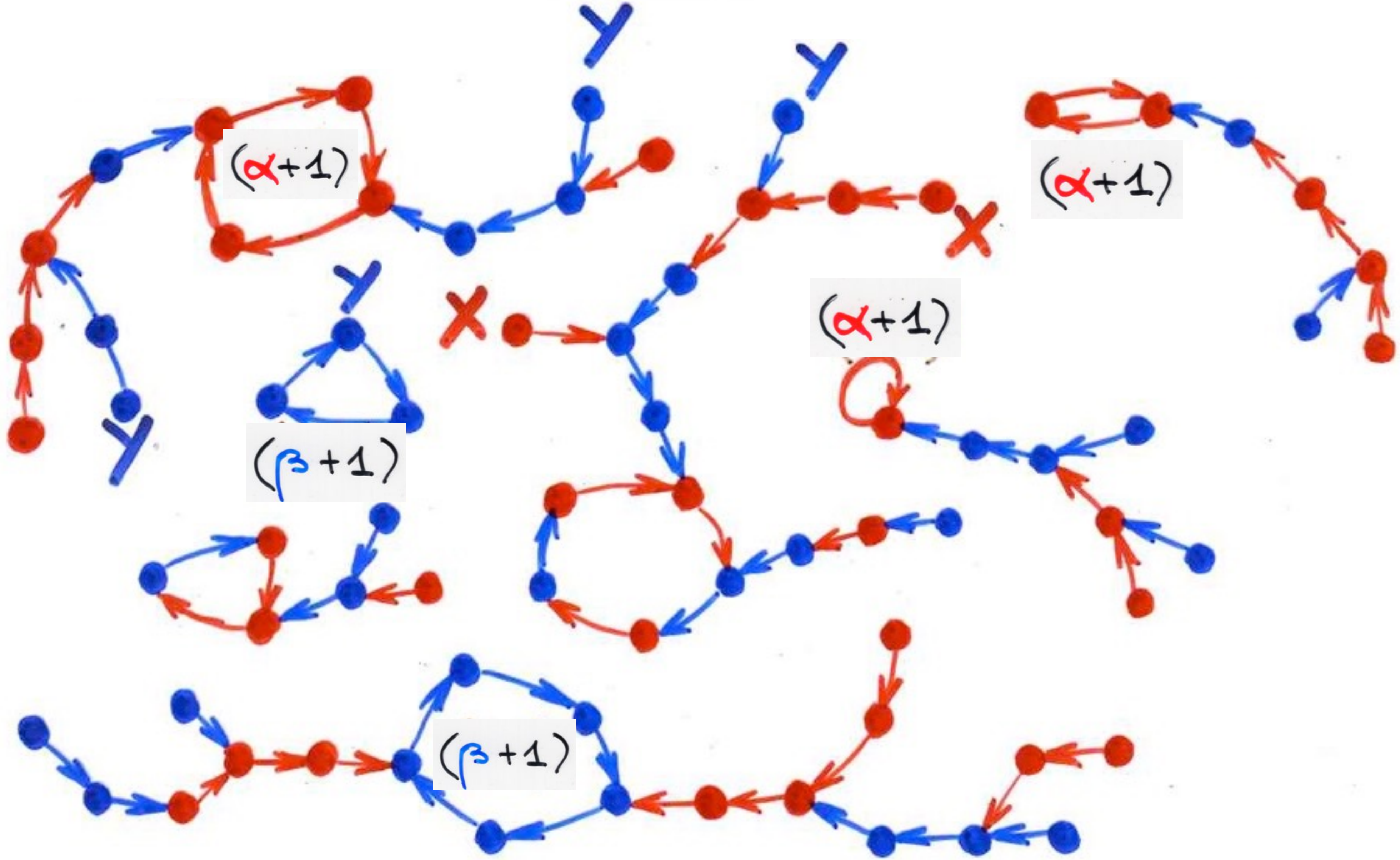


Laguerre configuration



$(f, g) \in \mathbf{L}[A, B]$

$$w(f, g) = (\alpha + 1)^{cyc(f)} (\beta + 1)^{cyc(g)} x^{|A|} y^{|B|}$$



$(f, g) \in \mathbf{L}[A, B]$

$$w(f,g) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)} \quad x^{|A|} y^{|B|}$$

Proposition $\quad |E| = n \quad (A, B)$

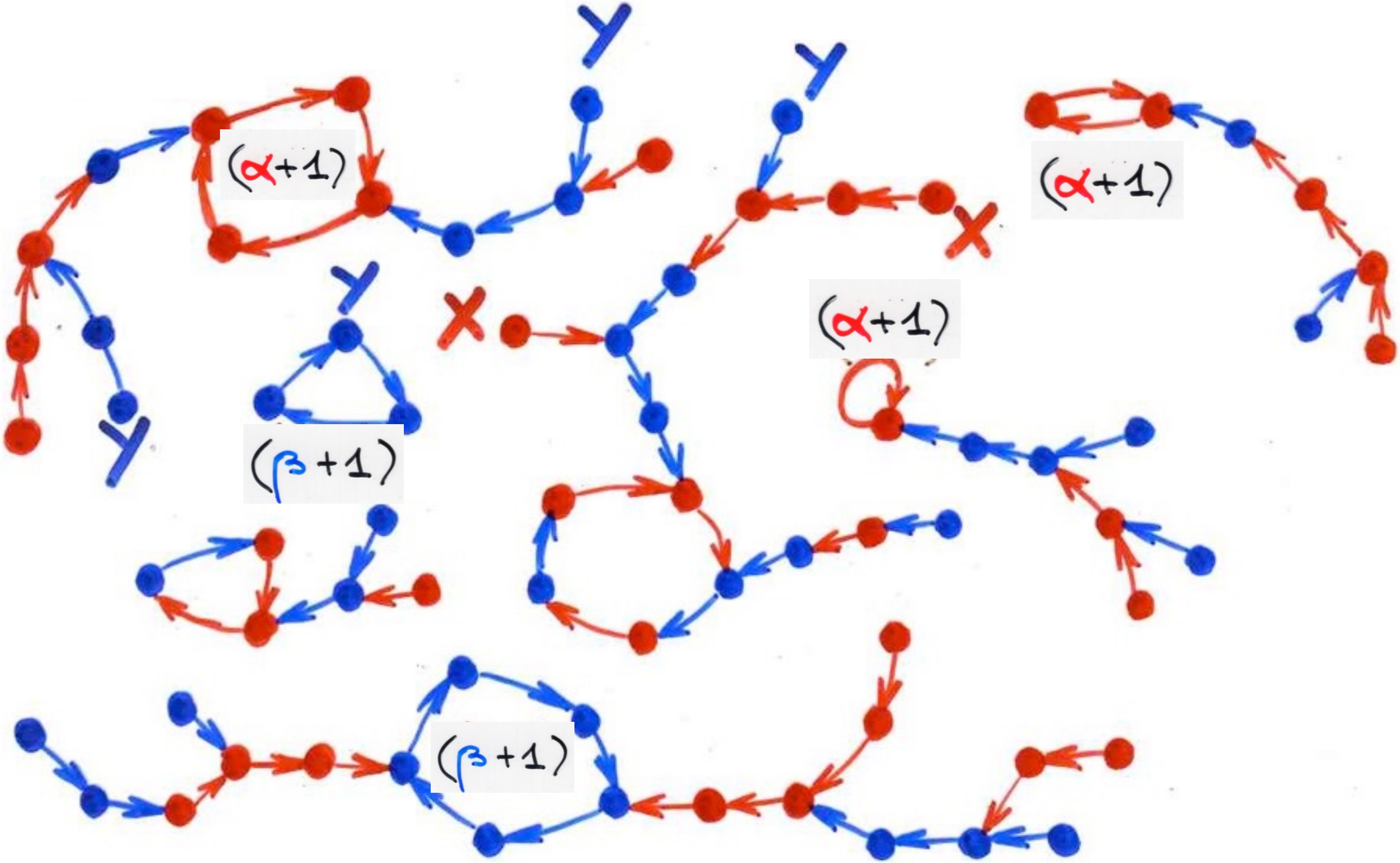
$$\mathcal{P}_n^{(\alpha,\beta)}(x,y) = \sum_{(f,g) \in \mathbf{J}[A,B] = \mathbf{L}[A,B] \times \mathbf{L}[B,A]} w(f,g)$$

Proposition $\quad \mathcal{R} = \left[1 - 2(x+y)t + (x-y)^2 t^2\right]^{1/2}$

$$\sum_{n \geq 0} \mathcal{P}_n^{(\alpha,\beta)}(x,y) \frac{t^n}{n!} = 2^{\alpha+\beta} \mathcal{R}^{-1} \left[1 - (x-y)t + \mathcal{R}\right]^{-\alpha} \left[1 - (y-x)t + \mathcal{R}\right]^{-\beta}$$

$(f, g) \in L[A, B]$

$$w(f, g) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)}\, x^{|A|}\, y^{|B|}$$

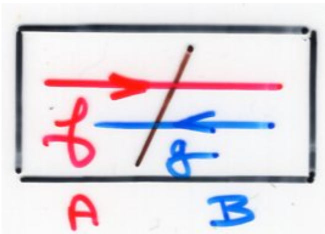


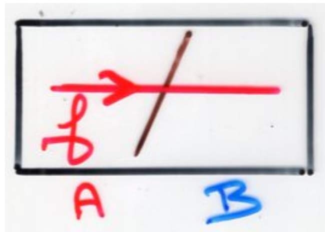
limit formula

example

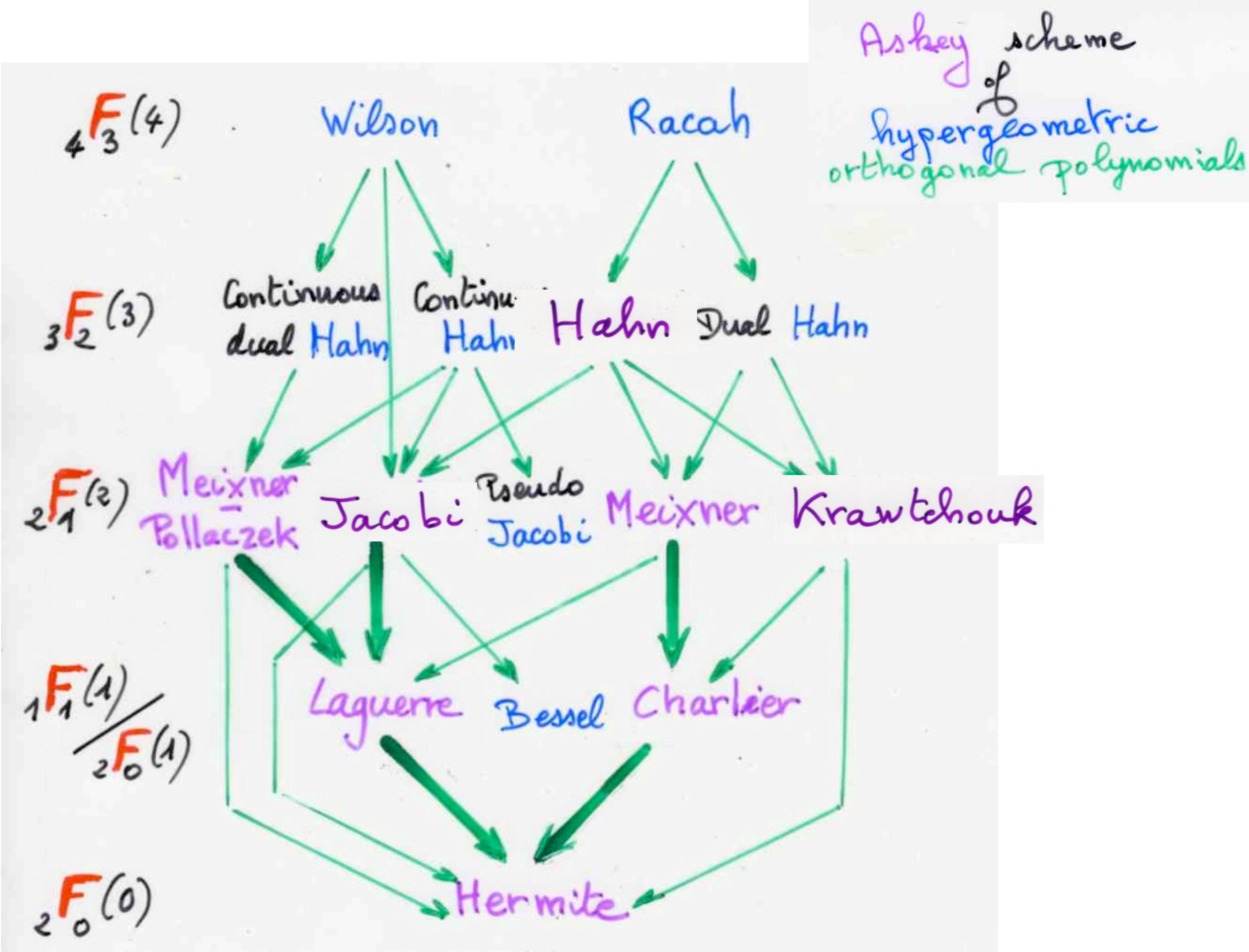
$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

Jacobi → Laguerre

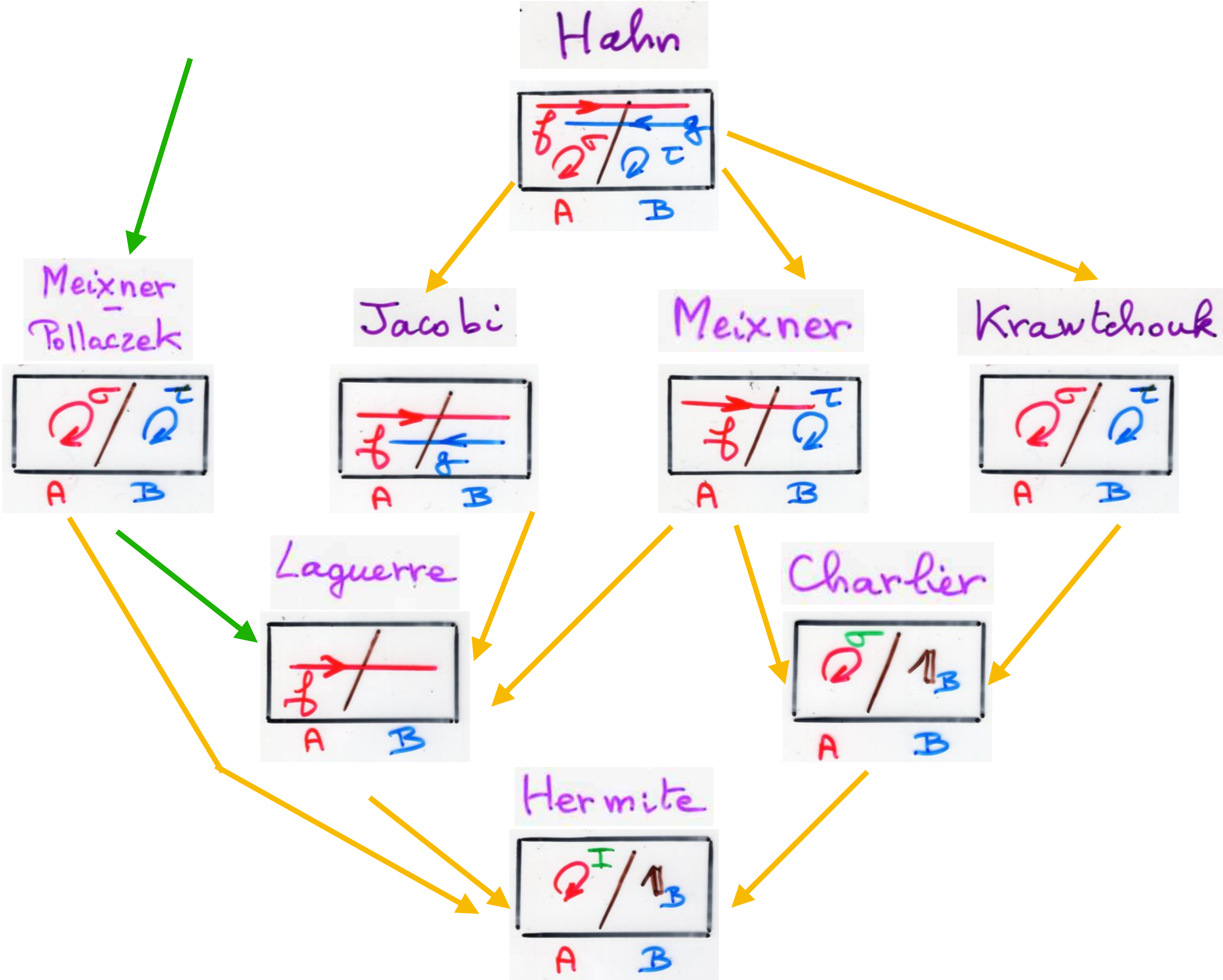




J. Labelle, Y.N. Yeh (1989)



Askey scheme of hypergeometric orthogonal polynomials
4F3(4)
Wilson
Racah
3F2(3)
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
2F1(2)
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
1F1(1) / 2F0(1)
Laguerre
Bessel
Charlier
2F0(0)
Hermite


Hahn
f
g
σ
τ
A
B
Meixner-Pollaczek
σ
τ
A
B
Jacobi
f
g
A
B
Meixner
f
τ
A
B
Krawtchouk
σ
τ
A
B
Laguerre
f
A
B
Charlier
σ
1B
A
B
Hermite
I
1B
A
B

(formal) orthogonal polynomials

$$f\big(P(x)Q(x)\big) = \int_{\mathbb{R}} P(x)Q(x)\, d\mu(x)$$

measure μ on $\mathbb{R}$

$$f(x^n) = \int_{\mathbb{R}} x^n \, d\mu(x)$$

moments problem

$$f(x^n) = \mu_n$$

moments

$\mathbb{K}$ ring

field $\mathbb{R}$, $\mathbb{C}$
or $\mathbb{Q}[\alpha, \beta, \ldots]$

$\mathbb{K}[x]$
polynomials in x

$\{P_n(x)\}_{n\geq 0}$
sequence of polynomials

$P_n(x) \in \mathbb{K}[x]$.

Definition $\{P_n(x)\}_{n\geq 0}$ sequence of polynomials orthogonal iff $\exists$ $f : \mathbb{K}[x] \to \mathbb{K}$ linear functional

(i) $\deg(P_n) = n$, for $n \geq 0$

degree

(ii) $f(P_k P_l) = 0$, for $k \neq l \geq 0$

(iii) $f(P_k^2) \neq 0$, for $k \geq 0$

$$f(x^n) = \mu_n$$

moments

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

moments of 1 st kind
(Tchebychev) 2 nd kind

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

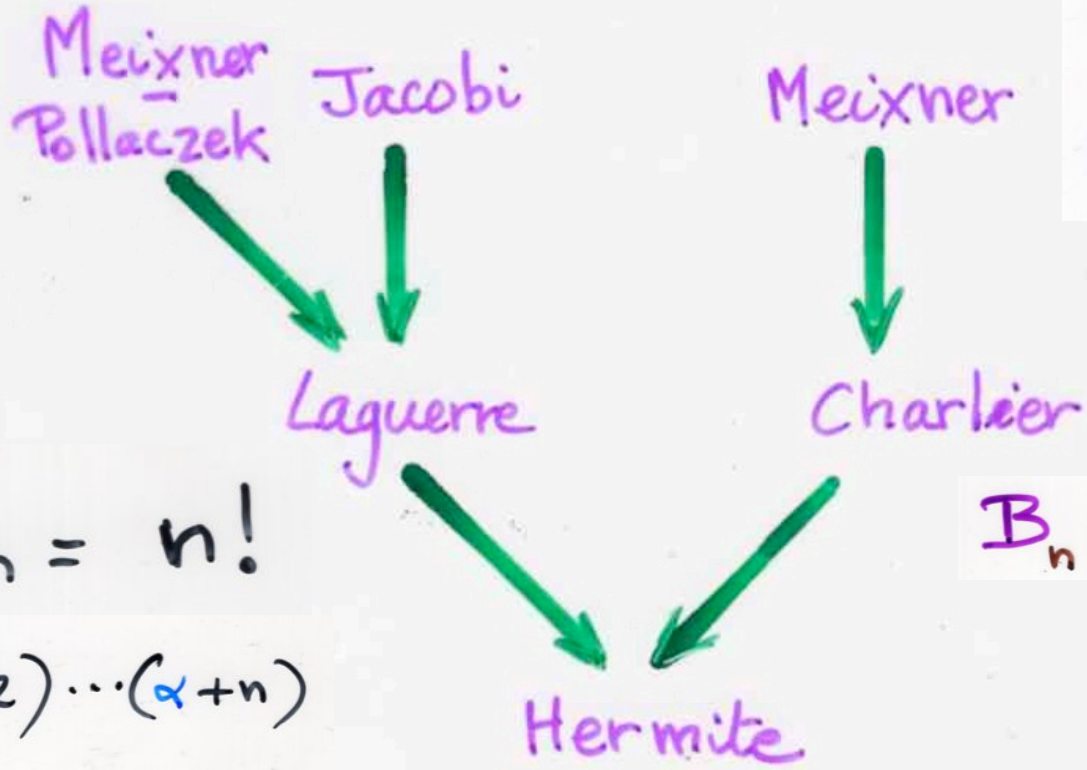
Catalan number

$$\frac{2}{\pi}\int_{-1}^{+1} x^{2n} (1-x^2)^{1/2}\, dx = \frac{1}{4^n} C_n$$

Catalan

E_{2n}

secant number



number of ordered partitions

$\mu_n = n!$

$(\alpha+1)(\alpha+2)\cdots(\alpha+n)$

B_n Bell number

number of partitions

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

Combinatorial theory of orthogonal polynomials

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k \geq 1$

(formal) Favard's Theorem

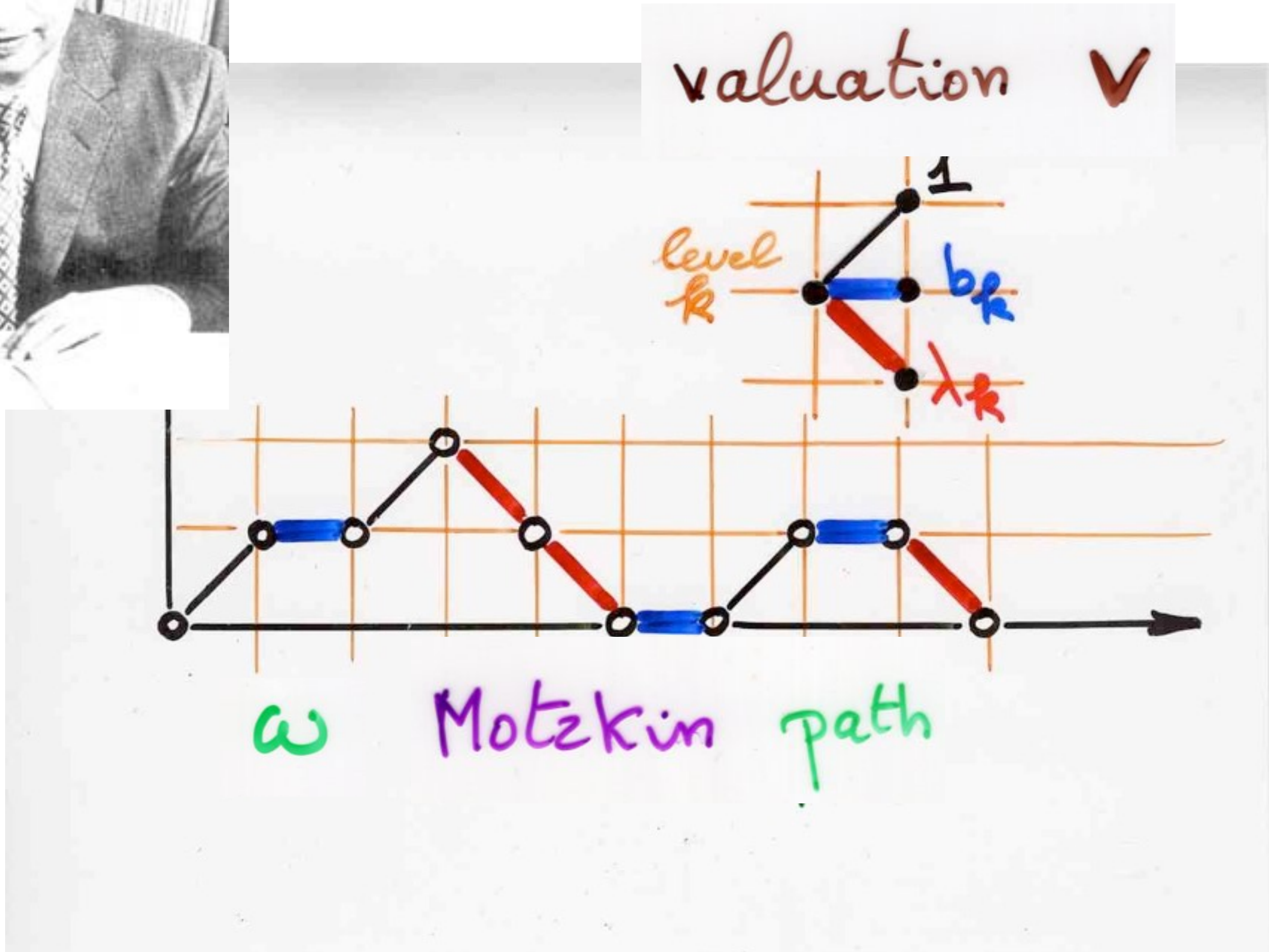
3-terms linear recurrence relation

$\Rightarrow$ orthogonality

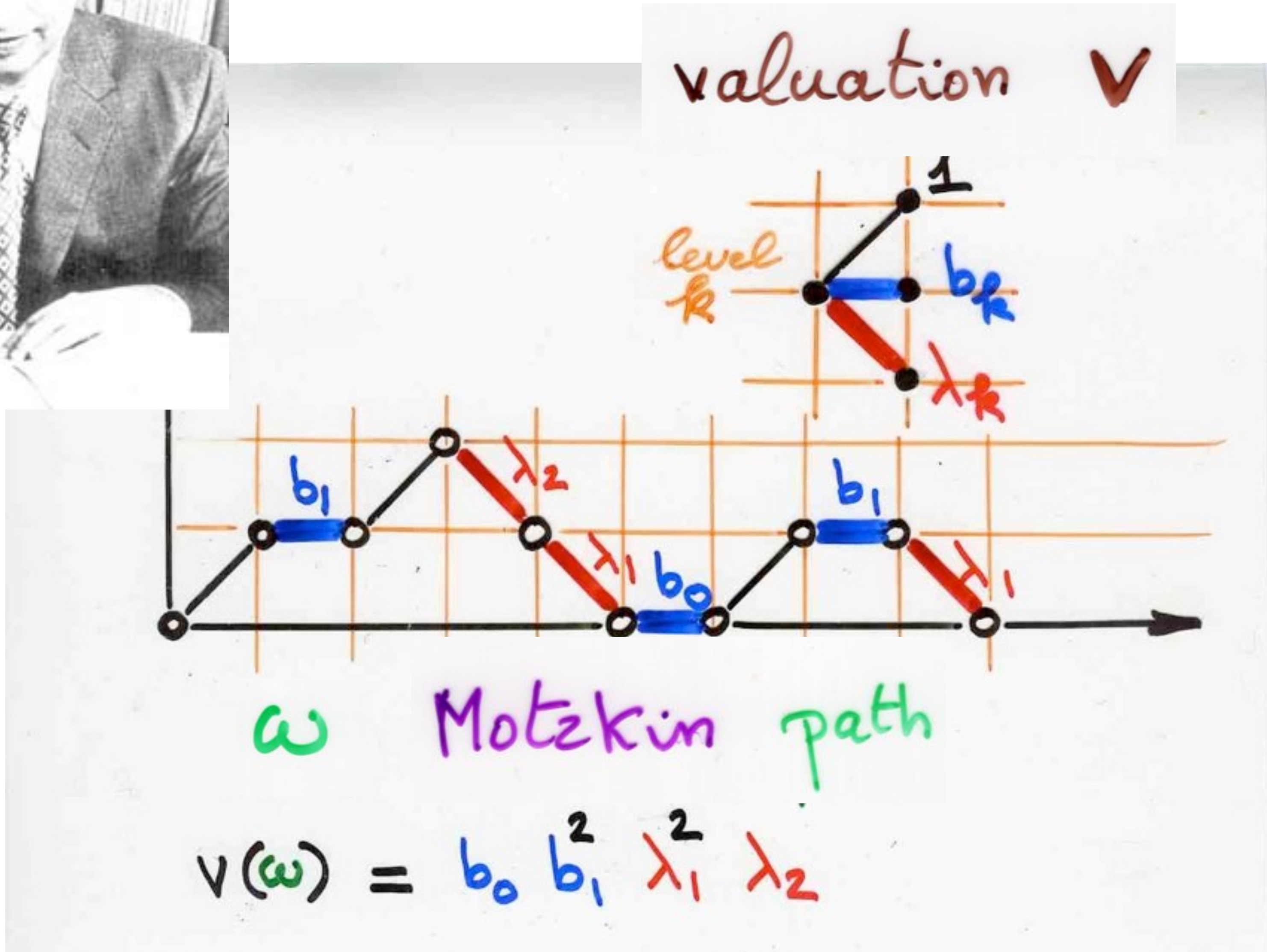
$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$
ring

