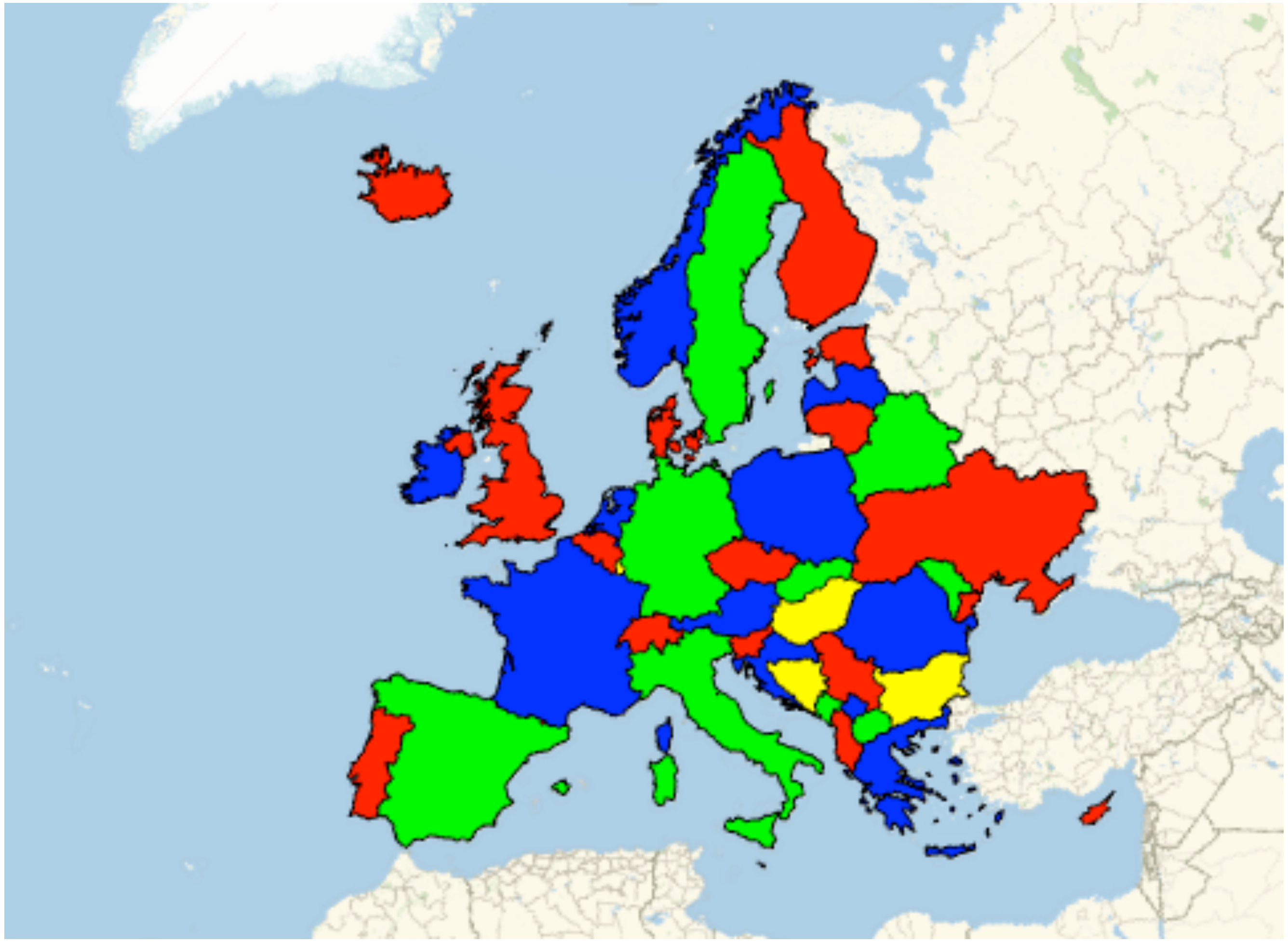
How to color a map with (-1) color ?

(first part)

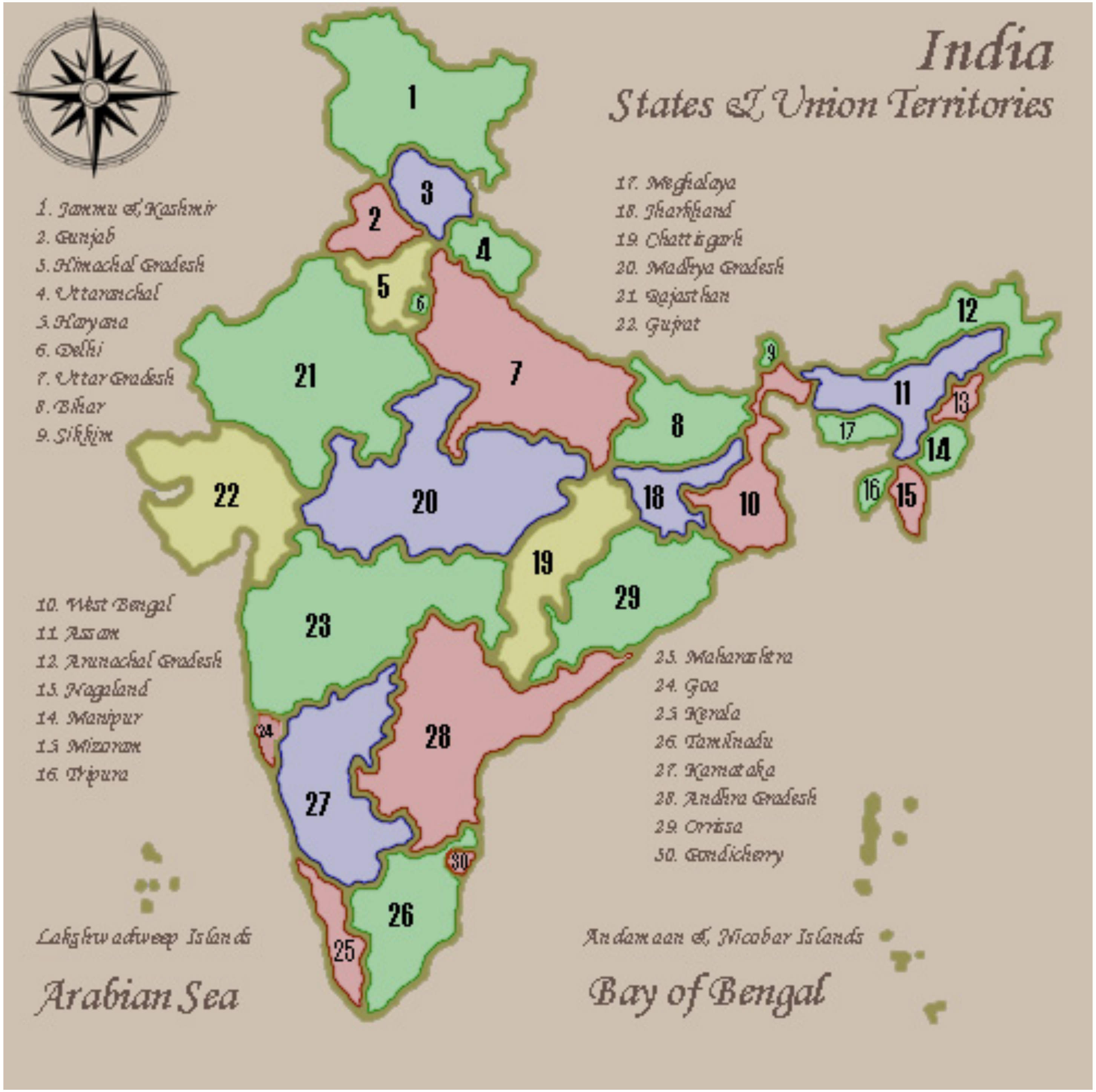
Amrita Vishwa Vidyapeetham
Amrita University, Coimbatore
7 March 2017

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.xavierviennot.org/xavier

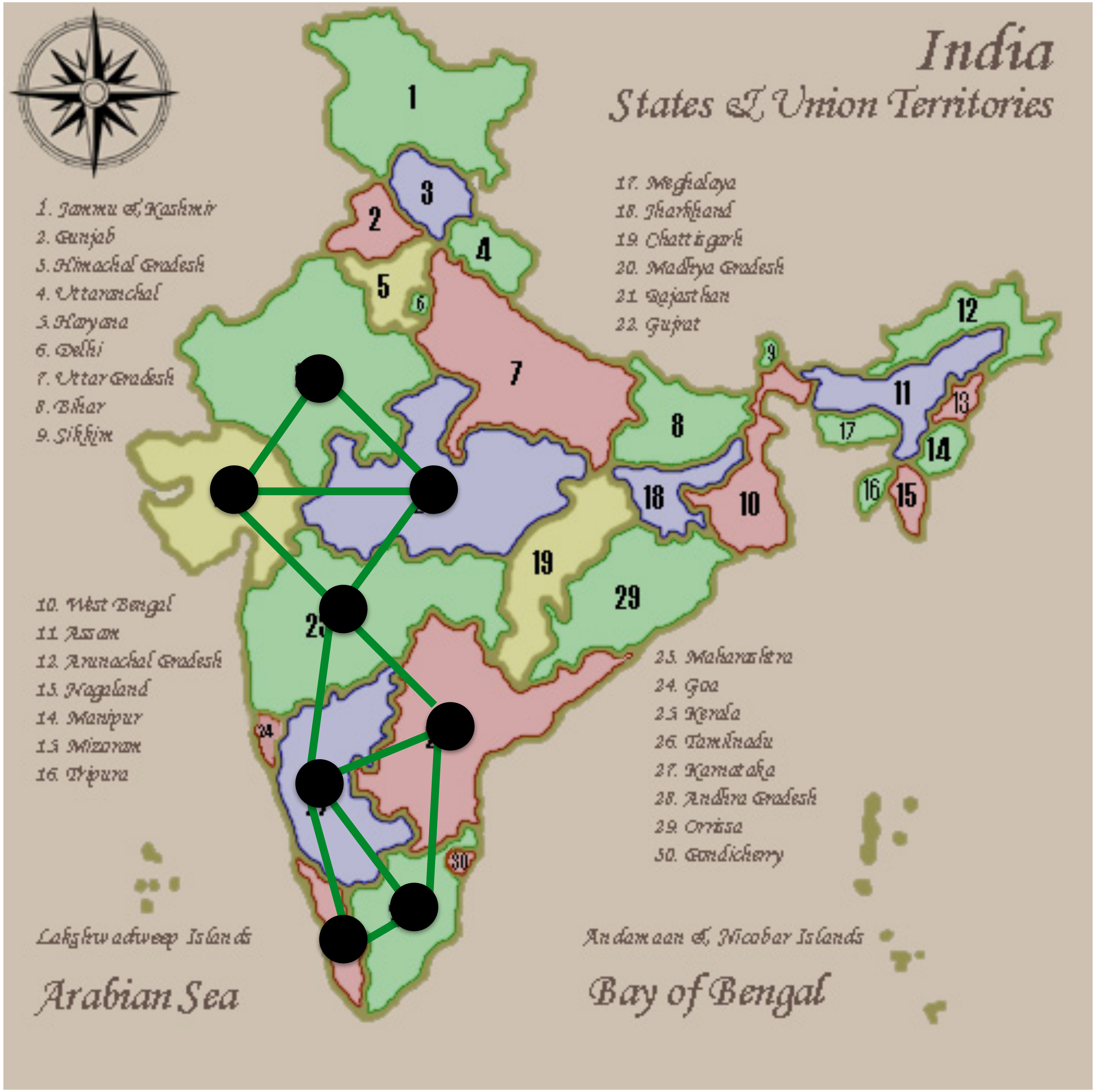


the four colors theorem

The Vertex Coloring Algorithm
Ashay Dharwadker



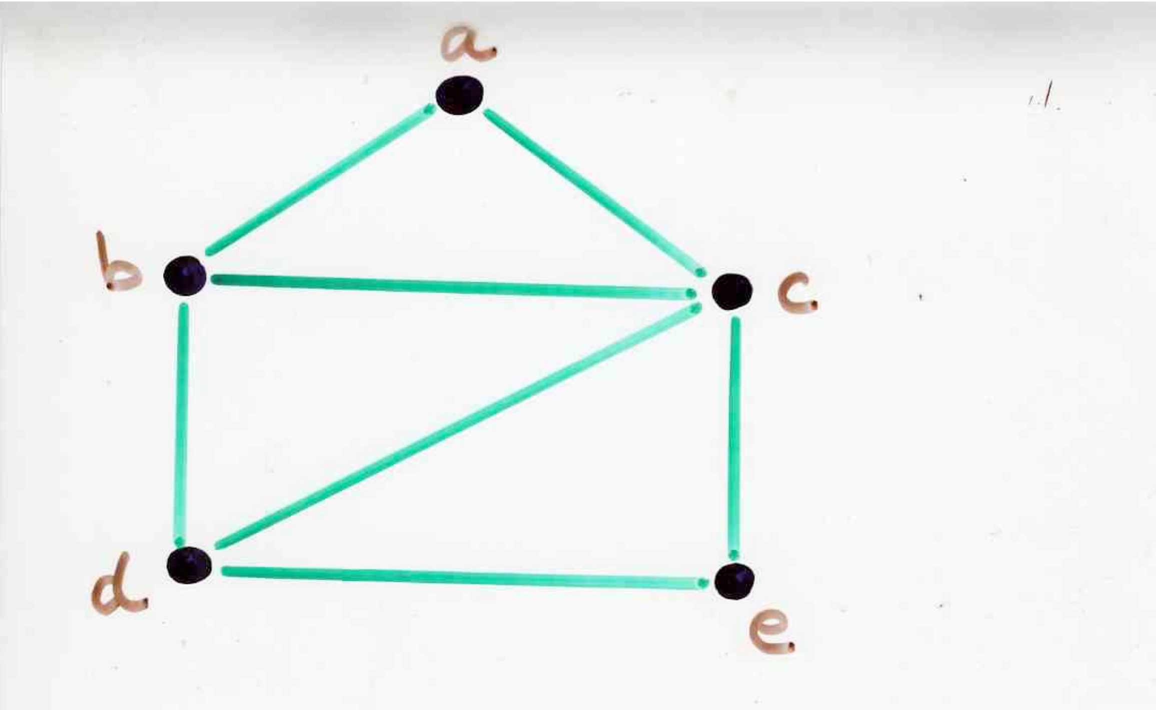
The Vertex Coloring Algorithm Ashay Dharwadker



graph $G = (V, E)$

$\gamma_G(\lambda)$

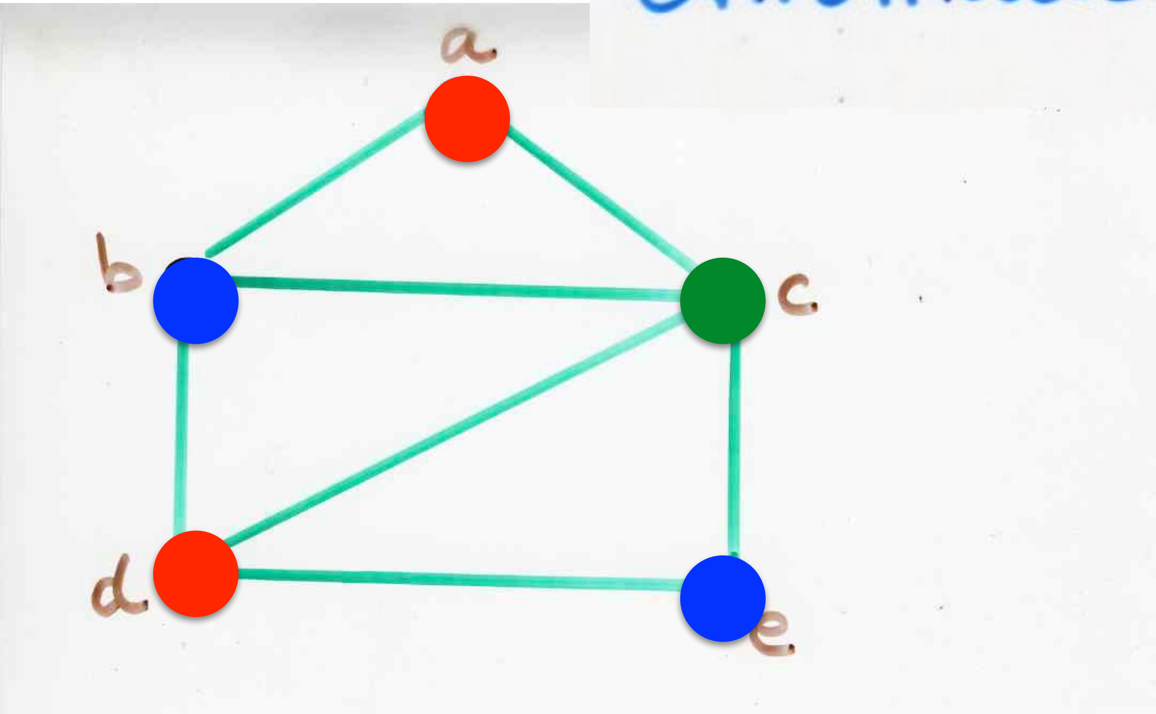
number of (proper) coloring of the graph G with λ colors



graph $G = (V, E)$

$\gamma_G(\lambda)$ number of (proper) coloring of the graph G with λ colors

chromatic polynomial



$\gamma_G(\lambda)$ chromatic polynomial

chromatic number $\nu(G)$

= smallest number ν
such that $\gamma_G(\nu) \neq 0$

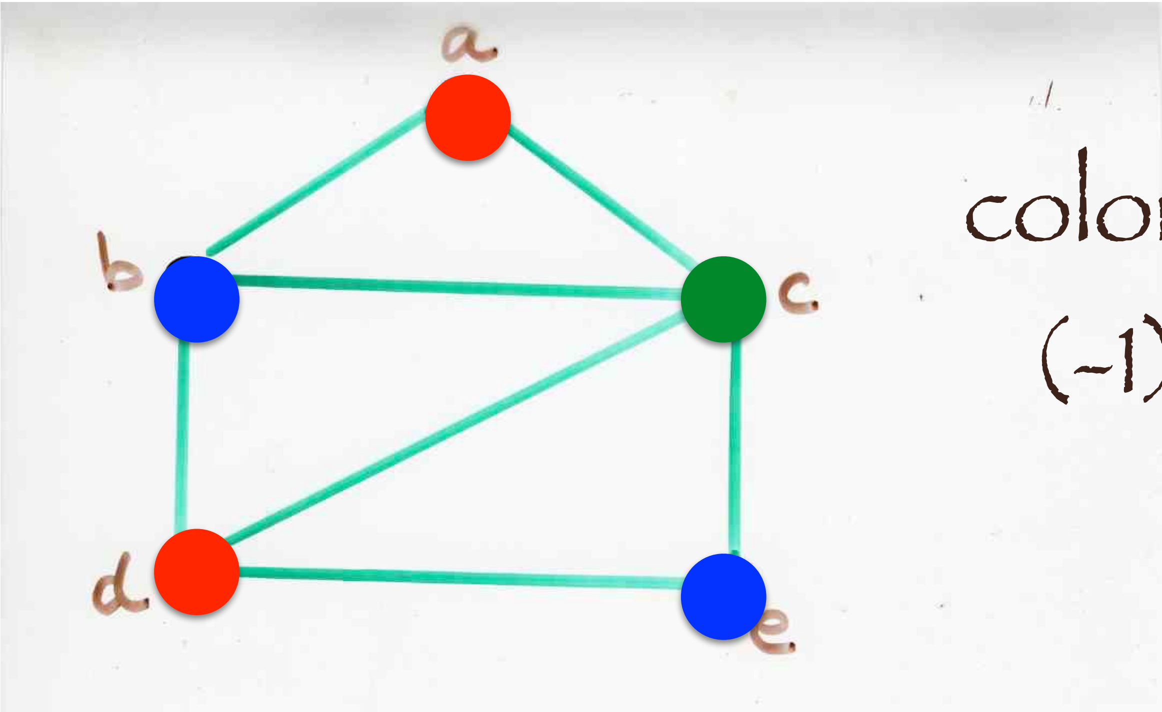
→ zeros of $\gamma_G(\lambda)$

The 4 colors theorem is
"almost" false ....

graph $G = (V, E)$

$\gamma_G(\lambda)$ number of (proper) coloring of the graph G with λ colors

chromatic polynomial



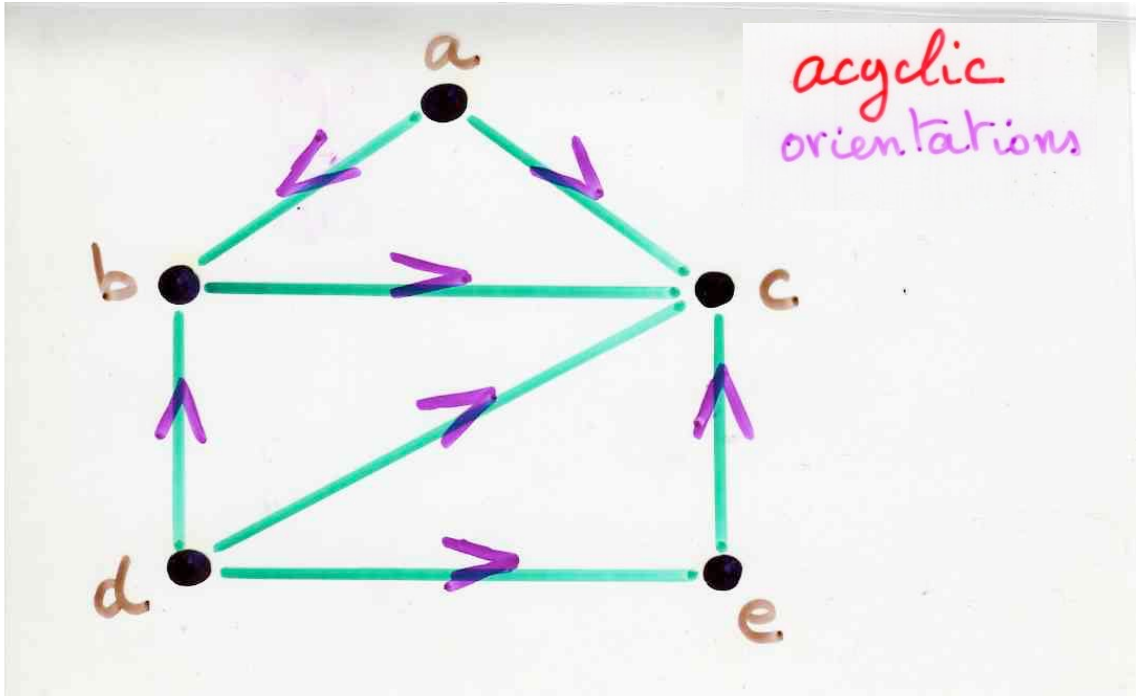
coloring with (-1) colors

?

$a(G)$

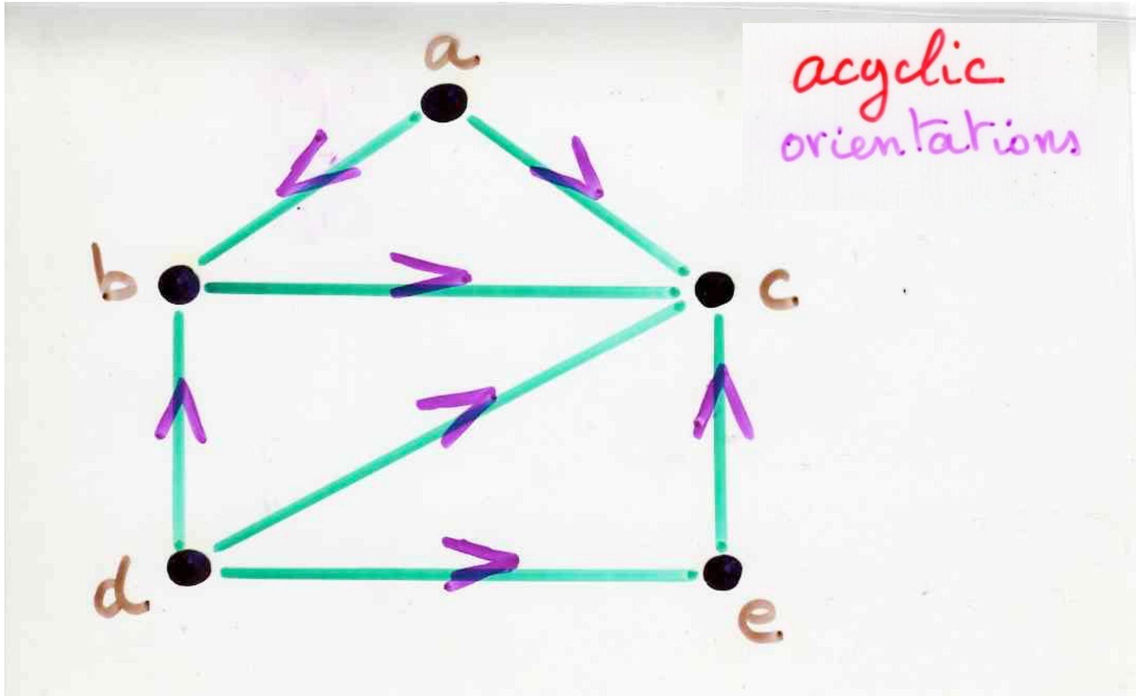
number of acyclic orientations of G

$n(G) = |V|$
number of vertices



Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \chi_G(-1)$$



algebraic graph theory

$G = (V, E)$

graph $\begin{cases} V & \text{vertices} \\ E & \text{(non-oriented) edges } \{u, v\} \end{cases}$

combinatorial properties of graphs ⟷ algebraic objects

- polynomials
- vector spaces
- power series
- ....

