Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 5
Orthogonal polynomials and exponential structures

Ch5b

IMSc, Chennai
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Back to Ch 5a

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \dots a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k} z^k $$

hypergeometric

Gauss (1812) hypergeometric series

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] $$

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} $$

Pfaff-Saalschütz
(1797) (1890)

Hermite

$$H_n(x) = (2x)^n \, {}_2F_0\left[\begin{matrix}-n/2, \; -(n-1)/2 \\ - \end{matrix}; -\frac{1}{x^2}\right]$$

Laguerre

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left(\begin{matrix}-n \\ \alpha+1\end{matrix}\middle| x\right)$$

$(\alpha > -1)$

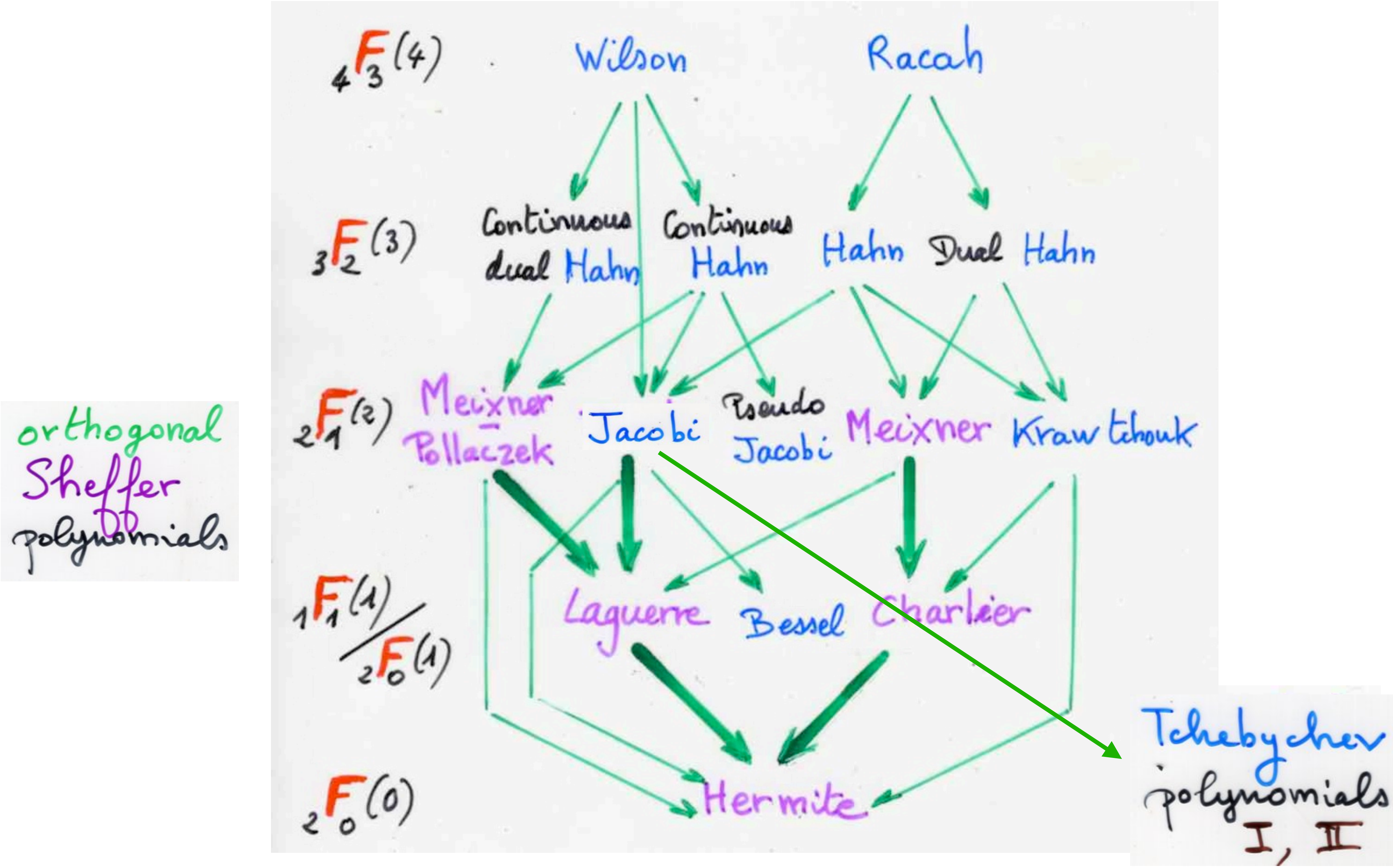
Charlier

$$C_n^{(a)}(x) = {}_2F_0\left[\begin{matrix}-n, \; -x \\ - \end{matrix}; a^{-1}\right]$$

Meixner

Meixner – Pollaczek

Askey scheme of hypergeometric orthogonal polynomials



Sheffer polynomials

$$\sum_{n\geq 0} T_n(x)\frac{t^n}{n!} = g(t)\exp\left(x f(t)\right)$$

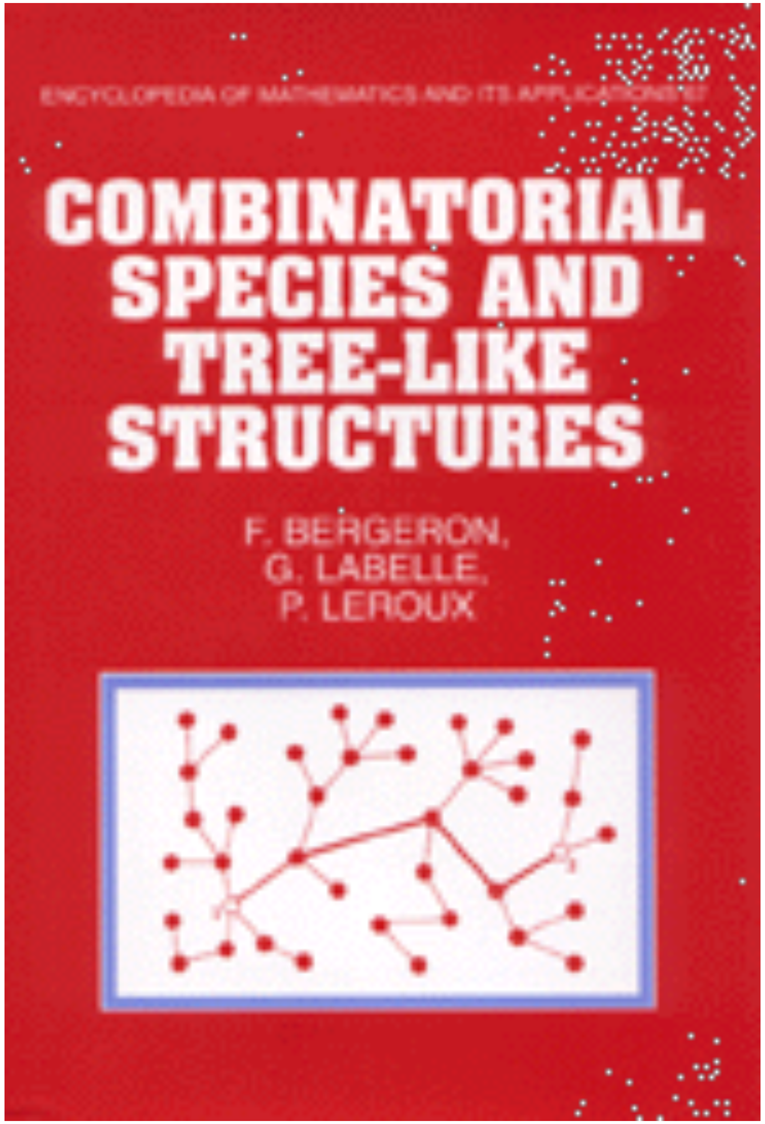
Hermite

Laguerre

Charlier

Meixner

Meixner-Pollaczek



Combinatorial model for exponential generating function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Species

(combinatorial) structures

UQAM Montréal Québec

(combinatorial) Hermite polynomials

$$\sum_{n \geq 0} He_n(x) \frac{t^n}{n!} = e^{\left(xt - \frac{t^2}{2}\right)}$$

"probabilists' Hermite polynomial" $He_n(x)$

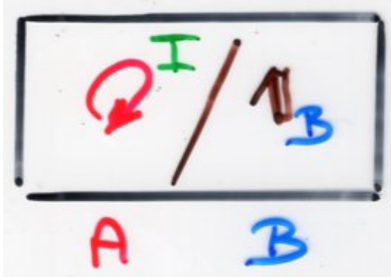
$$\sum_{n \geq 0} \tilde{L}_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

$$\sum_{n \geq 0} C_n^{(a)}(x) \frac{t^n}{n!} = e^t \left(1 - t/a\right)^x$$

(A, B)