μ_n ?



valuation v
1
level
k
b_k
λ_k
ω Motzkin path


valuation v
1
level
k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k \geq 1$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

length

$$f(x^n) = \mu_n$$

combinatorial proof

3-terms recurrence relation
implies orthogonaliy

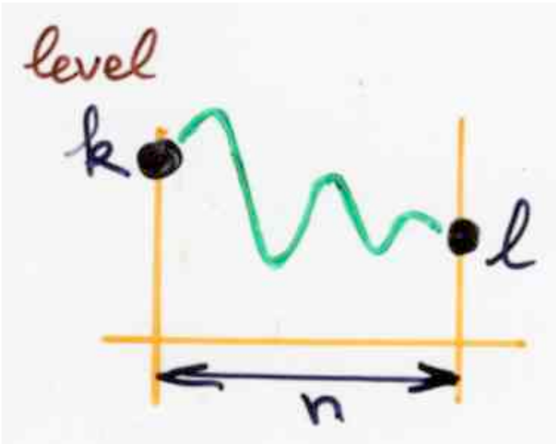
Theorem

(X.V. 1983)

$$f(P_k P_l x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_l$$

"Motzkin path" ω

$|\omega| = n$ level $k \rightsquigarrow l$

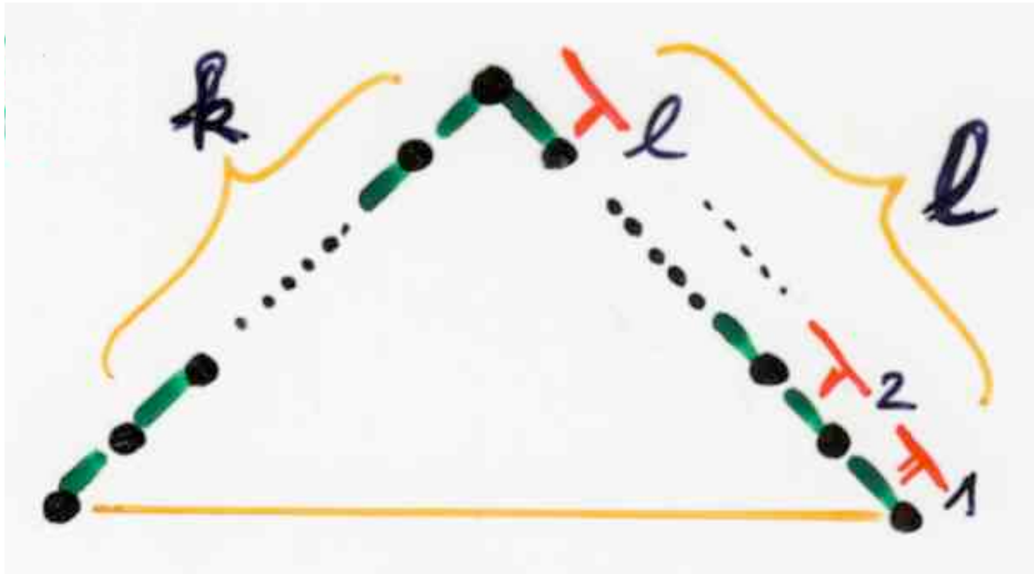


Corollary

$\Rightarrow$ orthogonality

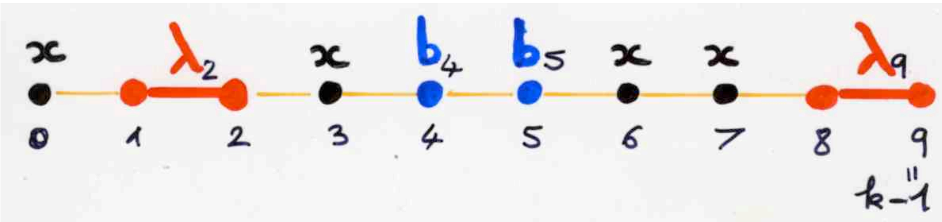
$n=0$

$$f(P_k P_\ell) = 0 \quad k \neq \ell$$
$$= \lambda_1 \cdots \lambda_\ell \quad k = \ell$$



orthogonal polynomial

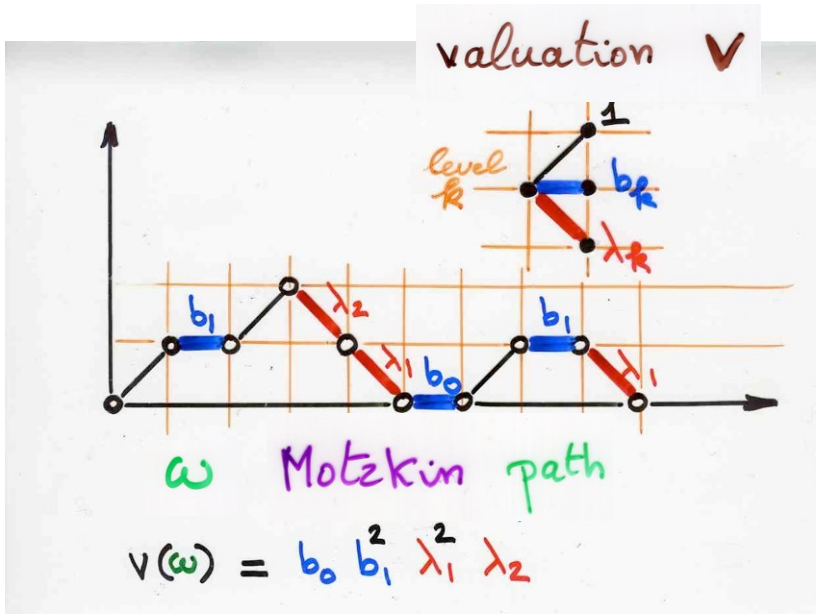
$$\{P_n(x)\}_{n\geq 0}$$



$$f(x^n) = \mu_n$$

moments μ_n

weighted Motzkin paths



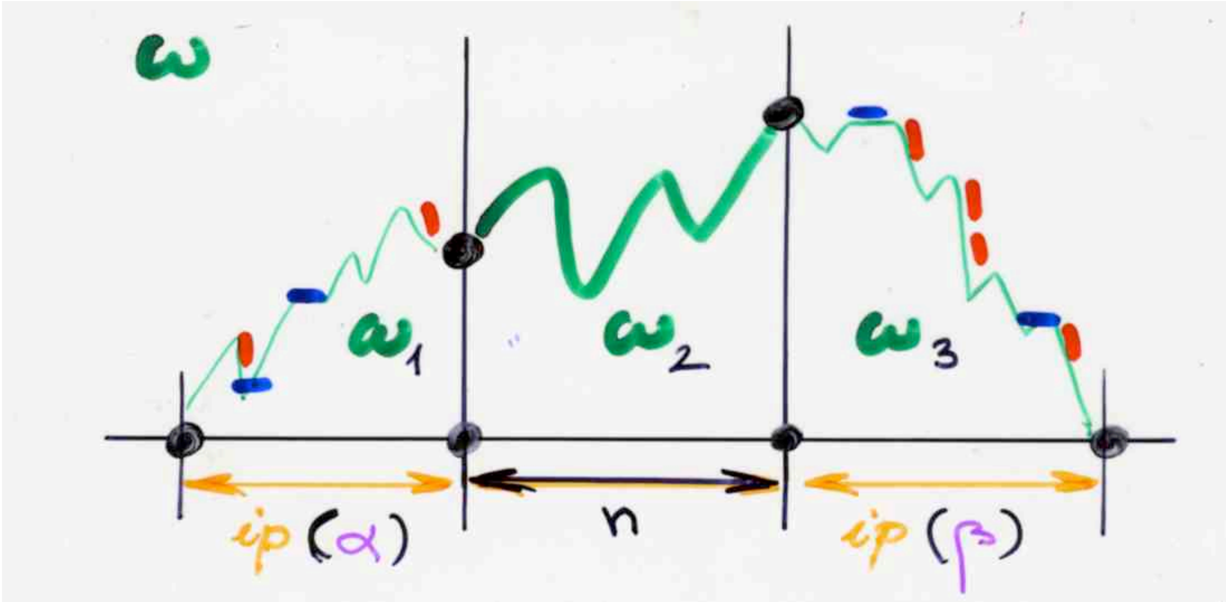
bijective proof

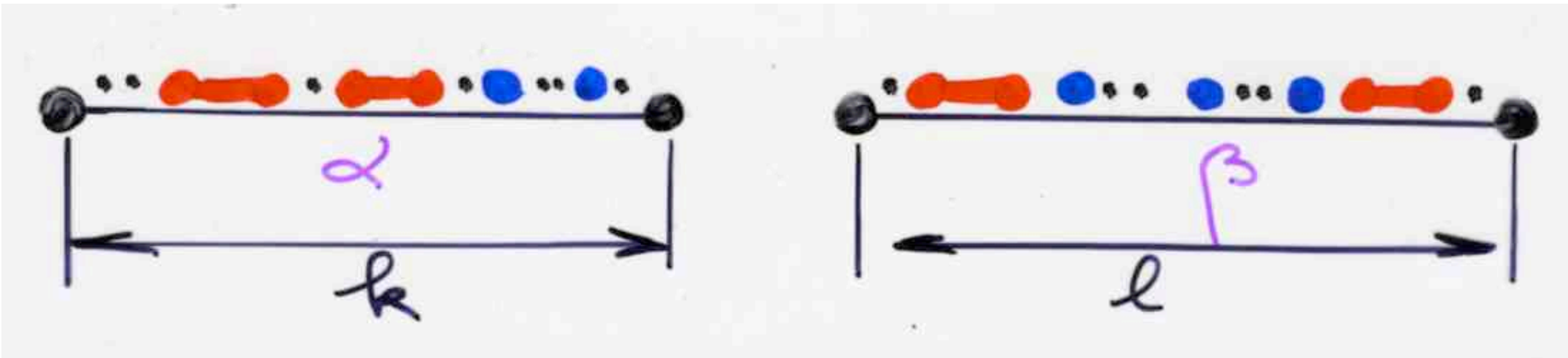
$$f(P_k P_l x^n) = \sum_{\alpha, \beta, \omega} (-1)^{|\alpha|+|\beta|} v(\alpha) v(\beta) v(\omega)$$

α pavage of $[0, k-1]$
β pavage of $[0, l-1]$
ω Motzkin path
(level $0 \rightsquigarrow 0$)

$$|\omega| = ip(\alpha) + ip(\beta) + n$$

$$(\alpha, \beta, \omega) \in E_{n,k,l}$$





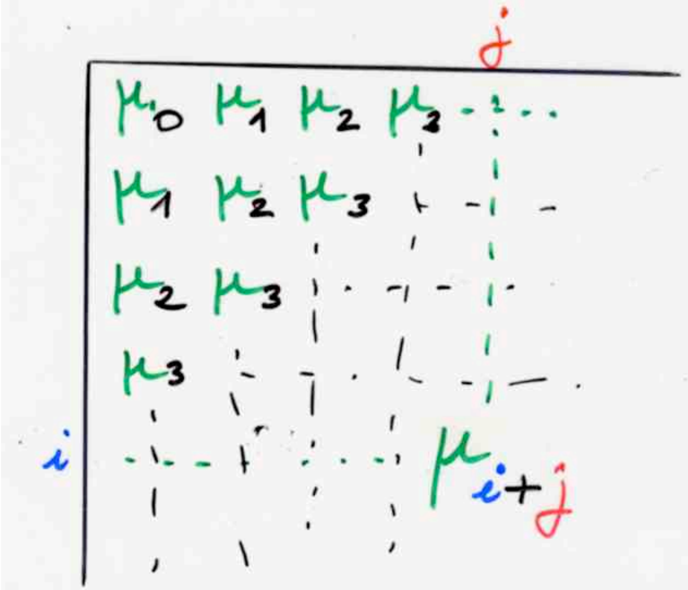
$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$

Hankel determinants

Hankel determinant

any minor of the matrix

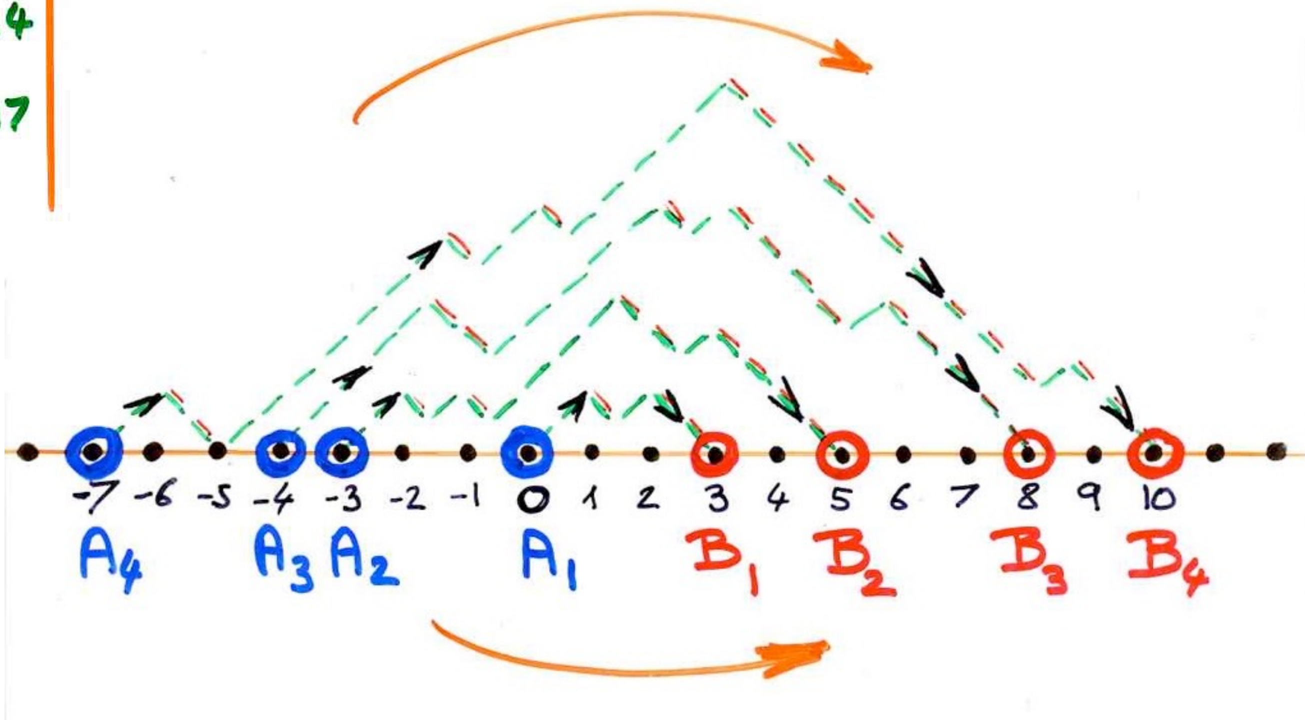
$\mathbf{H}\left(\{\mu_n\}_{n\geq 0}\right)$



LGV Lemma

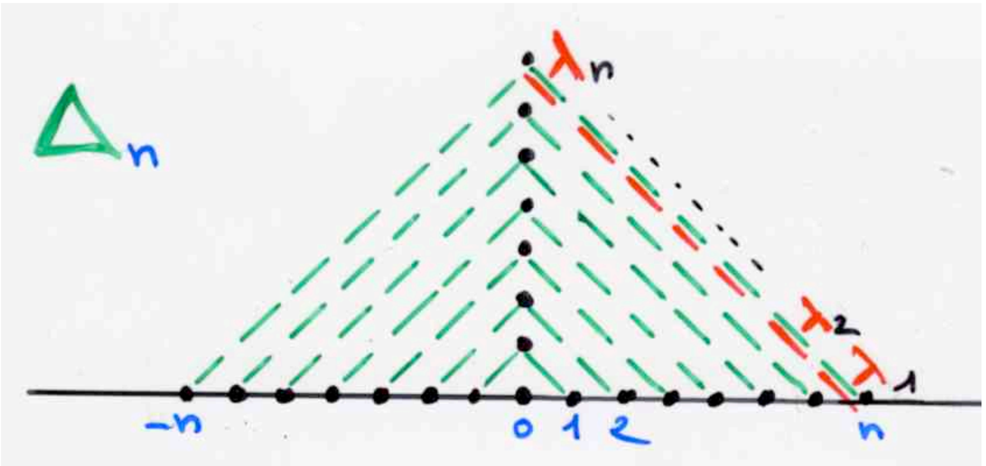
determinant $\leftrightarrow$ configuration of non-intersecting paths

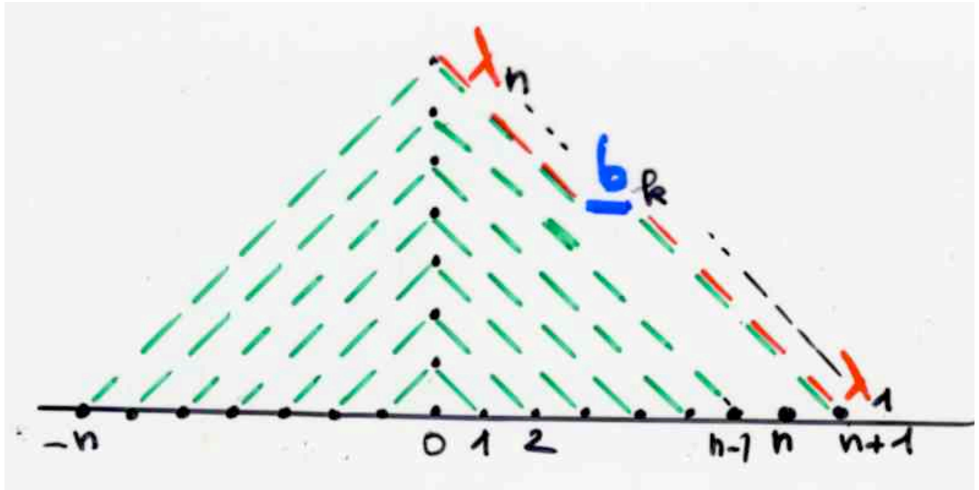
$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$



$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$

$$\chi_n = \det \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & & \mu_{2n-1} & \mu_{2n+1} \end{vmatrix}$$