N. Biggs "algebraic graph theory" (1974)

connection with
- Statistical physics
- Knots theory
- Lie algebra
- Heaps theory

some polynomials or numbers associated to a graph

characteristic polynomial of a graph G

$$A = (a_{ij})_{1 \leq i, j \leq n}$$

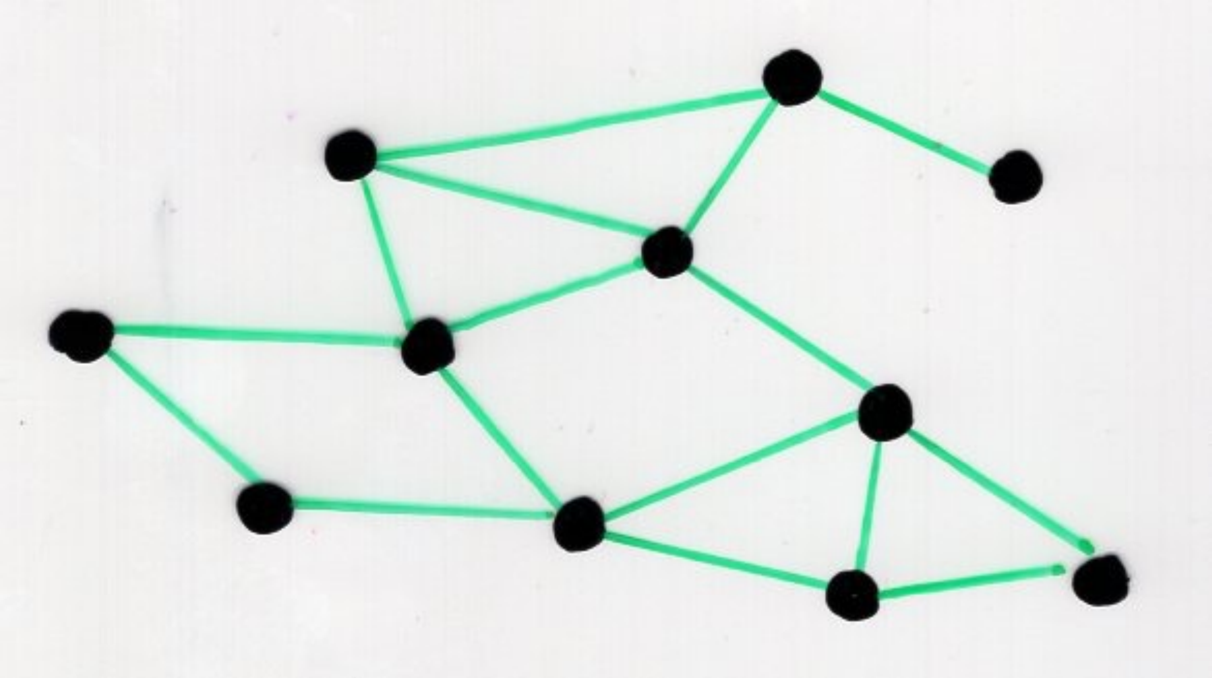
adjacency matrix

$$a_{ij} = \begin{cases} 1 & i \text{ — } j \text{ (edge)} \\ 0 & \text{no edge} \end{cases}$$

$$\chi(x) = \det(Ix - A)$$

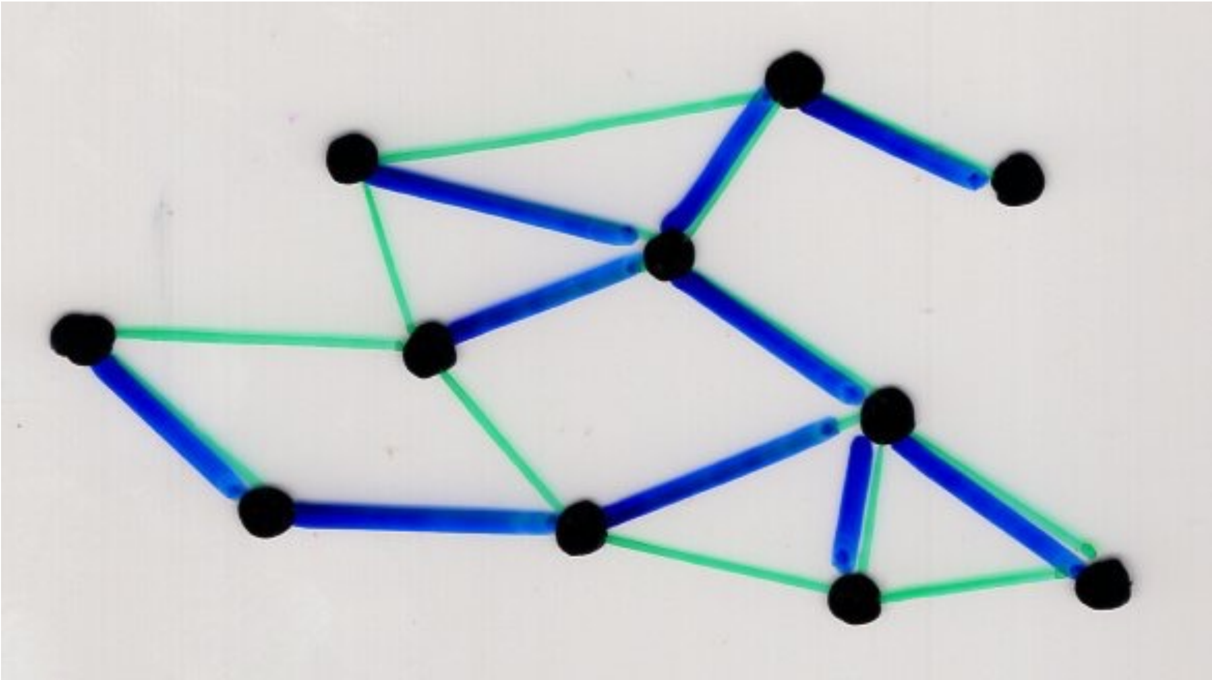
spanning tree

of a graph $G = (V, E)$



spanning tree

of a graph $G = (V, E)$



• number of spanning tree

Tutte polynomial

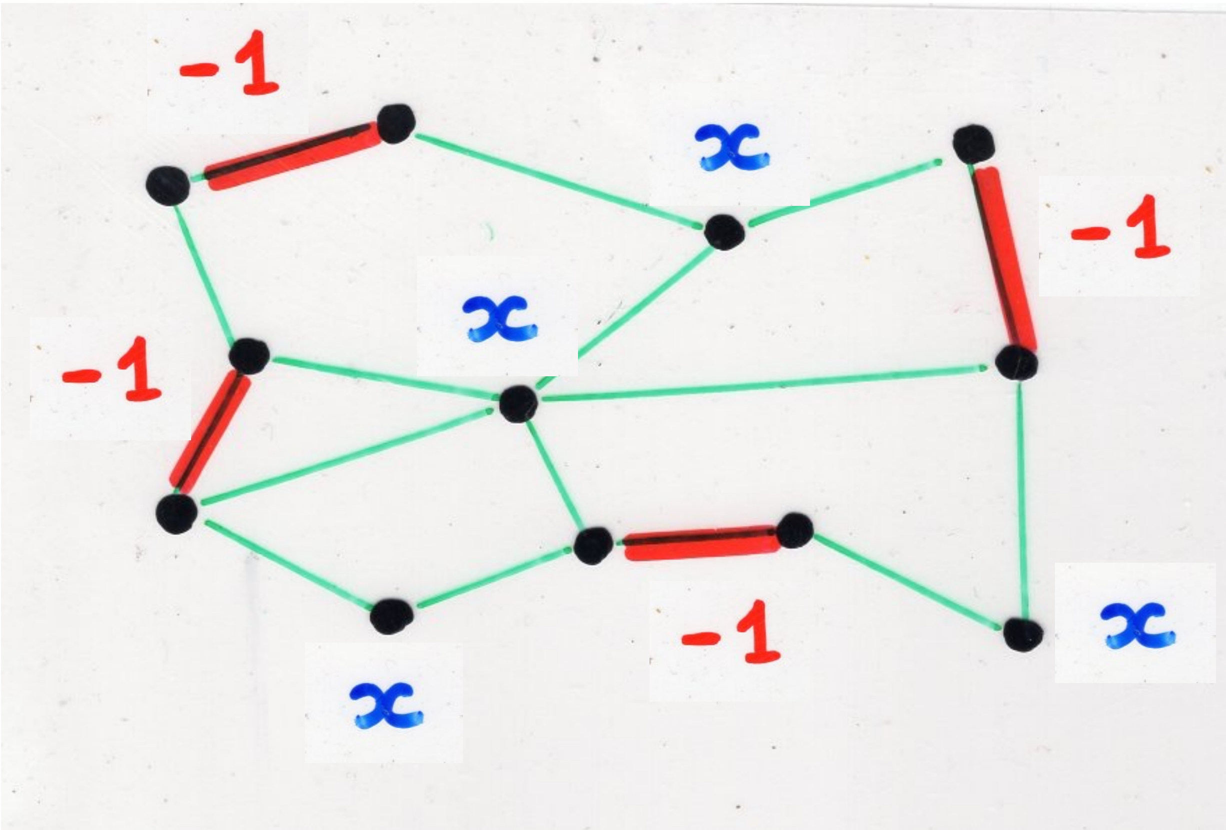
$$T(x,y) \quad \sum_{T} x^{i(T)} y^{e(T)}$$

spanning trees

→ Potts model

$T(1,1)$ = number of spanning trees

$T(2,0)$ = chromatic number



matching polynomial of a graph G

- number of perfect matchings

  constant term
  of the matching polynomial

→ Pfaffian, determinant ...
(for planar graph).

→ statistical mechanics
Ising model for magnetism

Ihara-Selberg zeta function of a graph

→ Ch5b

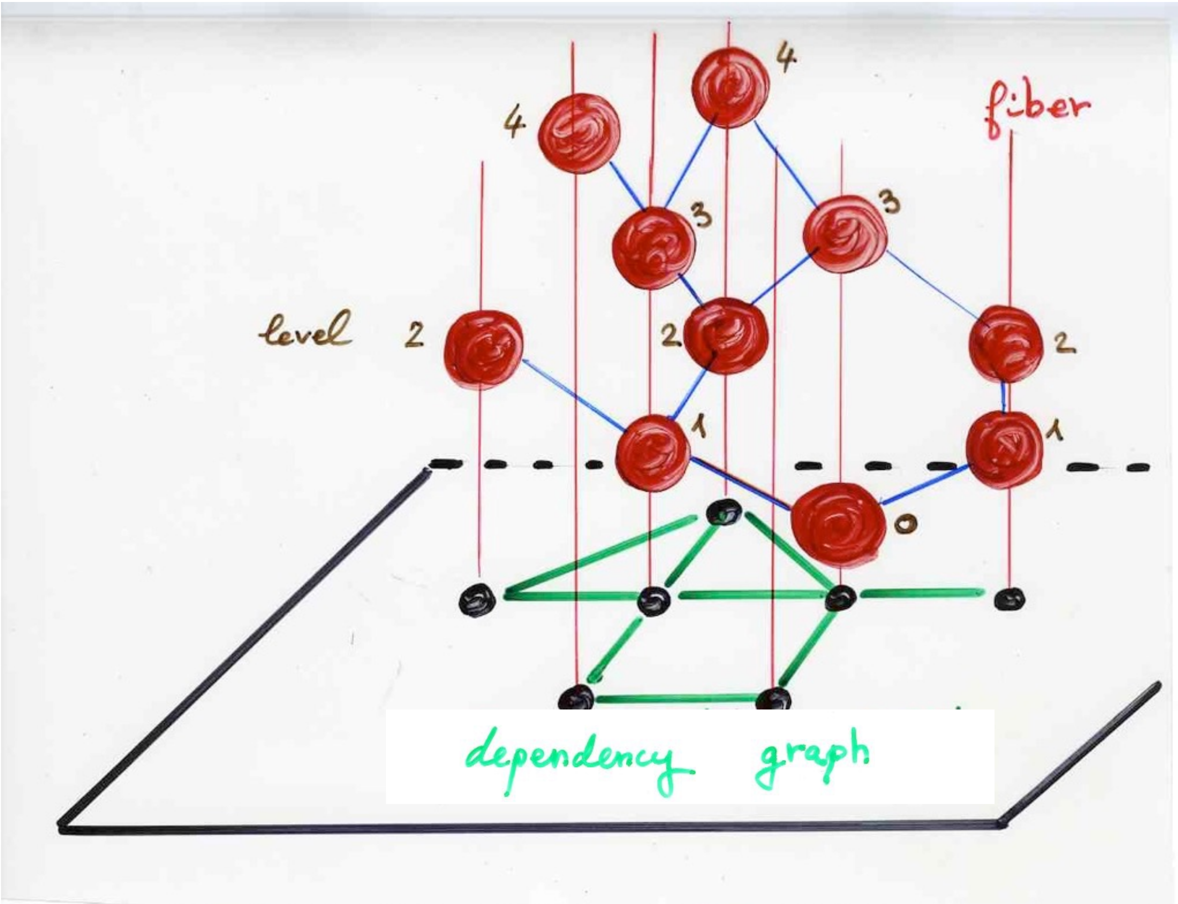
extension of Riemann zeta function

$$\sum_{n \geq 1} n^{-s}$$

Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$

heaps over a graph



proof using commutation (Cartier-Foata) monoid

from Gessel (1985)?

Commutation monoids

$a, b, c, d, \ldots$

letters
formal variables

$ad = da \qquad ab \neq ba$

$cd = dc \qquad ac \neq ca$

$bc = cb \qquad bd \neq db$

$a, b, c, d, \ldots$

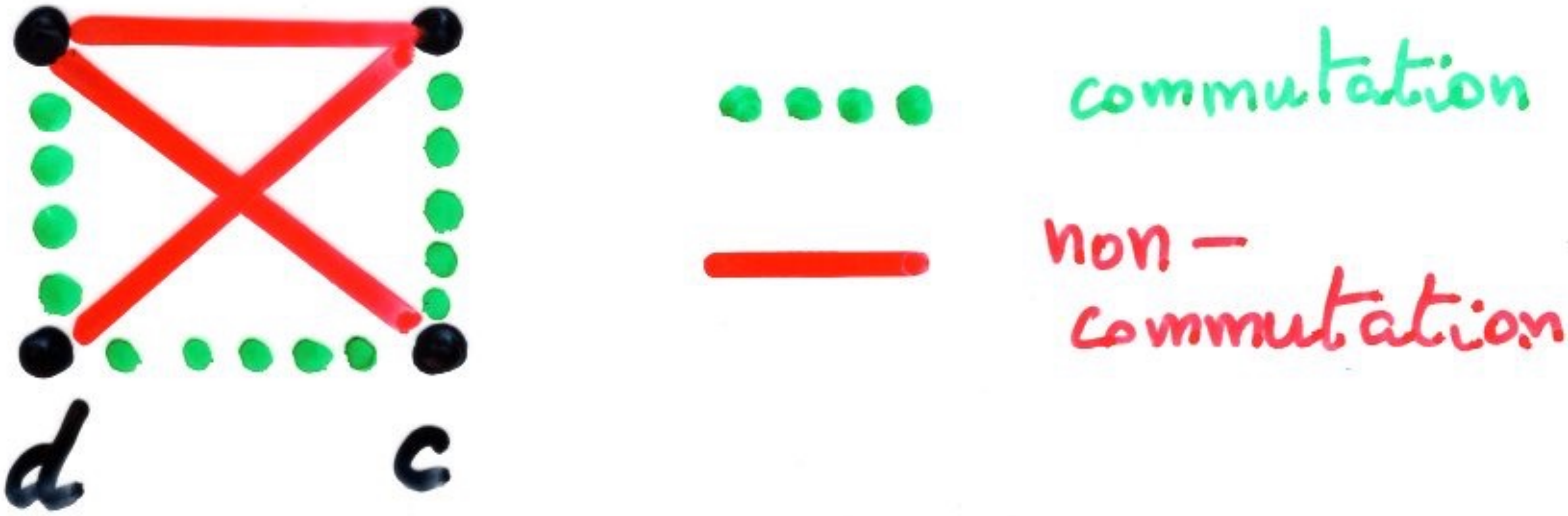
letters

formal variables

$ad = da \qquad ab \neq ba$

$cd = dc \qquad ac \neq ca$

$bc = cb \qquad bd \neq db$



abcad

word
monomial

$w = a\underline{bc}ad$

$ad = da$

$cd = dc$

$bc = cb$

abcad

word
monomial

$w = abcad$

$acbad$

$ad = da$

$cd = dc$

$bc = cb$

abcad

word
monomial

w = abcad — abcda

acbad

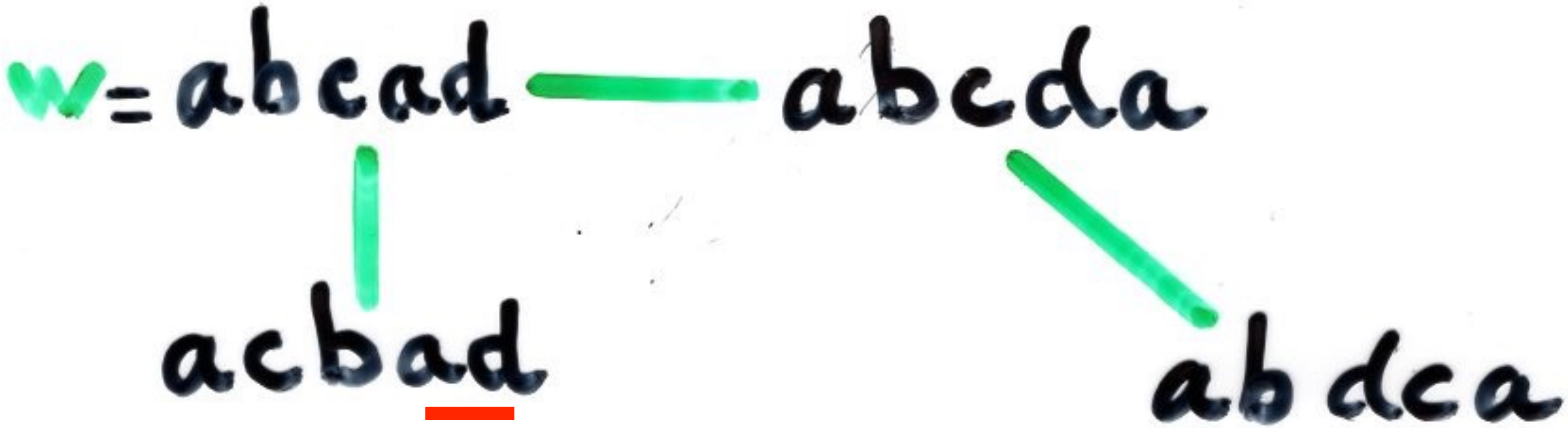
ad = da

cd = dc

bc = cb

abcad

word
monomial



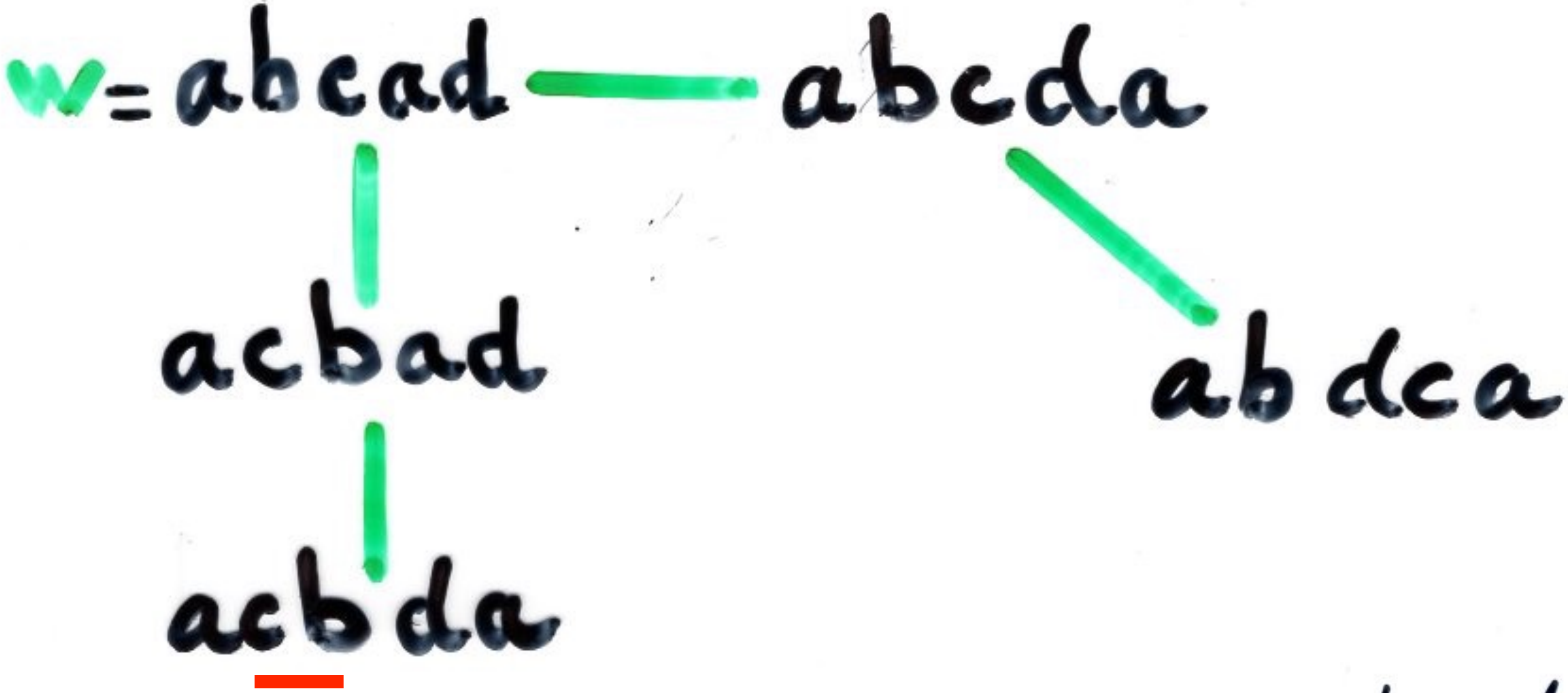
$ad = da$

$cd = dc$

$bc = cb$

abcad

word
monomial



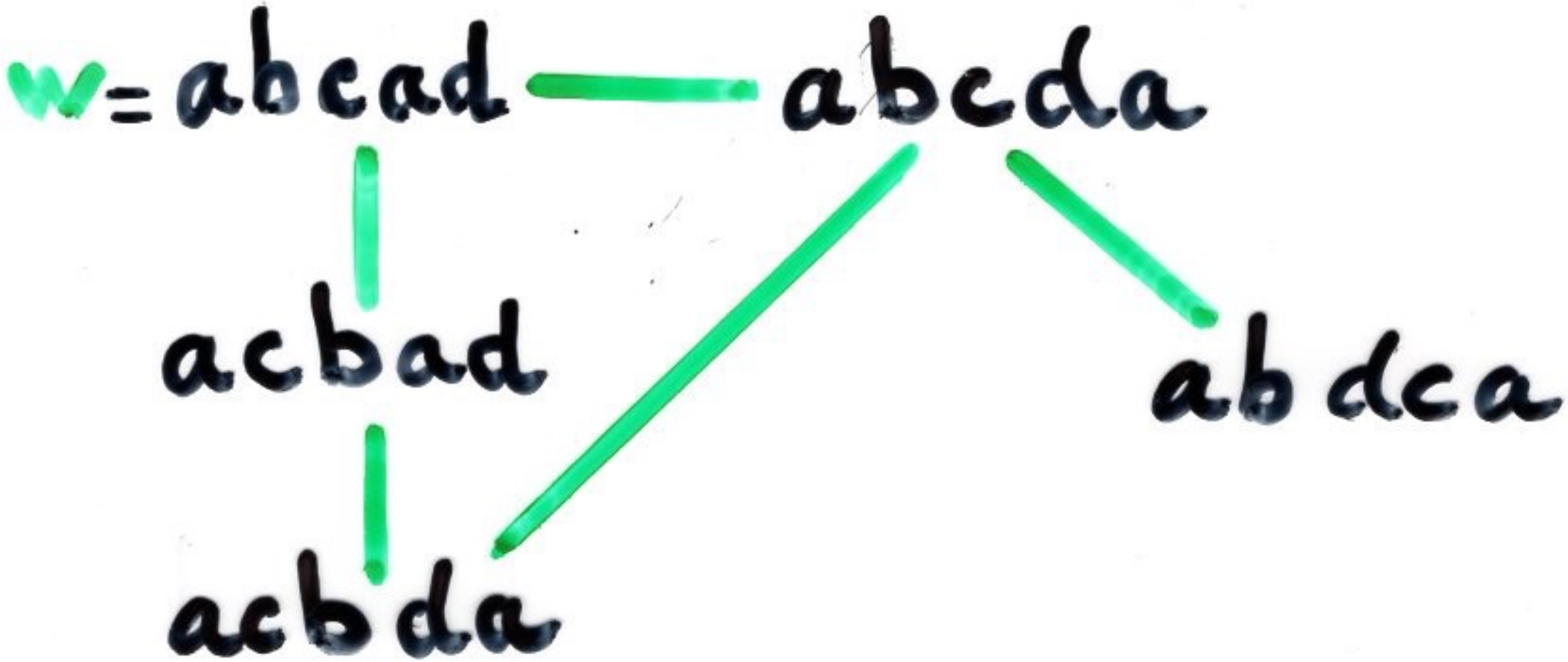
$ad = da$

$cd = dc$

$bc = cb$

abcad

word
monomial



$ad = da$

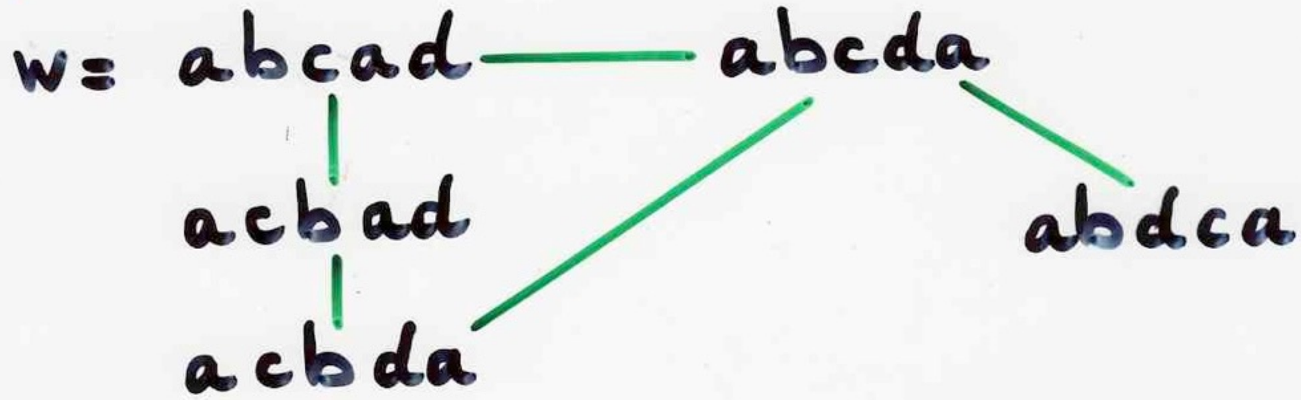
$cd = dc$

$bc = cb$

ex: $A = \{a, b, c, d\}$

$C \quad \begin{cases} ad = da \\ bc = cb \\ cd = dc \end{cases}$

equivalence class



Cartier-Foata monography
in SLC Séminaire Lotharingien
de Combinatoire
(2006)

