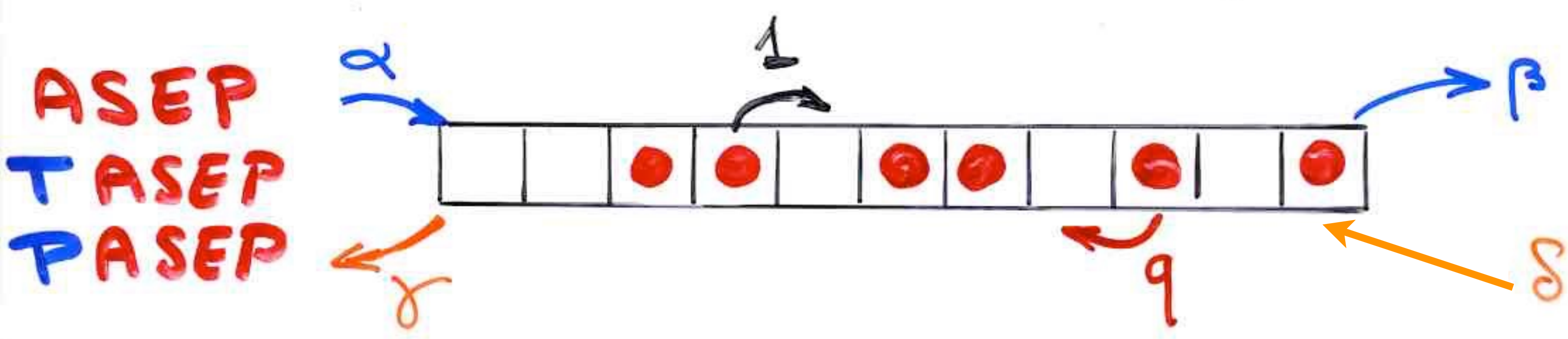
Empilements de Laguerre pour le PASEP

GT Combinatoire, LaBRI
24 Septembre 2018

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

PASEP

toy model in the physics of
dynamical systems far from equilibrium



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

$$\begin{cases} DE = qED + E + D \\ \langle W| (\alpha E - \gamma D) = \langle W| \\ (\beta D - \delta E) |V\rangle = |V\rangle \end{cases}$$

colum vector V

row vector W

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

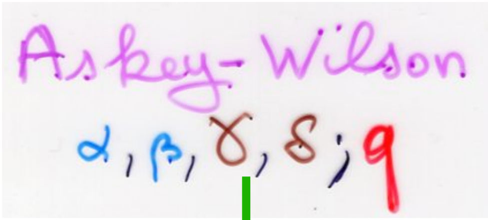
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

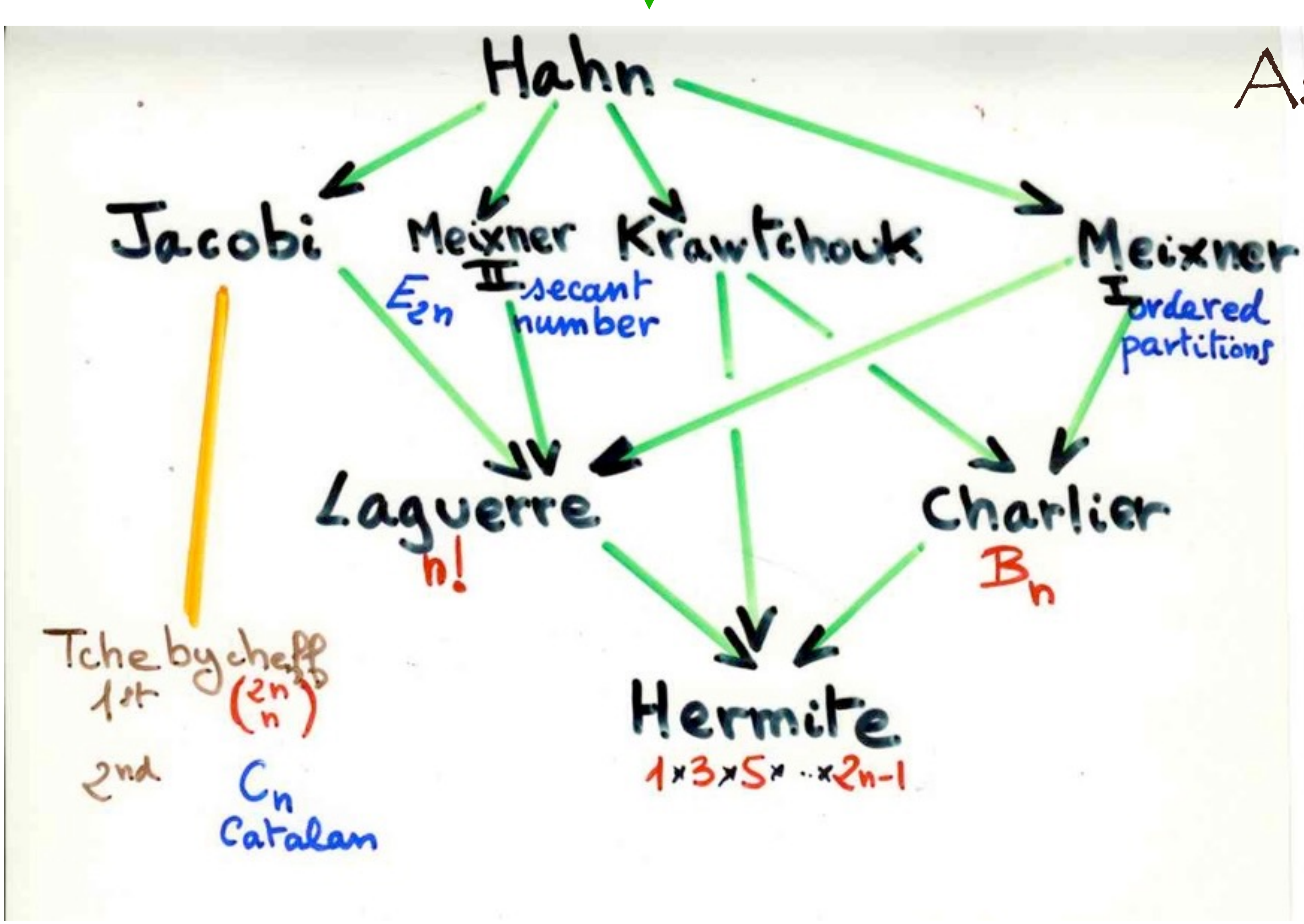
→ Uchiyama, Sasamoto, Wadati (2003)

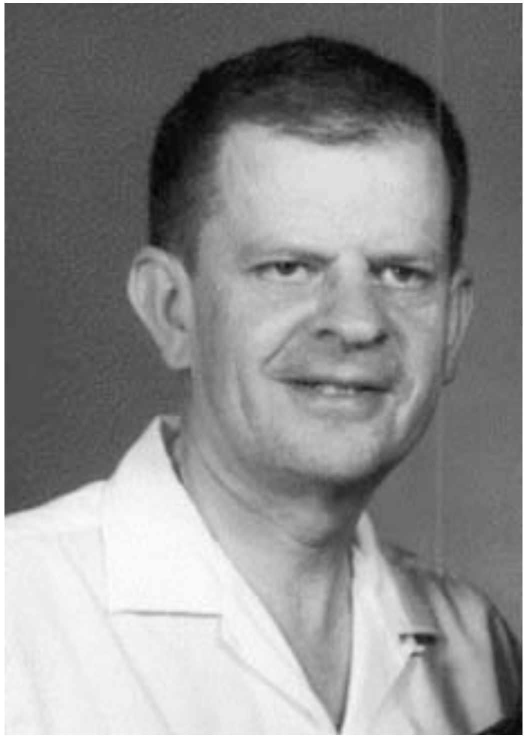
$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials



Askey tableau





Combinatorics of the PASEP

(1982) Shapiro, Zeilberger

(2004) Brak, Essam
(2005) Duchi, Schaeffer
(2006) Corteel Bunstein
Brak, Corteel, Essam, Paviainen, Rechnitzer
Corteel, Williams
(2007) Corteel, Nadeau Corteel, Williams
Steingrimsson Williams X.V.
(2008) X.V.
(2009) Corteel, Josuat-Vergès, Prellberg, Rubey
Josuat-Vergès Nadeau
(2010) Corteel, Williams

(2011) Josuat-Vergès Corteel, Kim
Corteel, Dasse-Hartaut
Corteel, Josuat-Vergès, Williams
Aval, Boussicault, Nadeau

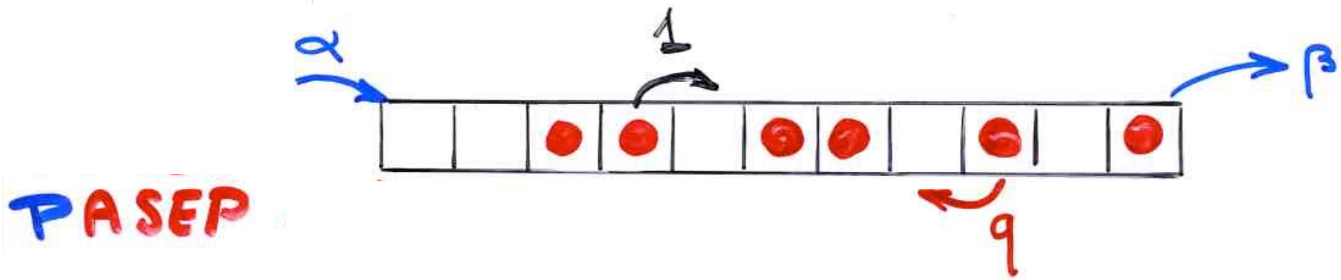
(2012) Corteel, Stanley, Stanton, Williams
(2013) Aval, Boussicault, Bouvel, Silimbani
Aval, Boussicault, Nadeau
Aval, Boussicault, Dasse-Hartaut

(2014) E. Jin
(2016) Aval, Boussicault, Delcroix-Oger, Hivert, Laborde-Zubieta
Corteel, Kim, Stanton Mandelshtam, X.V.
(2017) Corteel, Williams Mandelshtam, X.V.
Corteel, Mandelshtam, Williams
Corteel, Nunge
Laborde-Zubieta

alternative tableaux

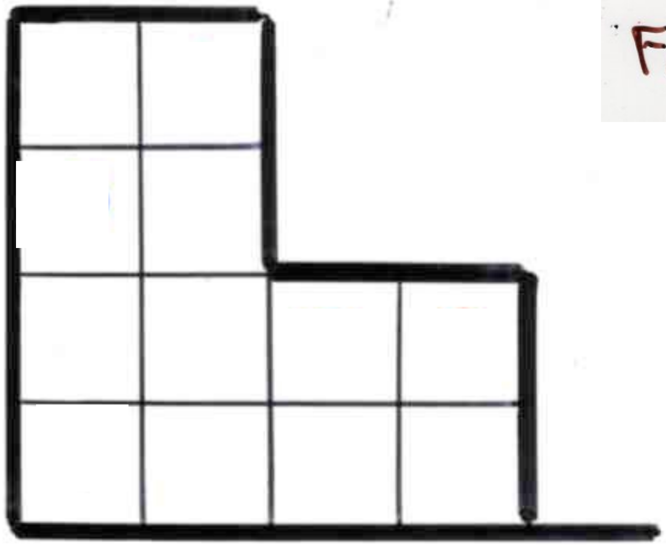
PASEP with 3 parameters

$\gamma = \delta = 0$ $\quad q, \alpha, \beta$



alternative tableau

Definition



Ferrers diagram **F**

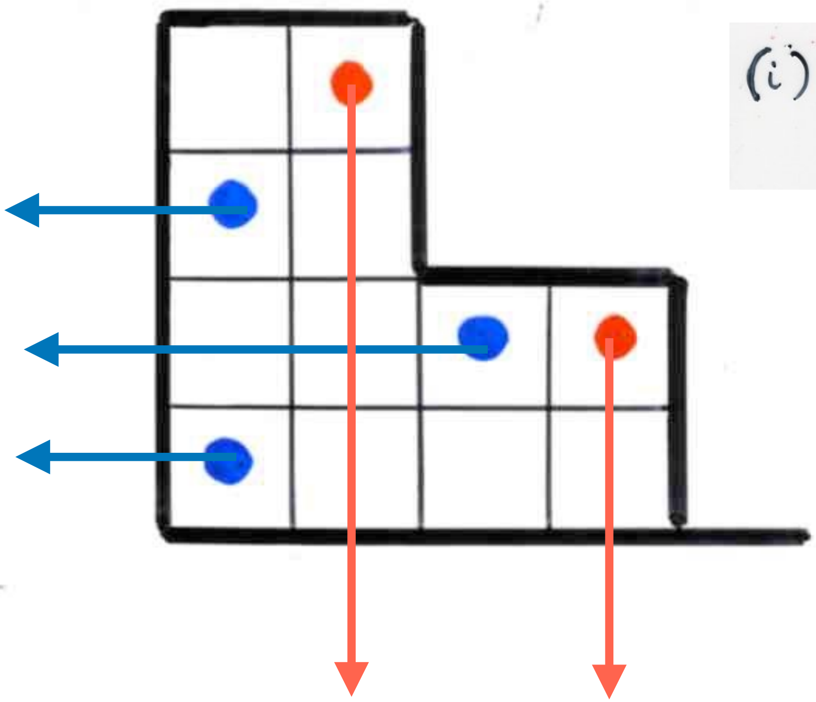
with possibly empty rows or columns

size of **F**

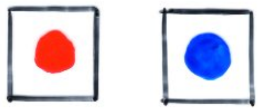
$\mathbf{n}$ = (number of rows) + (number of columns)

alternative tableau

Definition

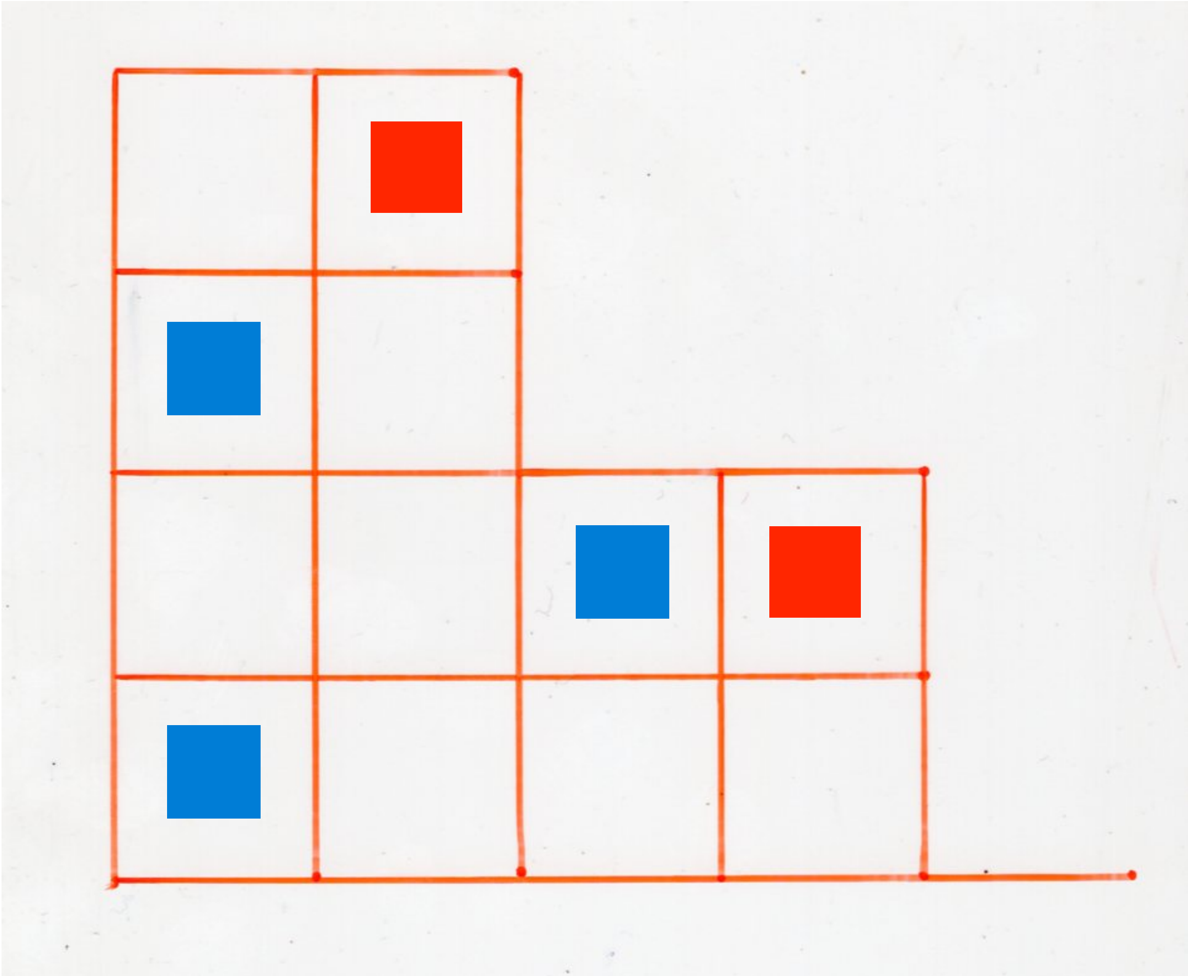


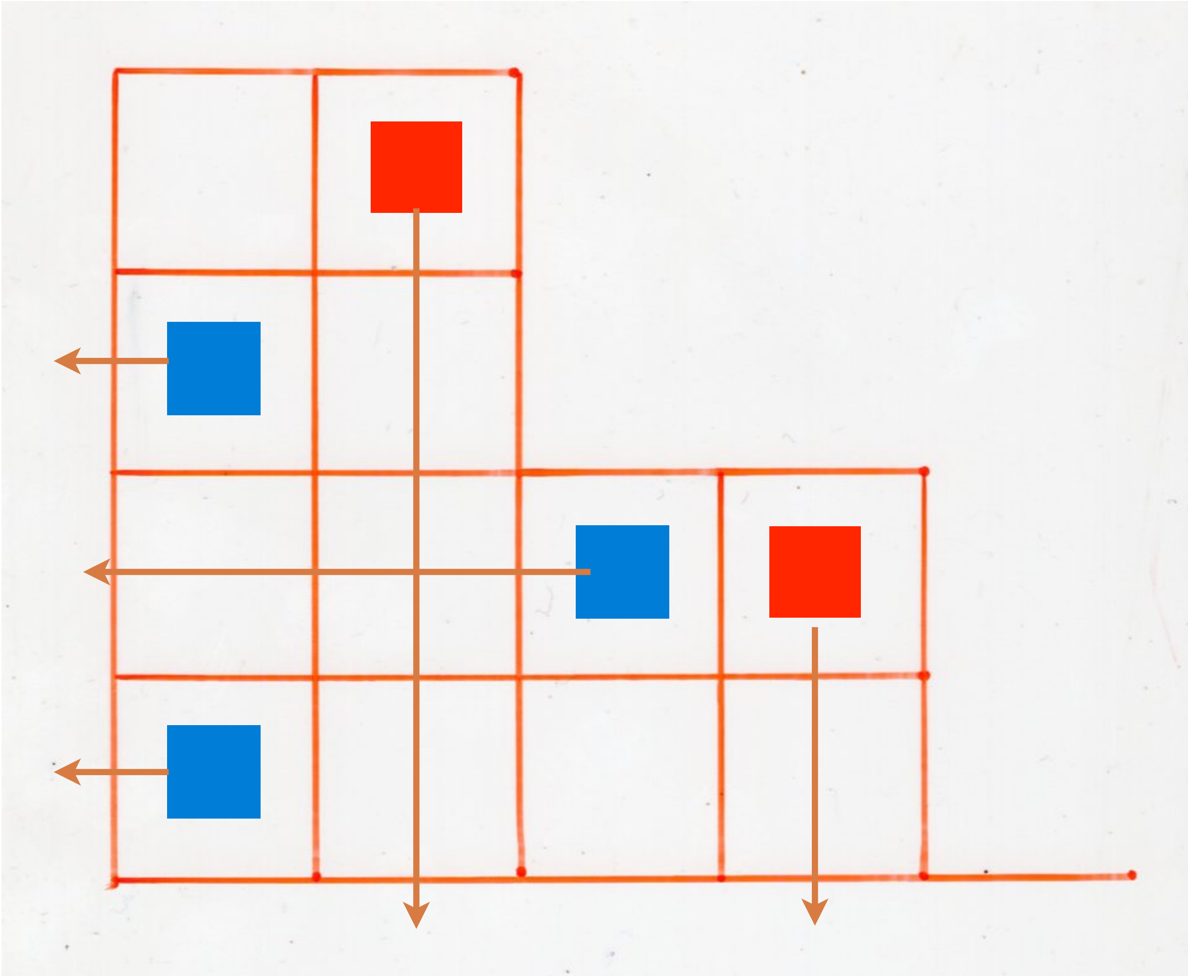
(i) some cells are coloured red or blue

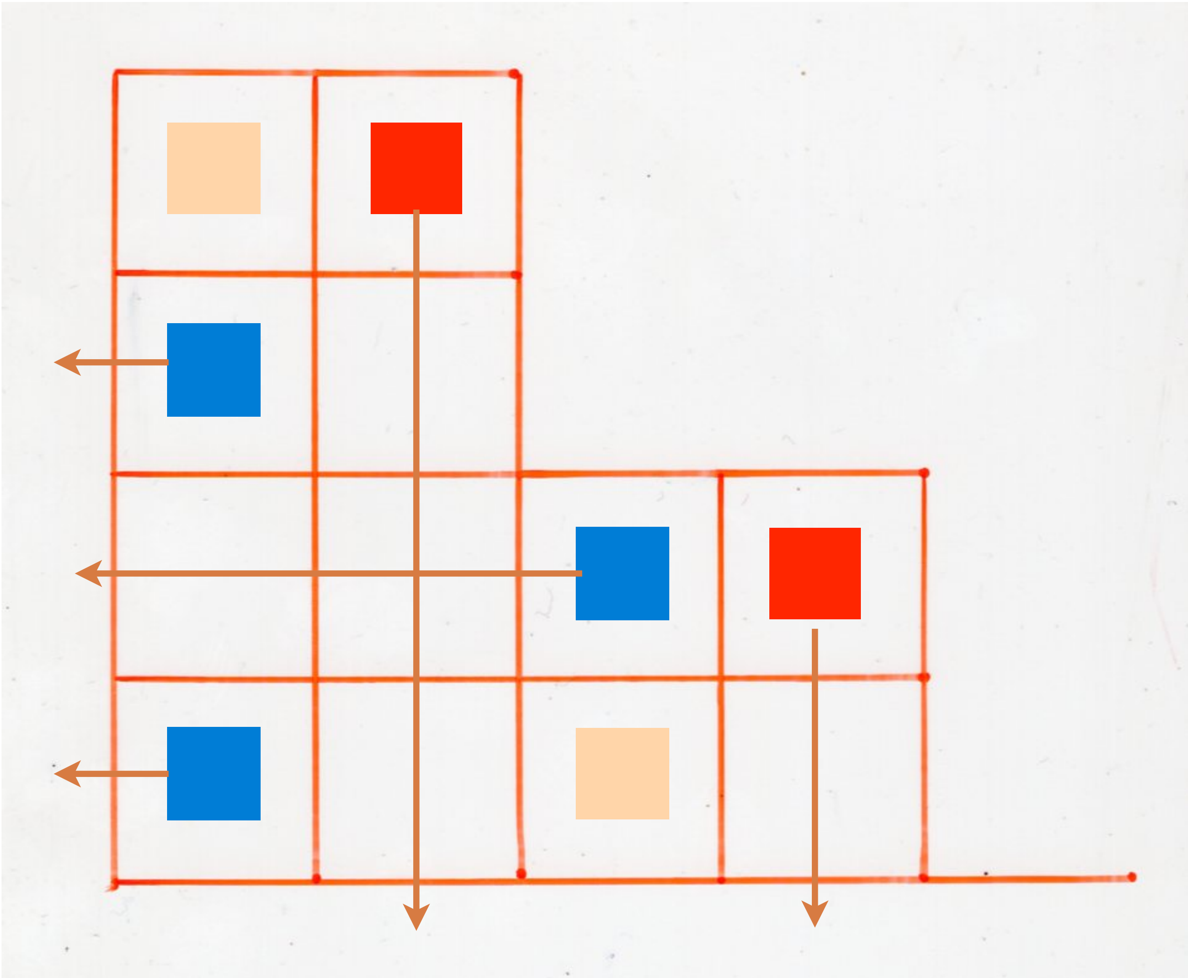


(ii) • no coloured cell at the left of a blue cell

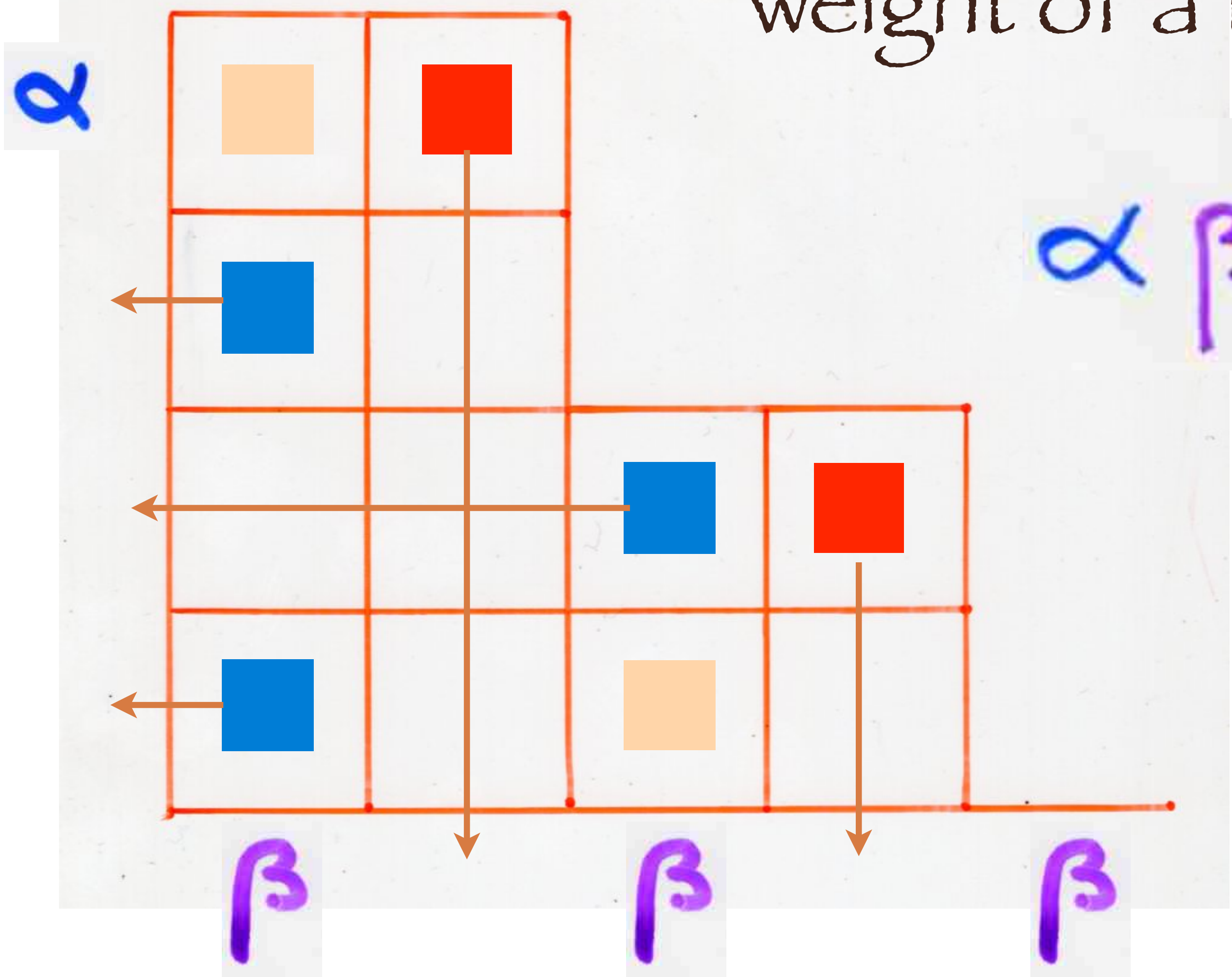
• no coloured cell below a red cell







weight of a tableau



$$\alpha \beta^3 q^2$$

$q \quad k(T) = \text{nb of cells } \square$

$\alpha \quad i(T) = \text{nb of rows without } \bullet$

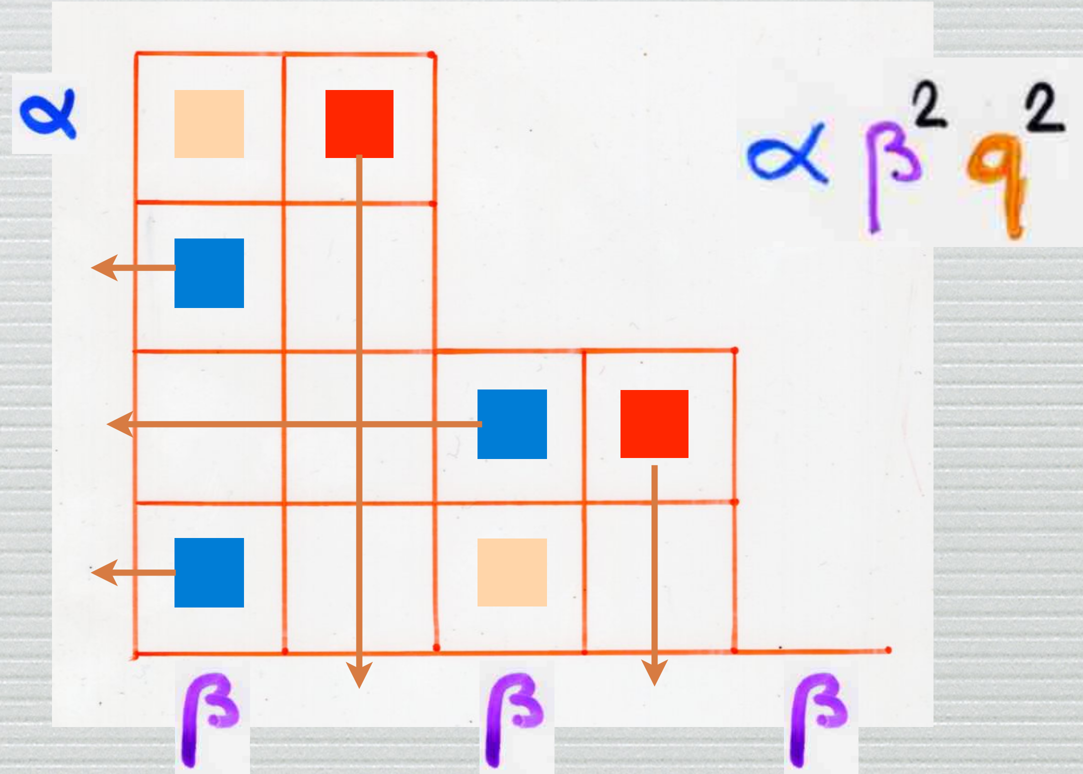
$\beta \quad j(T) = \text{nb of columns without } \bullet$

Partition function

$$\bar{Z}_N = Z_N(\alpha^{-1}; \beta^{-1}; q)$$

$$\bar{Z}_n$$

Sum of the weight of all tableaux of size n



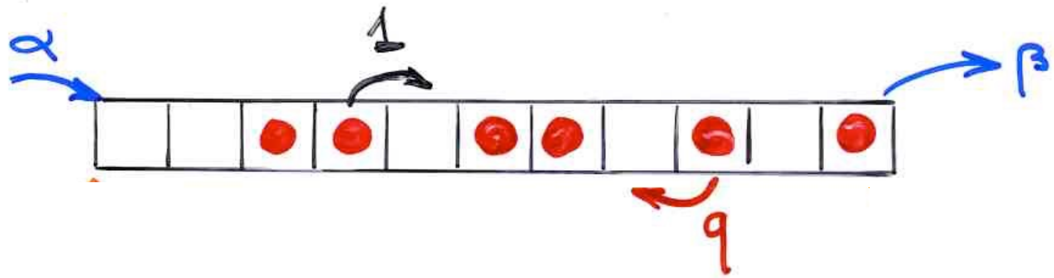
q — $k(T)$ = nb of cells $\square$

α — $i(T)$ = nb of rows without ●

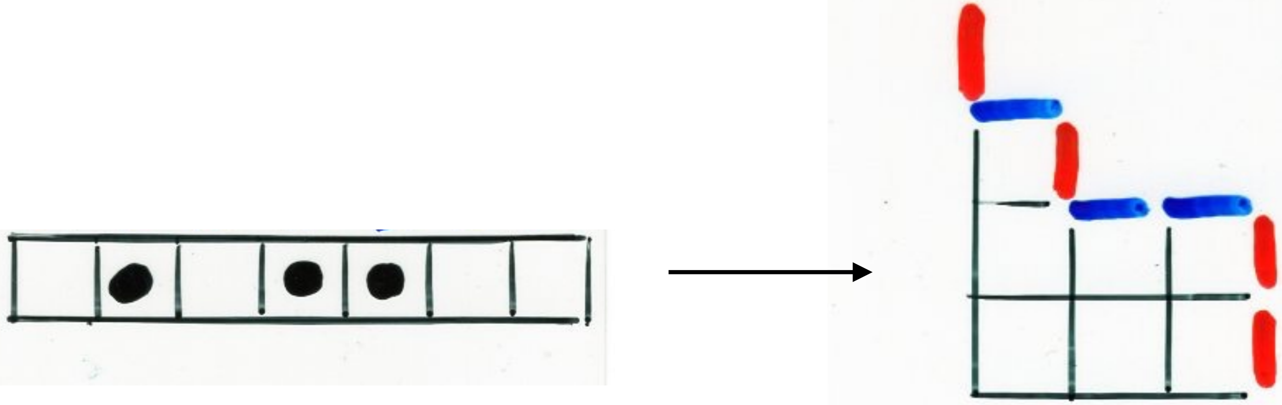
β — $j(T)$ = nb of columns without ●

computation of the
"stationary probabilities"

PASEP with 3 parameters q, α, β



PASEP



Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\mathrm{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

T: alternative tableaux, profile τ

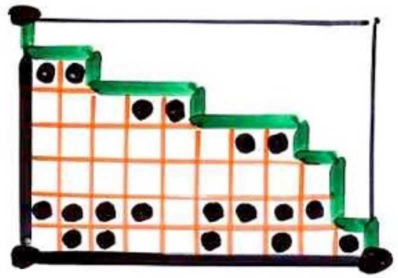
alternative tableau
X.V. (2008)

permutation tableau

S. Corteel, L. Williams
(2007, 2008, 2009)

permutation tableaux

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) $\begin{array}{ll} 1 & \cdots 0 \\ & \;\,\vdots \\ & \;\,1 \end{array}$ forbidden

□ = 0 ⊡ = 1

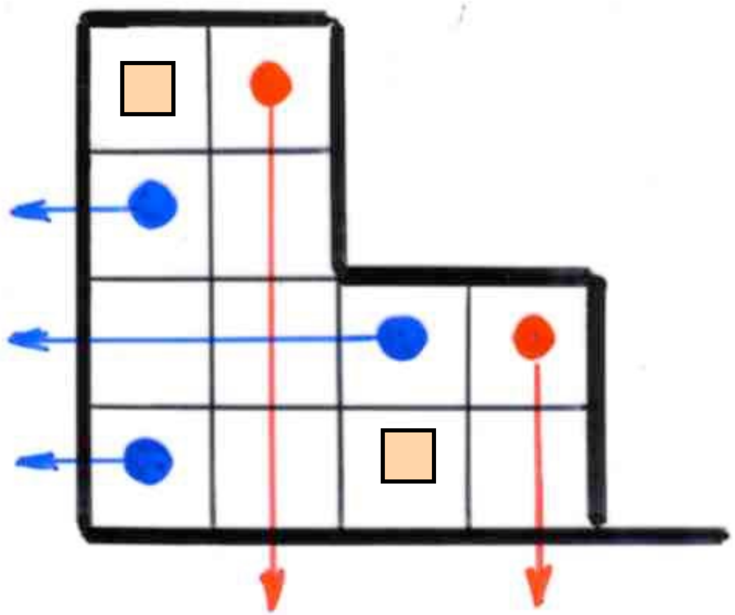
permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

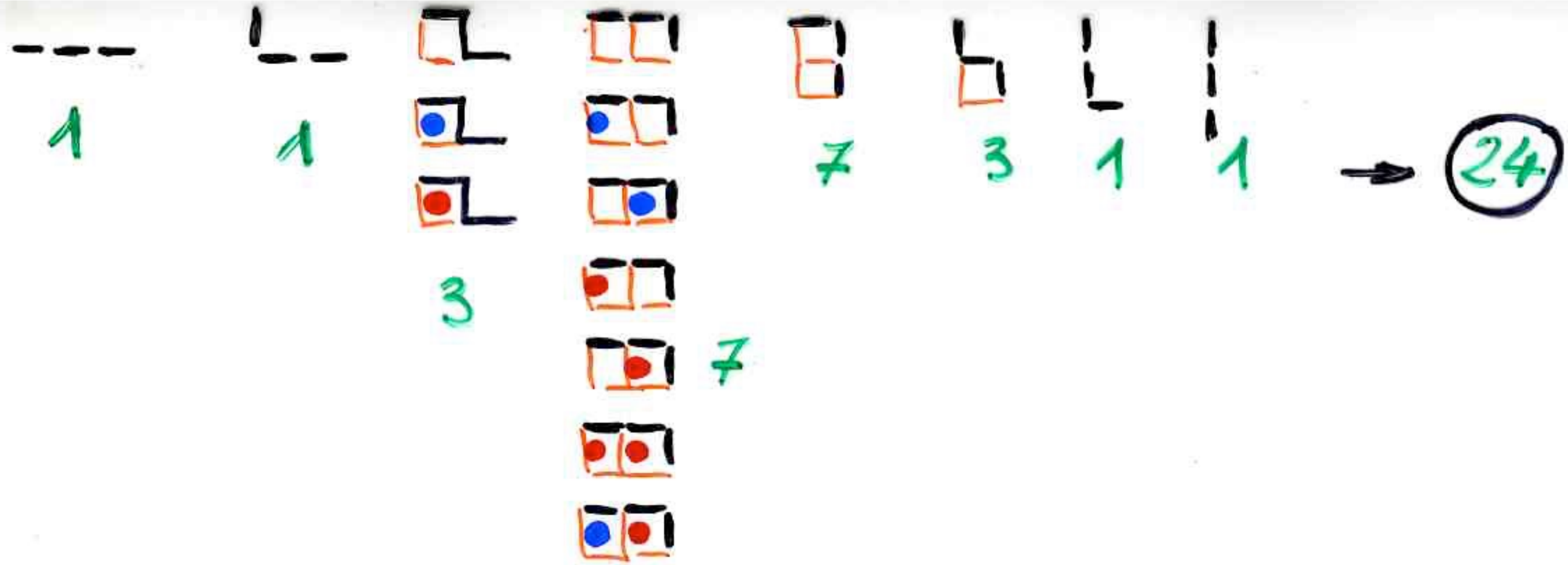
Enumeration of alternative tableaux



Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

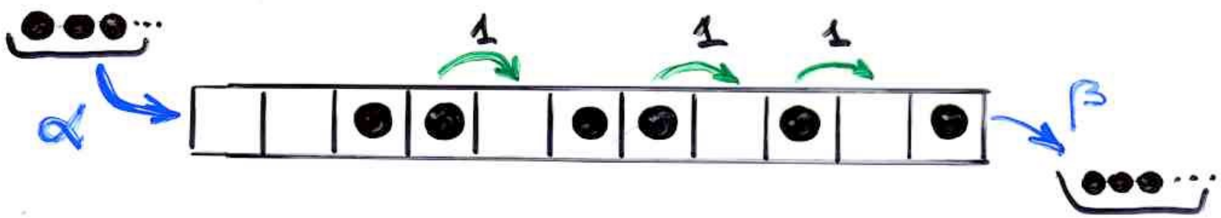
ex: n=2



$q = 0$ TASEP (α, β)

TASEP

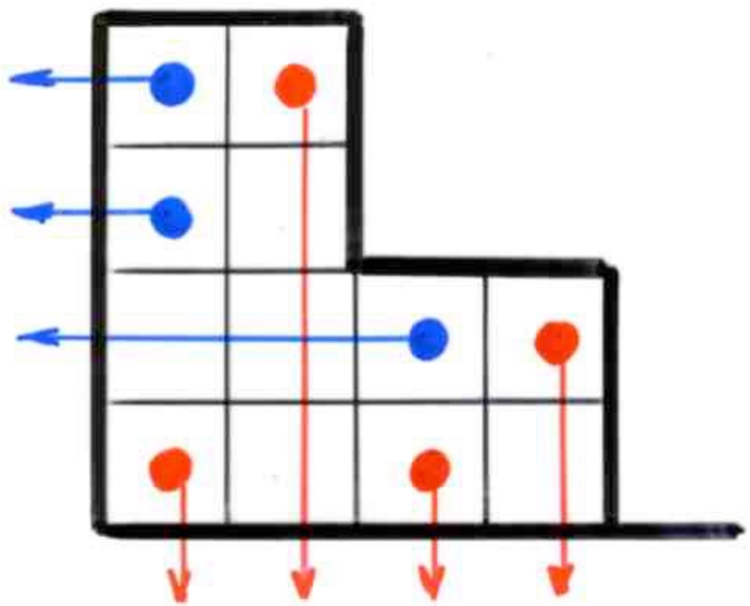
"totally asymmetric exclusion process"



Definition Catalan alternative tableau

alternative tableau T without cells $\square$

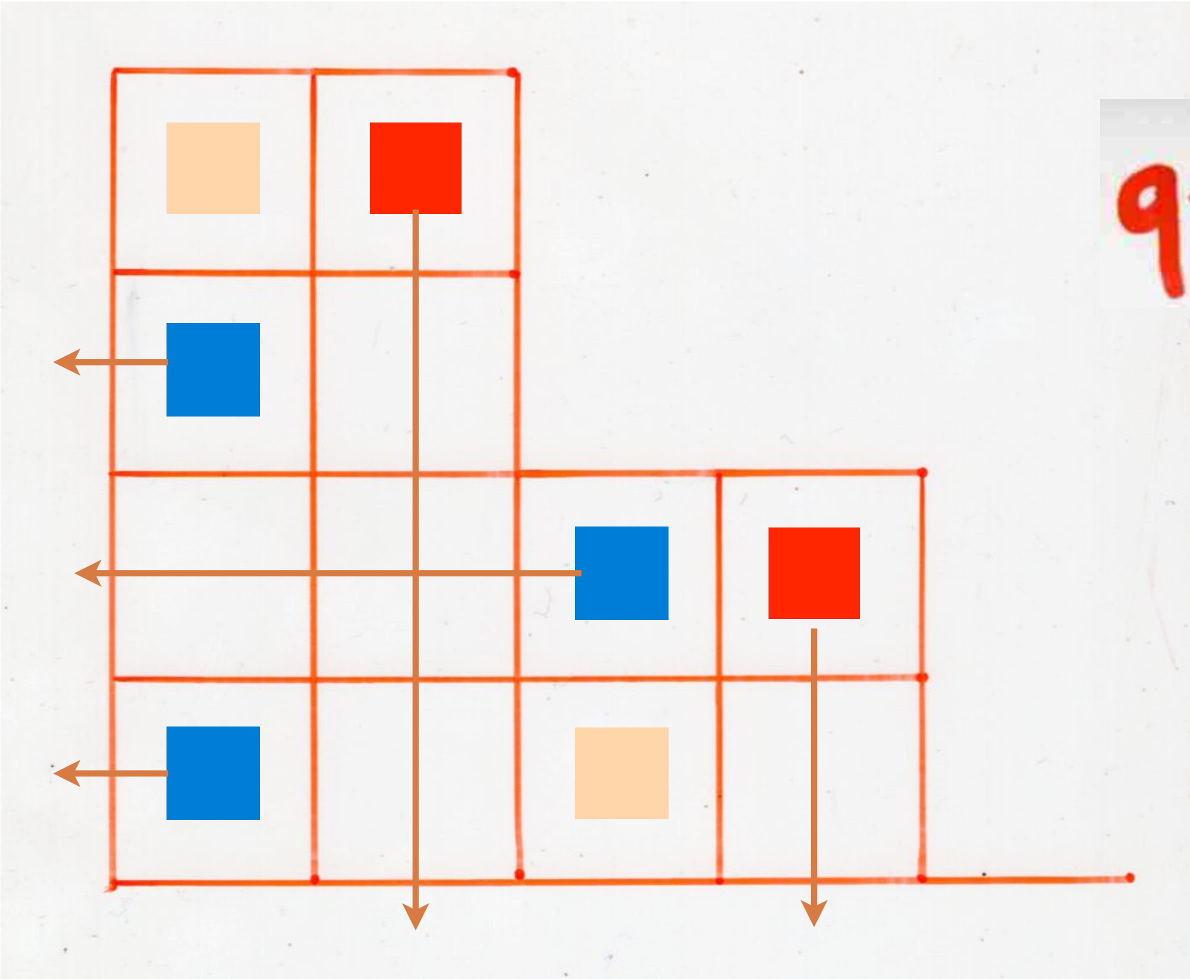
i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

q- analog of n!



q- analog of n!

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

Inv

number of inversions

Maj

Major index

Interpretation of the 3-parameters Partition function

q, α, β

Josuat-Vergès (2011)

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

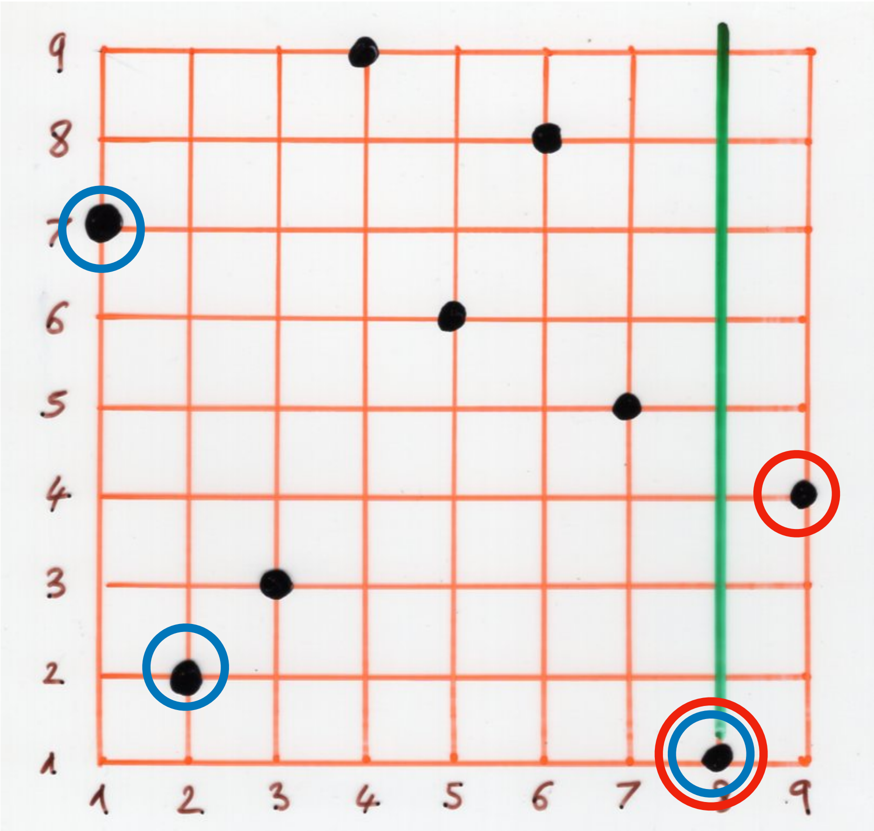
$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

$31\text{-}2(\sigma)$ = number of patterns 31-2



σ = 7 2 3 9 6 8 5 1 4

permutation

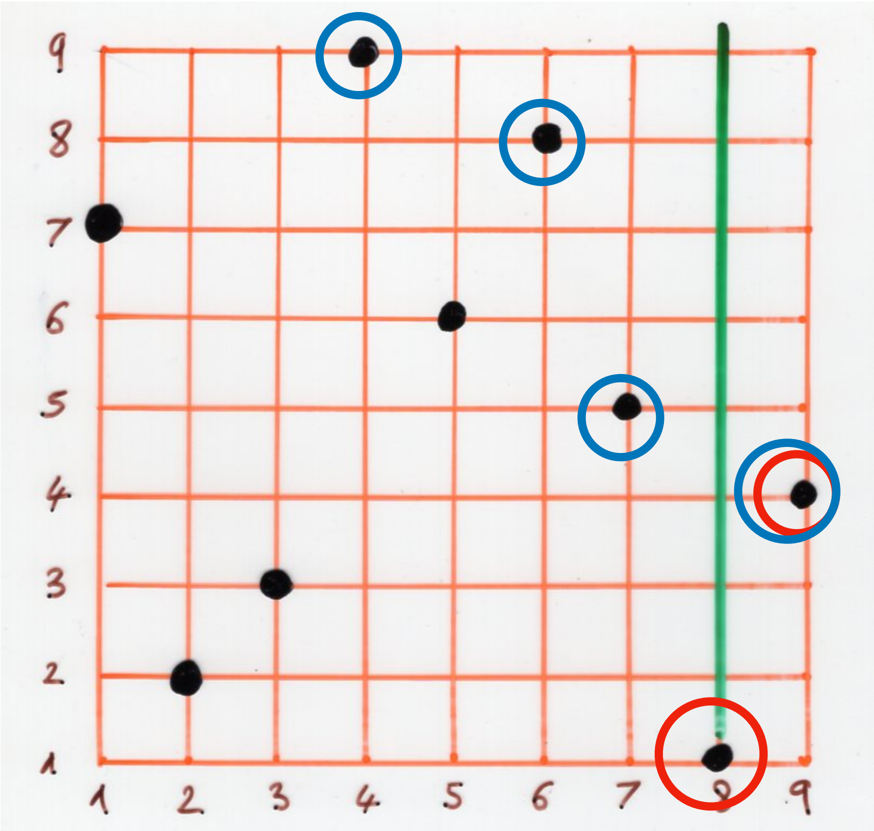
word

left-to-right

right-to-left

minimum

elements



σ = 7 2 3 9 6 8 5 1 4

permutation

word

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

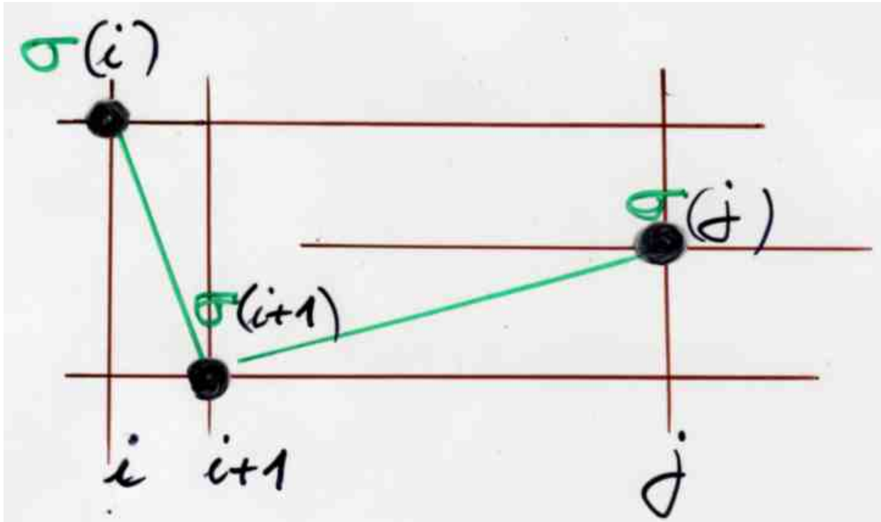
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

31-2



$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon- V.