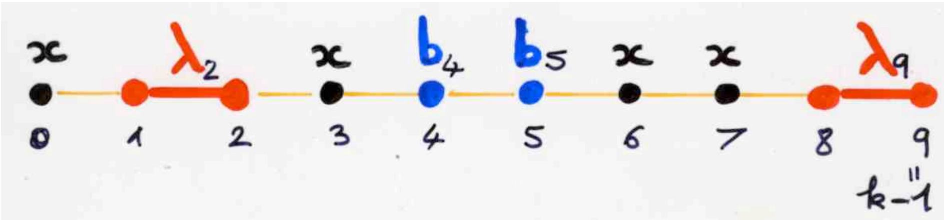


Duality

duality

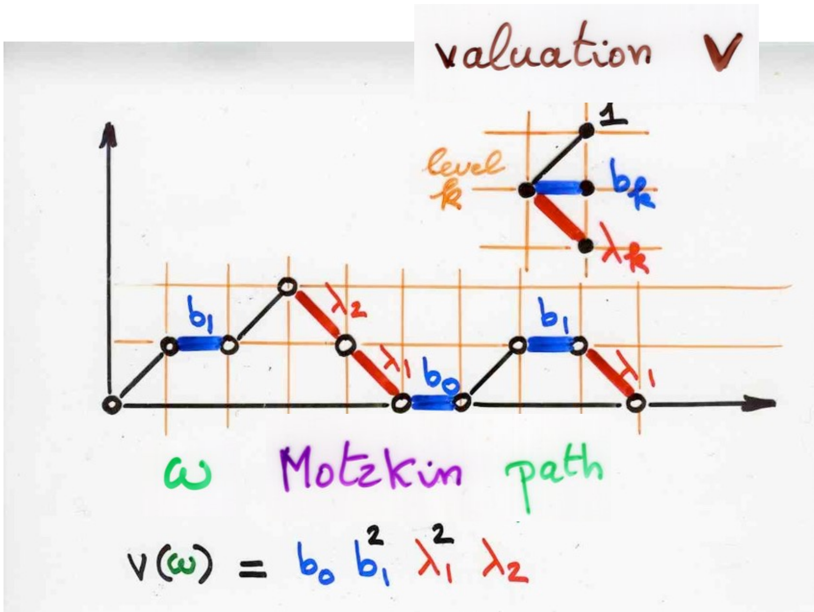
orthogonal polynomial

$\{P_n(x)\}_{n\geq 0}$

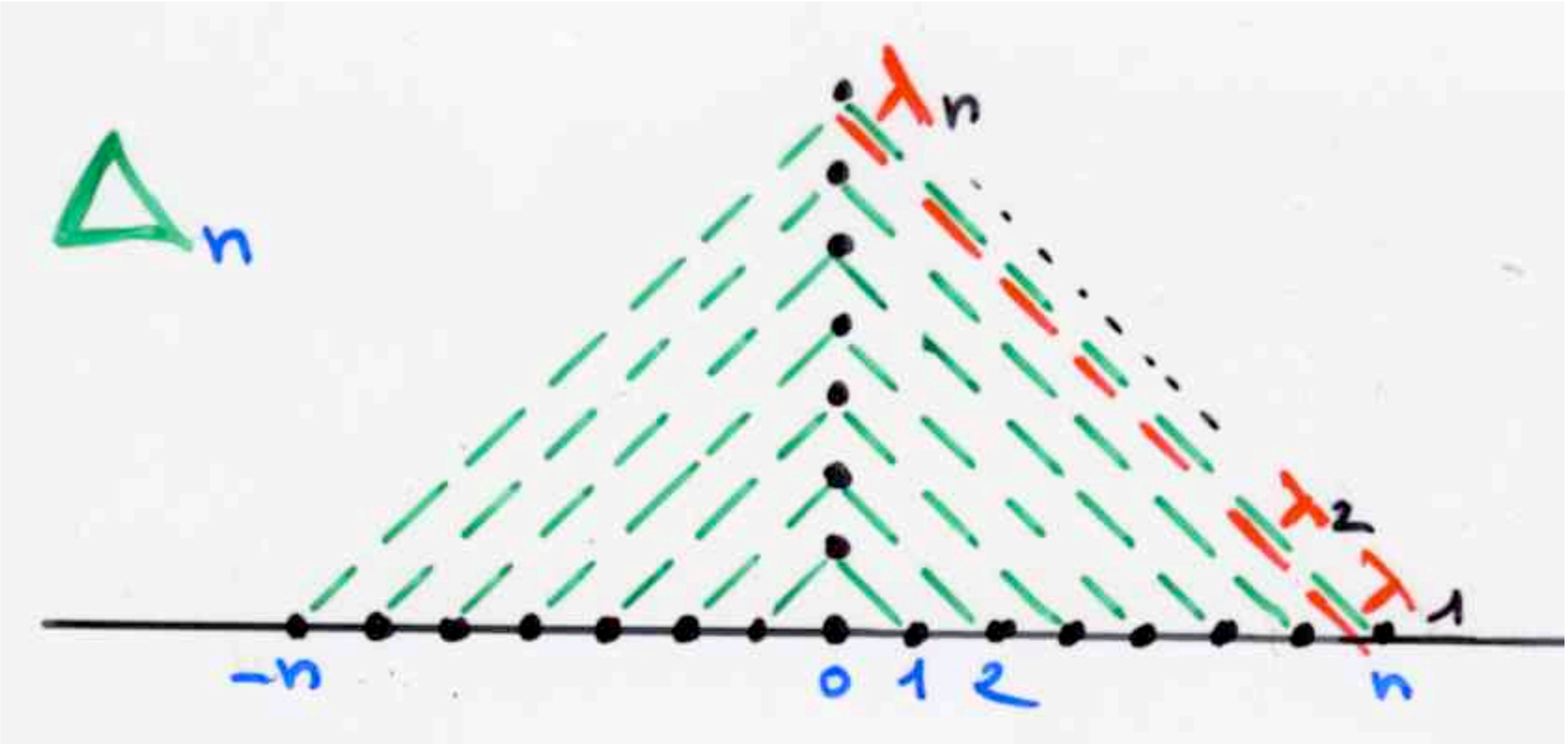


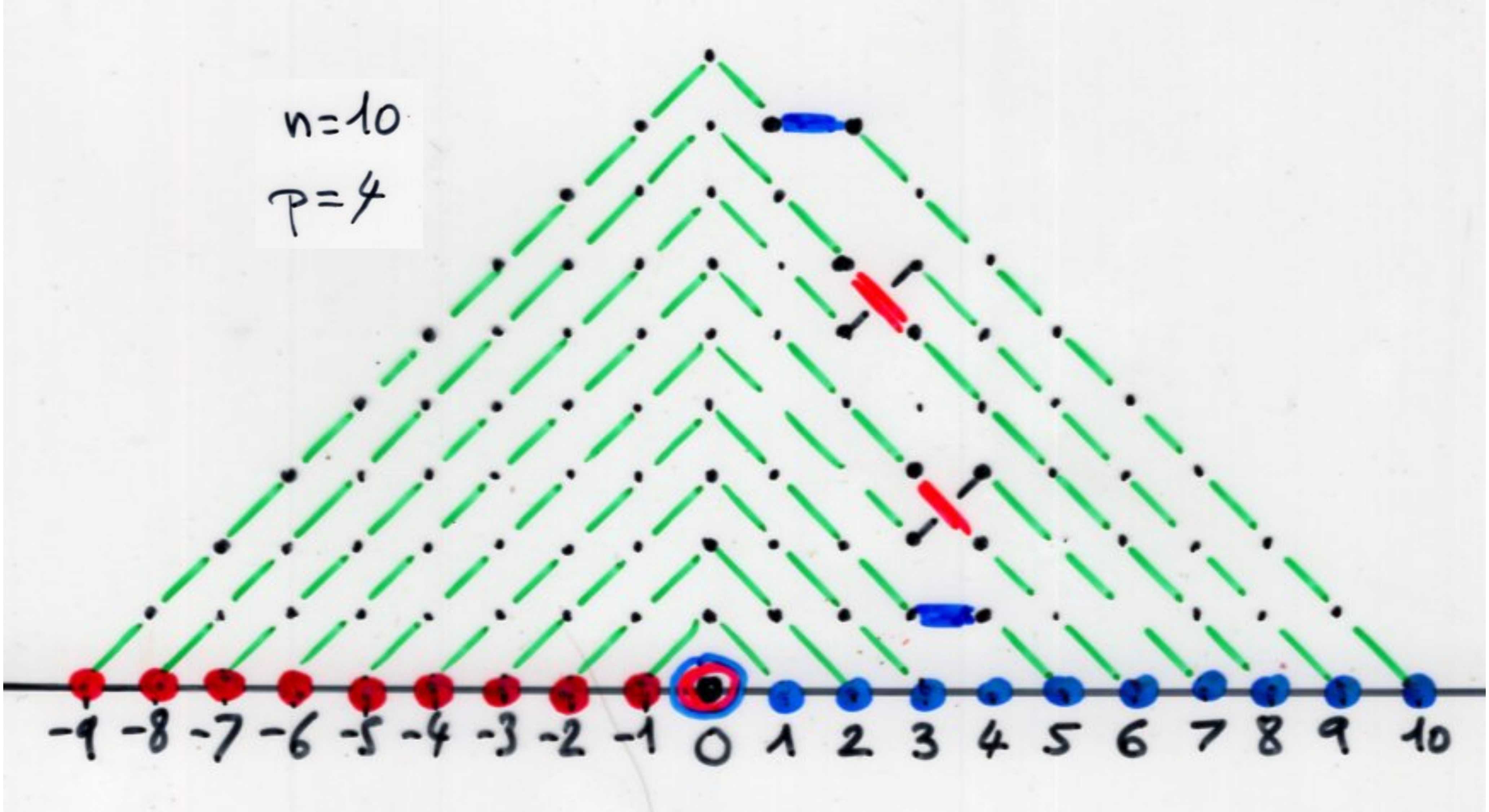
moments μ_n

weighted Motzkin paths

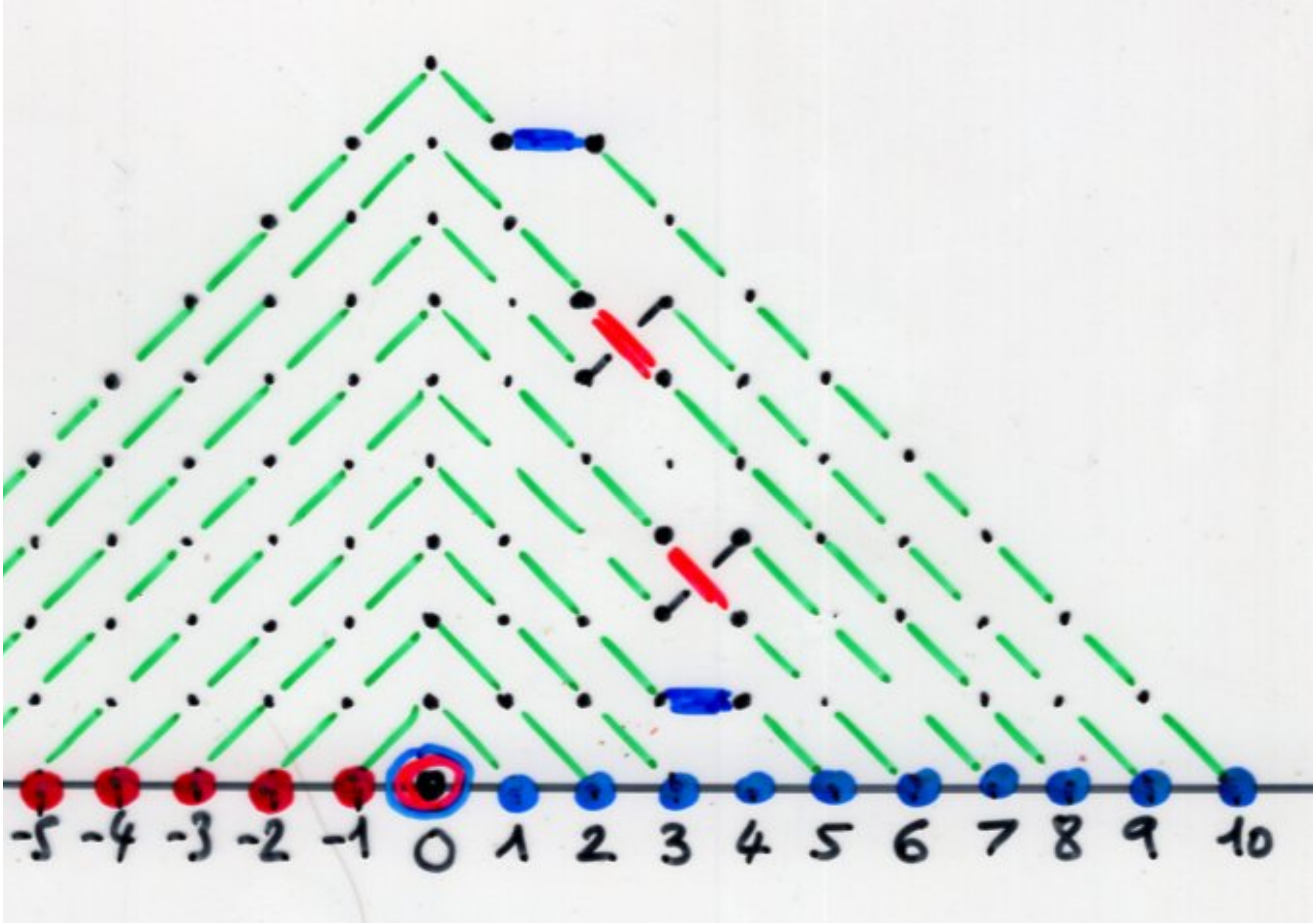


$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}$$



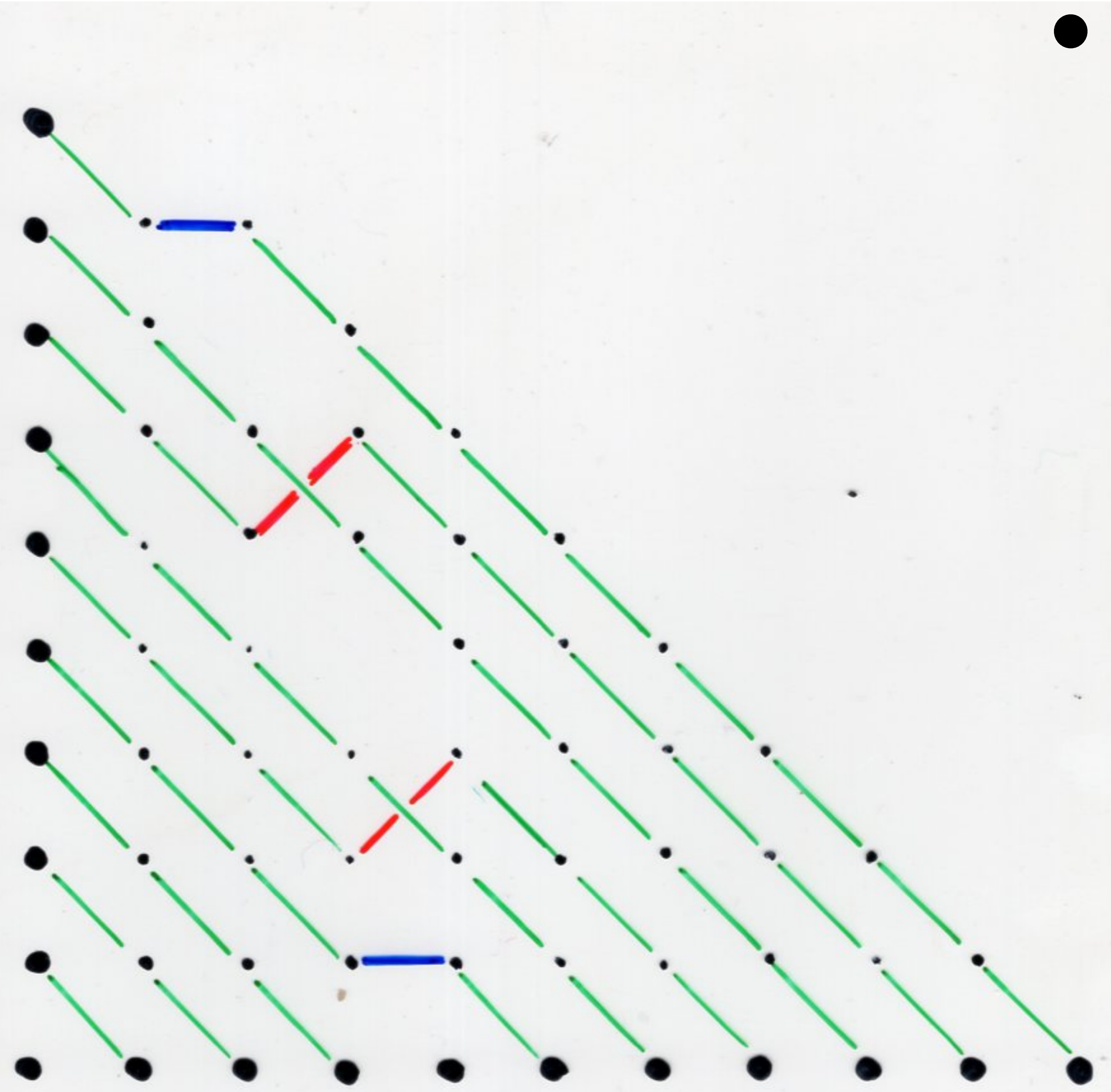


n=10
p=4
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10


-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

n=10

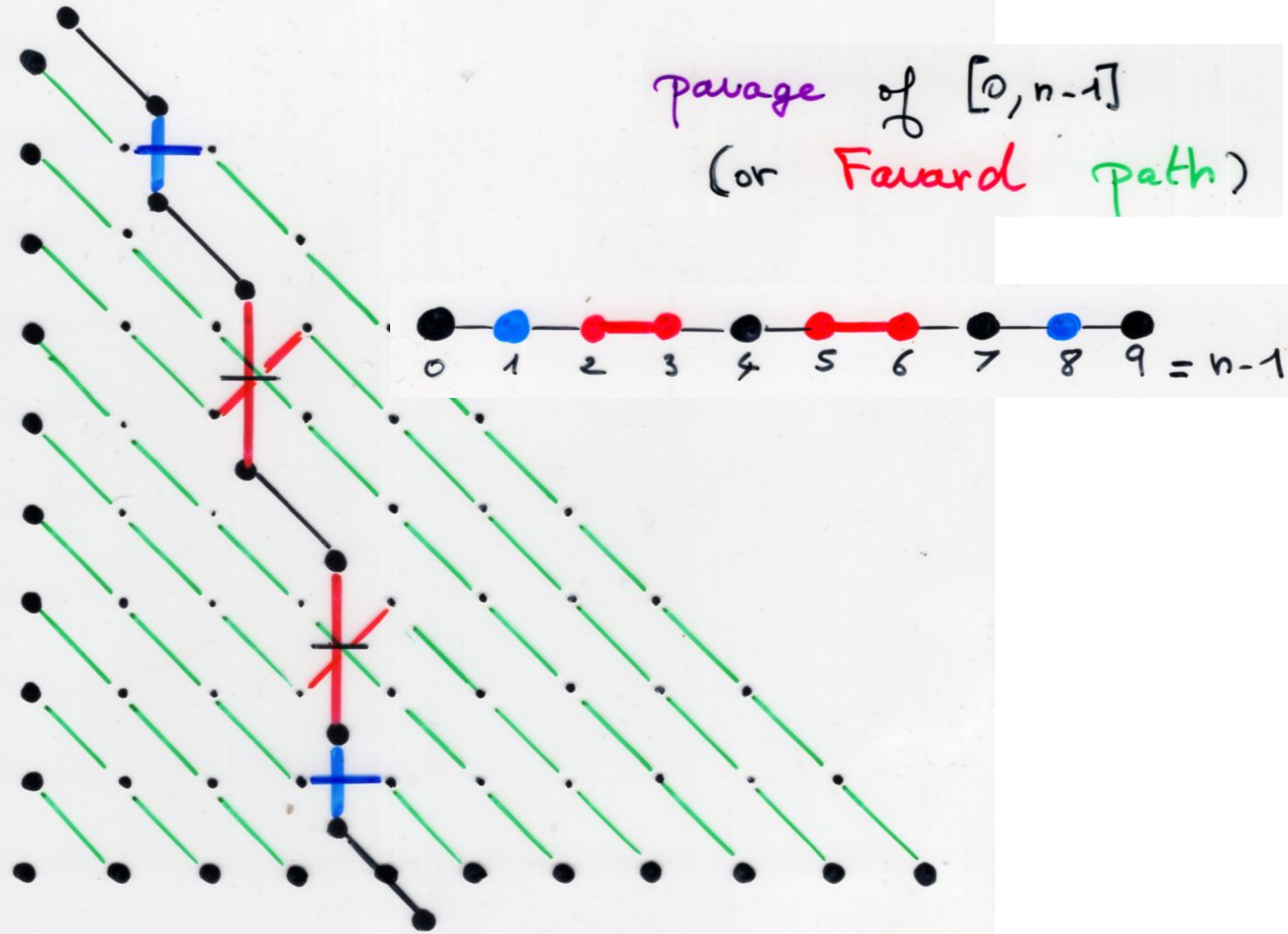
p=4



bijection: $\zeta = (\sigma; \omega_0, .., \omega_{n-1}) \longrightarrow \beta$

$n = 10$

$p = 4$



pavage of $[0, n-1]$

(or Favard path)

analytic continued fractions

continued fractions

Stieltjes

$$\underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cfrac{\cdots\cdots}{1-\cfrac{\lambda_k t}{\cdots}}}}}}_{S(t;\lambda)}$$

$$\cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\cdots\cdots\cfrac{}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

$J(t;b,\lambda)$

Jacobi continued fraction

$b=\{b_k\}_{k\geq 0}$ $\lambda=\{\lambda_k\}_{k\geq 1}$

The fundamental Flajolet Lemma

combinatorial interpretation of a
continued fraction with weighted paths

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \lambda_2 t^2 \cfrac{}{\cdots\cdots \cfrac{}{1 - b_k t - \lambda_{k+1} t^2 \cfrac{}{\cdots\cdots}}}}}$$

$$J(t; b, \lambda)$$

Jacobi continued fraction

$$b = \{b_k\}_{k \geq 0} \qquad \lambda = \{\lambda_k\}_{k \geq 1}$$

Philippe Flajolet fundamental Lemma

continued fractions

J-fraction

$$\sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

Motzkin path

$|\omega| = n$

Philippe Flajolet fundamental Lemma

orthogonal polynomials

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

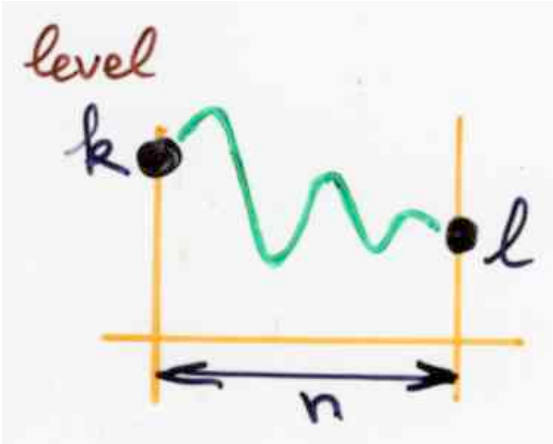
$$f(x^n) = \mu_n$$

moments

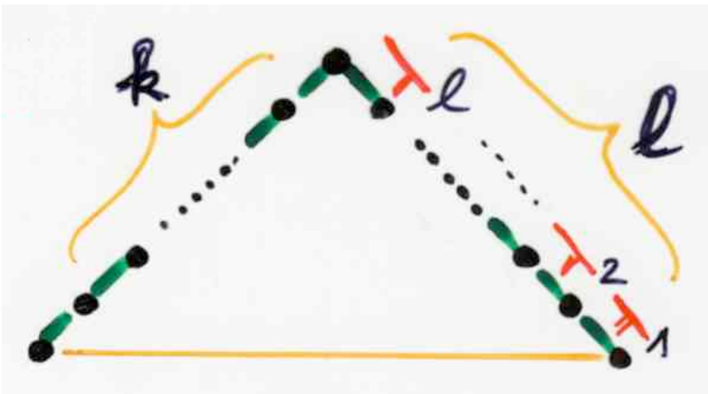
$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path
$|\omega| = n$

$$f(P_k P_\ell x^n) =$$



$$f(P_k P_\ell) = 0 \quad k \neq \ell$$
$$= \lambda_1 \cdots \lambda_\ell \quad k = \ell$$



classical theory

continued fractions

J-fraction

$$\mu_n = \sum_{\omega} v(\omega) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

Motzkin path
$|\omega| = n$

orthogonal polynomials

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path
$|\omega| = n$

same « essence »

for various bijective proofs

- Ramanuján's formula
  (Notebook, entry 17, Ch. 12)
- The "main theorem" Ch 1
  ⇒ Favard's theorem
- Convergents of continued fractions

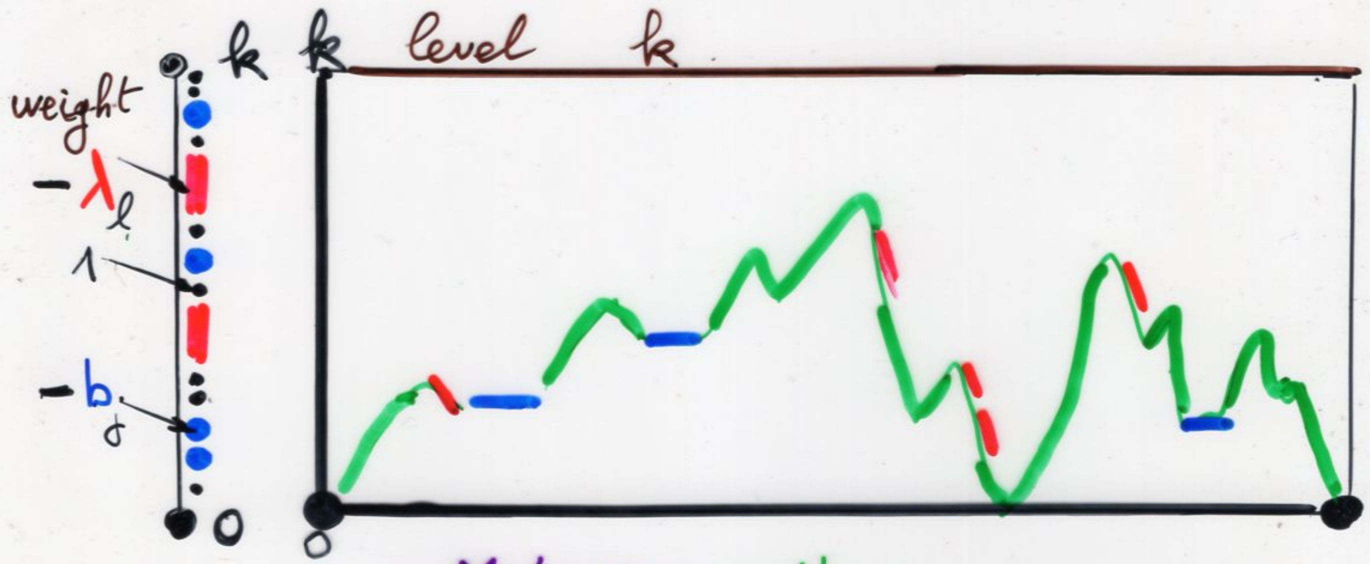
. . . .

same "essence" of the involution

sign-reversing, weight preserving

(with some variations

and different "border conditions"



Motzkin path

The notion of histories

example: Hermite histories

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

moments

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$\mu_{2n+1} = 0$

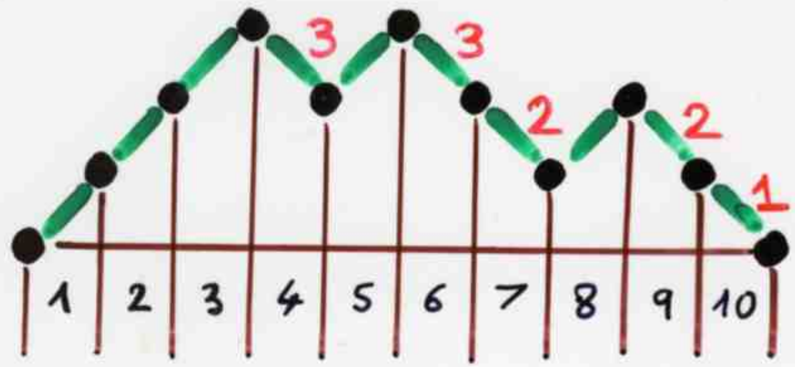
$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching

moments

Hermite polynomials



$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

chord diagrams

perfect matching

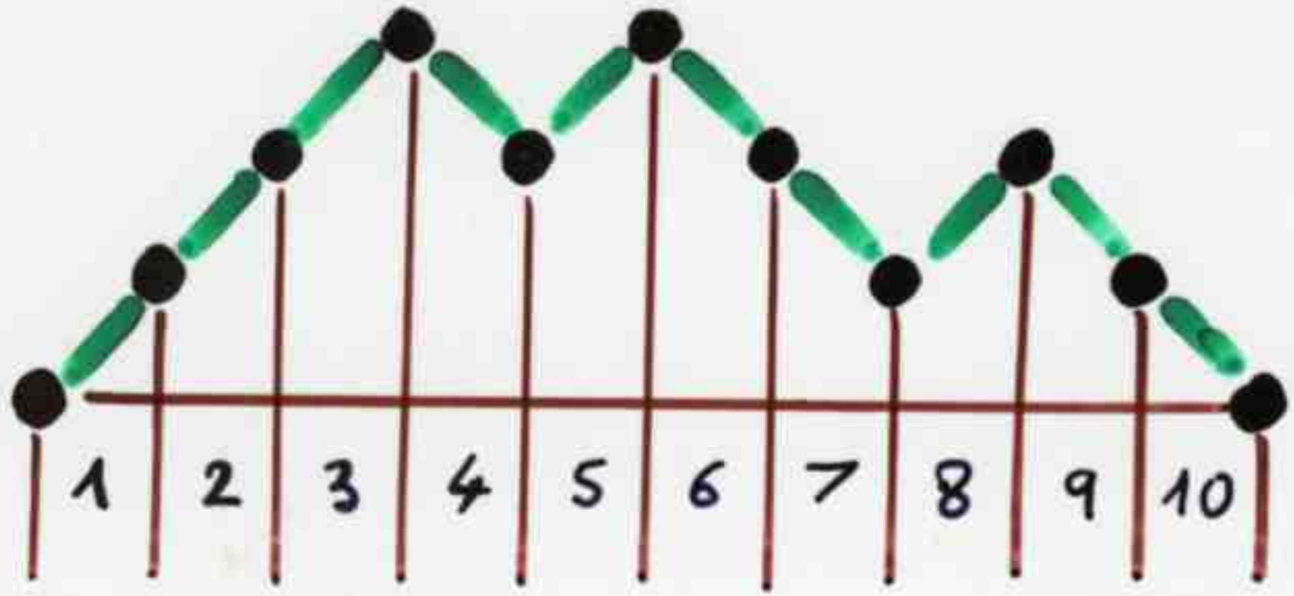
history

Françon (1978)

data structures
in
computer science

sequence
of
primitive
operations

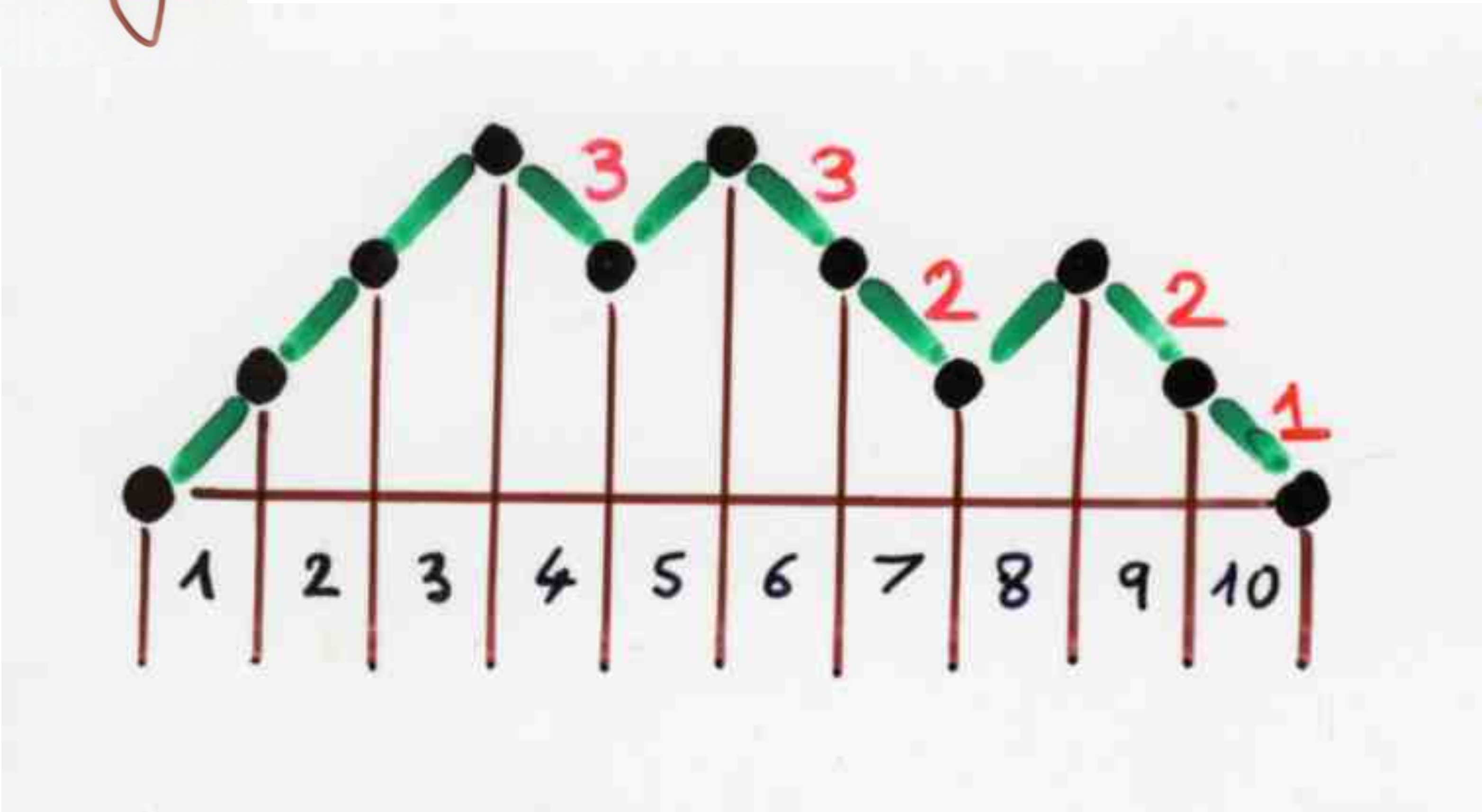
Hermite
history



Hermite history

Hermite polynomials

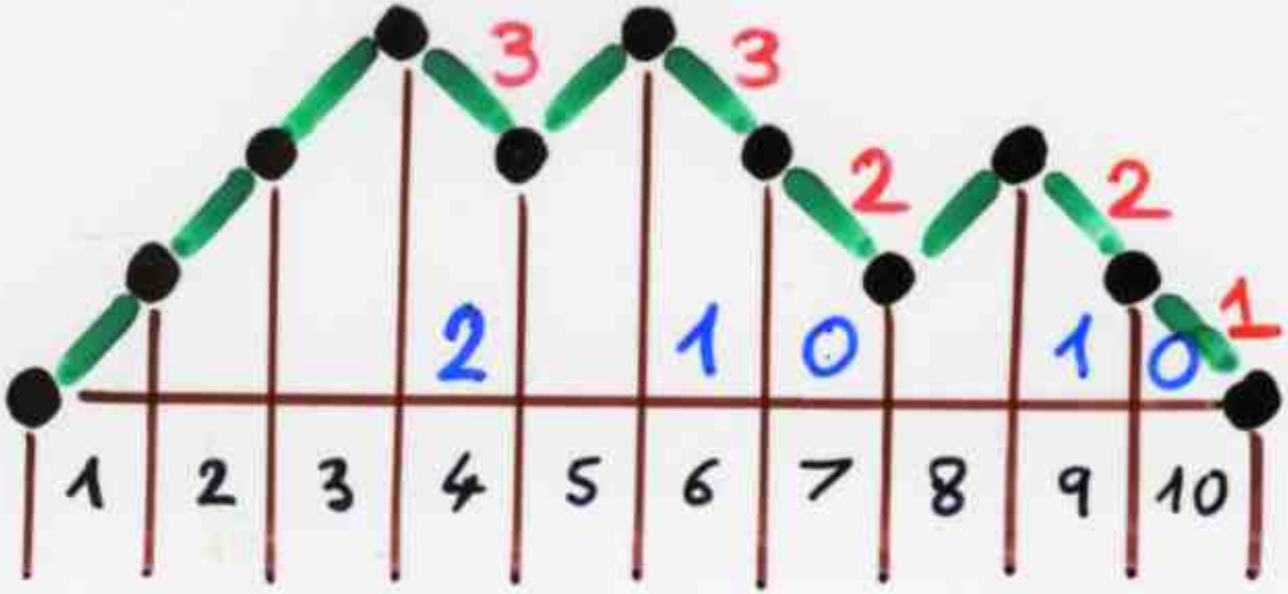
$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$