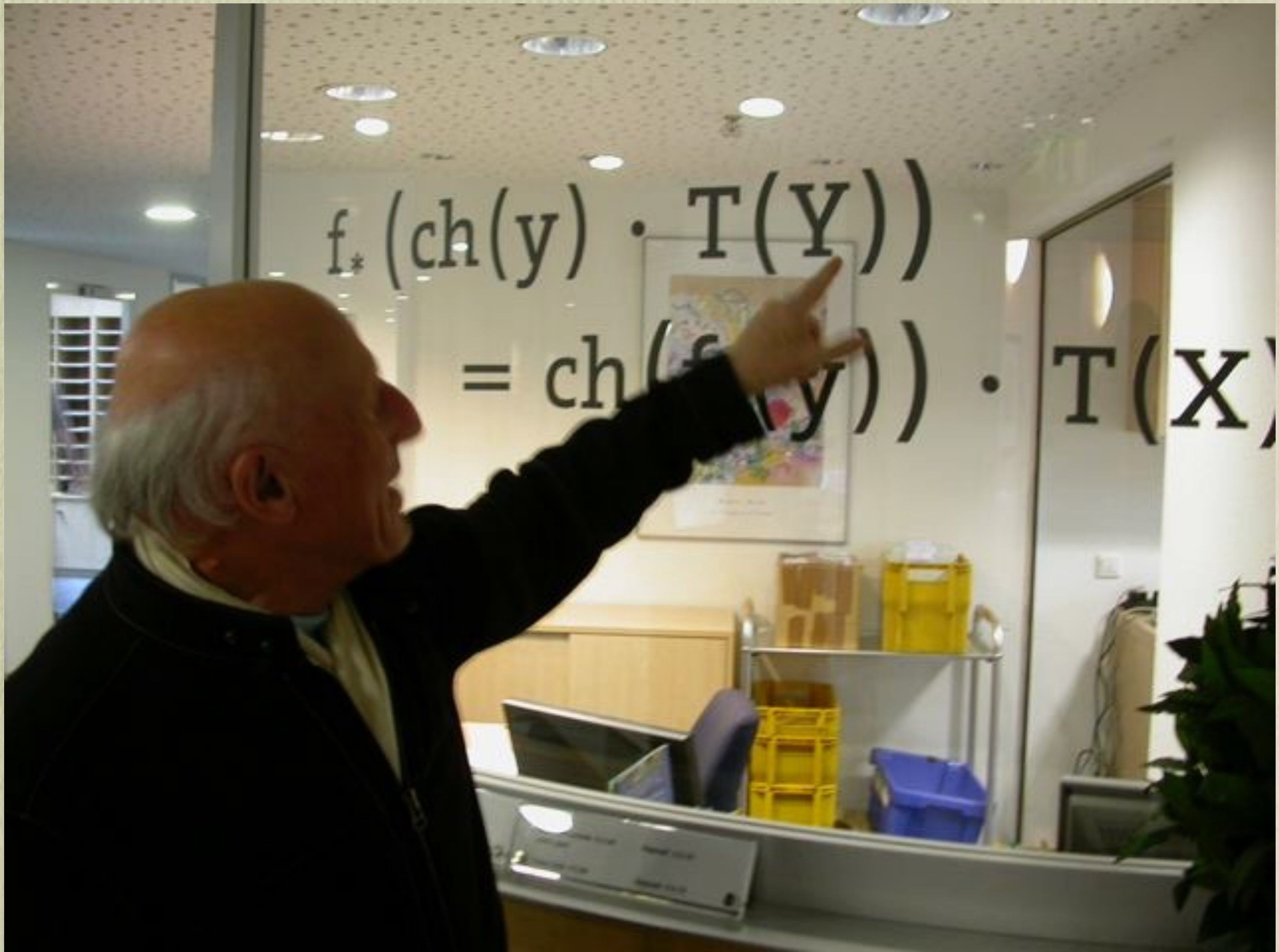
http://www.mat.univie.ac.at/~slc/

with an appendix
by C. Krattenthaler

Cartier-Foata **commutation** **monoid**

Lecture Note in Maths n°85 (1969)

"Problèmes combinatoires de
commutation et réarrangements"


f* (ch(y) · T(Y))

A, C)
such
form
mutatio of ∞, 1, 2, . . . , n
begi , an ele
Pro g d is a bijection of ∞S
ont

monoid M $(u,v) \to u \bullet v$

- associativity $(u \bullet v) \bullet w = u \bullet (v \bullet w)$
- neutral element $u \bullet e = e \bullet u$

examples

- $\mathbb{N}$ $+$, 0 addition
- $\mathbb{N}$ $\times$, 1 product

alphabet A

free monoid A^*

words $w = a_1 a_2 \cdots a_p$

product : concatenation

$\left.\begin{array}{l} u = a_1 \cdots a_p \\ v = b_1 \cdots b_q \end{array}\right\} uv = a_1 \cdots a_p b_1 \cdots b_q$

empty word

commutation monoid $A^*/\equiv_C$

$[w]$ equivalence class of the word $w \in A^*$

• product in the commutation monoid $A^*/\equiv_C$

$$[u] \cdot [v] = [uv]$$

independant of the choices of representants u and v

Trace monoids

Computer Science

model for parallelism

concurrency access to data structures

Trace

Mazurkiewicz (1977)

model of the logical behavior of safe Petri nets

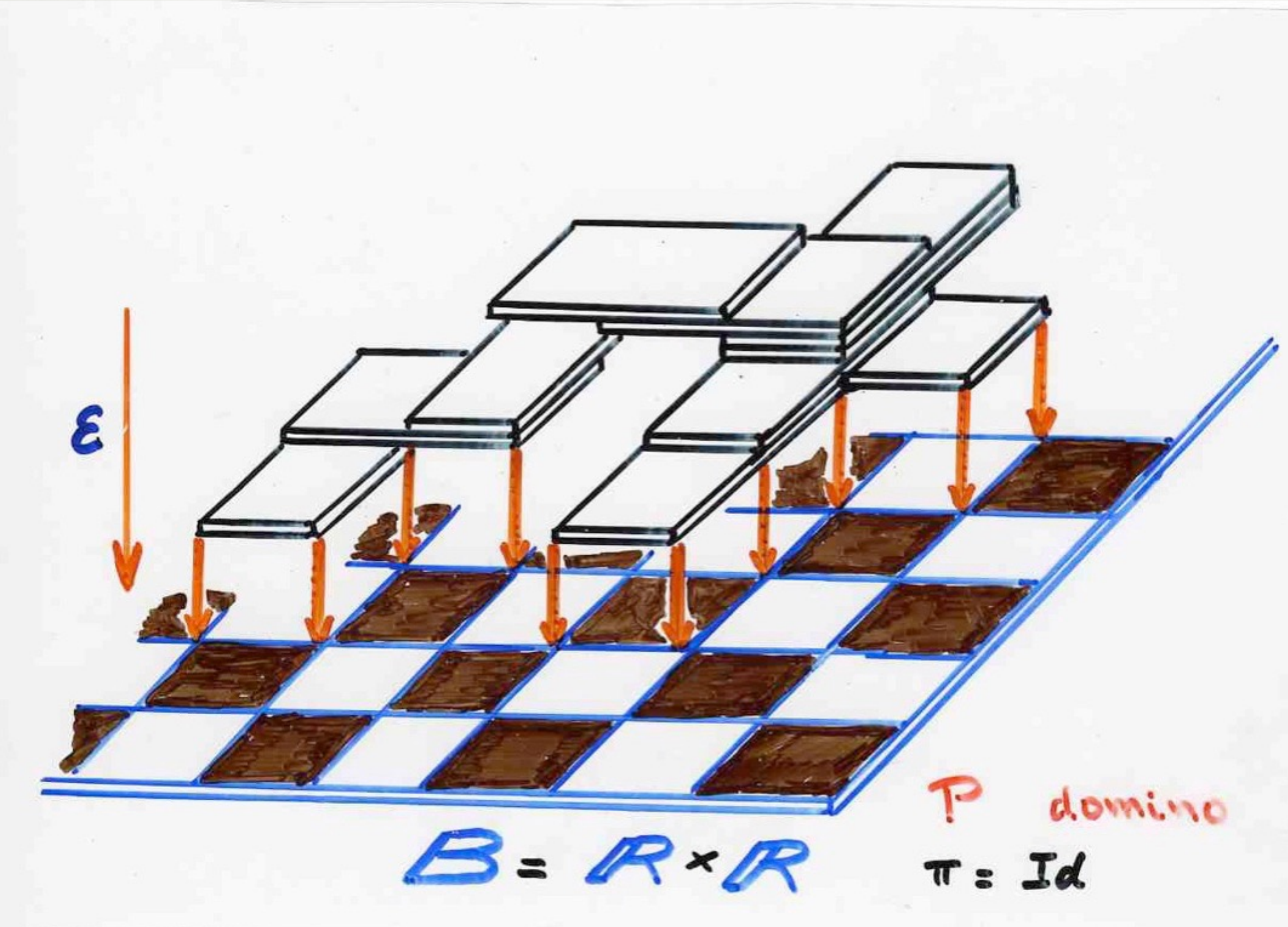
Dieckert, Rosenberg ed. (1995)

The book of traces

Heaps of pieces

(X.V. 1985)

Introduction
Heaps



heap definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P { symmetric, reflexive }
  (dependency relation)
- heap E, finite set of pairs
  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)
  projection (α), level (i)

(i)

(ii)

heap definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ { symmetric, reflexive }
  (**dependency** relation)
- heap E, finite set of pairs
  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)
  (α ← projection, i ← level)

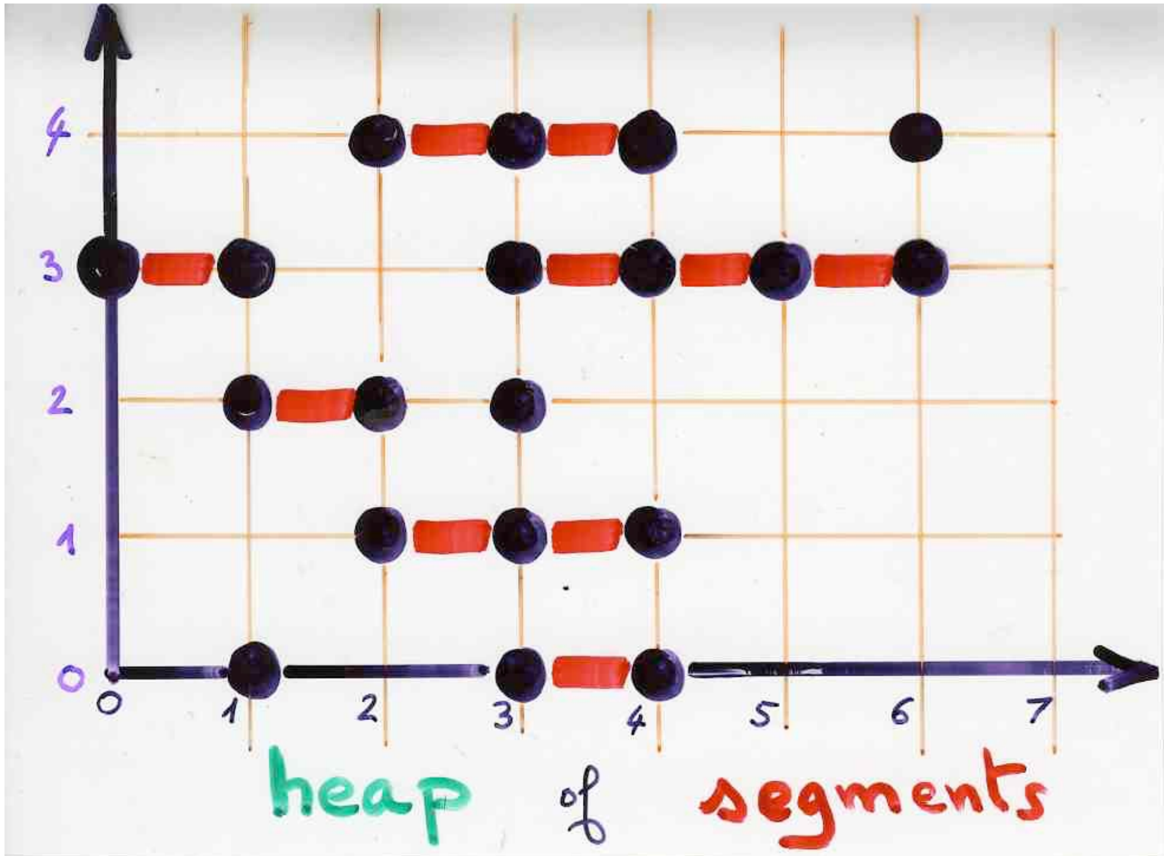
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \, \mathcal{C} \, \beta \implies i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \implies \exists \beta \in P$, $\alpha \, \mathcal{C} \, \beta$, $(\beta, i-1) \in E$

<u>ex</u>: heap of segments over $\mathbb{N}$

$P = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \leq a \leq b \}$

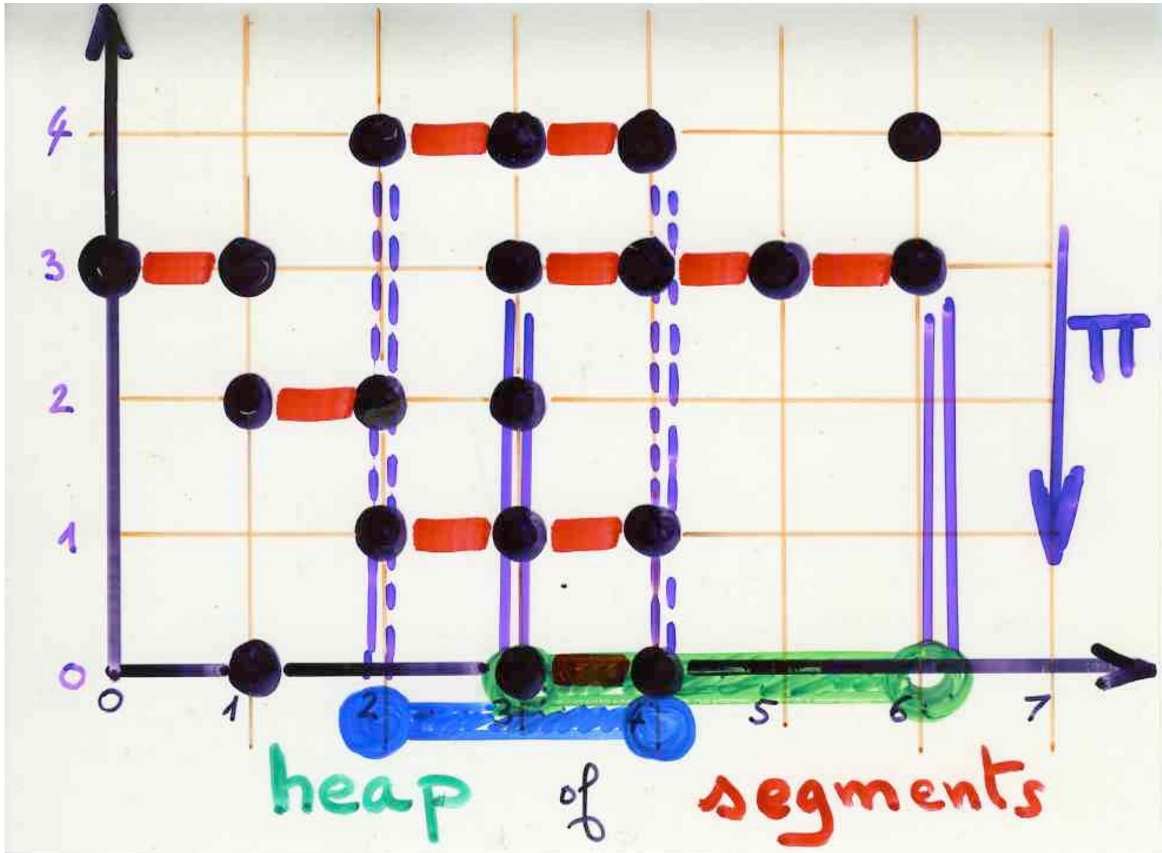
$\mathcal{C}$ $\quad [a,b] \, \mathcal{C} \, [c,d] \Leftrightarrow [a,b] \cap [c,d] \neq \emptyset$



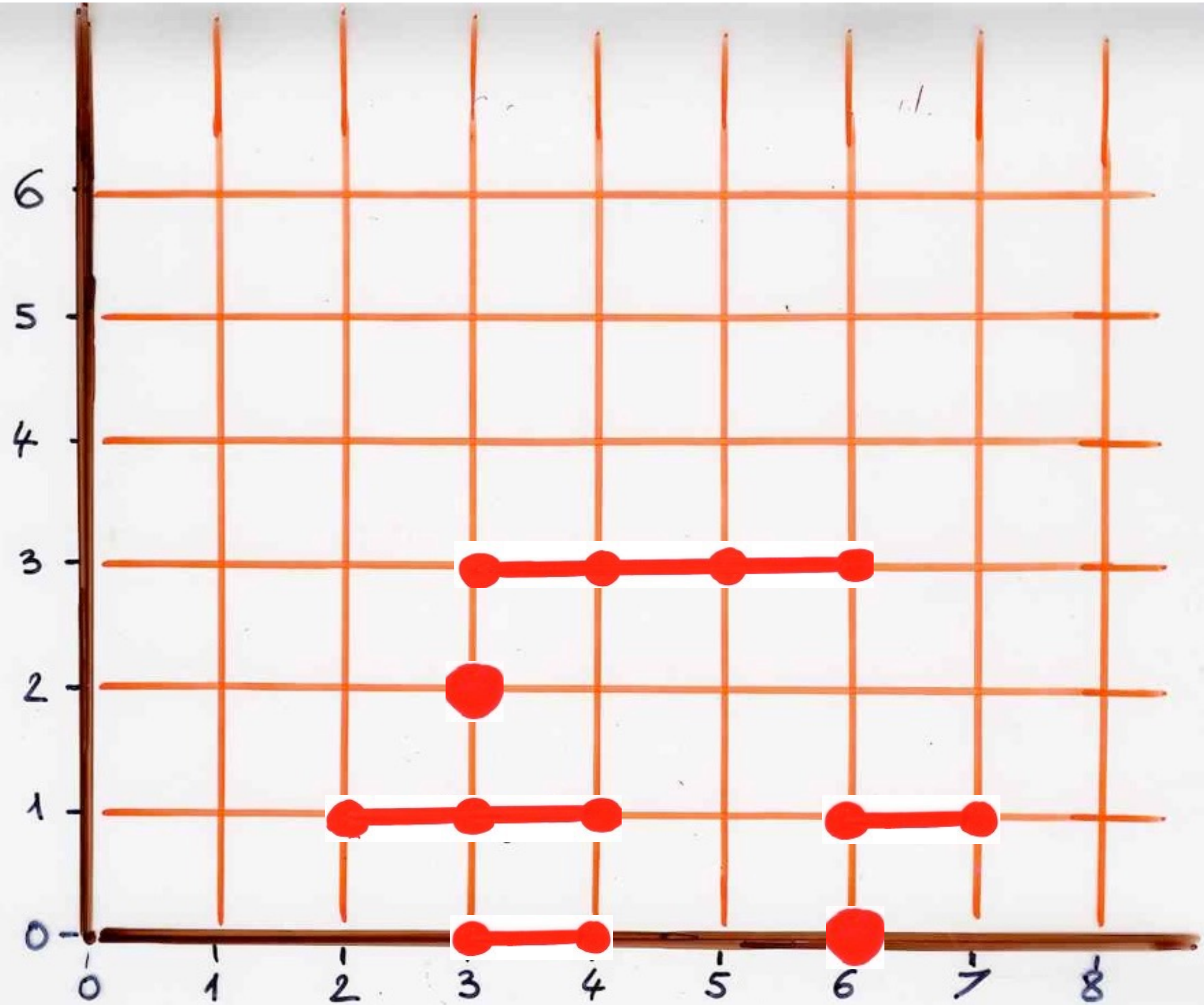
ex: heap of segments over $\mathbb{N}$

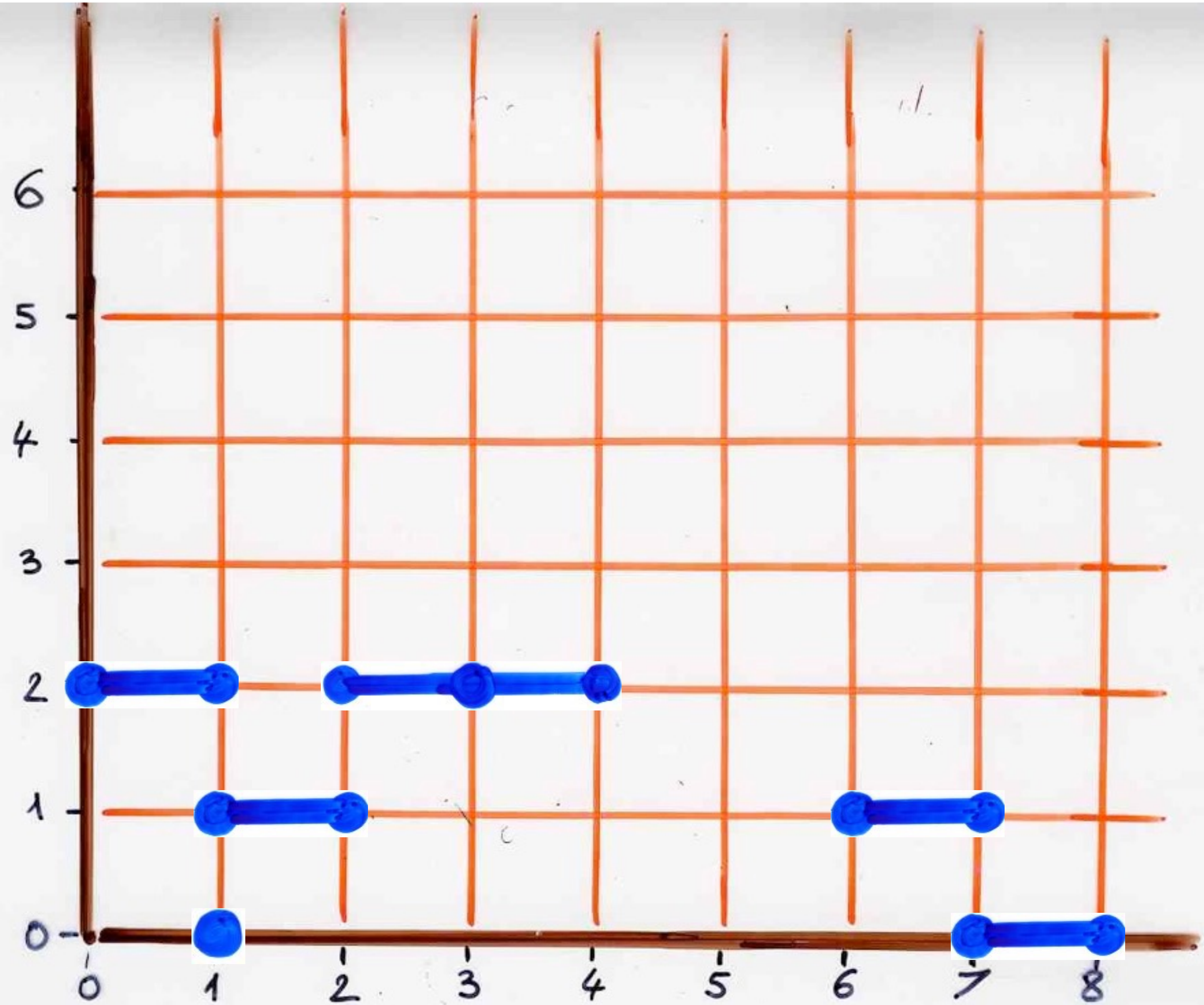
$$P = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \leq a \leq b \}$$

$$\mathcal{C} \qquad [a,b] \,\mathcal{C}\, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$