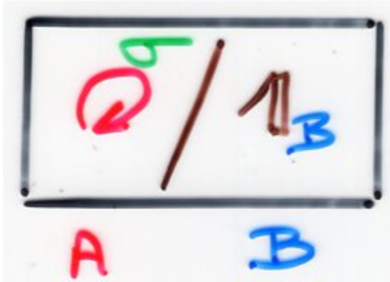
Hermite configurations

$$\mathbf{H}[A, B] = \mathbf{I}[A] \times \{1_B\}$$



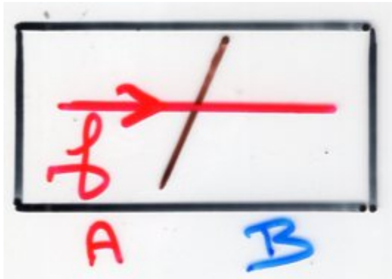
Charlier configurations

$$\mathbf{C}[A, B] = \mathbf{S}[A] \times \{1_B\}$$



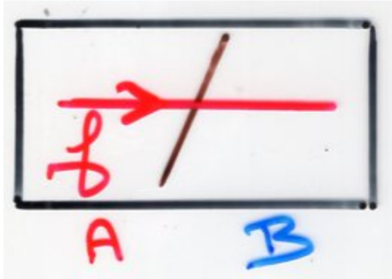
Laguerre configurations

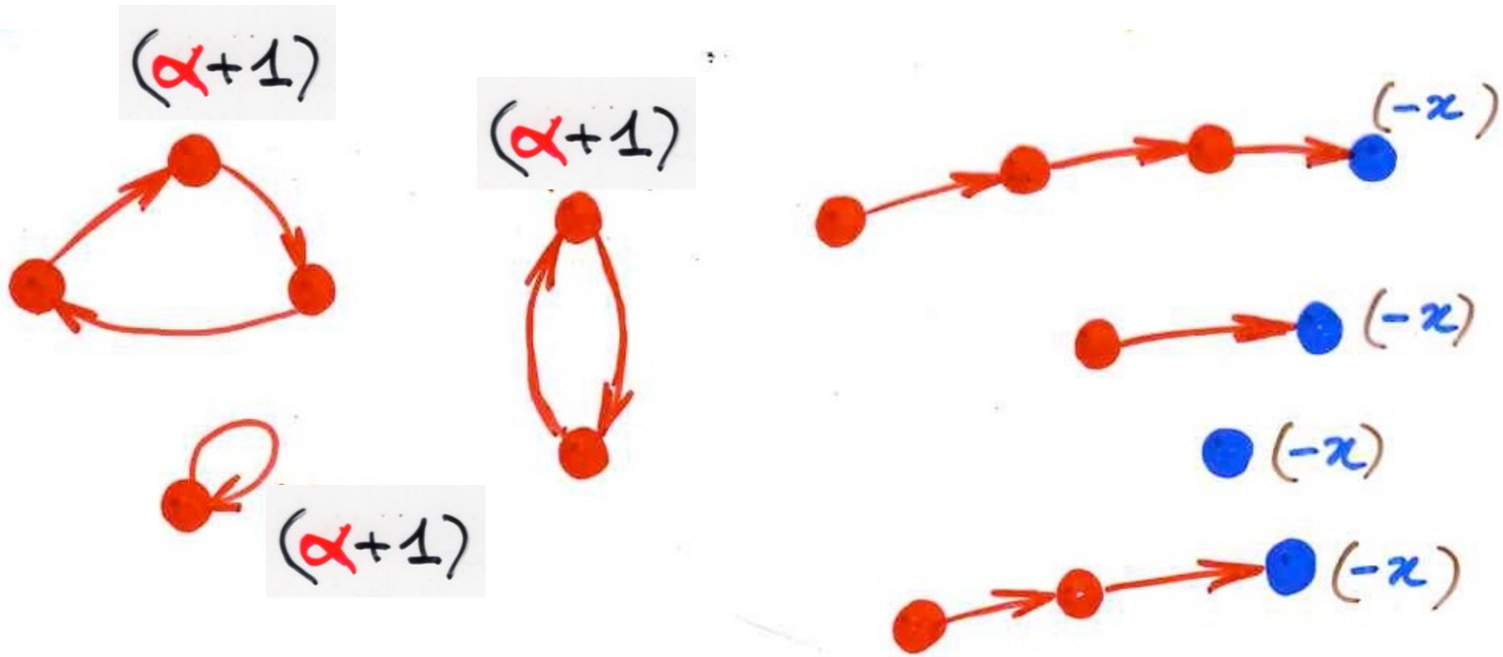
$$\mathbf{L}[A, B] = \left\{ \begin{array}{c} \text{injective map } f \\ \text{from } A \text{ to } A+B \end{array} \right\}$$



Laguerre configurations

$\mathbf{L}[A, B] = \{$ injective map f from A to $A+B$ $\}$





About the combinatorial proof of Mehler formula

Foata (1978)

