Course IMSc, Chennai, India

The Art of Bijective Combinatorics

2016-2019

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Epilogue

The essence of bijections:

from growth diagrams to
Laguerre heaps of segments for the PASEP

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

IMSc, Chennai
March 14, 2019

3,14

mirror website
www.imsc.res.in/~viennot

« Pie Day »

The Art of Bijective Combinatorics

IMSc, Chennai, India

Part I (2016)

An introduction to enumerative, algebraic and bijective combinatorics

Bijective proof ?

The Catalan garden

The n! garden

The Art of Bijective Combinatorics

IMSc, Chennai, India

Part II (2017)

Commutations and heaps of pieces with interactions in physics, mathematics and computer science

3 basic Lemma

Going back to the « sources »

- Linear algebra
- Algebraic graph theory
- Coxeter group
- Statistical physics
- Quantum Gravity
- Concurrency in computer science

The Art of Bijective Combinatorics
IMSc, Chennai, India

Part III (2018)

The Cellular ansatz:bijective combinatorics and quadratic algebra

Robinson-Schensted-Knuth,
Asymmetric Exclusion Process,
Tilings,
Alternating Sign Matrices ...
under the same roof

under the same roof

The Art of Bijective Combinatorics
IMSc, Chennai, India

Part IV (2019)

Combinatorial theory of orthogonal polynomials

Some Complements ....

The Art of Bijective Combinatorics
IMSc, Chennai, India

« Video-book »

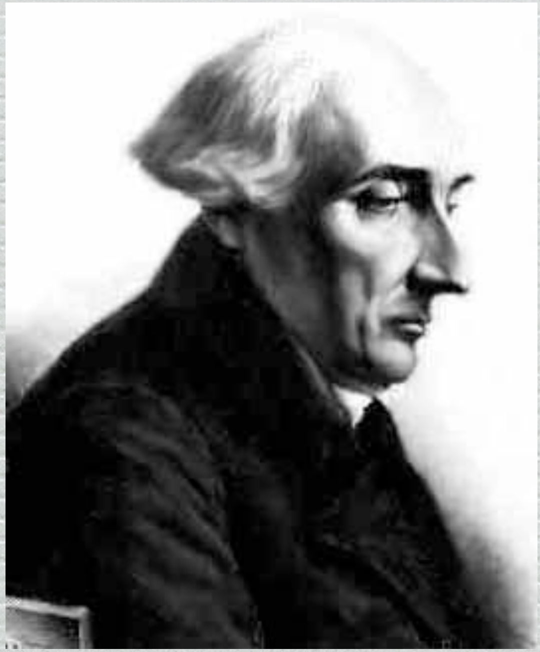
- videos (76 lectures)

www.viennot.org

mirror website

www.imsc.res.in/~viennot

“Lisez Euler,
dans ses écrits tout est clair, bien calculé,
ils regorgent de beaux exemples



et parce que l’on doit
toujours étudier les sources”

Joseph-Louis Lagrange
1736 - 1813

SLC81, Krattenthalerfest

9-12 September, Strobl, Austria

The essence of bijections:

The essence of bijections:

from growth diagrams to Laguerre heaps of segments for the PASEP

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org
mirror website
www.imsc.res.in/~viennot

SLC81, Krattenthalerfest
9-12 September, Strobl, Austria

growth diagrams

Laguerre

essence

heaps of segments

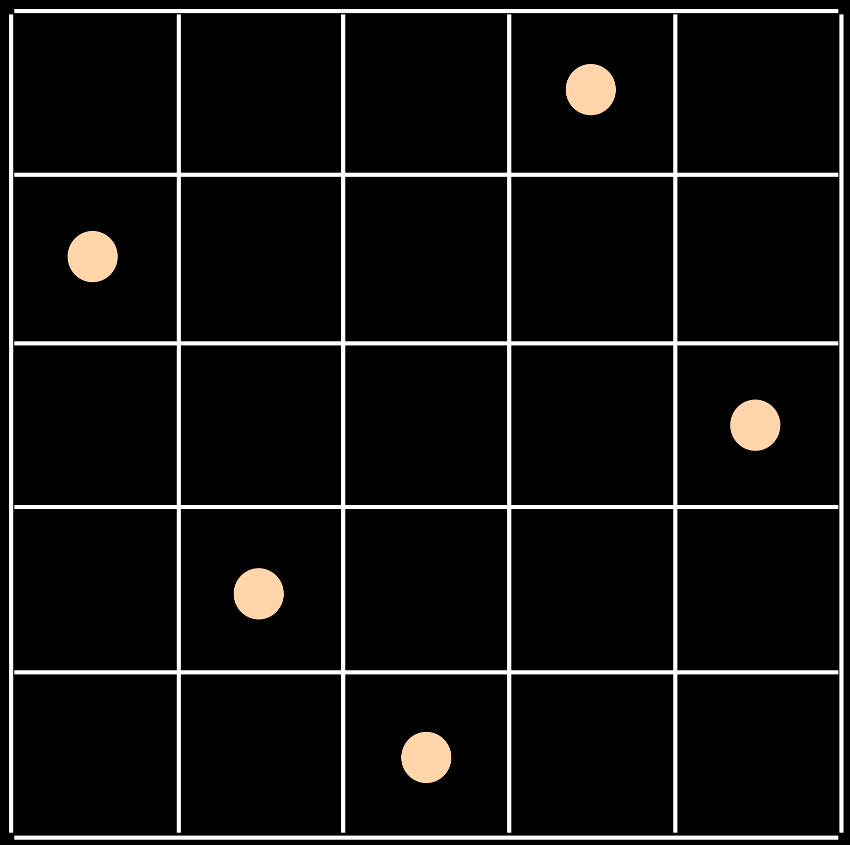
PASEP

growth diagrams

S. Fomin, 1986, 1994

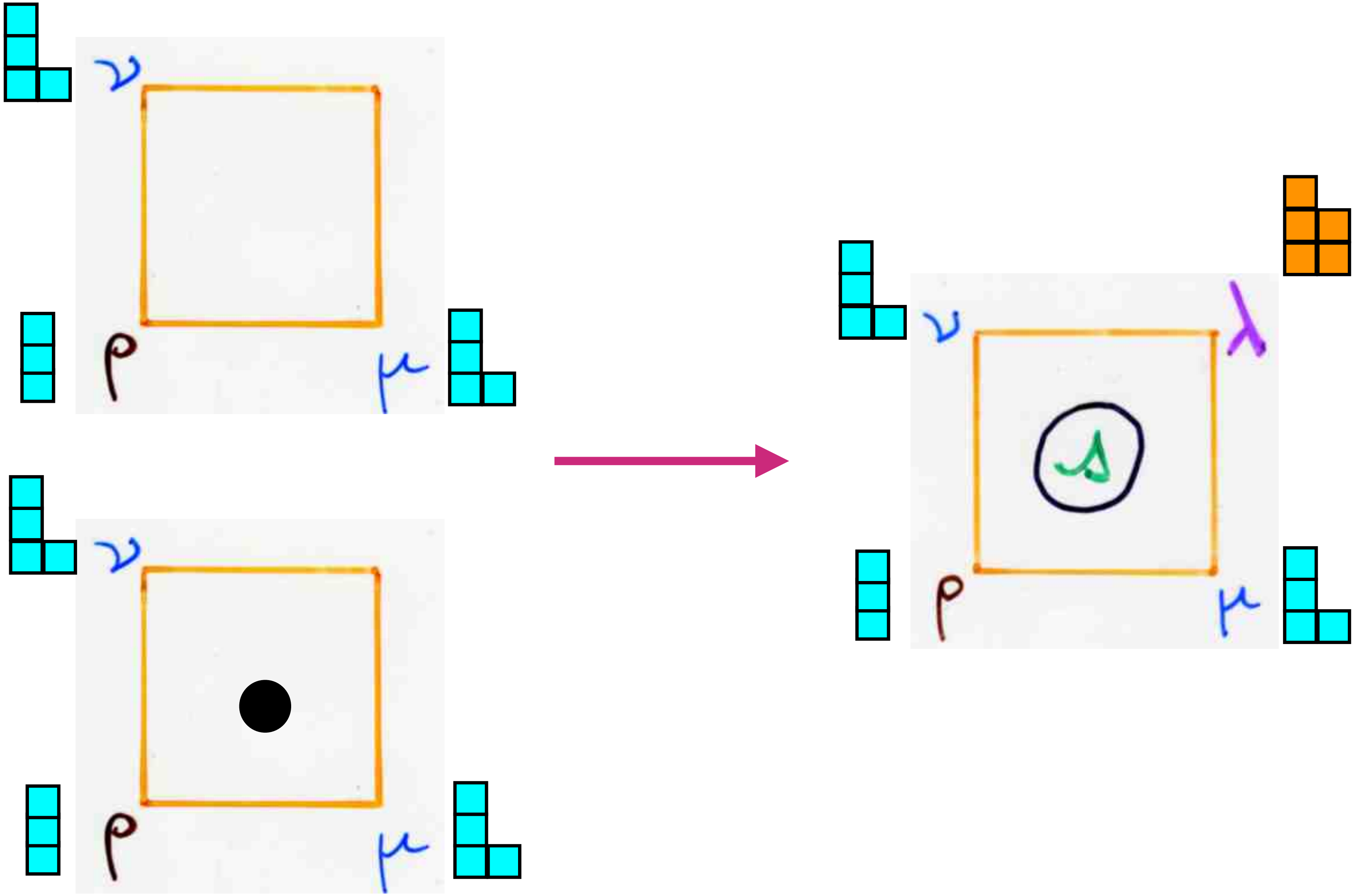
C.Krattenthaler

Banff, 2005

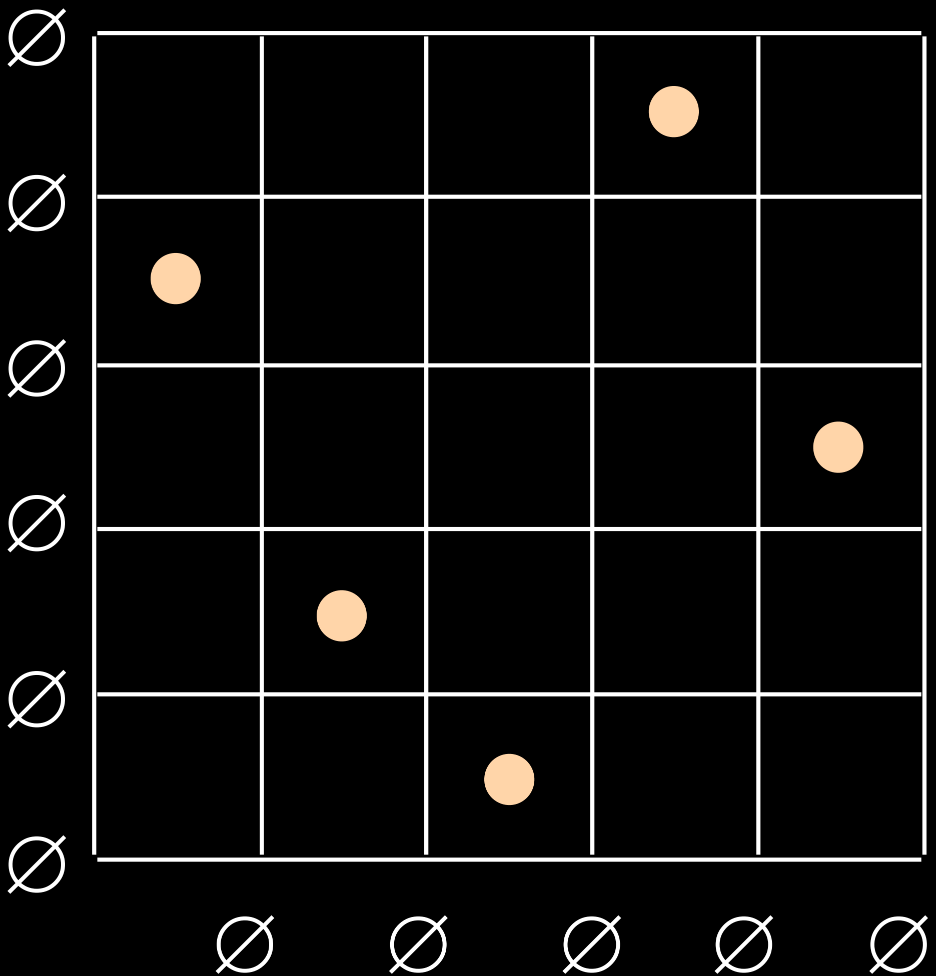


$\sigma = 4, 2, 1, 5, 3$

Fomin's "local rules" "growth diagrams"

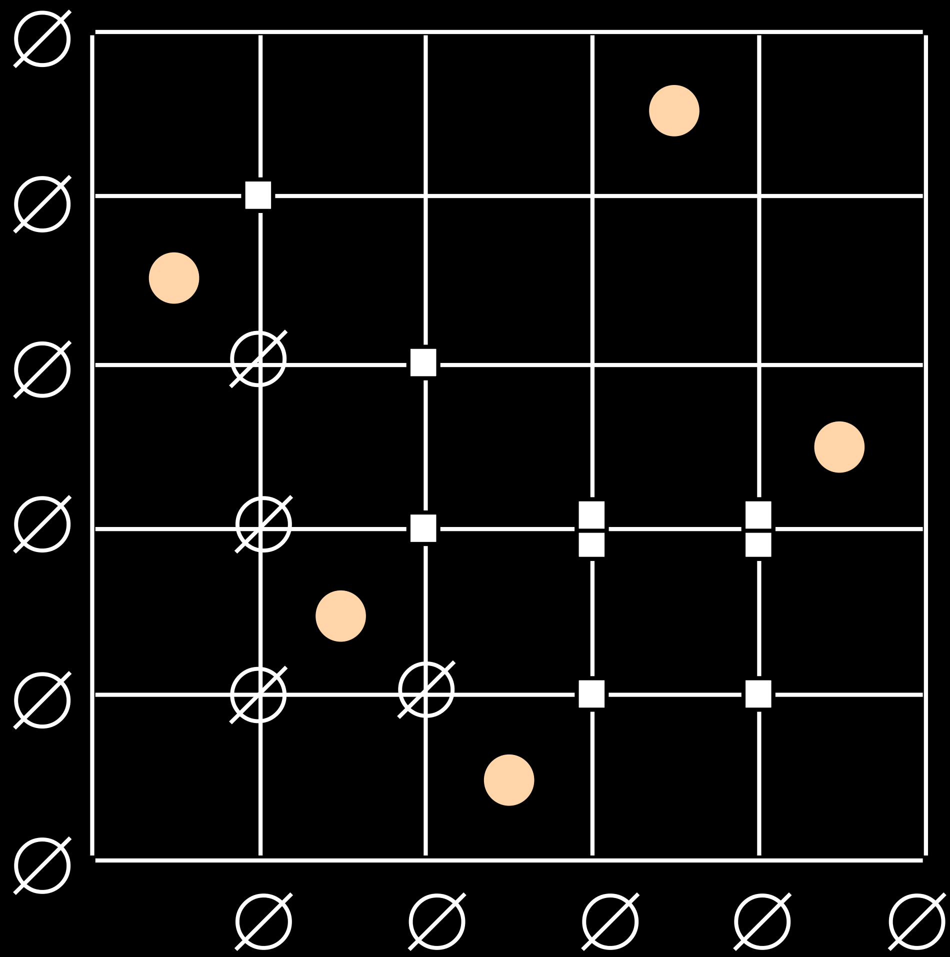


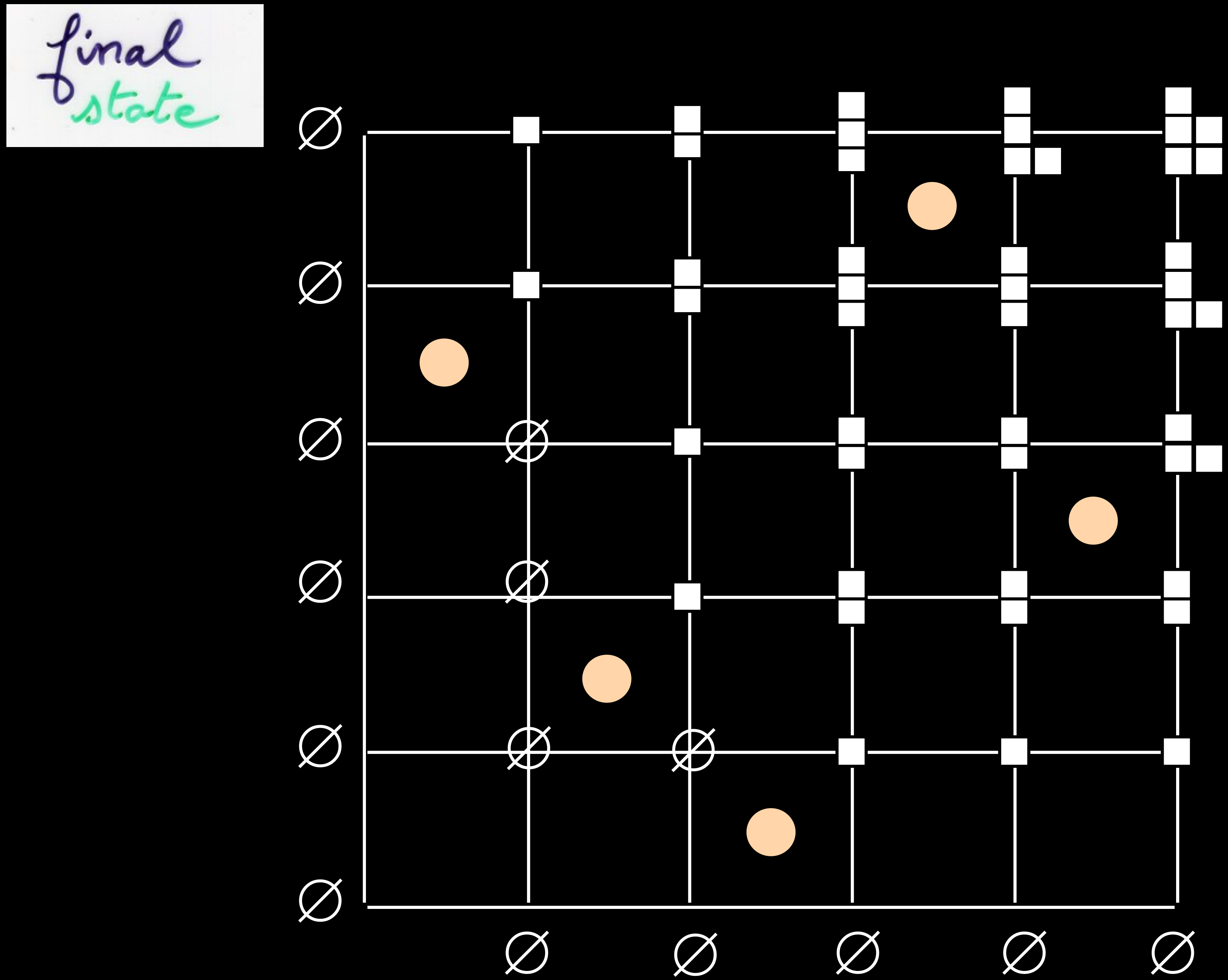
initial state

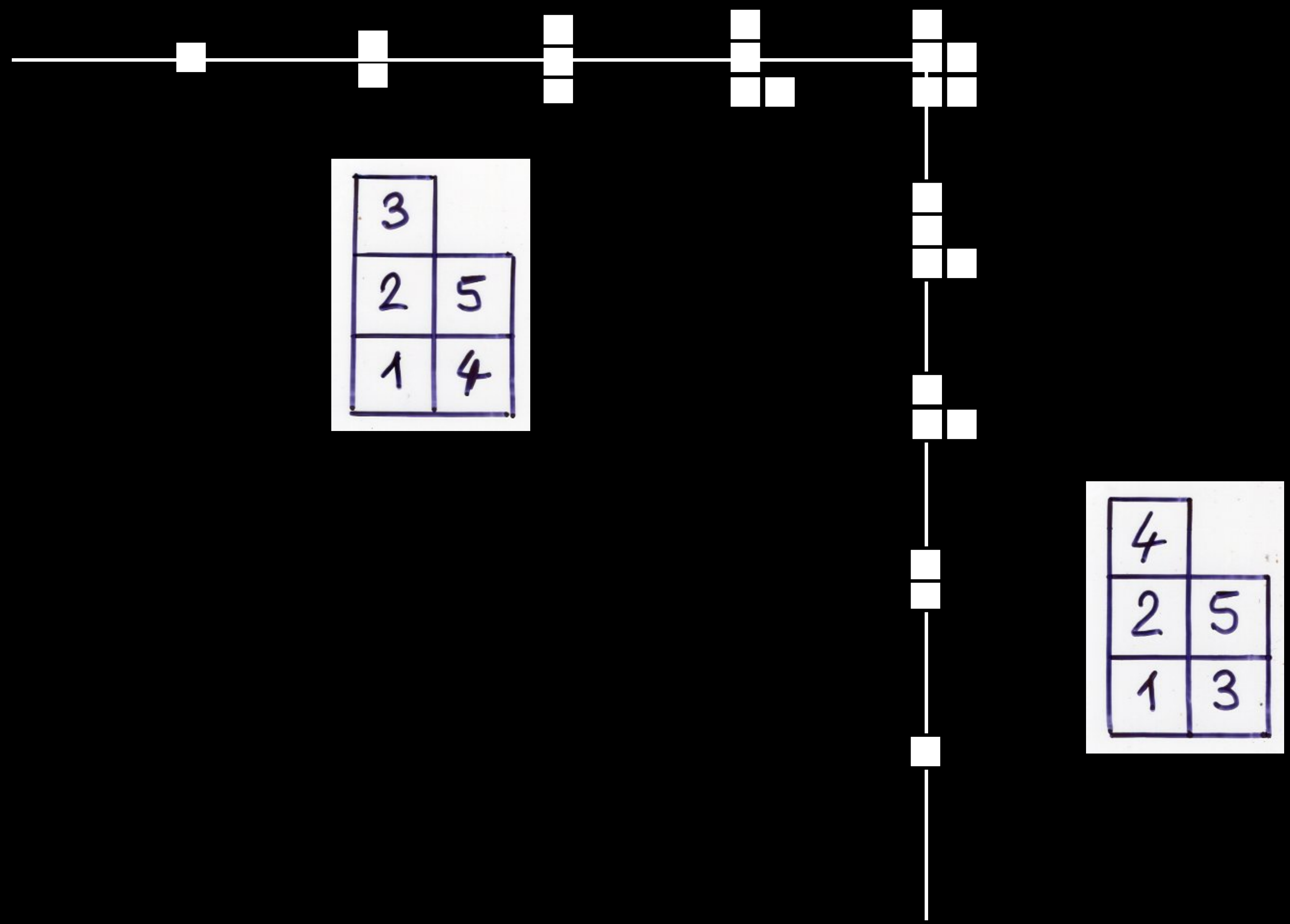


$\sigma = 4,2,1,5,3$

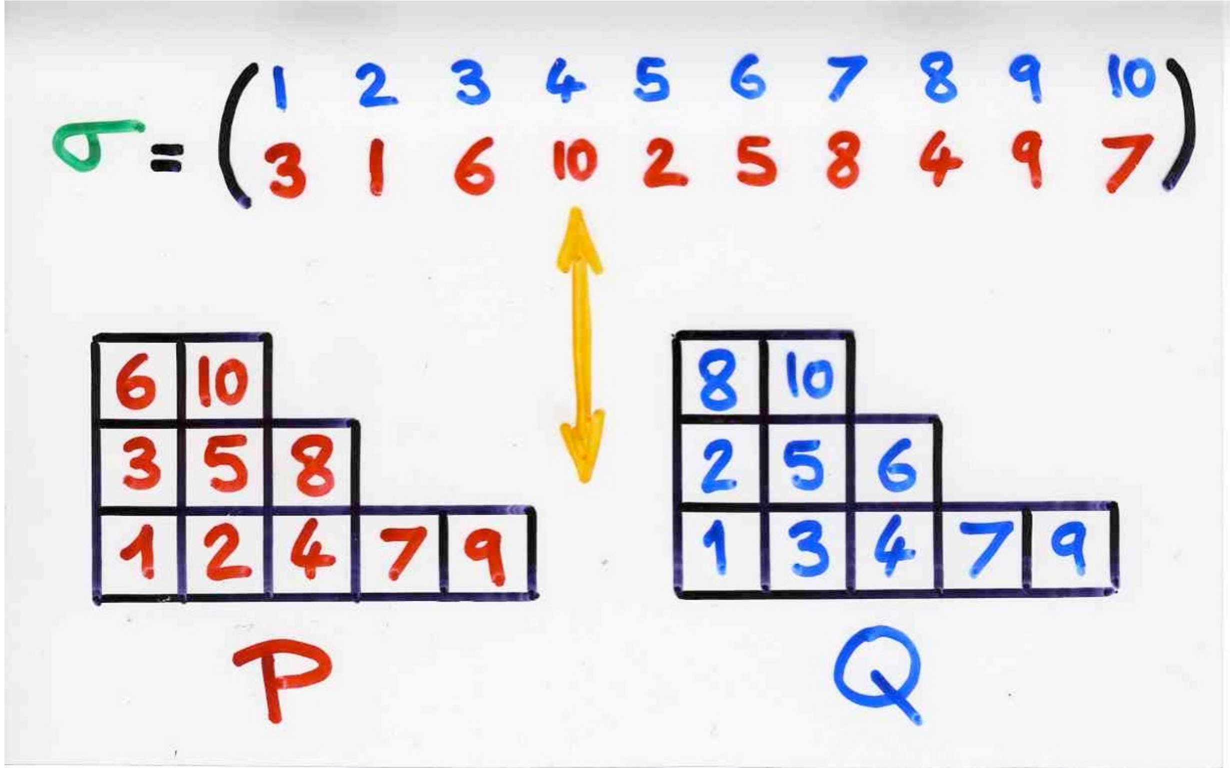
during the labeling process




final
state



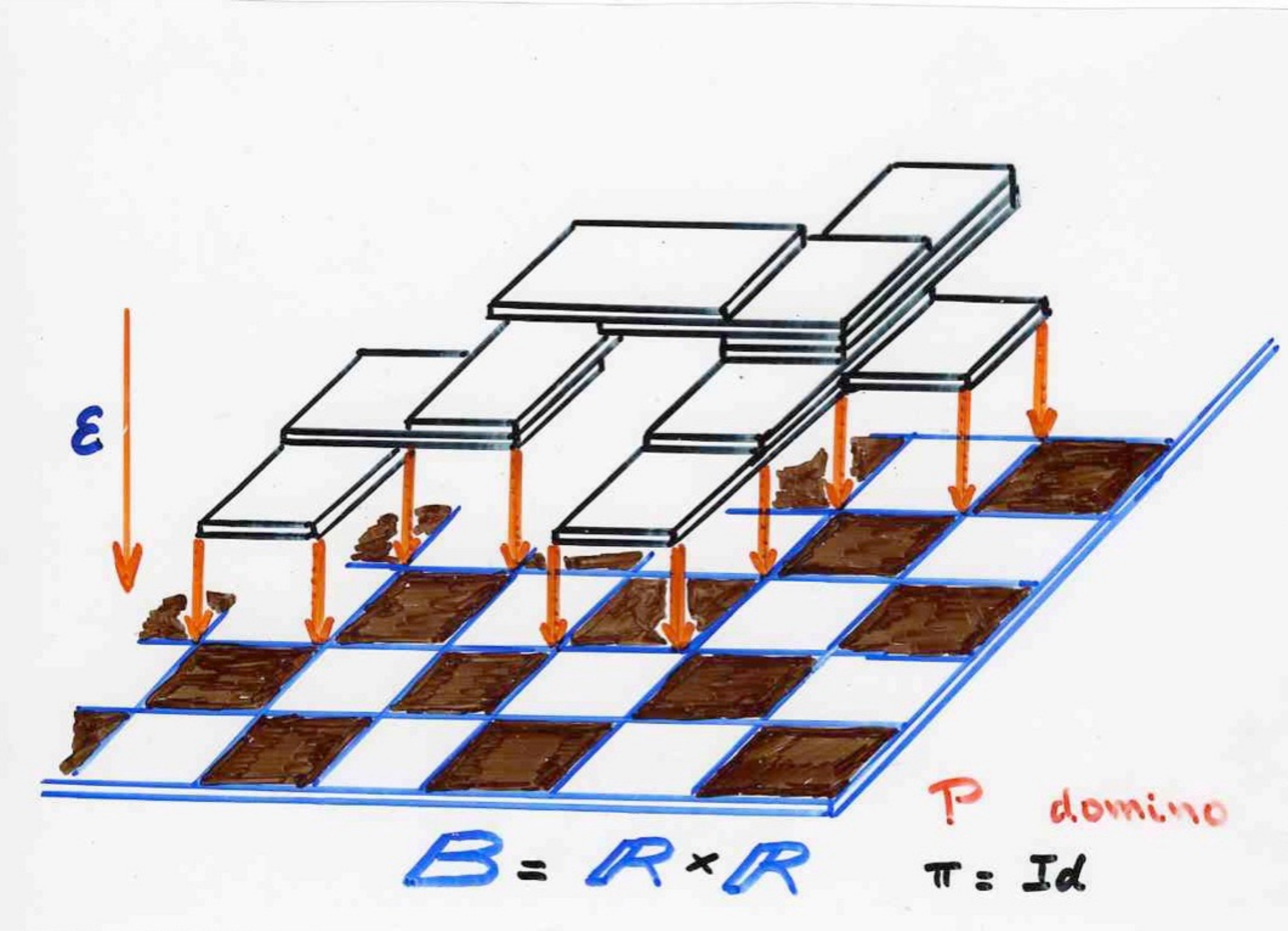
$\sigma = 4, 2, 1, 5, 3$

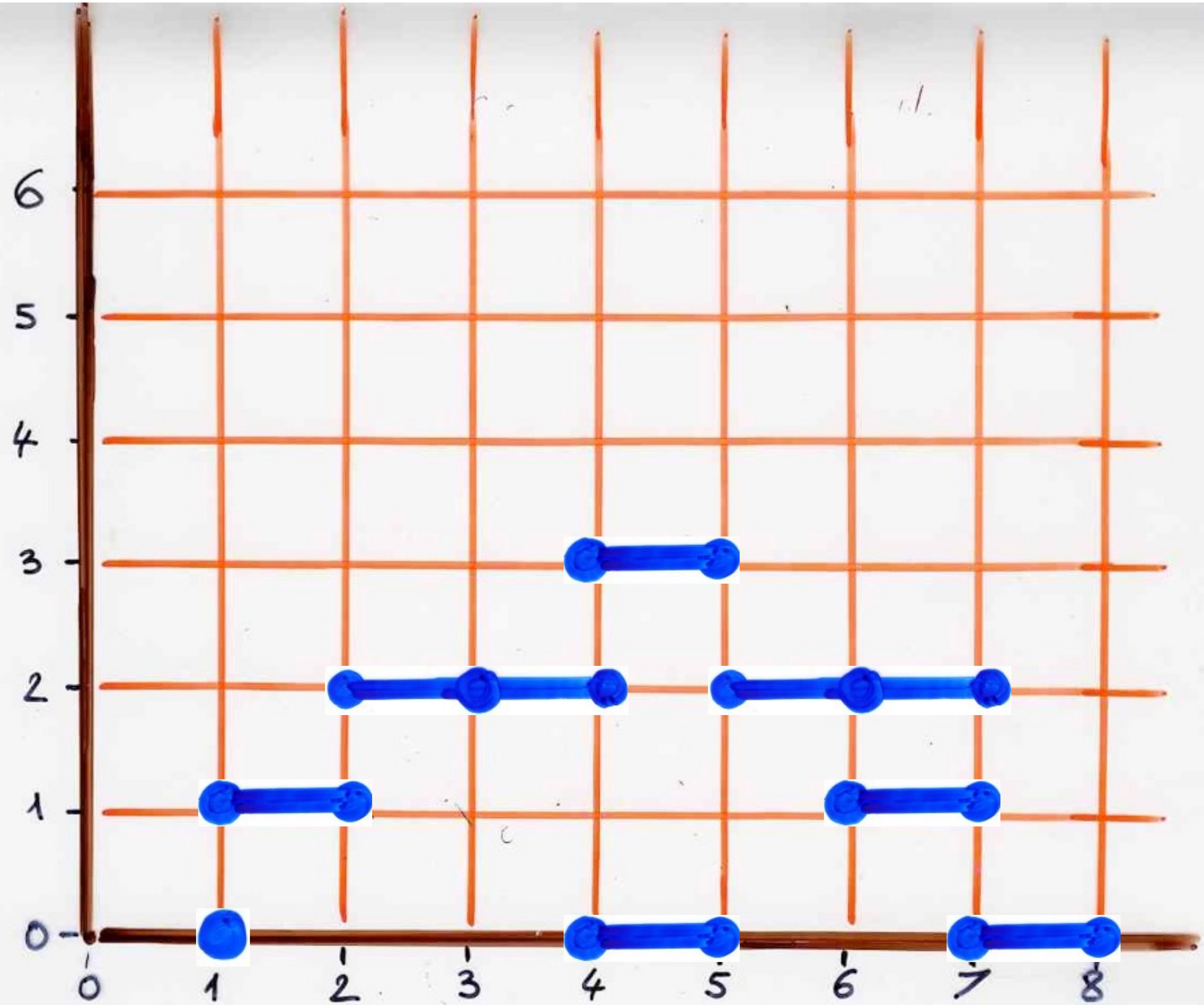


The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

heaps of segments

Introduction
Heaps





6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

Laguerre

Laguerre
polynomial

formal orthogonality

$$\mathfrak{f}(x^n) = \mu_n$$

$$\mathfrak{f}(P_k P_l) = 0$$

moments

$$k \neq l$$

$$\mathfrak{f}(PQ) = \int_a^b P(x)\, Q(x)\, d\mu$$

classical analysis

measure

Laguerre
polynomial

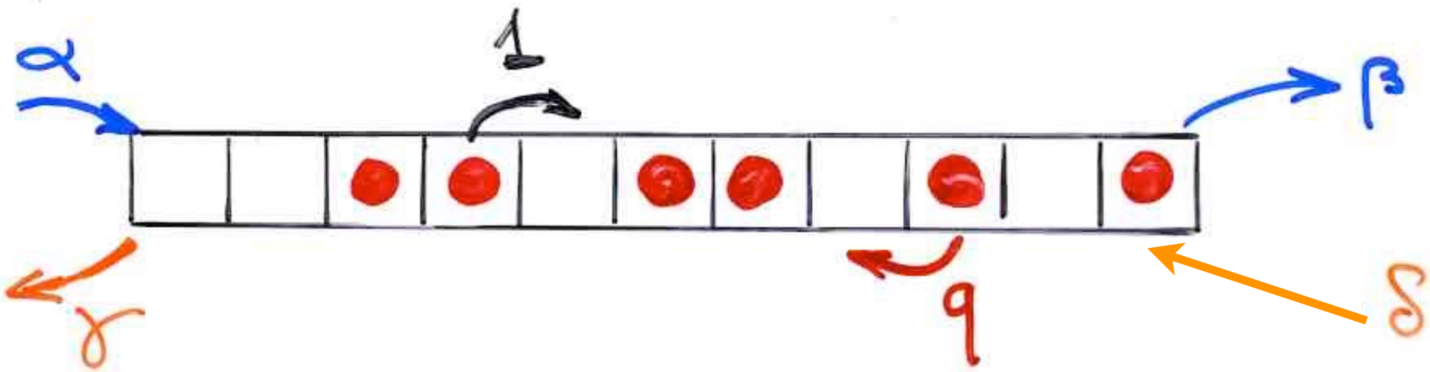
$$\mu_n = (n+1)!$$

moments

PASEP

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

$$\begin{cases} DE = qED + E + D \\ \langle W | (\alpha E - \gamma D) = \langle W | \\ (\beta D - \delta E) | V \rangle = | V \rangle \end{cases}$$

colum vector V

row vector W

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

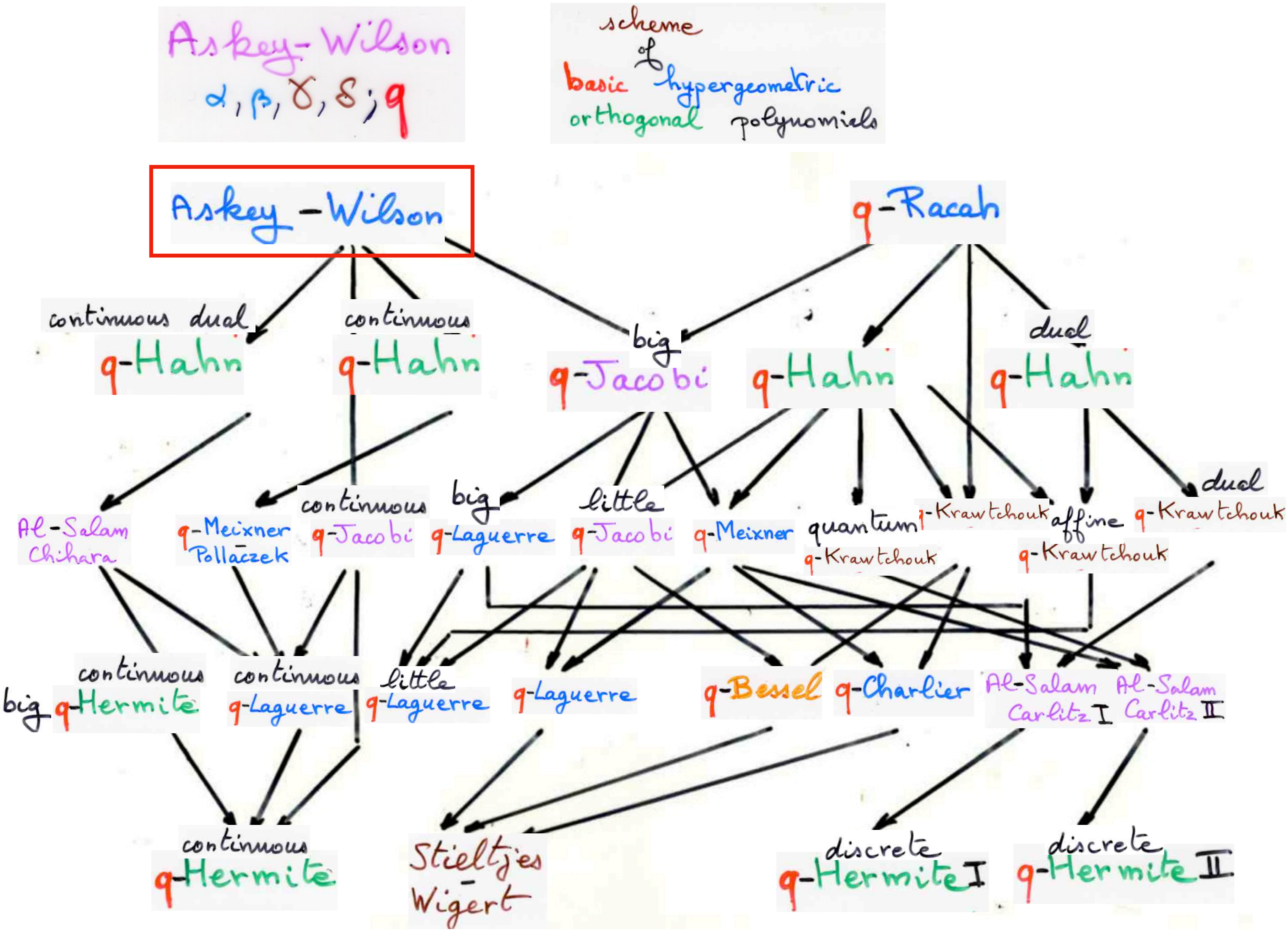
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials


Askey-Wilson
α, β, γ, δ ; q
scheme of basic hypergeometric orthogonal polynomials
Askey-Wilson
q-Racah
continuous dual q-Hahn
continuous q-Hahn
big q-Jacobi
q-Hahn
dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
continuous q-Jacobi
big q-Laguerre
little q-Jacobi
q-Meixner
quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes-Wigert
discrete q-Hermite I
discrete q-Hermite II

Combinatorial theory of orthogonal polynomials

combinatorial

theory of

orthogonal polynomials

moments X.V. (1983)

Françon, X.V. (1978)

and

continued fractions

Flajolet (1980)

Thm. (Favard)

- $\{P_n(x)\}_{n\geq 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

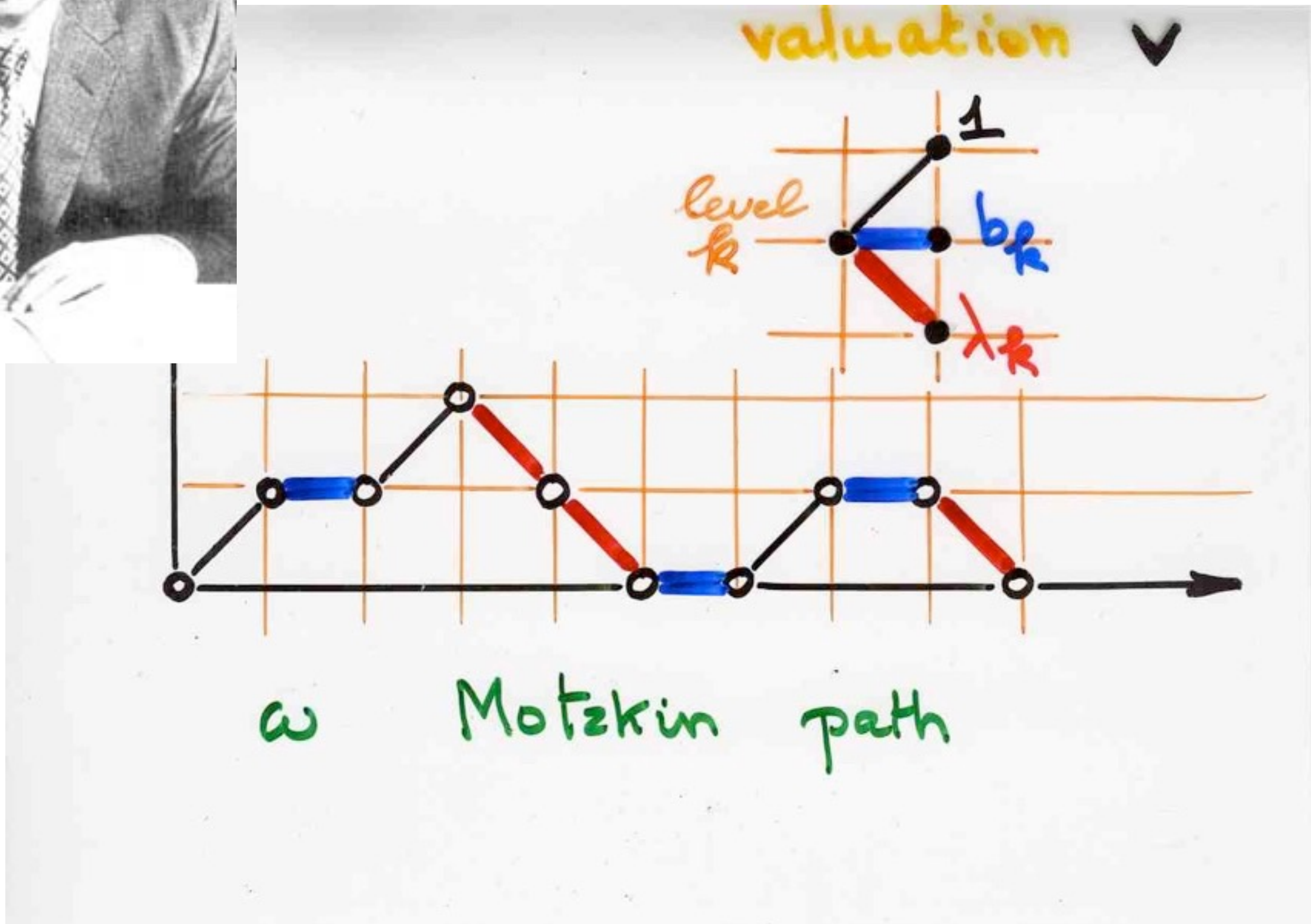
$(\forall k\geq 1)$

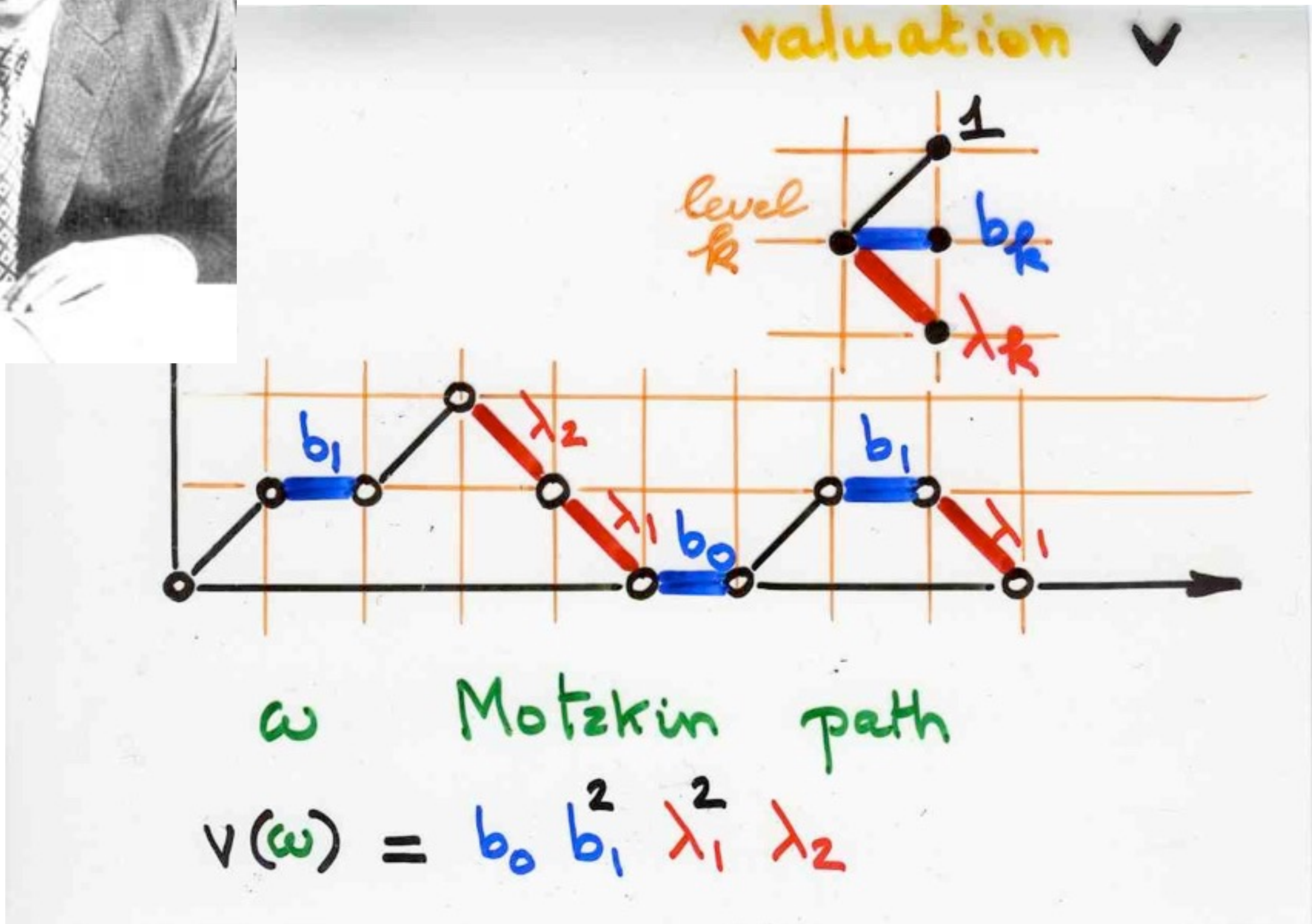
3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring

μ_n ?


valuation v
1
level k
b_k
λ_k
ω Motzkin path



valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 b_1^2 \lambda_1^2 \lambda_2$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}}$$

Philippe Flajolet
fundamental
Lemma

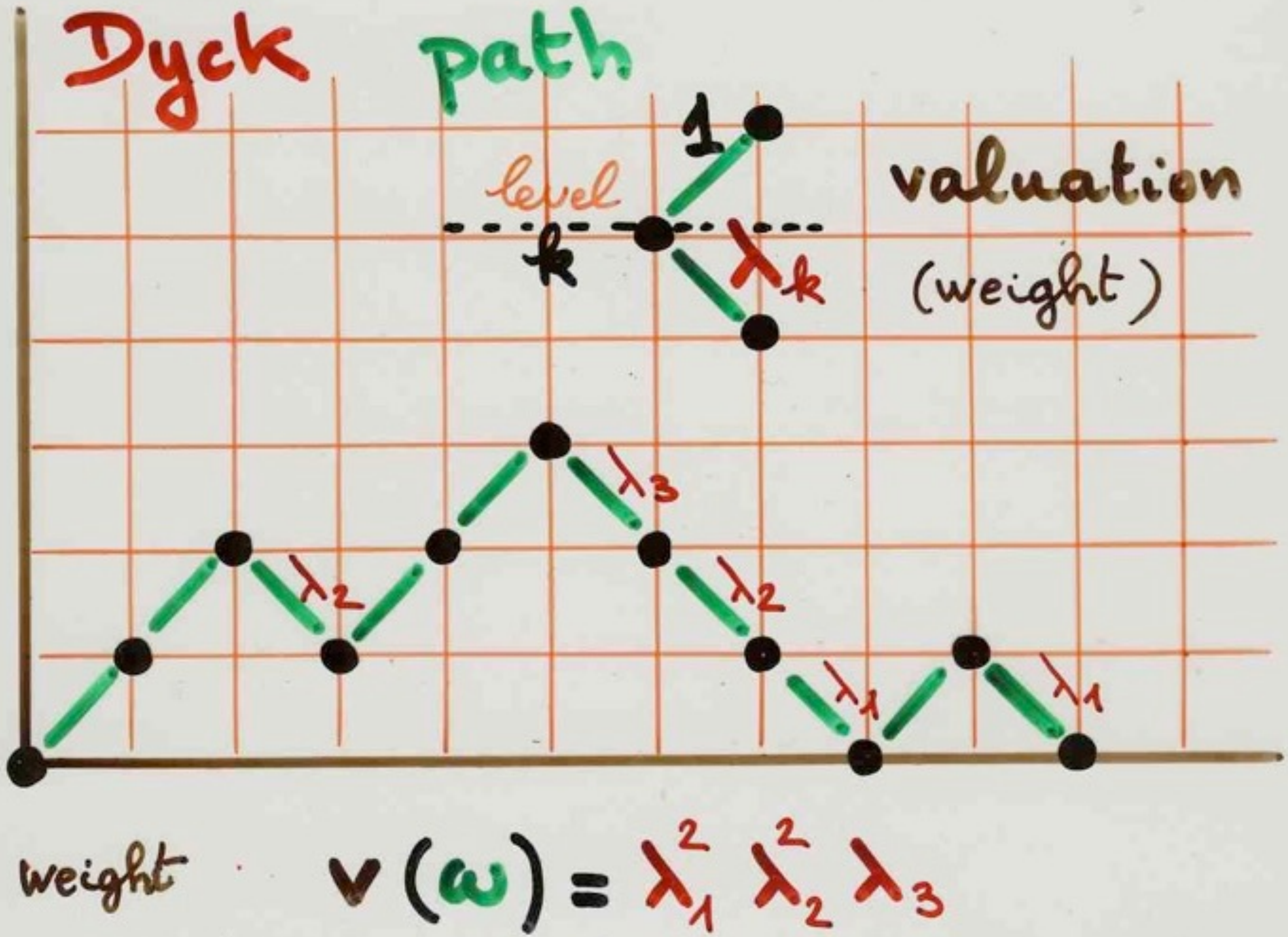
continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{\cdots\cdots\cdot \atop \overline{1 - \cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$\mu_0 = 1$

$S(t;\lambda)$

Stieltjès continued fraction



Dyck path
1
level
k
λ_k
valuation
(weight)
λ_3
λ_2
λ_2
λ_1
λ_1
weight $v(\omega) = \lambda_1^2 \lambda_2^2 \lambda_3$

The notion of histories

example: Hermite histories

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

Hermite polynomials

$$\cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}$$

moments
Hermite
polynomials

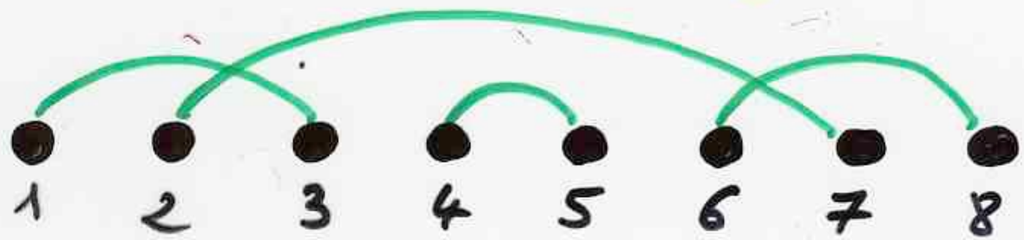
Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching



atque series infinita ita se habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis est huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

DE FRACTIONIBVS CONTINVIS.
DISSERTATIO.

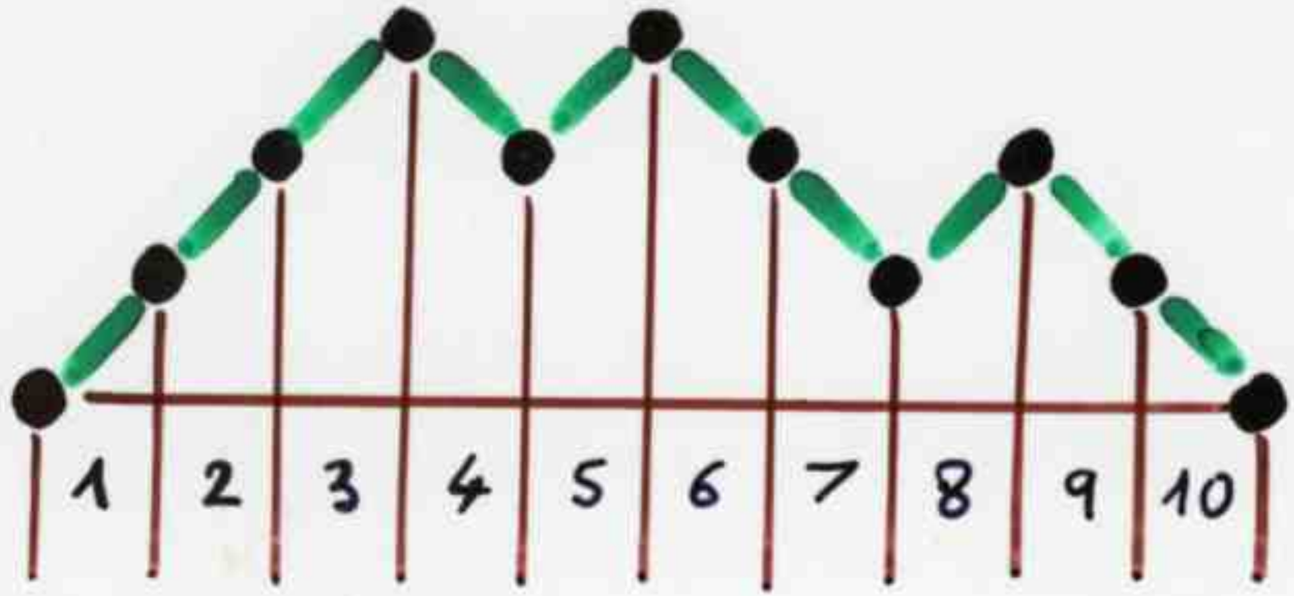
AVCTORE
Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.\,4.6.\,6.8.\,8.10.\,10}{3.3.\,5.5.\,7.7\,\,9.9.\,11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

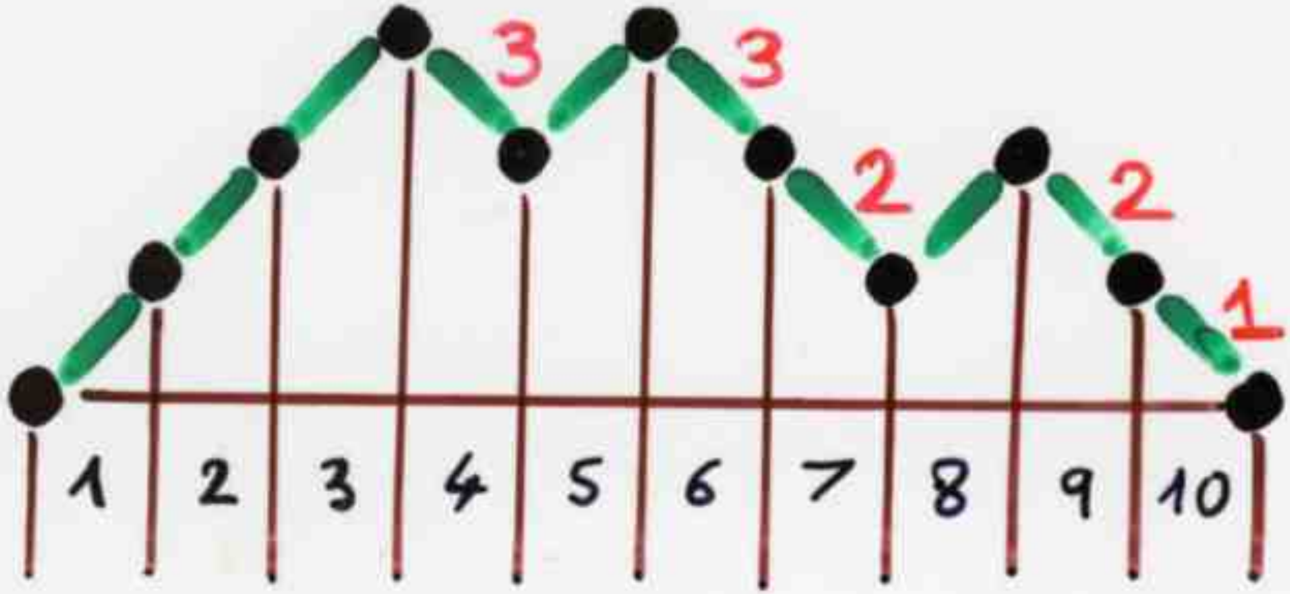
§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

Hermite history



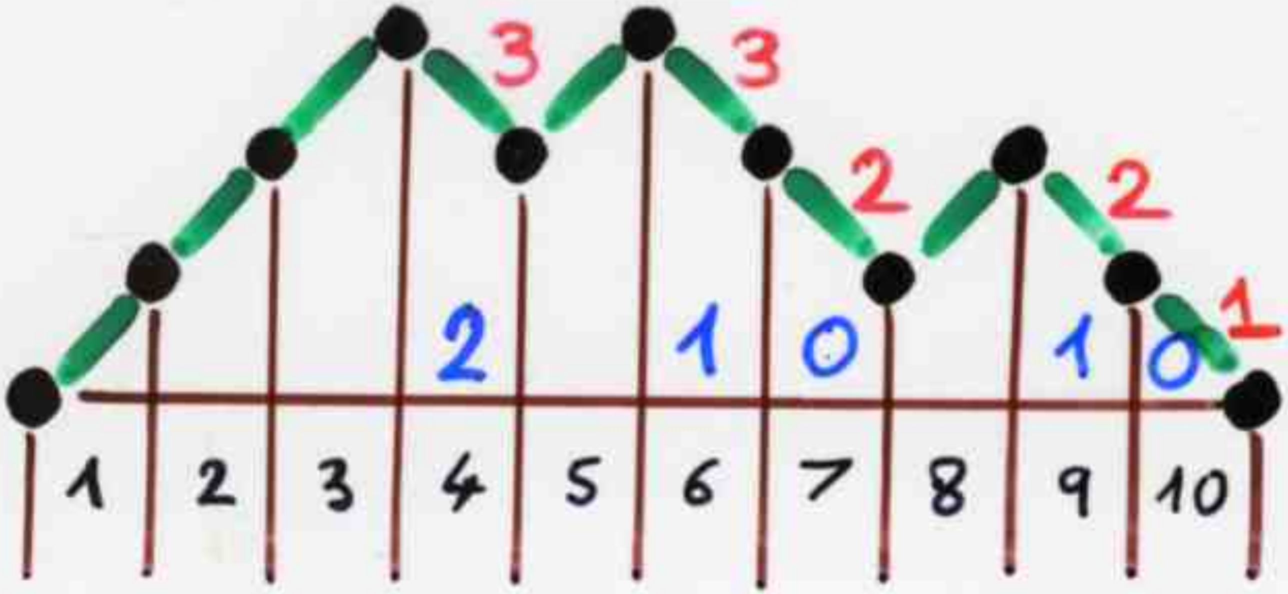
Hermite history

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

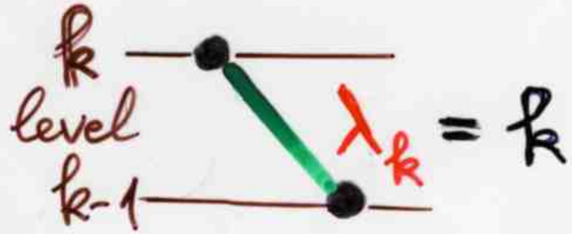


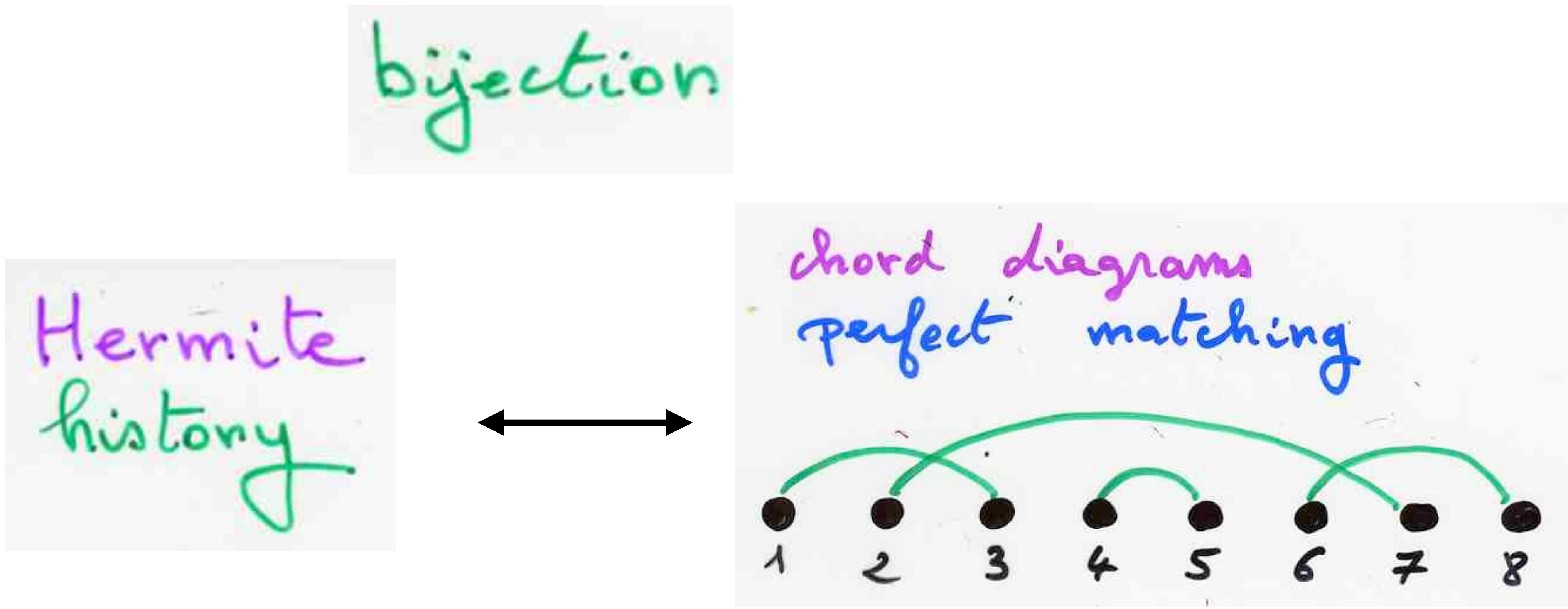
Hermite history

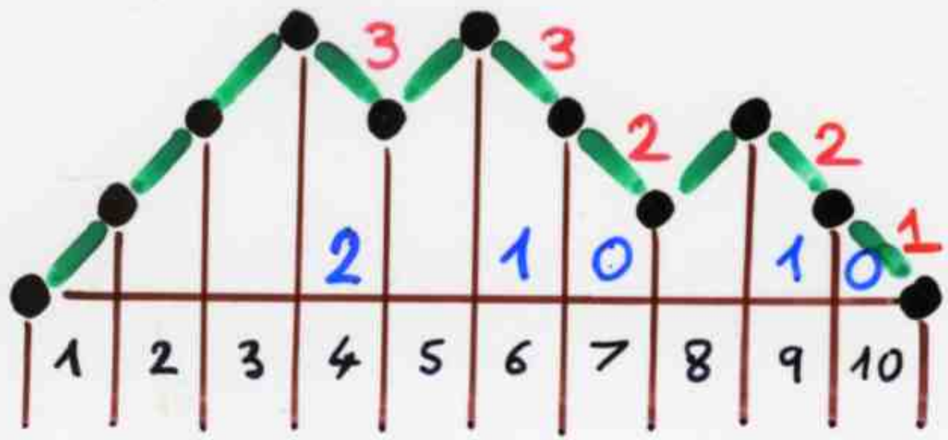
Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$