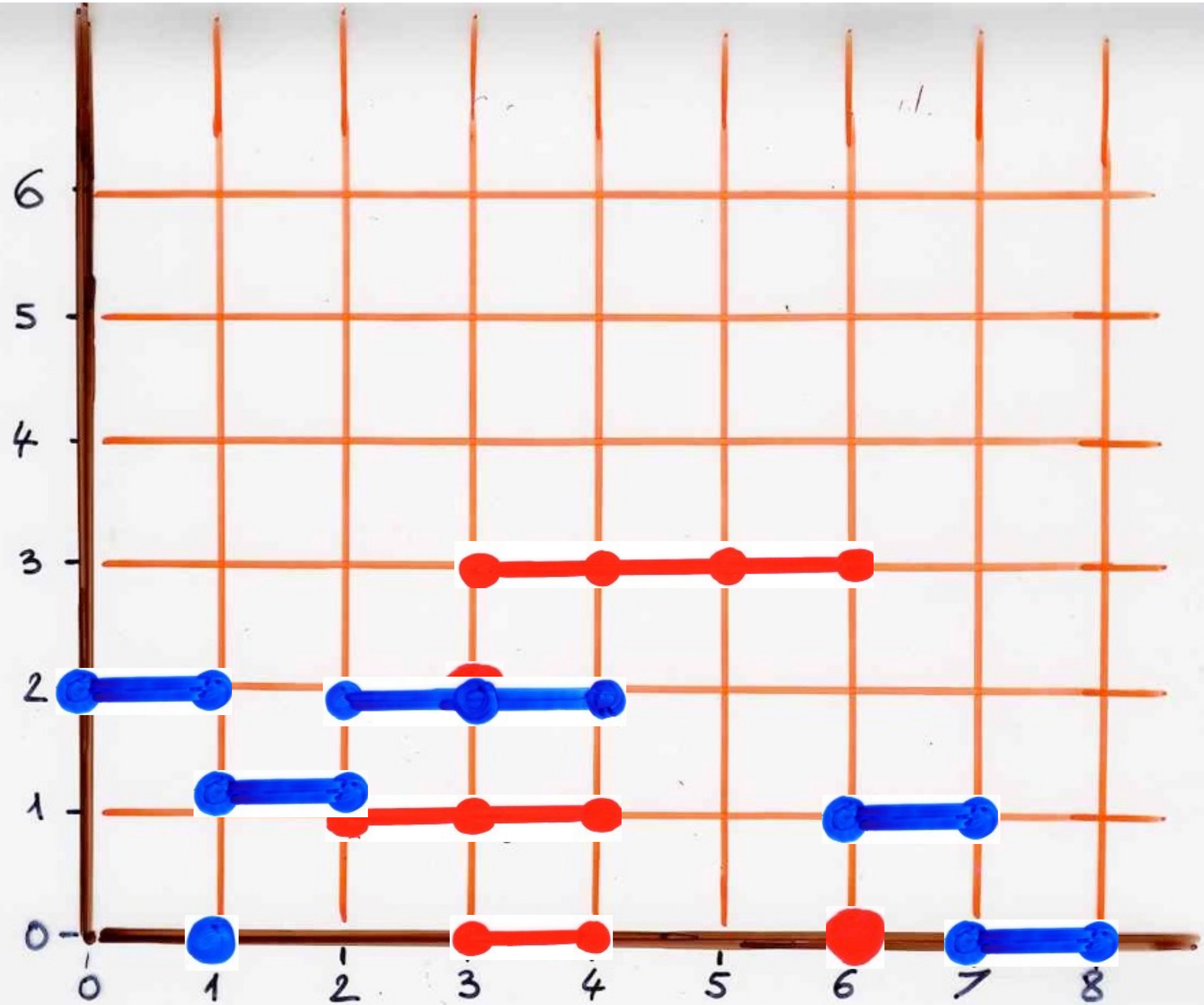


Heaps monoids

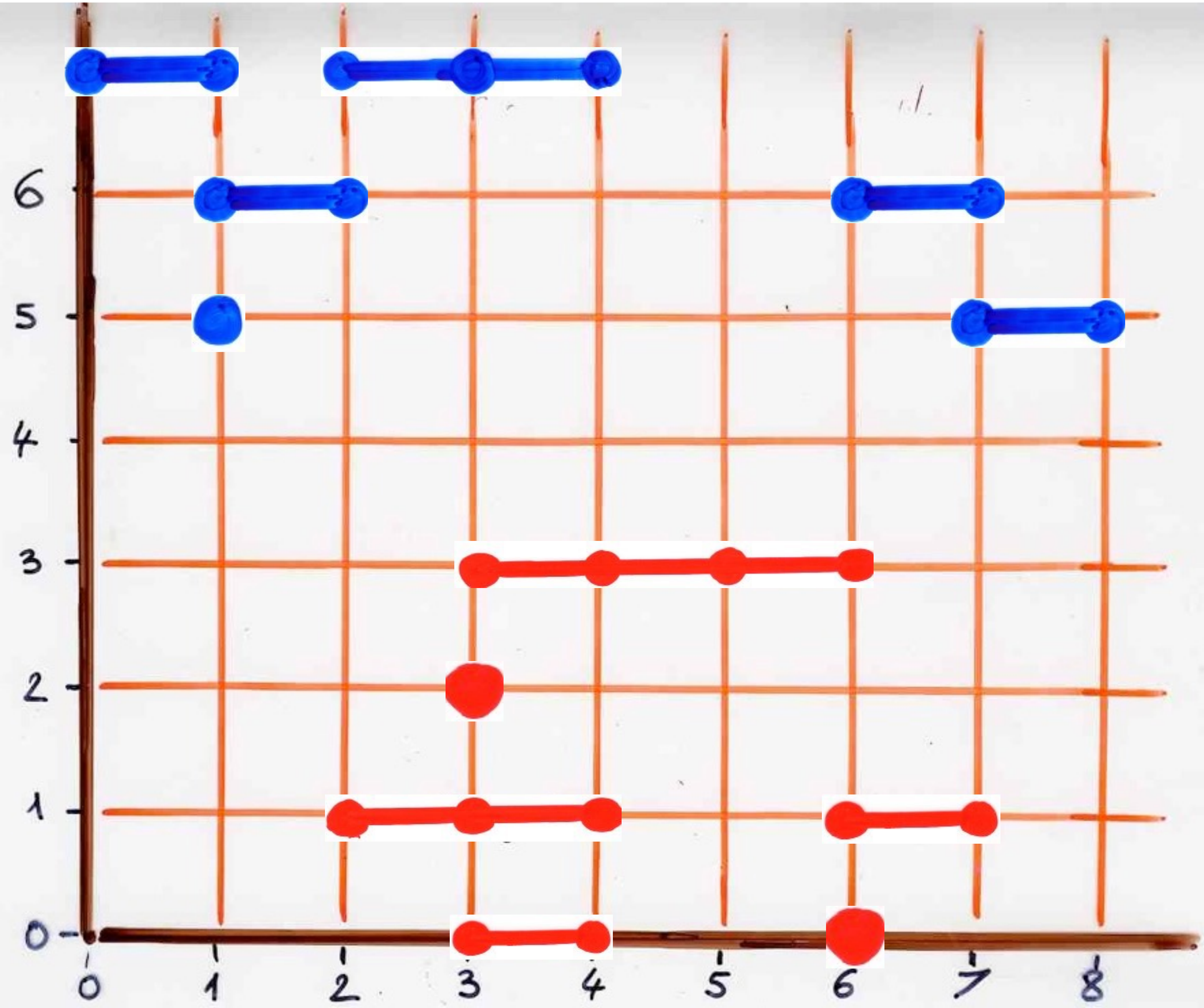

0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8



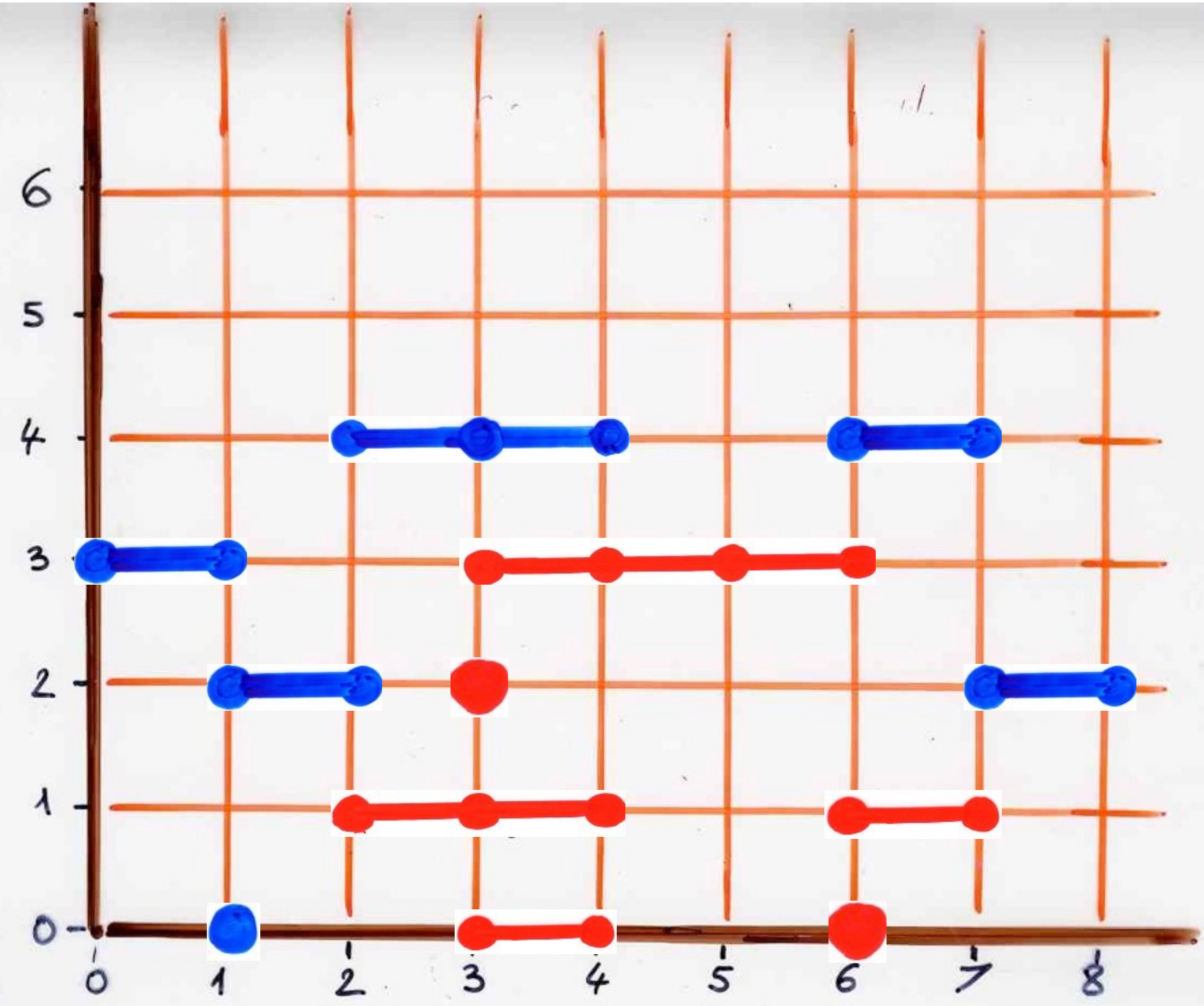
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

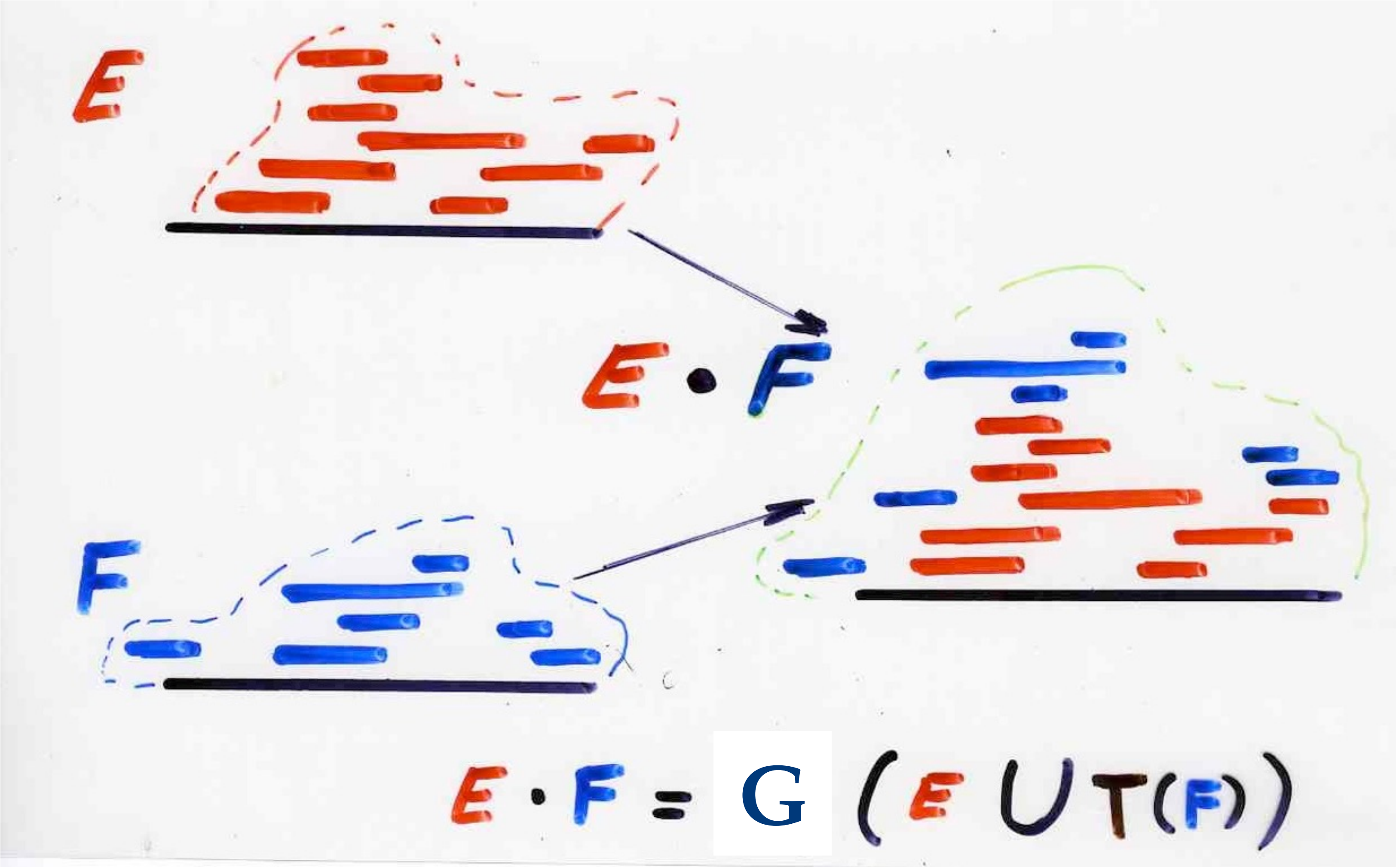


0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8



6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8


6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8


E
E • F
F
E · F = G (E ∪ T(F))

Equivalence commutation monoids and heaps monoids

example:
heaps of dimers

ex: heaps of dimers on $\mathbb{N}$

$$P = \{ [i, i+1] = \sigma_i , i \geq 0 \}$$

$\mathcal{C}$

C commutations

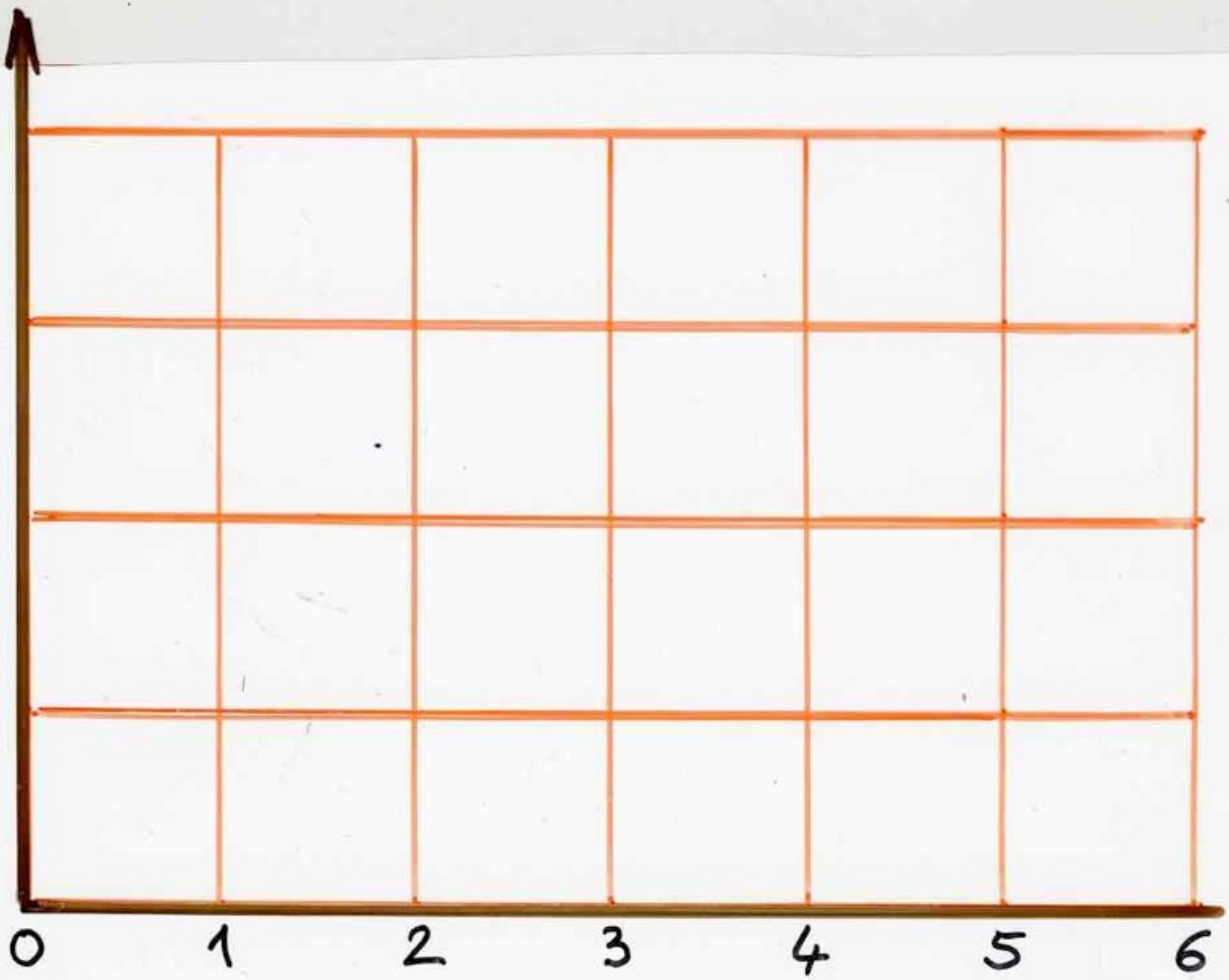
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{iff } |i-j| \geq 2$$

$$C = \overline{\mathcal{C}}$$

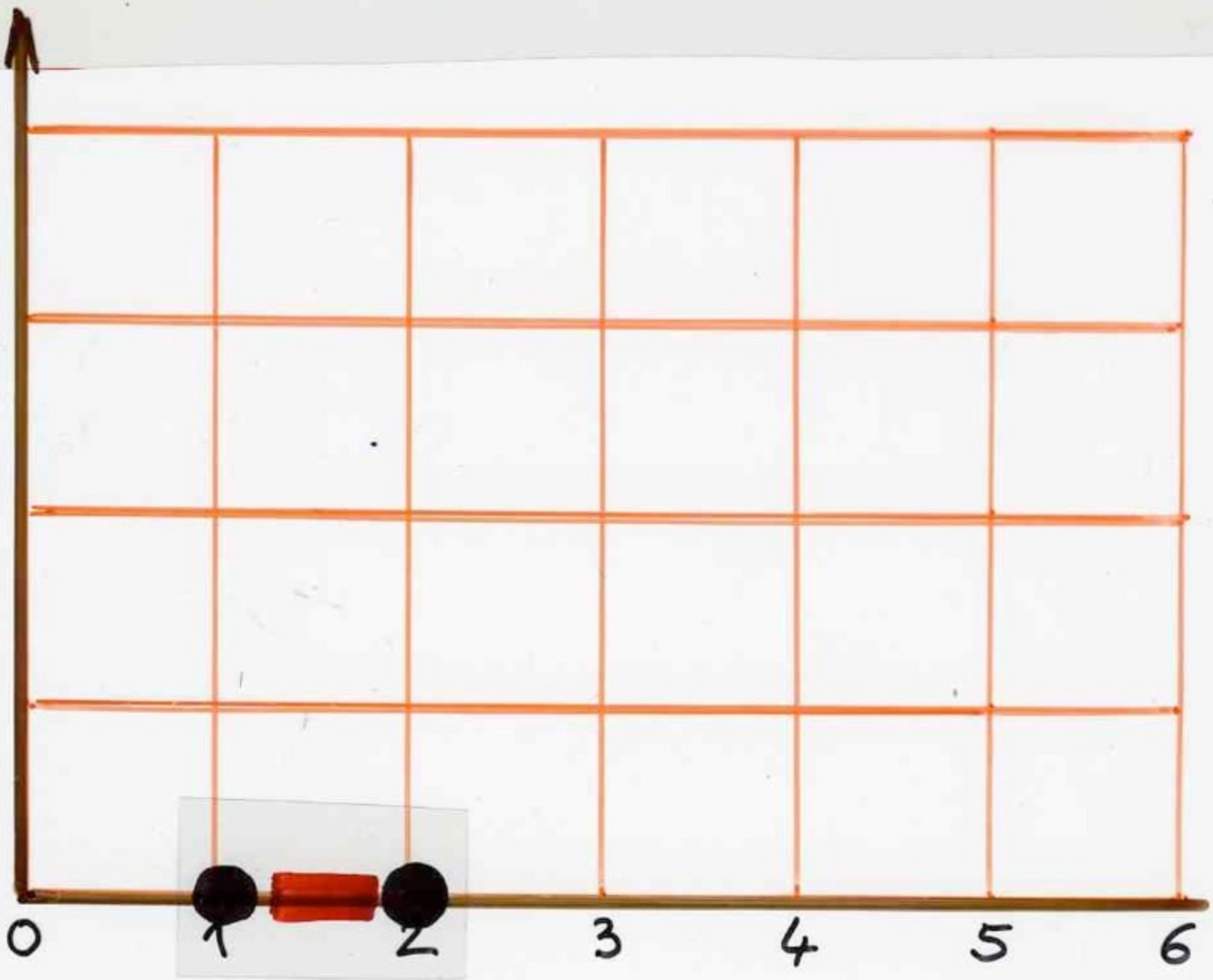
C: commutation relation

$\overline{\mathcal{C}}$: complementary of the dependency relation

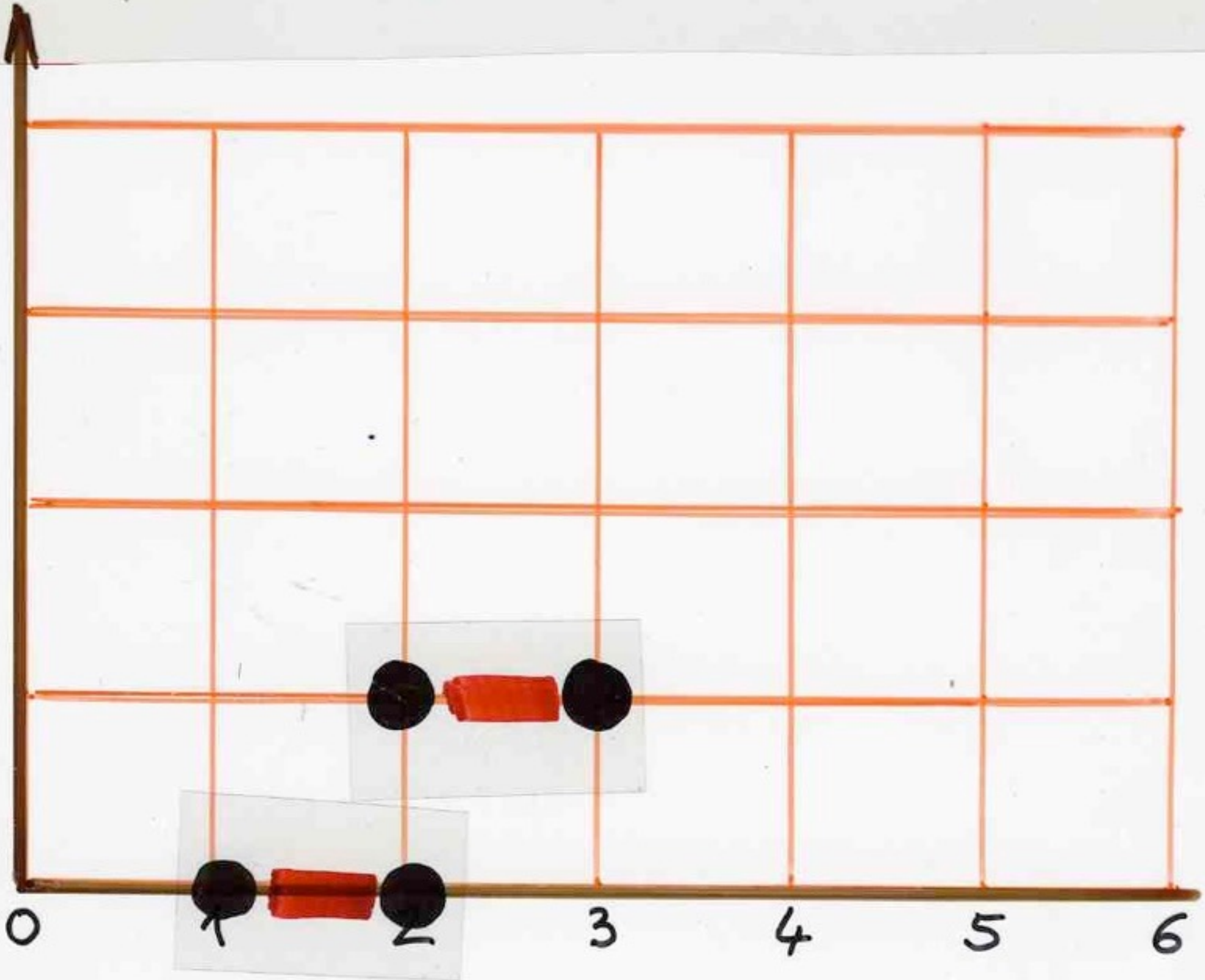
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



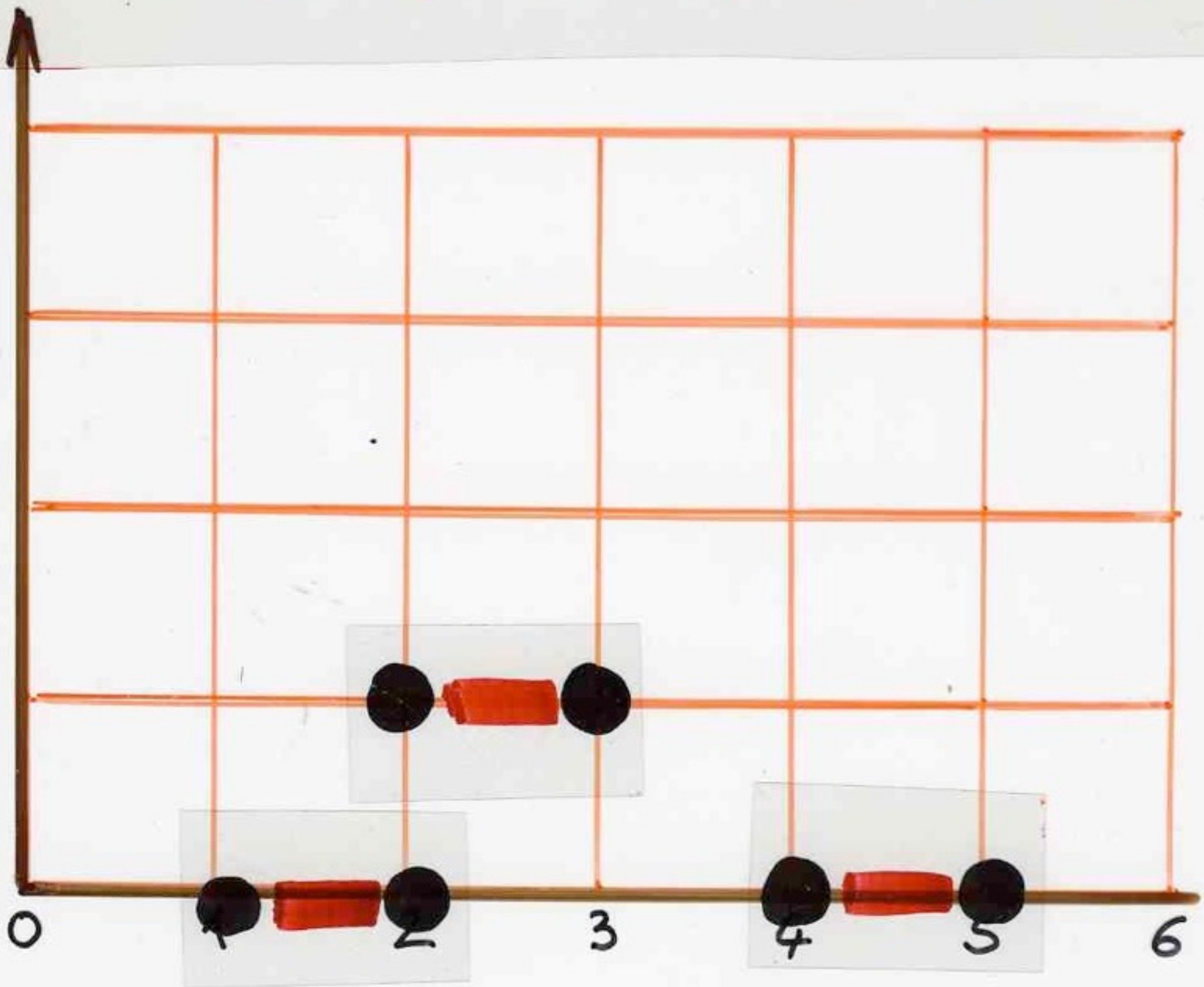
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