some historical remarks

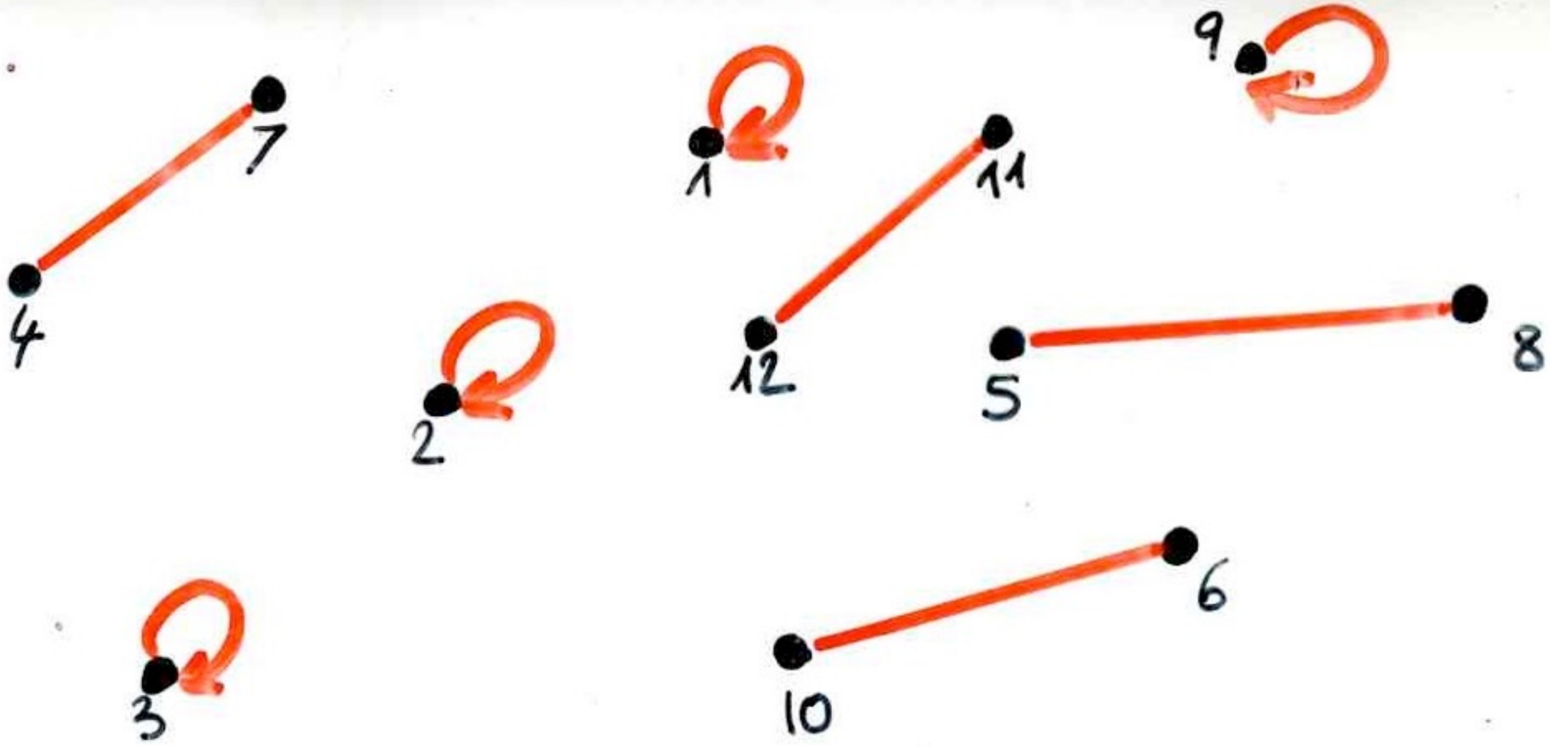
Mehler identity

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

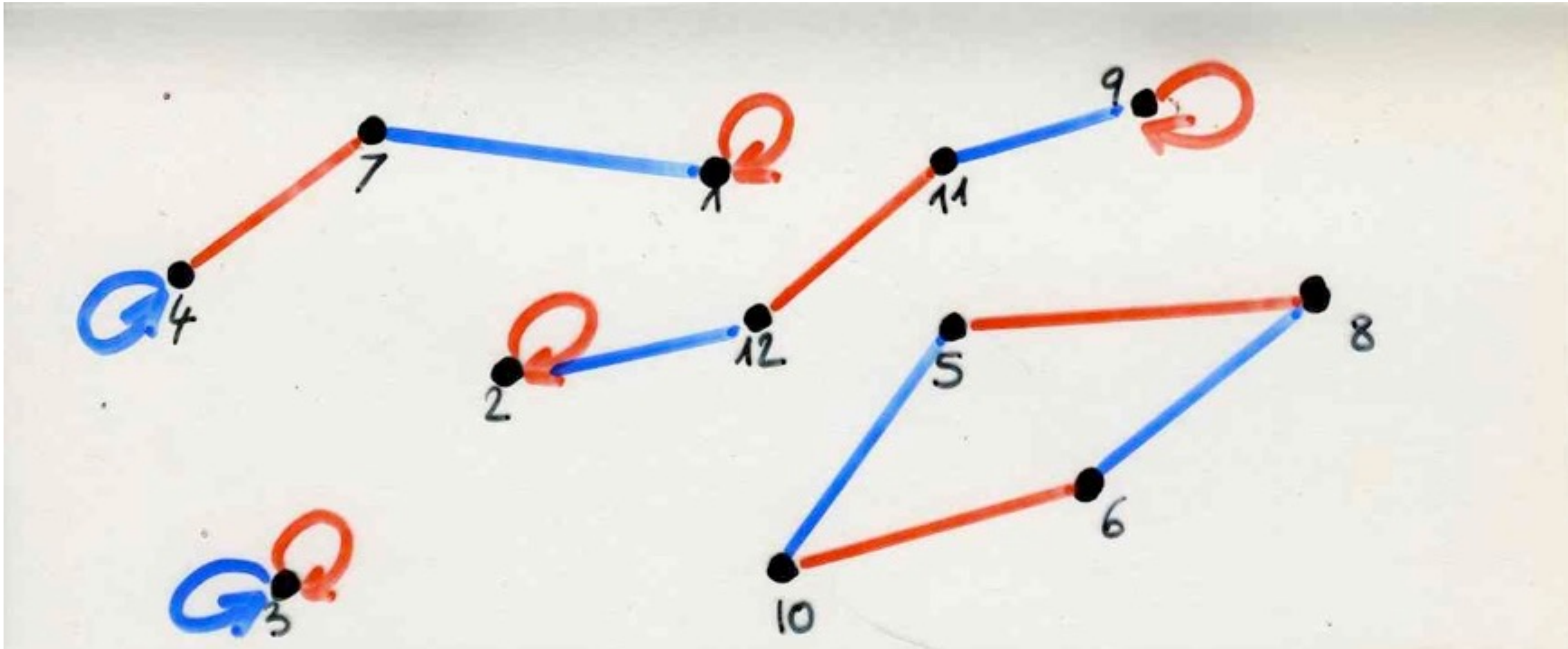
$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

"physicists" Hermite polynomial $H_n(x)$

$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

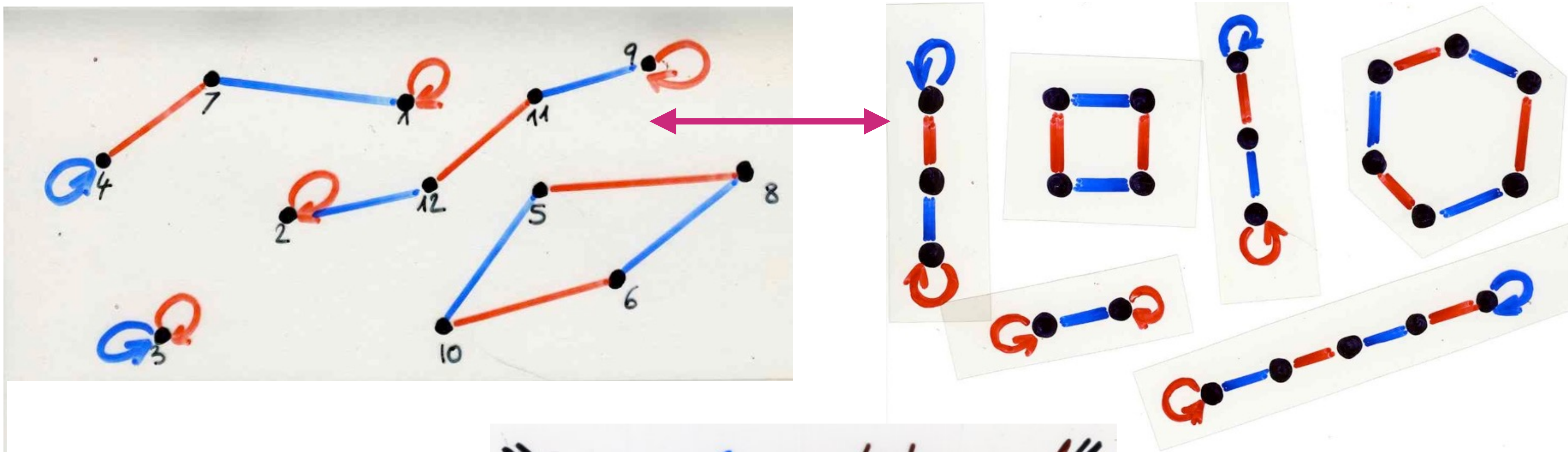


$$\sum_{n\geq 0} H_n(x) \frac{t^n}{n!}$$



$$\sum_{n\geq 0} H_n(x)\, H_n(y)\, \frac{t^n}{n!}$$

$$\sum_{n\geq 0} H_n(x)\,H_n(y)\,\frac{t^n}{n!} \quad = (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



Foata (1978)

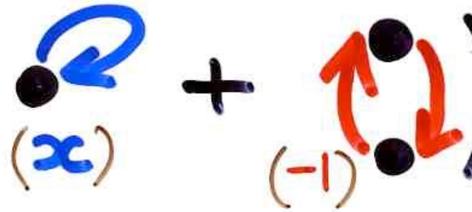
"composé partitionnel"
Foata (1974)

A. Joyal (1981)
seminal paper on the theory of species

$$\sum_{n\geq 0} H_n(x)\,H_n(y)\,\frac{t^n}{n!}$$

$$= (1-t^2)^{-1/2} \exp\left[\frac{2xyt - (x^2+y^2)t^2}{2(1-t^2)}\right]$$

exp(



)

$$\sum_{n\geq 0} H_n(x)\,\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

(combinatorial)
Hermite polynomials

Jackson (1941)

analytic proofs

$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp(2xt - t^2)$$

"physicists" Hermite polynomial $H_n(x)$

Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$$

integral form

$$H_n(x) = \frac{(-i)^n e^{x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(t^2/2)+itx}\, t^n\, dt$$

analytic proofs

- expanding both sides of the identity into power series
- formulae for summation of power series

$${}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}$$

Pfaff-Saalschütz (1797) (1890)

Watson (1933)

Rainville (1960)

- bilinear for Laguerre polynomials — Erdélyi (1939)

  Hille–Hardy identity — Erdélyi (1953) book

  Watson (1933) (1st proof) — Szegö (1939) book

- integral form — Mehler (1866)

  Watson (1933) (second proof) — Lebedev (1972) book

Institut Henri Poincaré (I.H.P.)
Paris

" Journées sur les méthodes
en mathétiques "
2-3 Avril 2003

Foata (2003)

analytic proofs

bijective proofs

bijective proofs

- positivity properties

Poisson kernel for Hermite polynomials

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

always ≥ 0 for every value of x and y

- multilinear extensions

Foata, Garsia (1979)