Al-Salam-Chihara polynomials

$$2x\, Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})\, Q_{n-1}(x)$$

Laguerre heaps of segments

non-ambiguous
trees

tree-like
tableaux
rectangular shape

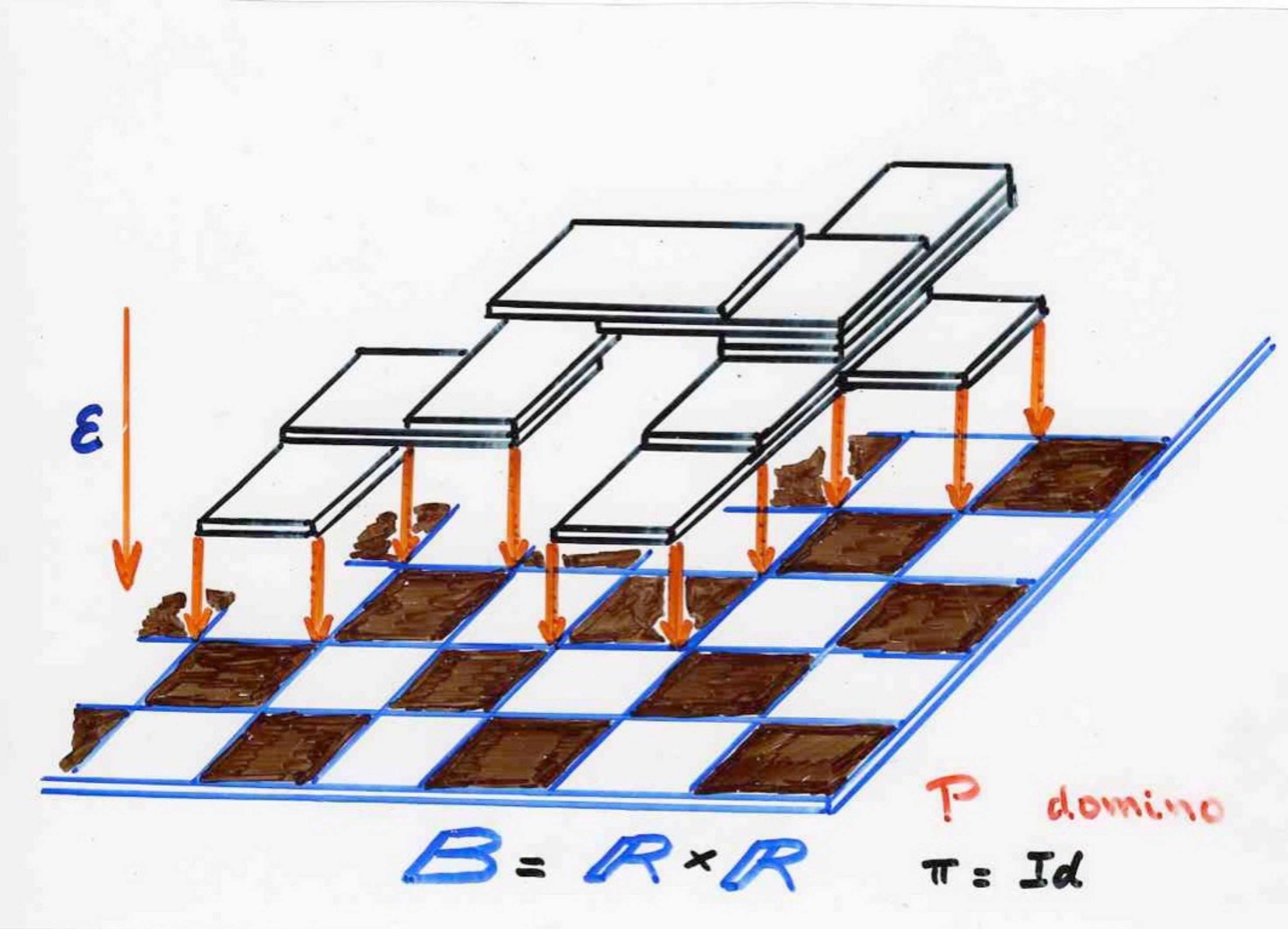
complete
non-ambiguous
trees

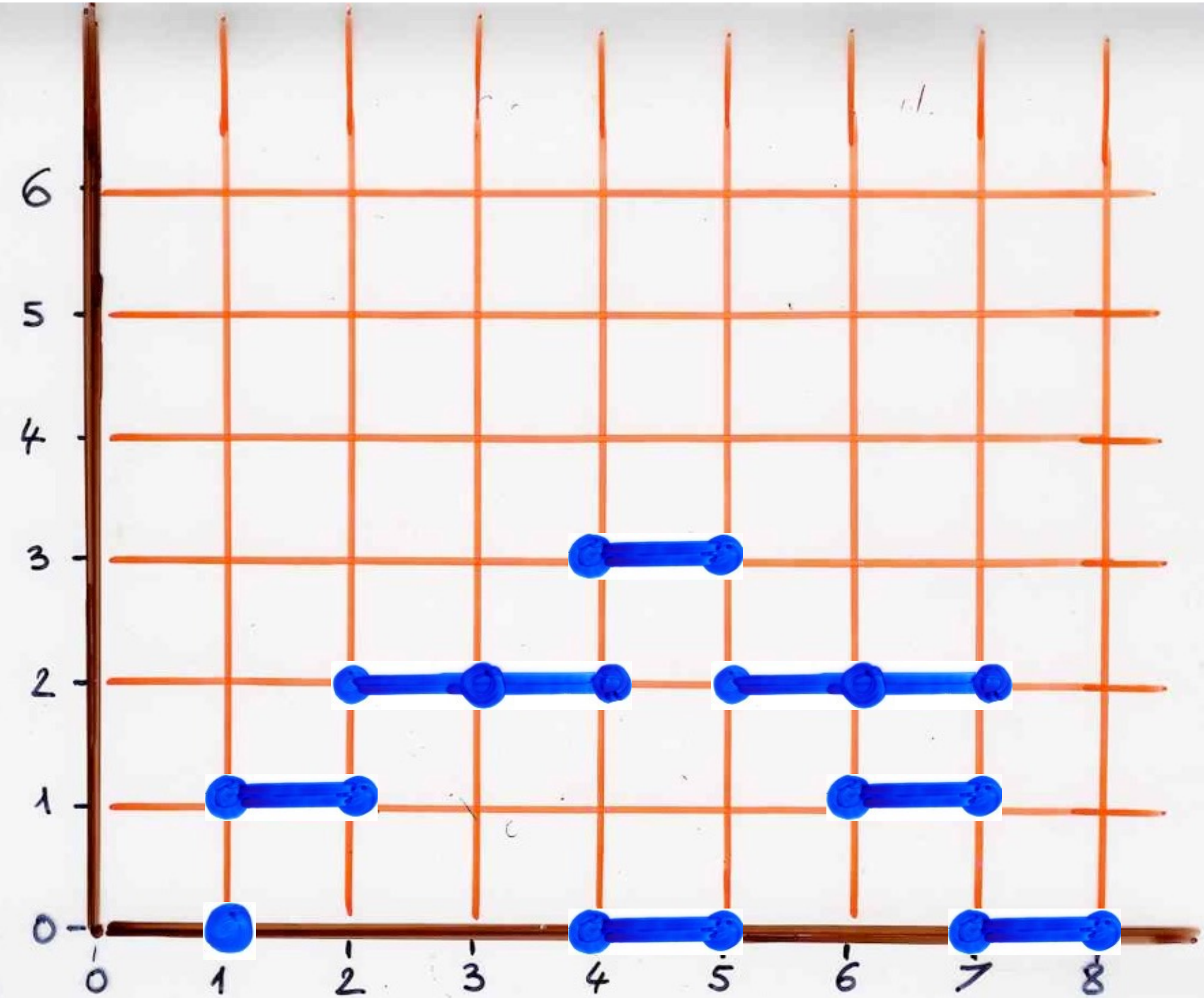
E. Jin (2014)

Bessel functions
heaps
logarithmic lemma

heaps of pieces

Introduction
Heaps





6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

heap definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases}\text{symmetric}\\\text{reflexive}\end{cases}$

  (dependency relation)
- heap E, finite set of pairs

  (α, i) $\alpha \in \mathbf{P}$, $i \in \mathbb{N}$ (called pieces)

  projection level

(i)

(ii)

heap definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases}\text{symmetric}\\ \text{reflexive}\end{cases}$

  (**dependency** relation)

- heap E, finite set of pairs

  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  α: projection, i: level

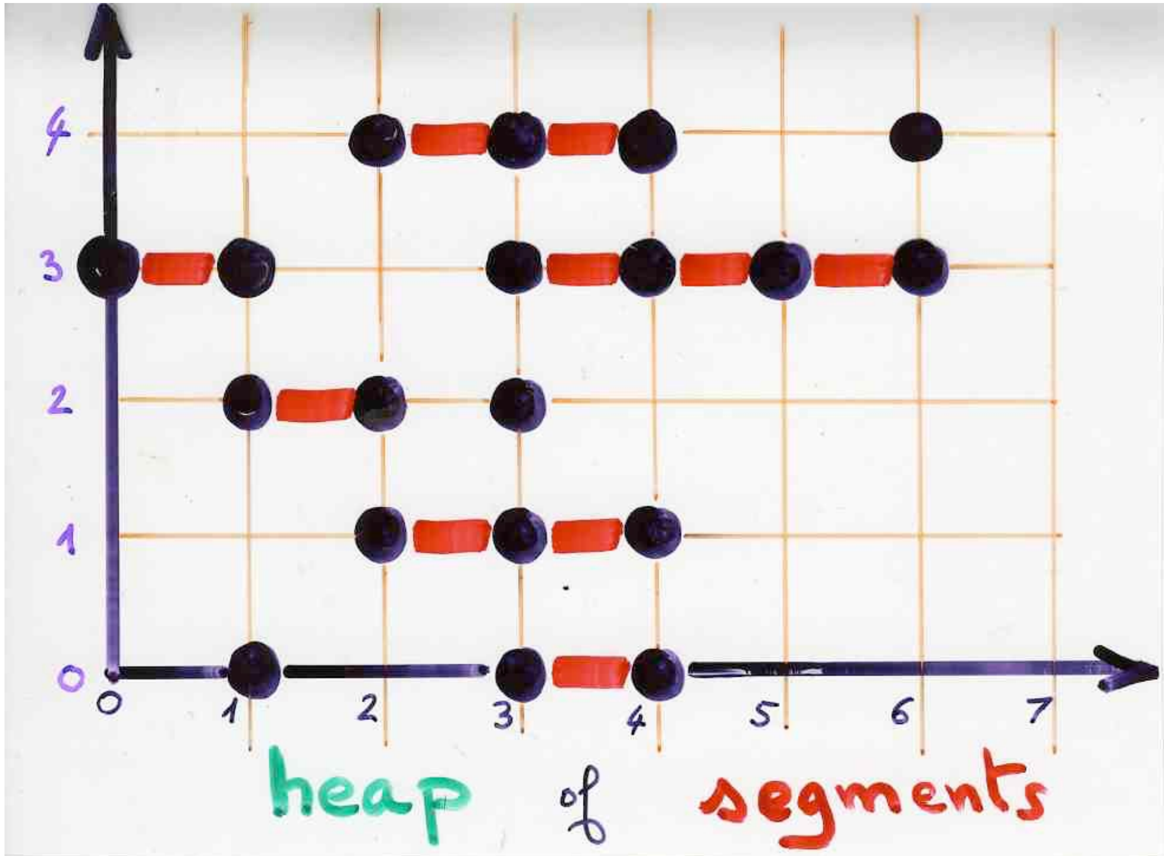
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \, \mathcal{C} \, \beta \implies i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \implies \exists \beta \in P$, $\alpha \, \mathcal{C} \, \beta$, $(\beta, i-1) \in E$

ex: heap of segments over $\mathbb{N}$

$$P = \{ [a,b] = \{a, a+1, \dots, b\} \;, 0 \leq a \leq b \}$$

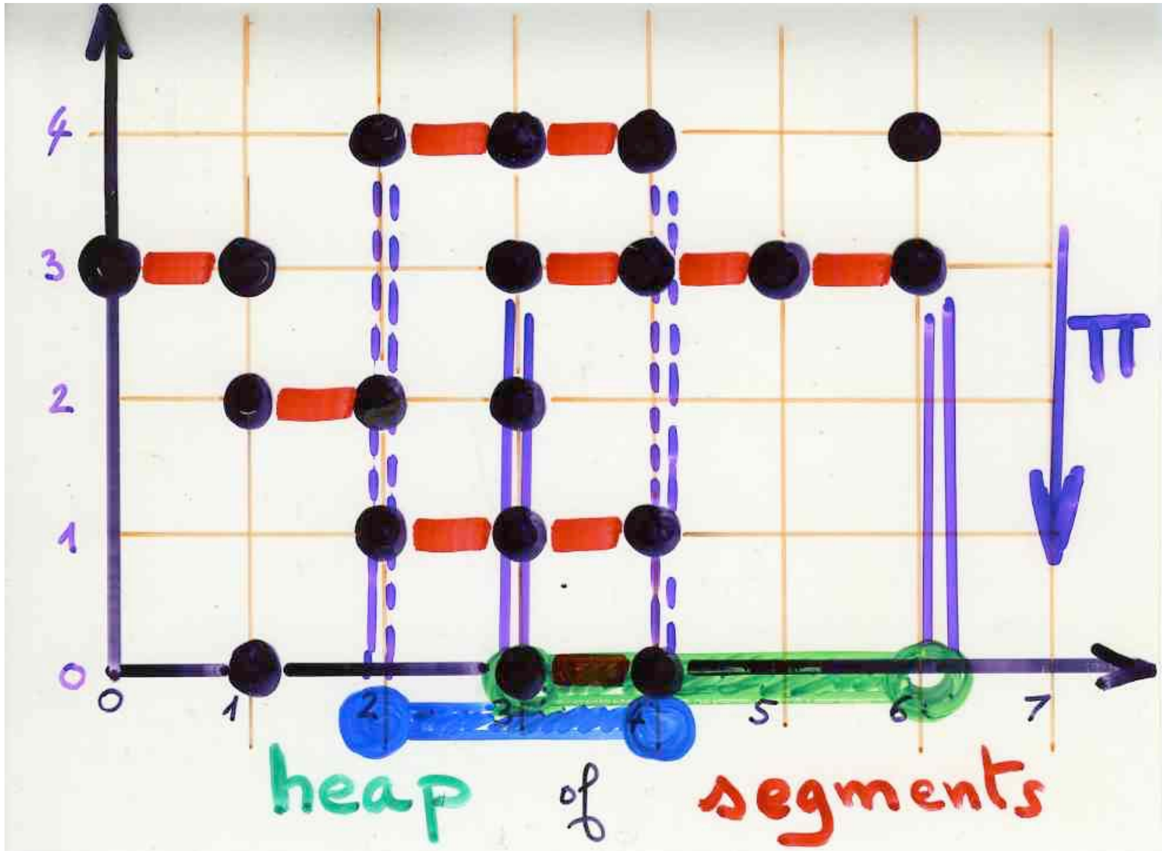
$$\mathcal{C} \qquad [a,b] \,\mathcal{C}\, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$

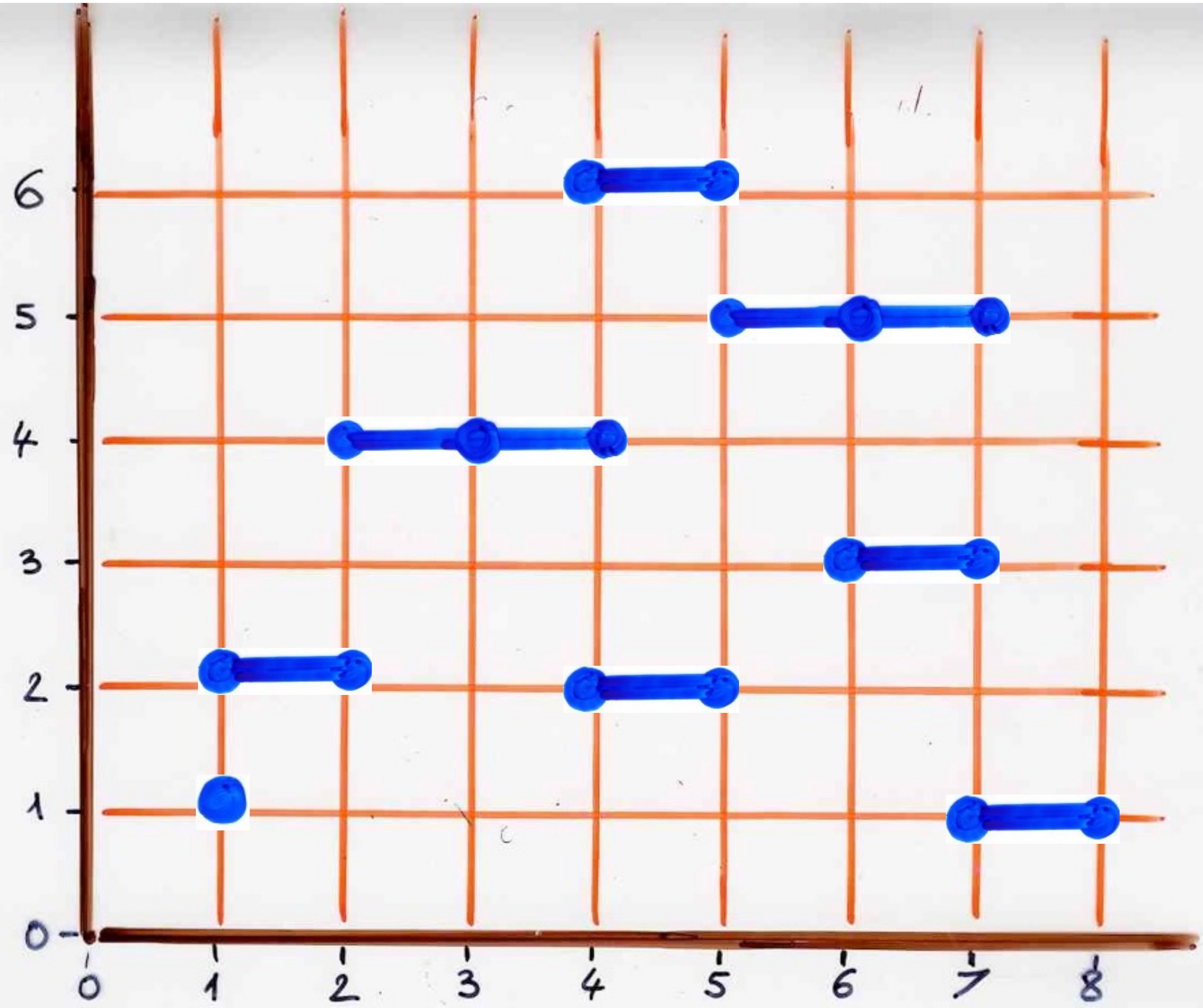


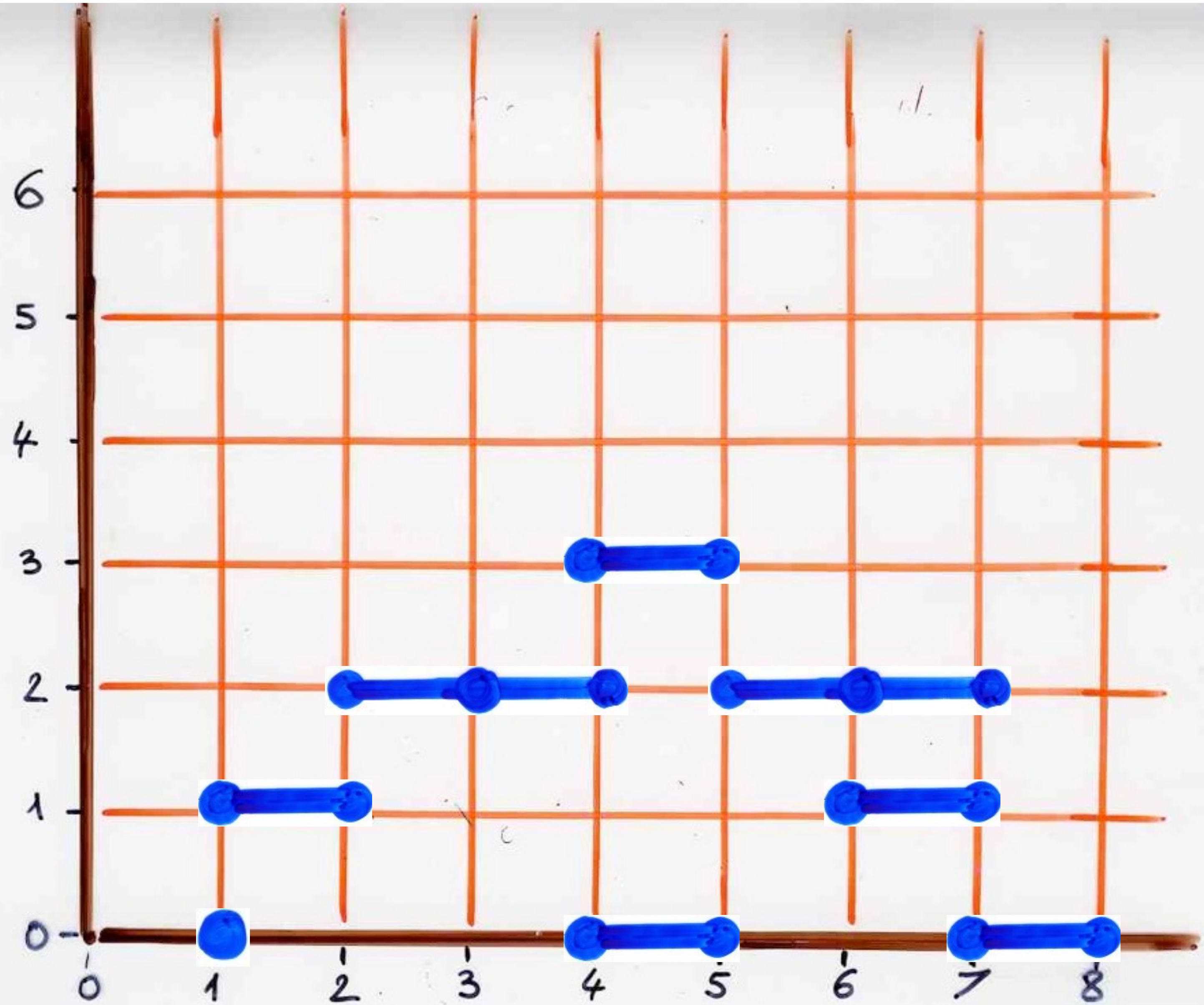
ex: heap of segments over $\mathbb{N}$

$P = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \le a \le b \}$

$\mathcal{C}$ $\quad [a,b] \,\mathcal{C}\, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$

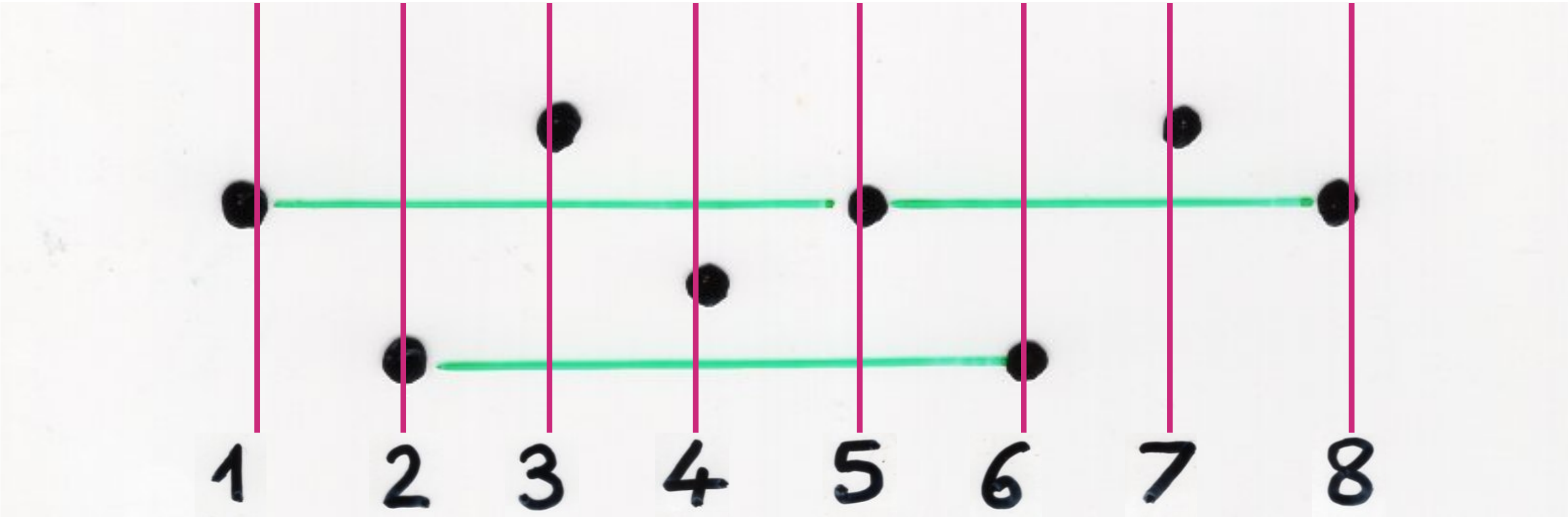



0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8



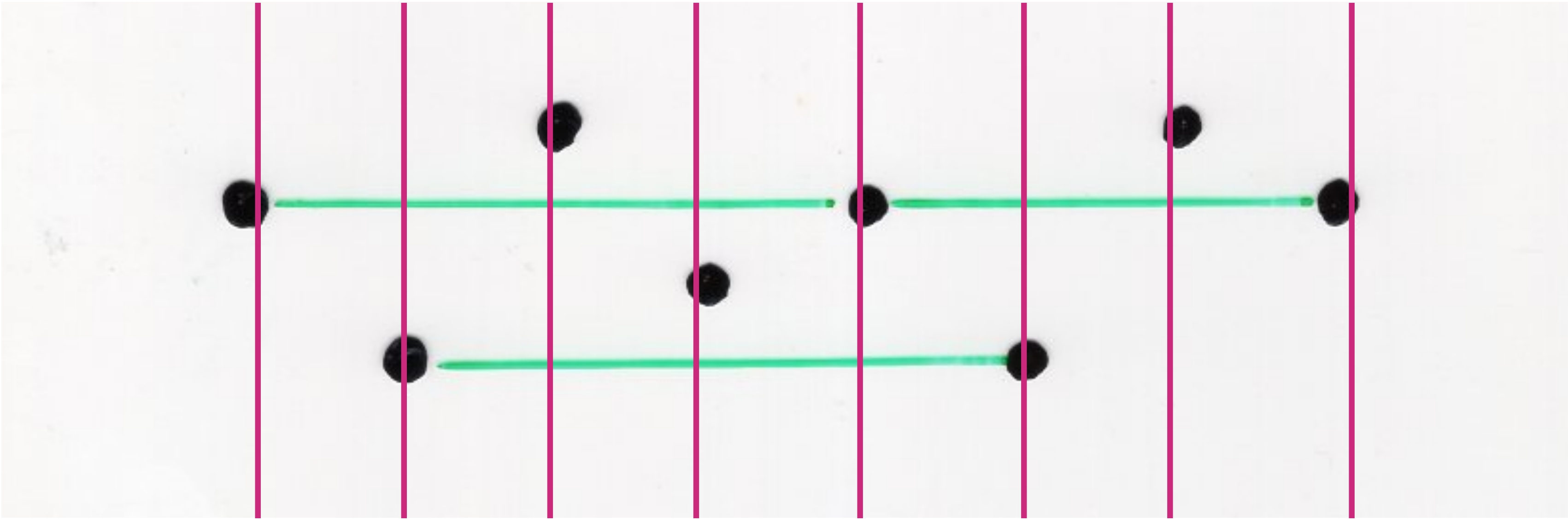
0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8

Laguerre heaps of segments



Definition heaps of pointed segments

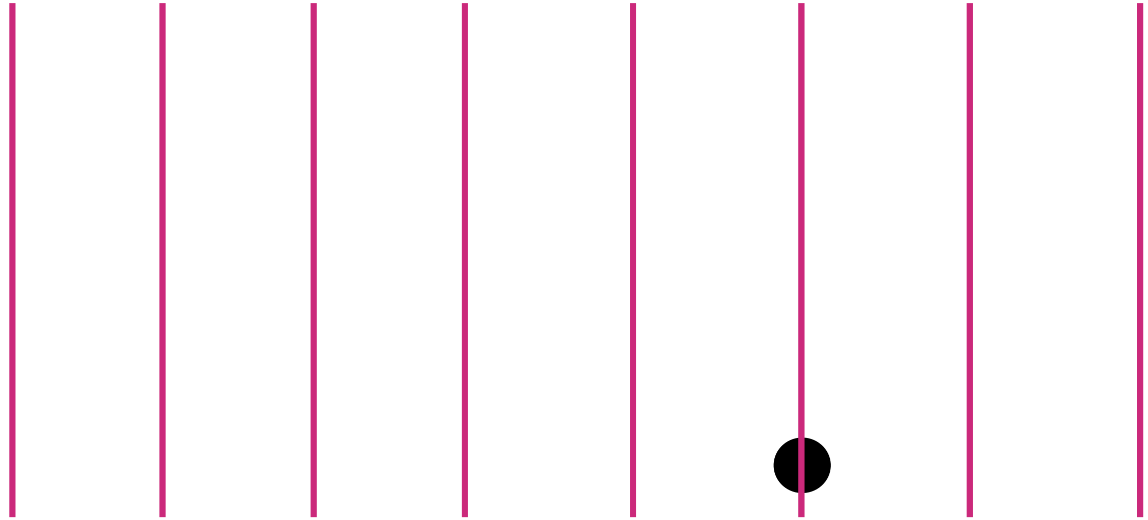
Definition multilinear heap of pointed segments