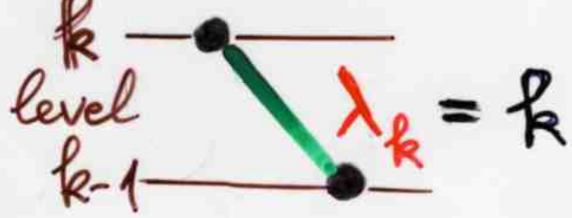
Hermite history

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

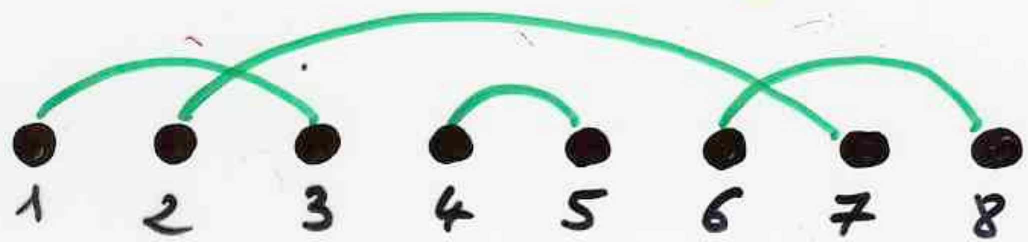


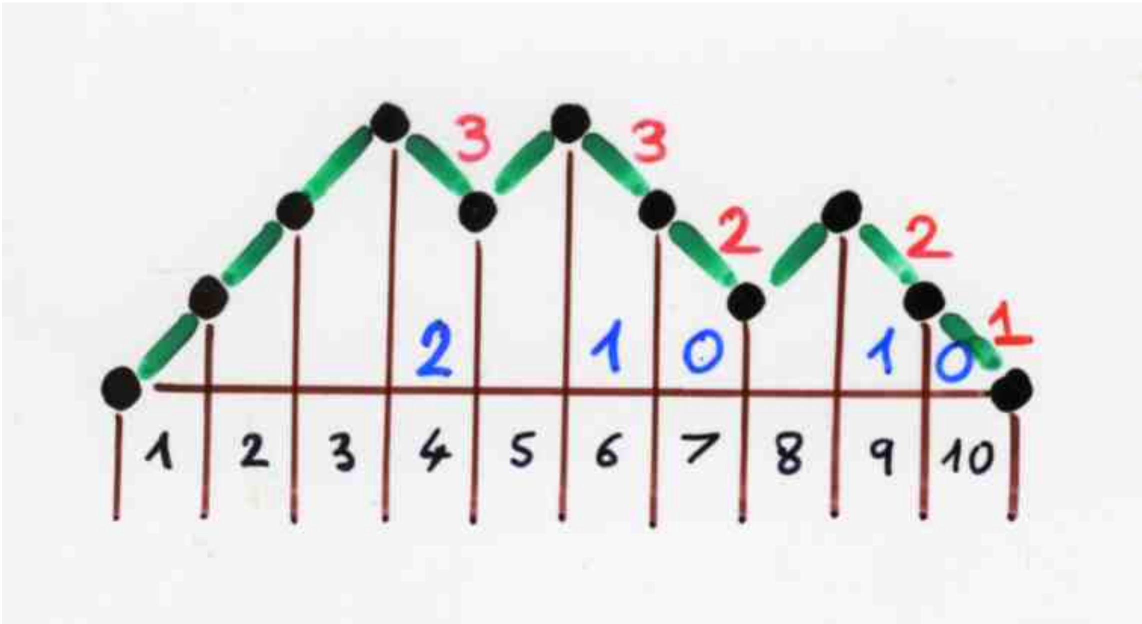
$$0 \leq i < \lambda_k = k$$



bijection

Hermite history ↔ chord diagrams / perfect matching

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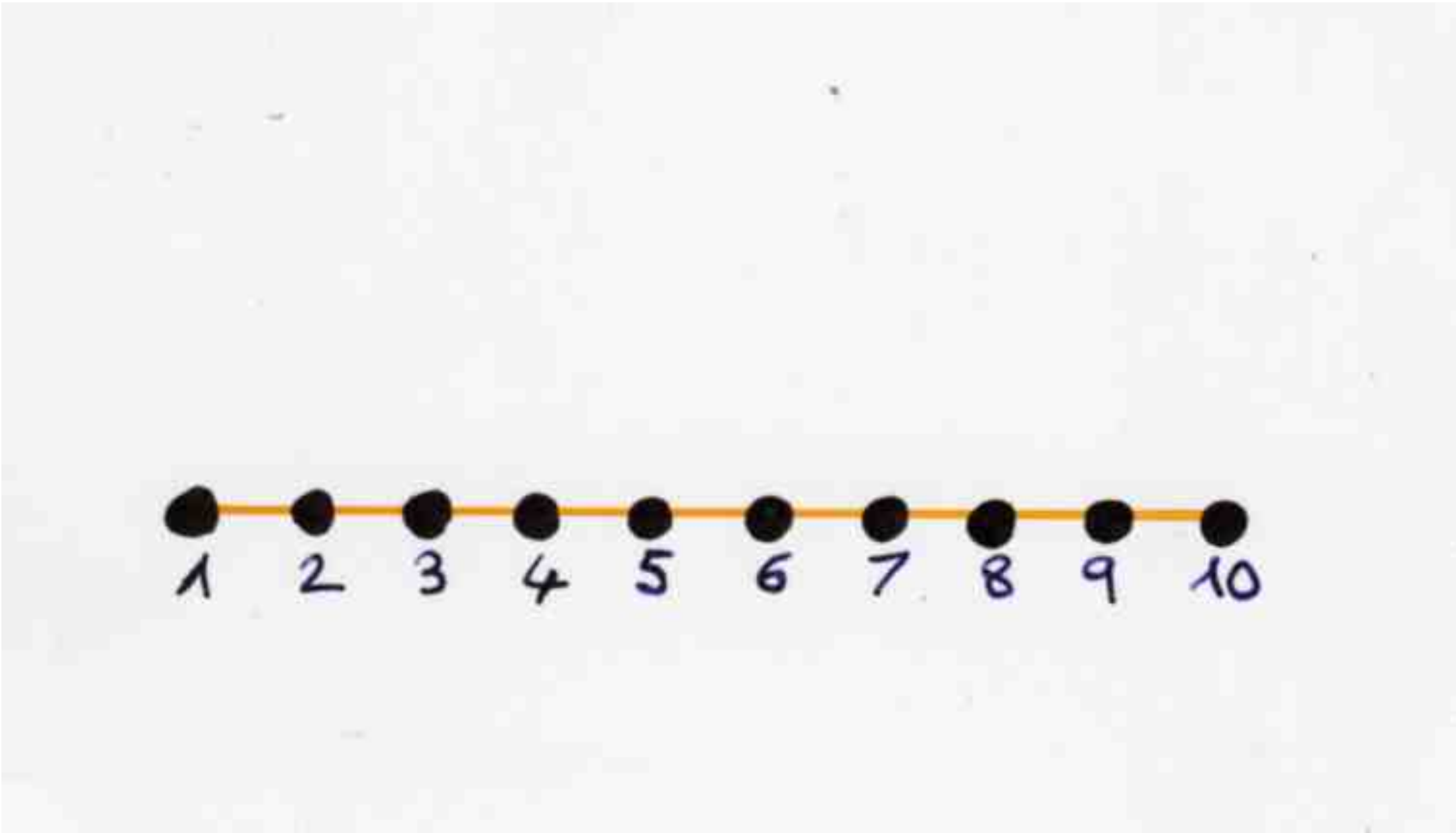
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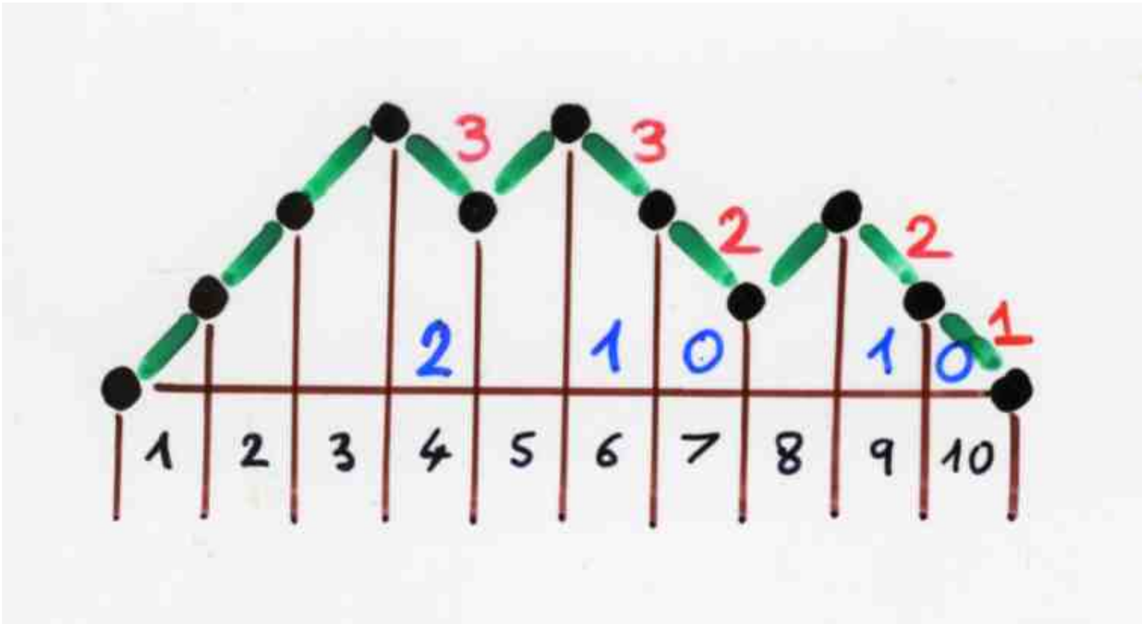
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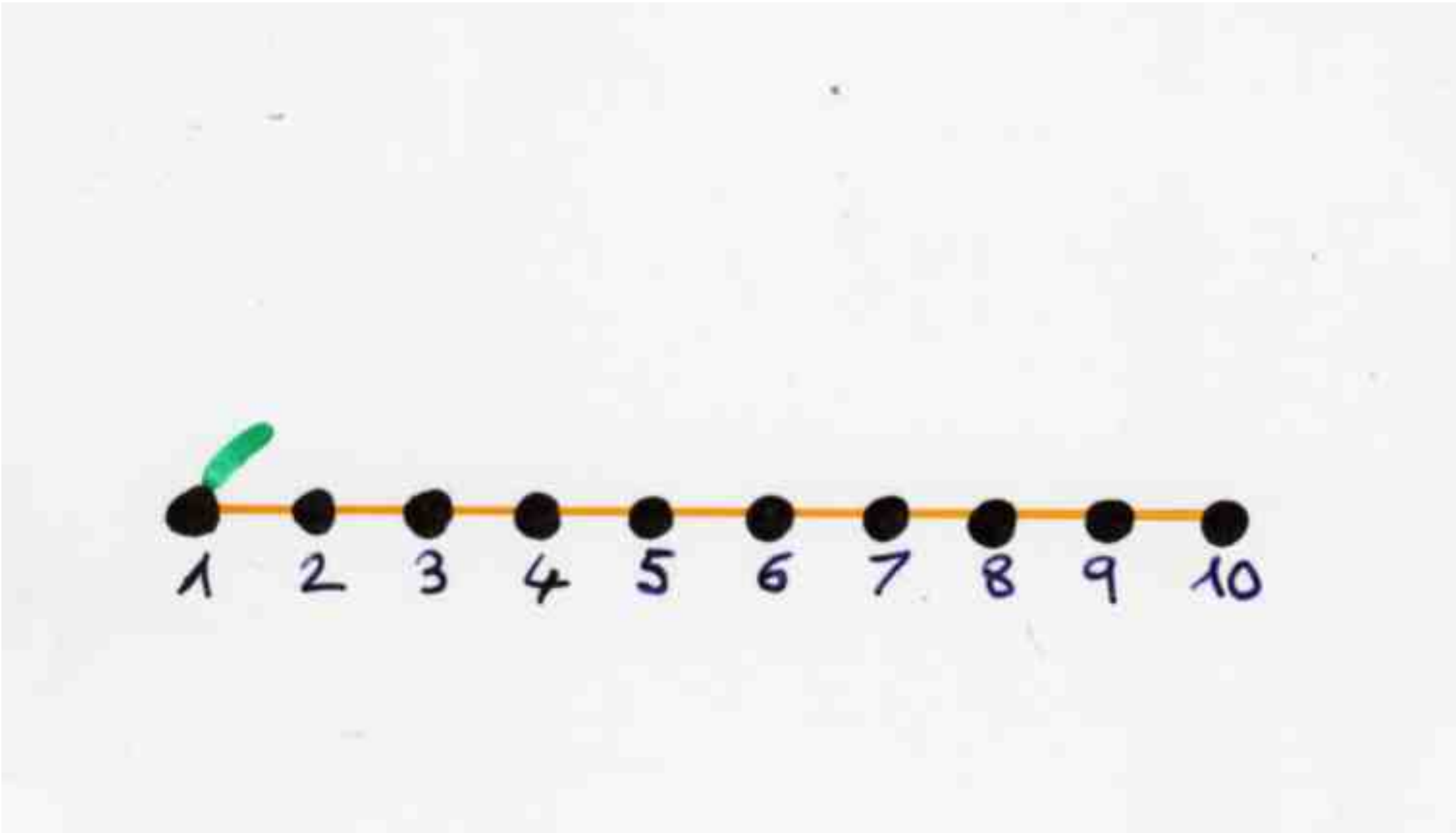
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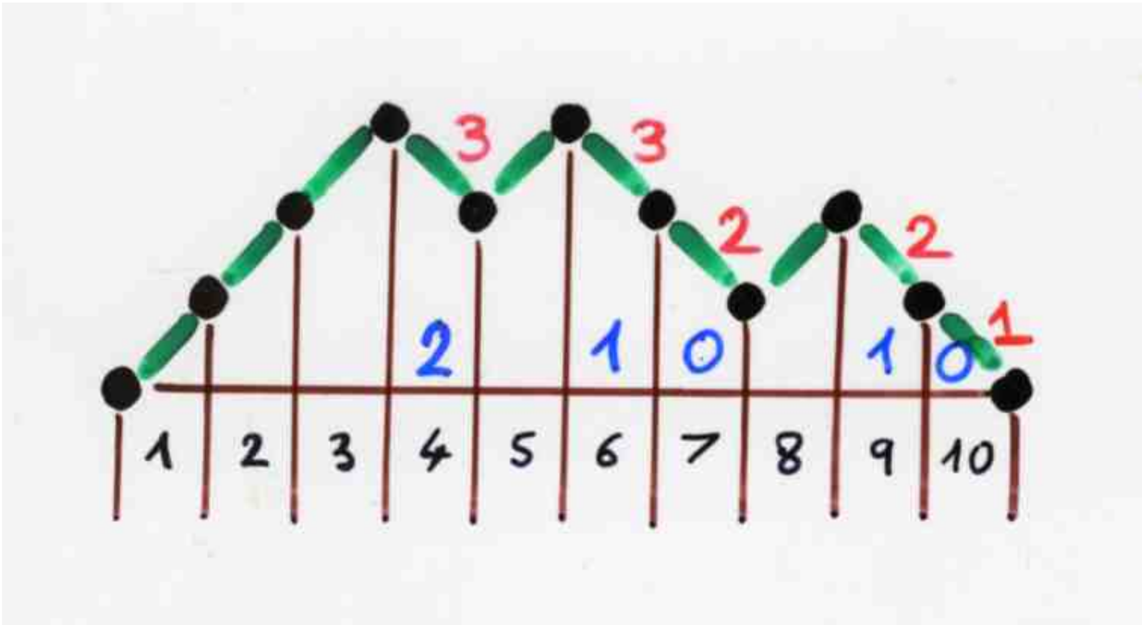
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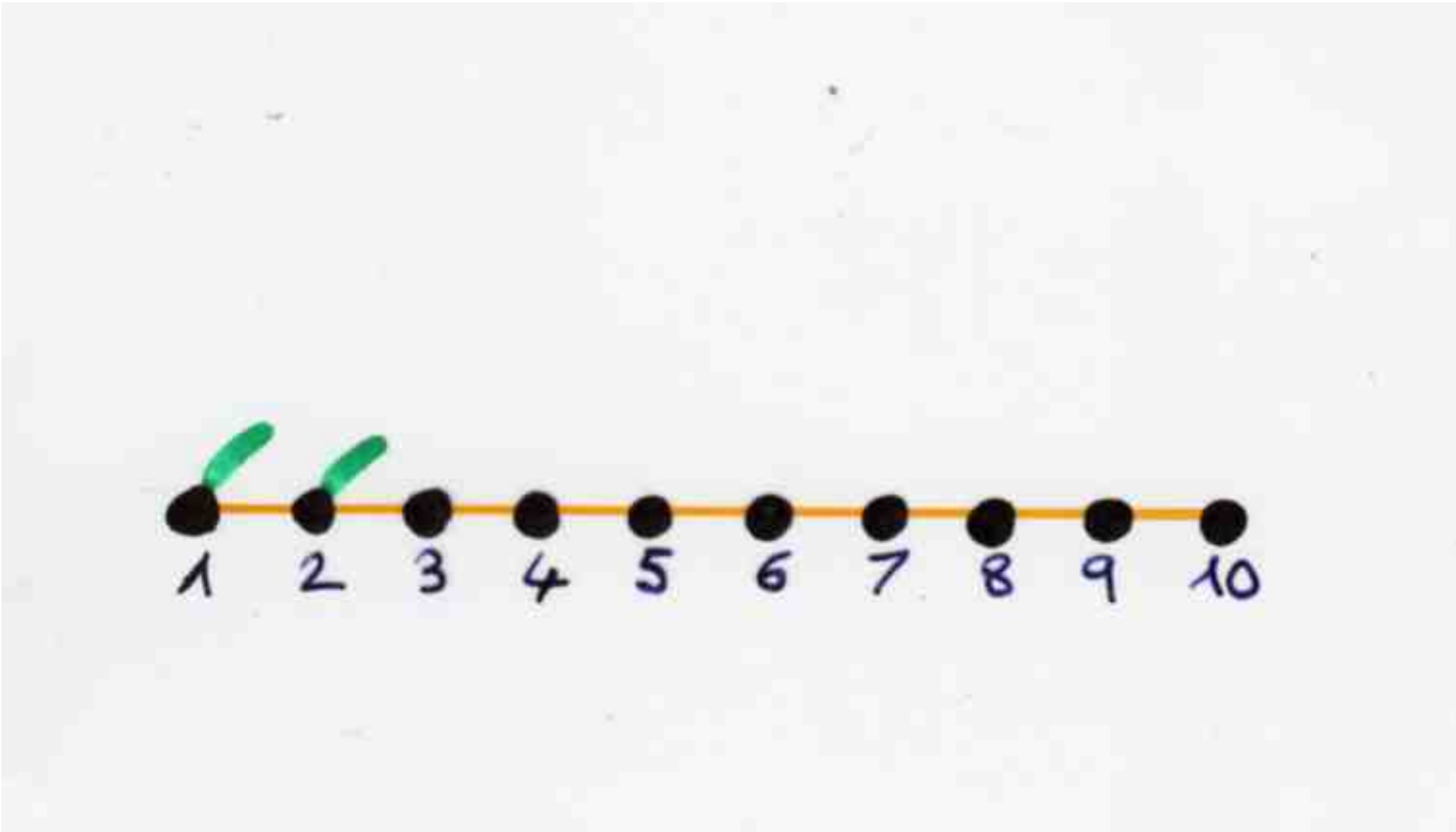
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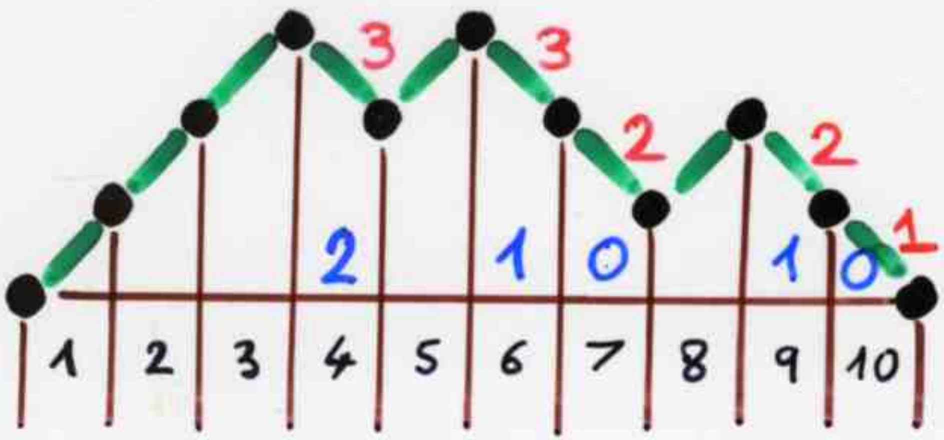
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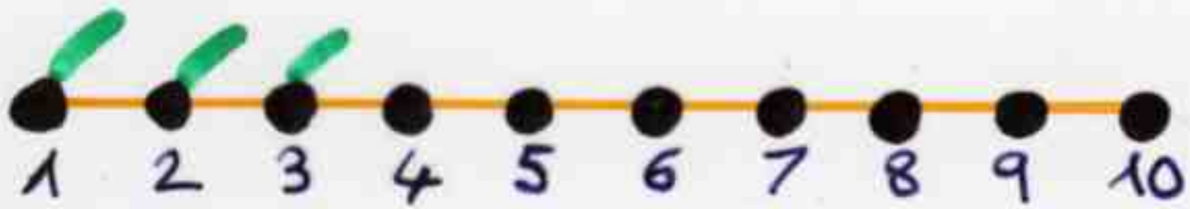
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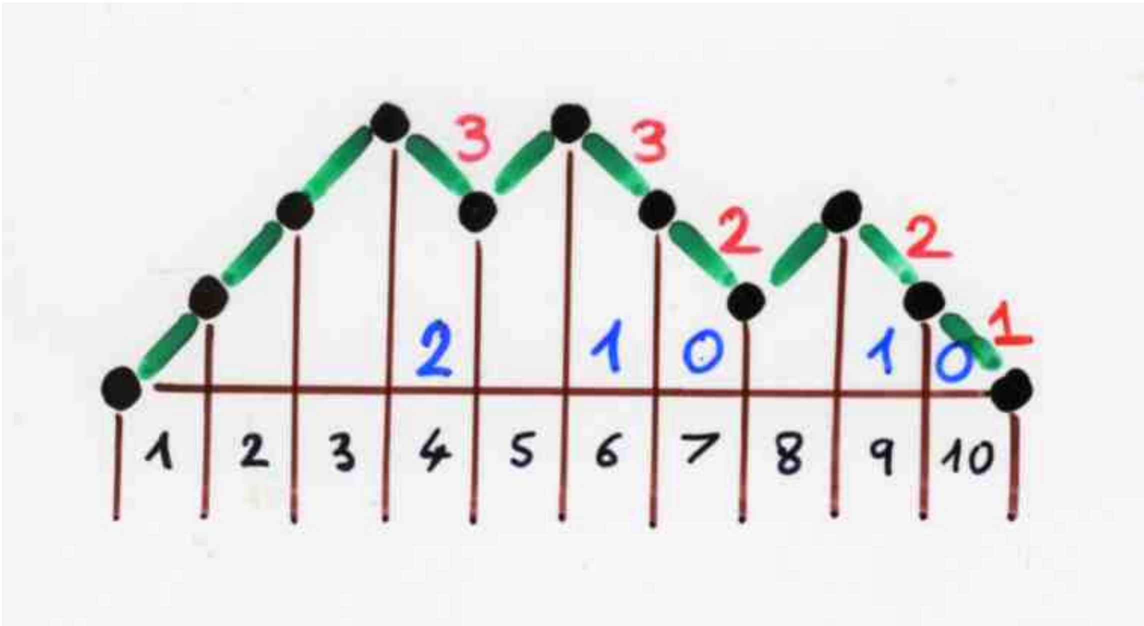

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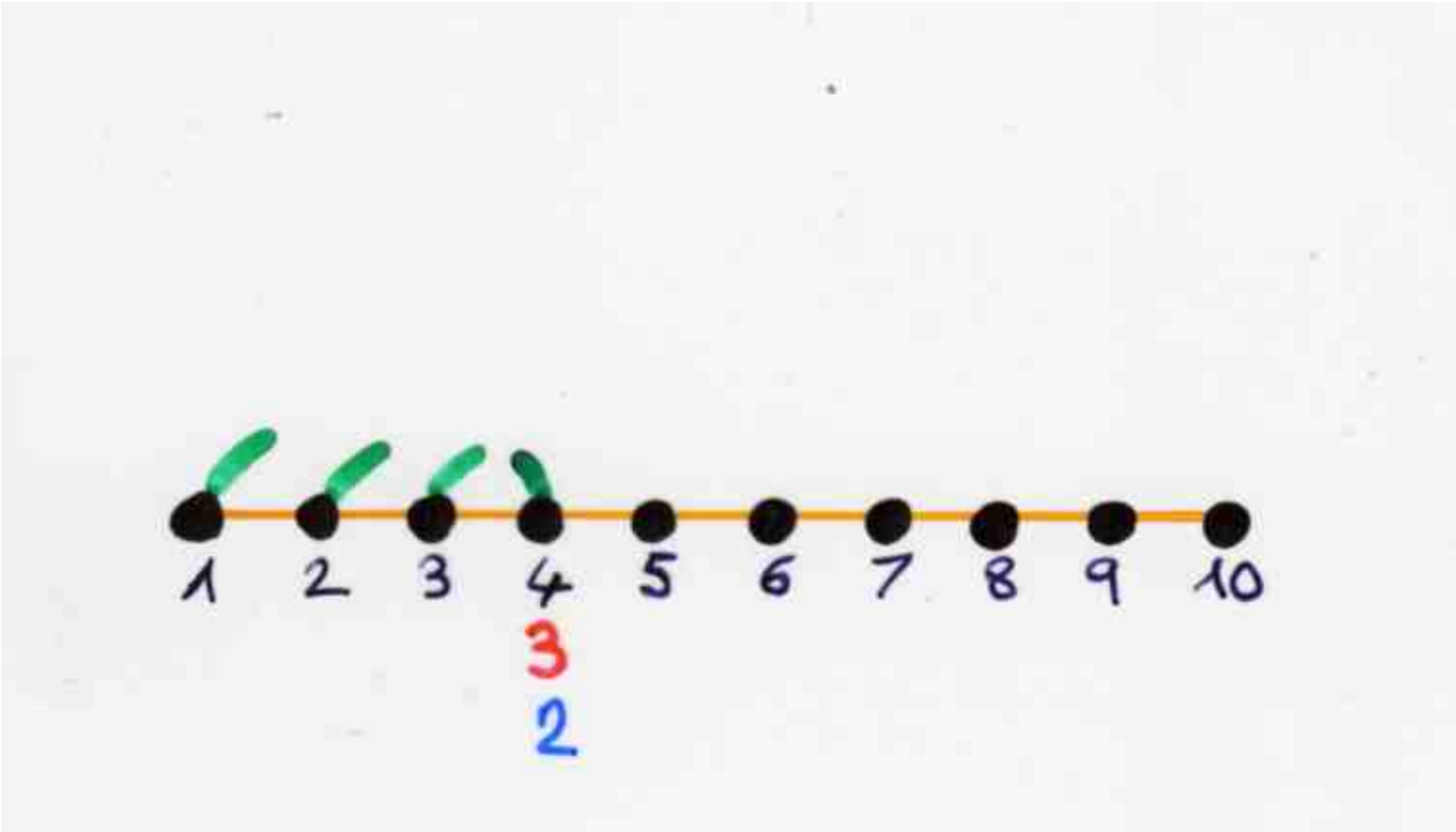

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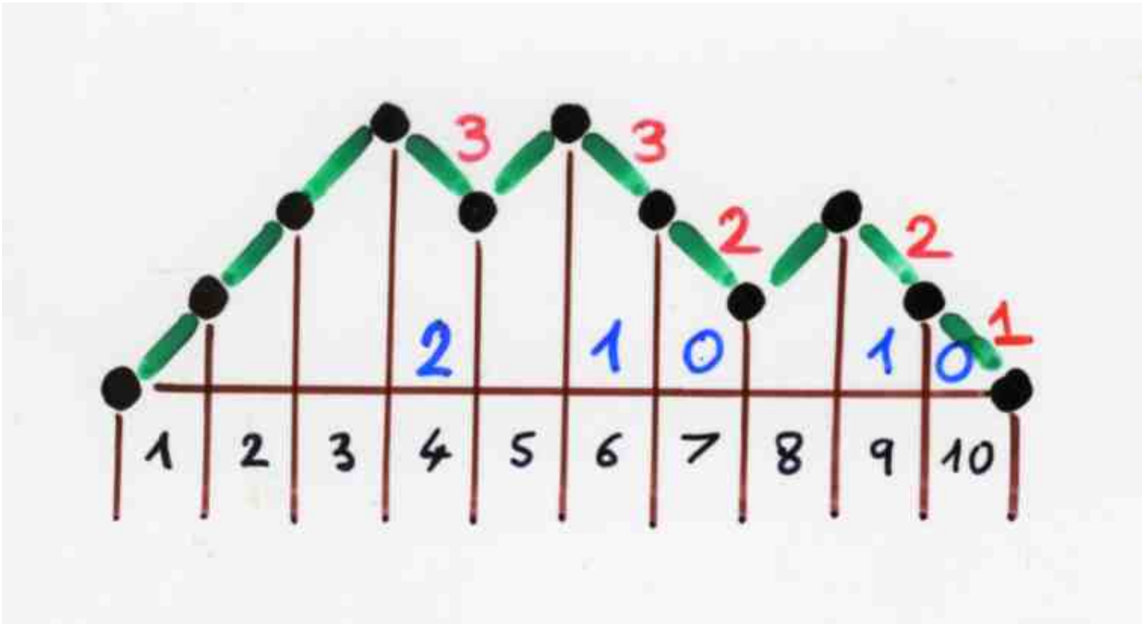