- analytic proofs

Kibble (1945) (Ph.D. thesis)

Slepian (1972)

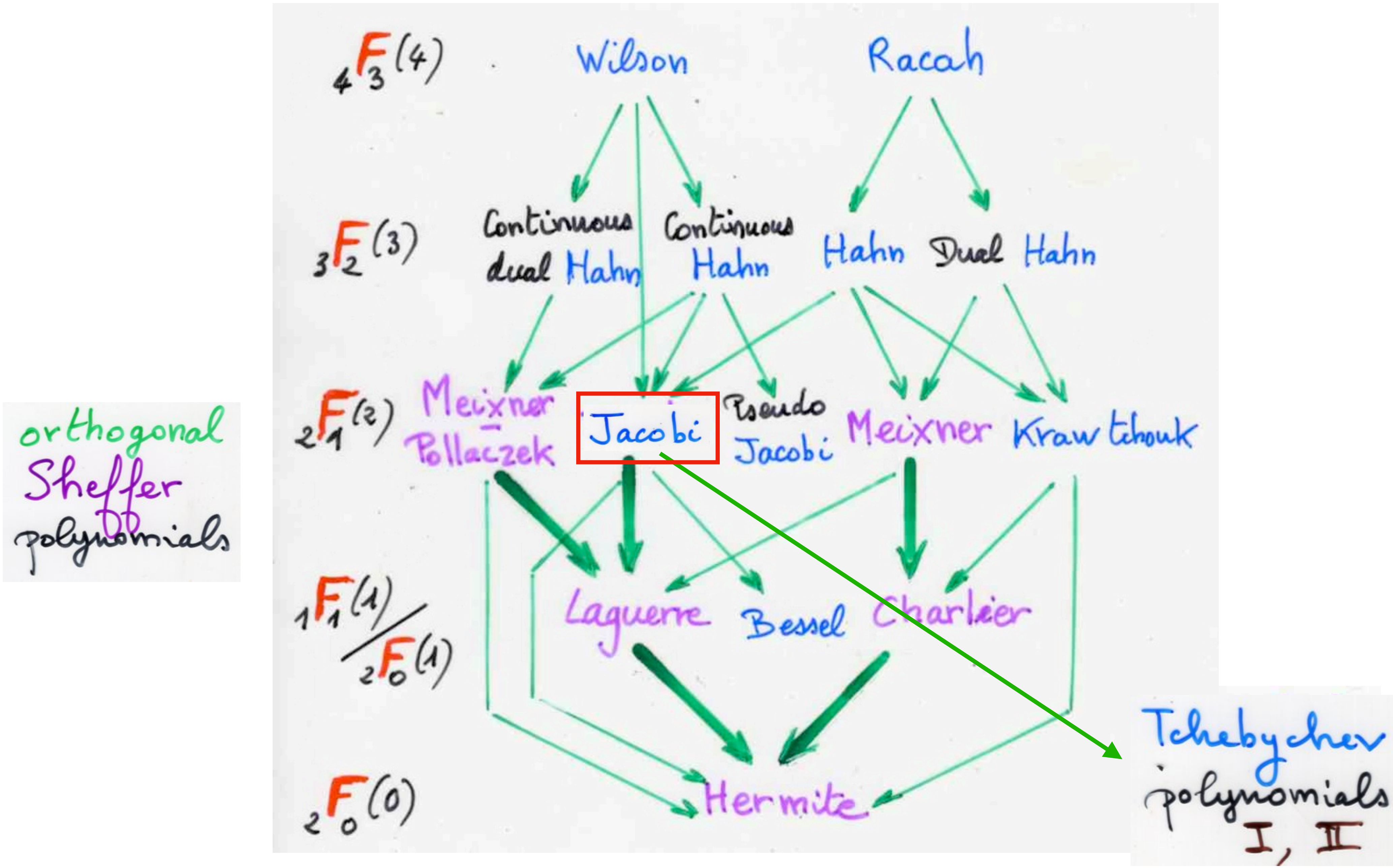
Louck (1981)

boson operators methods

Reminding Jacobi configurations

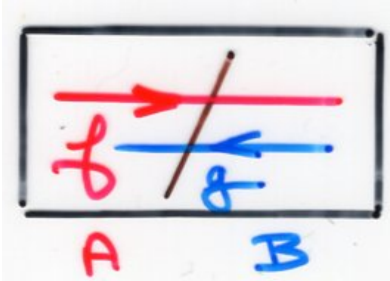
Foata, Leroux (1983)