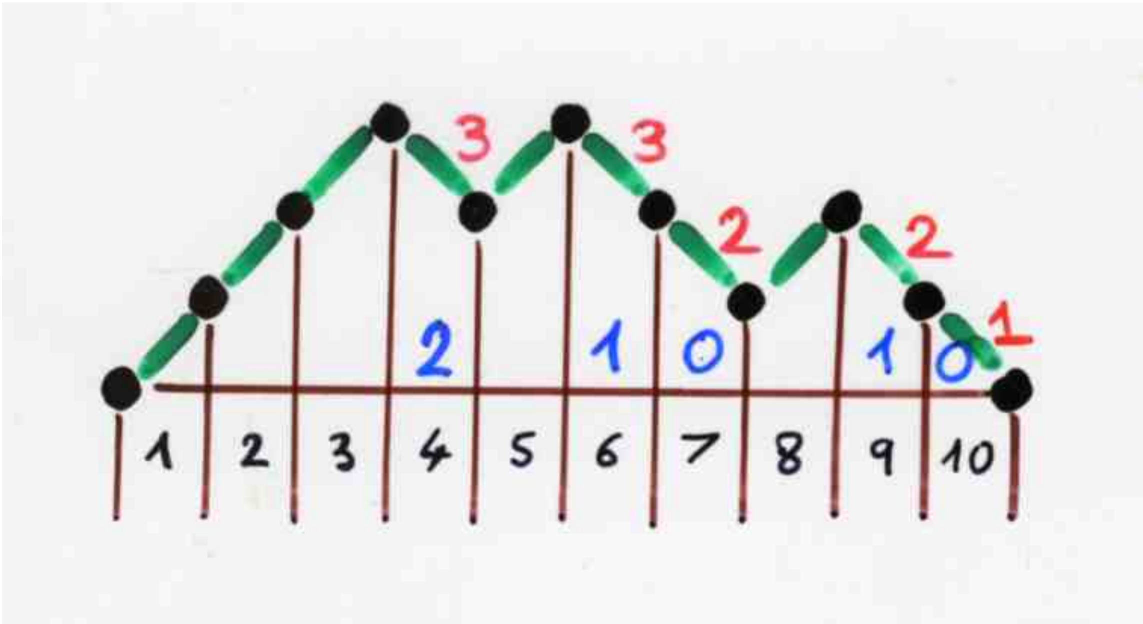
$$0 \leq i < \lambda_k = k$$

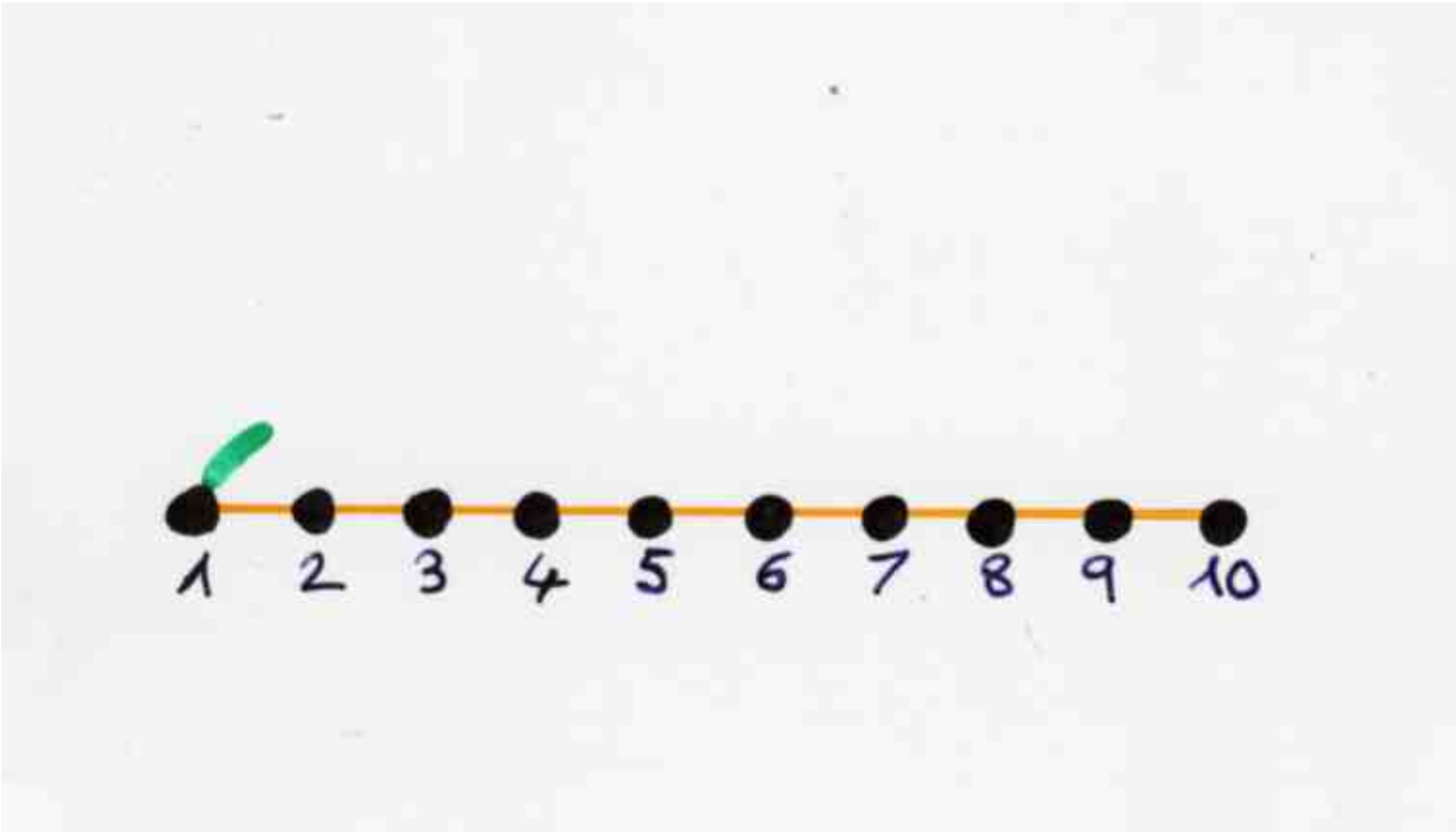


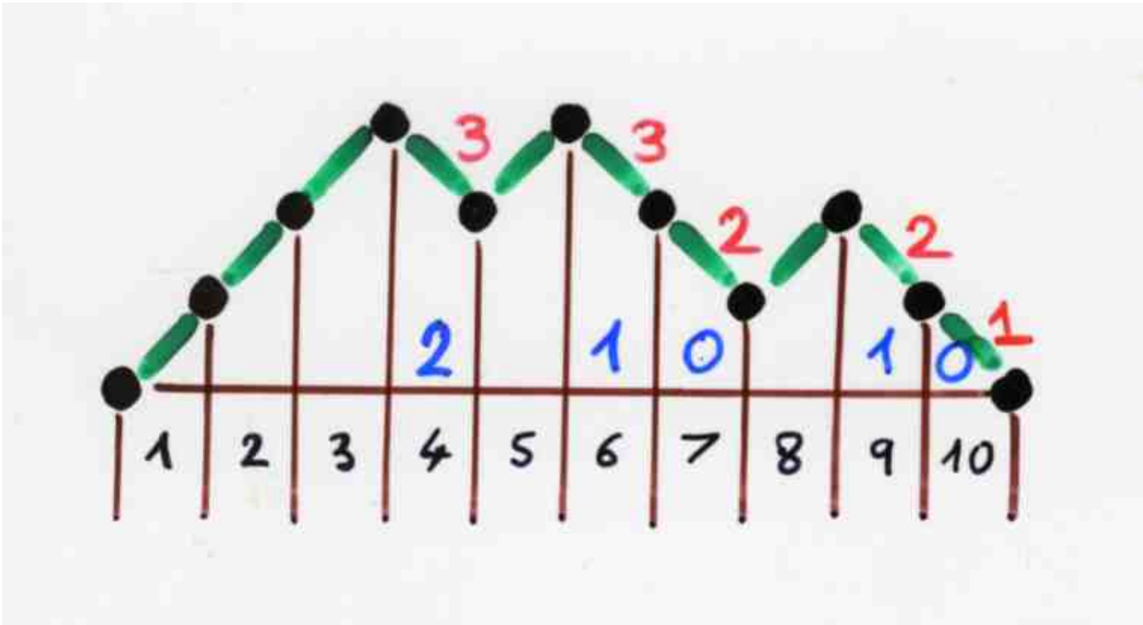

bijection
Hermite history
chord diagrams
perfect matching
1
2
3
4
5
6
7
8

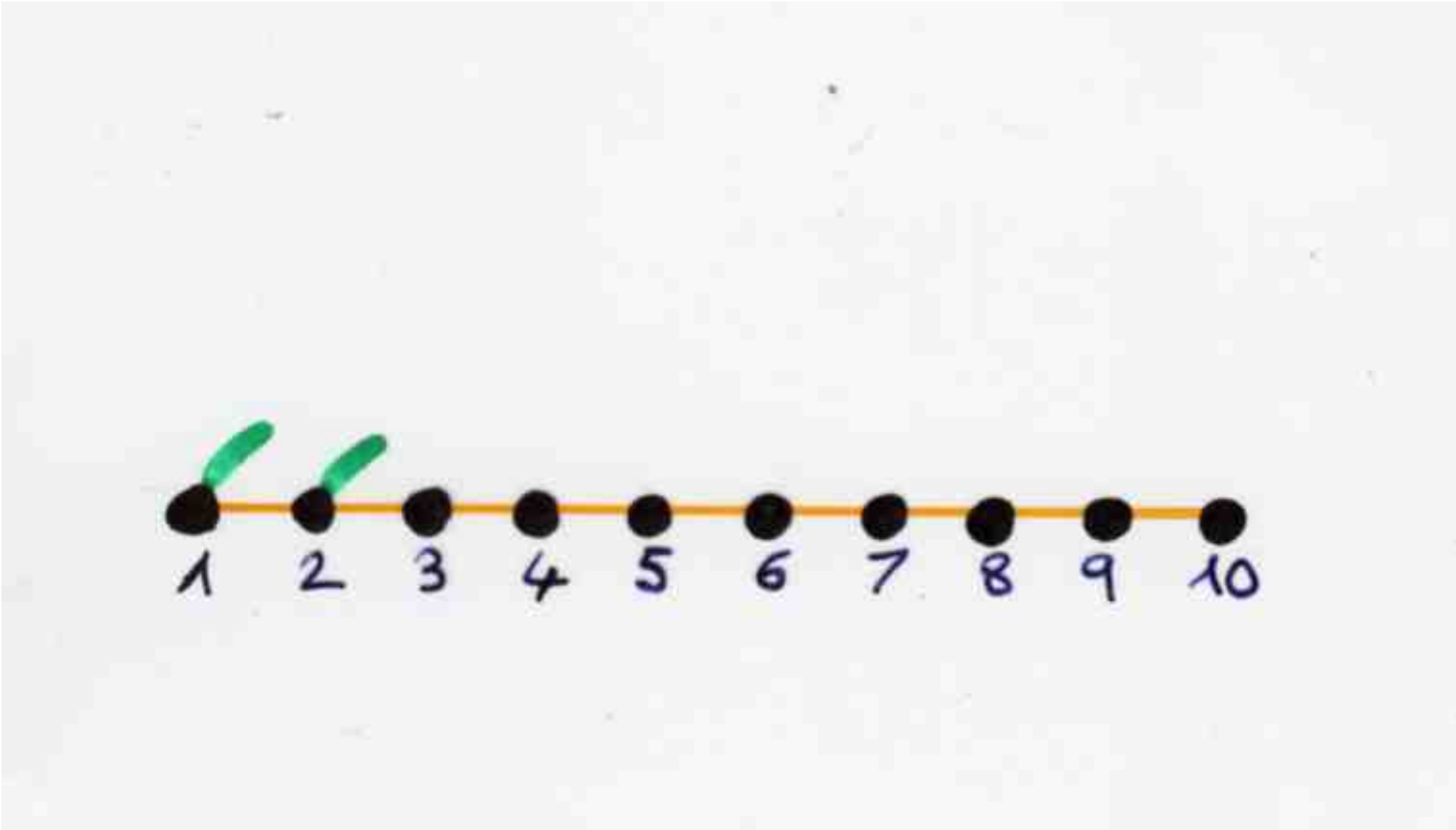

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

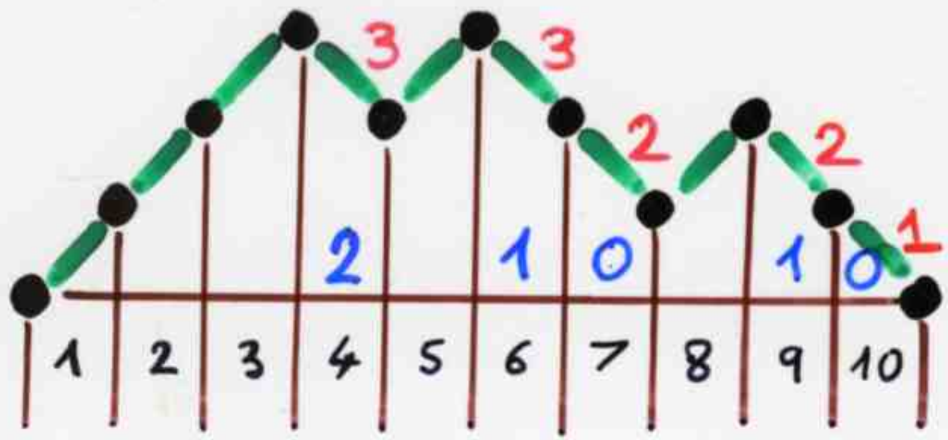

1 2 3 4 5 6 7 8 9 10

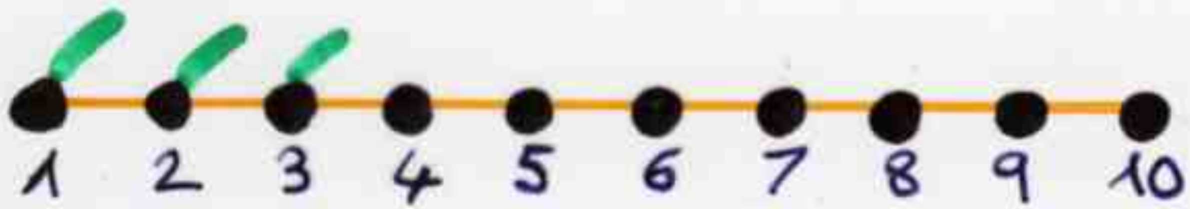

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

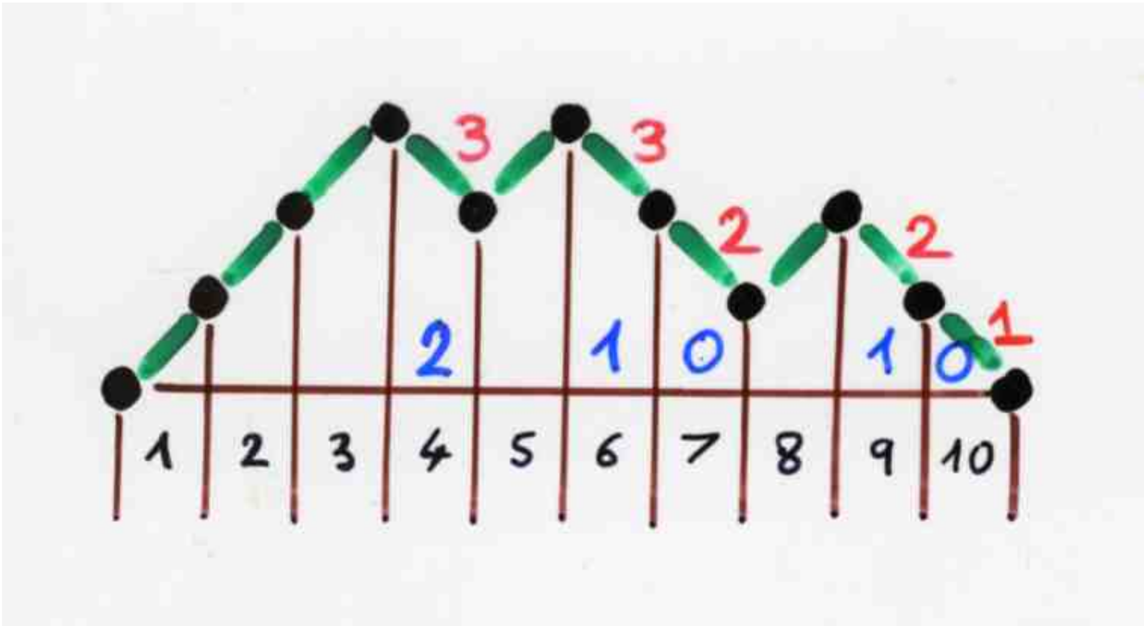

1
2
3
4
5
6
7
8
9
10

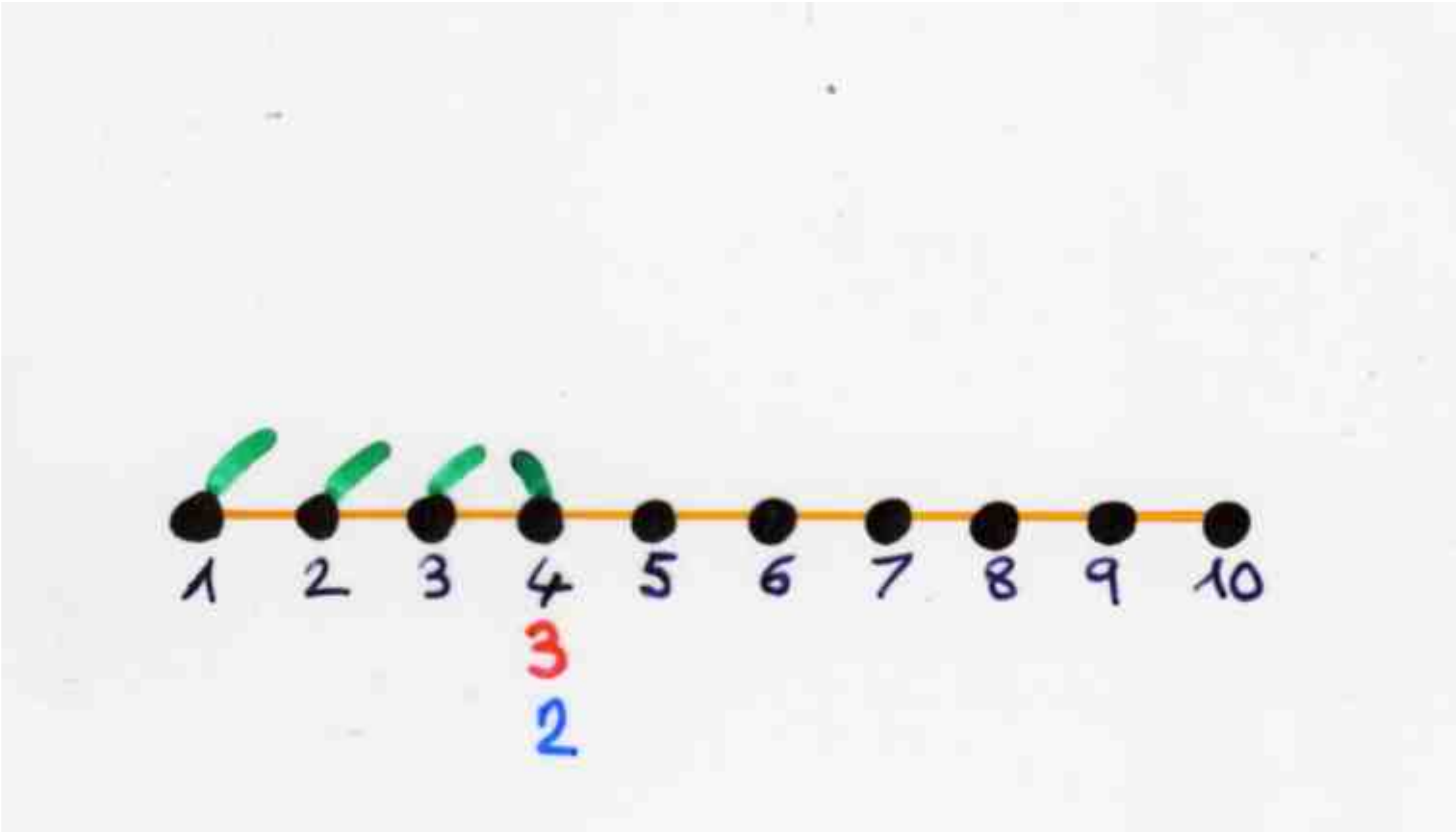

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

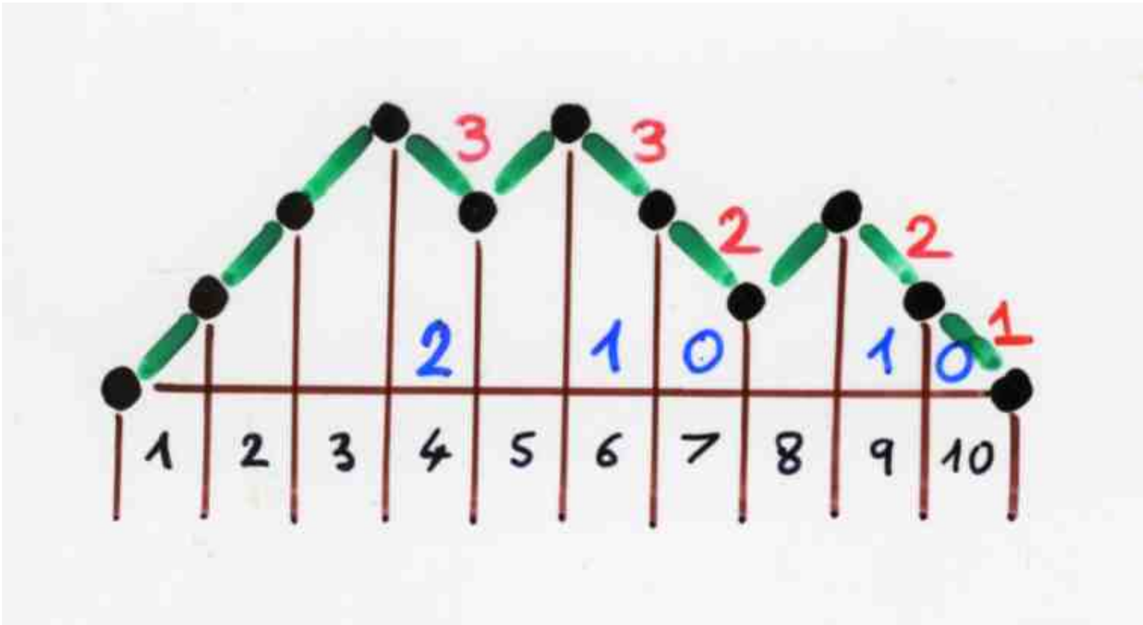

1 2 3 4 5 6 7 8 9 10

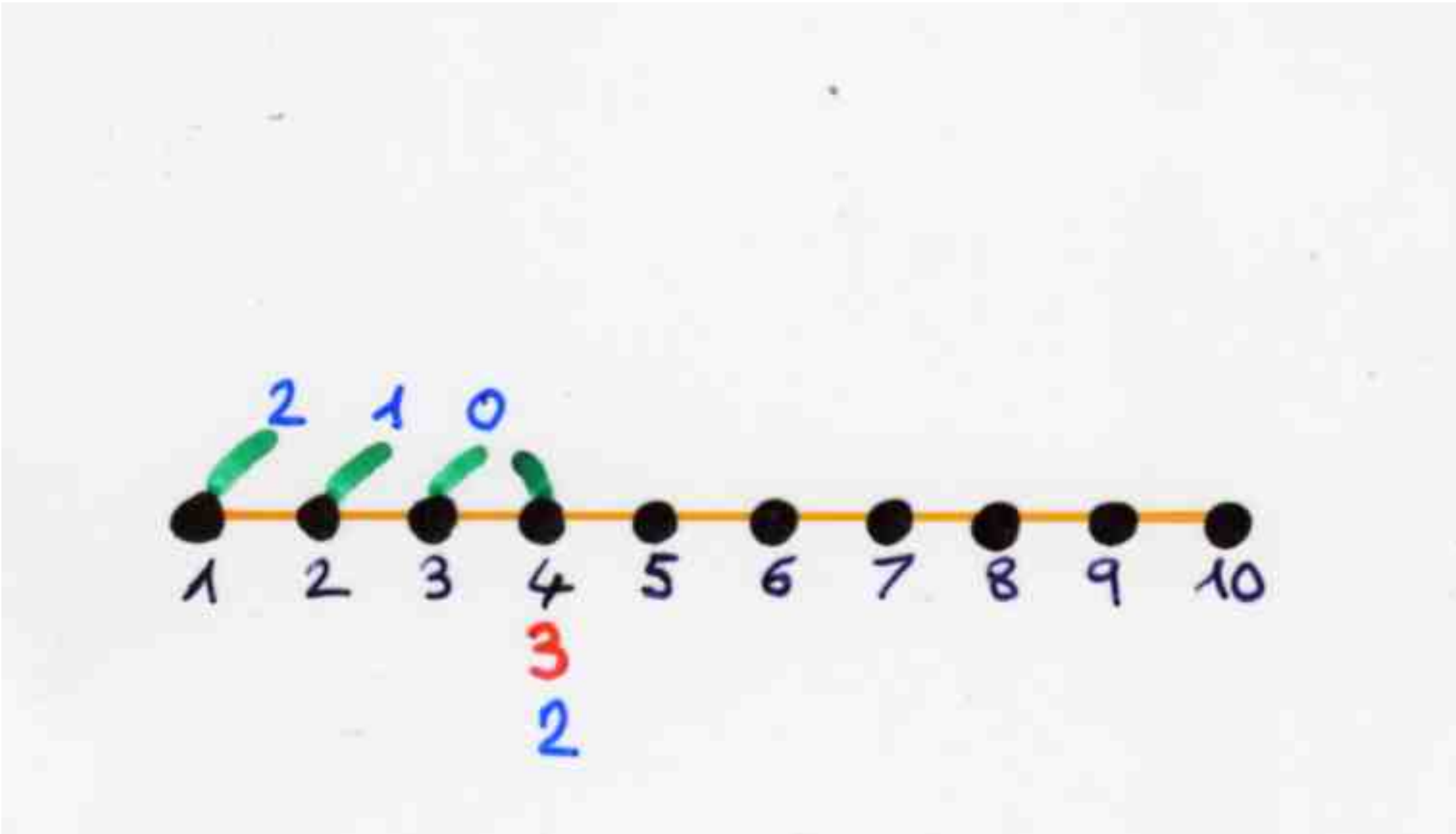

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

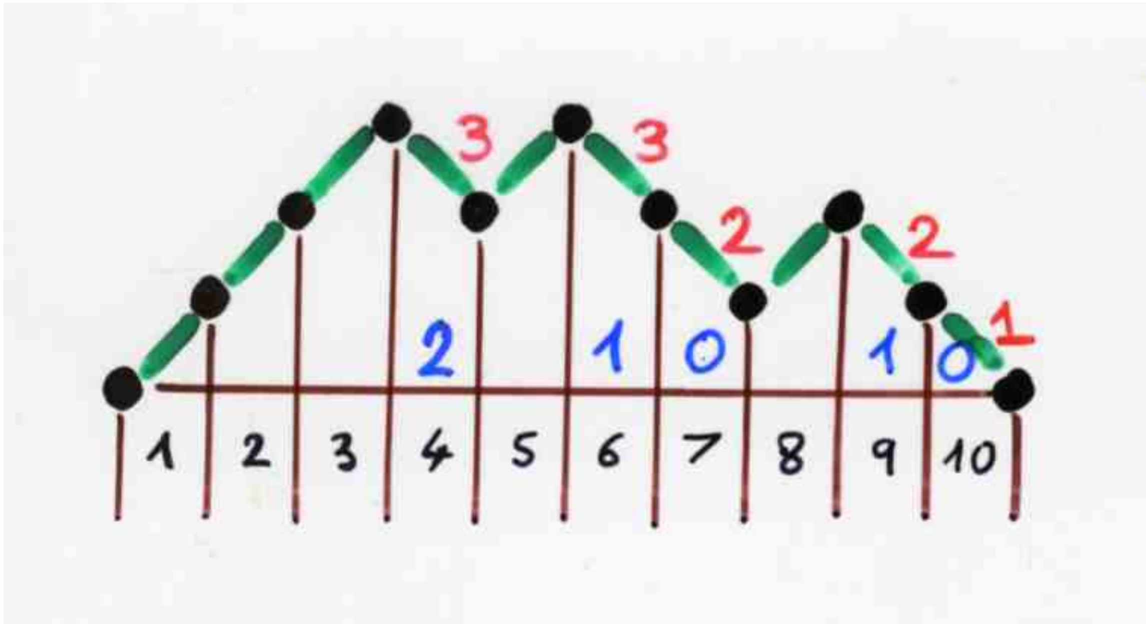

1
2
3
4
5
6
7
8
9
10

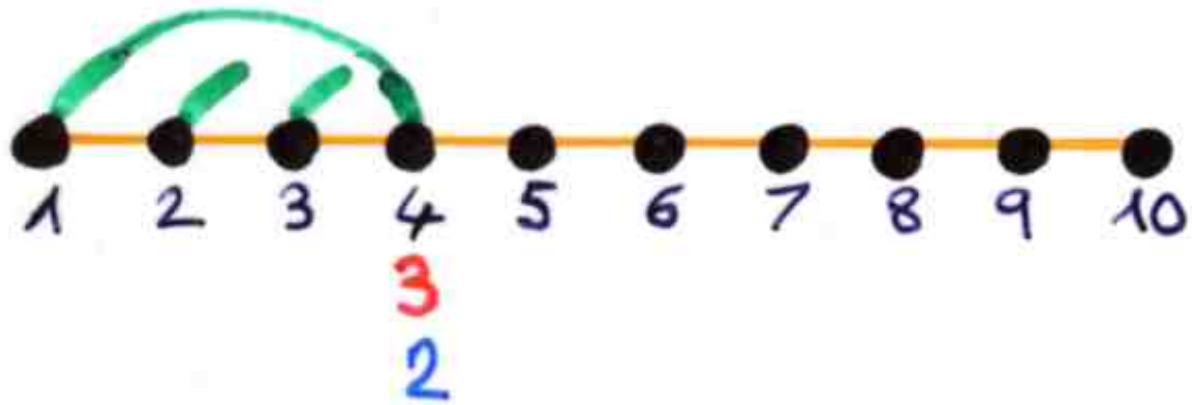

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

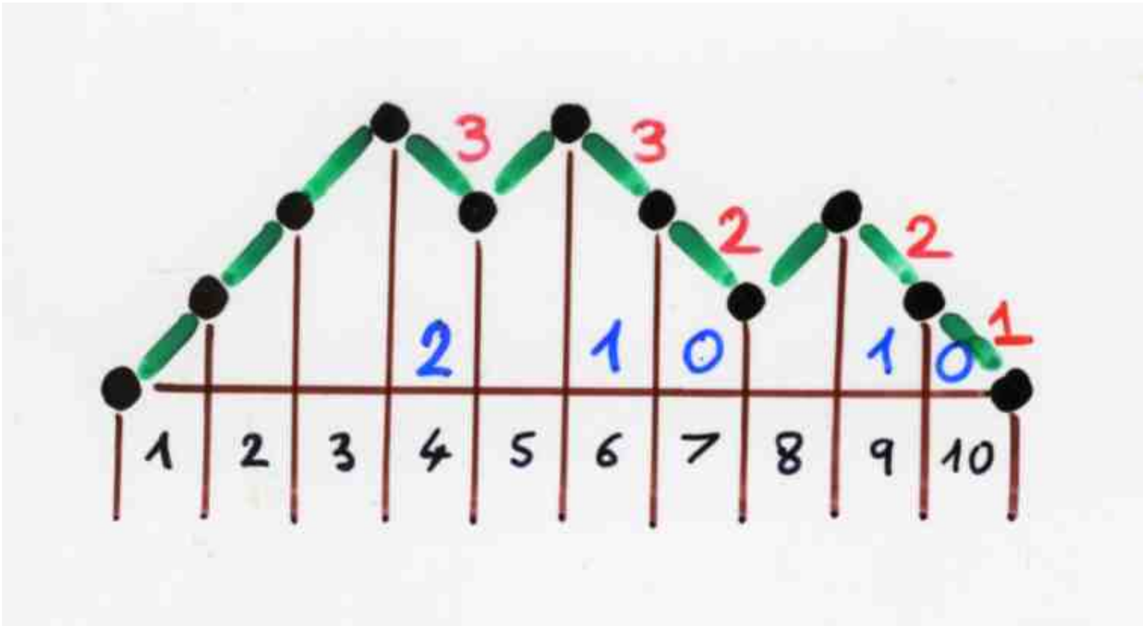

1 2 3 4 5 6 7 8 9 10
3
2

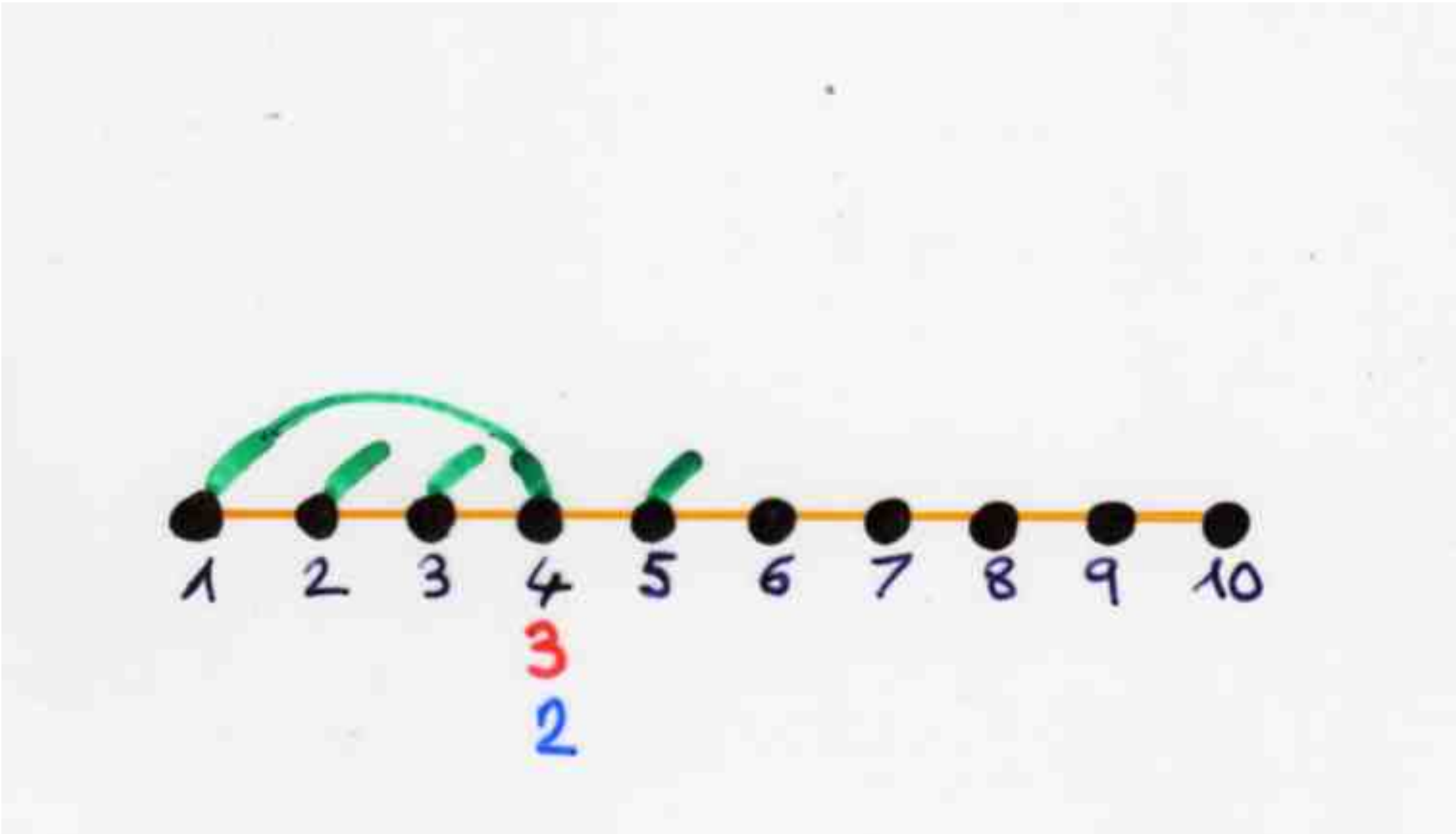

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

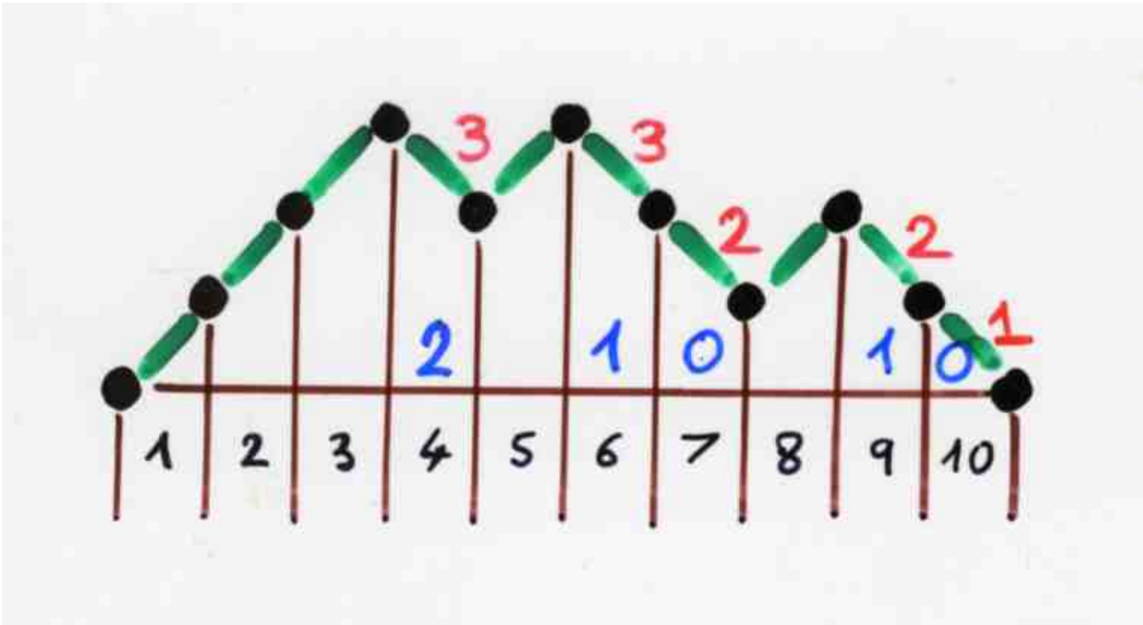

2 1 0
1 2 3 4 5 6 7 8 9 10
3
2

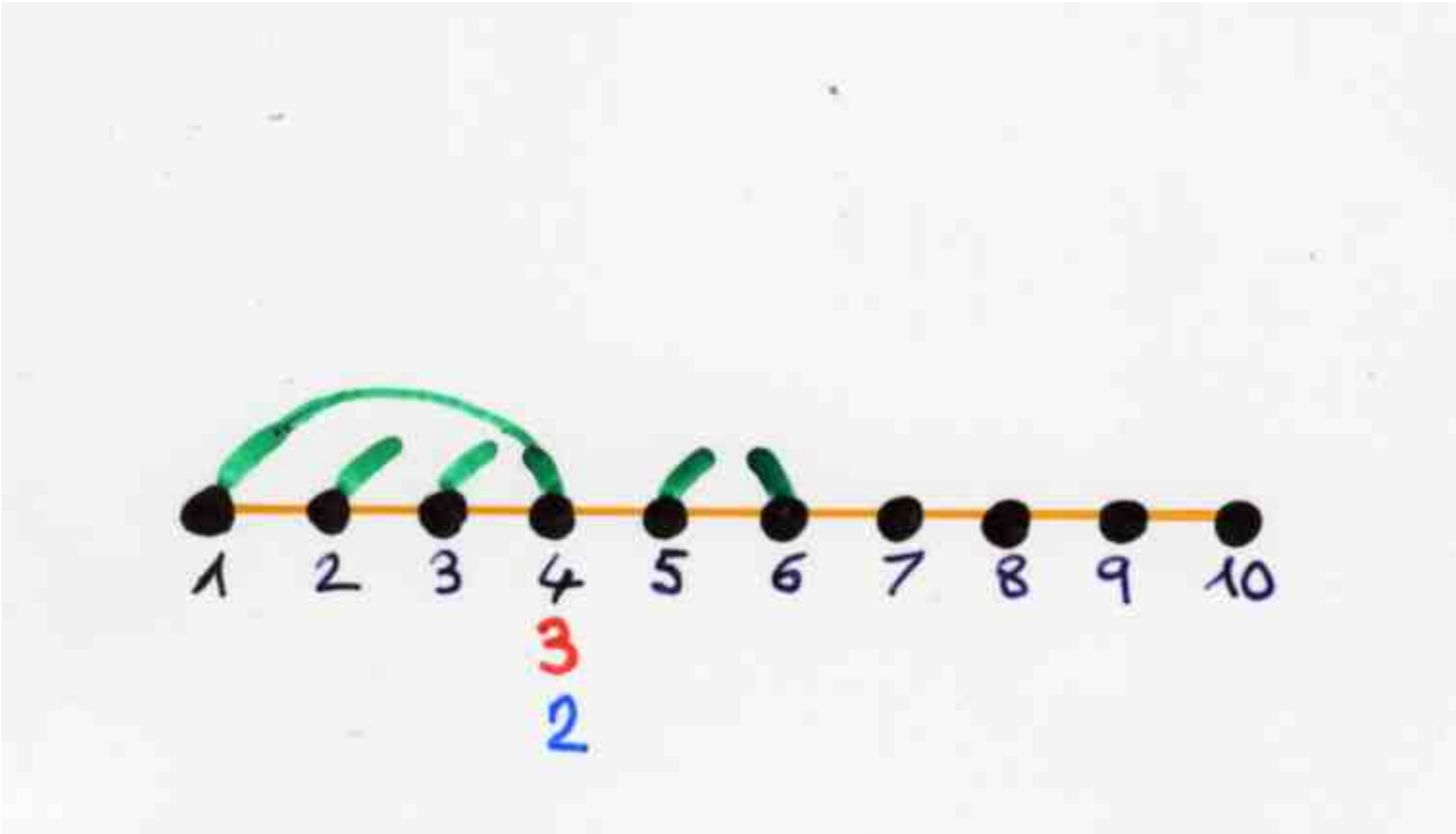

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

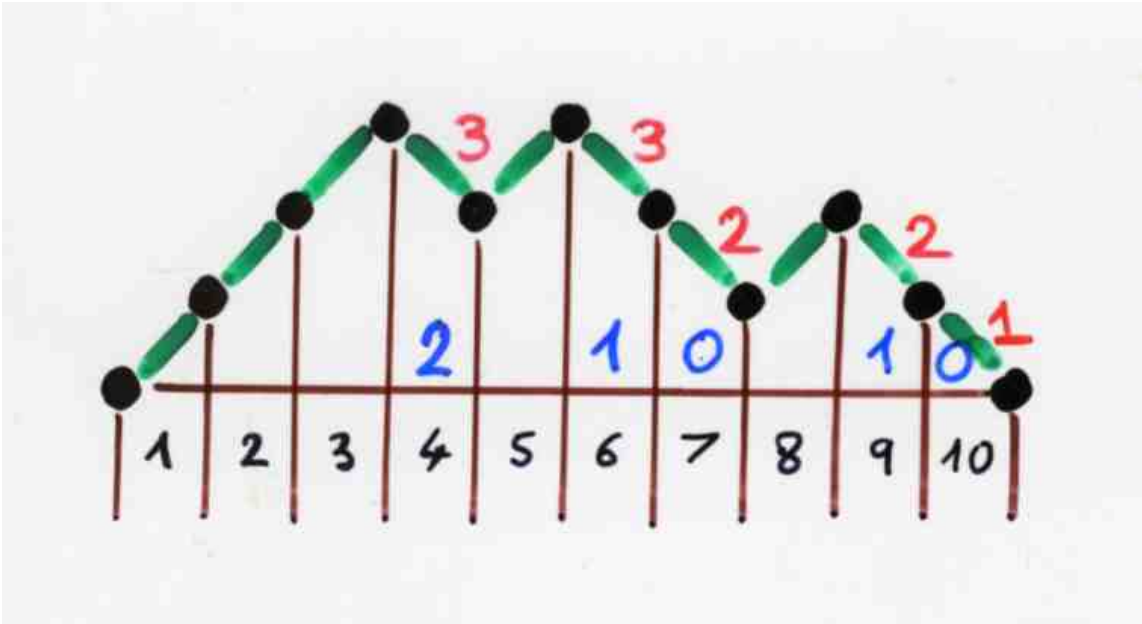

1 2 3 4 5 6 7 8 9 10
3
2

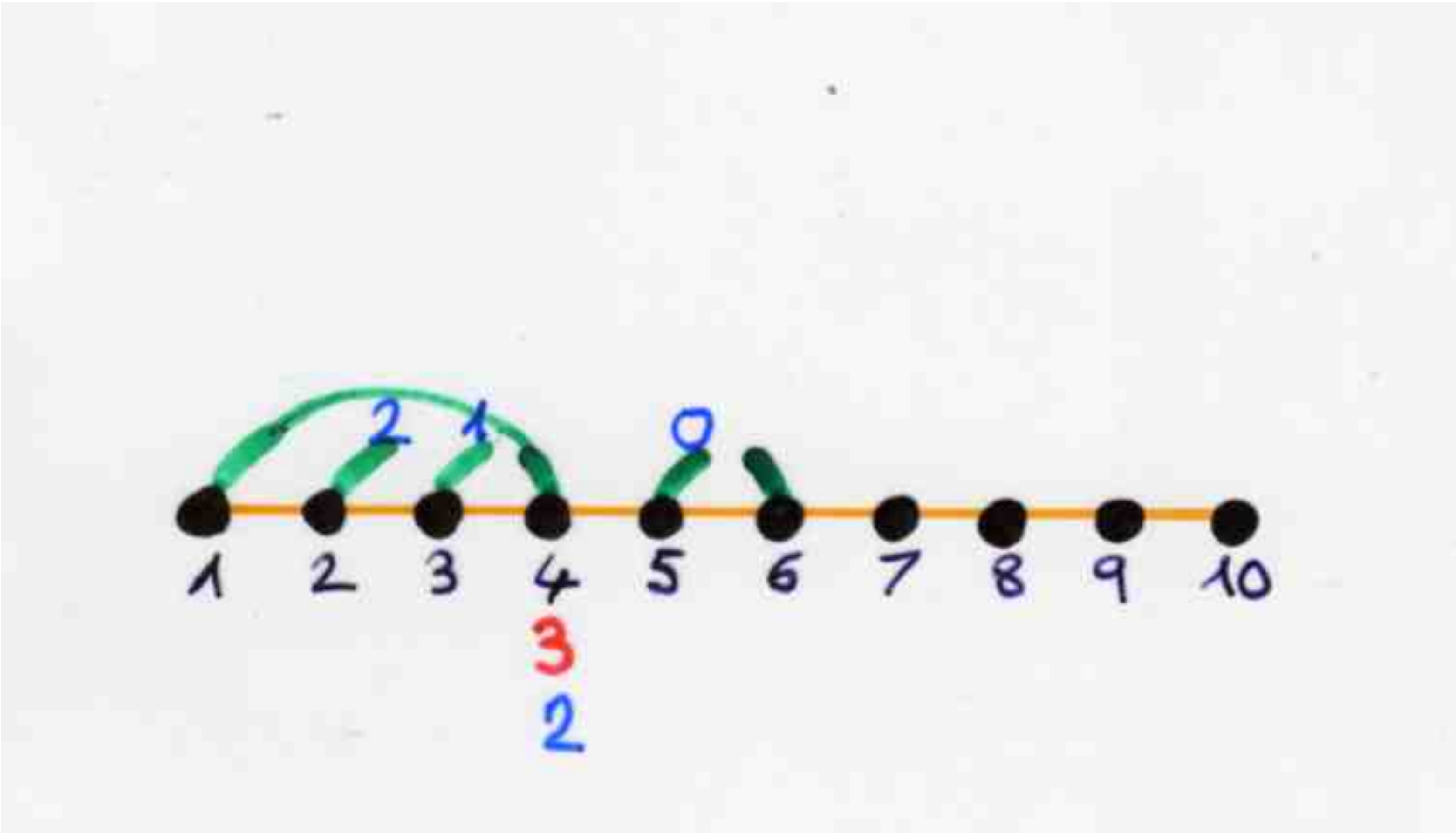

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

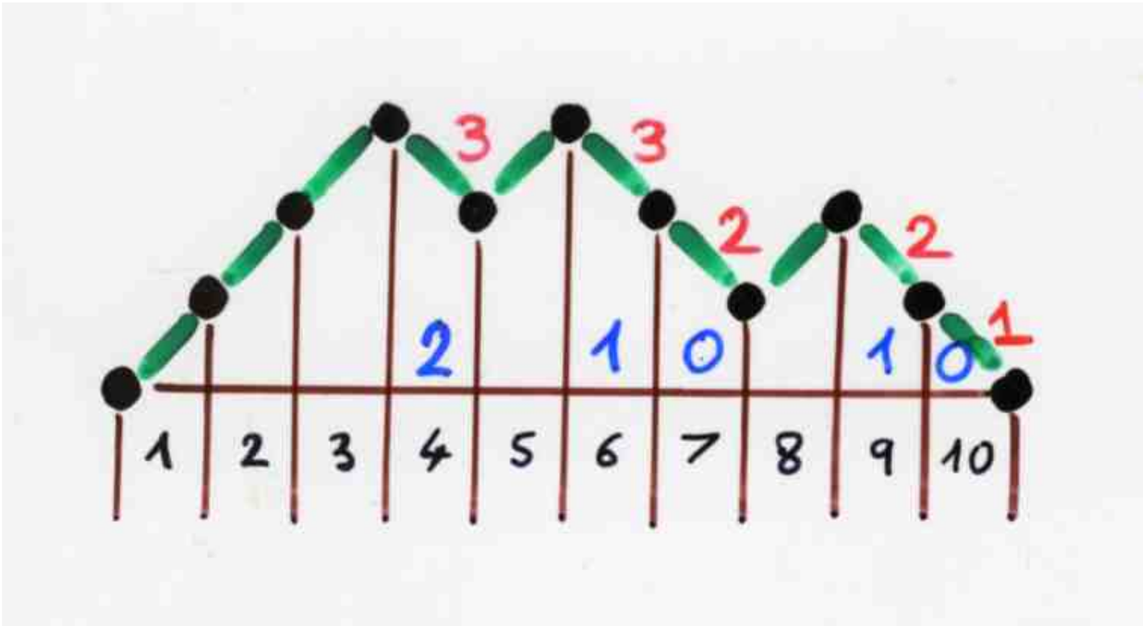

1
2
3
4
5
6
7
8
9
10
3
2

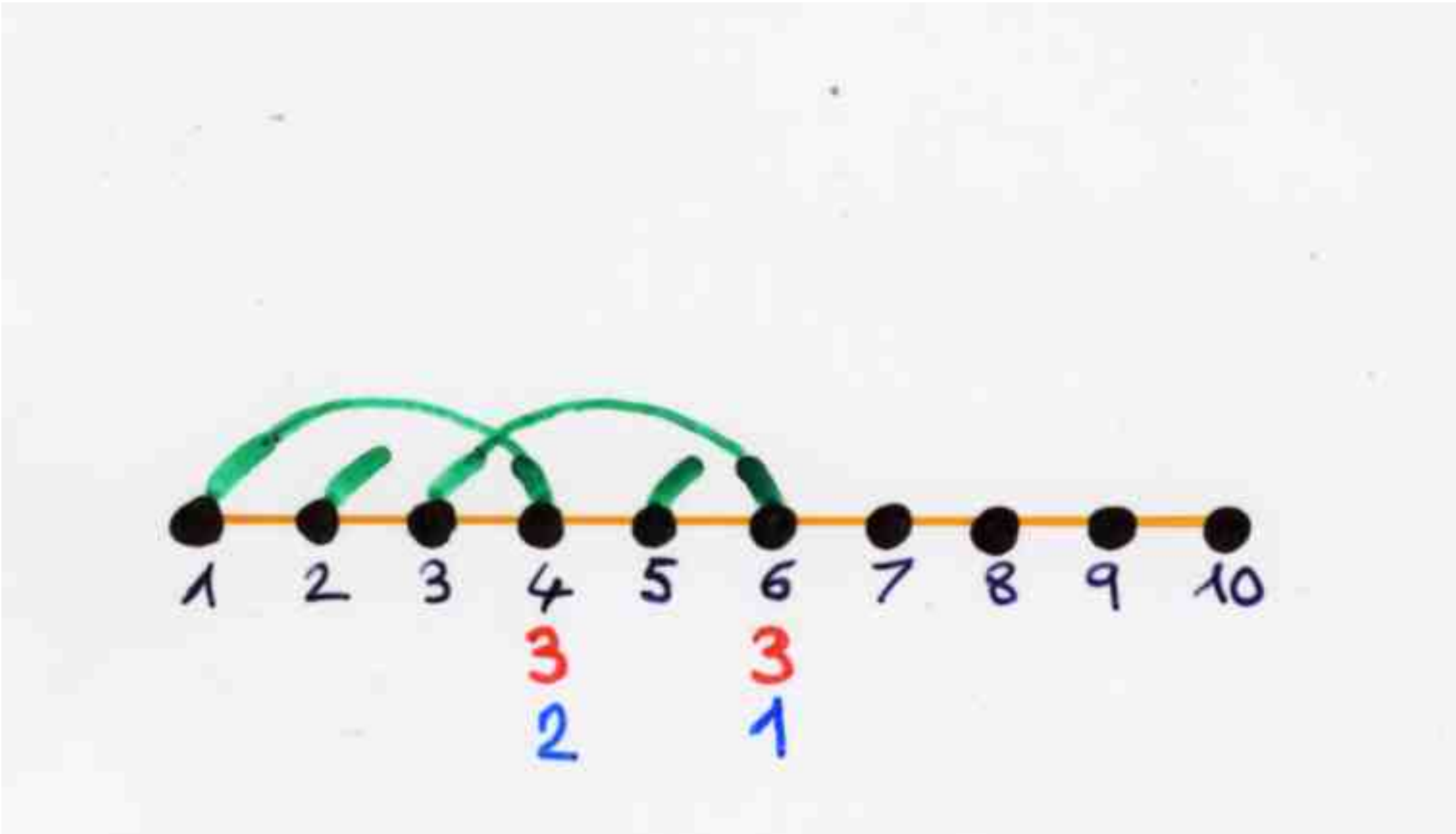

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

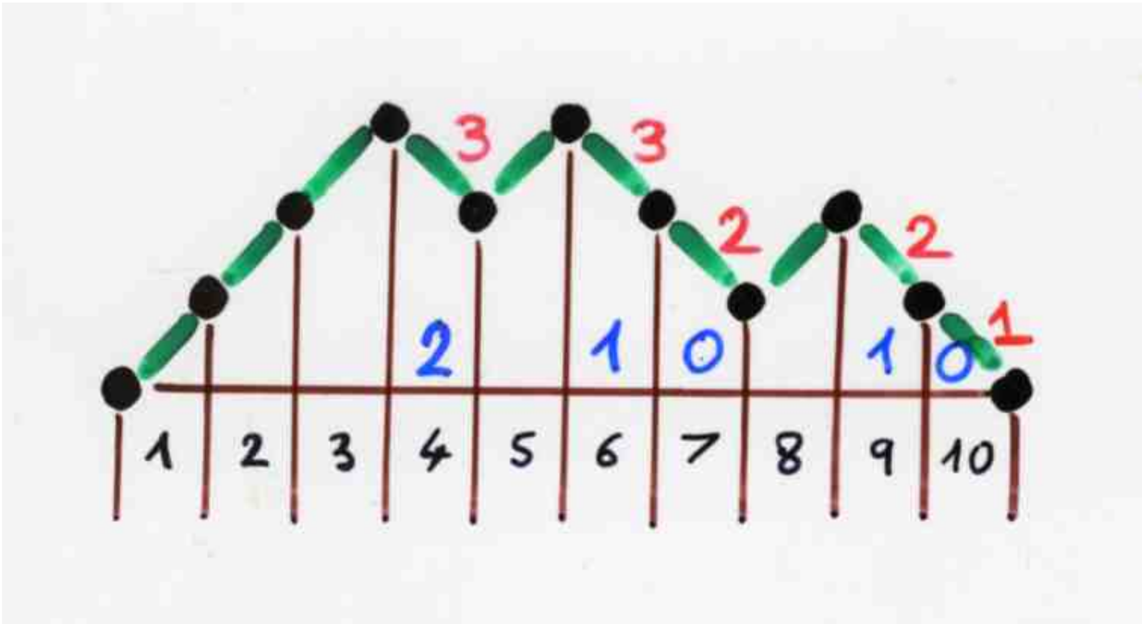

1
2
3
4
5
6
7
8
9
10
3
2

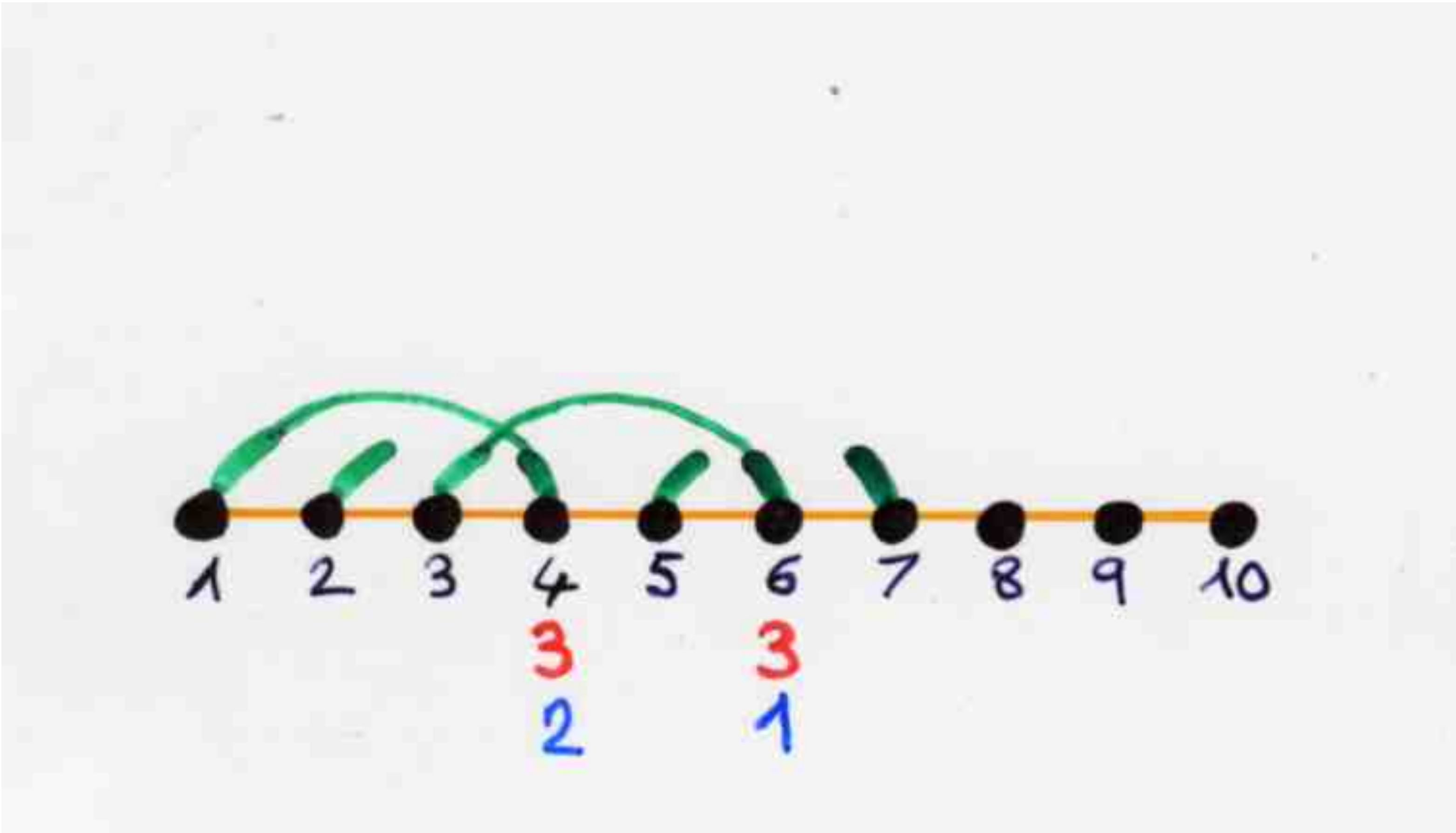

3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10

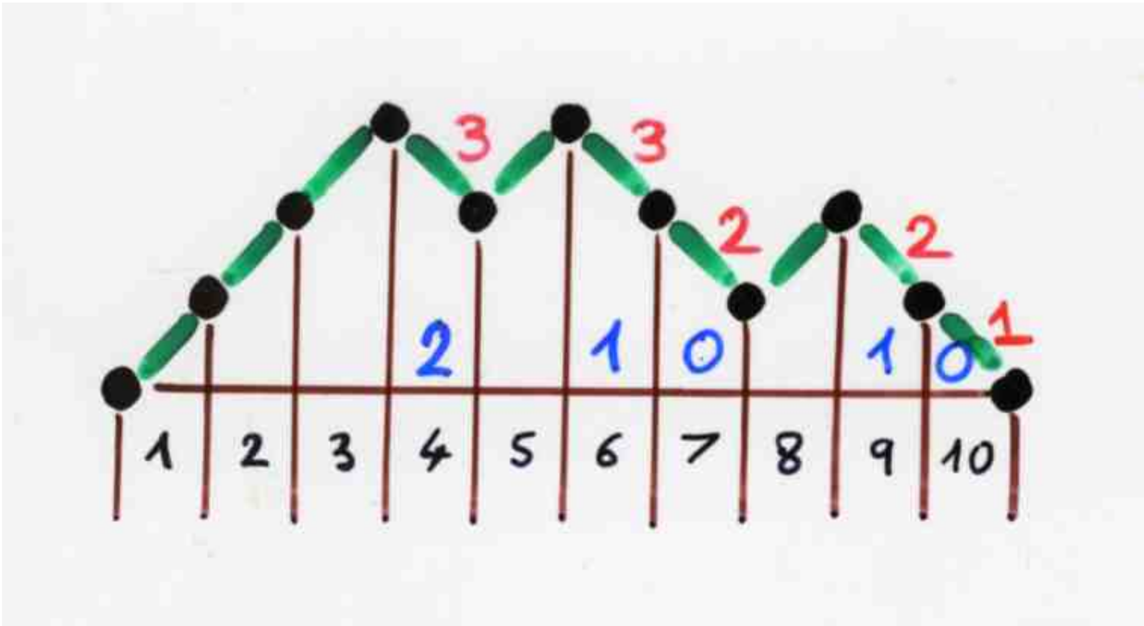

2
1
0
1
2
3
4
5
6
7
8
9
10
3
2


3 3 2 2 1
2 1 0 1 0
1 2 3 4 5 6 7 8 9 10

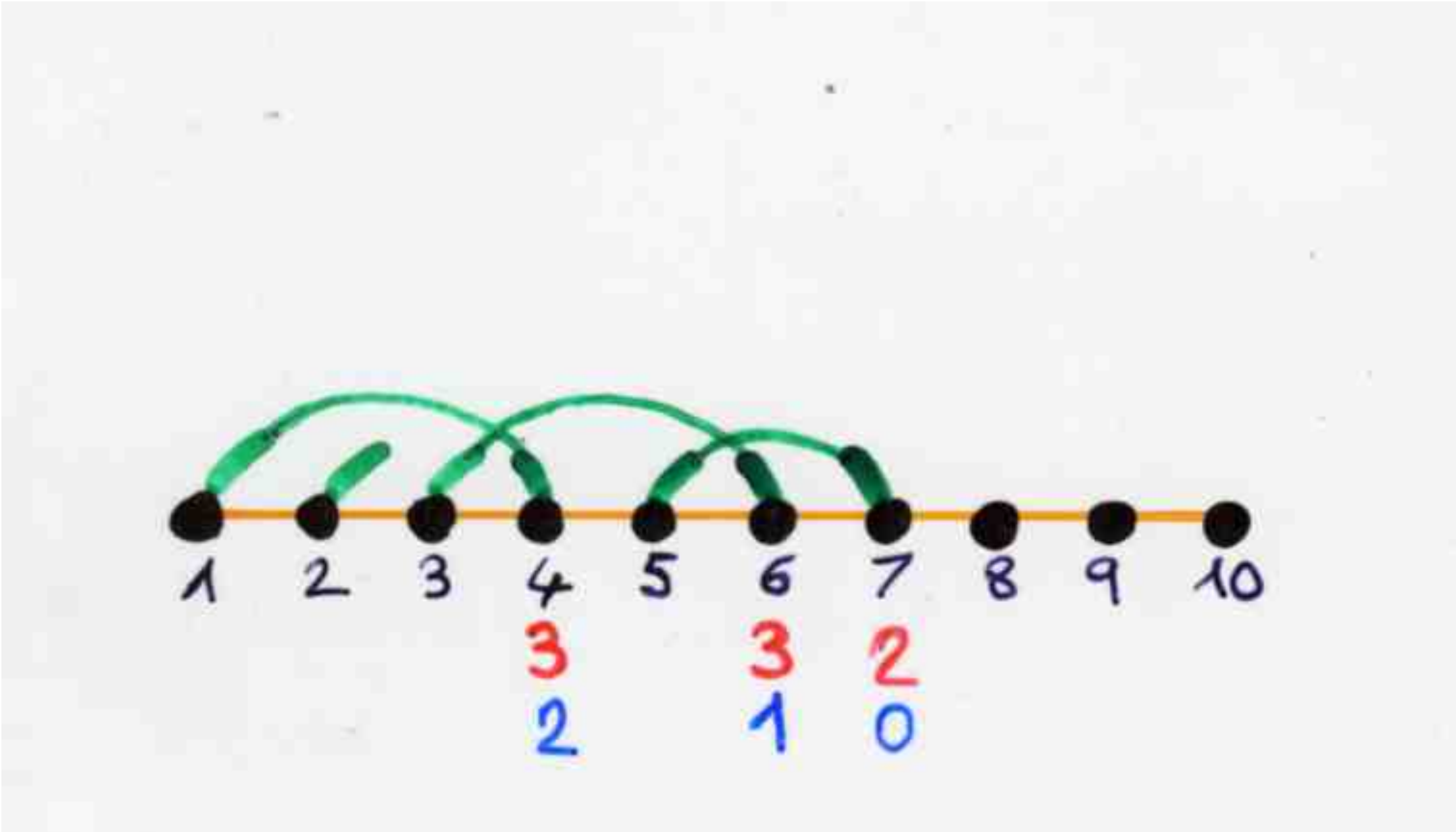

1 2 3 4 5 6 7 8 9 10
3 3
2 1


3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

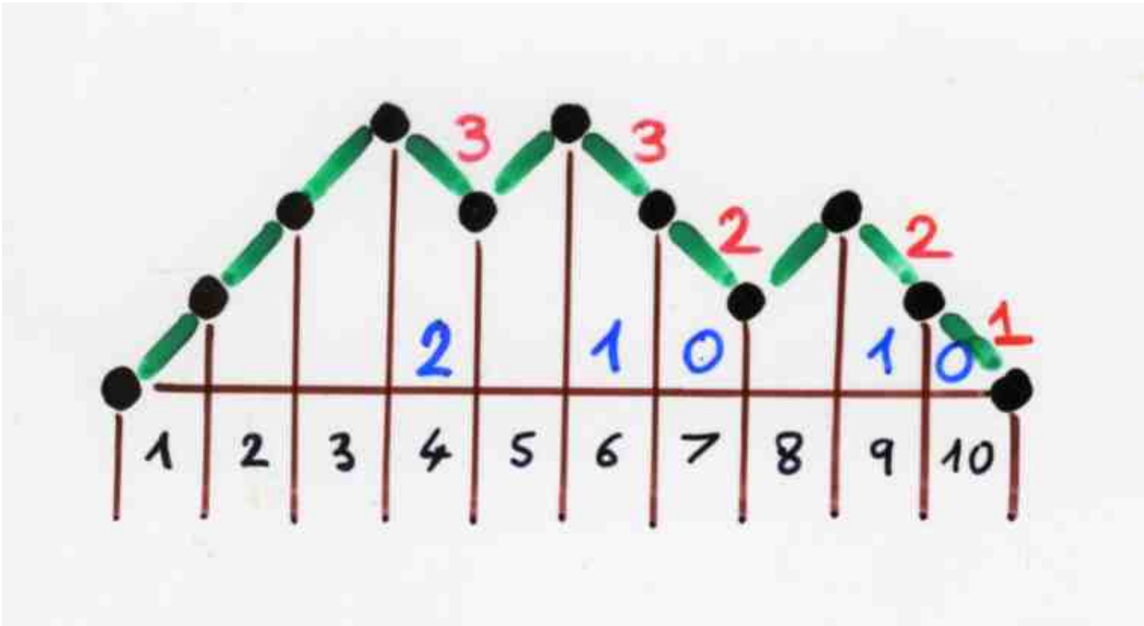

1 2 3 4 5 6 7 8 9 10
3
2
3
1

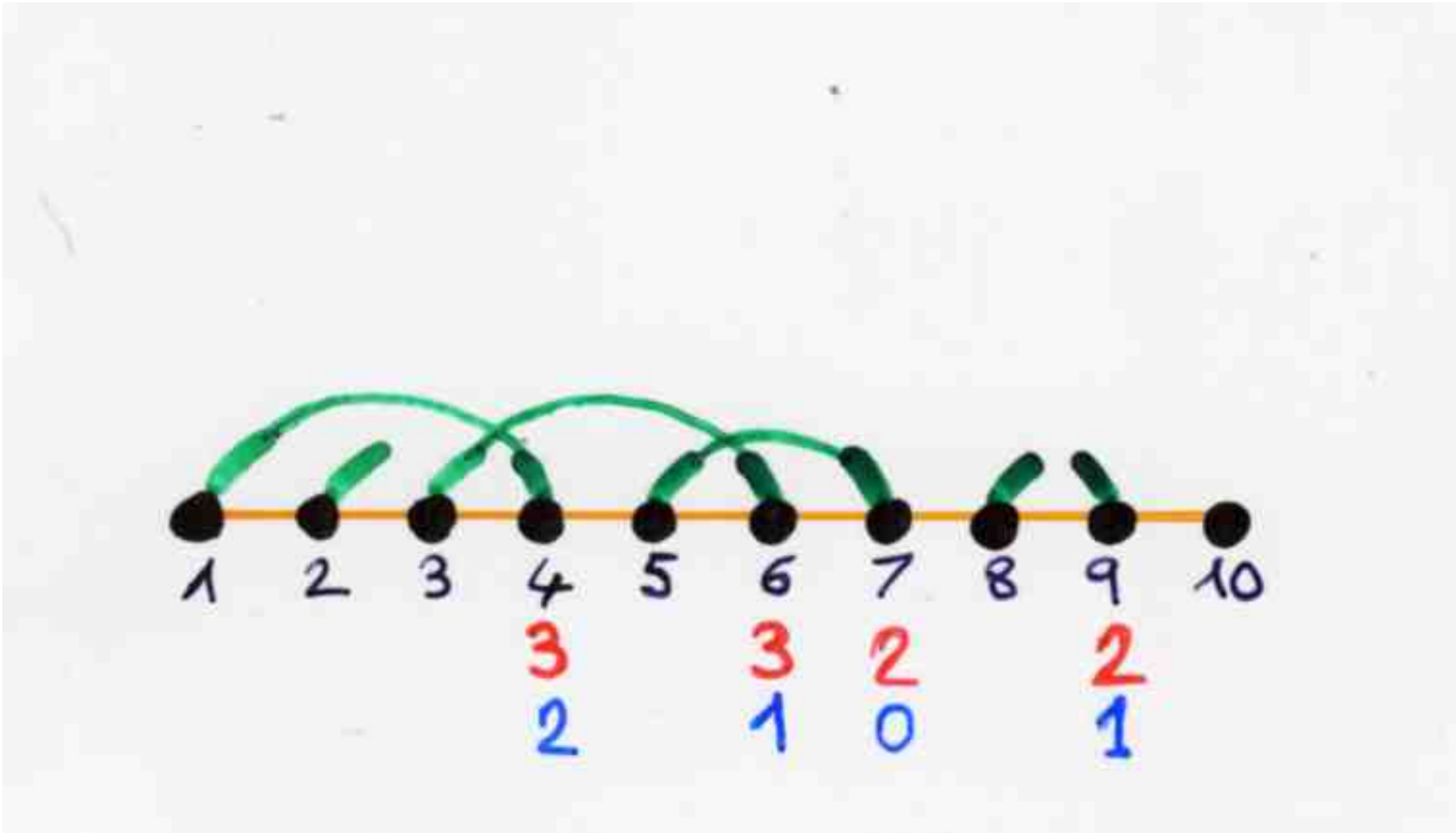


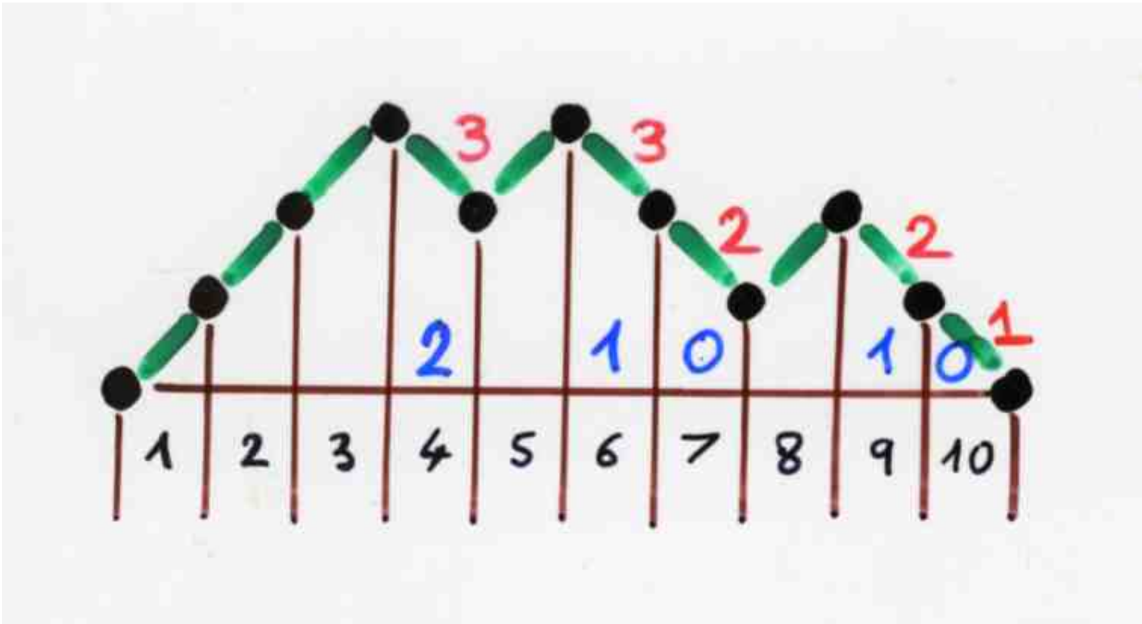
3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10

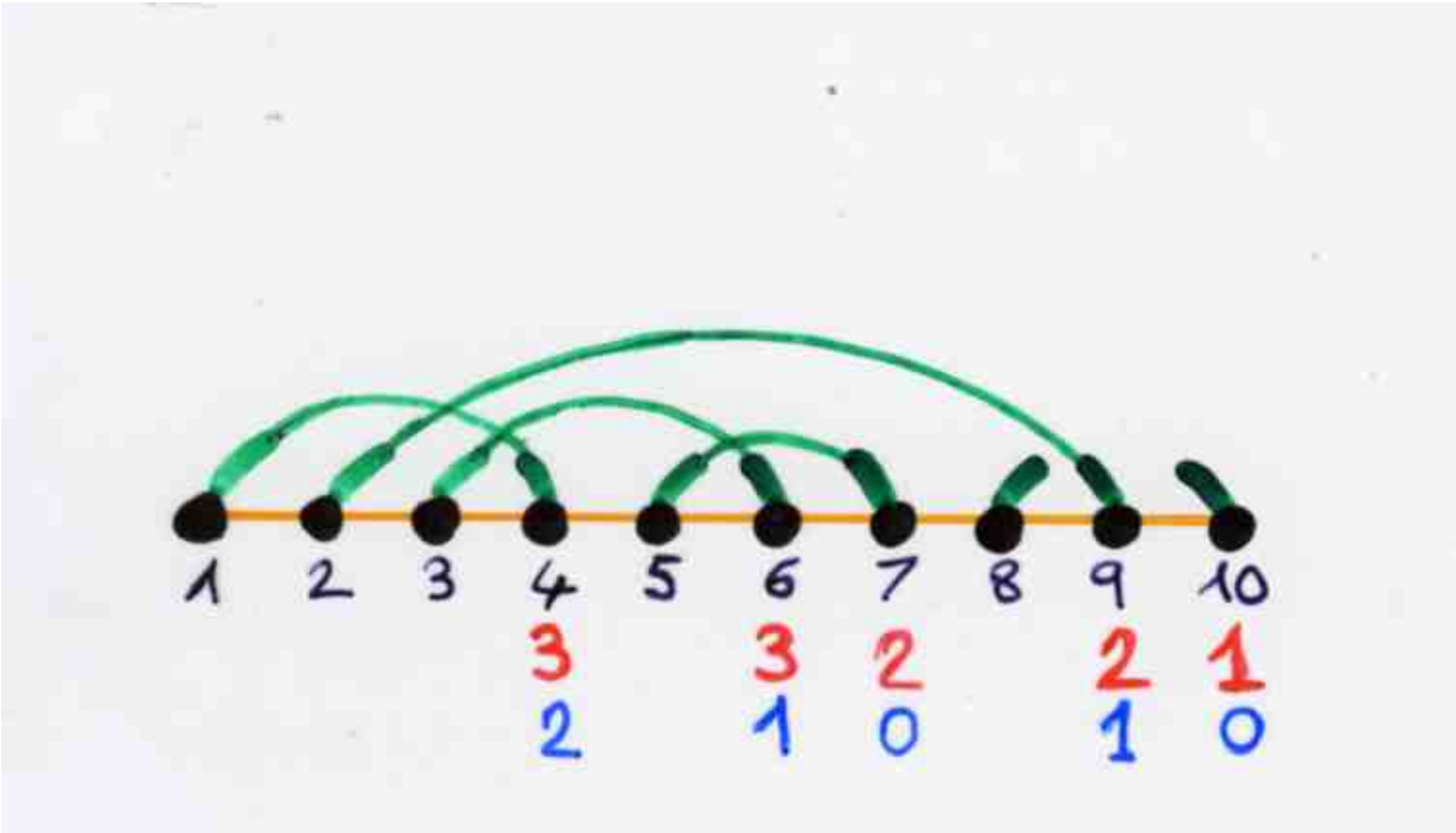


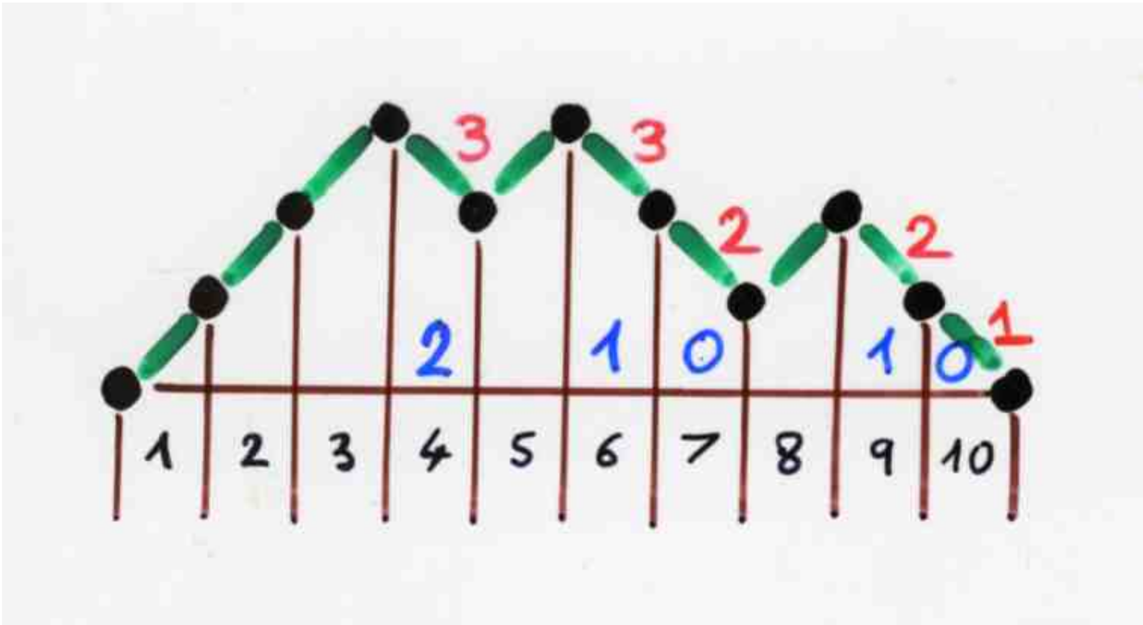
1 2 3 4 5 6 7 8 9 10
3 3 2
2 1 0

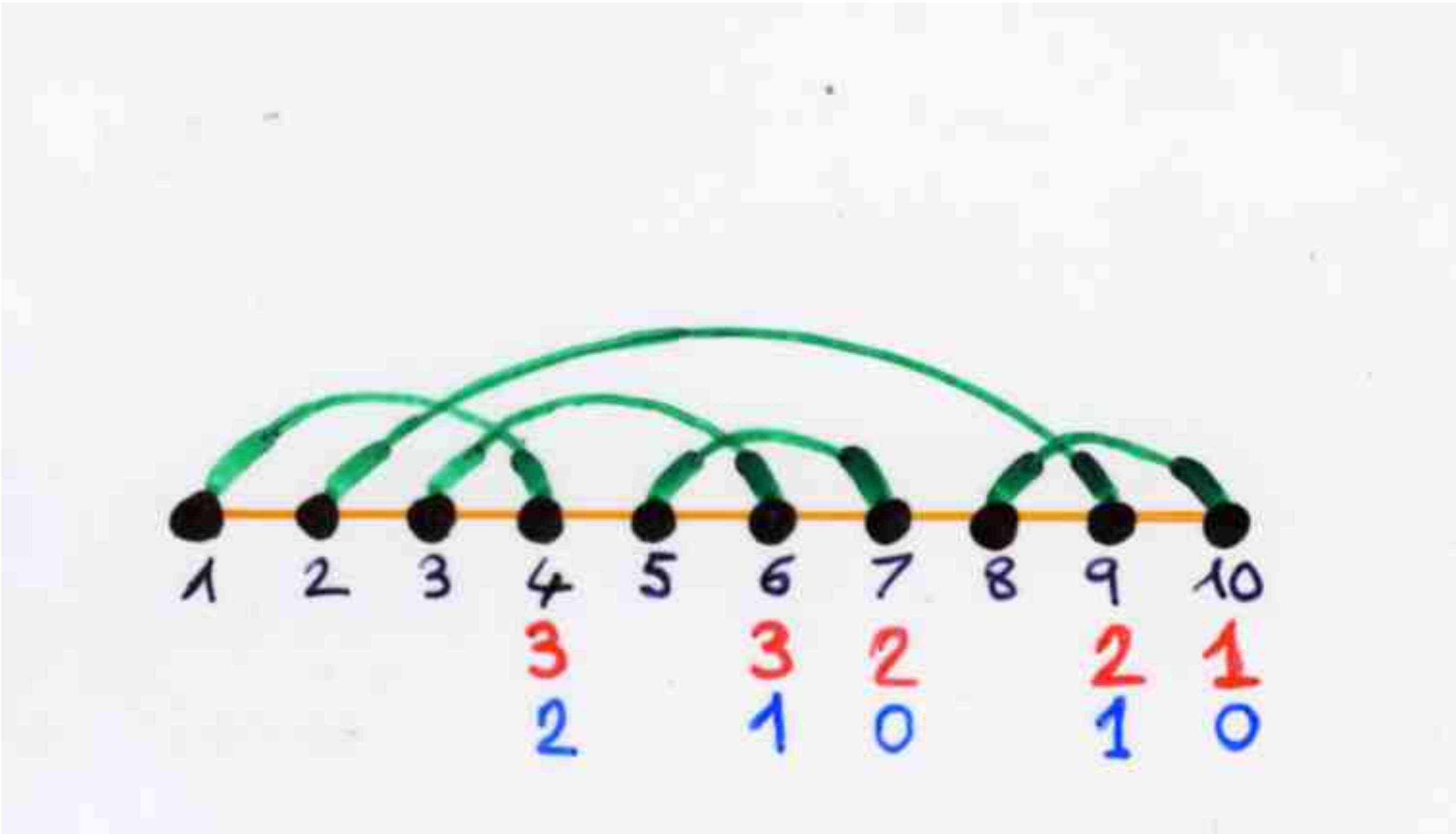

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10
3 3 2 2
2 1 0 1


3
3
2
2
1
2
1
0
1
0
1
2
3
4
5
6
7
8
9
10


1
2
3
4
5
6
7
8
9
10
3
3
2
2
1
2
1
0
1
0

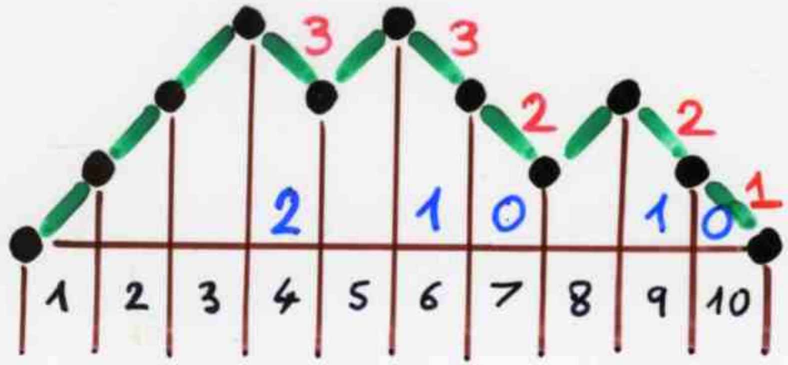

3
3
2
2
1
2
1
0
1
0
1 2 3 4 5 6 7 8 9 10


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

q-analog of Hermite histories

$\lambda_k = [k]_q$

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

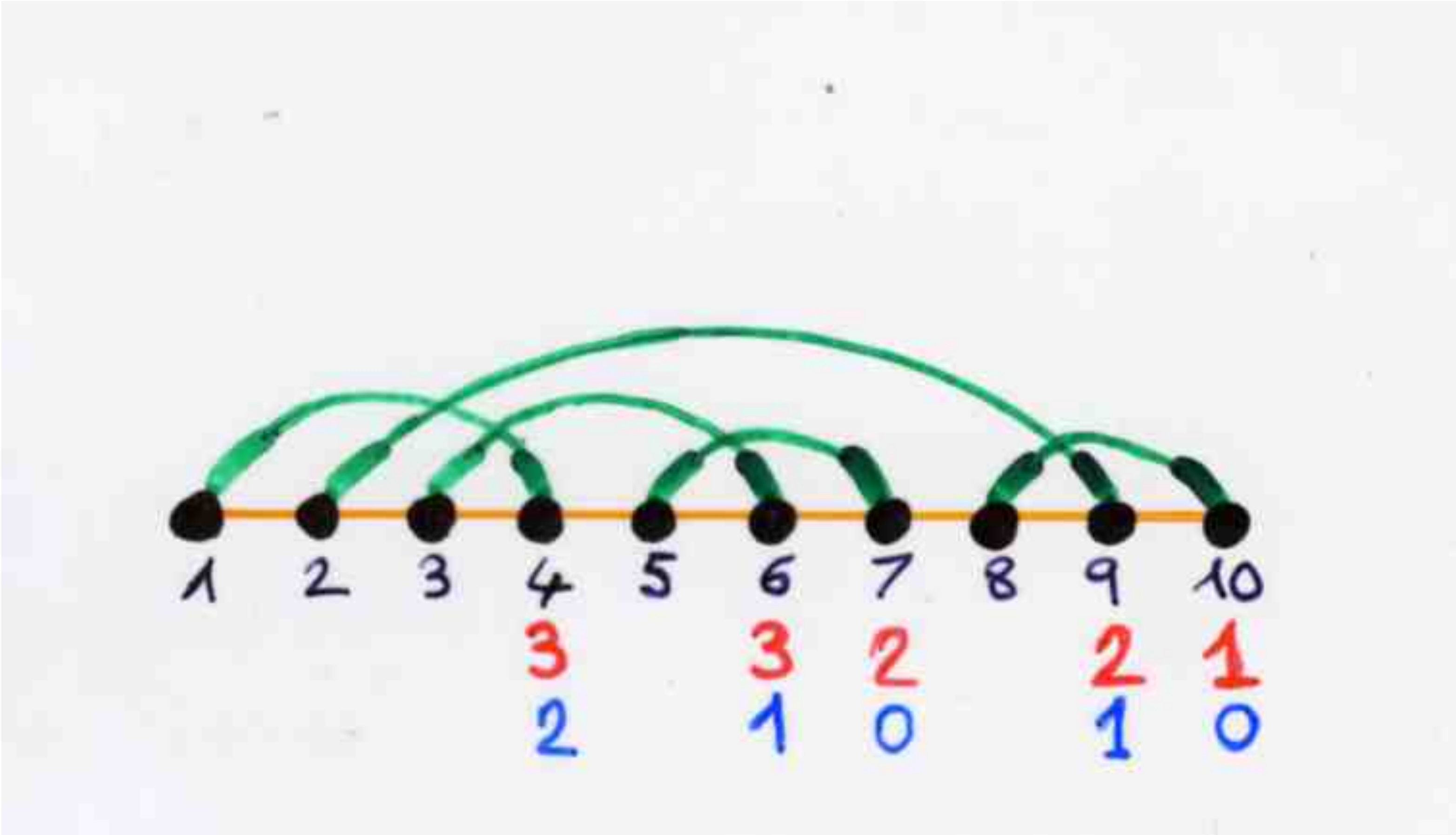


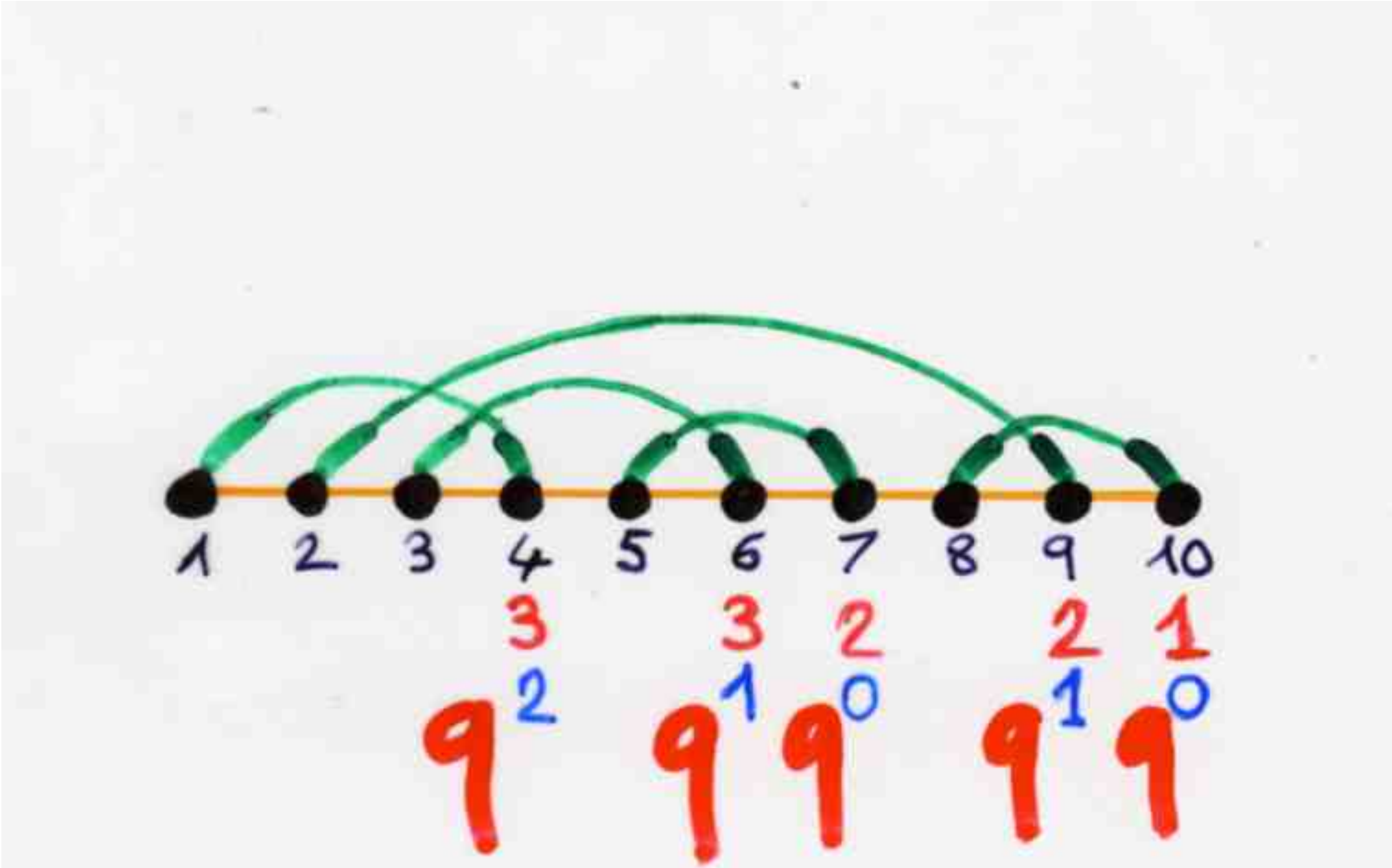
Hermite history related to ω

ω

Dyck path

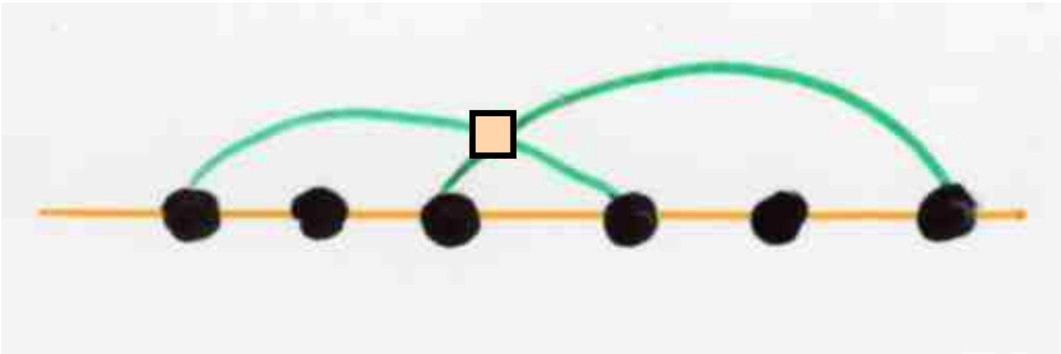
$$v_q(h) \qquad q^{2+1+0+1+0} \qquad = q^4$$


1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
2 1 0 1 0

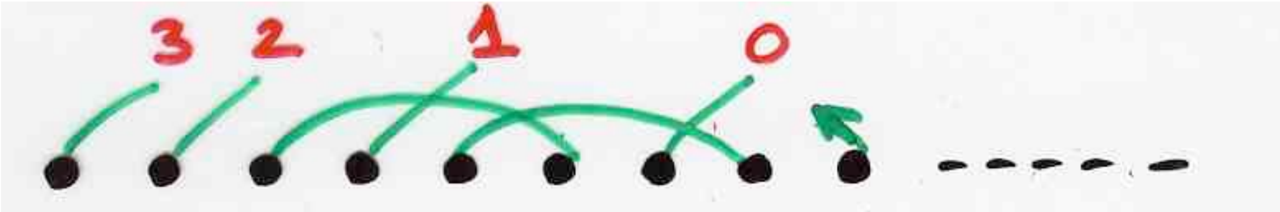


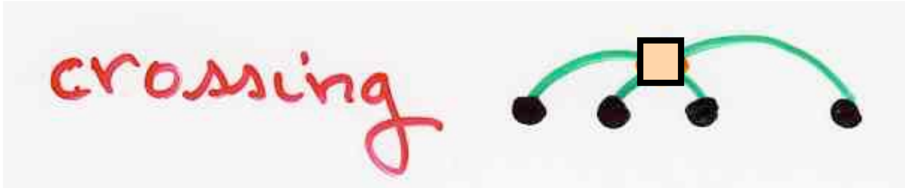
$$q^{2+1+0+1+0} = q^{4}$$

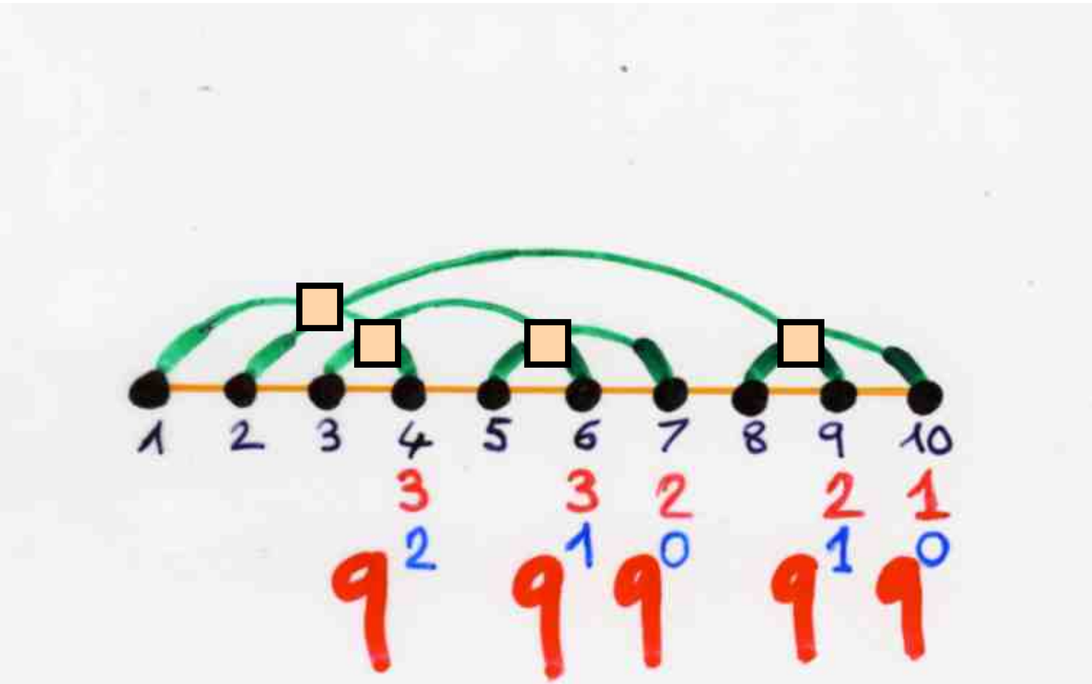
$$v_q(h)$$



crossing







$$v_q(h)$$

$$q^{2+1+0+1+0} = q^{4}$$

Combinatorics of the PASEP

(1982) Shapiro, Zeilberger

(2004) Brak, Essam

(2005) Duchi, Schaeffer

(2006) Corteel   Burstein

Brak, Corteel, Essam, Parviainen, Rechnitzer

Corteel, Williams

(2007) Corteel, Nadeau   Corteel, Williams

Steingrimsson Williams   X.V.

(2008) X.V.

(2009) Corteel, Josuat-Vergès, Prellberg, Rubey

Josuat-Vergès   Nadeau

(2010) Corteel, Williams

(2011) Josuat-Vergès Corteel, Kim

Corteel, Dasse-Hartaut

Corteel, Josuat-Vergès, Williams

Aval, Boussicault, Nadeau

(2012) Corteel, Stanley, Stanton, Williams

(2013) Aval, Boussicault, Bouvel, Silimbani

Aval, Boussicault, Nadeau

Aval, Boussicault, Dasse-Hartaut

(2014) E. Jin

(2016) Aval, Boussicault, Delcroix-Oger, Hivert, Laborde-Zubieta

Corteel, Kim, Stanton Mandelshtam, X.V.

(2017) Corteel, Williams Mandelshtam, X.V.

Corteel, Mandelshtam, Williams

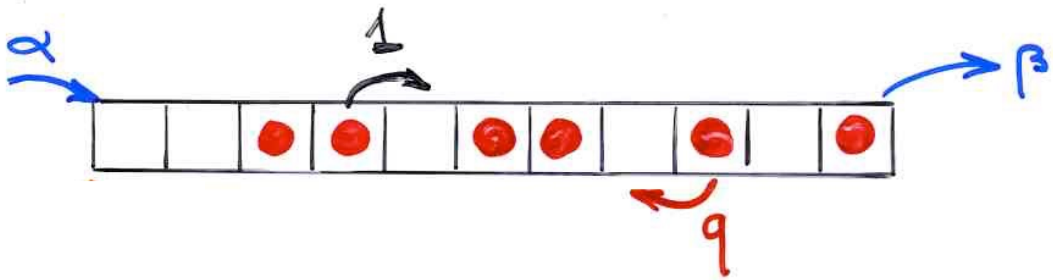
Corteel, Nunge

Laborde-Zubieta

alternative tableaux

PASEP with 3 parameters

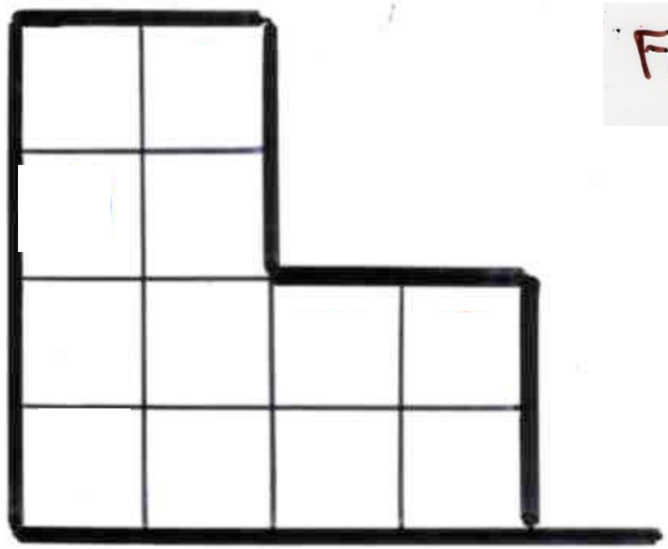
$\gamma = \delta = 0$ $\quad q, \alpha, \beta$



PASEP

alternative tableau

Definition



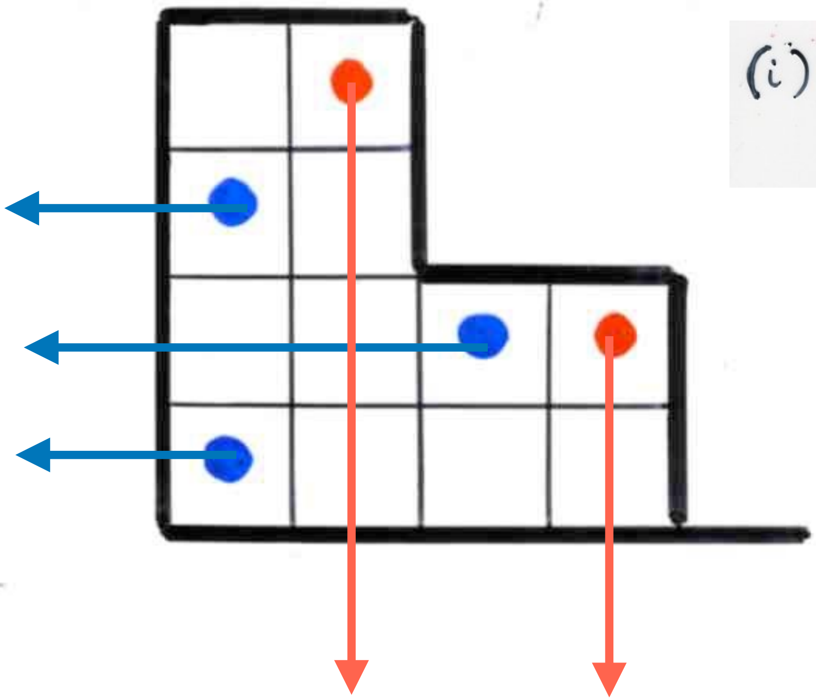
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

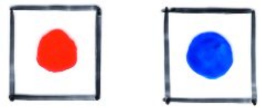
n = (number of rows) + (number of columns)

alternative tableau



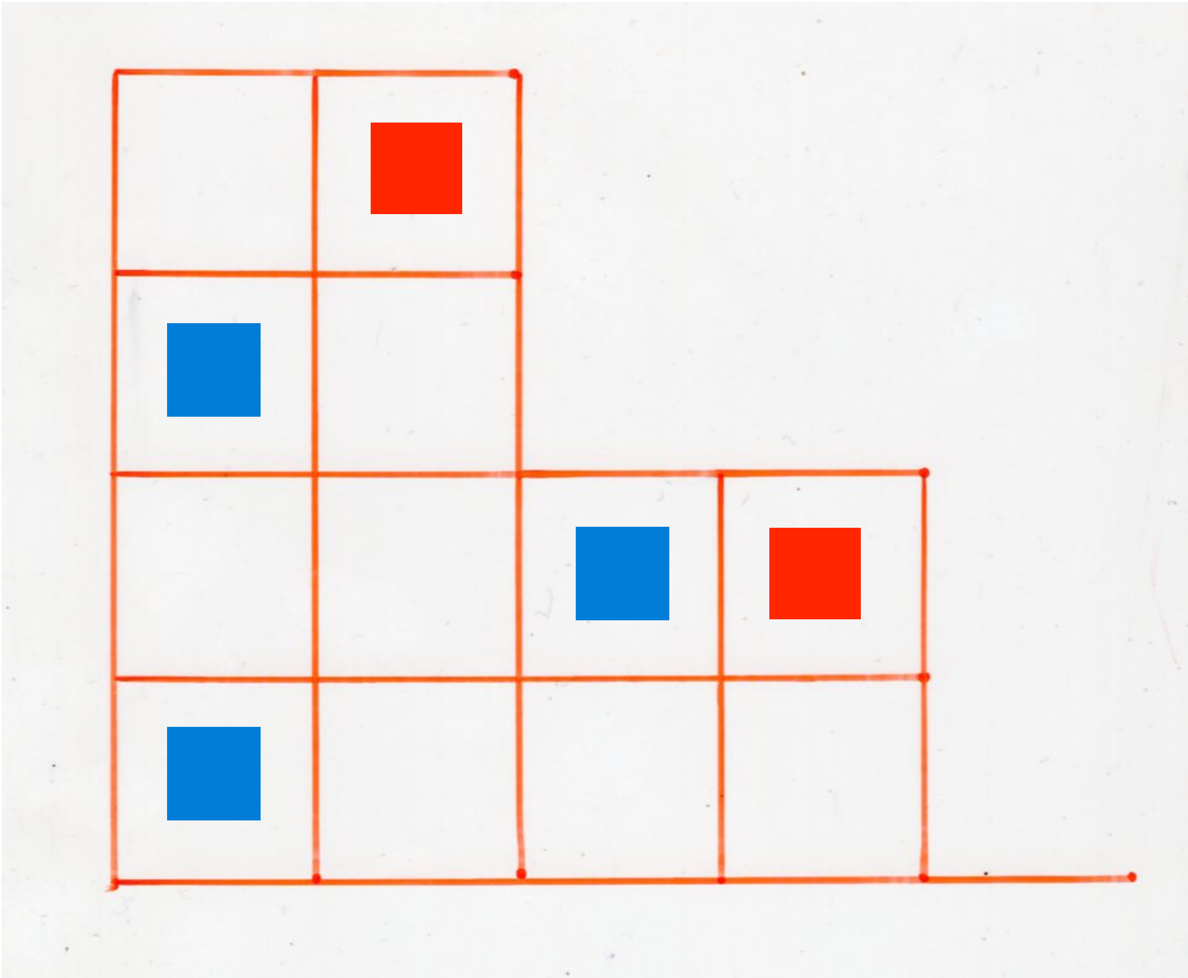
Definition

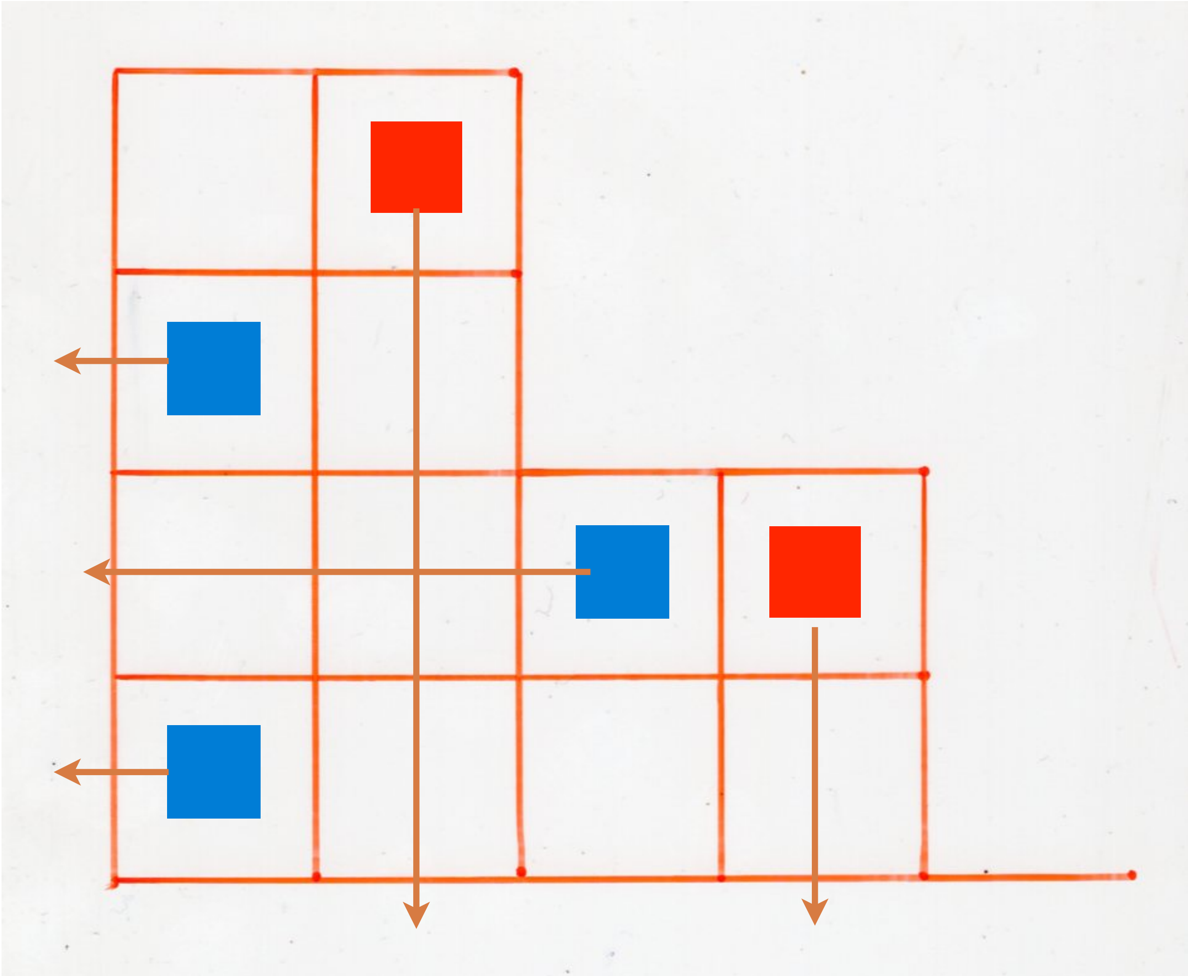
(i) some cells are coloured red or blue

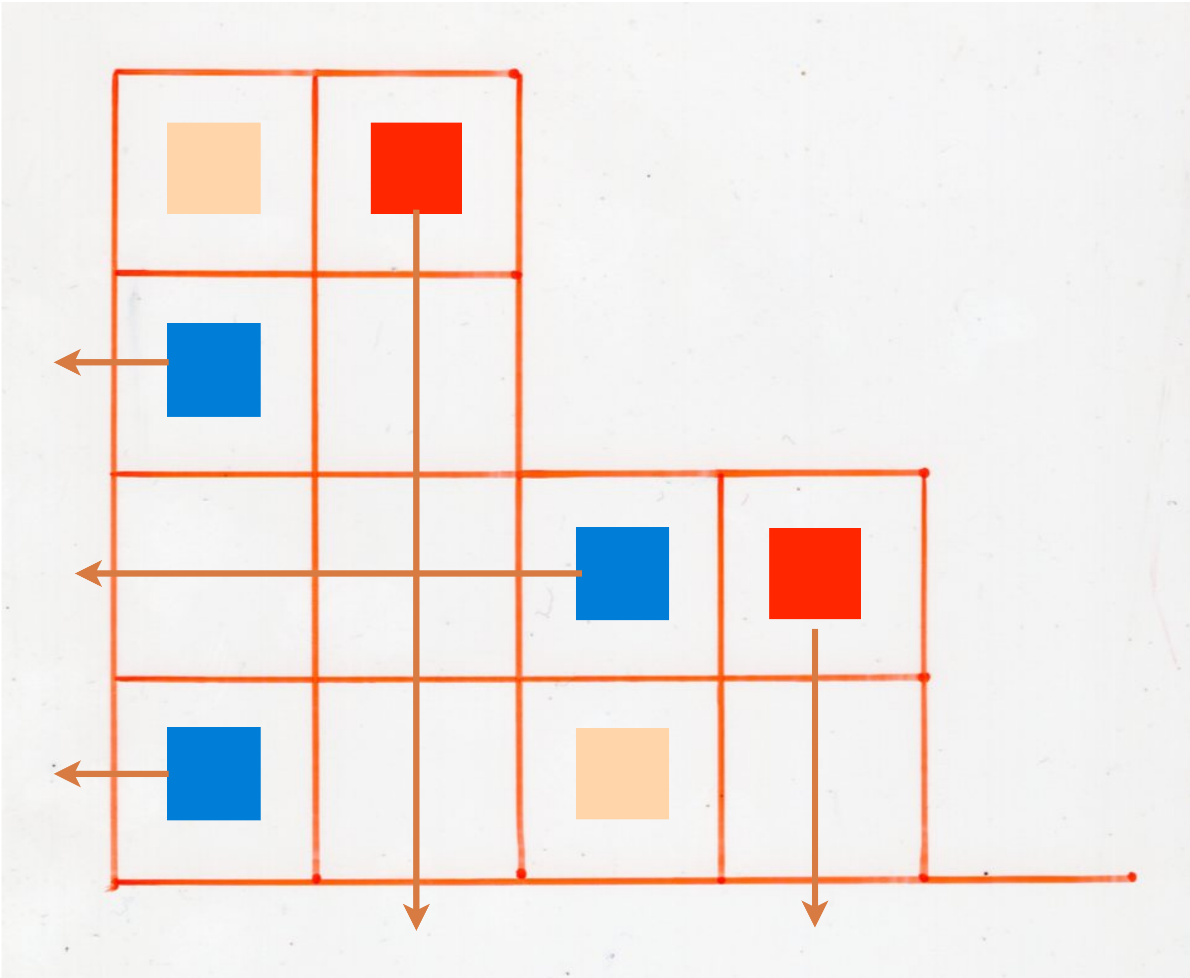


(ii) • no coloured cell at the left of a blue cell

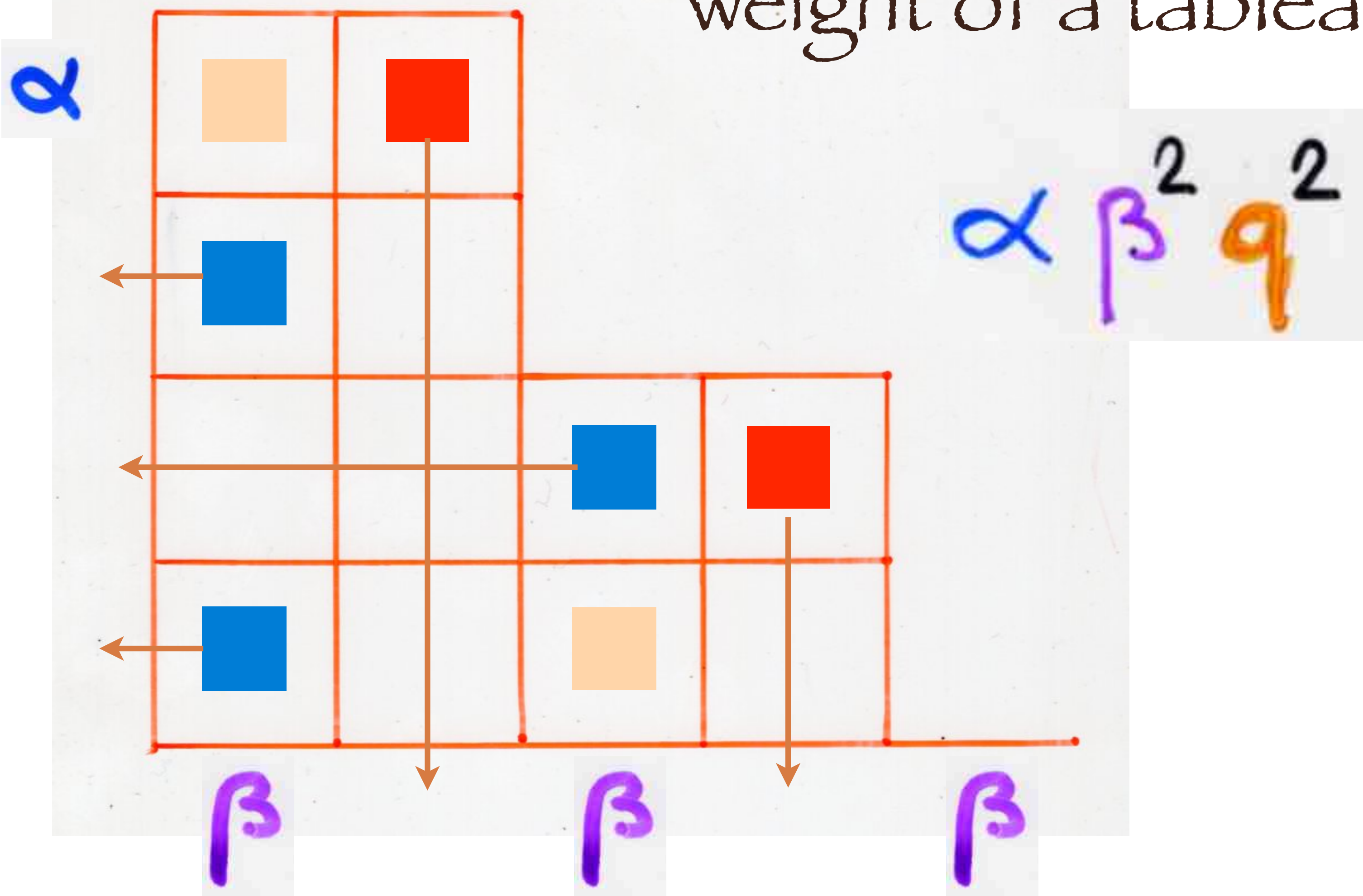
• no coloured cell below a red cell







weight of a tableau



q $k(T)$ = nb of cells ▢

α $i(T)$ = nb of rows without ●

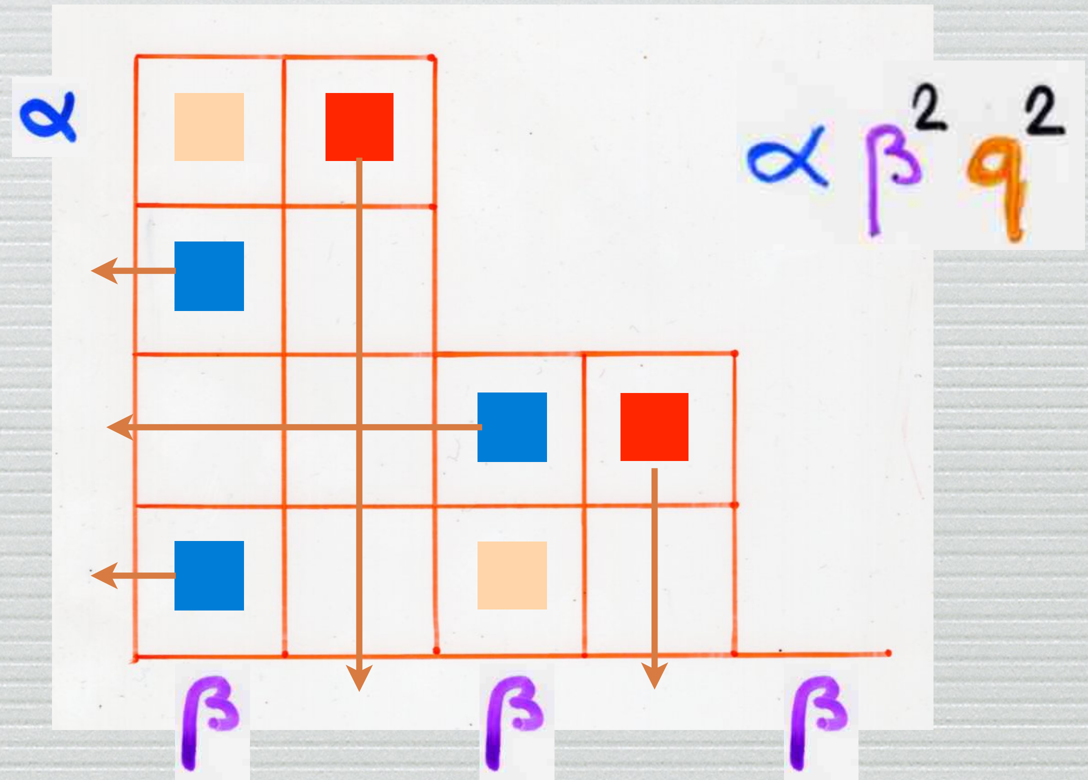
β $j(T)$ = nb of columns without ●

Partition function

$\bar{Z}_N = Z_N(\alpha^{-1}; \beta^{-1}; q)$

Z_n

Sum of the weight of all tableaux of size n



q $k(T)$ = nb of cells □

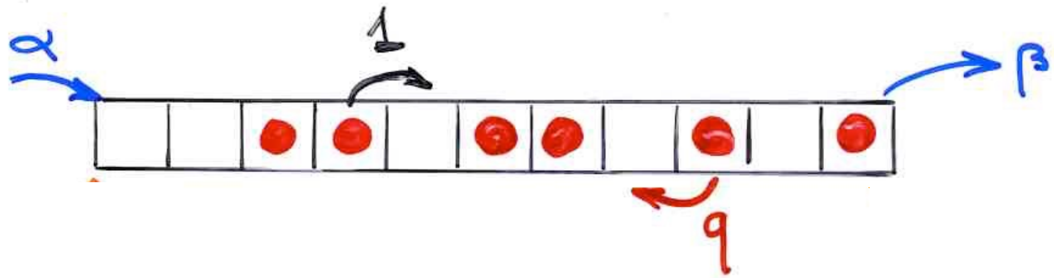
α $i(T)$ = nb of rows without ●

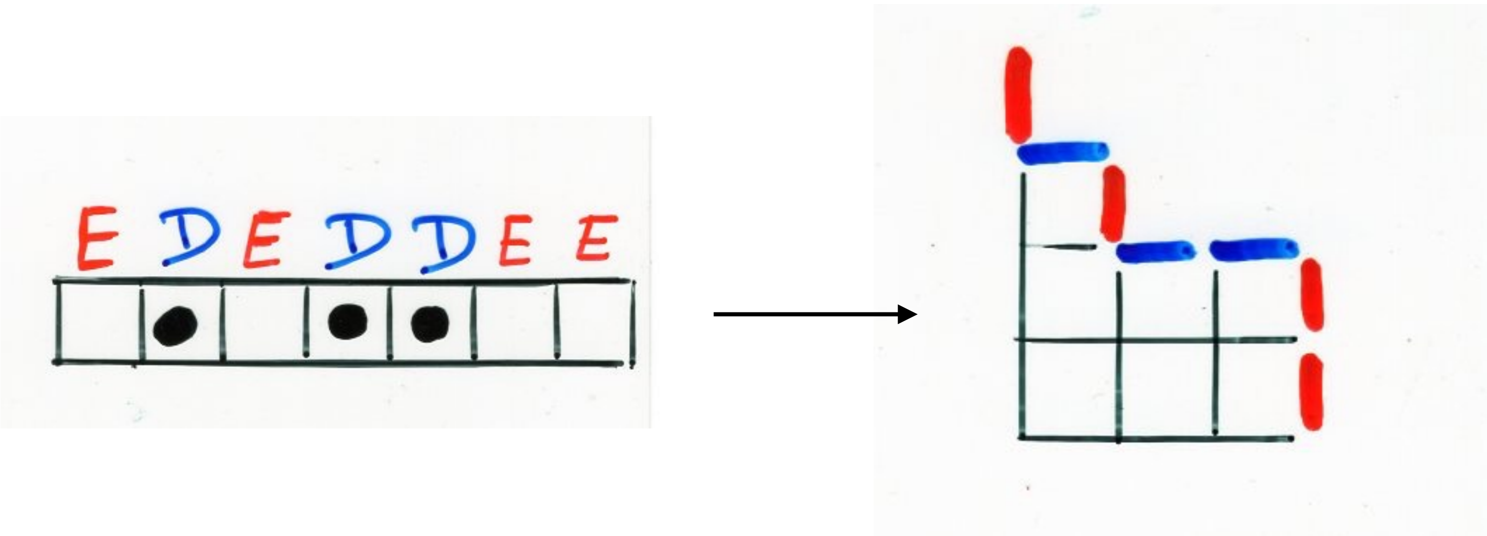
β $j(T)$ = nb of columns without ●

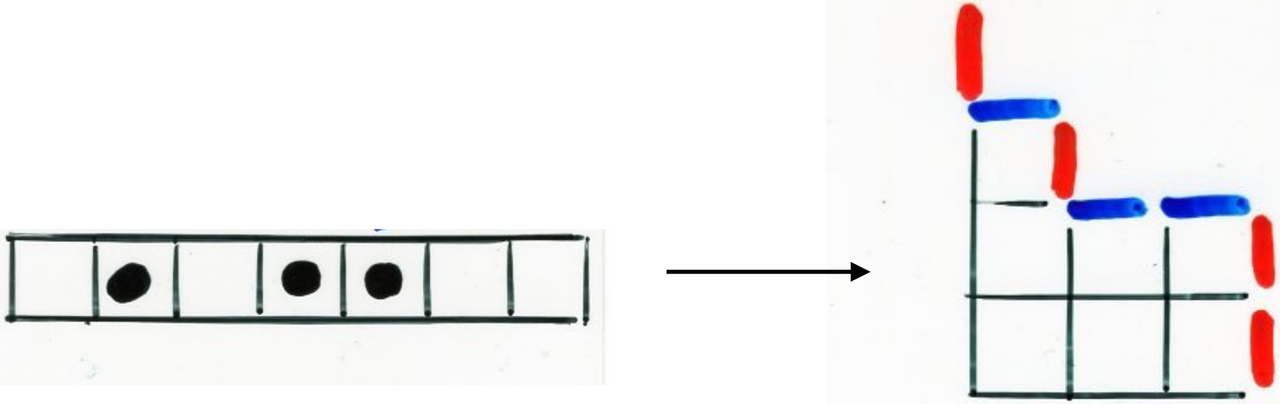
computation of the "stationary probabilities"

PASEP with 3 parameters q, α, β

PASEP







Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_{\tau}(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

T alternative tableaux profile τ

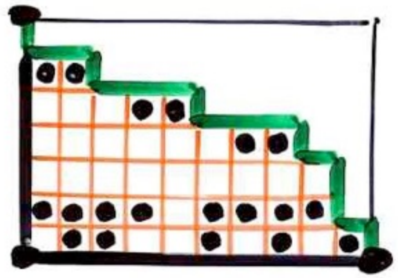
alternative tableau
X.V. (2008)

permutation tableau

S. Corteel, L. Williams
(2007, 2008, 2009)

permutation tableaux

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0 ⊡ = 1

filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) 1 ---- 0 forbidden
           ¦
           1

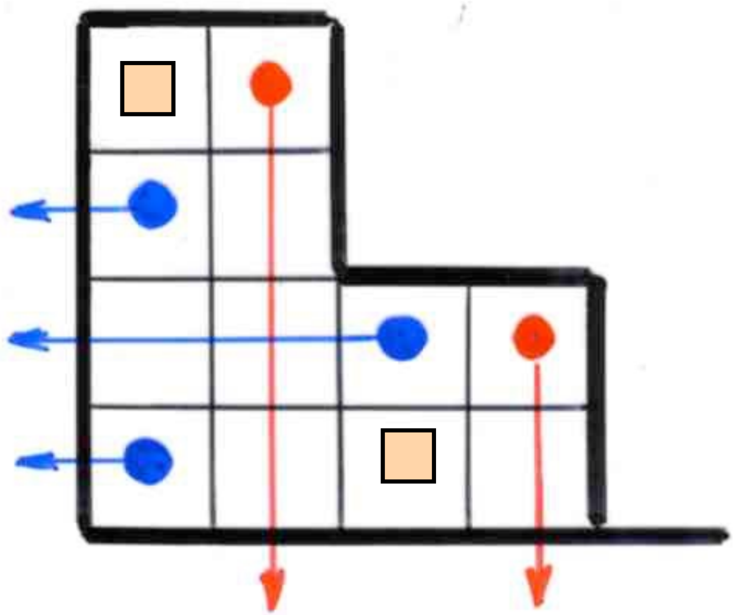
permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

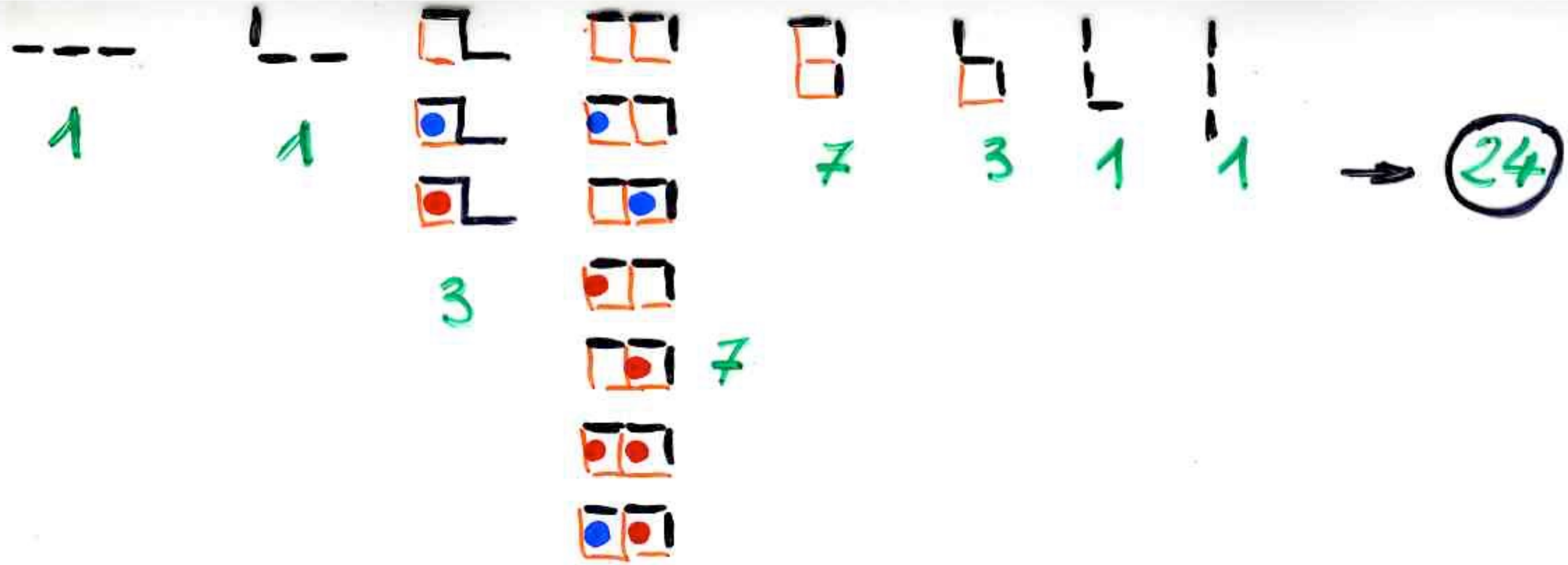
Enumeration of alternative tableaux



Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

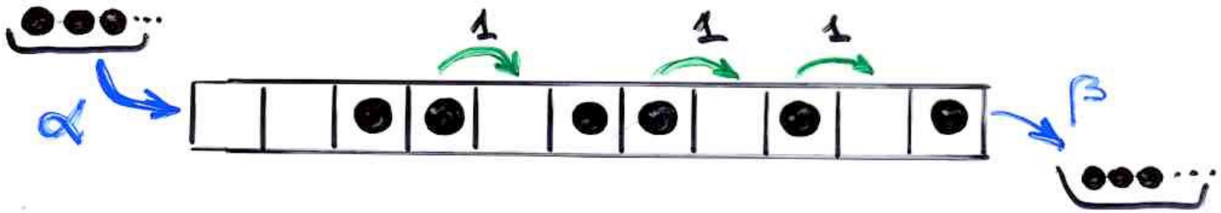
ex: n=2



$q = 0$ TASEP (α, β)

TASEP

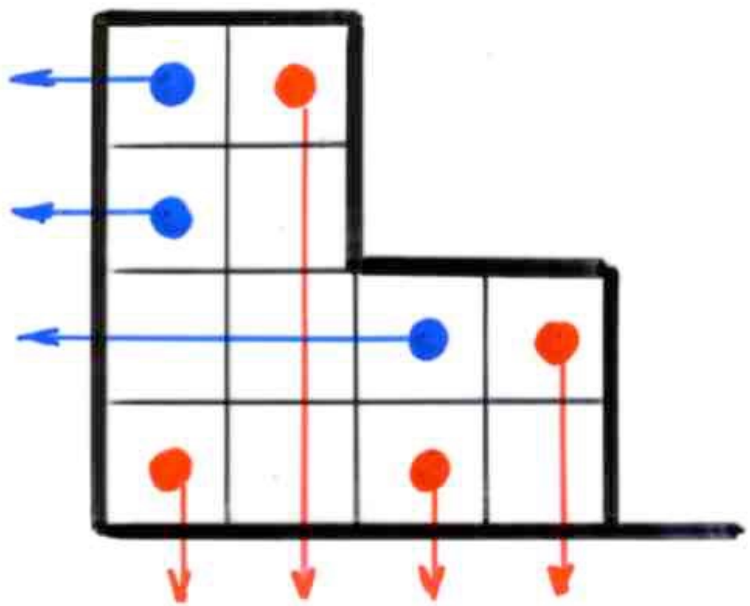
"totally asymmetric exclusion process"



Definition Catalan alternative tableau

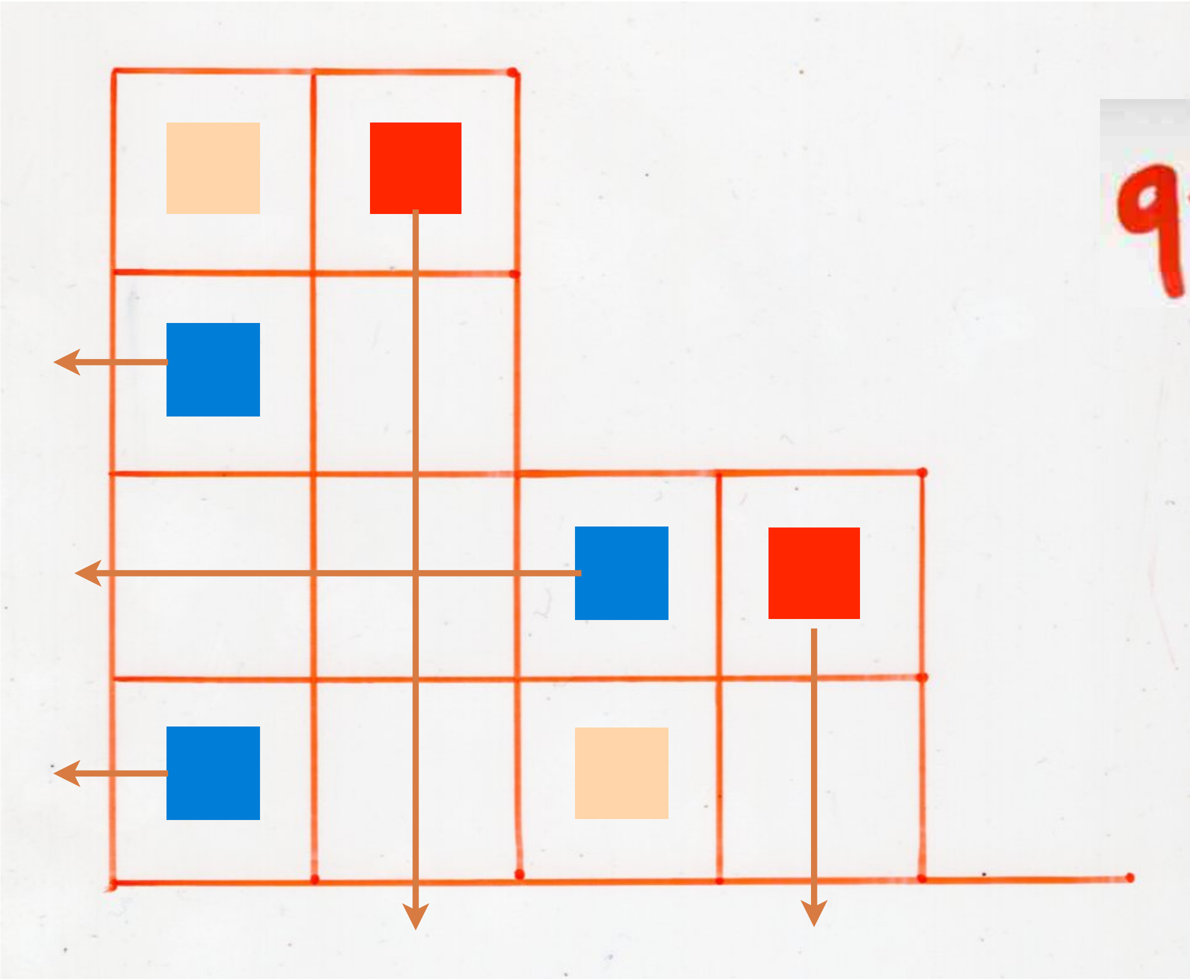
alternative tableau T without cells ☐

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



q- analog of n!

q-analog of n!

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

Inv

number of inversions

Maj

Major index

Interpretation of the 3-parameters
Partition function

q, α, β

q-analogue of n! ?

First bijection: tableaux— permutations

Corteel, Williams (2007)

Steingrimsson, Williams (2007)

bijection

permutation tableaux → permutations

α, β

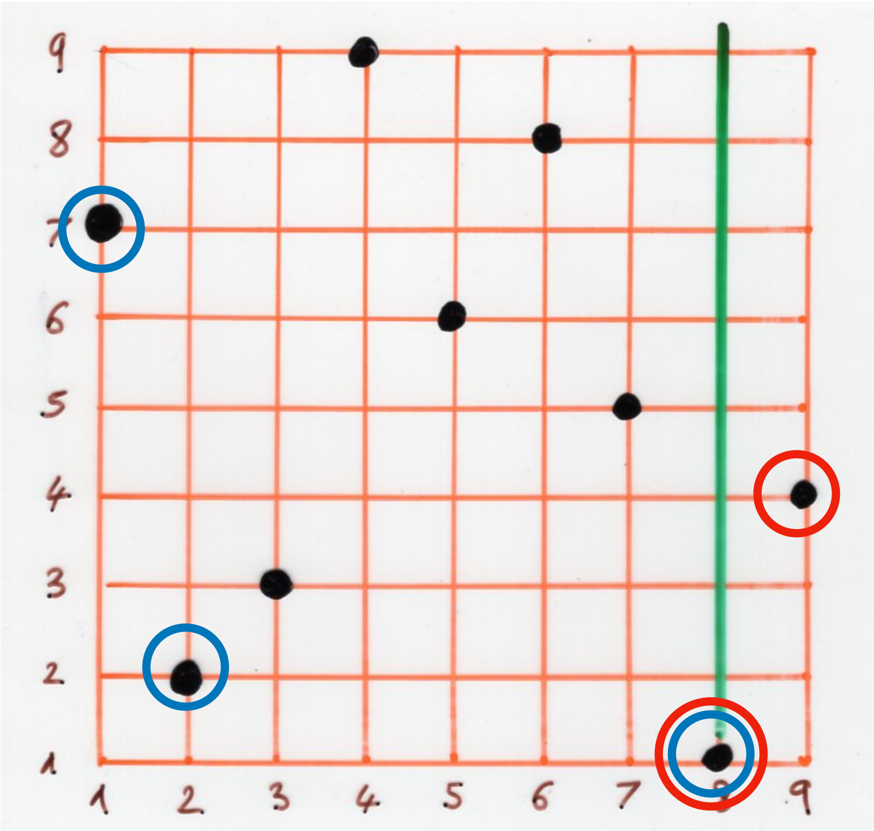
"special"

left-to-right

right-to-left

minimum

elements



"special"

σ = 7 2 3 9 6 8 5 1 4

permutation

word

left-to-right

right-to-left

minimum

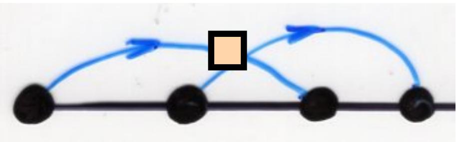
elements

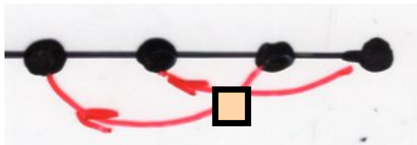
9

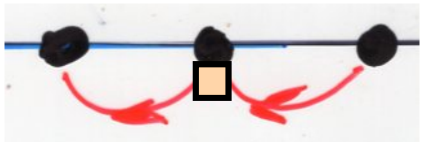
number of crossings of a permutation

Corteel (2007)

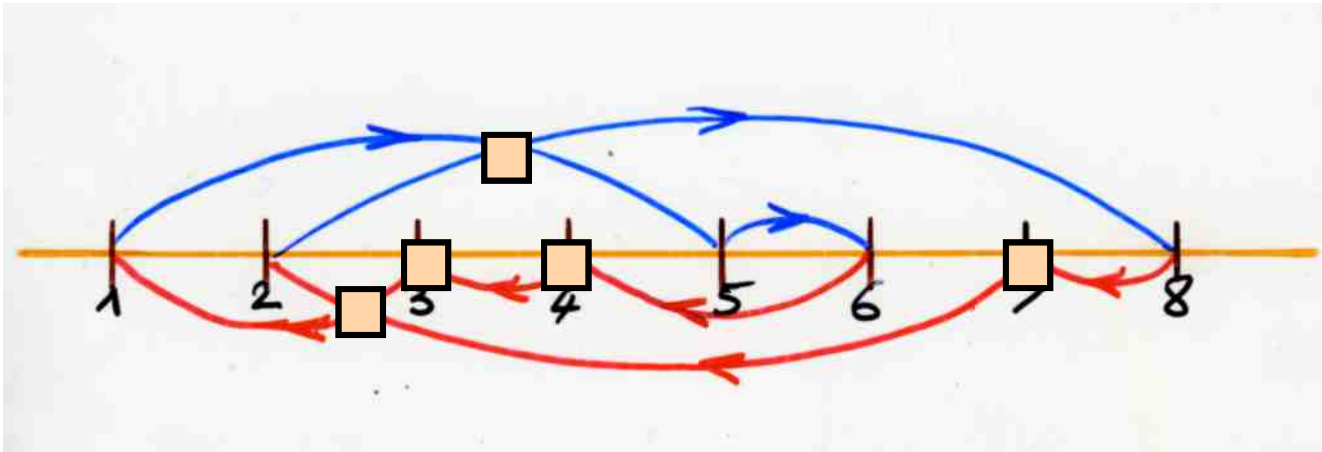
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$







(strict) exceedances



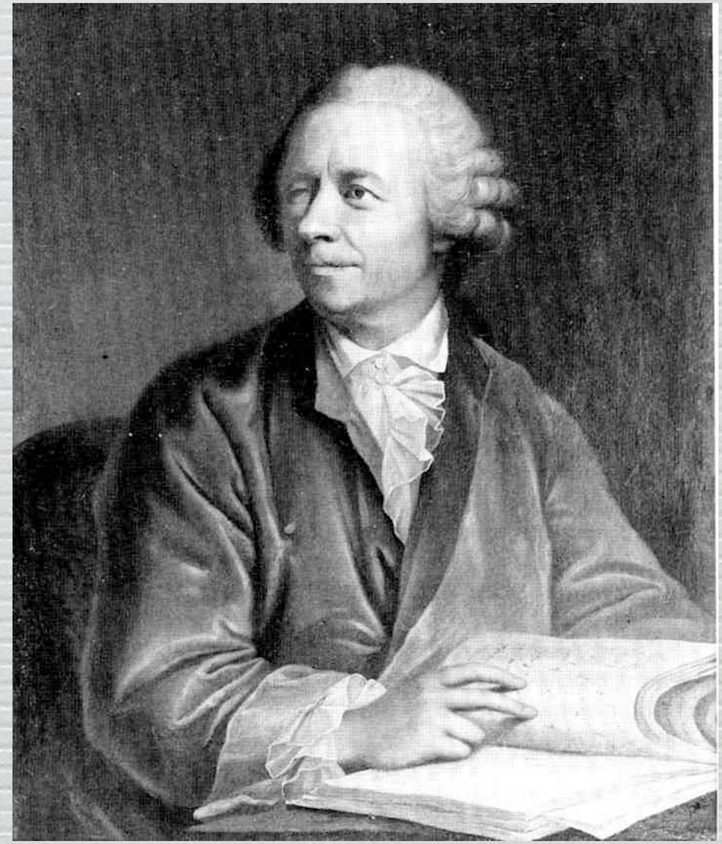
DE FRACTIONIBVS CONTINVIS. DISSERTATIO.