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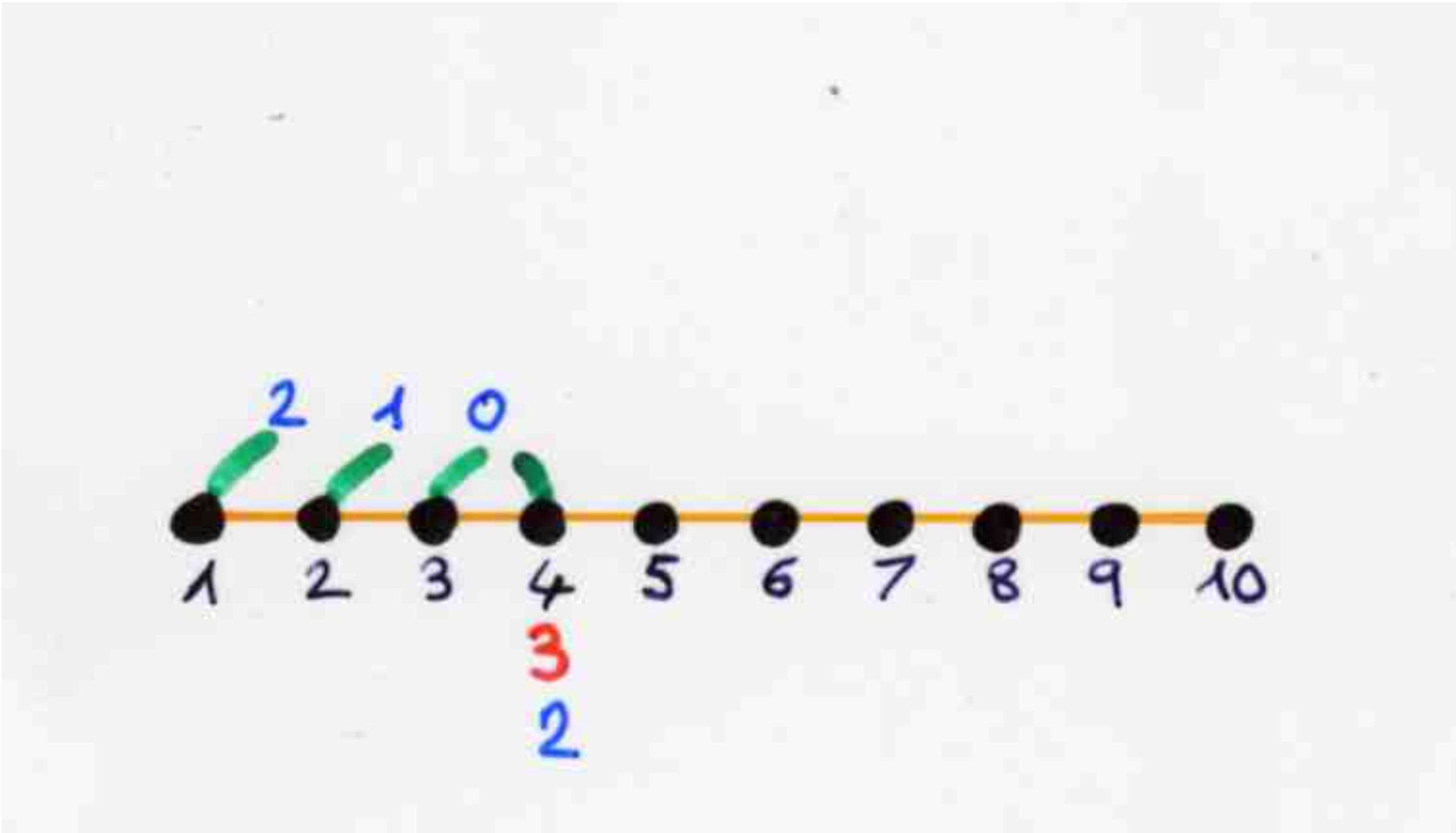

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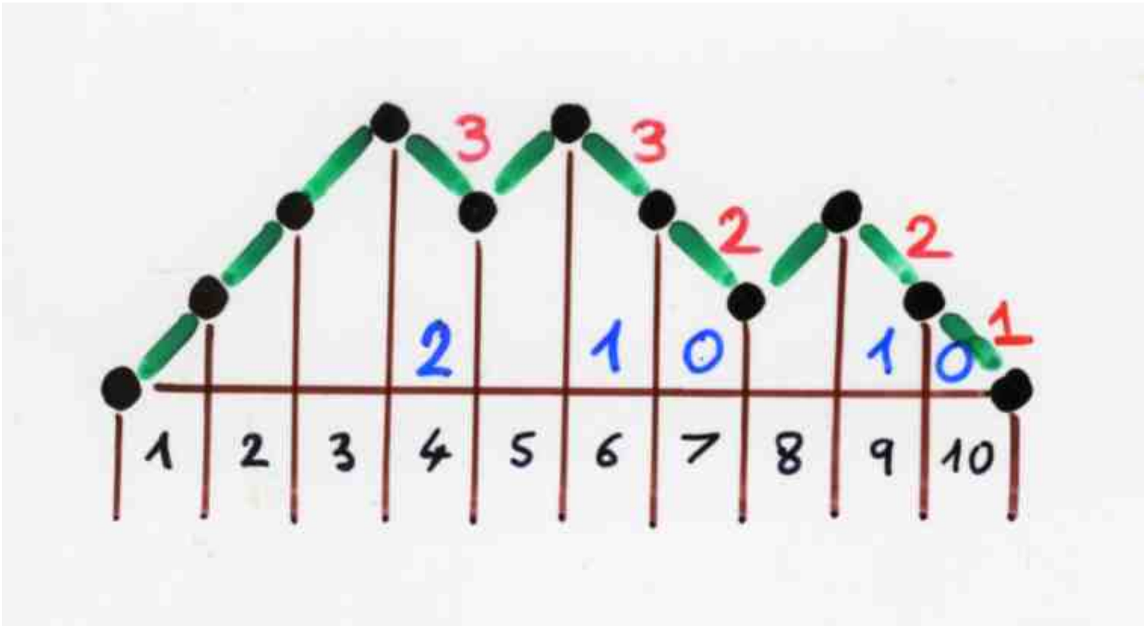
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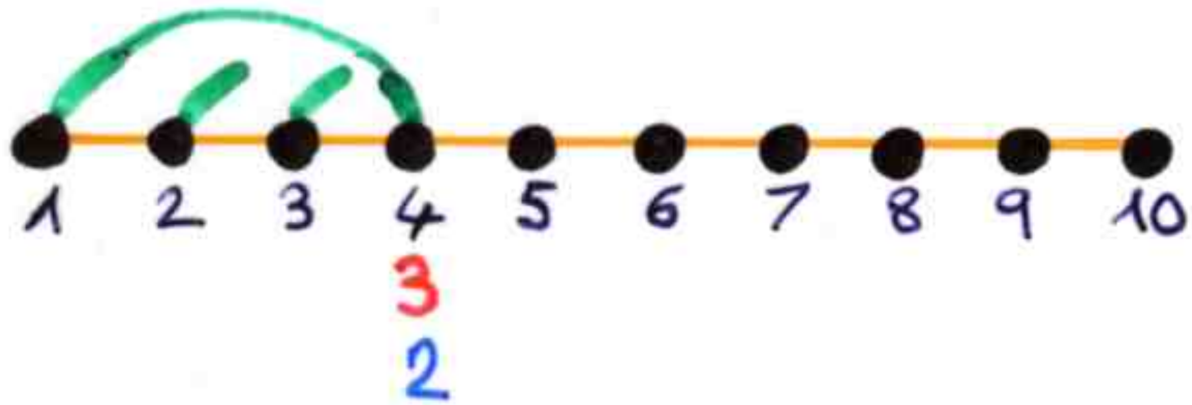
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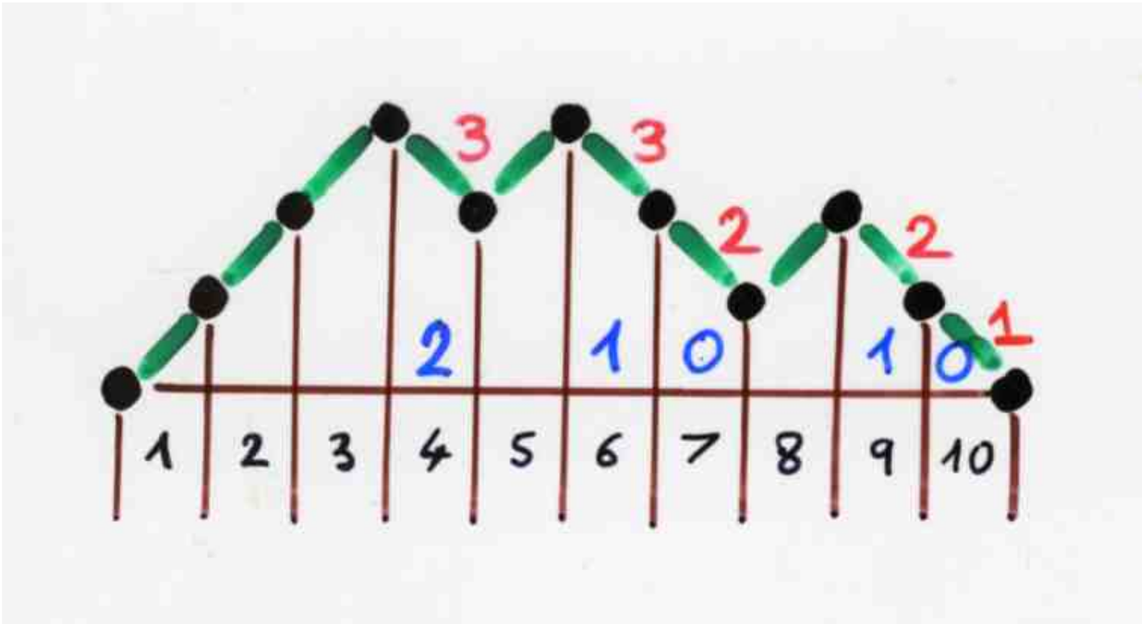
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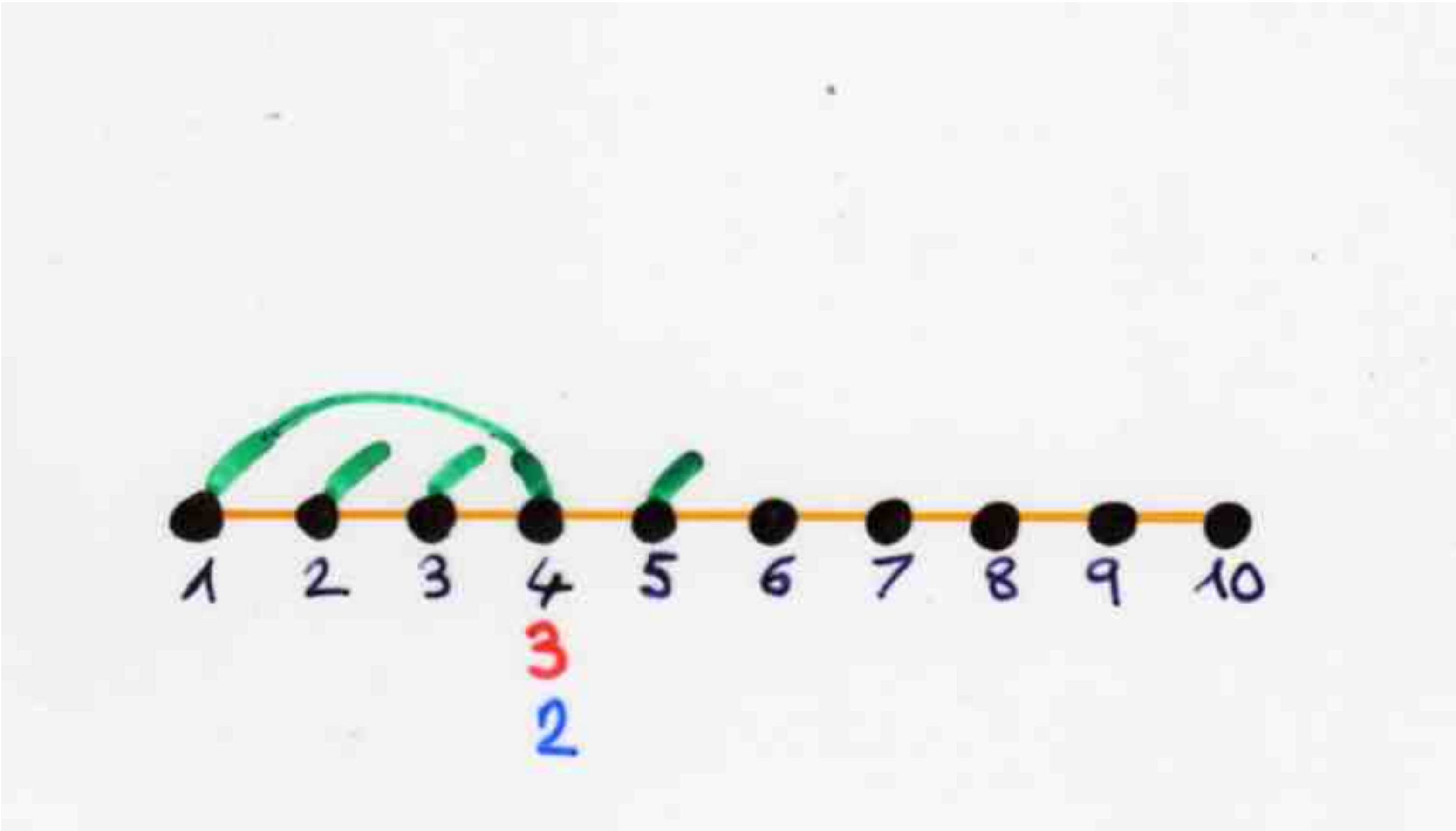
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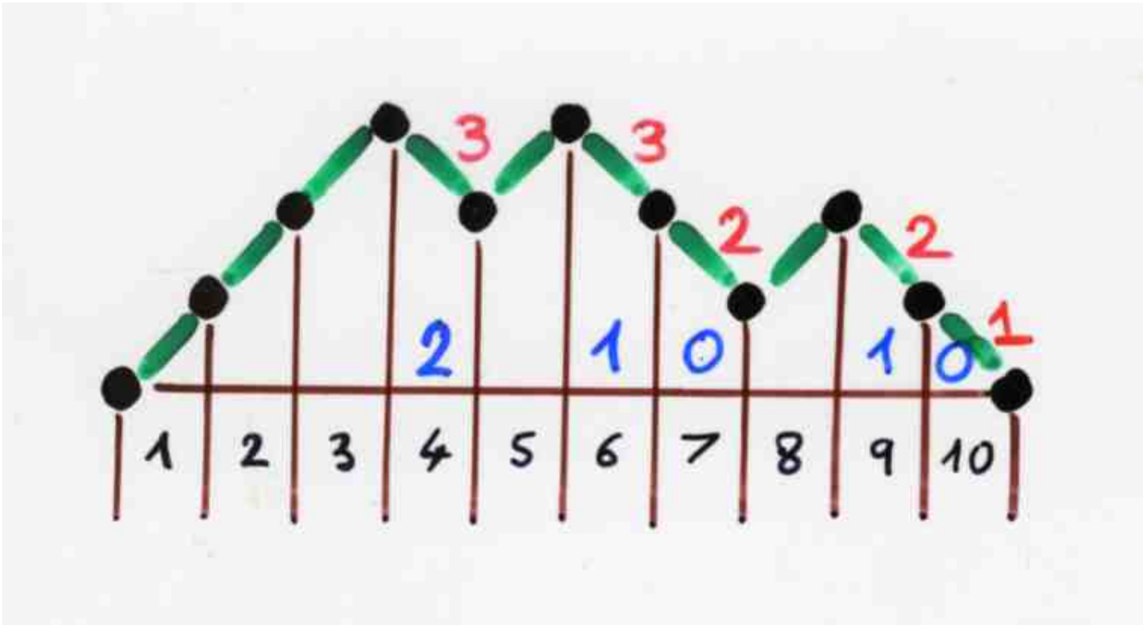

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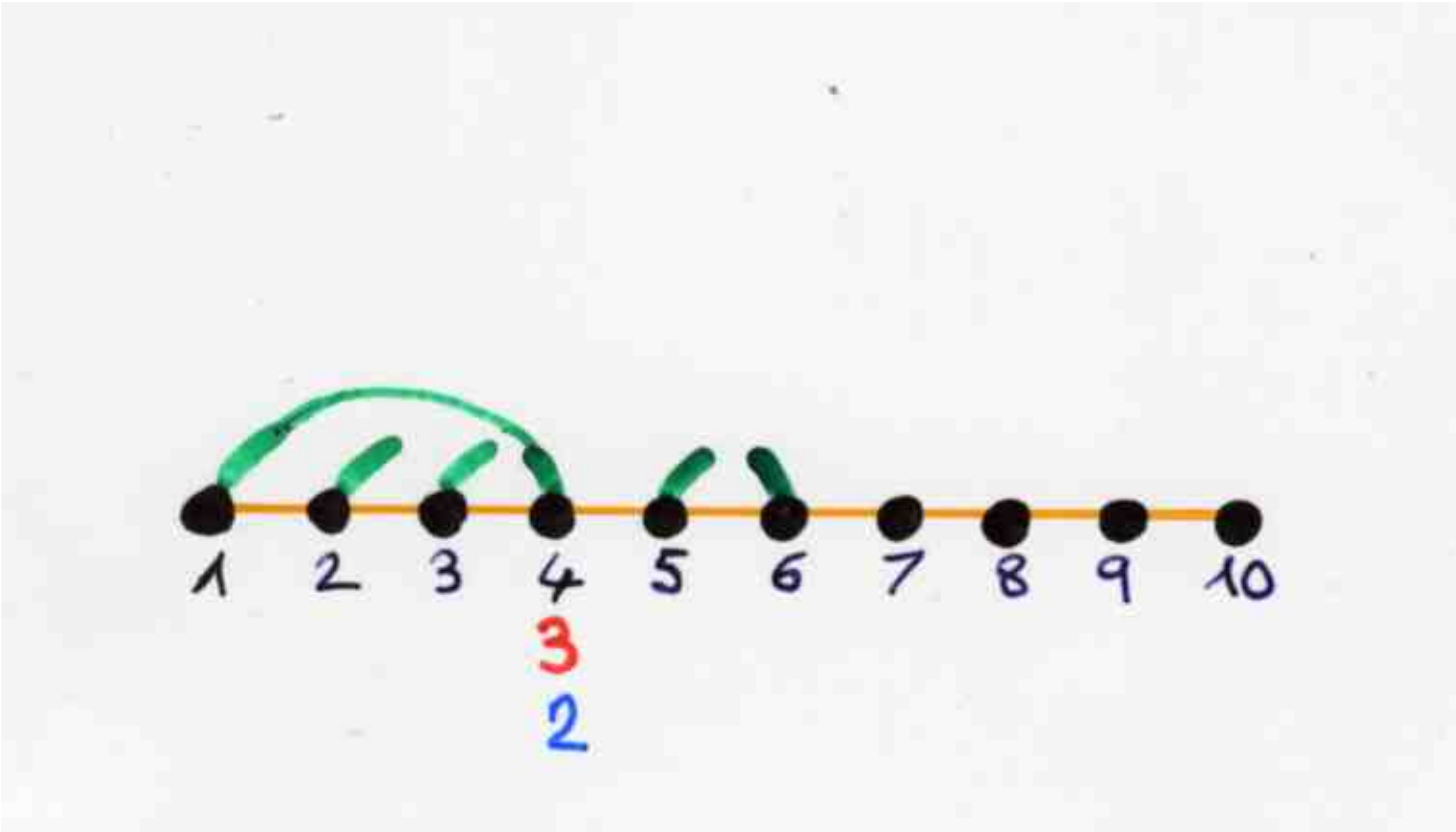

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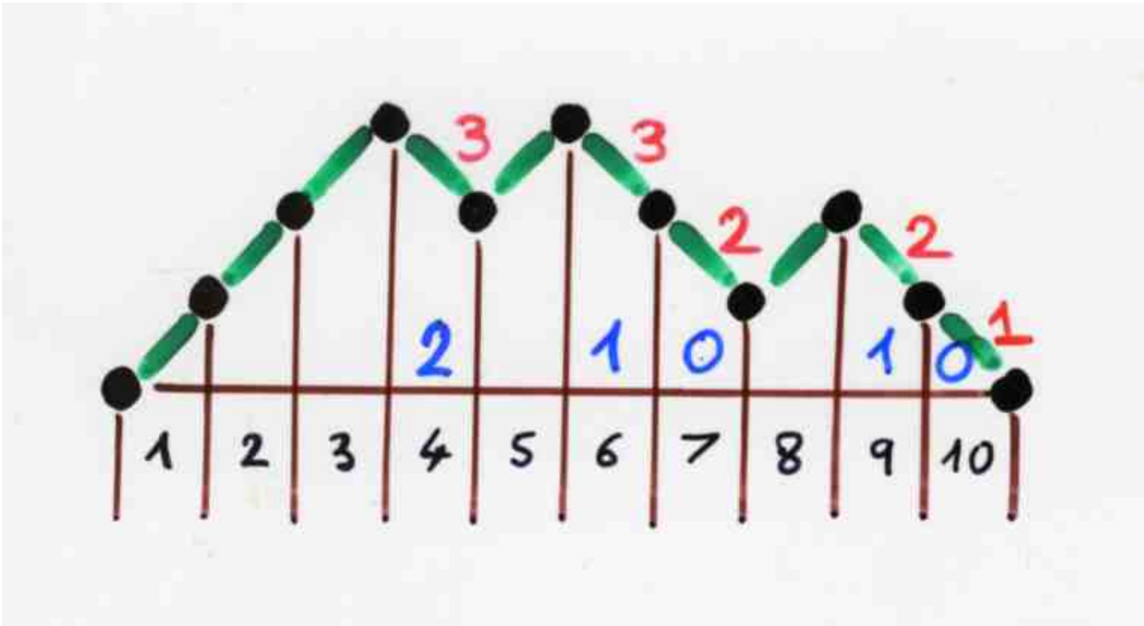

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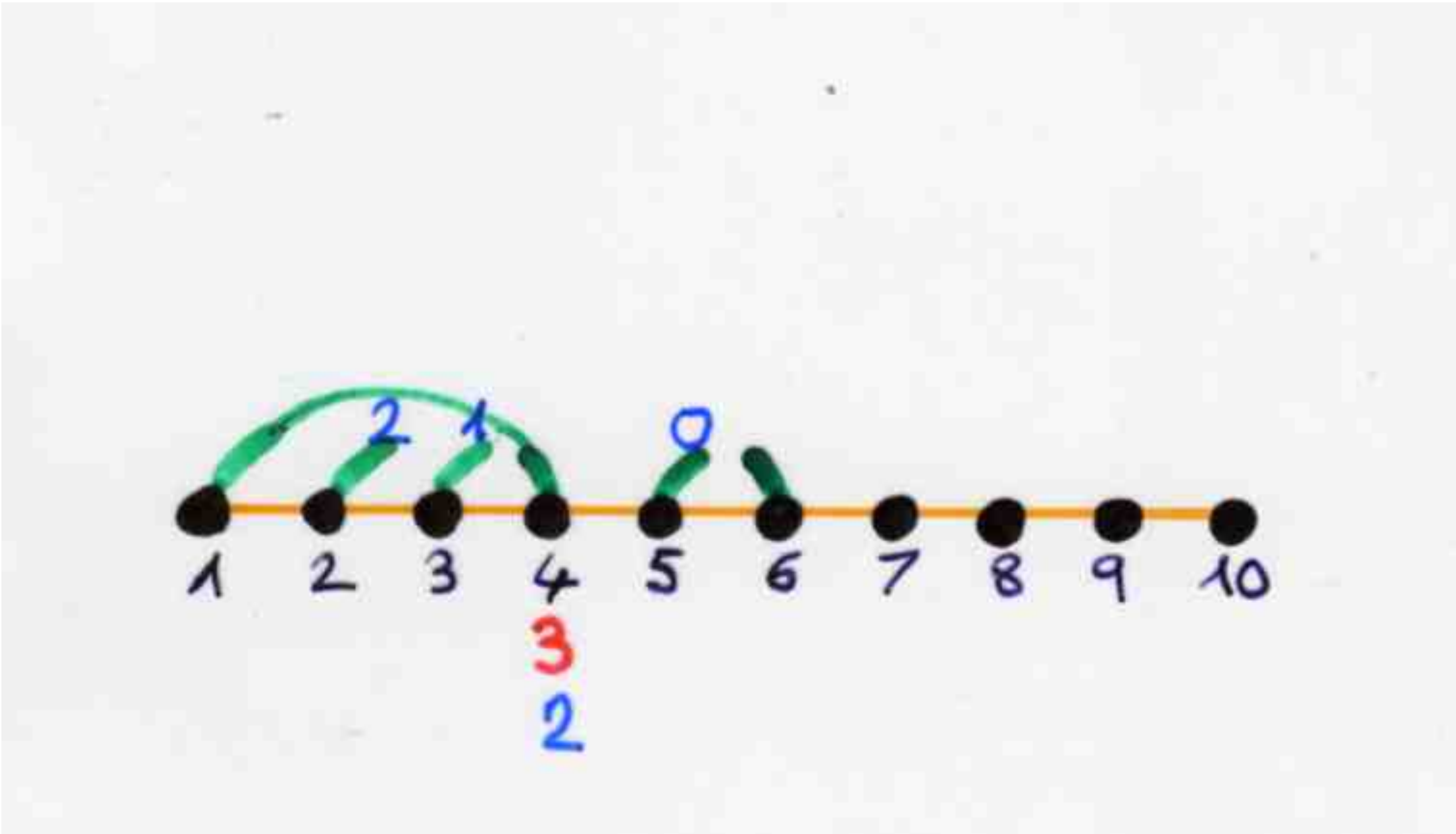