Bijection

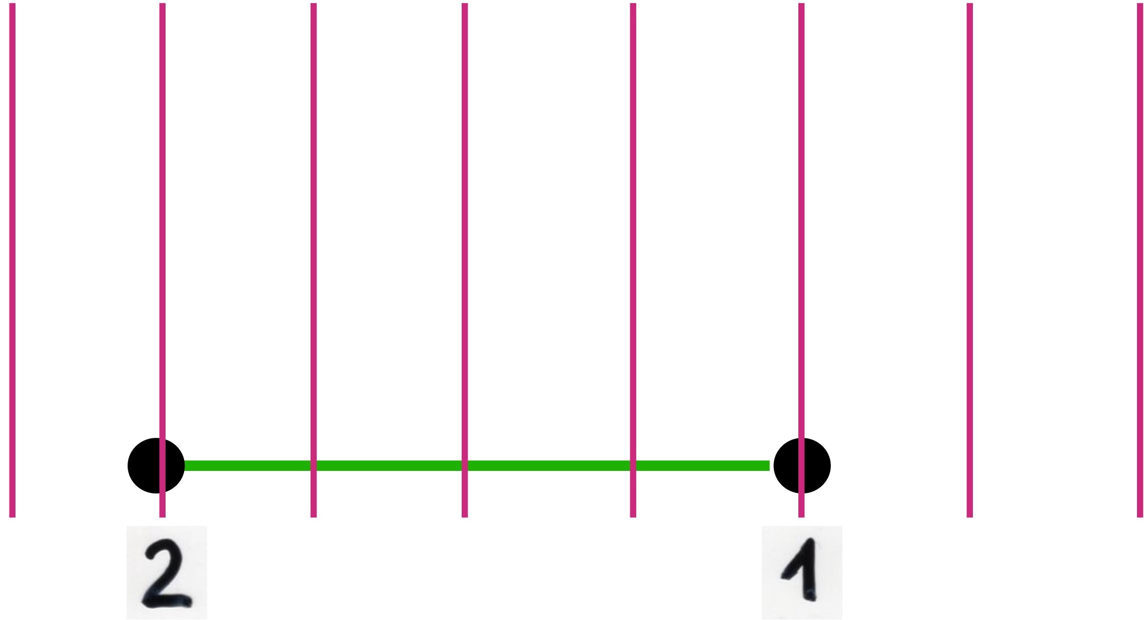
permutations

Laguerre heaps of segments

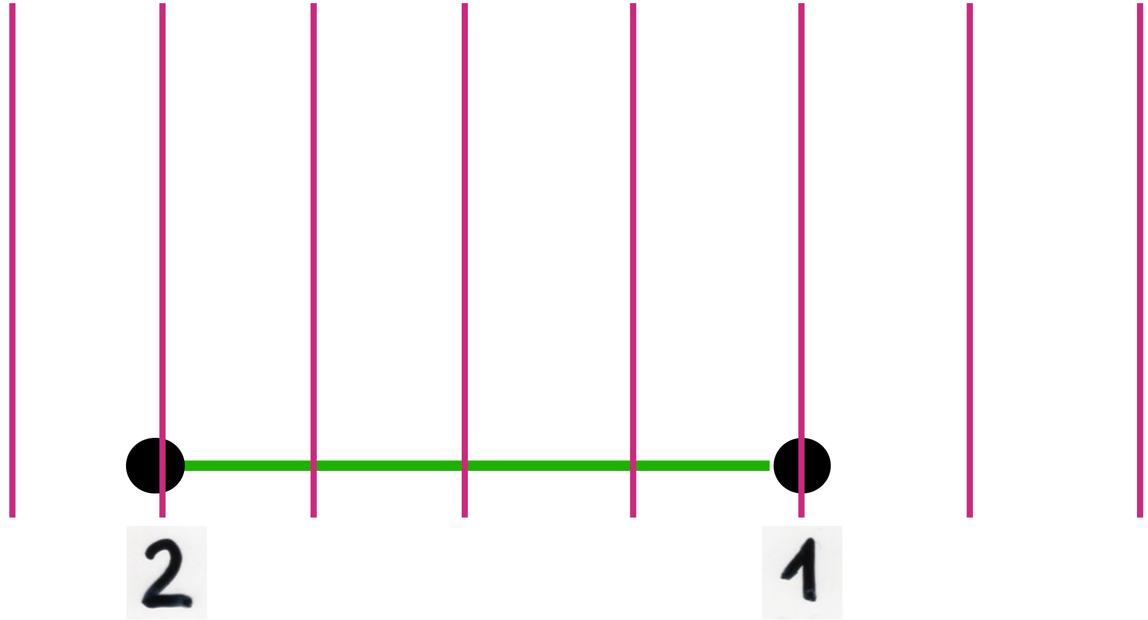


1

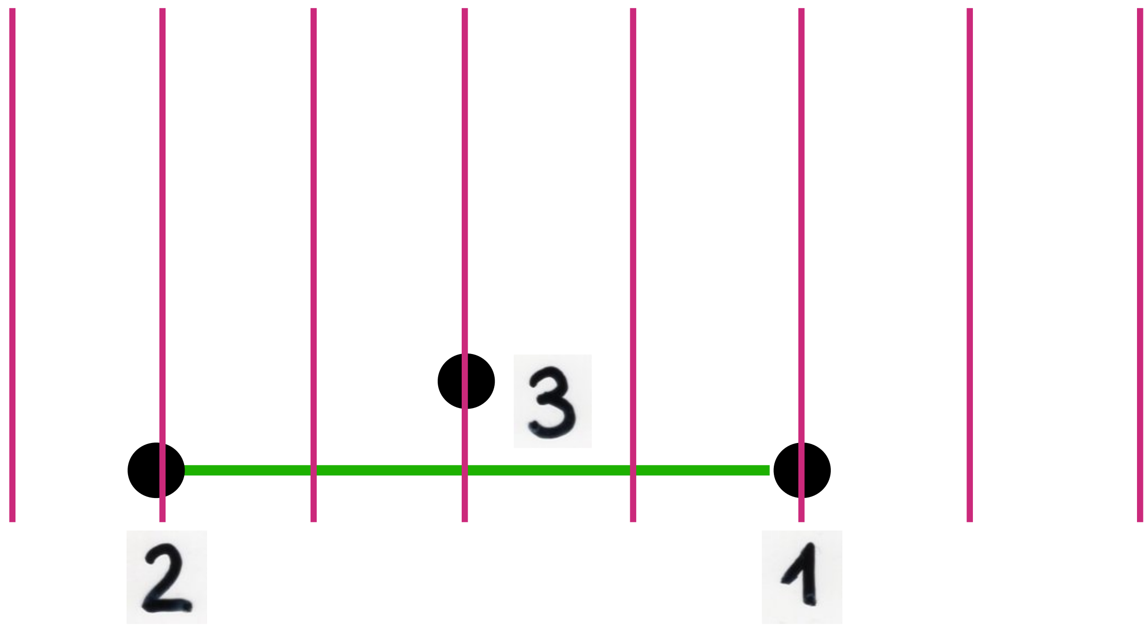
6 2 7 3 5 1 8 4



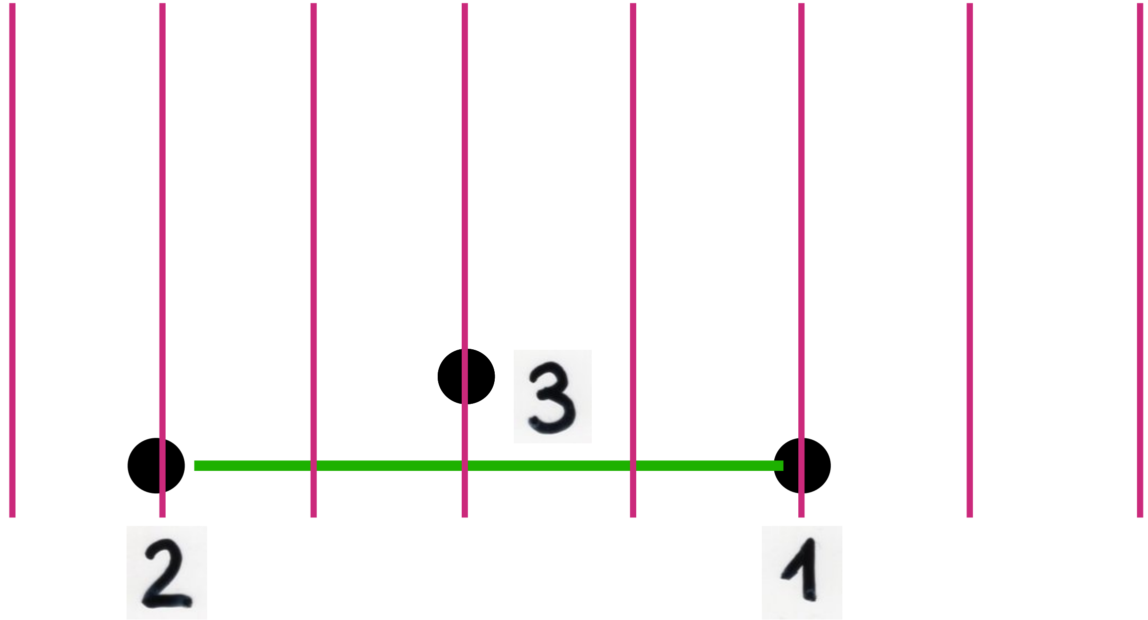
6 2 7 3 5 1 8 4



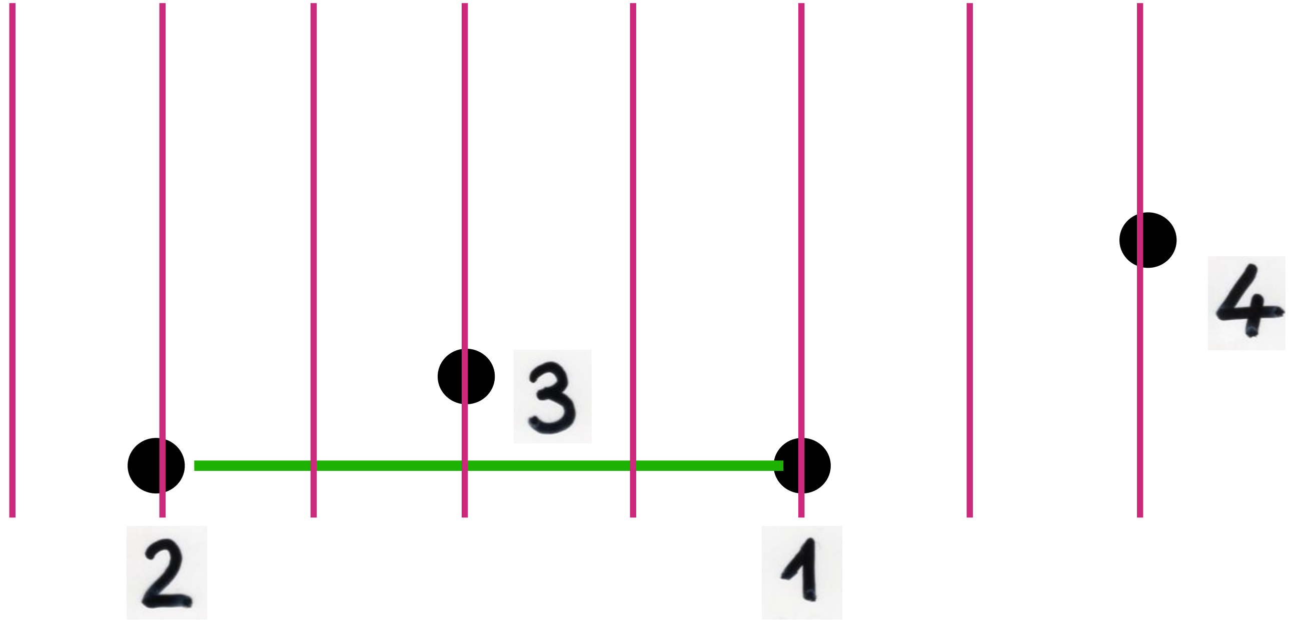
6 2 7 3 5 1 8 4



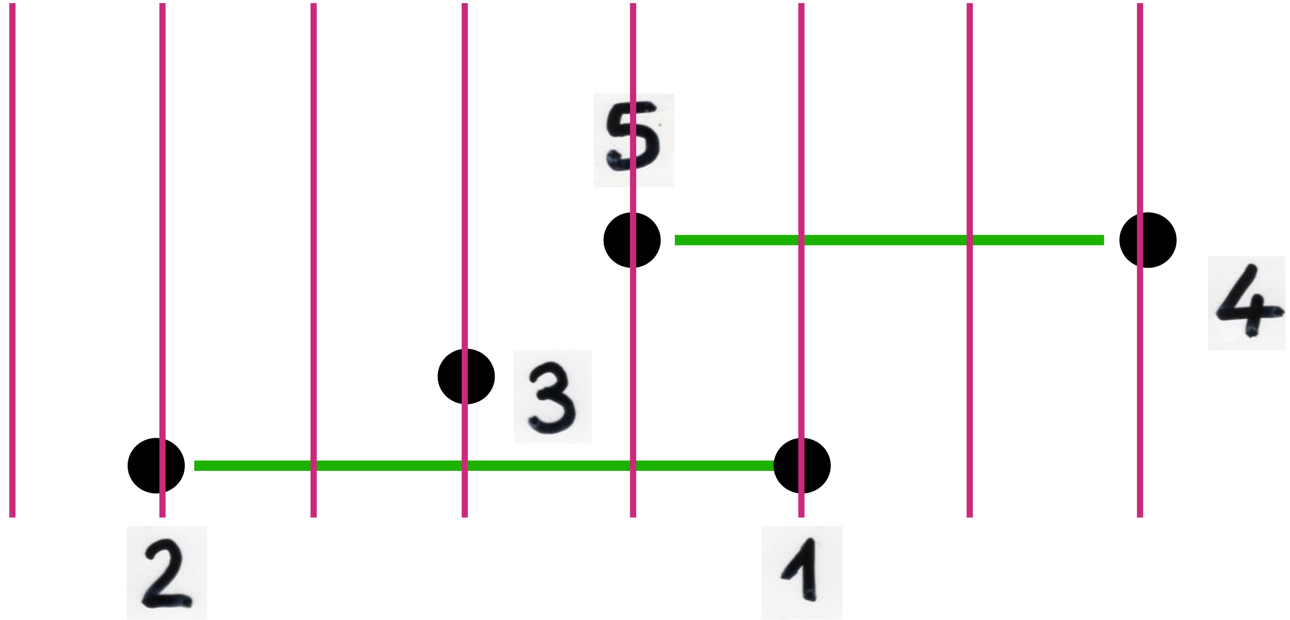
6 2 7 3 5 1 8 4



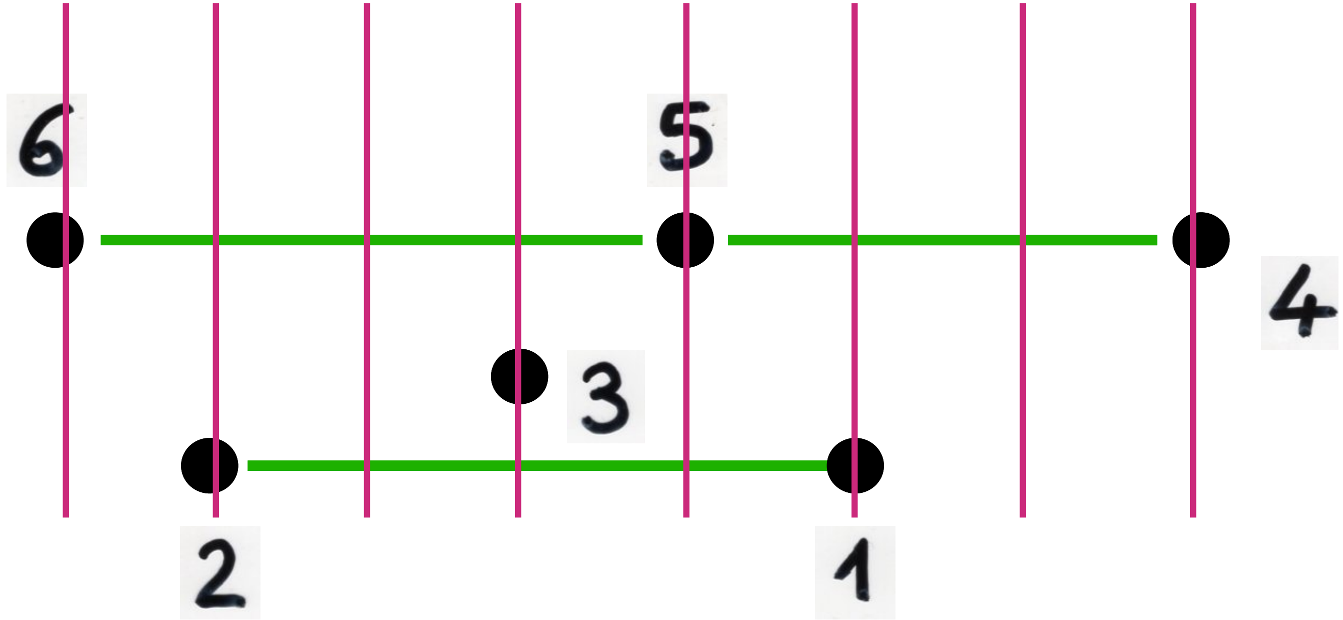
6 2 7 3 5 1 8 4



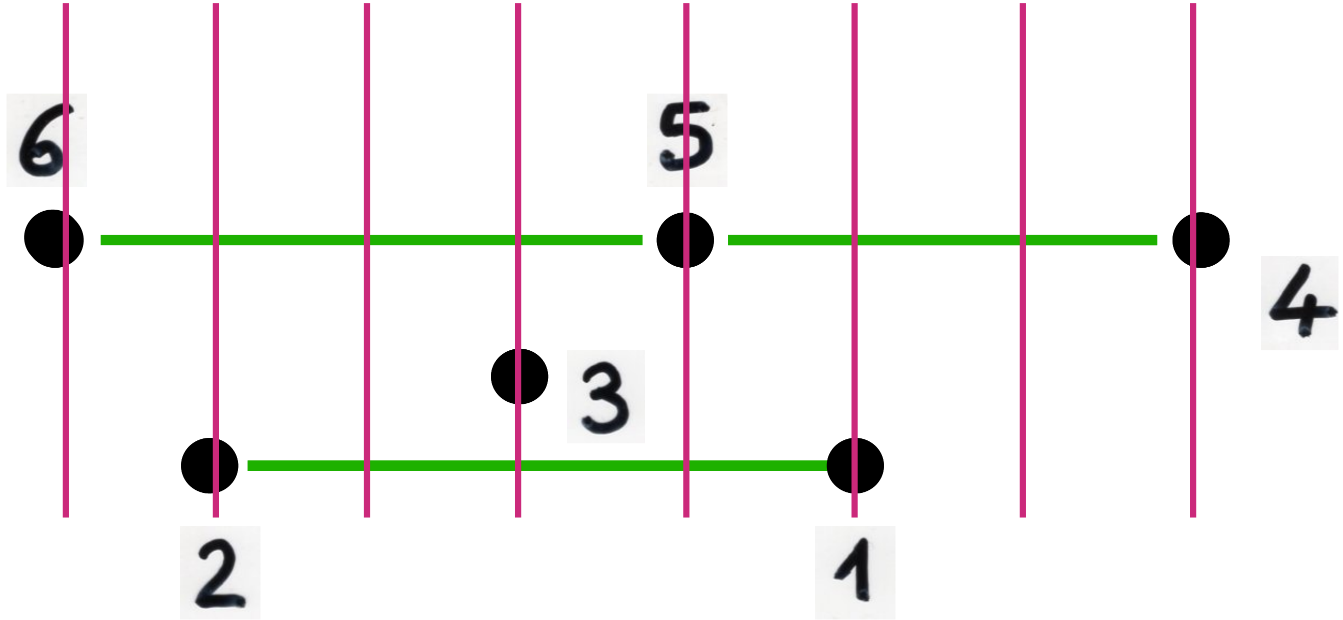
6 2 7 3 5 1 8 4



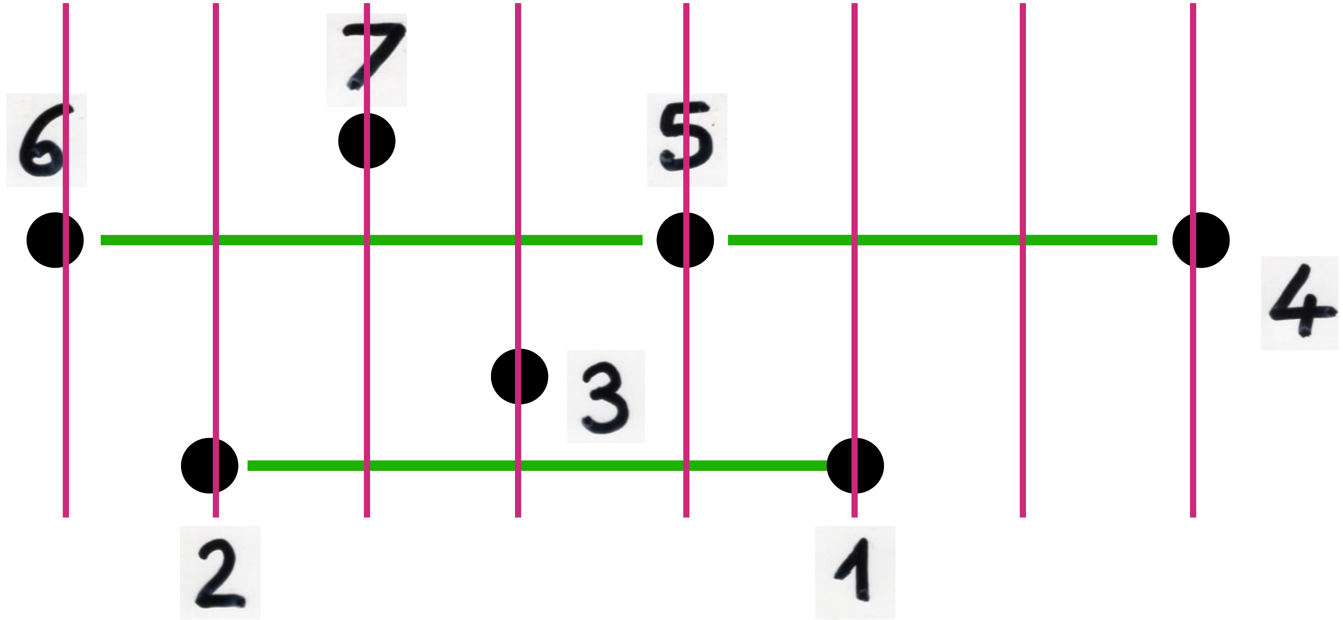
6 2 7 3 5 1 8 4



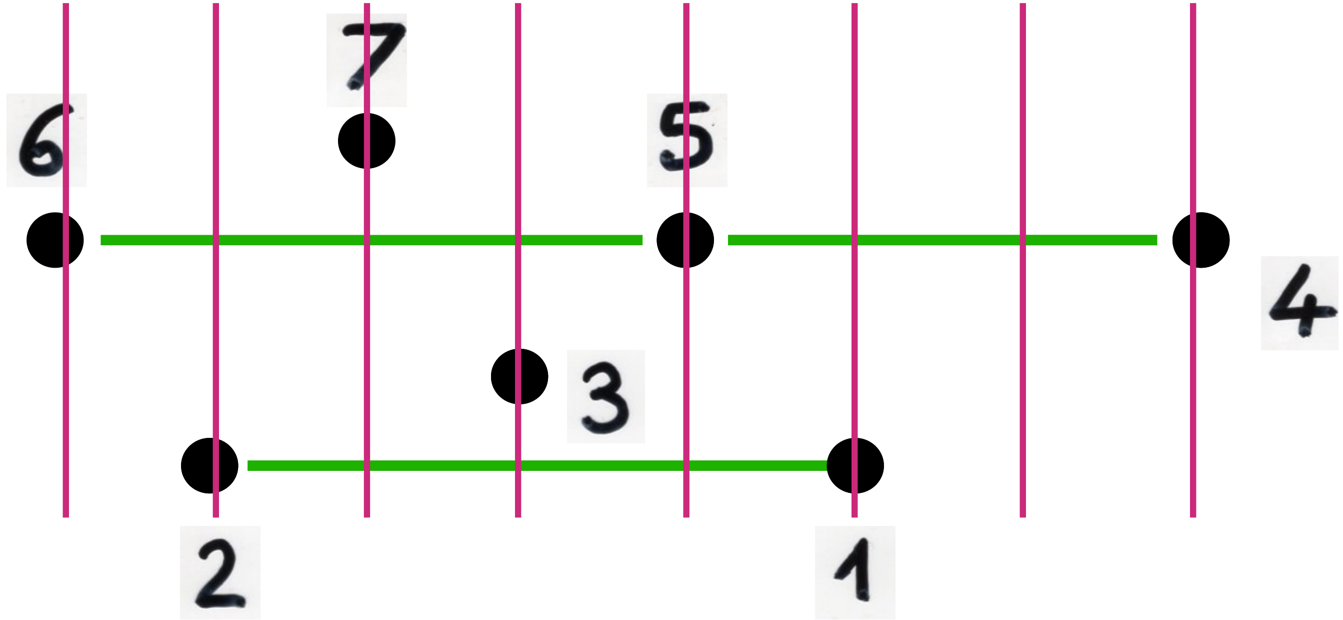
6 2 7 3 5 1 8 4



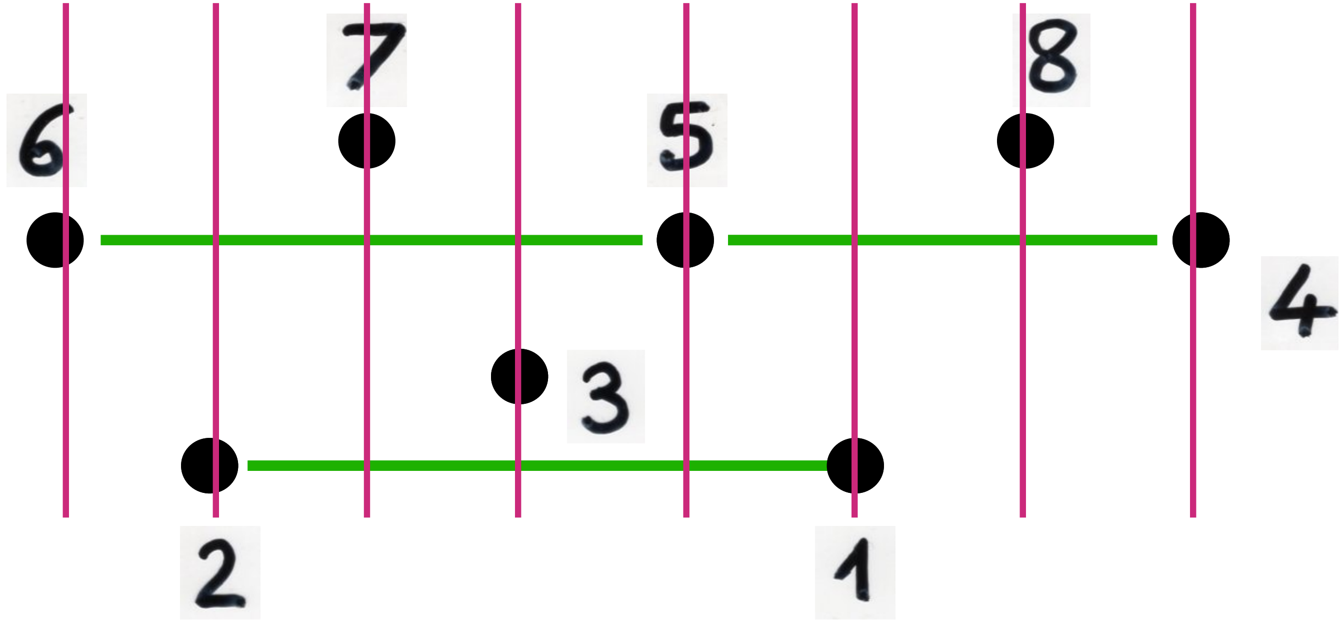
6 2 7 3 5 1 8 4



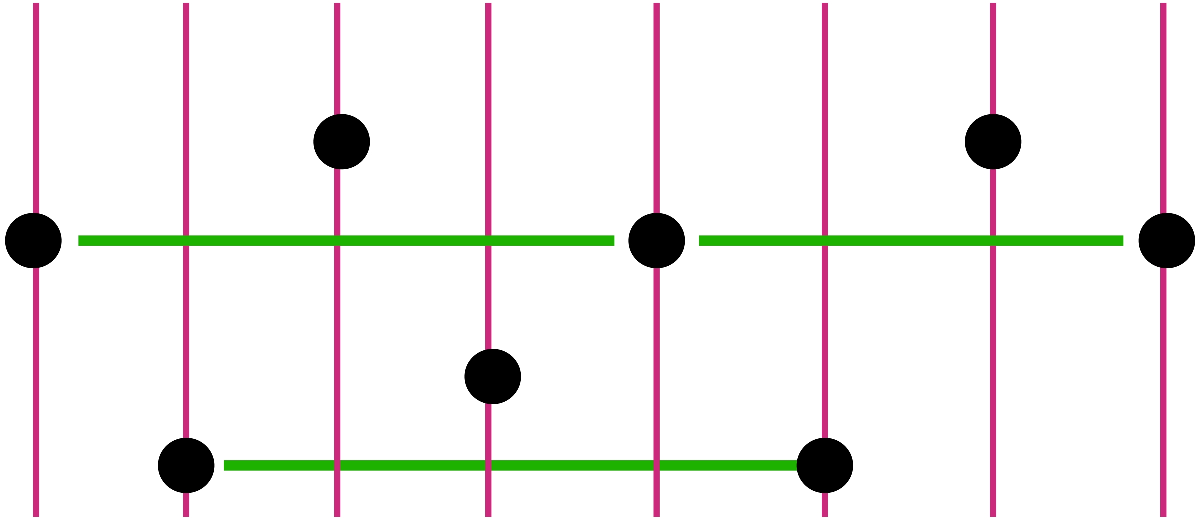
6 2 7 3 5 1 8 4



6 2 7 3 5 1 8 4



6 2 7 3 5 1 8 4

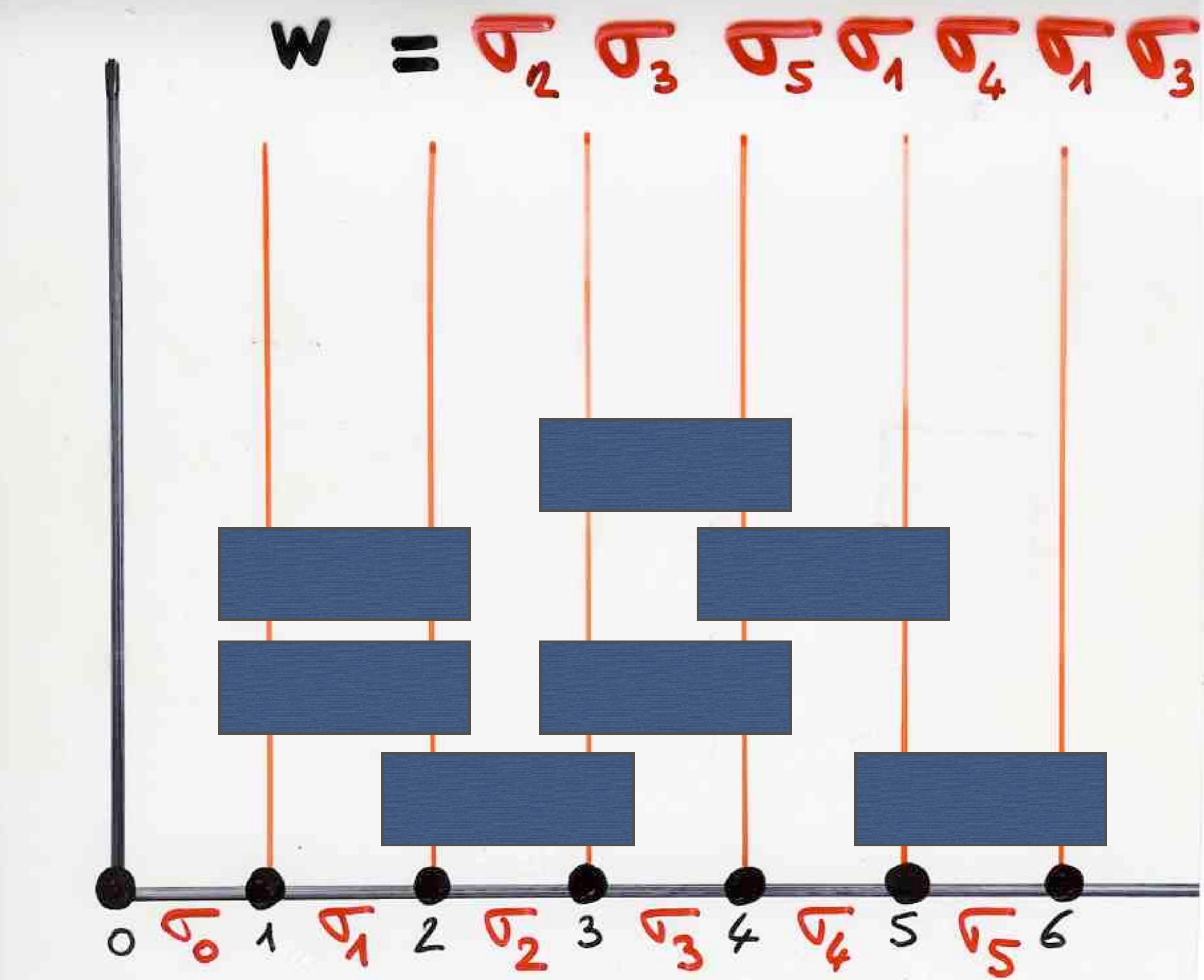


6 2 7 3 5 1 8 4

Reverse bijection

Lexicographic normal form of a heap

From ABJC 2, Ch 1b, p44


w = σ2 σ3 σ5 σ1 σ4 σ1 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

ex: heaps of dimers on $\mathbb{N}$

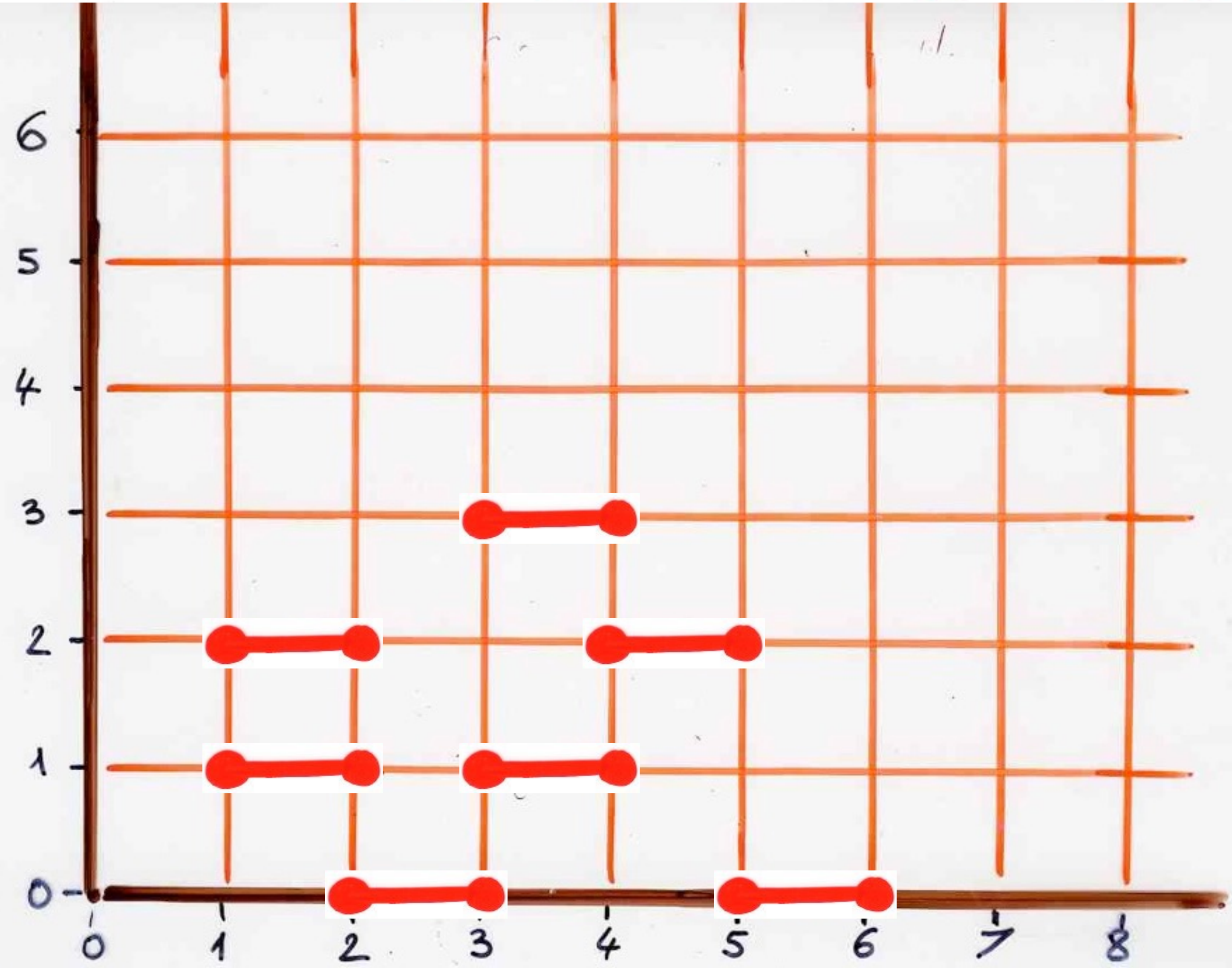
$$P = \{ [i, i+1] = \sigma_i , i \geq 0 \}$$

$\mathcal{C}$

C commutations

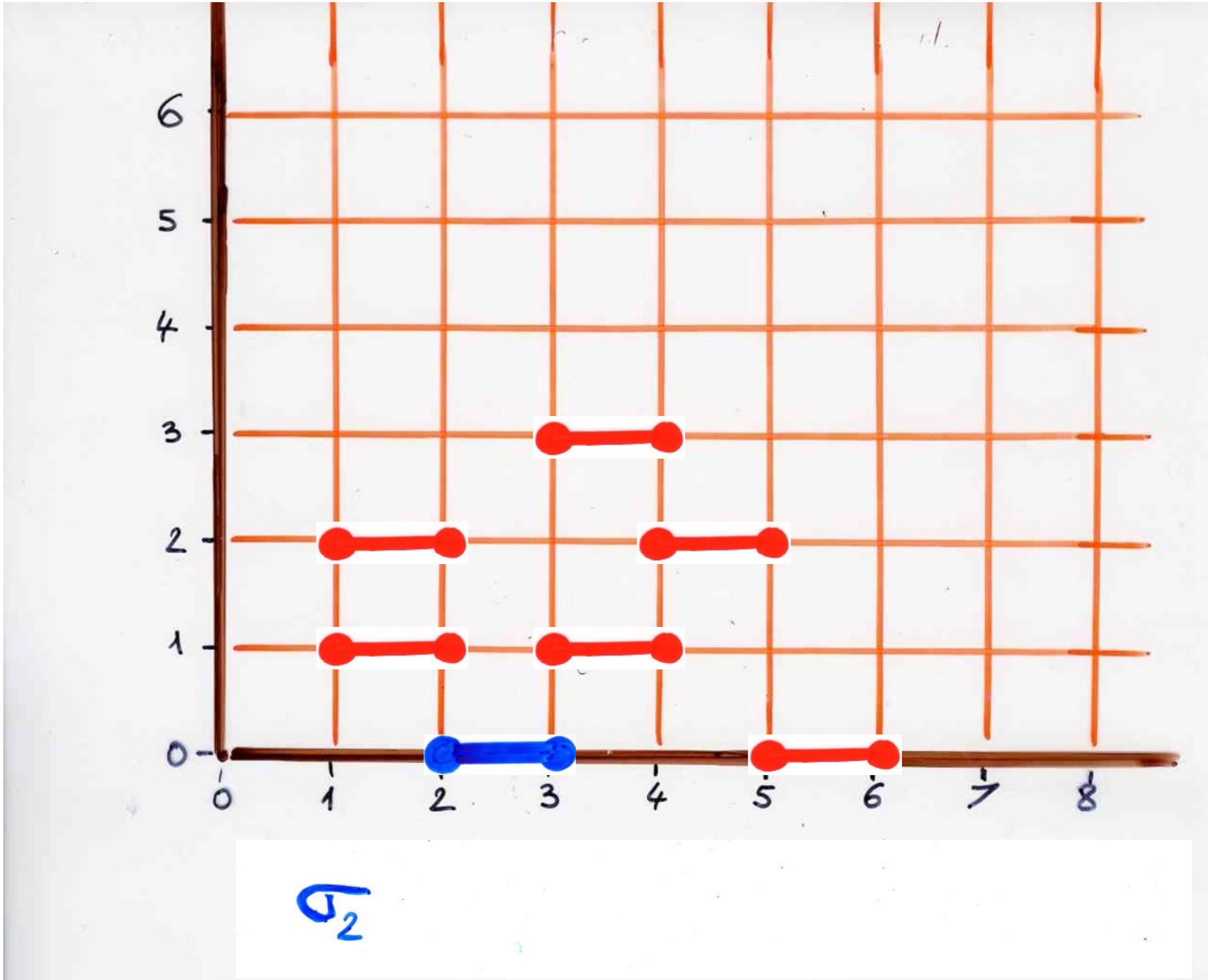
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ iff } |i-j| \geq 2$$

example $w = \sigma_2 \, \sigma_3 \, \sigma_5 \, \sigma_1 \, \sigma_4 \, \sigma_1 \, \sigma_3$

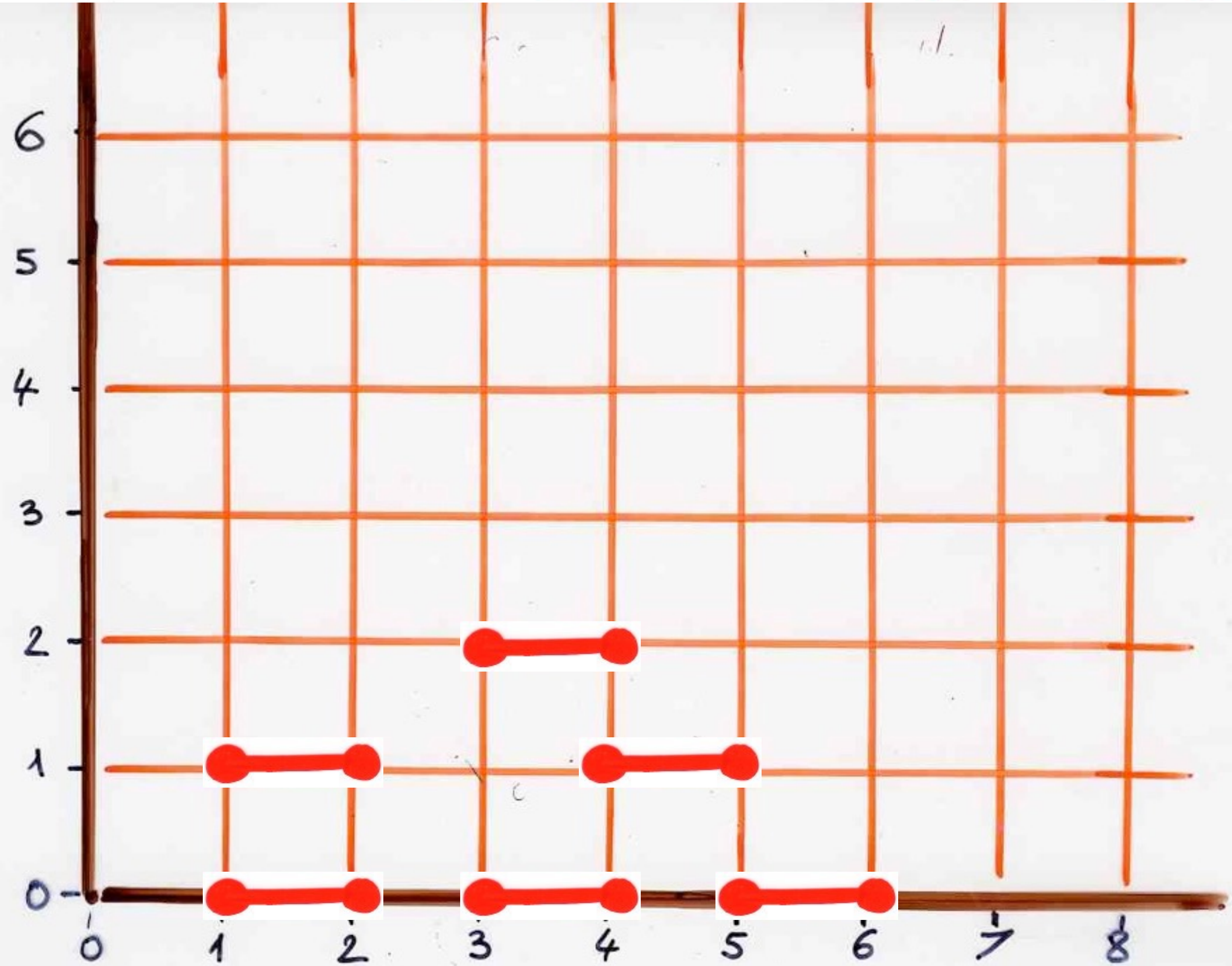


$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_5$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$

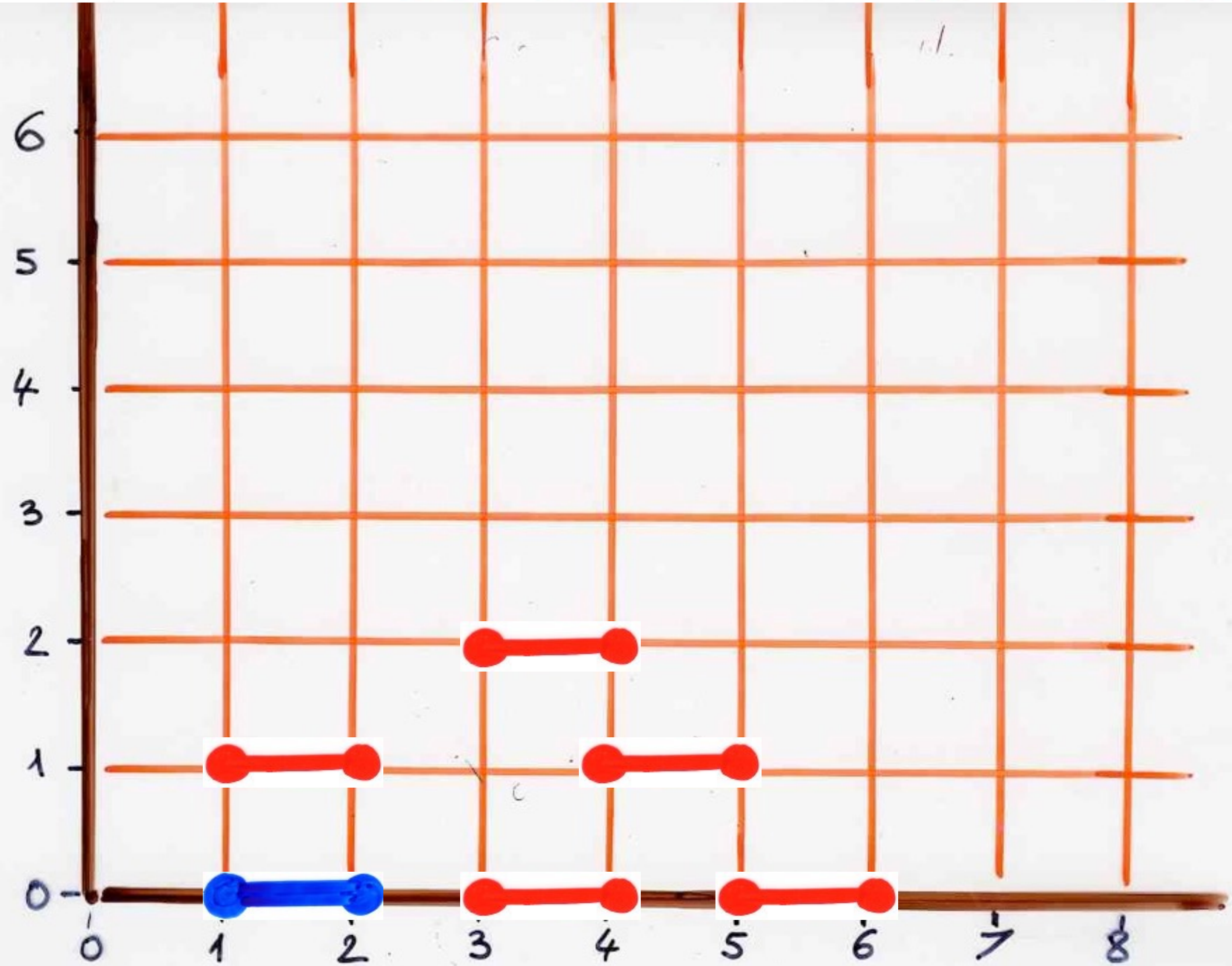


example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



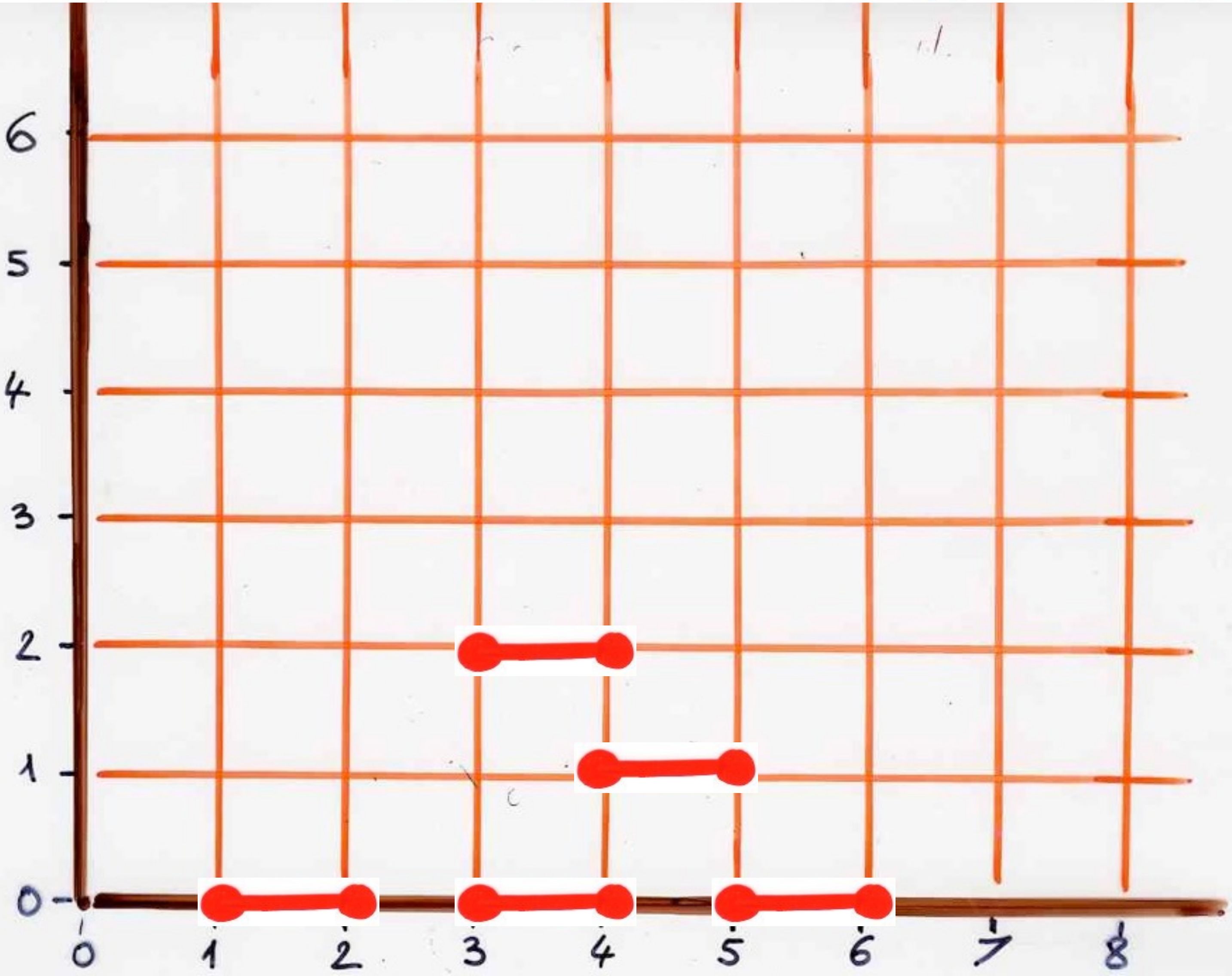
σ_2

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



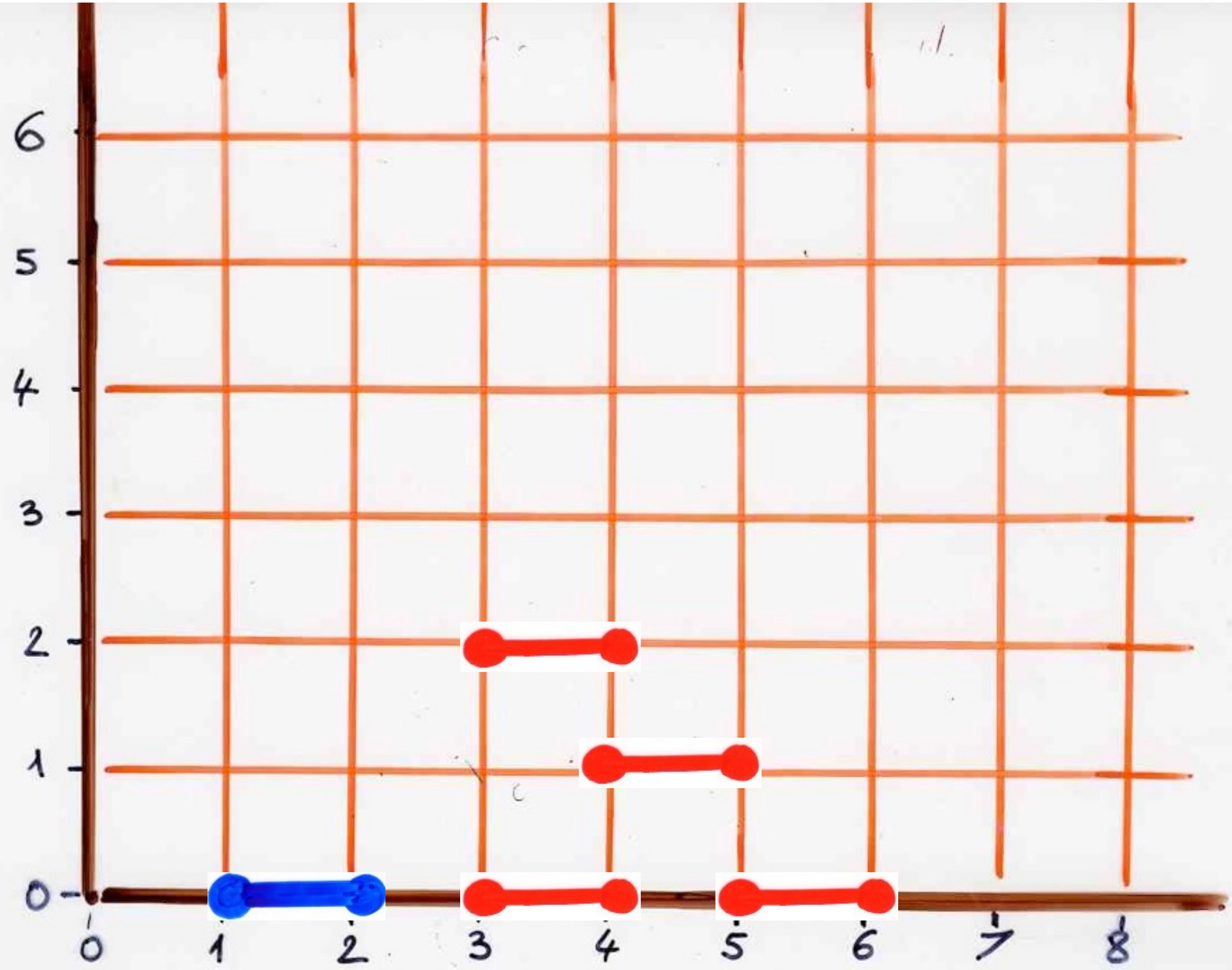
$\sigma_2 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



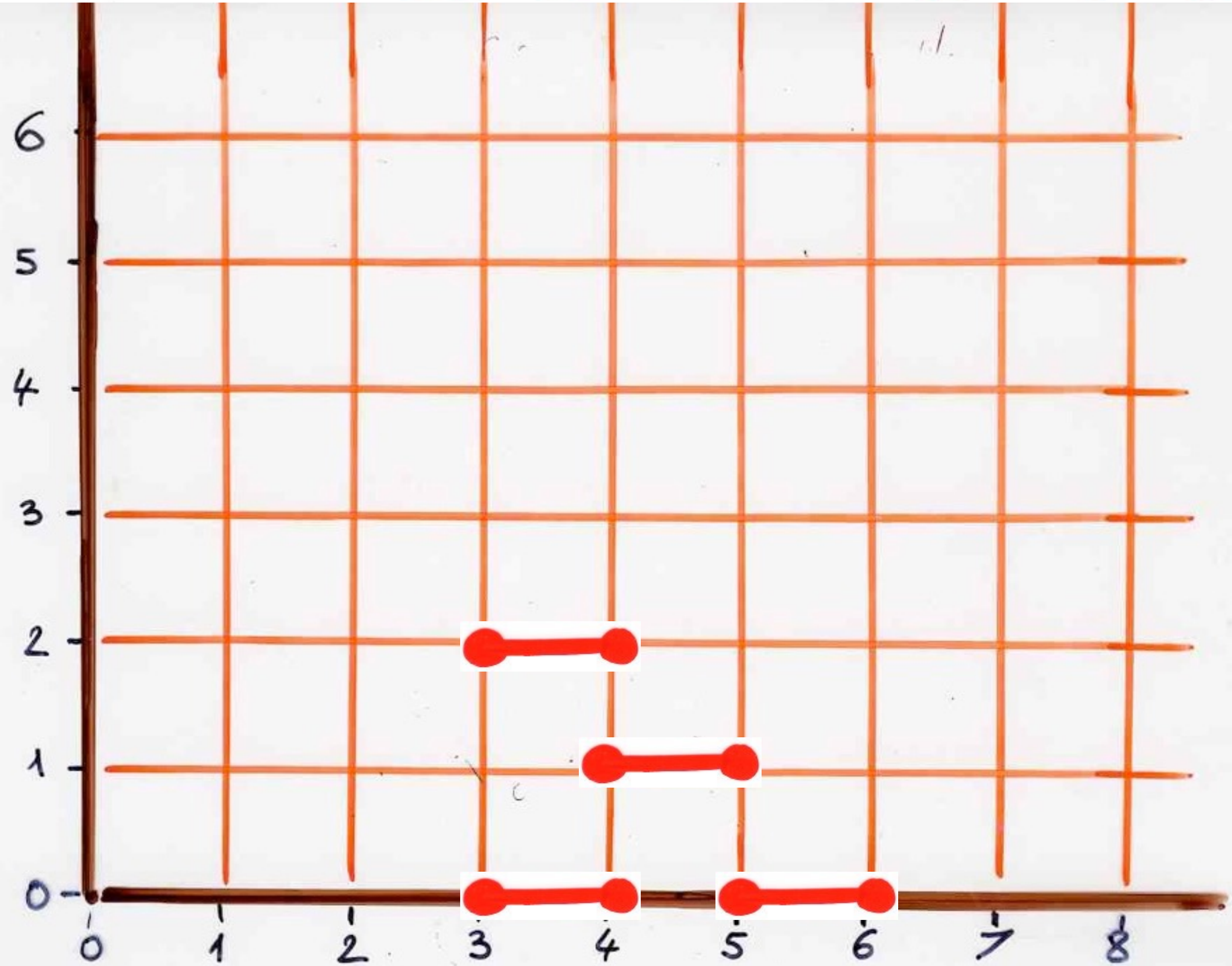
$\sigma_2 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



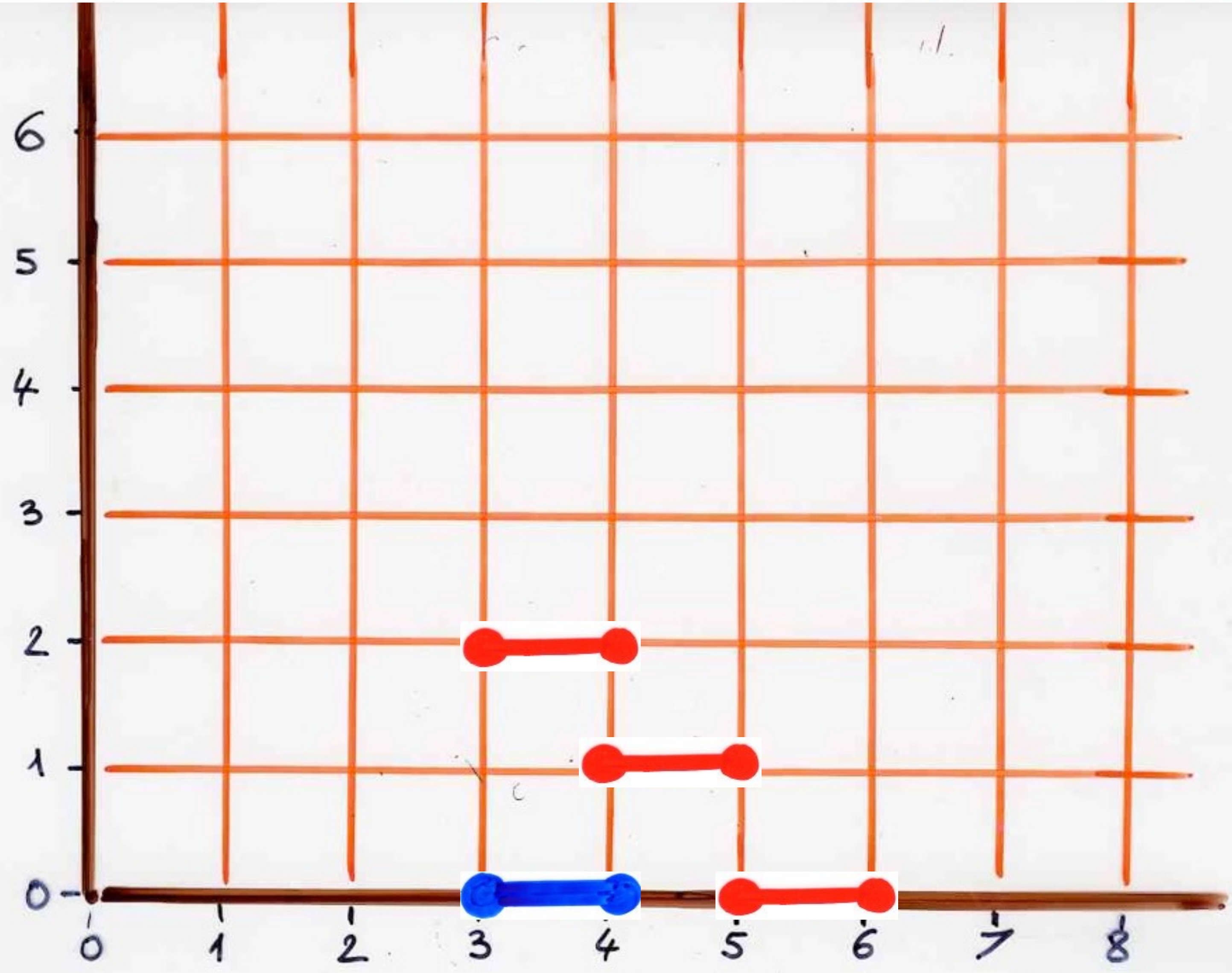
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



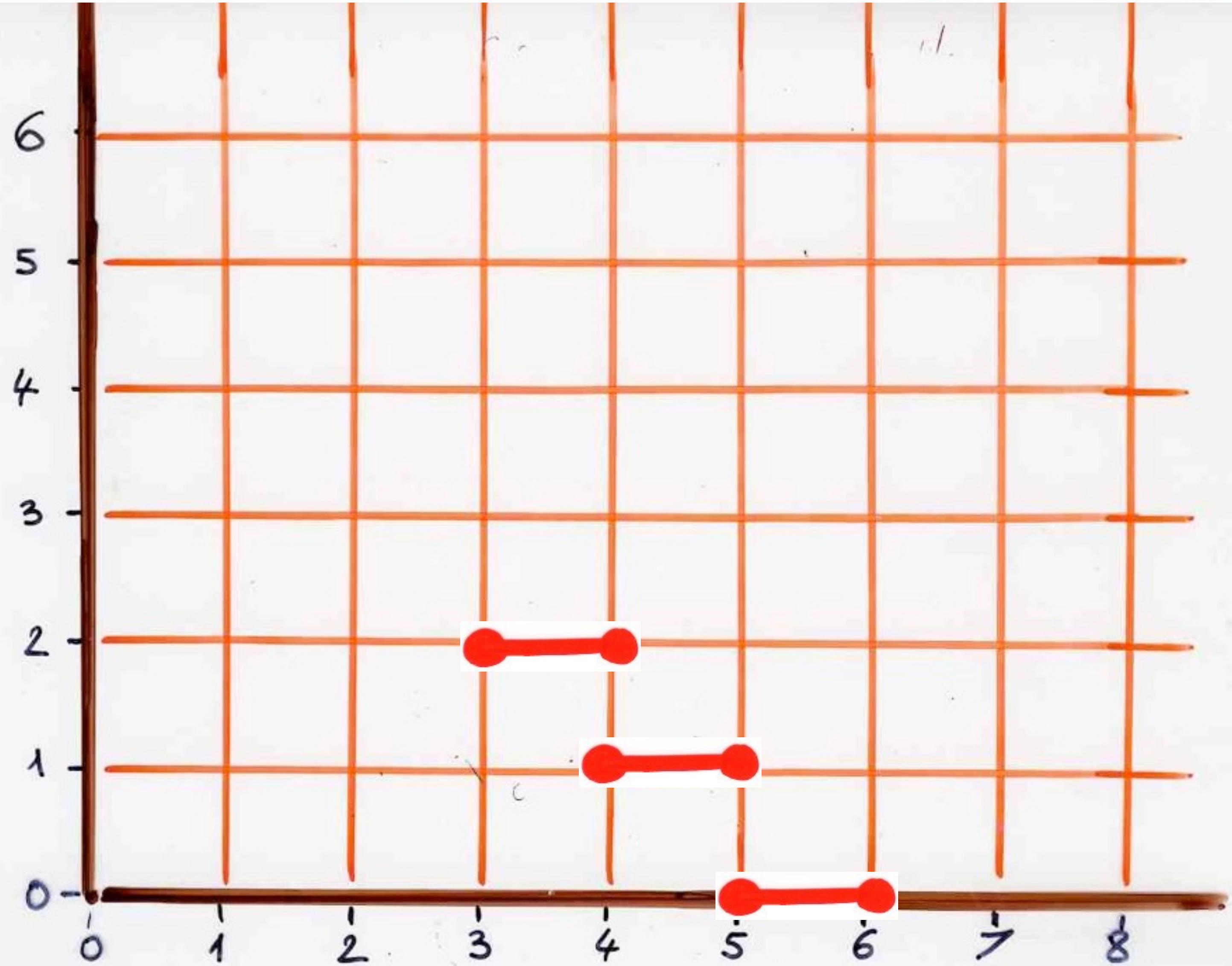
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



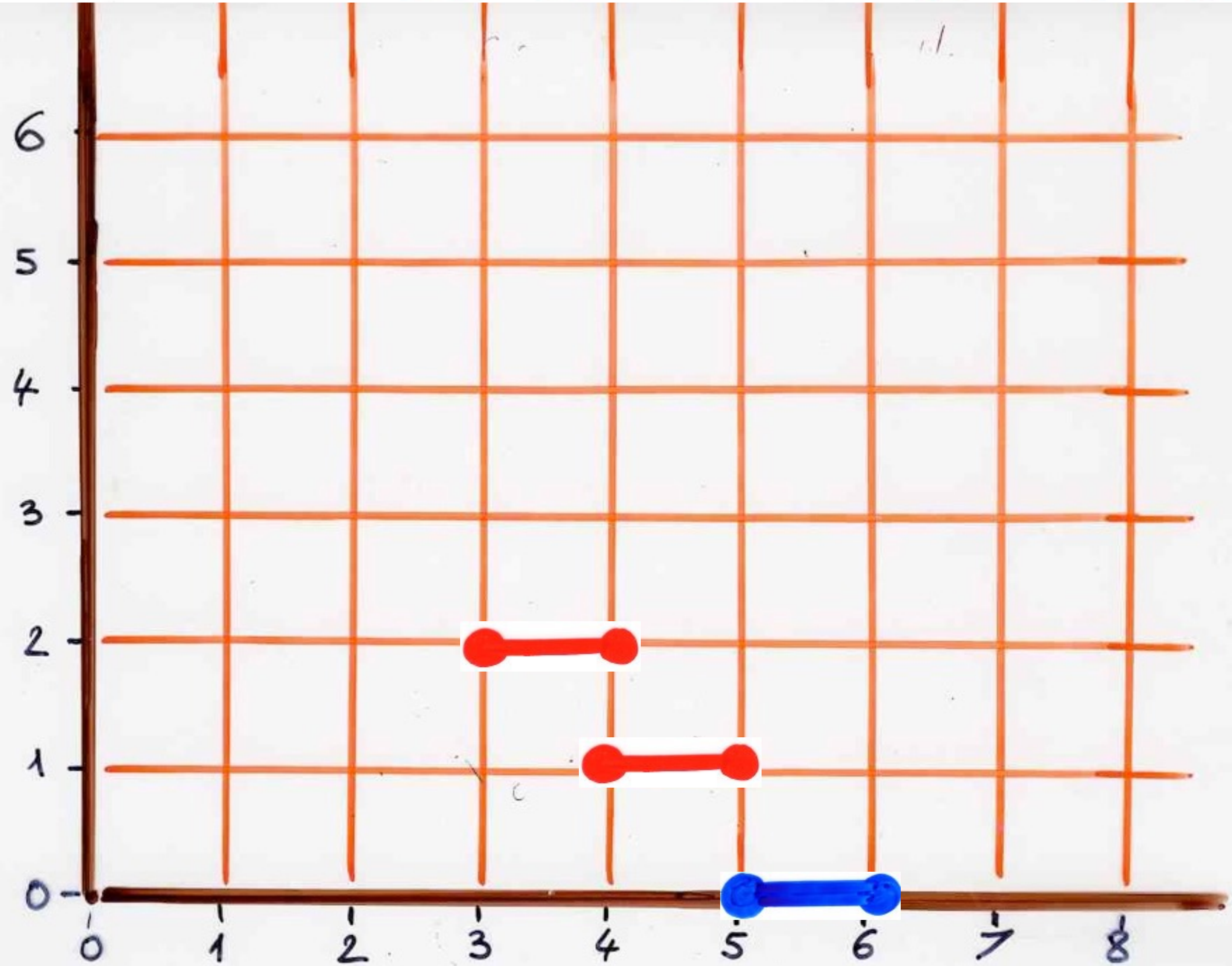
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



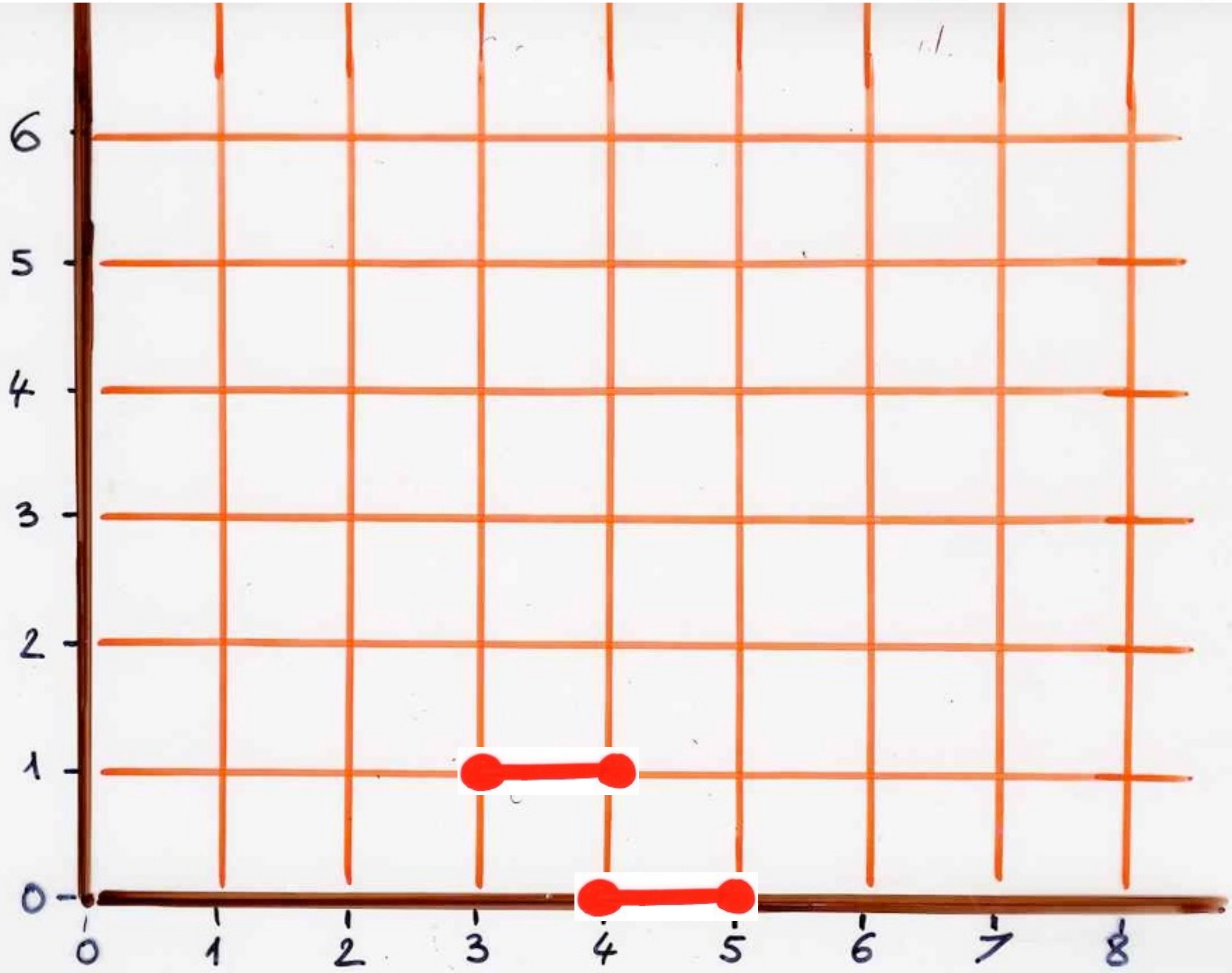
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