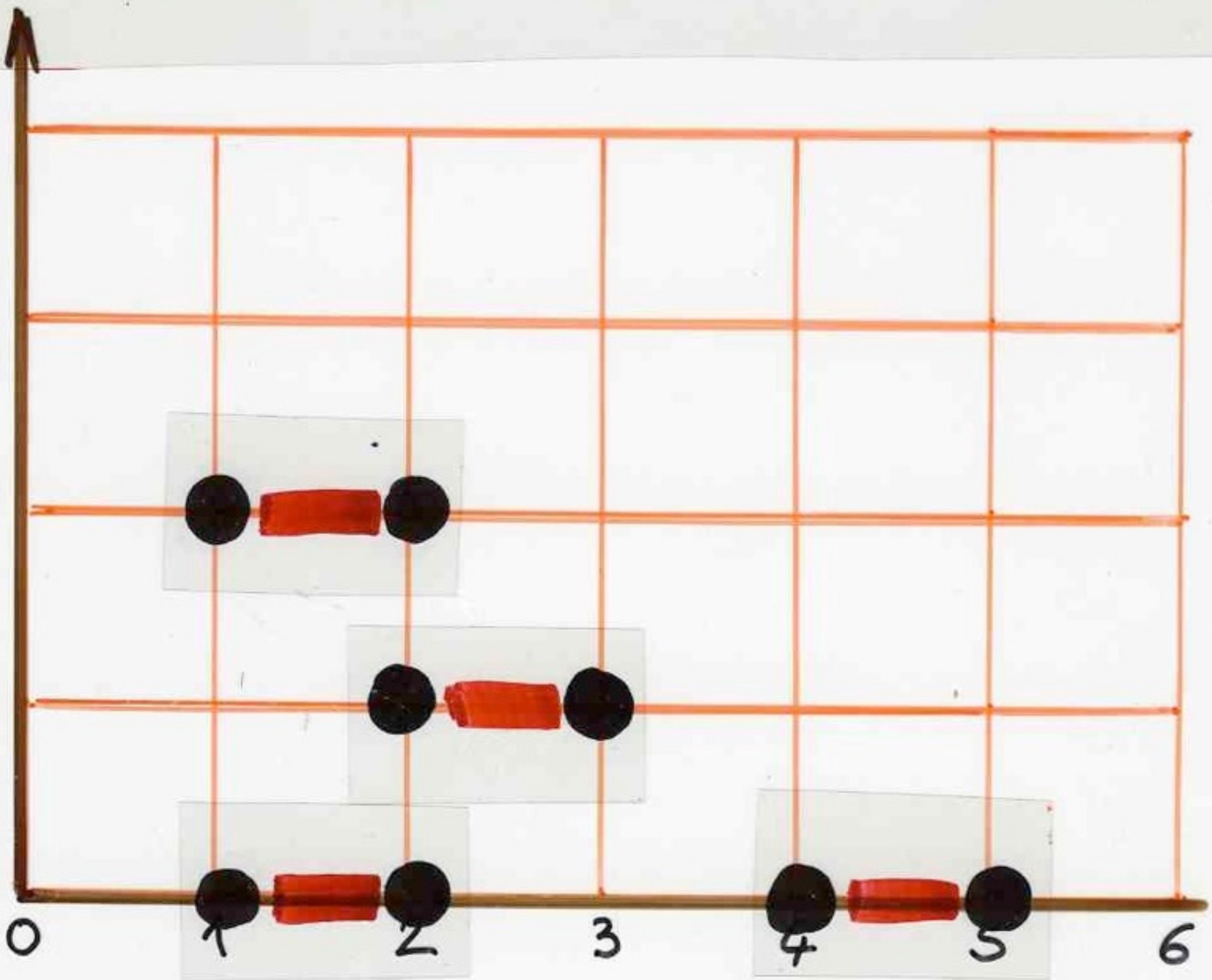
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



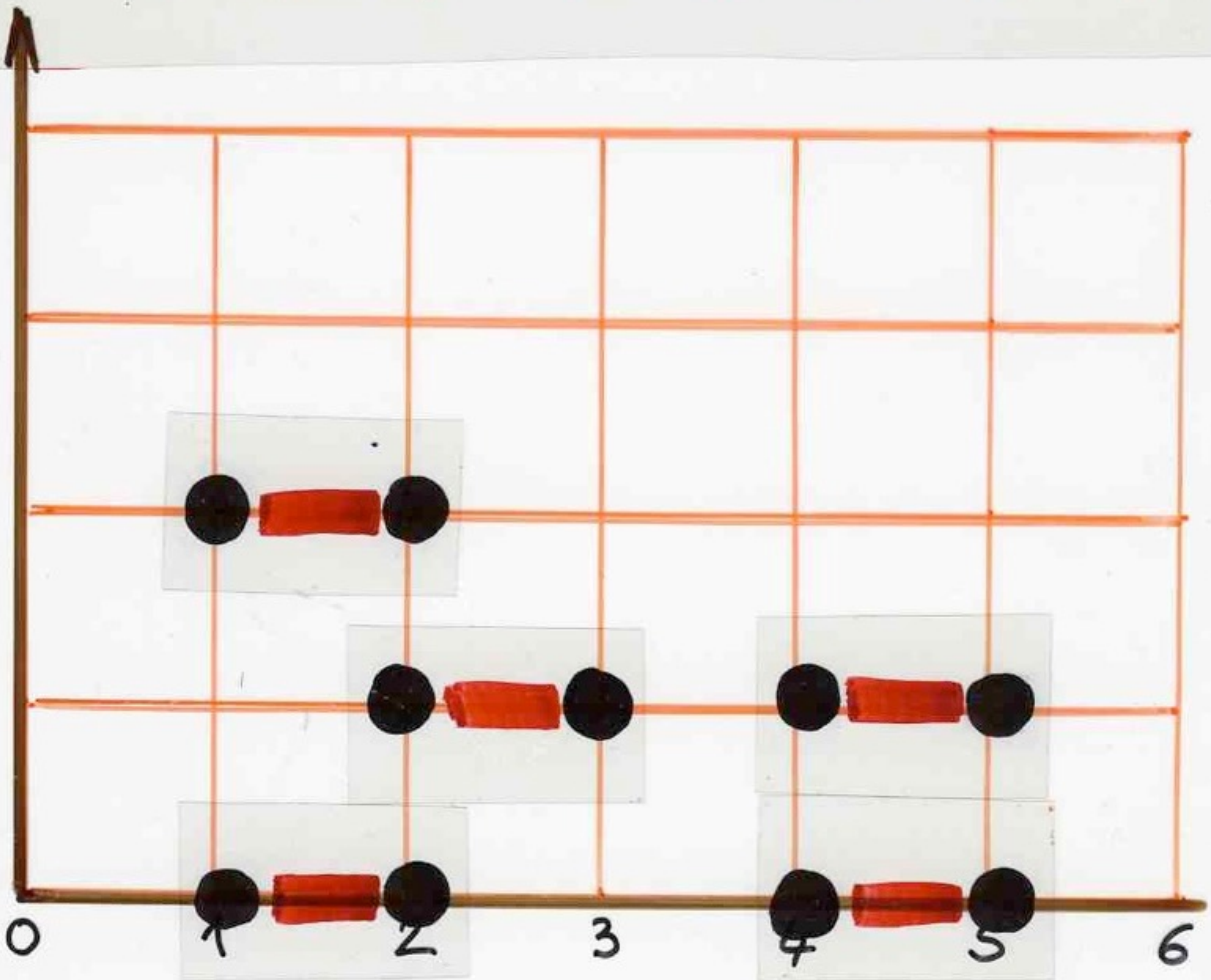
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



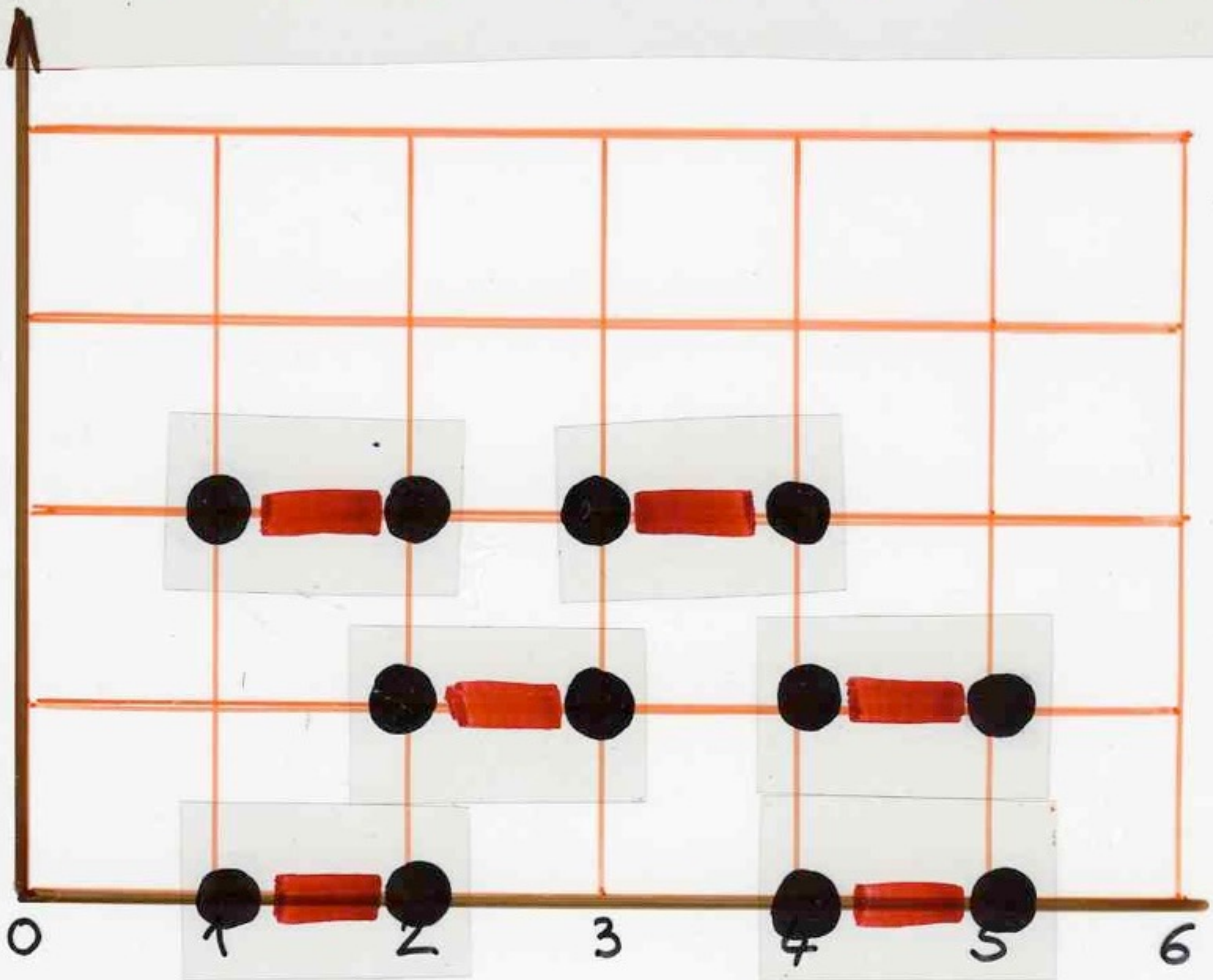
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



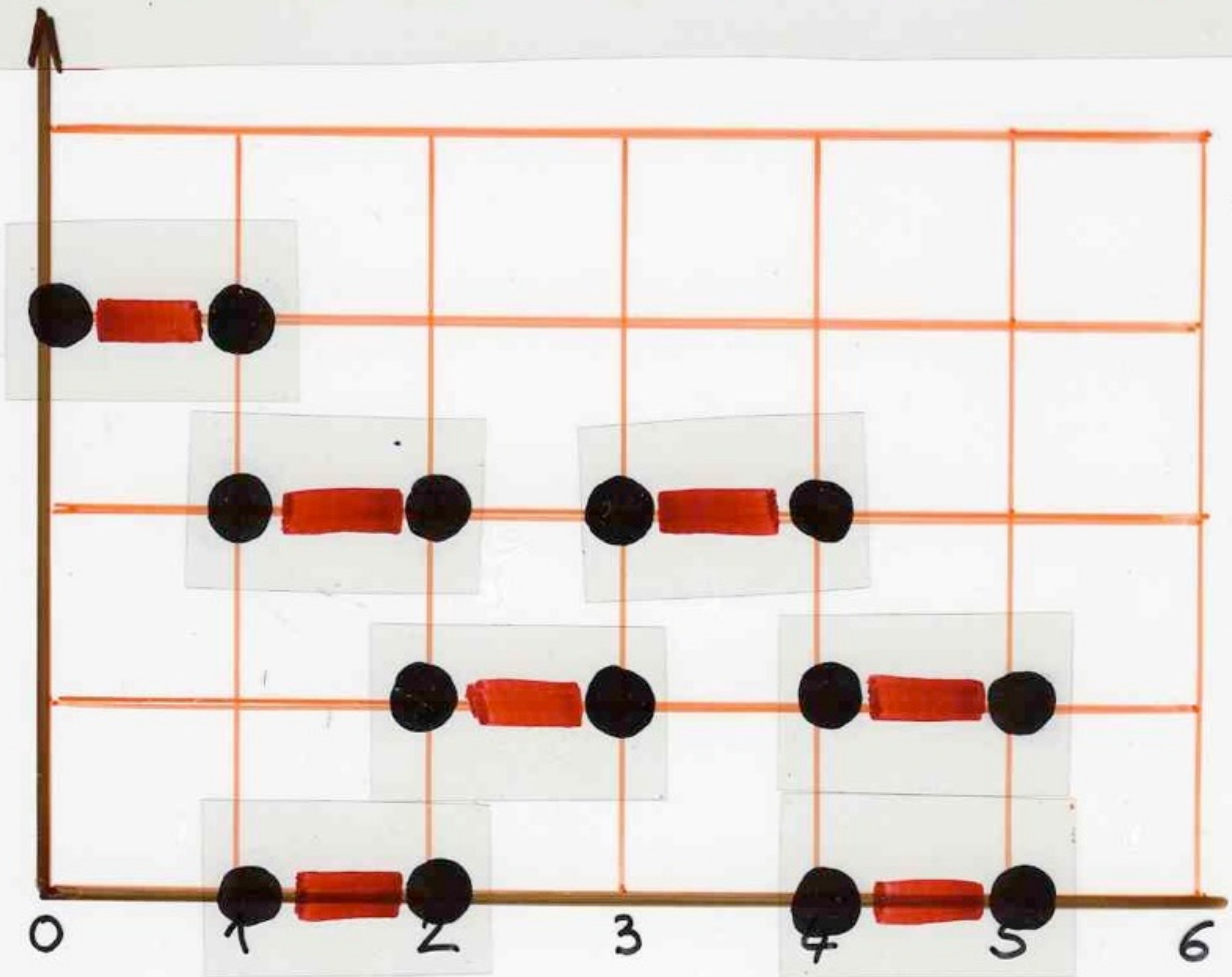
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



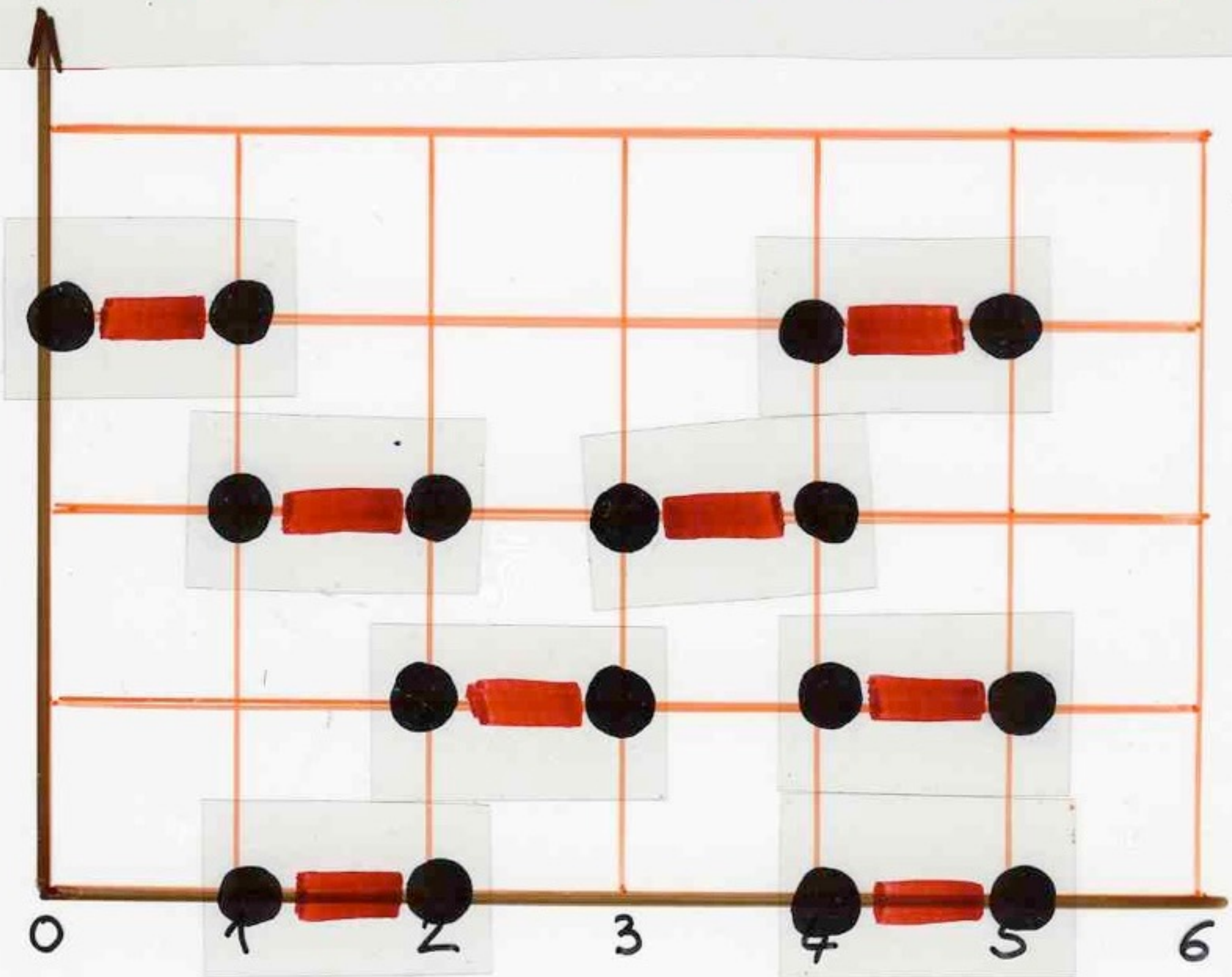
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

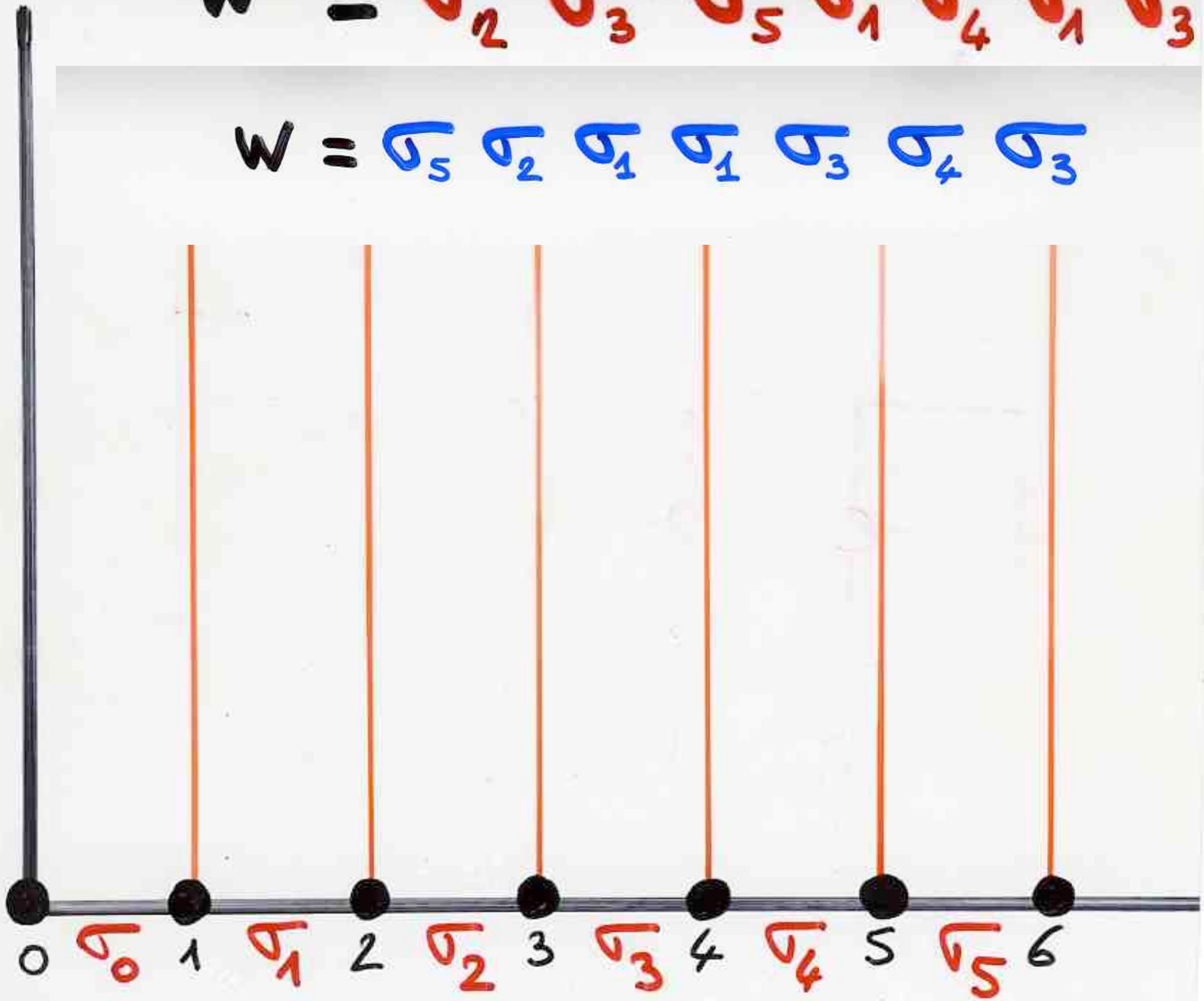


$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



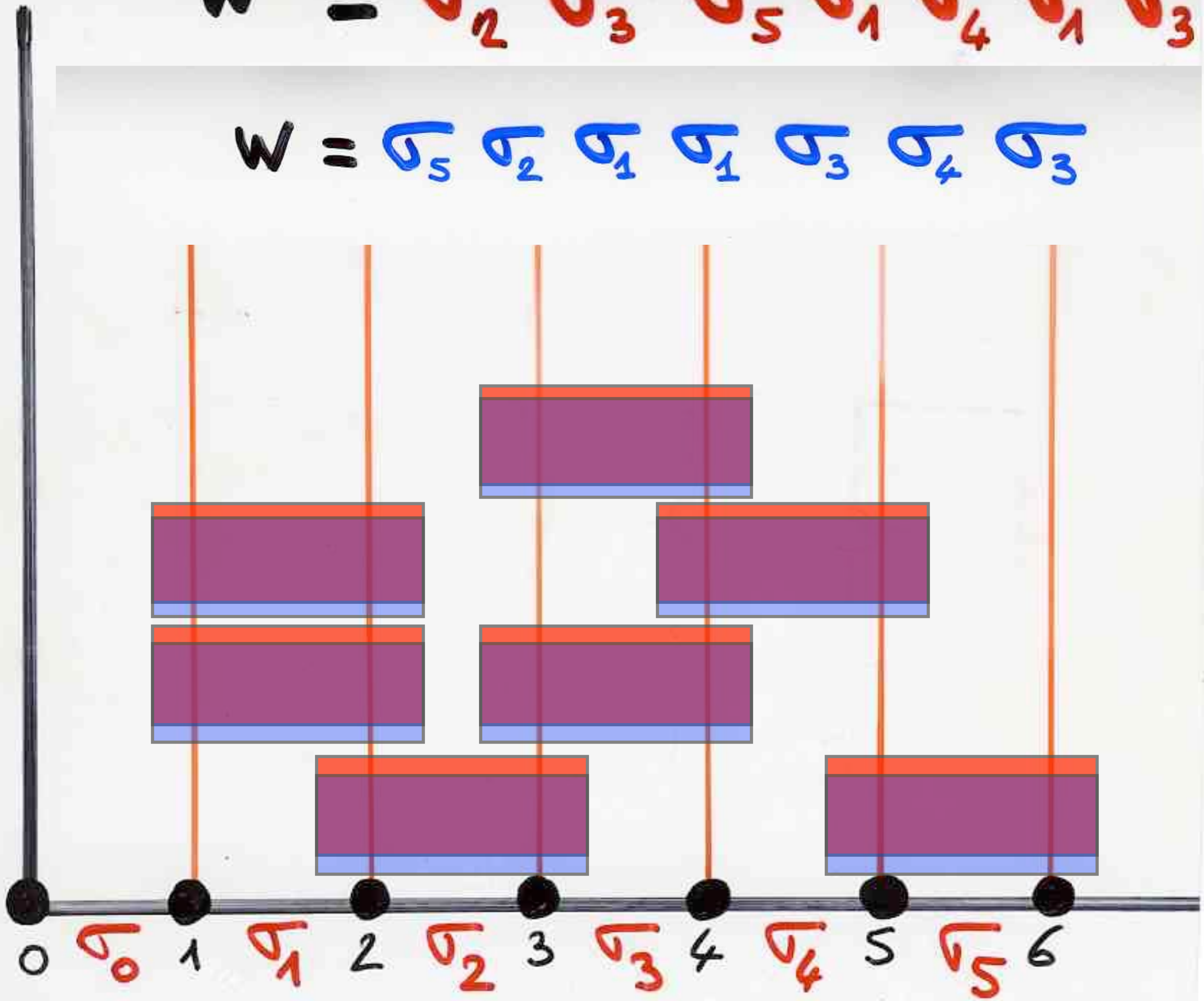
$$W = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$$