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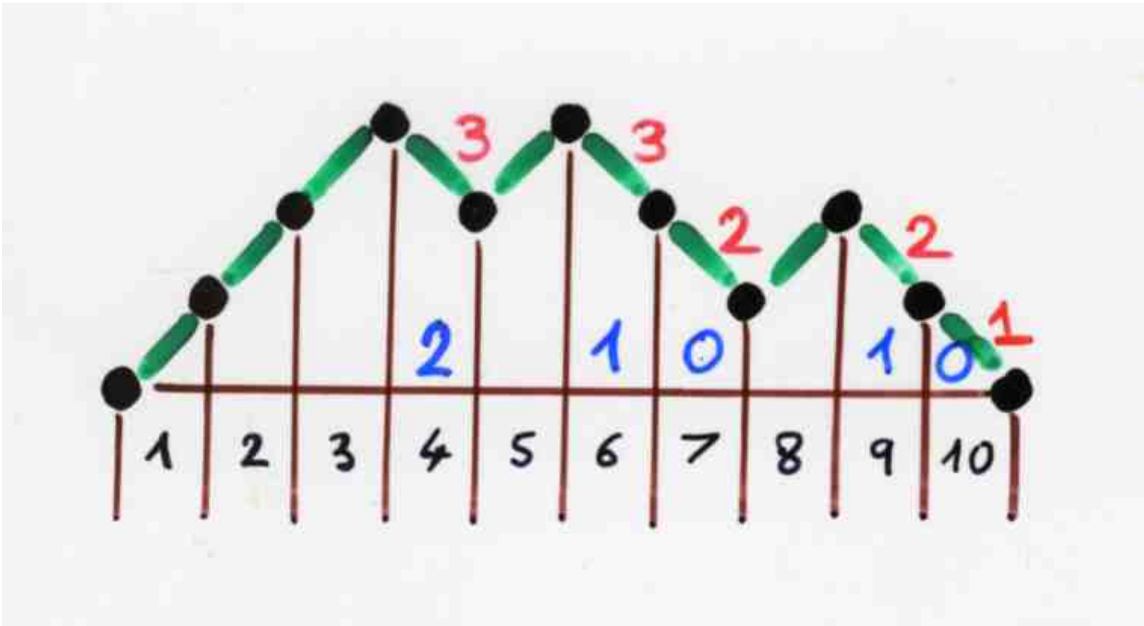
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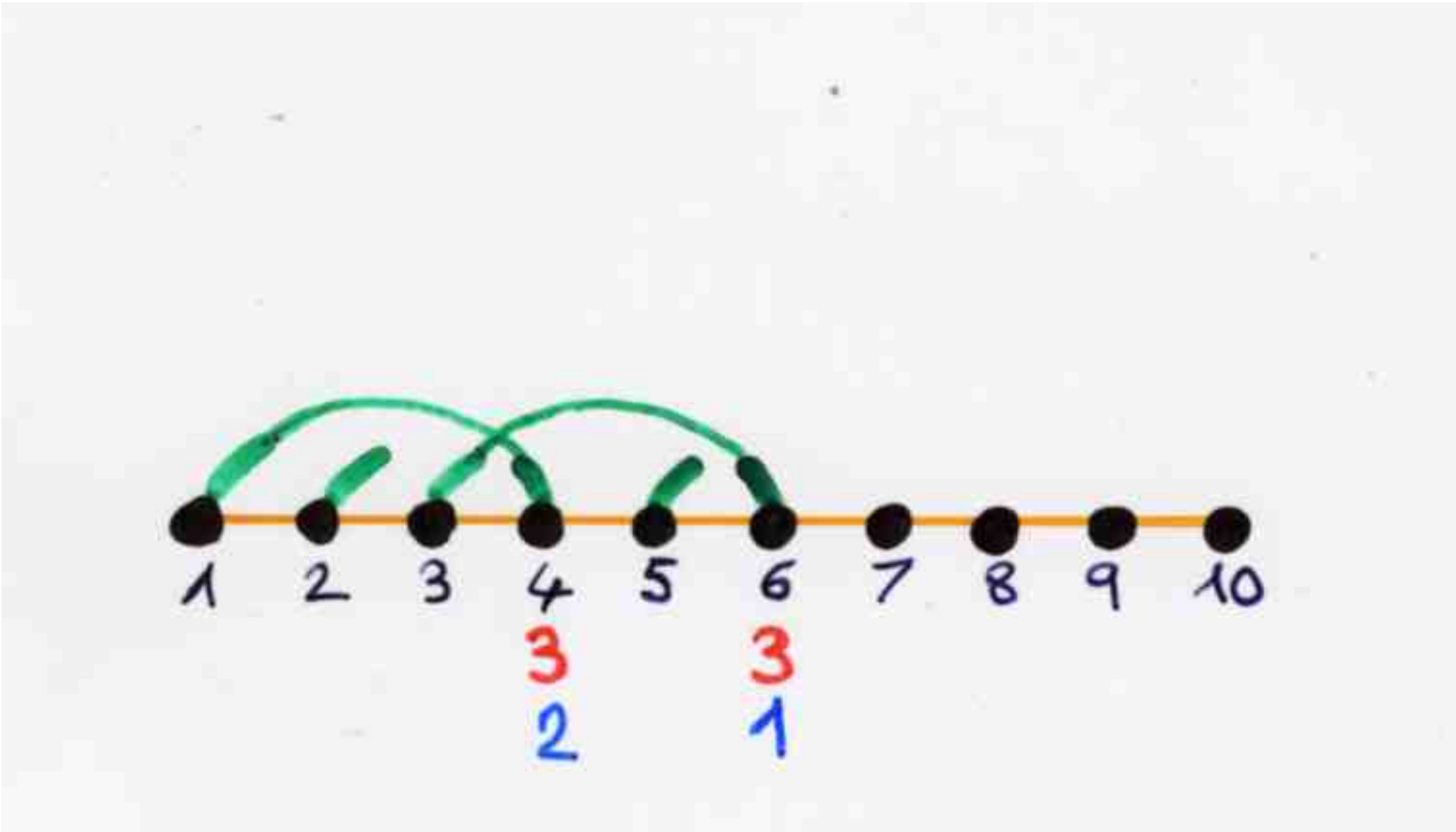
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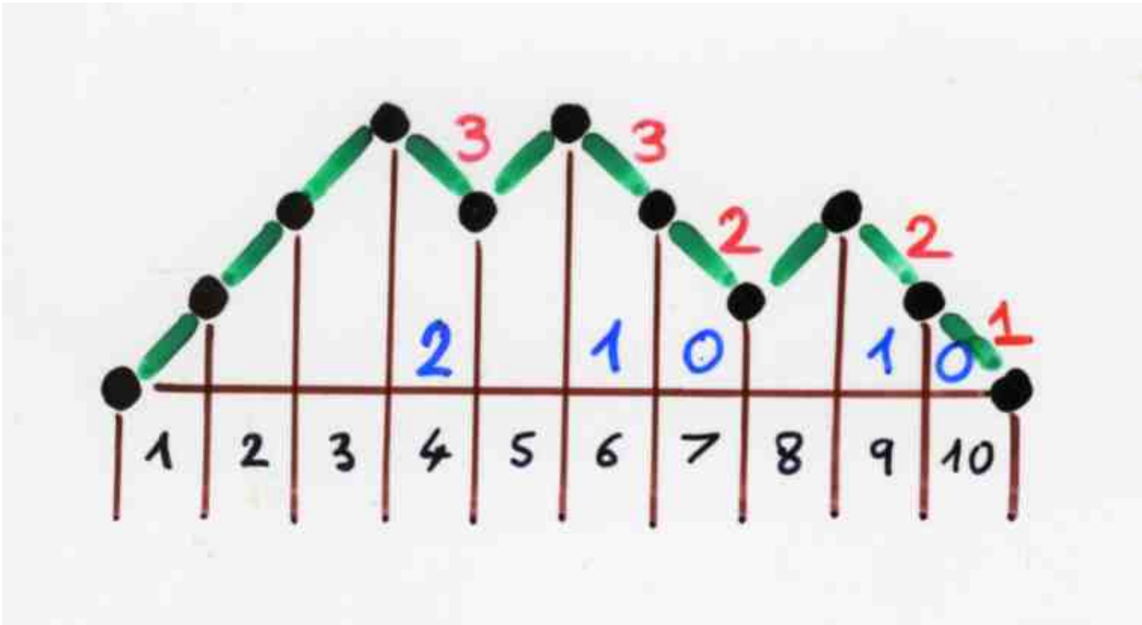
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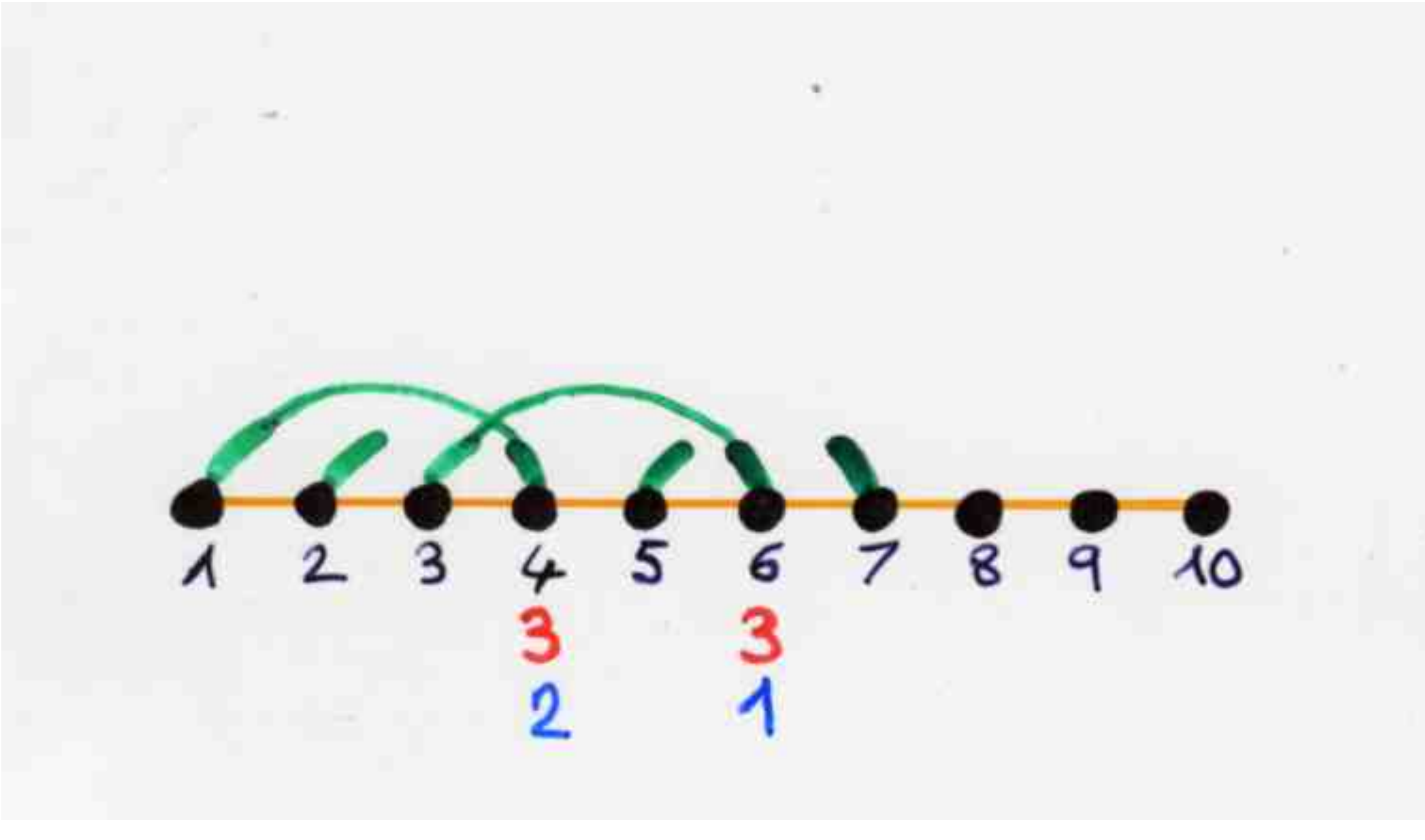
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10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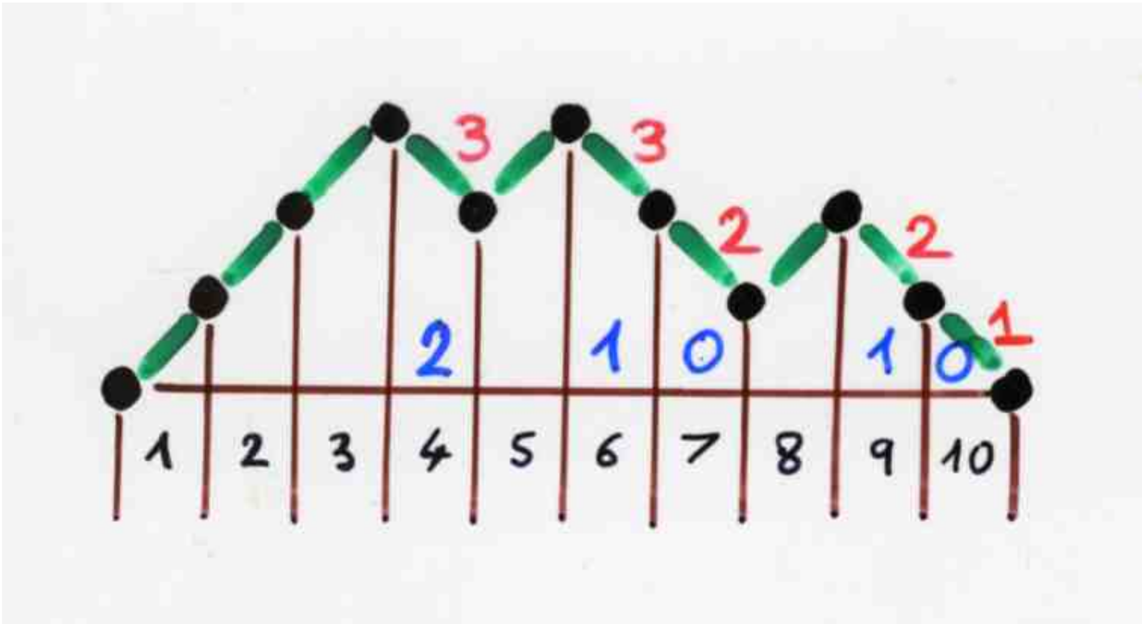
3 3
2 1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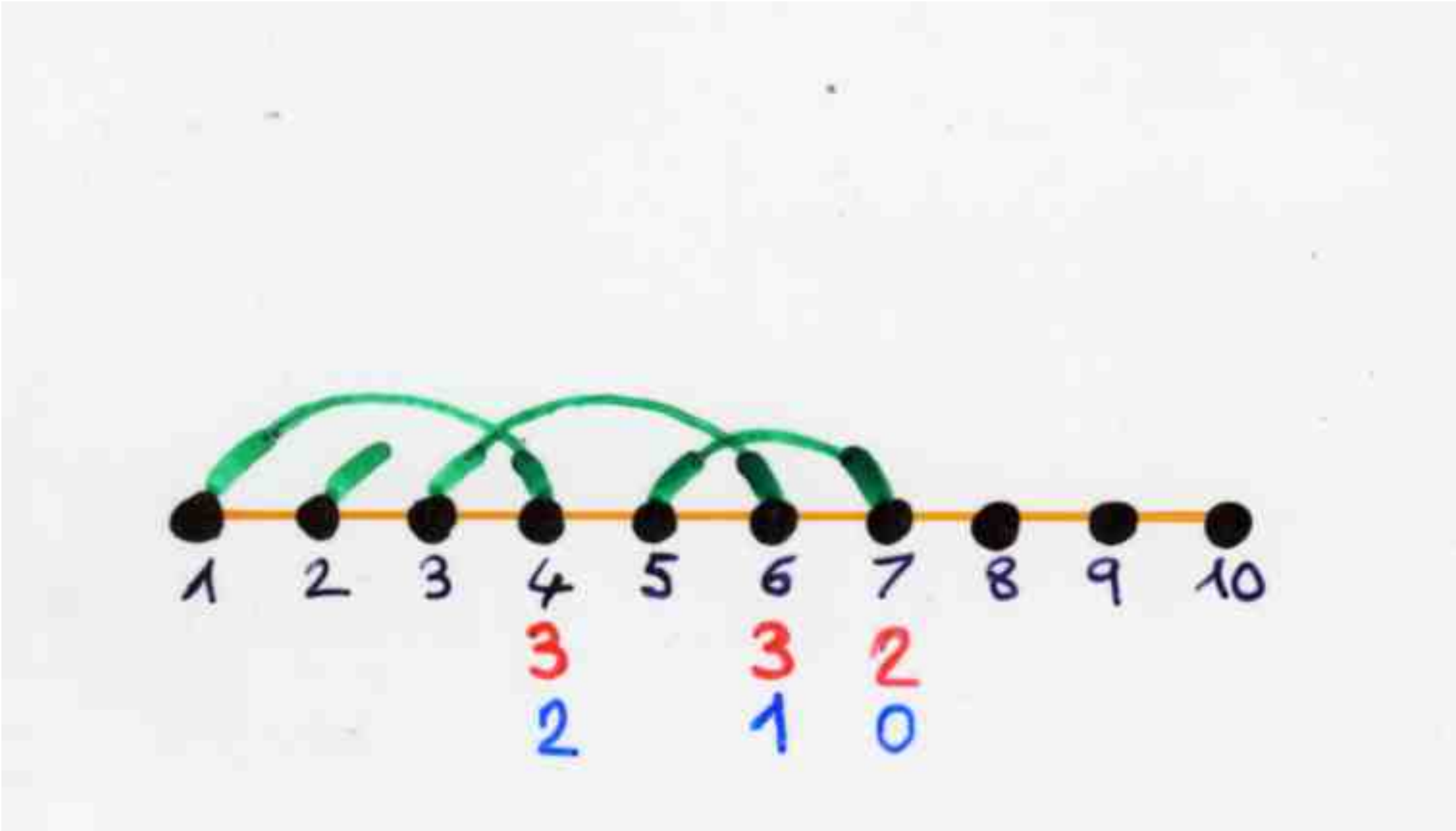
1 2 3 4 5 6 7 8 9 10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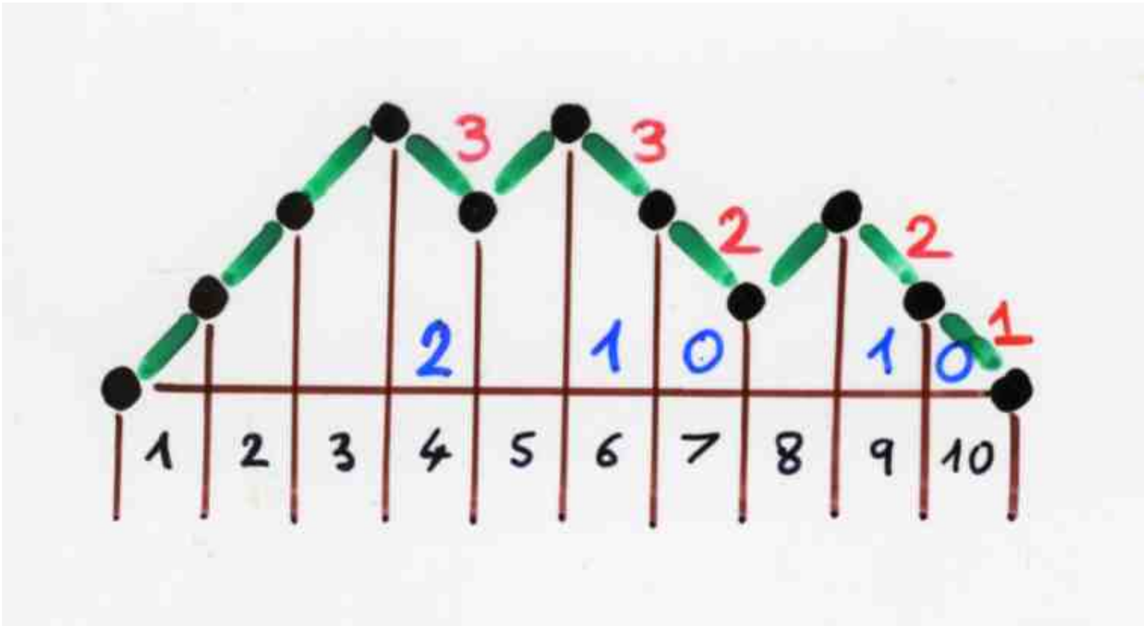
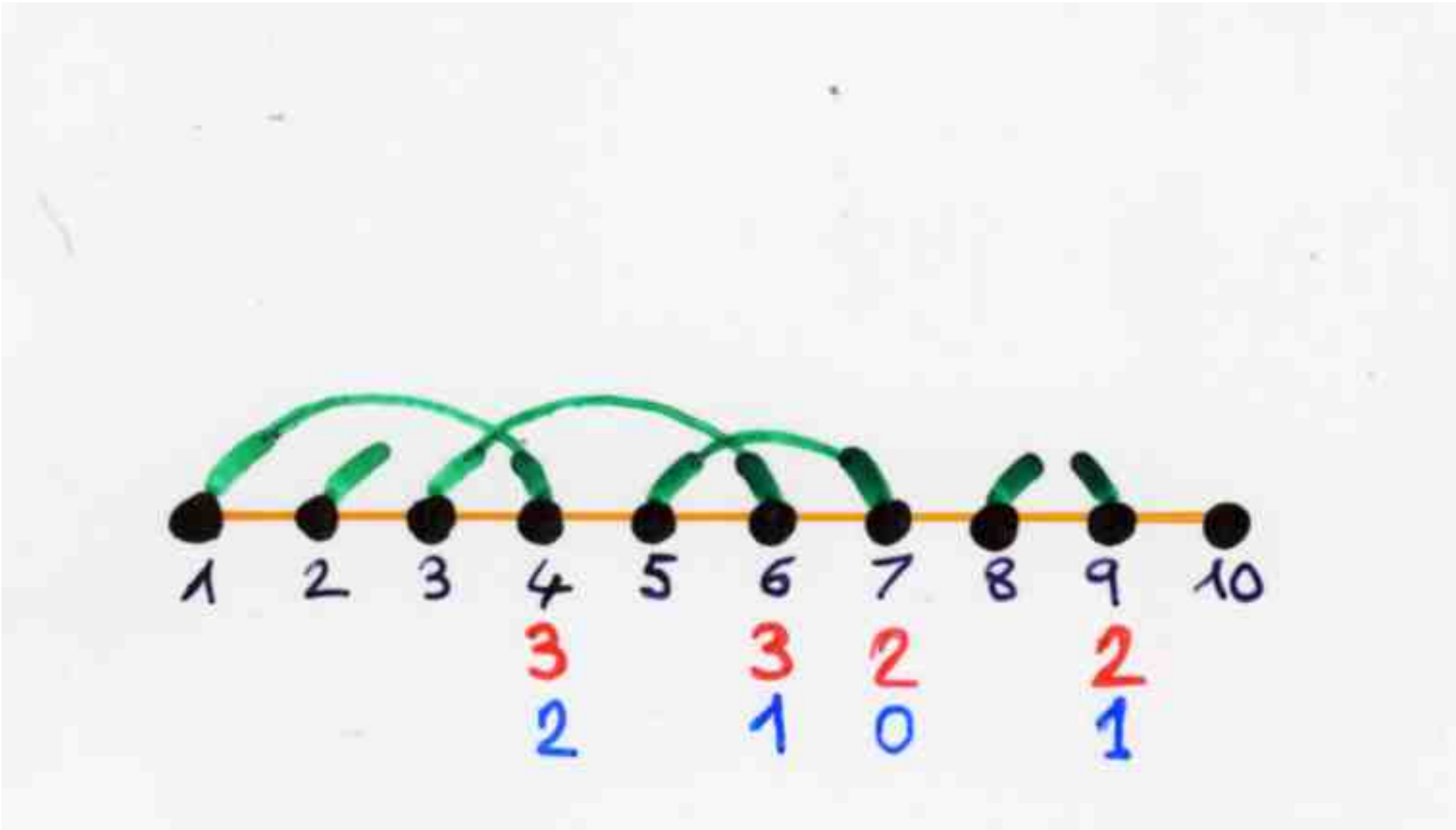
3 3 2
2 1 0


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10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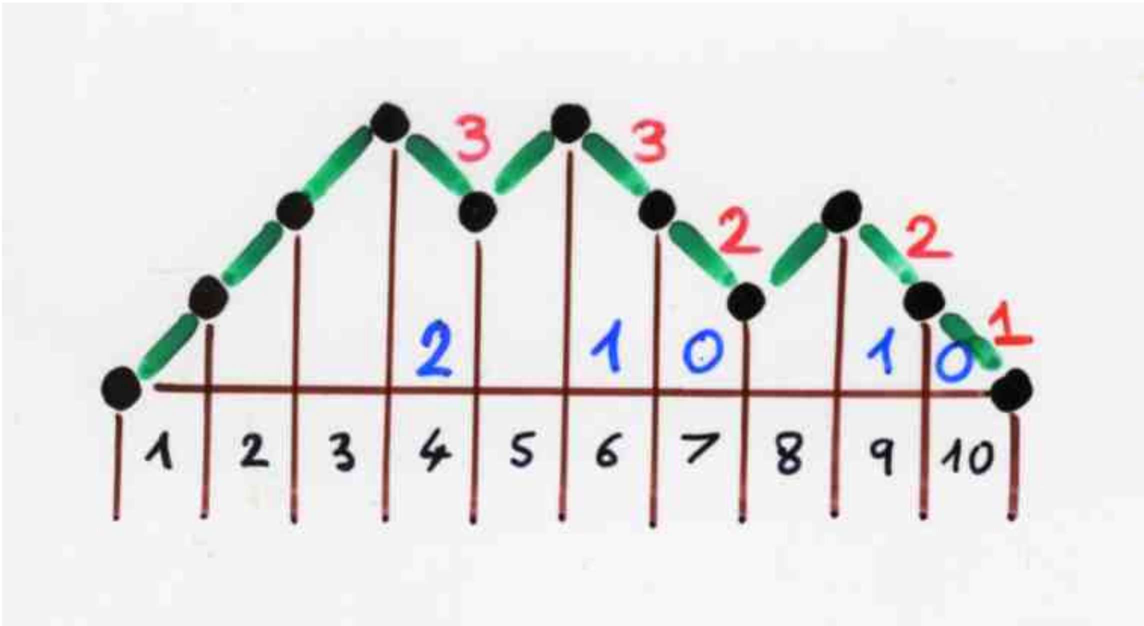
3 3 2 2
2 1 0 1