AVCTORE
Leonh. Euler.

§. 1.

Varii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4\ 4.6\ 6.8\ 8.10\ 10}{3.3\ 5.5\ 7.7\ 9.9\ 11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-



§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

9

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

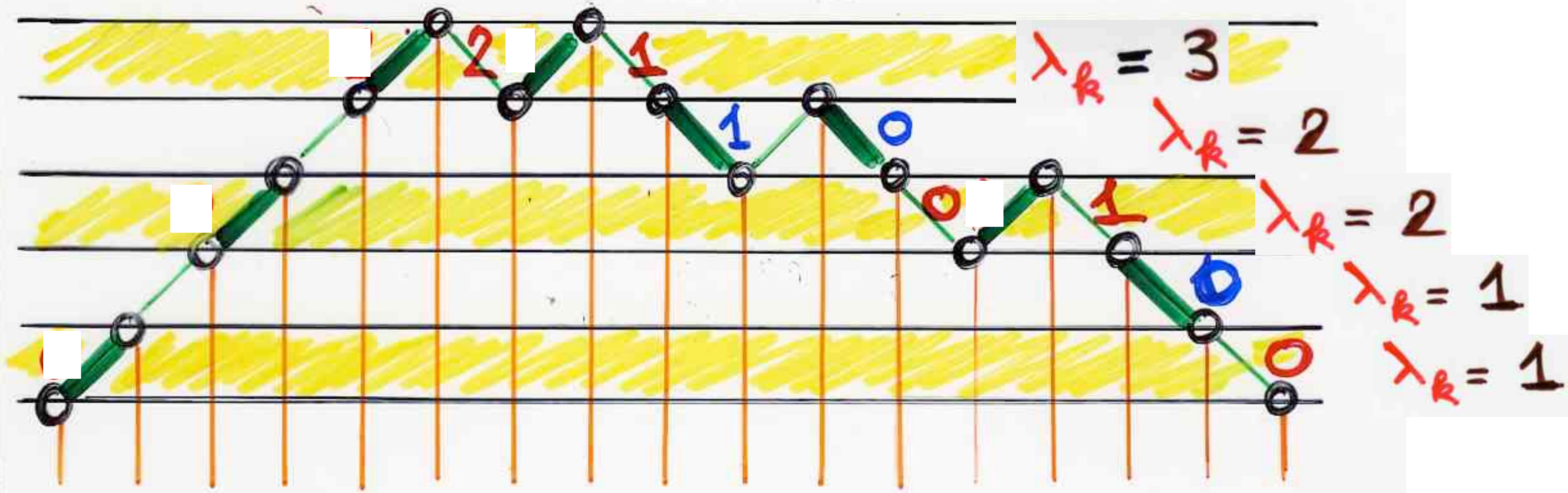
$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \ldots}}}}}}$$

$$\lambda_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$$

$$\sum_{n\geq 0} (n!)_q \, t^n = \cfrac{1}{1-(1)t - \cfrac{(1)t}{1-(1+q)t - \cfrac{(1+q)t}{1-(1+q+q^2)t - \cdots}}}$$

subdivided
Laguerre
histories

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$



subdivided Laguerre history

bijection permutations
subdivided Laguerre histories

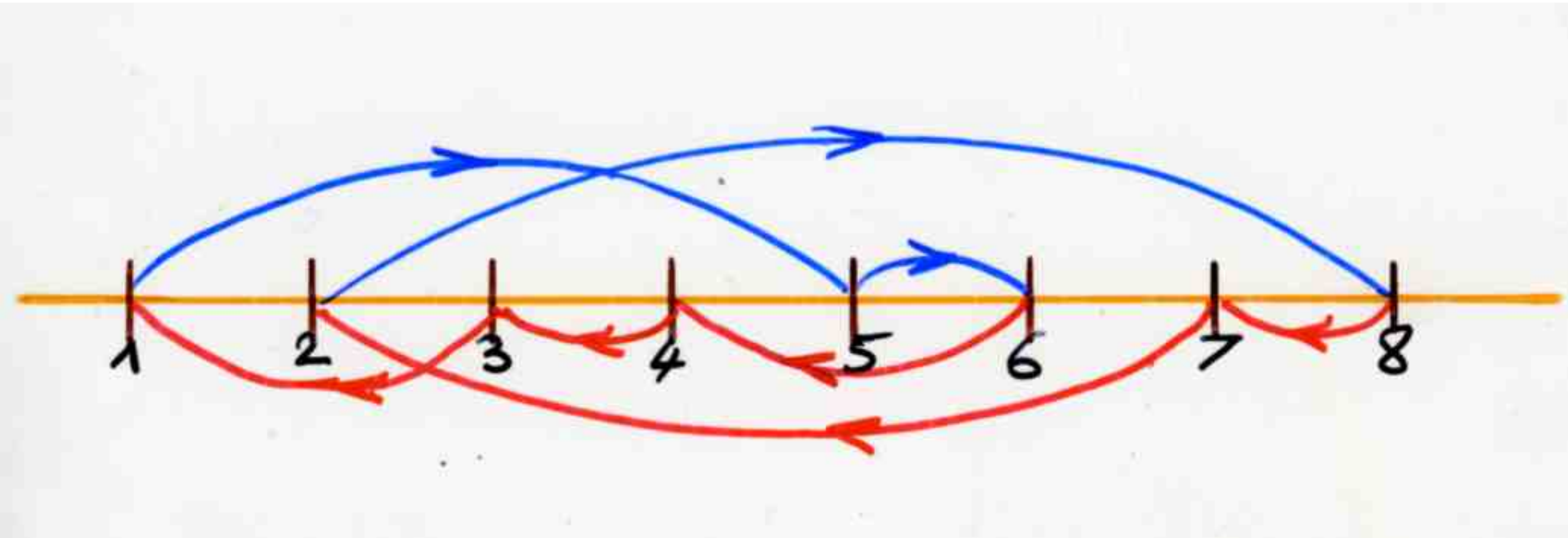
A. de Médicis, X.V.
(1994)

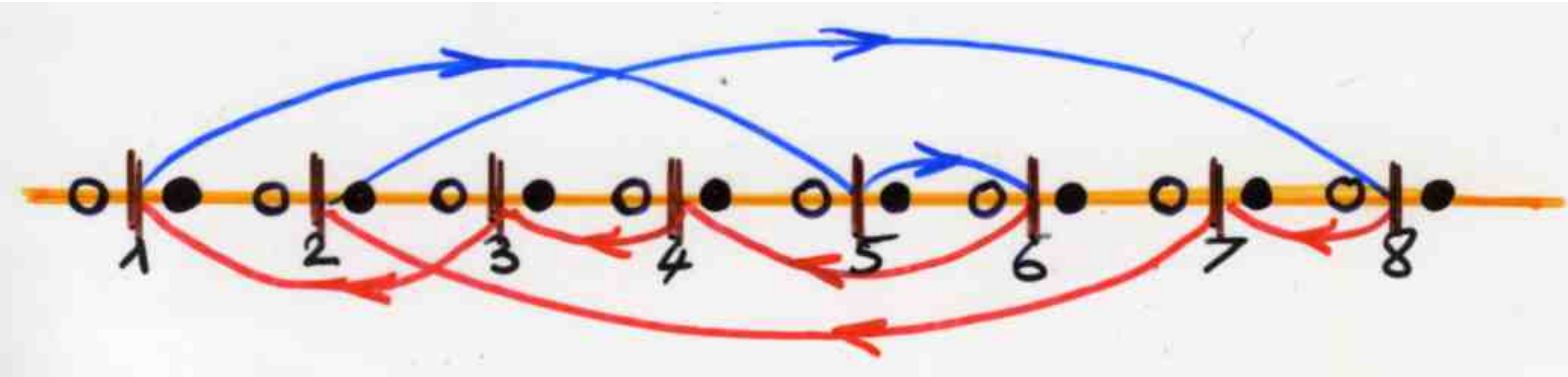
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

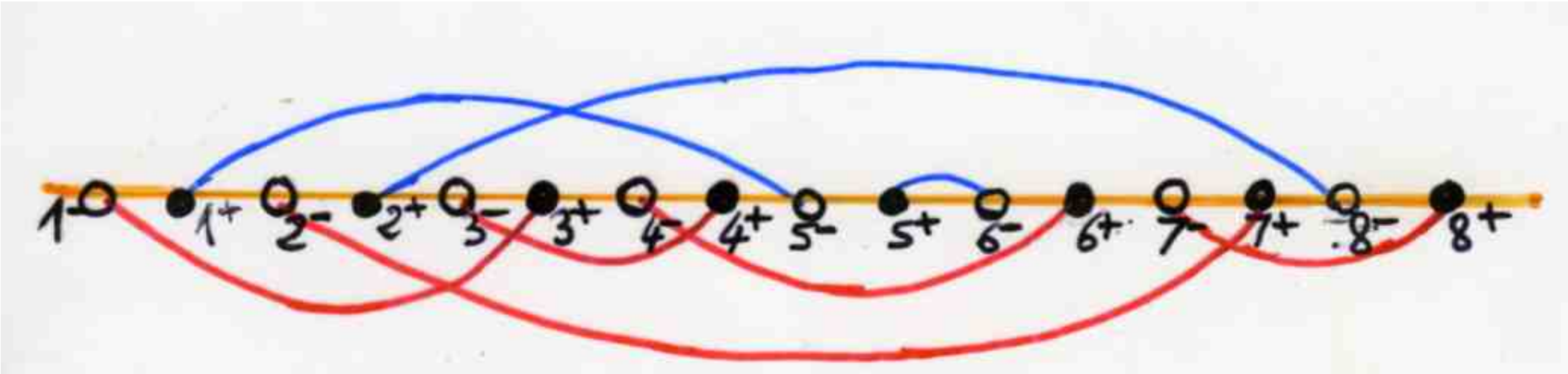
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

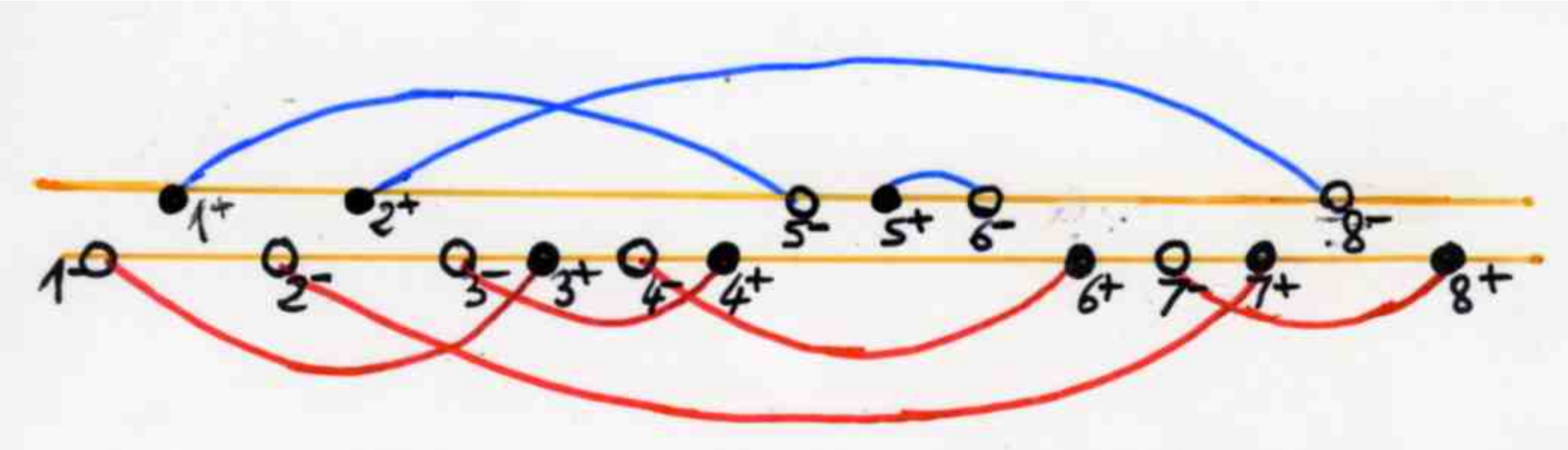
(strict) exceedances

$$i < \sigma(i)$$

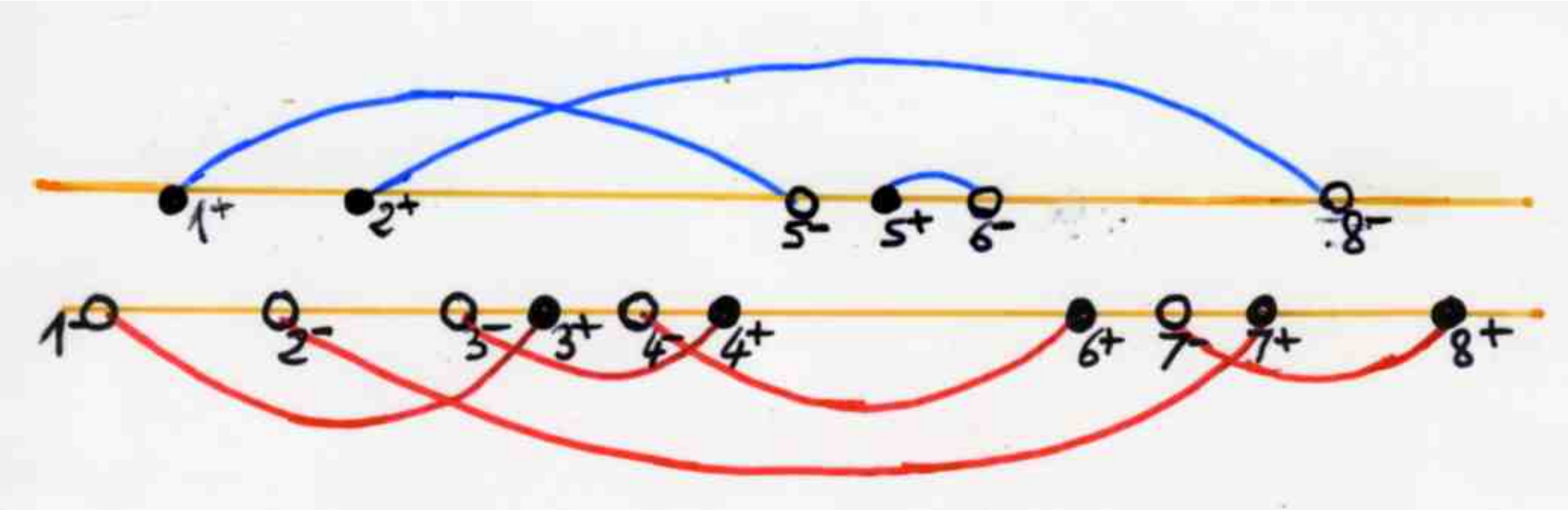

1
2
3
4
5
6
7
8


1
2
3
4
5
6
7
8

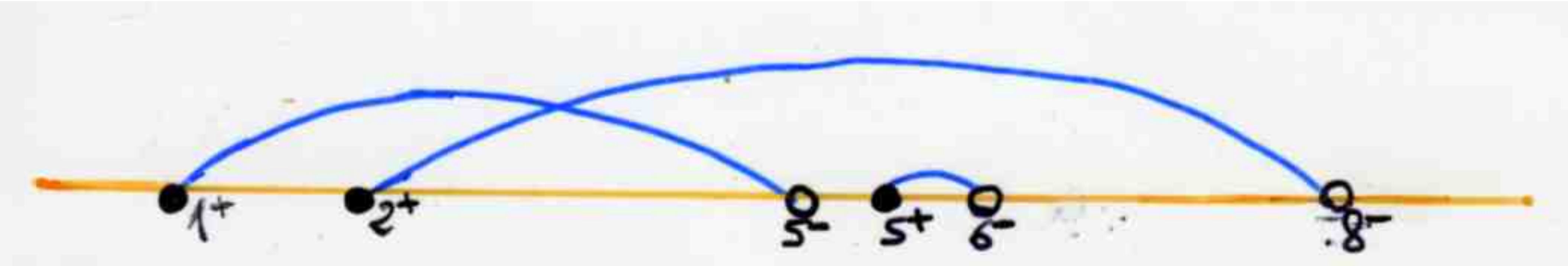

1- 1+ 2- 2+ 3- 3+ 4- 4+ 5- 5+ 6- 6+ 7- 7+ 8- 8+



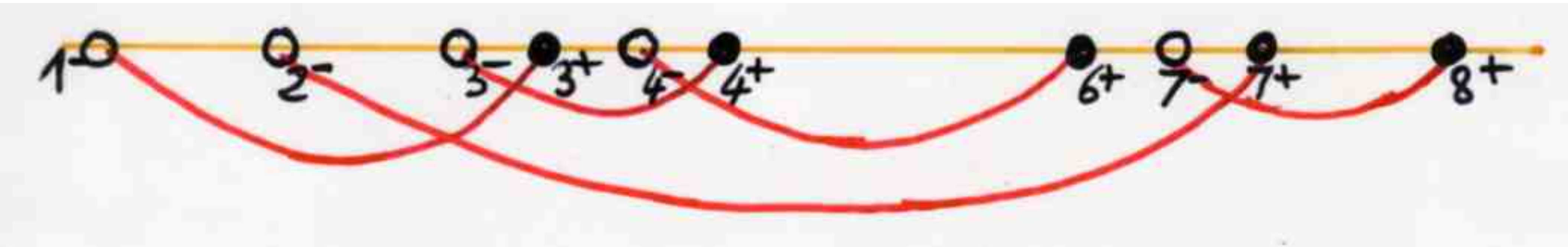
1+
2+
5-
5+
6-
8-
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+



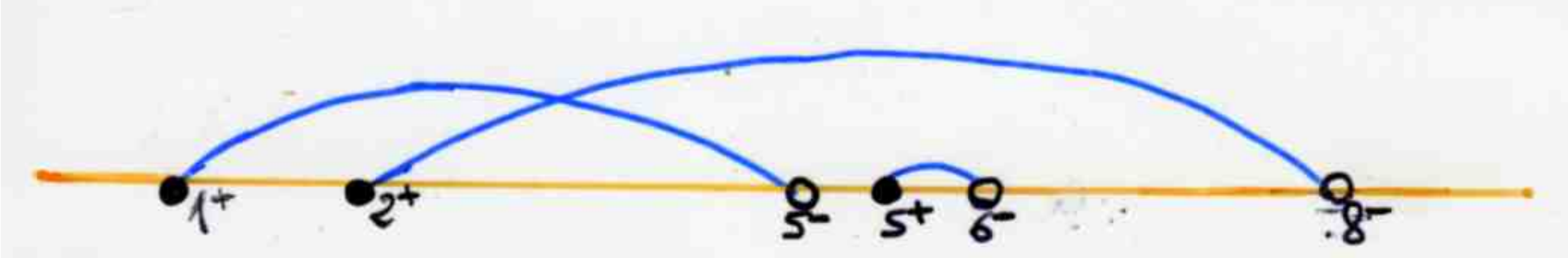
1+
2+
5-
5+
6-
8-
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+

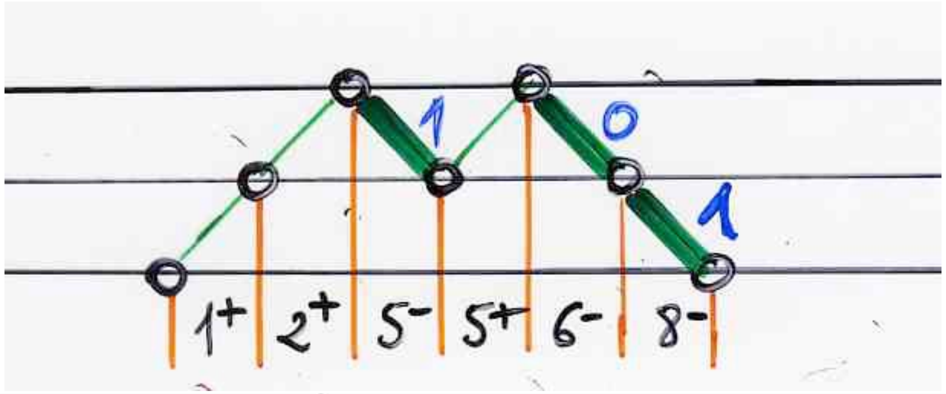


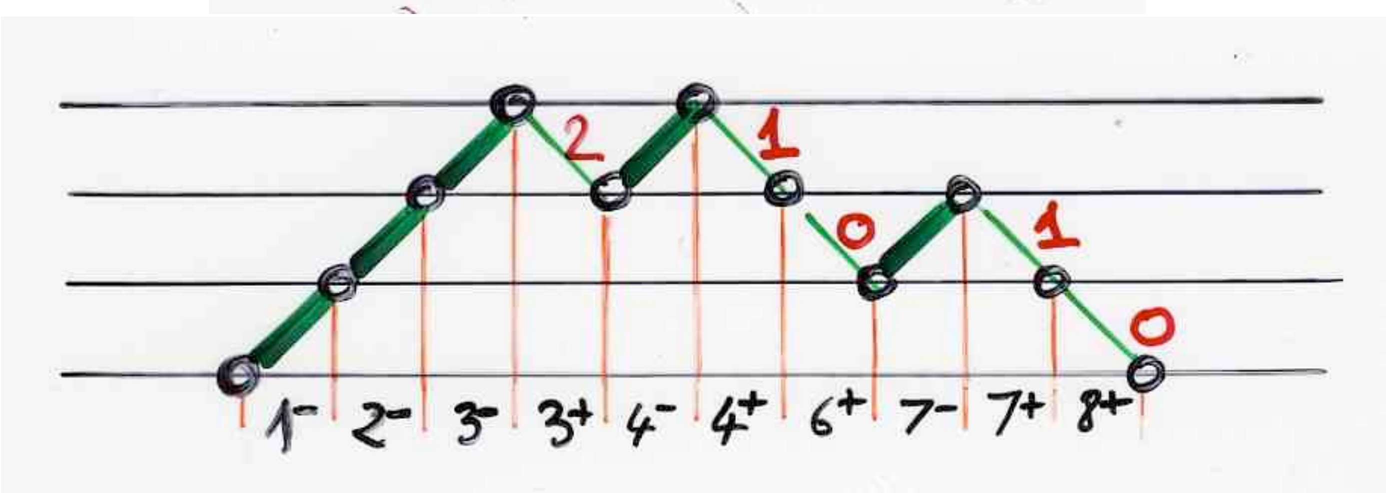
1+
2+
5-
5+
6-
8-

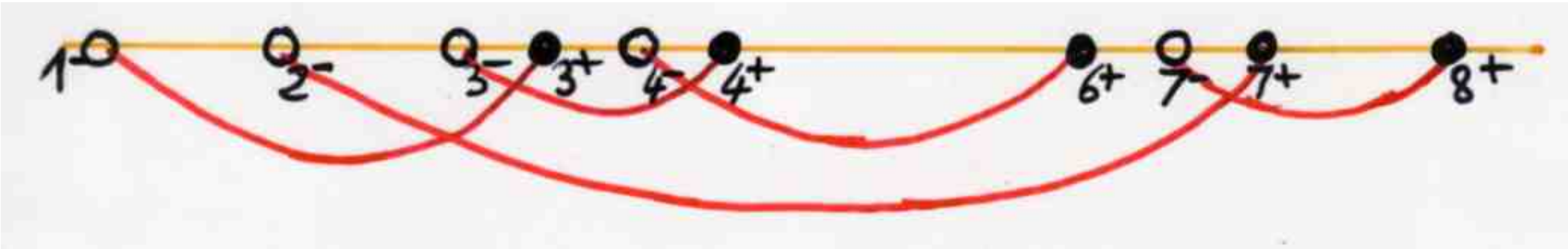


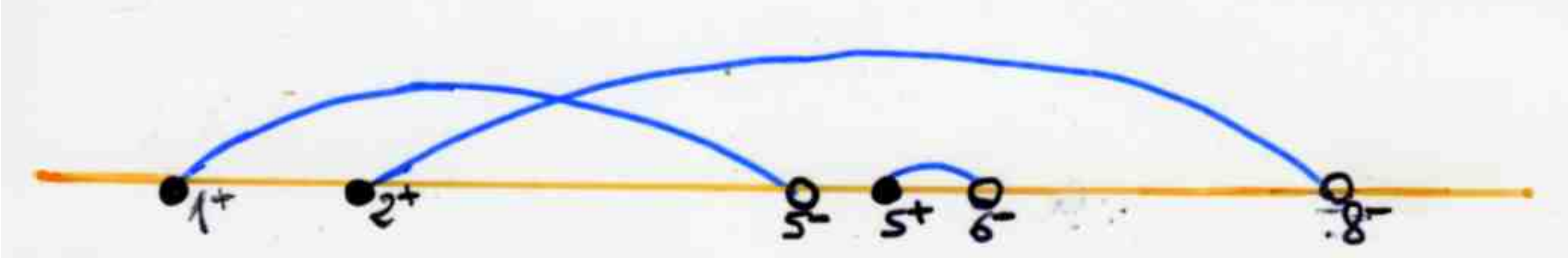
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+

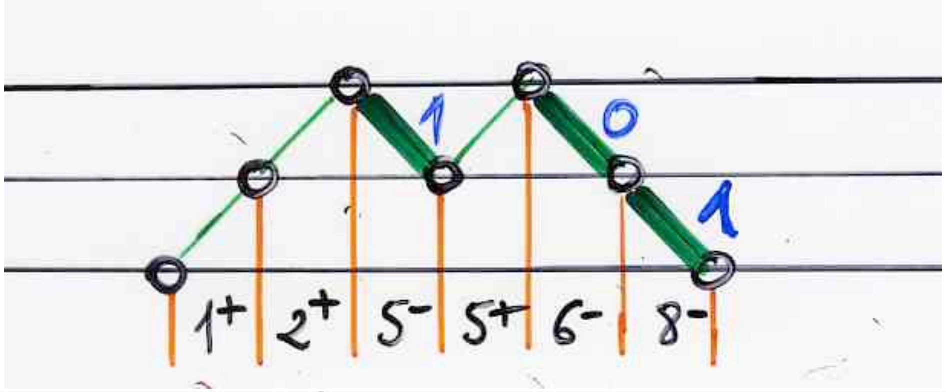

1+ 2+ 5- 5+ 6- 8-

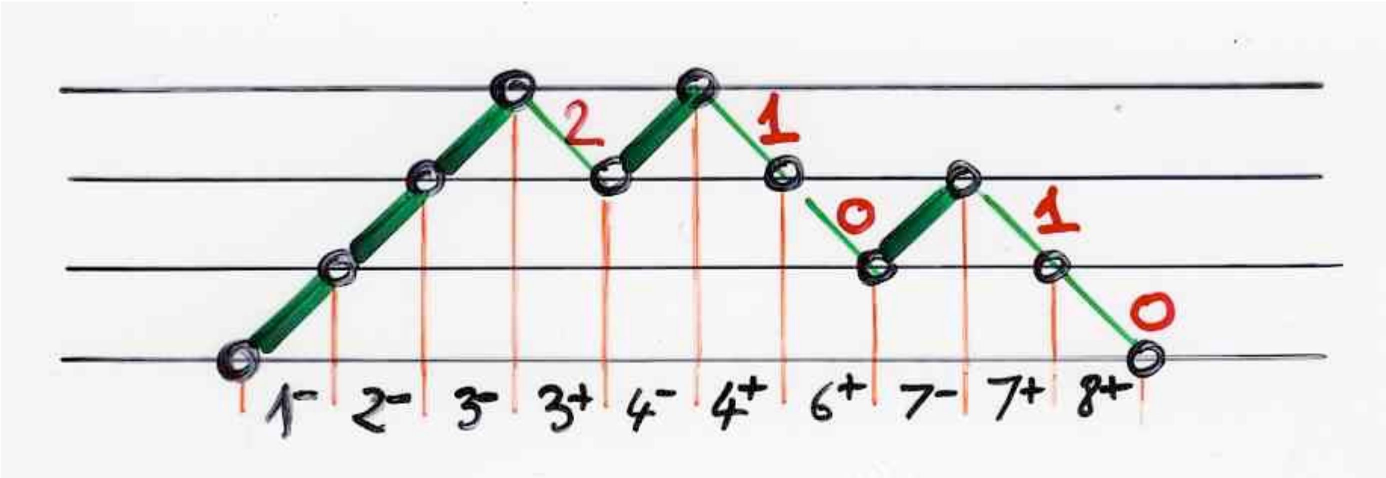

1
0
1
1+ 2+ 5- 5+ 6- 8-

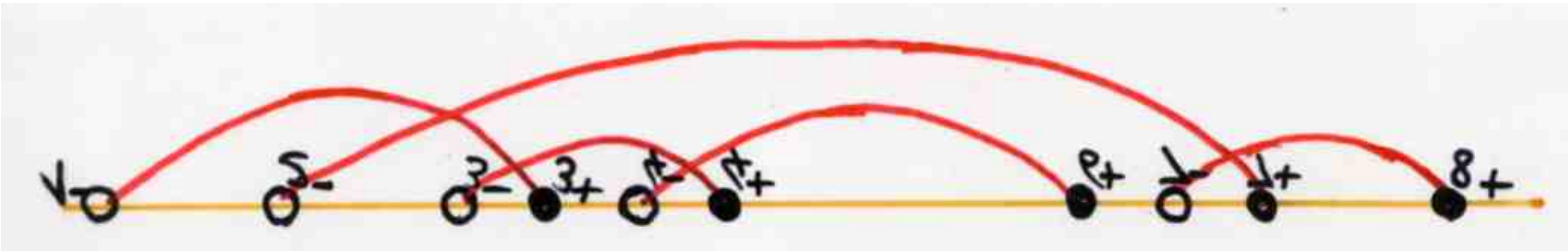

2
1
0
1
0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

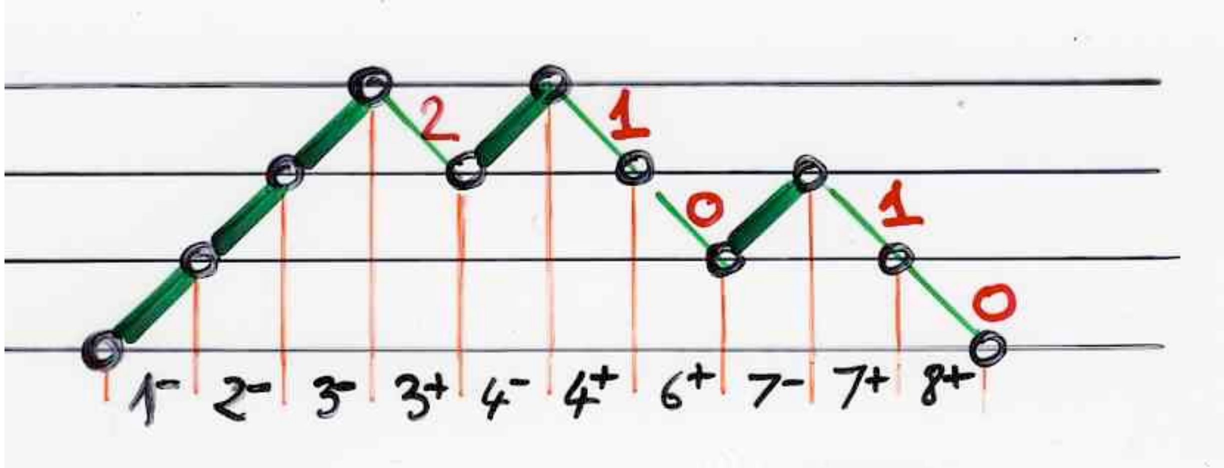

1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+


1+
2+
5-
5+
6-
8-

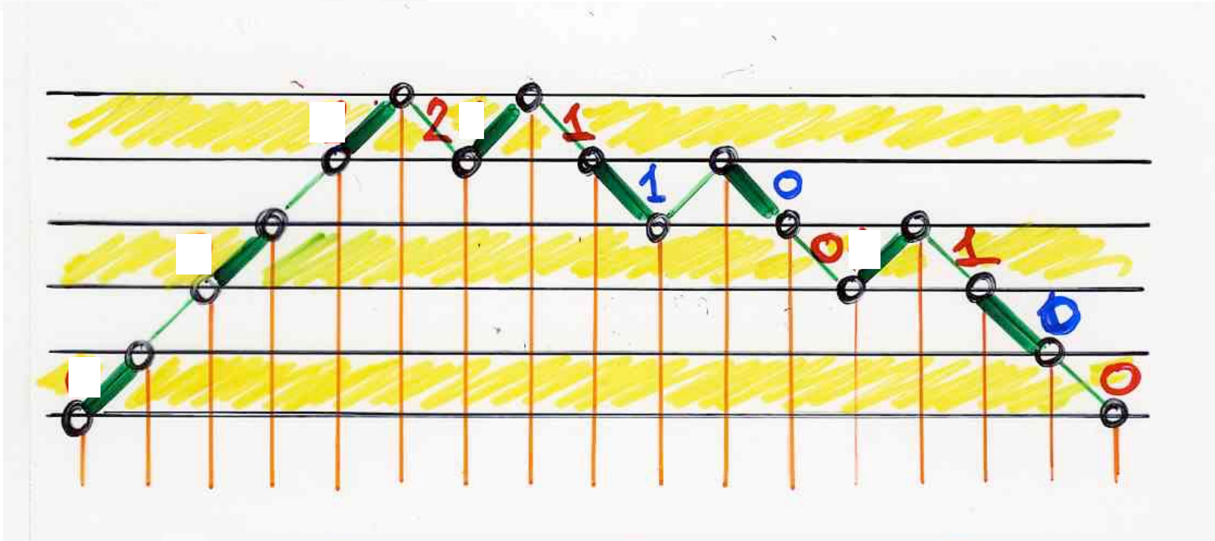

1
0
1
1+ 2+ 5- 5+ 6- 8-


2
1
0
1
0
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

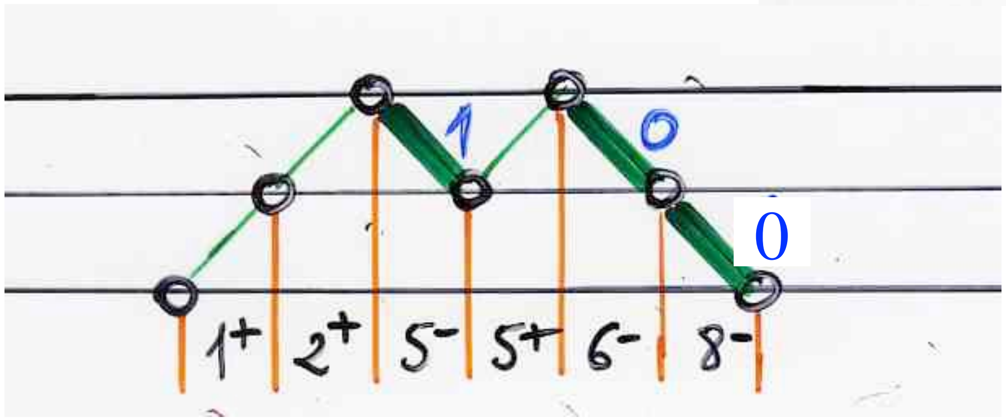

3+
8+

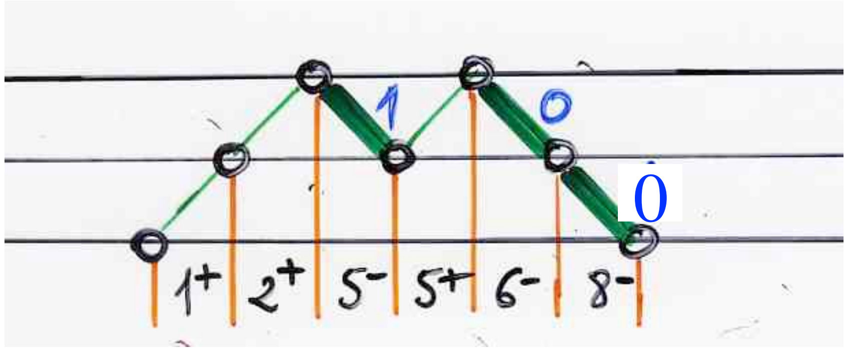


pair of two Hermite histories ("shuffle") =

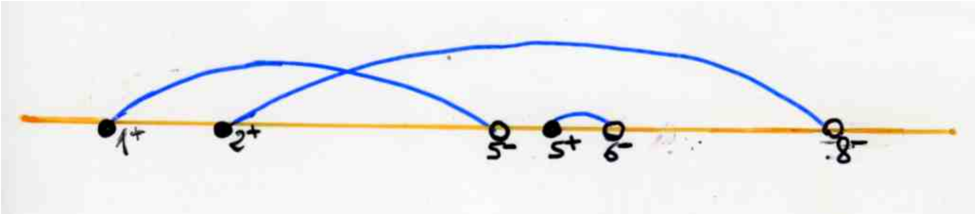


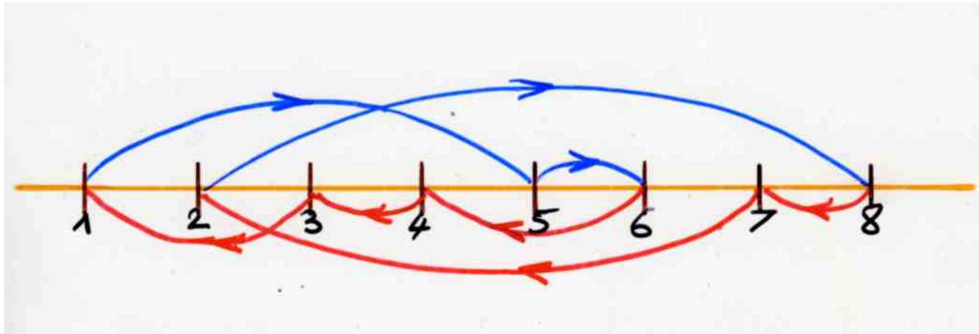
subdivided Laguerre history

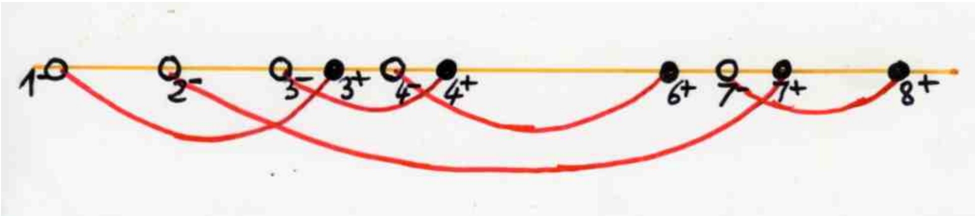


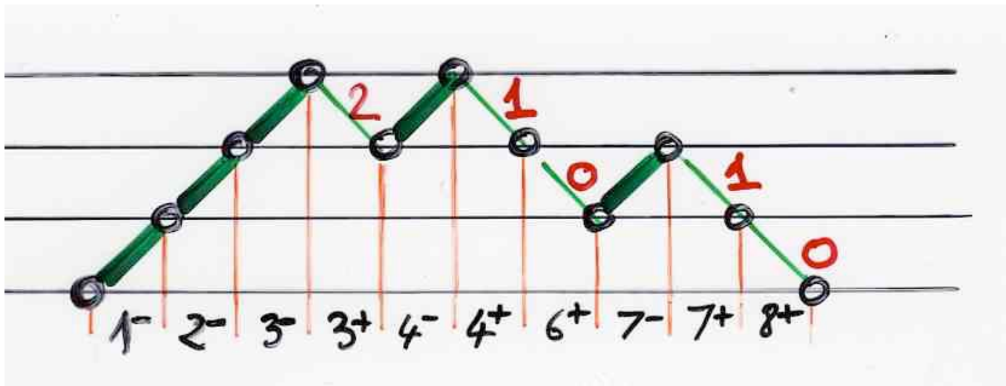


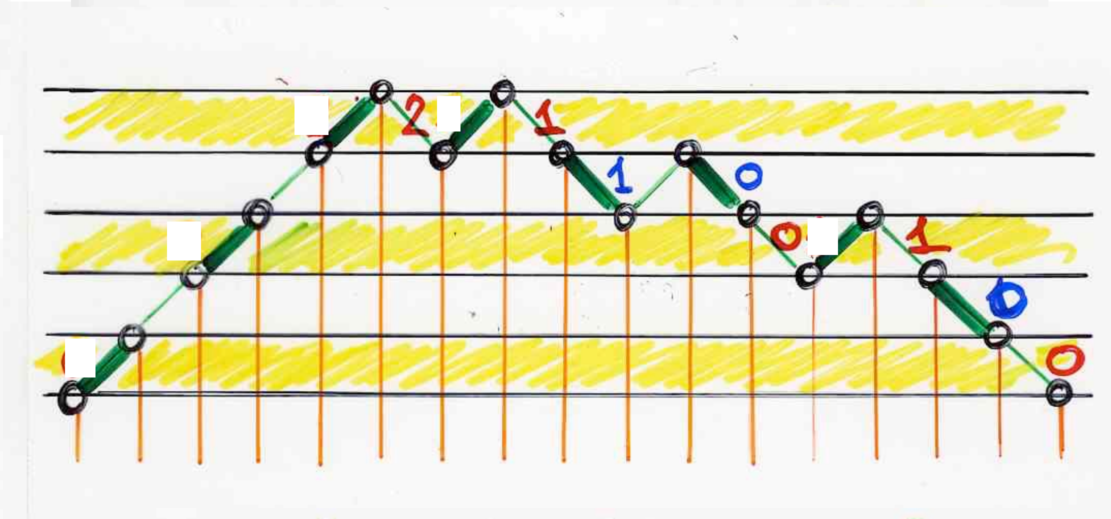
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$



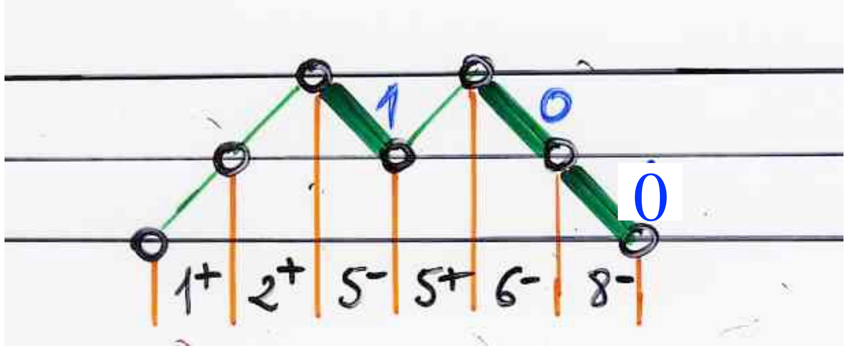




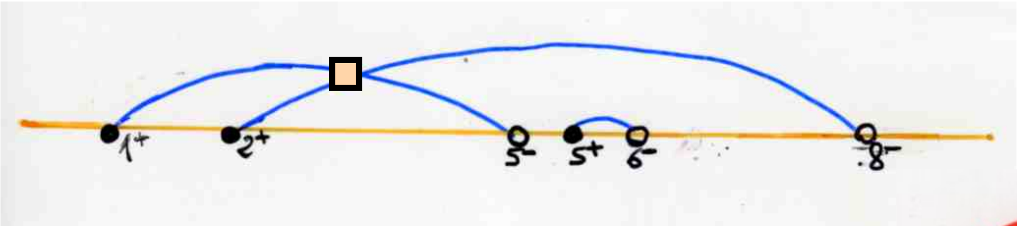


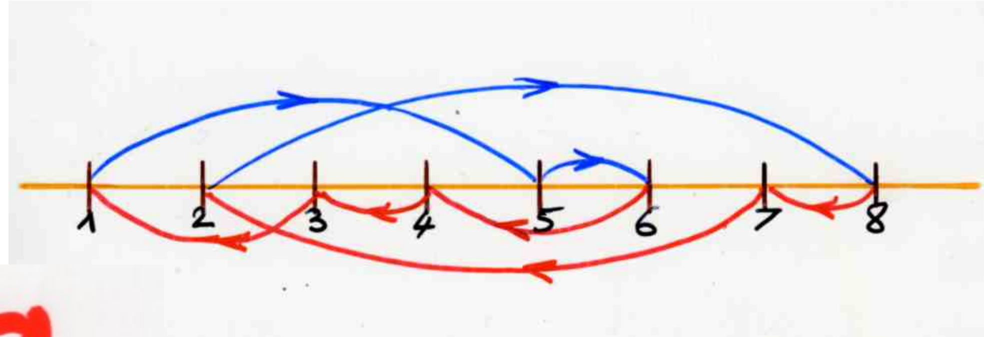


subdivided Laguerre history



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

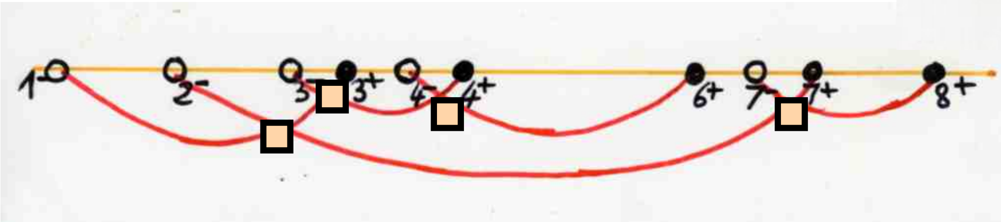


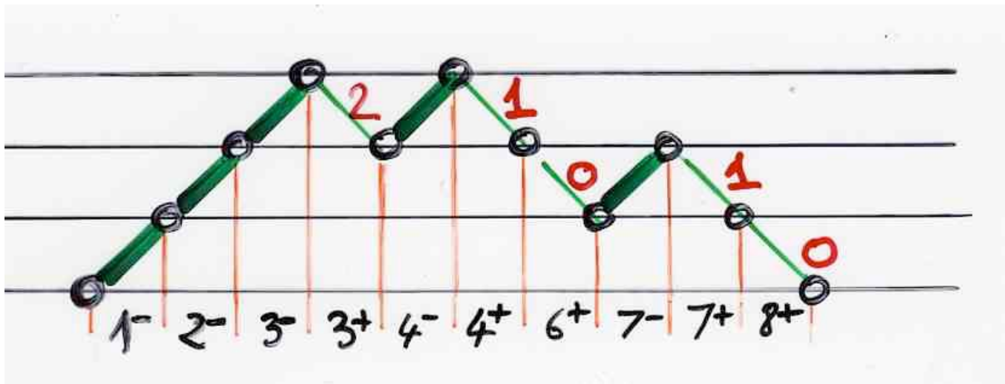


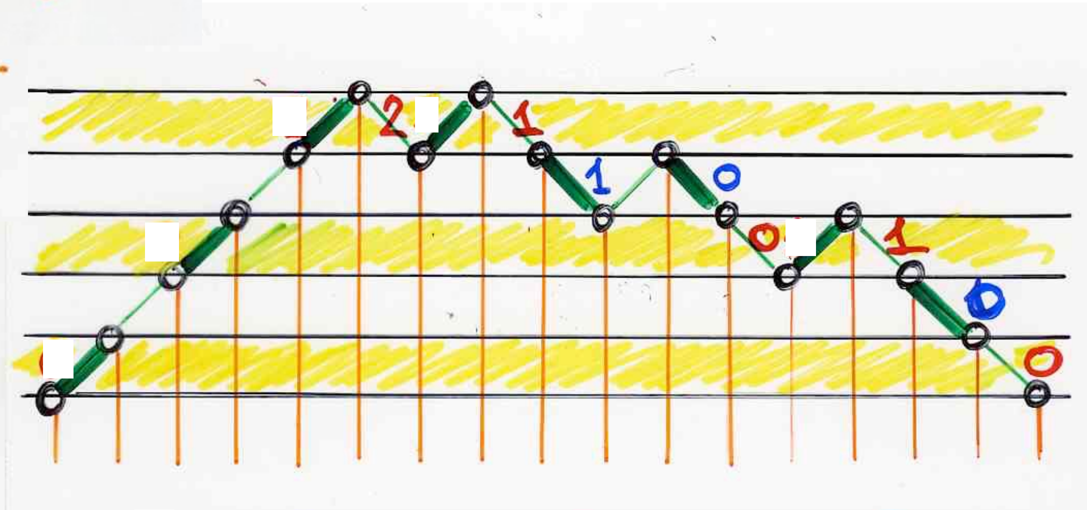
nb of crossings

9

nb of crossings







subdivided Laguerre history

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\cdots}}}}}}$$

$$\lambda_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$$

$$\sum_{n \geq 0} (n!)_q \, t^n = \cfrac{1}{1 - (1)t \cfrac{}{1 - (1)t \cfrac{}{1 - (1+q)t \cfrac{}{1 - (1+q)t \cfrac{}{1 - (1+q+q^2)t \cfrac{}{1 - \cdots}}}}}}$$

Pairs of Hermite histories ⟷ permutations τ ⟷ permutation tableaux

exceedances

↕

subdivided Laguerre histories

q

Interpretation of the 3-parameters
Partition function

q, α, β

Second bijection: tableaux— permutations

bijection
Corteel, Nadeau (2007)

equivalent to

"exchange-fusion"
"exchange-delete"
algorithm X.V. (2007)

~~"special"~~

q, α, β

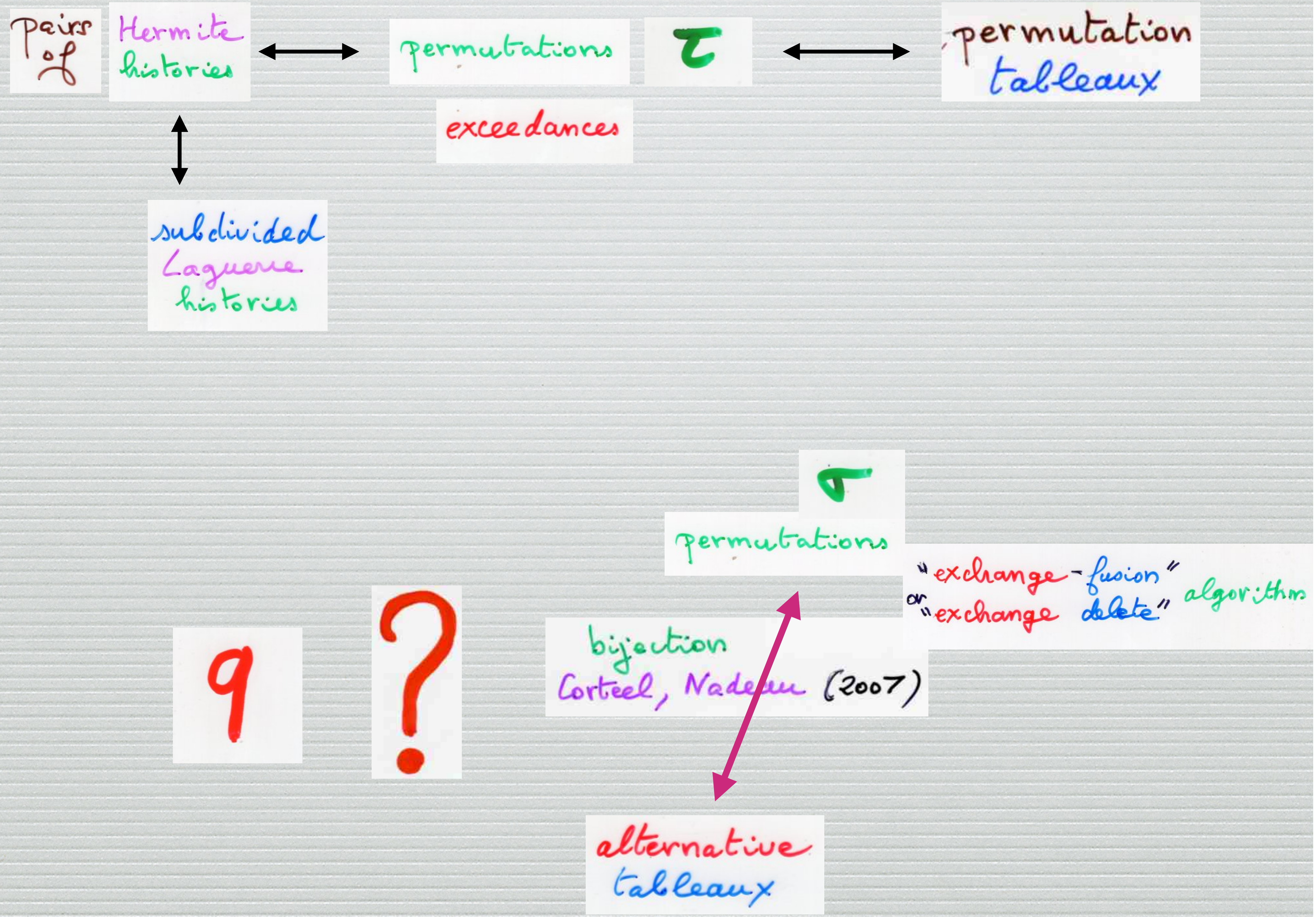
left-to-right
right-to-left

minimum

elements

generating polynomial:

$$xy(x+y)(x+1+y)\ldots(x+n-1+y)$$


Pairs of
Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
σ
permutations
"exchange-fusion"
or "exchange delete" algorithm
bijection
Corteel, Nadeau (2007)
q
?
alternative tableaux

q, α, β

Josuat-Vergès (2011)

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma)$ = number right-to-left maxima

$t(\sigma)$ = number right-to-left minima

$31\text{-}2(\sigma)$ = number of patterns 31-2

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

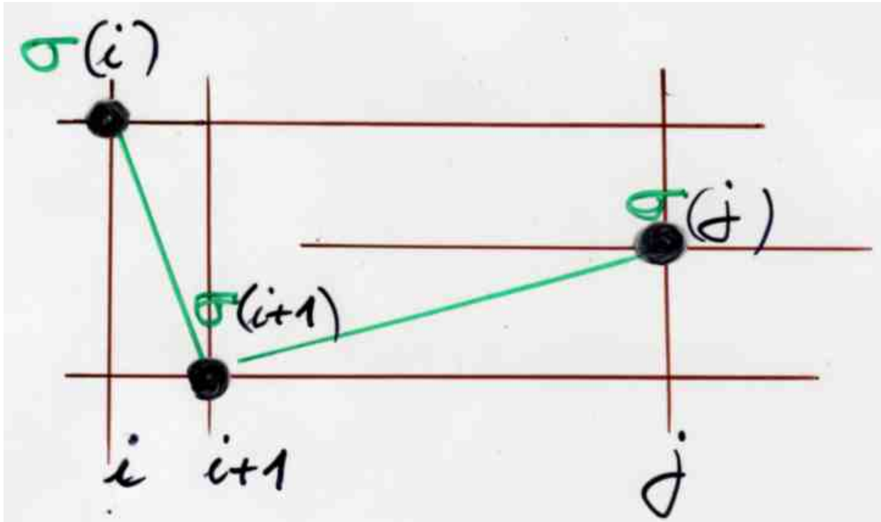
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

31-2



$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Al-Salam-Chihara polynomials

$$2x Q_n(x) = Q_{n+1}(x) + (a+b) q^n Q_n(x) + (1-q^n)(1-abq^{n-1}) Q_{n-1}(x)$$

Laguerre histories

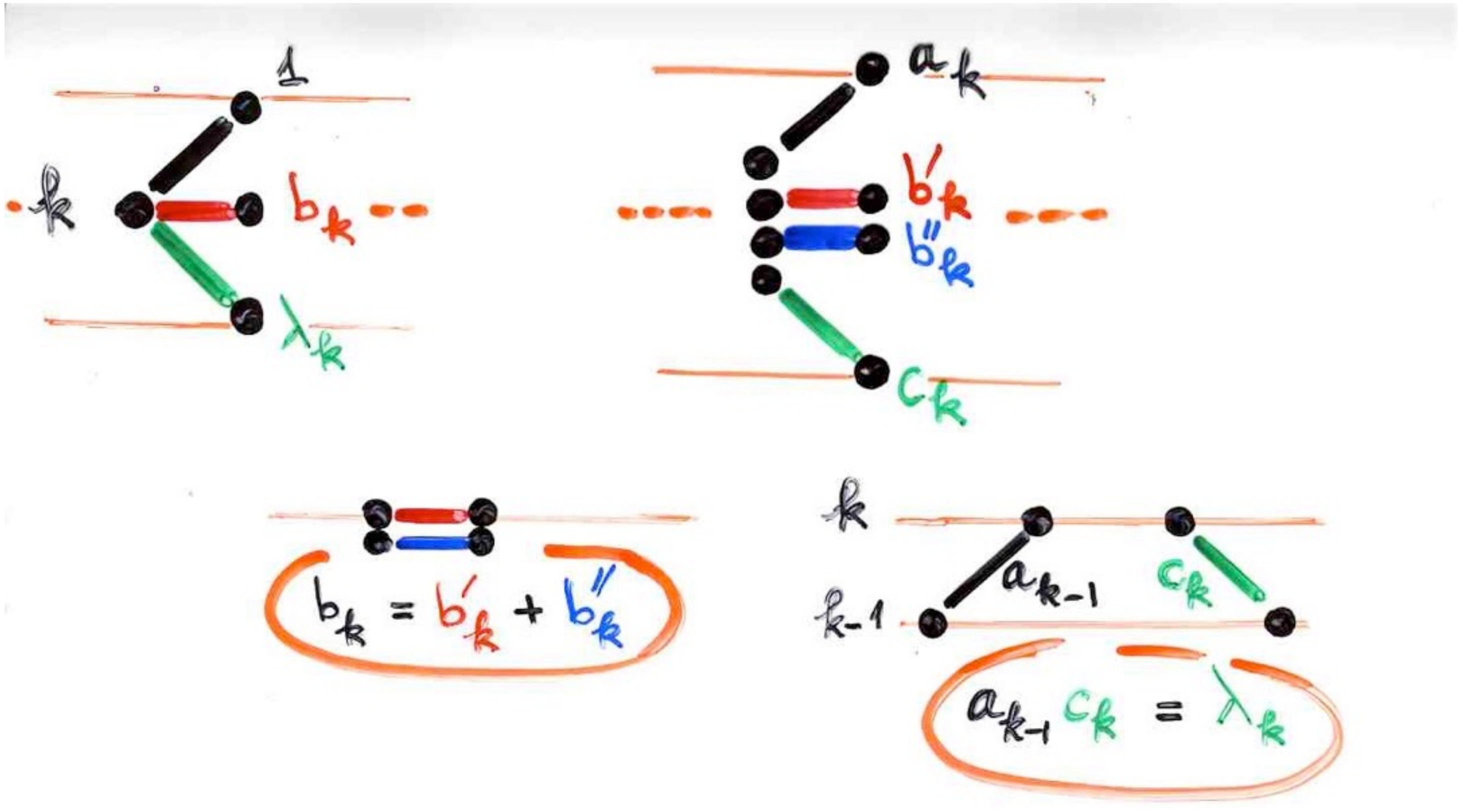
The FV bijection

Françon, X.V. (1978)

Laguerre polynomial

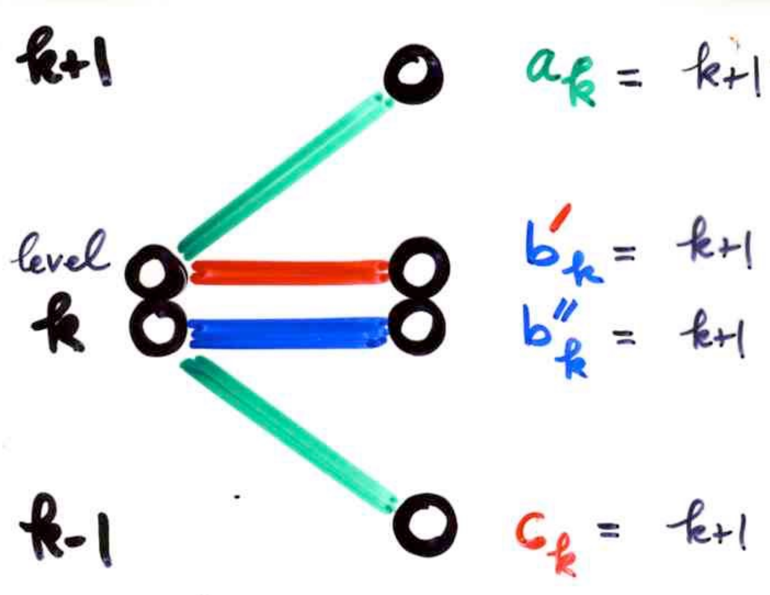
$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

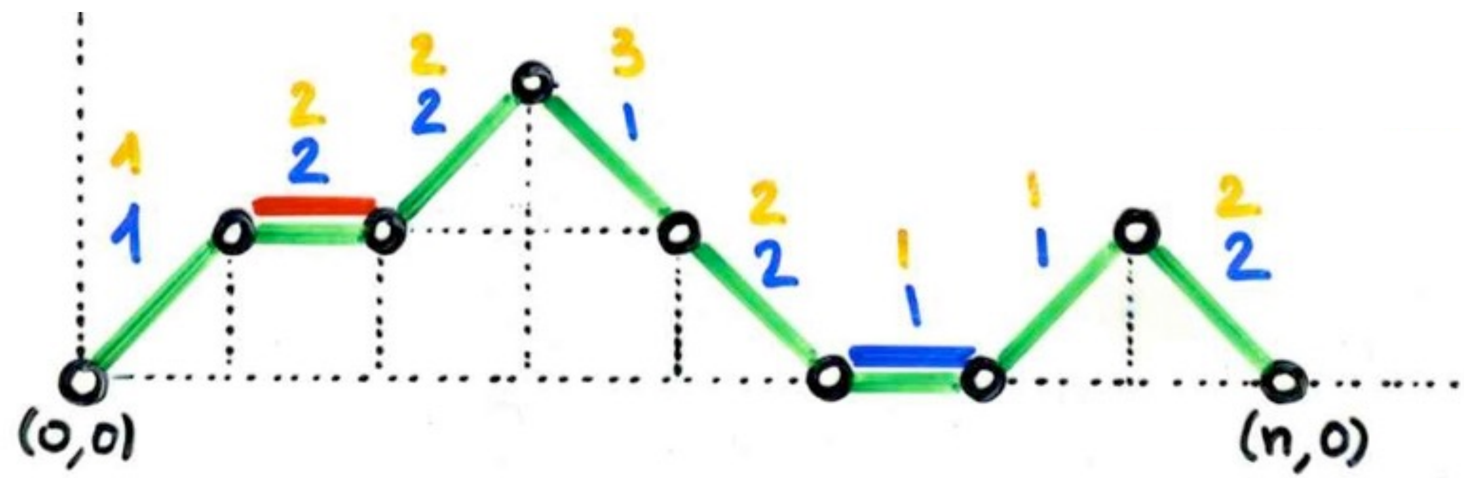
$$\mu_n = (n+1)!$$

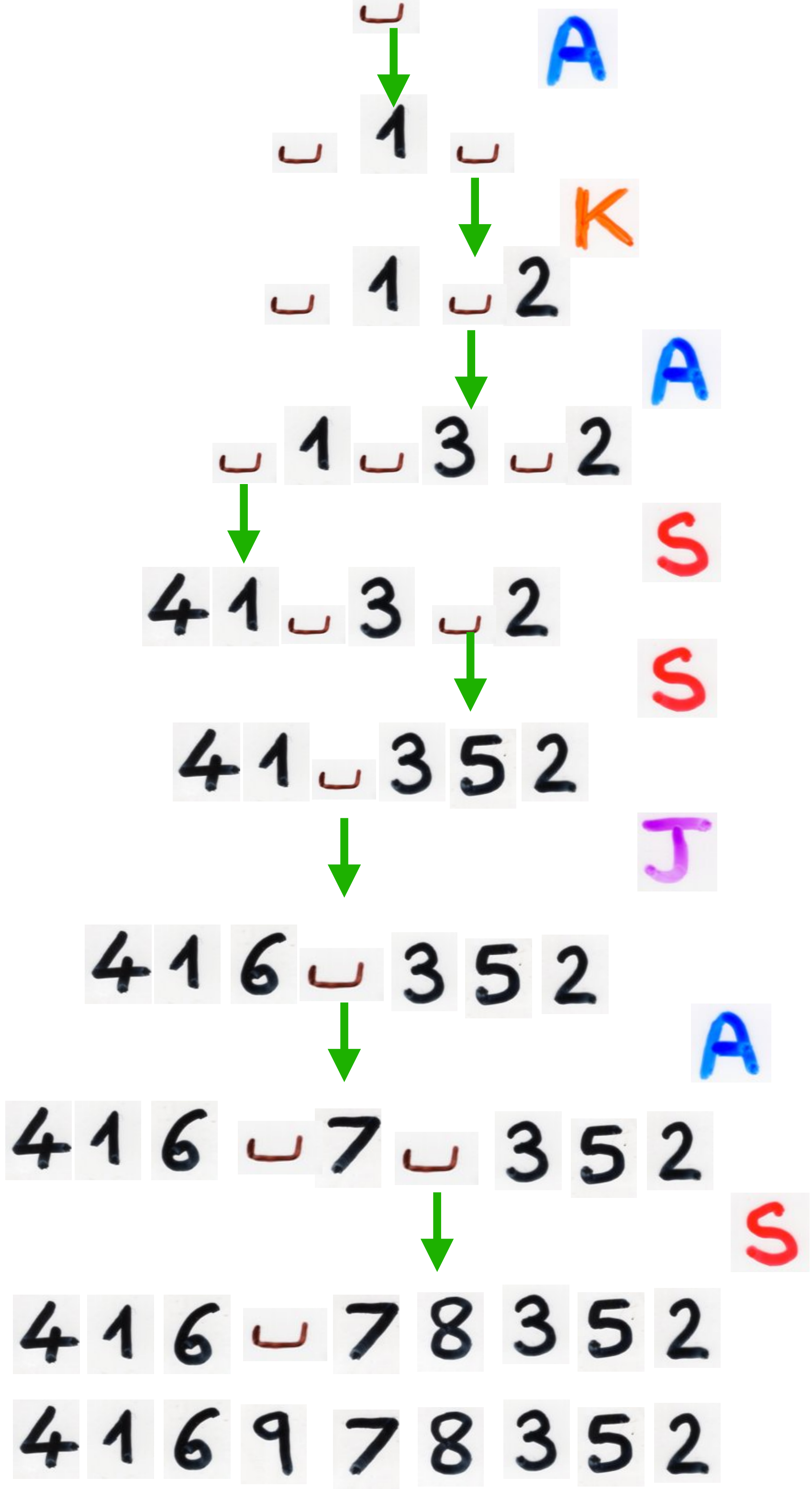


$$b_k = (2k+2)$$

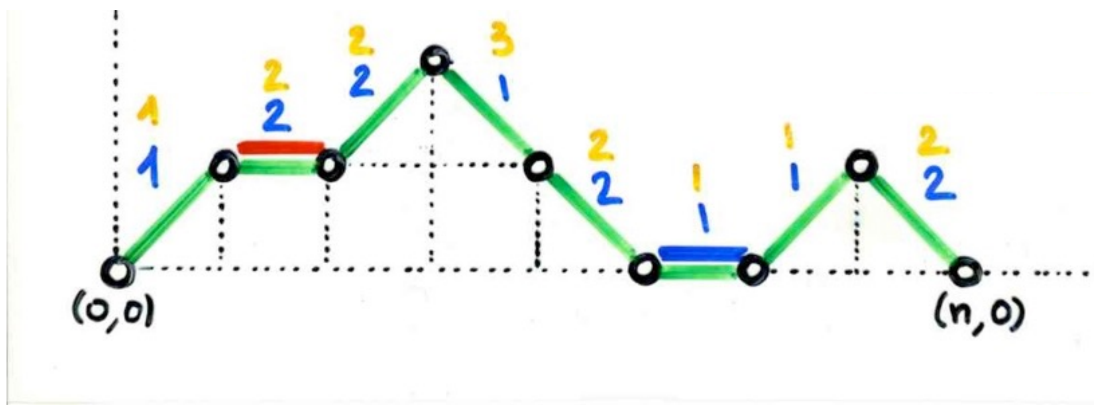
$$\lambda_k = k(k+1)$$


k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1


(0,0)
(n,0)



Laguerre histories

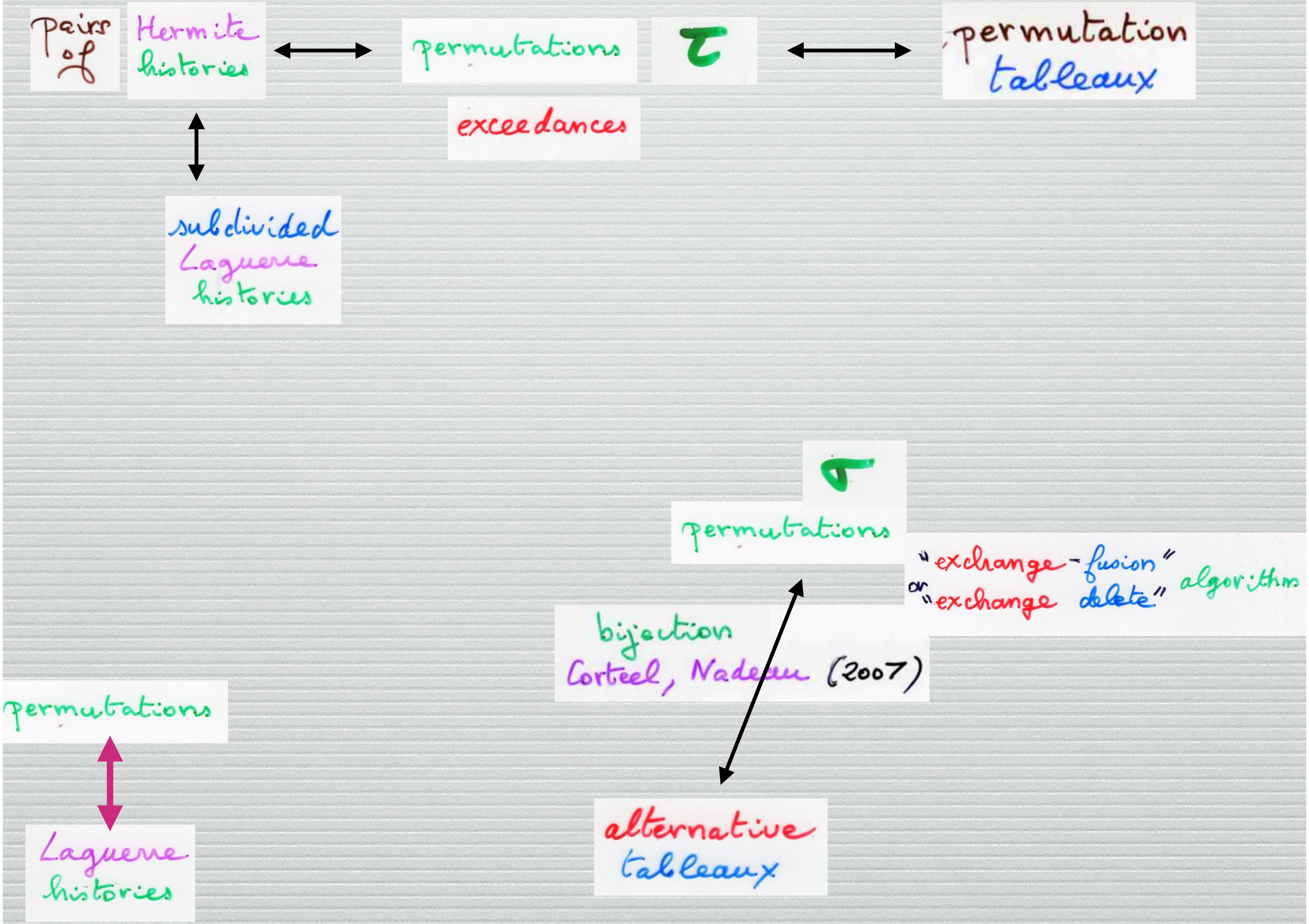


$$\langle k|\, A = (k+1)\, \langle (k+1)|$$

$$\langle k|\, K = (k+1)\, \langle k|$$

$$\langle k|\, J = (k+1)\, \langle k|$$

$$\langle k|\, S = (k+1)\, \langle (k-1)|$$



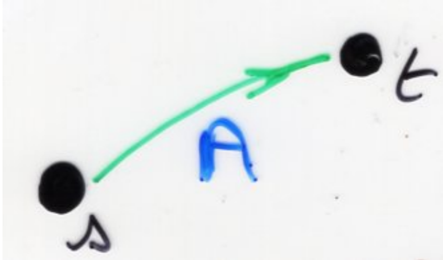
Pairs of
Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
σ
permutations
"exchange-fusion"
or "exchange delete"
algorithm
bijection
Corteel, Nadeau (2007)
alternative tableaux
permutations
Laguerre histories

The essence of the parameter 31-2

the philosophy of « histories »

and its q-analogues

S states



operators

history

weight

$V_A(s,t)$ = number of possibilities to apply A

$H = h_1 h_2 \cdots h_n$

sequence of operators

initial state s_0

$P = (p_1, p_2, \ldots, p_n)$

$s_i \xrightarrow{h_i} s_{i+1}$

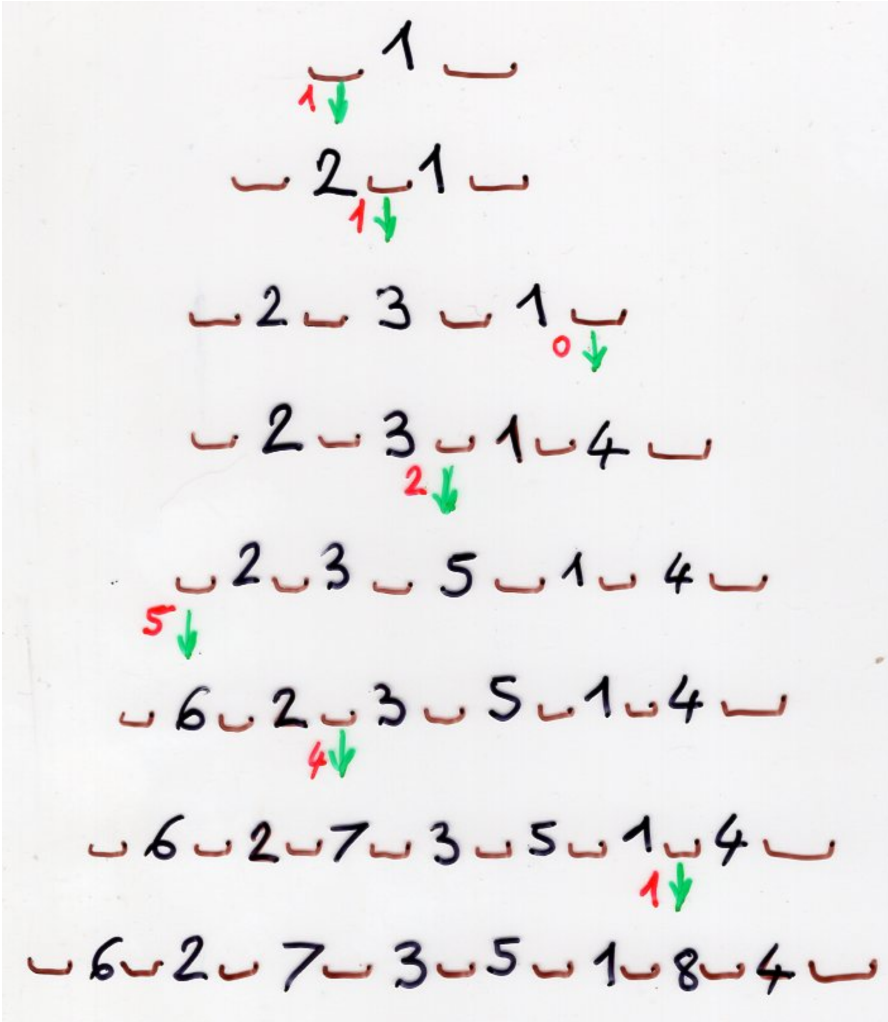
$0 \leq p_i < V_{h_i}(s_i, s_{i+1})$

q- weight

$$V_q(H) = q^{\left(\sum_{i=1}^{n} p_i\right)}$$

Inv

number of inversions



Maj

Major index

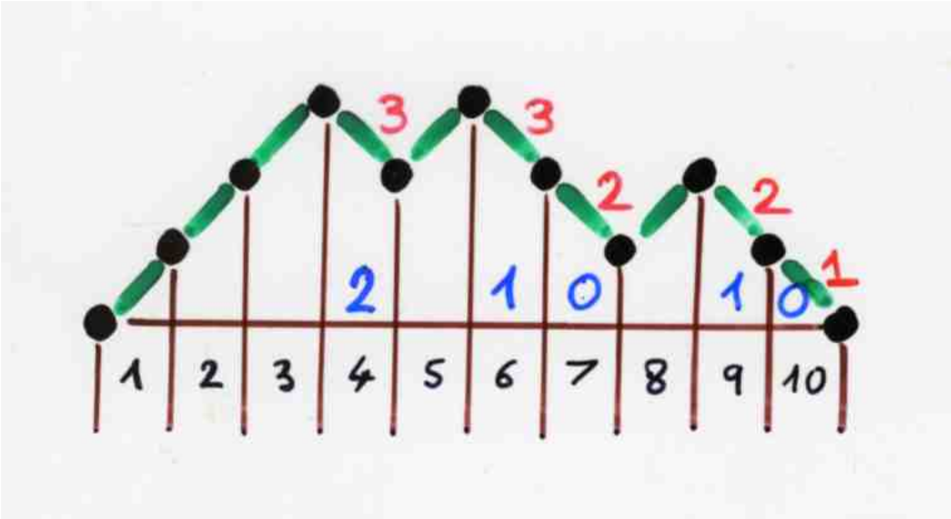
σ = 6 2 7 3 5 1 8 4

$$\mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma(i) > \sigma(i+1)}} i$$

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}$$

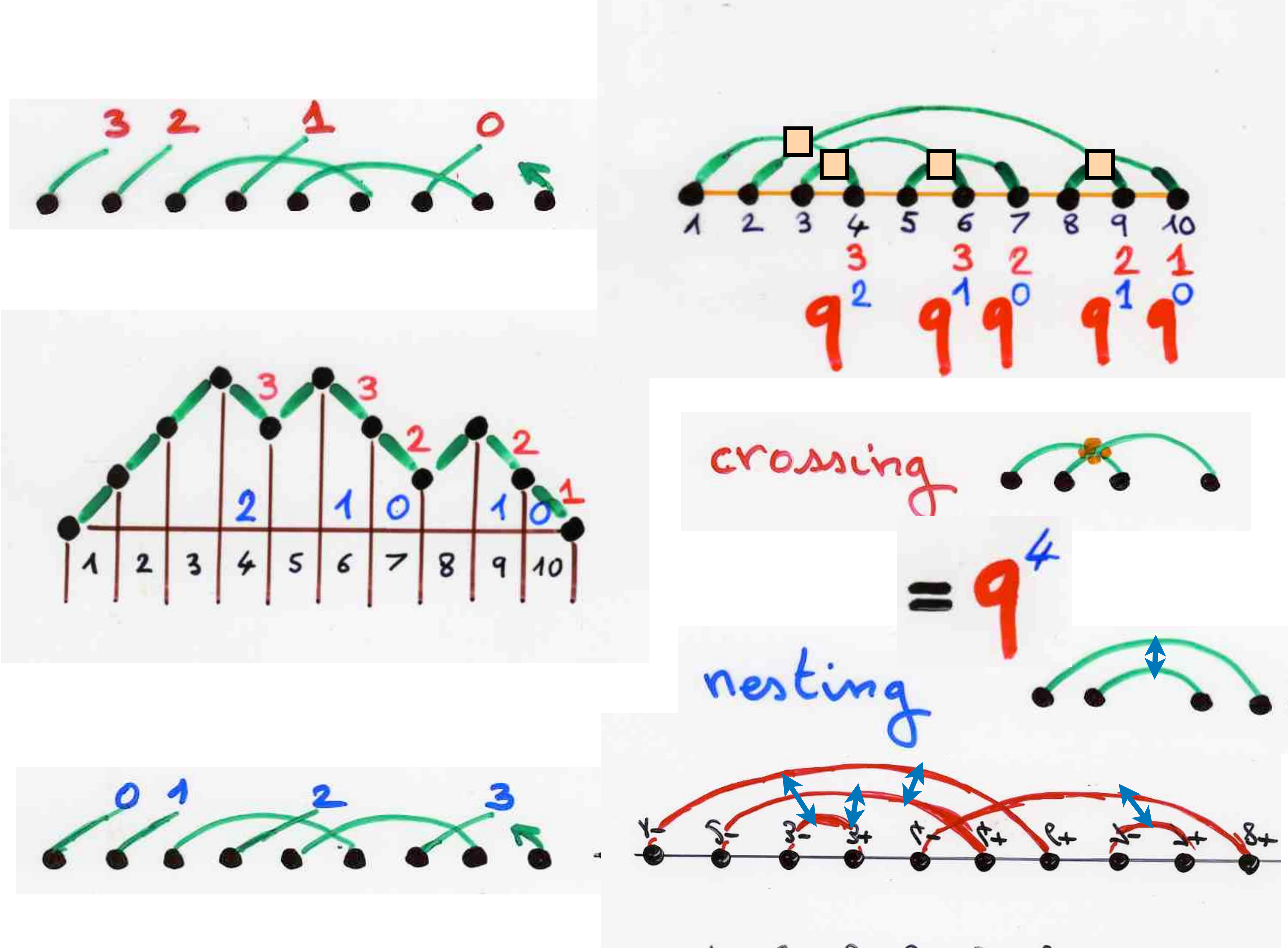
Mahonian
distribution

Hermite history



crossing

nesting

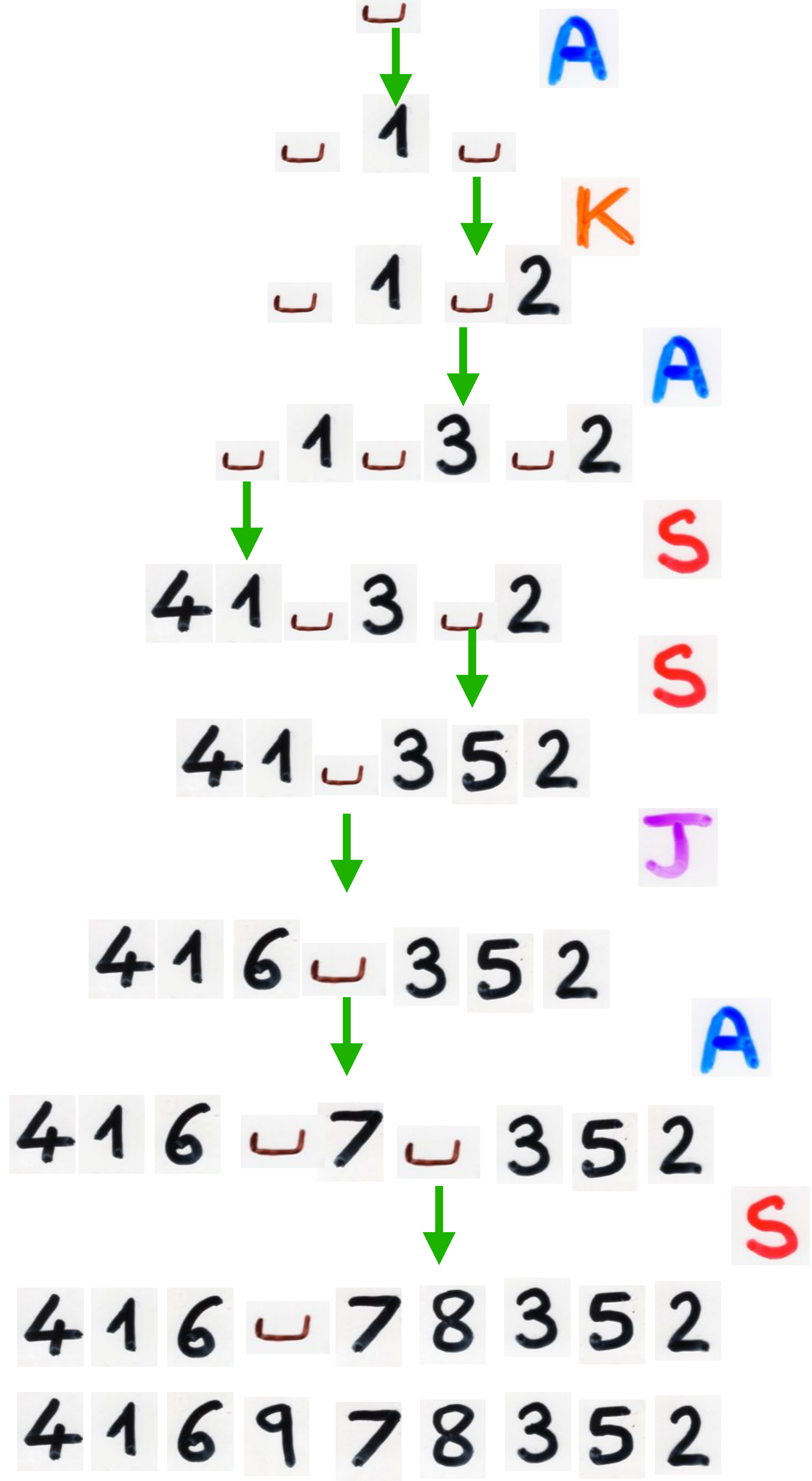


3 2 1 0
1 2 3 4 5 6 7 8 9 10
3 3 2 2 1
q2 q1 q0 q1 q0
crossing
= q4
nesting
0 1 2 3

q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$



"q-analogue" of Laguerre histories

choice function

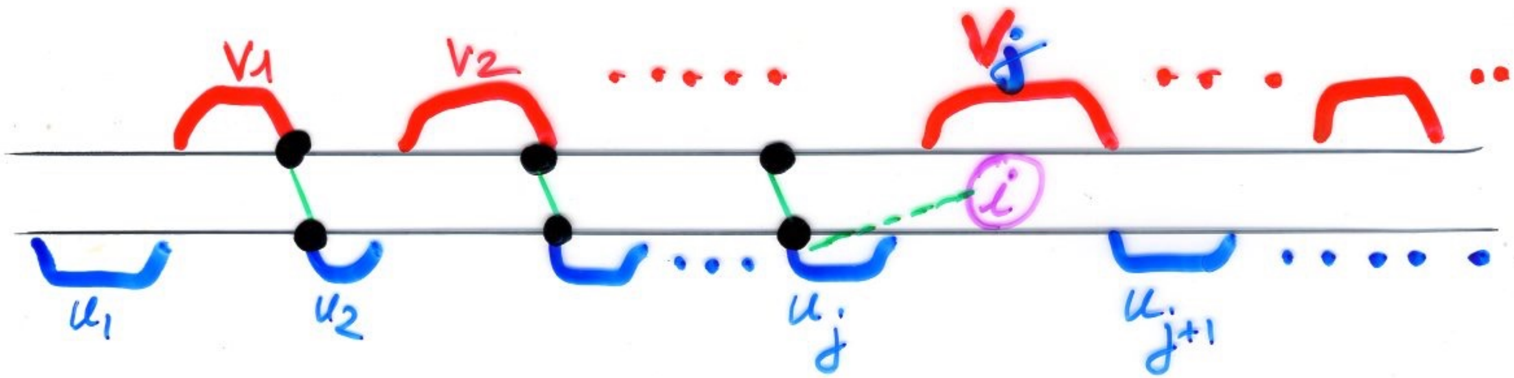
$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories q^4

weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n} (p_i - 1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern (31-2)



q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q-Laguerre restricted histories

$$\mu_n = n!$$

q - Laguerre II

if $\mu_n = [n!]_q$

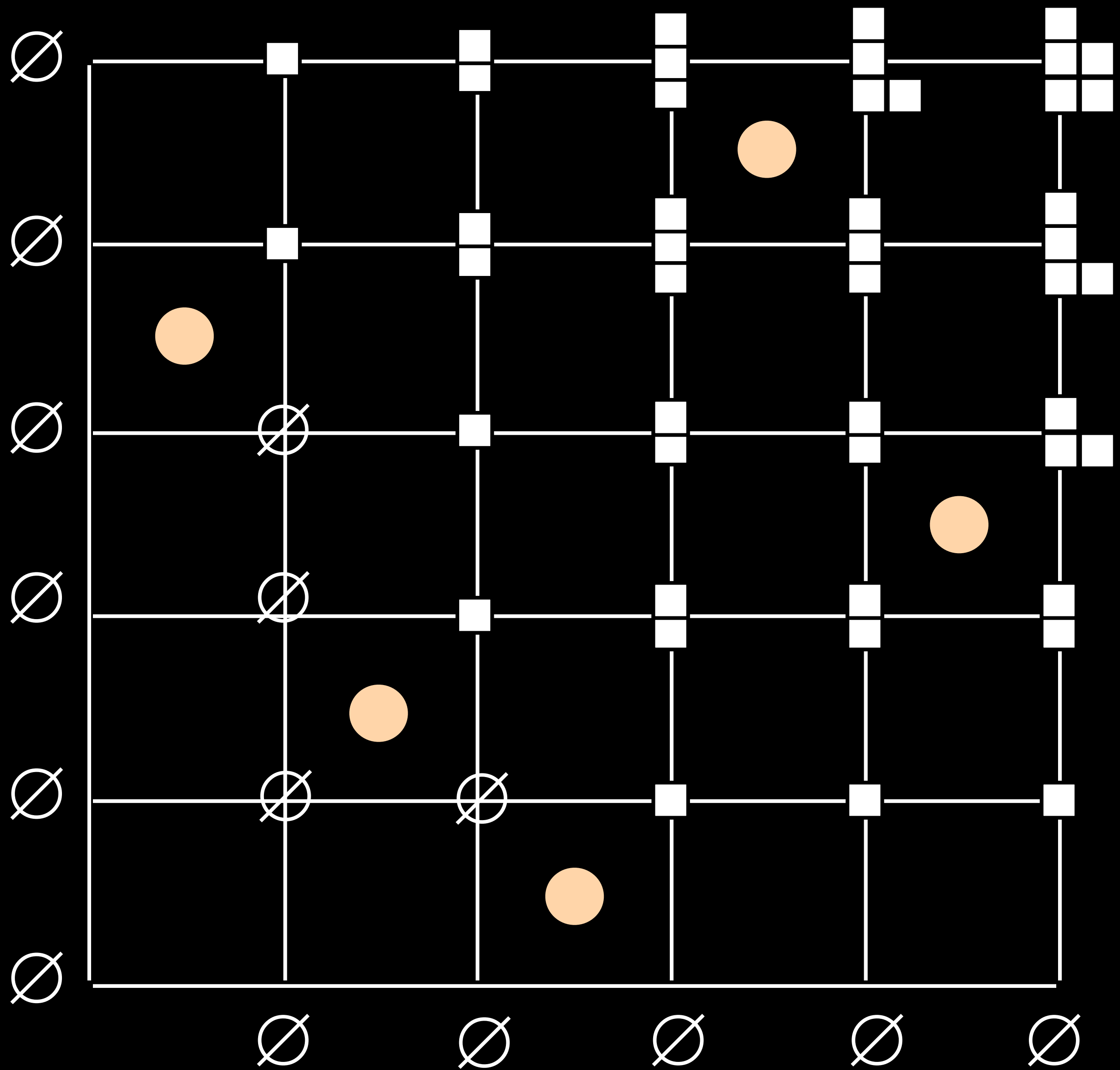
then
$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

The essence of bijections

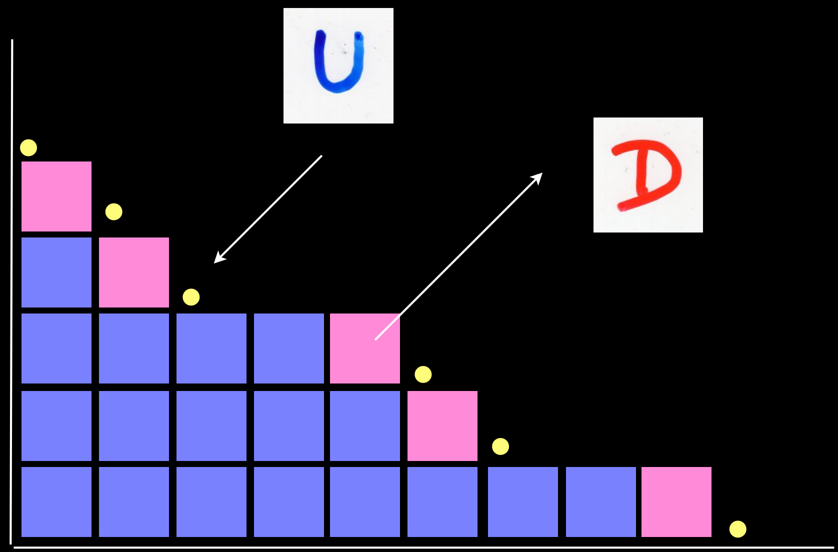
growth diagrams and the RS bijection

S. Fomin, 1986, 1994

Сергей Владимирович Фомин

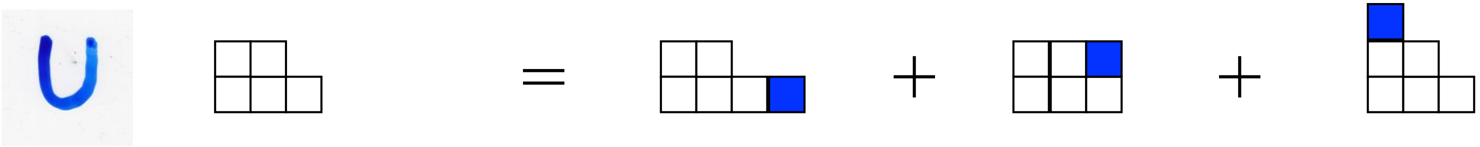


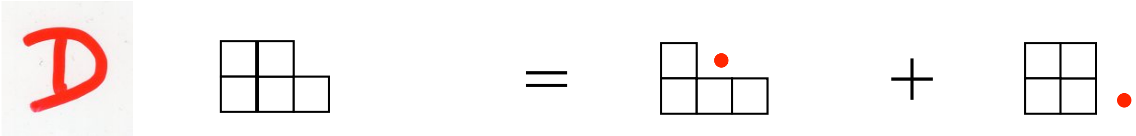
operators U and D

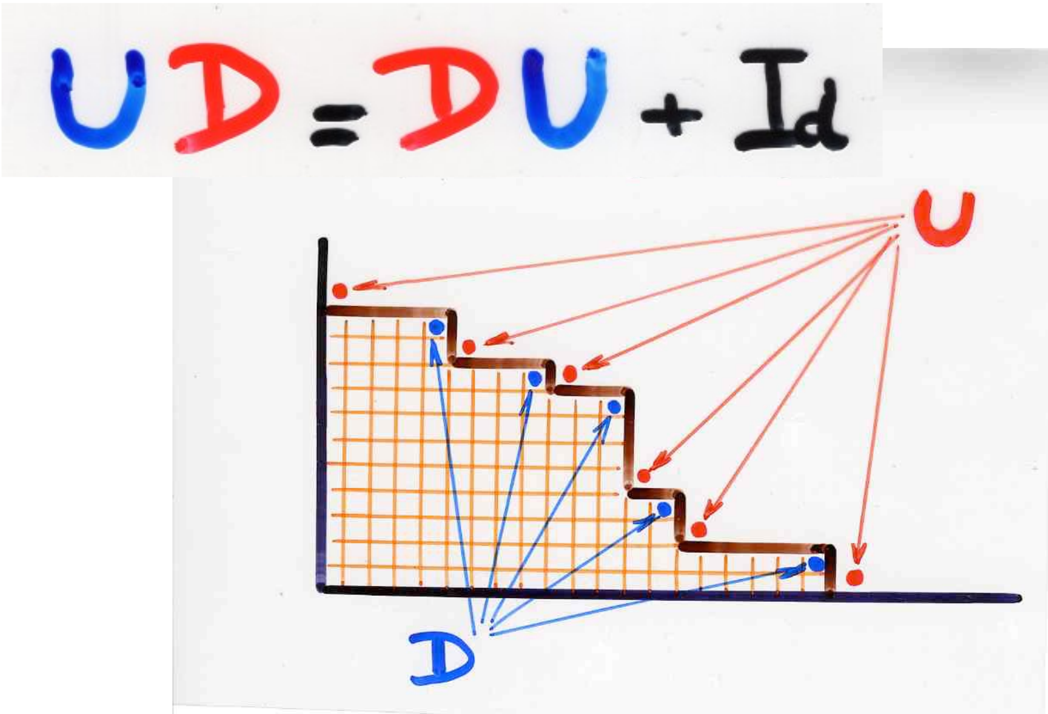


Young lattice

U adding / D deleting a cell in a Ferrers diagram


U
=
+
+


D
=
+


UD = DU + Id
U
D

$$UD = DU + I_d$$

normal ordering in physics

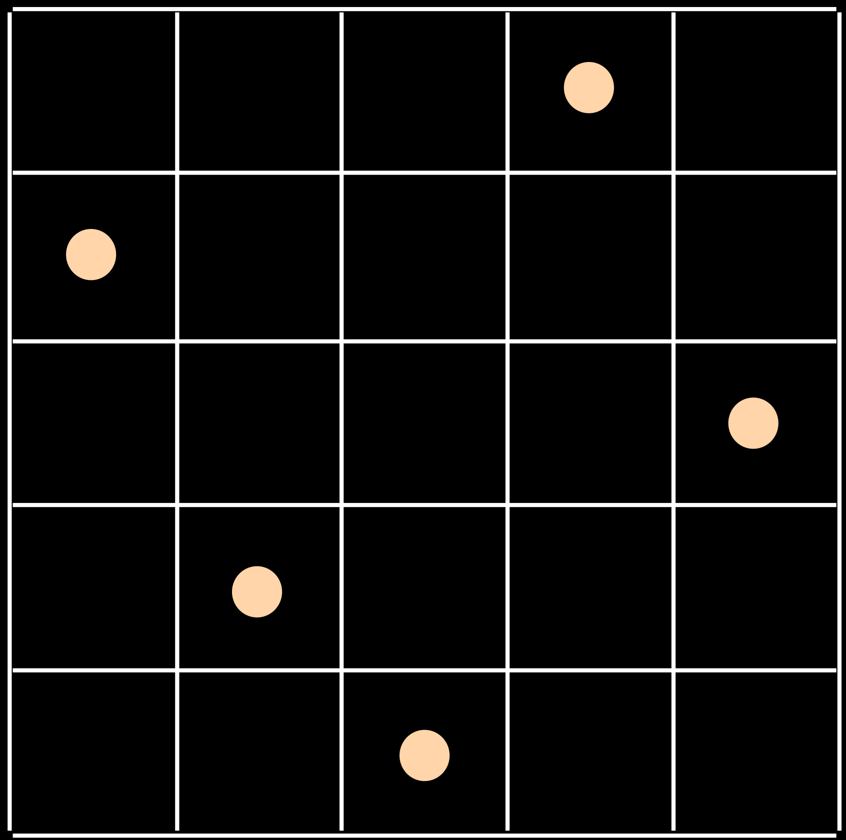
Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