$$W = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$$



$$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$$

$$w = \sigma_5 \sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_4 \sigma_3$$



$$P \subseteq Heap(P, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

$$\varphi : P^* \longrightarrow Heap(P, \mathcal{C})$$

$$w = \alpha_1 \alpha_2 \cdots \alpha_n \longrightarrow \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n$$

word — heap

$$C = \overline{\mathcal{C}}$$

C: commutation relation

$\overline{\mathcal{C}}$: complementary of the dependency relation

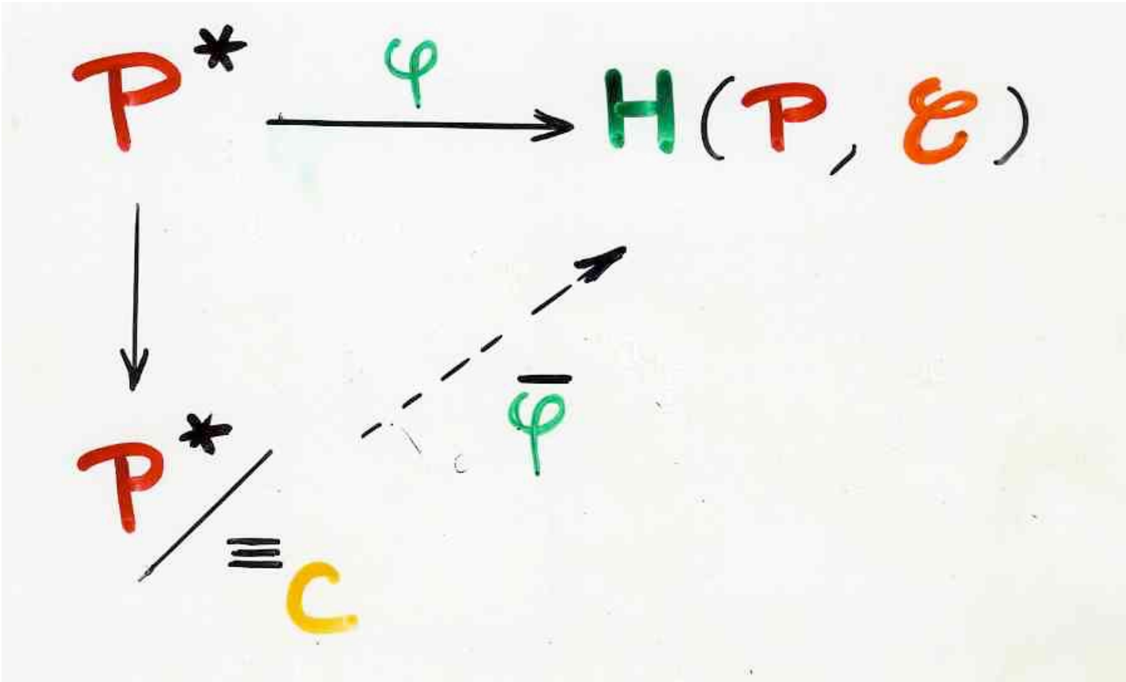
<u>Lemma 1</u>

$$u \equiv_C v \Rightarrow \varphi(u) = \varphi(v)$$

<u>Lemma 2</u>

$$\varphi(u) = \varphi(v) \Rightarrow u \equiv_C v$$

Definition $\overline{\varphi}([u]) = \varphi(u)$



Proposition $\overline{\varphi}$ is an isomorphism of monoids

$$\mathrm{Heap}(P, \mathcal{E}) \simeq P^{*} / \equiv_{C}$$

heaps monoid — commutation monoid

$$C = \overline{\mathcal{E}}$$ complementary relation

Symmetric group $\mathfrak{S}_n$

S_n

$n!$ permutations

$$\sigma_i = (i, i+1) \qquad i=1,2,\dots,n-1$$

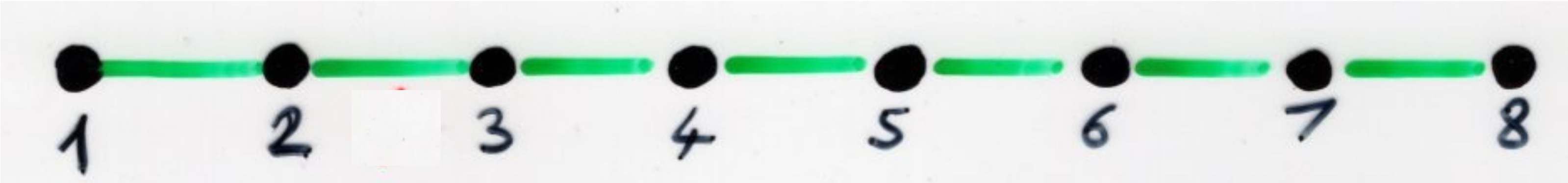
transposition of two consecutive elements

$$\begin{cases} \text{(i)} & \sigma_i \sigma_j = \sigma_j \sigma_i , \qquad |i-j| \geq 2 \\ \text{(ii)} & \sigma_i^2 = 1 , \\ \text{(iii)} & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} . \end{cases}$$

Moore-Coxeter

Yang-Baxter

Coxeter graph



Heaps as Posets

Poset (partially ordered set)

$(E, \preccurlyeq)$

$\preccurlyeq$ order relation

$\preccurlyeq$ order relation on E

- reflexive $x \preccurlyeq x$ all $x \in E$
- antisymmetric $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$
- transitive $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$

for all $x, y, z \in E$

Poset (partially ordered set)

$(E, \preccurlyeq)$

$\preccurlyeq$ order relation

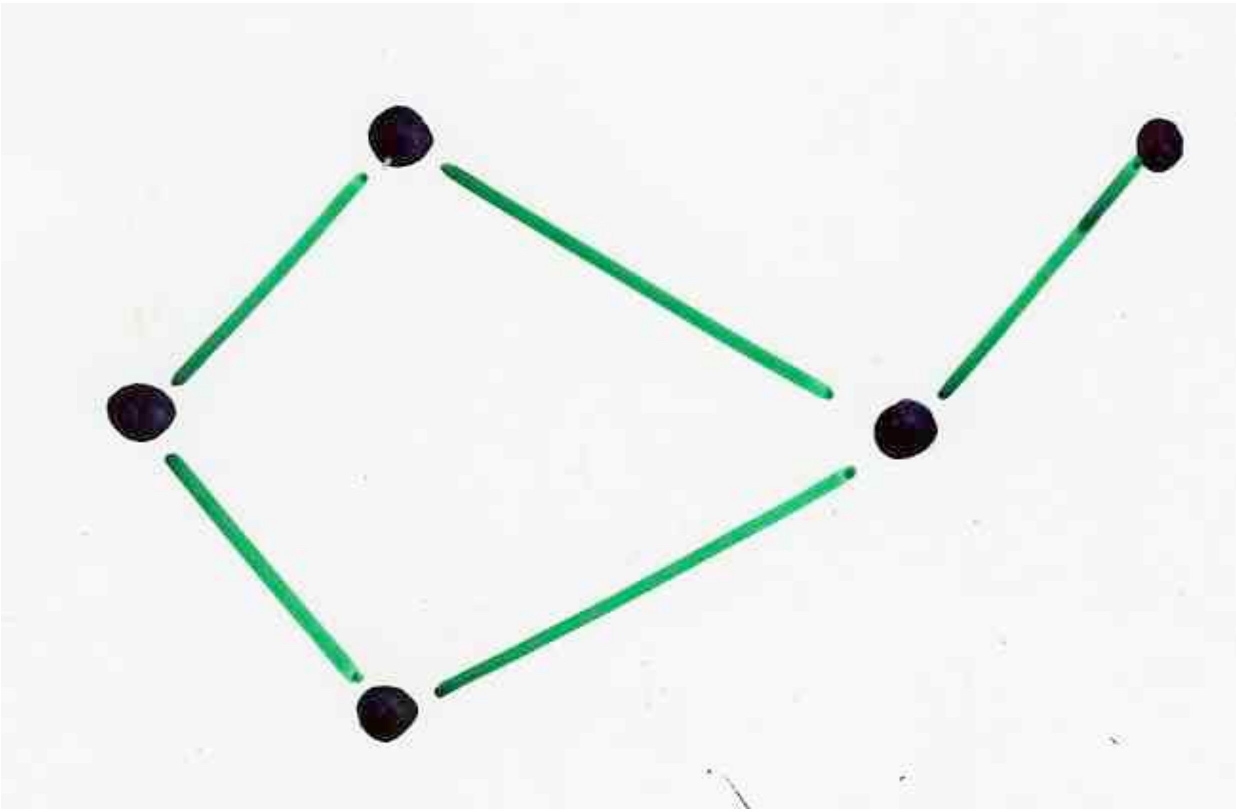
covering relation

$x, y \in E$, y covers x

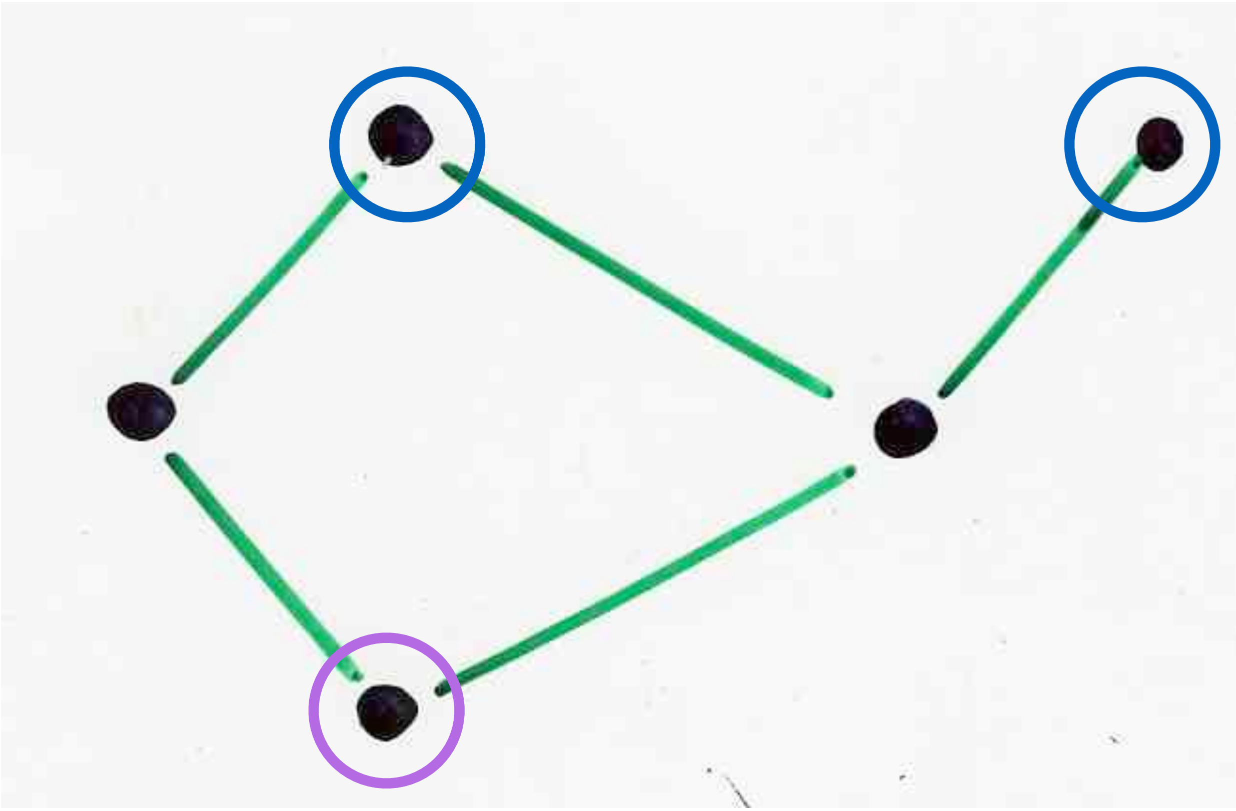
iff $x \prec y$ (strict) and $x \preccurlyeq z \preccurlyeq y \Rightarrow \begin{cases} z = x \\ \text{or} \\ z = y \end{cases}$

the interval $[x, y]$ is reduced to $\{x, y\}$

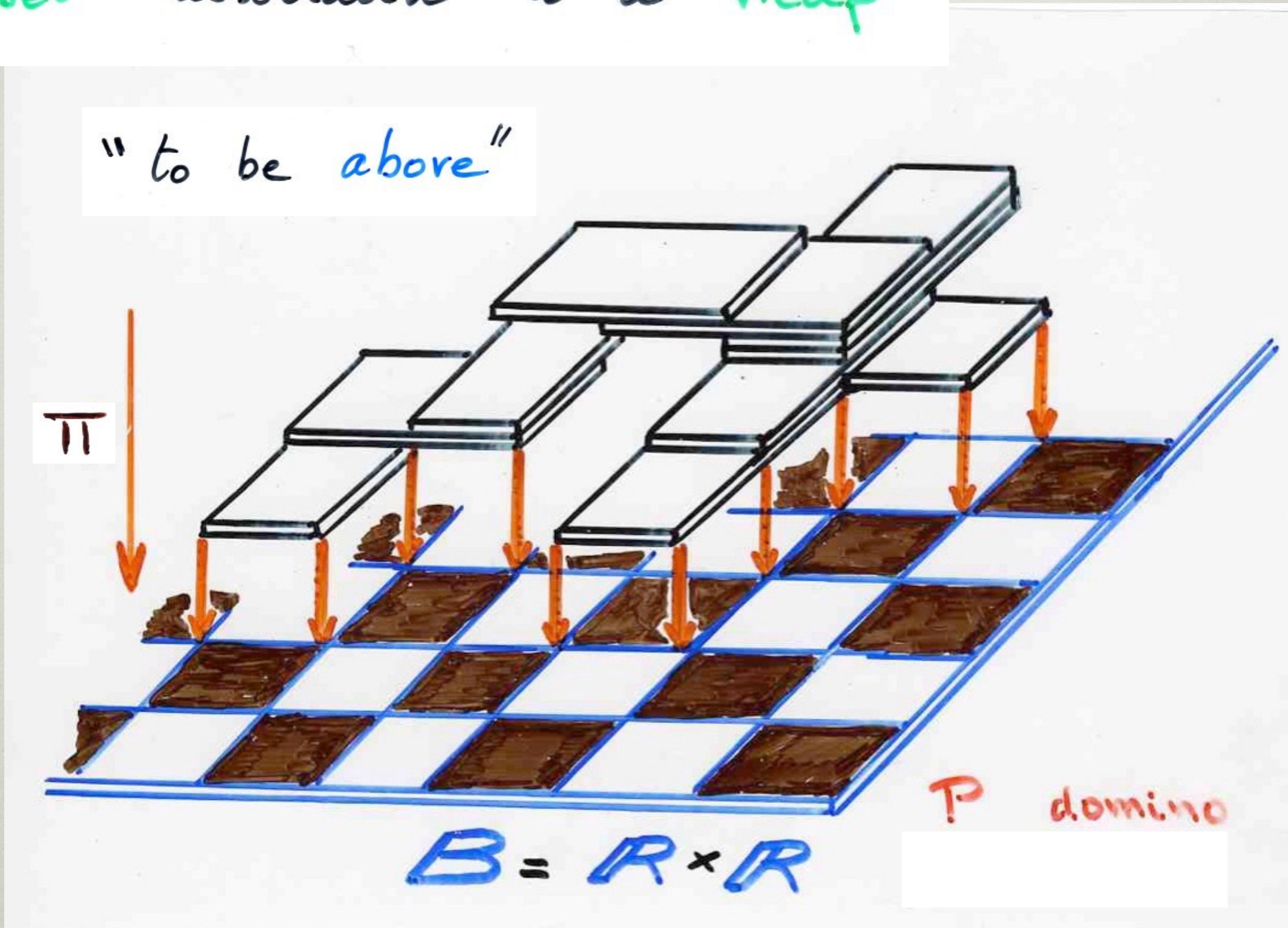
Hasse diagram of a poset



minimal
maximal
element of a poset



poset associated to a heap

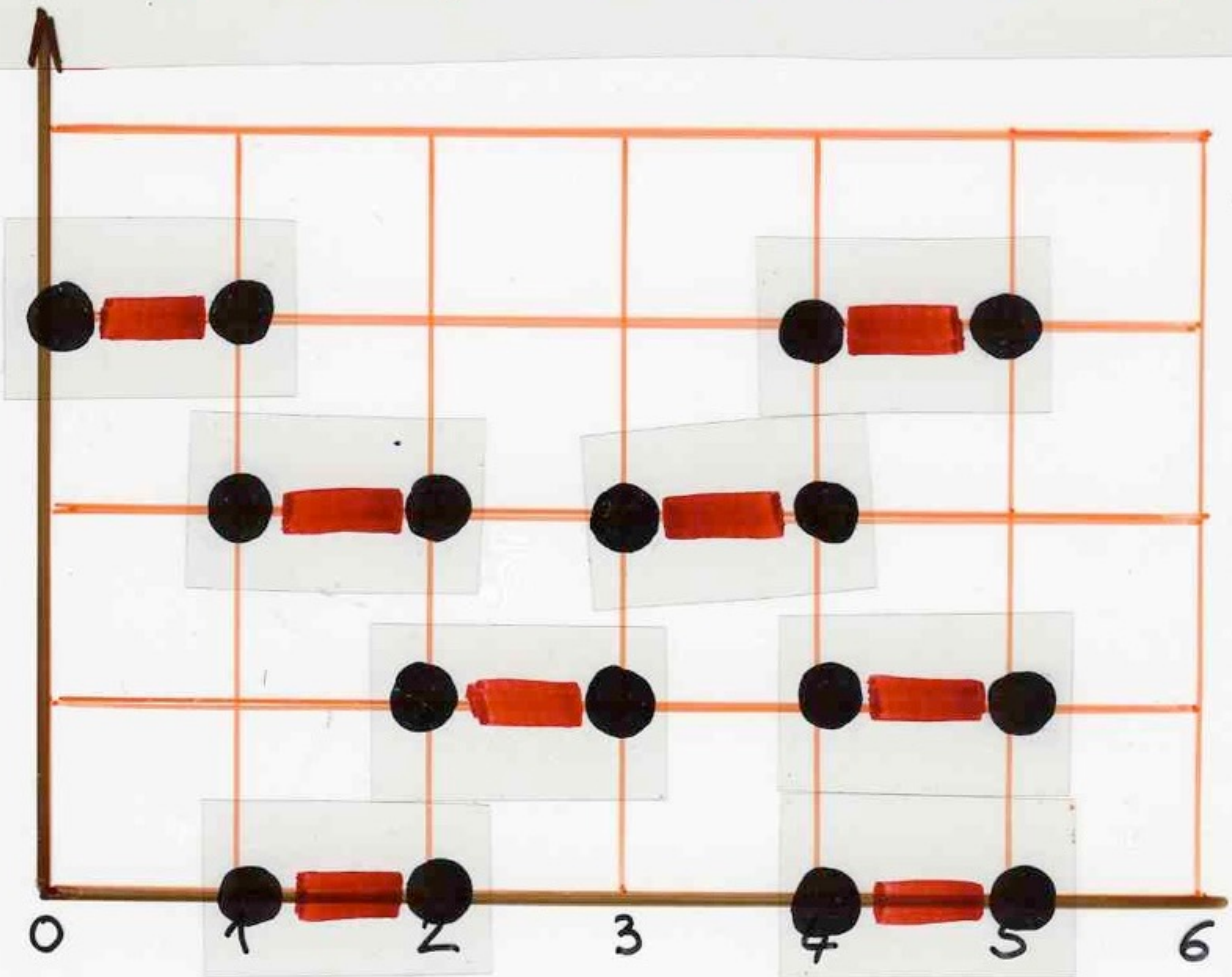


Def. **<u>Poset</u>** $(E, \preceq)$ associated to a heap E

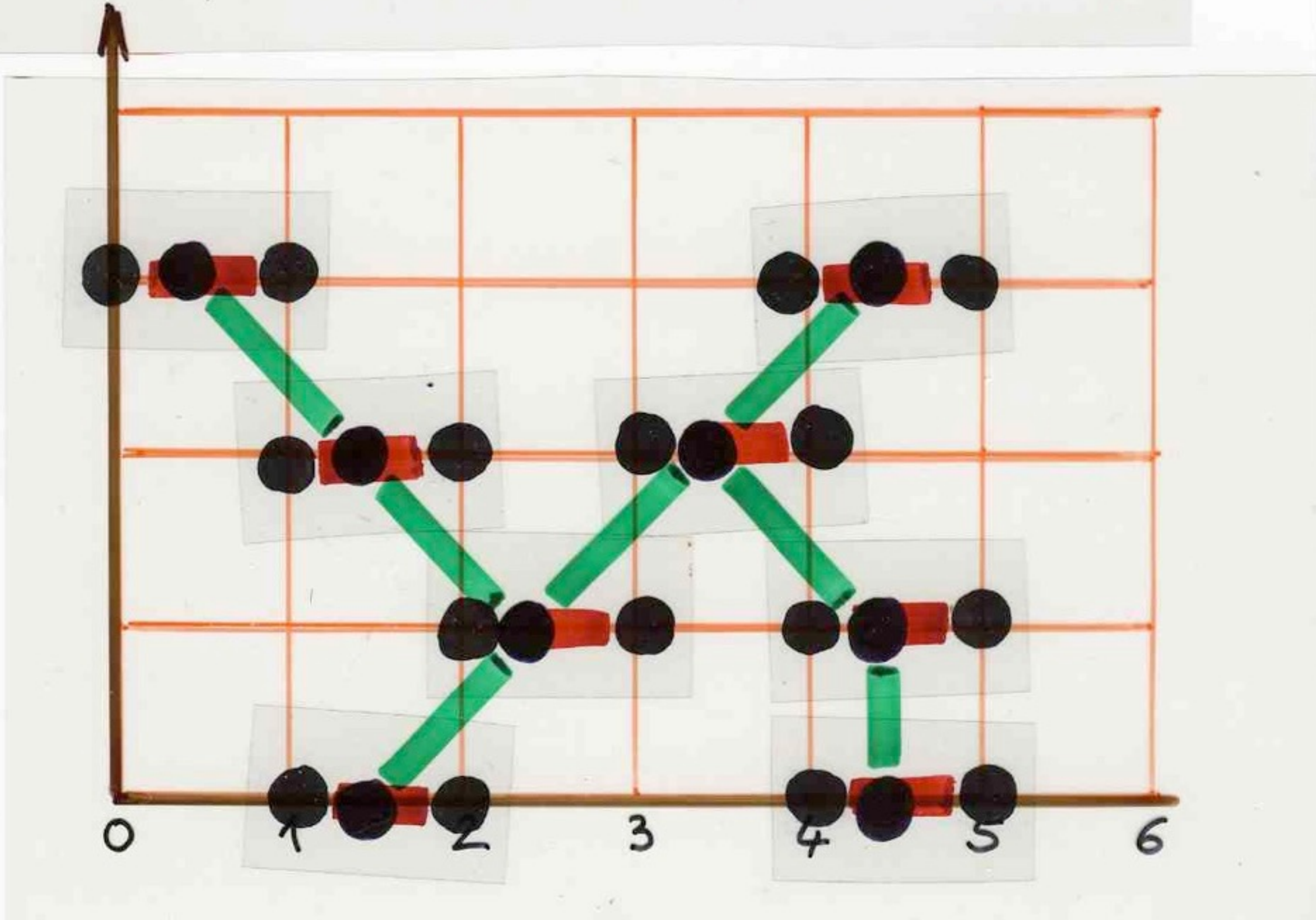
$\preceq$ transitive closure of the relation $\preceq_{\mathcal{C}}$