3 3 2 2 1
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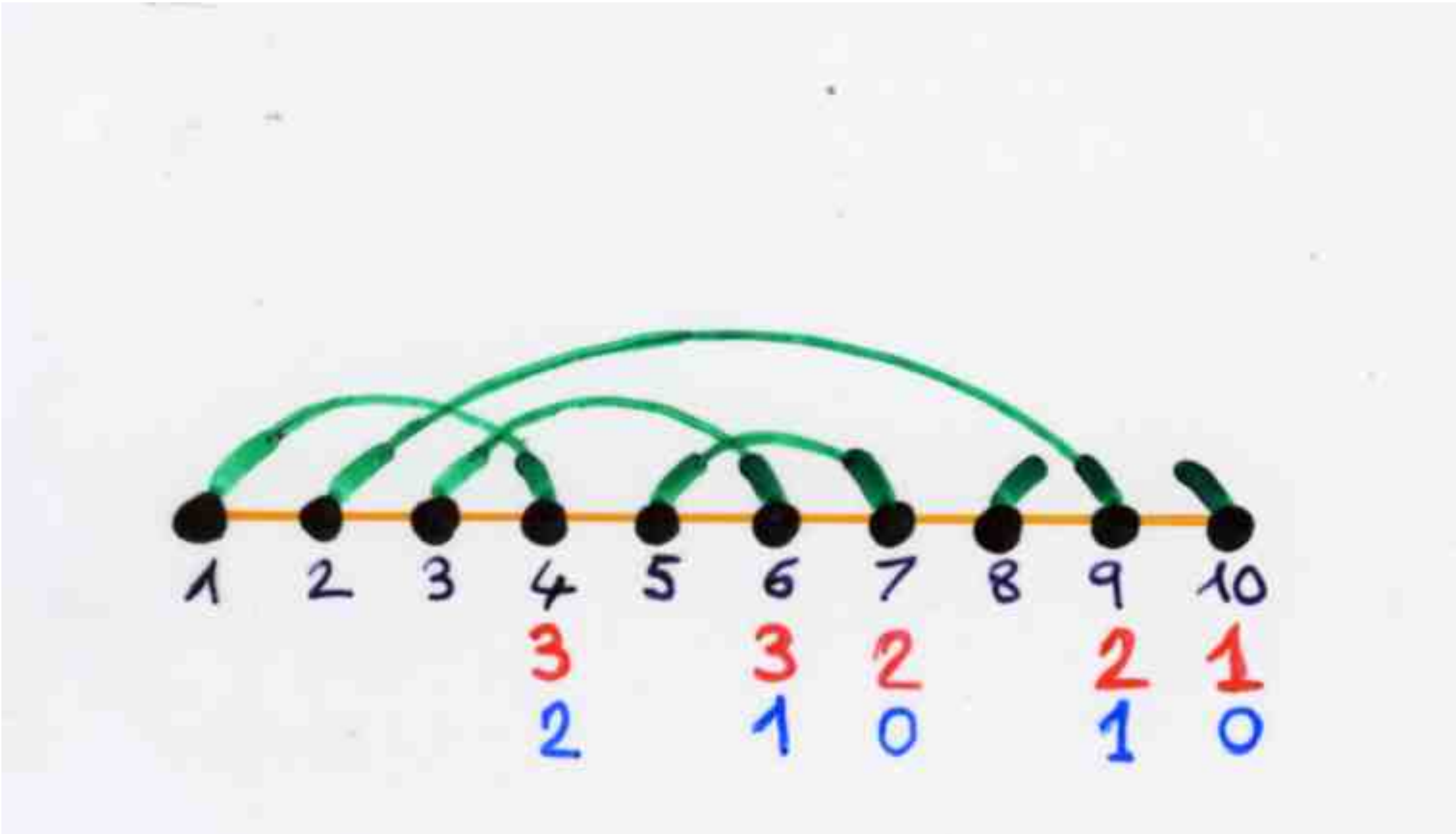
1 2 3 4 5 6 7 8 9 10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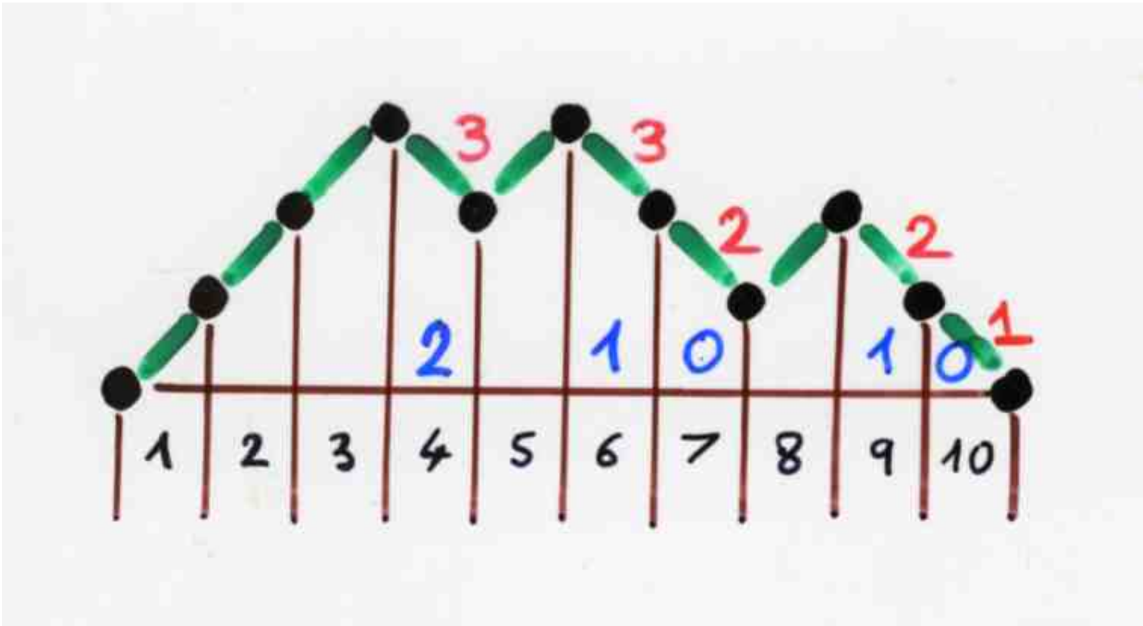
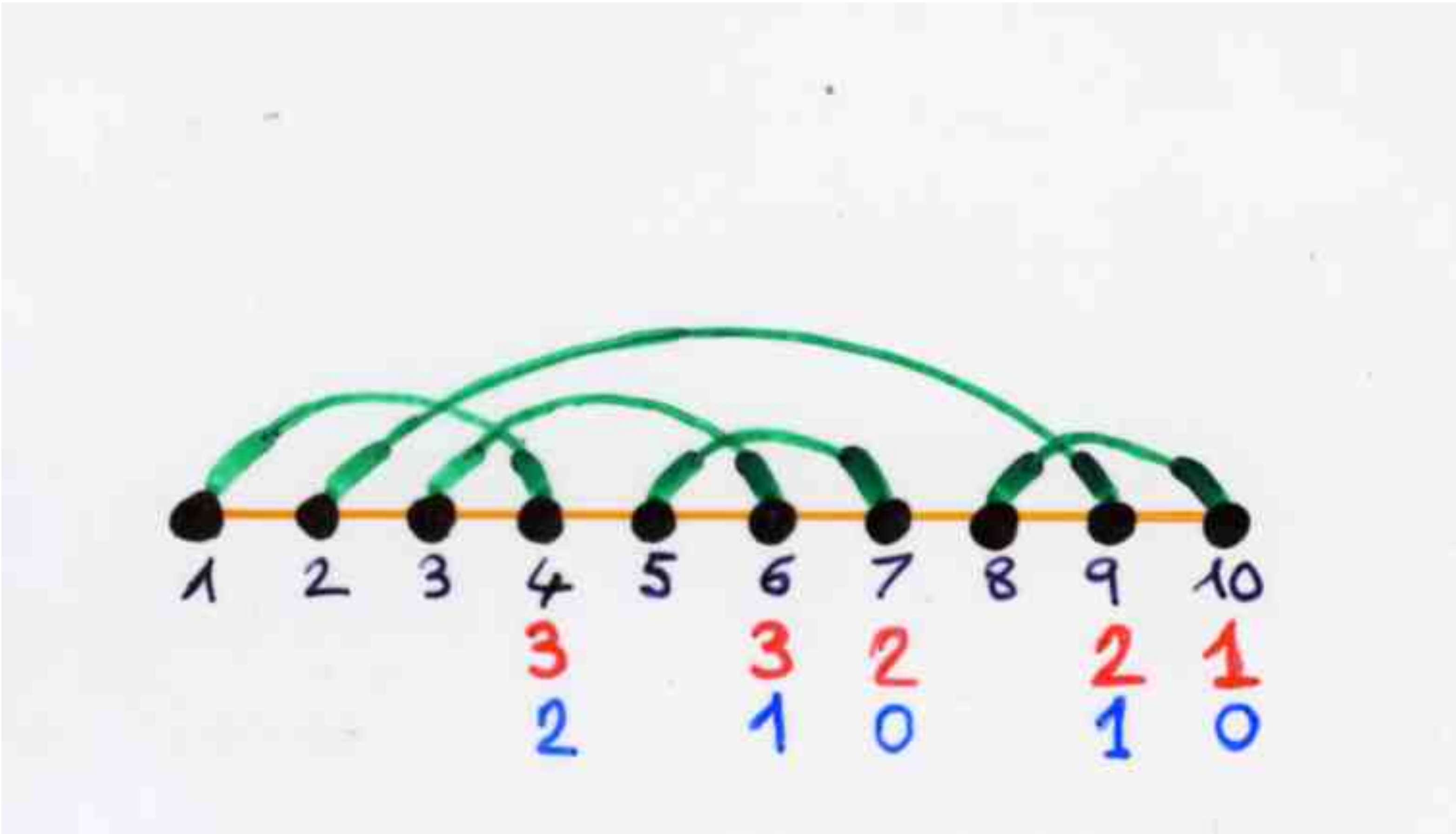
3 3 2 2 1
2 1 0 1 0


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

Laguerre histories

The FV bijection

Laguerre polynomials

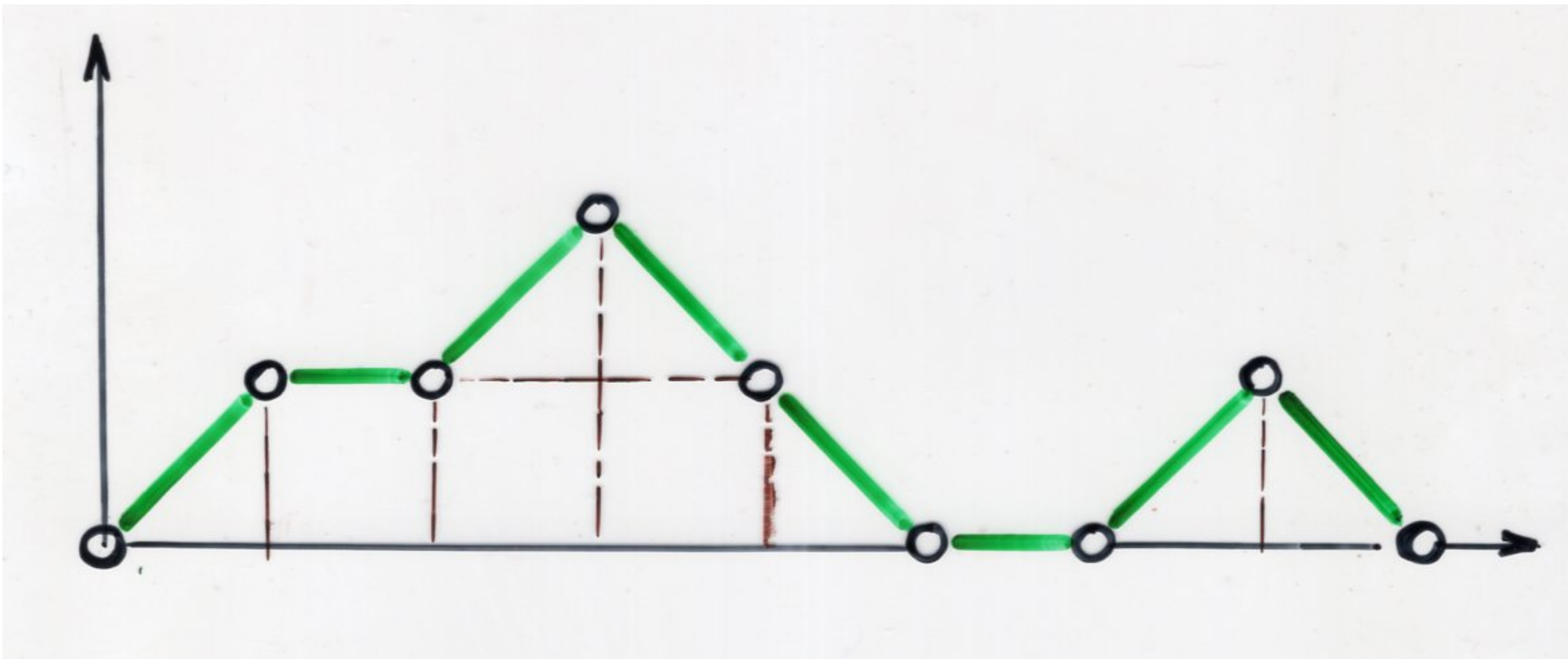
$$b_k = (2k+2)$$

$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$

Laguerre history $h = (\omega_c, p)$

Motzkin path

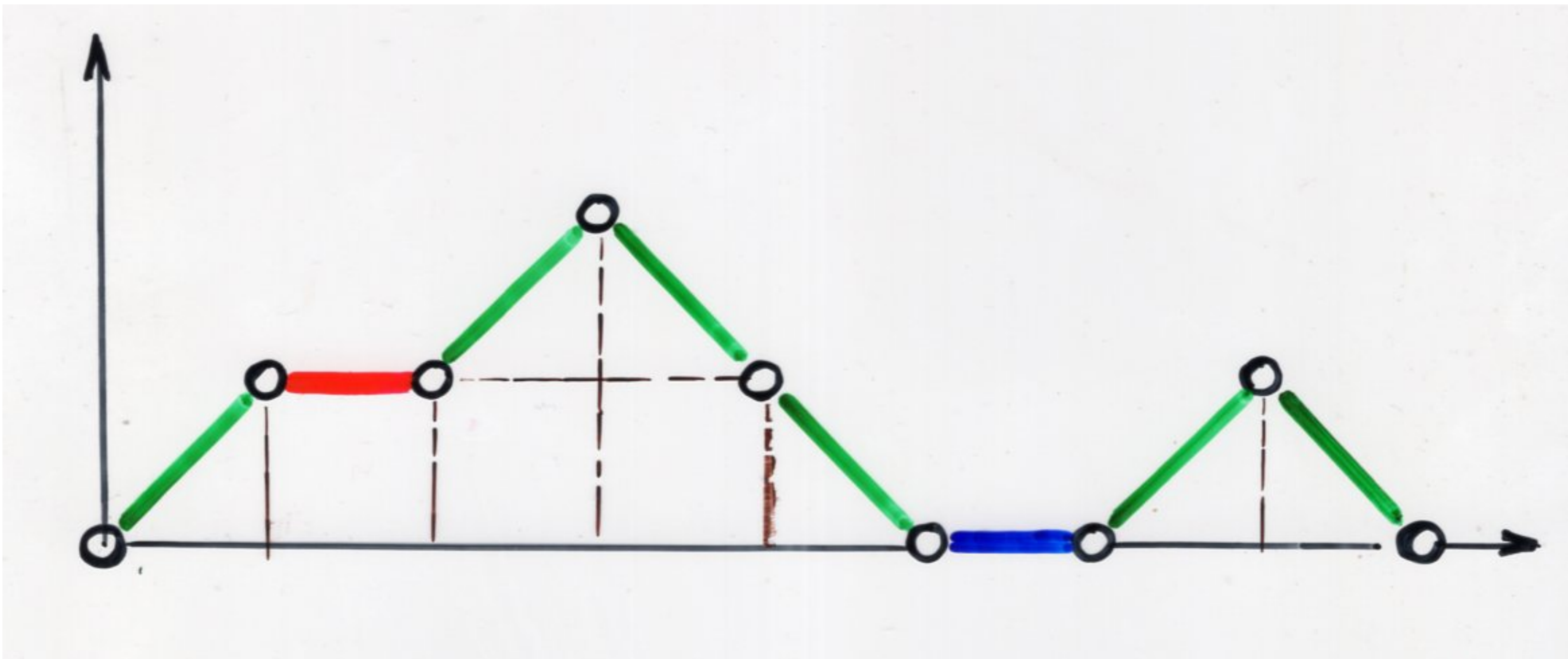


Laguerre history

$$h = (\omega_c, p)$$

Motzkin path

2 colors East steps

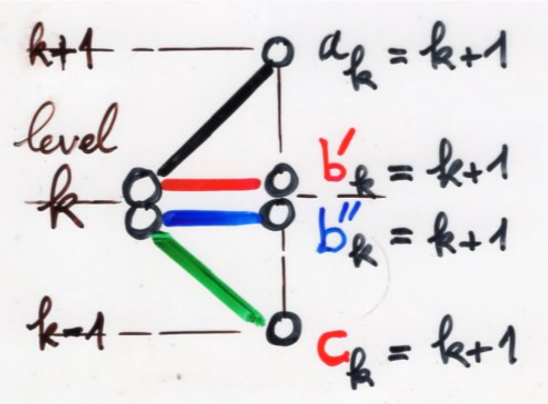


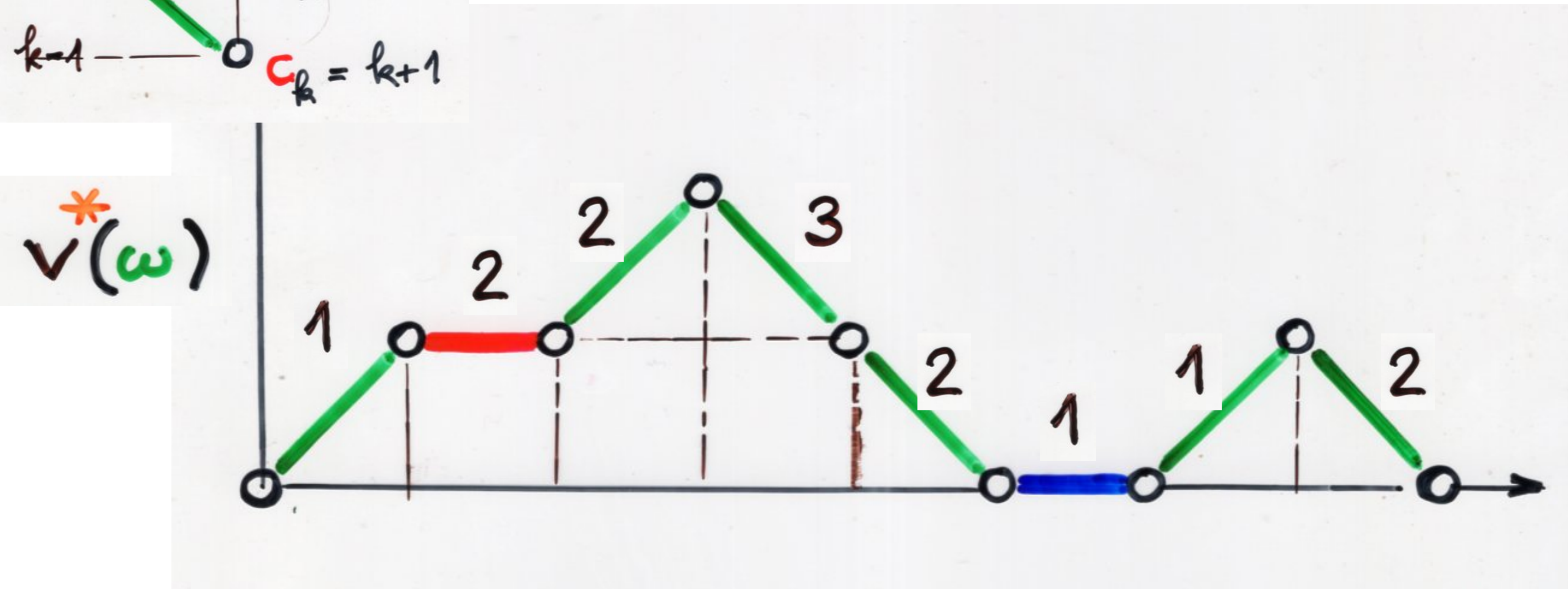
Laguerre history

$h = (\omega_c, p)$

Motzkin path

2 colors East steps





Laguerre history

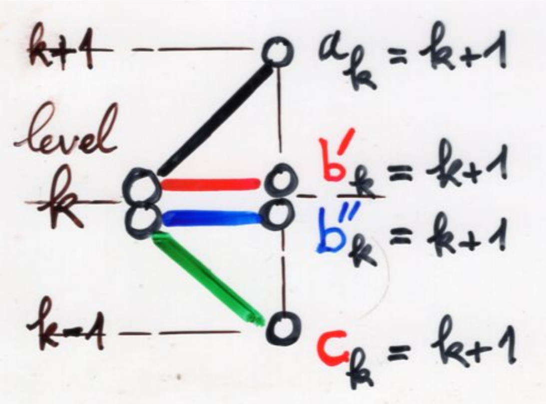
$$h = (\omega_c, p)$$

Motzkin path → ω

2 colors East steps → c

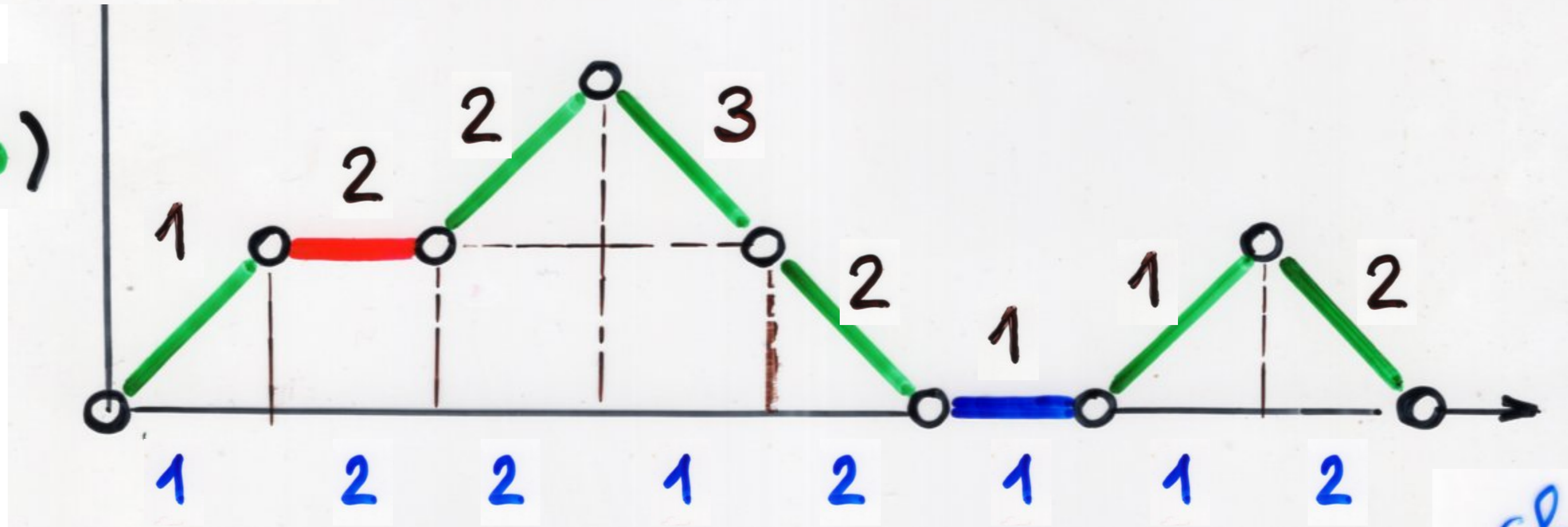
$p = (p_1, \dots, p_n)$ → p

$1 \leq p_i \leq v(\omega_i)$



$\omega = (\omega_1 \cdots \omega_n)$

$v^*(\omega)$



choice function

bijection

$h = (\omega_c ; \underbrace{(p_1, \dots, p_n)}_{p})$ $\quad |\omega| = n$ $\longleftrightarrow$ permutations $\sigma \in \mathfrak{S}_{n+1}$

Laguerre histories

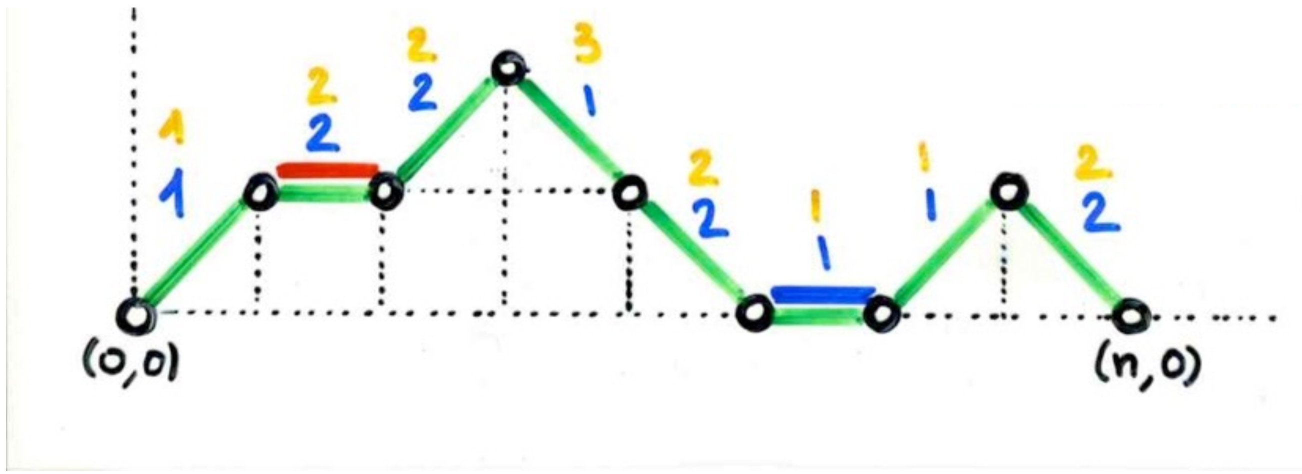
$(n+1)!$

$|h| = |\omega|$

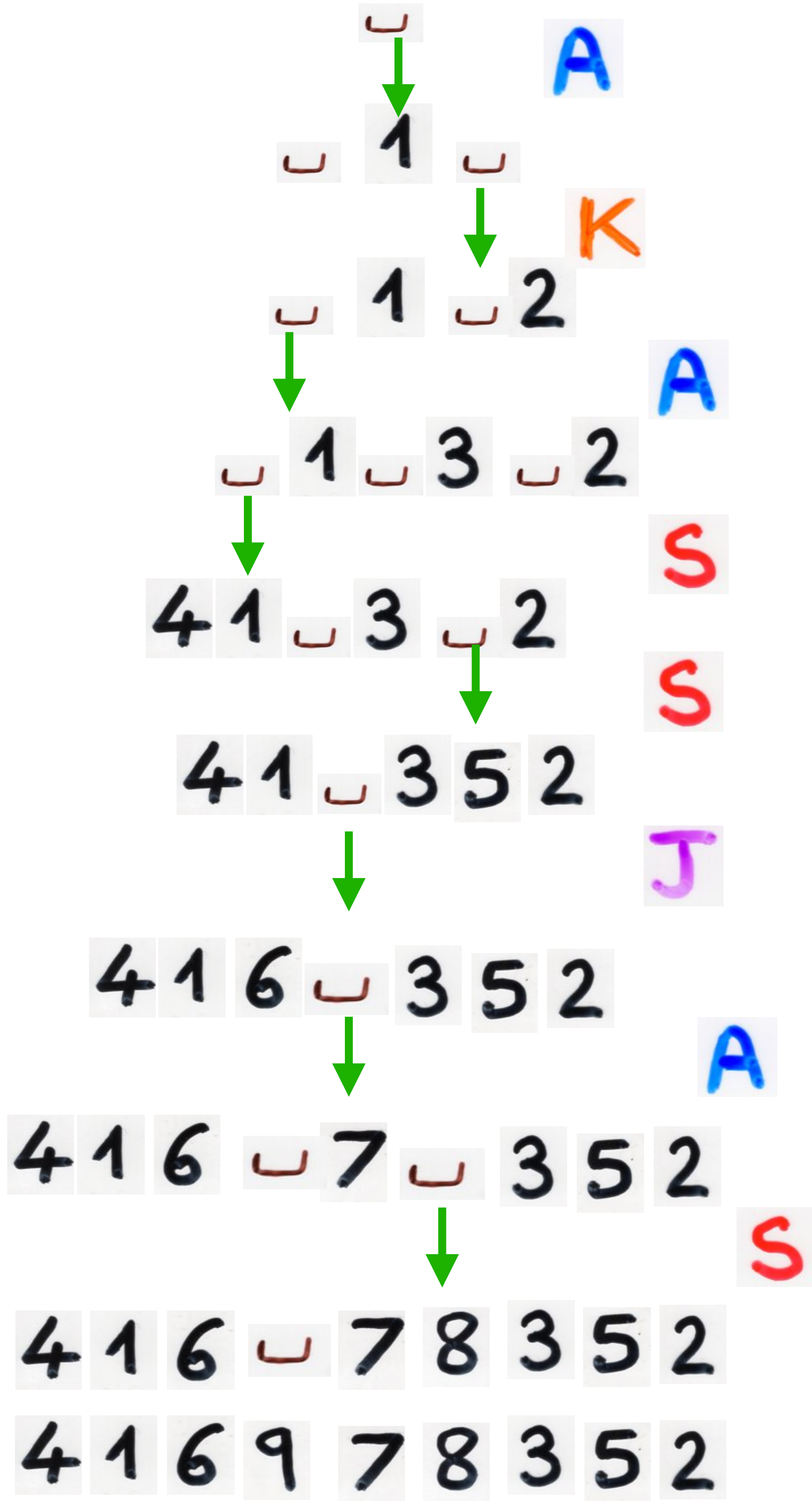
length of the history

J. Françon, X.V. (1979)

Laguerre history




1 1 1
2 2 2
3 2 2
4 3 1
5 2 2
6 1 1
7 1 1
8 2 2


A
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J
A
S
1
1 2
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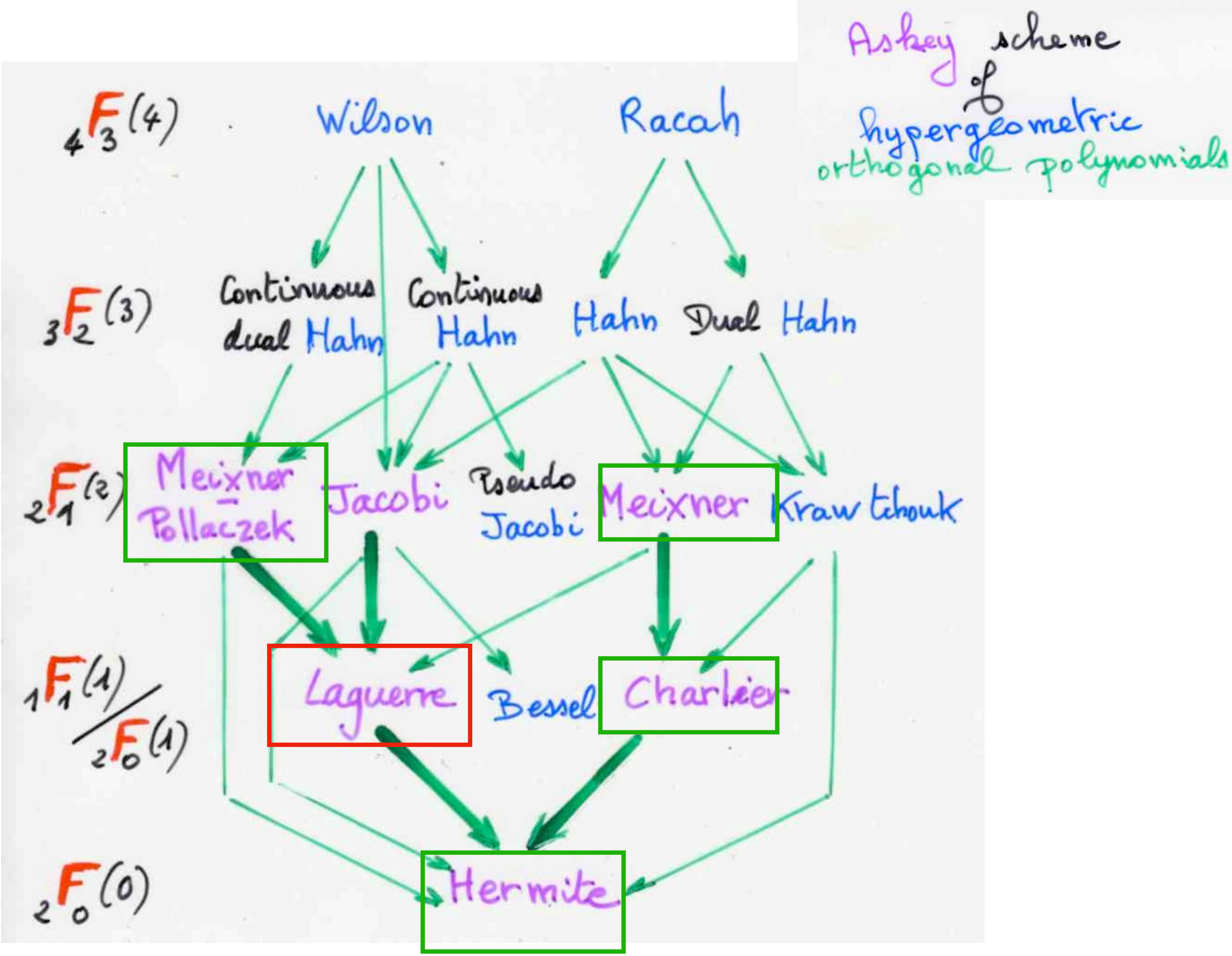
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2
4 1 6 7 3 5 2
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2