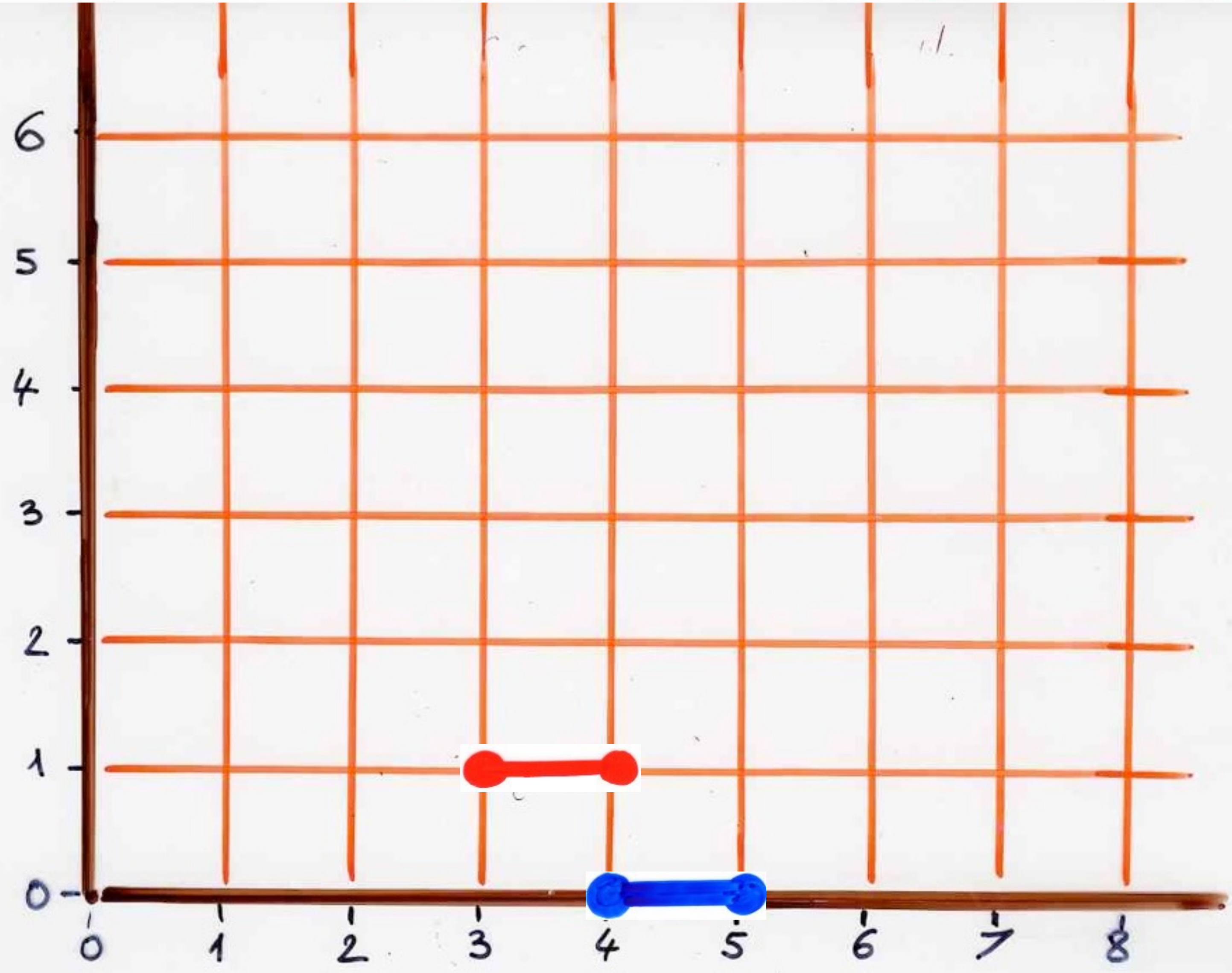
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



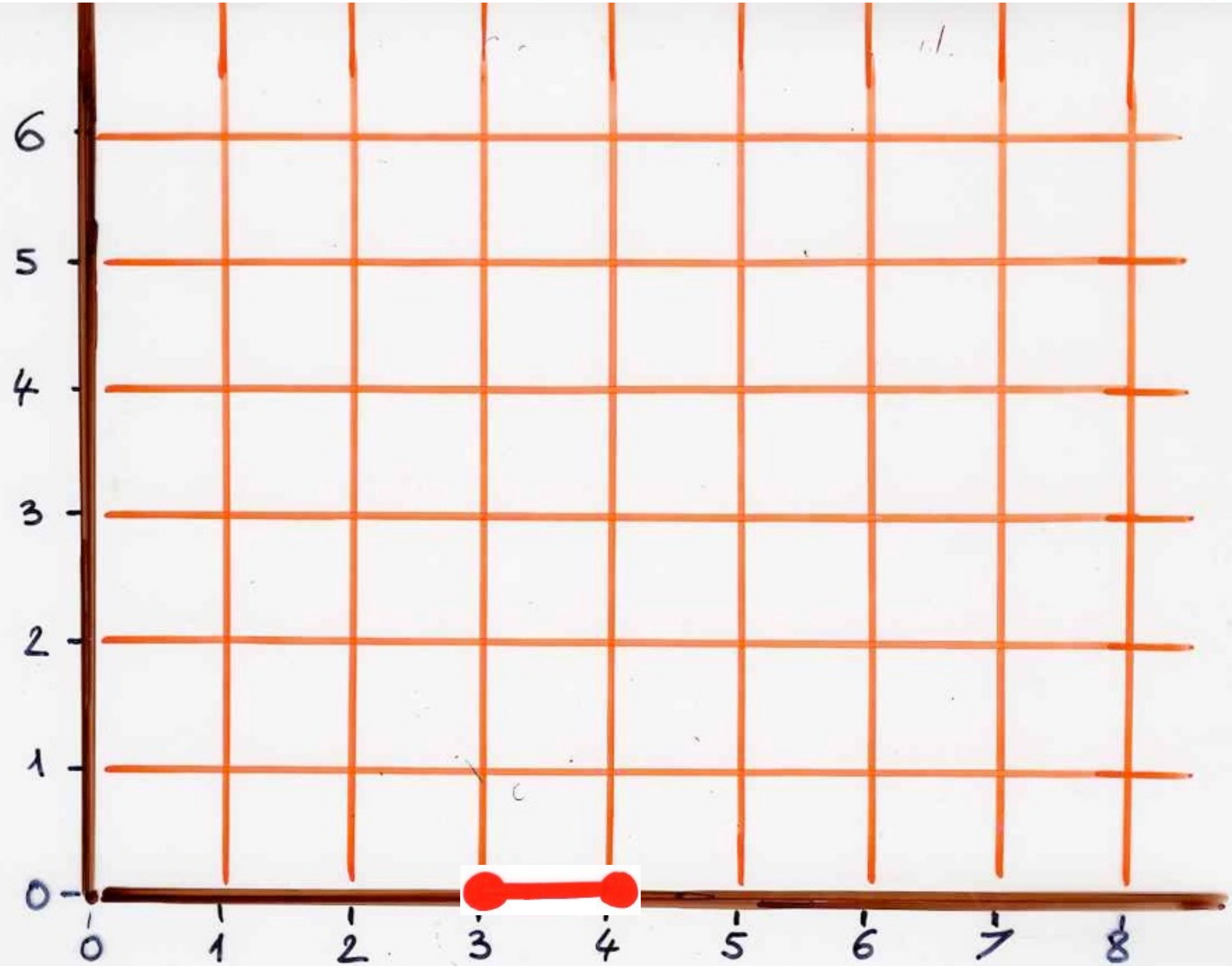
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



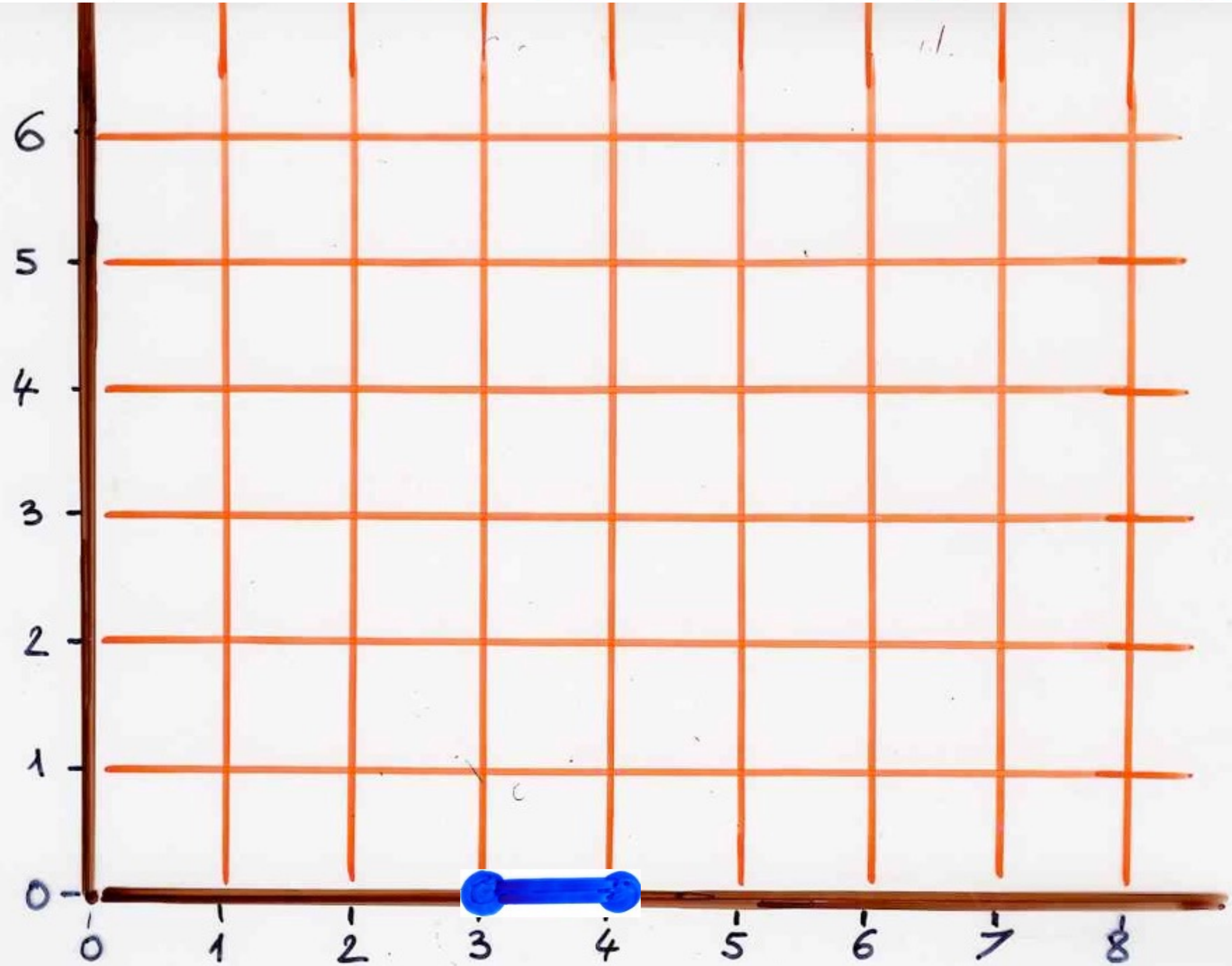
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4$

example $\quad w = \sigma_2 \, \sigma_3 \, \sigma_5 \, \sigma_1 \, \sigma_4 \, \sigma_1 \, \sigma_3$



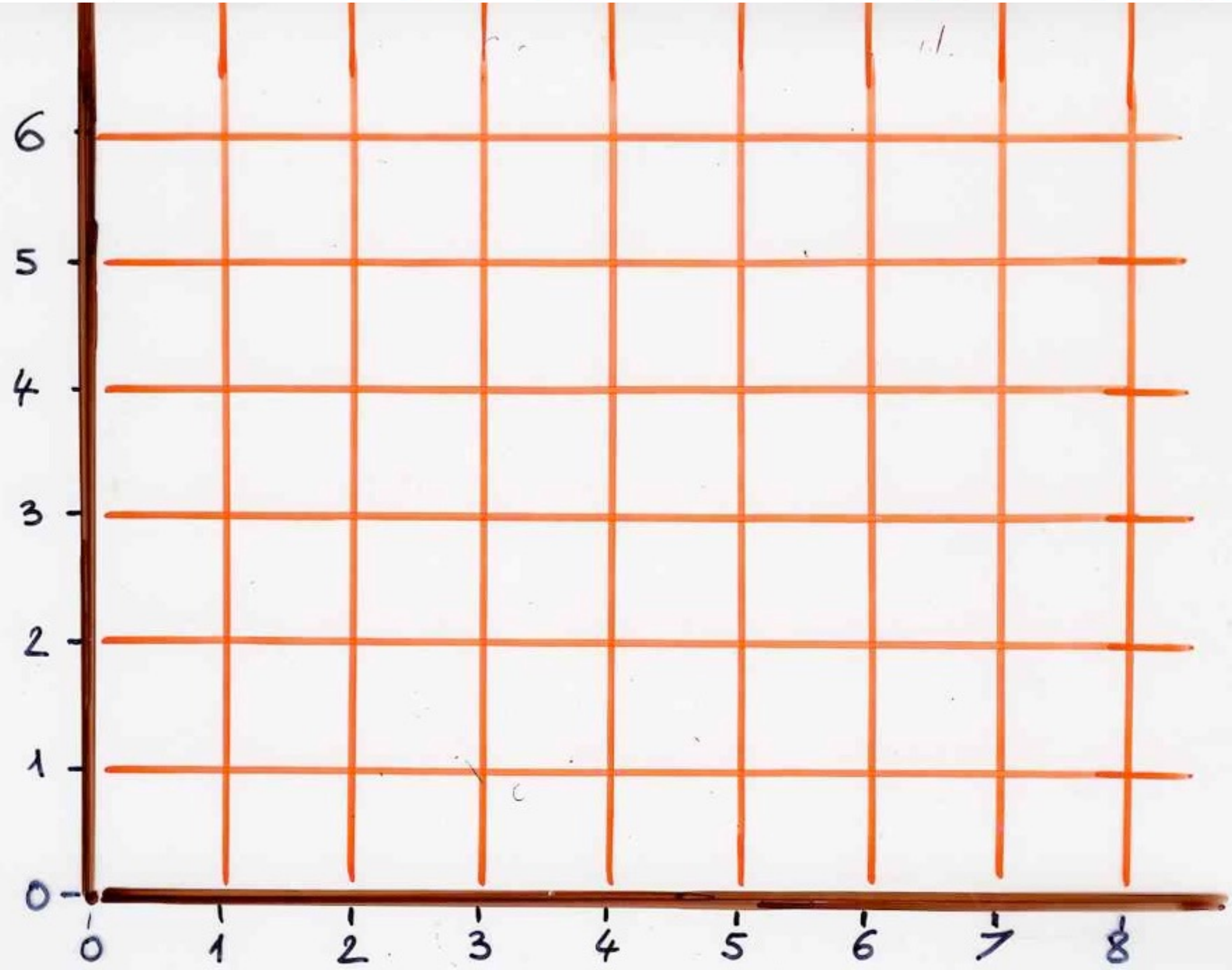
$\sigma_2 \, \sigma_1 \, \sigma_1 \, \sigma_3 \, \sigma_5 \, \sigma_4$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



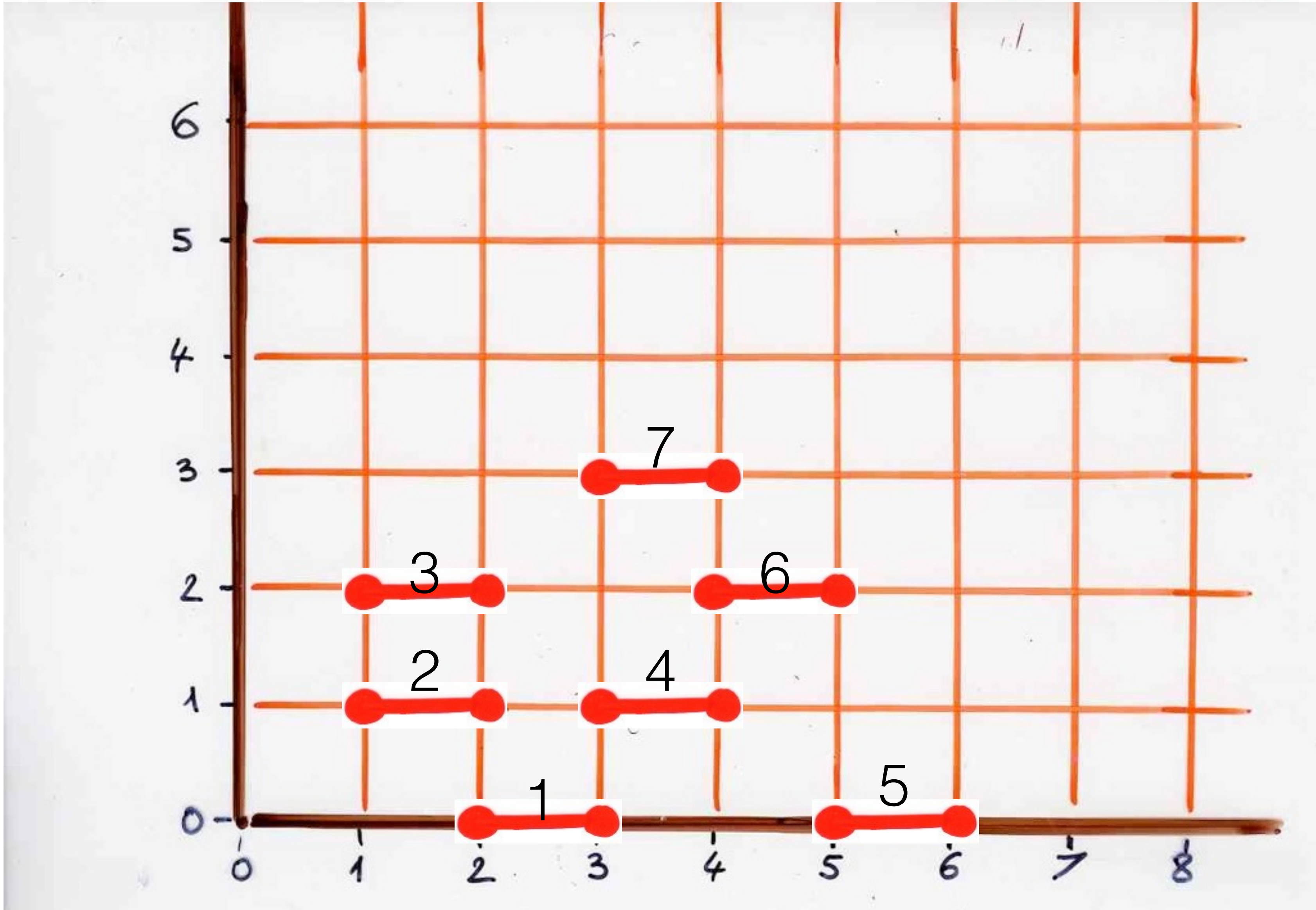
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$

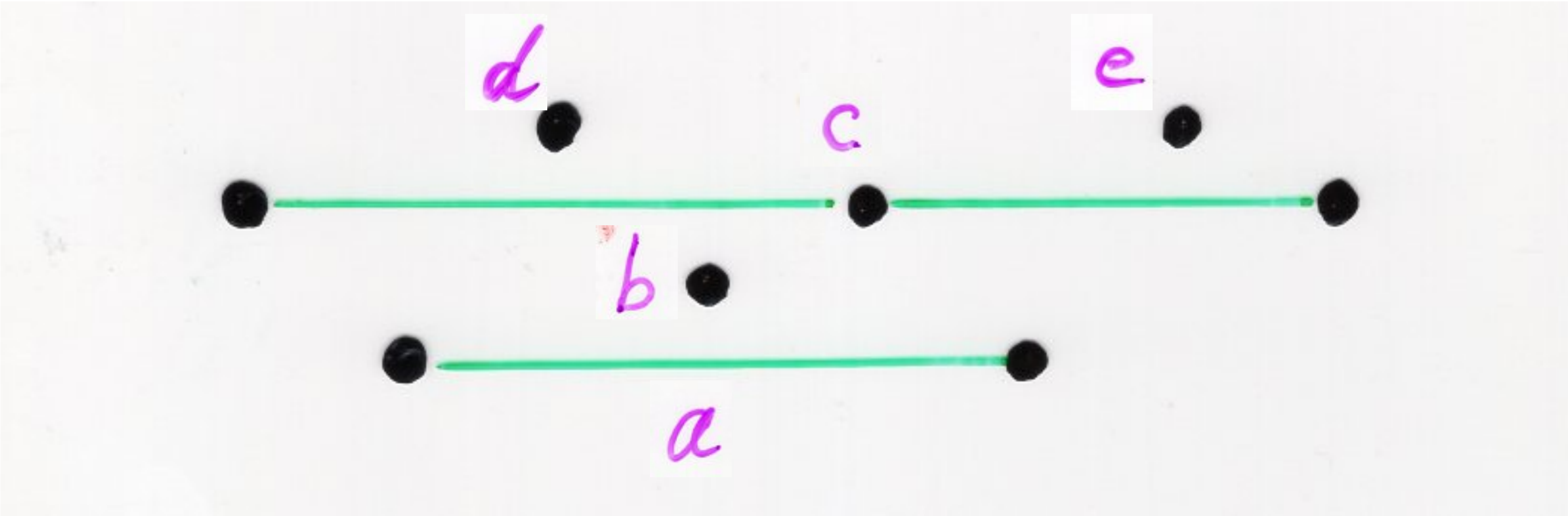


$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4 \sigma_3$

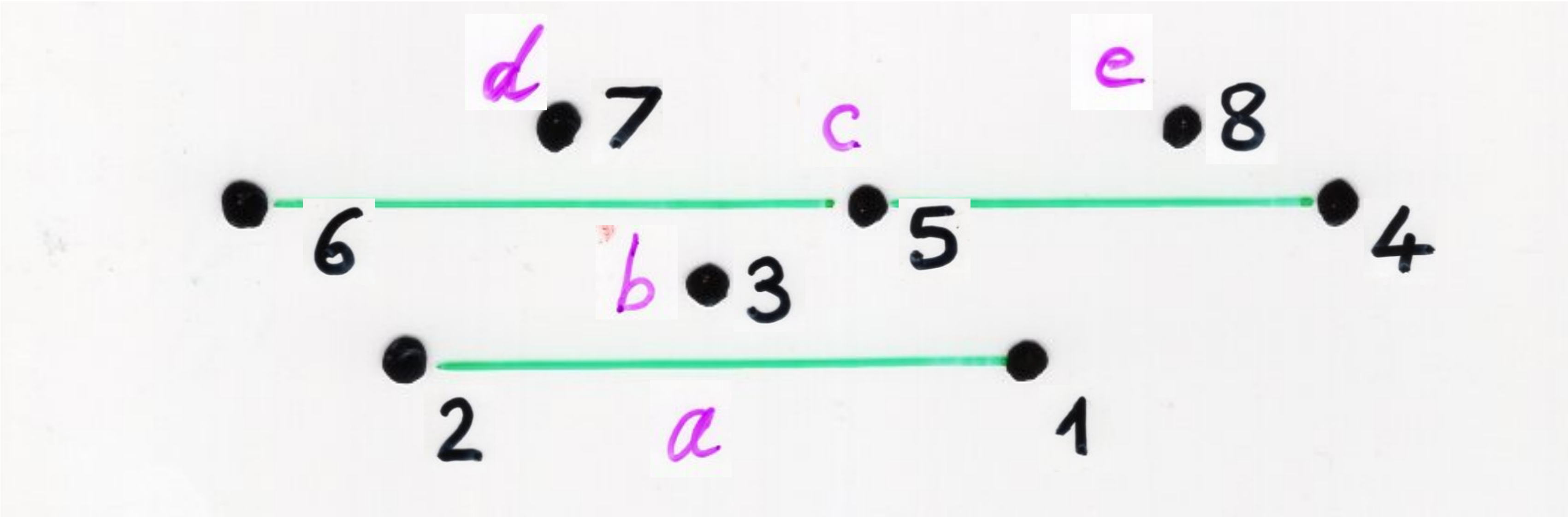
Reverse bijection

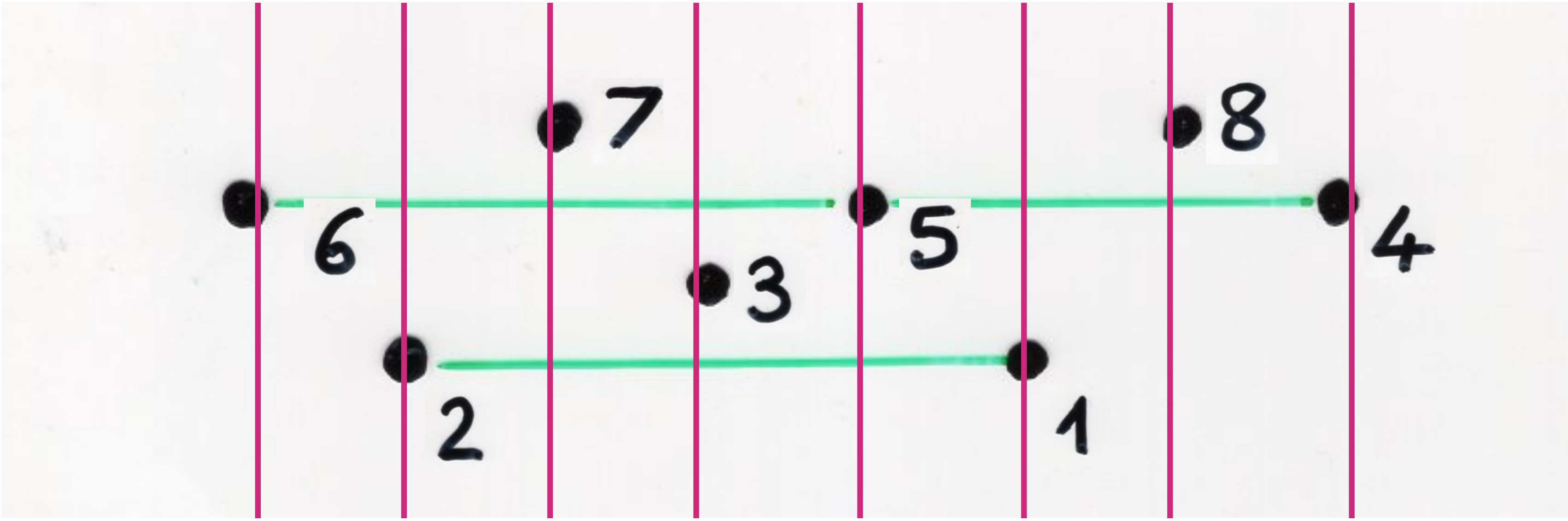
permutations

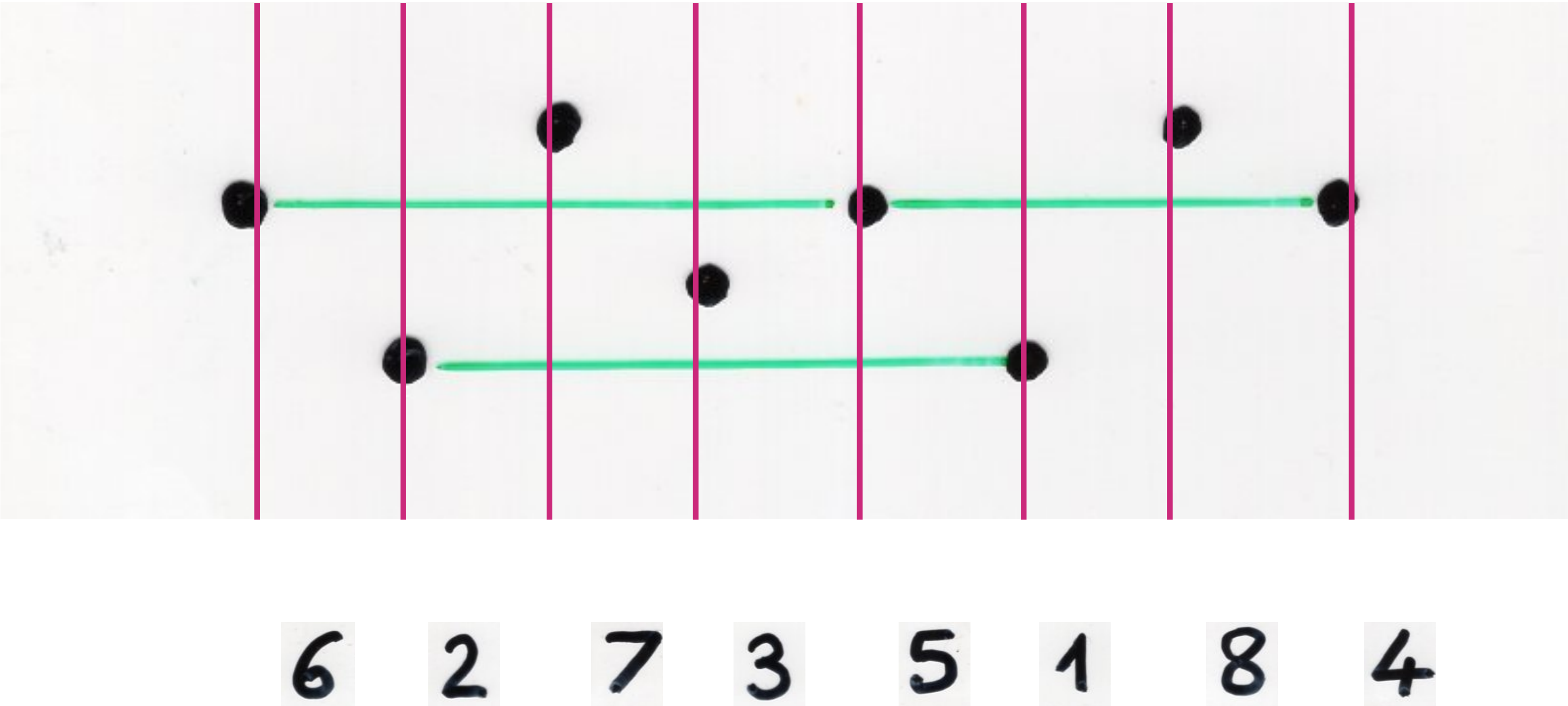
Laguerre heaps of segments



d
e
c
b
a


d
7
c
e
8
6
5
4
b
3
2
a
1


7
8
6
5
4
3
2
1


6 2 7 3 5 1 8 4

Proof of Josuat-Vergès proposition

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

__Proposition__

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

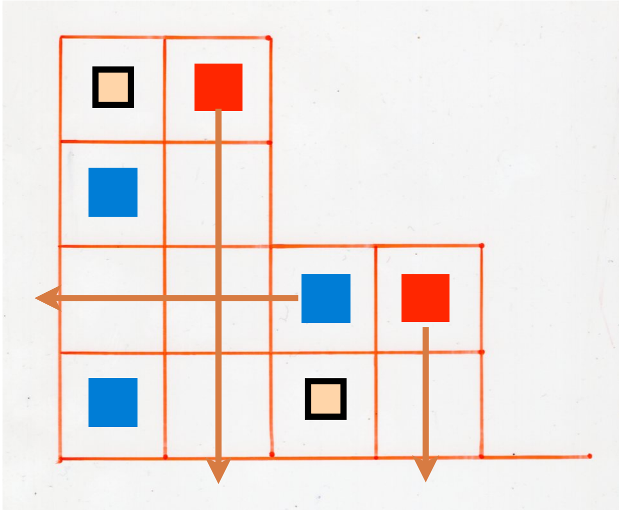
$31\text{-}2(\sigma)$ = number of patterns 31-2

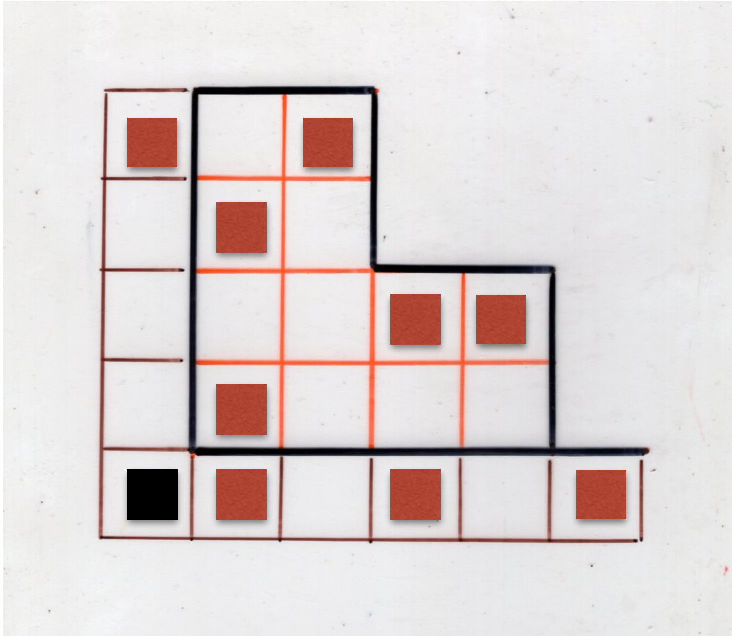
- direct bijection (with tree-like tableaux)

Aval, Boussicault, Nadeau (2011)

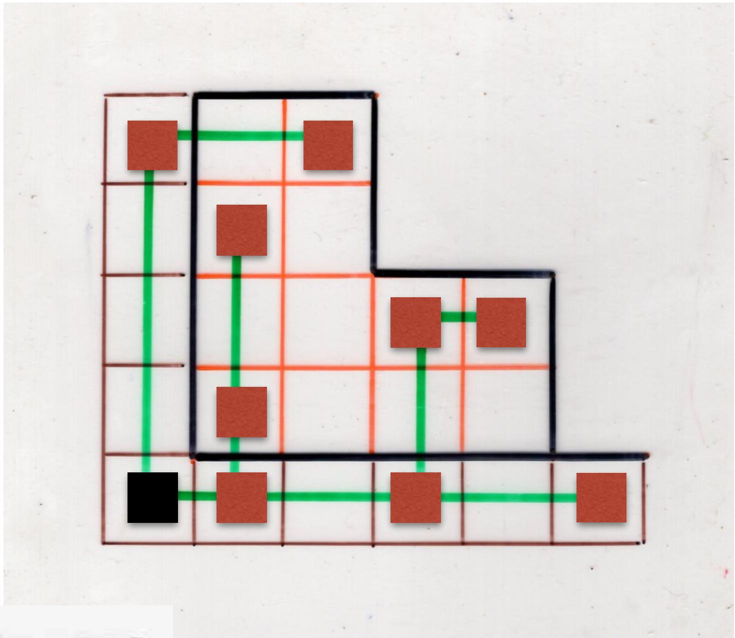
tableaux size $(n+1)$ $\longleftrightarrow$ (tableaux size n, $1 \leq i \leq n+1$)

$(n+1)!$



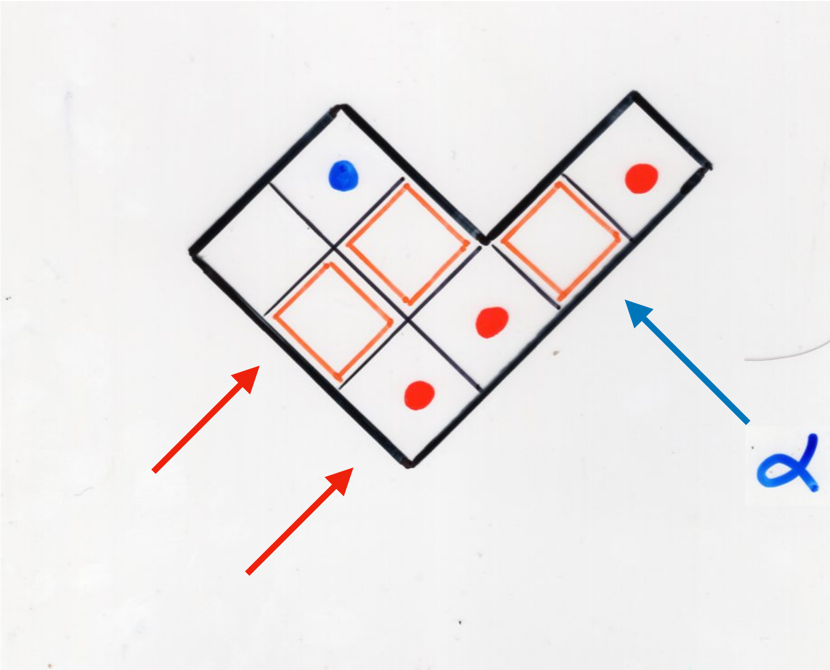


alternative tableaux



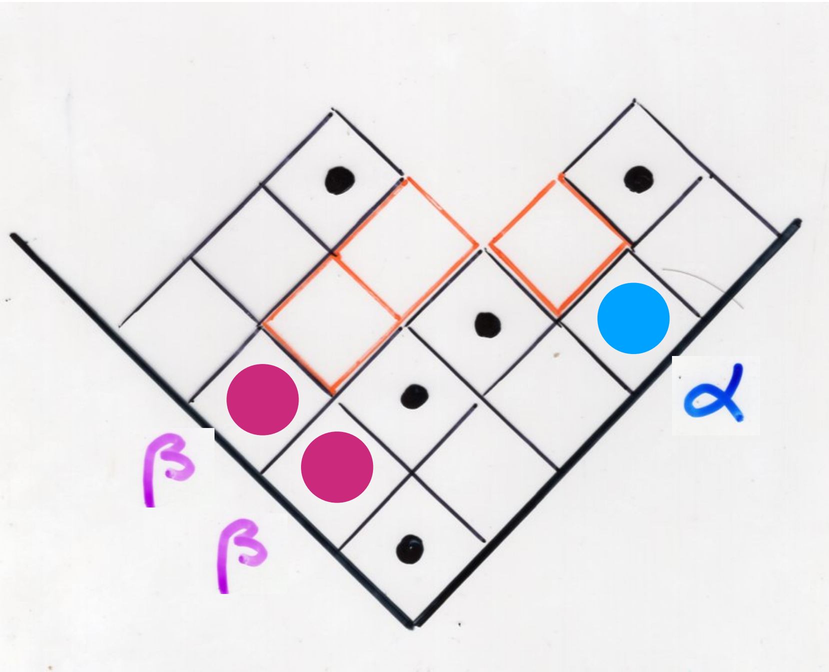
tree-like tableaux

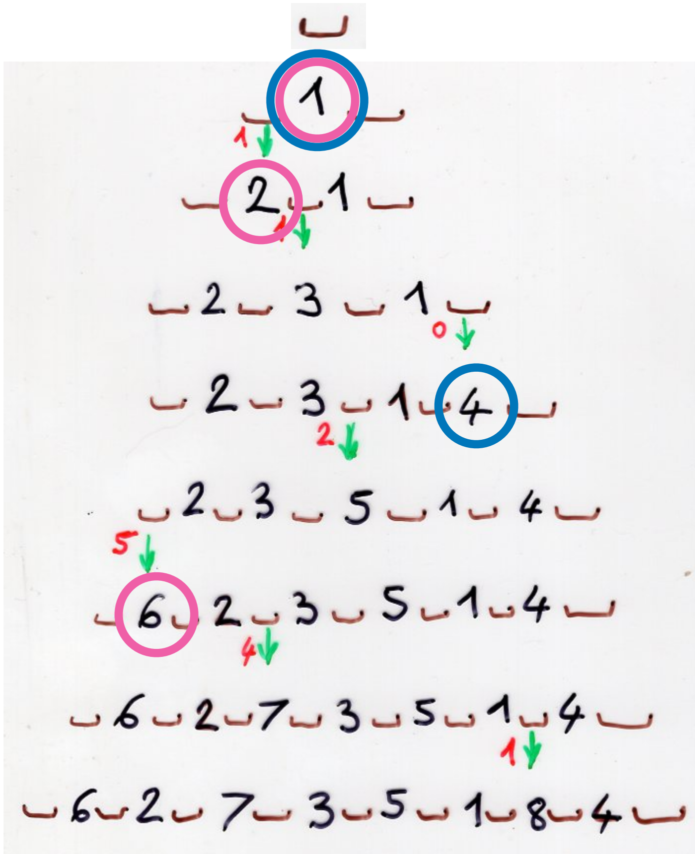
Aval, Boussicault, Nadeau (2013)



bijection

$f \longrightarrow T$

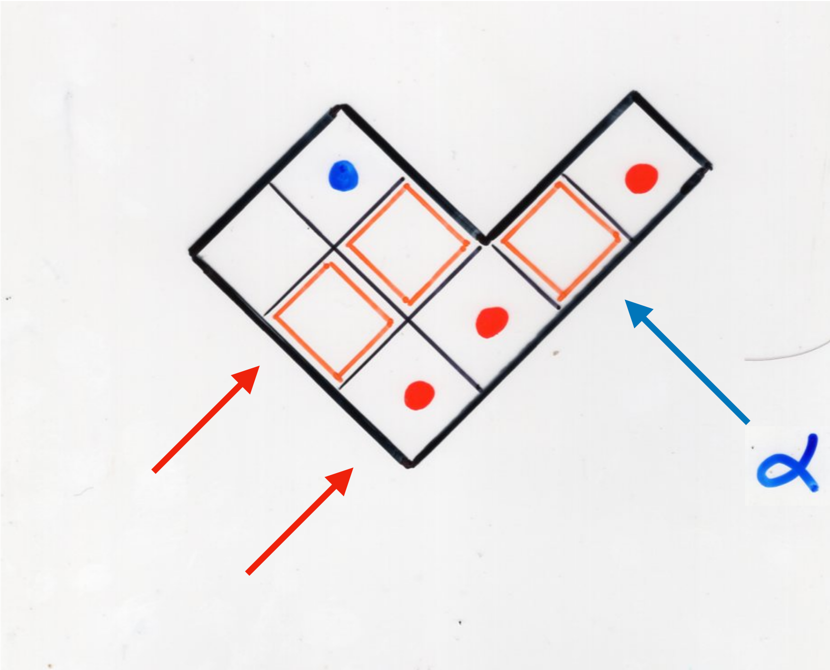


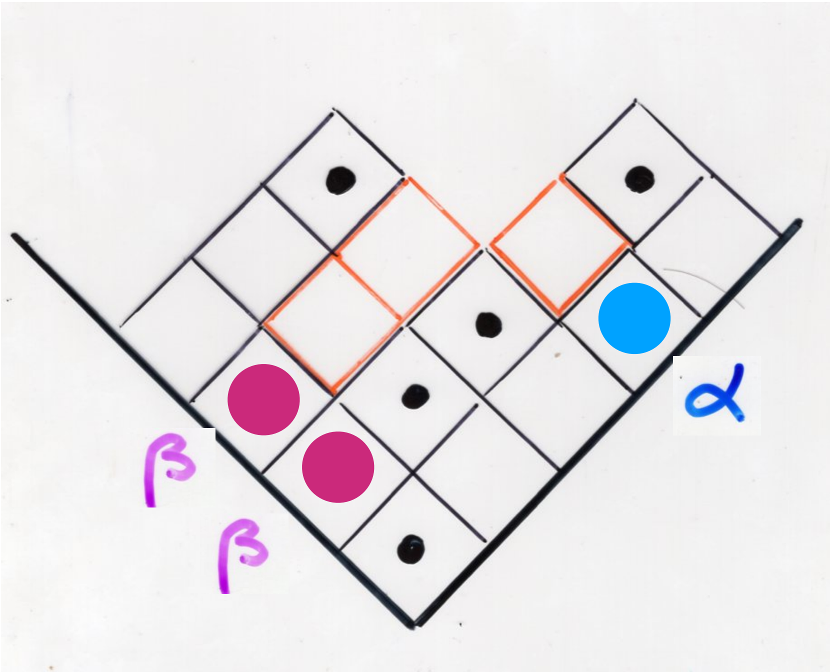


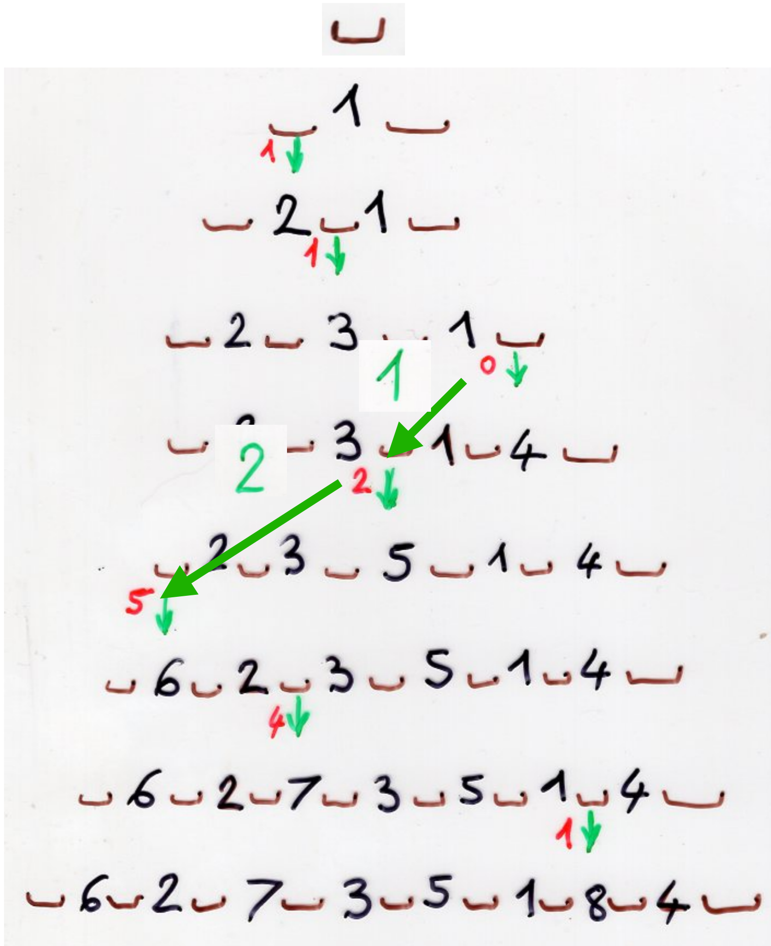
number of crossings

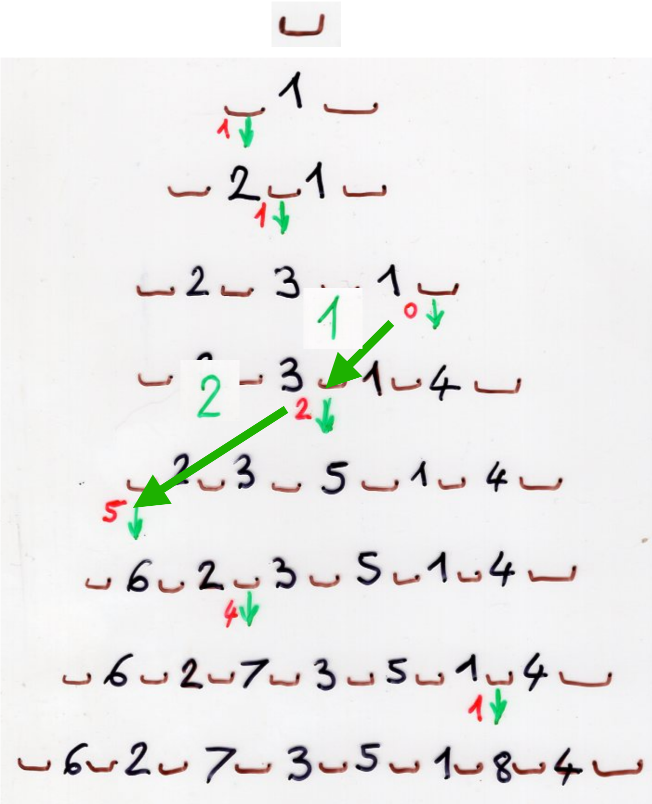
$cr(T)$

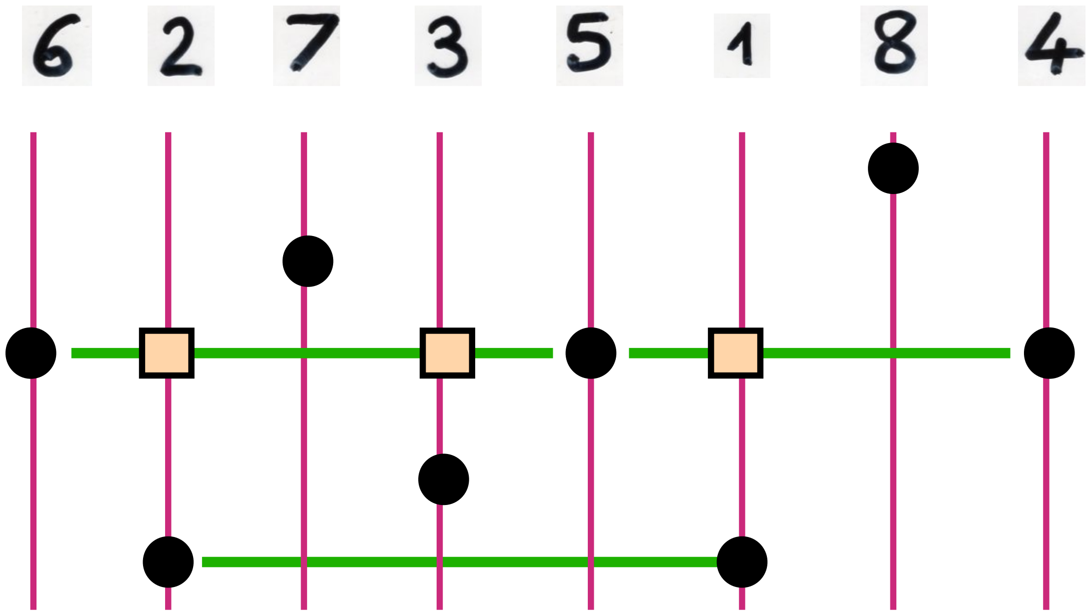
$$= \sum_{1 \leq i \leq (n-1)} \max\left[\left(f(i+1) - f(i) \right) - 1, 0 \right]$$