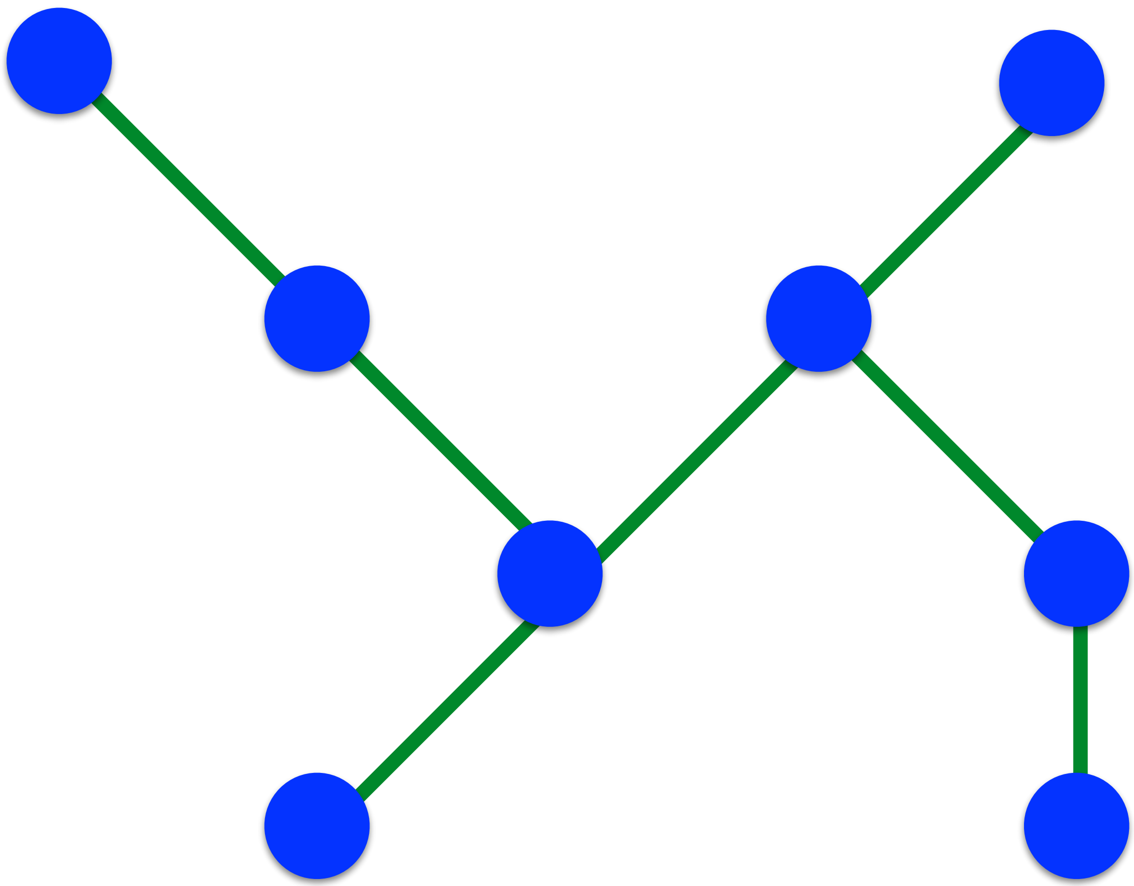
$$(\alpha, i) \preceq_{\mathcal{C}} (\beta, j) \iff \alpha \, \mathcal{C} \, \beta \,, \; i < j$$

$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

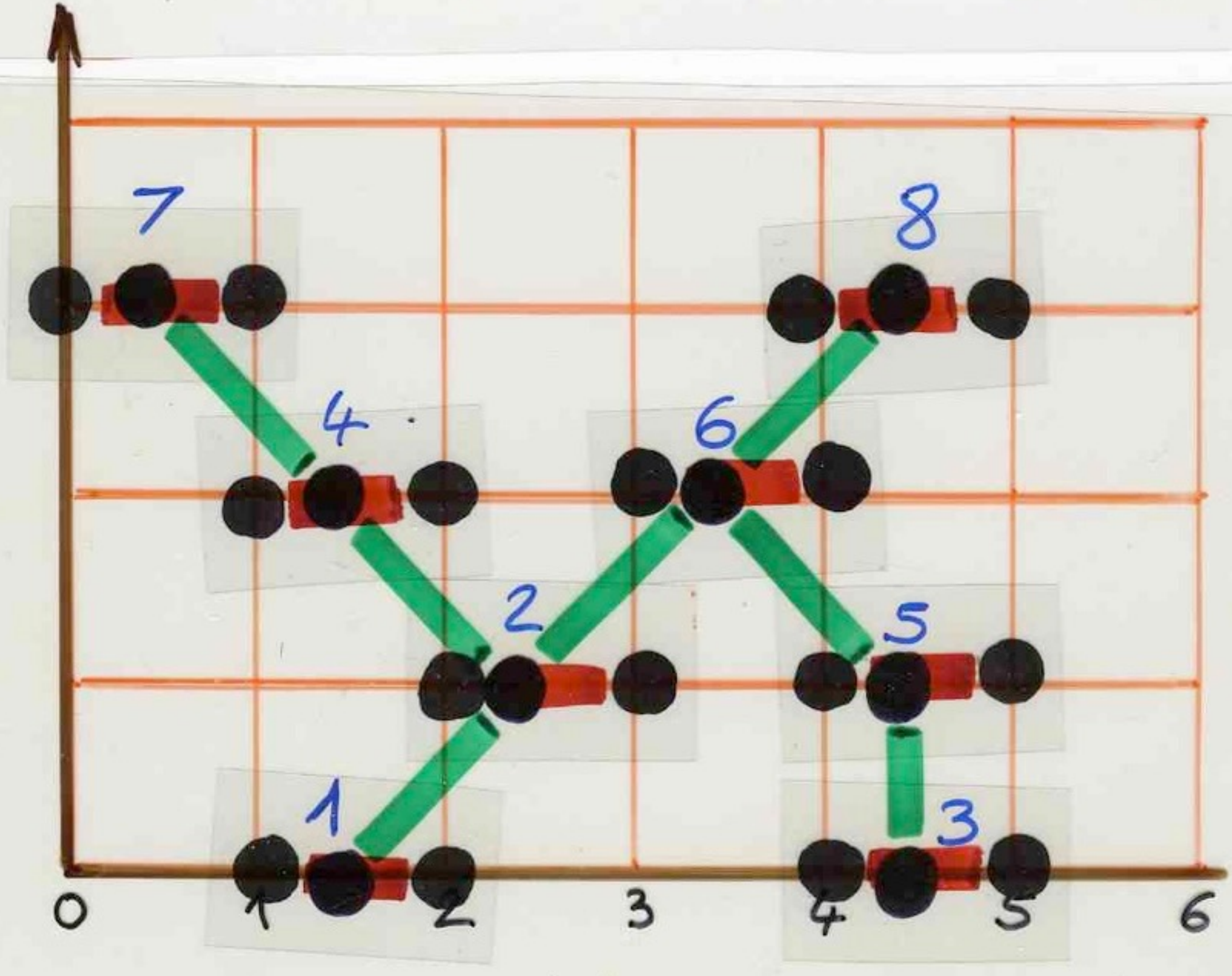


$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$





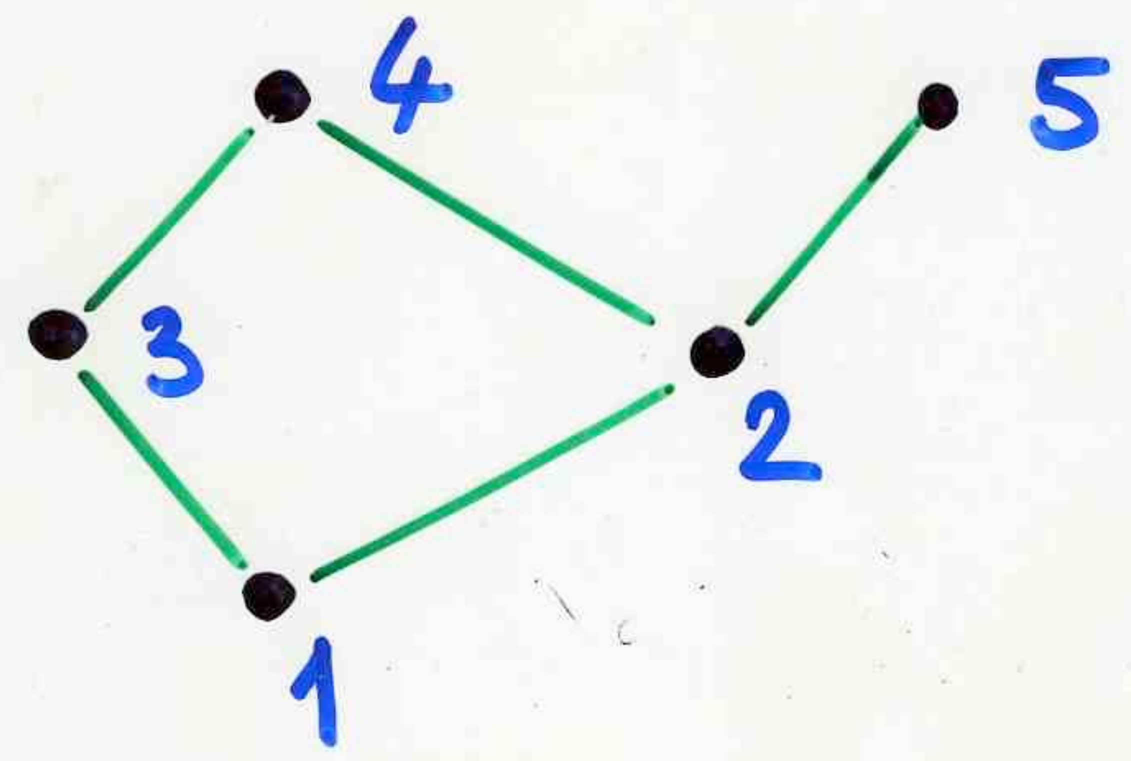
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

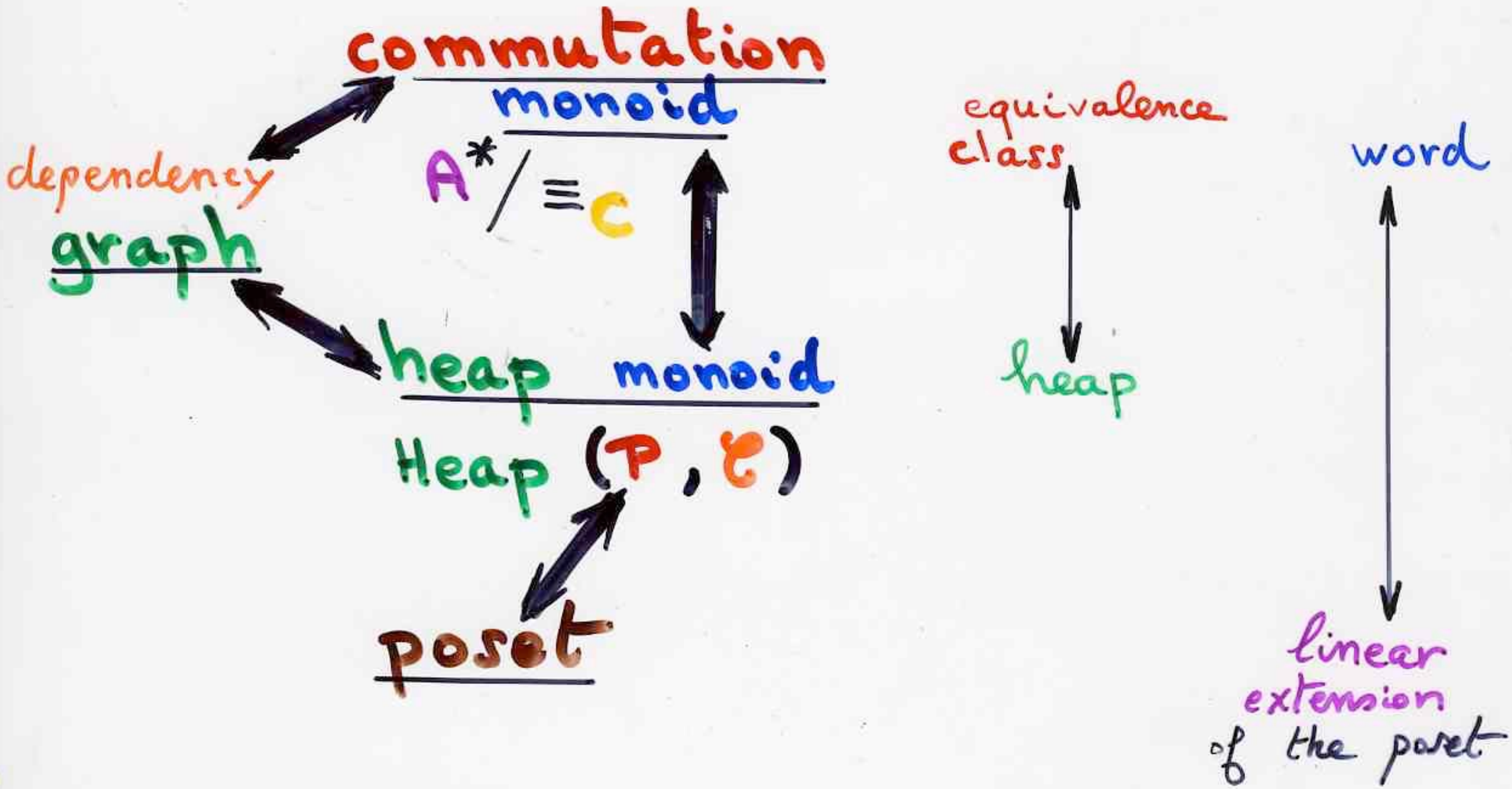


linear extension of a poset $(E, \preccurlyeq)$

Def- $f : E \longrightarrow [1, n]$ bijection

$x \preccurlyeq y \Rightarrow f(x) \leq f(y)$


4
5
3
2
1


commutation monoid
$A^*/\equiv_C$
dependency graph
heap monoid
Heap (P, C)
poset
equivalence class
heap
word
linear extension of the poset

second definition of heap

E heap of pieces in P

- P set (of basic pieces)
- $\mathscr{C}$ dependency relation on P
  symmetric and reflexive

- is a poset with order relation $\preccurlyeq$
- $E \xrightarrow{\pi} P$ π projection ("to be above")

(i) $\alpha, \beta \in E$, $\pi(\alpha)\, \mathscr{C}\, \pi(\beta) \Rightarrow$ or $\begin{matrix} \alpha \preccurlyeq \beta \\ \beta \preccurlyeq \alpha \end{matrix}$

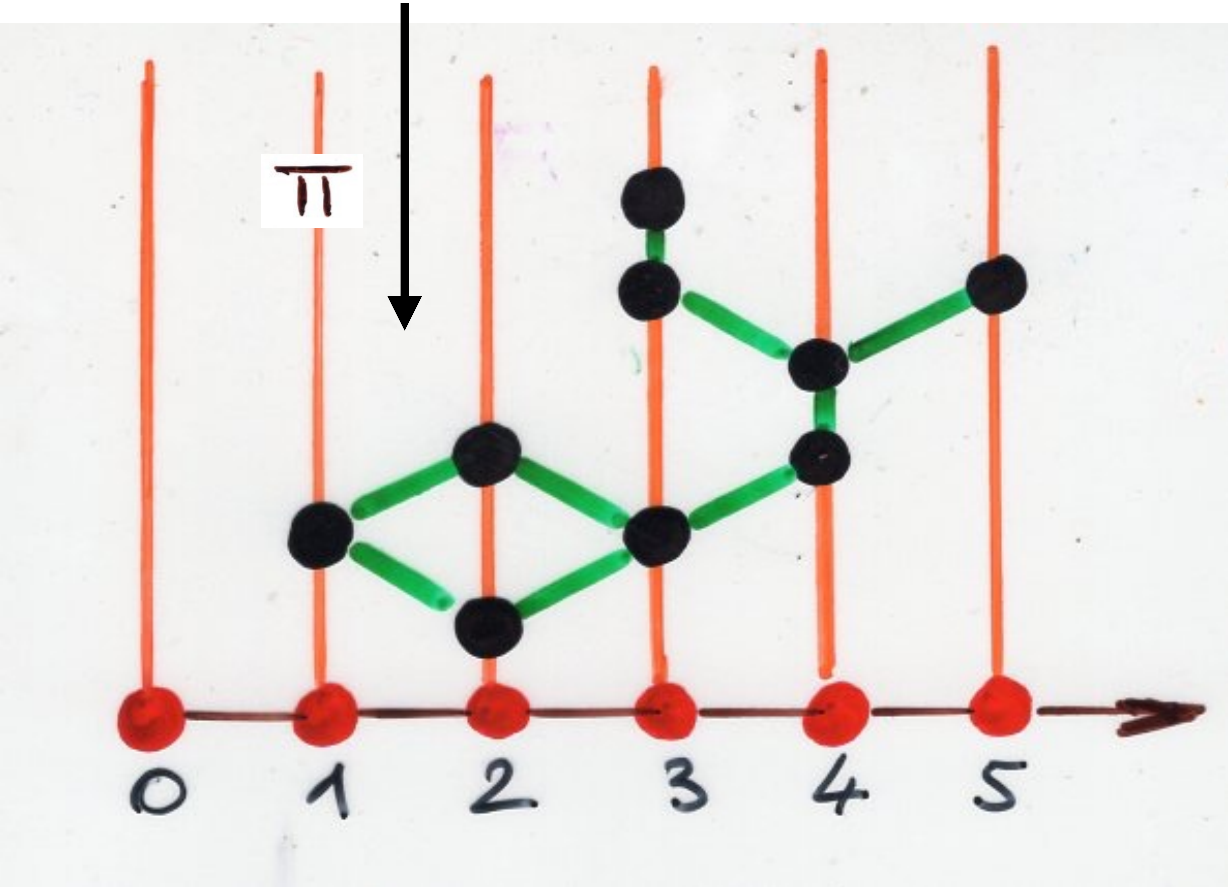
(ii) $\alpha, \beta \in E$, $\alpha \preccurlyeq \beta$, β covers α

$\Rightarrow \pi(\alpha)\, \mathscr{C}\, \pi(\beta)$

(i) $\alpha, \beta \in E$, $\pi(\alpha)\, \mathcal{C}\, \pi(\beta) \Rightarrow \alpha \preceq \beta$ or $\beta \preceq \alpha$

(ii) $\alpha, \beta \in E$, $\alpha \preceq \beta$, β covers α

$$\Rightarrow \pi(\alpha)\, \mathcal{C}\, \pi(\beta)$$



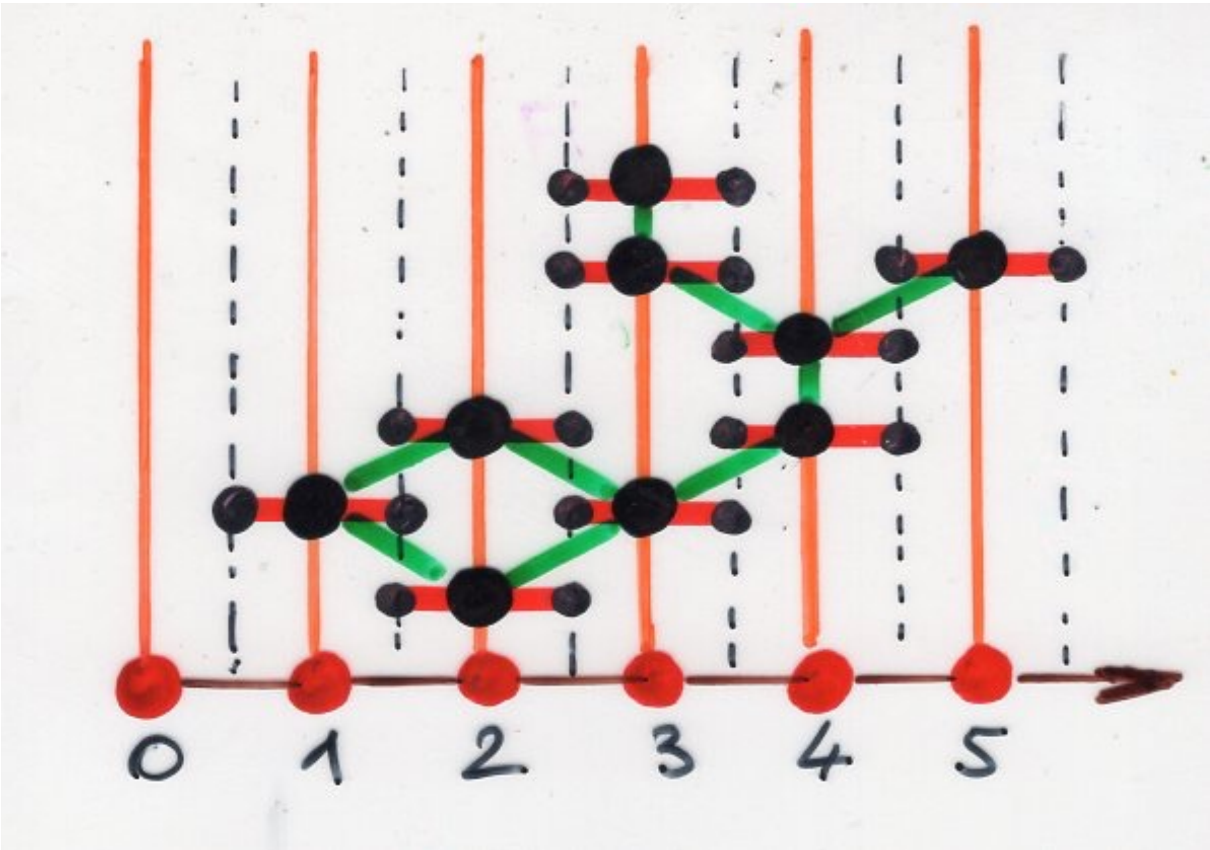
$P = N$

$i\, \mathcal{C}\, j \Leftrightarrow |i-j| \leq 1$

(i) $\alpha, \beta \in \mathbf{E},\ \pi(\alpha)\, \mathcal{C}\, \pi(\beta) \Rightarrow$ $\alpha \preceq \beta$ or $\beta \preceq \alpha$

(ii) $\alpha, \beta \in \mathbf{E},\ \alpha \preceq \beta,\ \beta$ covers α

$$\Rightarrow \pi(\alpha)\, \mathcal{C}\, \pi(\beta)$$



$$P = \mathbb{N}$$

$$i\, \mathcal{C}\, j \Leftrightarrow |i-j| \leq 1$$

equivalent definition

(i) $\alpha, \beta \in E$, $\Pi(\alpha) \mathcal{C} \Pi(\beta) \Rightarrow \begin{cases} \alpha \preccurlyeq \beta \\ \text{or } \beta \preccurlyeq \alpha \end{cases}$

(ii') $\preccurlyeq$ is the transitive closure of the relation in (i)

$\alpha \preccurlyeq \beta$ and $\Pi(\alpha) \mathcal{C} \Pi(\beta)$

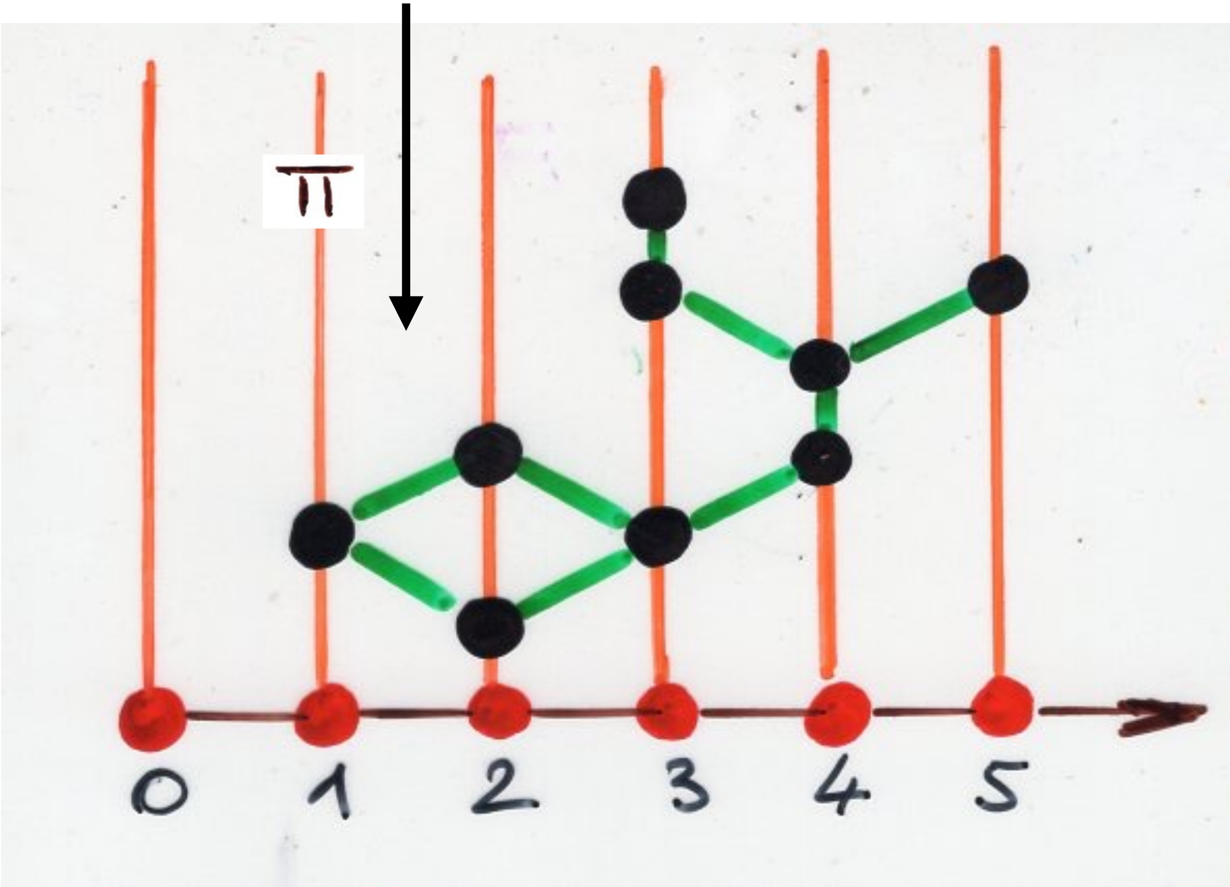
i.e. $\alpha \preccurlyeq \beta \Leftrightarrow \exists\, \alpha_1 = \alpha \preccurlyeq \alpha_2 \preccurlyeq \cdots \preccurlyeq \alpha_k = \beta$,
with $\Pi(\alpha_i) \mathcal{C} \Pi(\alpha_{i+1})$ for $i = 1, \ldots, k-1$.

heaps over a graph

(i) $\alpha, \beta \in \mathbf{E}$, $\pi(\alpha) \,\mathcal{C}\, \pi(\beta) \Rightarrow \alpha \preceq \beta$ or $\beta \preceq \alpha$