Askey scheme of hypergeometric orthogonal polynomials
${}_4F_3(4)$
Wilson
Racah
${}_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
${}_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
${}_1F_1(1)$ / ${}_2F_0(1)$
Laguerre
Bessel
Charlier
${}_2F_0(0)$
Hermite

Sheffer polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\exp\left(x f(t)\right)$$

$\{P_n(x)\}_{n\geq 0}$ orthogonal polynomials

Meixner (1934)

are Sheffer polynomials

$\Leftrightarrow$ $\{P_n(x)\}_{n\geq 0}$ are one of the 5 possible types:

Hermite

Laguerre

Charlier

Meixner

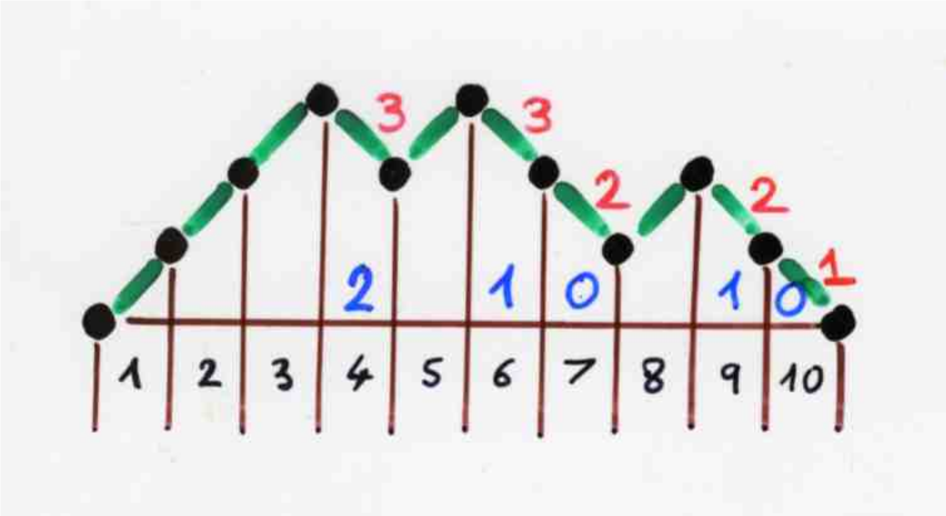
Meixner - Pollaczek

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\dots\times(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\frac{(1+c)k+\beta c}{(1-c)}$	$\frac{ck(k-1+\beta)}{(1-c)^2}$	$\sum_{\sigma\in\mathfrak{S}_n} \frac{\beta^{s(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

Some q-analogues of orthogonal polynomials

$$\lambda_k = [k]_q$$

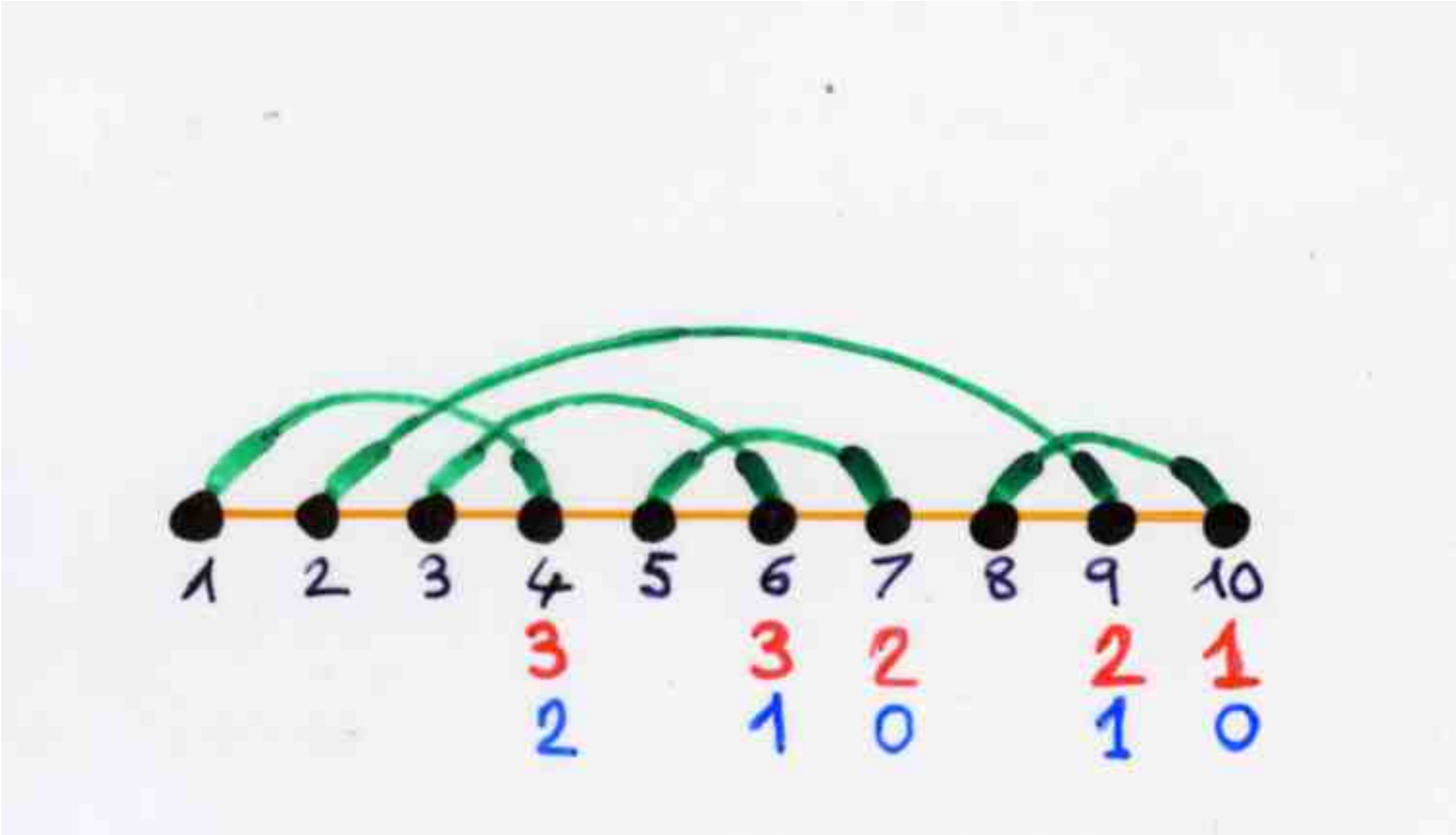
$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$



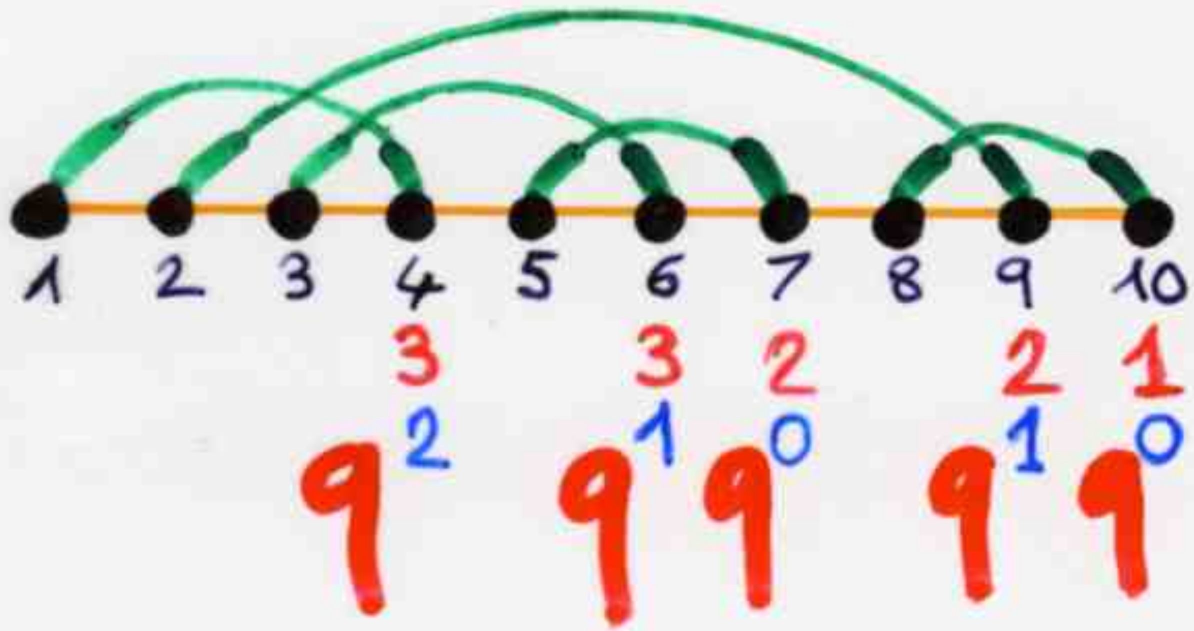
Hermite history related to ω

ω
Dyck path

$$v_q(h) \qquad q^{2+1+0+1+0} \qquad = q^4$$

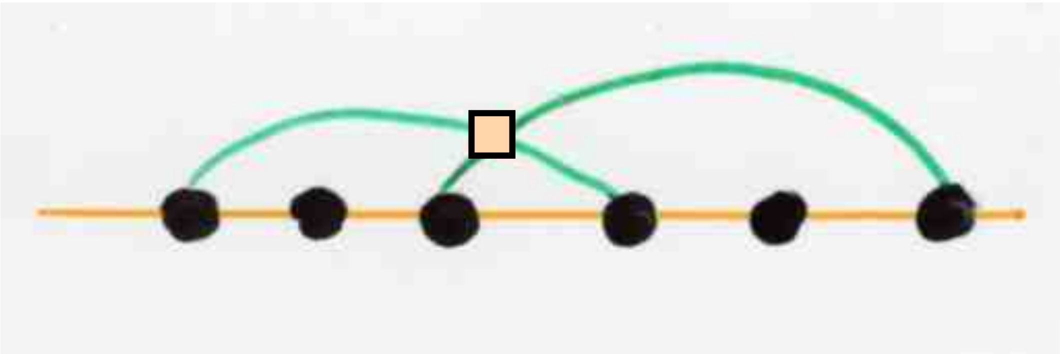


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

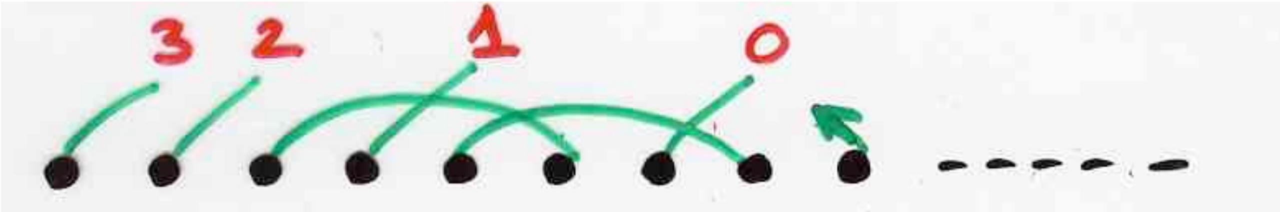


$$q^{2+1+0+1+0} = q^{4}$$

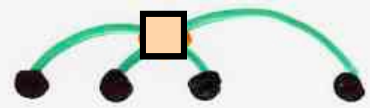
$$v_q(h)$$

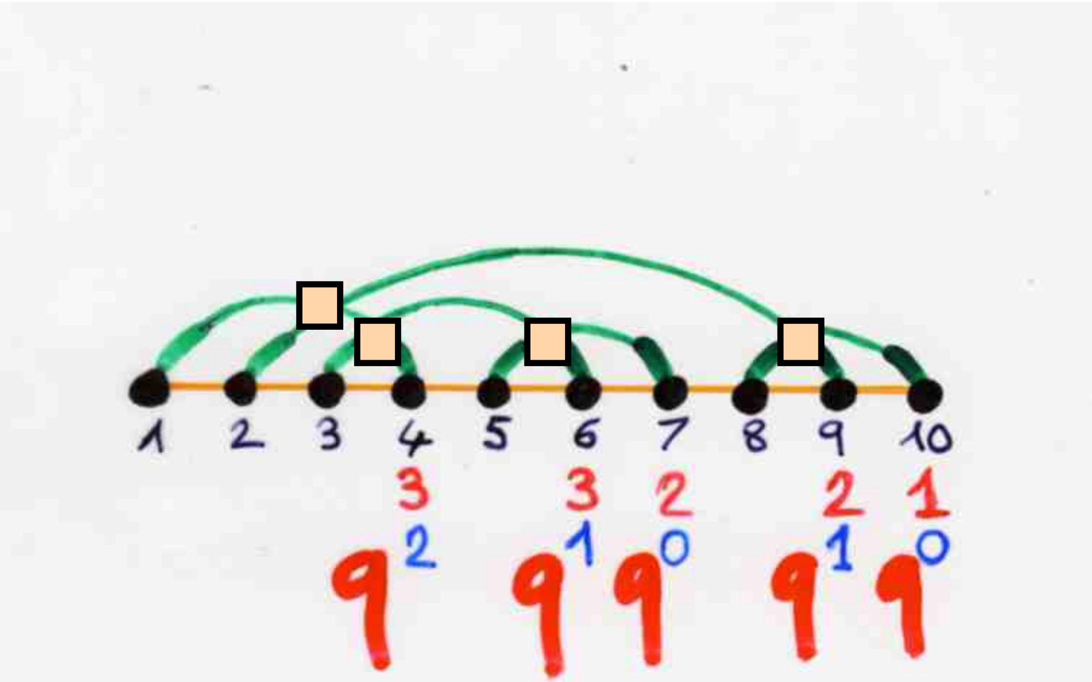


crossing



crossing





$v_q(h)$

$q^{2+1+0+1+0} = q^4$

Epilogue

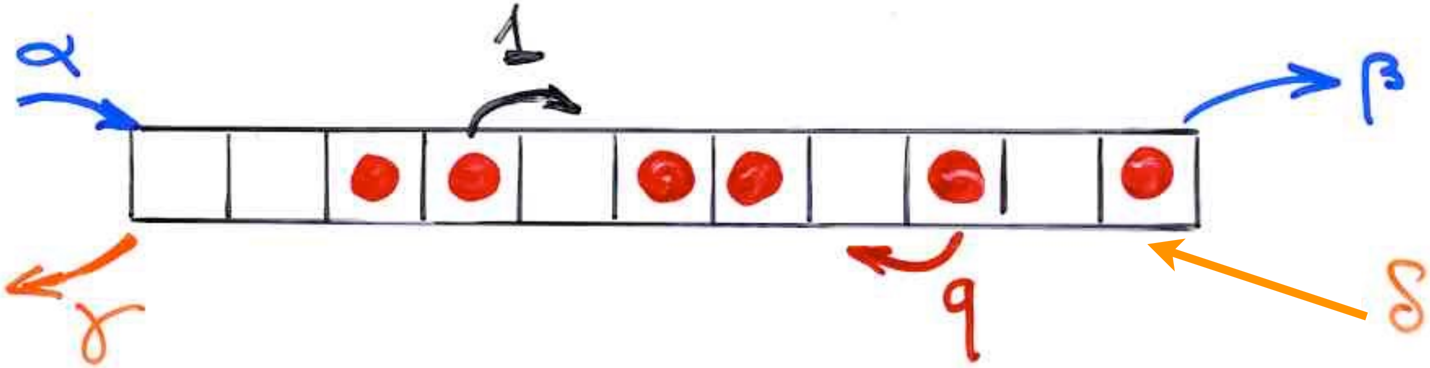
PASEP

and

orthogonal polynomials

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

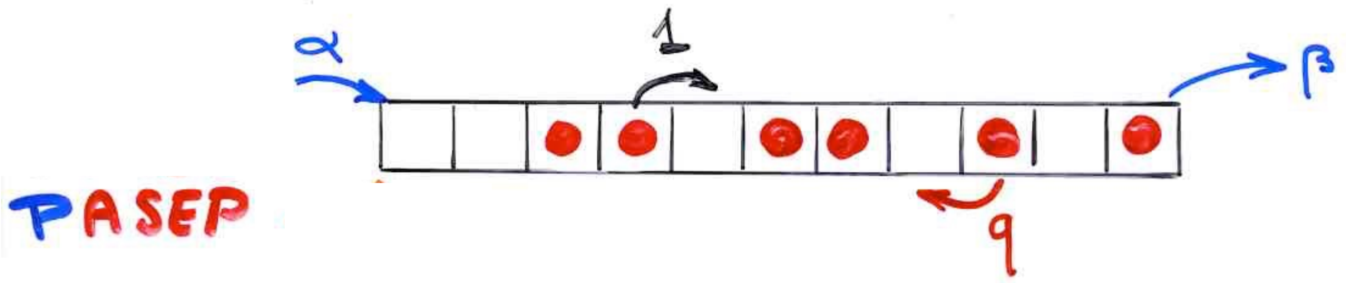
$$\begin{cases} DE = qED + E + D \\ \langle W | (\alpha E - \gamma D) = \langle W | \\ (\beta D - \delta E) | V \rangle = | V \rangle \end{cases}$$

colum vector V

row vector W

PASEP with 3 parameters

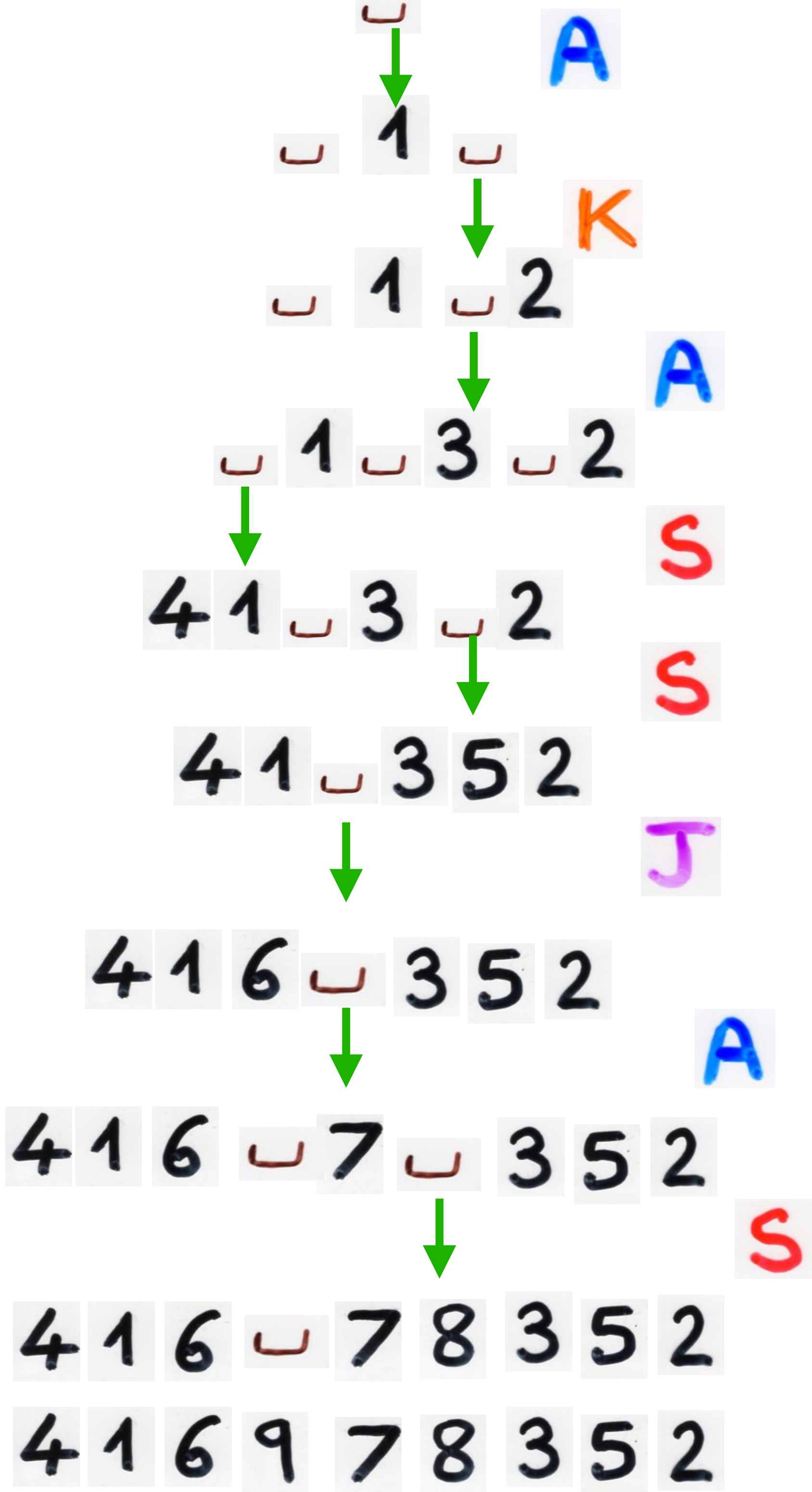
$\gamma = \delta = 0$ $\quad q, \alpha, \beta$



$$\begin{cases} DE = qED + E + D \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

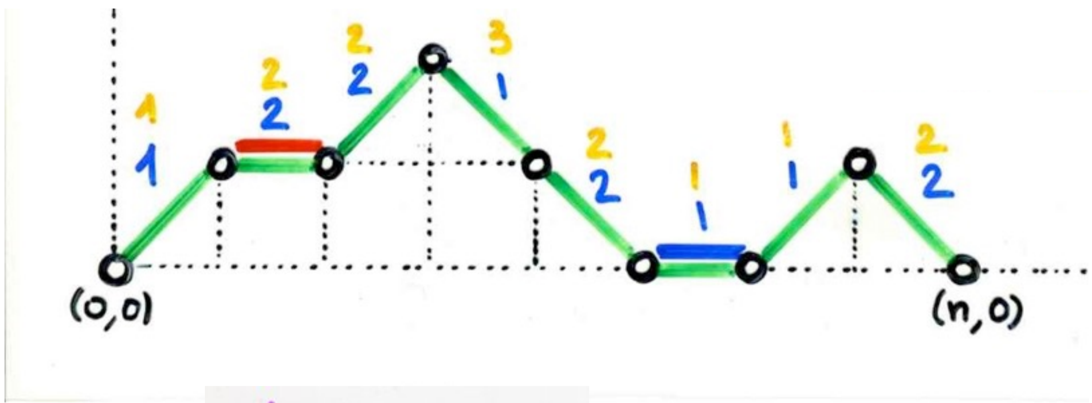
$\bar{\beta} = \dfrac{1}{\beta}$

$\bar{\alpha} = \dfrac{1}{\alpha}$



$$D = A + K$$
$$E = S + J$$

$$DE = ED + E + D$$



Laguerre histories

PASEP with 3 parameters

Z_n partition function = moments of q-Laguerre polynomials

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q, α, β

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

9

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cdots}}}}}}$$

$$\lambda_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$$

$$\sum_{n \geq 0} (n!)_q \, t^n = \cfrac{1}{1 - (1)t - \cfrac{}{1 - (1)\,t - \cfrac{}{1 - (1+q)t - \cfrac{}{1 - (1+q)t - \cfrac{}{1 - (1+q+q^2)t - \cfrac{}{1 - \ldots}}}}}}$$

subdivided
Laguerre
histories

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\quad\quad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

Pairs of

Hermite histories

$$UD = qDU + I$$

Hermite polynomials

↕

permutations

$$DE = qED + E + D$$

Laguerre histories

Laguerre polynomials

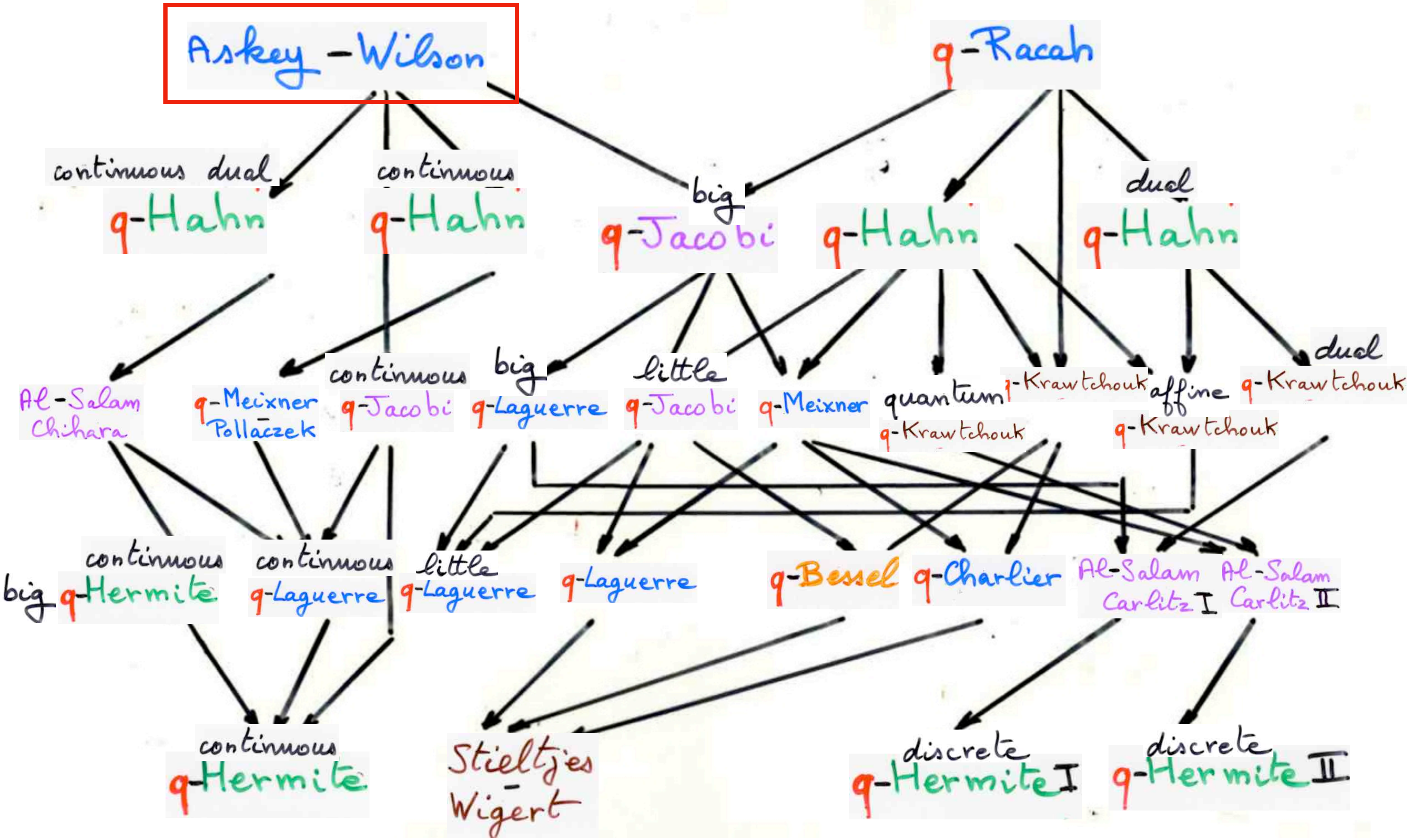
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009) staircase tableaux scheme of basic hypergeometric orthogonal polynomials



The Art of Bijective Combinatorics

Part IV. Combinatorial theory of orthogonal polynomials and continued fractions (2019)

« Video-book »

- videos
- slides
- www.viennot.org

mirror website

www.imsc.res.in/~viennot

Thank you !