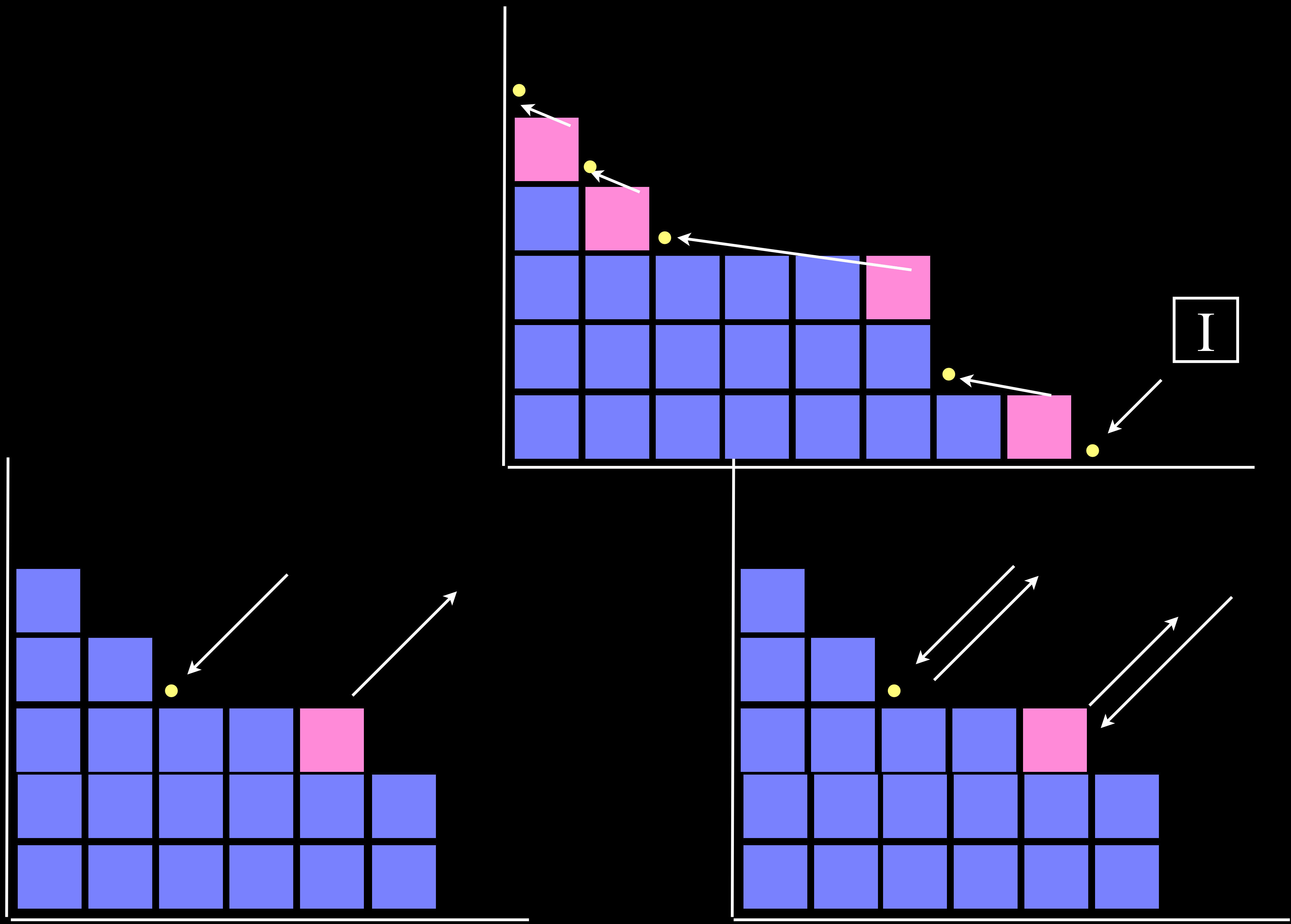
$$UD = qDU + I$$

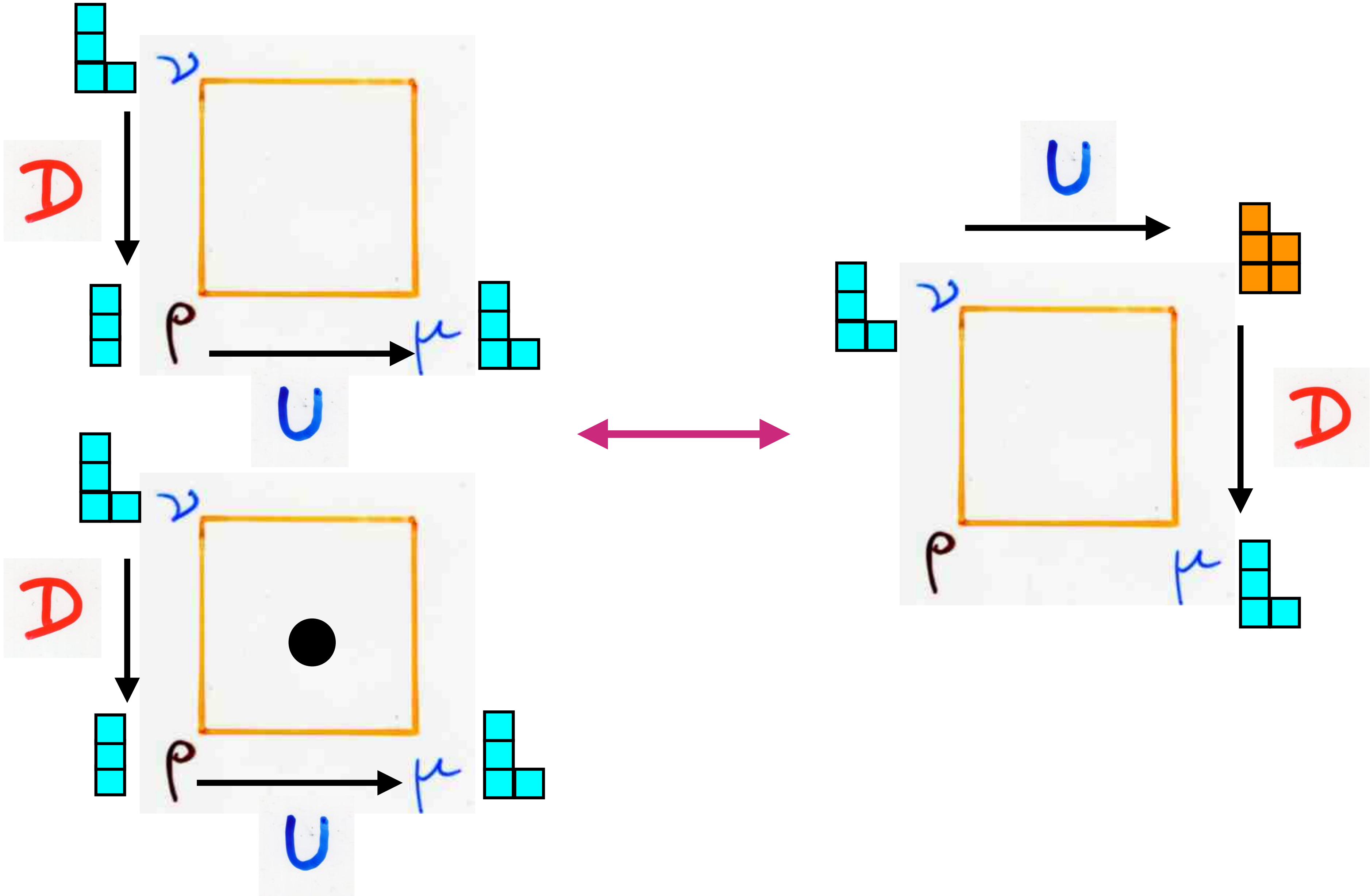
normal ordering

rook placement

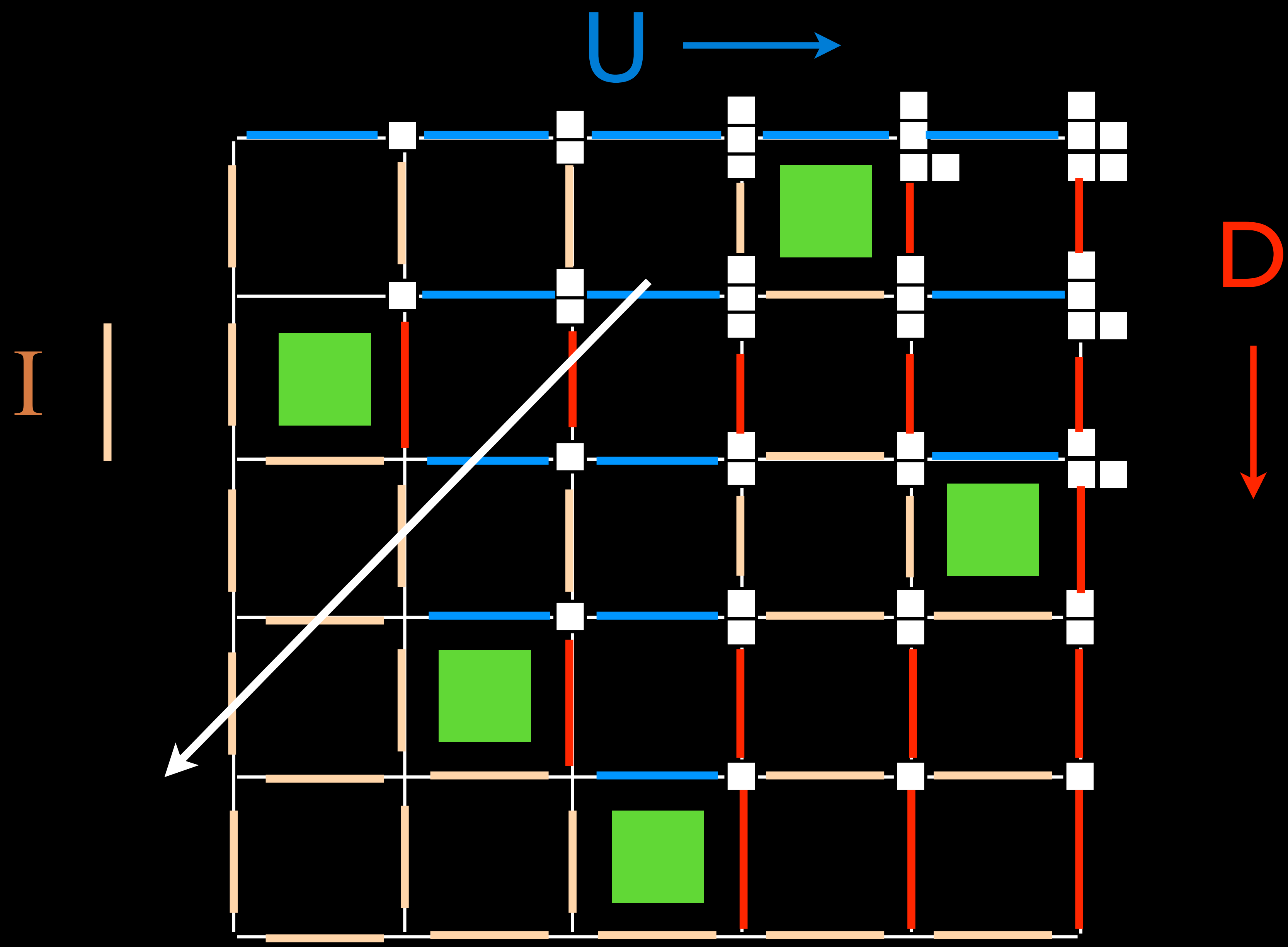


$$\sigma = 4,2,1,5,3$$


I



ν
D
ρ
μ
U
ν
D
ρ
μ
U
U
ν
D
ρ
μ



This "propagation" algorithm is exactly the reverse of Fomin's "growth diagrams"

The PASEP algebra

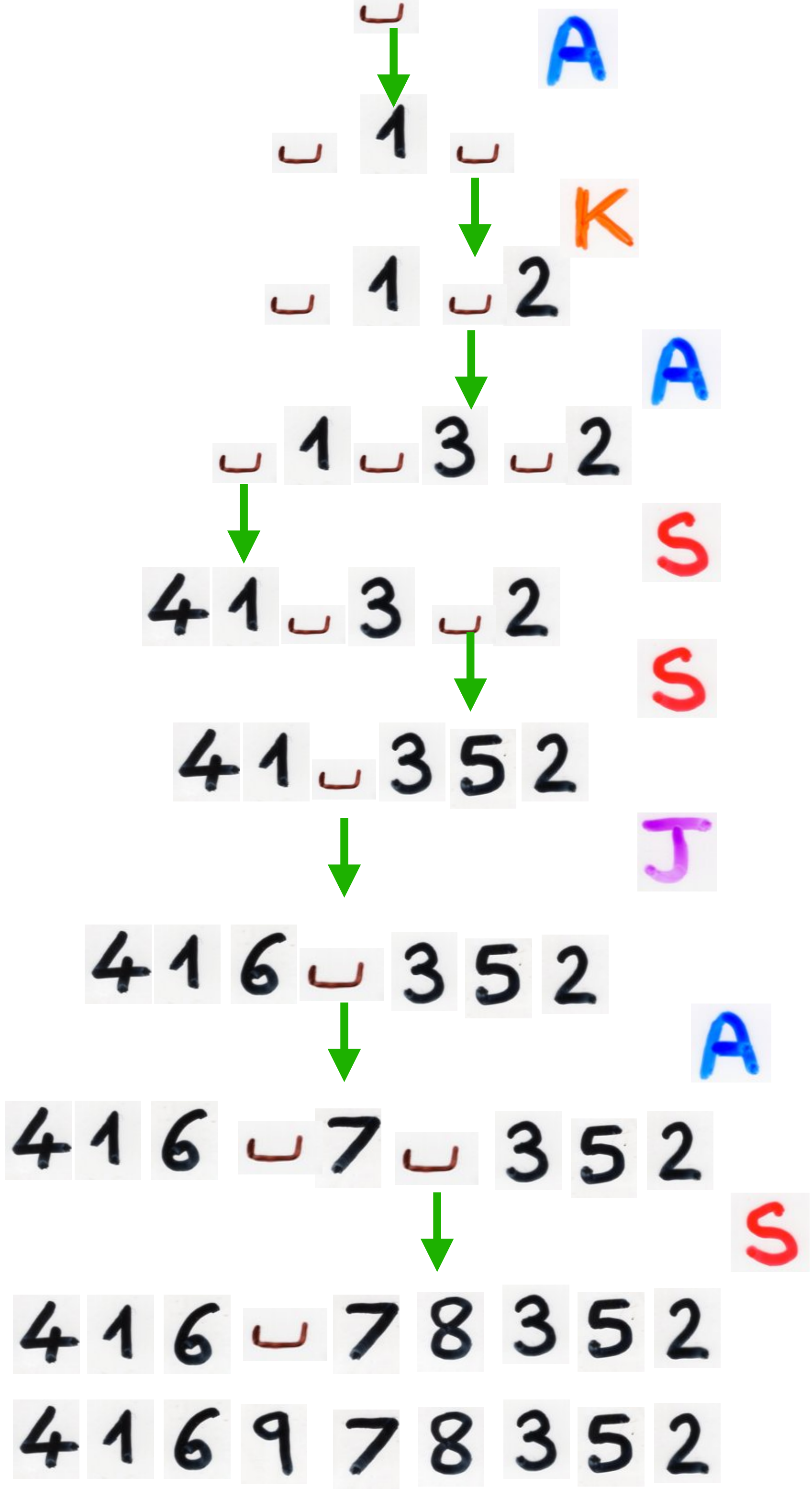
$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

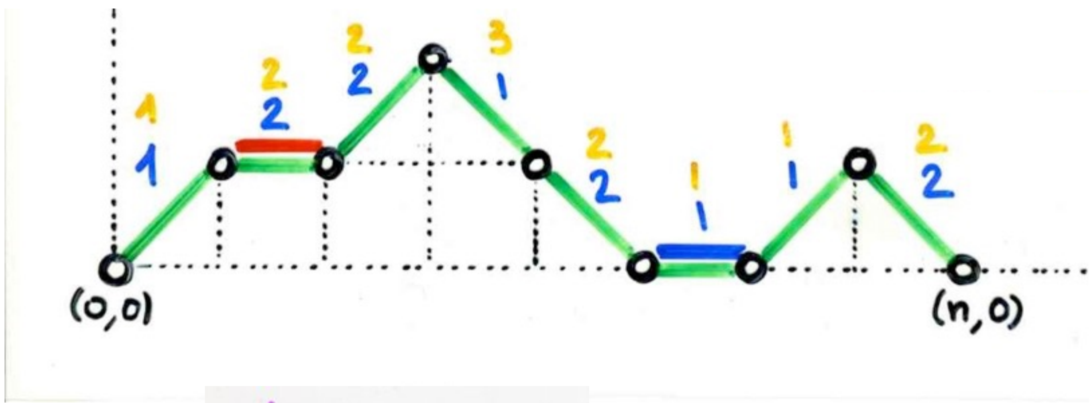
tableau

unique

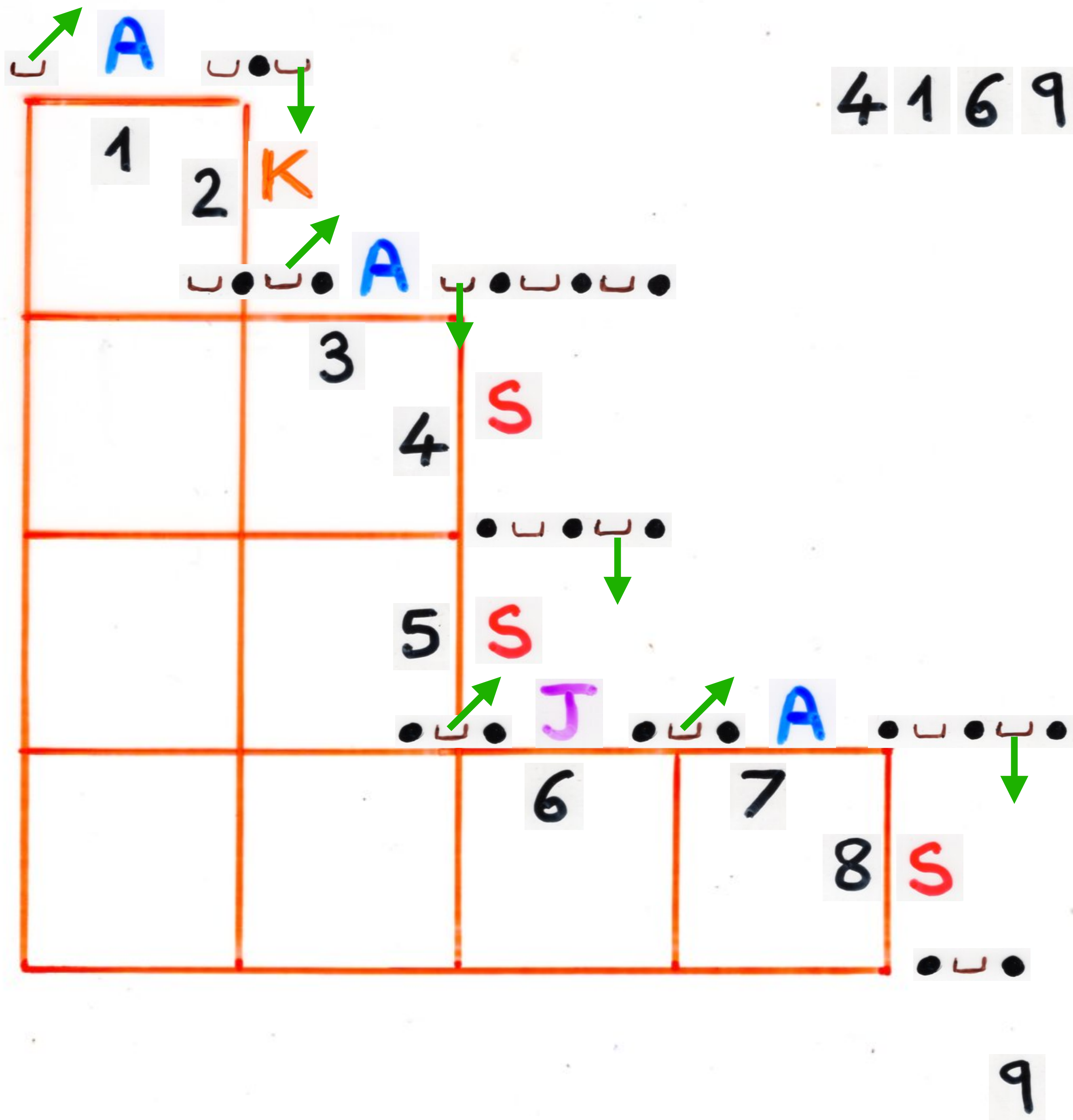


$$D = A + K$$
$$E = S + J$$

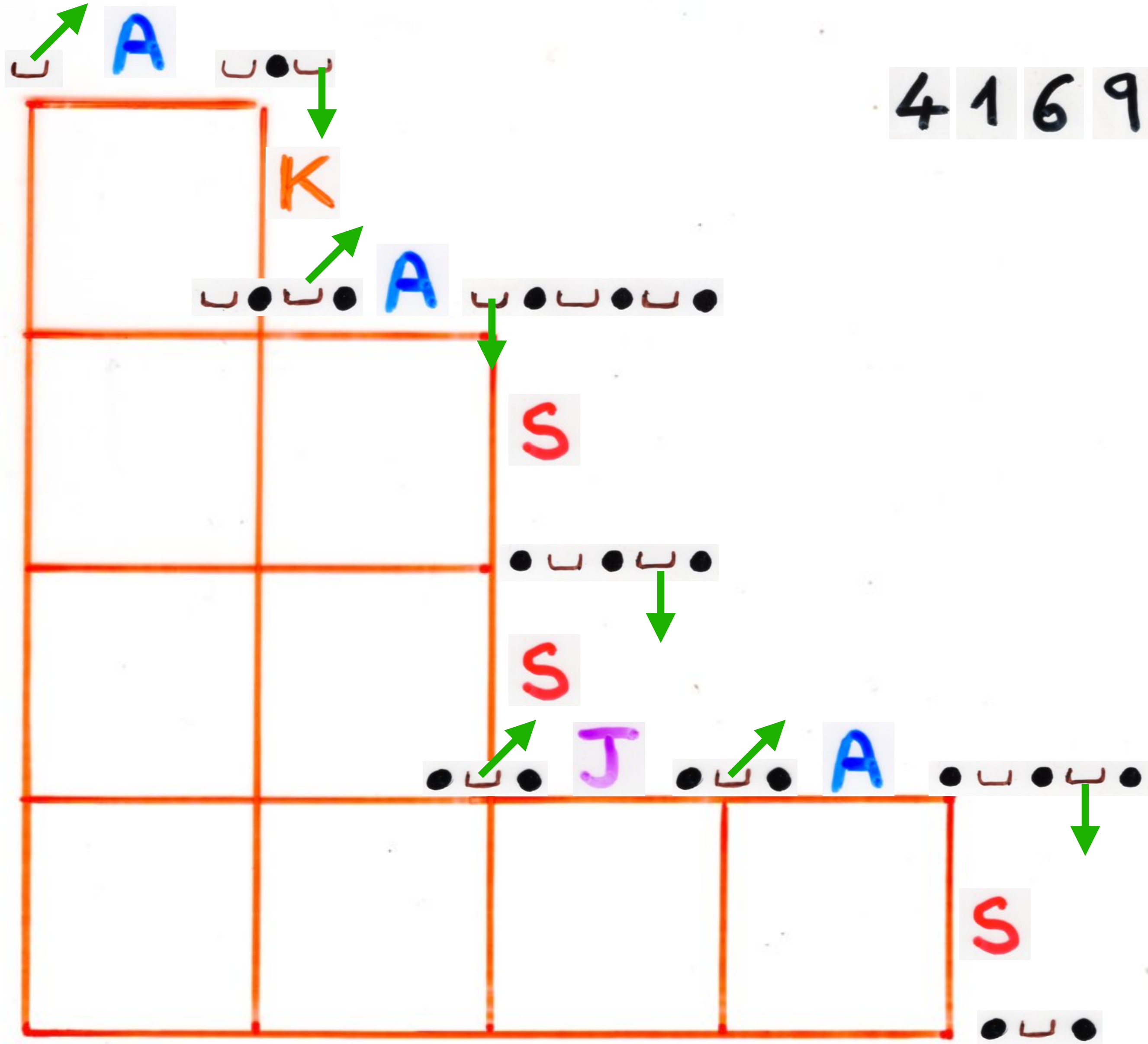
$$DE = ED + E + D$$



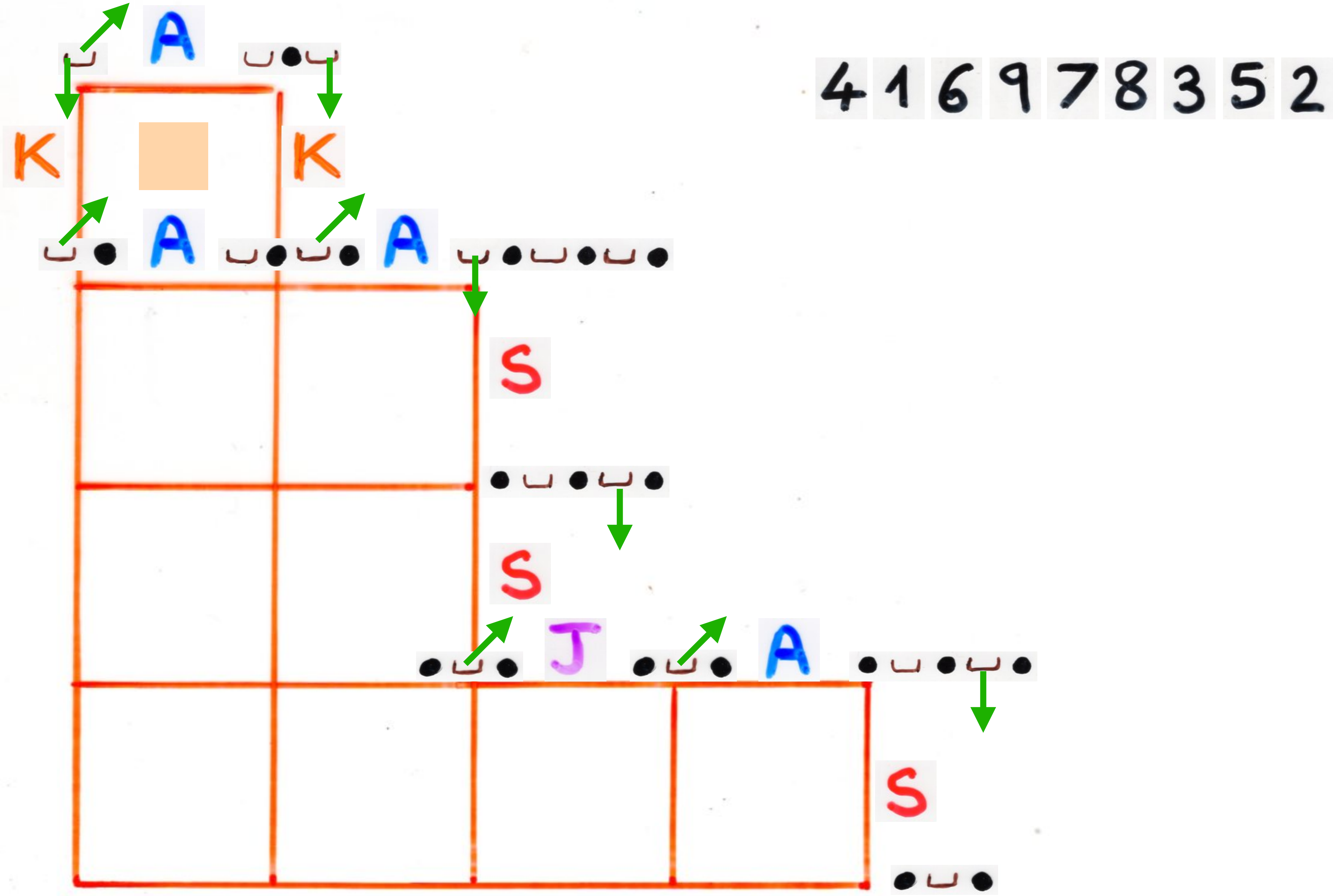
Laguerre histories



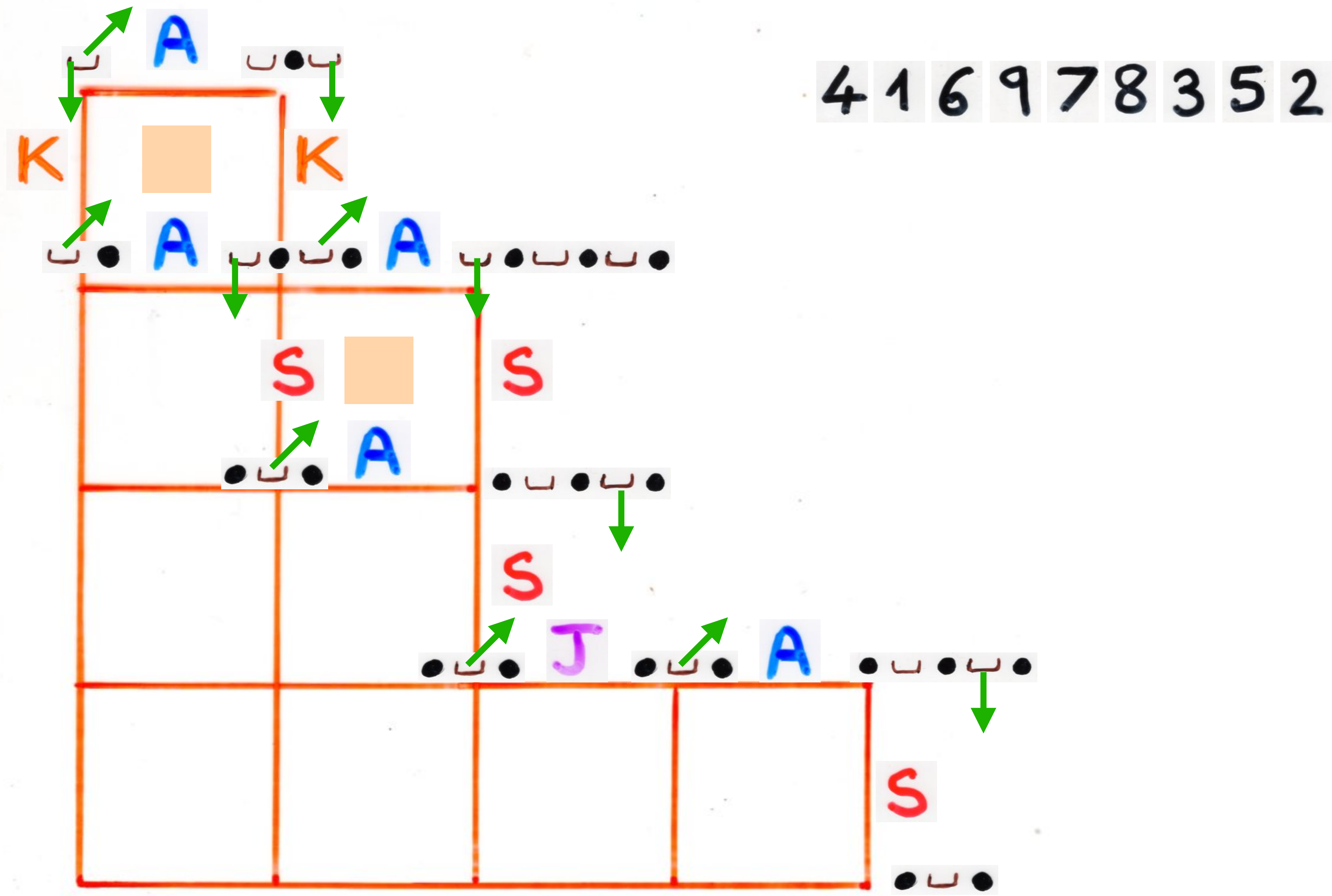
4 1 6 9 7 8 3 5 2



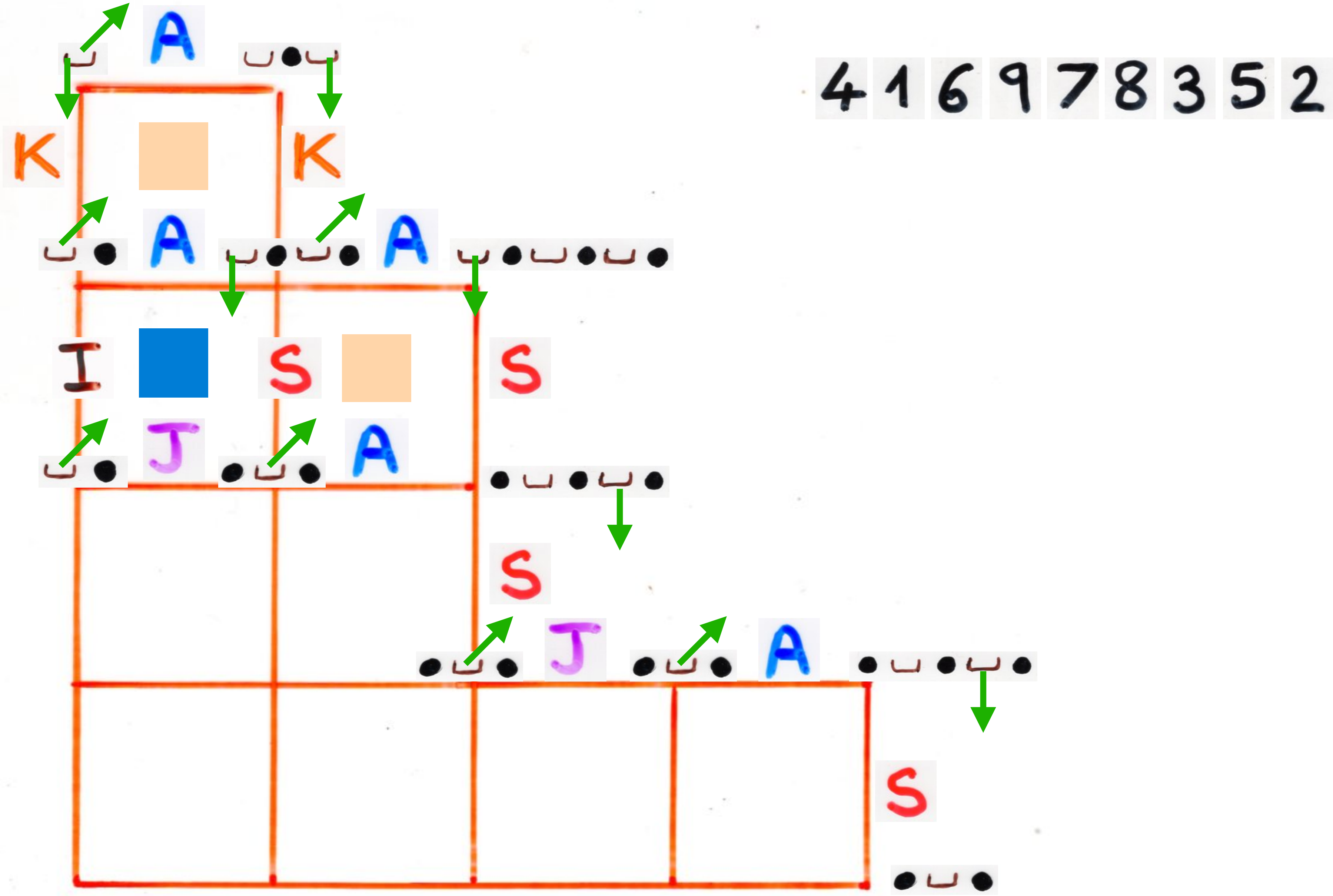
4 1 6 9 7 8 3 5 2



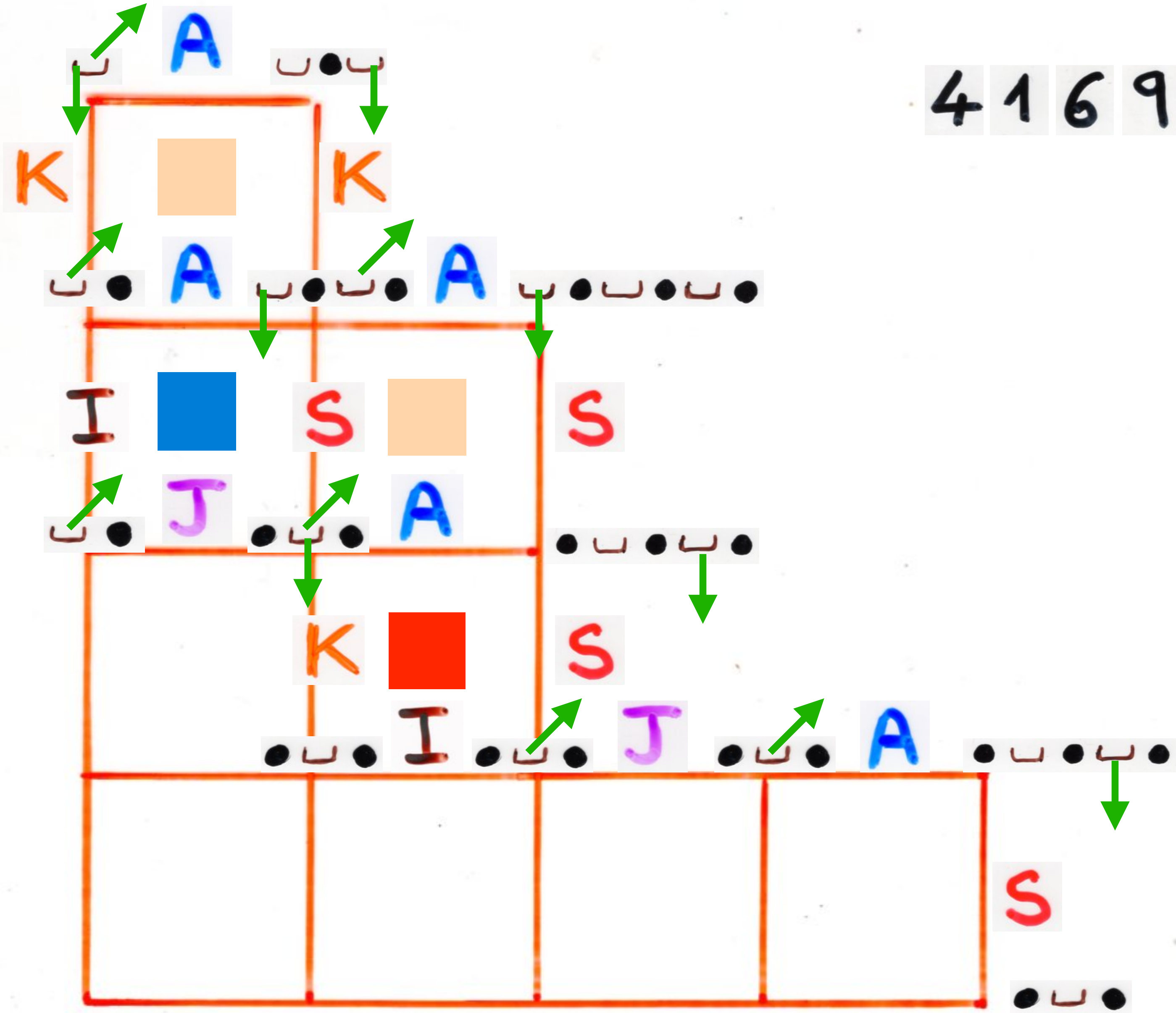
A
K
K
A
A
S
S
J
A
S
4 1 6 9 7 8 3 5 2



A
K
K
A
A
S
S
A
S
J
A
S
4 1 6 9 7 8 3 5 2

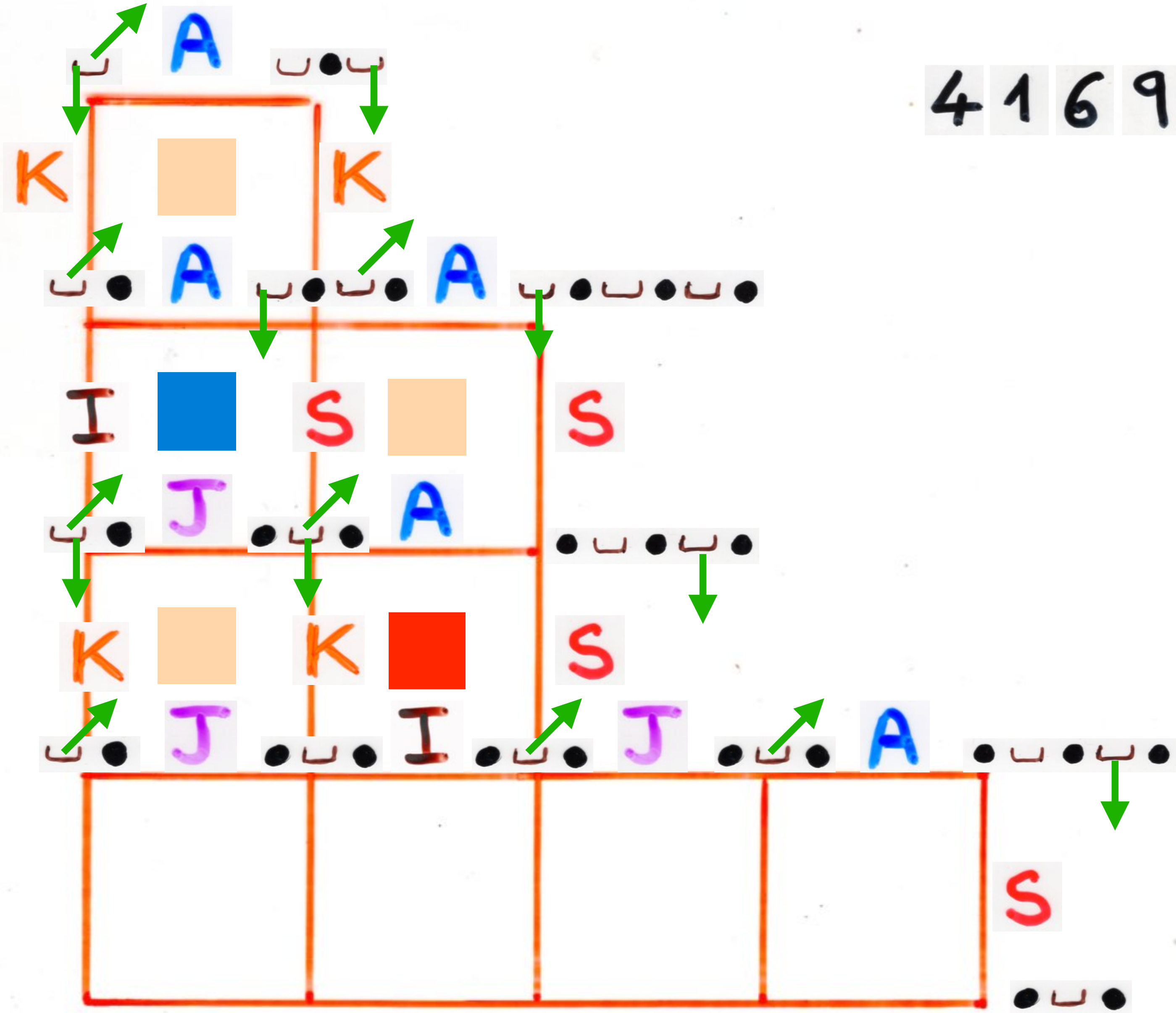


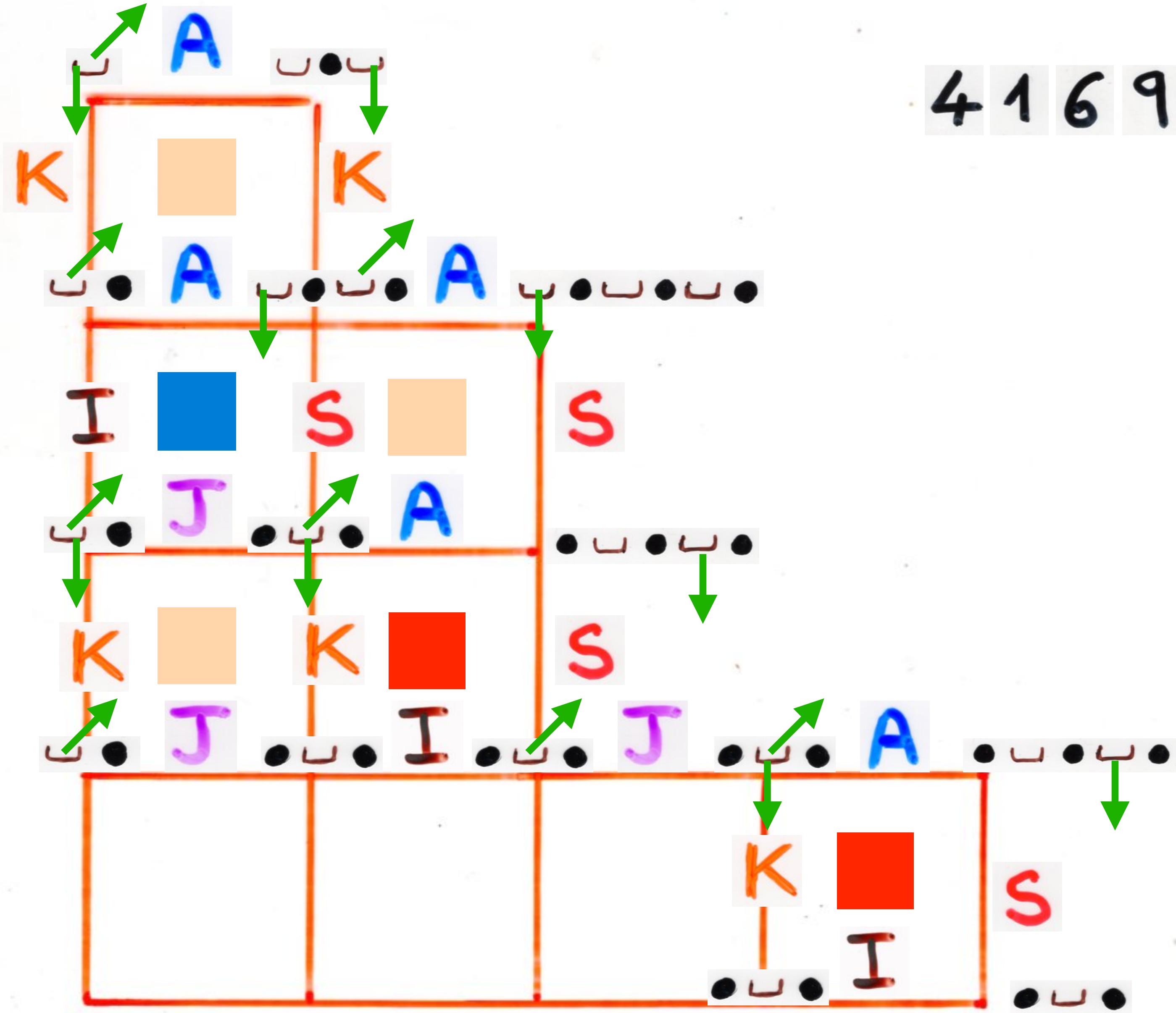
4 1 6 9 7 8 3 5 2



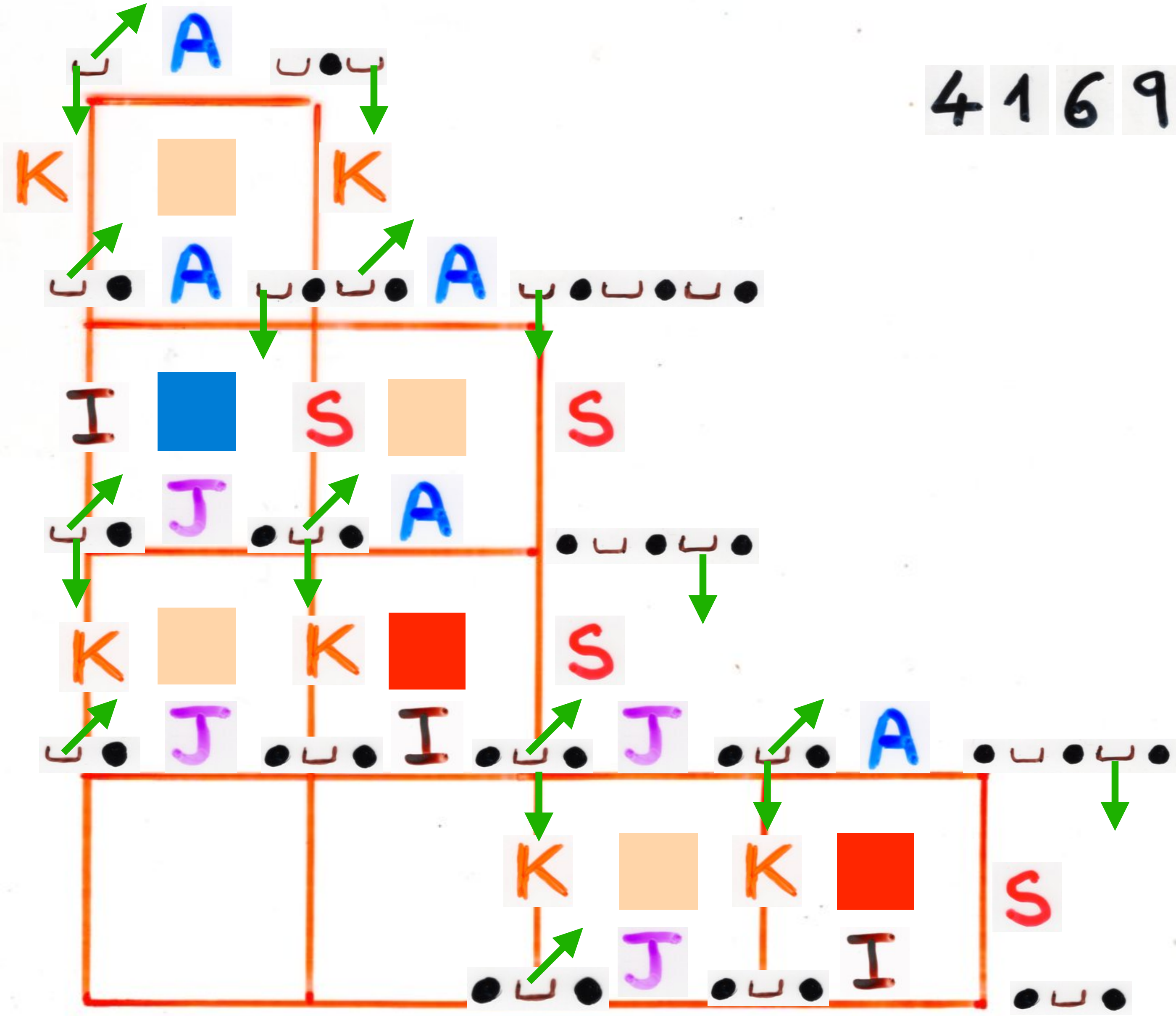
4 1 6 9 7 8 3 5 2

4 1 6 9 7 8 3 5 2

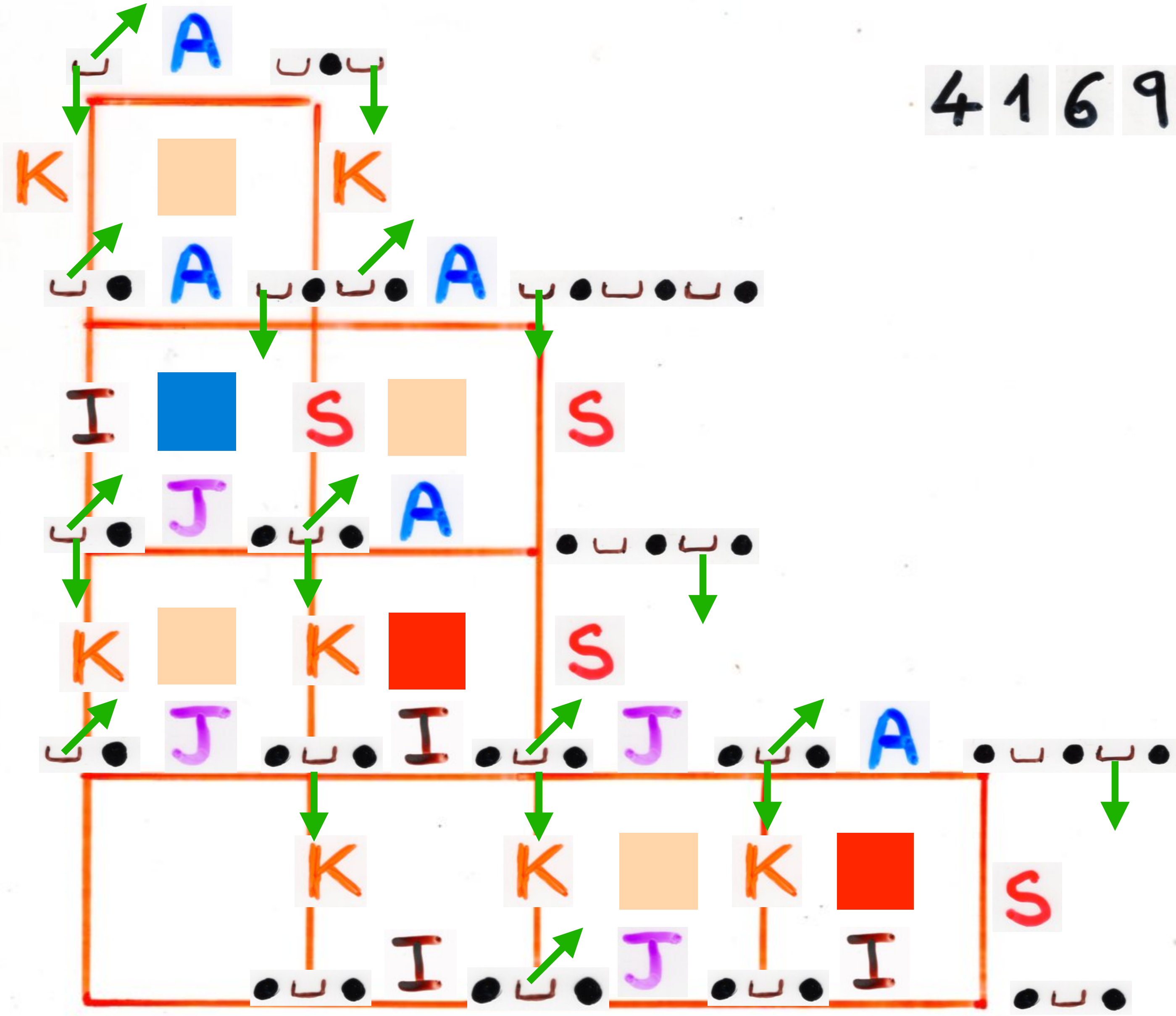




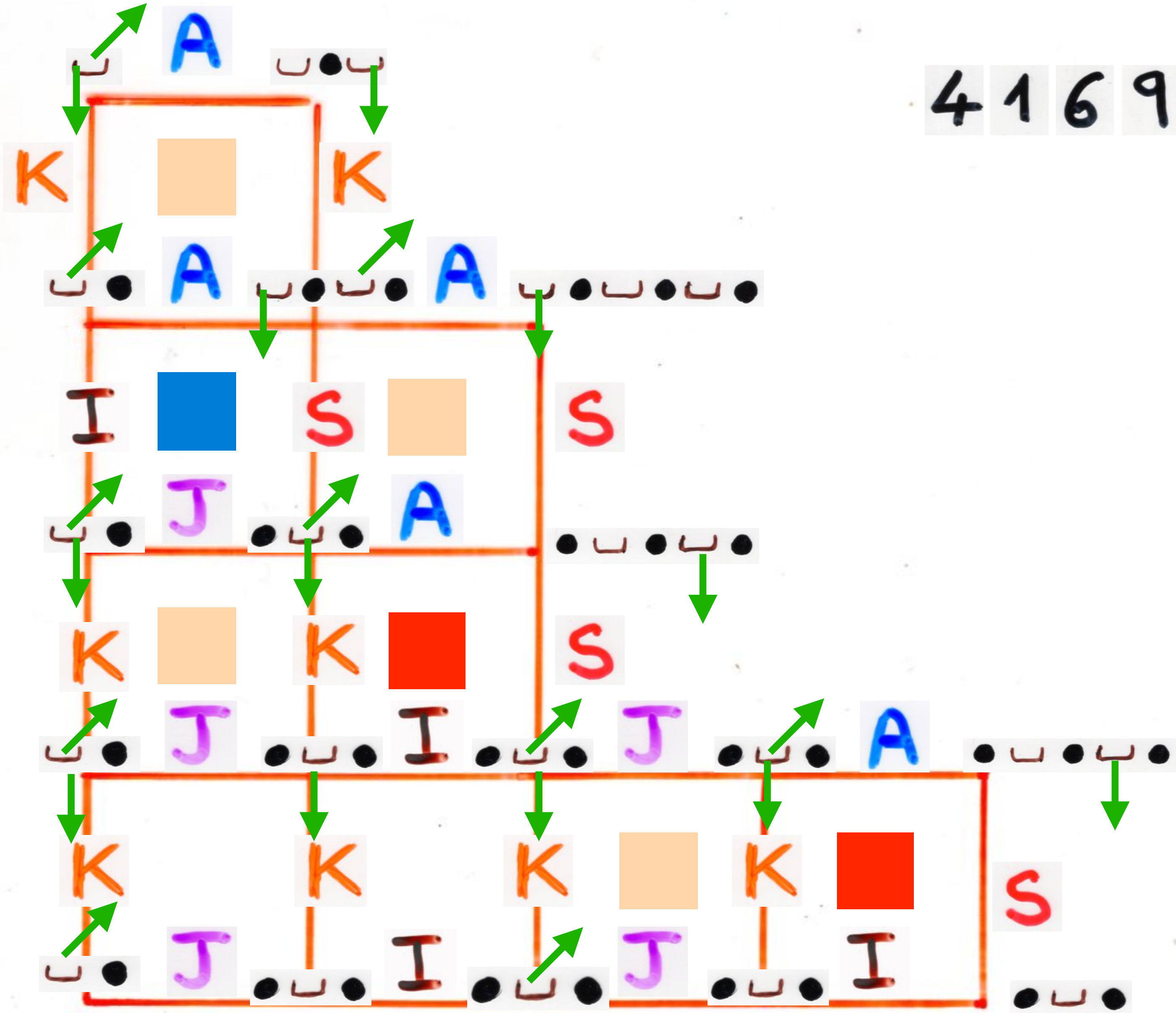
4 1 6 9 7 8 3 5 2



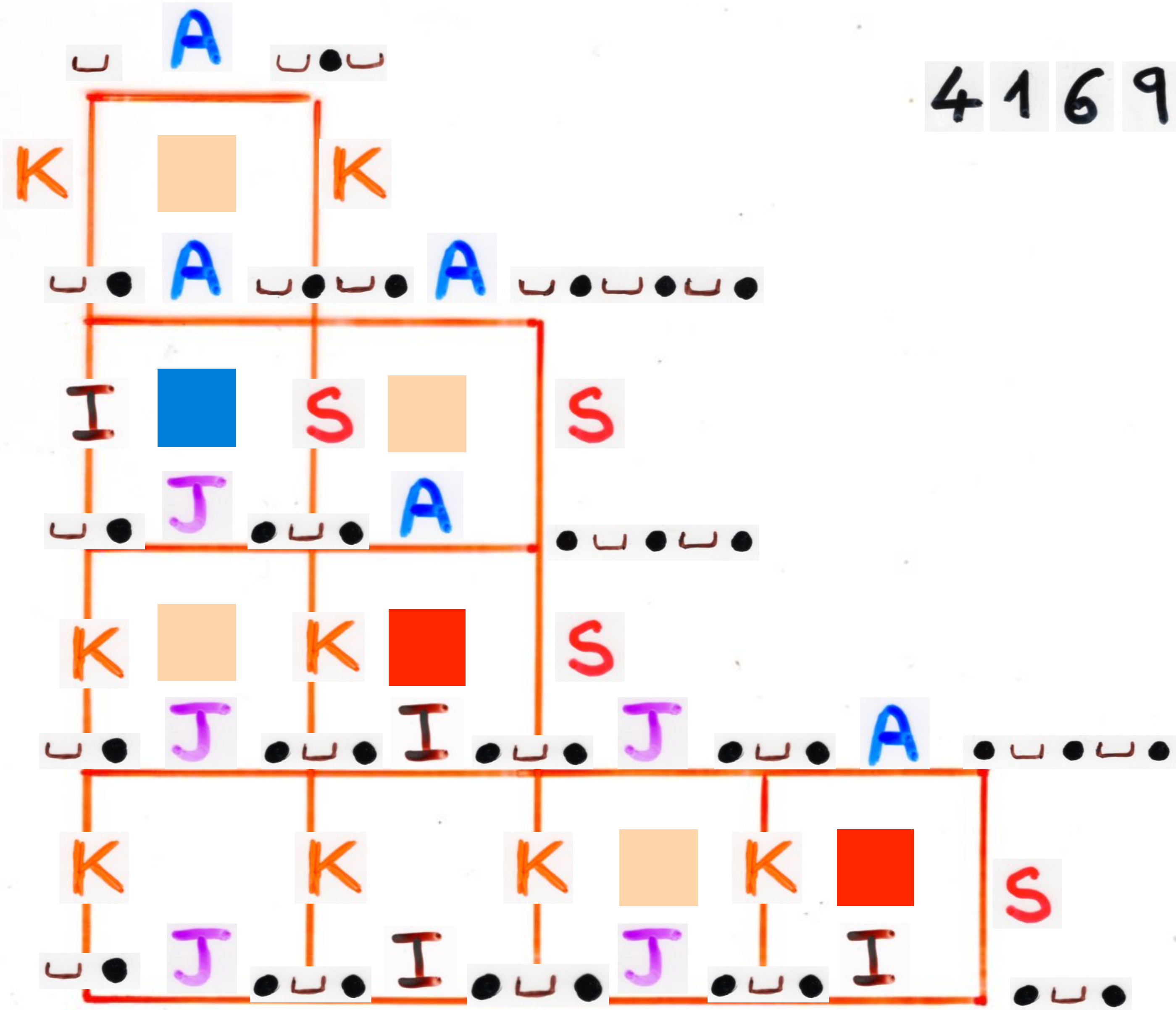
4 1 6 9 7 8 3 5 2



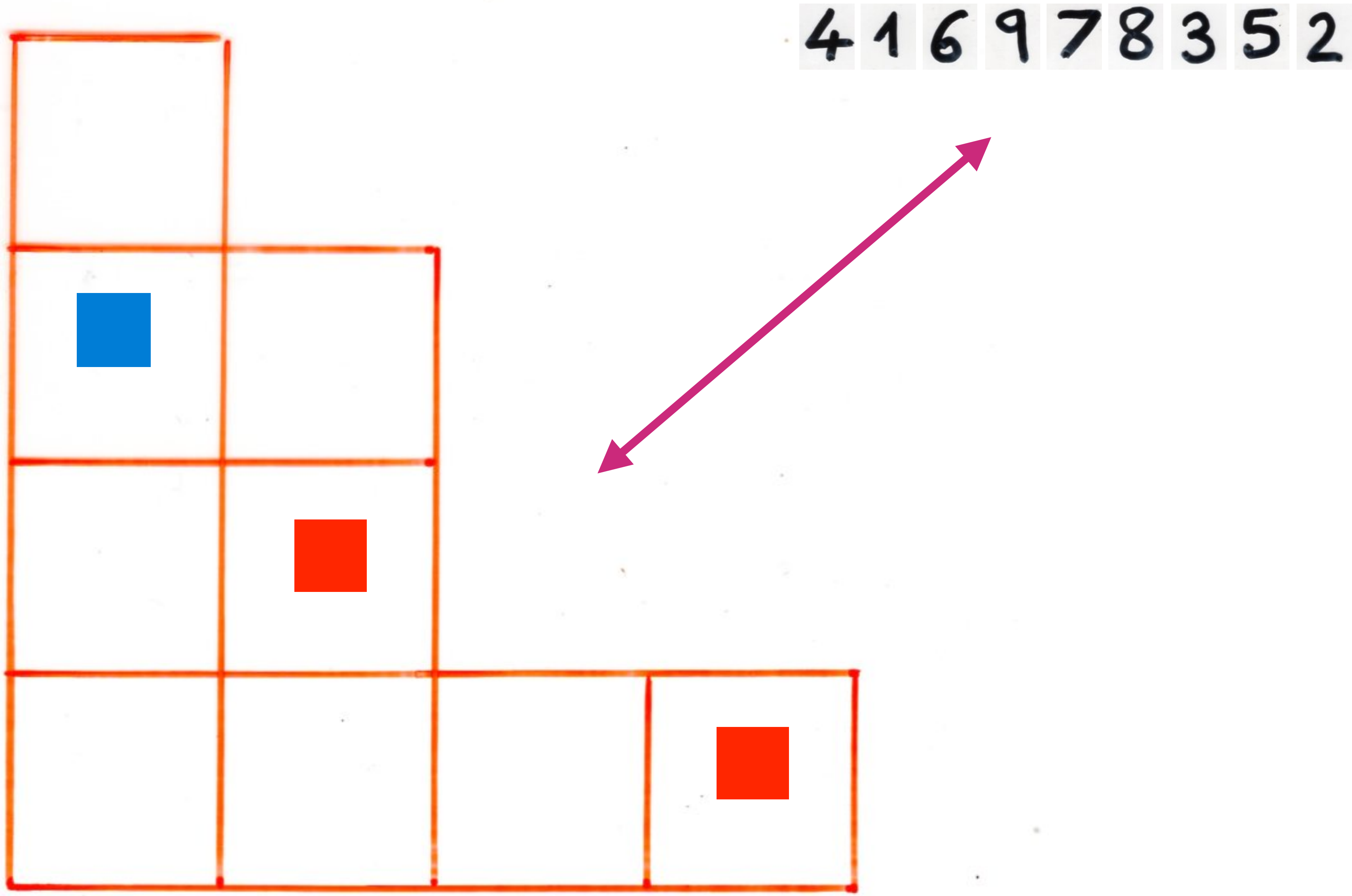
4 1 6 9 7 8 3 5 2

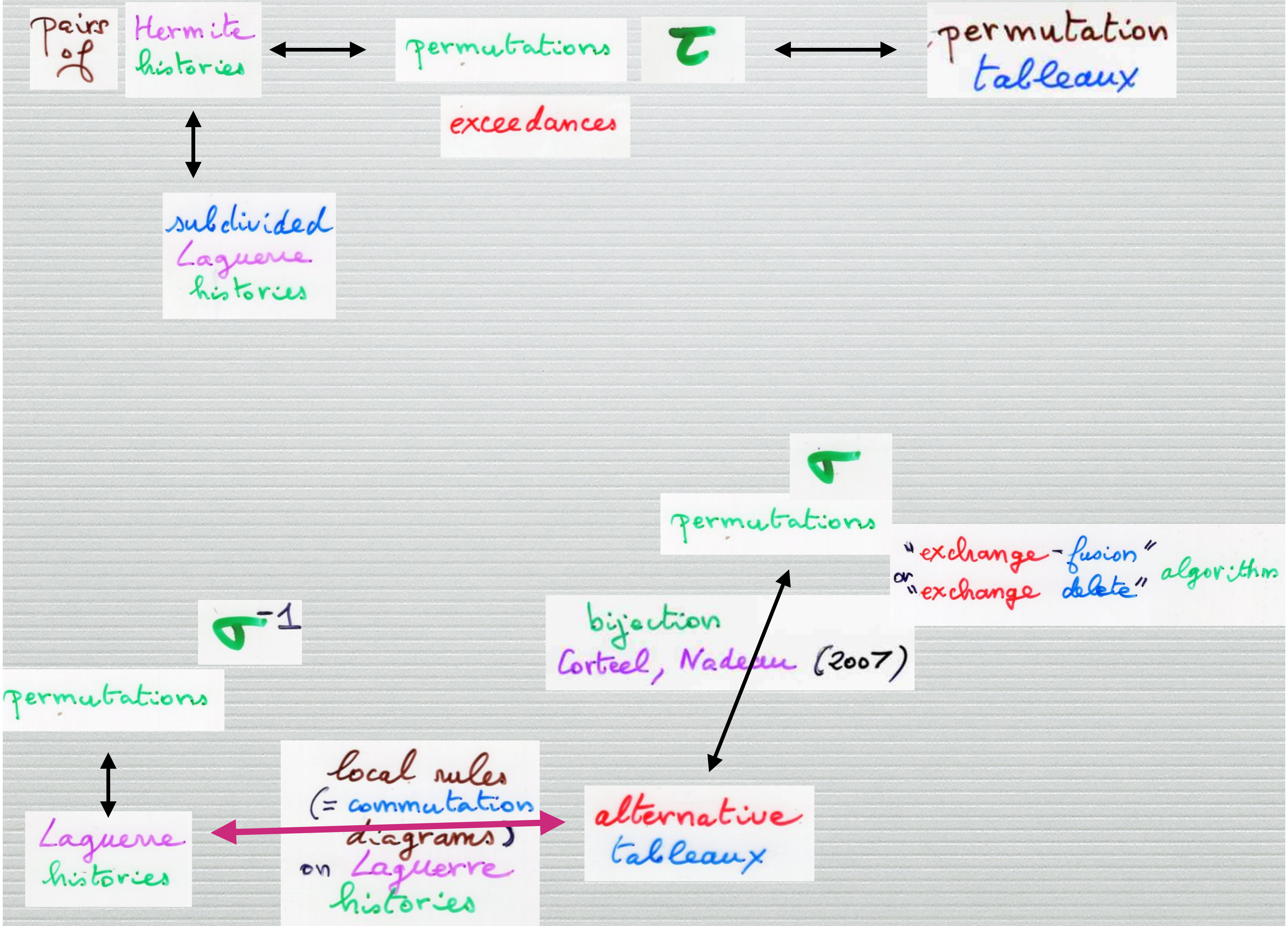


4 1 6 9 7 8 3 5 2



416978352


4 1 6 9 7 8 3 5 2



Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
σ
permutations
"exchange-fusion"
or "exchange delete" algorithm
bijection
Corteel, Nadeau (2007)
σ⁻¹
permutations
local rules
(= commutation diagrams)
on Laguerre histories
Laguerre histories
alternative tableaux

q, α, β

?

Josuat-Vergès (2011)

Third bijection

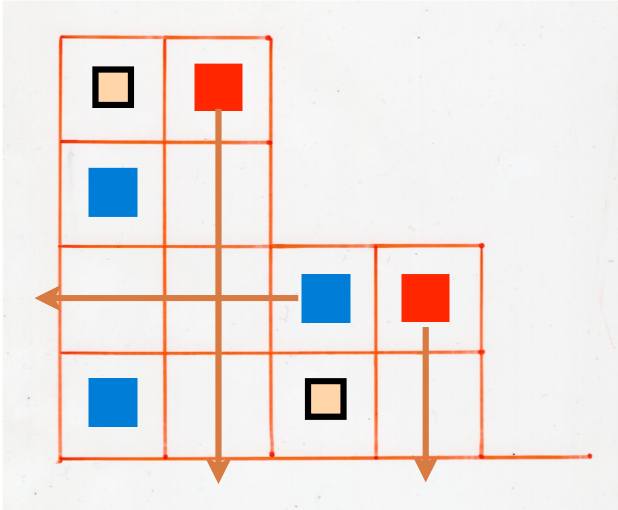
Tableaux — permutations

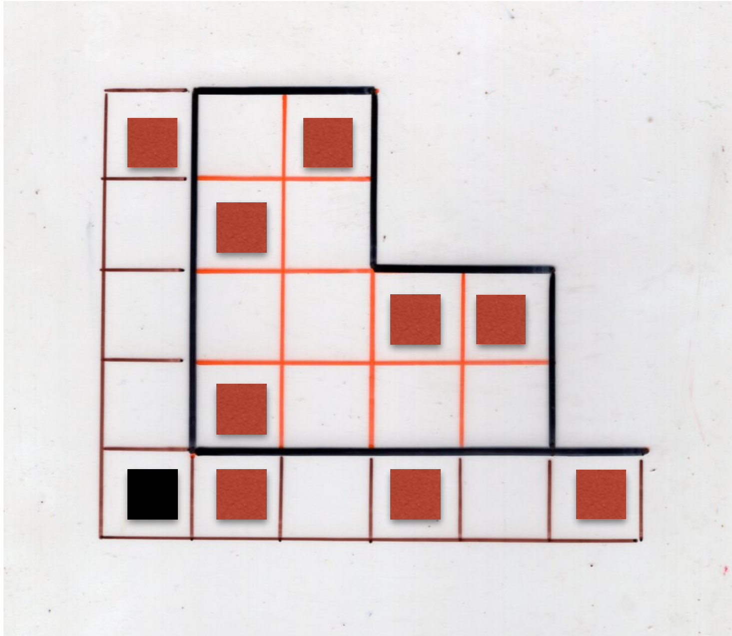
- direct bijection (with tree-like tableaux)

Aval, Boussicault, Nadeau (2011)

tableaux size $(n+1)$ $\longleftrightarrow$ $\left(\text{tableaux size } n, \; 1 \leq i \leq n+1\right)$

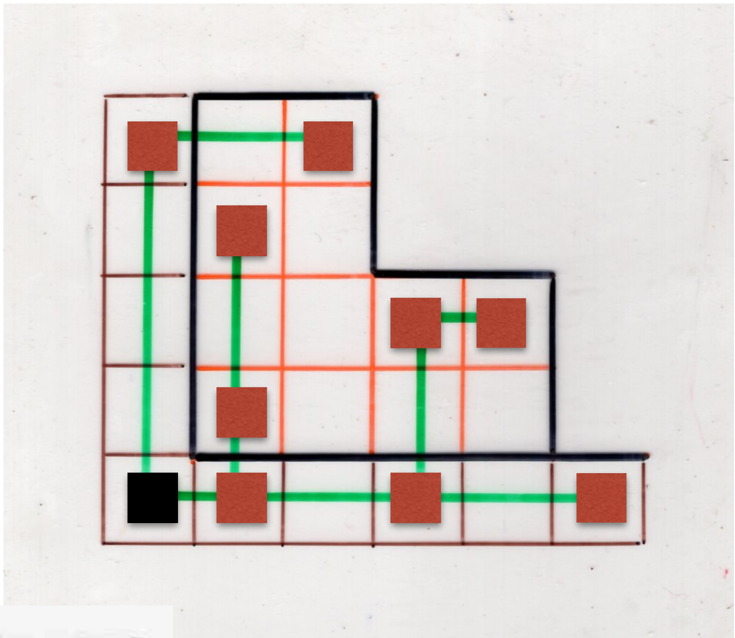
$(n+1)!$





alternative tableaux

tree-like tableaux



Aval, Boussicault, Nadeau (2013)

q = 0

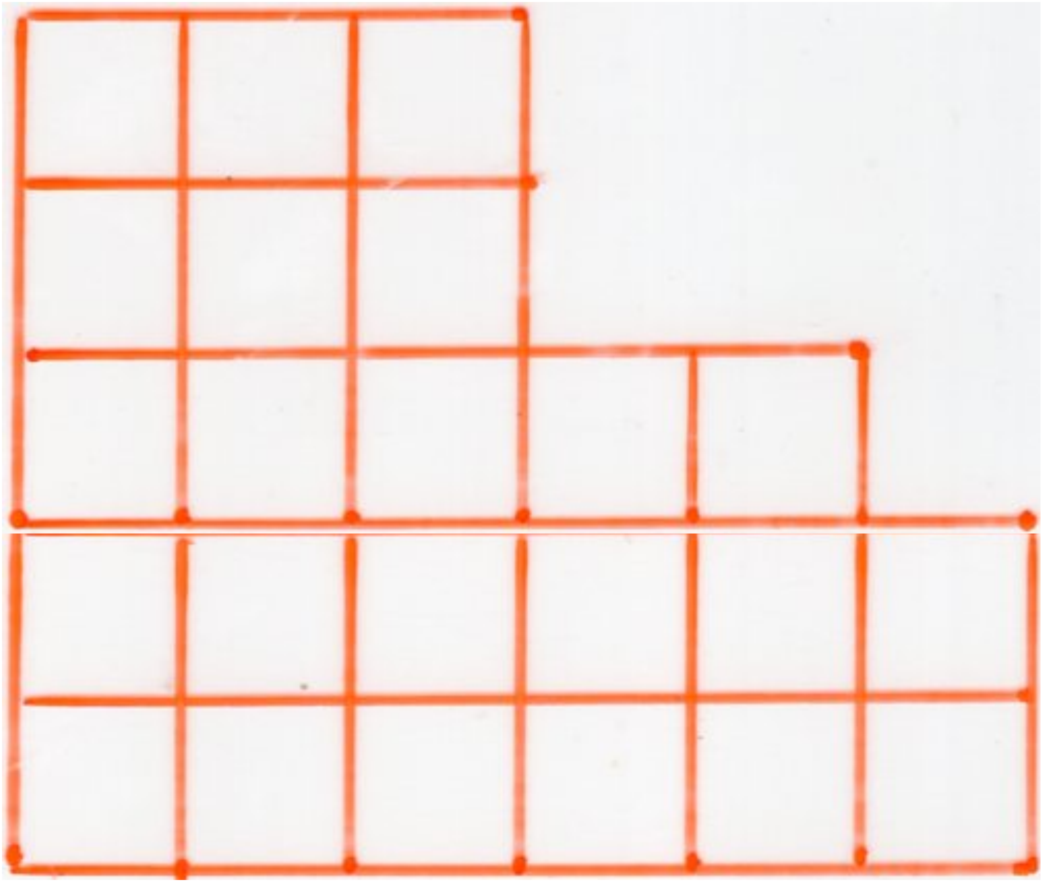
v-trees

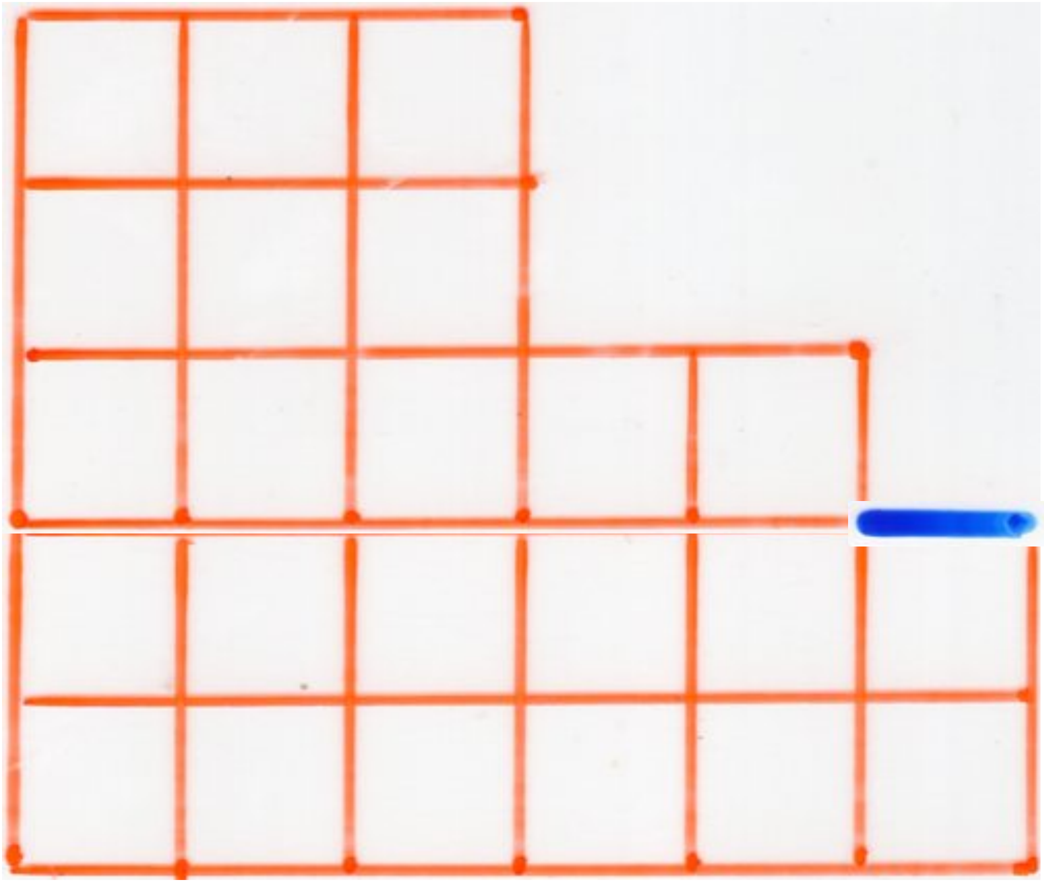
Ceballos, Padrol, Sarmiento (2016) (2018)

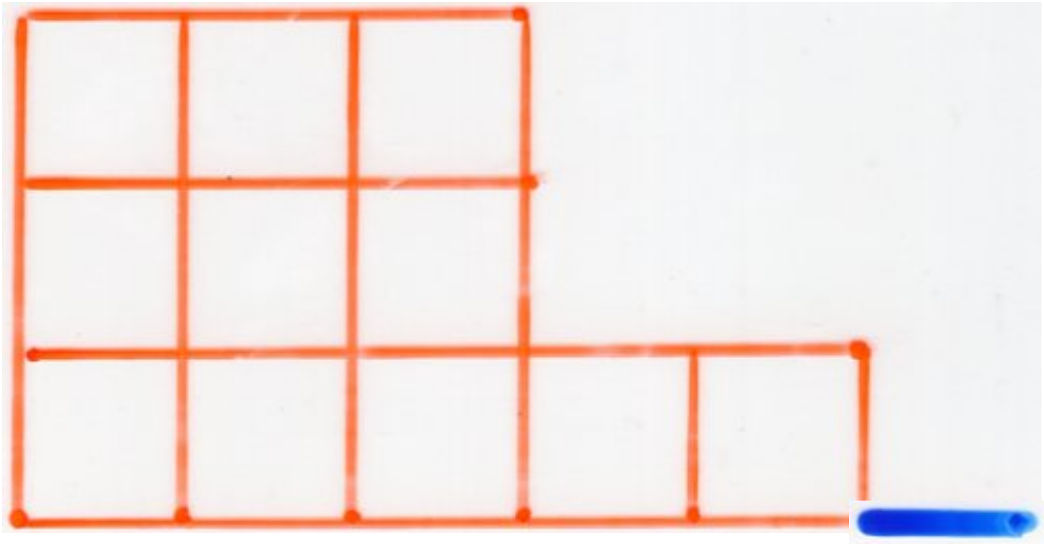
Catalan

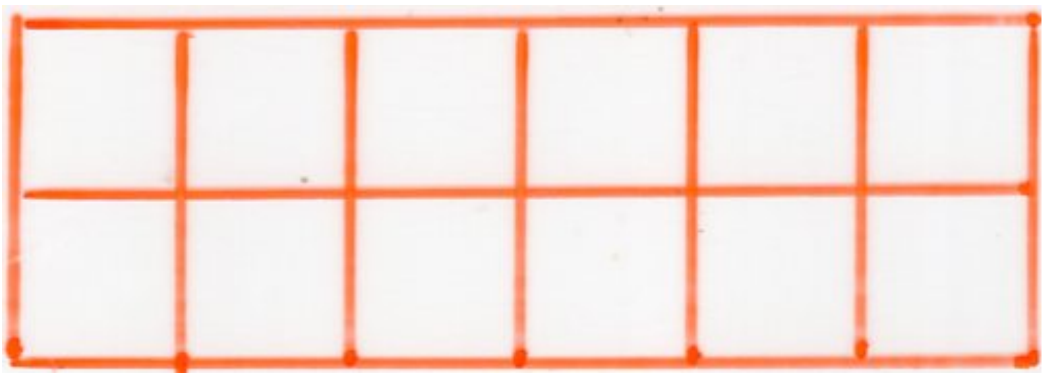
tree-like tableaux

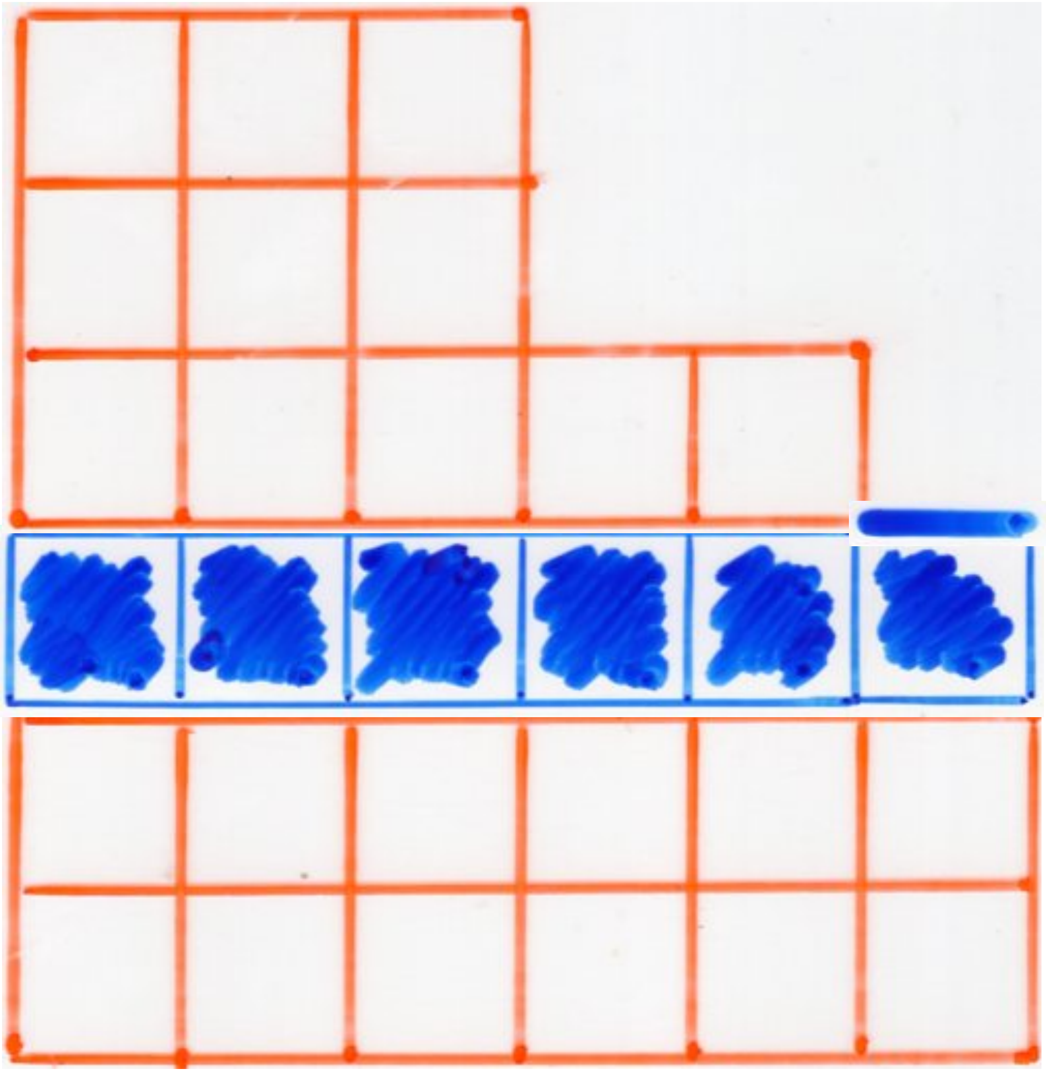
extensions of Tamari lattice

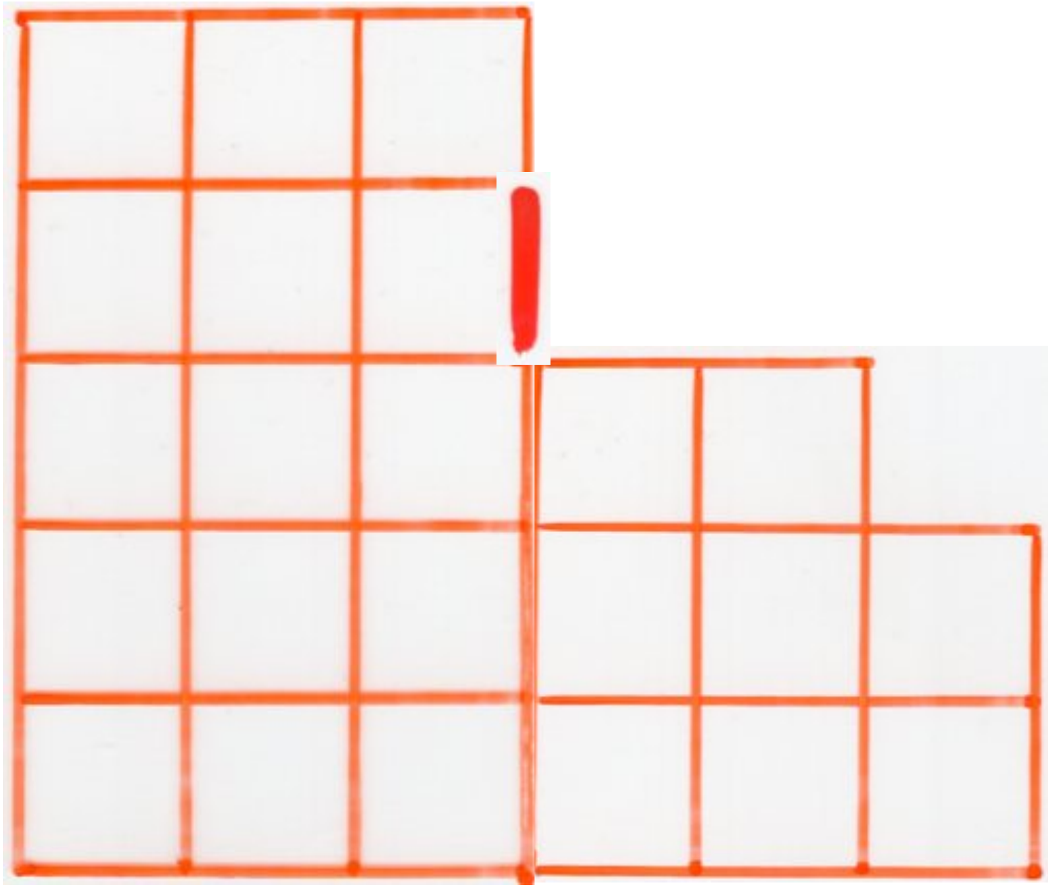


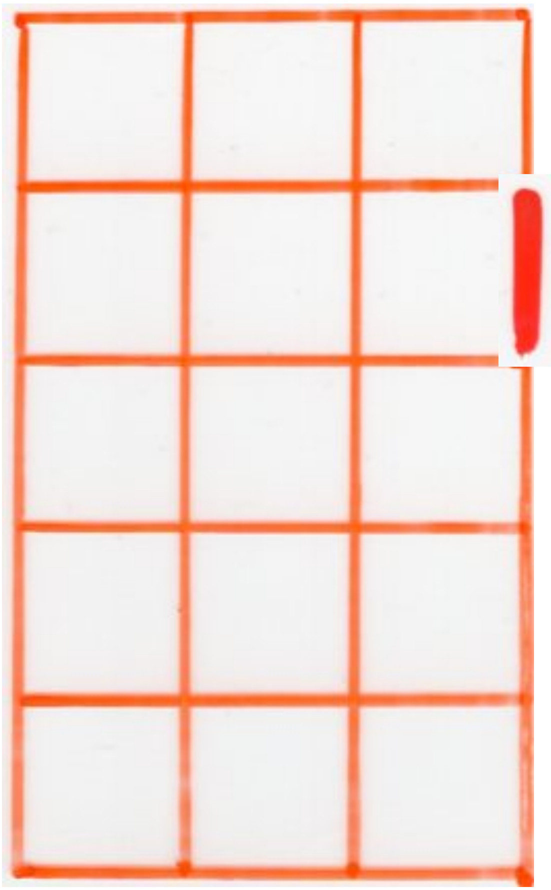


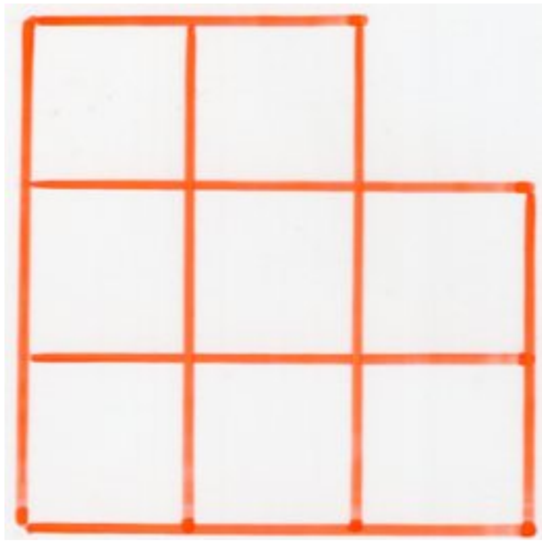


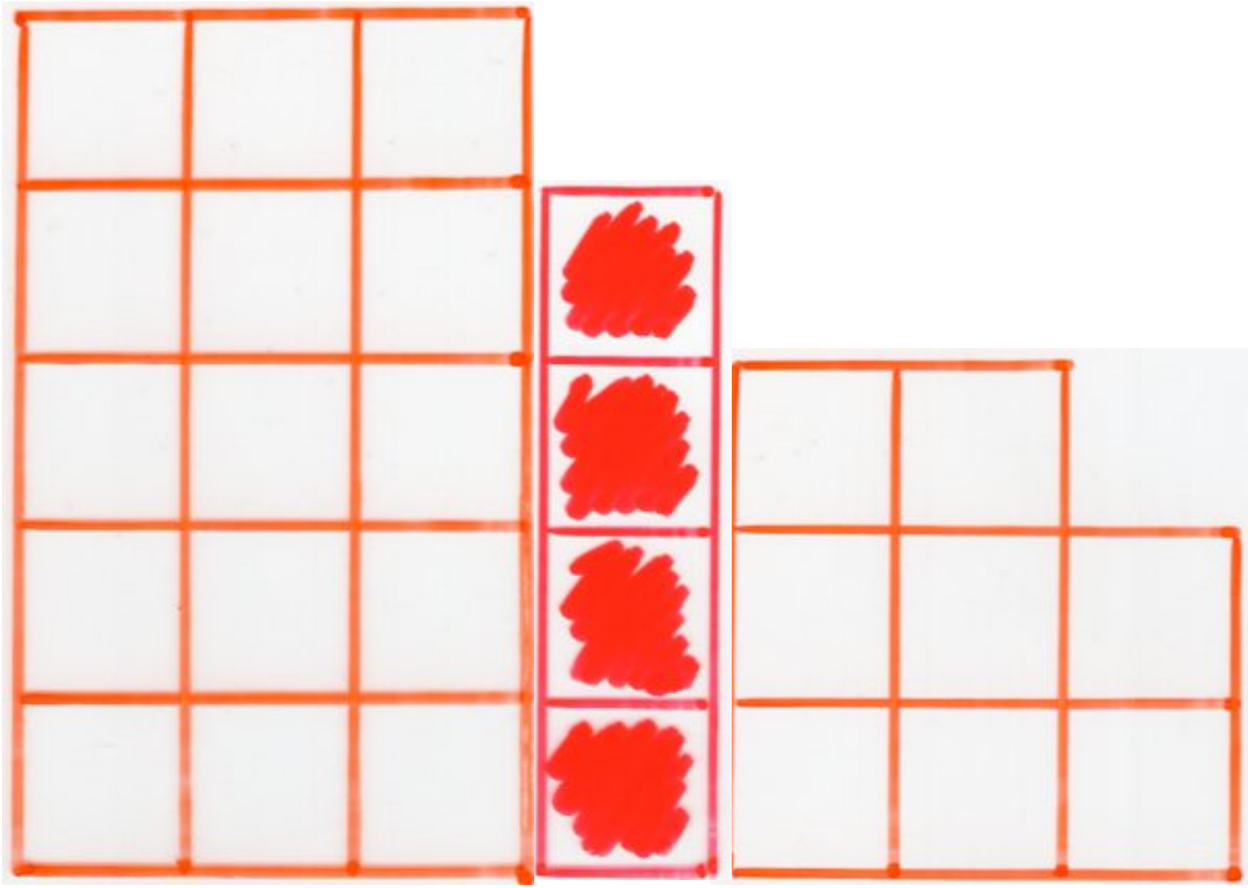


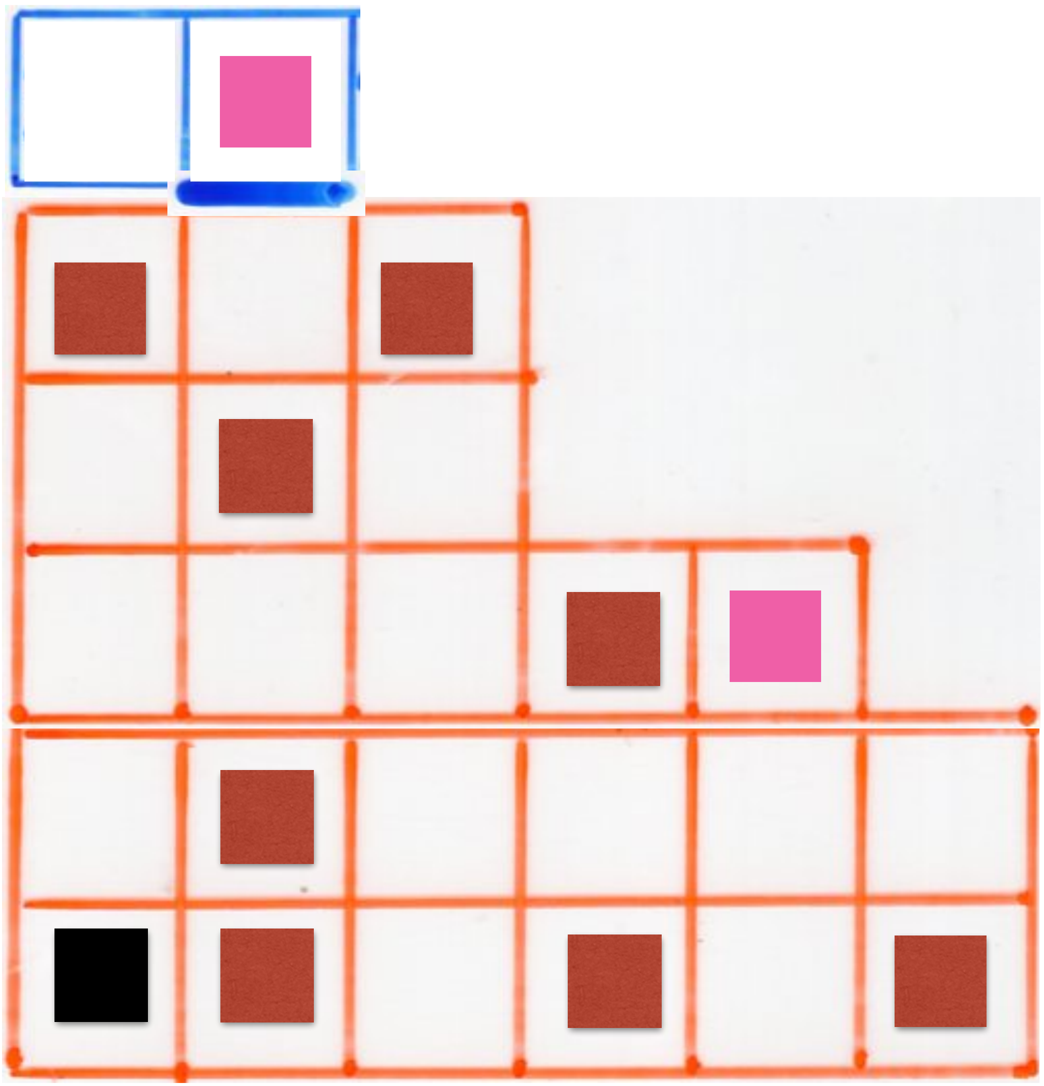


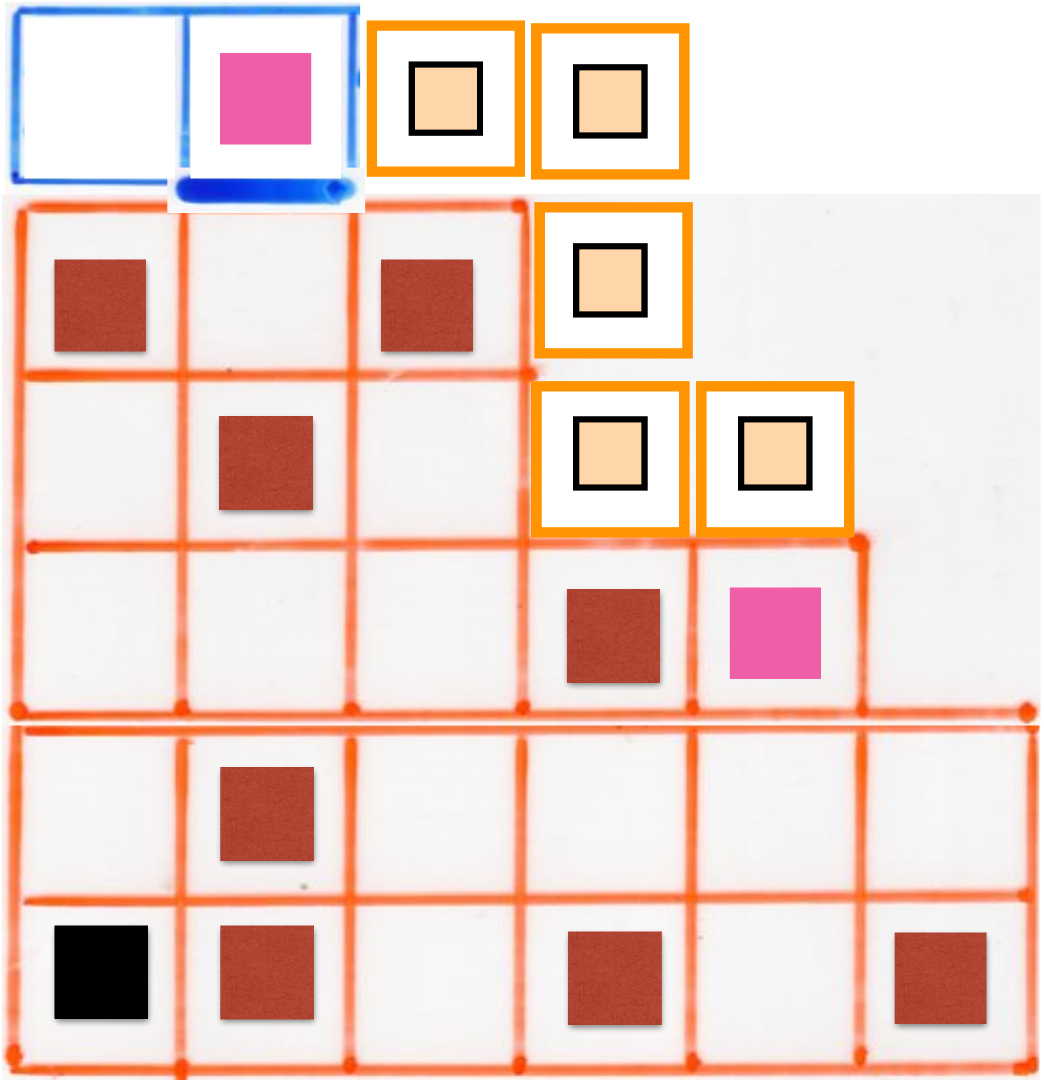


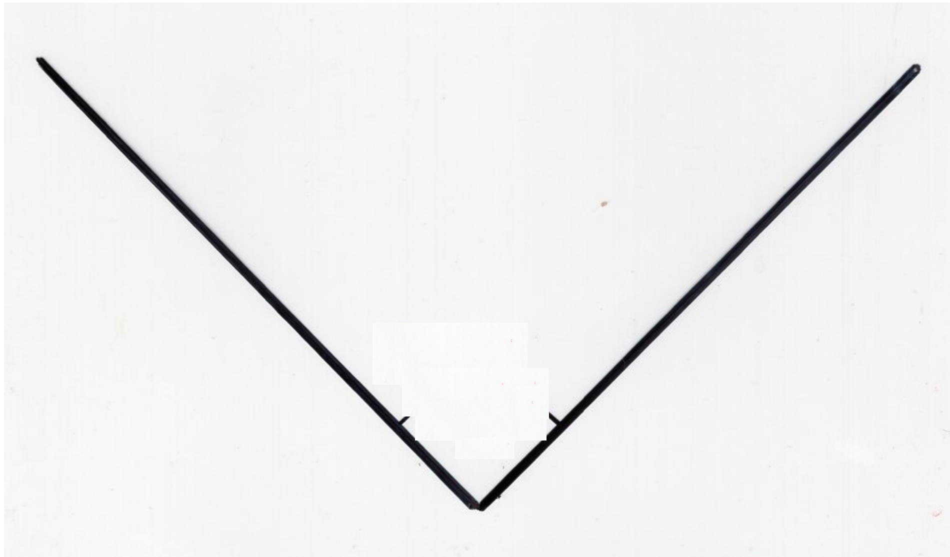




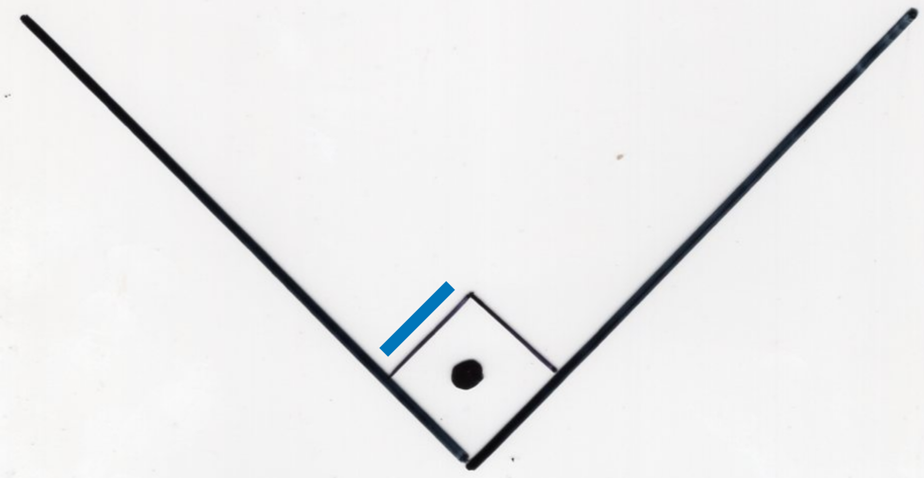




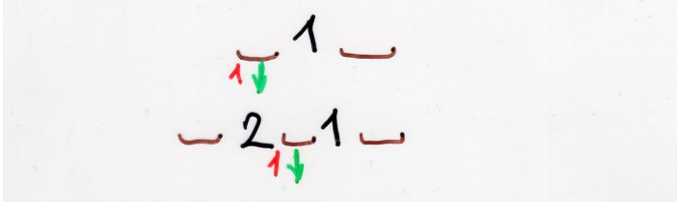


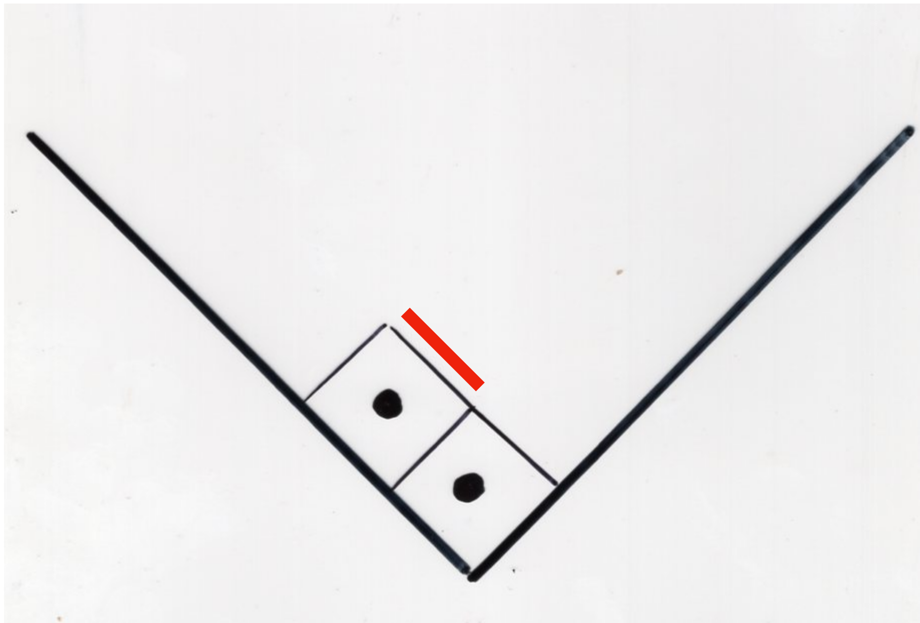


$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$

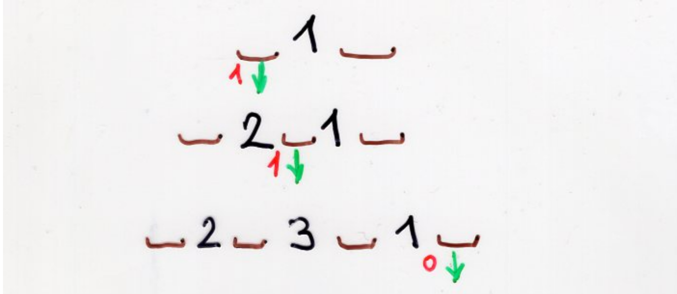


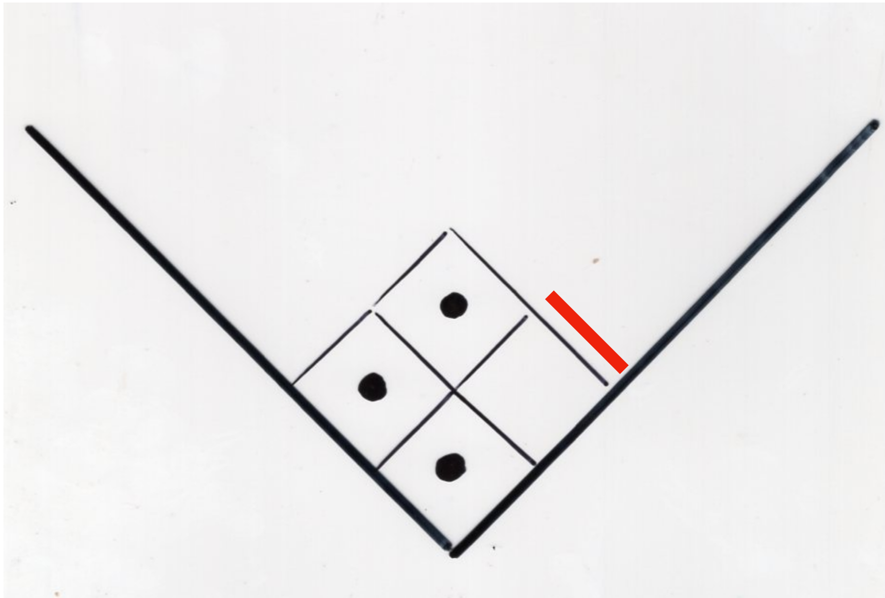
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