Askey scheme of hypergeometric orthogonal polynomials



Jacobi configurations

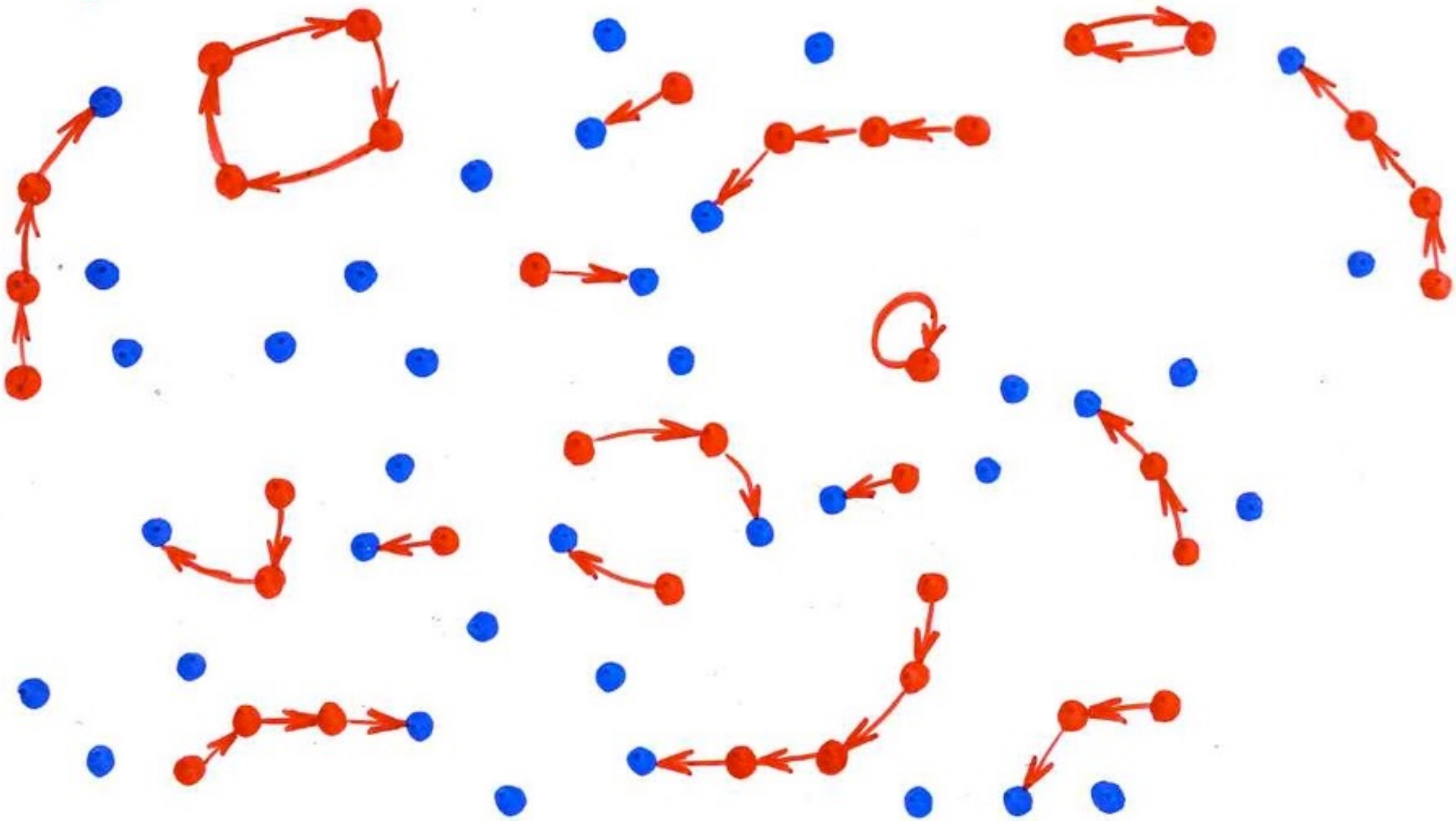
$$J[A,B] = L[A,B] \times L[B,A]$$



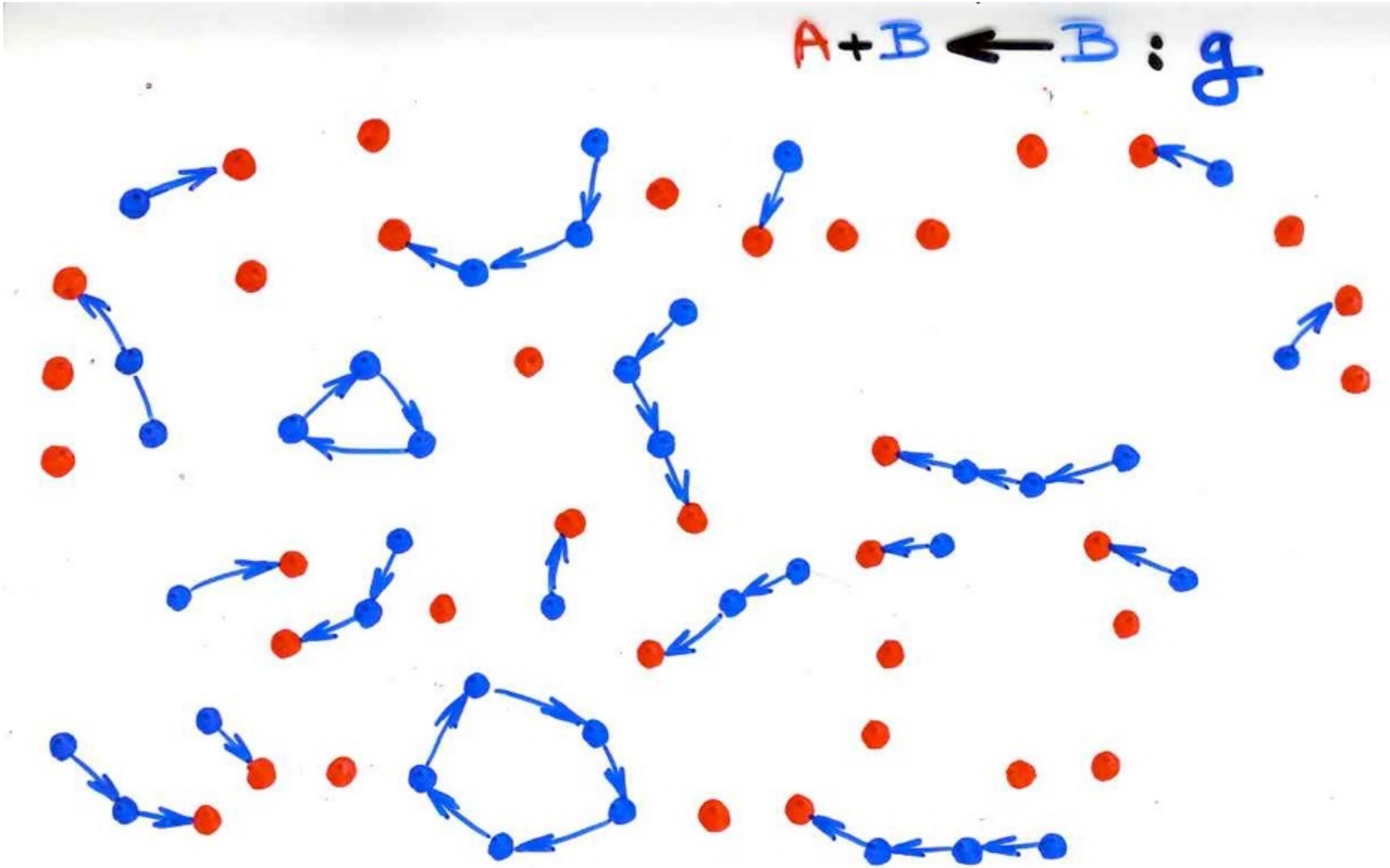
Foata, Leroux (1983)

(A, B)

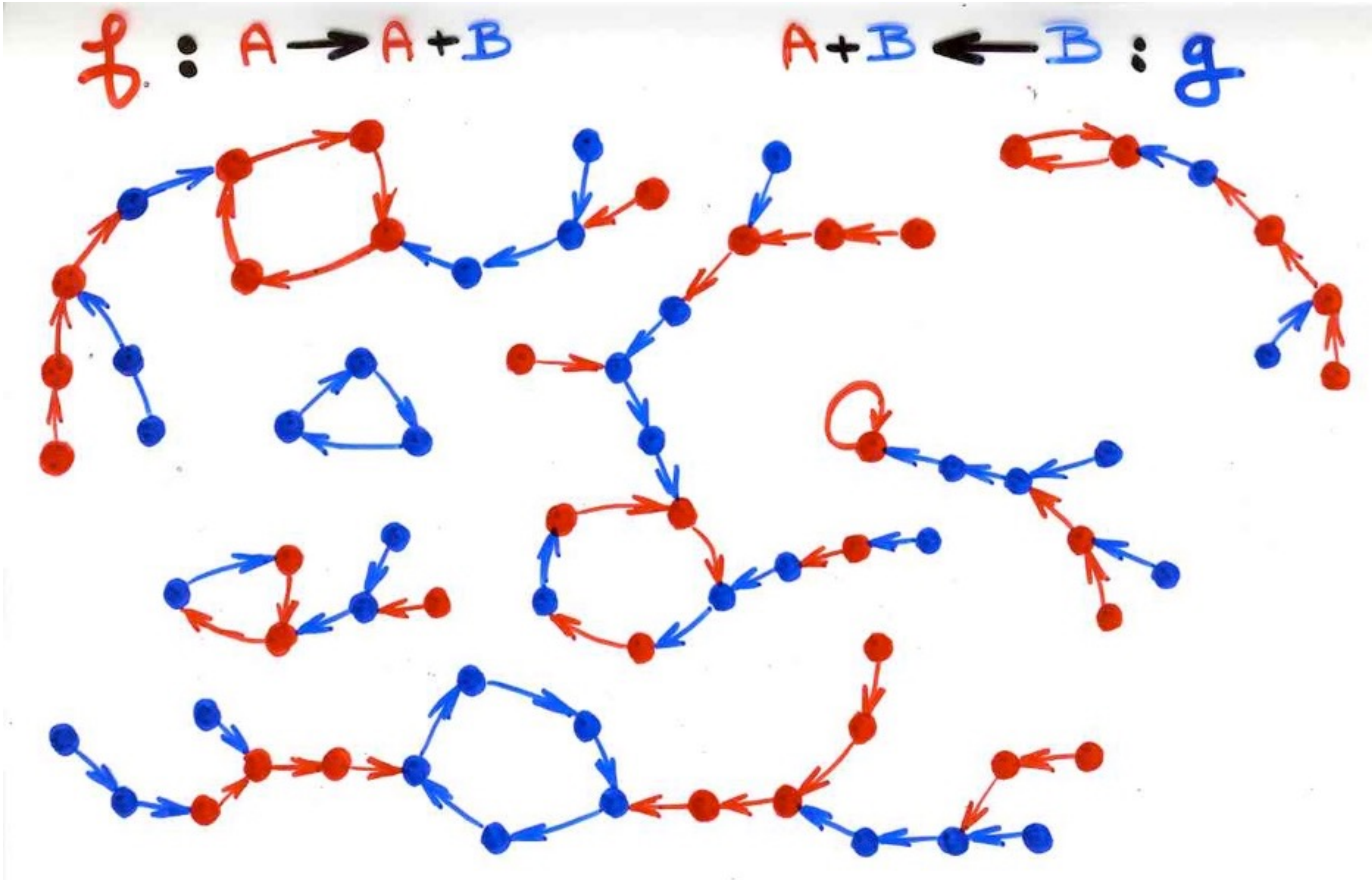
$f : A \to A + B$



(A, B)