(ii) $\alpha, \beta \in \mathbf{E}$, $\alpha \preceq \beta$, β covers α

$$\Rightarrow \pi(\alpha) \,\mathcal{C}\, \pi(\beta)$$

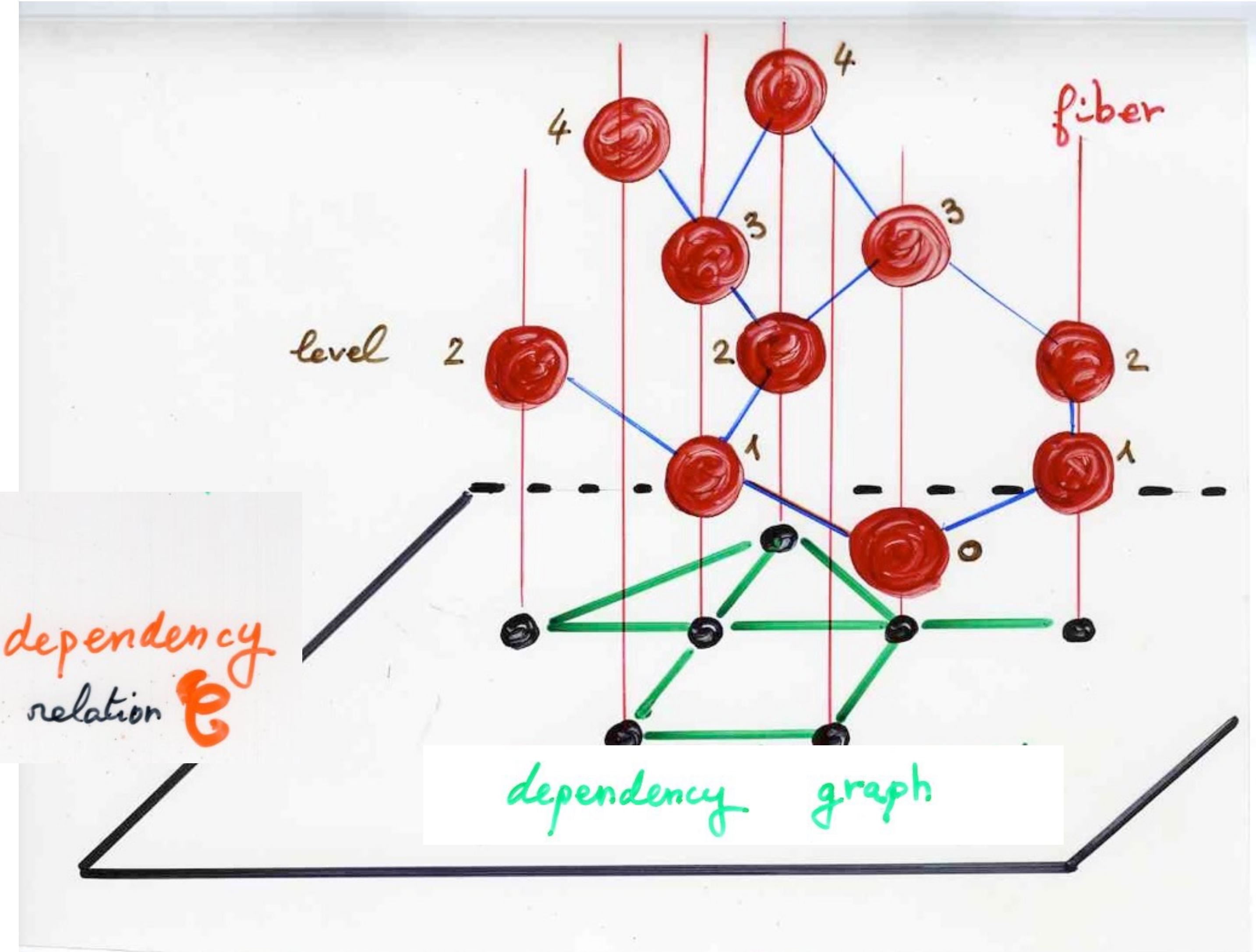


$\mathcal{P} = \mathbb{N}$

$i \,\mathcal{C}\, j \iff |i-j| \leq 1$

$G = (V, E) \longrightarrow$ heap monoid

$$\mathbf{H}(G) = \mathbf{H}(V, E)$$



$G = (V, E)$

V: basic pieces

E: dependency relation $\mathcal{E}$

finite poset $(H, \preccurlyeq)$

labeling map π

$H \xrightarrow[\pi]{} V$

(i) $\alpha, \beta \in E$, $\pi(\alpha) \, \mathcal{C} \, \pi(\beta) \Rightarrow \begin{cases} \alpha \preccurlyeq \beta \\ \text{or} \\ \beta \preccurlyeq \alpha \end{cases}$

(ii′) $\preccurlyeq$ is the transitive closure of the relation in (i)

$\alpha \preccurlyeq \beta$ and $\pi(\alpha) \, \mathcal{C} \, \pi(\beta)$

can be rewritten

can be rewritten as :

(i)′

for every vertex $s \in V$
$H_s = \pi^{-1}(\{s\})$ is a chain

fiber over $s \in V$

for any edges $\{s,t\}$ of G
$H_{s,t} = \pi^{-1}(\{s,t\})$ is a chain

fiber over $\{s,t\}$ edge of G

chain = totally ordered subset of H

(ii)′

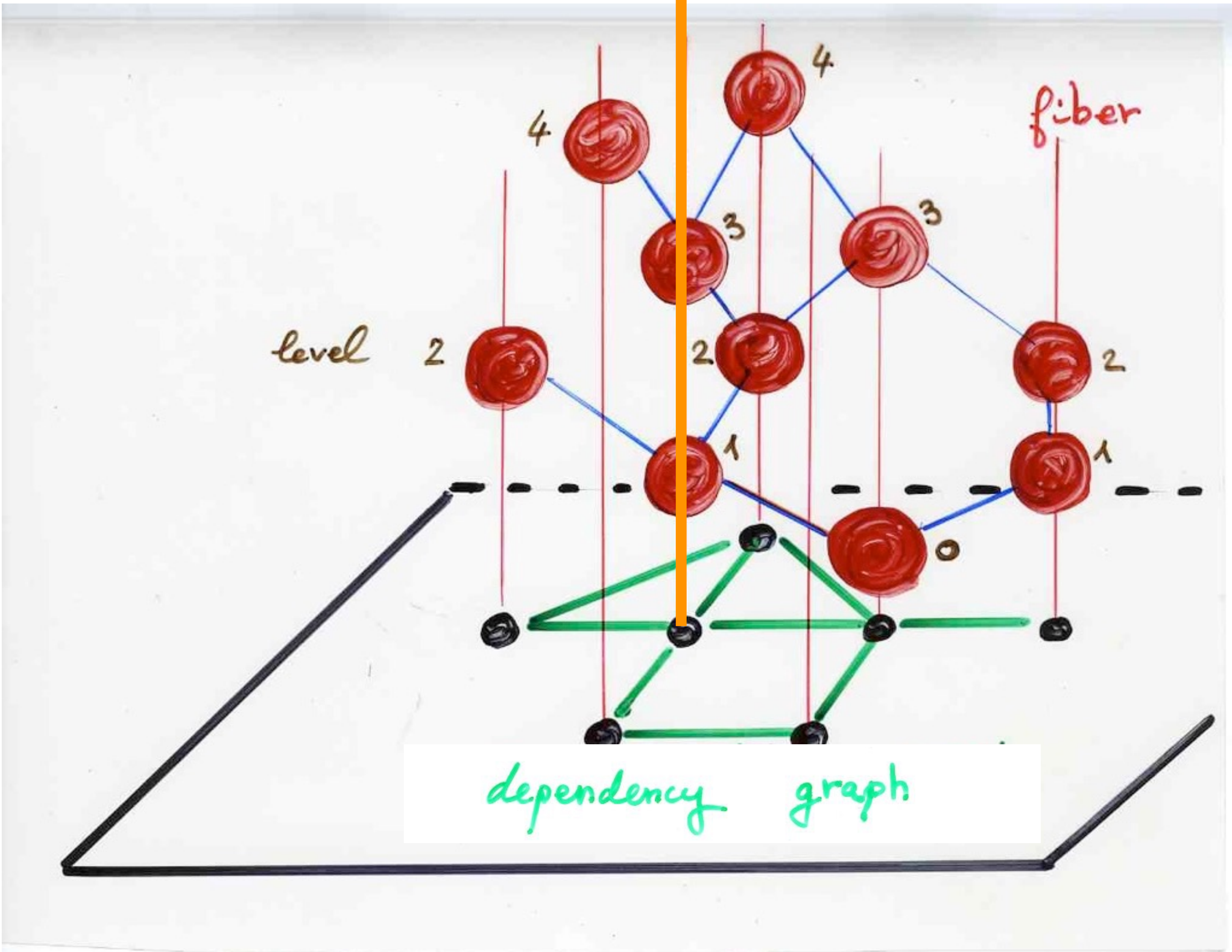
The order relation $\preccurlyeq$ is the transitive closure of the relations given by all chains of (i)′
H_s $H_{s,t}$

(i.e. the smallest partial ordering containing these chains)

$G = (V, E) \longrightarrow$ heap monoid

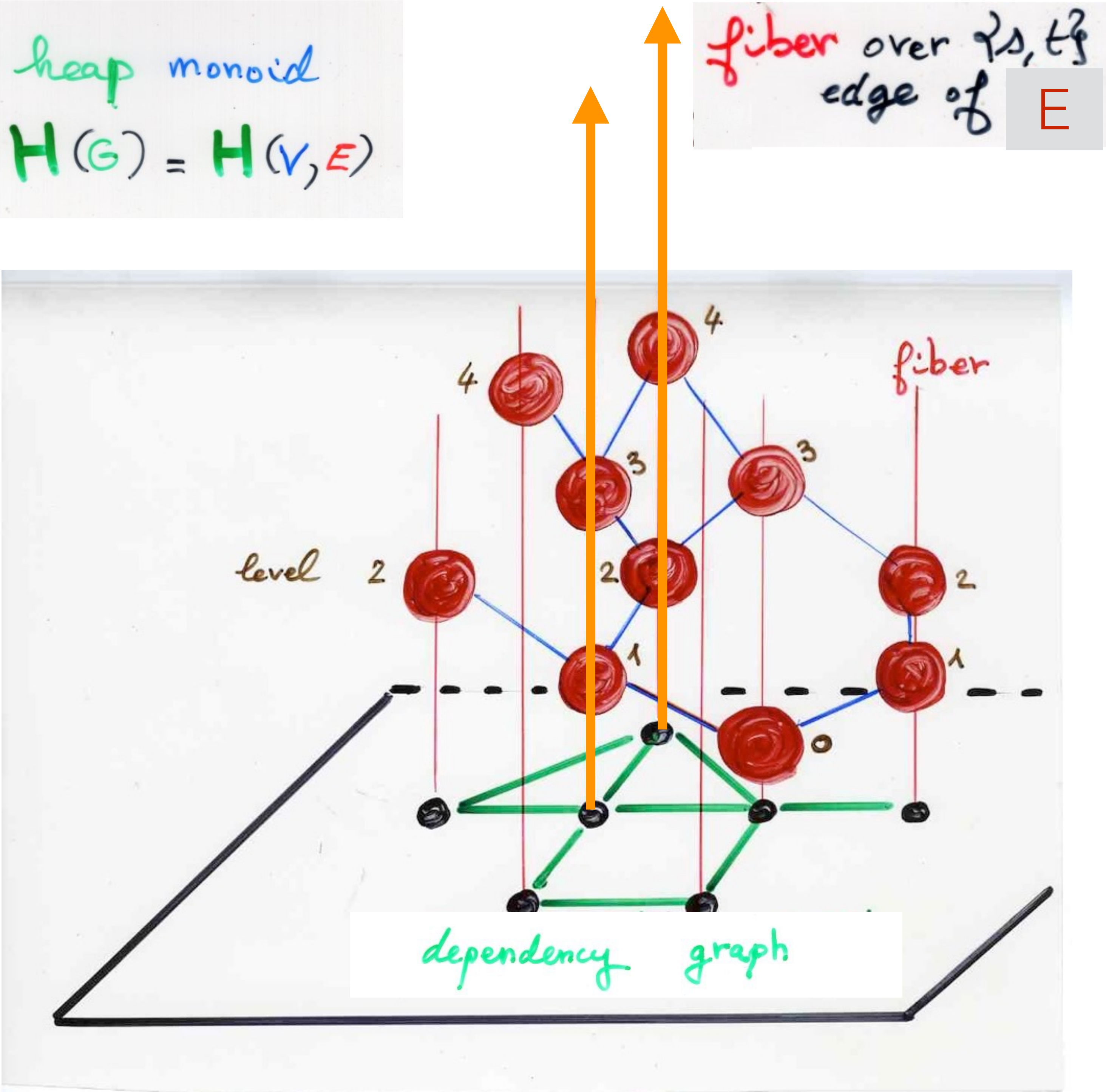
$\mathbf{H}(G) = \mathbf{H}(V, E)$

fiber over $s \in V$



$G = (V, E) \longrightarrow$ heap monoid

$\mathbf{H}(G) = \mathbf{H}(V, E)$



the inversion lemma

1/D

the inversion lemma

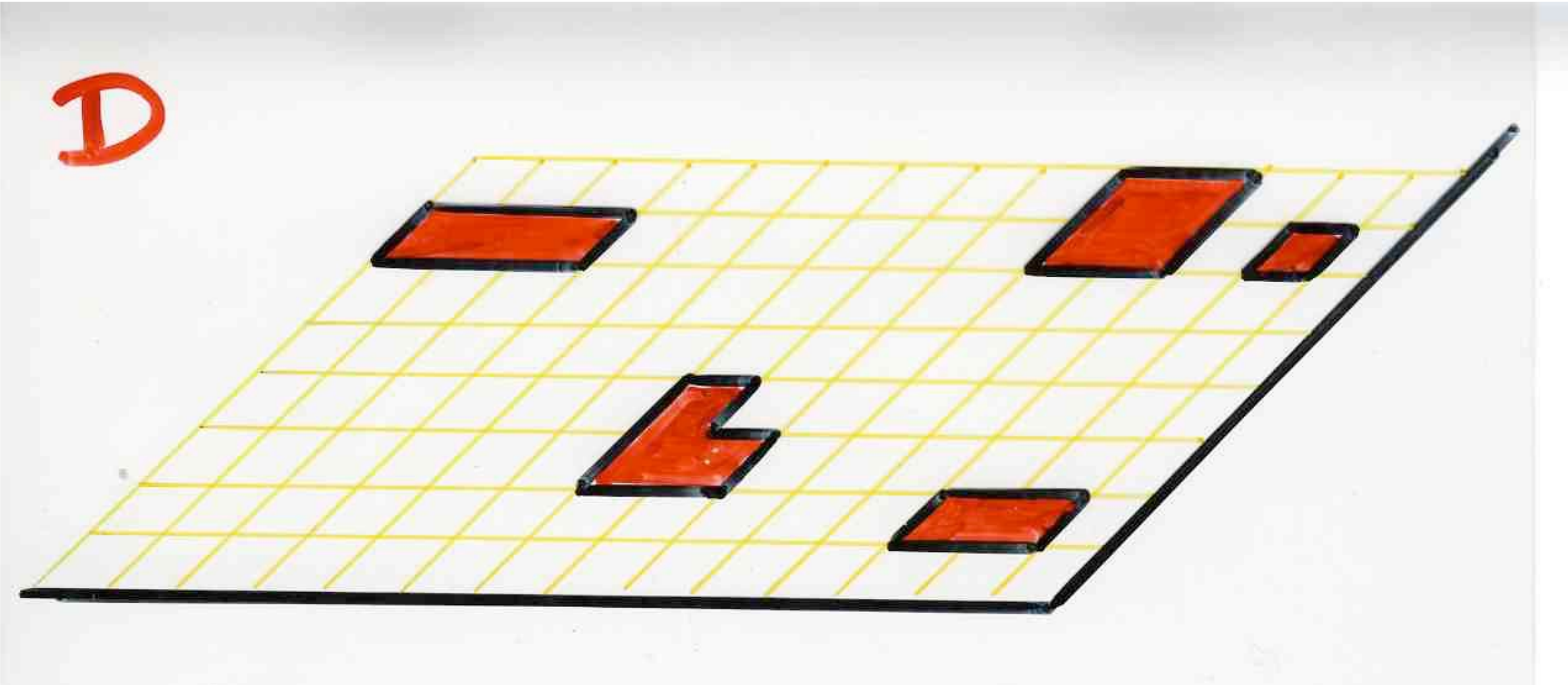
$$(\text{Heaps}) = \frac{1}{(\text{Trivial heaps})}$$

all pieces (α, i)
at level 0

trivial heap

F

all pieces (α, i) at level 0



weight
valuation

$v(E)$

- $v : P \longrightarrow \mathbb{K}[x, y, \dots]$

  basic piece

- $v(\alpha, i) = v(\alpha)$

  piece

- $v(E) = \prod_{(\alpha, i) \in E} v(\alpha\ i)$

  heap

the inversion lemma

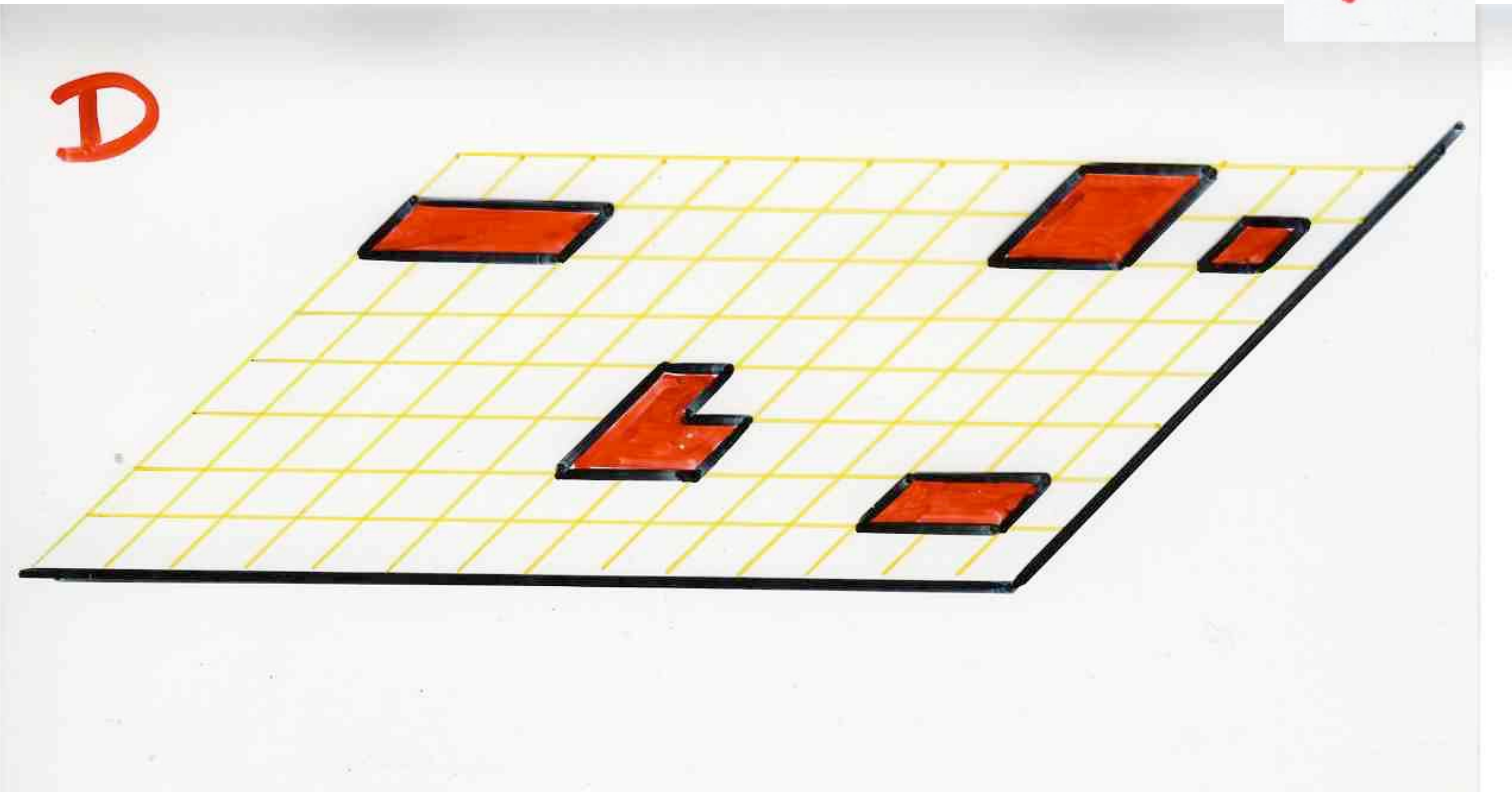
$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right)}$$

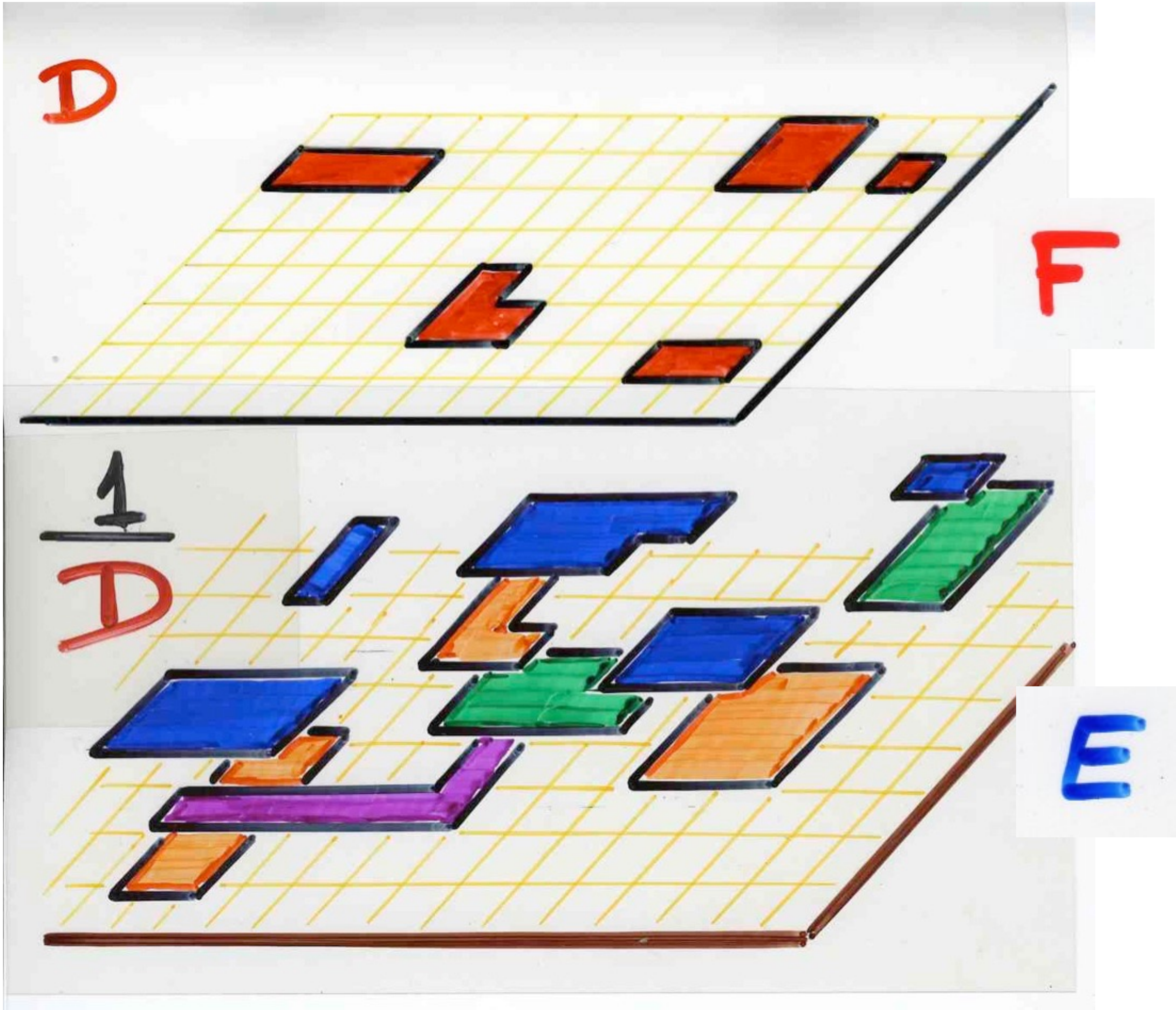
the inversion lemma

$$\left(\sum_{\substack{E\\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F\\ \text{trivial heaps}}} (-1)^{|F|} v(F)\right)}$$

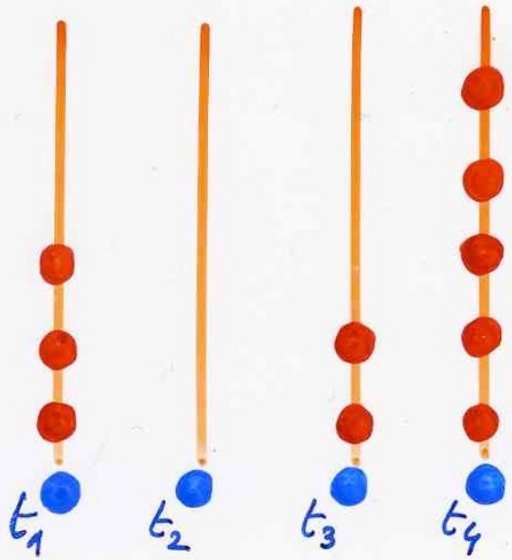
D

F

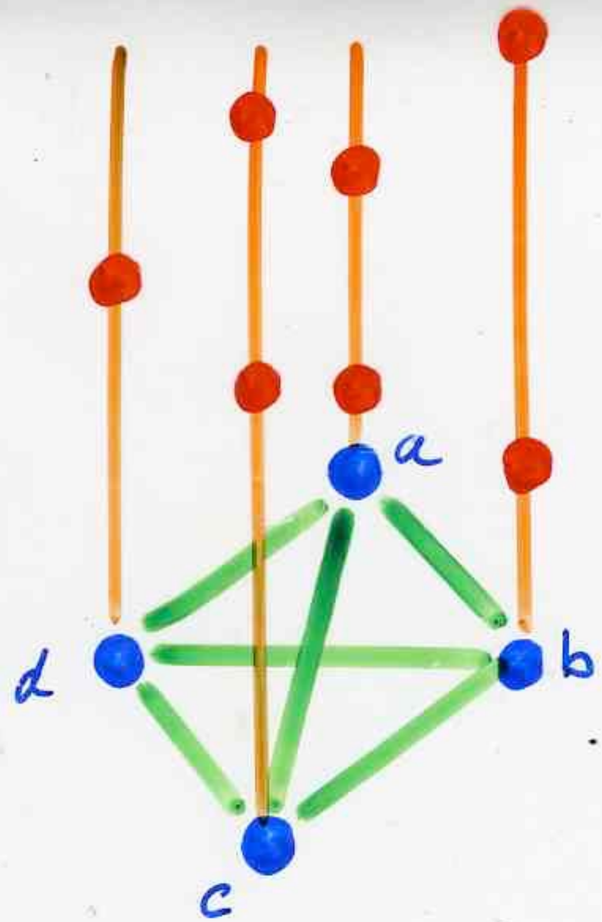



D
F
1
D
E

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n \cdots$$



$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} = \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \geq 0} t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} t_4^{\alpha_4}$$



$$X = \{a, b, c, d\}$$

$$\frac{1}{1 - X} = X^*$$

$$\left(= \sum_{w \in X^*} w \right)$$