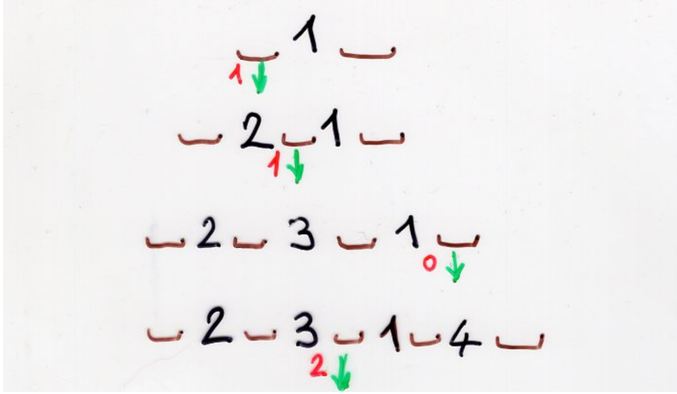


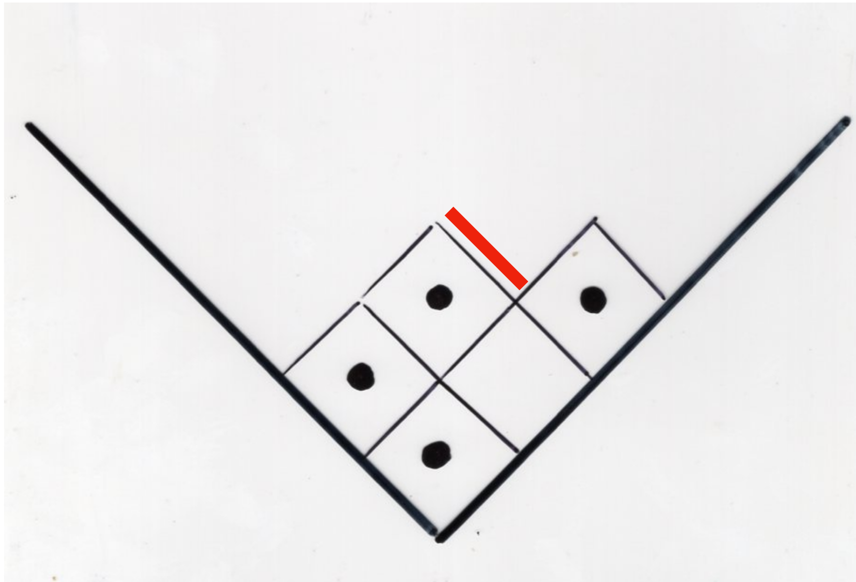
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



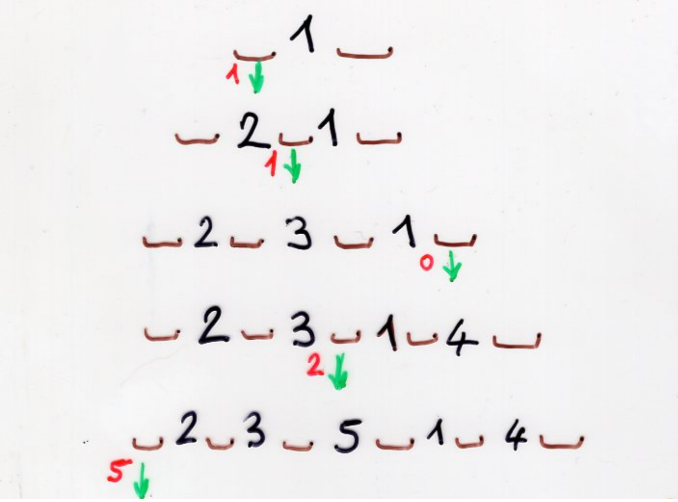


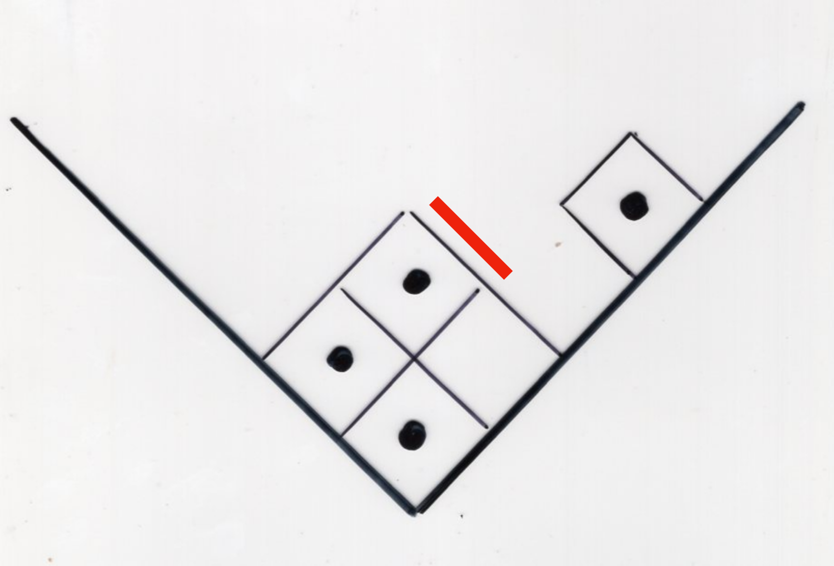
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



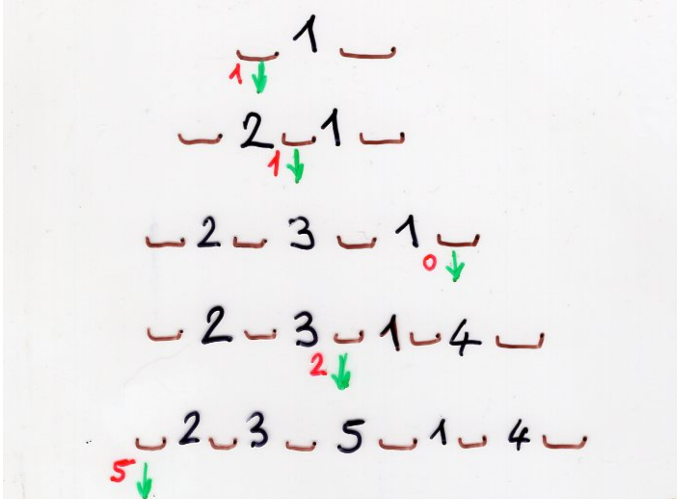


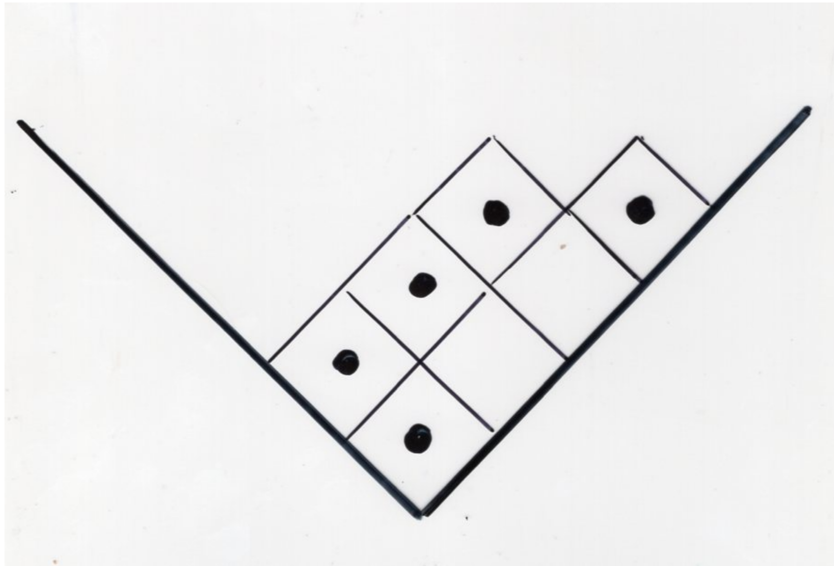
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



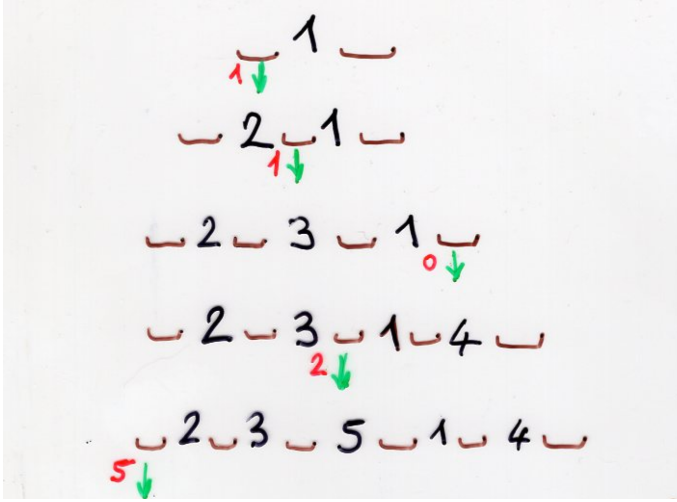


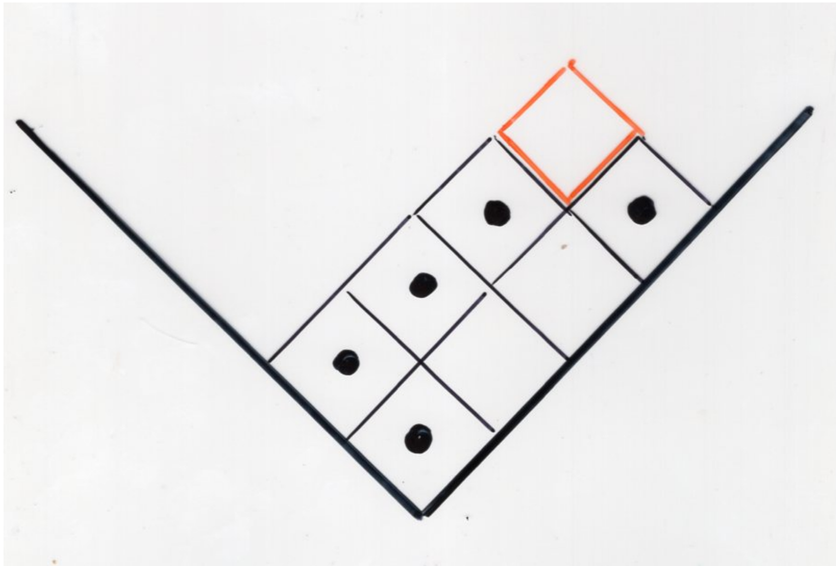
$$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$$



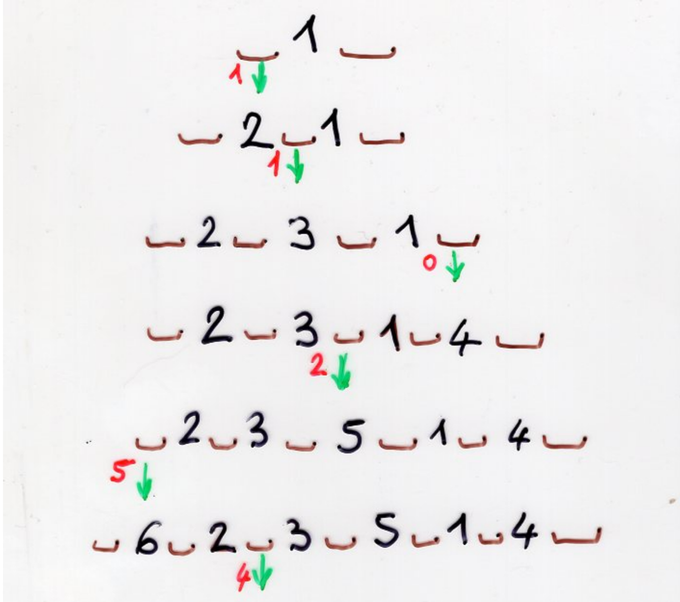


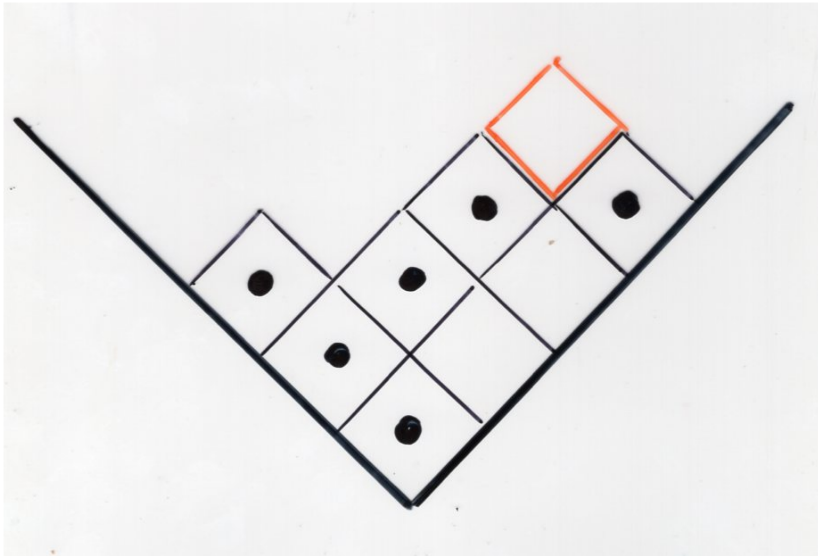
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



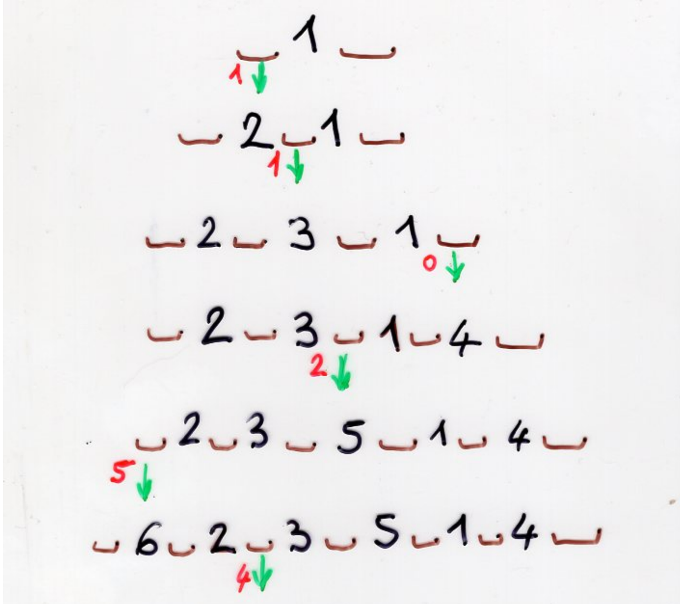


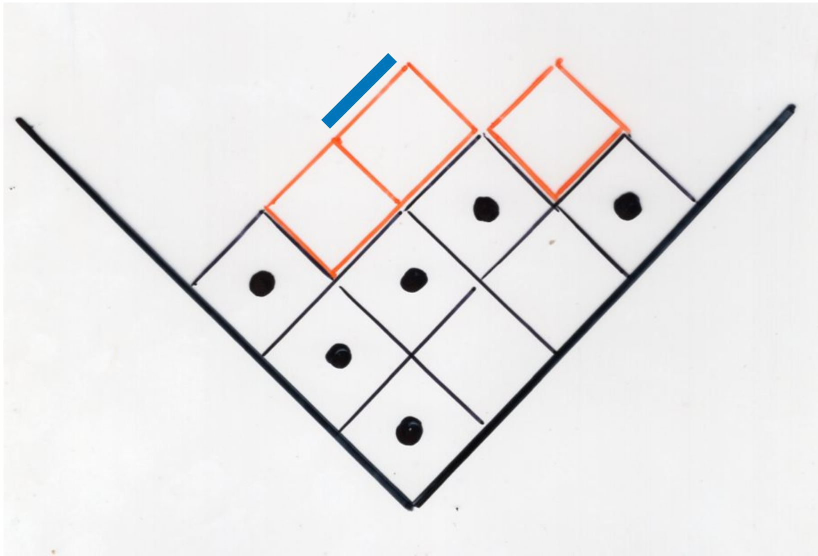
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



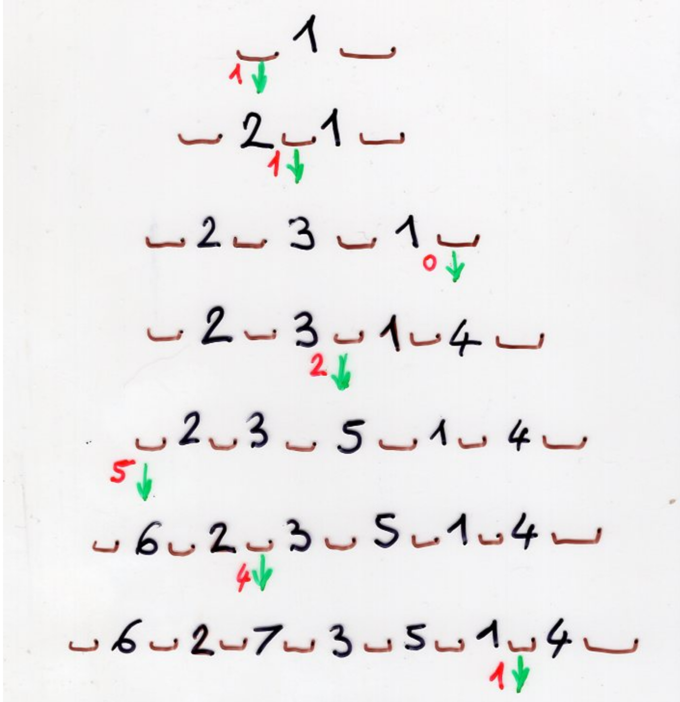


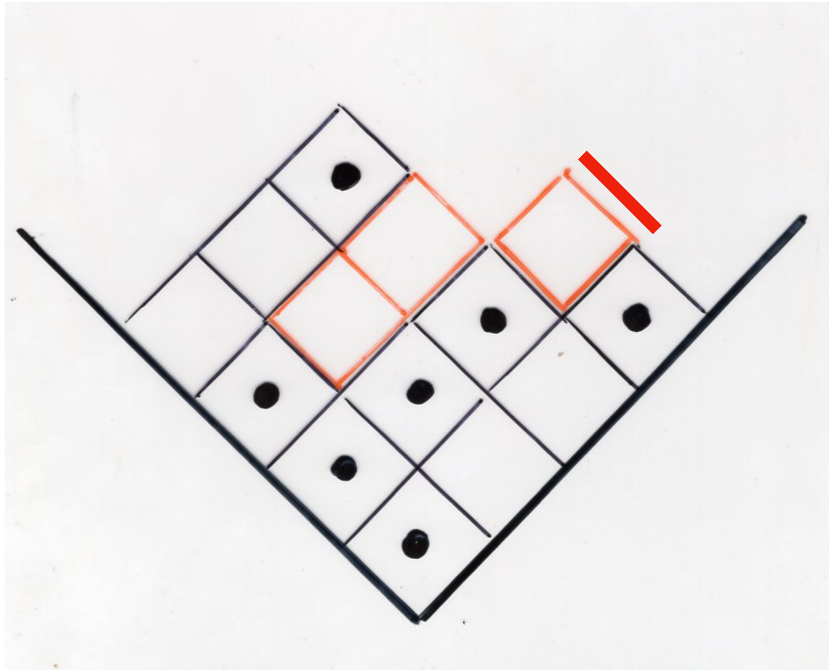
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



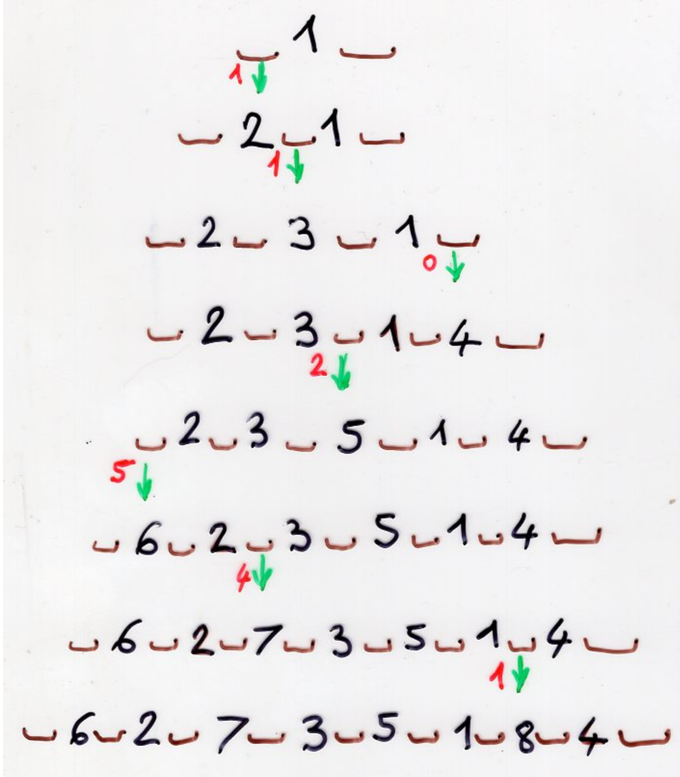


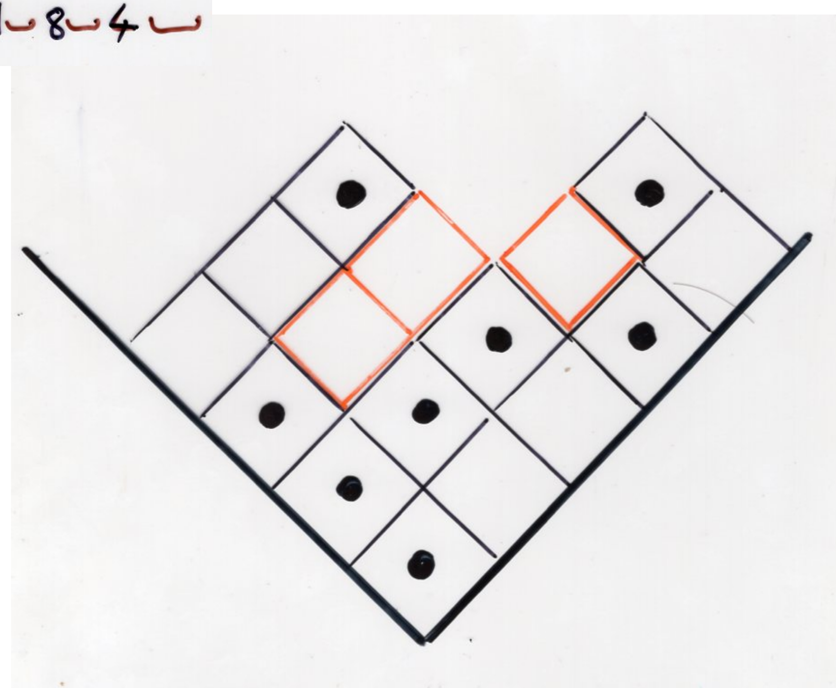
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



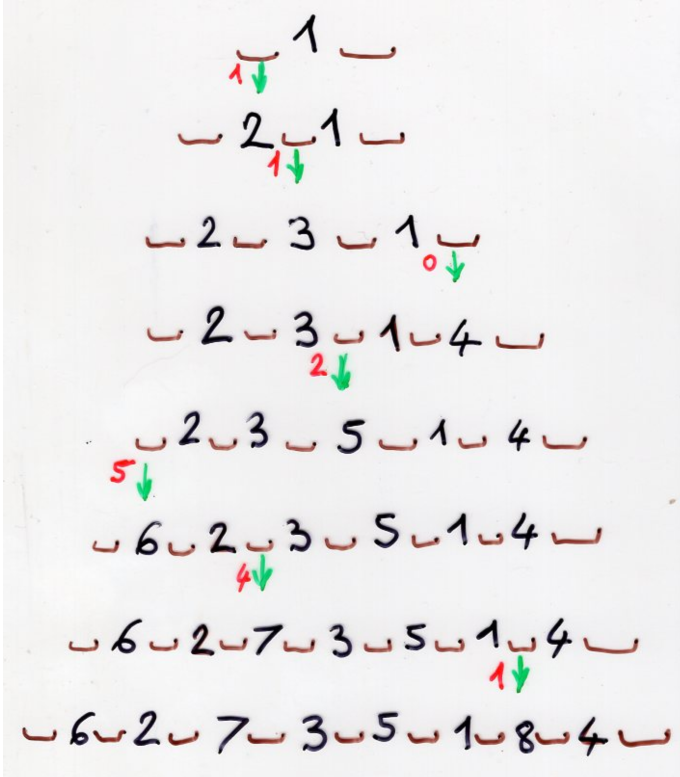


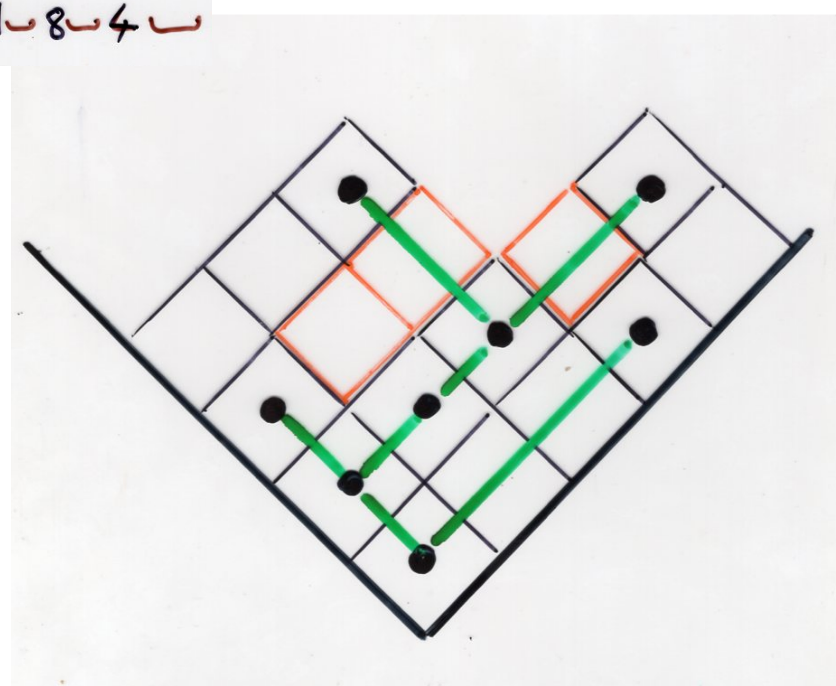
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



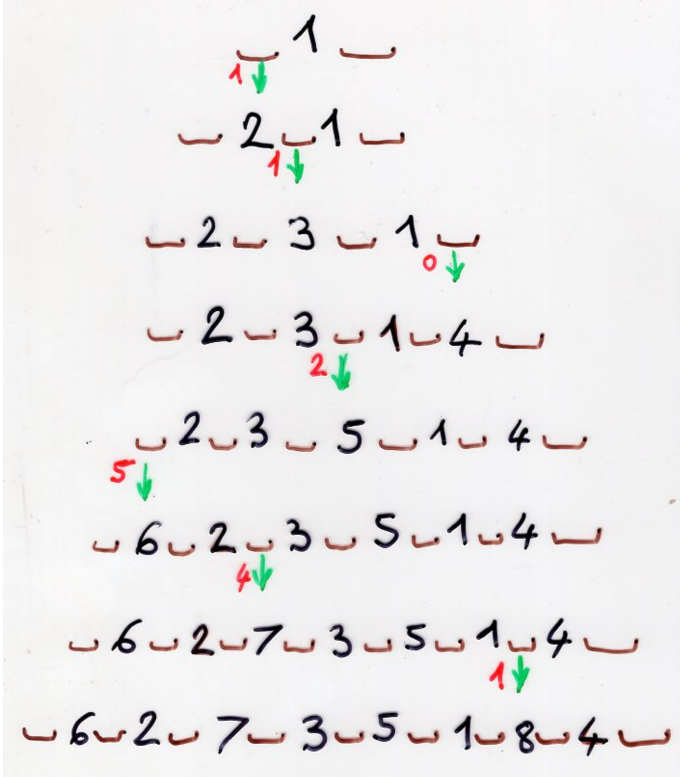


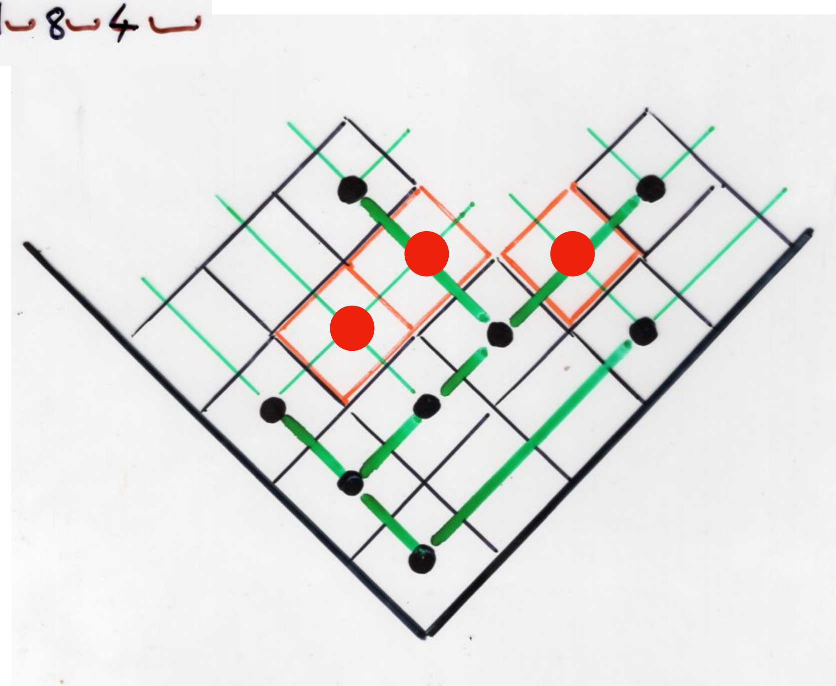
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



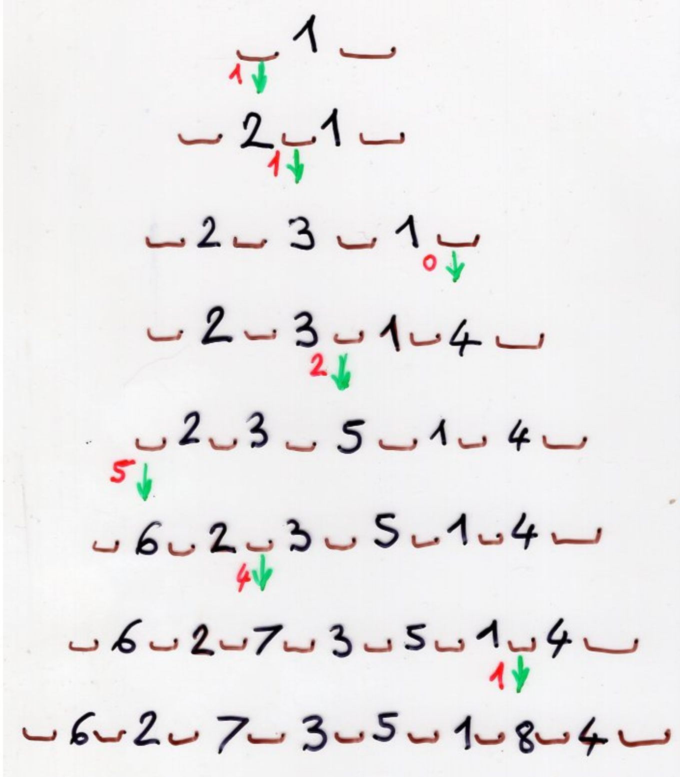


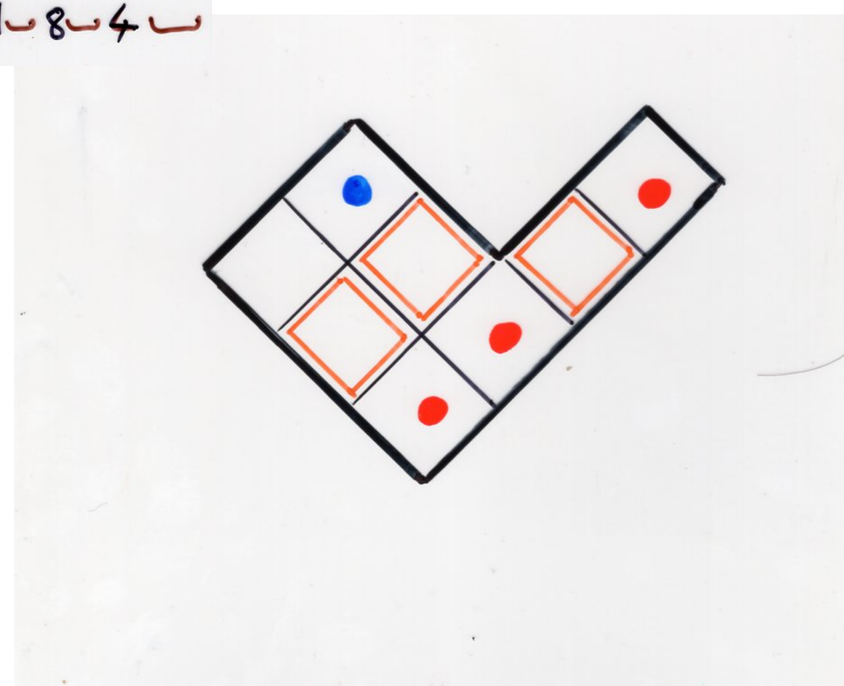
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$





$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$





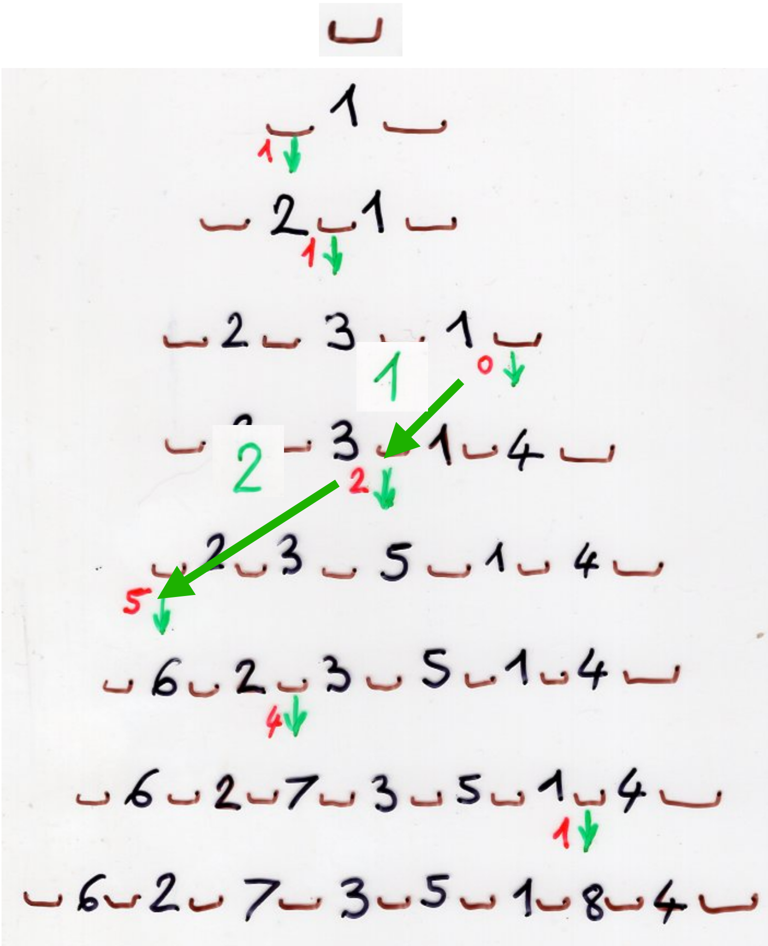
σ = 6 2 7 3 5 1 8 4

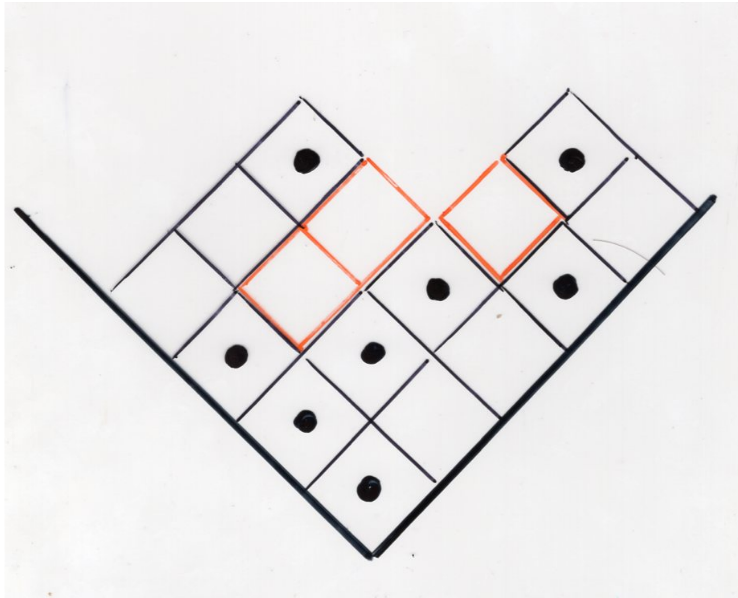
bijection

$f \longrightarrow T$

number of crossings

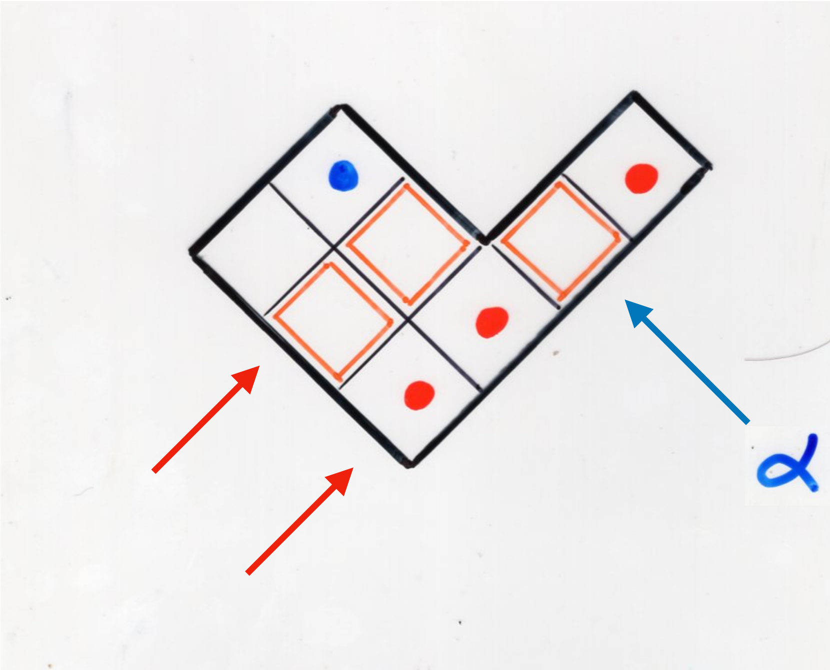
$cr(T)$

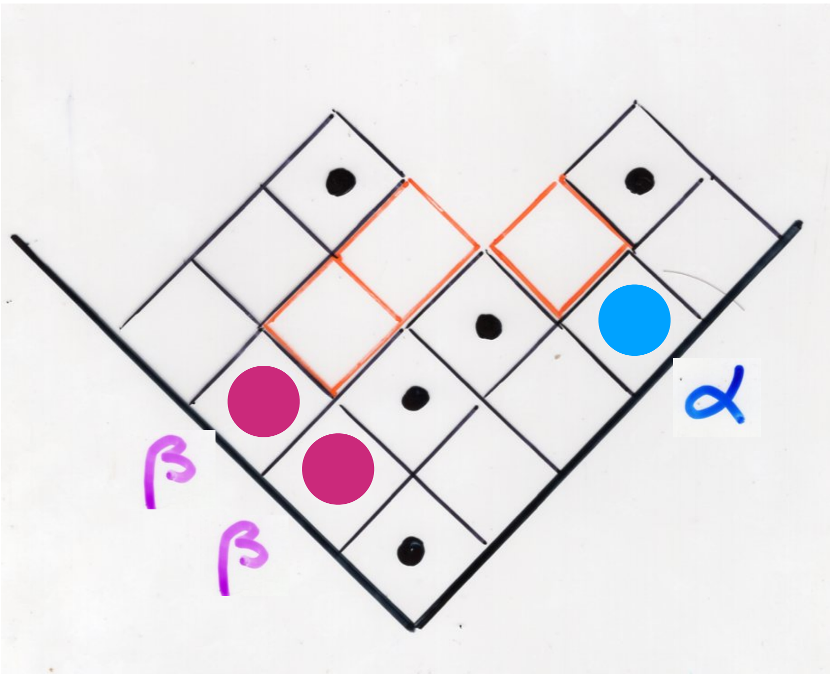


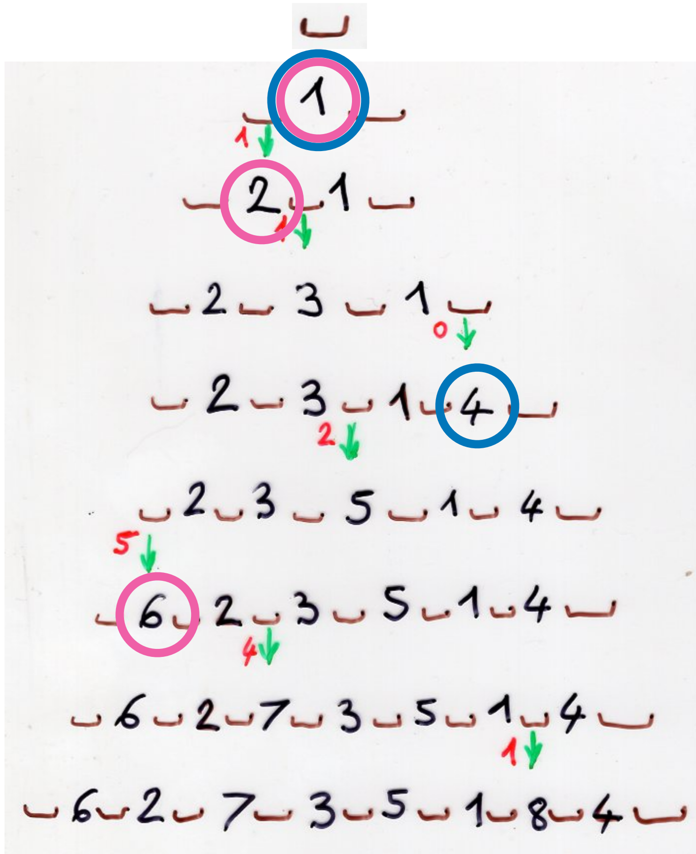


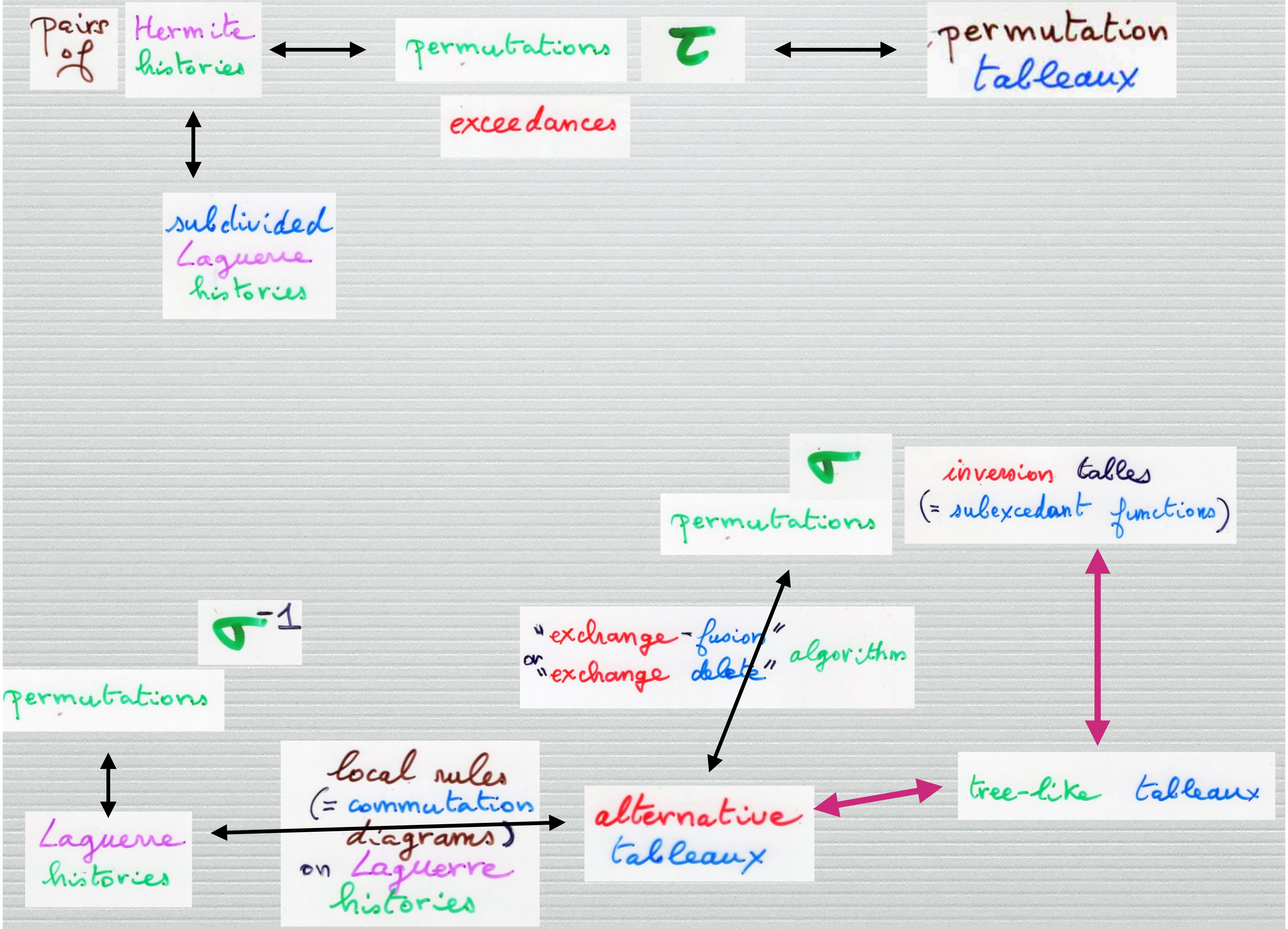
= sum of the length of all rim-hooks added in the insertion algorithm

$$= \sum_{1 \leq i \leq (n-1)} \max\left[\left(f(i+1) - f(i)\right), 0\right] \quad -1$$


α


β
β
α


1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4



pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
σ
permutations
inversion tables
(= subexcedant functions)
σ⁻¹
"exchange-fusion" algorithm
or "exchange delete"
permutations
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

Laguerre heaps of segments

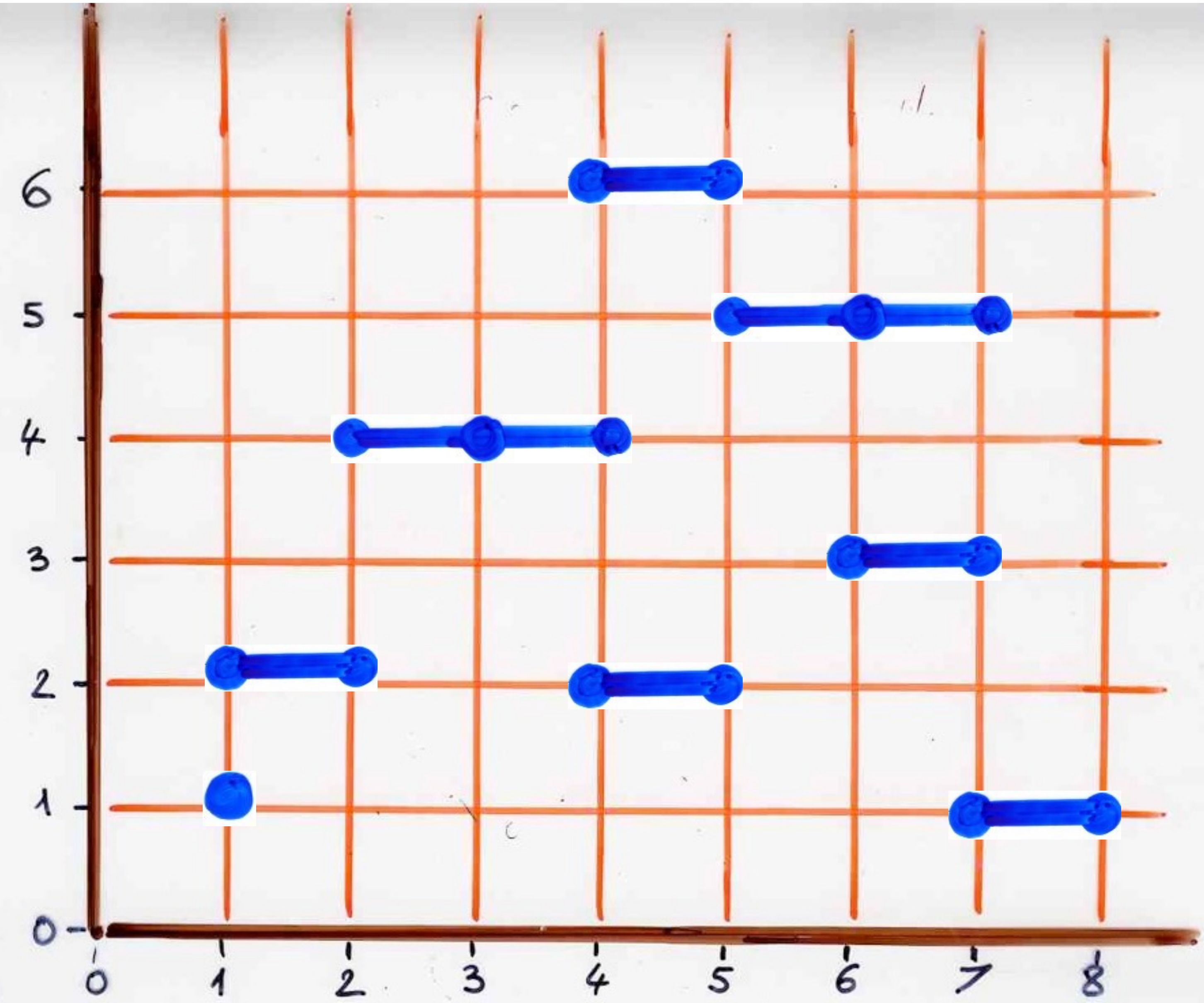
non-ambiguous
trees

tree-like
tableaux
rectangular shape

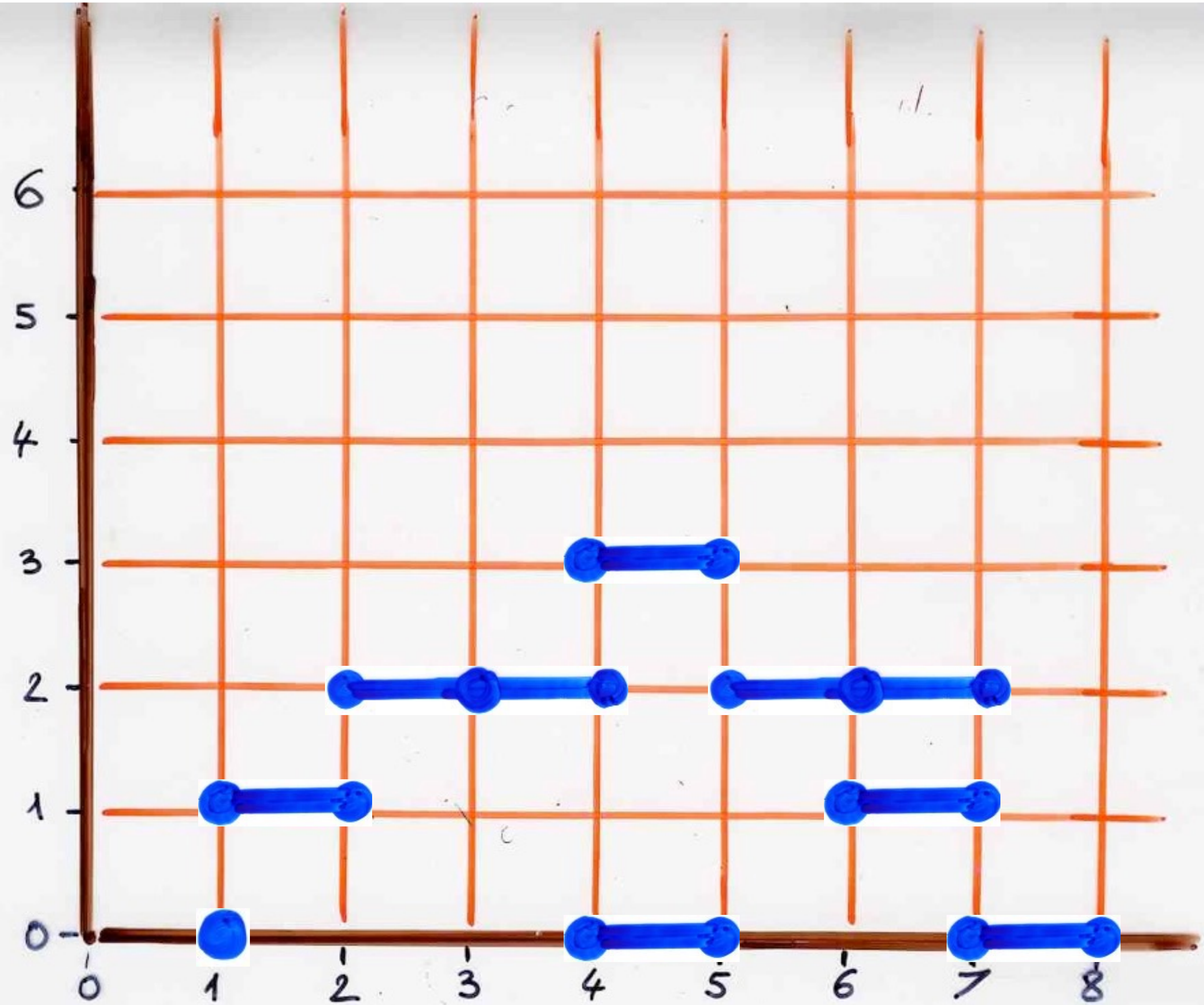
complete
non-ambiguous
trees

E. Jin (2014)

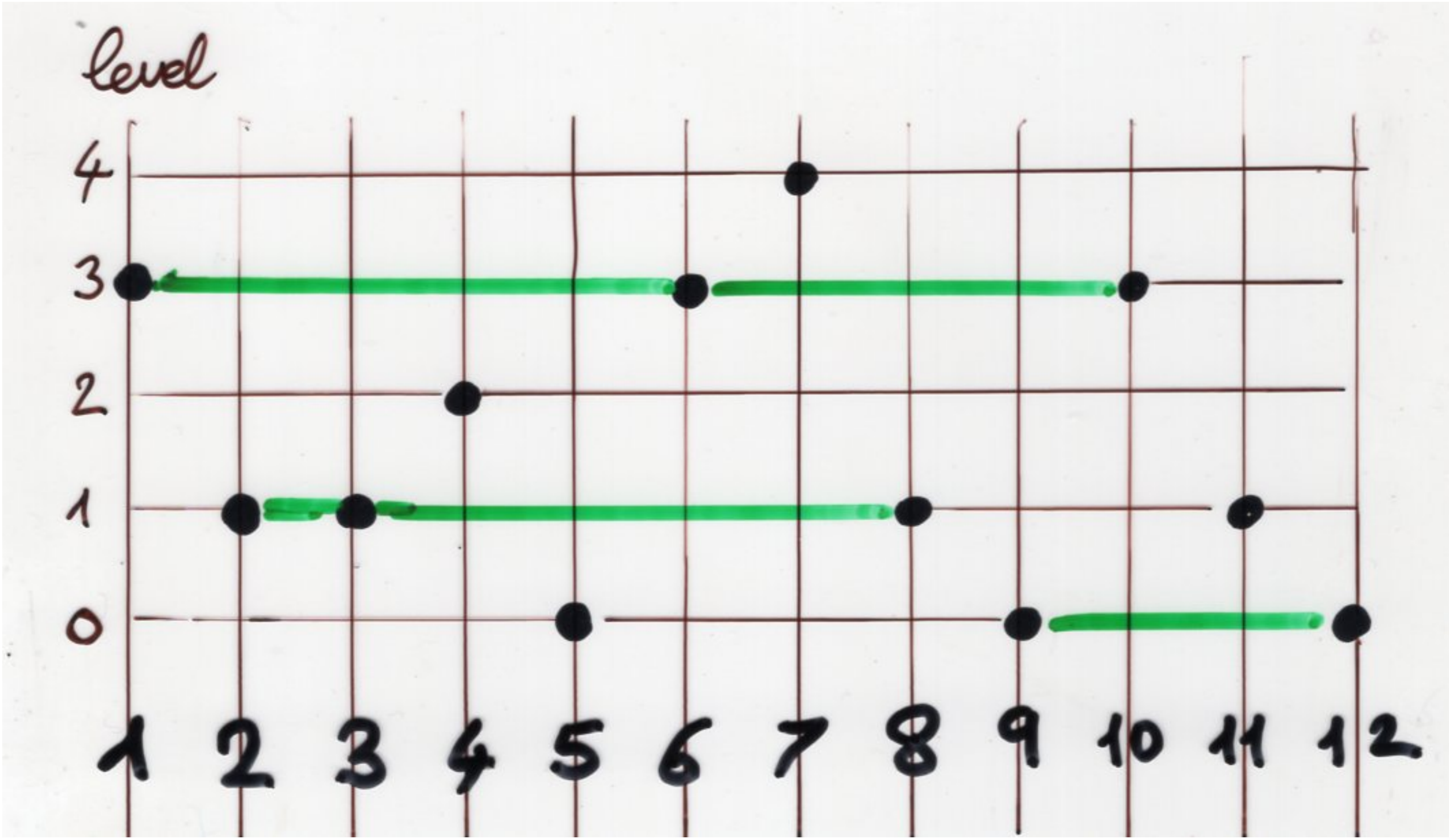
Bessel functions
heaps
logarithmic lemma



0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8



6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8



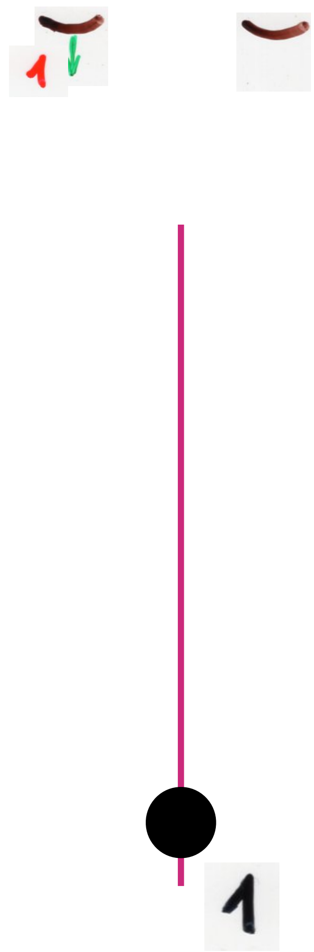
Laguerre heaps of segments

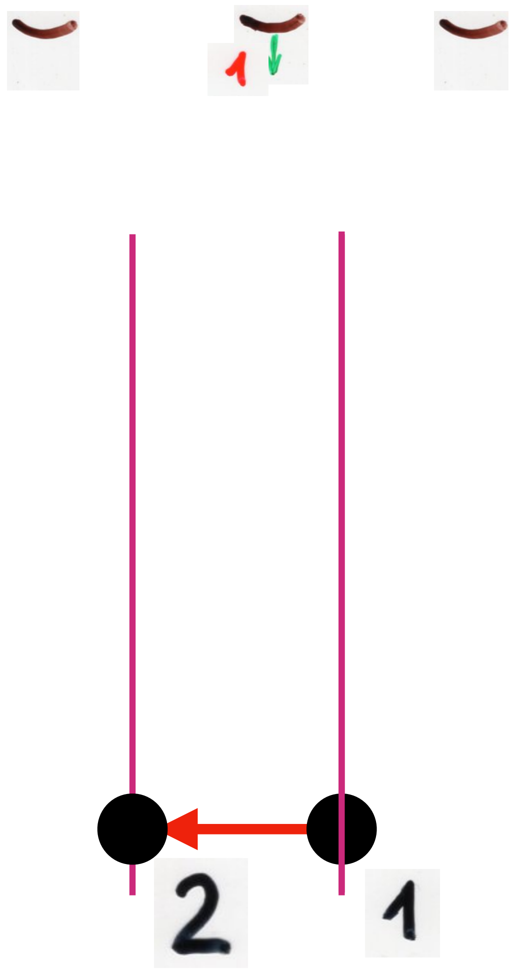
Bijection

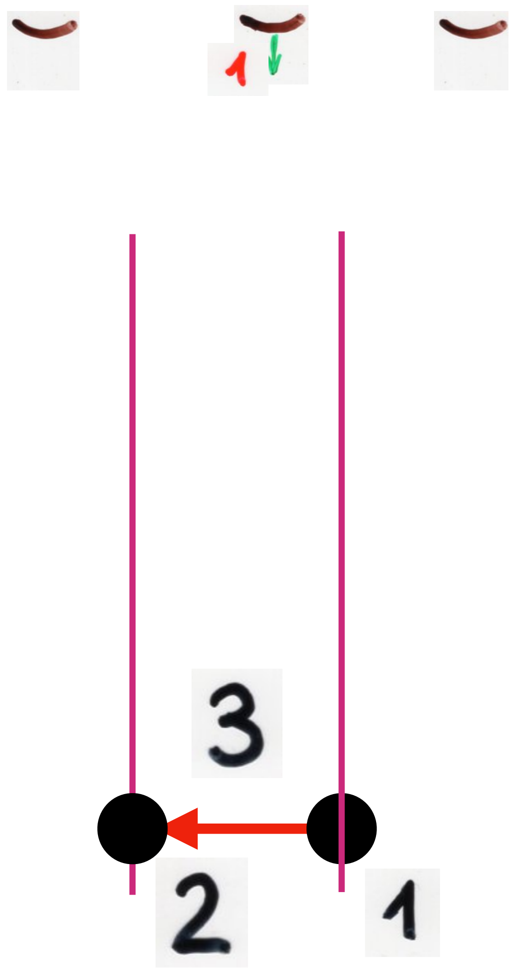
Permutions

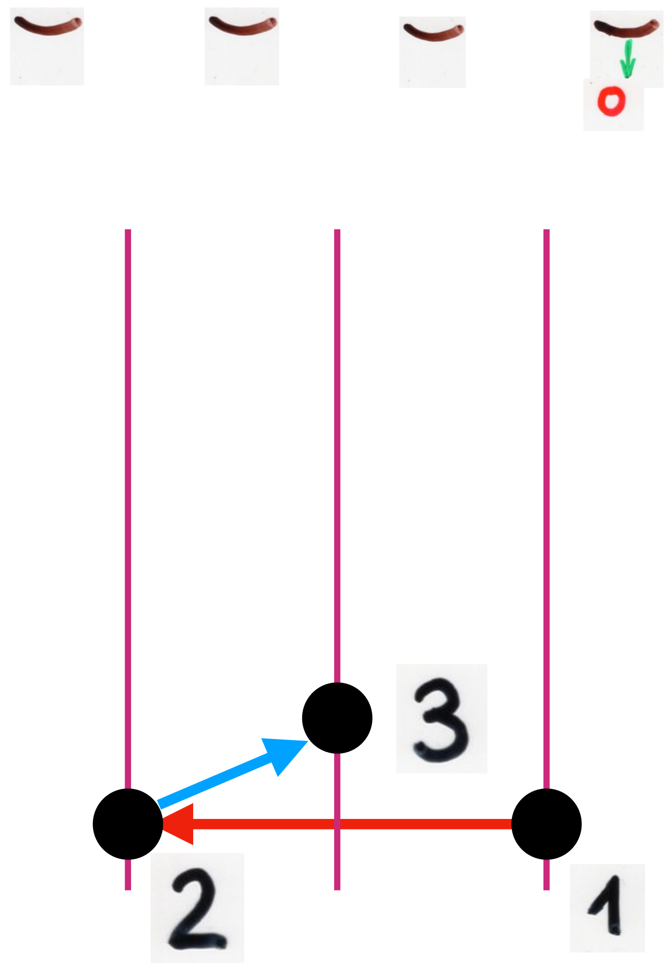
↕

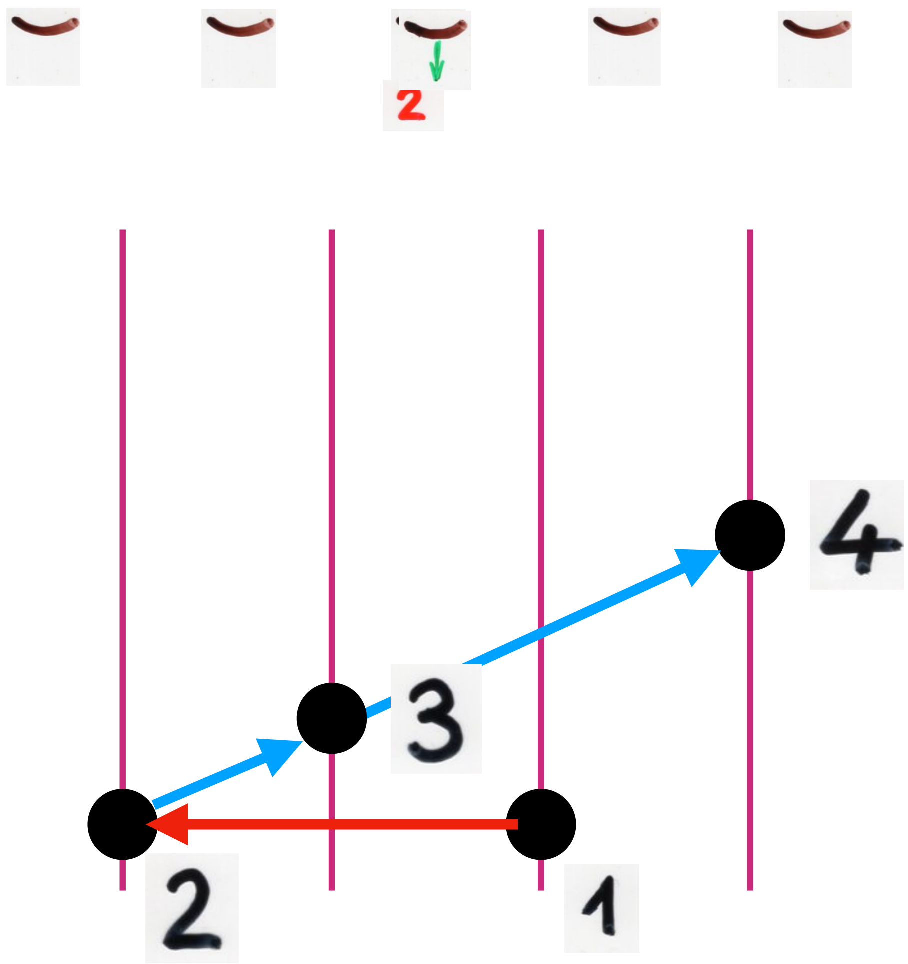
Laguerre heaps of segments

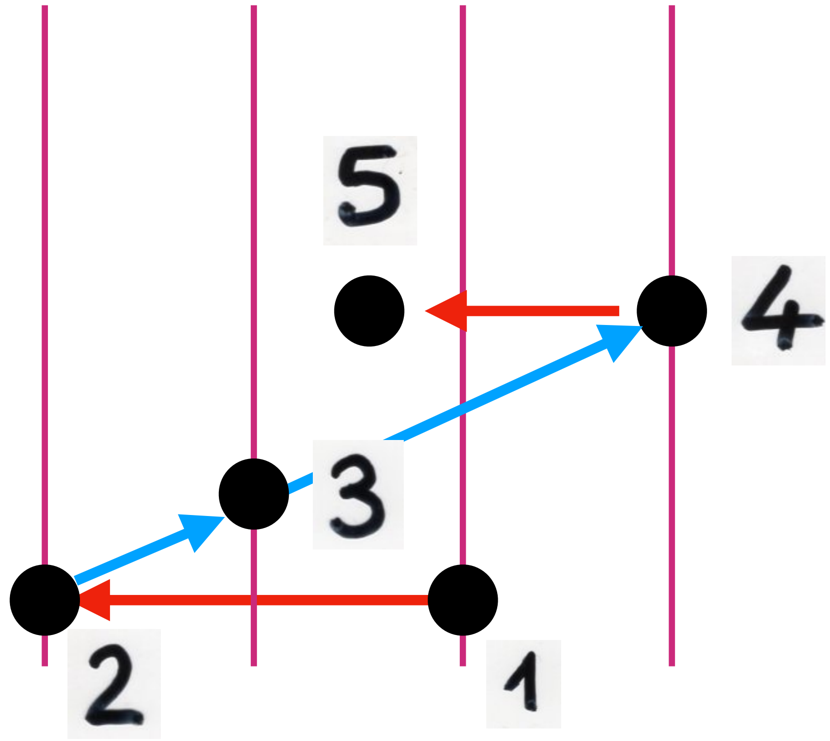

1
1

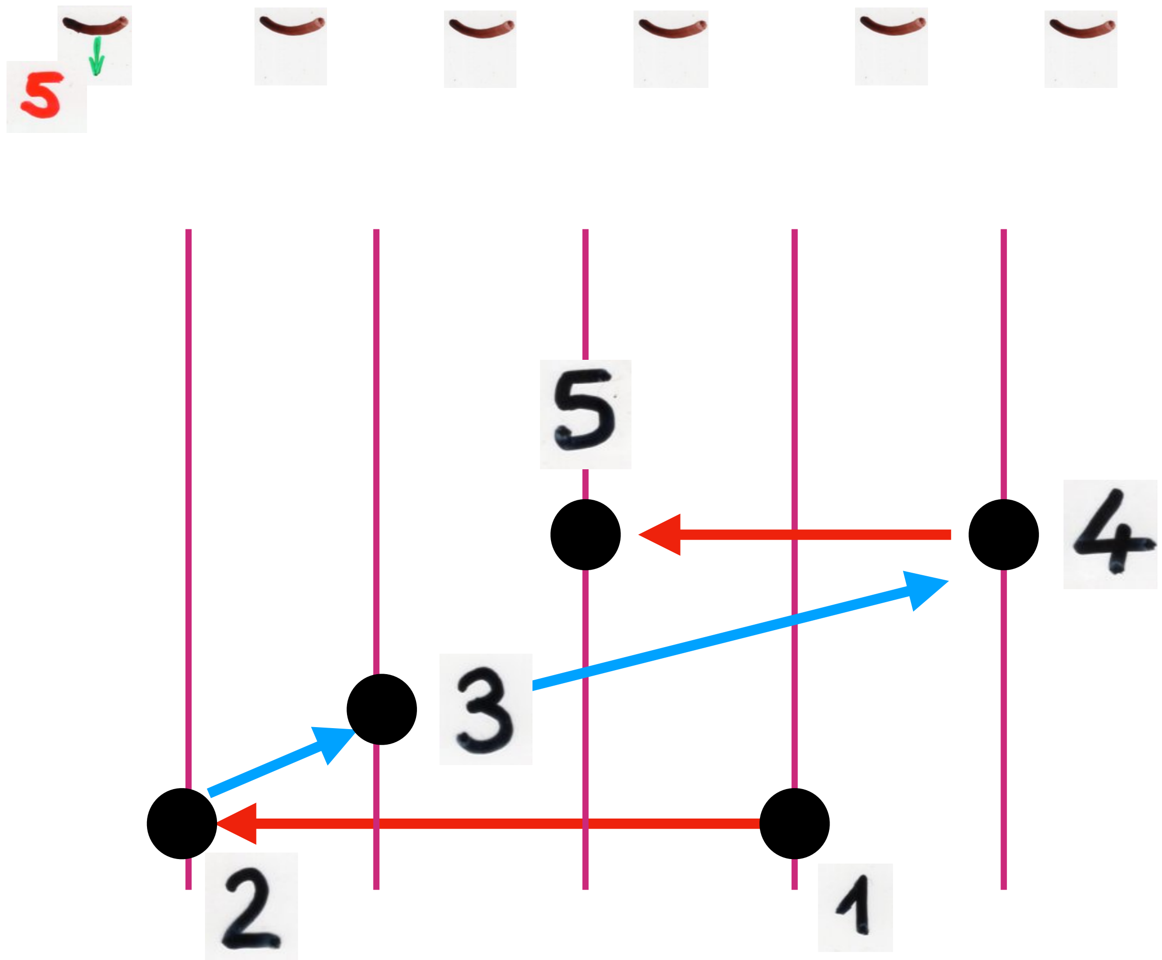

1
2
1

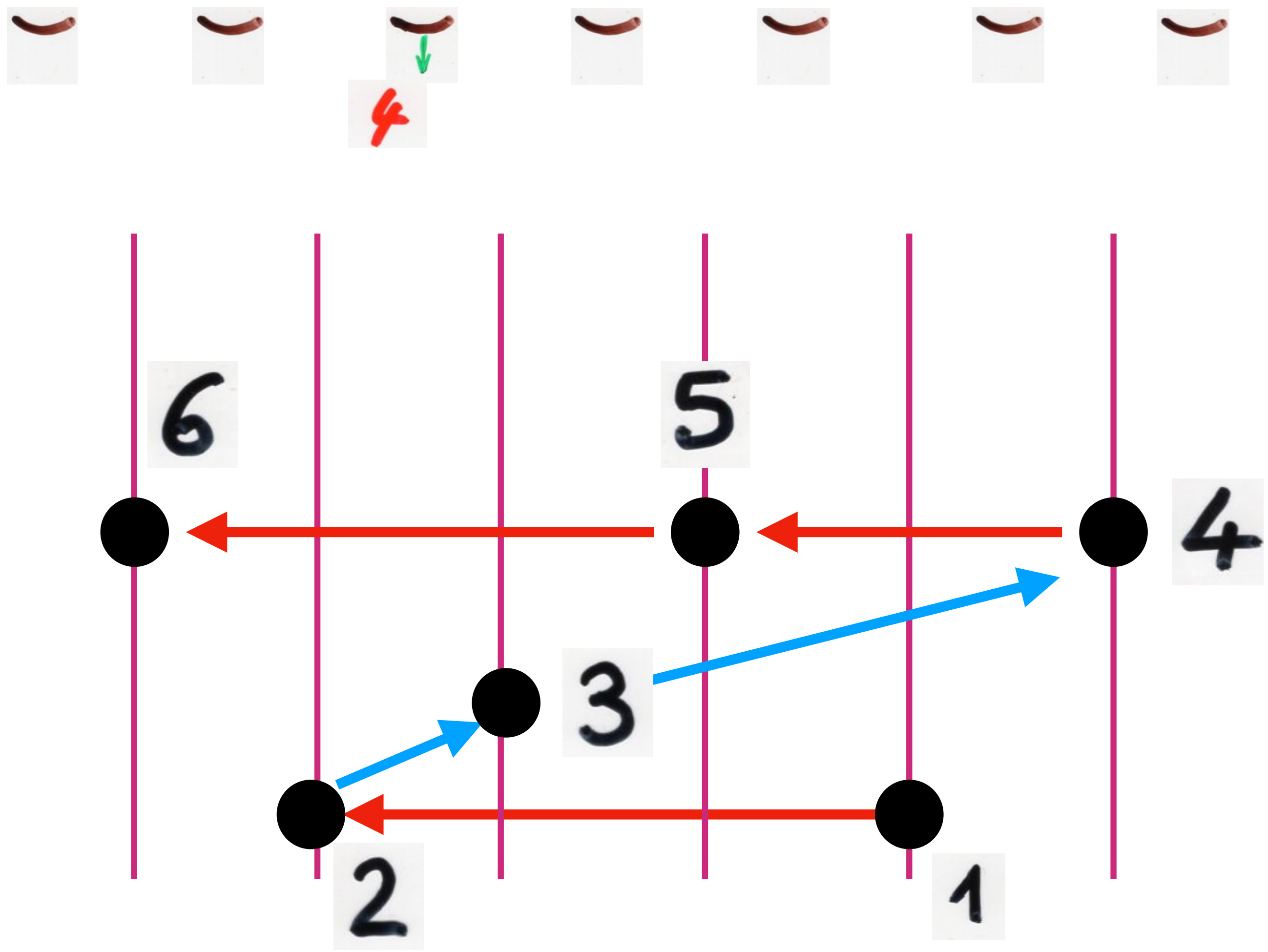

1
3
2
1

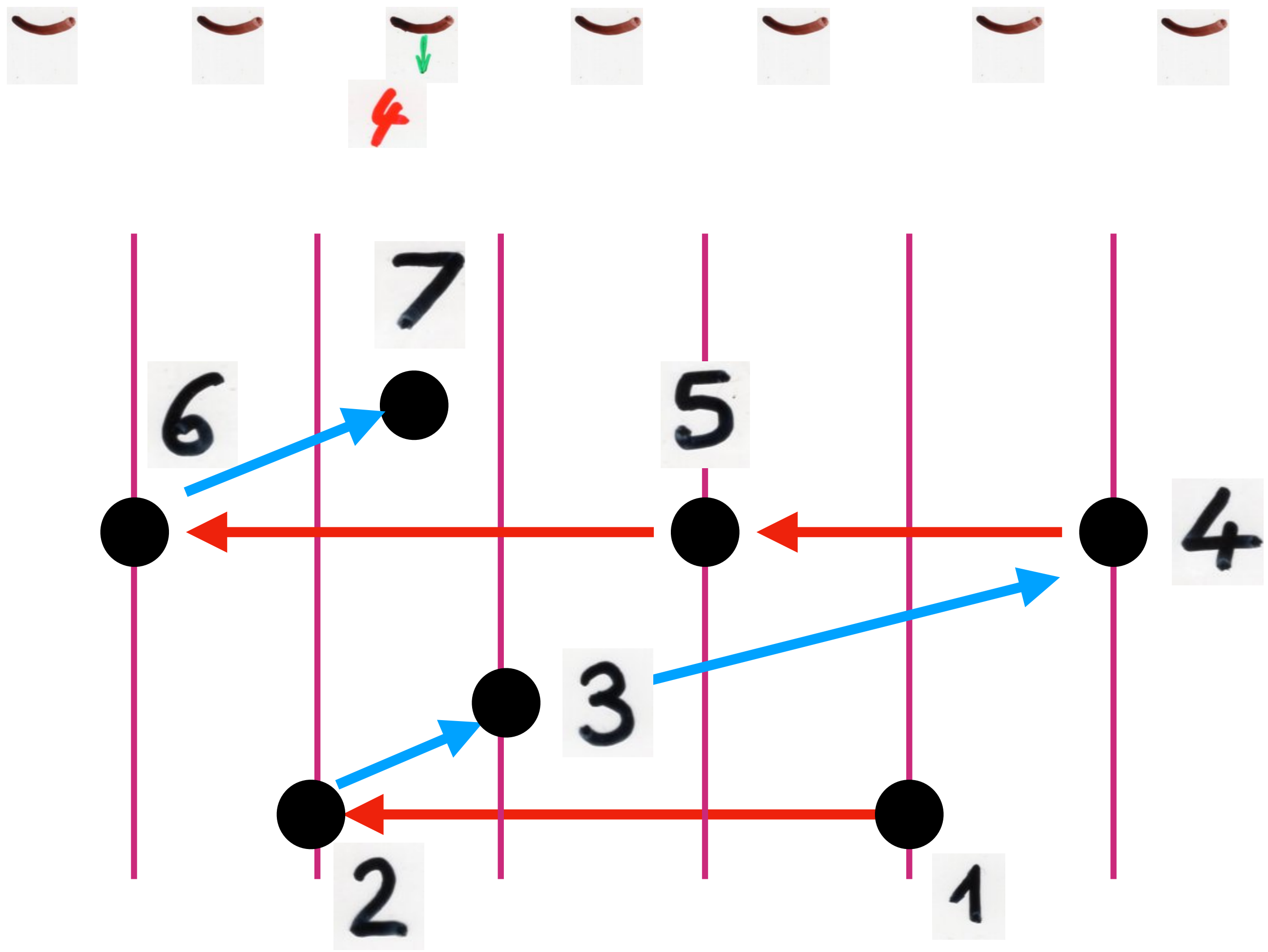

3
2
1


2
4
3
2
1

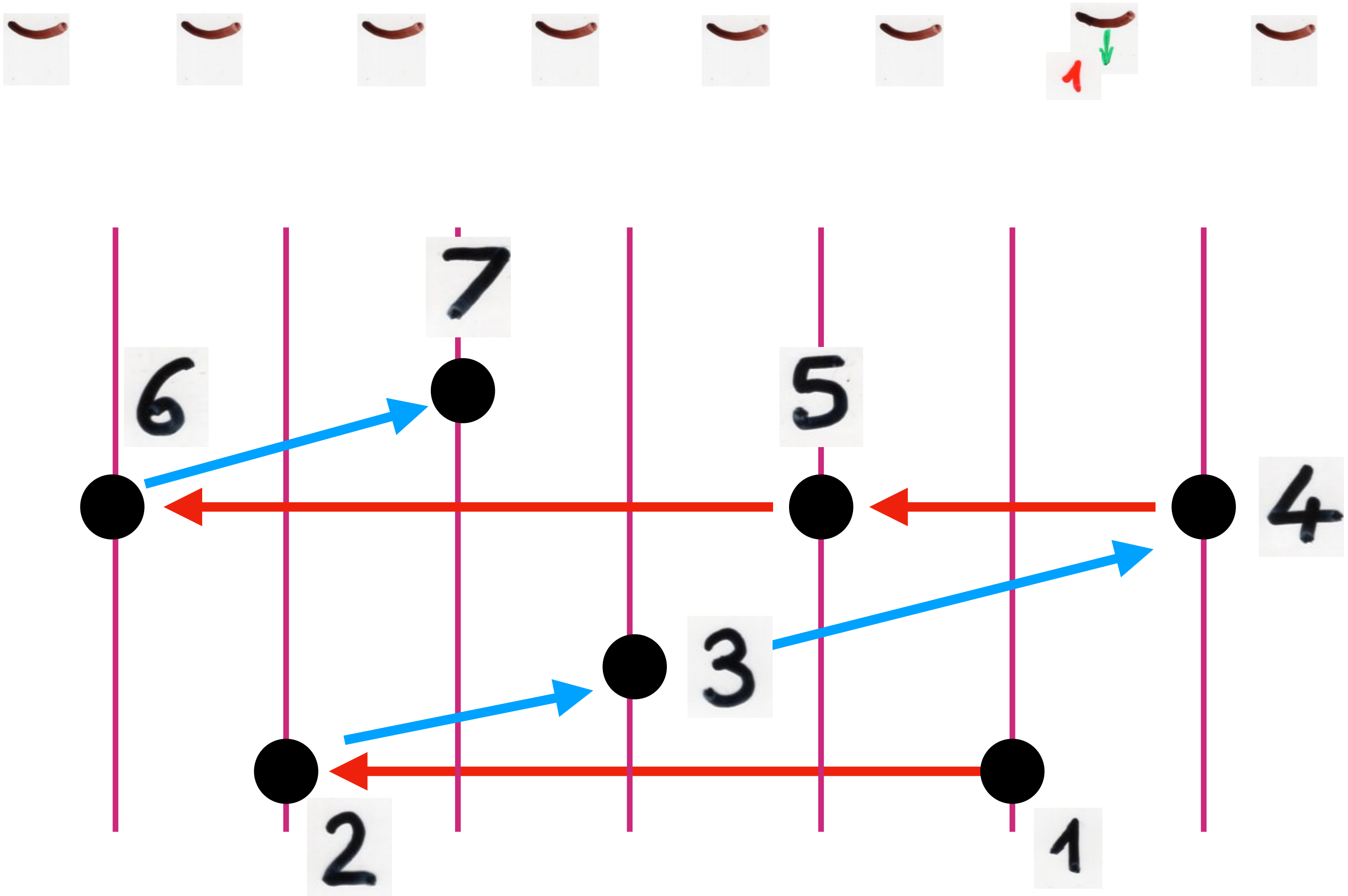

5
4
3
2
1


5
5
4
3
2
1

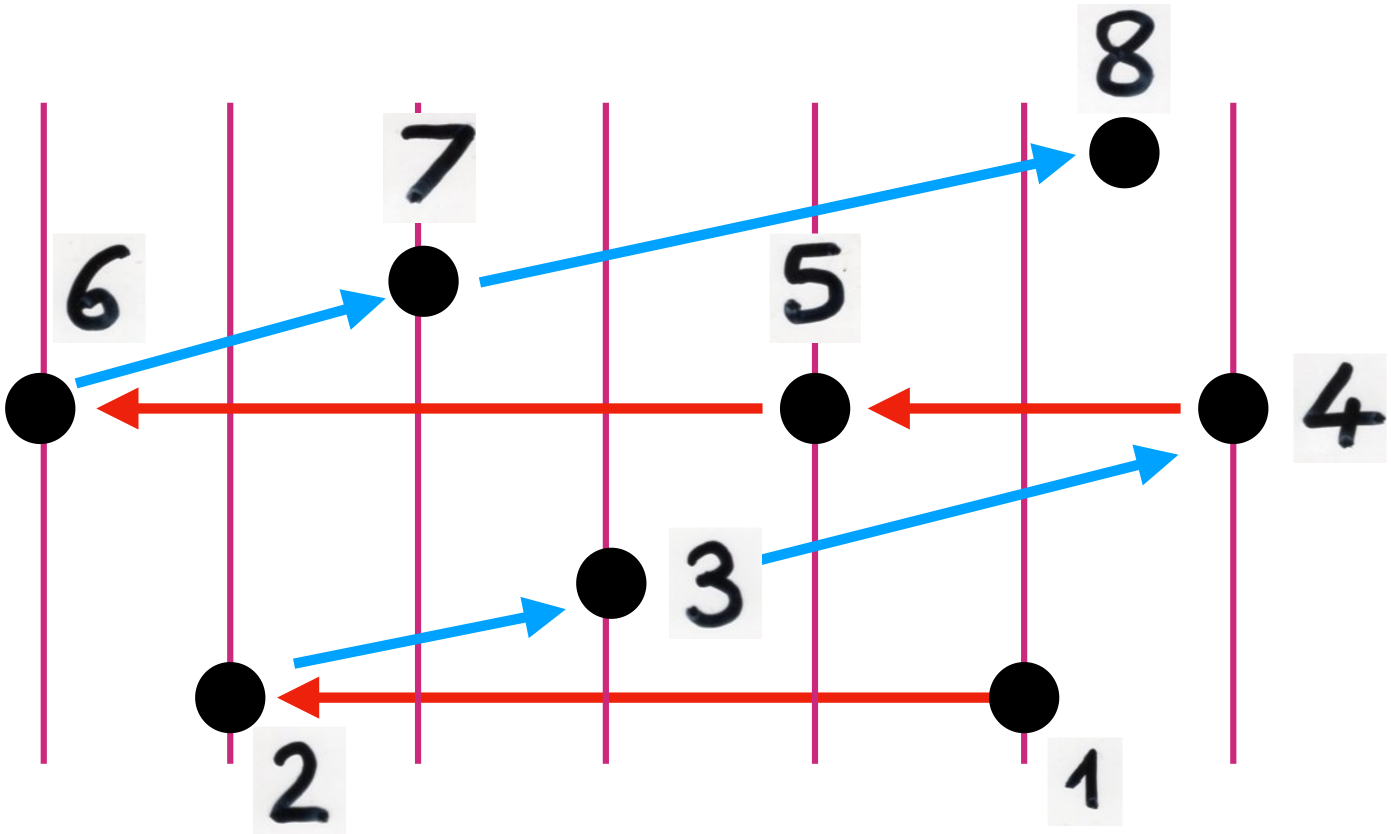

4
6
5
4
3
2
1



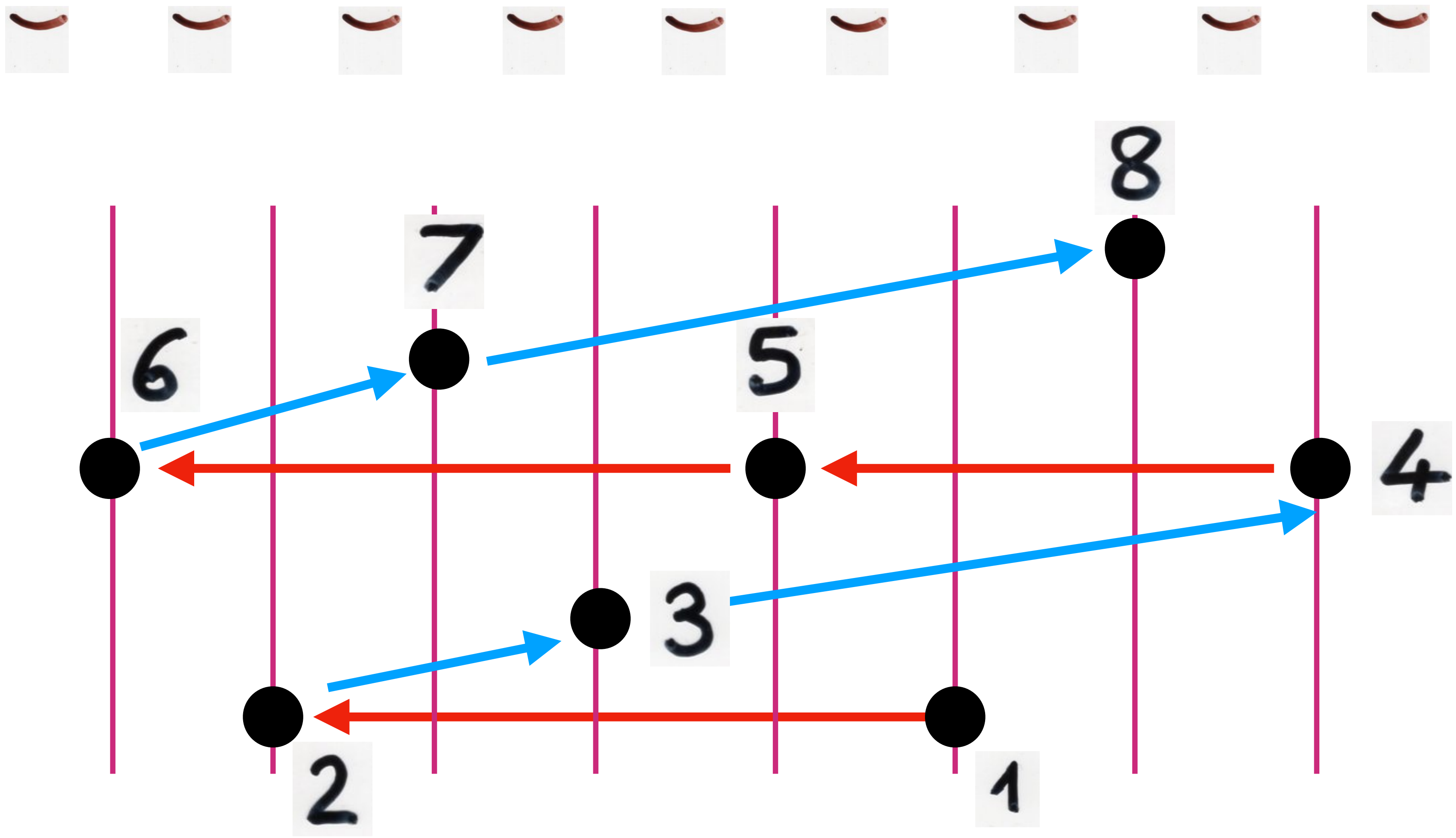
4
7
6
5
4
3
2
1

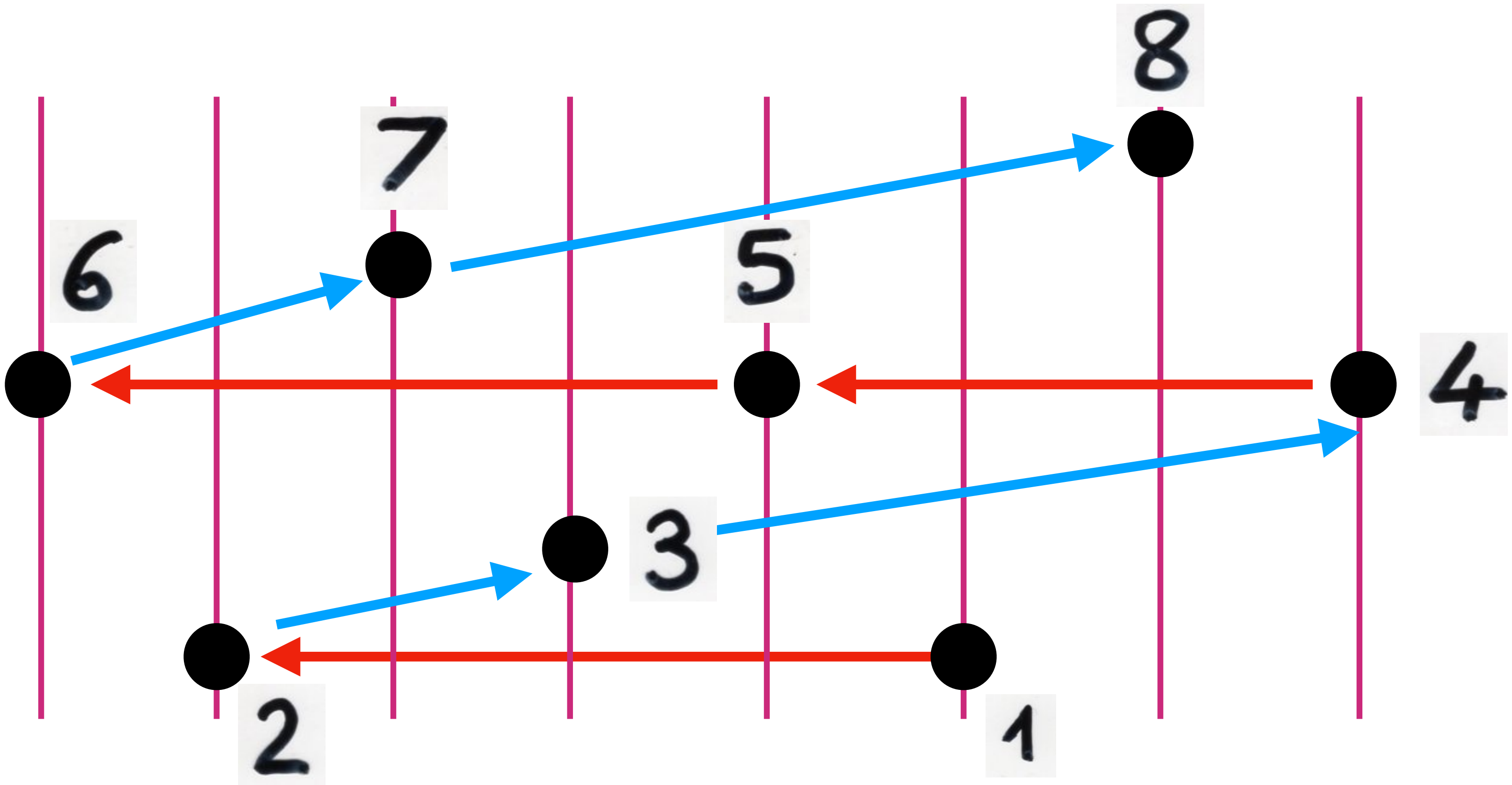


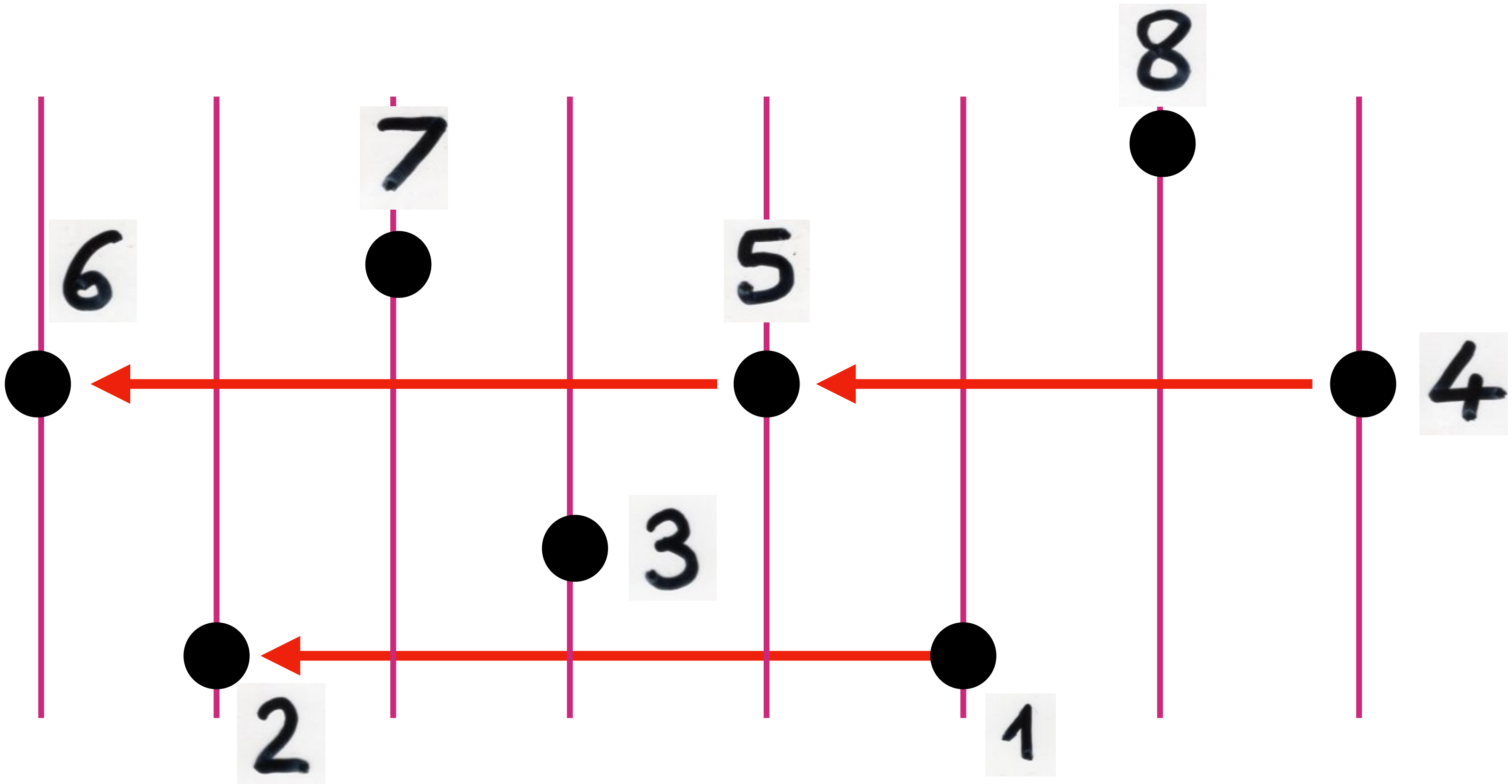
1
7
6
5
4
3
2
1

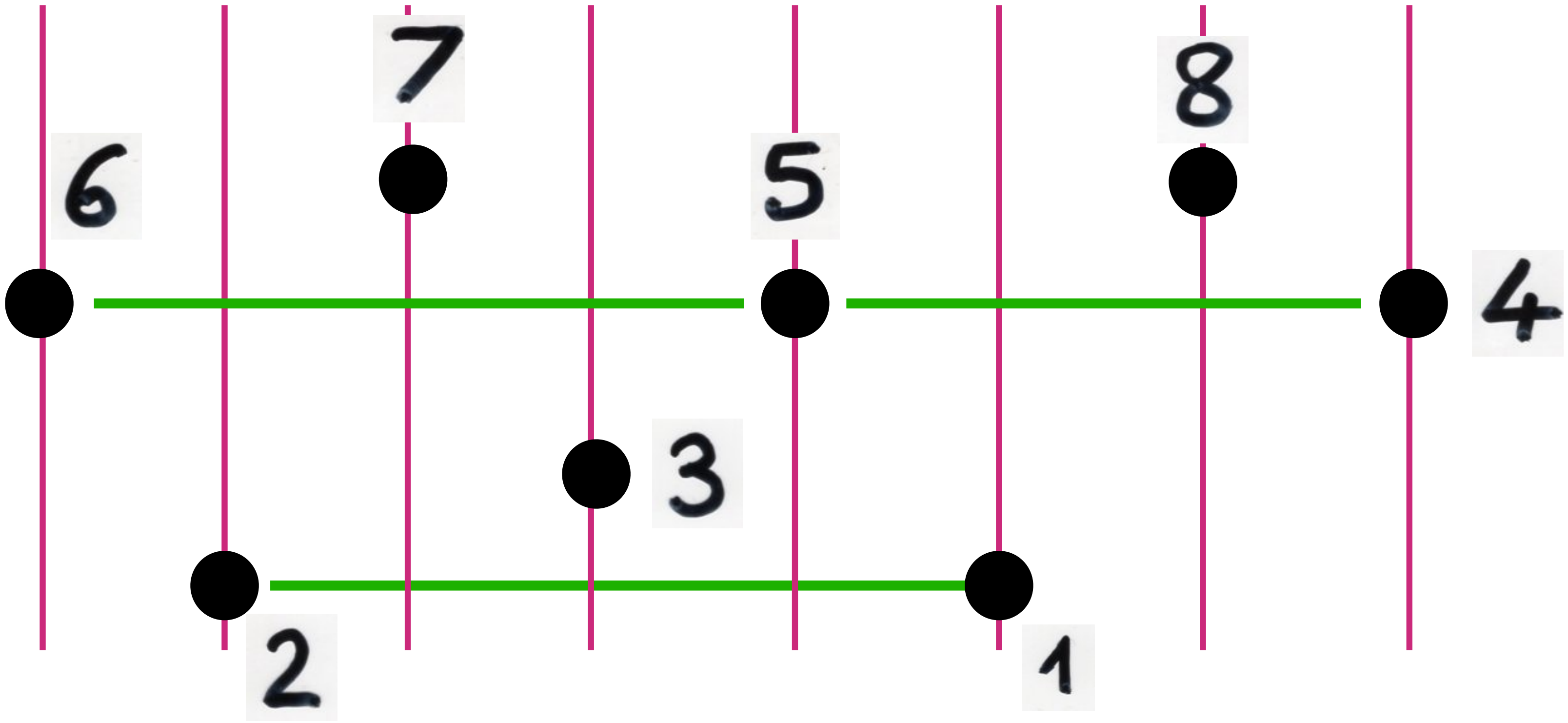


8
7
6
5
4
3
2
1

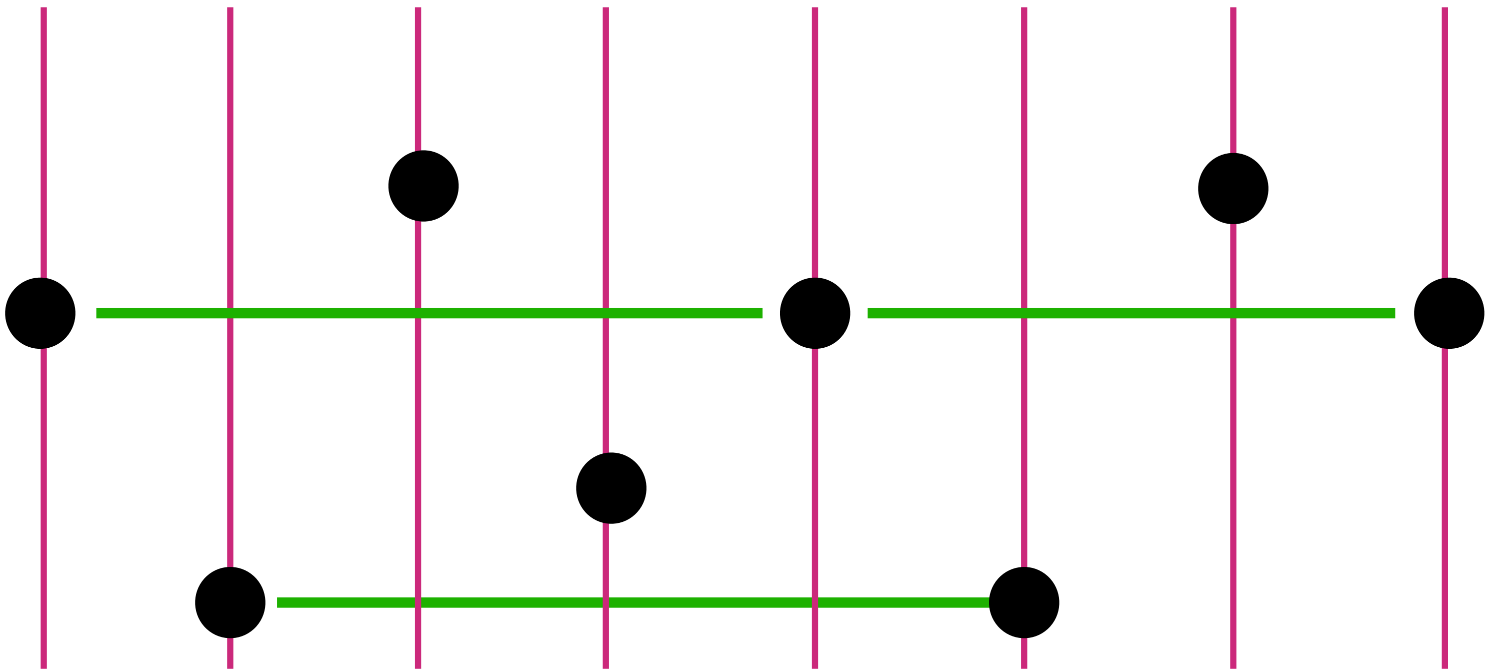

8
7
6
5
4
3
2
1

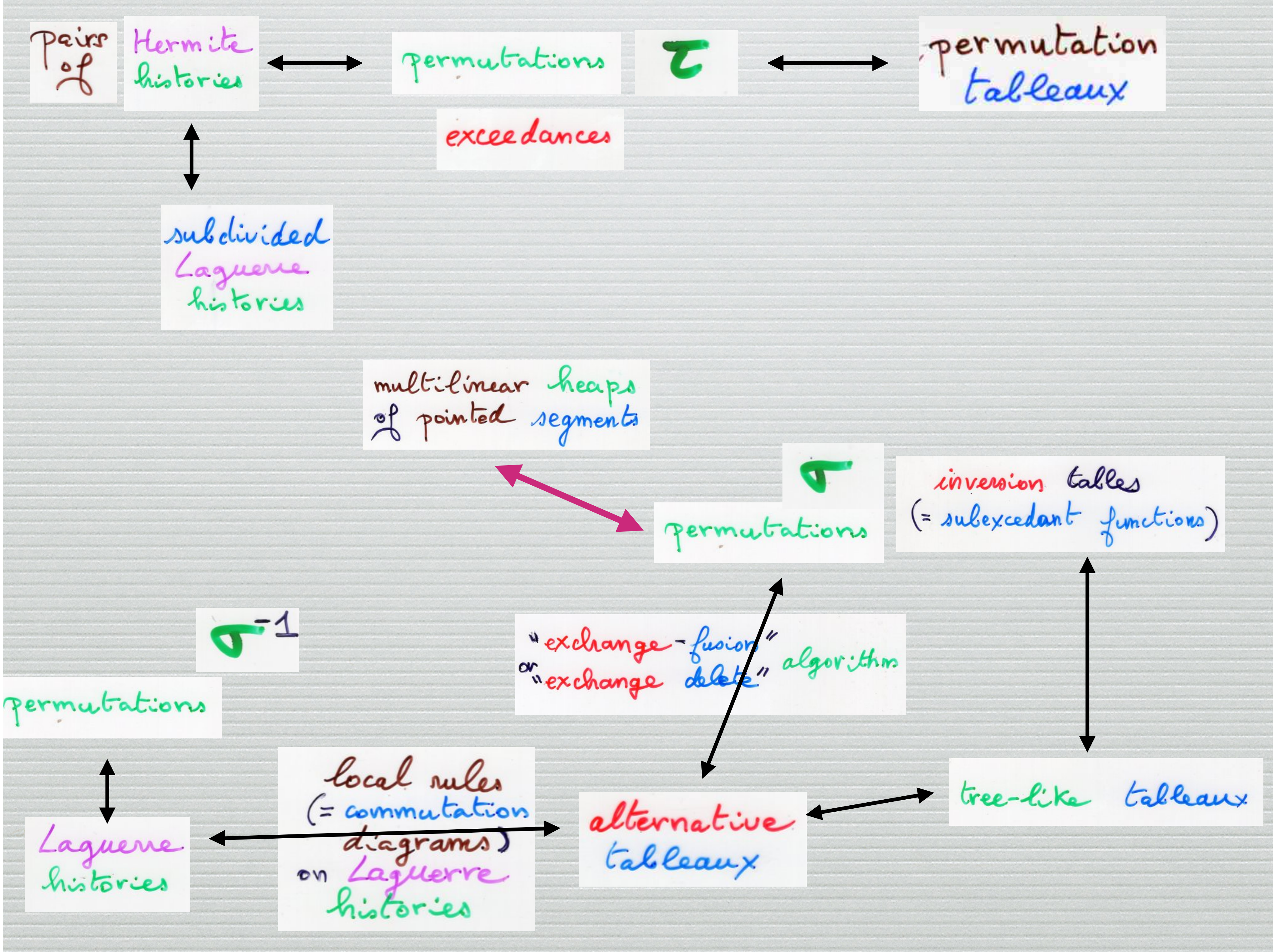

8
7
6
5
4
3
2
1

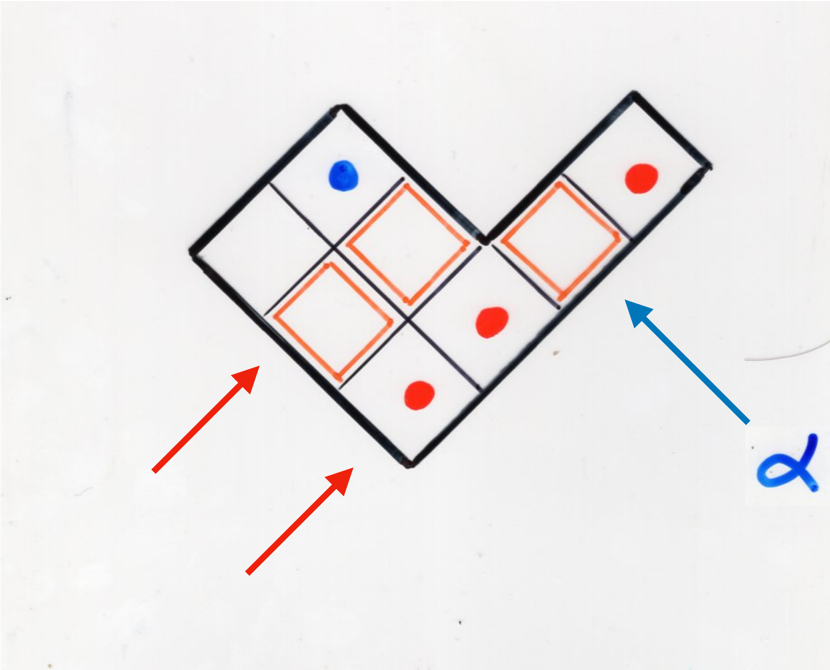

8
7
6
5
4
3
2
1

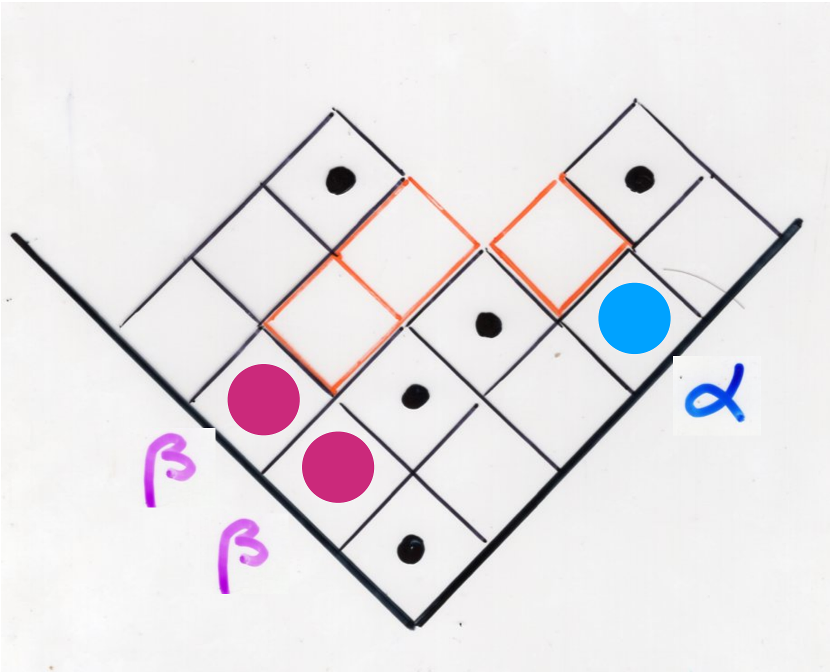

7
8
6
5
4
3
2
1

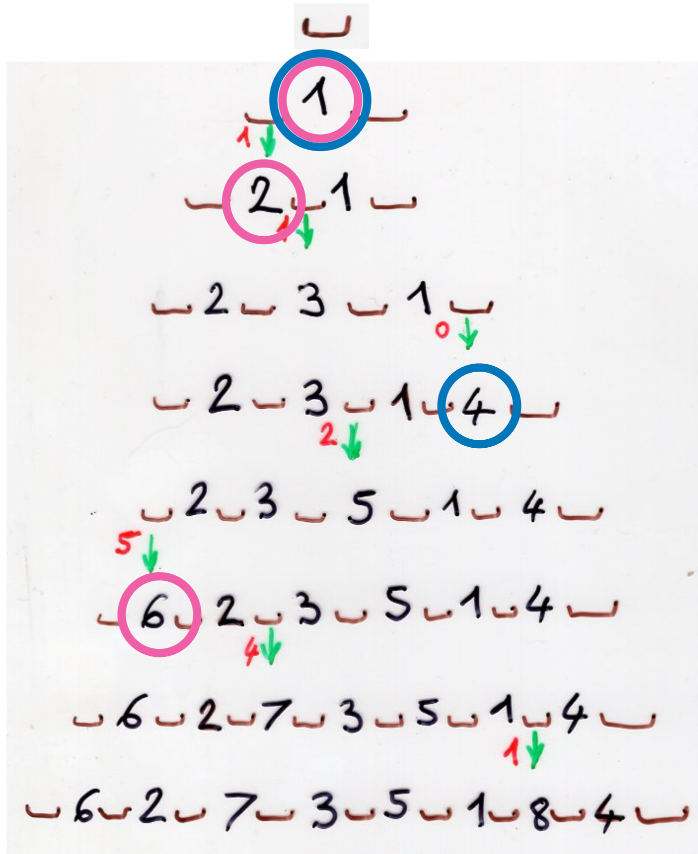
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