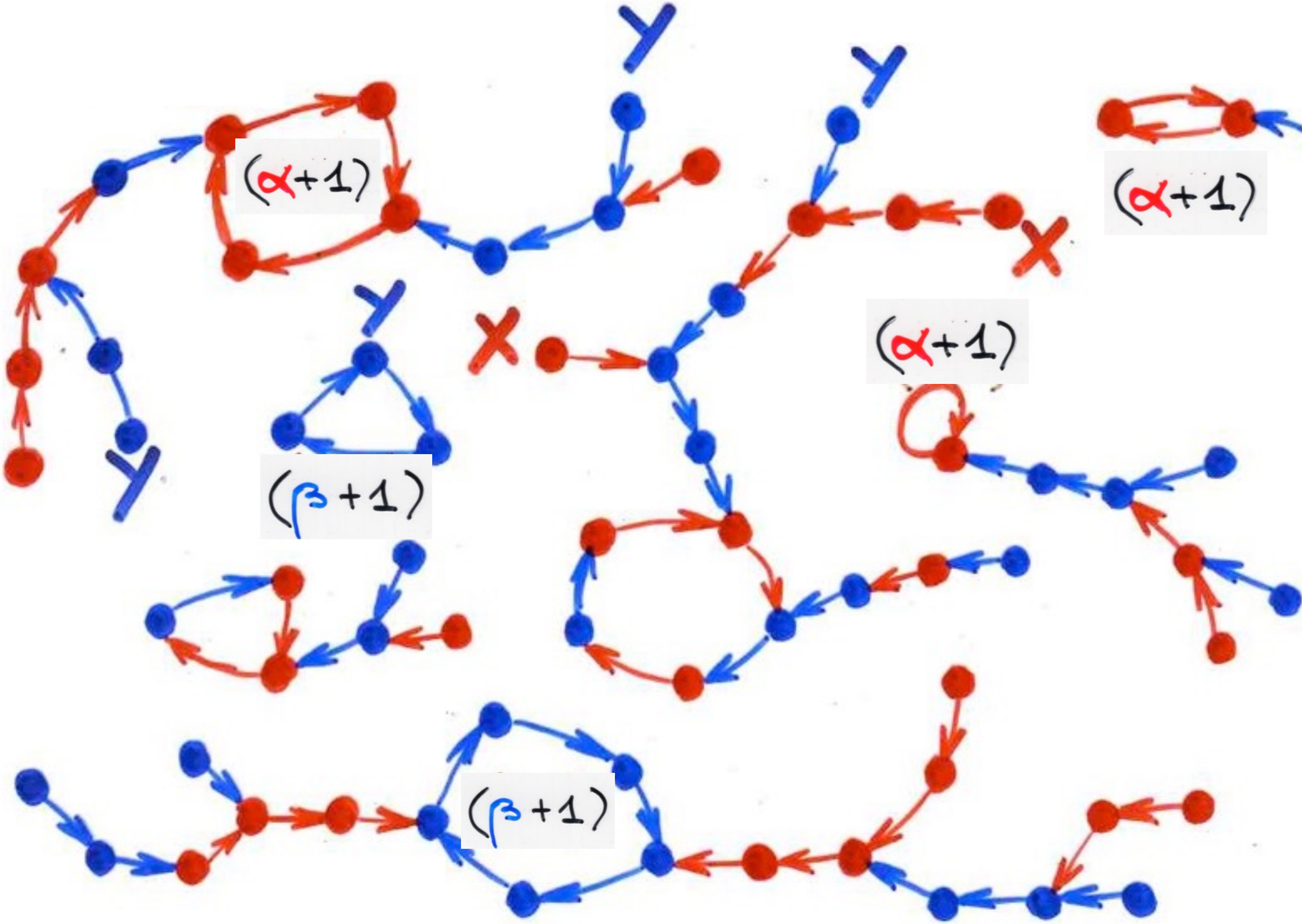


(A, B)



$(f, g) \in L[A, B]$

$$w(f,g) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)}\, x^{|A|}\, y^{|B|}$$



Proposition

$|E| = n \qquad (A, B)$

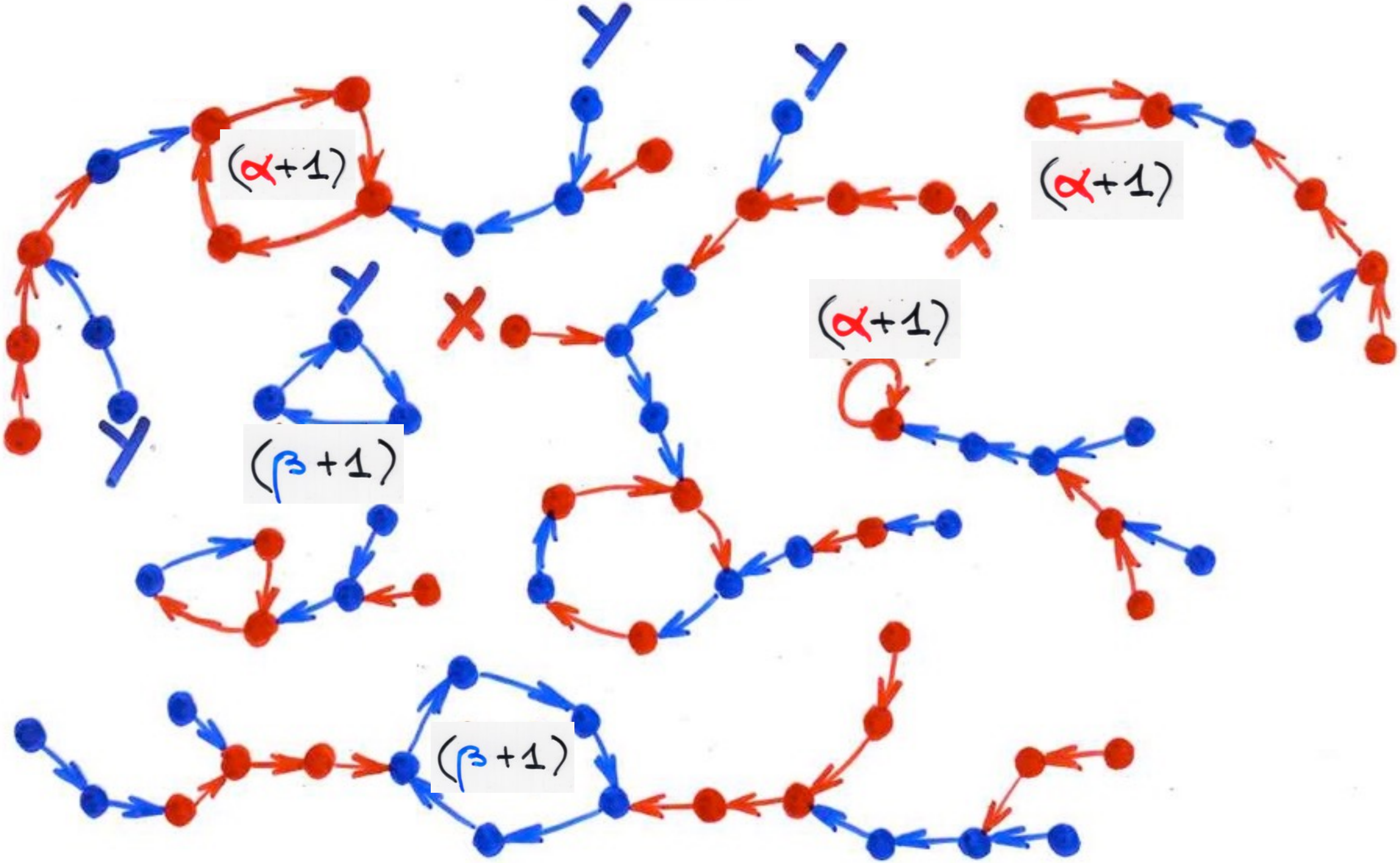
$$\mathcal{P}_n^{(\alpha,\beta)}(x, y) = \sum_{(f,g) \in \mathbf{J}[A,B] = \mathbf{L}[A,B] \times \mathbf{L}[B,A]} w(f, g)$$