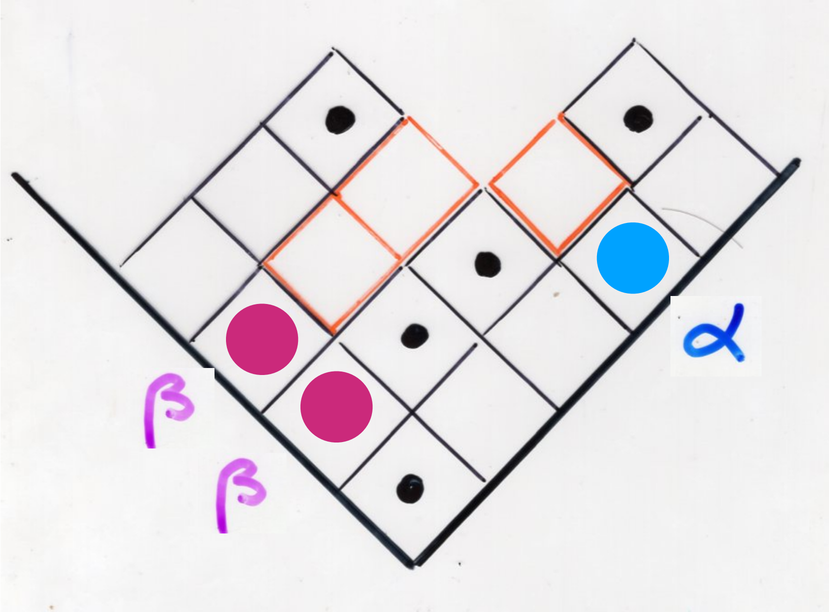


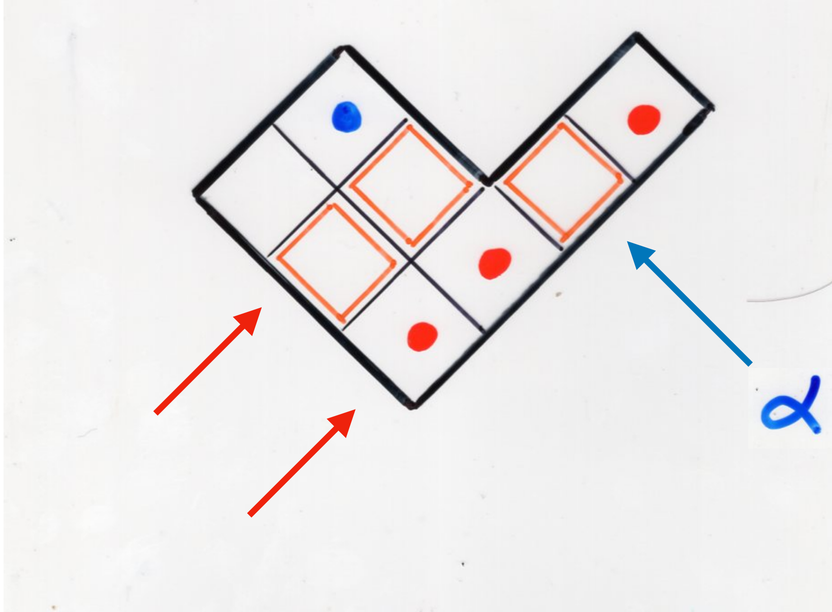


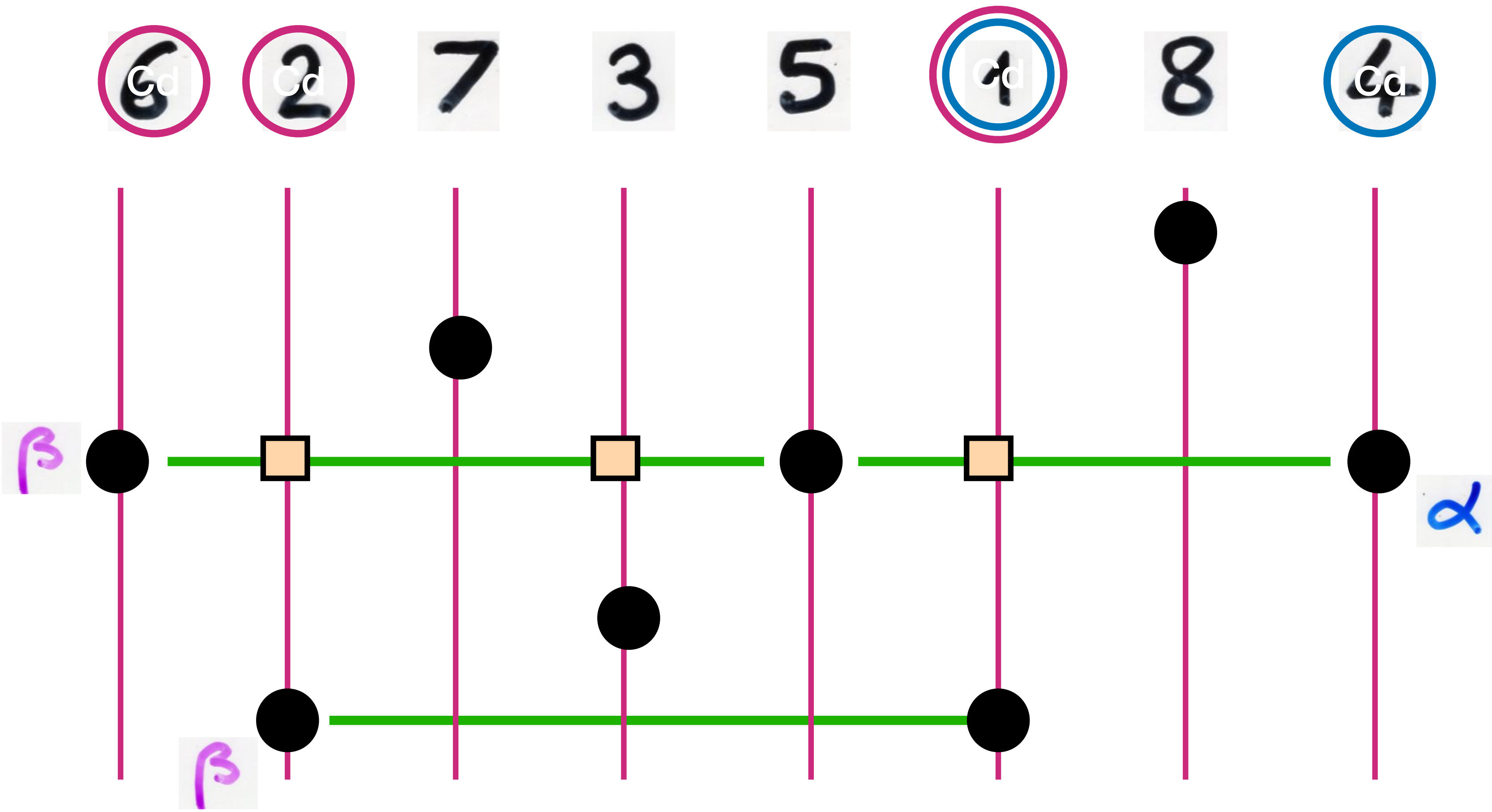


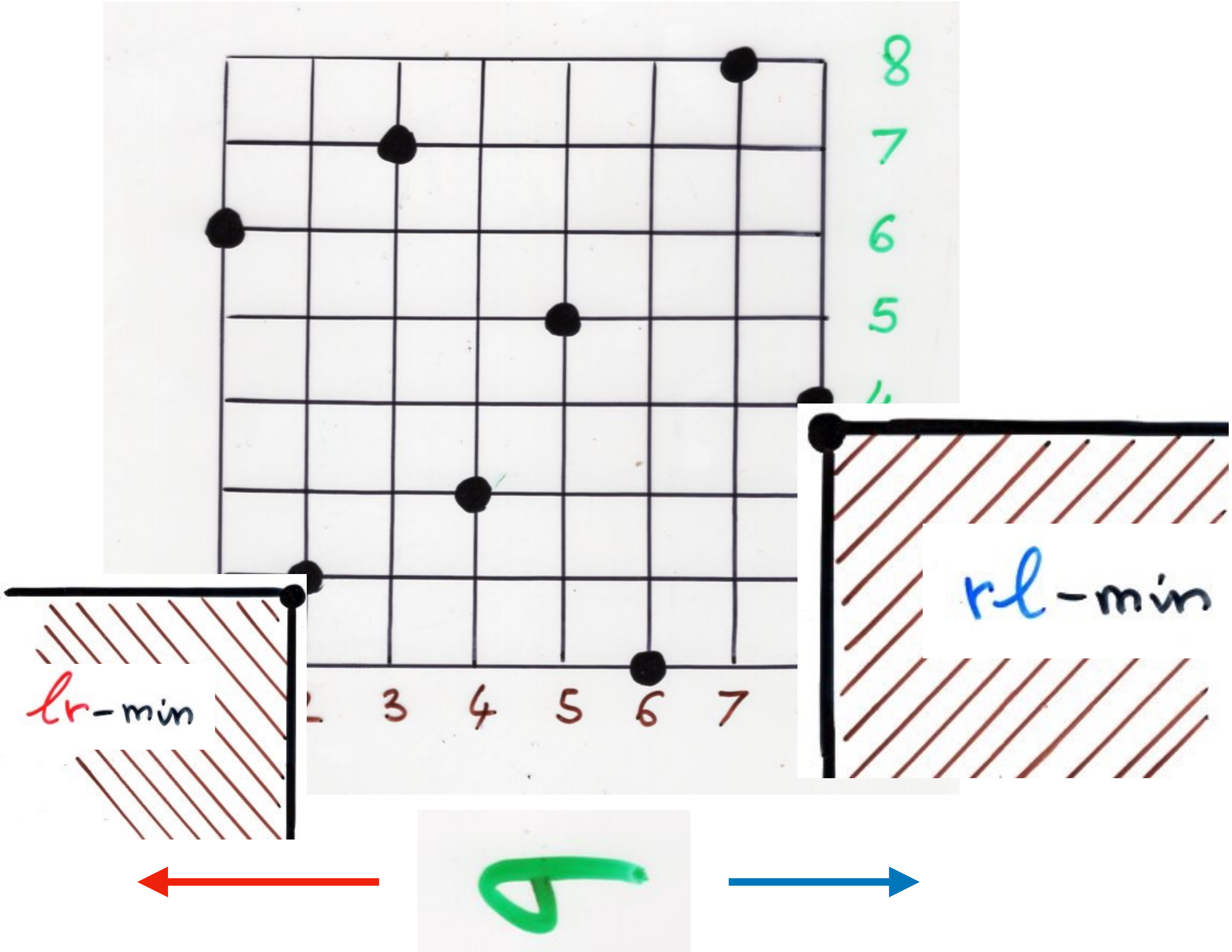

1
2 1
2 3 1
3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4


6 2 7 3 5 1 8 4

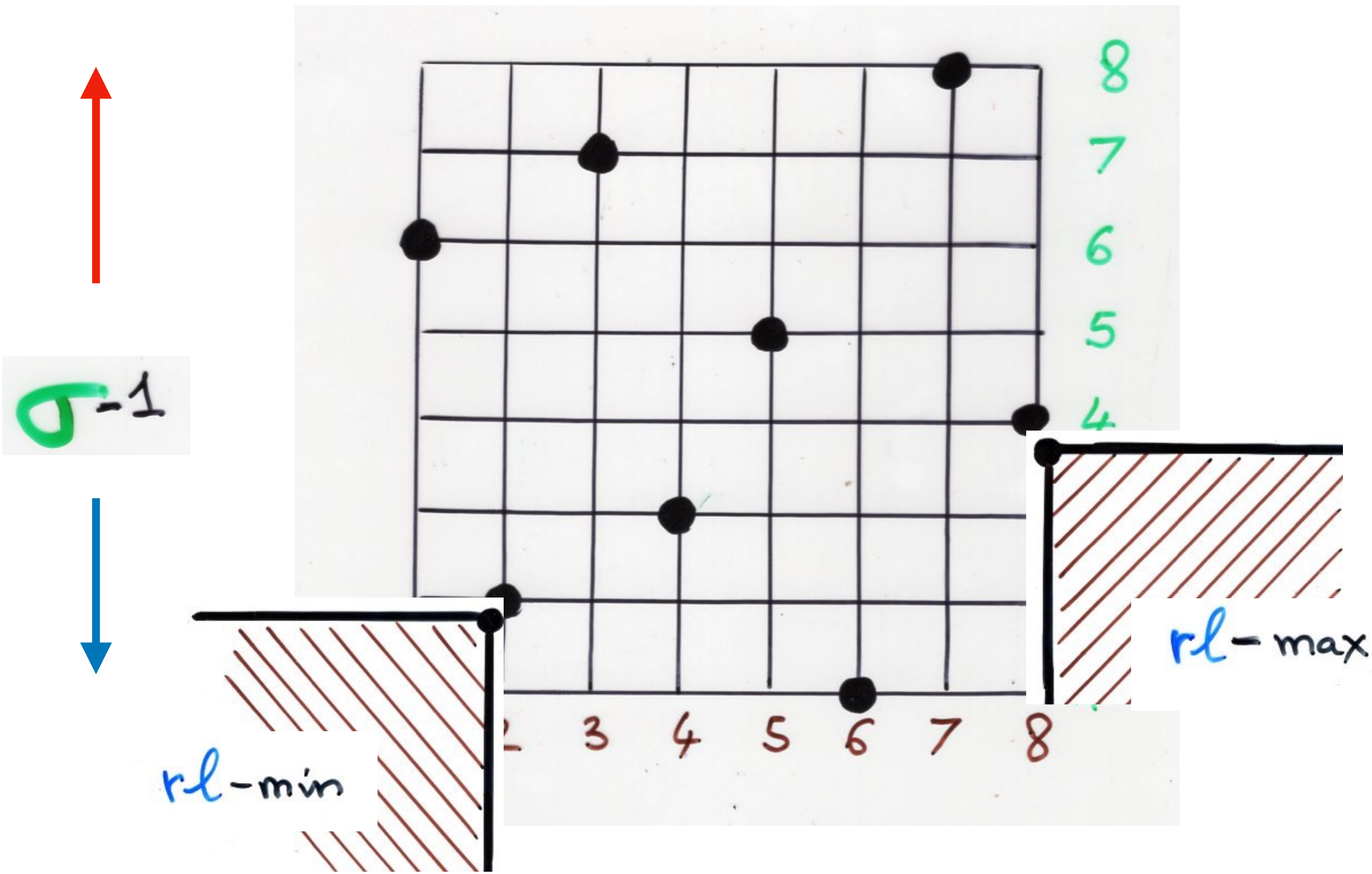

α
β
β


α


6
2
7
3
5
1
8
4
β
α
β



8
7
6
5
4
2 3 4 5 6 7
lr-min
rl-min

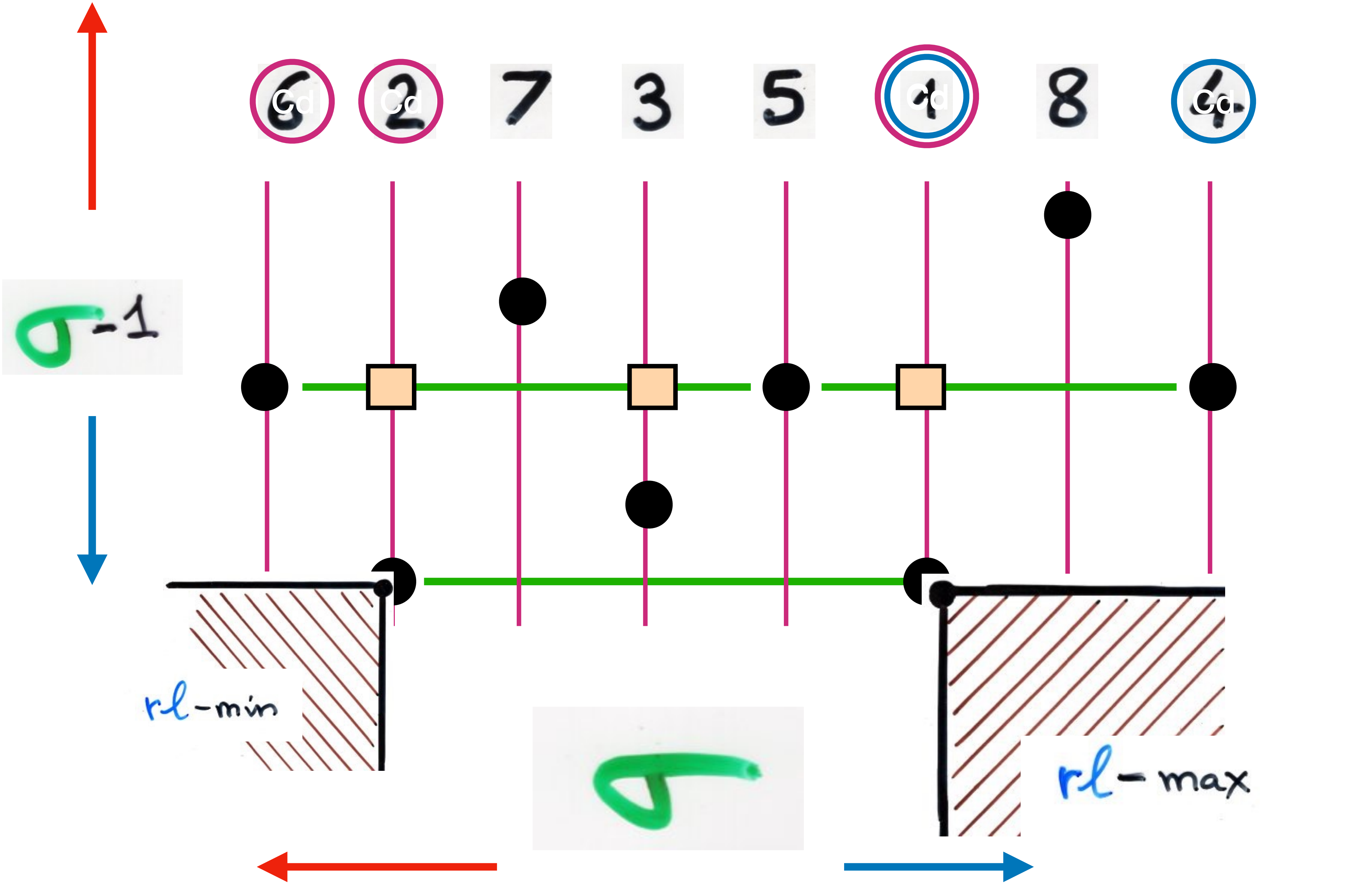


σ^{-1}
rl-min
rl-max
8
7
6
5
4
2 3 4 5 6 7 8

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

Josuat-Vergès (2011)



Combinatorial theory of orthogonal polynomials

combinatorial
theory of
orthogonal polynomials

moments X.V. (1983)
Françon, X.V. (1978)

and
continued fractions
Flajolet (1980)

formal orthogonality

$$\mathfrak{f}(x^n) = \mu_n$$

$$\mathfrak{f}(P_k P_l) = 0$$

moments

$k \neq l$

$$\mathfrak{f}(PQ) = \int_a^b P(x)\, Q(x)\, d\mu$$

classical analysis

measure

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

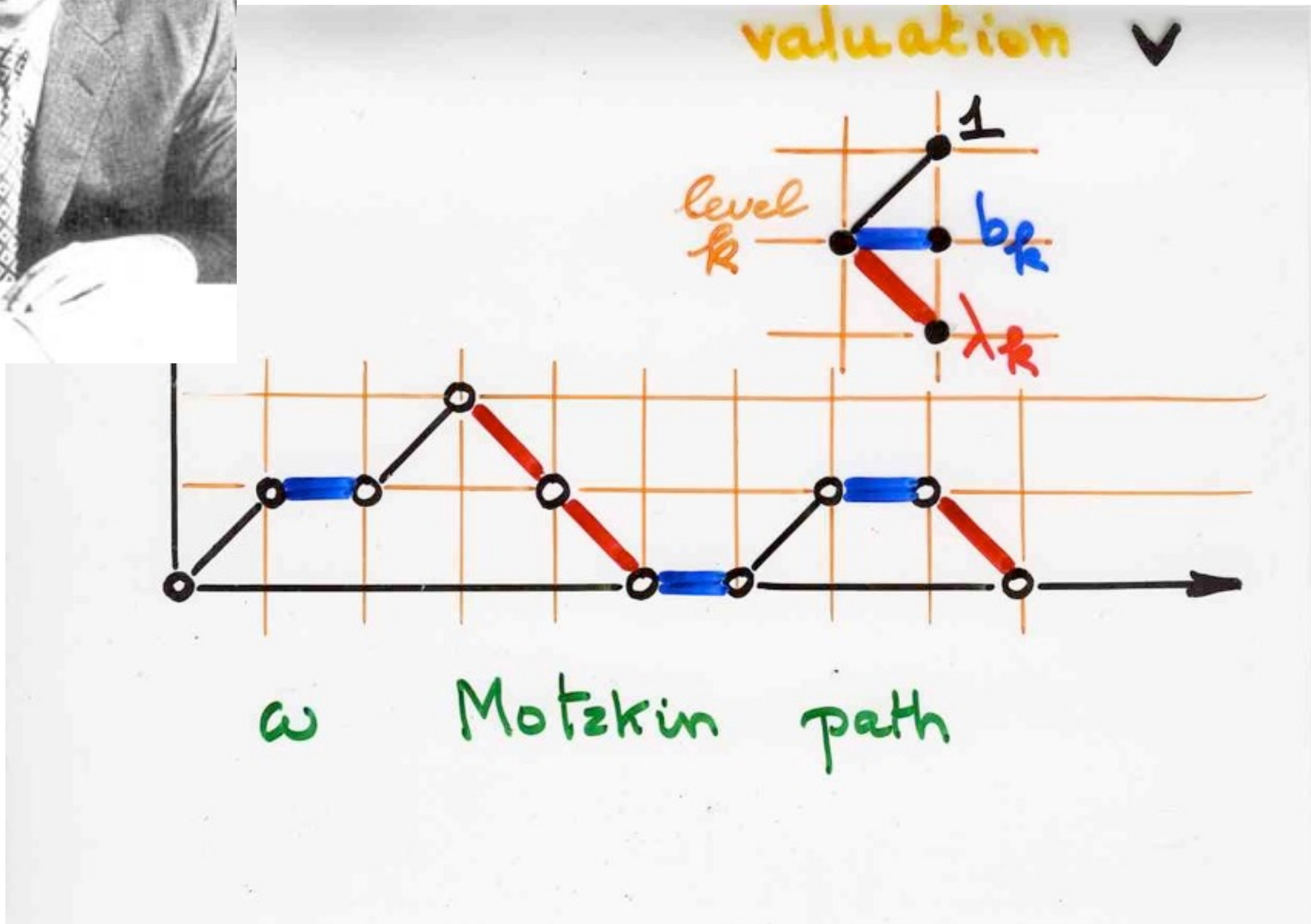
$(\forall k\geqslant 1)$

3 terms linear recurrence relation

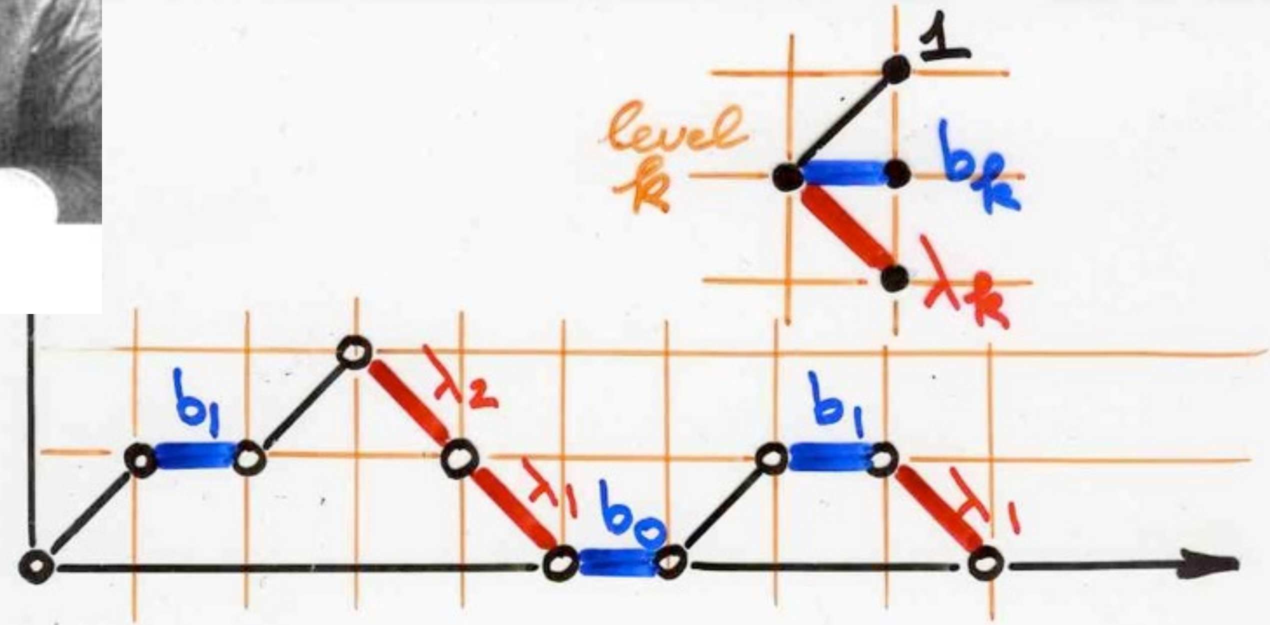
$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring

μ_n ?


valuation v
1
level k
b_k
λ_k
ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

Laguerre histories

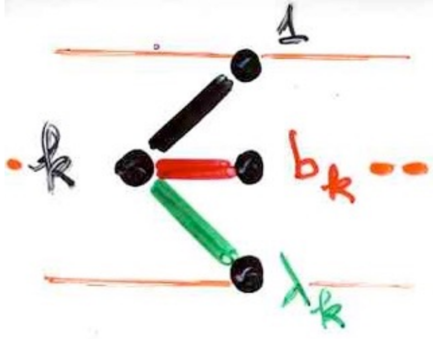
The FV bijection

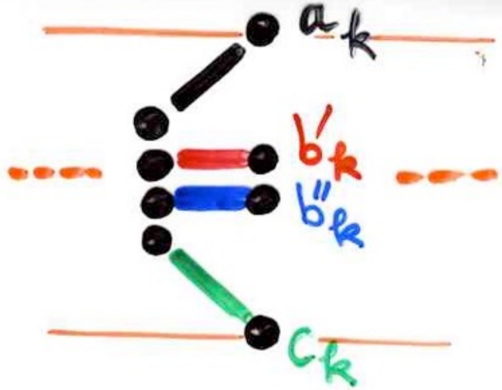
Françon, X.V. (1978)

Laguerre
polynomial

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$



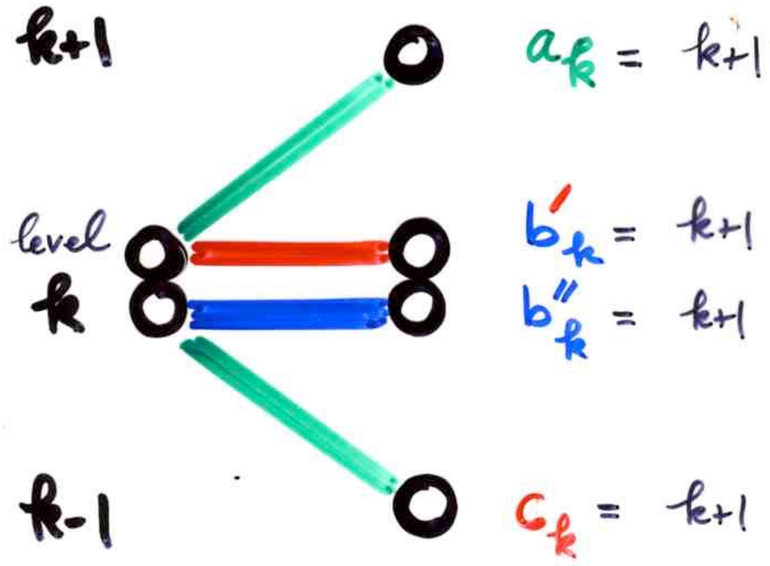


$$b_k = b'_k + b''_k$$

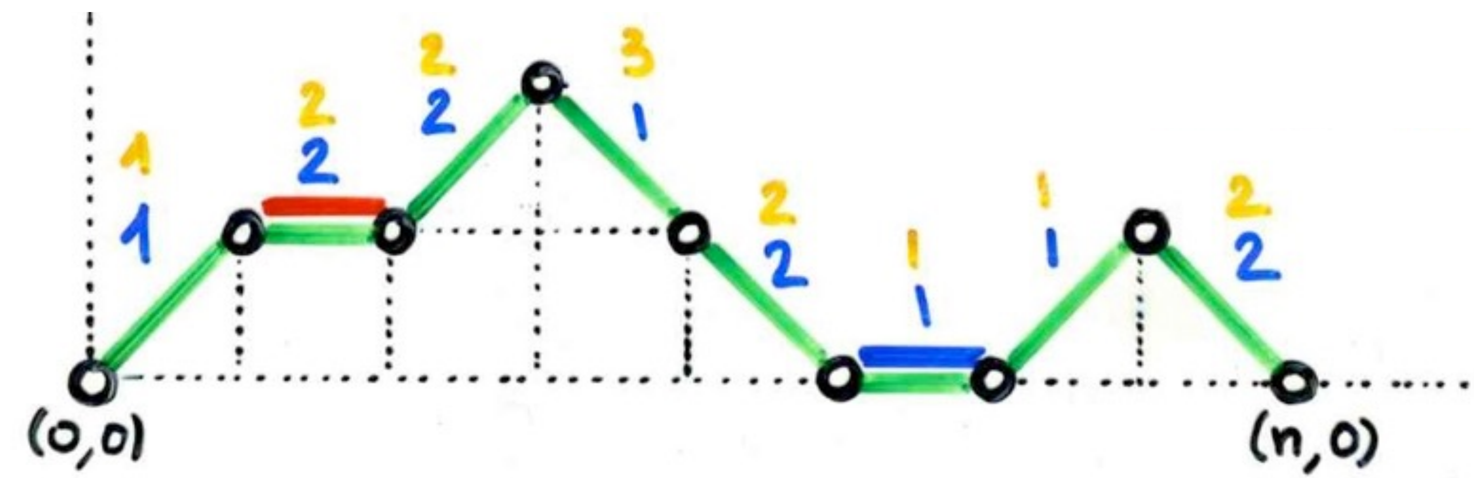
$$a_{k-1}\, c_k = \lambda_k$$

$$b_k = (2k+2)$$

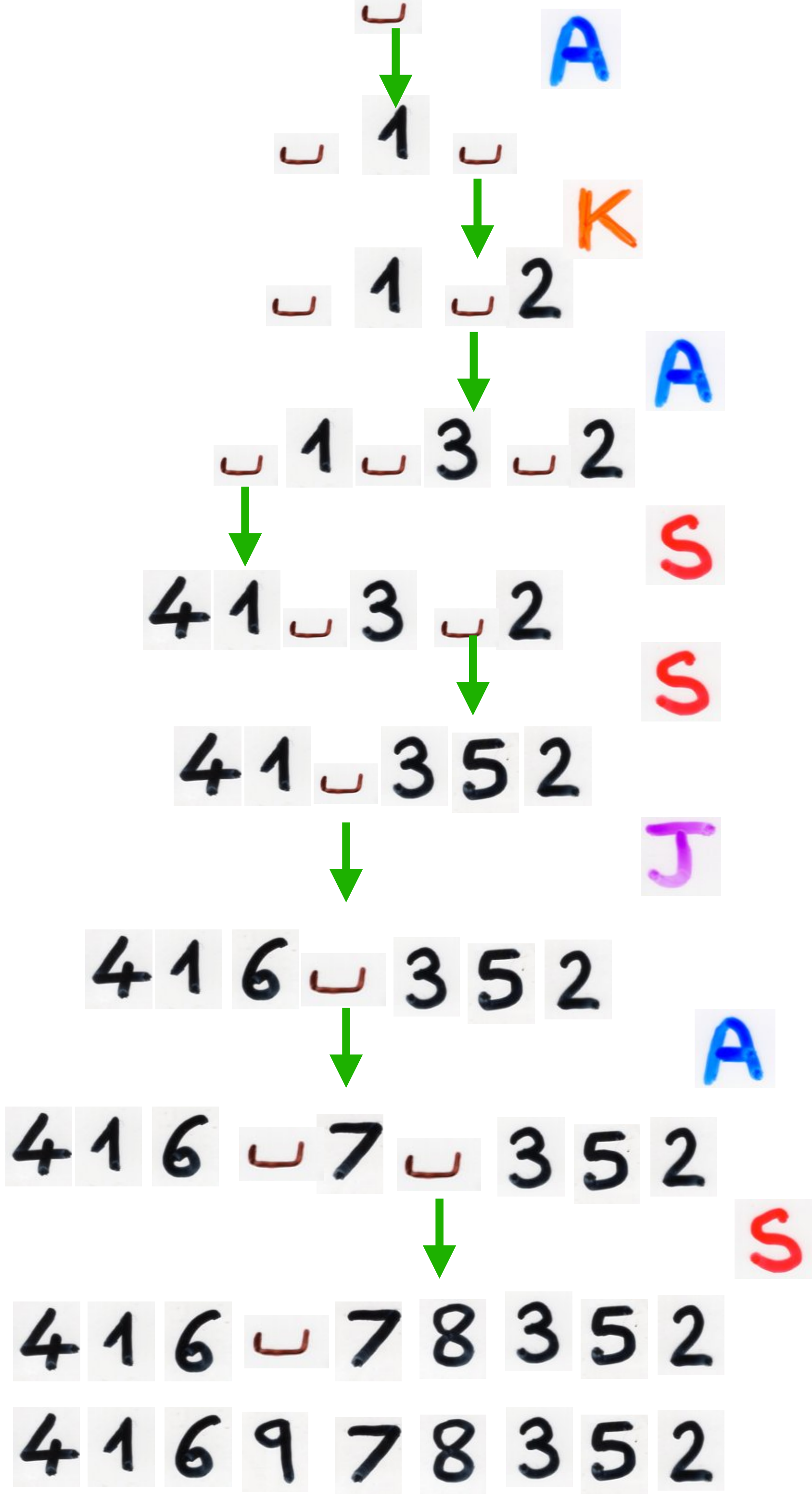
$$\lambda_k = k(k+1)$$



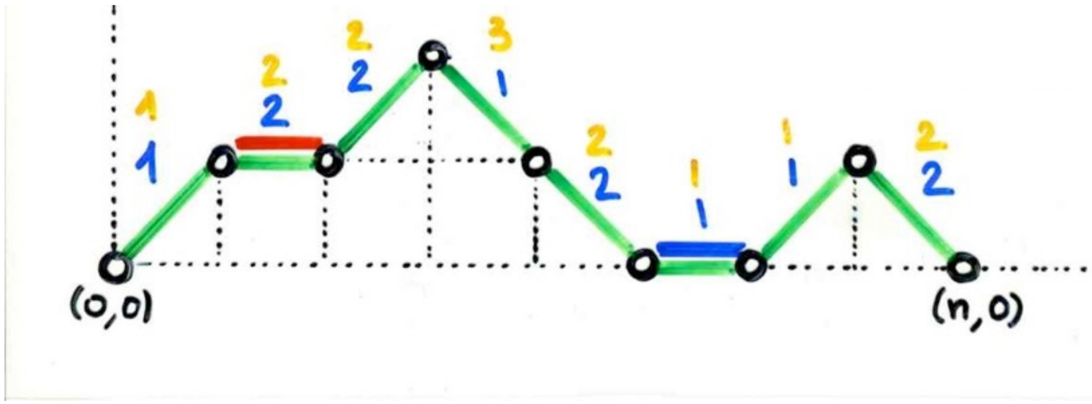
k+1
level
k
k-1
$a_k = k+1$
$b'_k = k+1$
$b''_k = k+1$
$c_k = k+1$



(0,0)
(n,0)



Laguerre histories



$$\langle k|\, A = (k+1)\, \langle (k+1)|$$

$$\langle k|\, K = (k+1)\, \langle k|$$

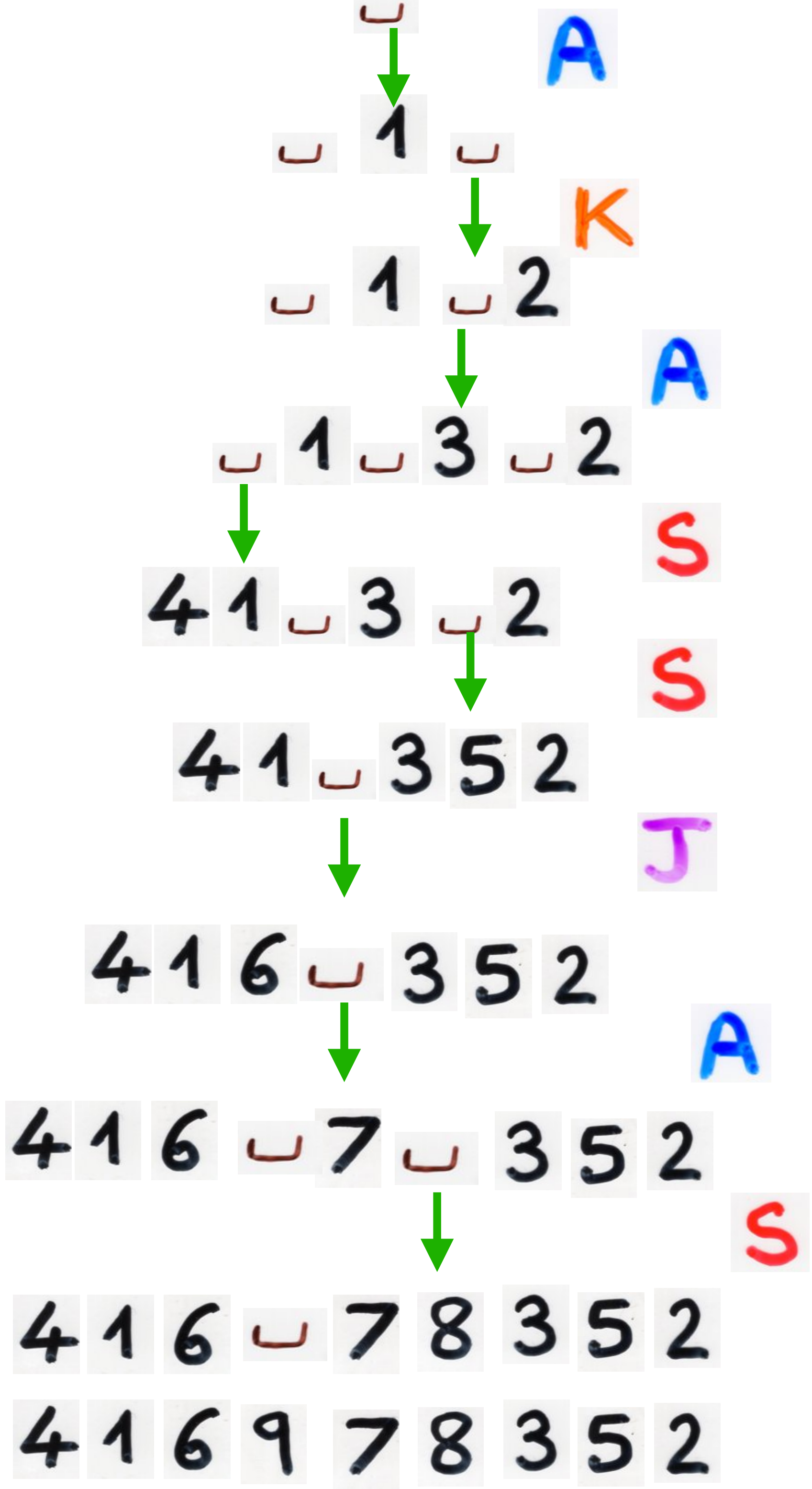
$$\langle k|\, J = (k+1)\, \langle k|$$

$$\langle k|\, S = (k+1)\, \langle (k-1)|$$

q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$



"q-analogue" of Laguerre histories

choice function

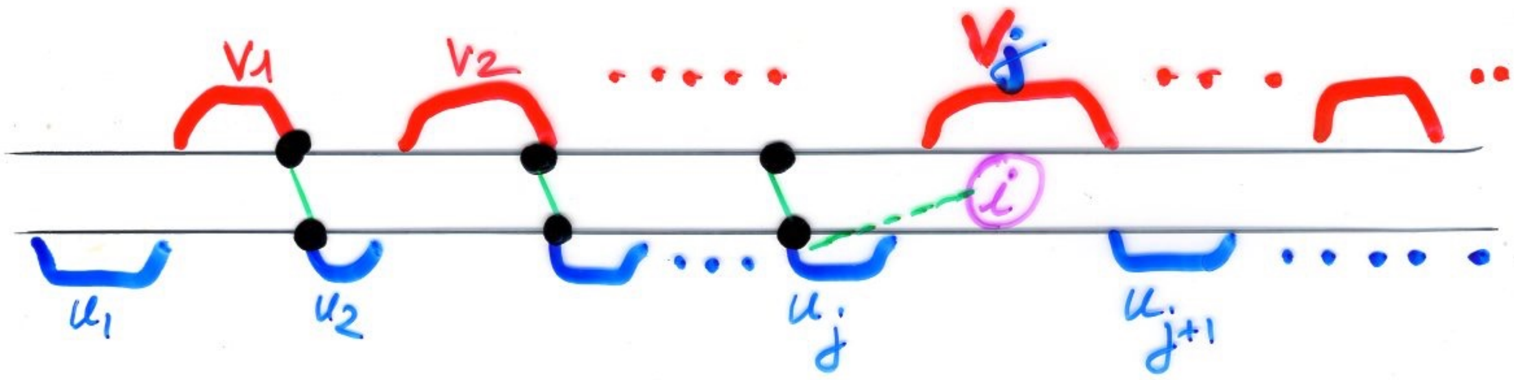
$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories $\quad q^4$

weighted q- Laguerre histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]}$$

this is also q^m where m is the number of subsequences (a,b,c) of σ having the pattern (31-2)



q-Laguerre
polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q-Laguerre
restricted
histories

$$\mu_n = n!$$

Bijection (restricted) Laguerre histories

(of the inverse permutation)

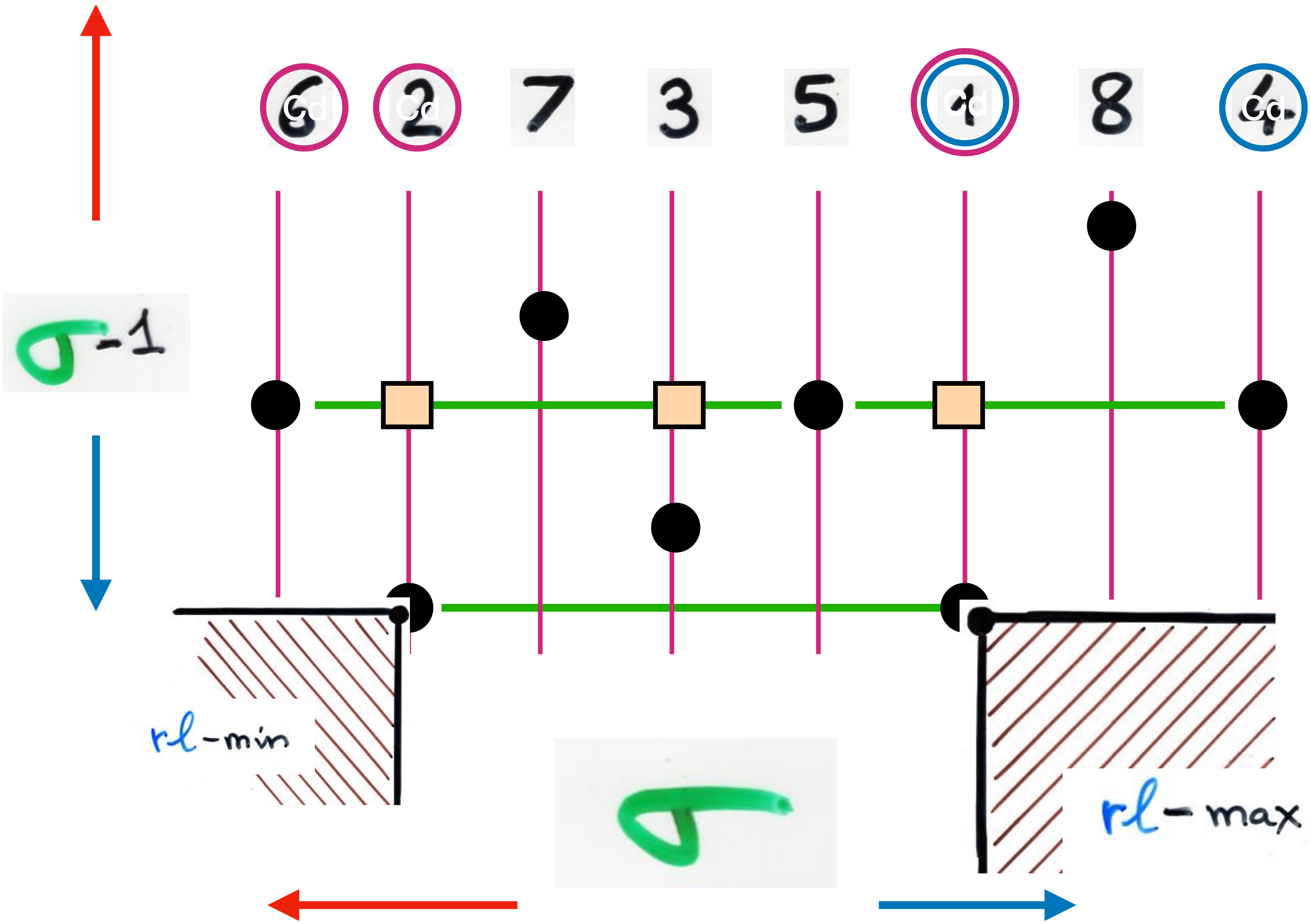
σ^{-1}

and Laguerre heaps of segments

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

Josuat-Vergès (2011)