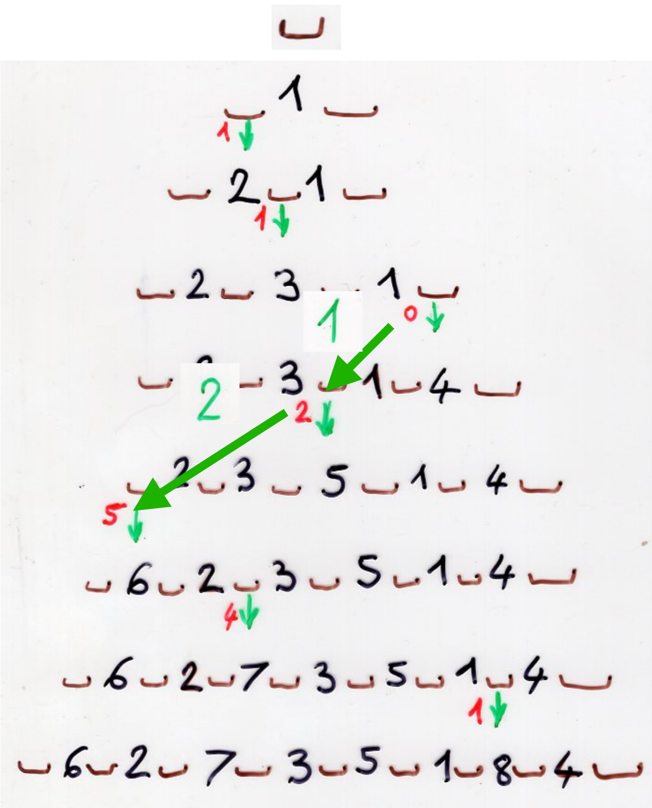


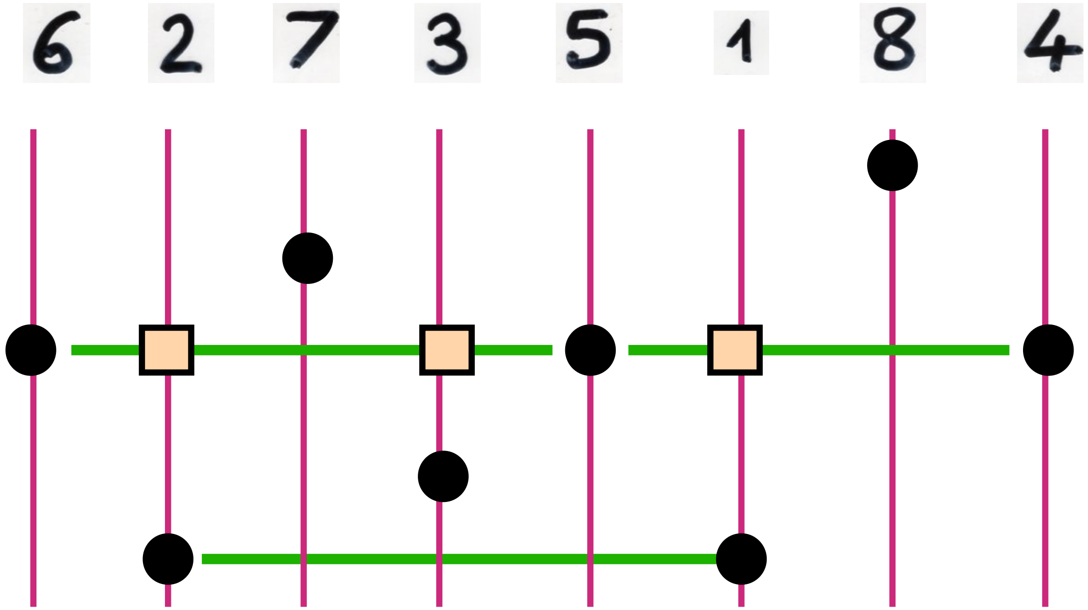

Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
multilinear heaps of pointed segments
σ
permutations
inversion tables
(= subexcedant functions)
σ⁻¹
"exchange-fusion" or "exchange delete" algorithm
permutations
local rules
(= commutation diagrams)
on Laguerre histories
Laguerre histories
alternative tableaux
tree-like tableaux

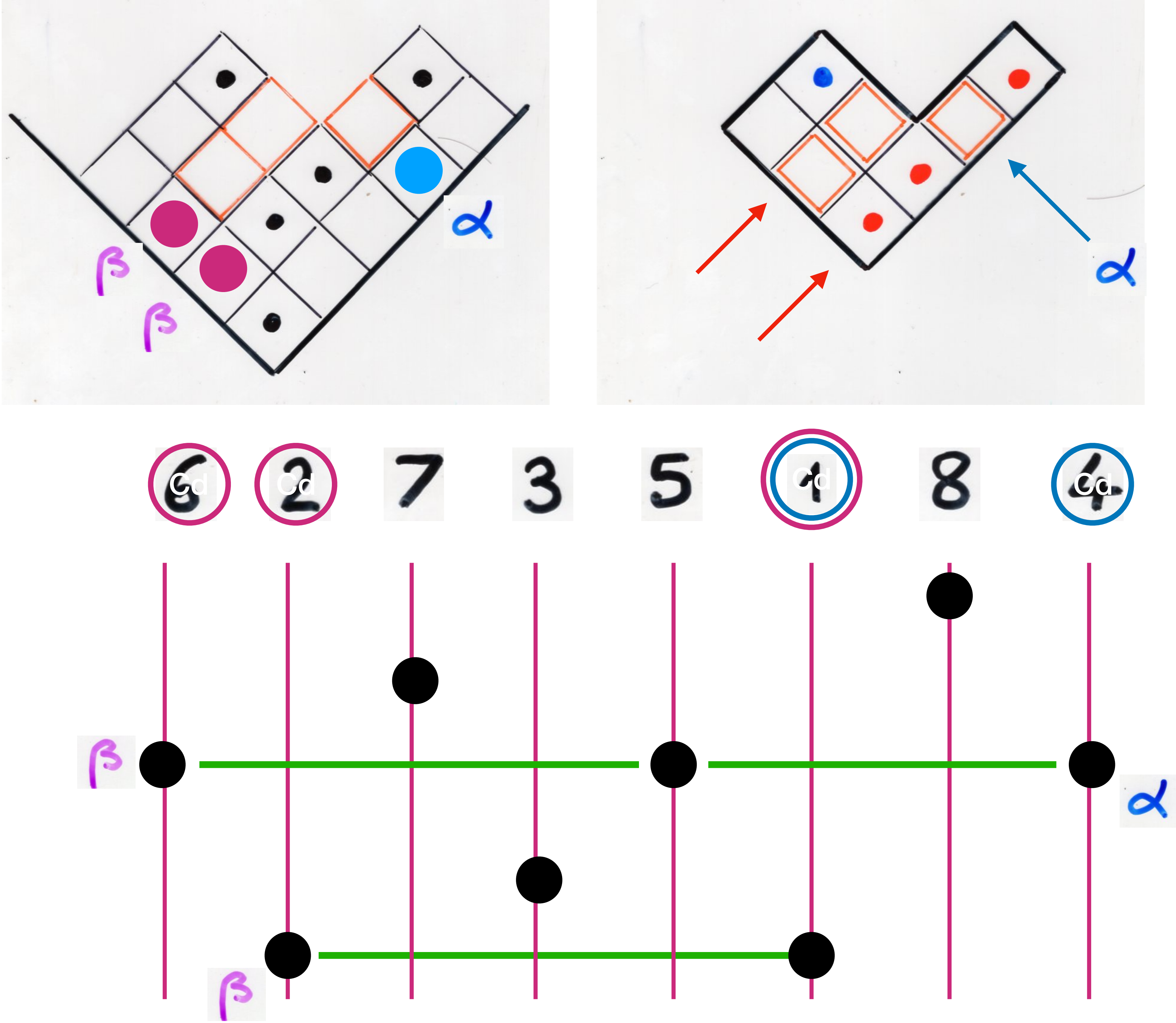

α


α
β
β

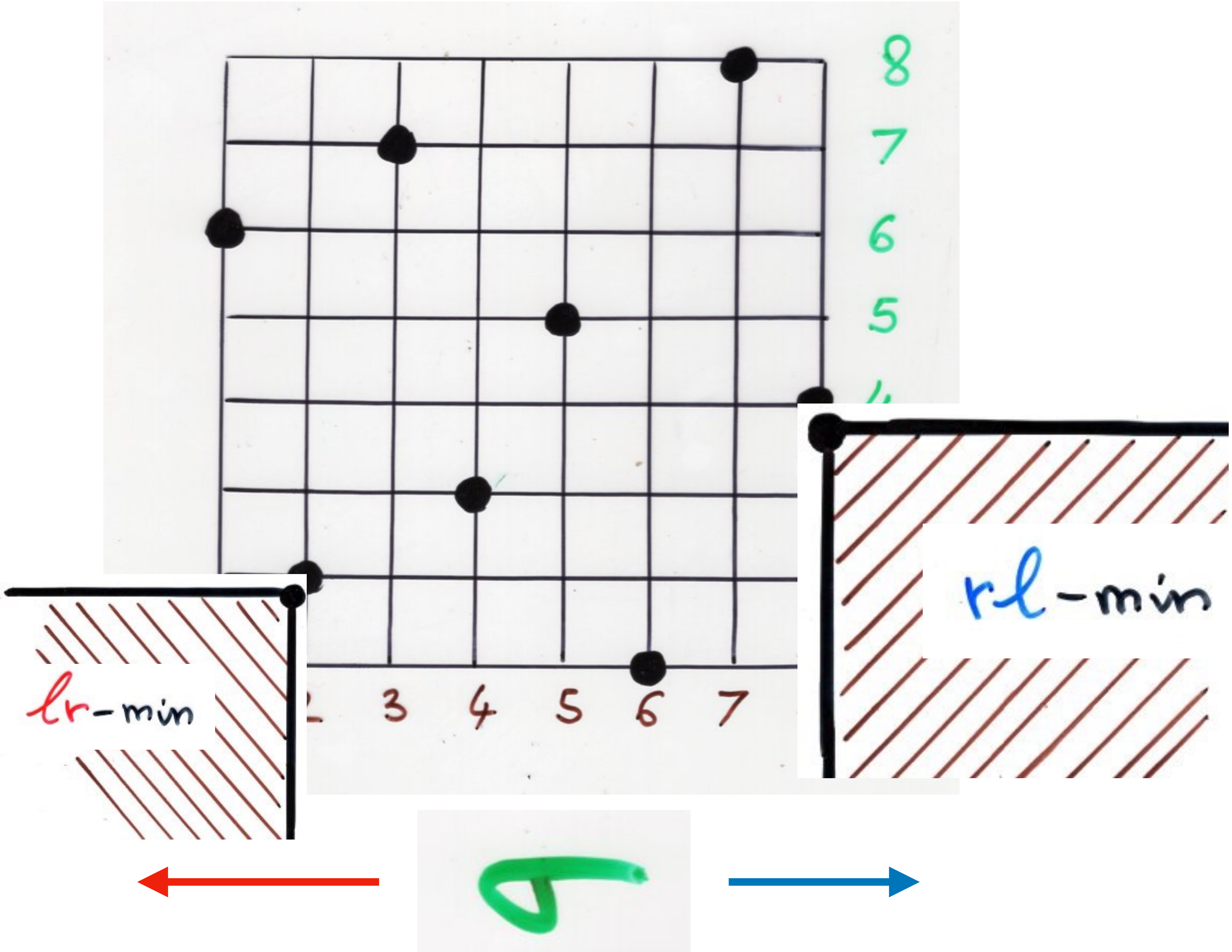

1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4


1
2 1
2 3 1
3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4

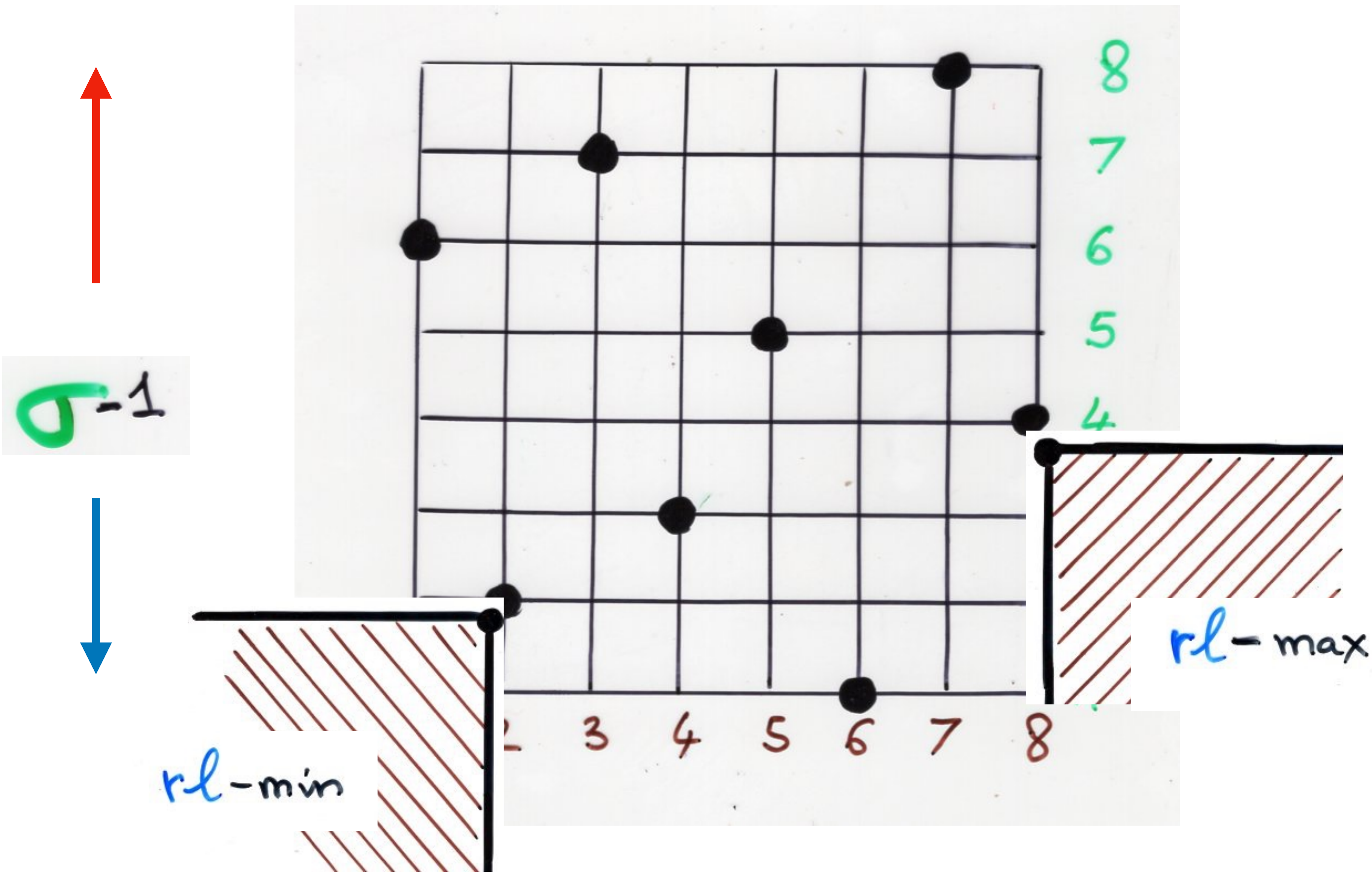

6 2 7 3 5 1 8 4



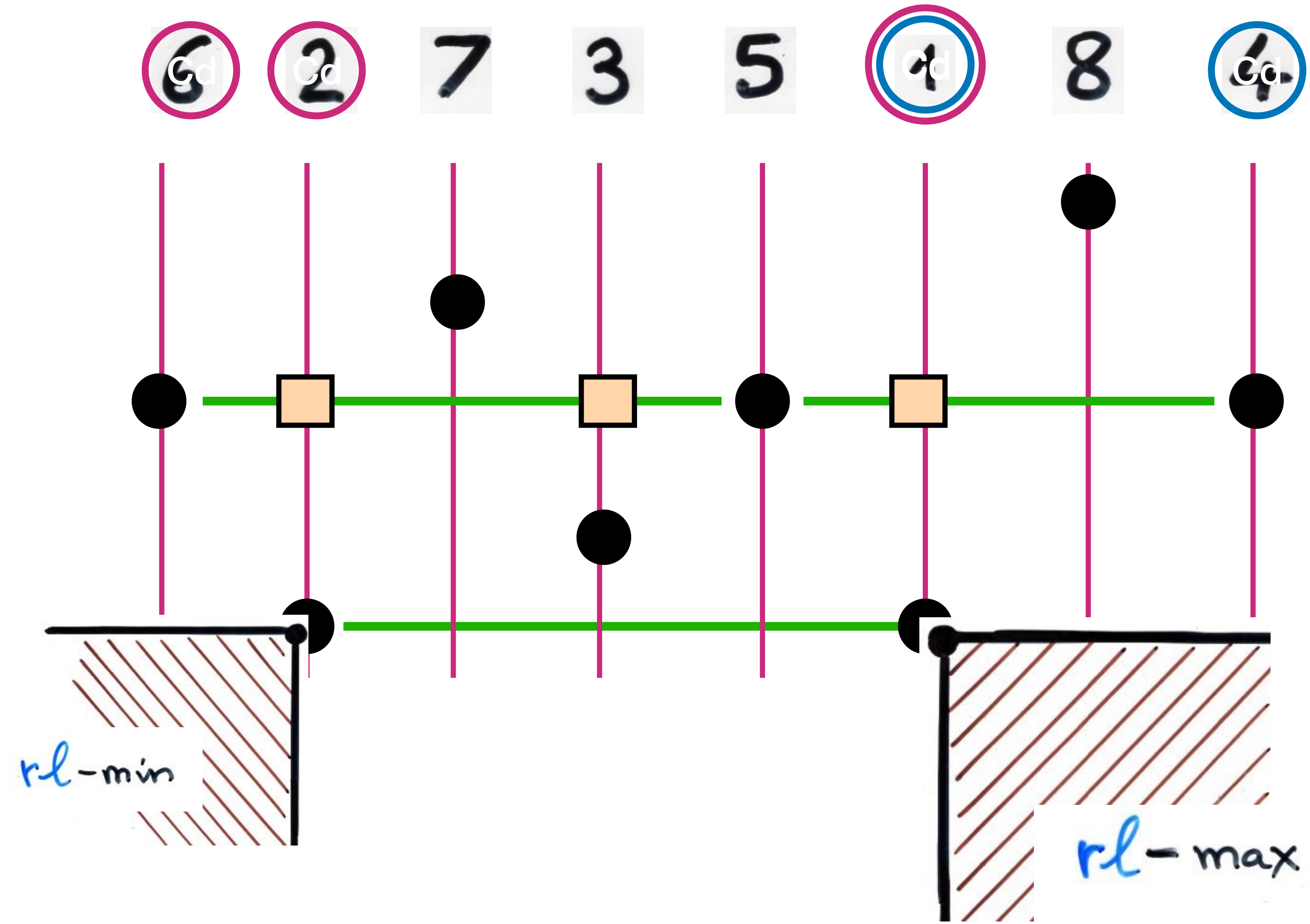
β
β
α
α
6
2
7
3
5
1
8
4
Cd
Cd
Cd
Cd
β
α
β



8
7
6
5
2 3 4 5 6 7
lr-min
rl-min



σ-1
8
7
6
5
4
3
4
5
6
7
8
rl-min
rl-max


6
2
7
3
5
1
8
4
rl-min
rl-max

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Bijection (restricted) Laguerre histories

(of the inverse permutation)

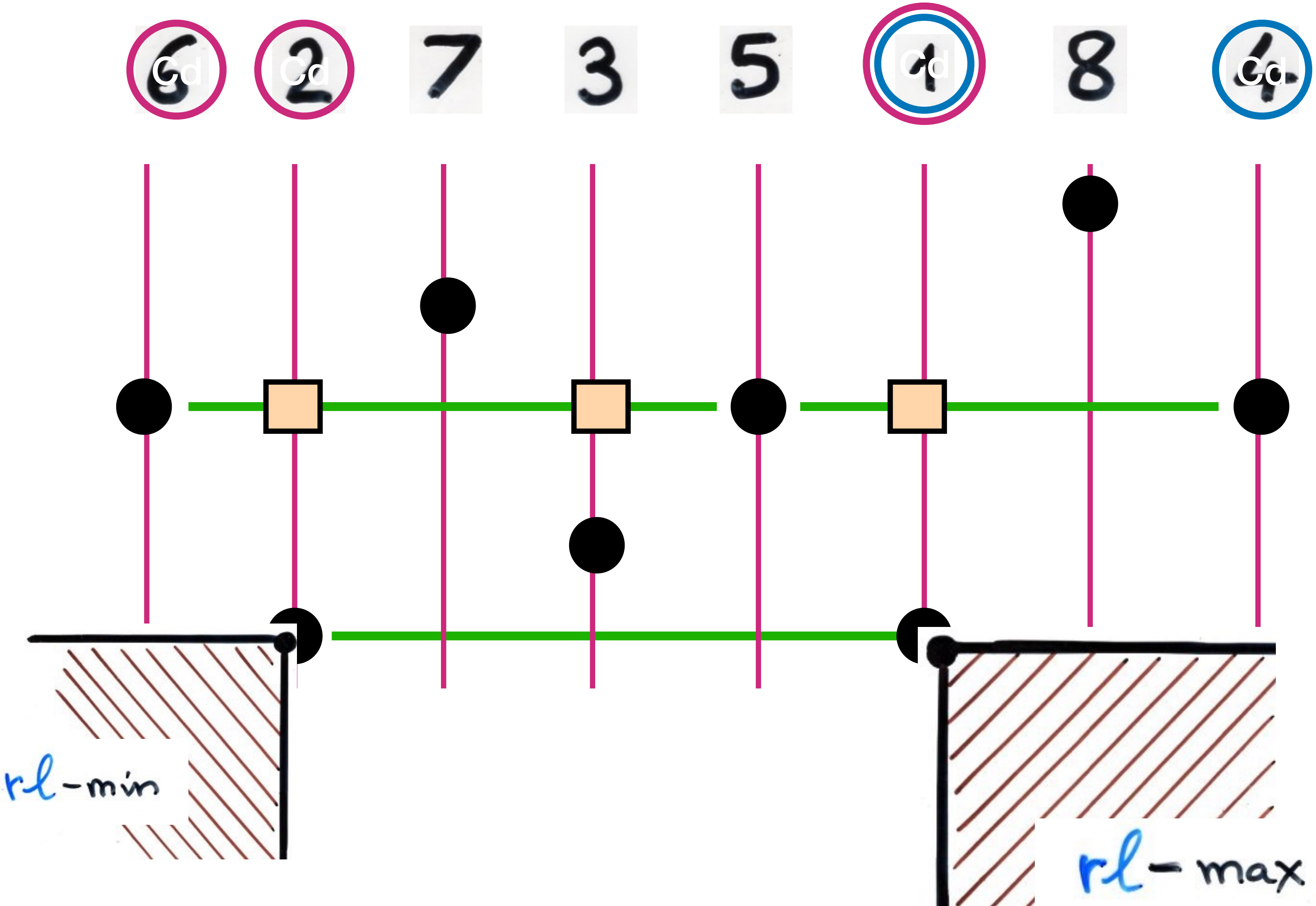
σ^{-1}

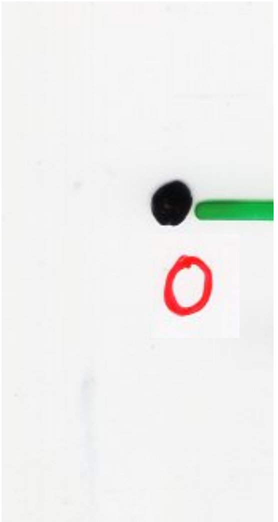
and Laguerre heaps of segments

Josuat-Vergès (2011)

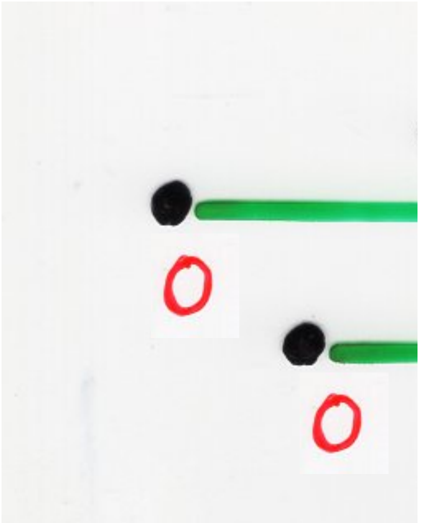
Proposition

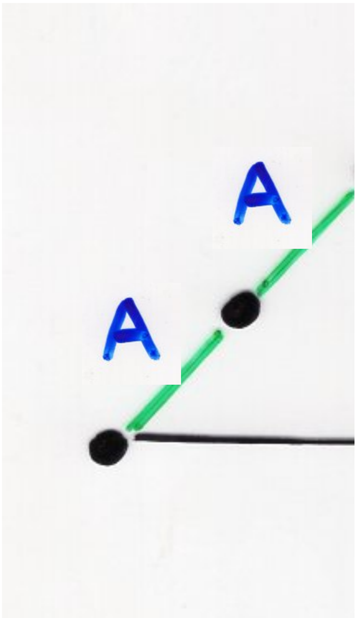
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

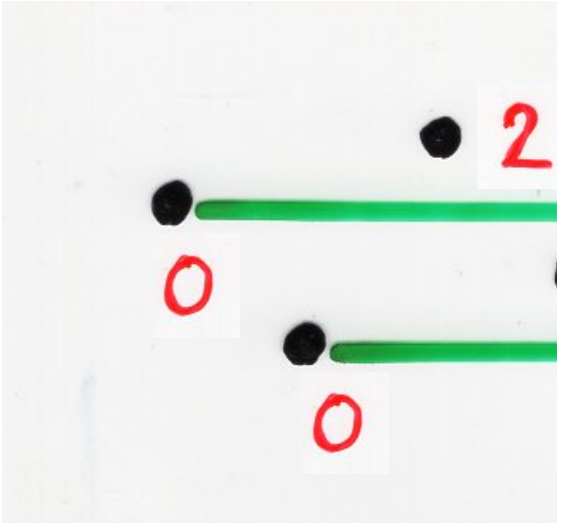



O

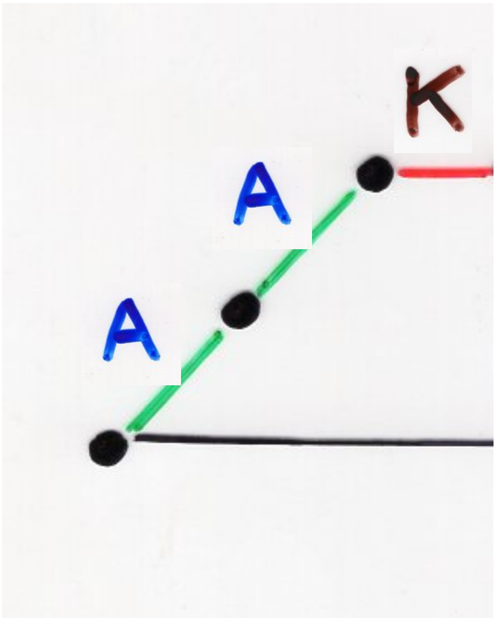

A


O
O

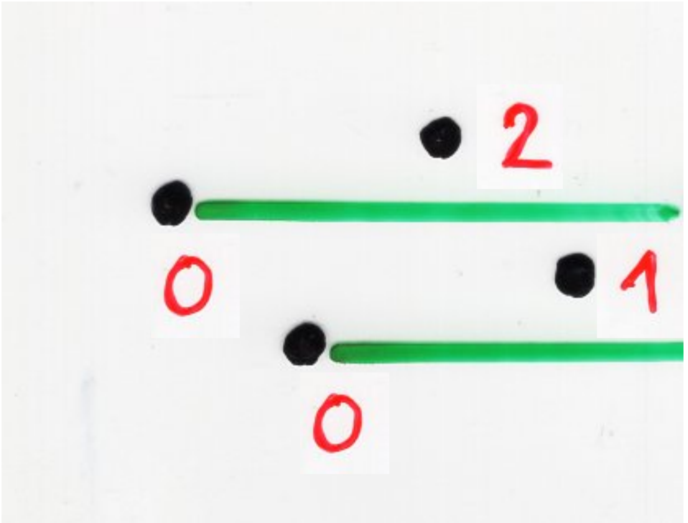

A
A



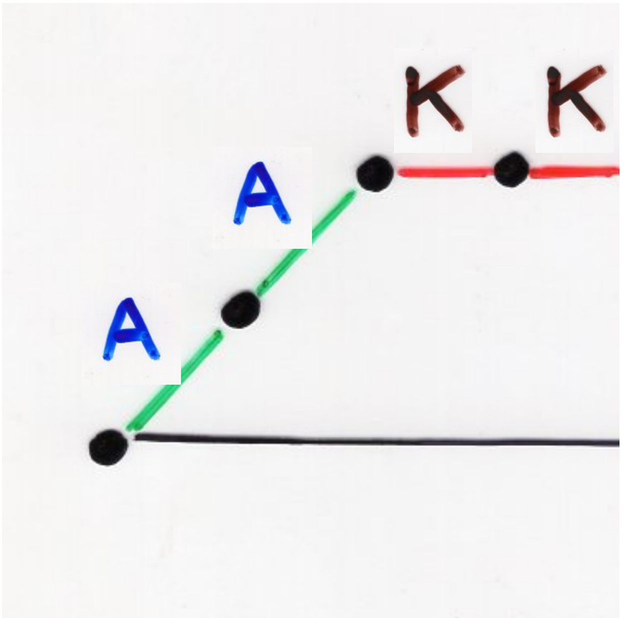
2
0
0



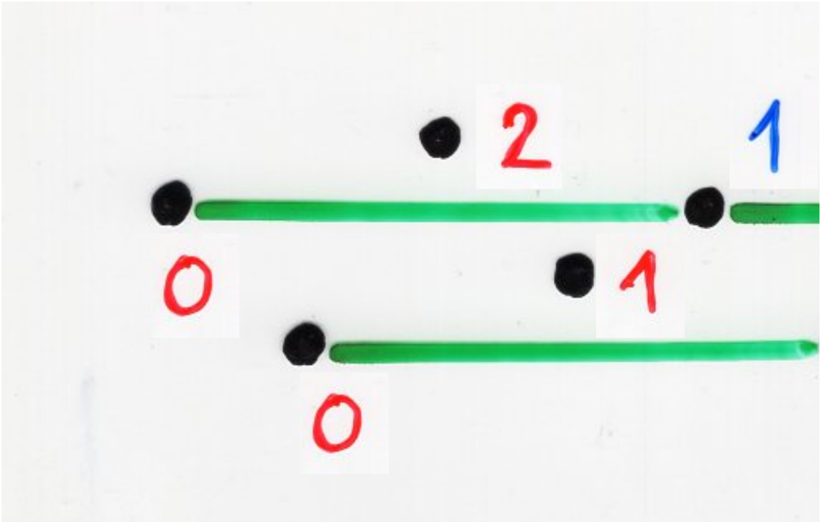
K
A
A

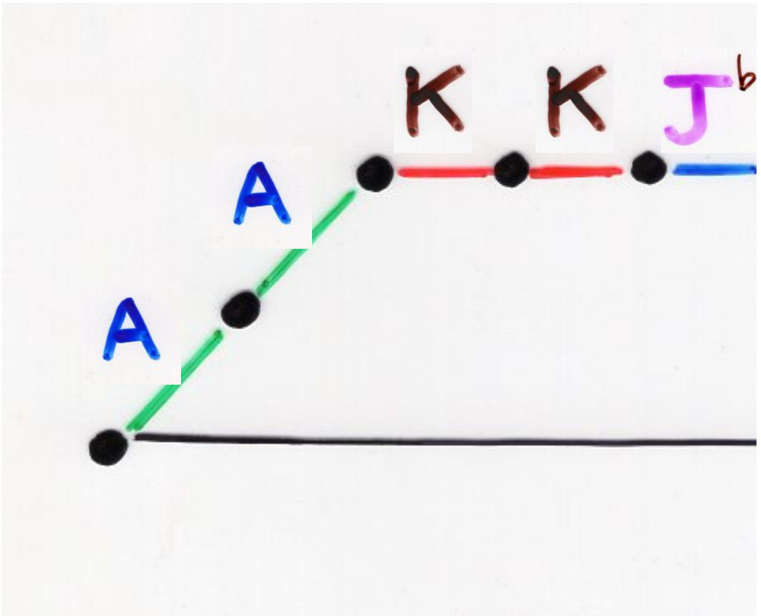


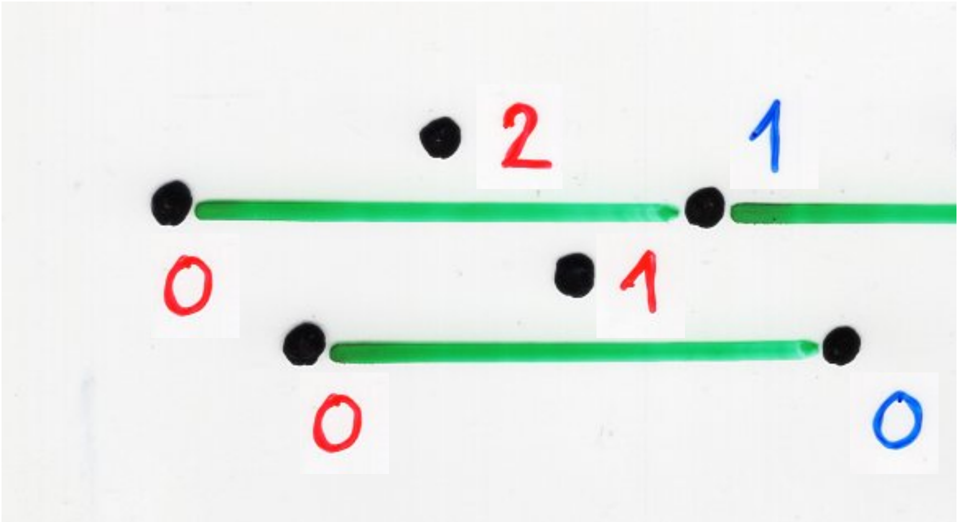
2
0
1
0

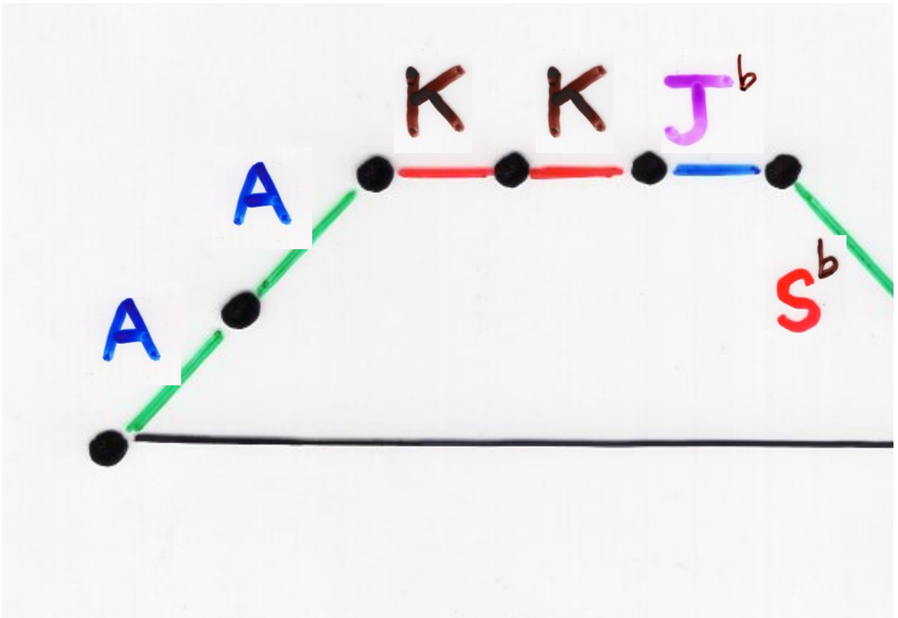


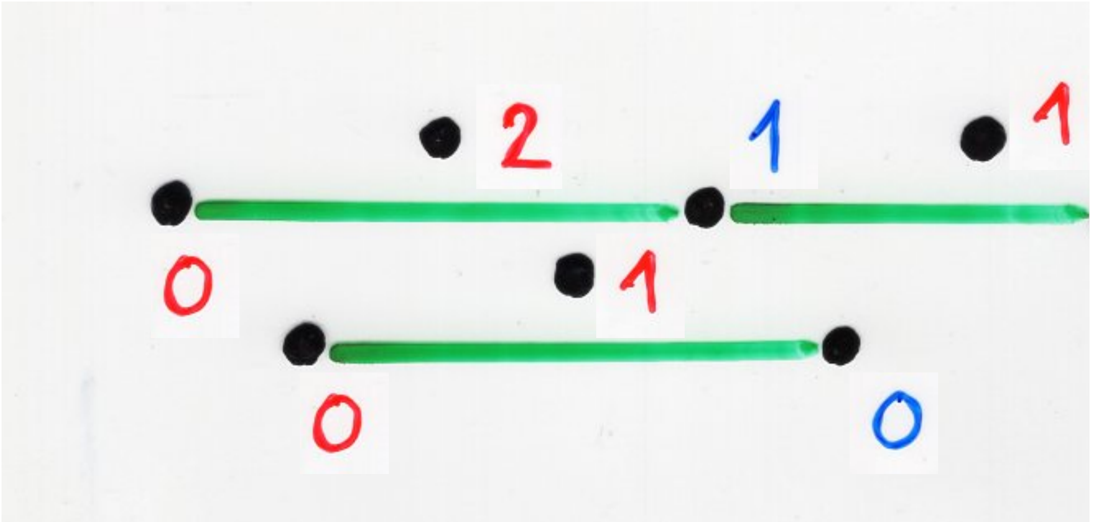
K
K
A
A

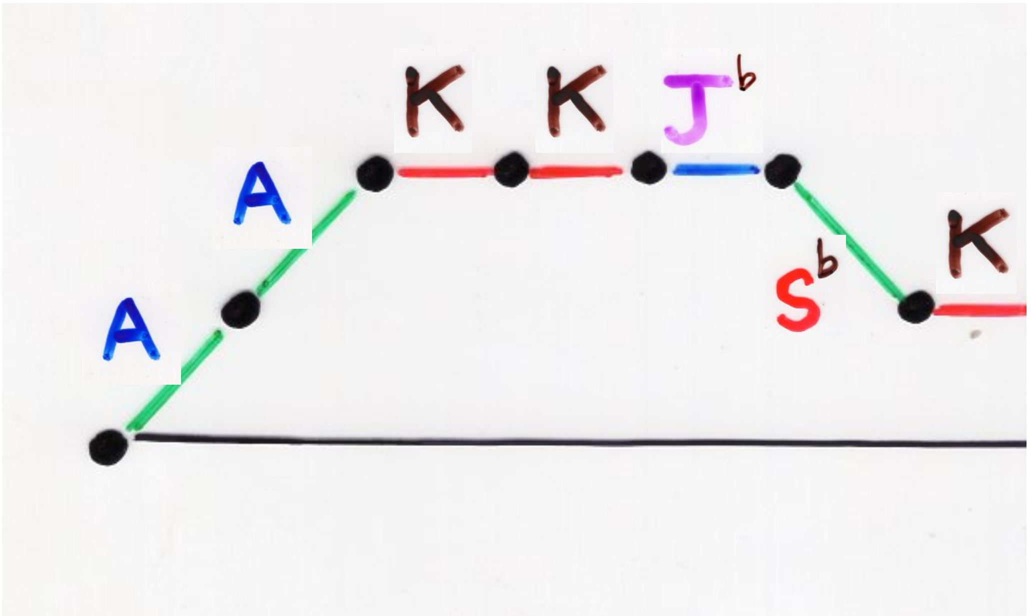

2
1
0
1
0

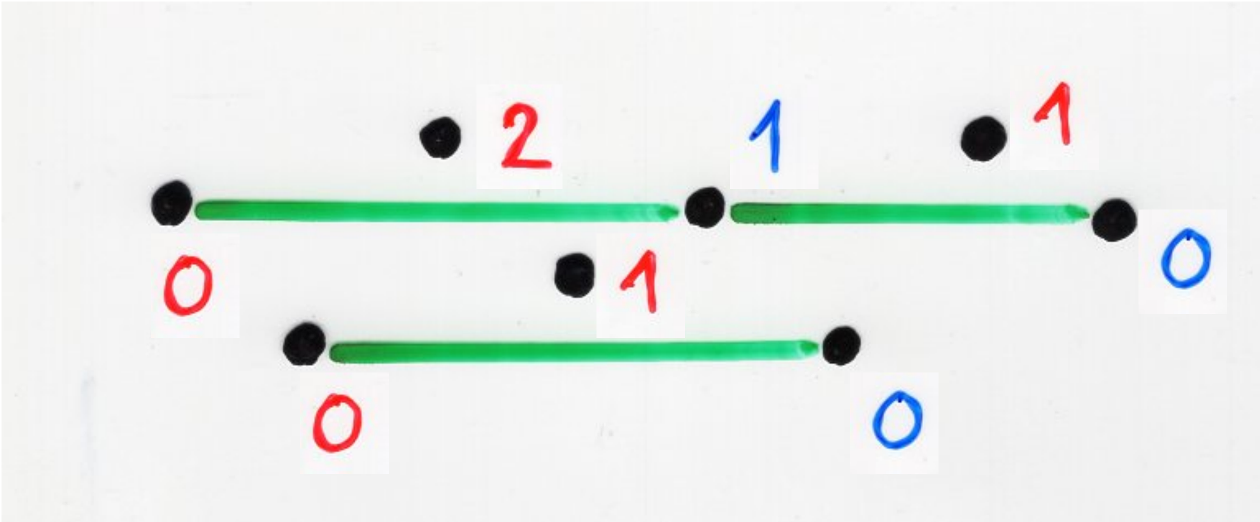

K
K
J
b
A
A


2
1
0
1
0
0

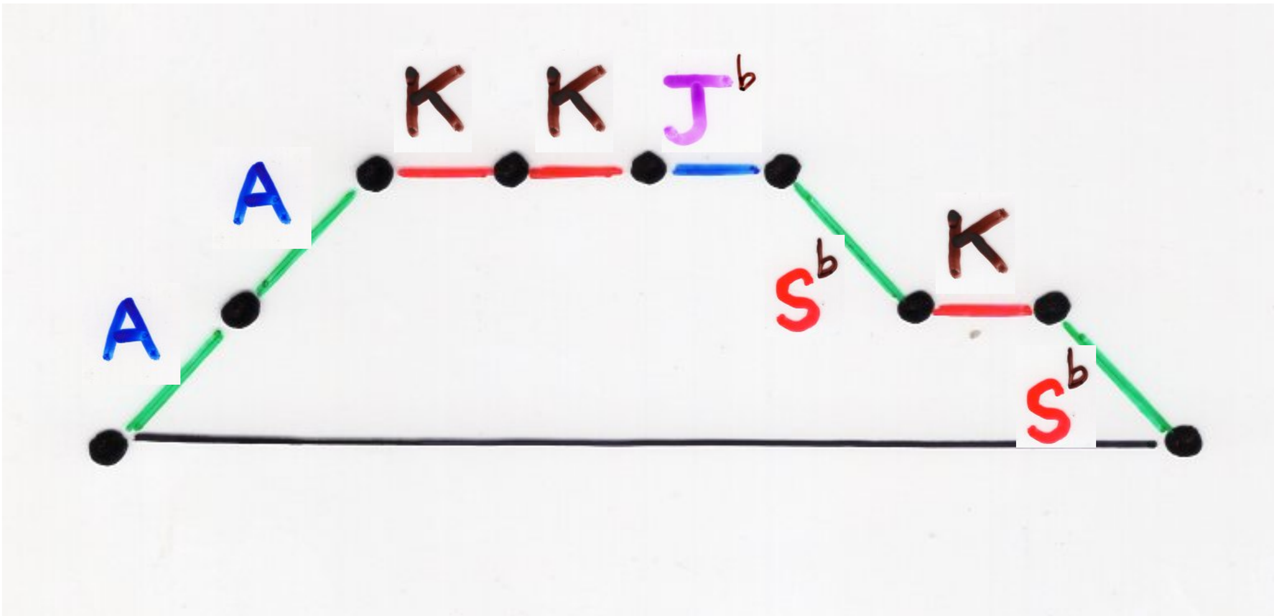

K
K
J
b
A
A
S
b


2
1
1
0
1
0
0

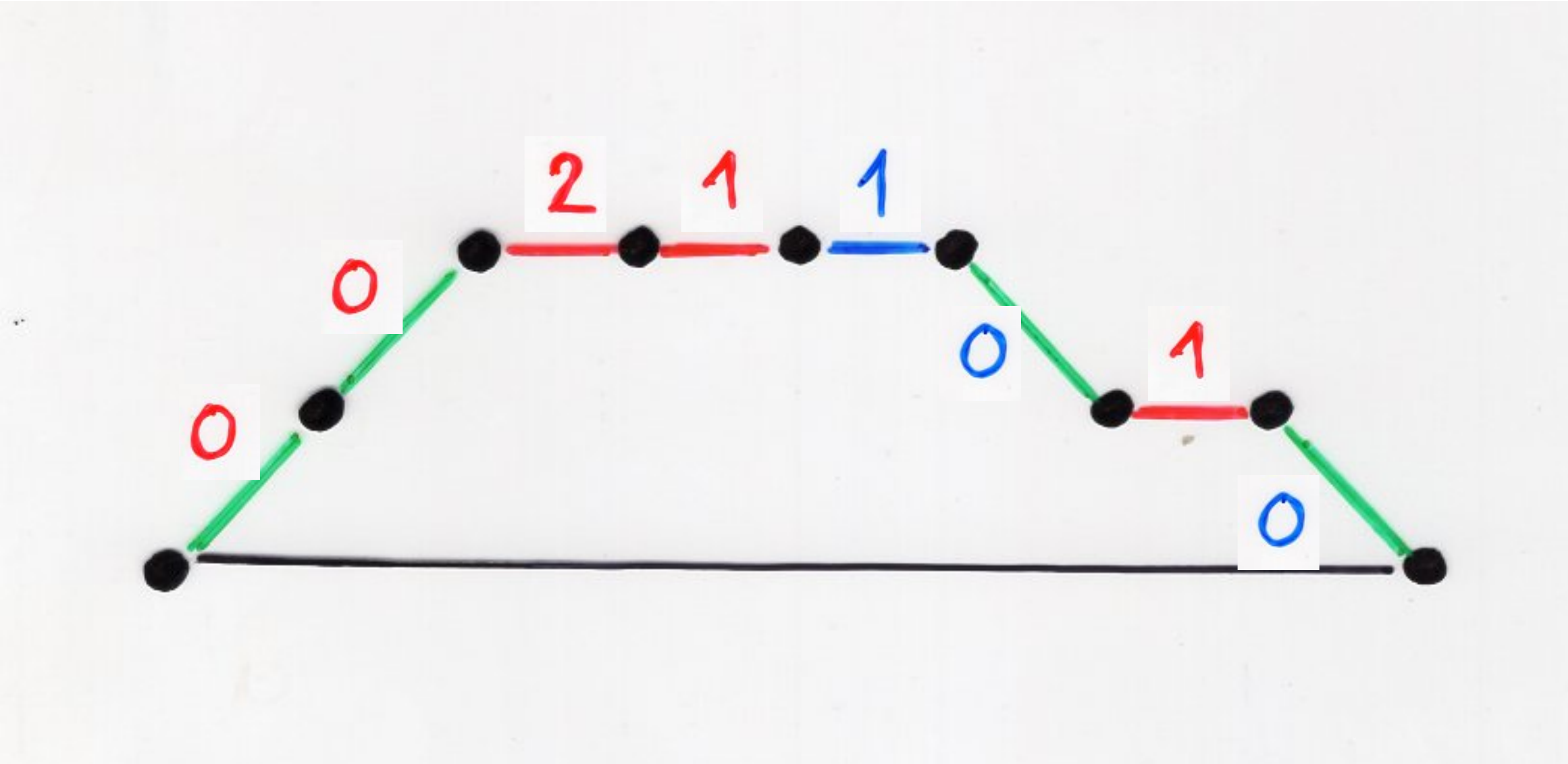

K
K
J b
A
S b
K
A

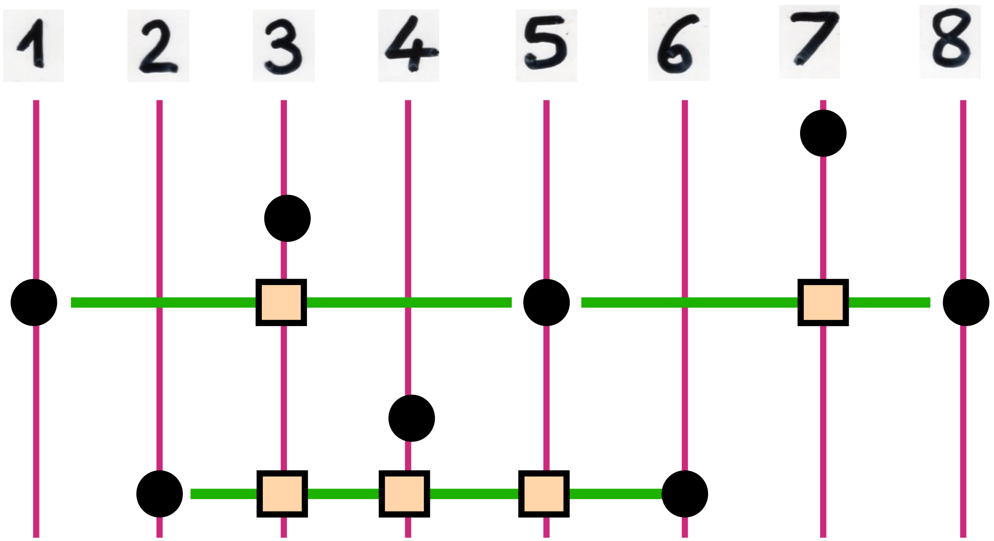


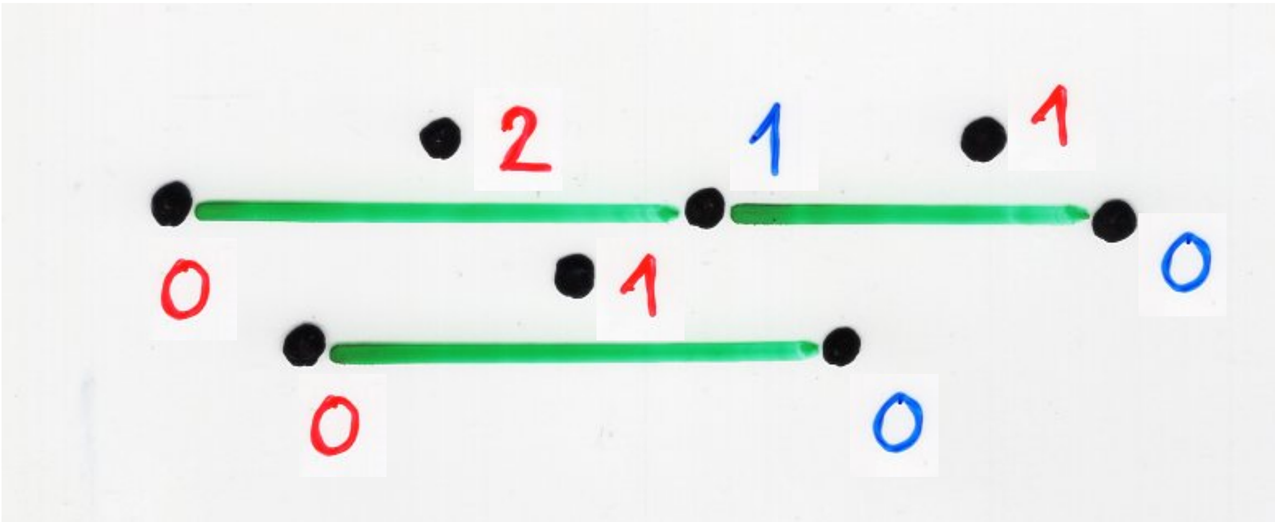
2
1
1
0
0
1
0
0

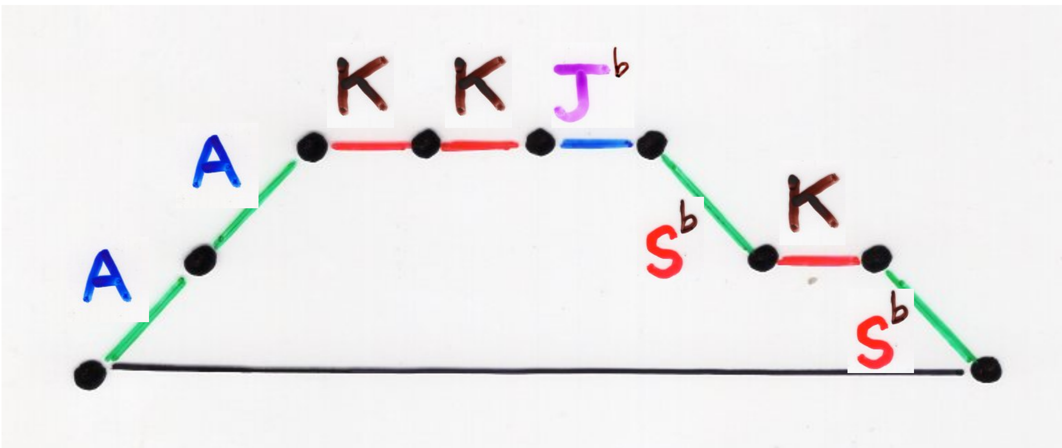


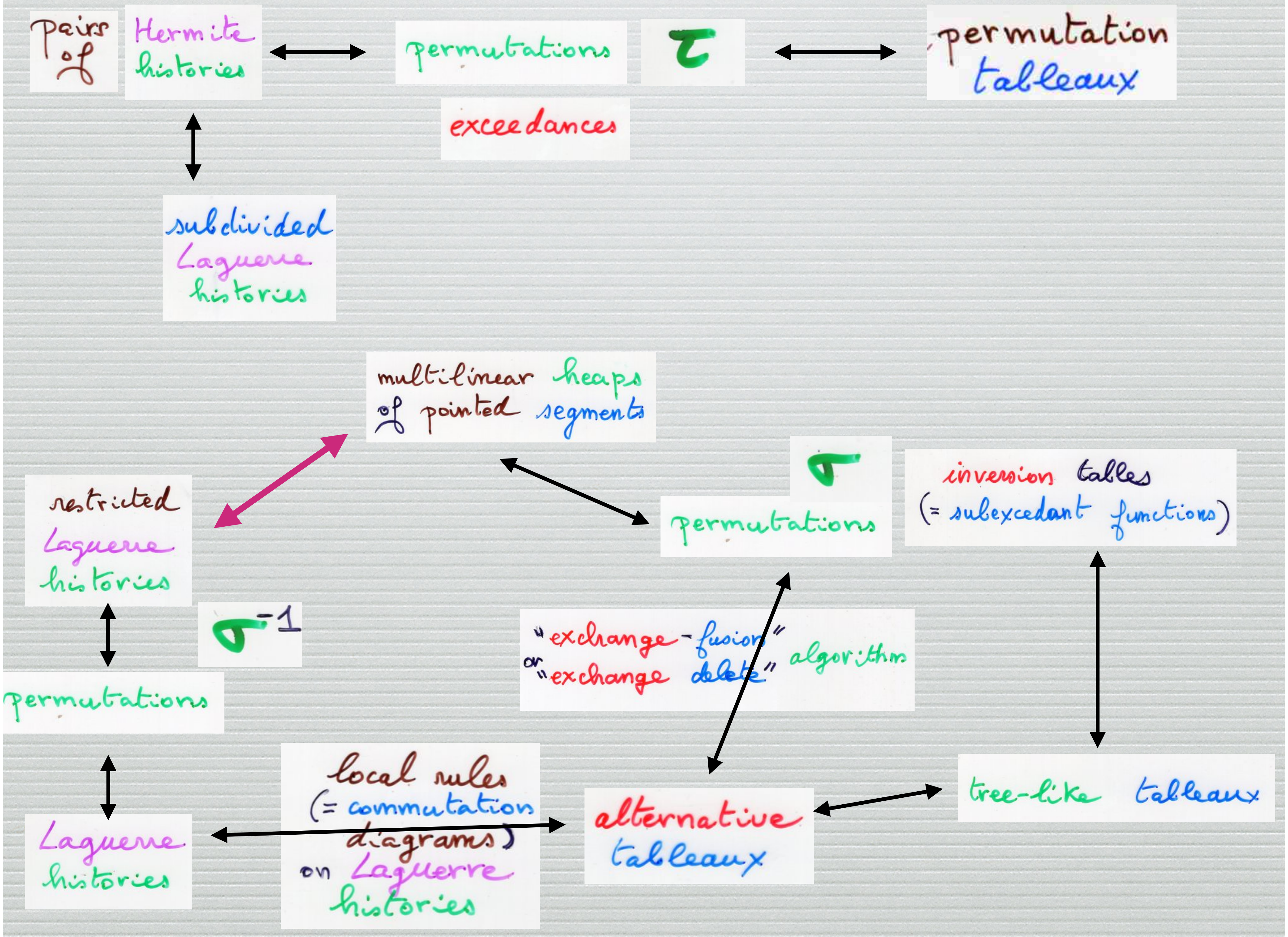
K
K
J^b
A
A
S^b
K
S^b

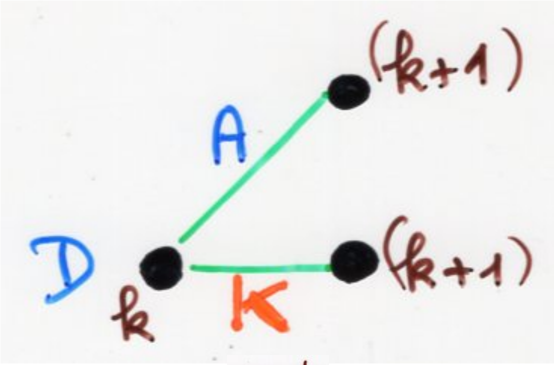

0
0
2
1
1
0
1
0

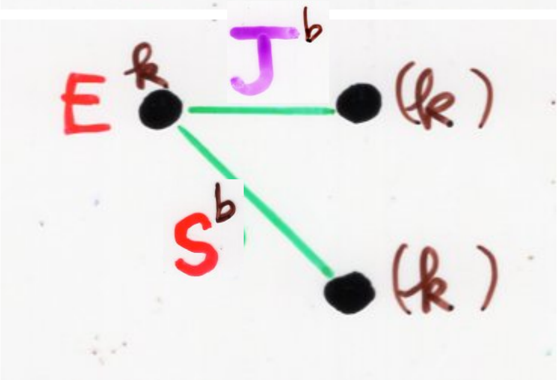

1
2
3
4
5
6
7
8


2
1
1
0
0
1
0
0


K
K
J b
A
A
S b
K
S b


Pairs of Hermite histories
permutations τ
exceedances
permutation tableaux
subdivided Laguerre histories
multilinear heaps of pointed segments
restricted Laguerre histories
σ
permutations
inversion tables (= subexcedant functions)
σ⁻¹
permutations
"exchange-fusion" or "exchange delete" algorithm
local rules (= commutation diagrams) on Laguerre histories
Laguerre histories
alternative tableaux
tree-like tableaux





D, E "restricted"

$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

$$\mu_n = n!$$

$$b_k = (2k+1)$$
$$\lambda_k = k^2$$

restricted
Laguerre
histories

$$\sigma(1) = (n+1)$$

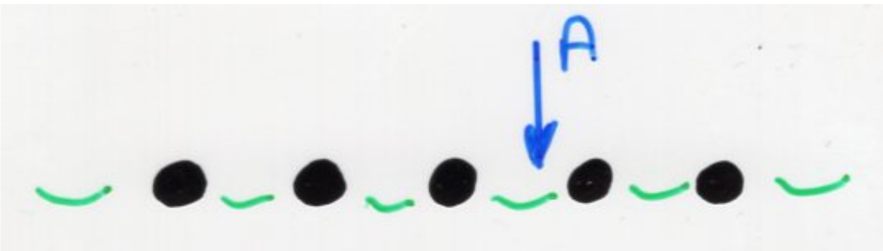
dictionnary data structure

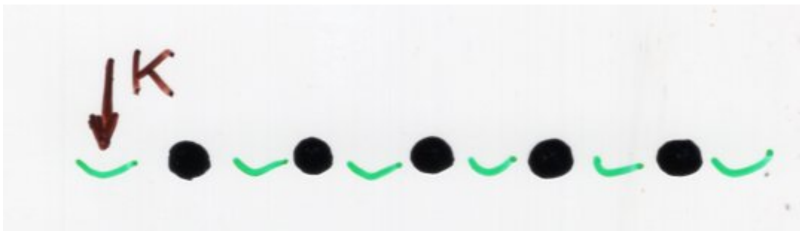
add or delete any element

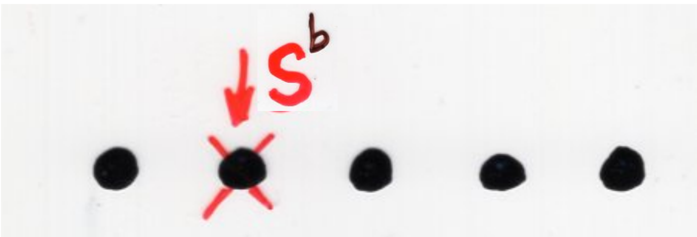
ask questions

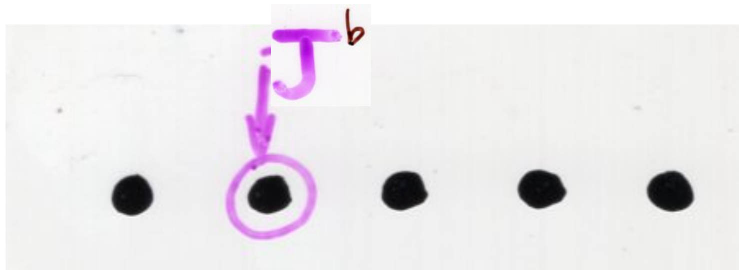
J^b positive

K negative









$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

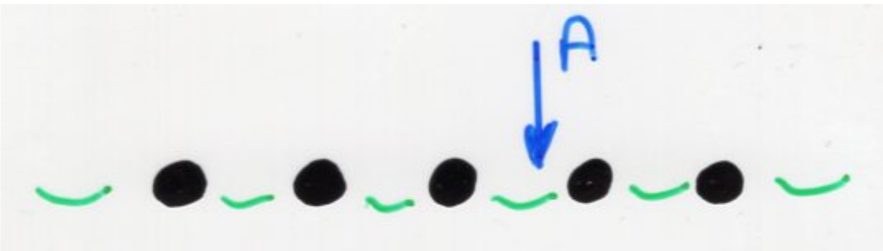
$$DE = ED + E + D$$

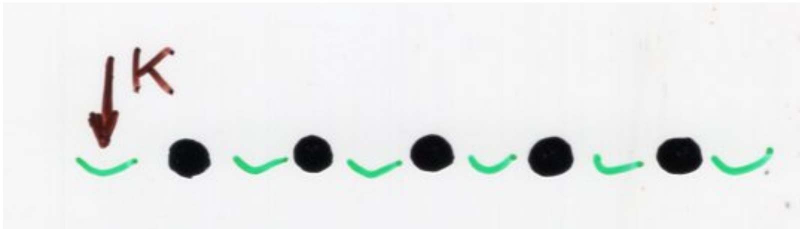
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

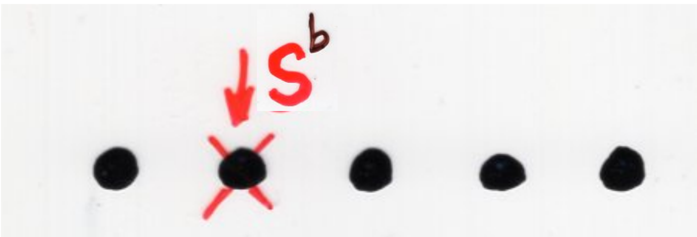
$$J^b\,|k\rangle = k\,|k\rangle$$

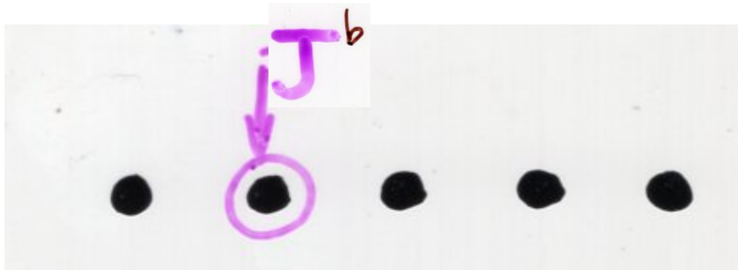
$$K\,|k\rangle = (k+1)\,|k\rangle$$

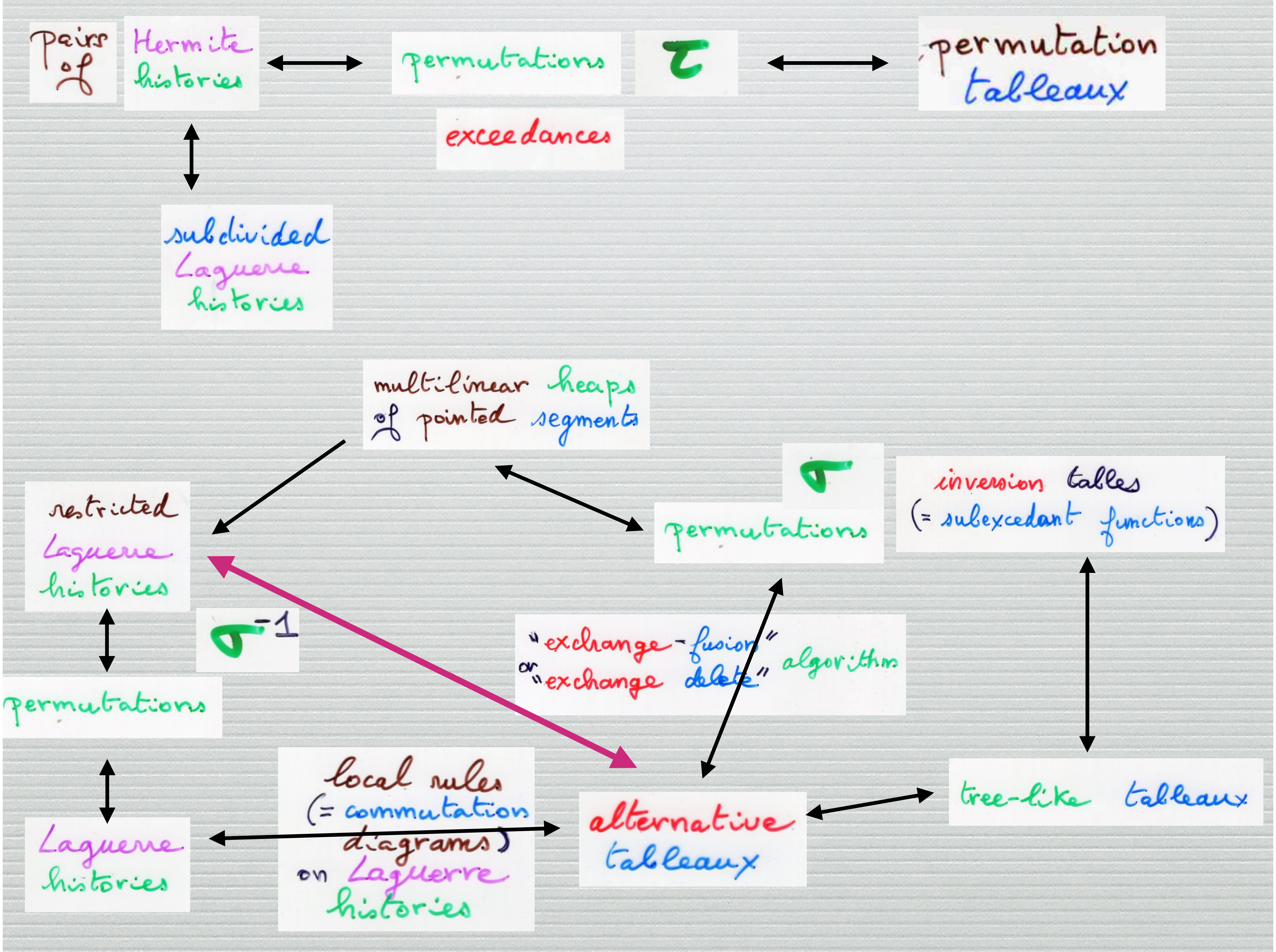
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$










Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
multilinear heaps of pointed segments
σ
permutations
inversion tables (= subexcedant functions)
restricted Laguerre histories
σ^{-1}
"exchange-fusion" or "exchange delete" algorithms
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

What about

the approach by physicists ?

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

$$UD = qDU + I$$

$$DE = qED + E + D$$

$$\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$$

$$\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$$

$$D = (D_{i,j})_{i,j \in \mathbb{N}}$$

$$E = (E_{i,j})_{i,j \in \mathbb{N}}$$

$$(1-q)\, D_{i,i} = 1 + \tilde{\beta} q^i$$

$$(1-q)\, D_{i,i+1} = 1 - \tilde{\alpha}\tilde{\beta} q^i$$

$$(1-q)\, E_{i,i} = 1 + \tilde{\alpha} q^i$$

$$(1-q)\, E_{i+1,i} = 1 - q^{i+1}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(\quad , q)\, B_n(\tilde{\alpha}, \tilde{\beta}, y, q)$$

$$R_{N,n}(\quad q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha}, \tilde{\beta}, \quad q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (\quad \tilde{\beta})^{n-k}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(y,q) B_n(\tilde{\alpha},\tilde{\beta},y,q)$$

$$R_{N,n}(1,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha},\tilde{\beta},y,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (y\tilde{\beta})^{n-k}$$

Josuat-Vergès (2011)

$$R_{N,n}(y,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-y)^i q^{\binom{i+1}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \sum_{j=0}^{N-n-2i} y^j \left(\binom{N}{j}\binom{N}{n+2i+j} - \binom{N}{j-1}\binom{N}{n+2i+j+1} \right)$$

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

<u>Proposition</u>

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

Al-Salam-Chihara polynomials

$$2x\, Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1}) Q_{n-1}(x)$$

PASEP
with 5 parameters

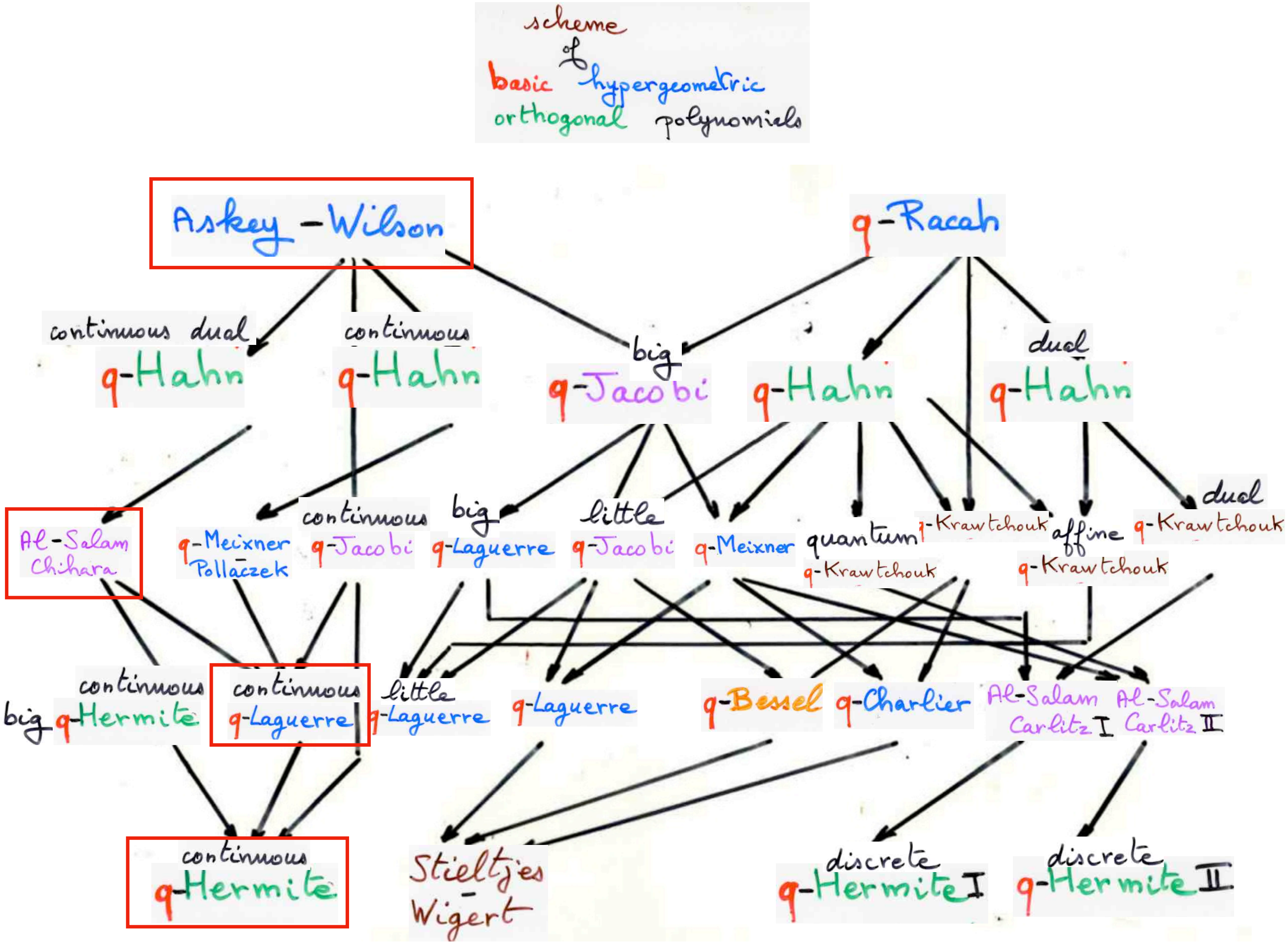
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009) staircase tableaux



scheme of basic hypergeometric orthogonal polynomials
Askey-Wilson
q-Racah
continuous dual q-Hahn
continuous q-Hahn
big q-Jacobi
q-Hahn
dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
continuous q-Jacobi
big q-Laguerre
little q-Jacobi
q-Meixner
quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes Wigert
discrete q-Hermite I
discrete q-Hermite II

Z_n partition function

Askey-Wilson polynomials

$$P_n(x) = P_n(x; a, b, c, d \mid q)$$

$$P_n(x) = a^{-n} (ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k}$$

$$(a_1, a_2, \ldots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k)$$

${}_4\phi_3$ basic hypergeometric function

$$A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) = 2x P_n(x)$$

$$A_n = \frac{1 - q^{n-1}abcd}{(1-q^{2n-1}abcd)(1-q^{2n}abcd)}$$

$$B_n = \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)}$$

$$\left[(1+q^{2n-1}abcd)(qs+abcd\,s') - q^{n-1}(1+q)abcd(s+qs')\right]$$

$$C_n = \frac{(1-q^n)(1-q^{n-1}ab)(1-q^{n-1}ac)(1-q^{n-1}ad)(1-q^{n-1}bc)(1-q^{n-1}bd)(1-q^{n-1}cd)}{(1-q^{2n-2}abcd)(1-q^{2n-1}abcd)}$$

$$s = a+b+c+d \qquad s' = a^{-1}+b^{-1}+c^{-1}+d^{-1}$$

$$D = \frac{1}{1-q}(1+d)$$

$$E = \frac{1}{1-q}(1+e)$$

$$de - qed = (1-q)\,\mathrm{Id}$$

$$DE = qED + E + D$$

$$d = \begin{bmatrix} d_0^{\natural} & d_0^{\sharp} & 0 & & \\ d_0^{\flat} & d_1^{\natural} & d_1^{\sharp} & & 0 \\ 0 & d_1^{\flat} & d_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$e = \begin{bmatrix} e_0^{\natural} & e_0^{\sharp} & 0 & & \\ e_0^{\flat} & e_1^{\natural} & e_1^{\sharp} & & 0 \\ 0 & e_1^{\flat} & e_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$D = \frac{1}{1-q}(1+d)$$

$$de - qed = (1-q)\,Id$$

$$E = \frac{1}{1-q}(1+e)$$

$$DE = qED + E + D$$

$$UD = qDU + I$$

$$DE = qED + E + D$$

$$d = \begin{bmatrix} d_0^{\natural} & d_0^{\sharp} & 0 & & \\ d_0^{\flat} & d_1^{\natural} & d_1^{\sharp} & & 0 \\ 0 & d_1^{\flat} & d_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$e = \begin{bmatrix} e_0^{\natural} & e_0^{\sharp} & 0 & & \\ e_0^{\flat} & e_1^{\natural} & e_1^{\sharp} & & 0 \\ 0 & e_1^{\flat} & e_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$d_k^4 = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[bd(a+c) + (b+d)q - abcd(b+d)q^{k-1}$$

$$-\Big(bd(a+c) + (abcd)(b+d)\Big)q^{k}$$

$$- bd(a+c)q^{k+1} + ab^2cd^2(a+c)q^{2k-1} + abcd(b+d)q^{2k}\Big]$$

$$e_k^4 = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[ac(b+d) + (a+c)q - abcd(a+c)q^{k-1}$$

$$-\Big(ac(b+d) + abcd(a+c)\Big)q^{k}$$

$$- ac(b+d)q^{k+1} + a^2bc^2d(b+d)q^{2k-1} + abcd(a+c)q^{2k}\Big]$$

$$d_k^{\sharp} = \frac{1}{1-q^k ac}\mathcal{A}_k$$

$$d_k^{\flat} = -\frac{q^k bd}{1-q^k bd}\mathcal{A}_k$$

$$e_k^{\sharp} = -\frac{q^k ac}{1-q^k ac}\mathcal{A}_k$$

$$e_k^{\flat} = \frac{1}{1-q^k bd}\mathcal{A}_k$$

$$A_k = \left[\frac{(1-q^{k-1}abcd)(1-q^{k+1})(1-q^k ab)(1-q^k ad)}{(1-q^{2k-1}abcd)(1-q^{2k}abcd)^2}\;\frac{(1-q^k ac)(1-q^k bc)(1-q^k bd)(1-q^k cd)}{(1-q^{2k+1}abcd)}\right]^{1/2}$$

$$\langle W| = h_0^{1/2}(1, 0, 0, \ldots)$$

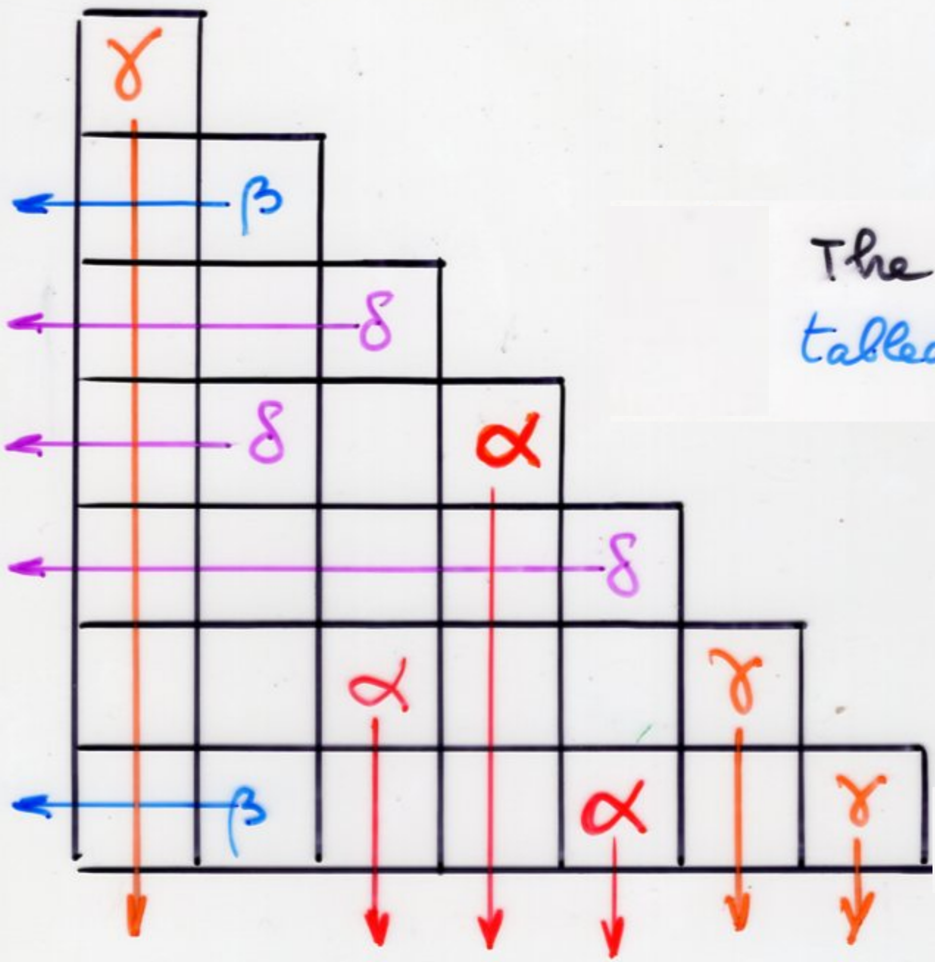
$$|V\rangle = h_0^{1/2}(1, 0, 0, \ldots)^T$$

$$h_0 = \left[(1-q)(1-q^2)\ldots\right]^{-1/2}$$

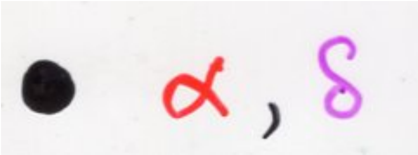
$$(q;q)_\infty$$

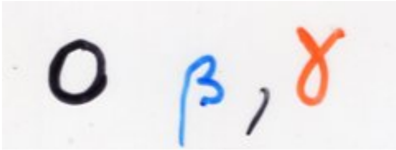
S. Corteel, L. Williams (2009)

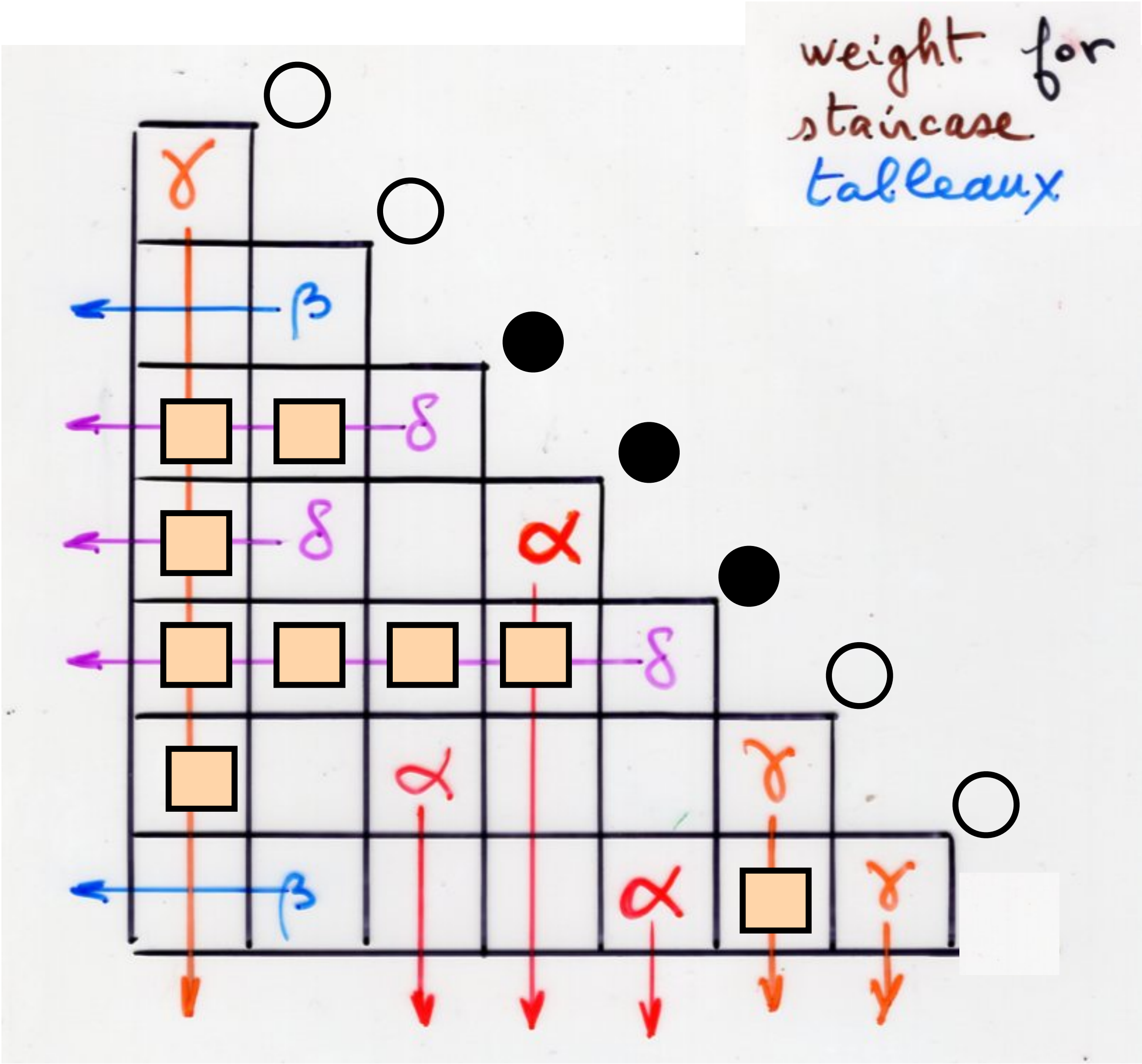
staircase tableaux

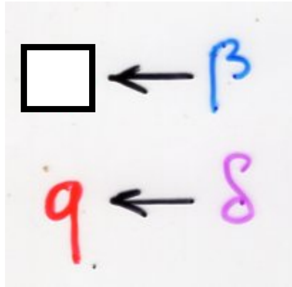


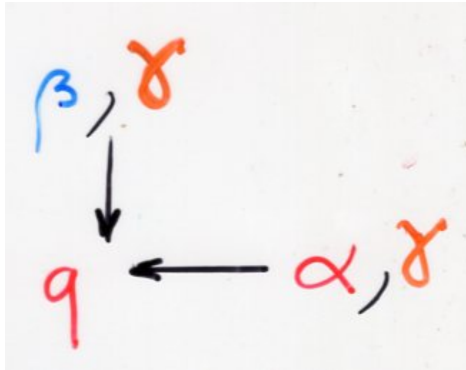
The number of staircase tableaux of size n is $4^n \, n!$

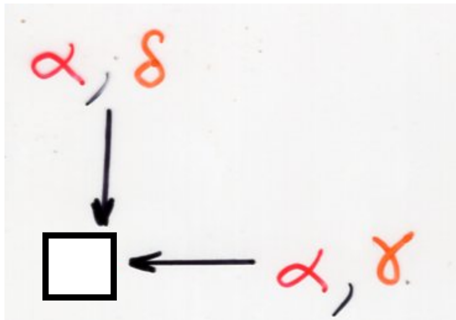

α, δ


β, γ


weight for
staircase
tableaux
γ
β
δ
δ
α
δ
α
γ
β
α
γ


β
q
δ


β, γ
q
α, γ


α, δ
α, γ

$\tau = (\tau_1, \dots, \tau_n)$

$$Z_\tau = \sum_{T} v(T)$$

profile of T

T staircase tableaux size n

S. Corteel, L. Williams (2009)

$$Z_n(\alpha, \beta, \gamma, \delta; q) = \sum_{T} v(T)$$

partition function

T staircase tableaux size n

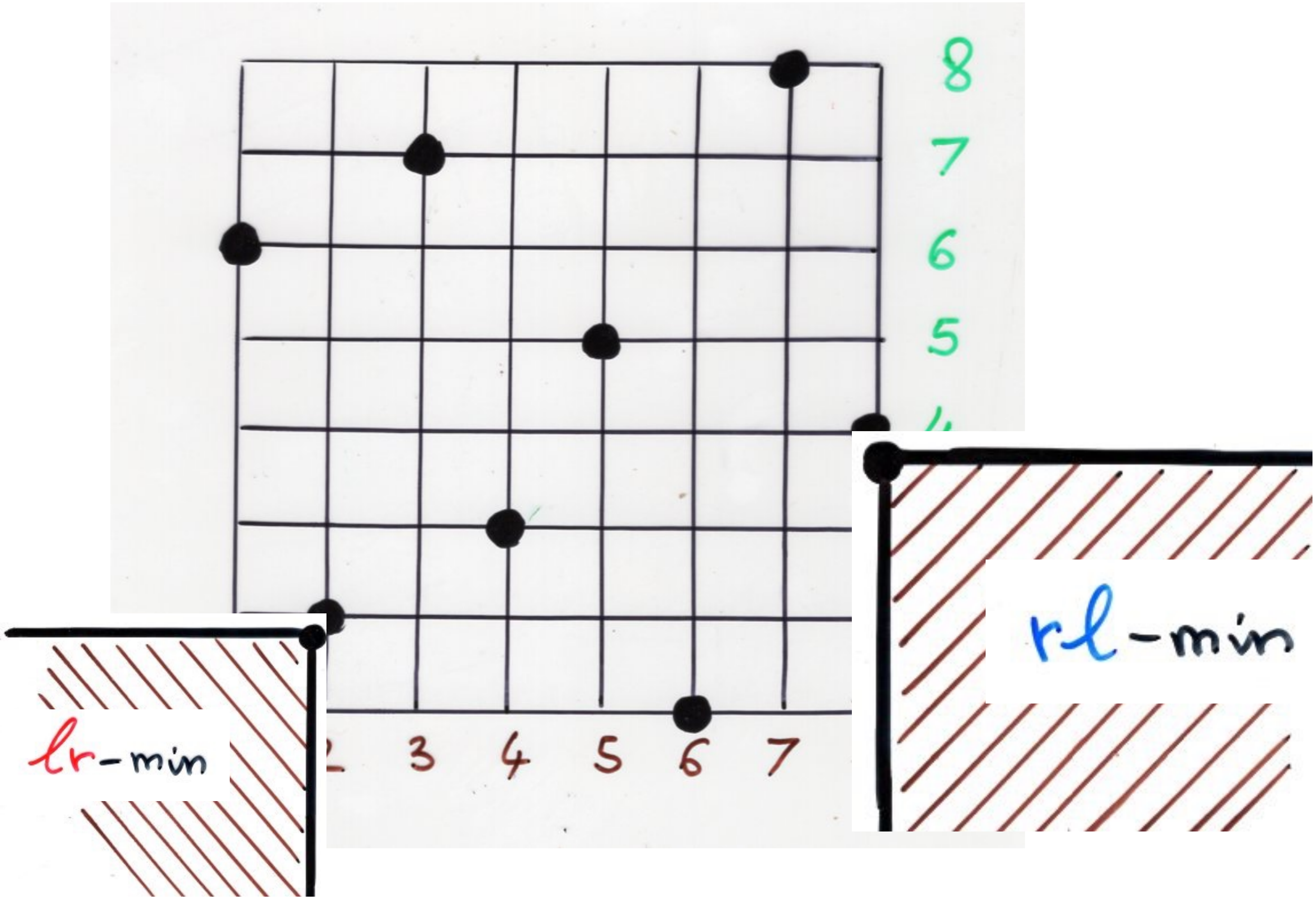
→ expression for the moments of the Askey-Wilson polynomials

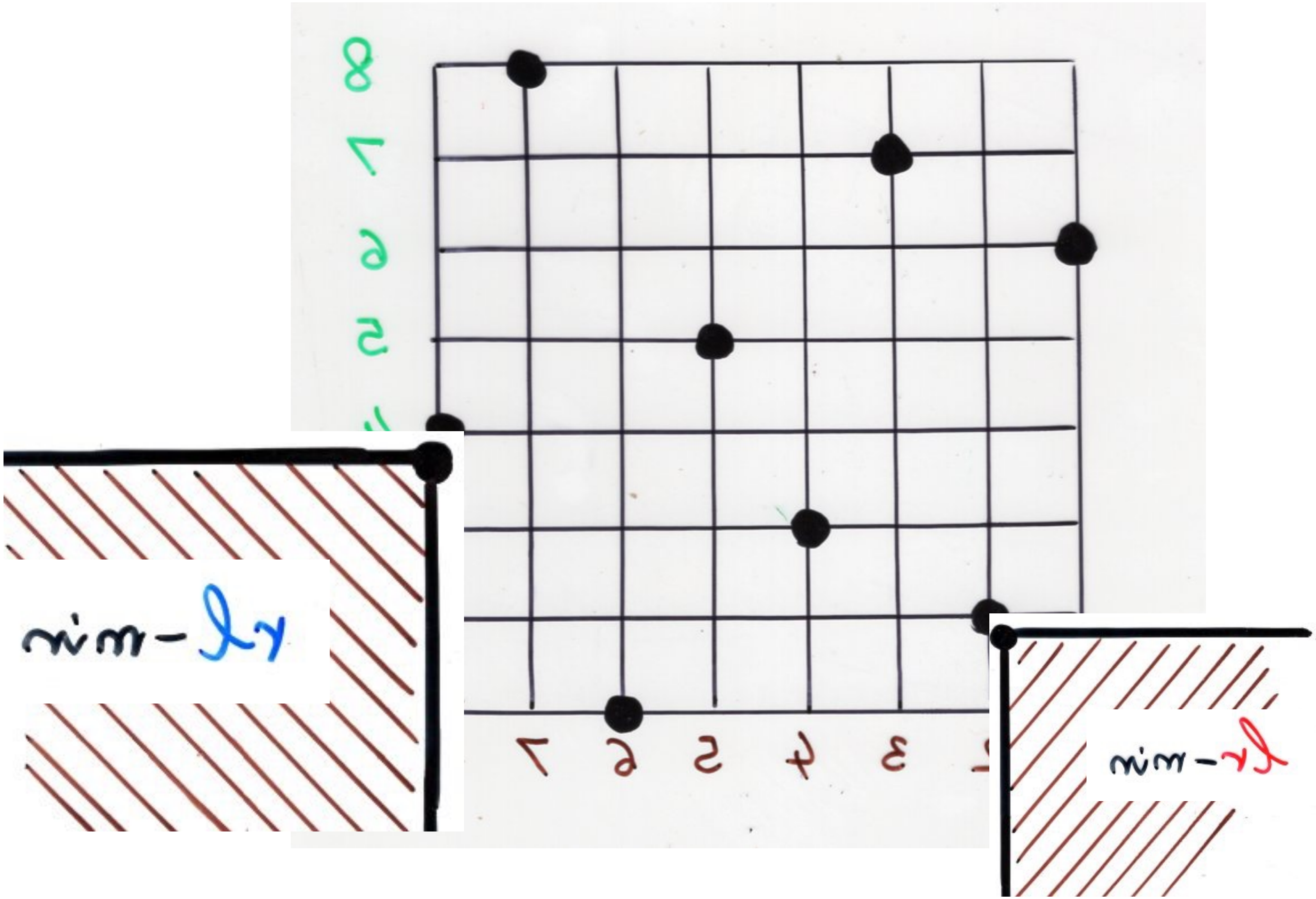
S. Corteel, L. Williams
R. Stanley, D. Stanton
(2010)

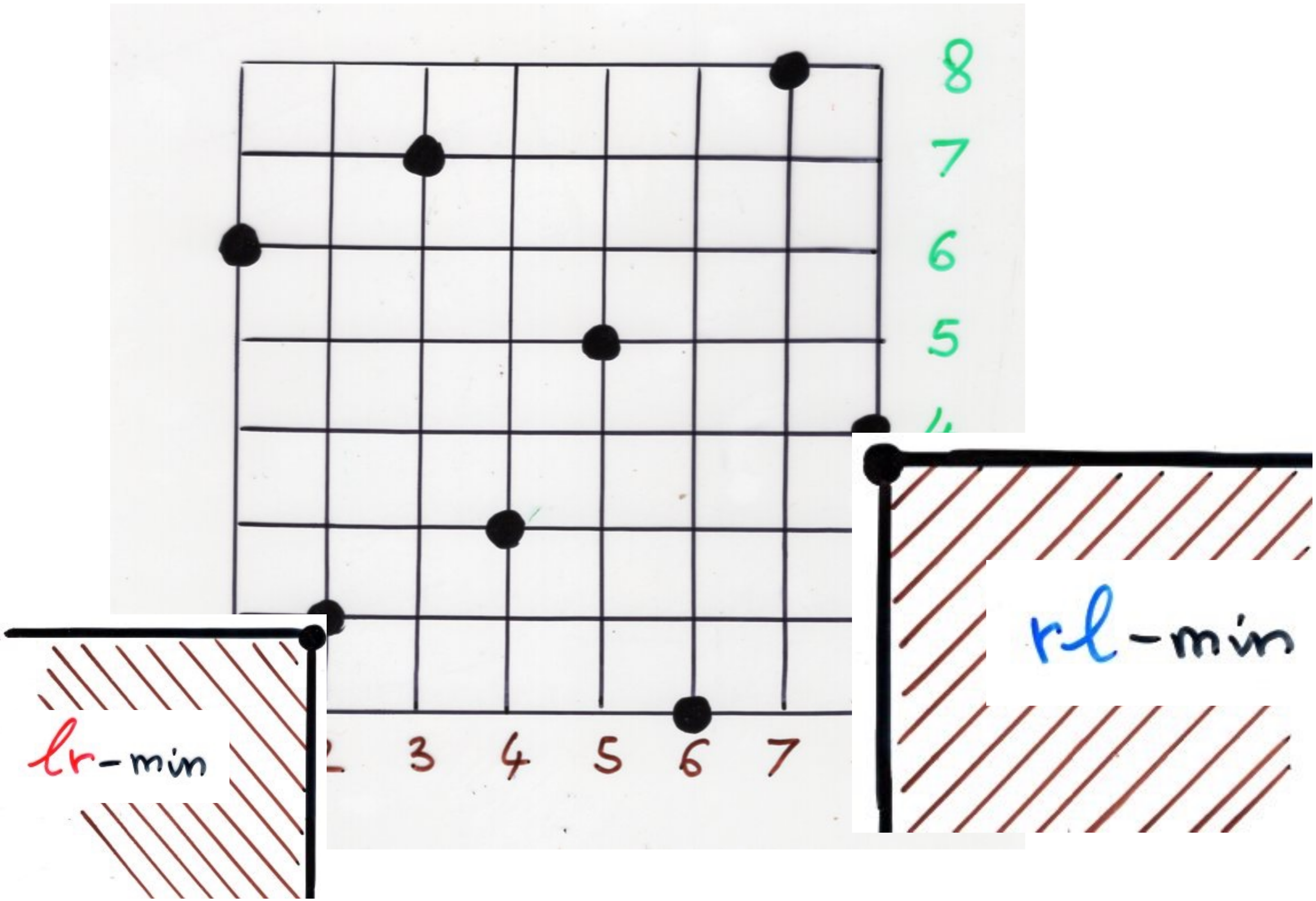
Why to insist
on the 3 parameters model ?