Proposition

$$\mathcal{R} = \left[1 - 2(x+y)t + (x-y)^2 t^2\right]^{1/2}$$

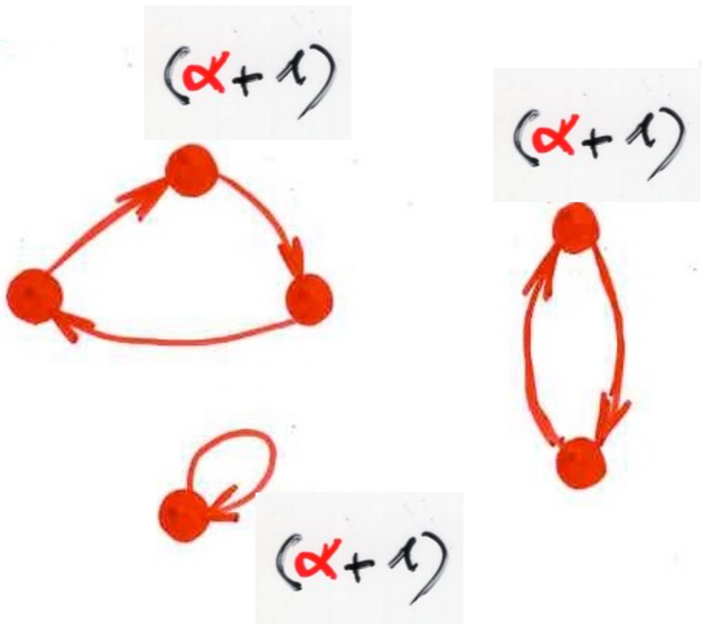
$$\sum_{n \geq 0} \mathcal{P}_n^{(\alpha,\beta)}(x, y) \frac{t^n}{n!} = 2^{\alpha+\beta} \mathcal{R}^{-1} \left[1 - (x-y)t + \mathcal{R}\right]^{-\alpha} \left[1 - (y-x)t + \mathcal{R}\right]^{-\beta}$$

$$\phi_w(t) = \phi_\alpha(t)\,\phi_\beta(t)\,\phi_m(t)$$



$$\frac{1}{(1-t)^{(\alpha+1)}} = \exp\left(\log \frac{1}{(1-t)^{(\alpha+1)}}\right)$$