Laguerre histories

$$\langle k|\, A = (k+1)\,\langle (k+1)|$$

$$\langle k|\, K = (k+1)\,\langle k|$$

$$\langle k|\, J = (k+1)\,\langle k|$$

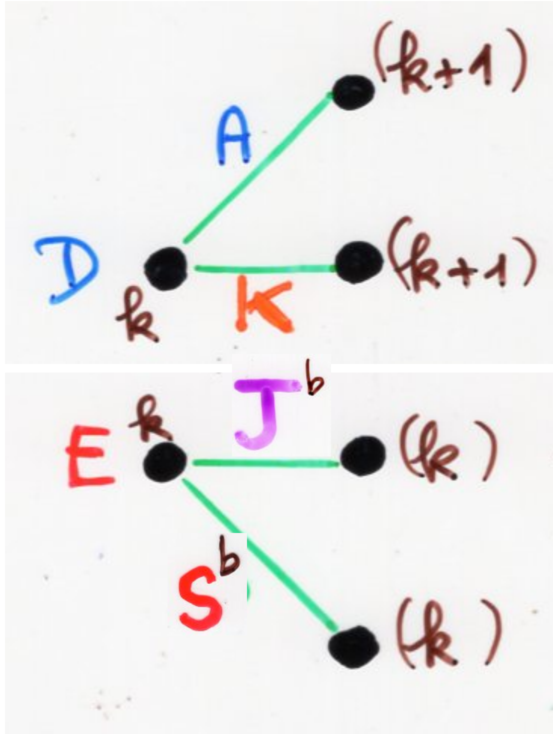
$$\langle k|\, S = (k+1)\,\langle (k-1)|$$

$$\mu_n = (n+1)!$$

$$b_k = (2k+2)$$

$$\lambda_k = k(k+1)$$

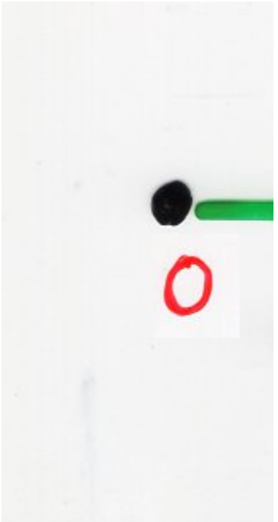
restricted Laguerre histories



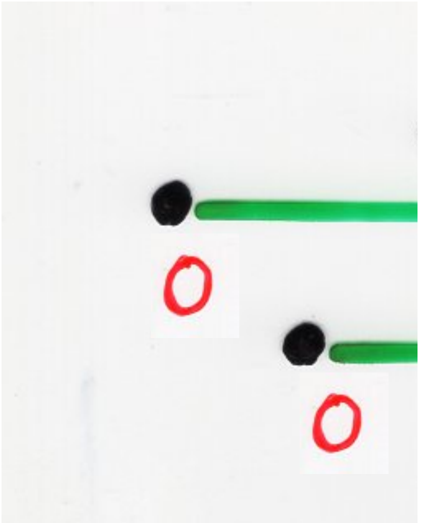
$$\mu_n = n!$$

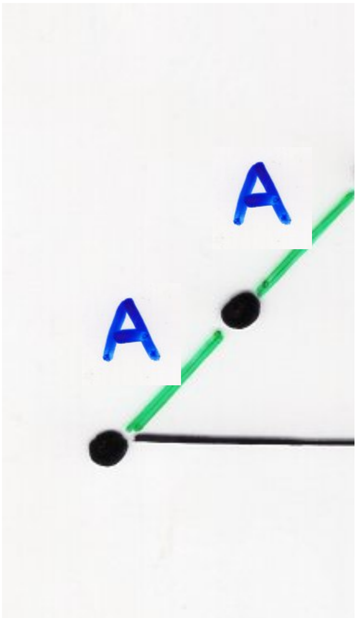
$$b_k = (2k+1)$$

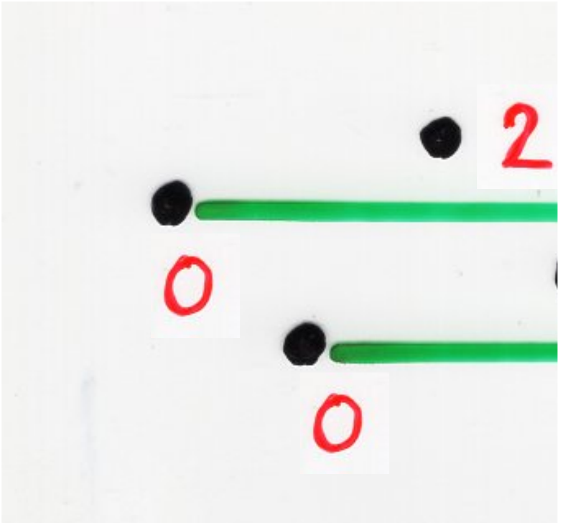
$$\lambda_k = k^2$$

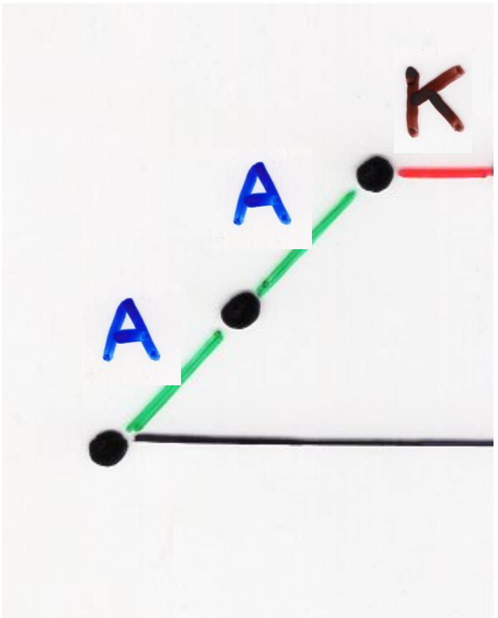

O

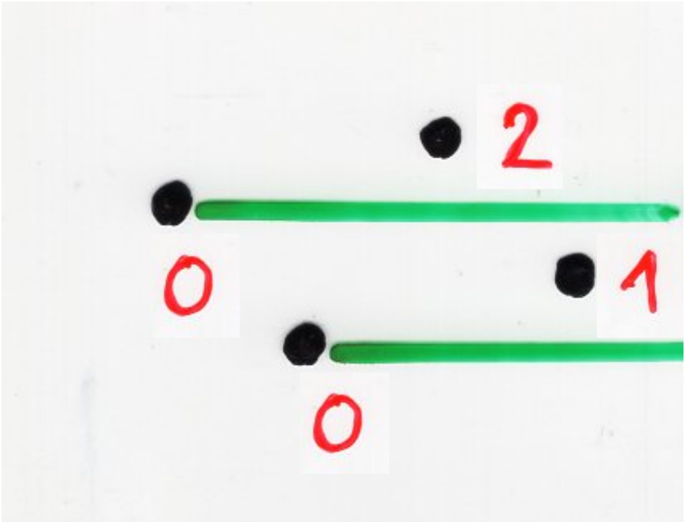

A

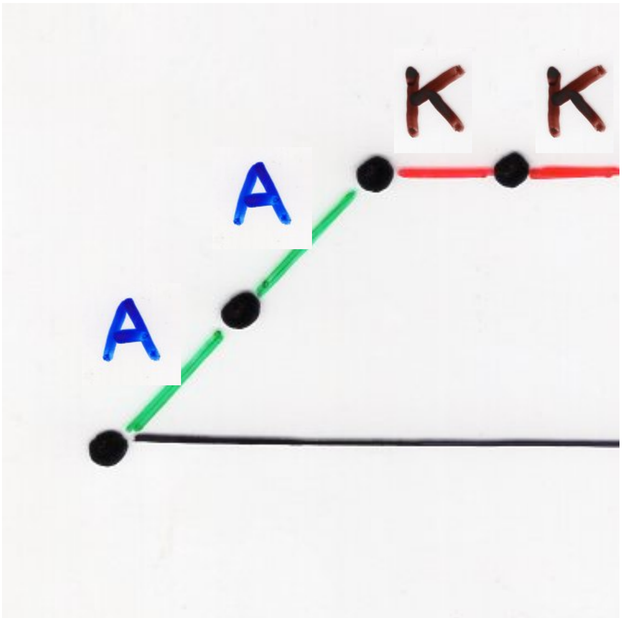

O
O

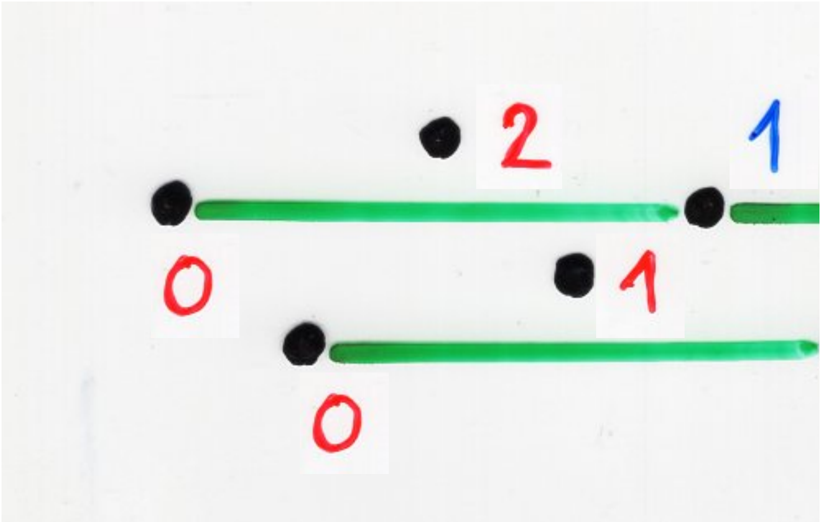

A
A

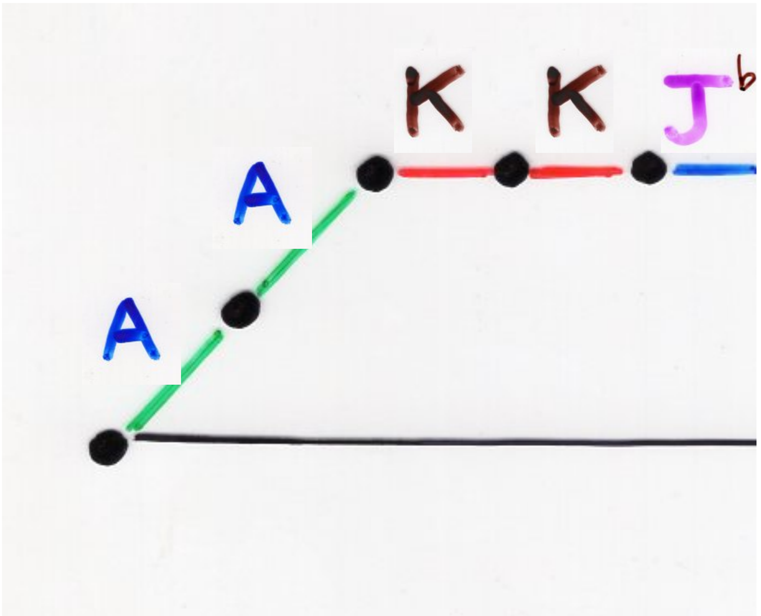

2
0
0

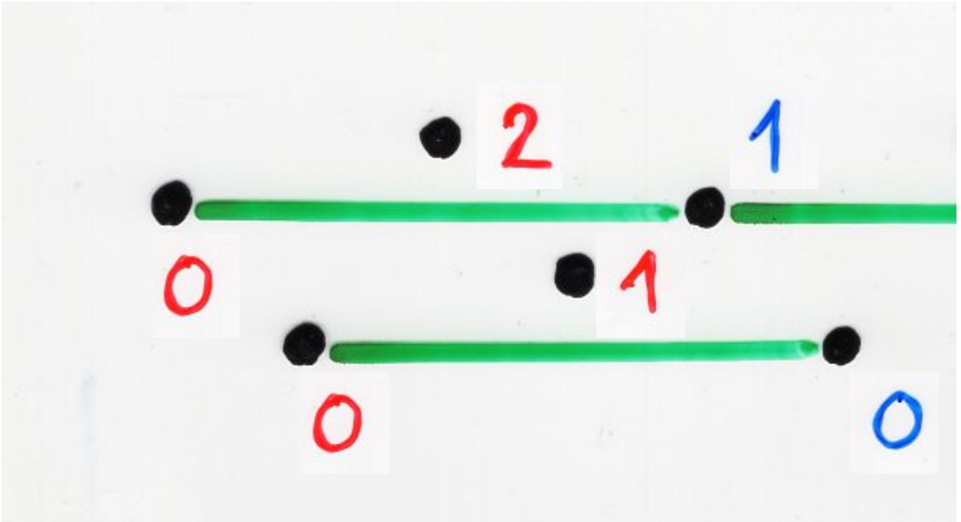

K
A
A

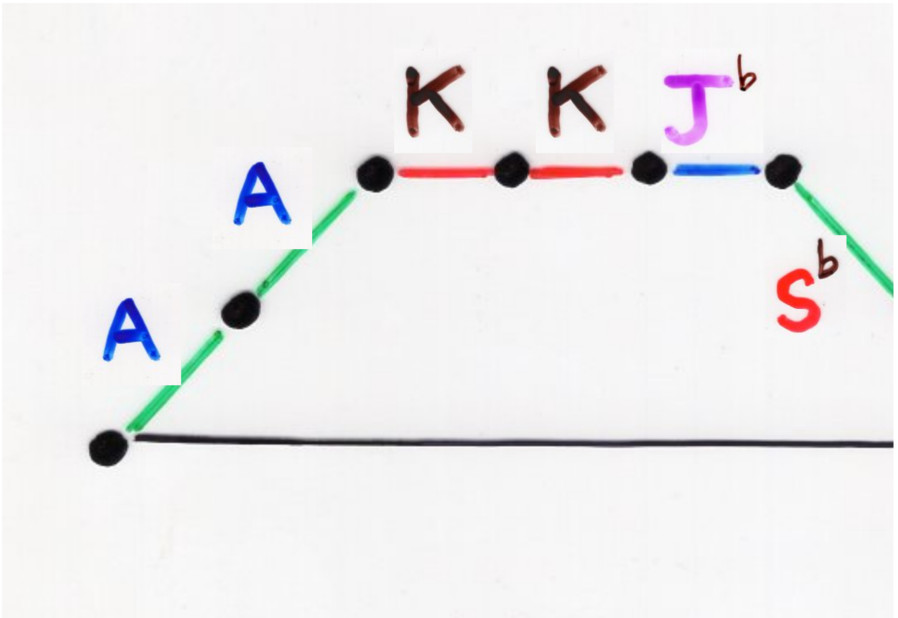

2
0
1
0

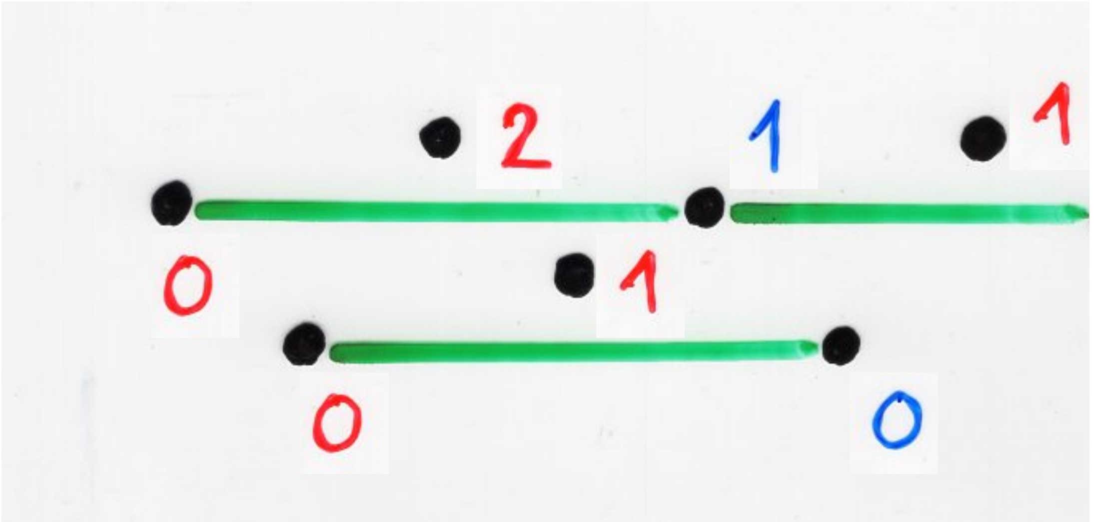

K
K
A
A


2
1
0
1
0

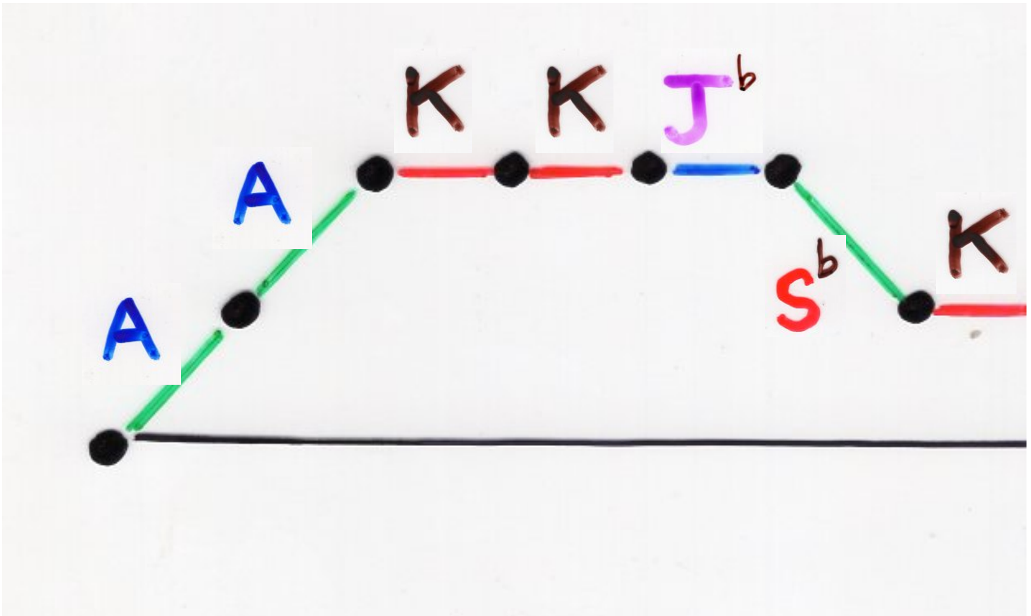

K
K
J
b
A
A


2
1
0
1
0
0

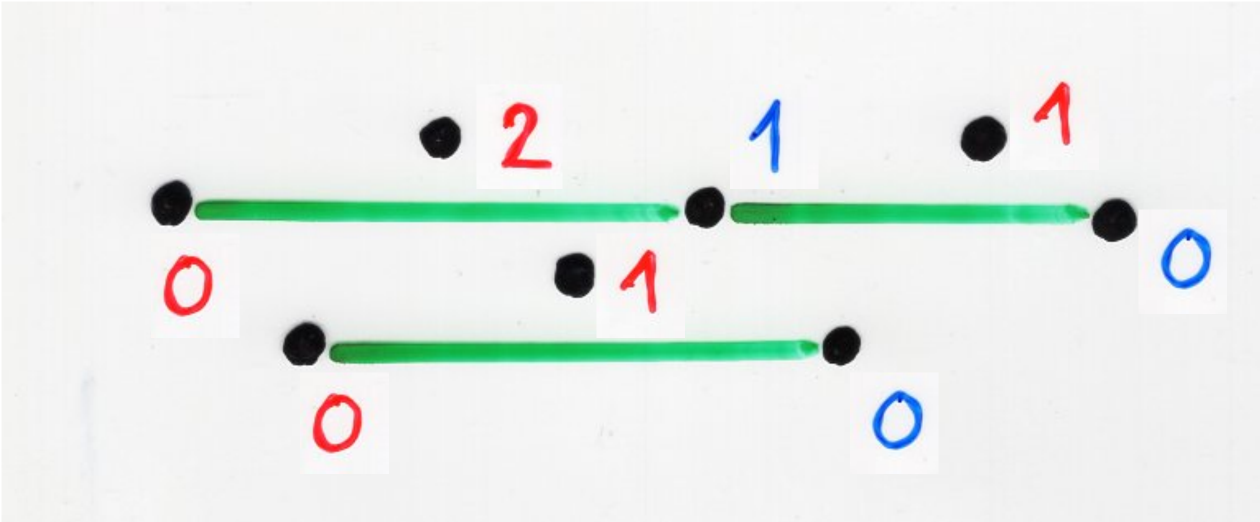

K
K
J
b
A
A
S
b

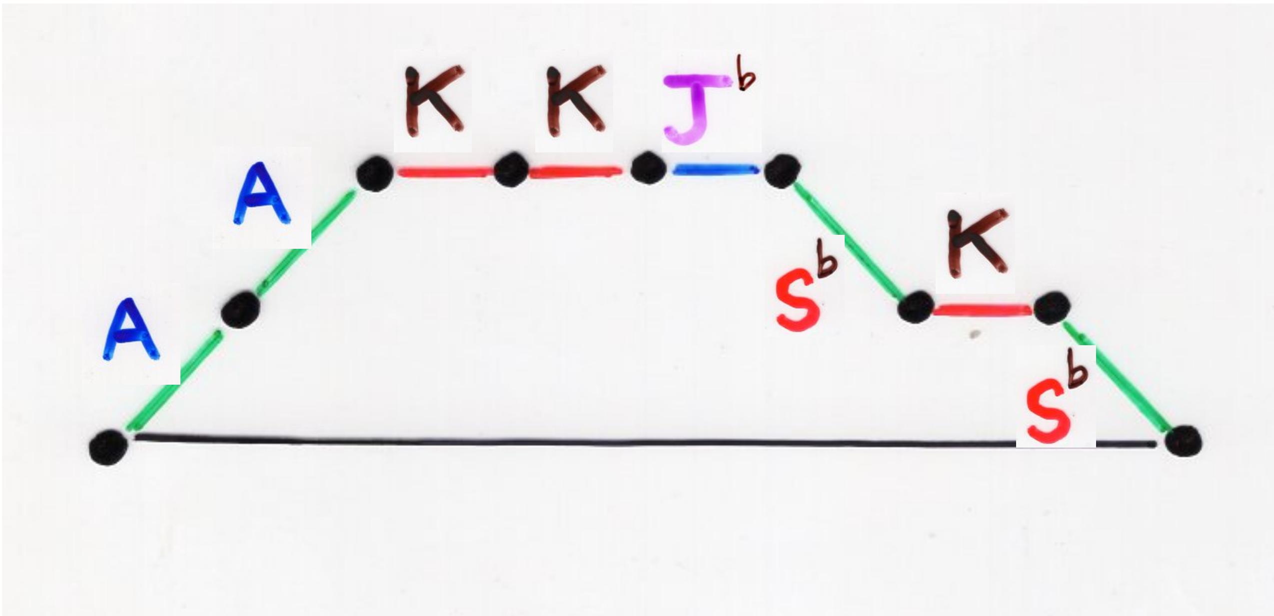


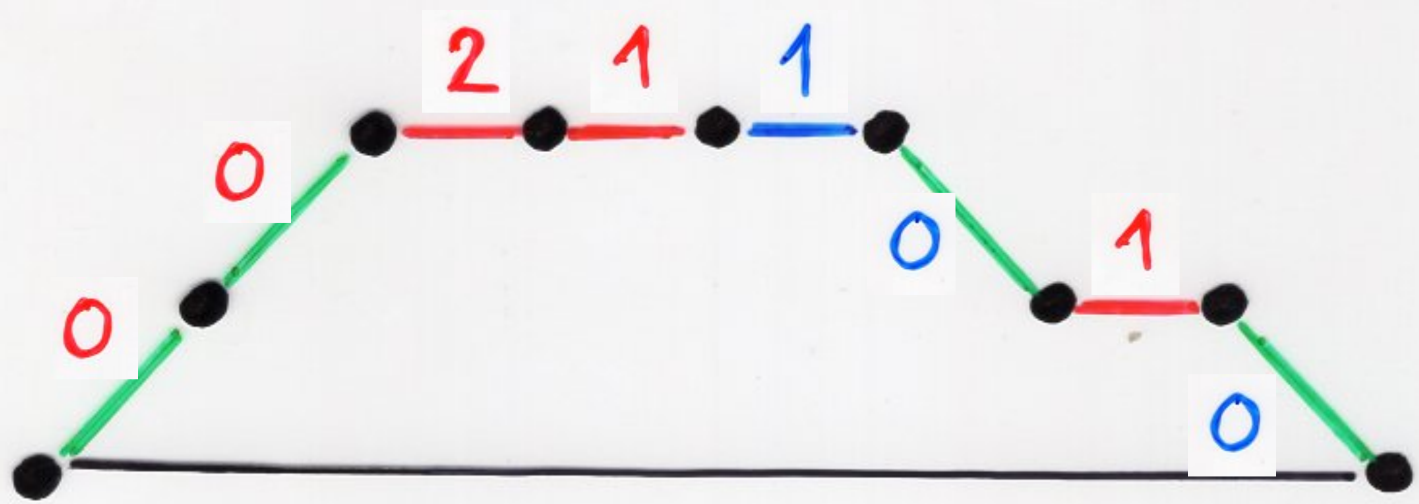
2
1
1
0
1
0
0



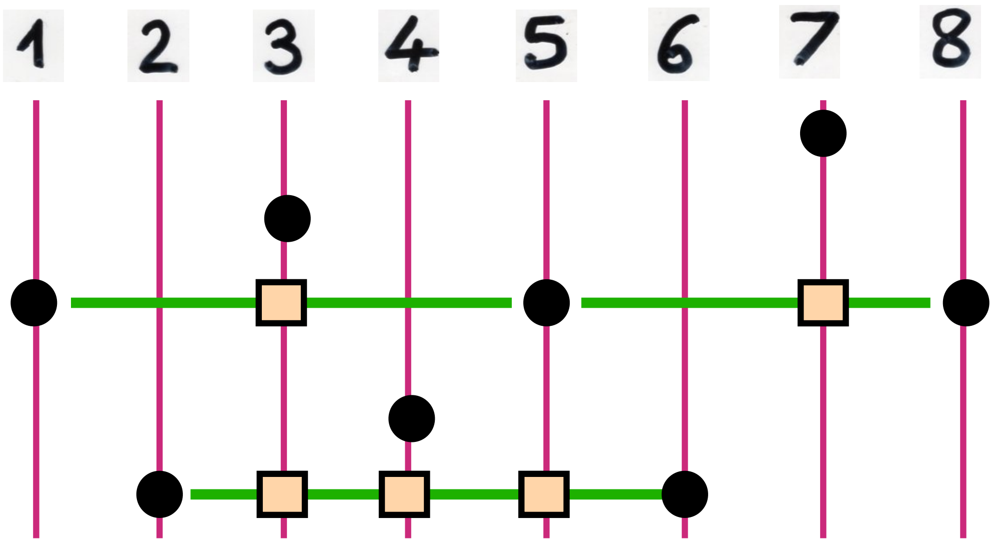
K
K
J b
A
S b
K
A

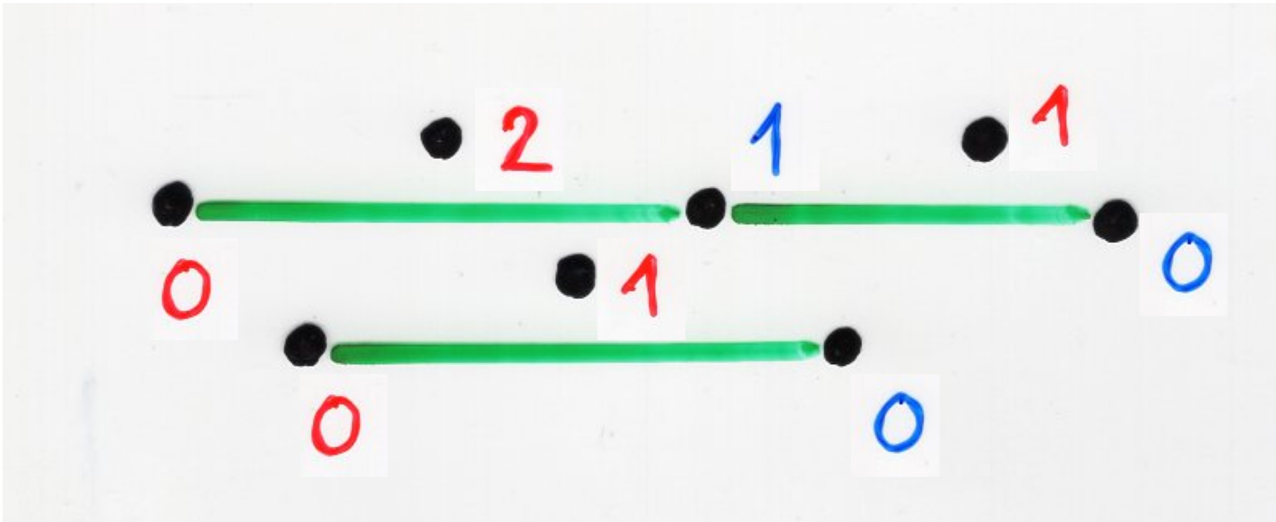

2
1
1
0
1
0
0
0

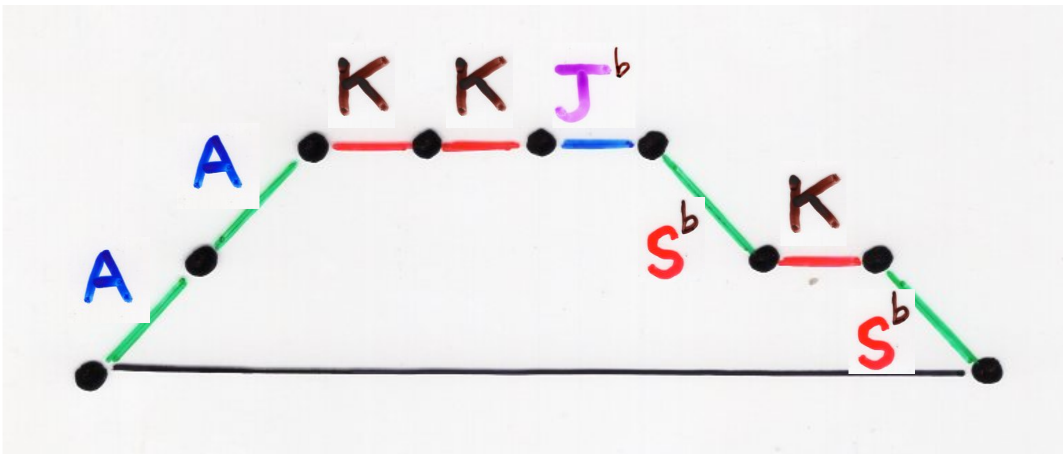

K
K
J^b
A
A
S^b
K
S^b



0
0
2
1
1
0
1
0


1
2
3
4
5
6
7
8


2
1
1
0
1
0
0
0


K
K
J^b
A
A
S^b
K
S^b

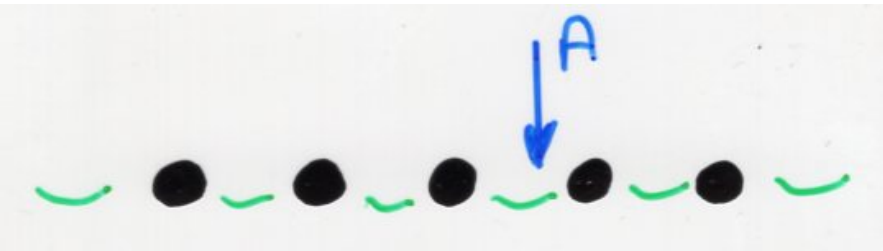
dictionnary data structure

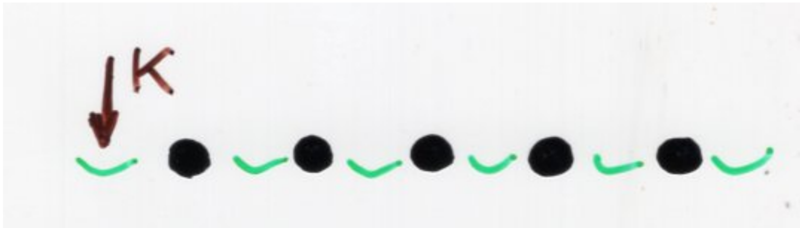
add or delete any element

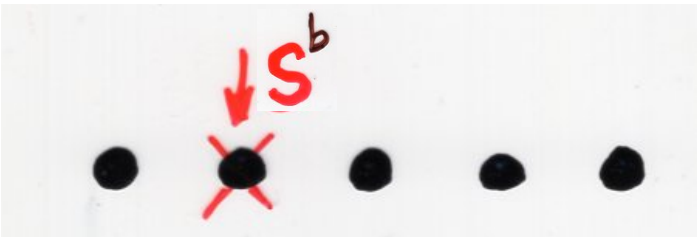
ask questions

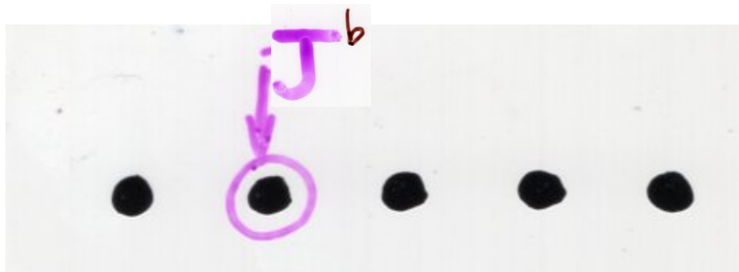
J^b positive

K negative







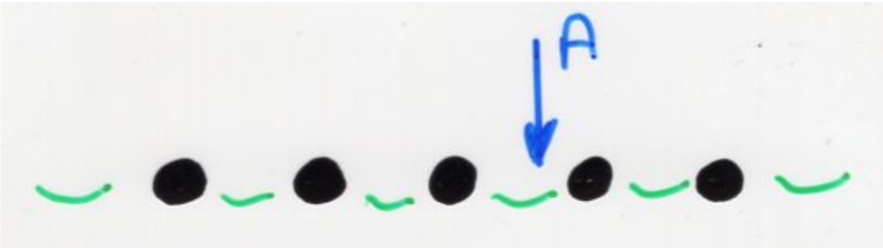


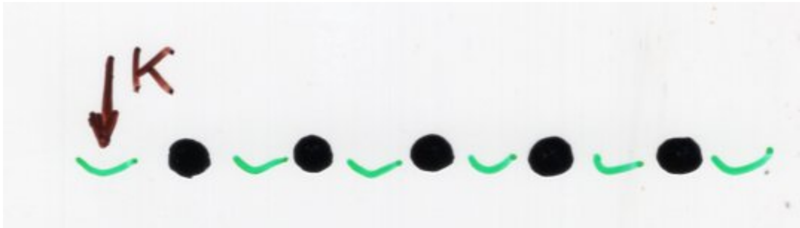
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

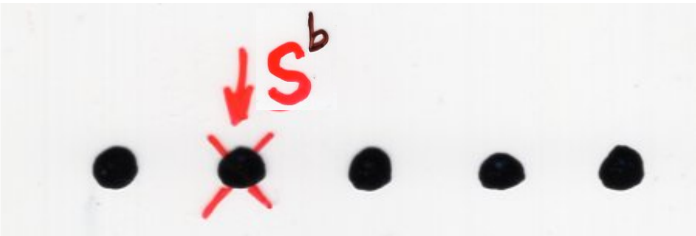
$$J^b\,|k\rangle = k\,|k\rangle$$

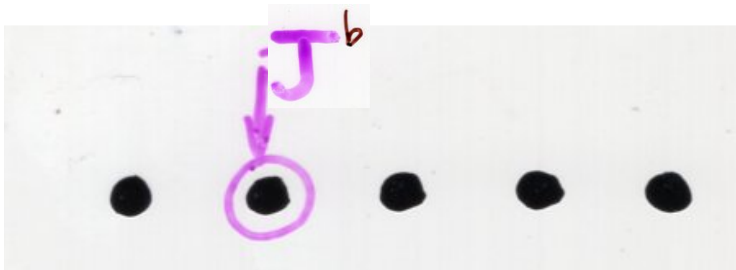
$$K\,|k\rangle = (k+1)\,|k\rangle$$

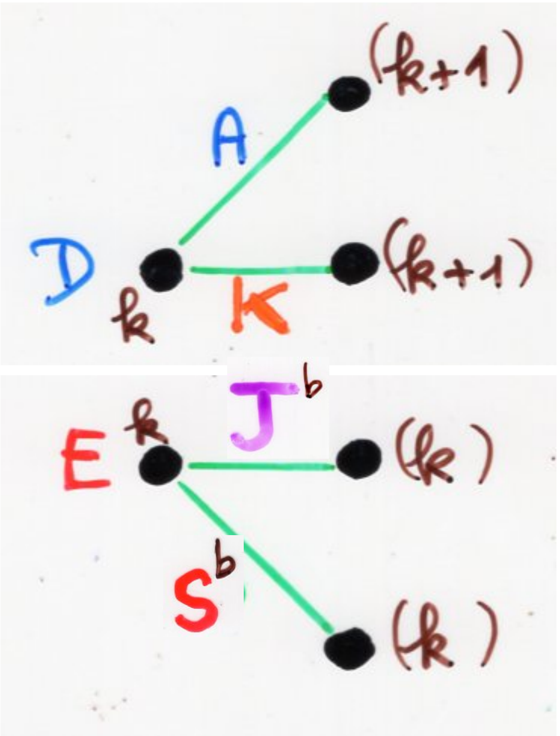
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$











D, E "restricted"

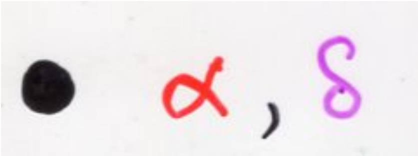
$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

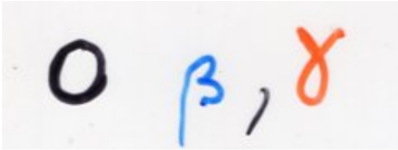
$$DE = ED + E + D$$

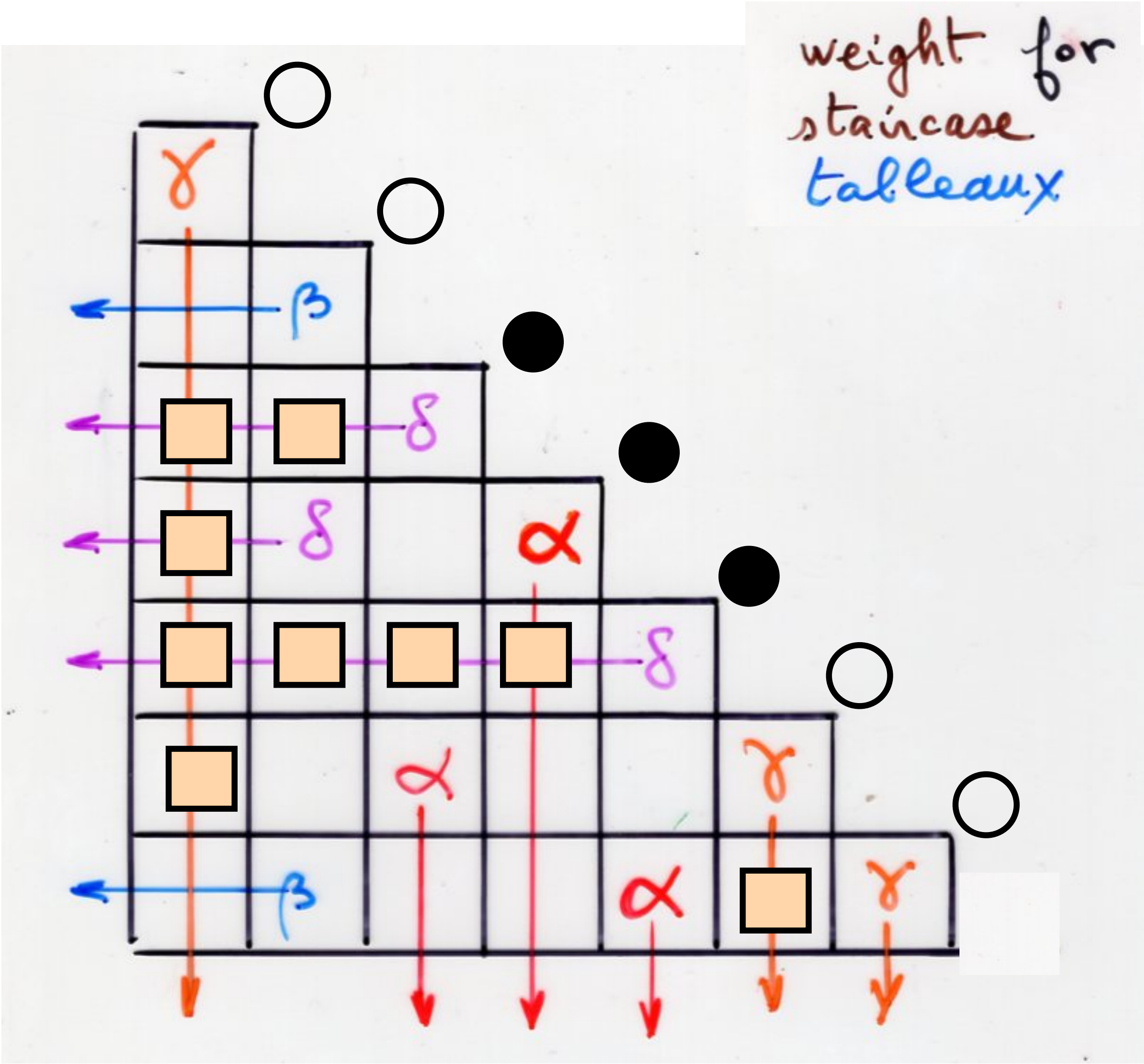
with the « cellular ansatz »:

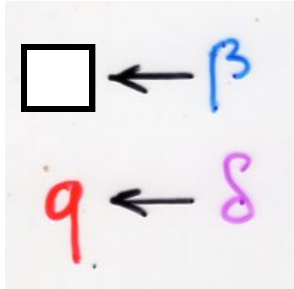
bijection Laguerre heaps — alternative tableaux

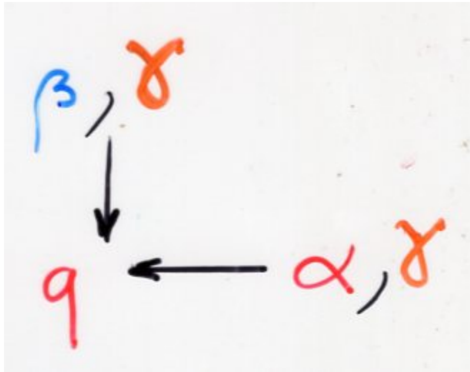
Why to insist on the 3 parameters model ?

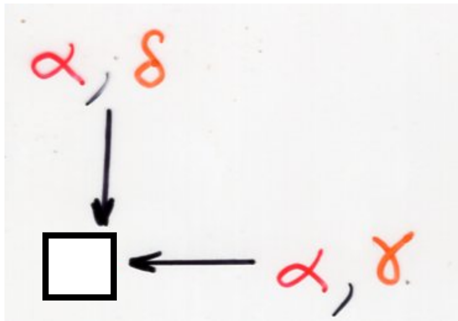

α , δ


β , γ


weight for
staircase
tableaux
γ
β
δ
δ
α
δ
α
γ
β
α
γ


β
q
δ


β , γ
q
α , γ


α , δ
α , γ

$\tau = (\tau_1, \dots, \tau_n)$

$$Z_\tau = \sum_{T} v(T)$$

staircase tableaux size n

profile of T

S. Corteel, L. Williams (2009)

$$Z_n(\alpha, \beta, \gamma, \delta; q) = \sum_{T} v(T)$$

partition function

staircase tableaux size n

→ expression for the moments of the Askey-Wilson polynomials

S. Corteel, L. Williams

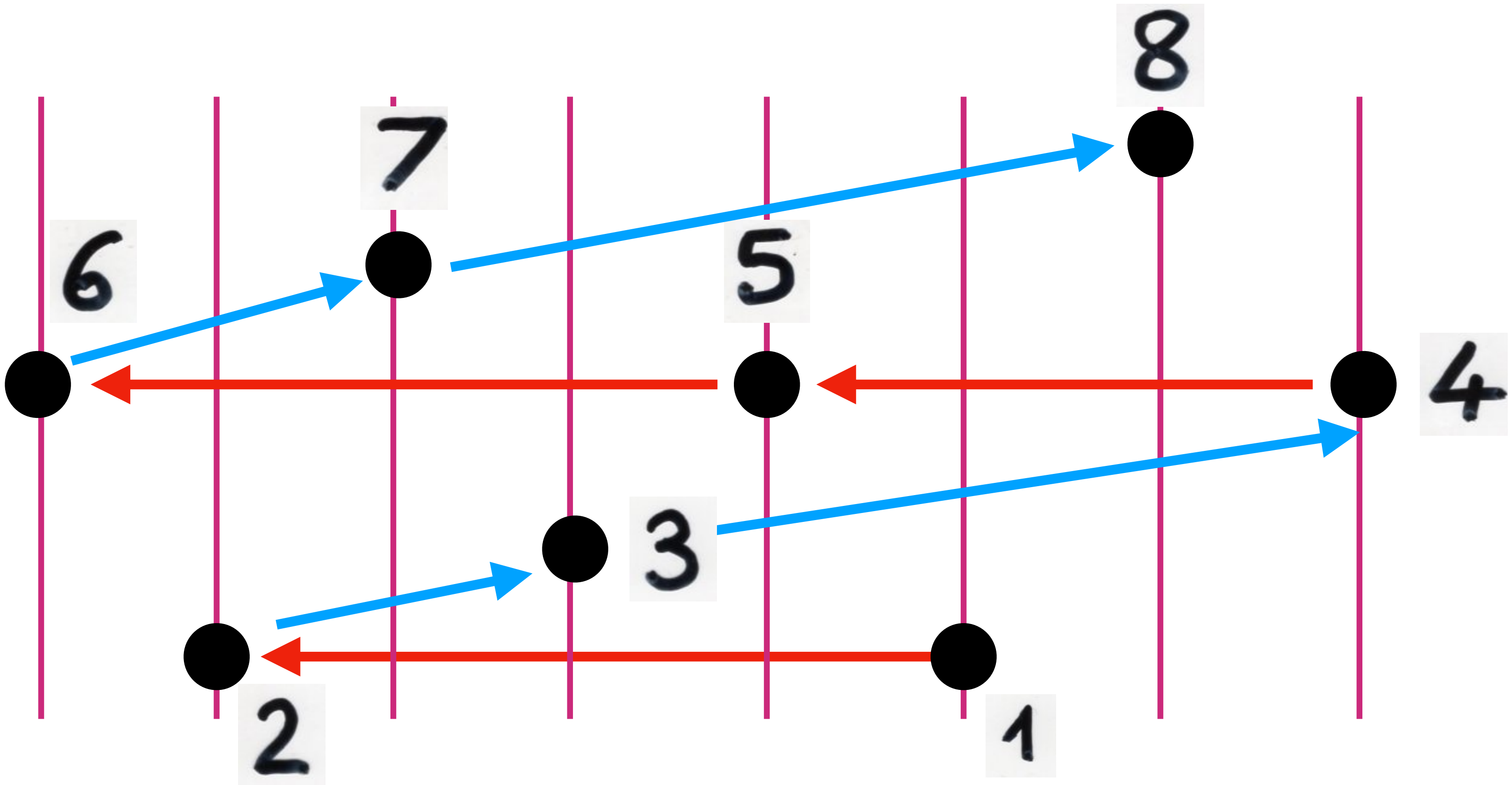
R. Stanley, D. Stanton

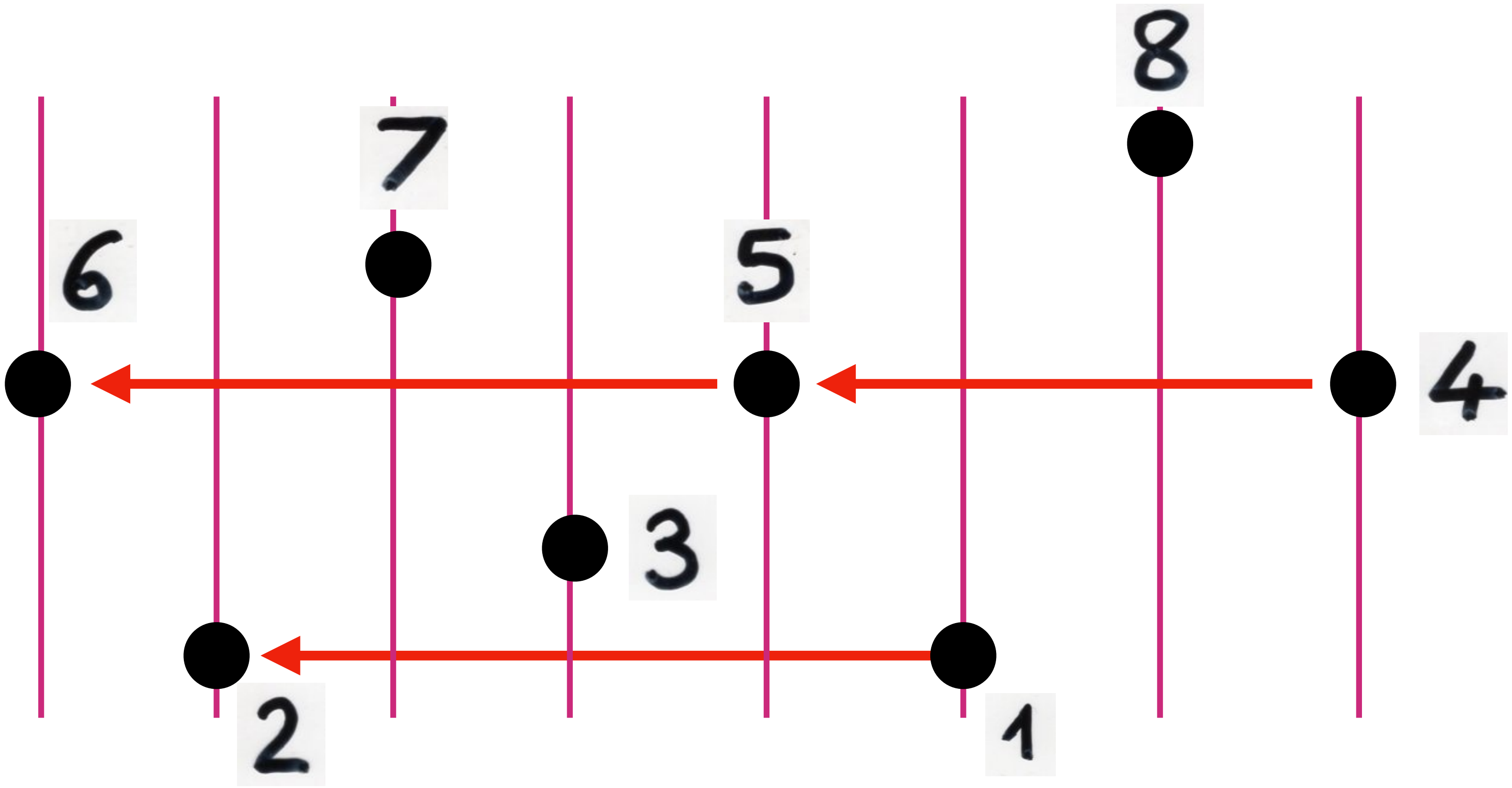
(2010)

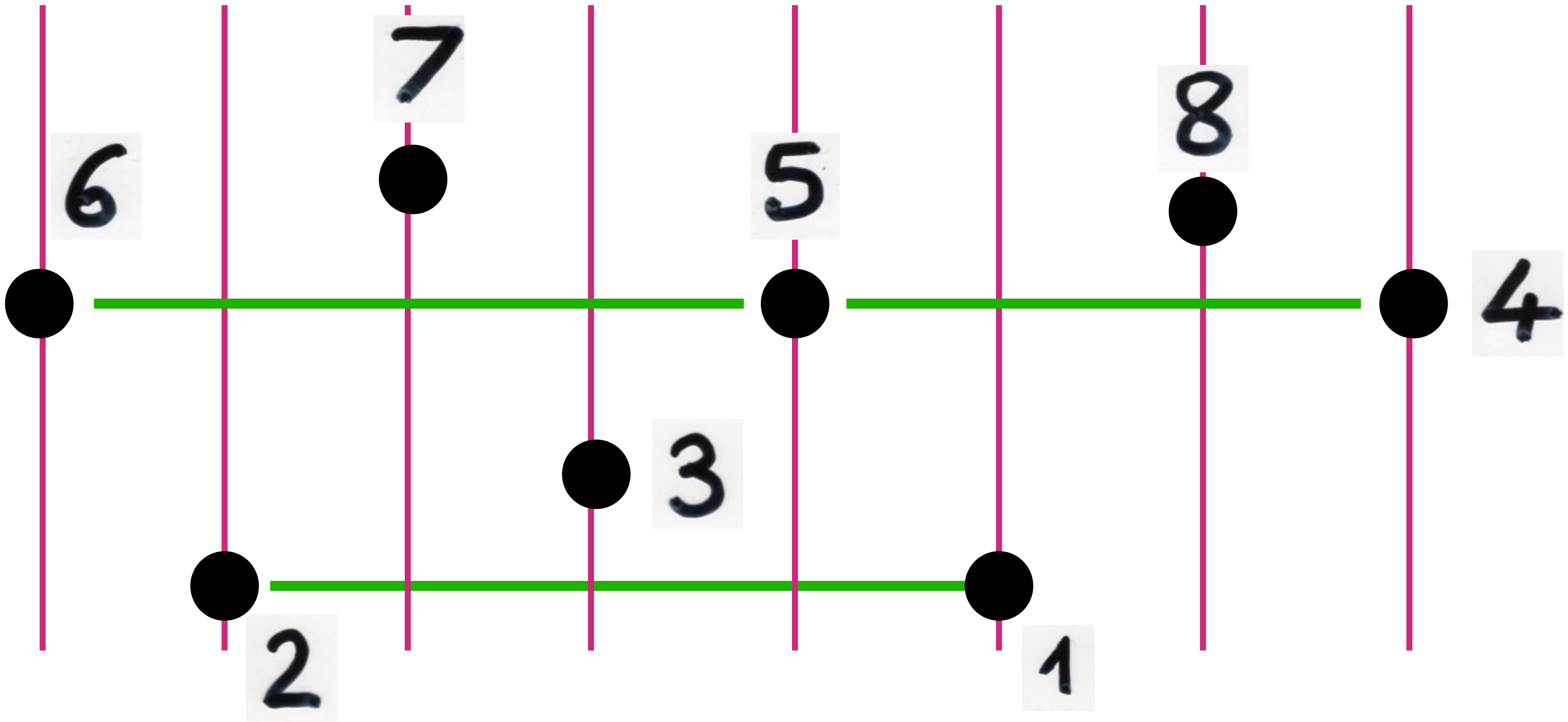
Why to insist
on the 3 parameters model ?

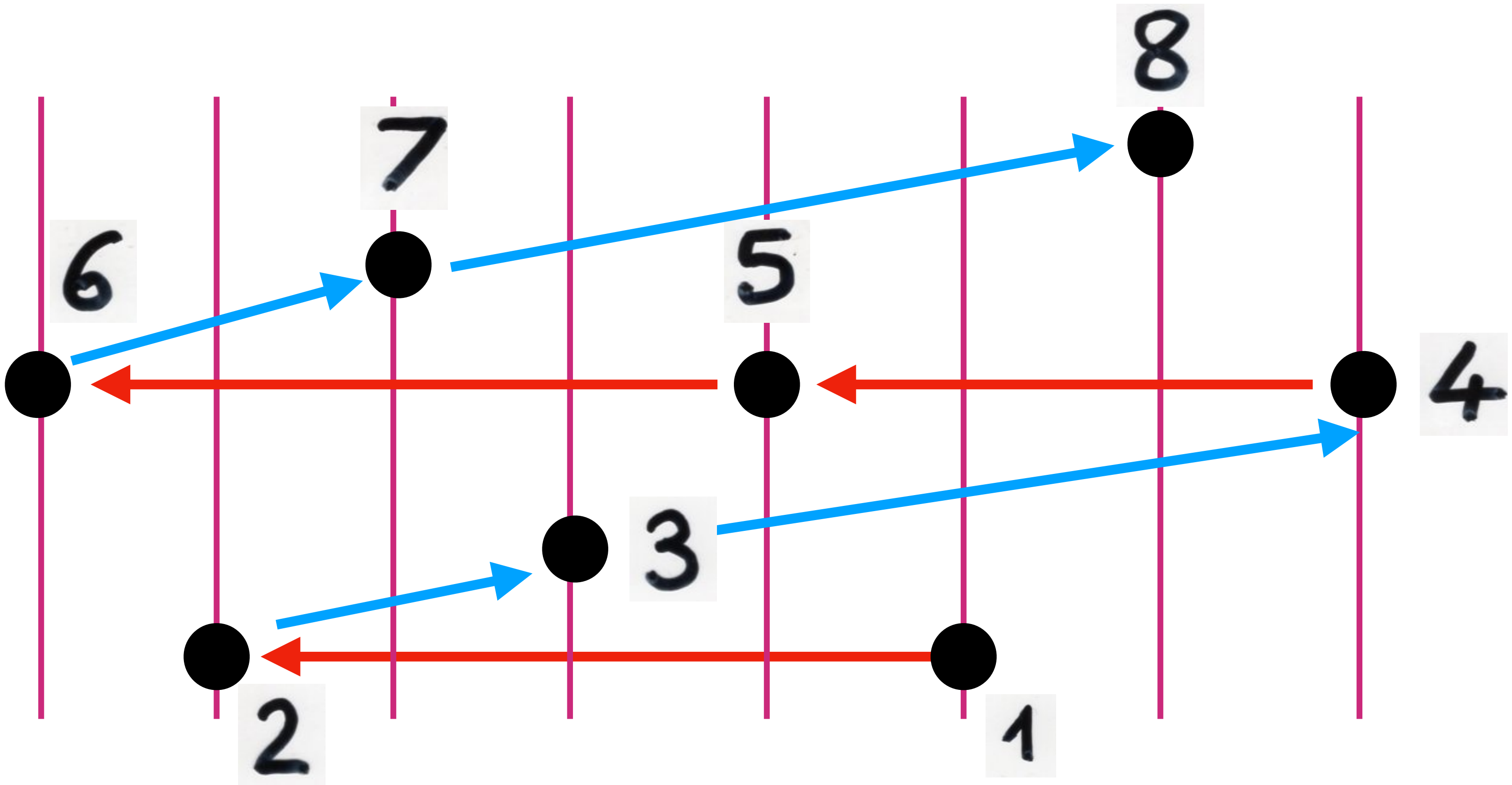
symmetries !