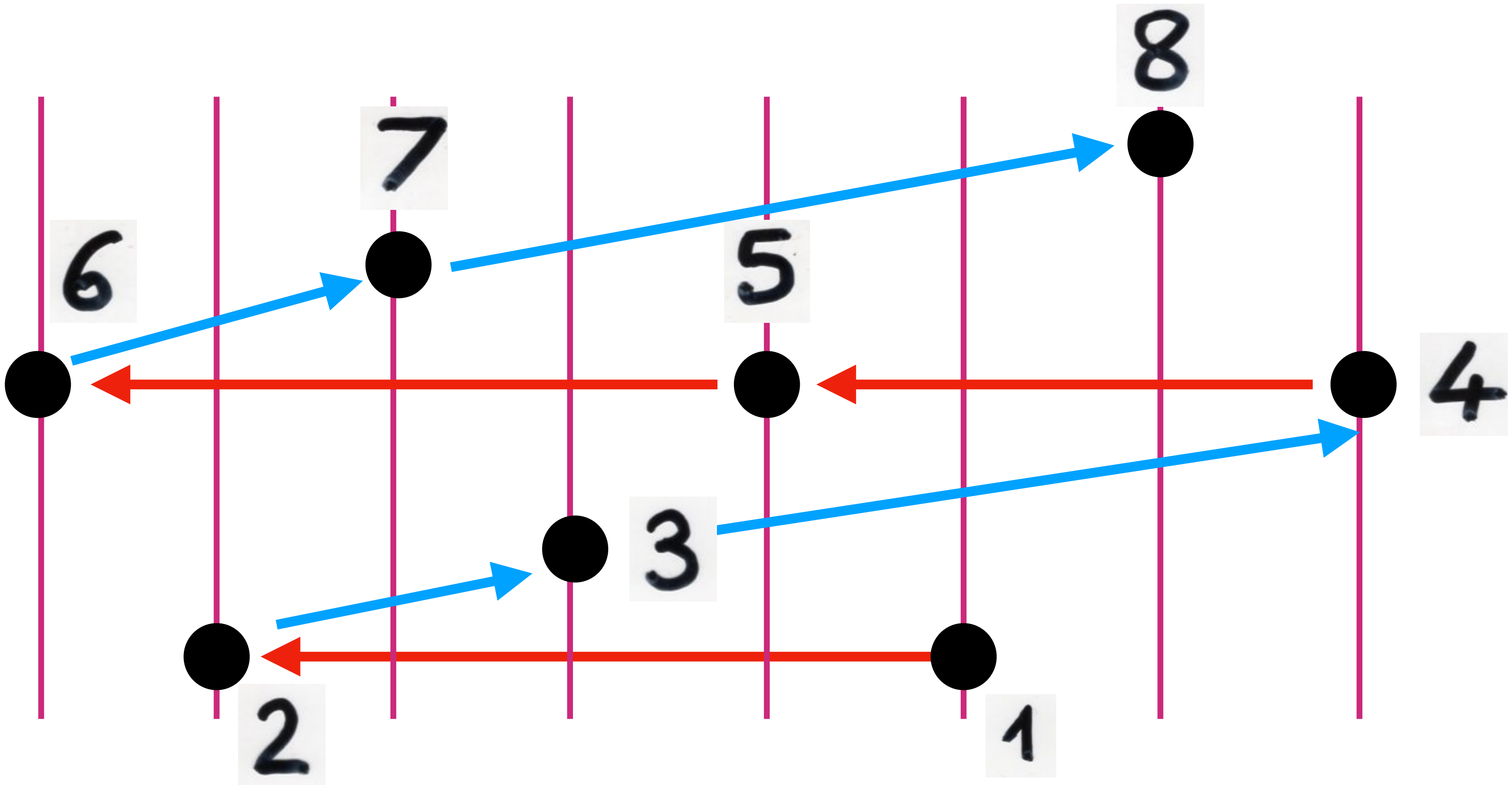
symmetries !

my dream ...

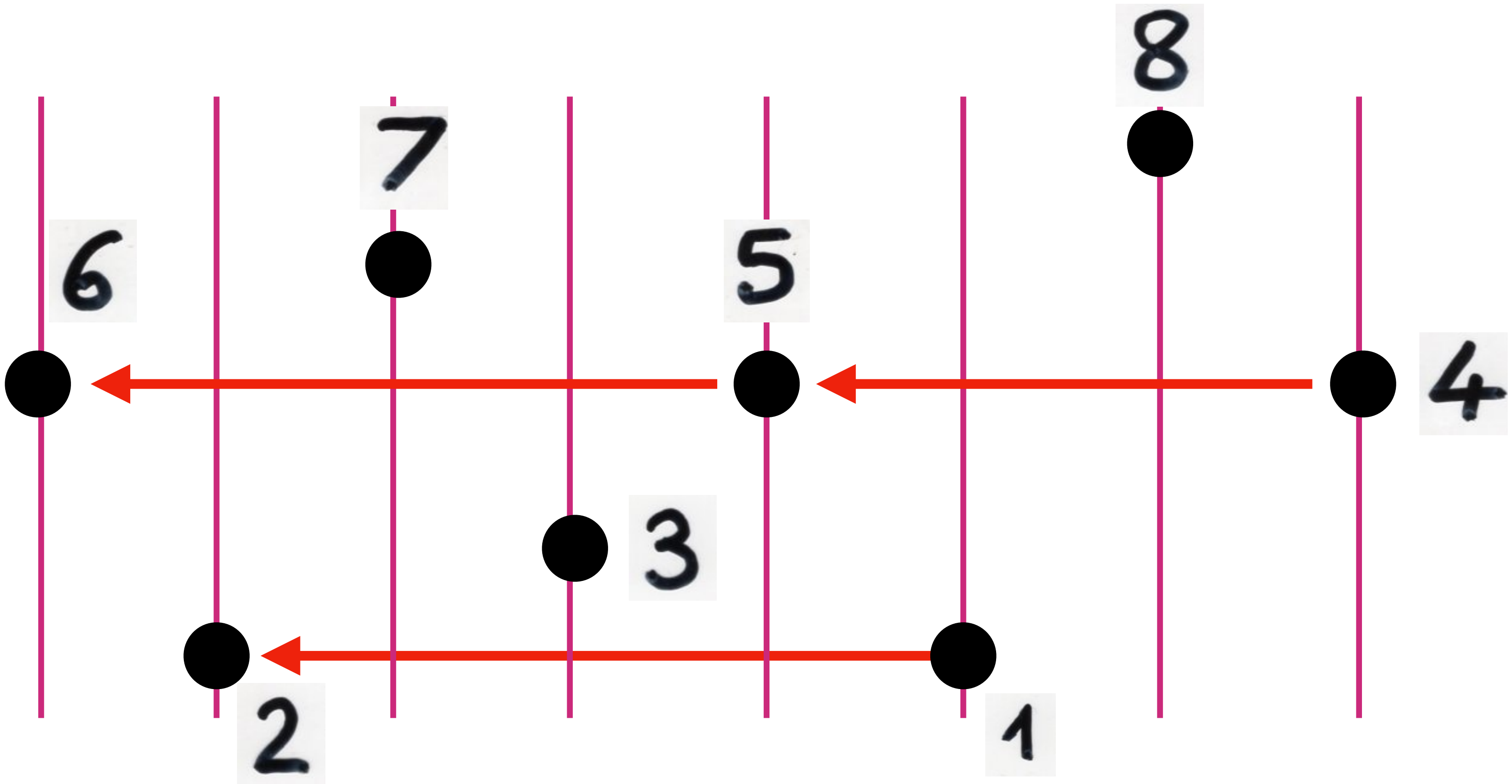

8
7
6
5
4
lr-min
2 3 4 5 6 7
rl-min

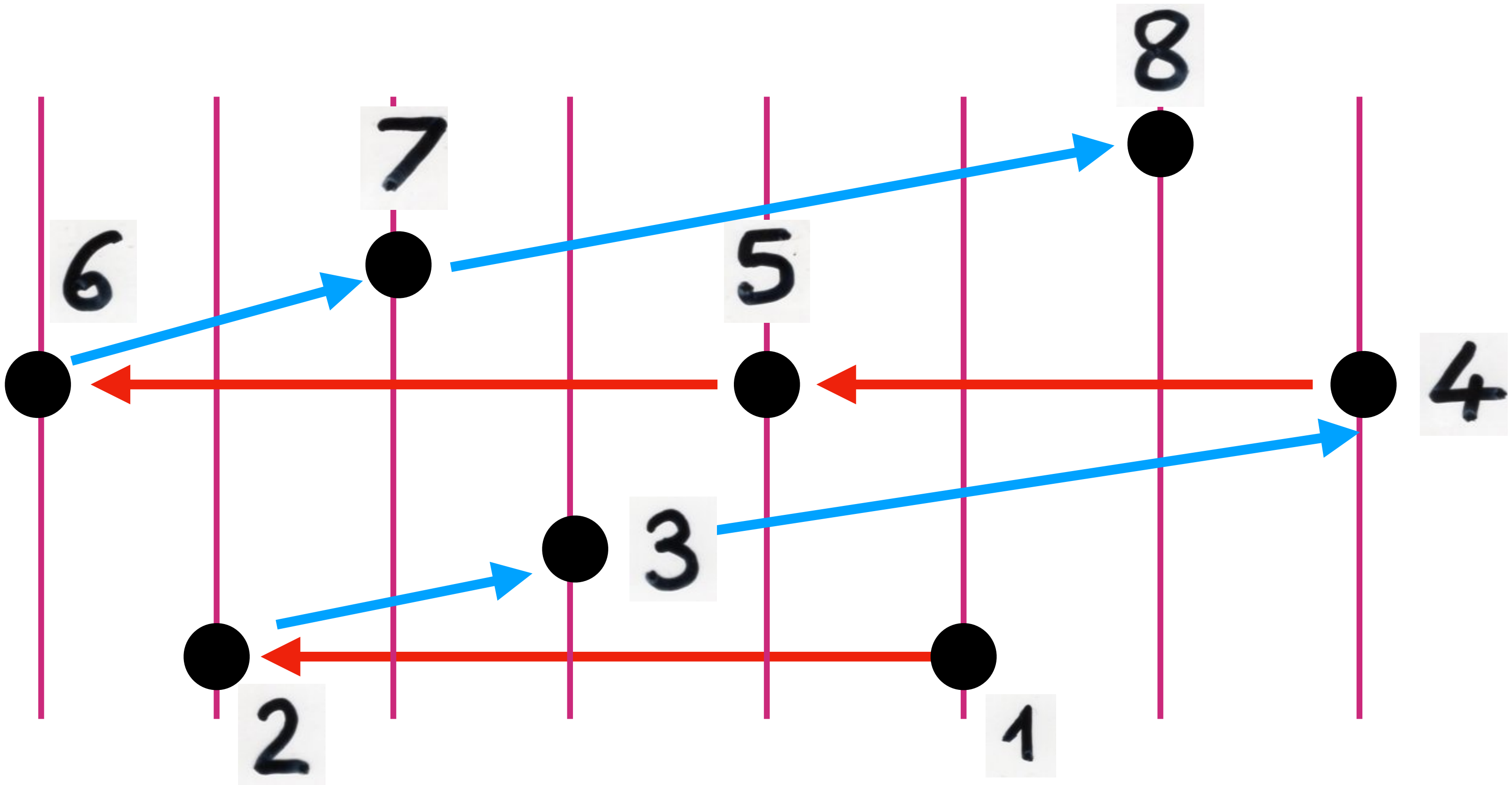


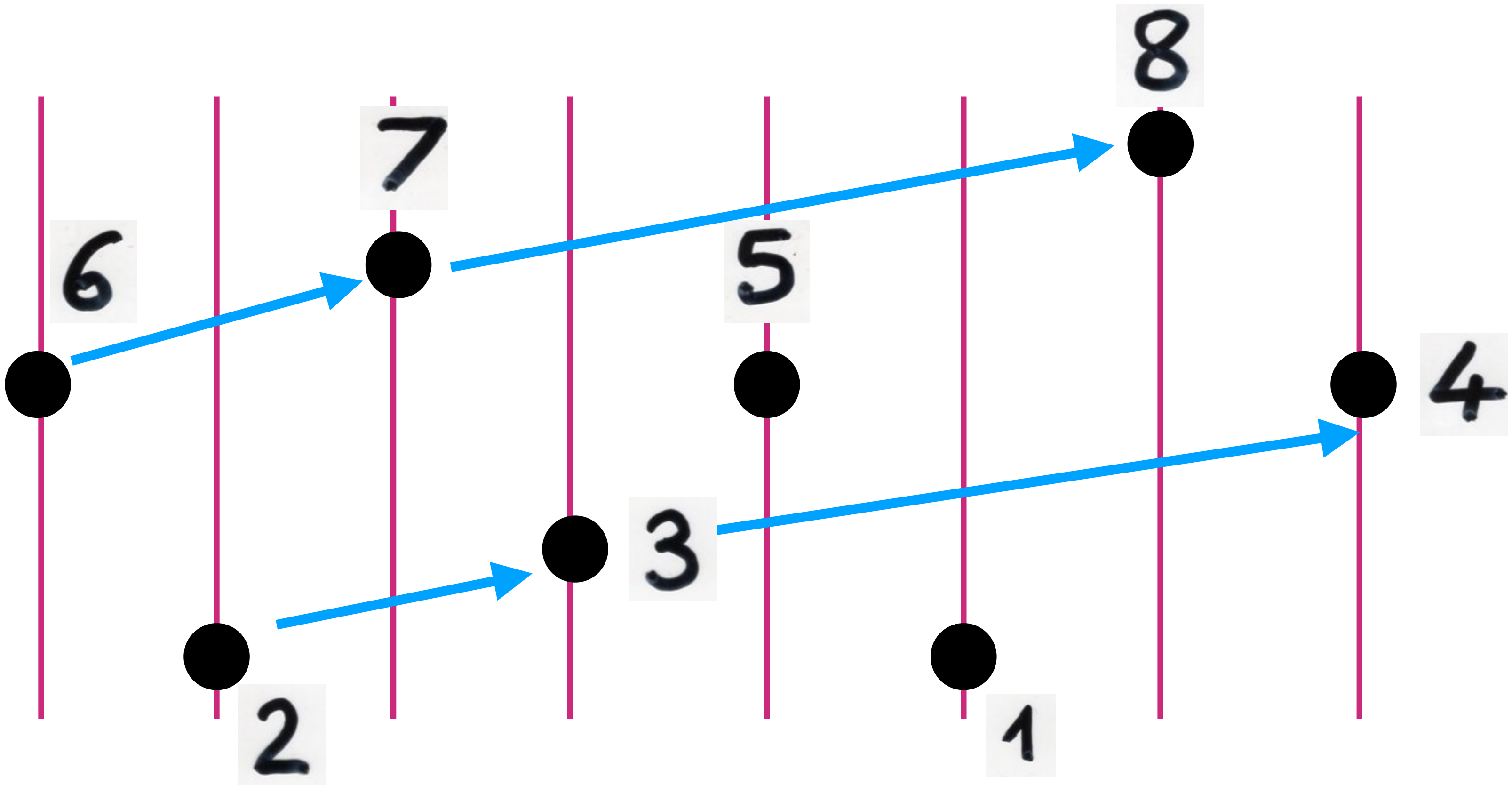

8
7
6
5
4
2 3 4 5 6 7
lr-min
rl-min



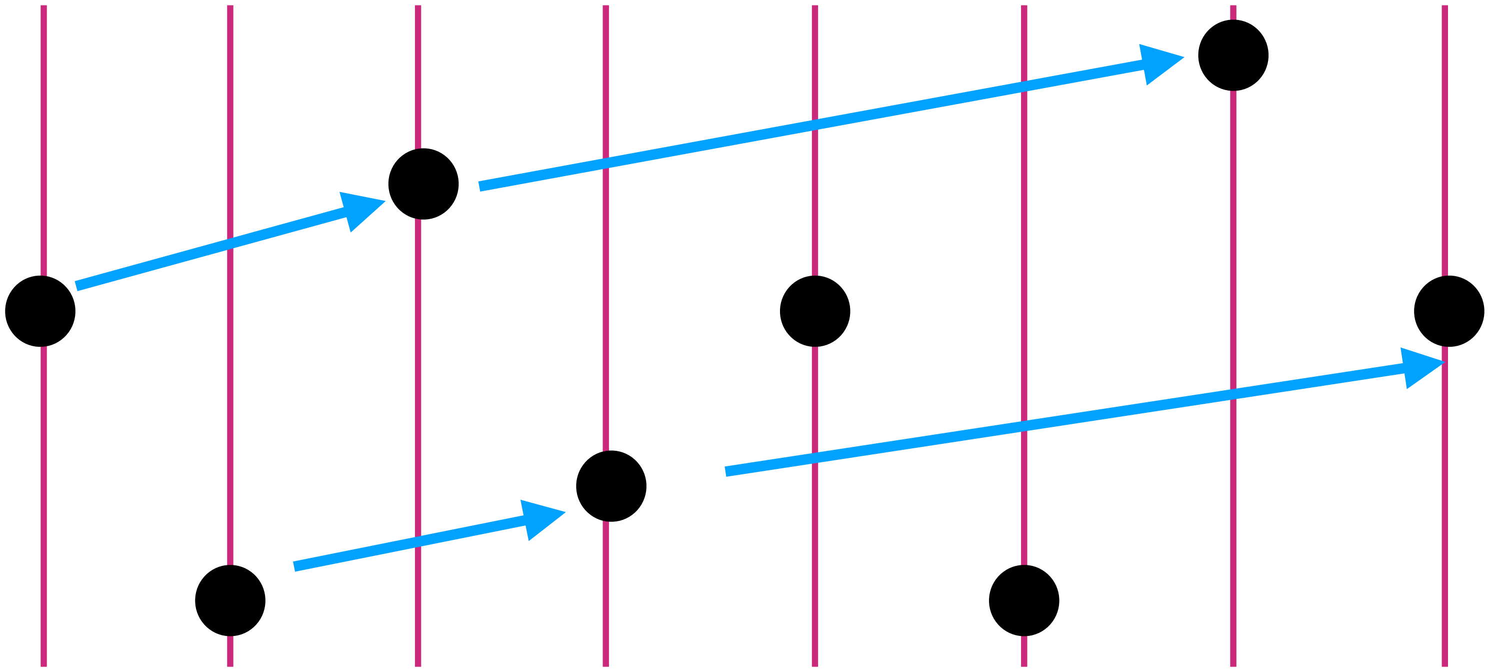
8
7
6
5
4
3
2
1

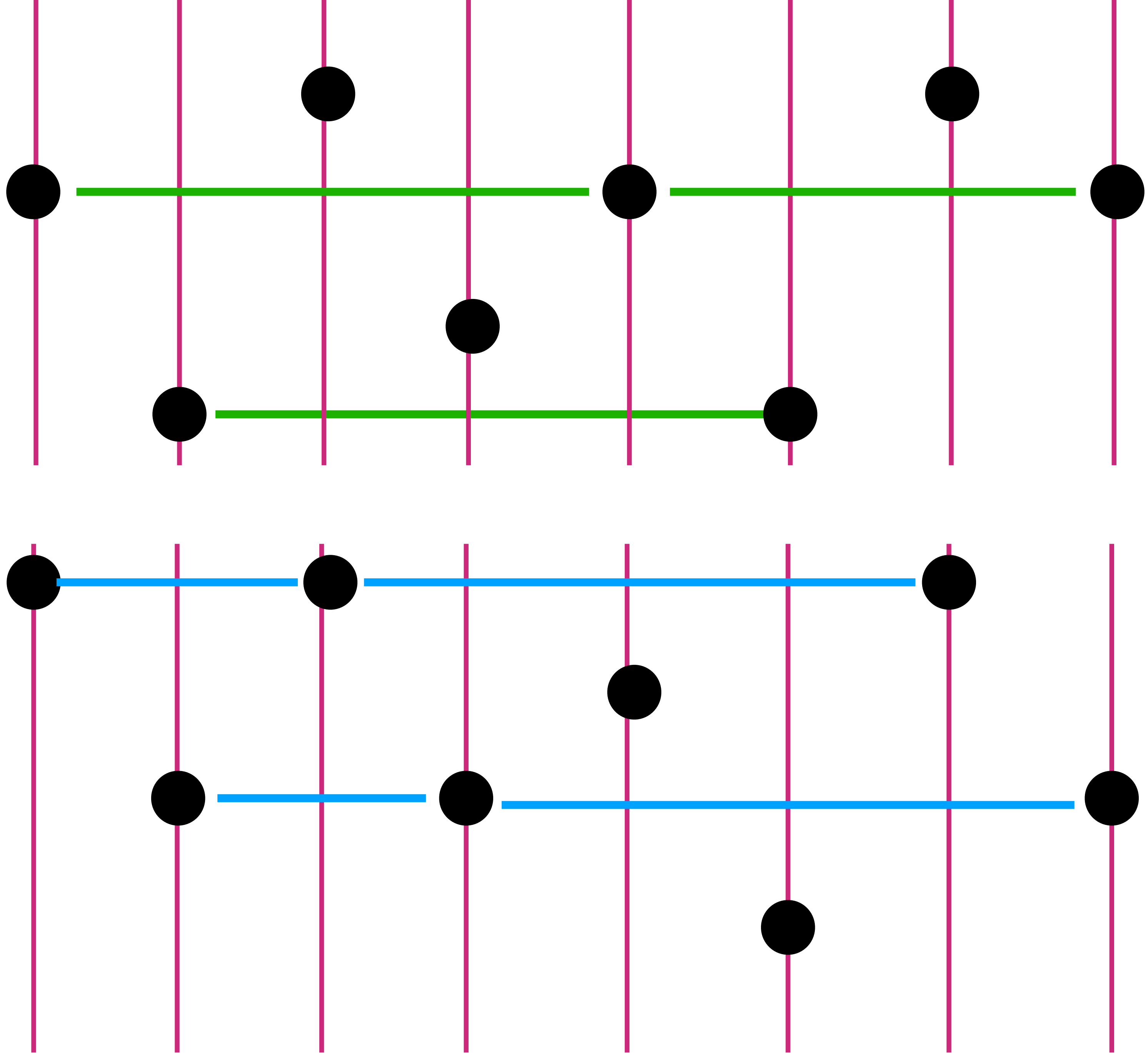

8
7
6
5
4
3
2
1


8
7
6
5
4
3
2
1



8
7
6
5
4
3
2
1





Askey-Wilson integral

The Askey-Wilson integral

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta =$$

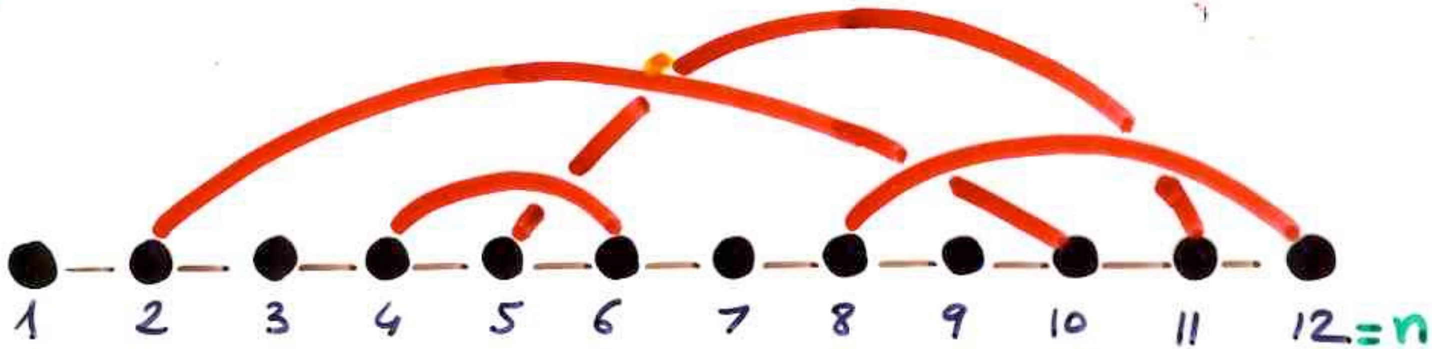
$$\frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

The Askey-Wilson integral

integral of the product
of 4 q-Hermite polynomials
(type II)

Ismail, Stanton, V. (1986)

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_k(\cos\theta|q)\,H_l(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty=(q)_k\,\delta_{kl}$$

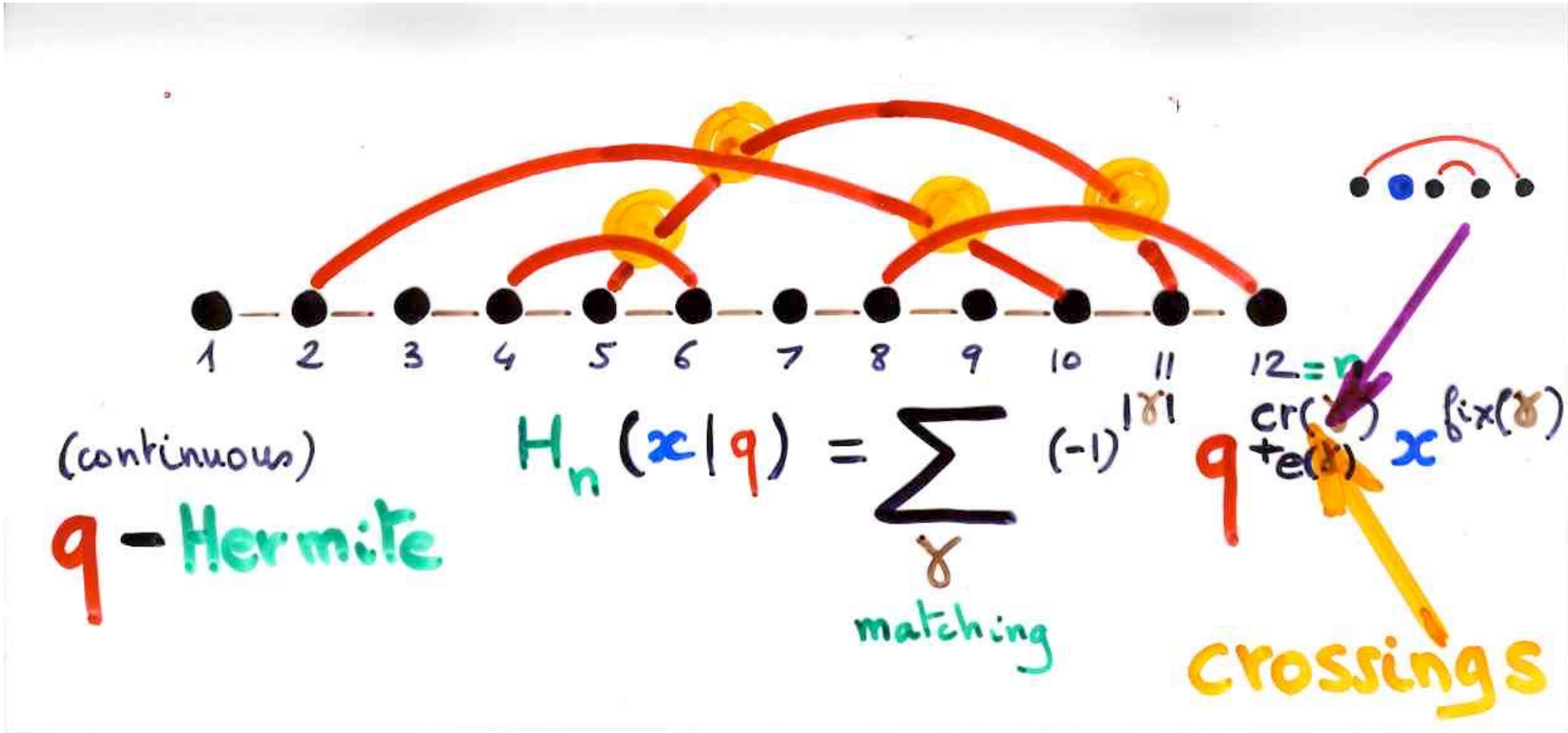


(continuous)

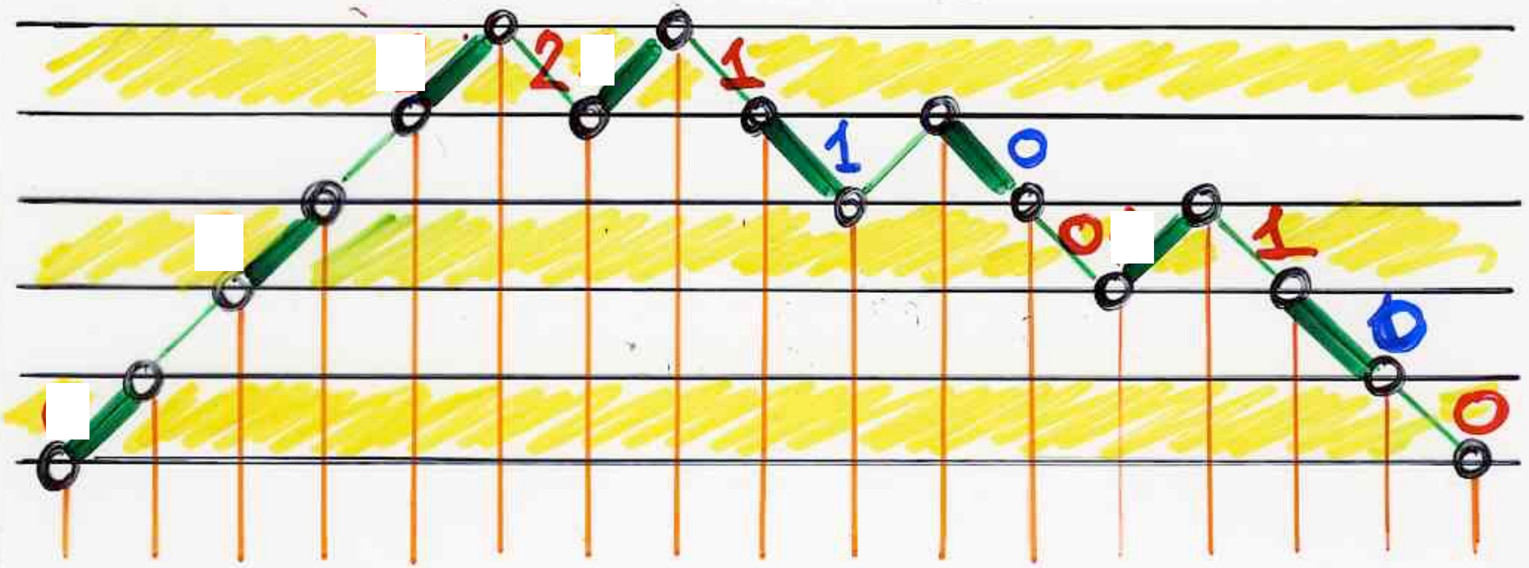
q-Hermite

$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} q^{cr(\gamma)+e(\gamma)} x^{fix(\gamma)}$$


nesting

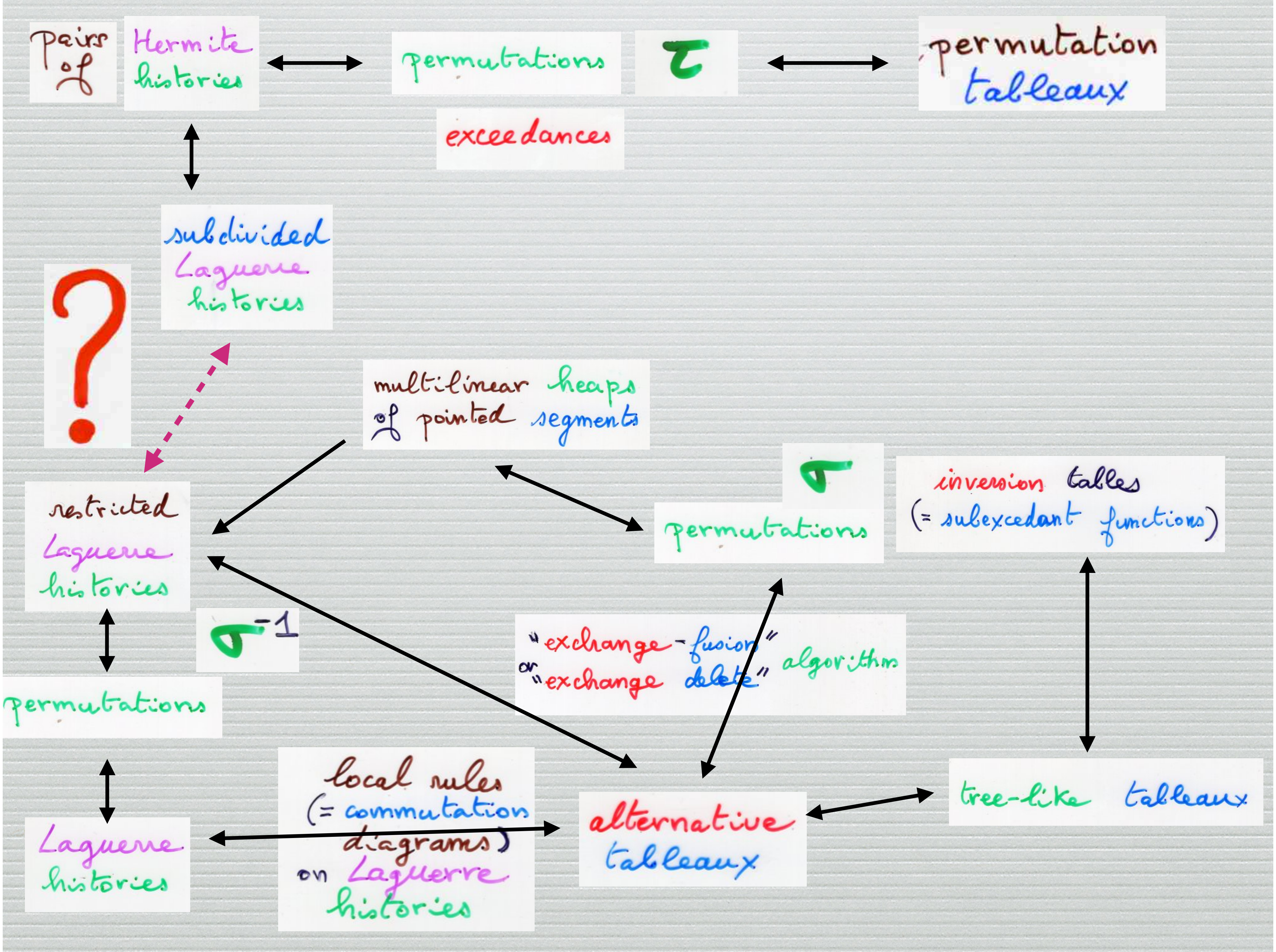

1 2 3 4 5 6 7 8 9 10 11 12=n
(continuous)
q-Hermite
$H_n(x|q) = \sum_{\gamma} (-1)^{|\gamma|} q^{cr(\gamma)+e(\gamma)} x^{fix(\gamma)}$
matching
crossings

$$UD = qDU + I$$



subdivided Laguerre history

$$DE = qED + E + D$$



Pairs of Hermite histories
permutations
τ
exceedances
permutation tableaux
subdivided Laguerre histories
?
multilinear heaps of pointed segments
σ
permutations
inversion tables
(= subexcedant functions)
restricted Laguerre histories
σ-1
permutations
"exchange-fusion"
or "exchange delete" algorithms
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

Pairs of Hermite histories

subdivided Laguerre histories

?

restricted Laguerre histories

permutations

Laguerre histories

$$UD = qDU + I$$

Hermite polynomials

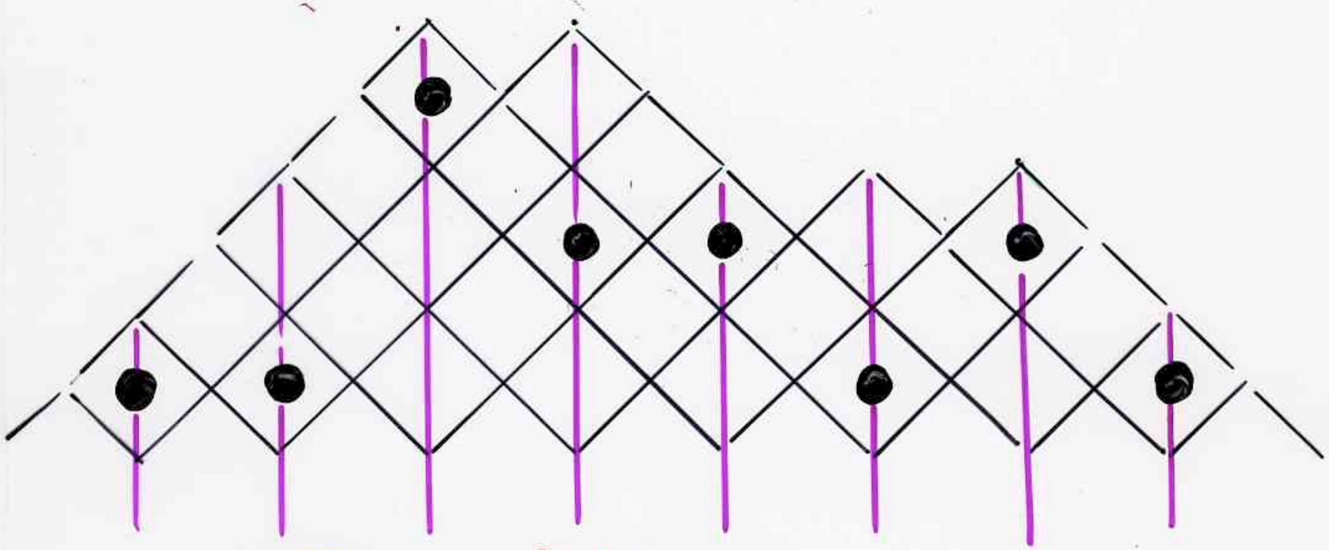
$$DE = qED + E + D$$

Laguerre polynomials

Dyck tableaux

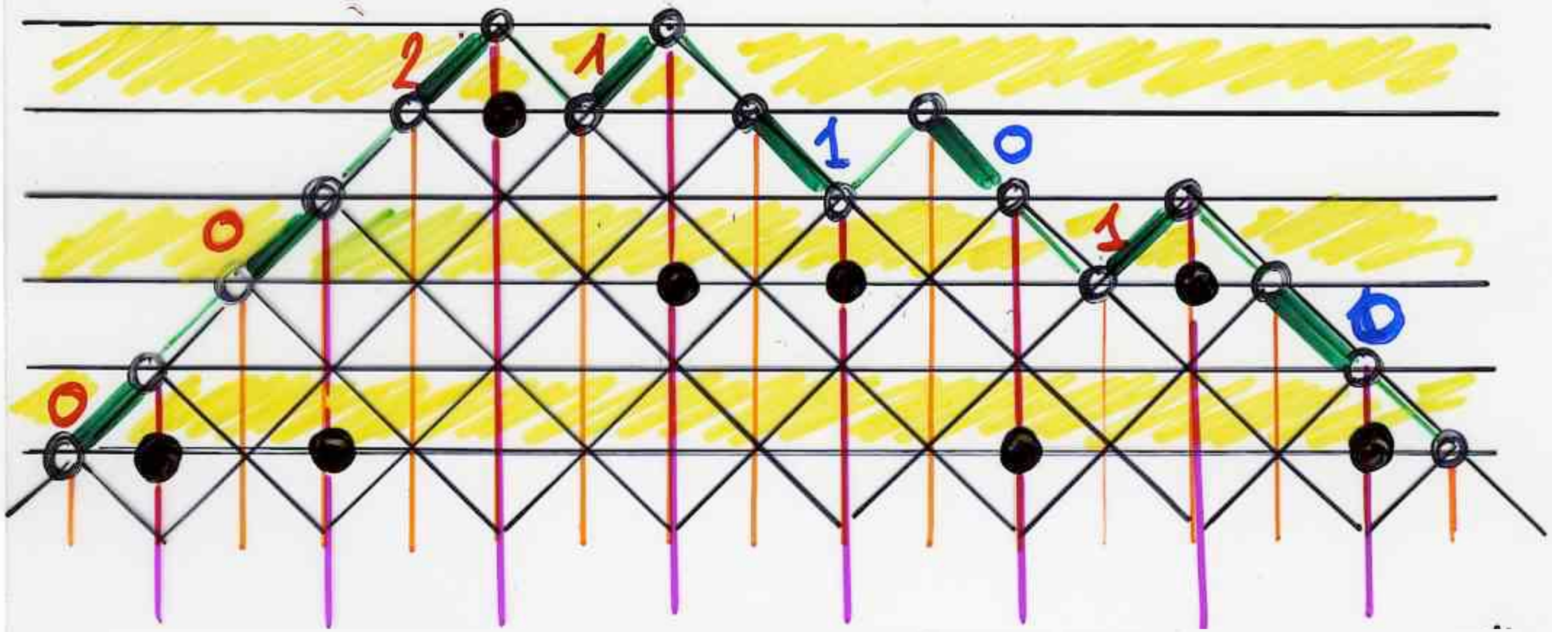
as

subdivided Laguerre histories

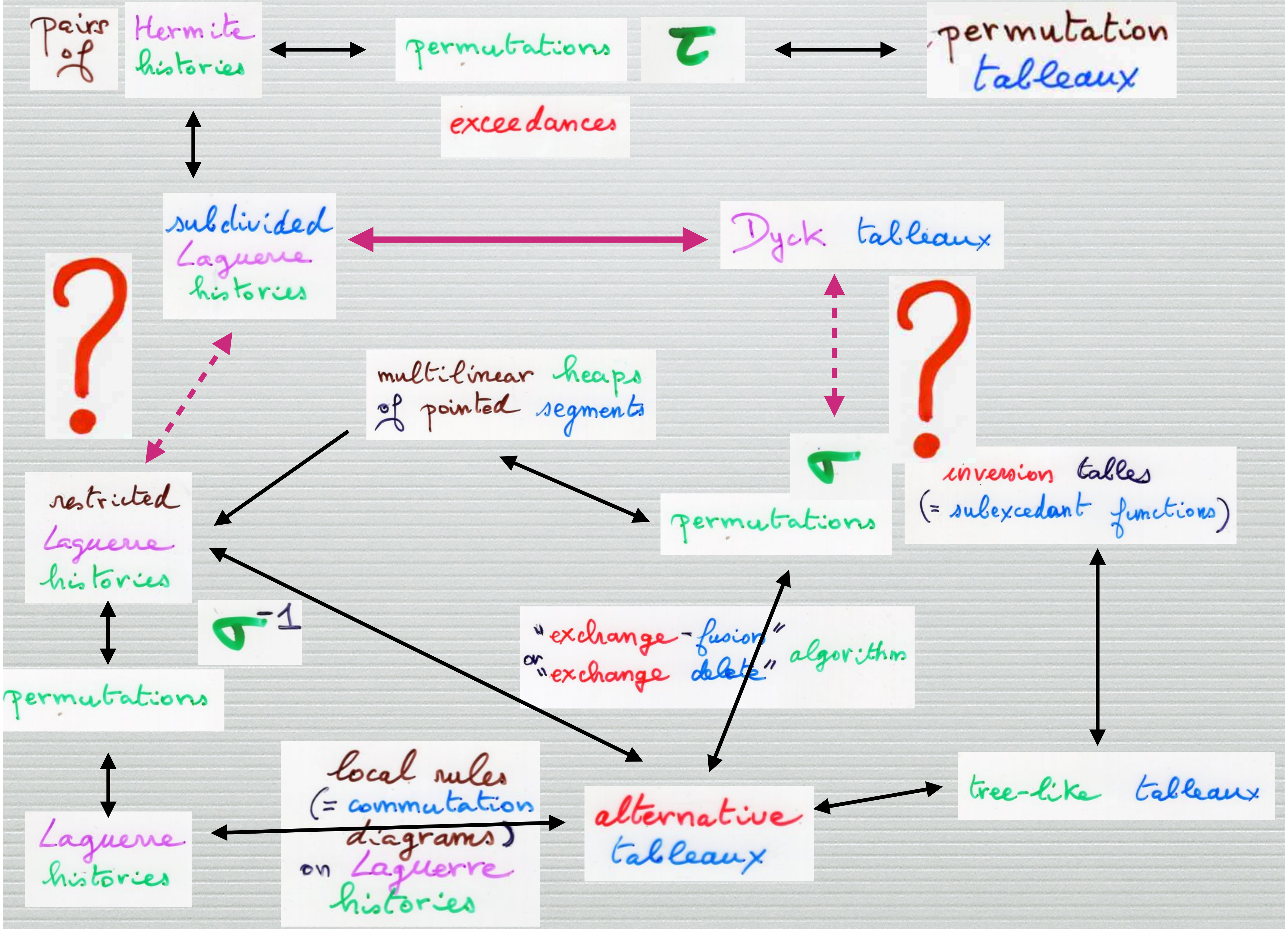


Dyck tableau

J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)



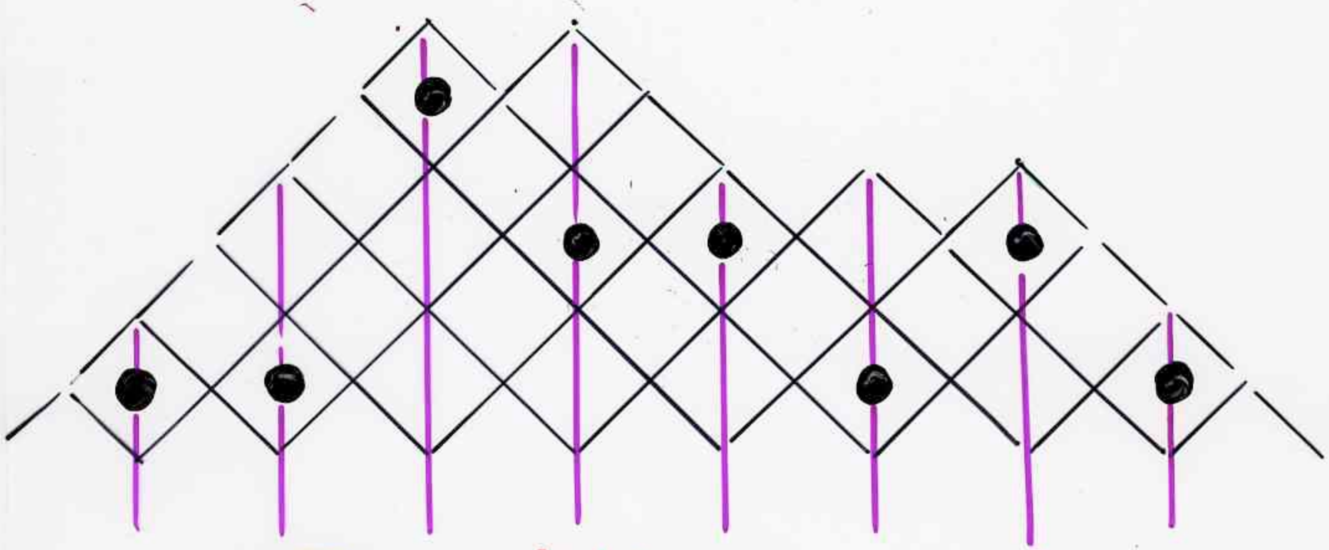
Dyck tableau as a subdivided Laguerre history



Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
Dyck tableaux
?
multilinear heaps of pointed segments
?
σ
inversion tables (= subexcedant functions)
restricted Laguerre histories
permutations
σ⁻¹
"exchange-fusion" or "exchange delete" algorithms
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

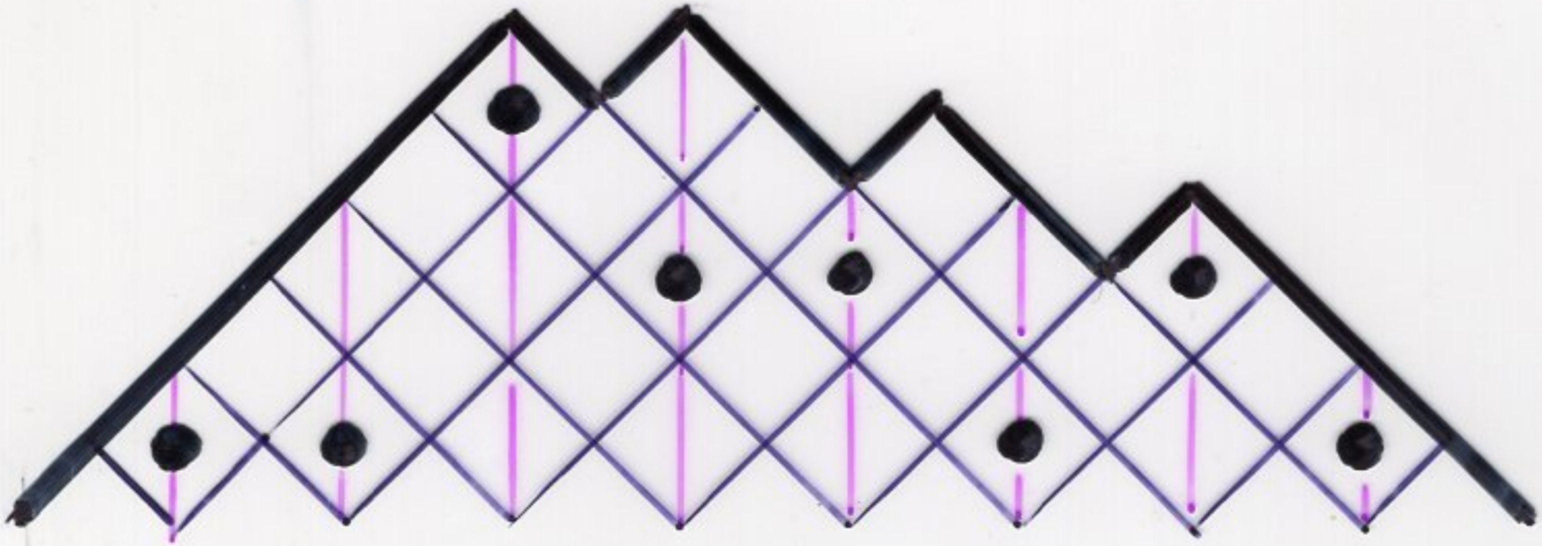
(direct) bijection

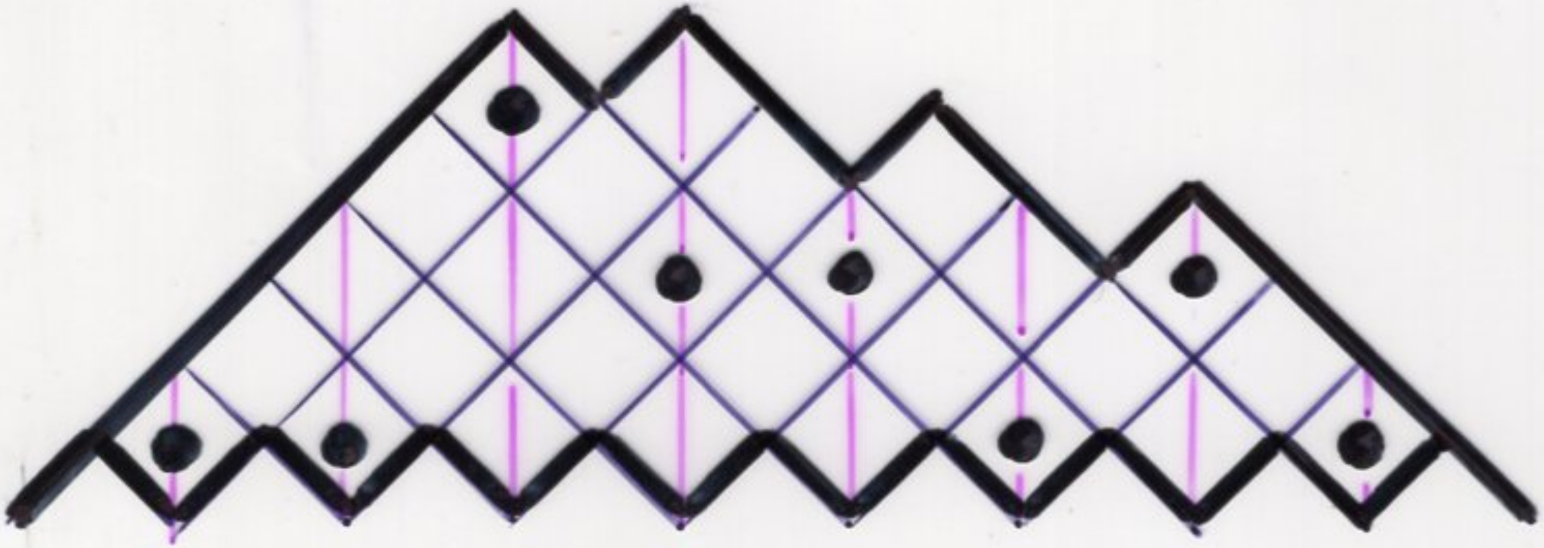
permutations ⟶ Dyck tableaux



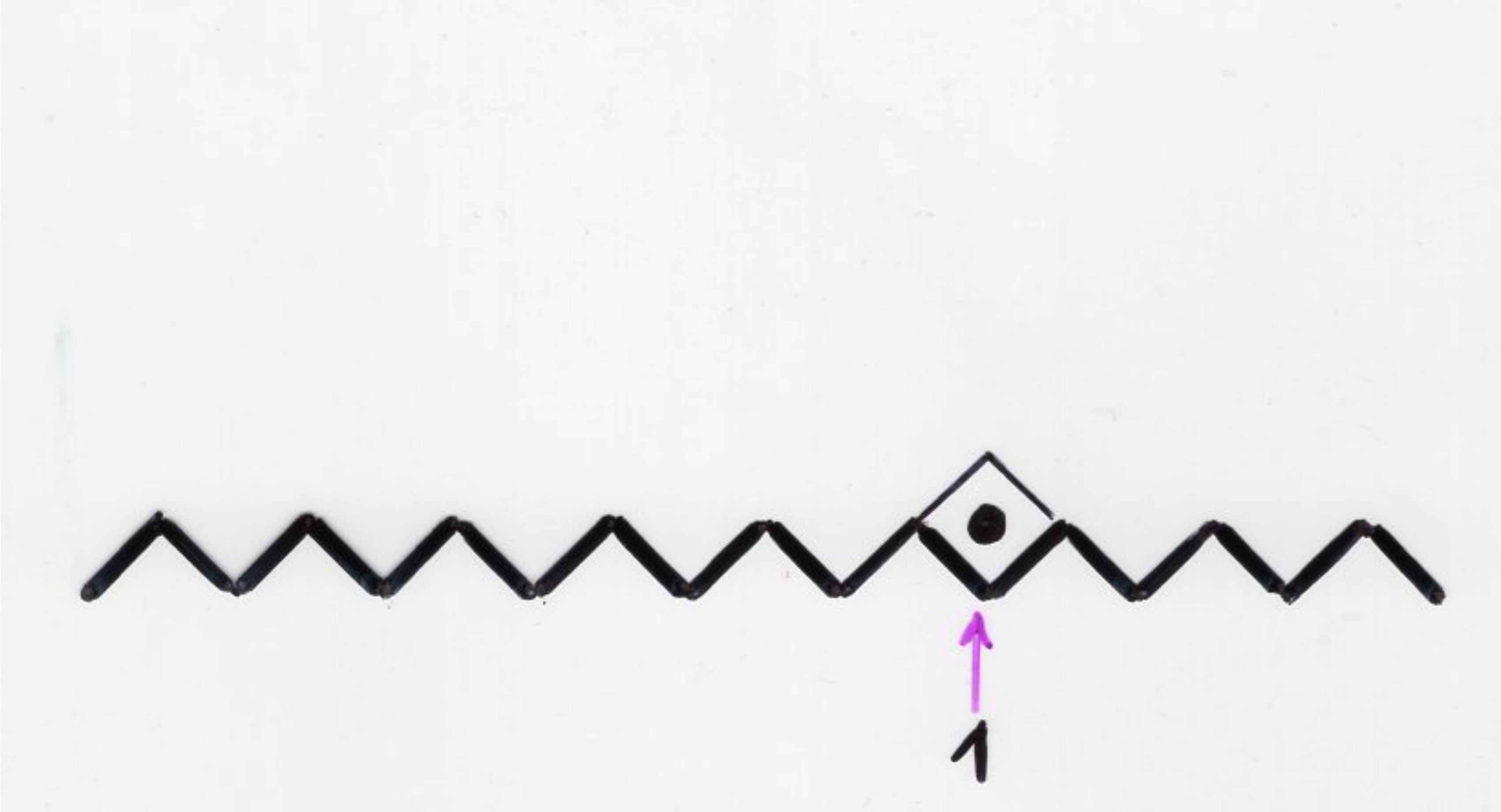
Dyck tableau

J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)

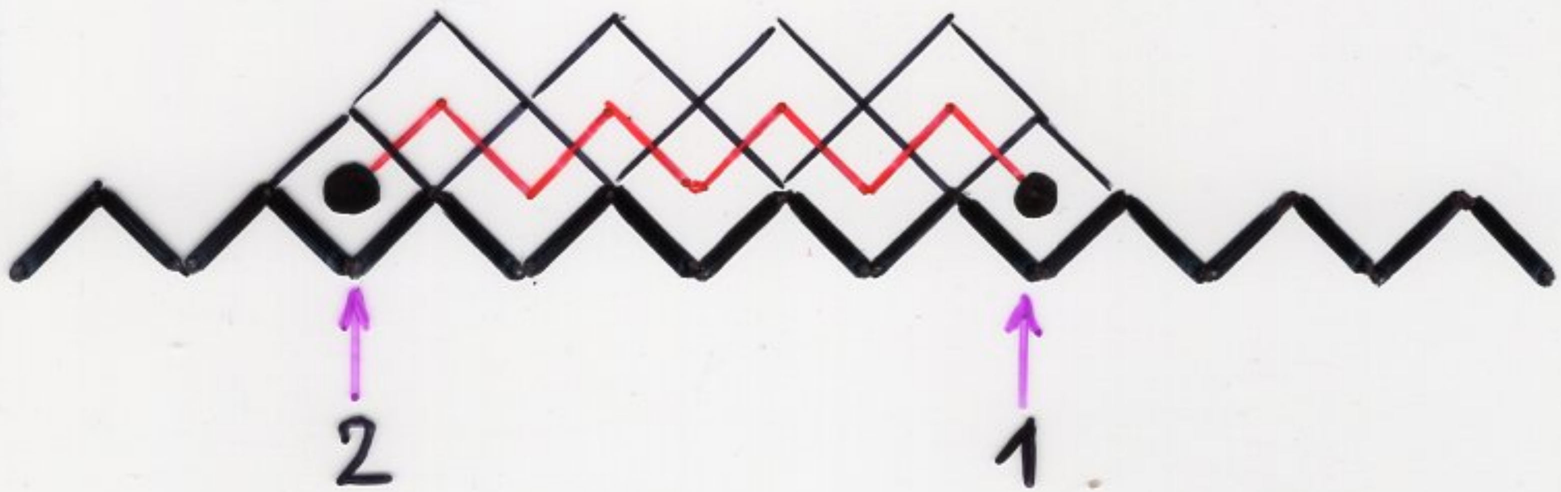




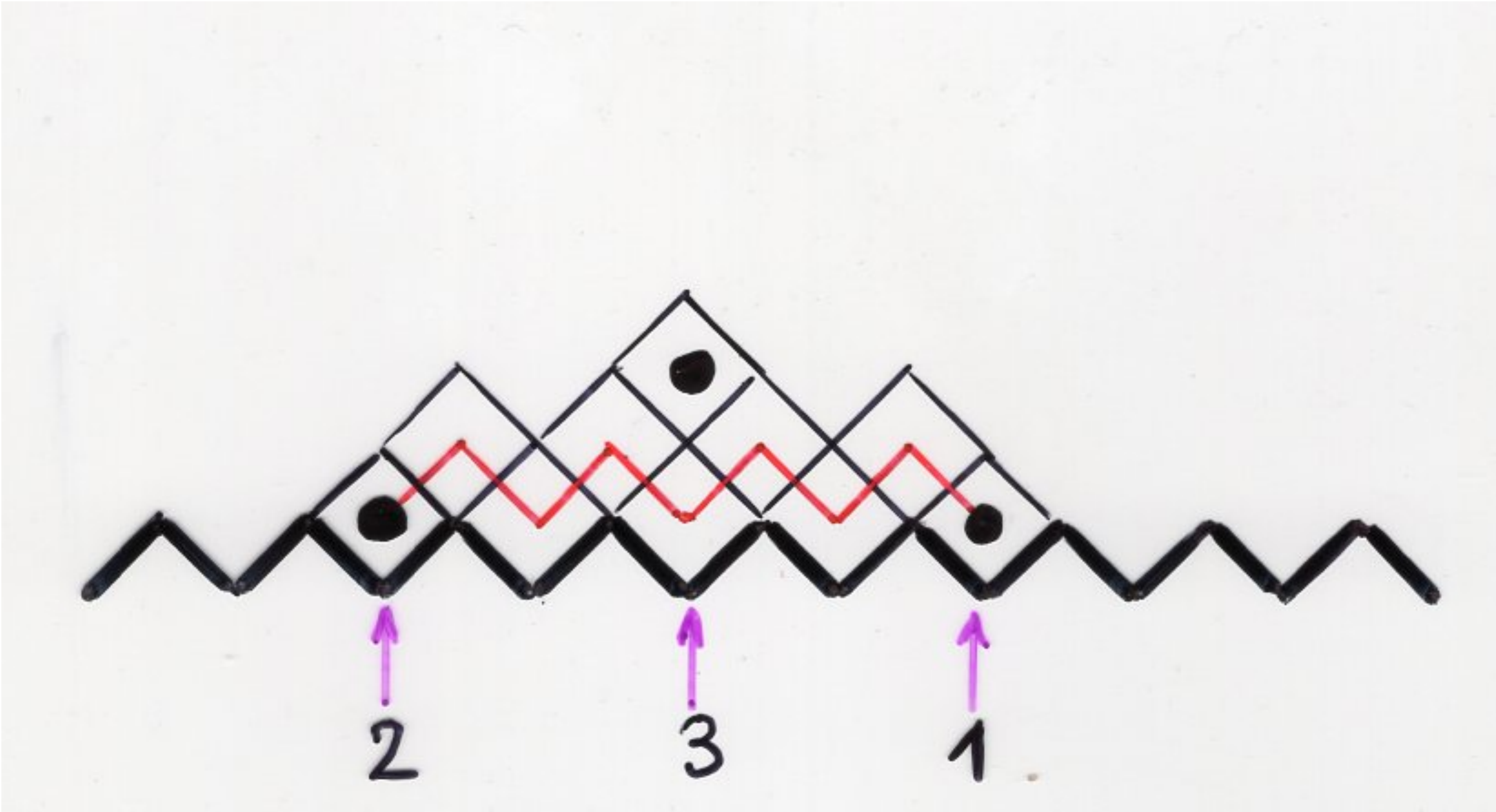
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



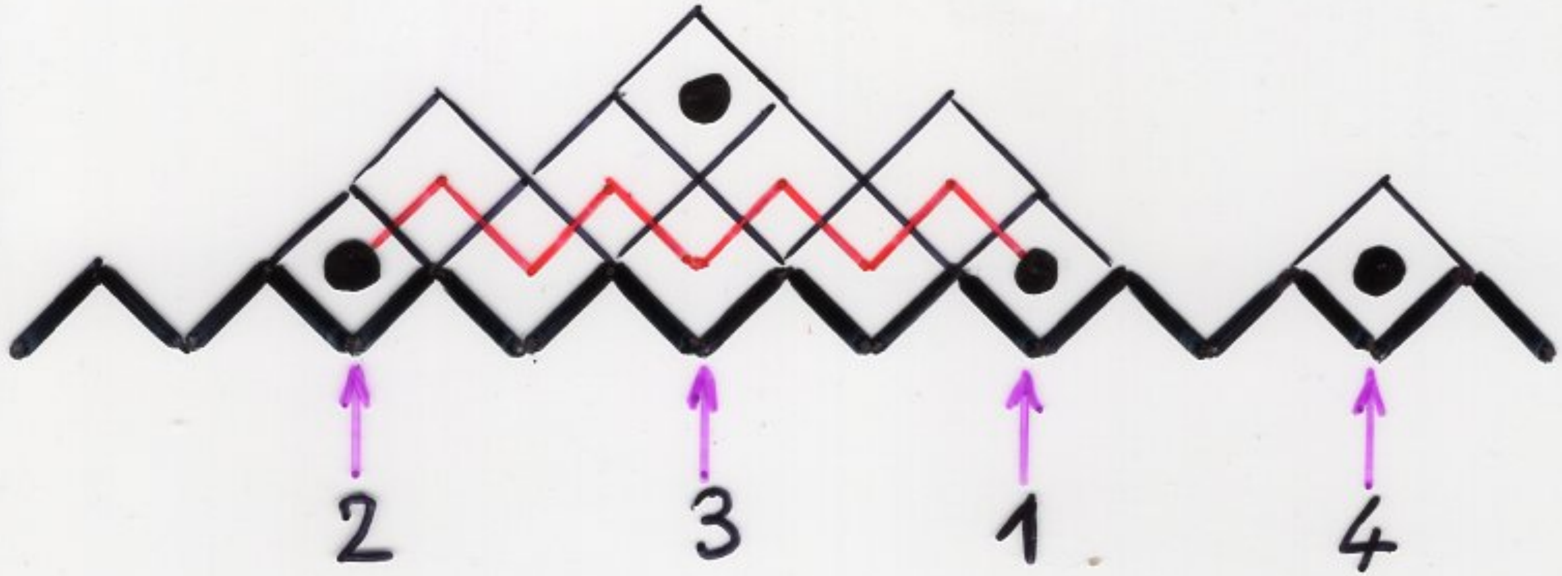
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



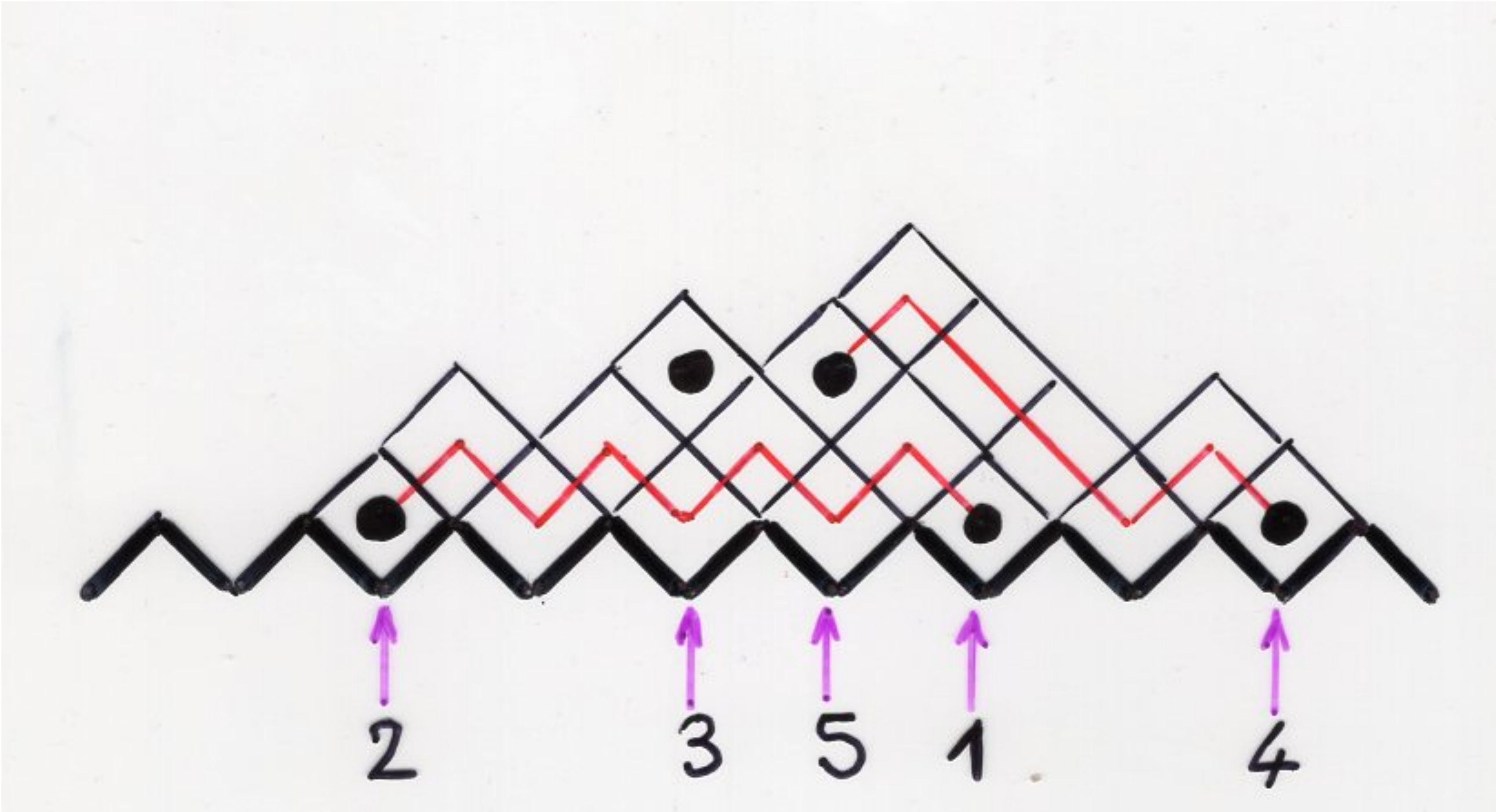
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



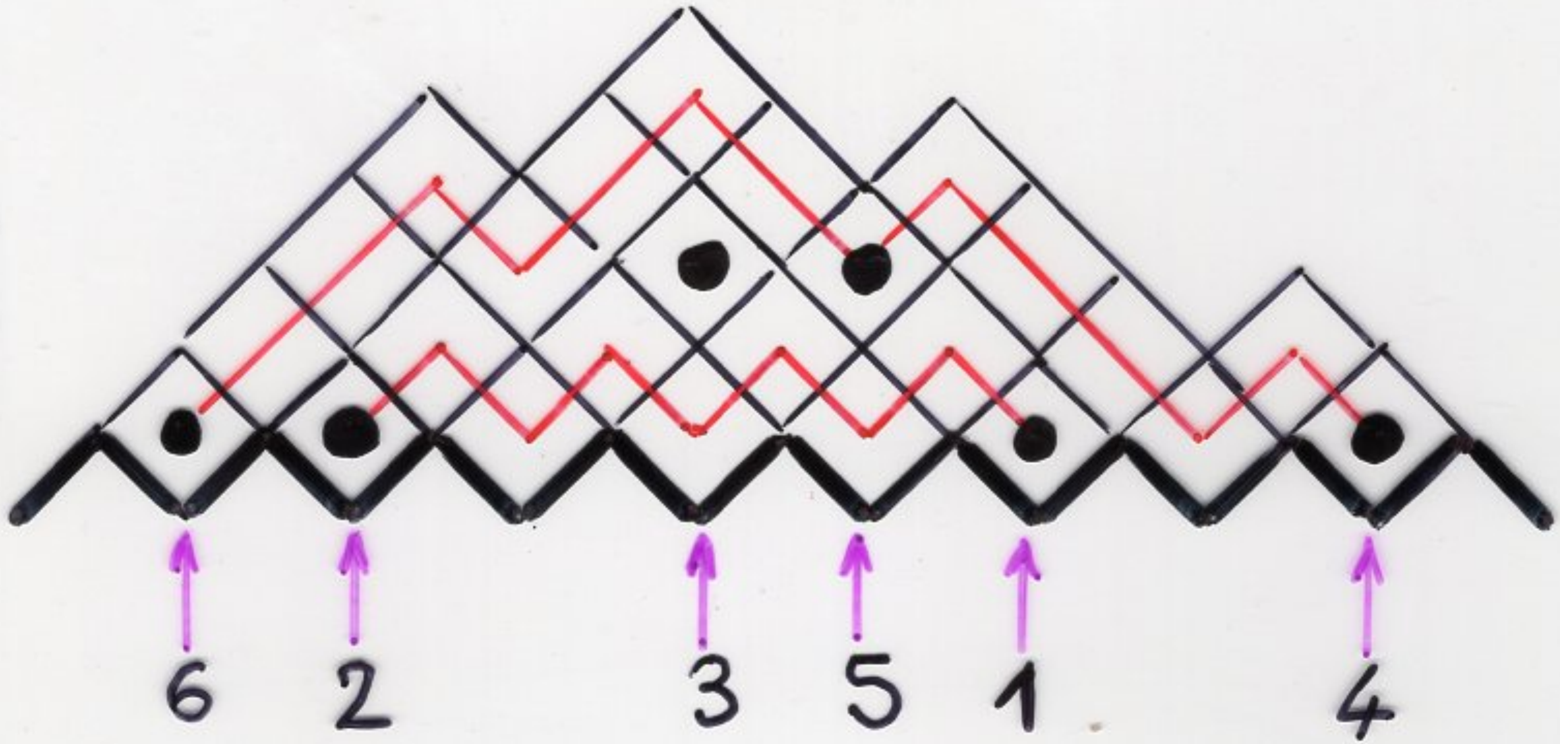
$$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$$



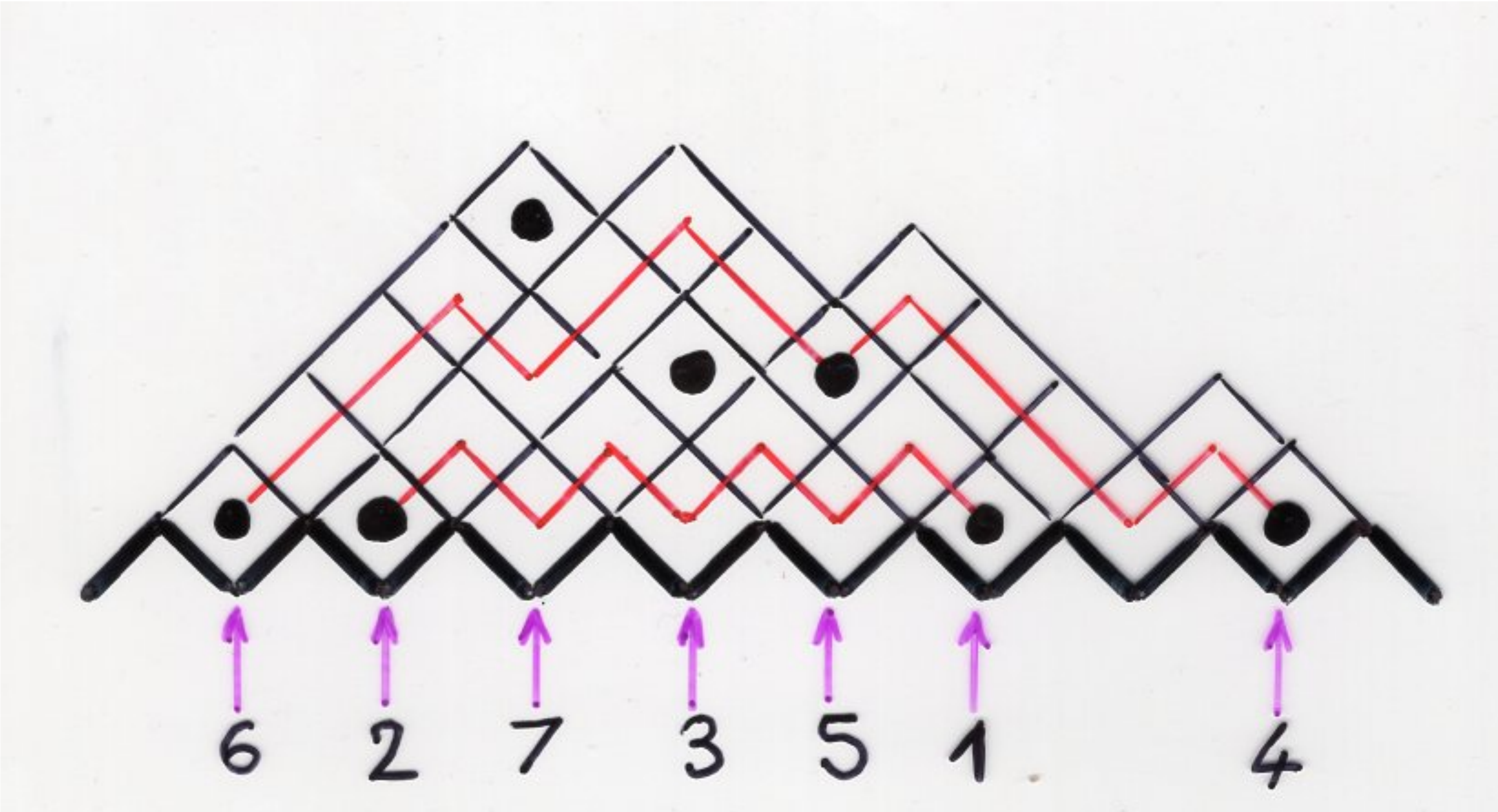
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



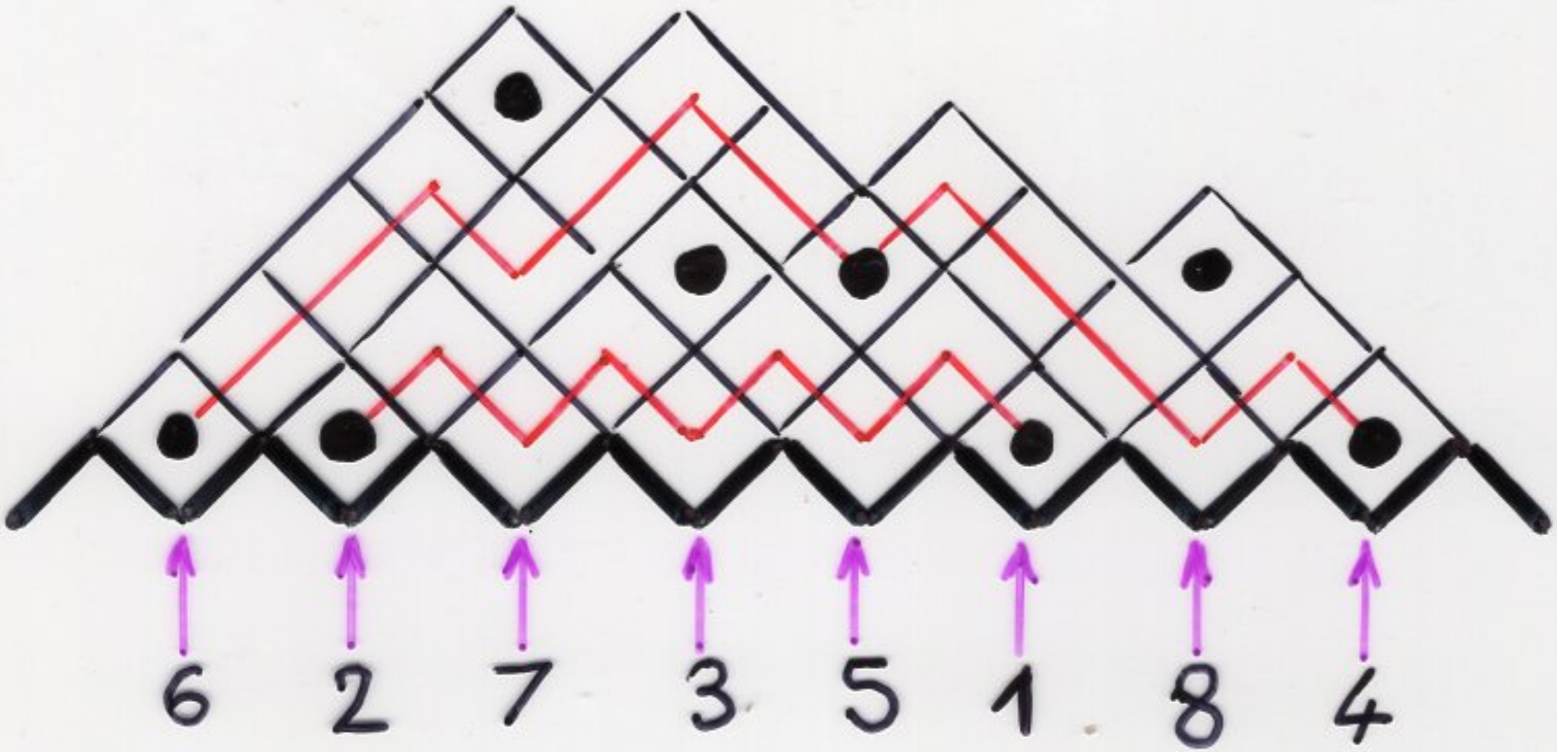
$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$

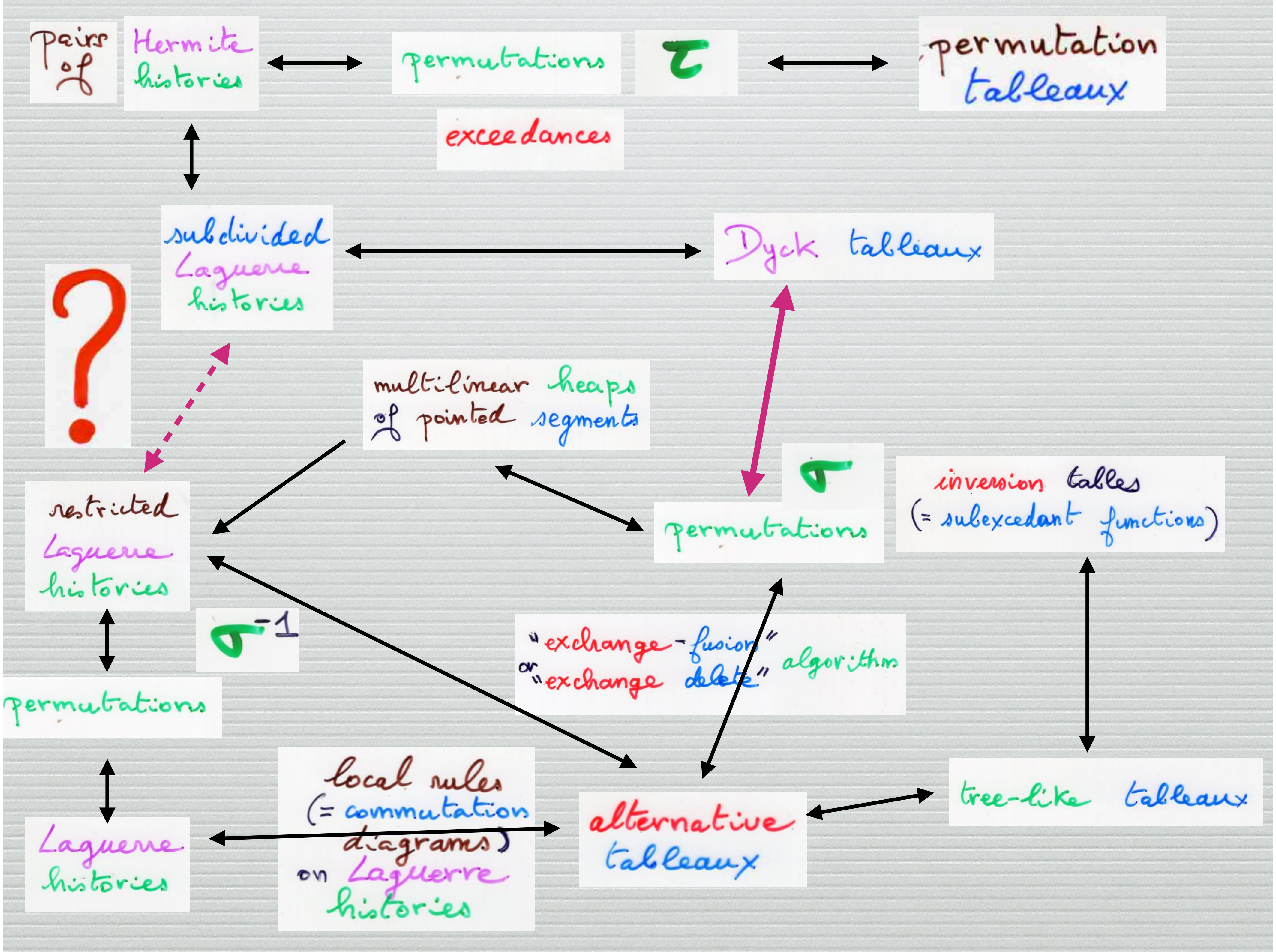


$\sigma = 6 \; 2 \; 7 \; 3 \; 5 \; 1 \; 8 \; 4$



$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$


6 2 7 3 5 1 8 4



Pairs of Hermite histories
permutations τ
permutation tableaux
exceedances
subdivided Laguerre histories
Dyck tableaux
?
multilinear heaps of pointed segments
σ
permutations
inversion tables (= subexcedant functions)
restricted Laguerre histories
σ⁻¹
"exchange-fusion" or "exchange delete" algorithms
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

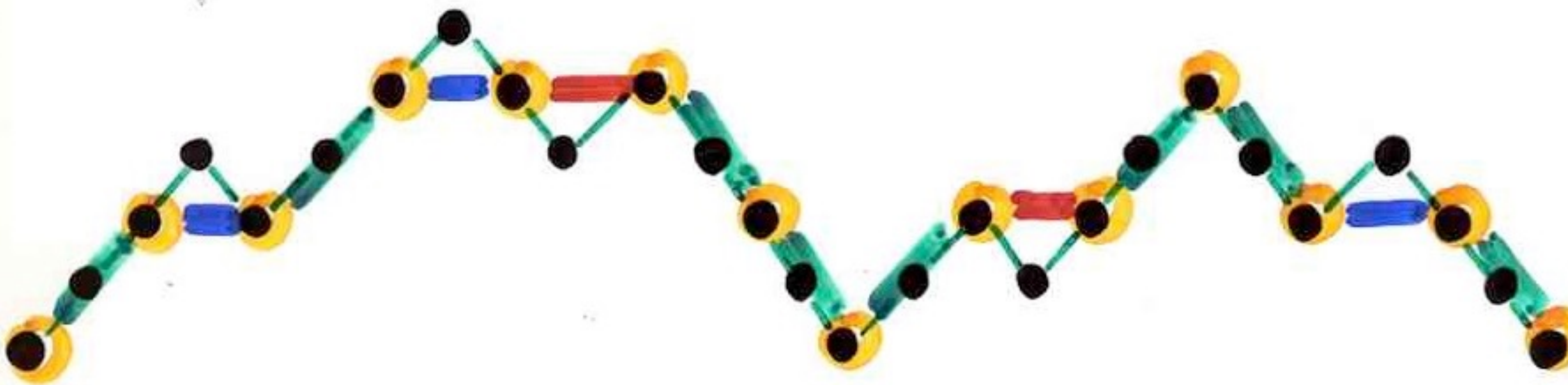
contractions

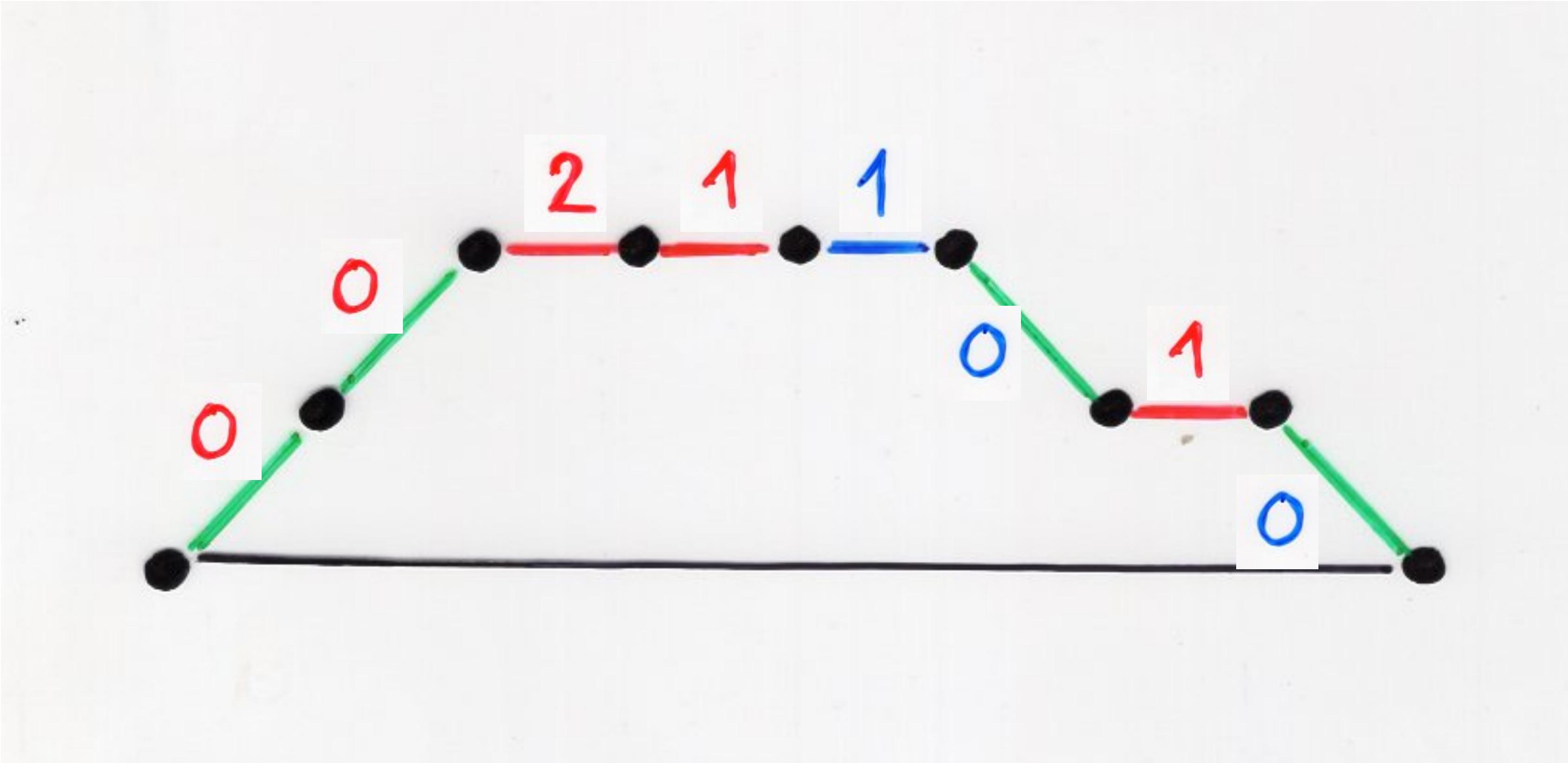
in

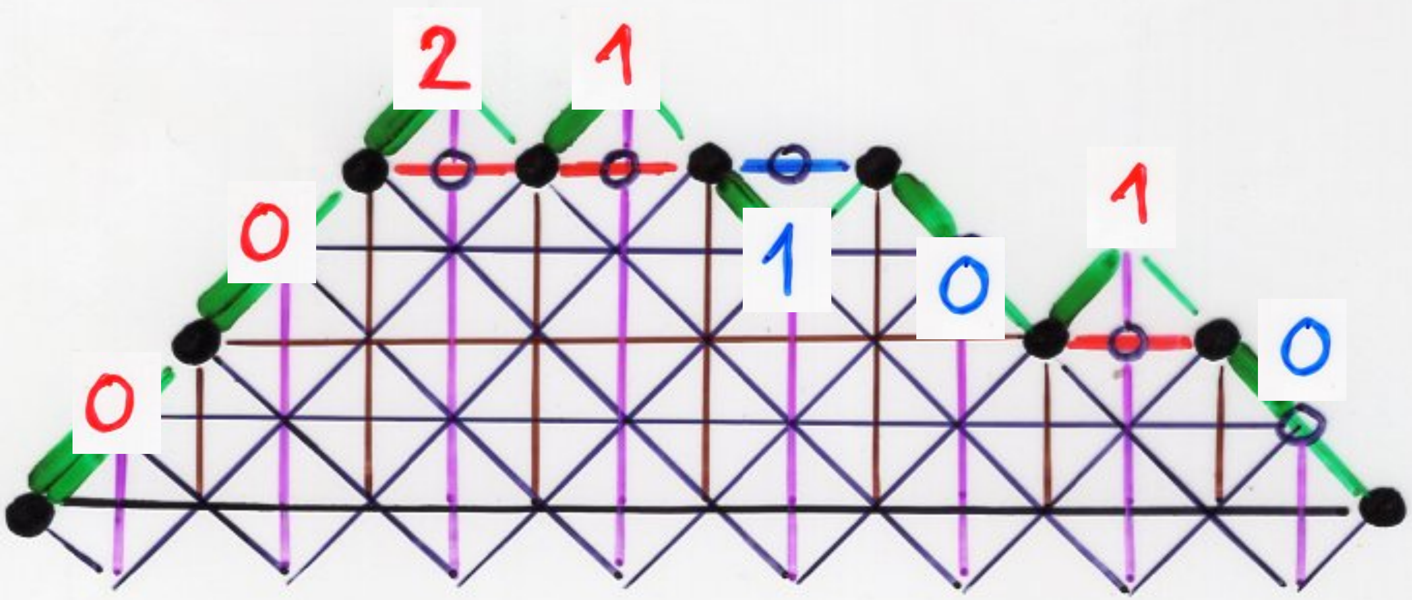
continued fractions

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t \cfrac{}{1 - 1t \cfrac{}{1 - 2t \cfrac{}{1 - 2t \cfrac{}{1 - 3t \cfrac{}{\cdots}}}}}}$$

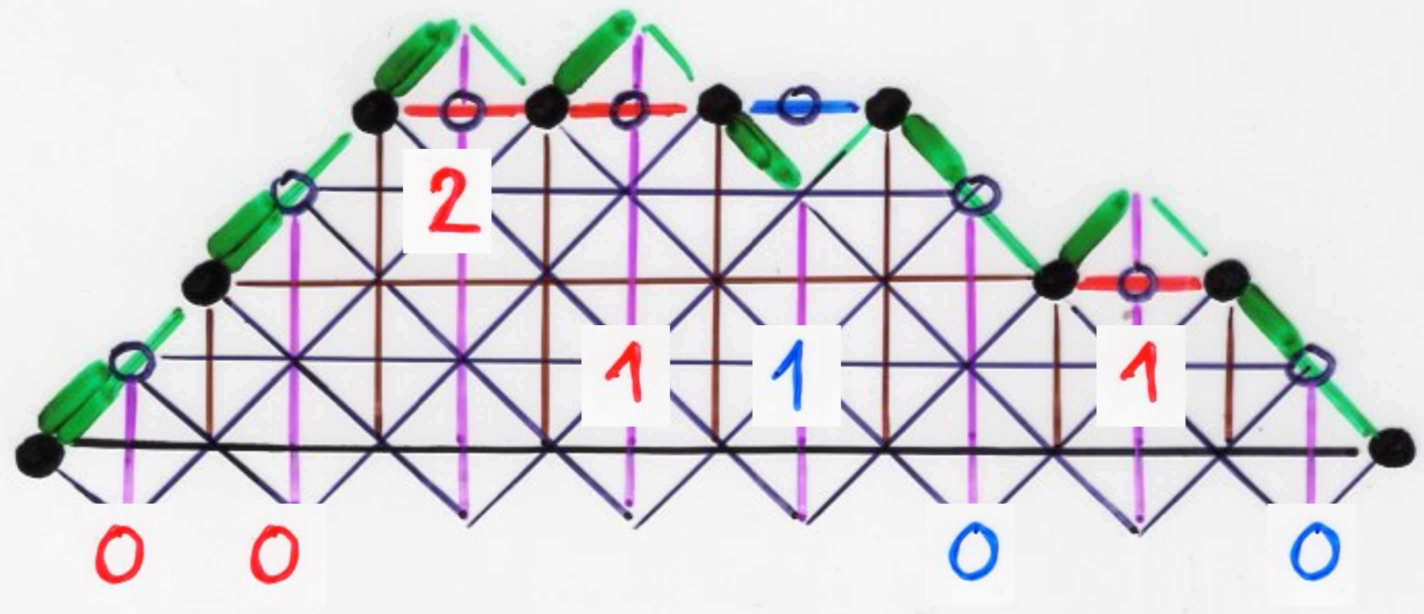
$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t - 1^2 t^2 \cfrac{}{1 - 3t - 2^2 t^2 \cfrac{}{1 - 5t - 3^2 t^2 \cfrac{}{\cdots}}}}$$