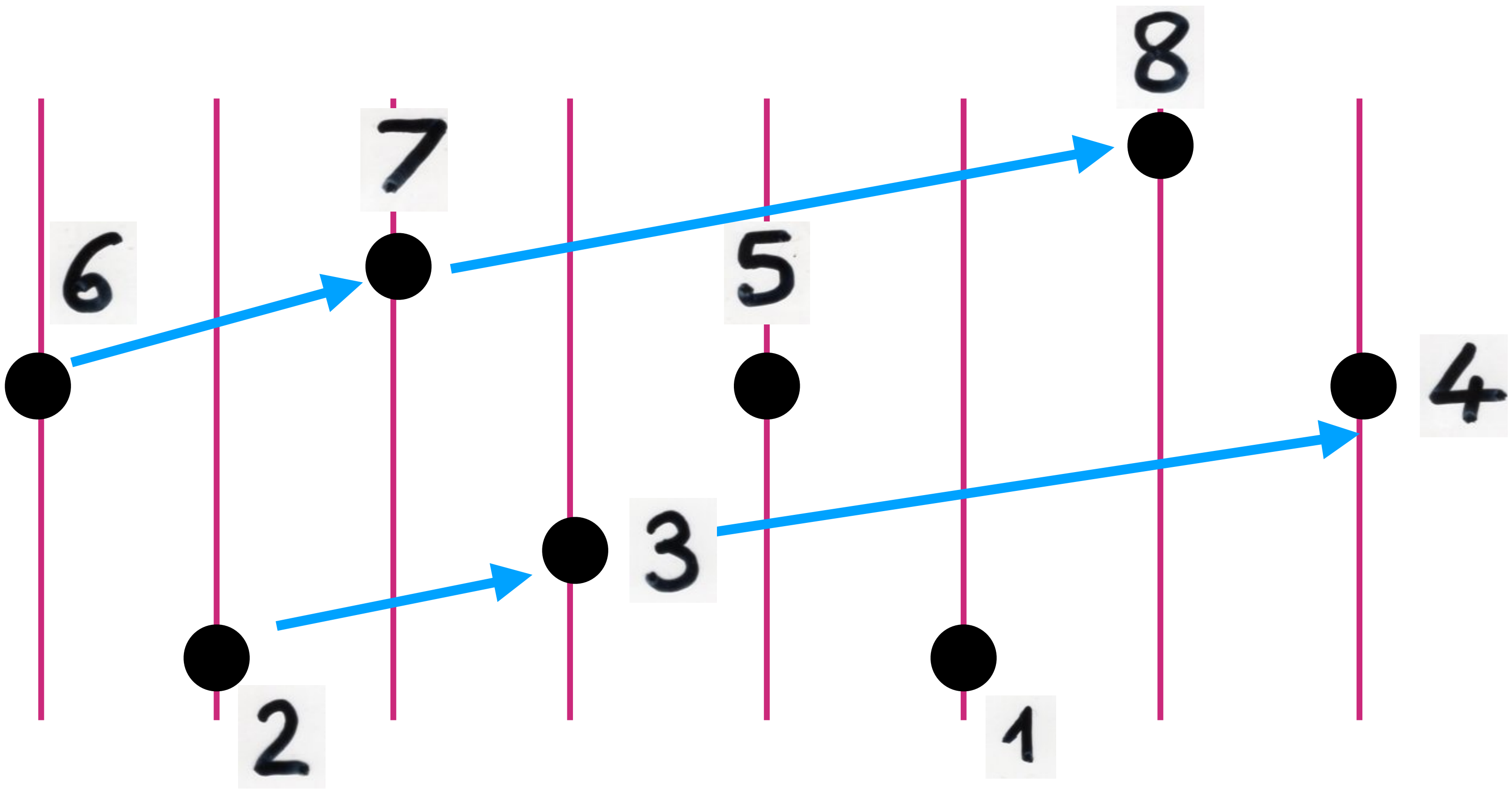
my dream ...


8
7
6
5
4
3
2
1

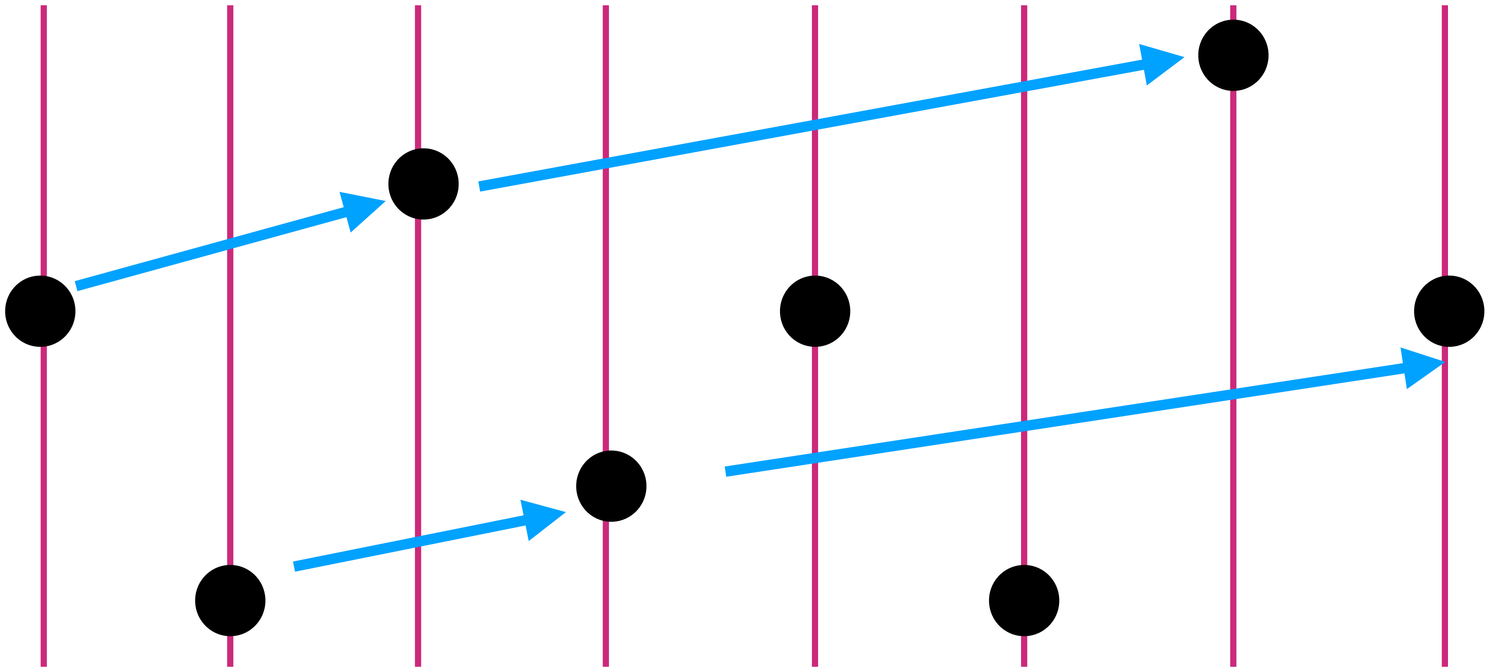

8
7
6
5
4
3
2
1

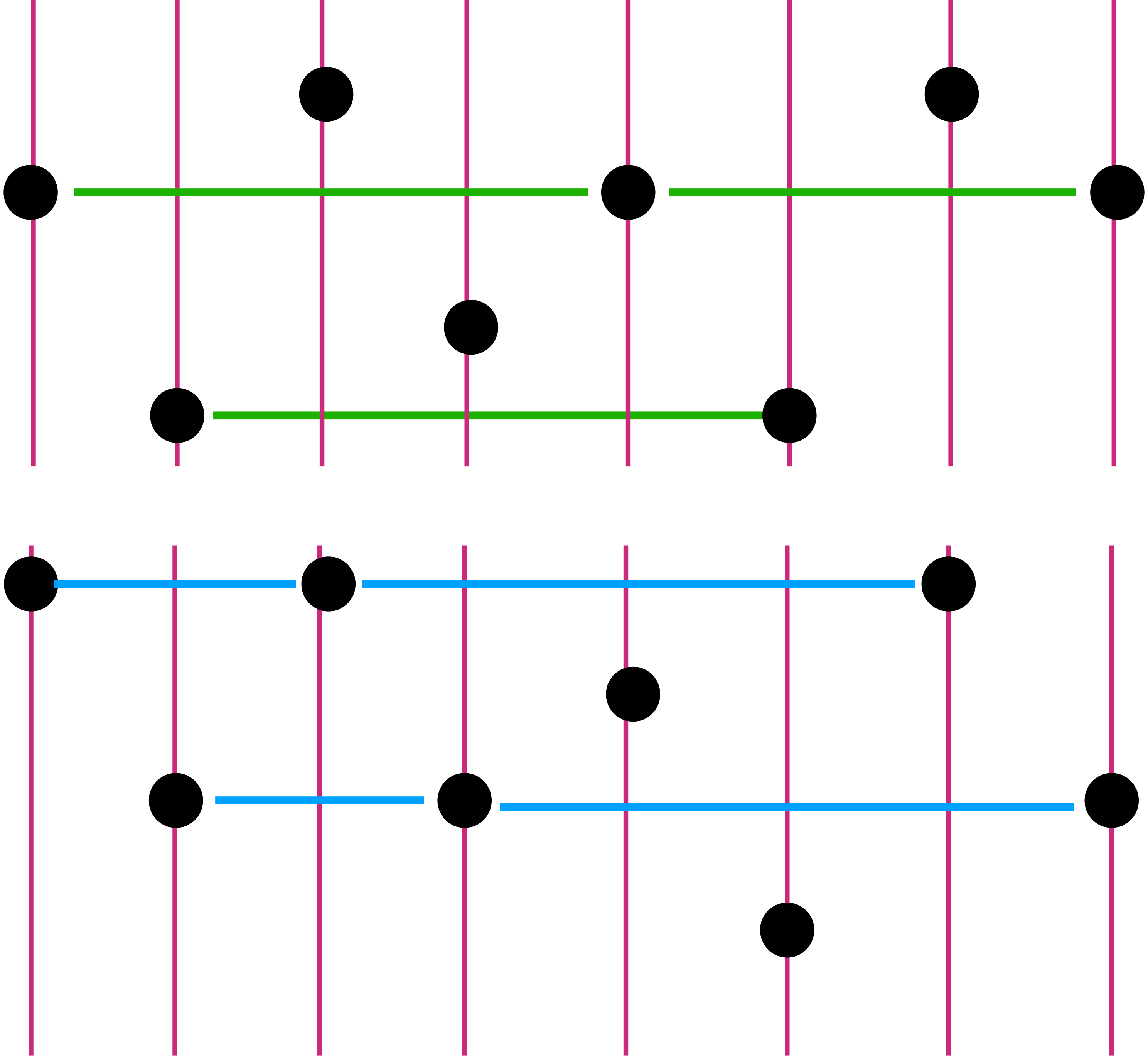

7
6
5
8
4
3
2
1


8
7
6
5
4
3
2
1



8
7
6
5
4
3
2
1



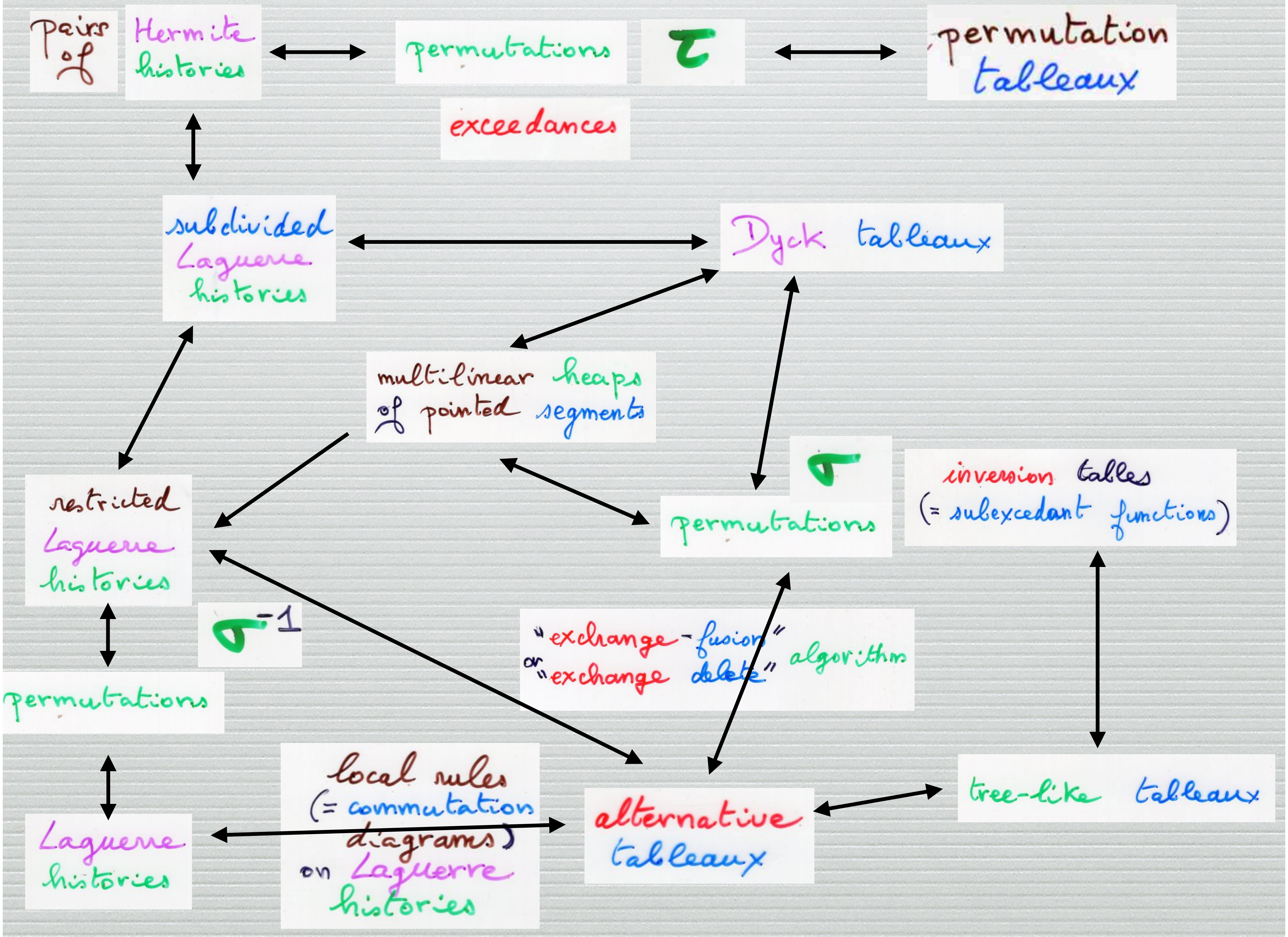


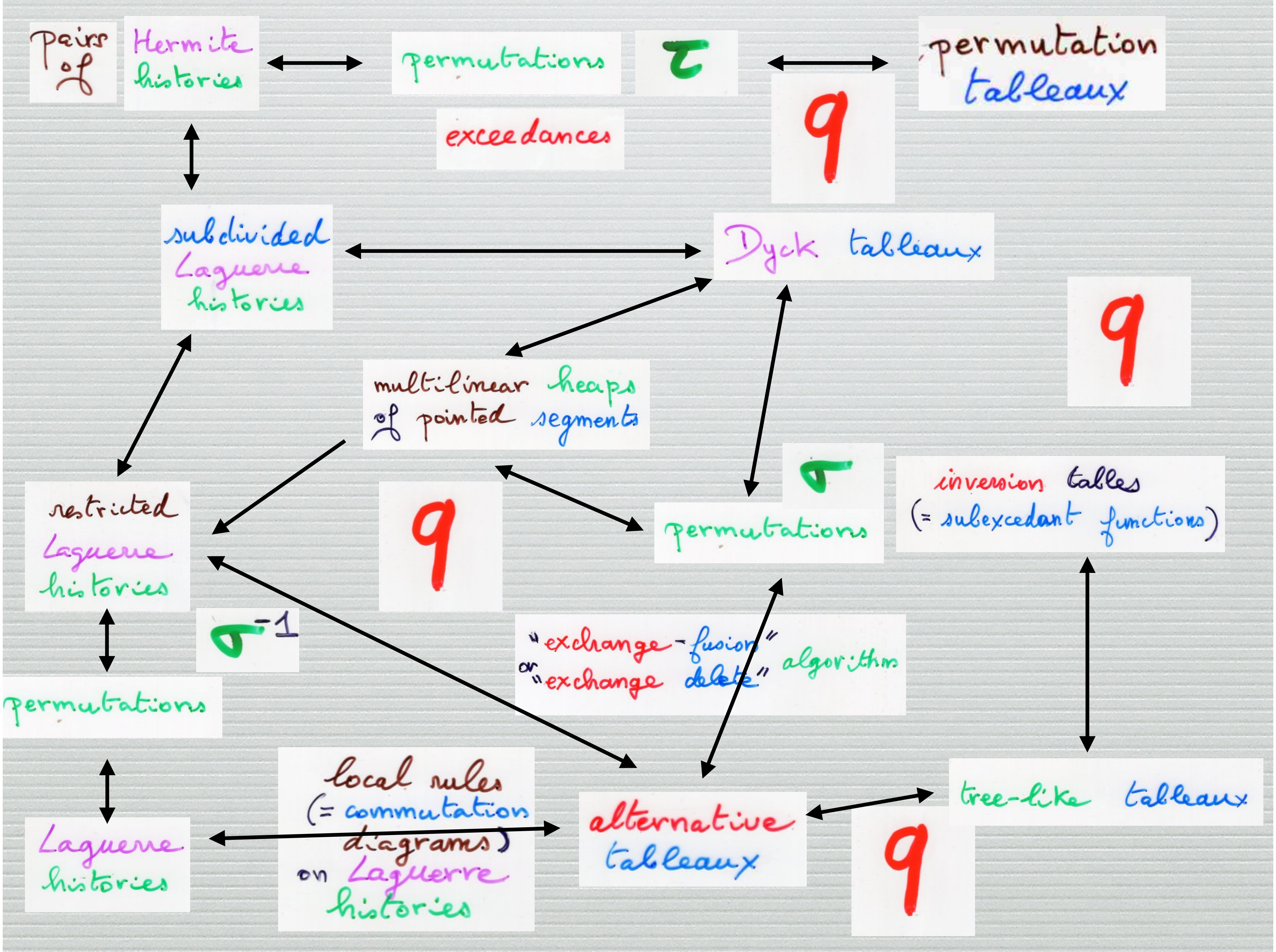
Epilogue

The « essence » of bijections …

SLC81, Krattenthalerfest

9-12 September, Strobl, Austria


Pairs of Hermite histories
permutations
τ
exceedances
permutation tableaux
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
inversion tables
(= subexcedant functions)
restricted Laguerre histories
σ⁻¹
permutations
"exchange-fusion"
or "exchange delete" algorithm
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories


Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
q
subdivided Laguerre histories
Dyck tableaux
q
multilinear heaps of pointed segments
restricted Laguerre histories
q
σ
permutations
inversion tables (= subexcedant functions)
σ-1
"exchange-fusion" or "exchange delete" algorithm
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
q
Laguerre histories

pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

Dyck tableaux

contraction of paths

multilinear heaps of pointed segments

σ

permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

"exchange-fusion" or "exchange delete" algorithm

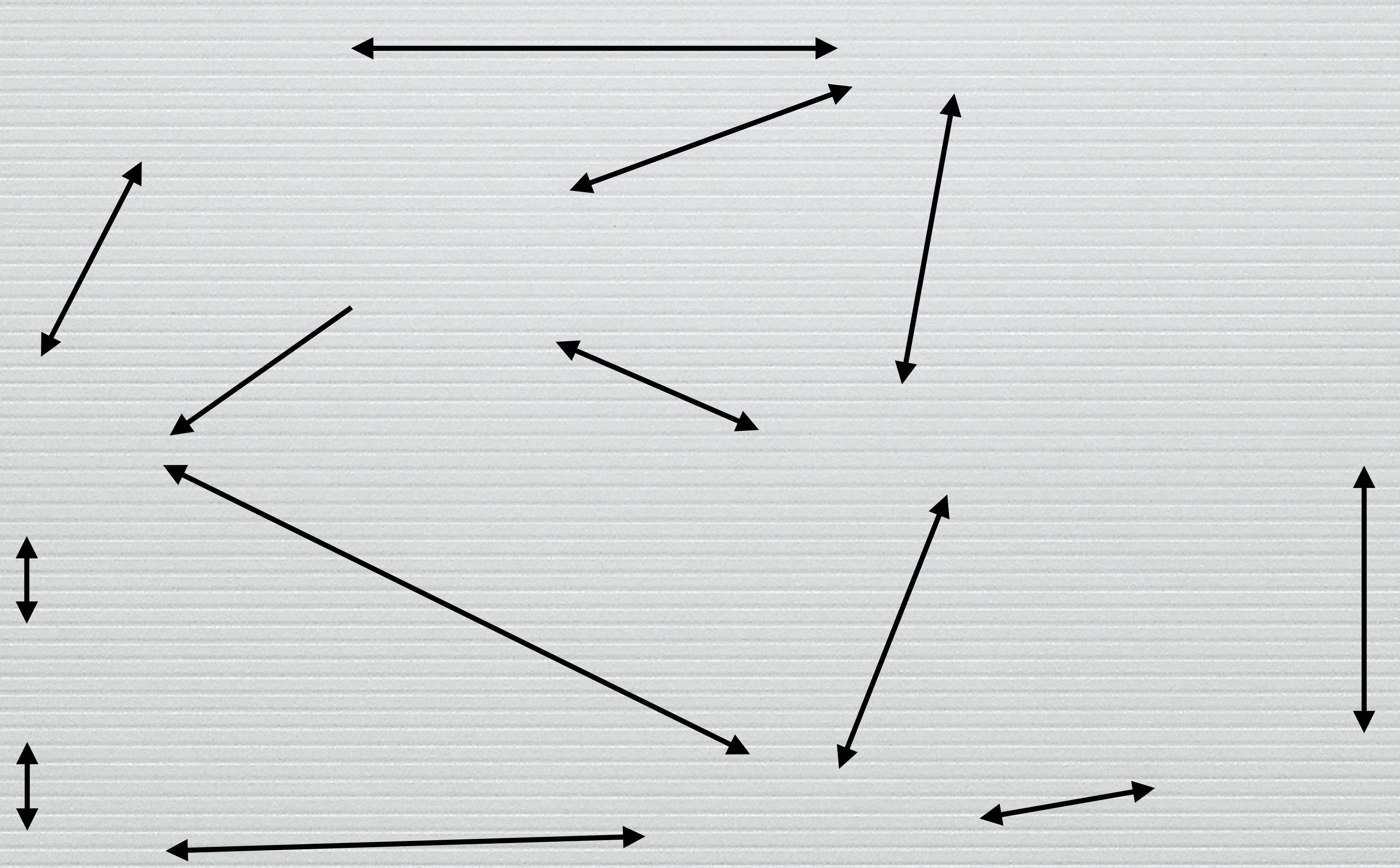
permutations

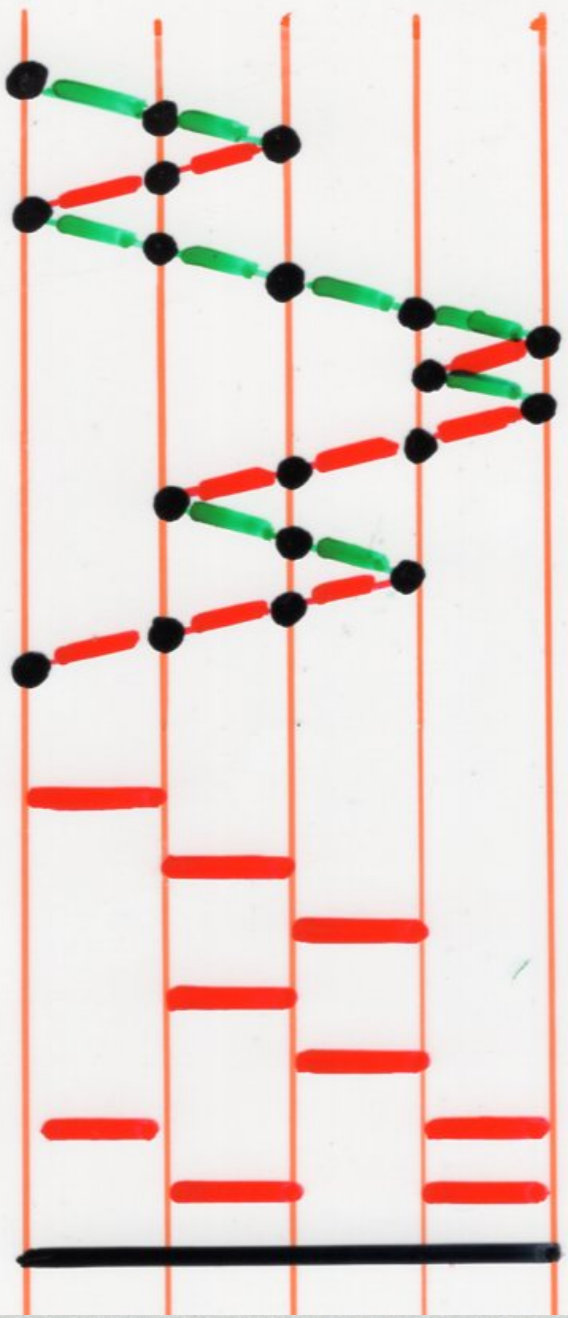
tree-like tableaux

alternative tableaux

Laguerre histories

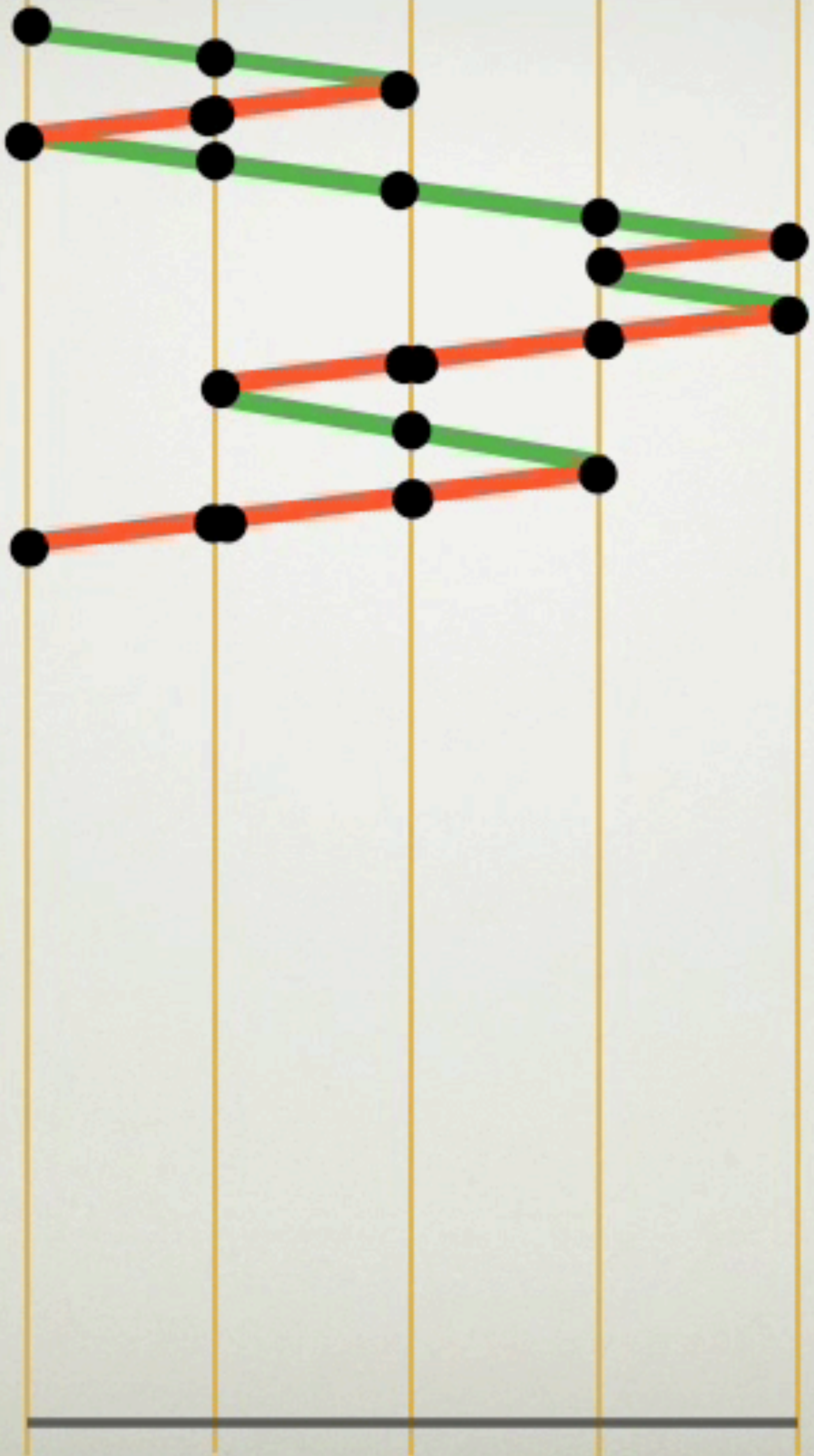
The «essence» of bijections ...

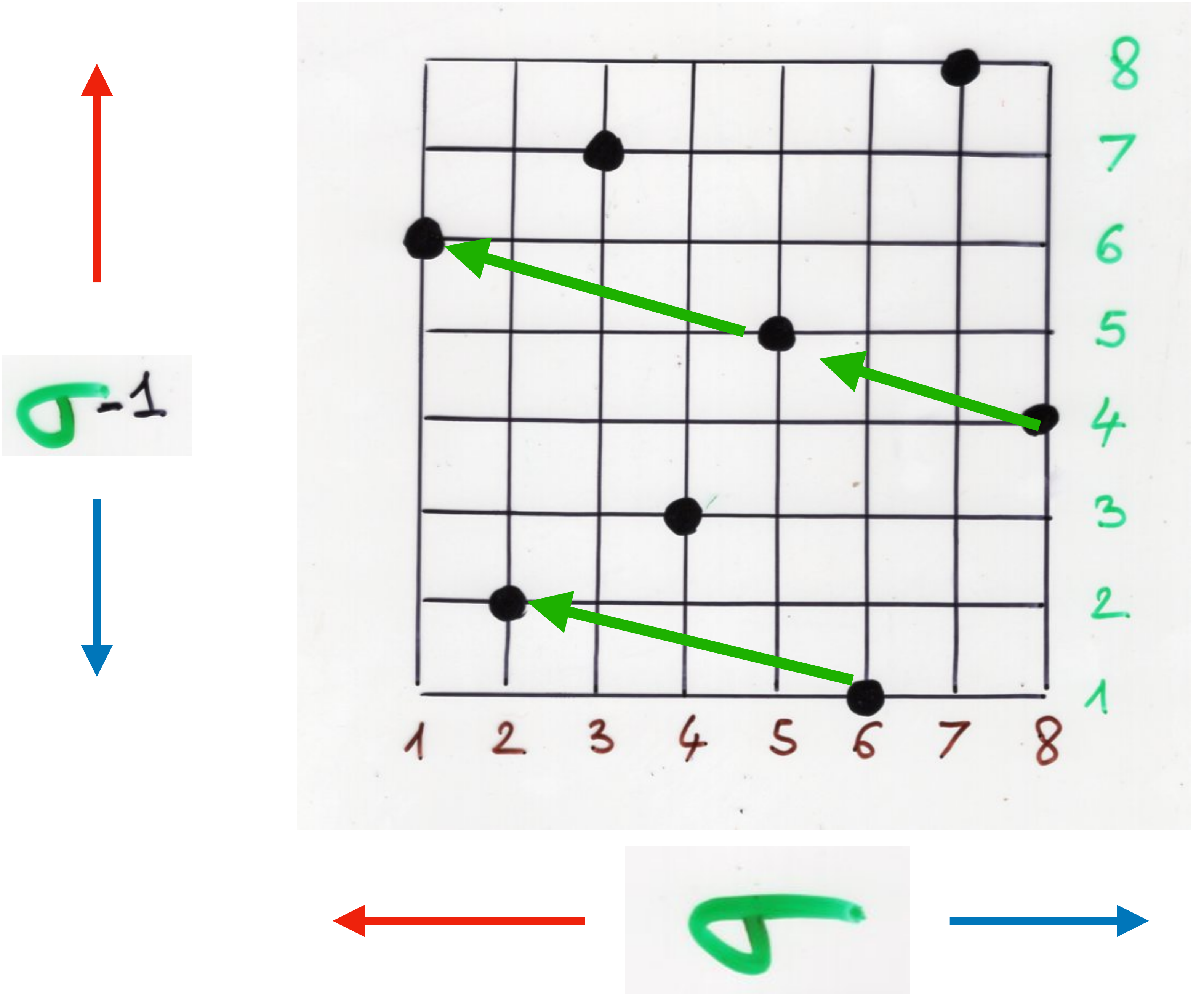




from a Dyck path

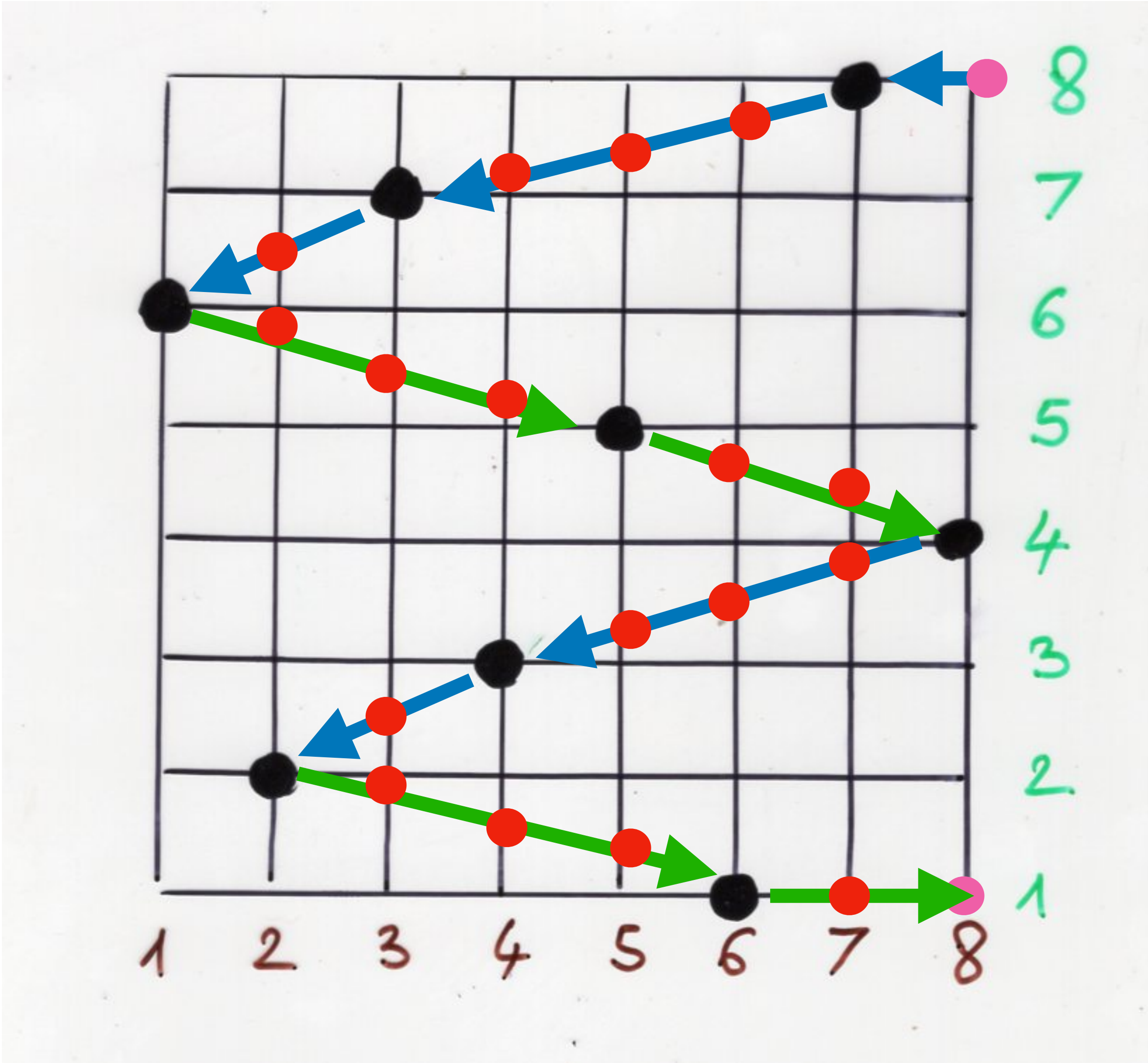
to a semi-pyramid of dimers on $\mathbb{N}$





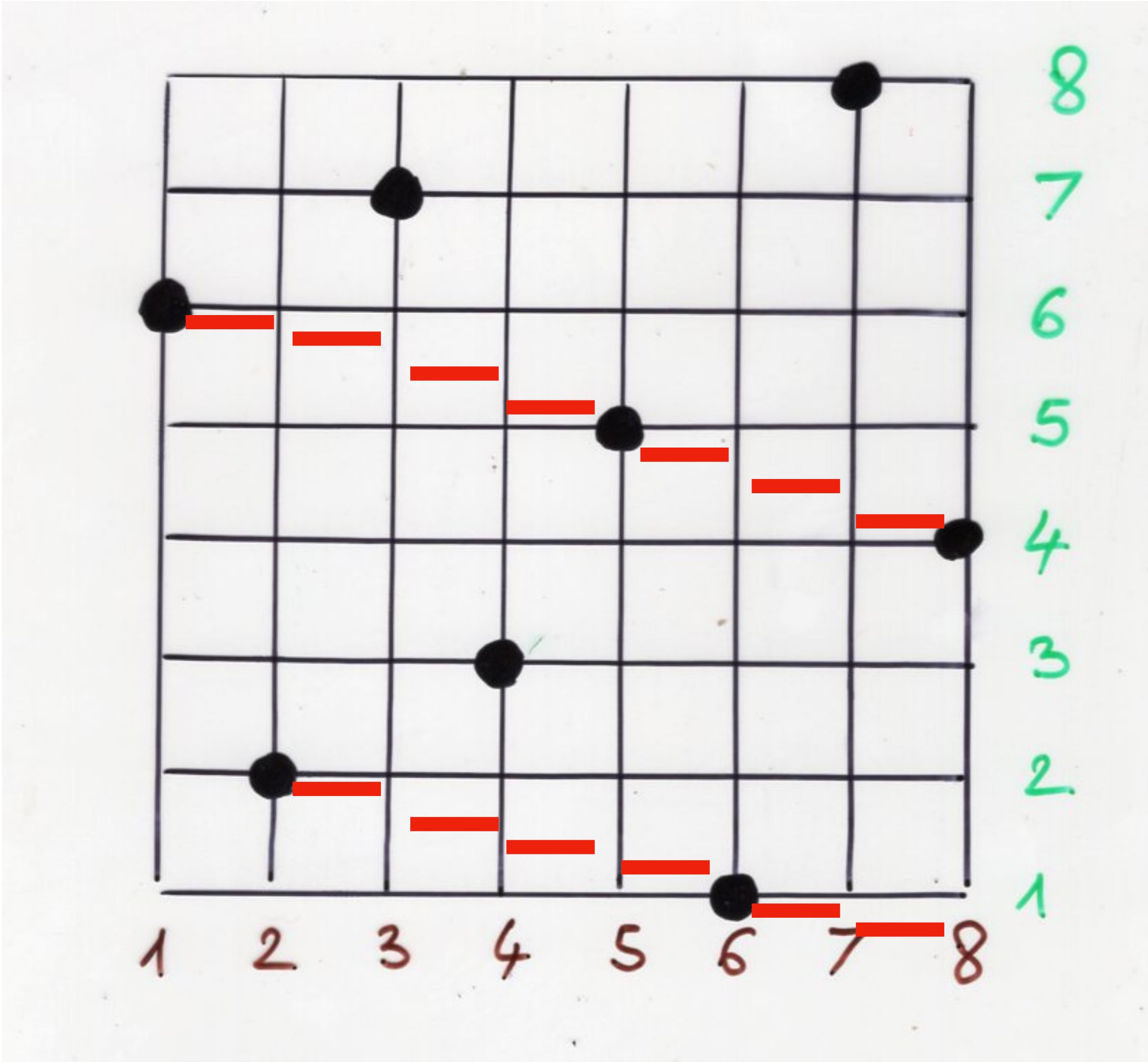
1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8
σ^{-1}
σ

σ^{-1}

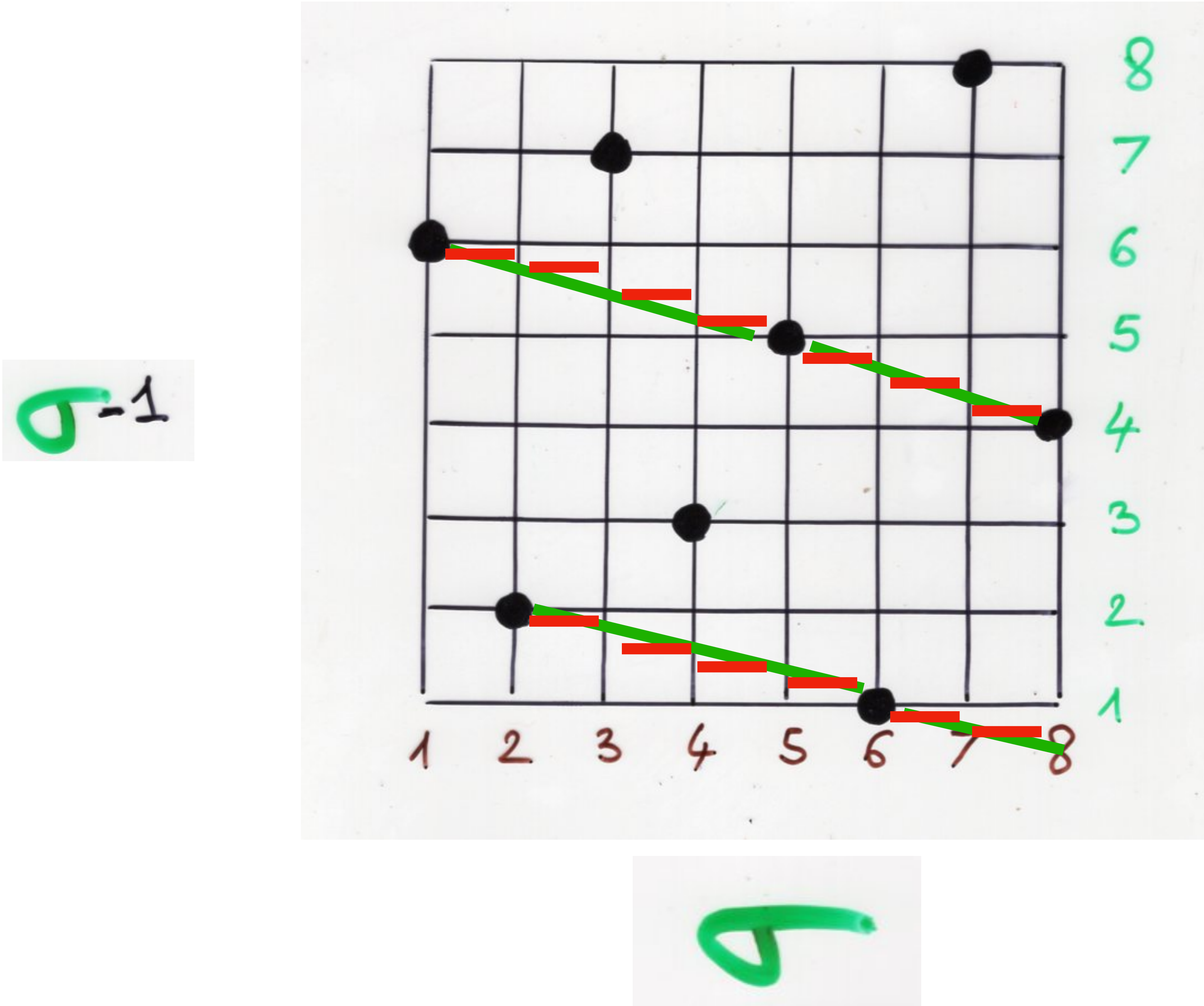


σ

σ^{-1}

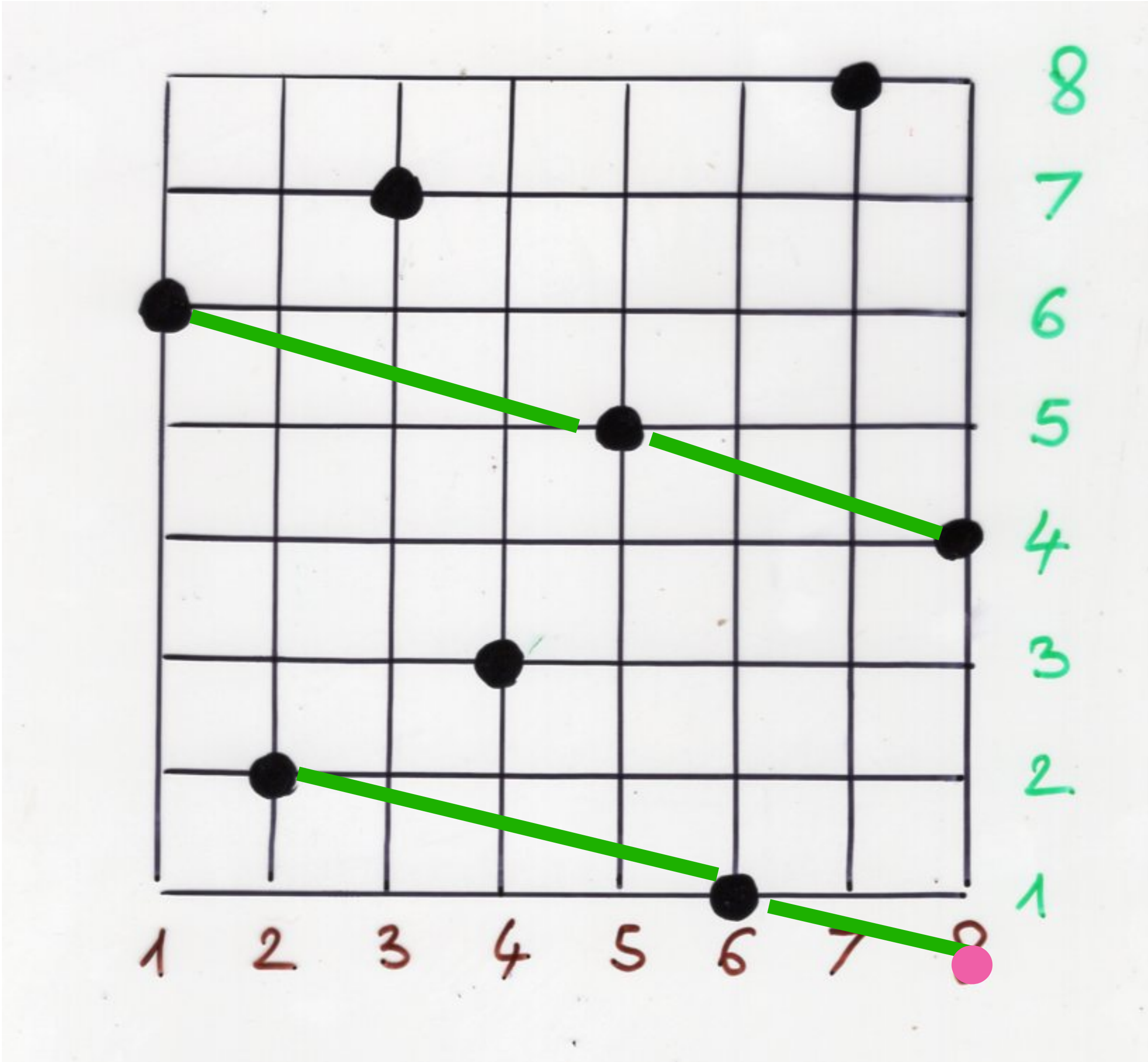


σ



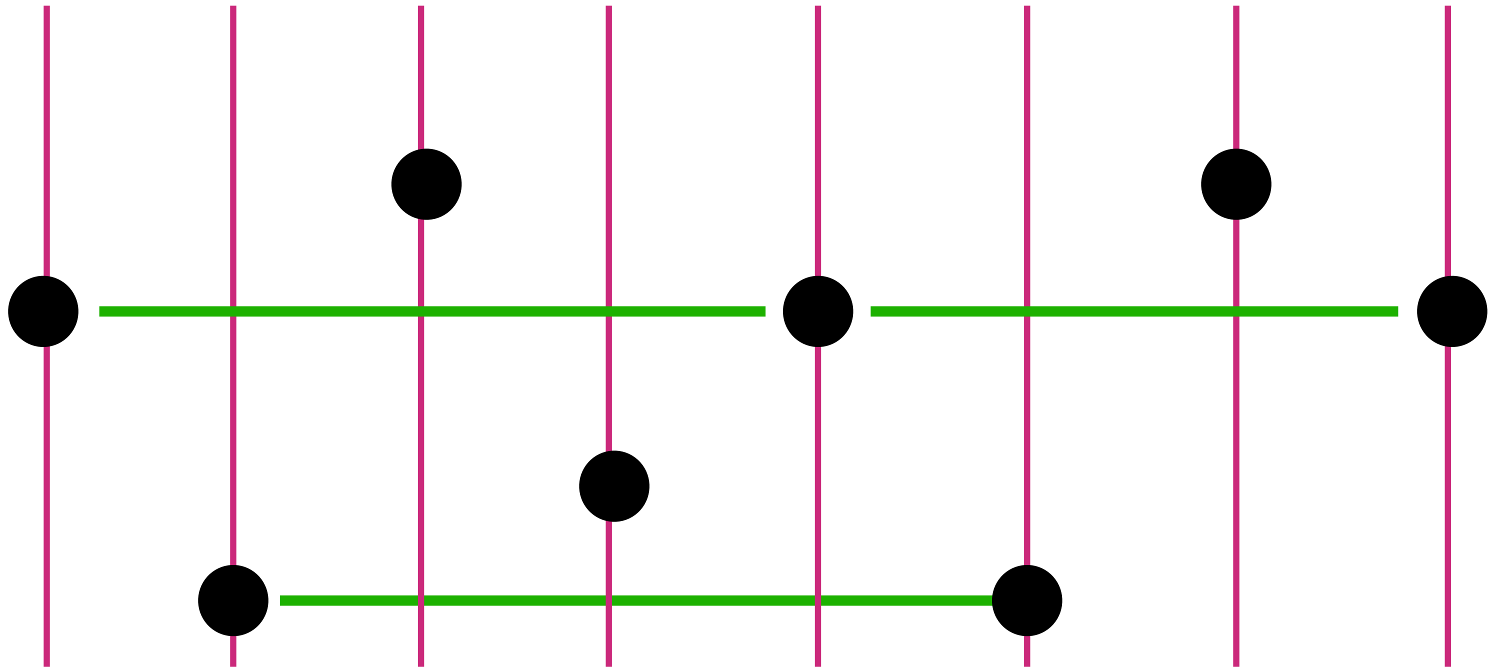
σ^{-1}
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ