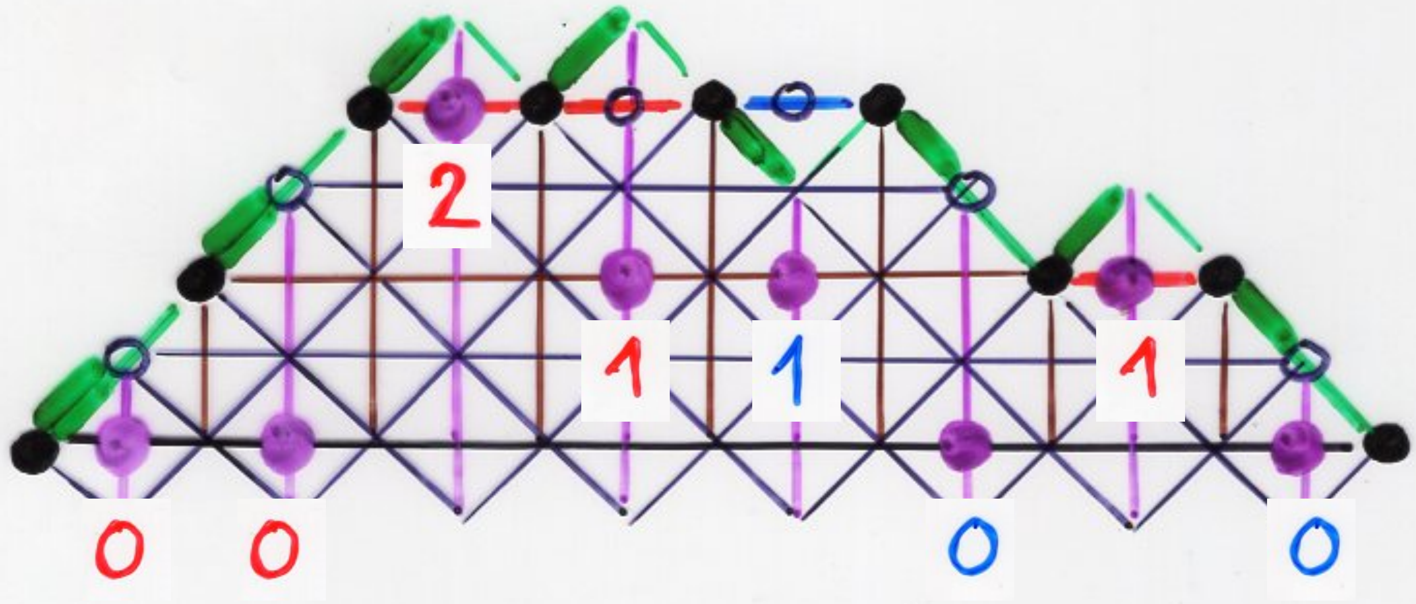


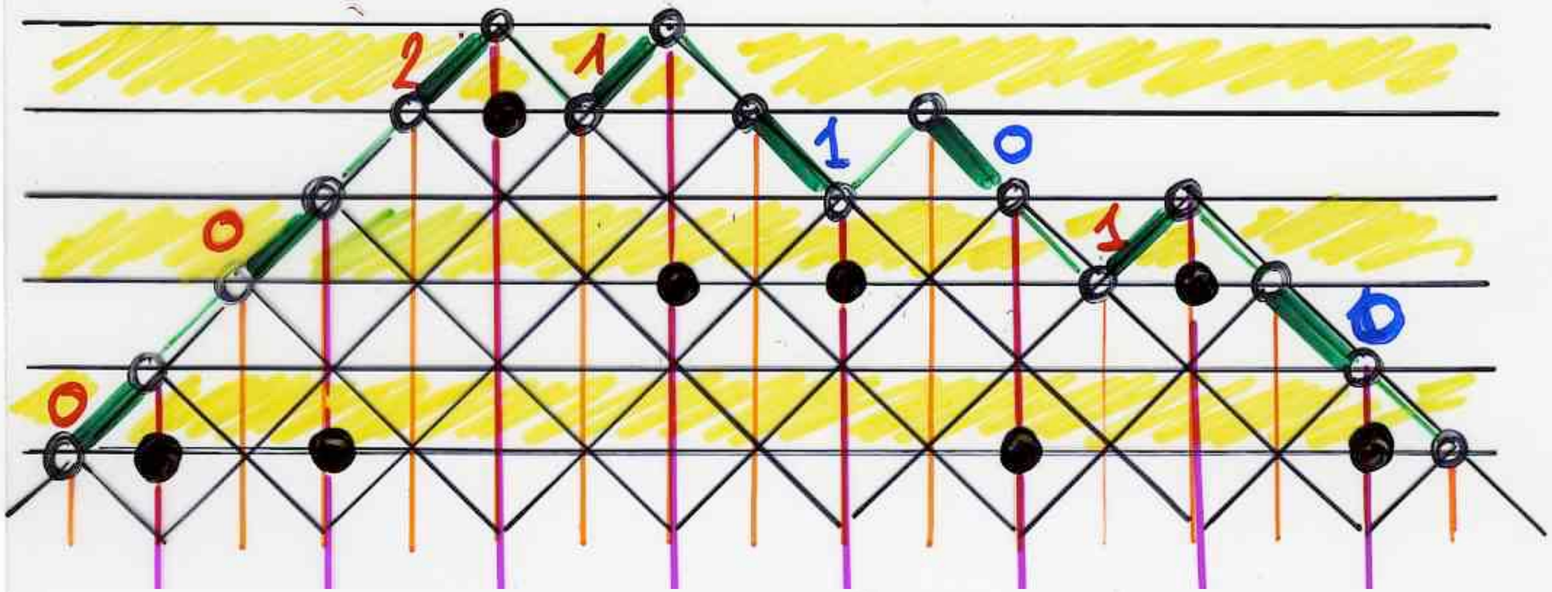

0
0
2
1
1
0
1
0



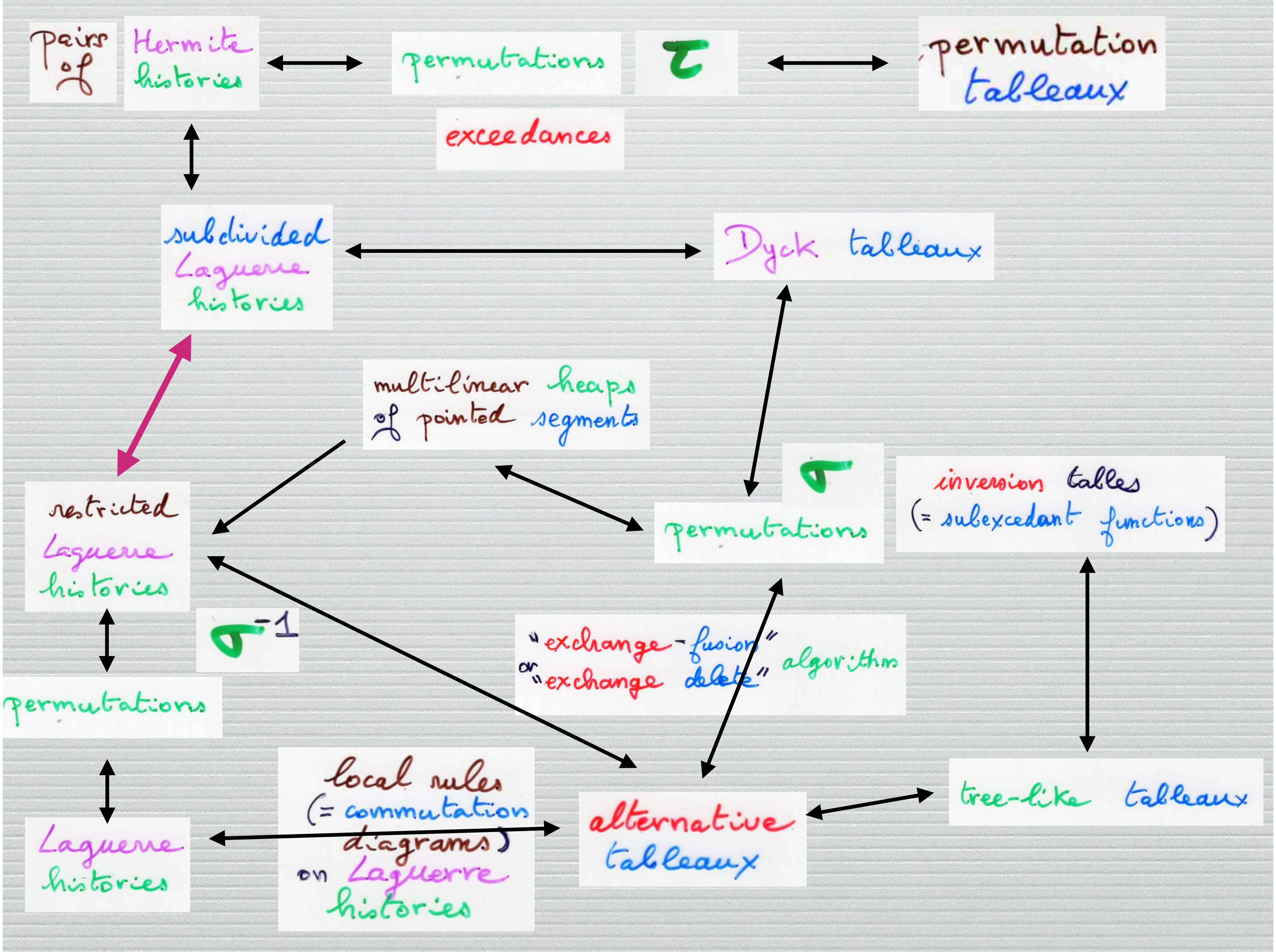
2
1
0
1
0
1
0
0


2
1
1
1
0
0
0
0


2
1
1
1
0
0
0
0



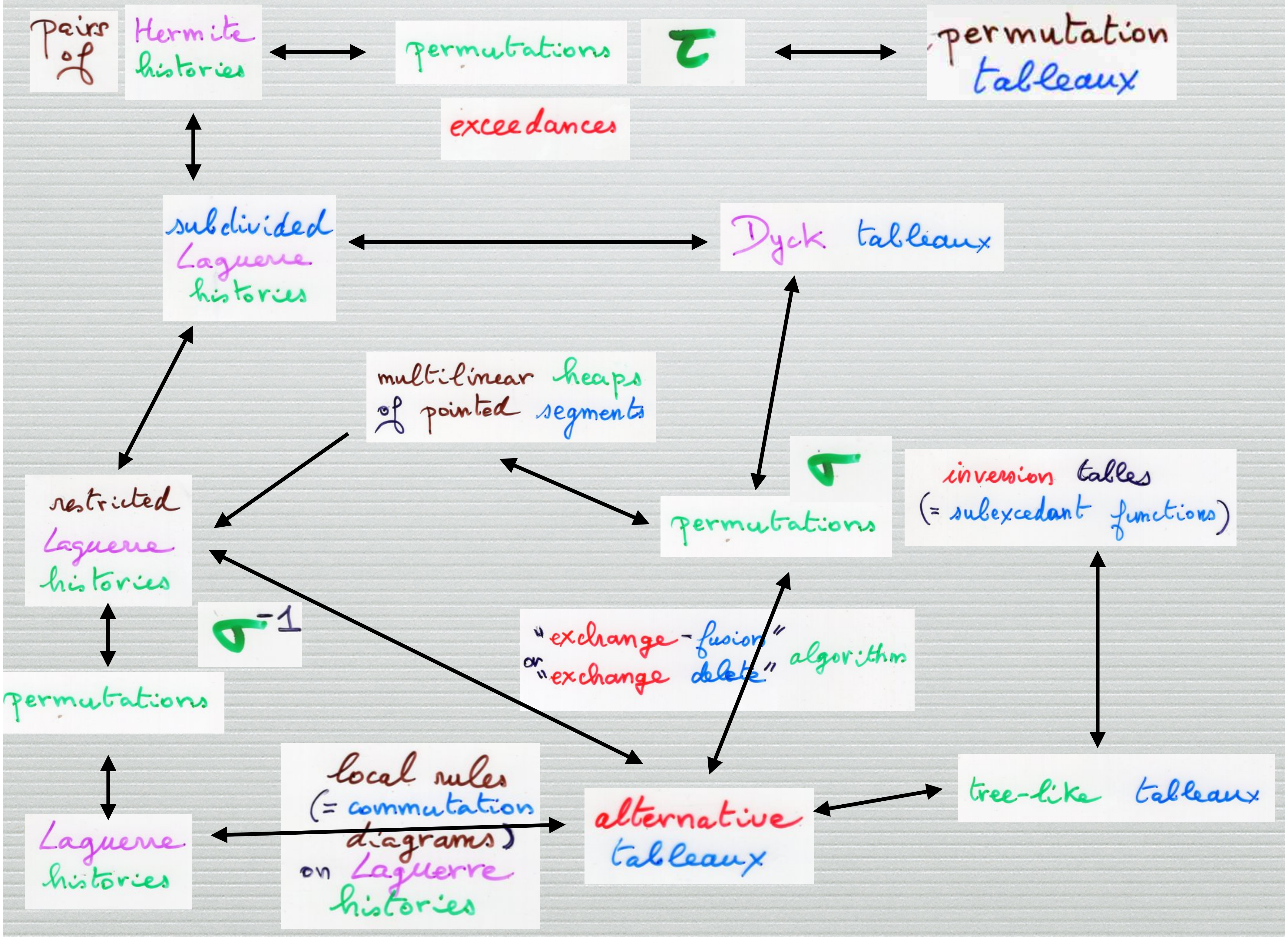
Dyck tableau as a subdivided Laguerre history



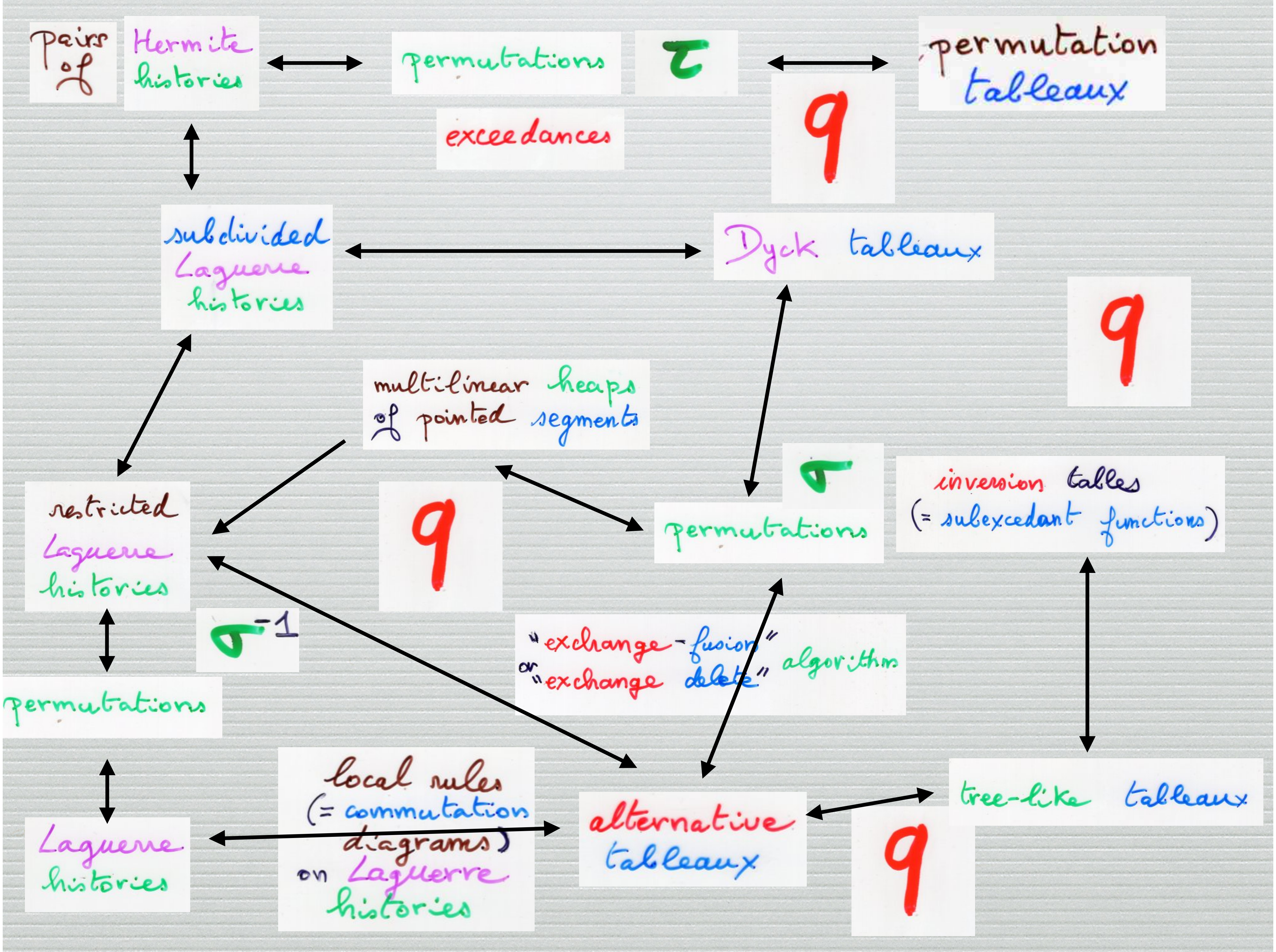
pairs of Hermite histories
permutations τ
exceedances
permutation tableaux
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
inversion tables (= subexcedant functions)
restricted Laguerre histories
σ^{-1}
permutations
"exchange-fusion" or "exchange delete" algorithms
local rules (= commutation diagrams) on Laguerre histories
Laguerre histories
alternative tableaux
tree-like tableaux

Epilogue

The « essence » of bijections …



Pairs of
Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
inversion tables
(= subexcedant functions)
restricted Laguerre histories
σ^{-1}
"exchange-fusion"
or "exchange delete" algorithm
permutations
local rules
(= commutation diagrams)
on Laguerre histories
Laguerre histories
alternative tableaux
tree-like tableaux



Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
q
subdivided Laguerre histories
Dyck tableaux
q
multilinear heaps of pointed segments
σ
inversion tables
(= subexcedant functions)
restricted Laguerre histories
q
permutations
$σ^{-1}$
"exchange-fusion"
or "exchange delete" algorithm
permutations
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories
q

pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

Dyck tableaux

contraction of paths

multilinear heaps of pointed segments

σ

permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

"exchange-fusion" or "exchange delete" algorithm

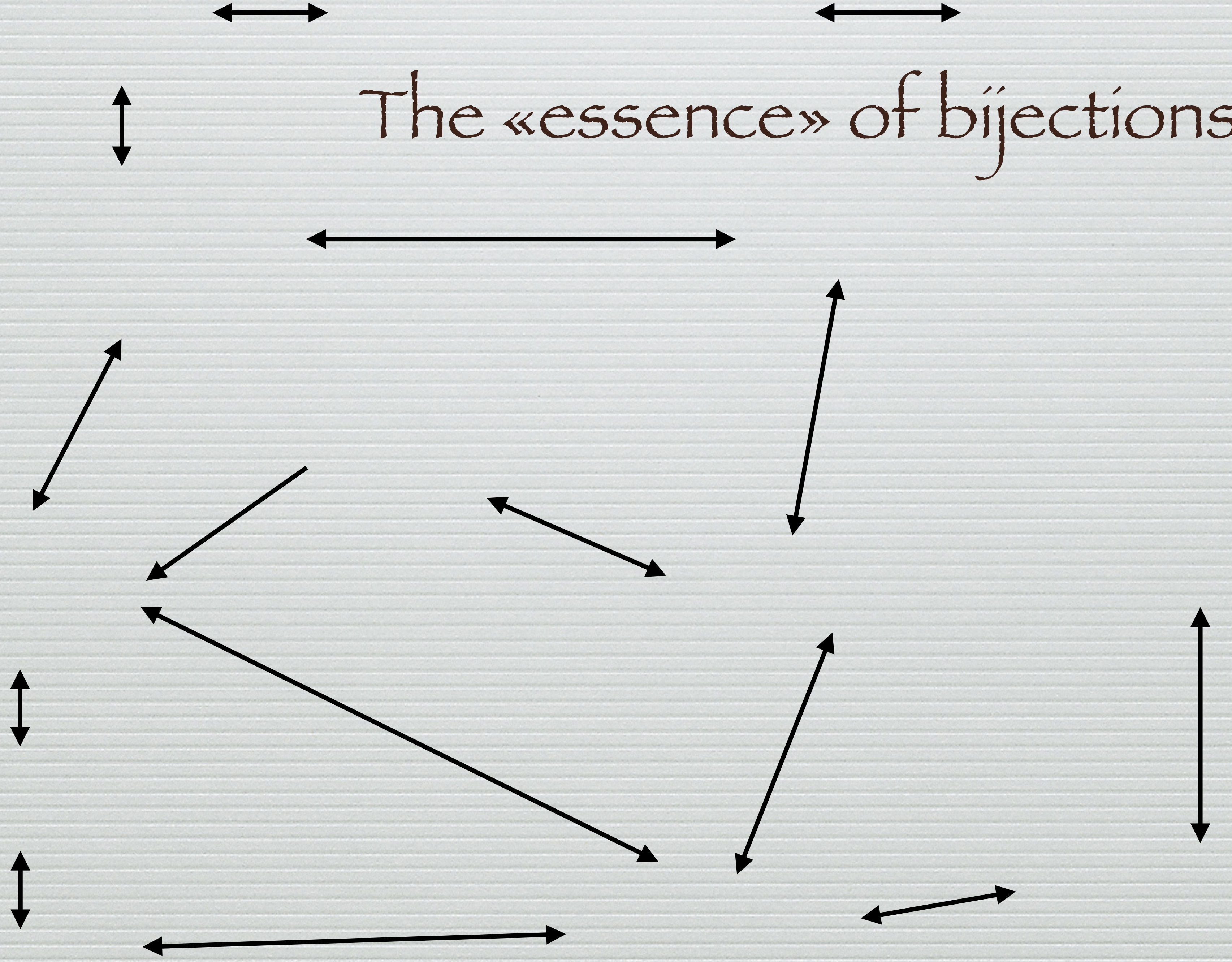
permutations

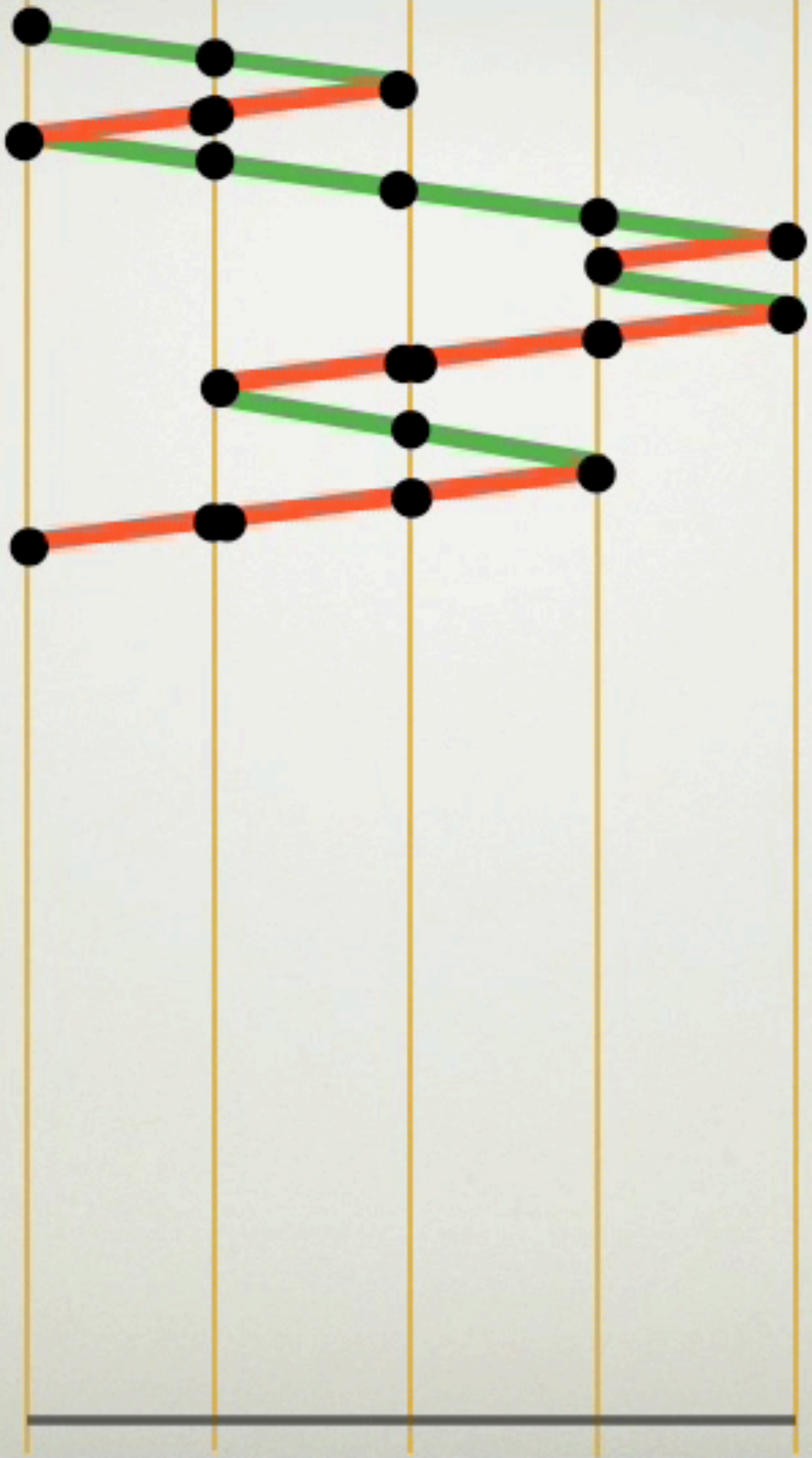
tree-like tableaux

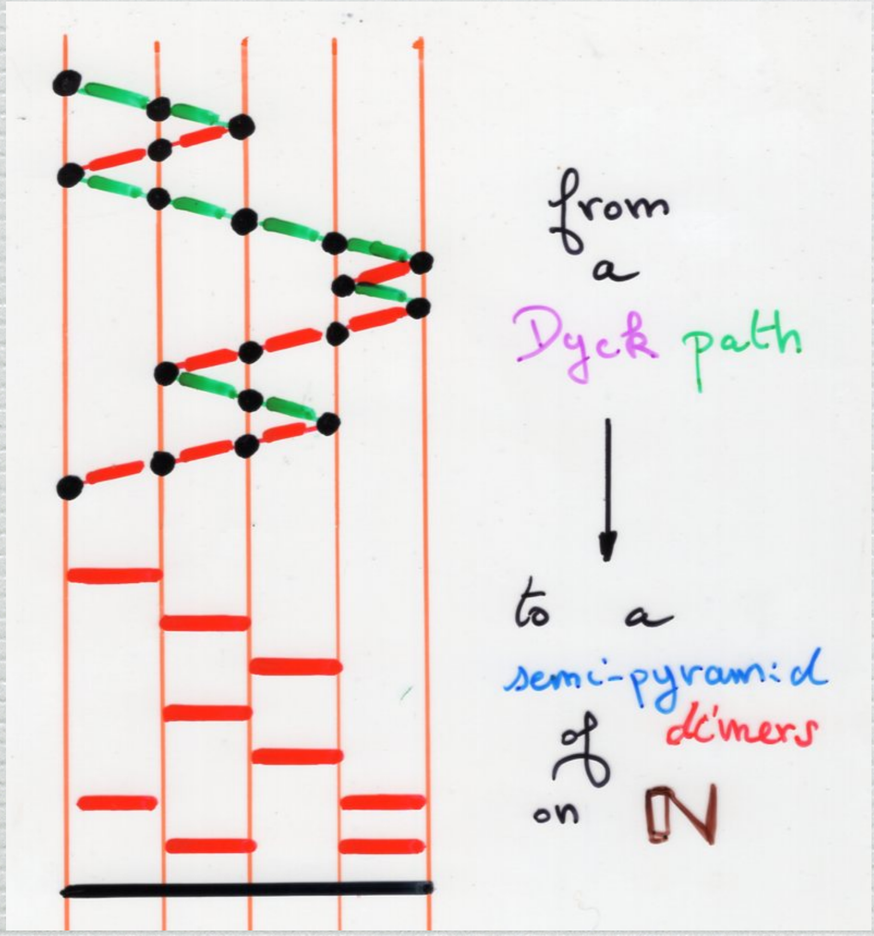
alternative tableaux

Laguerre histories

The «essence» of bijections …







Video with violon:

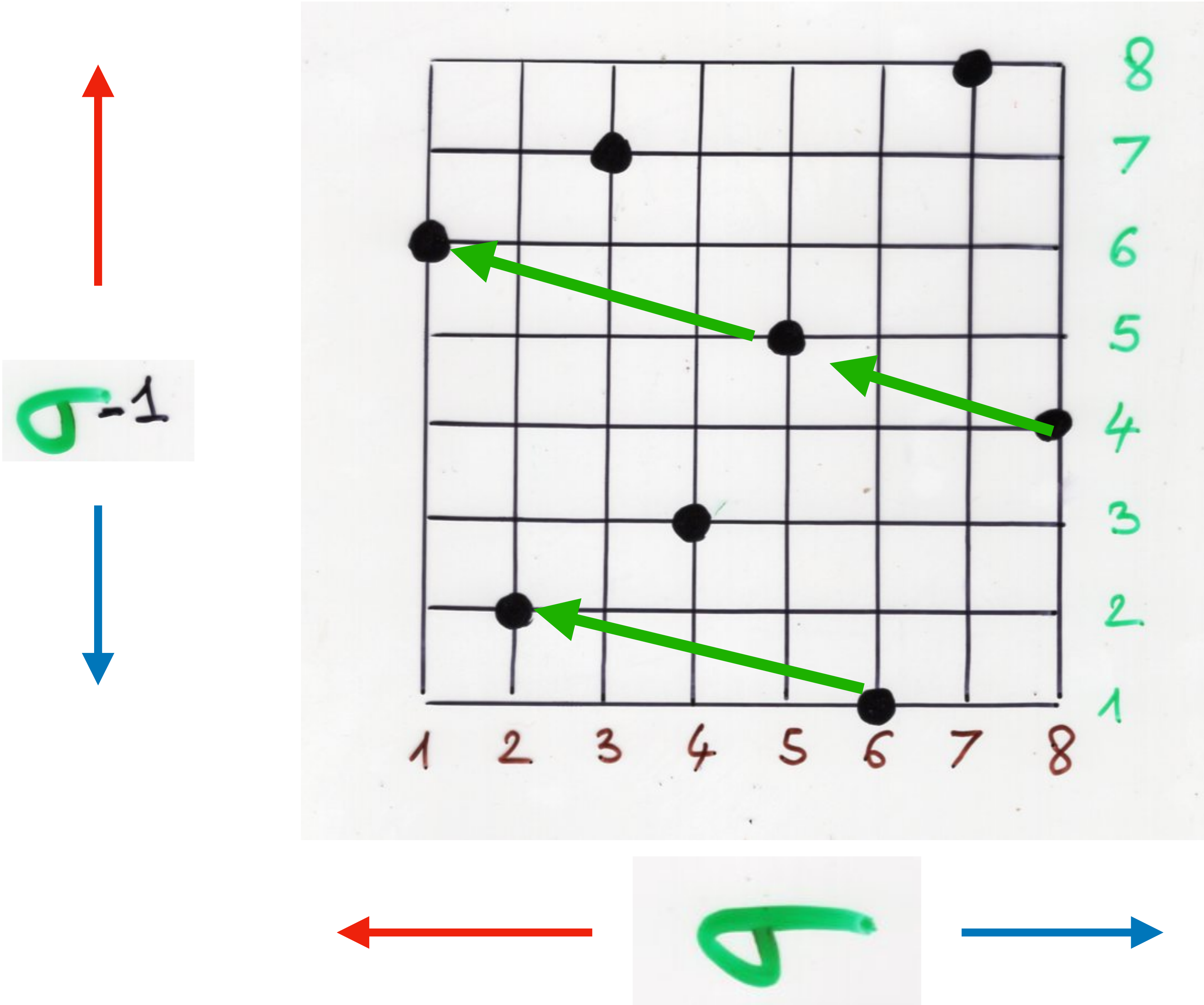
violonist: Gérard Duchamp

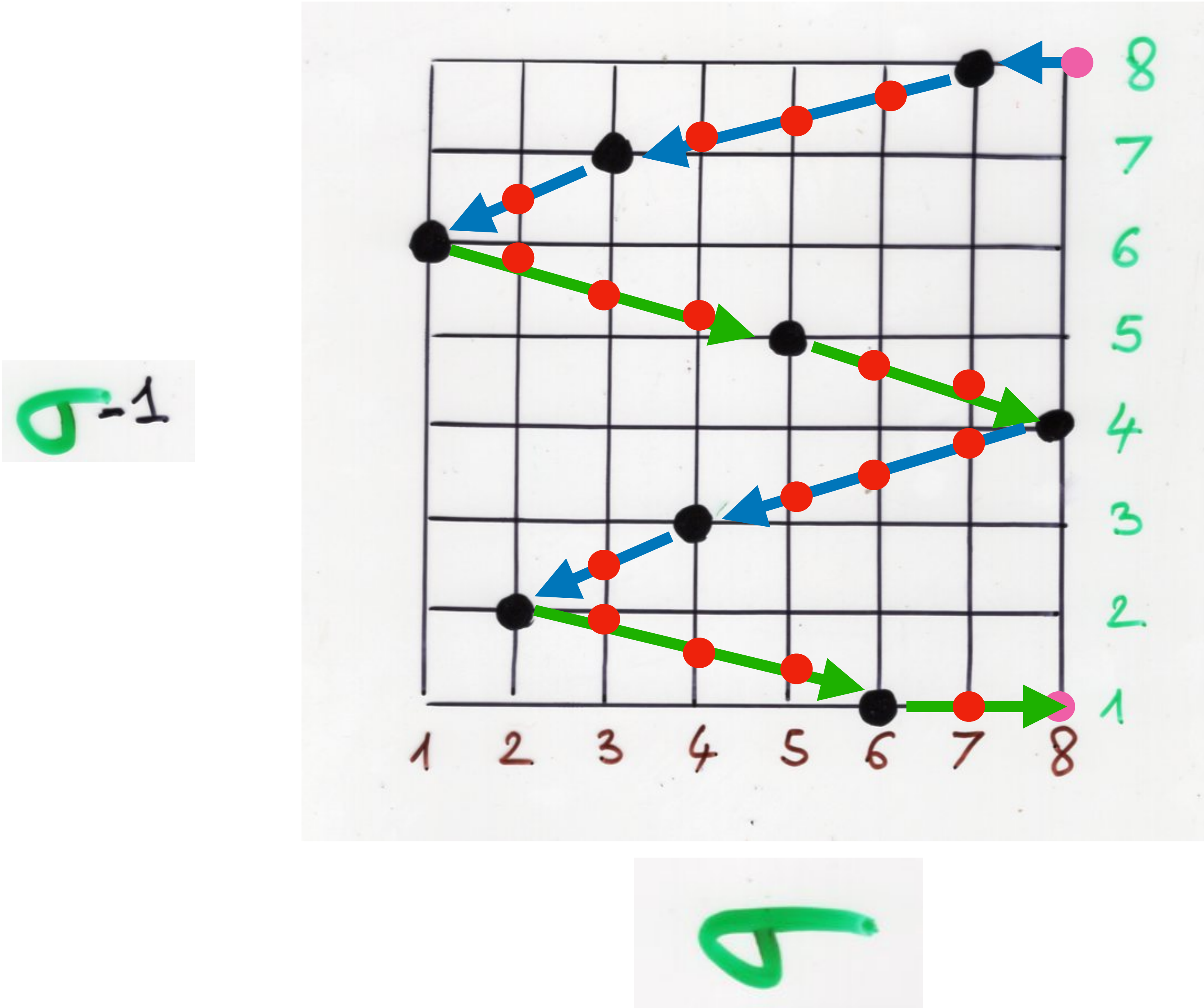
(association Cont'Science)

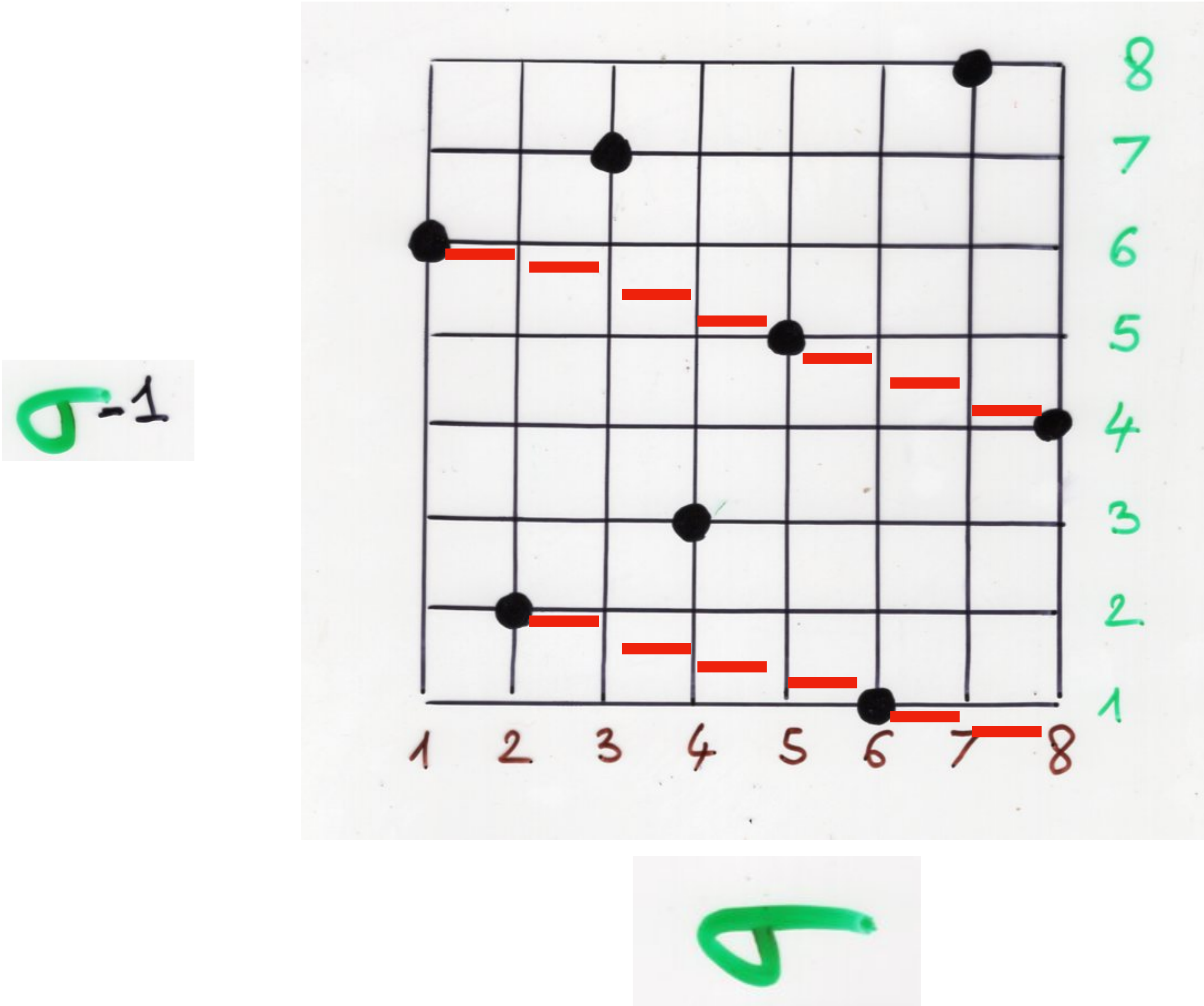
Bijection paths — heaps,

see « the art of bijective combinatorics » II, Ch3b p 26-40,

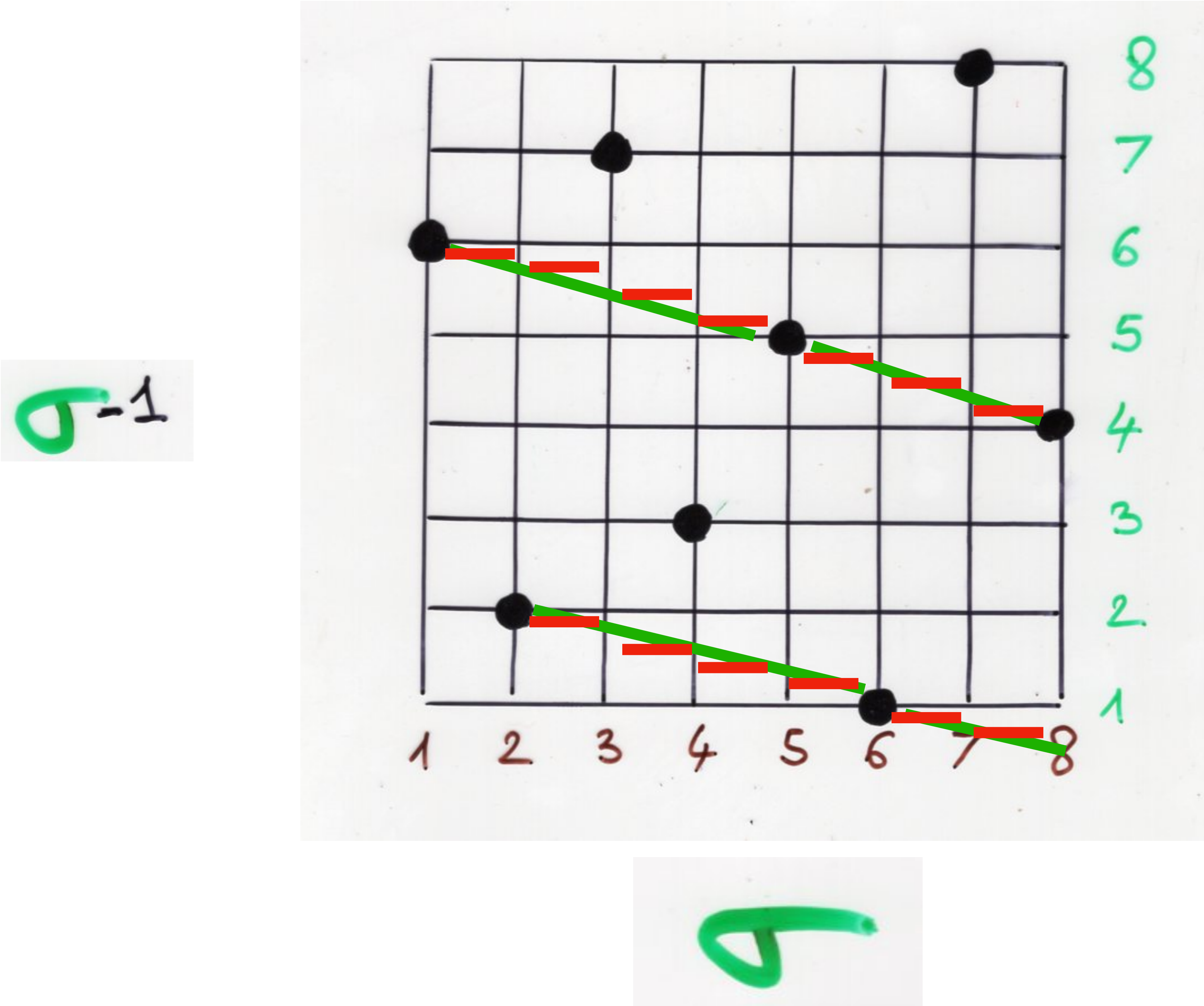
and p 42,60 in the case of Dyck paths.


8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ-1
σ

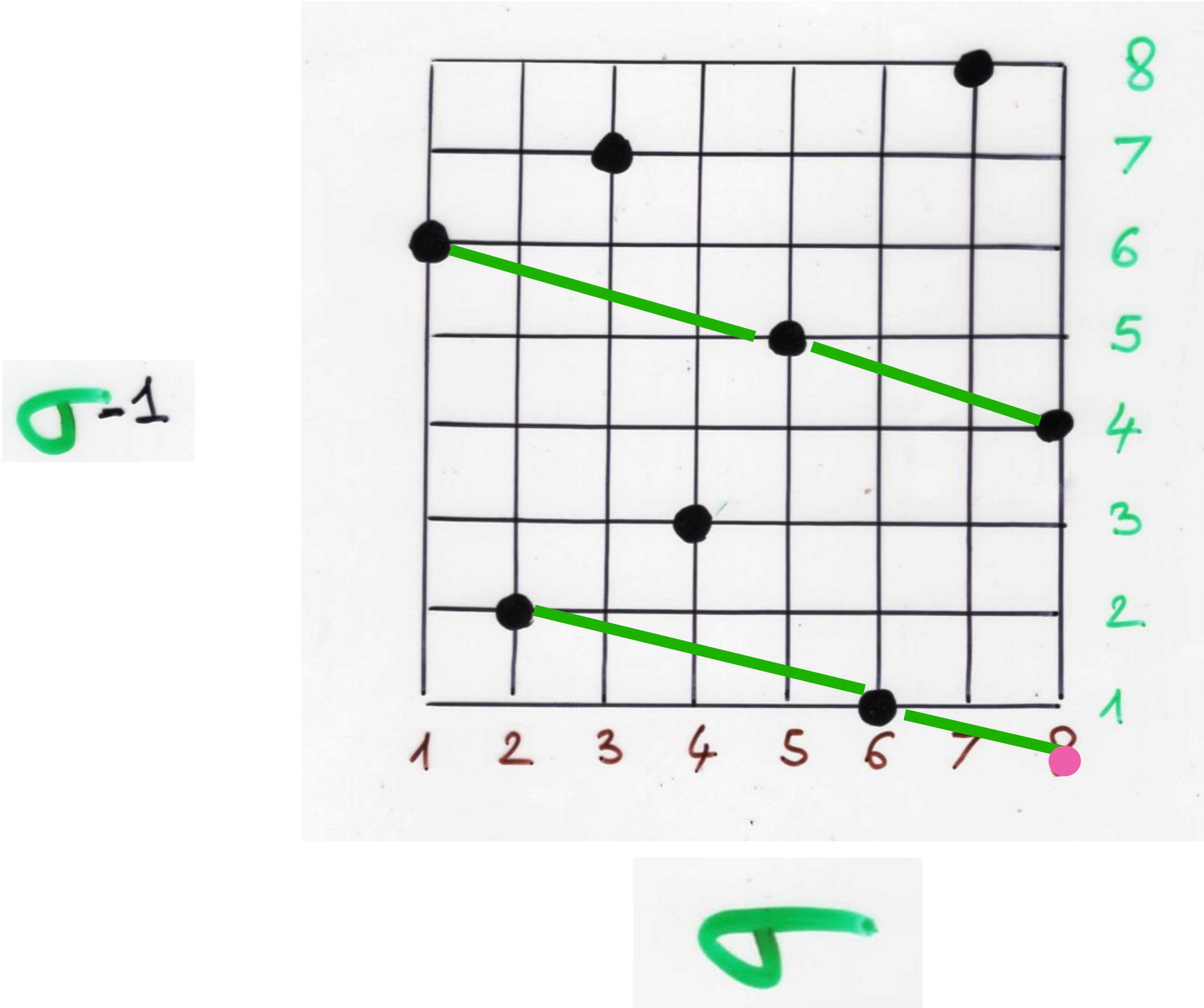

8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ-1
σ



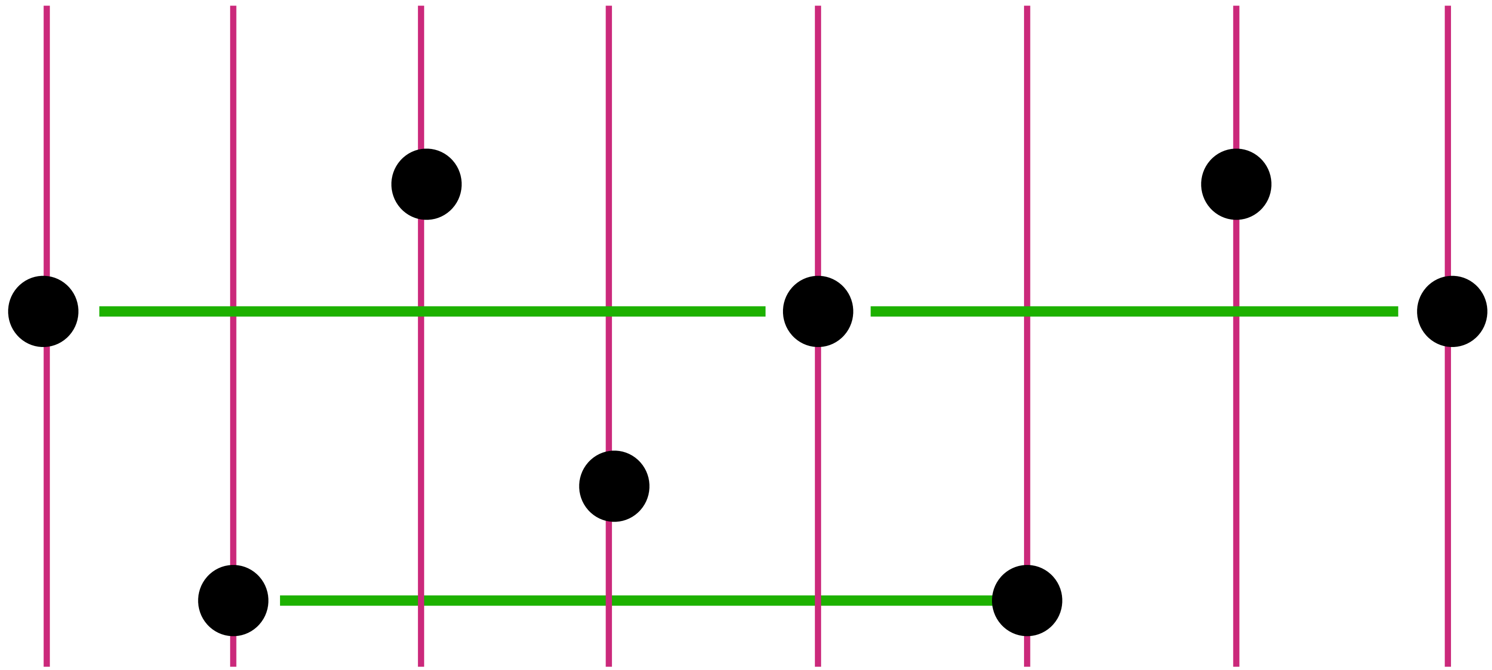
σ-1
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ

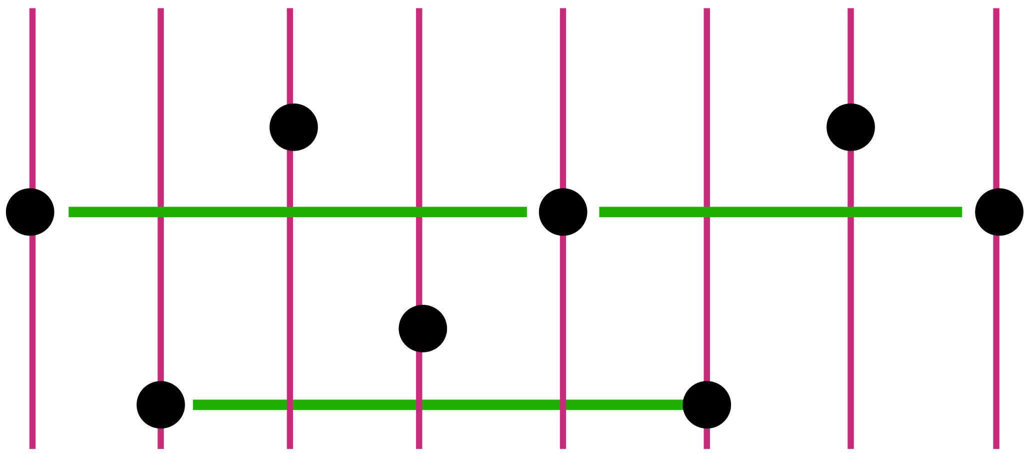


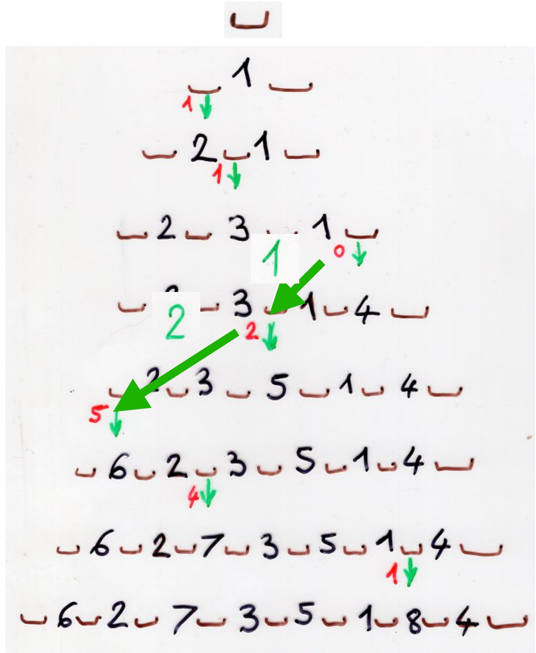
σ⁻¹
σ
1 2 3 4 5 6 7 8
8 7 6 5 4 3 2 1


σ^{-1}
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ

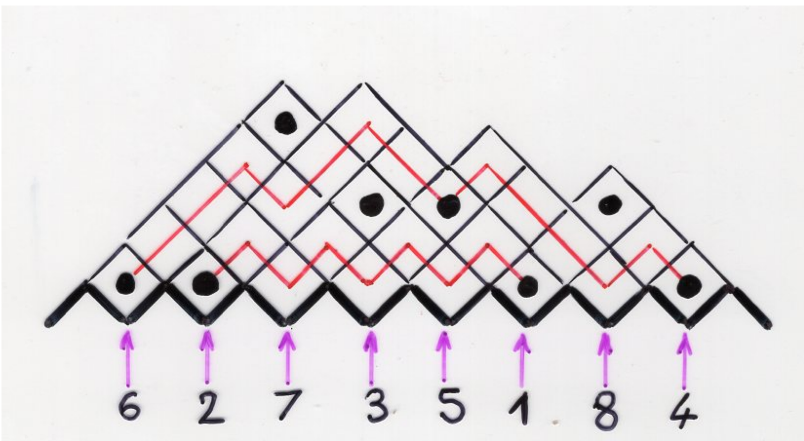
σ = 6 2 7 3 5 1 8 4

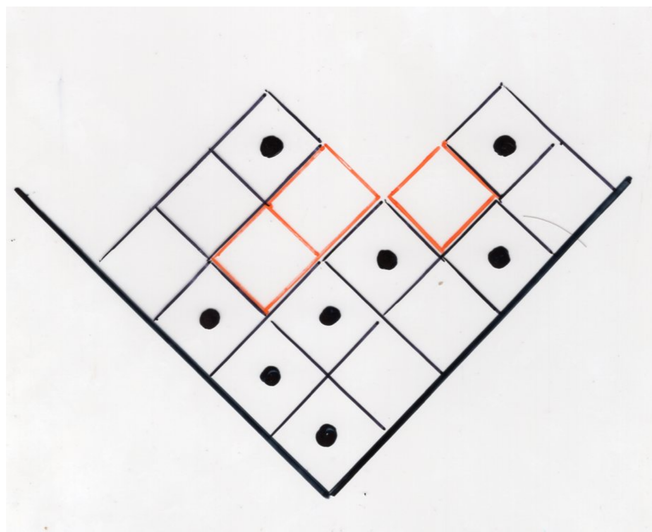


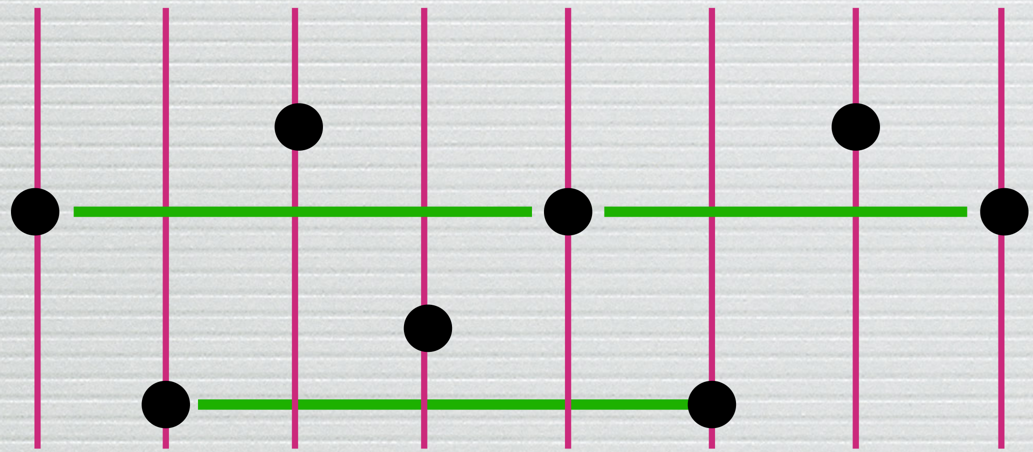




σ = 6 2 7 3 5 1 8 4





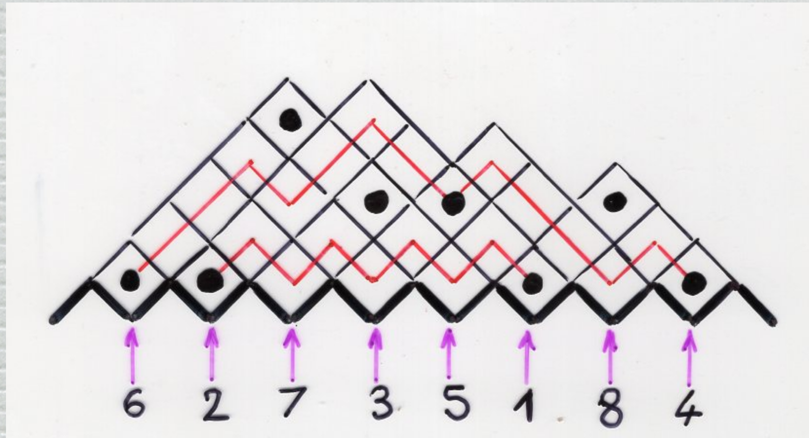


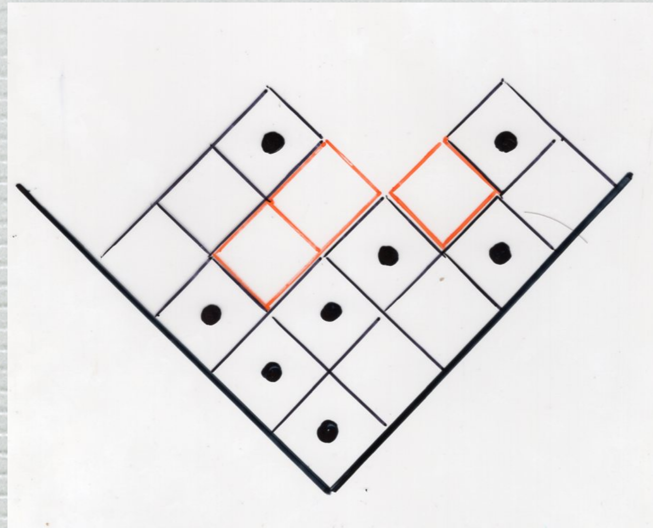
Next slides:

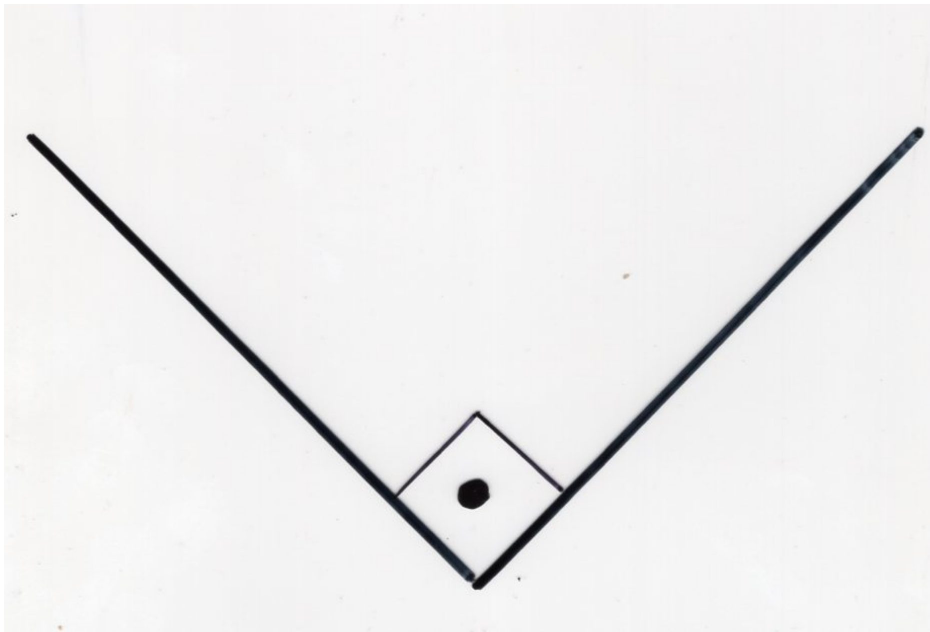
Video with violon:

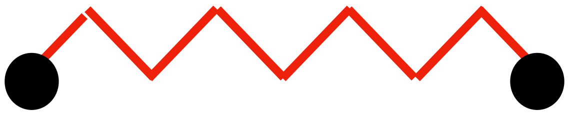
violonist: Gérard Duchamp

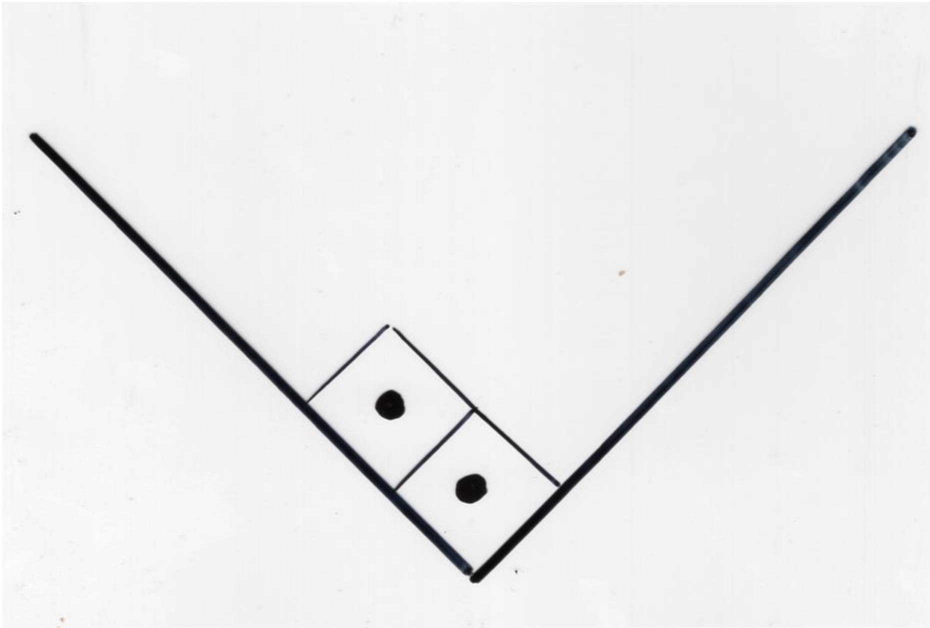
The « essence » of 3 bijections in parallel

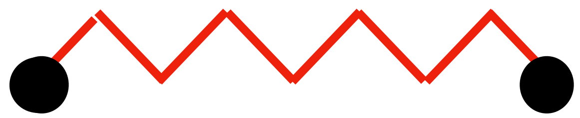


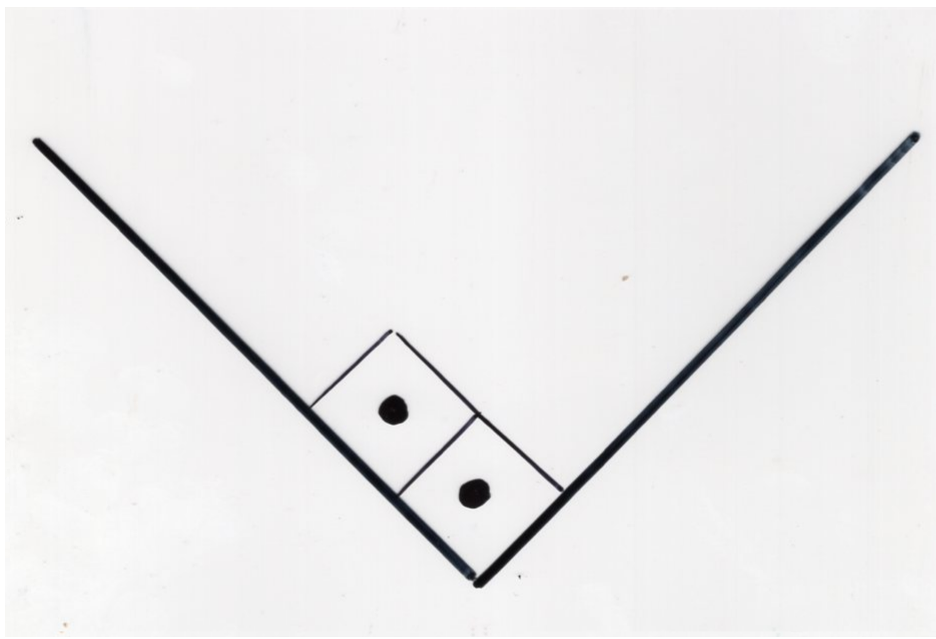


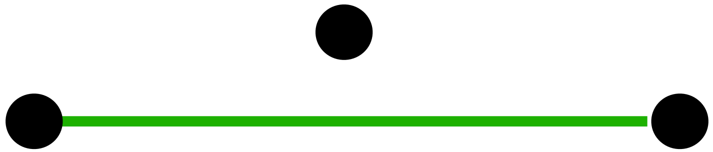


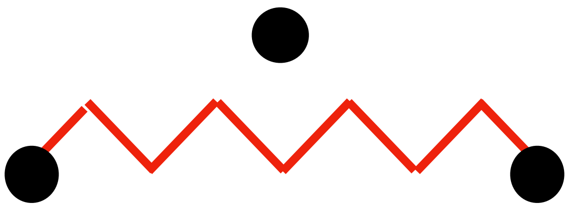


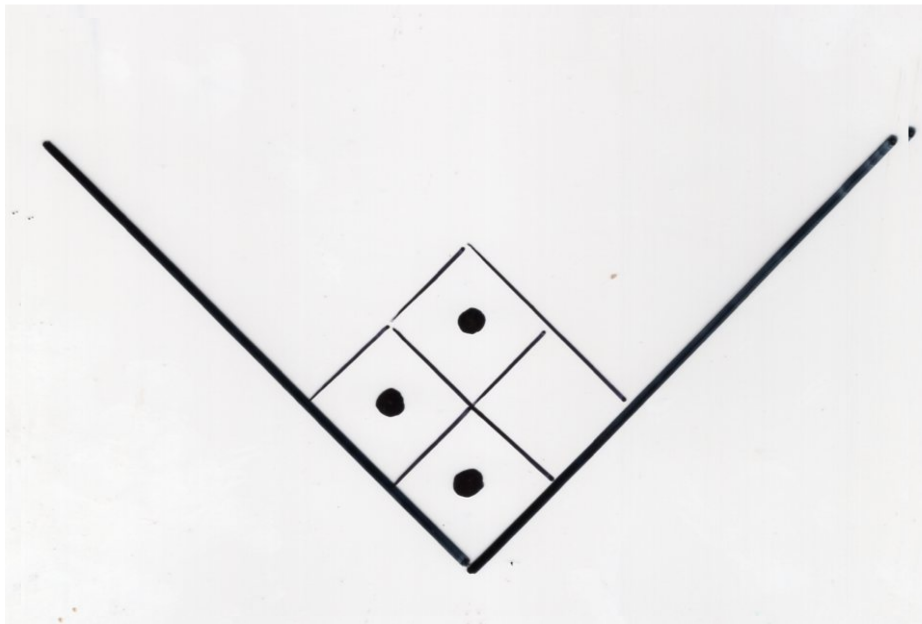


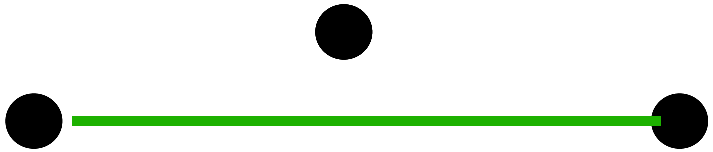


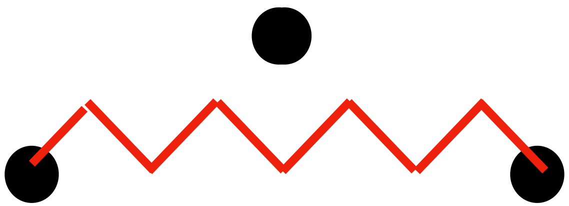


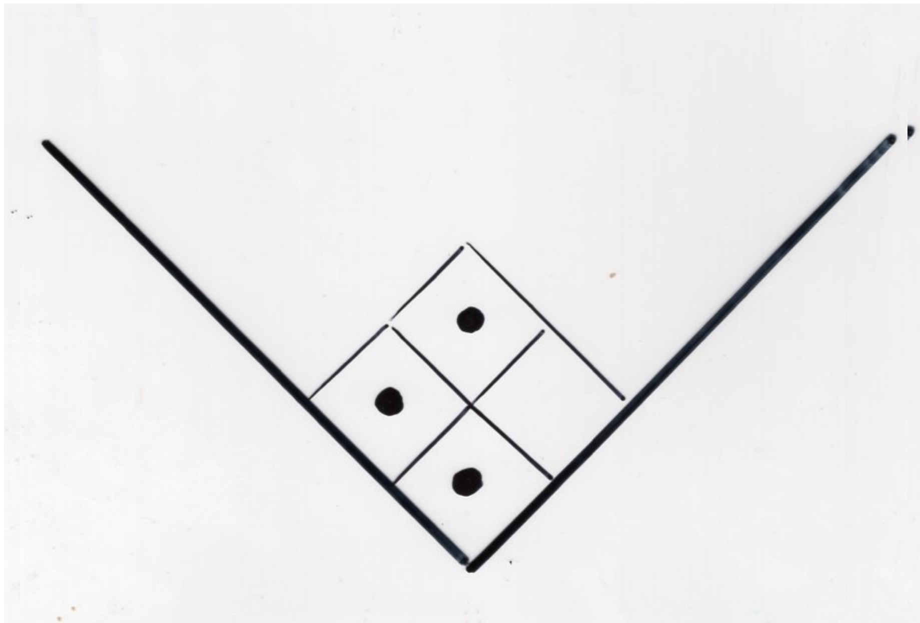


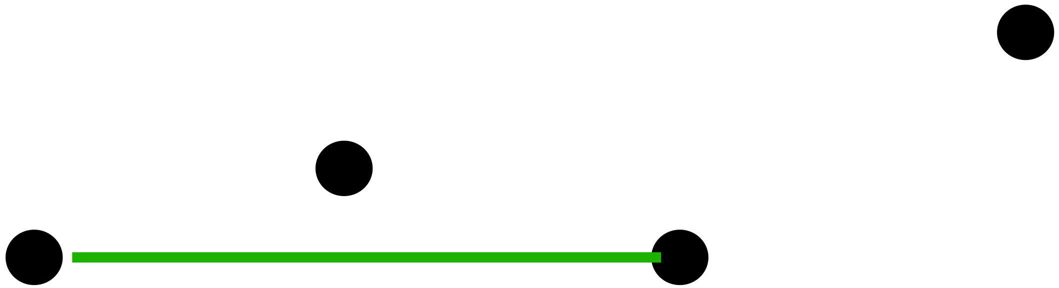


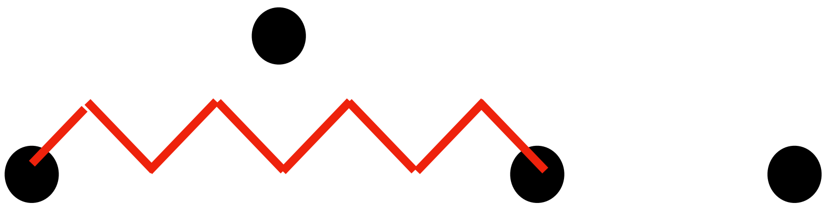


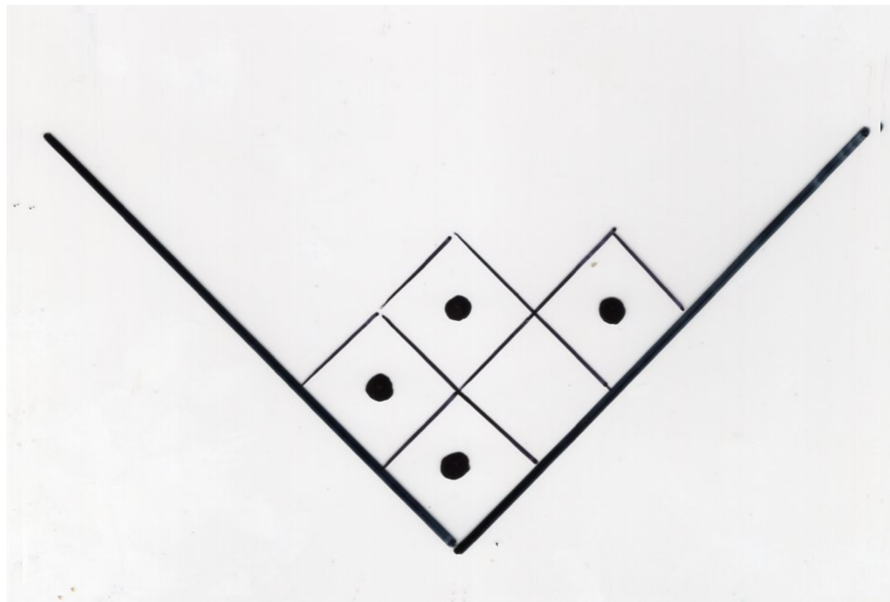


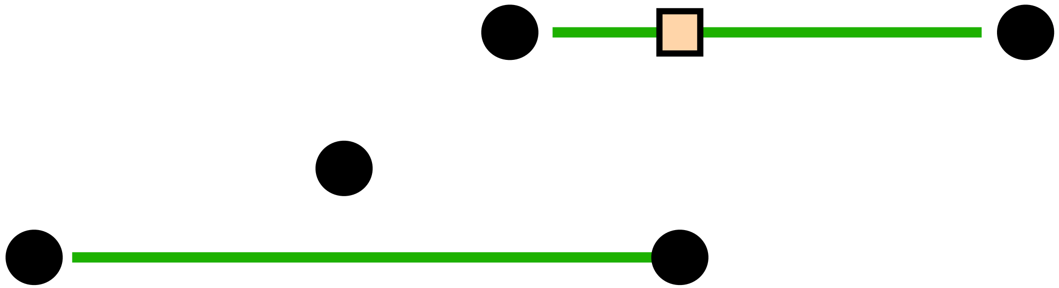


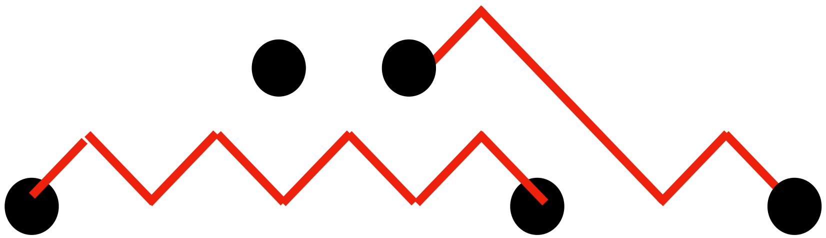


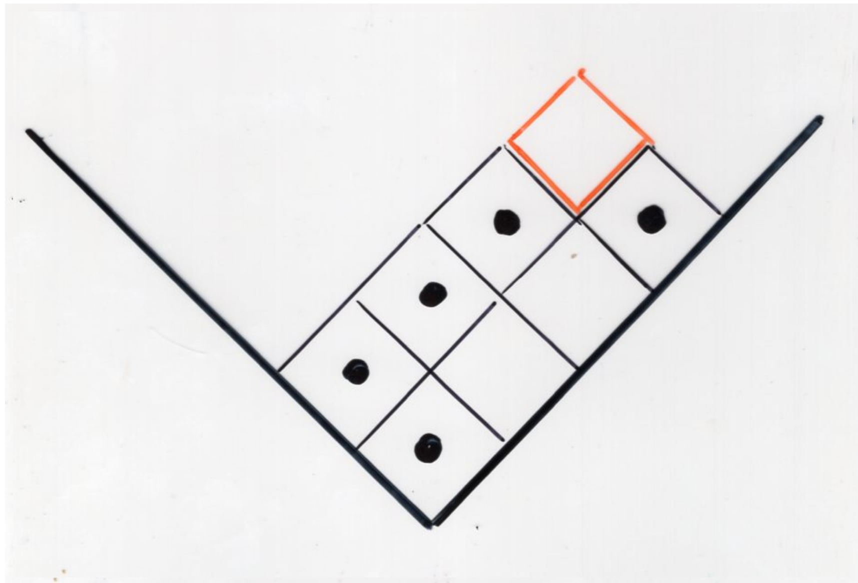


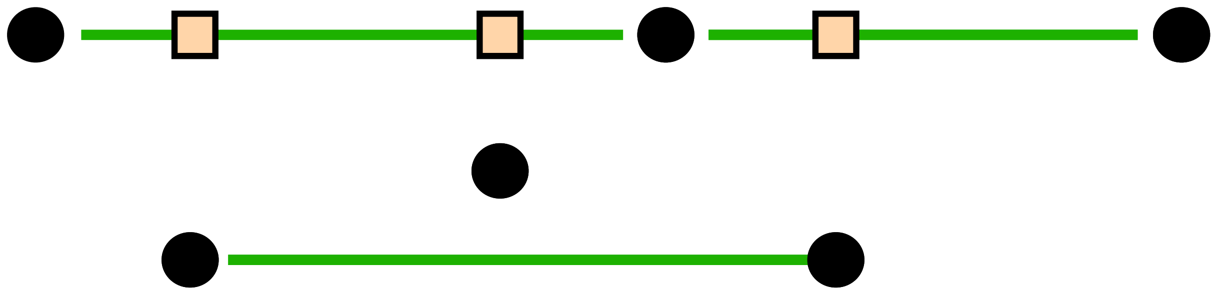


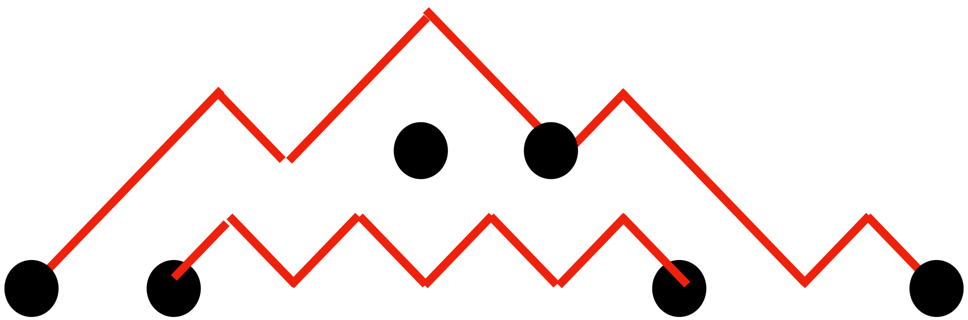


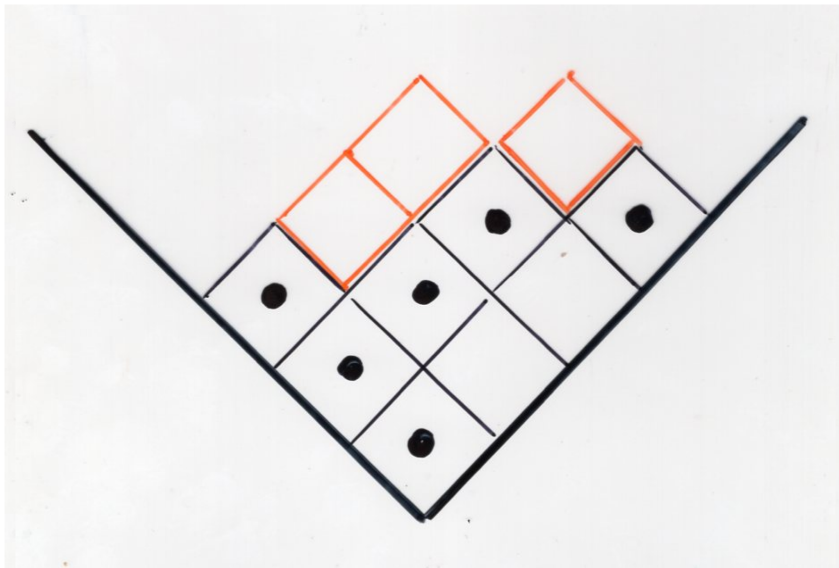


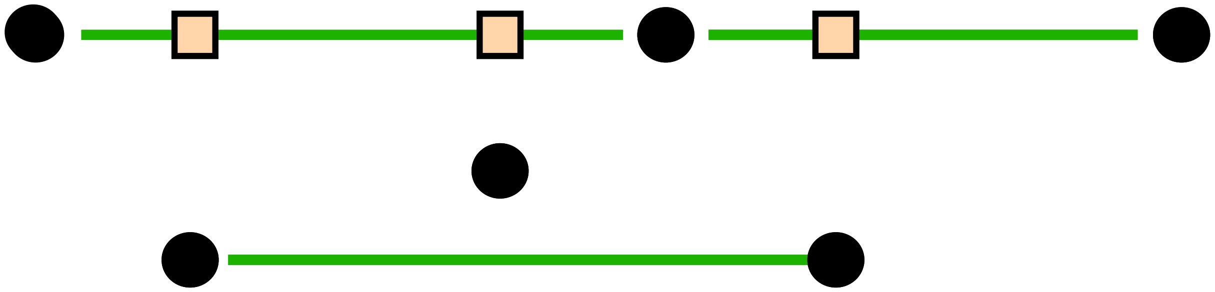


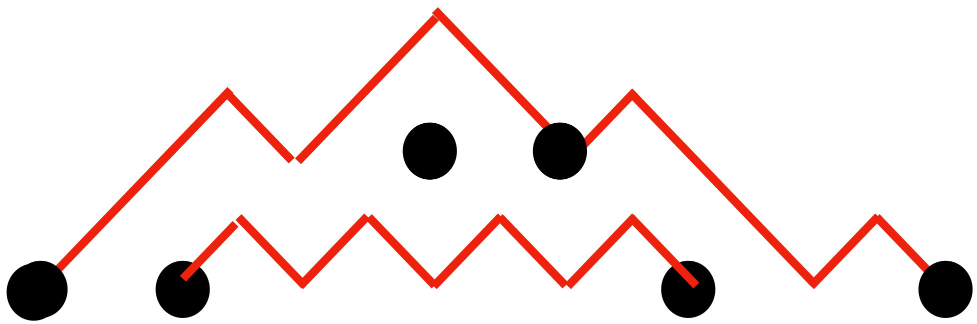


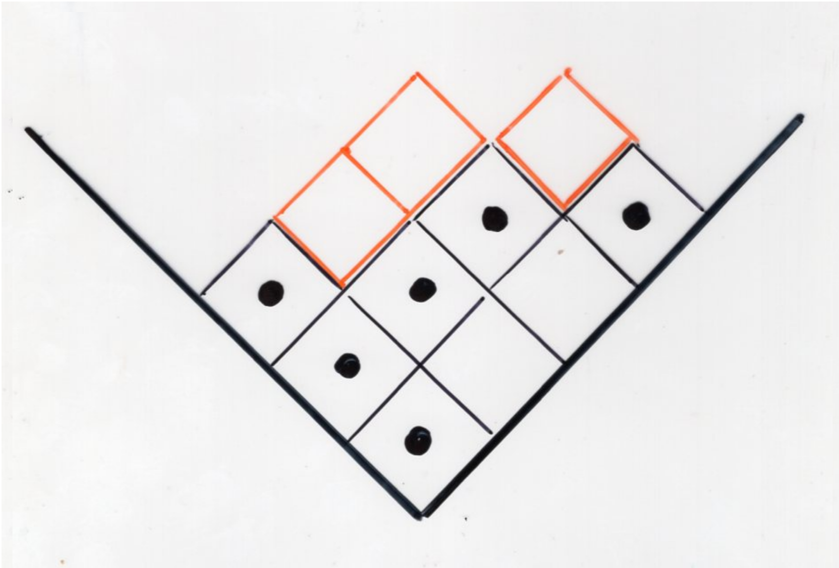


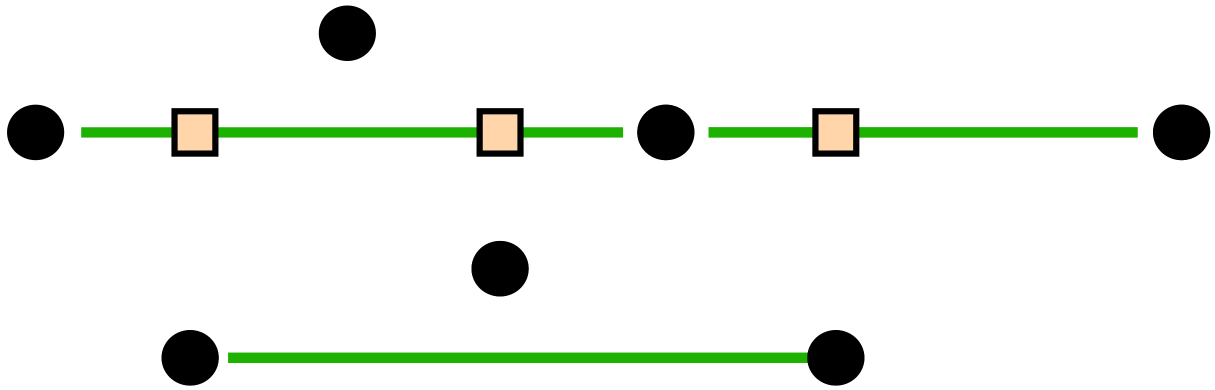


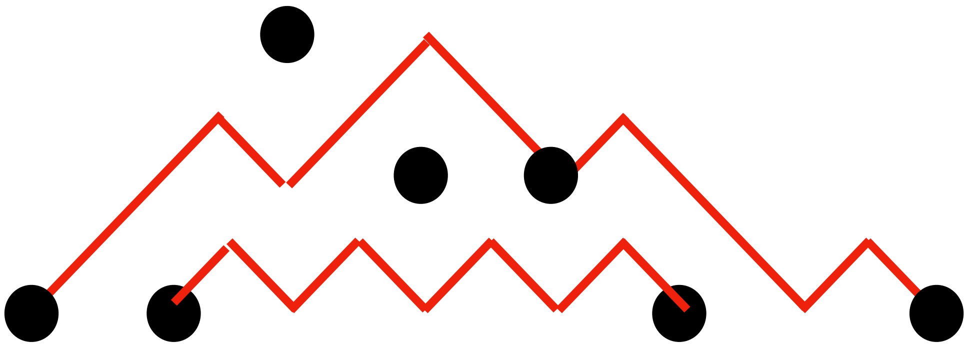


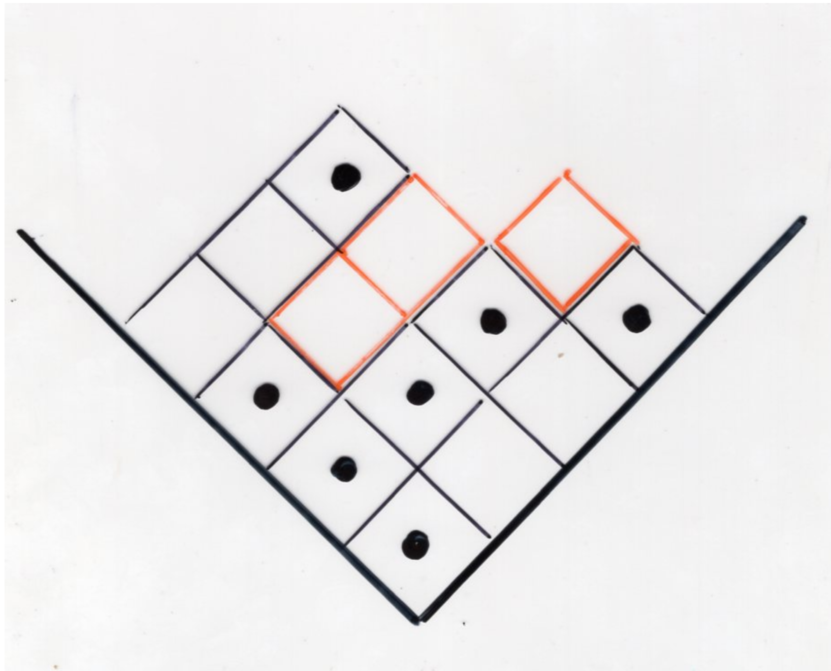


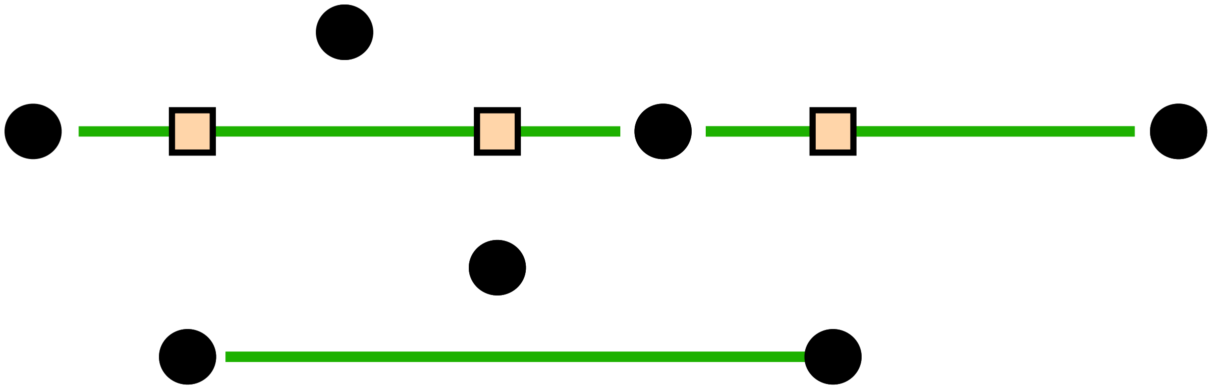


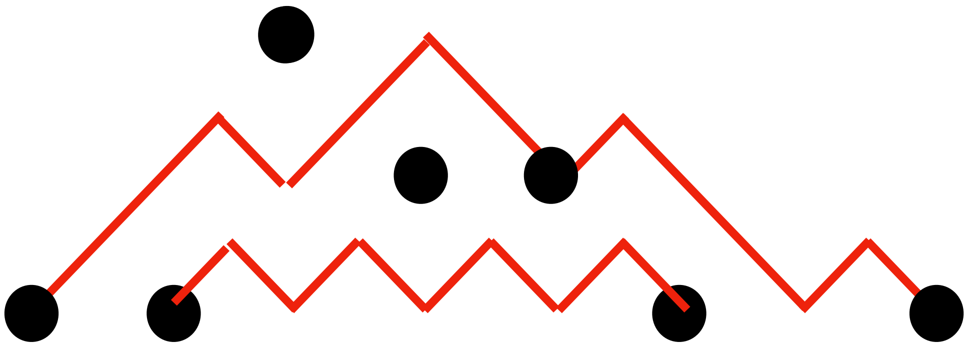


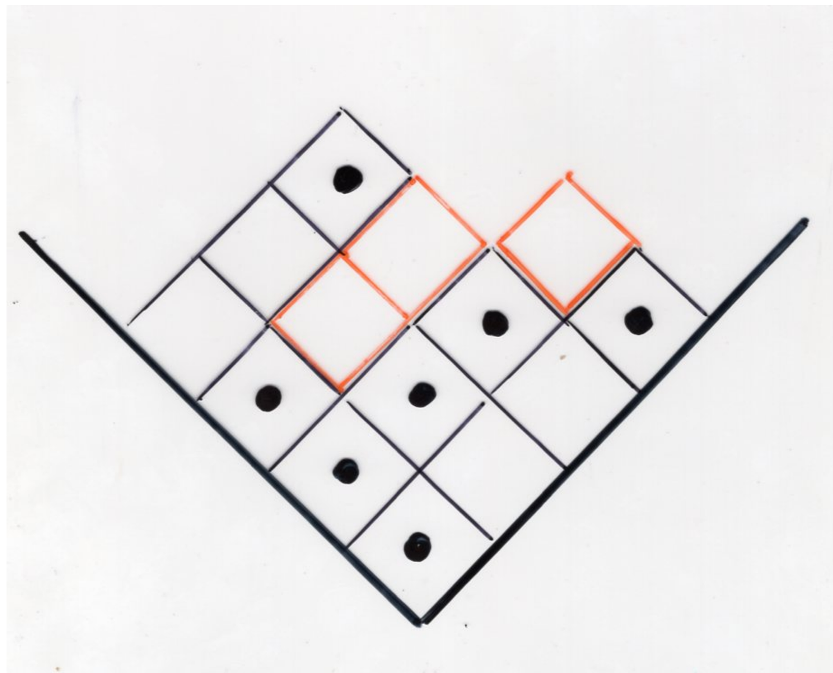


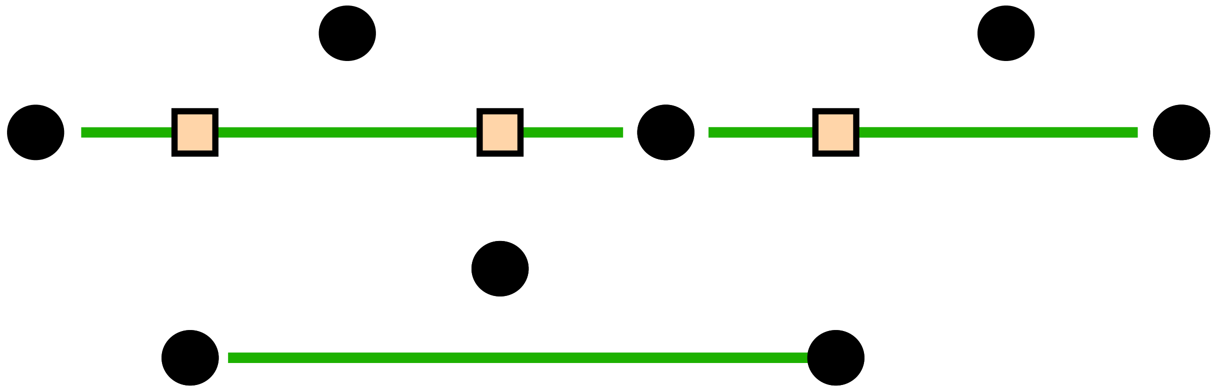


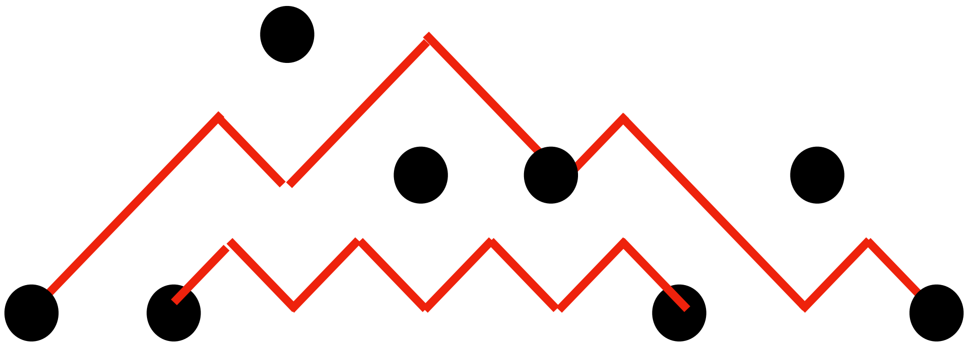


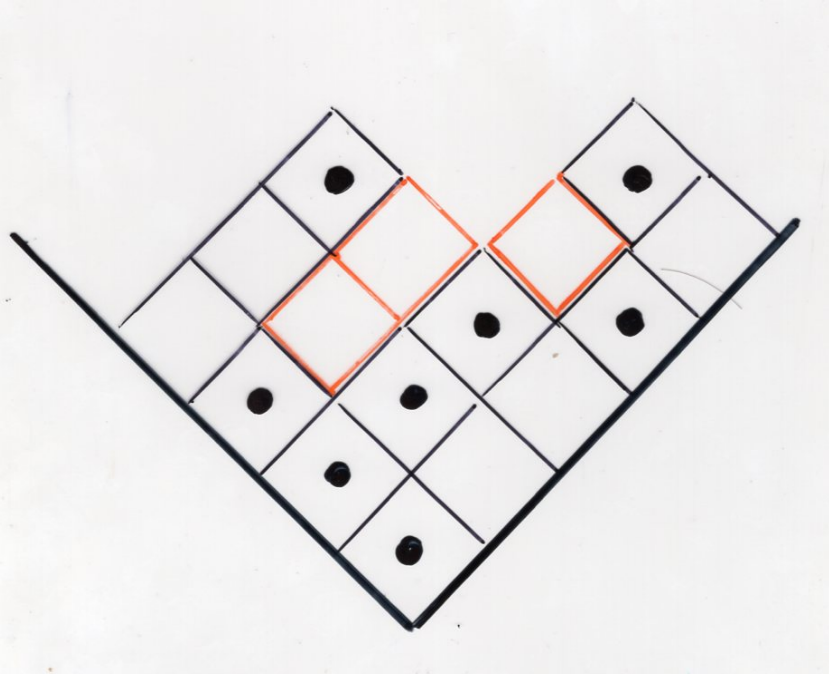


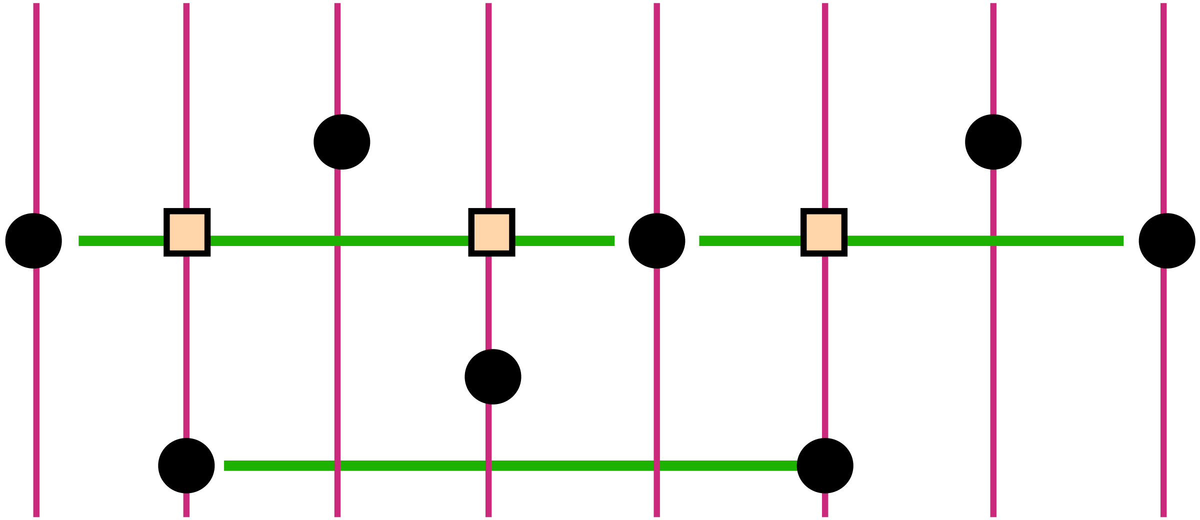


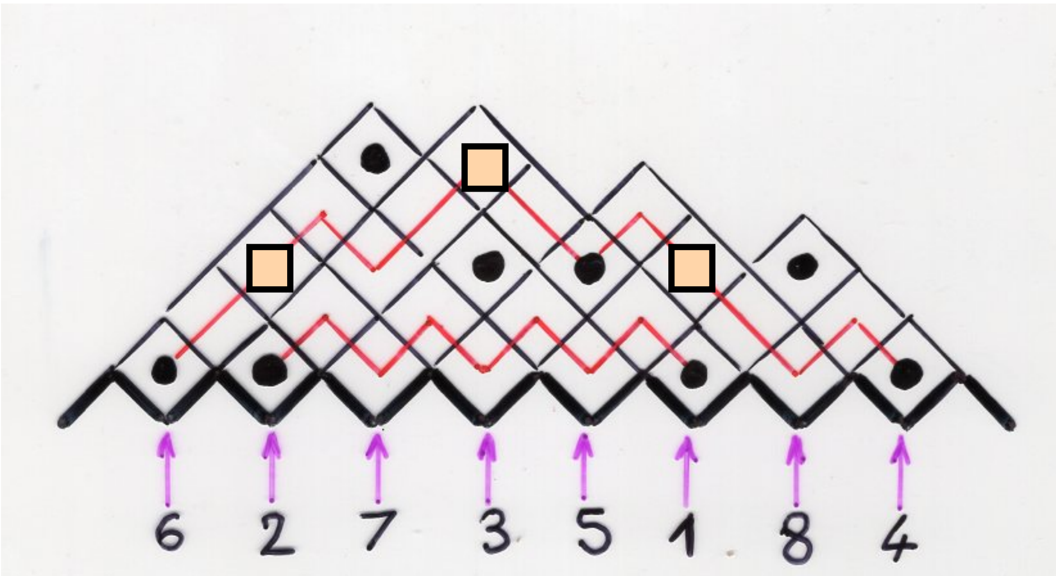


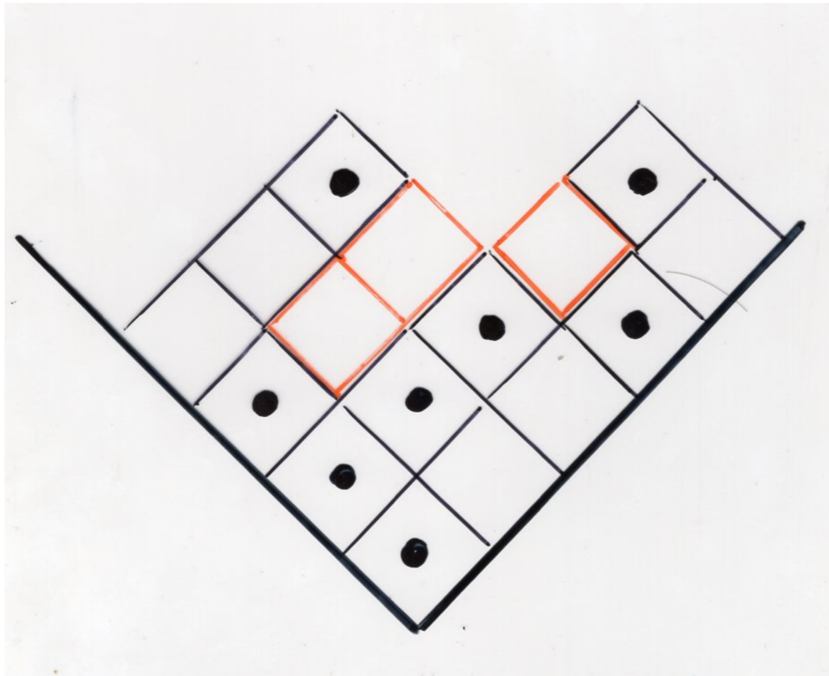









6 2 7 3 5 1 8 4



The «essence» of bijections ...

The end of the bijective course

Thank you very much !

for all of you, students, professors, friends,

For the videos:

Gayathri and Kirubananth

special thanks to Suresh and Amri

ॐ सरस्वत्यै नमः।