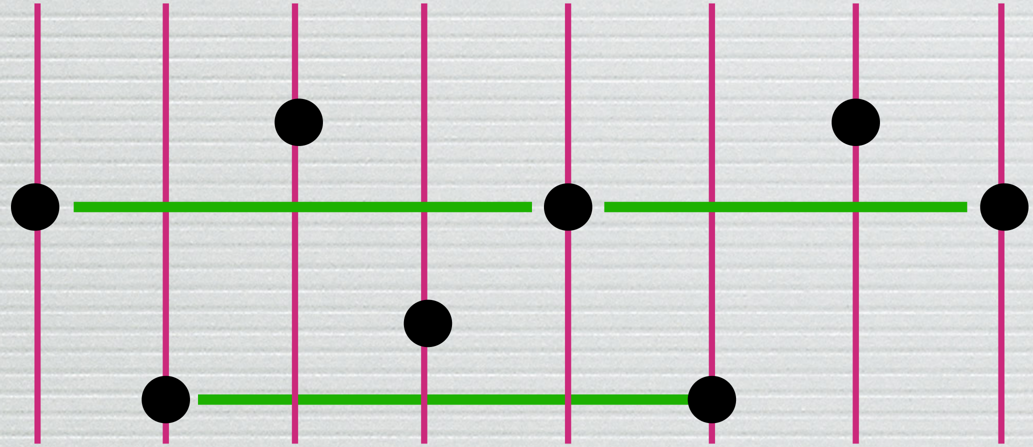
σ^{-1}



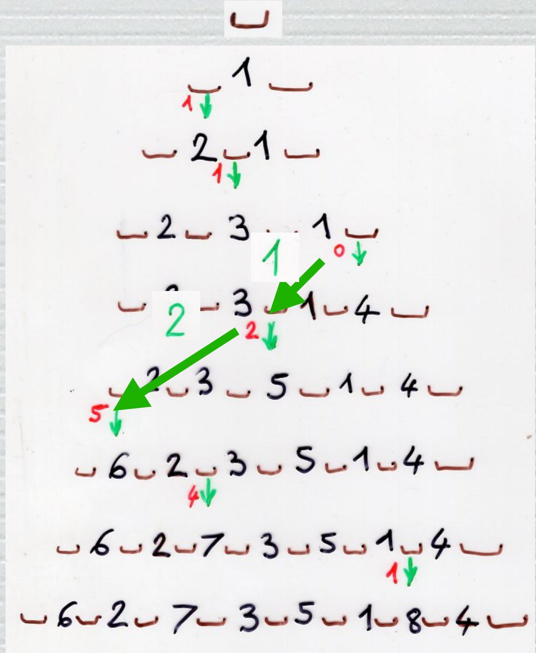
σ

$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$

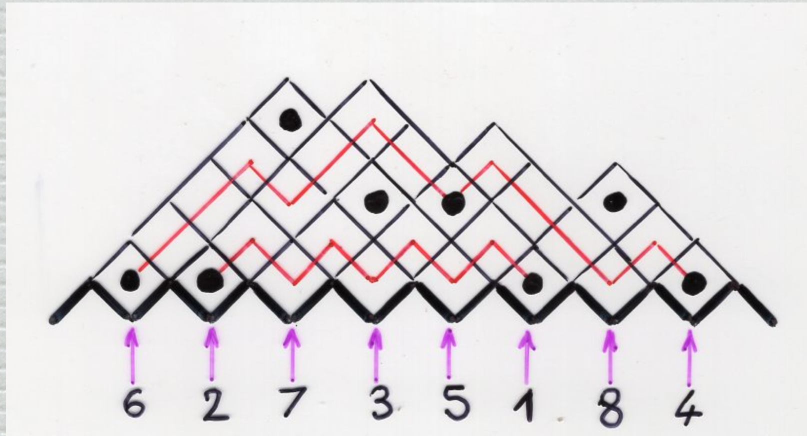


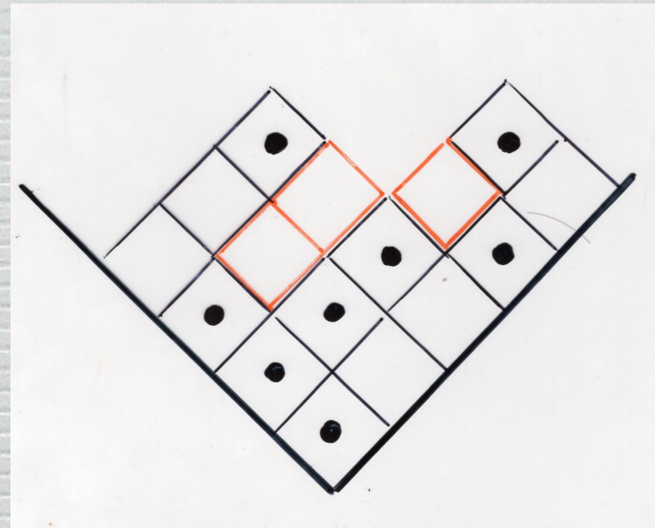


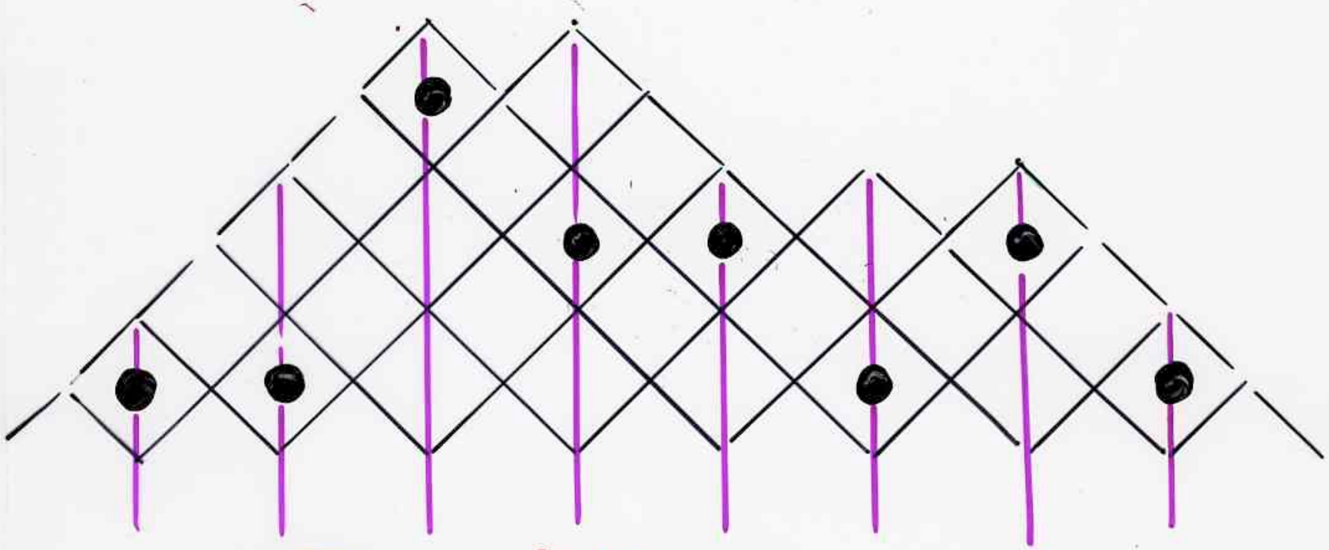
$\sigma = 6 \; 2 \; 7 \; 3 \; 5 \; 1 \; 8 \; 4$



The « essence » of 3 bijections in parallel

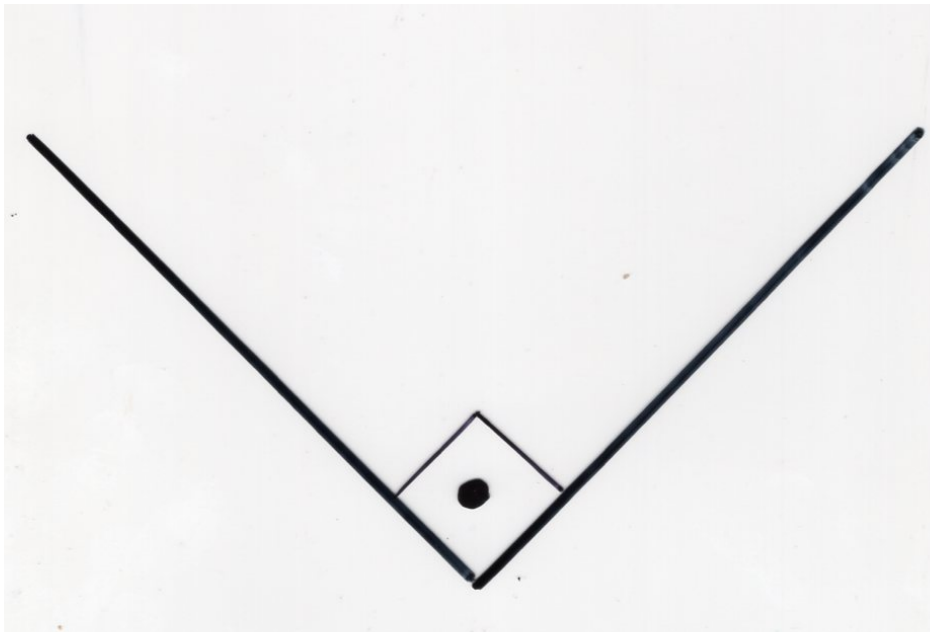


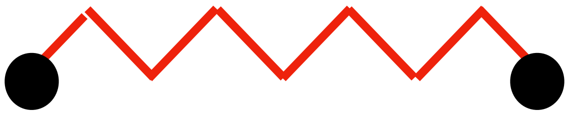


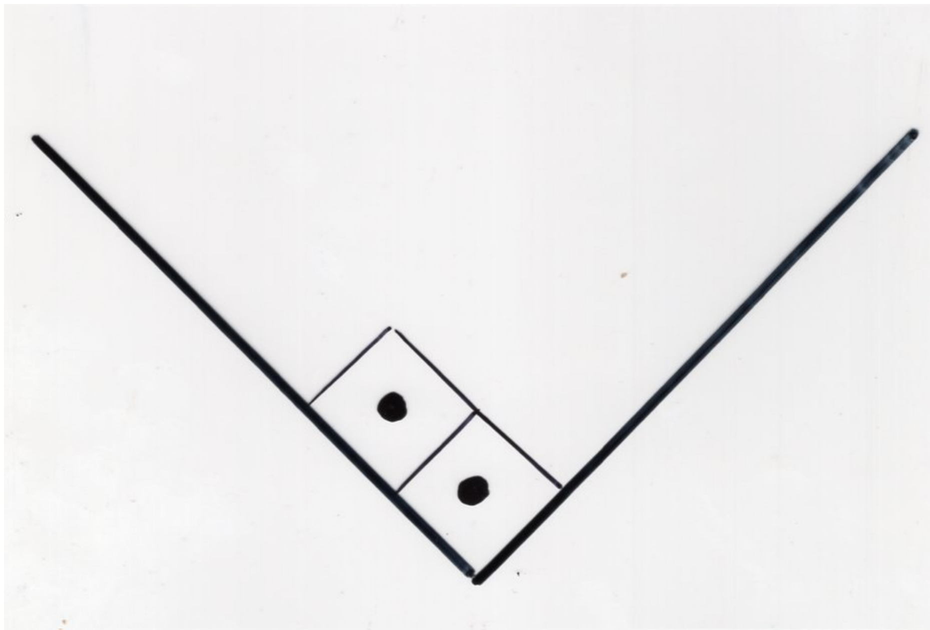


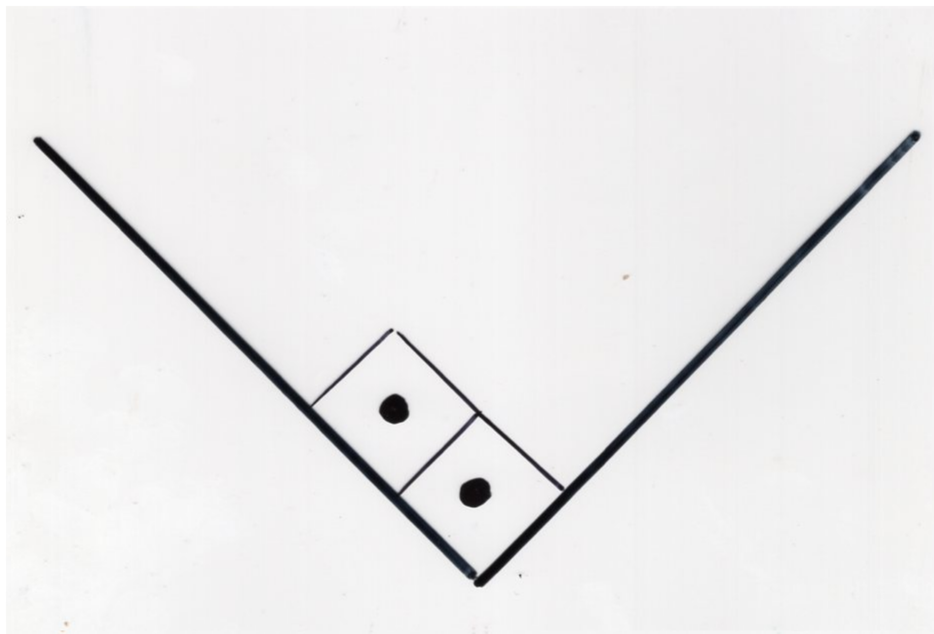
Dyck tableau

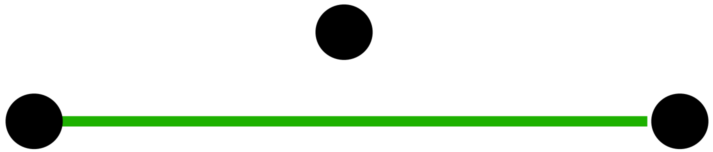
J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)

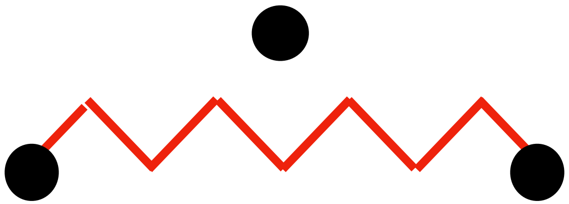


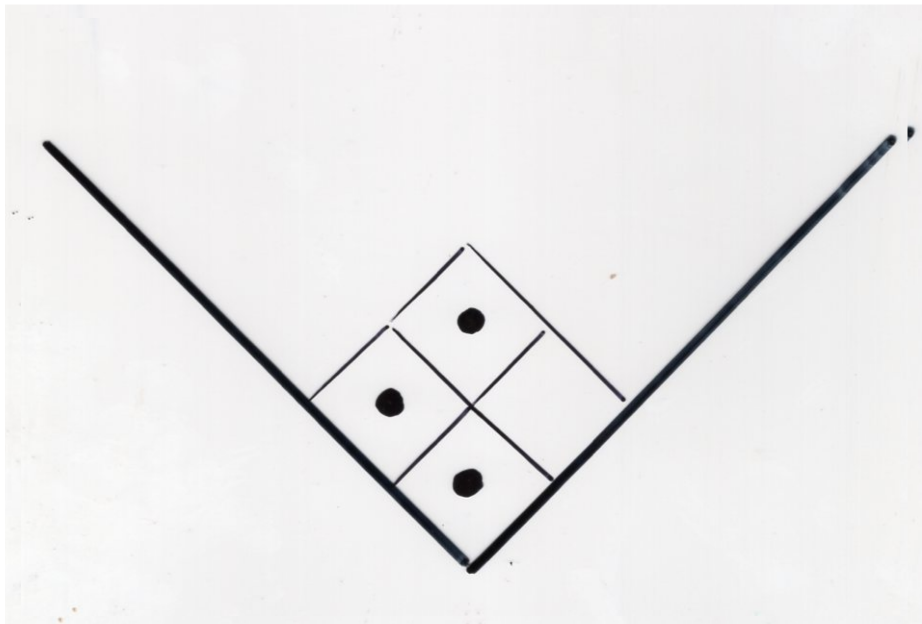


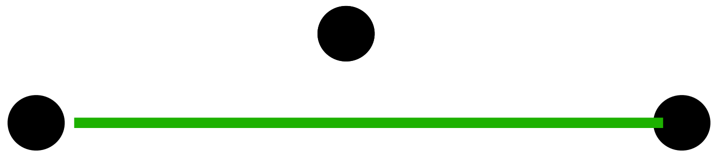


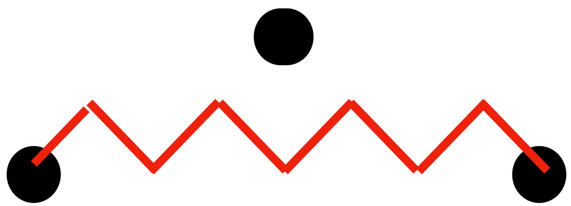


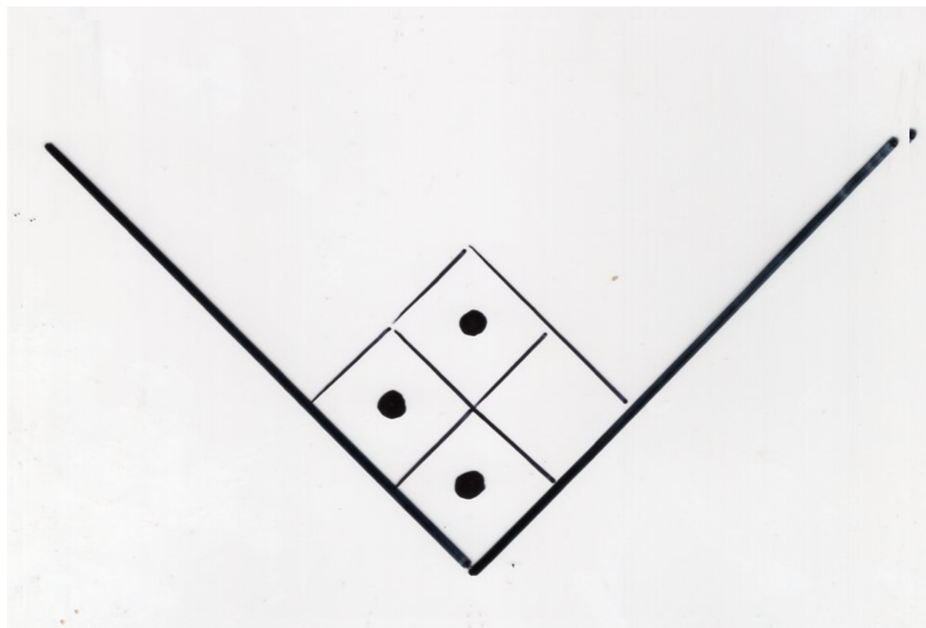


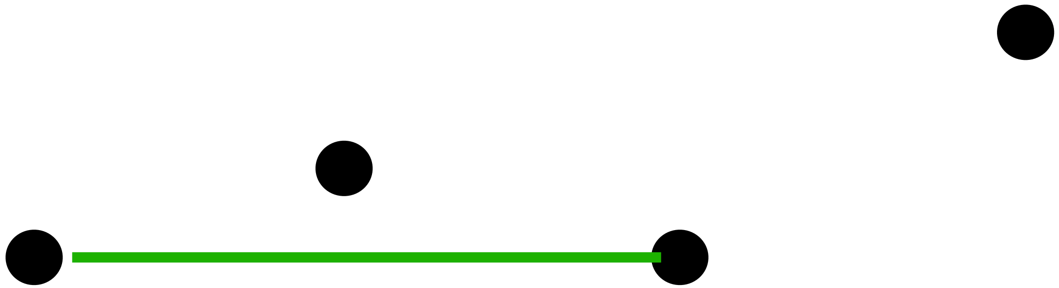


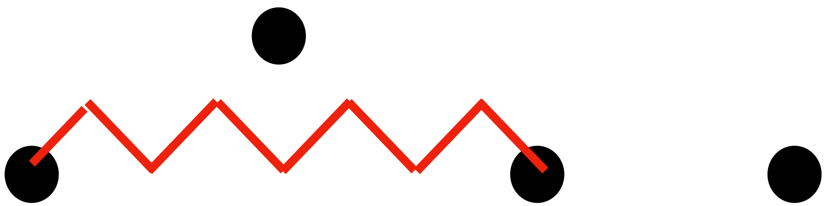


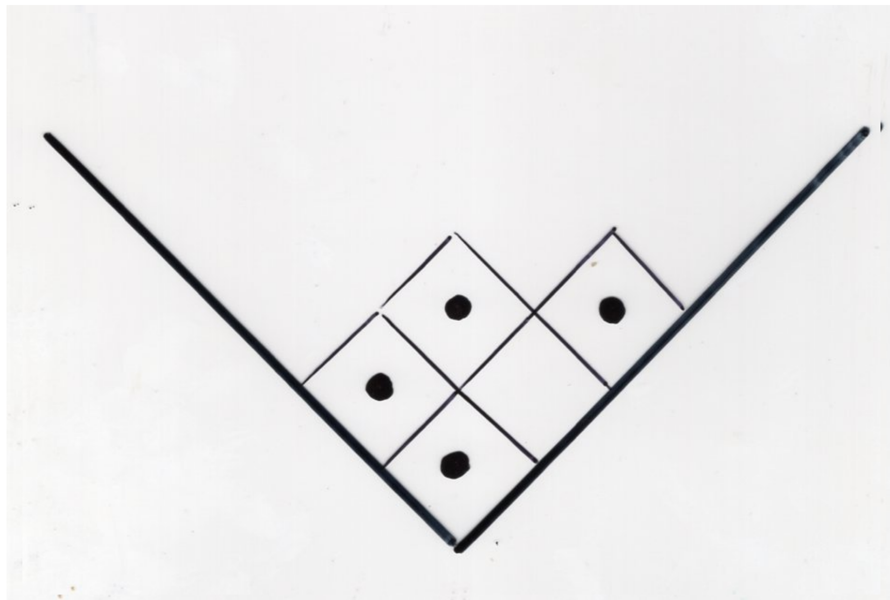


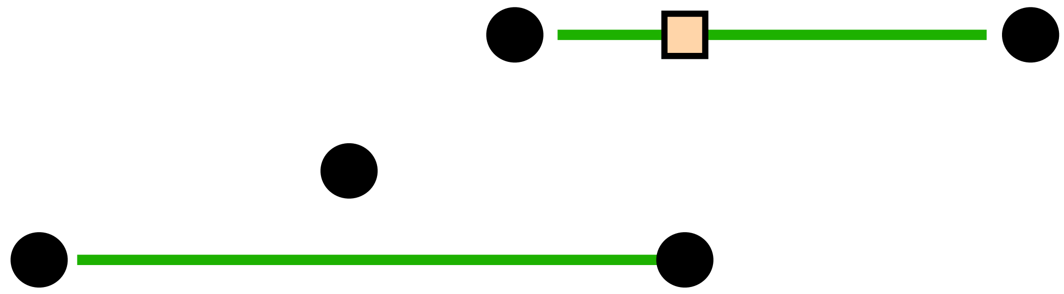


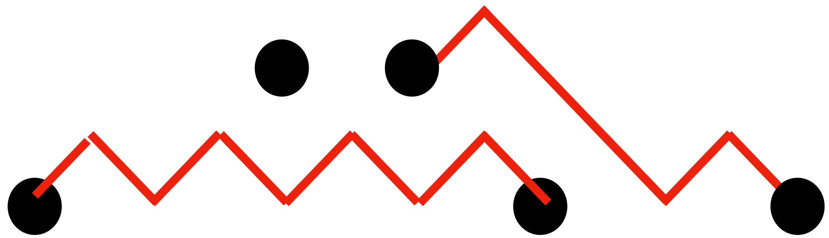


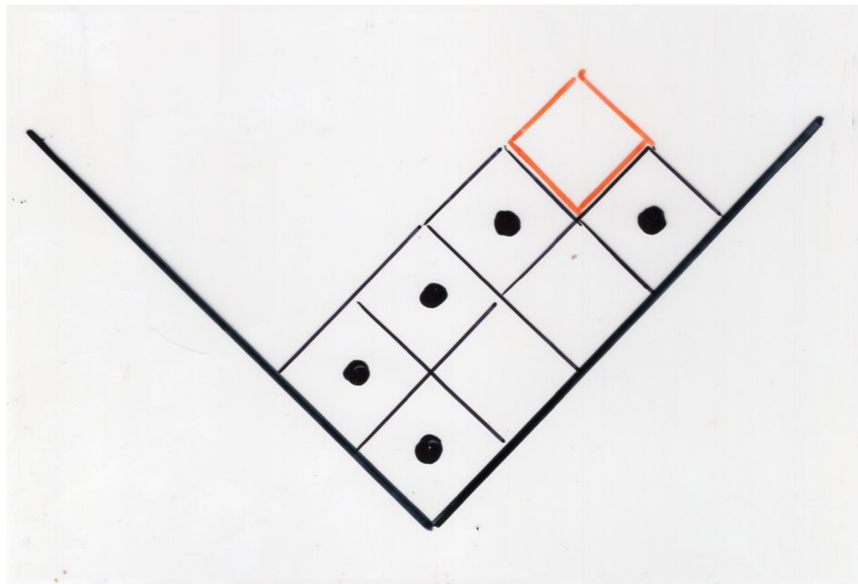


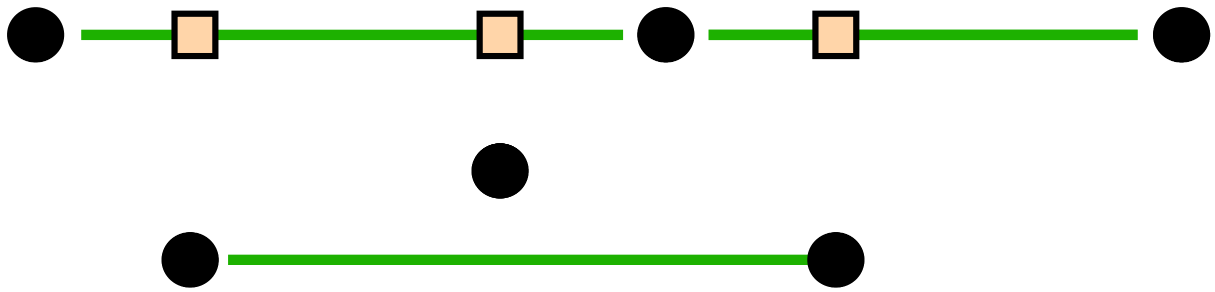


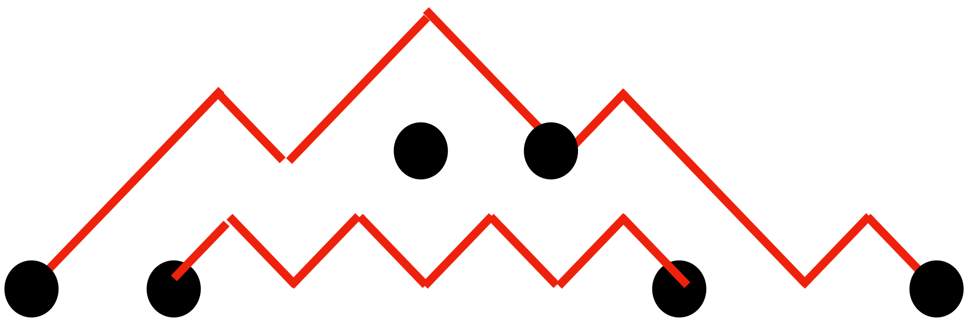


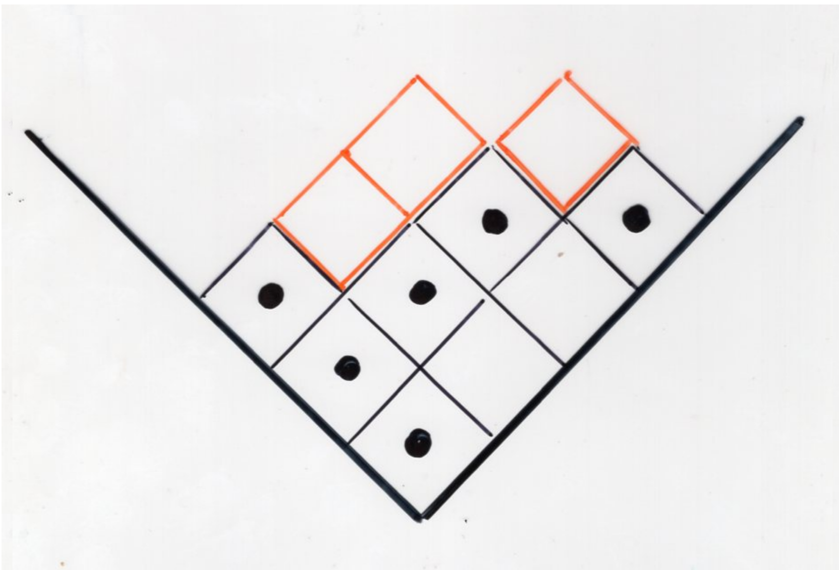


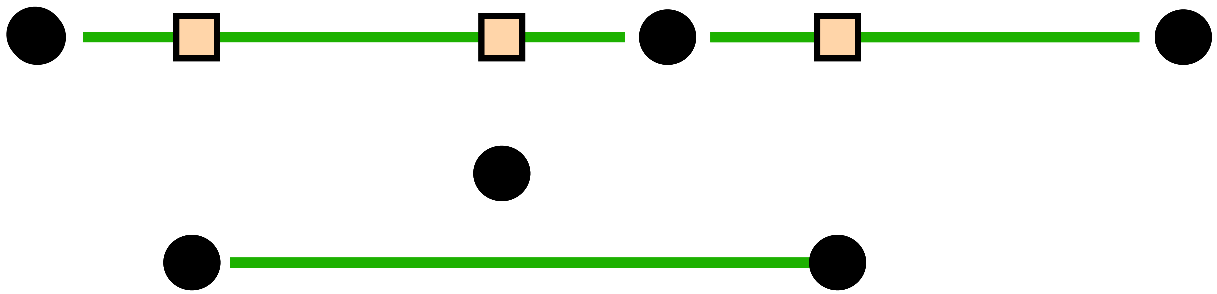


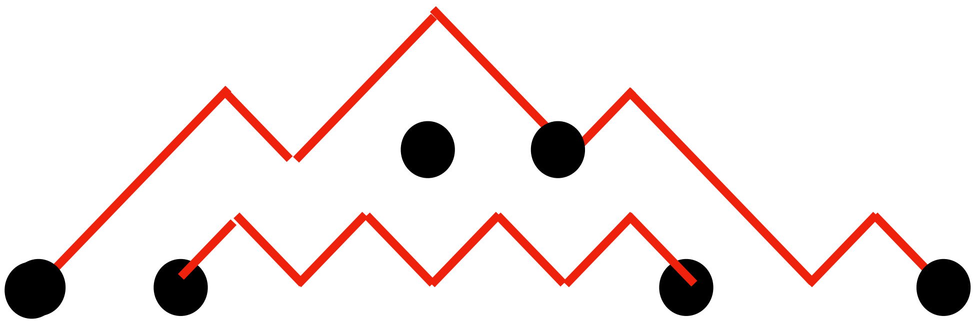


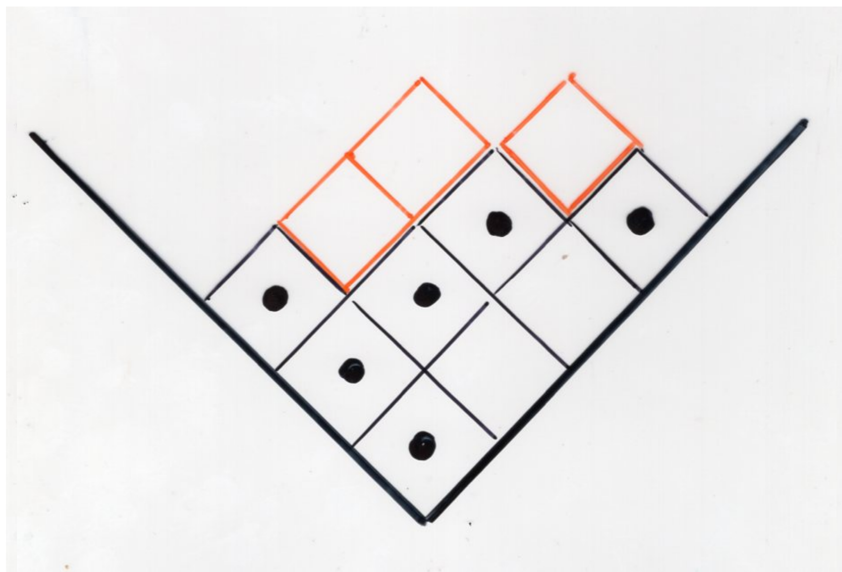


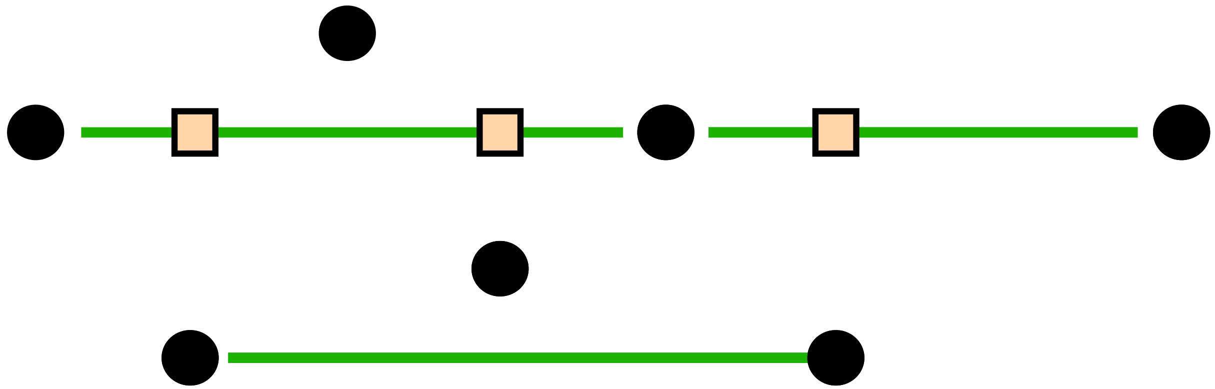


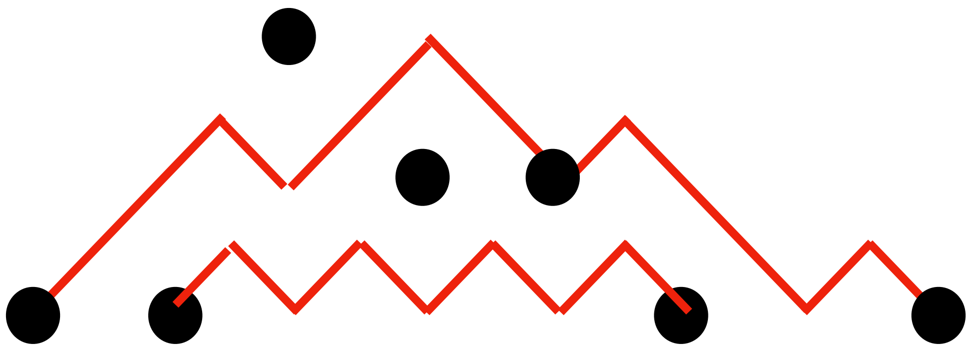


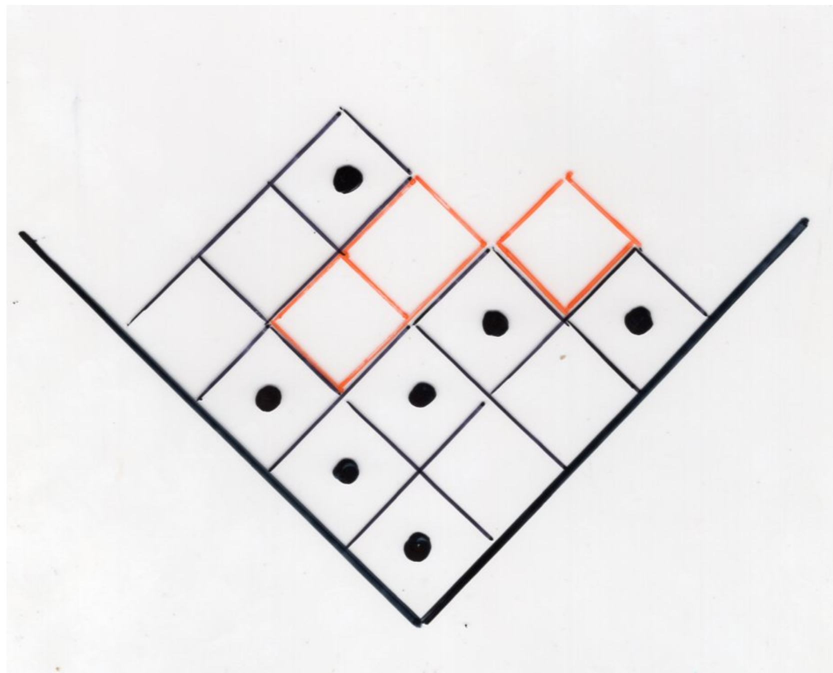


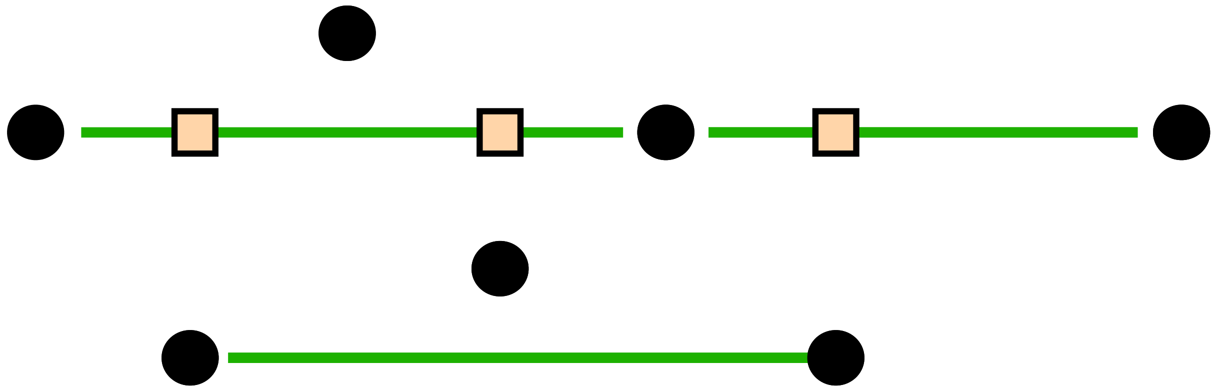


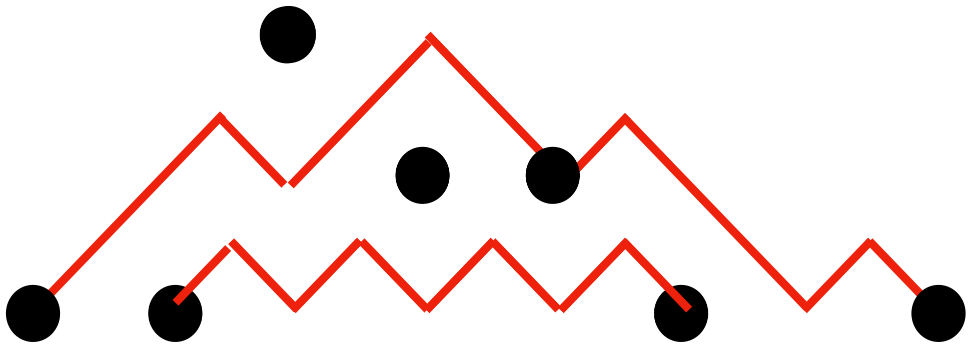


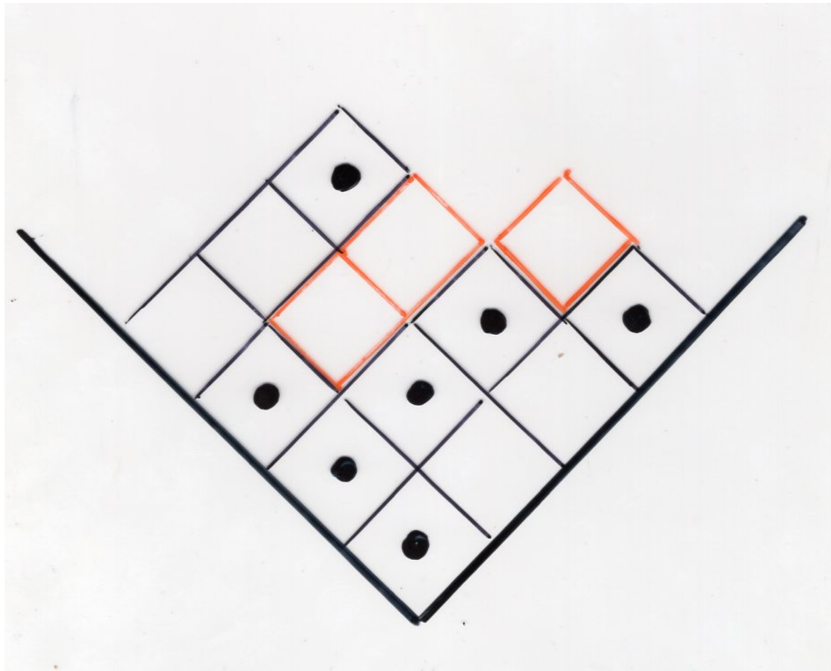


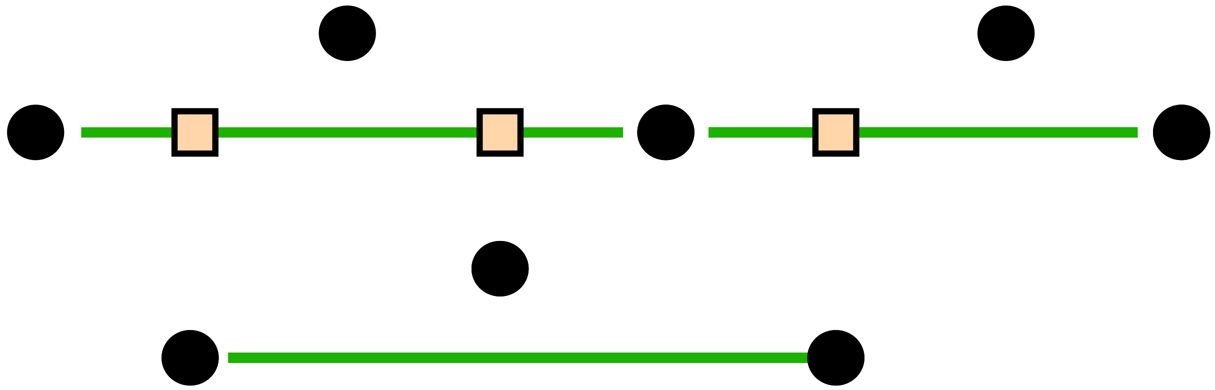


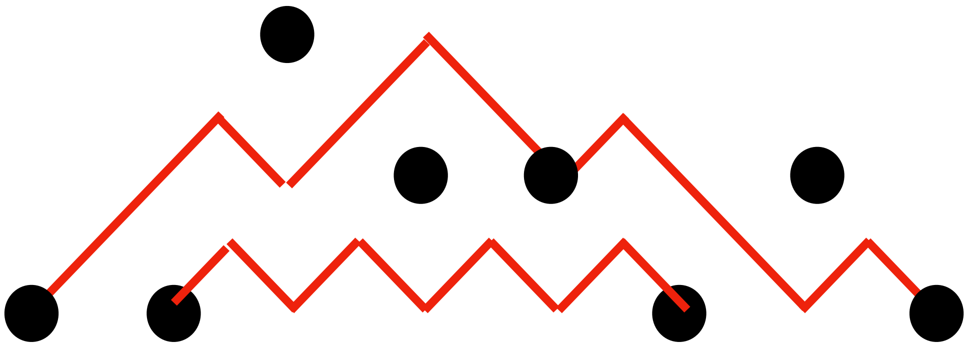


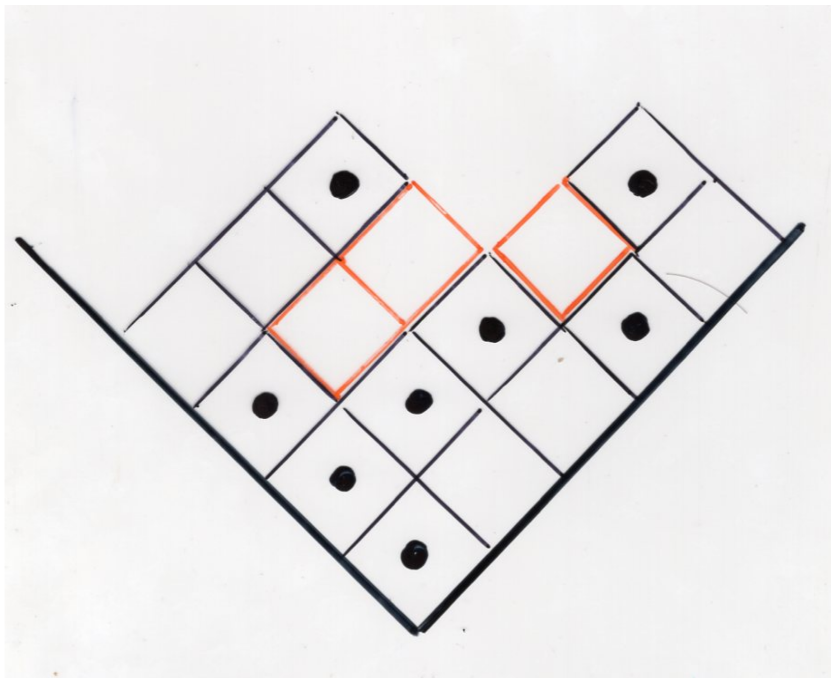


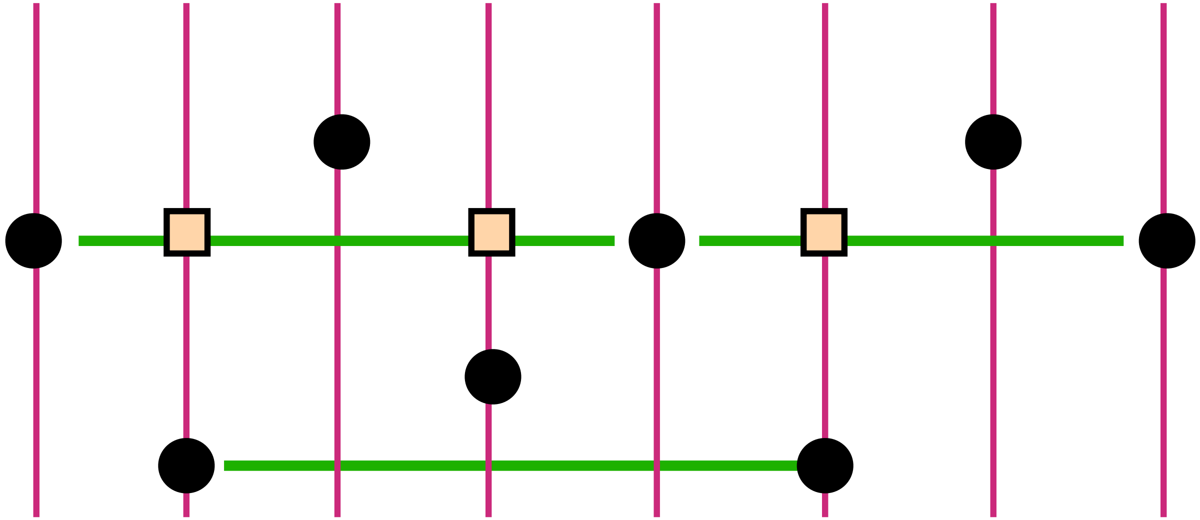


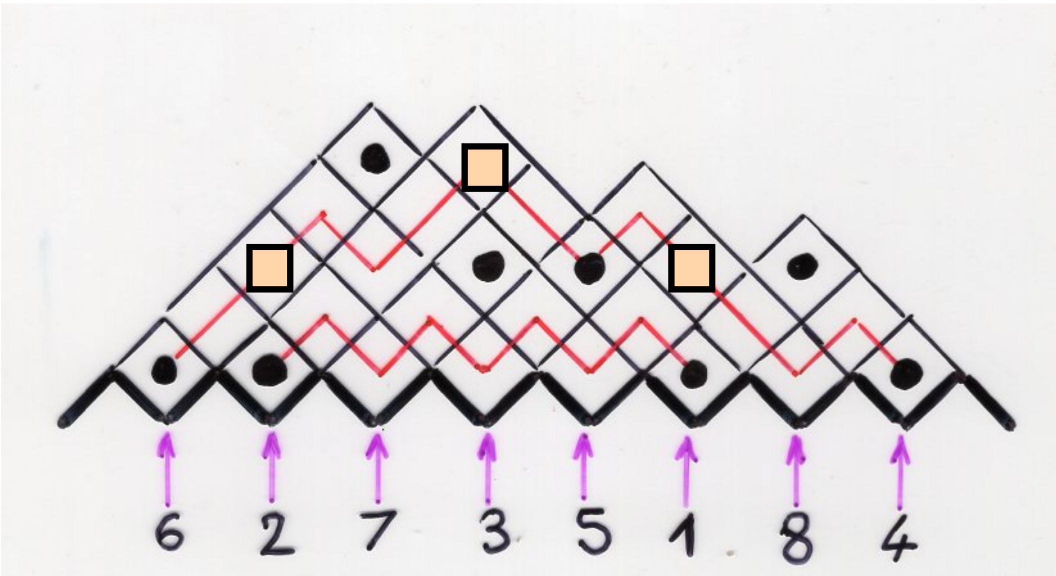


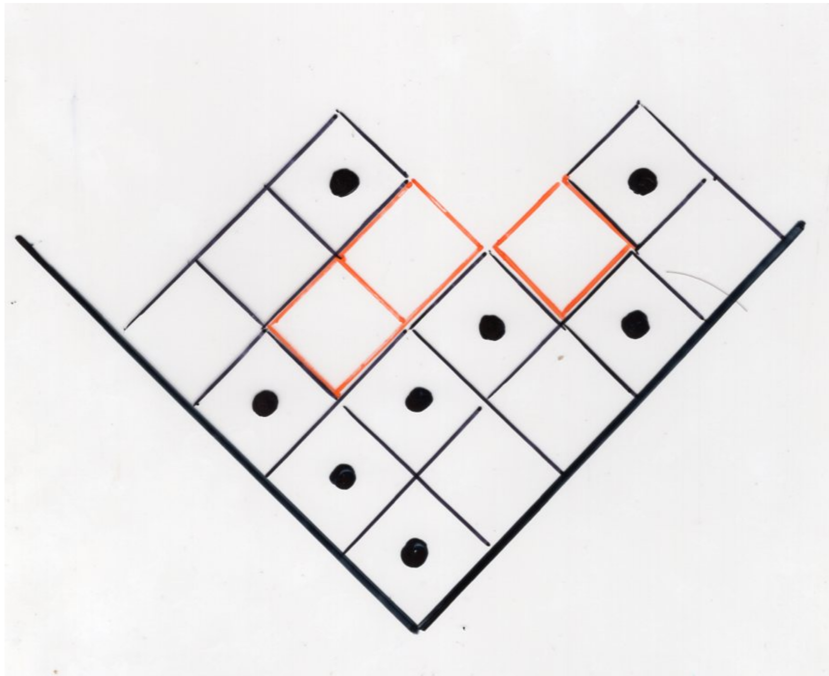









6 2 7 3 5 1 8 4



The «essence» of bijections …

see the V-book:

The Art of Bijective Combinatorics

Part III, Ch5b, 5c Tableaux and orthogonal polynomials

Part II, Ch1a, 1b Heaps of pieces and commutations

« Video-book »

- videos

- slides

- www.viennot.org

IMSc, Chennai, India

Part I (2016)

Part II (2017)

Part III (2018)