$$\exp\left((\alpha+1) \log \frac{1}{(1-t)}\right)$$

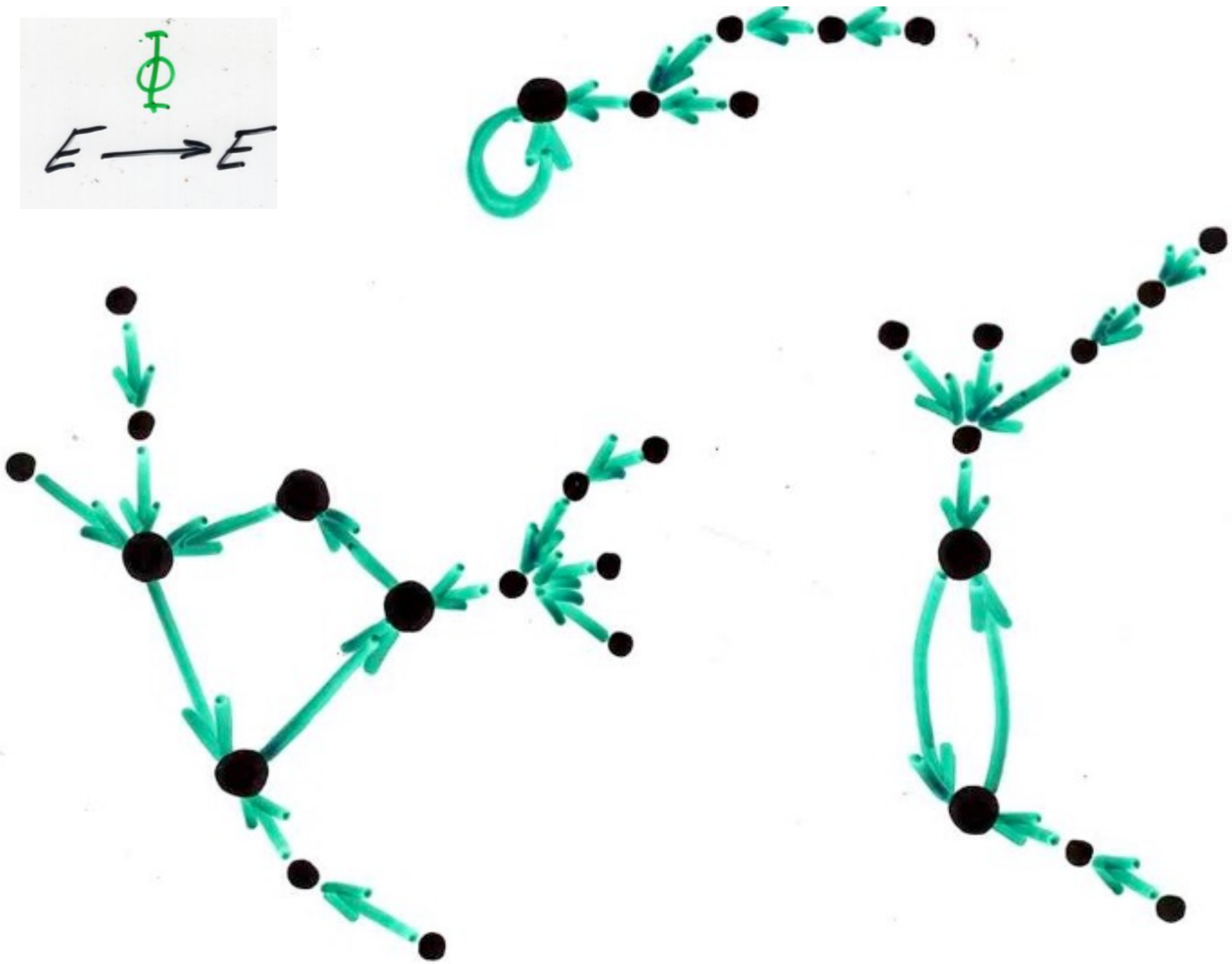


endofunction

n^n

$$E \xrightarrow{\Phi} E$$

species



arborescence A = rooted (Cayley) tree

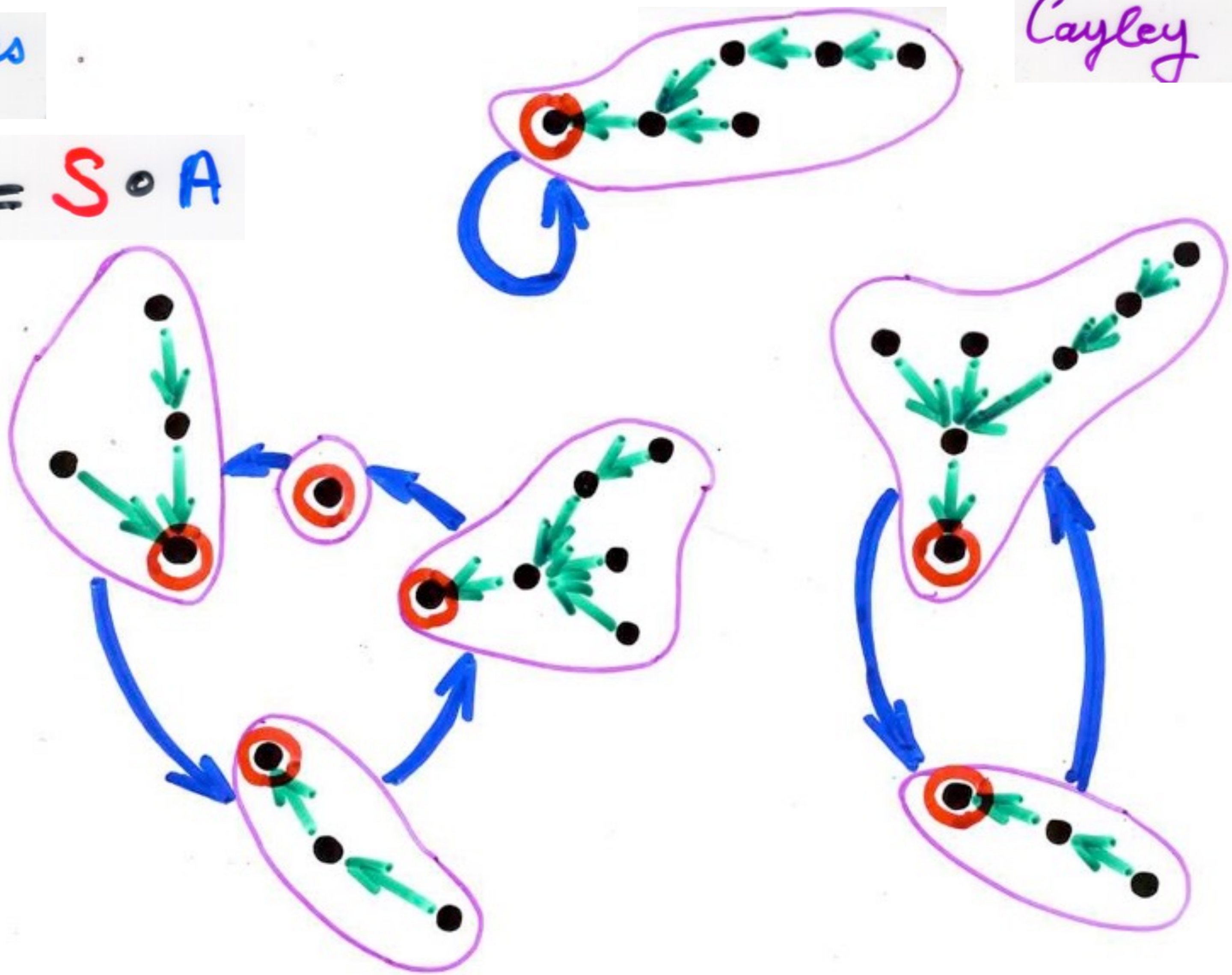
$A = \mathcal{A}^{\bullet}$

species

Cayley tree $\mathcal{A}$

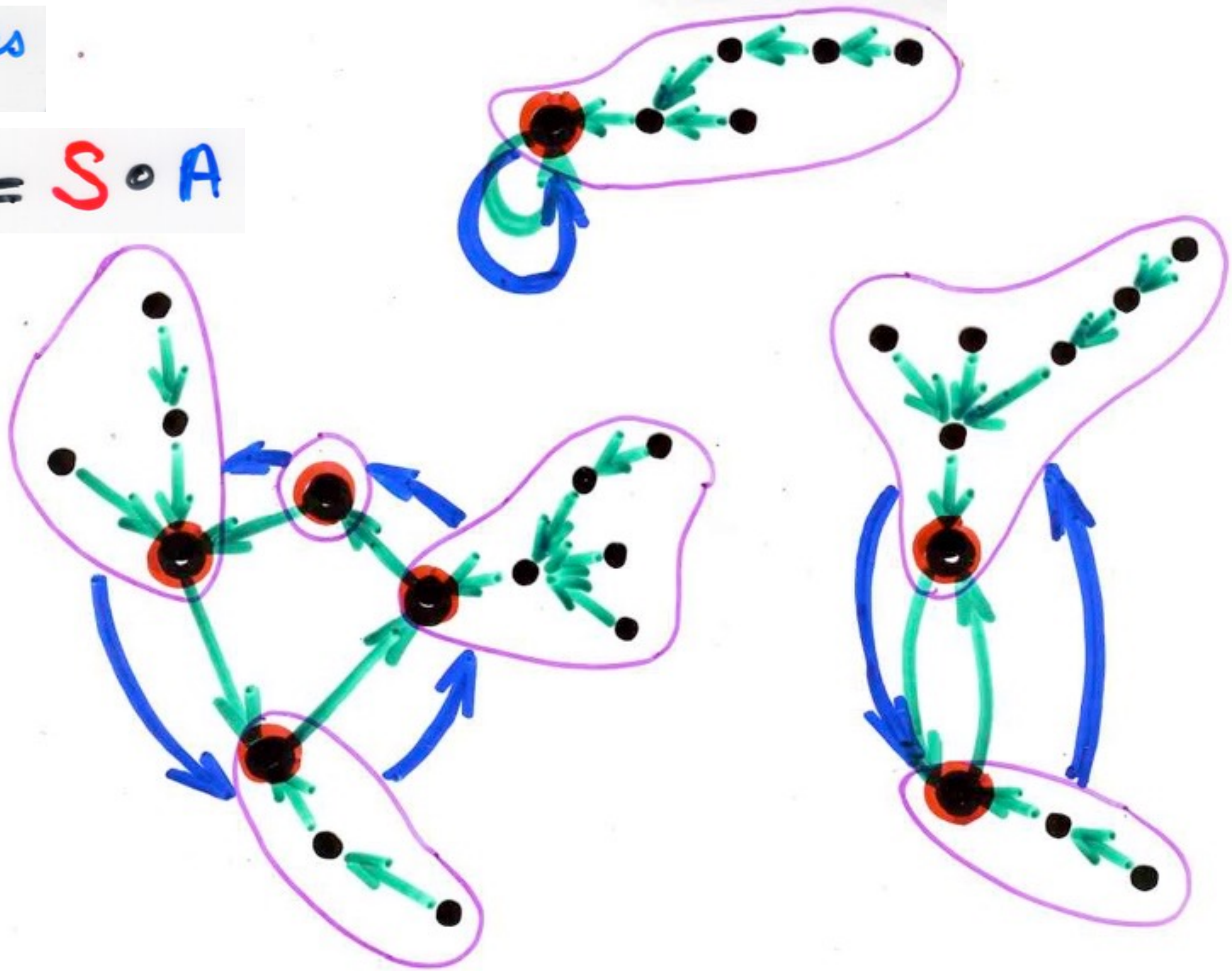
n^{n-2}

$End = S \circ A$



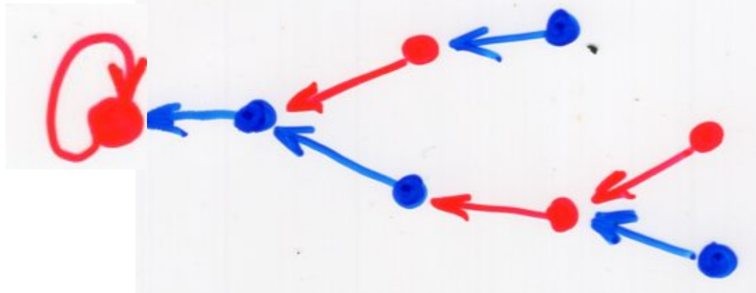
species

$End = S \circ A$



$C_a(t)$ $C_b(t)$

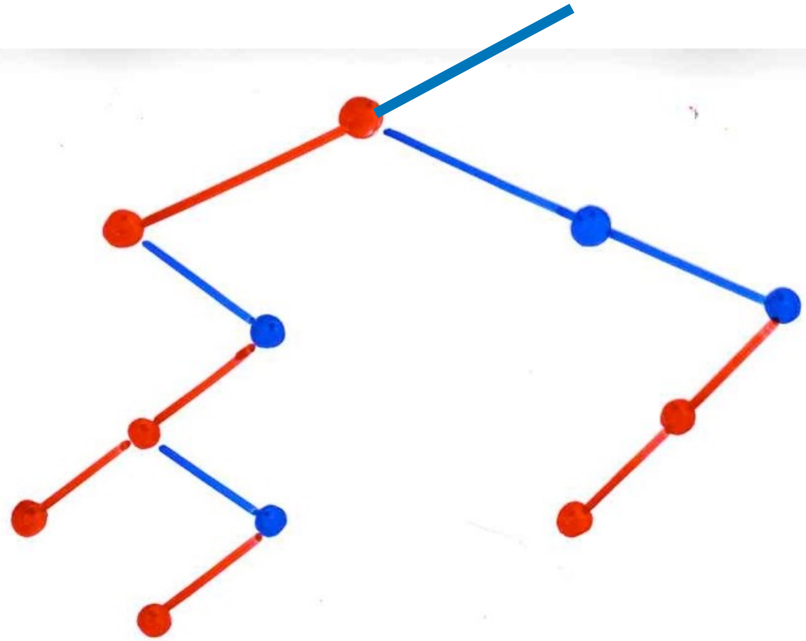
connected Jacobi configuration
type a with unique cycle of length 1

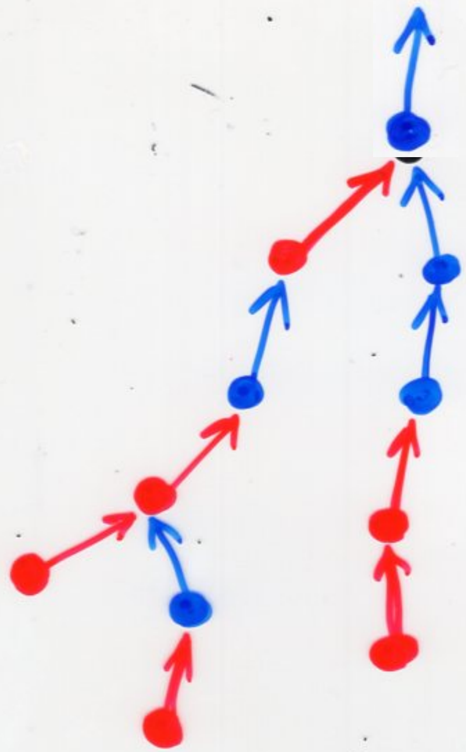


$$y = \sum_{n \geq 0} c_n t^n$$

$$= \sum_{n \geq 0} (n! c_n) \frac{t^n}{n!}$$

"labeled" binary tree





Proposition

$$\sum_{n\geq 0} \mathcal{P}_n^{(\alpha,\beta)}(X,Y)\,\frac{t^n}{n!} = 2^{\alpha+\beta}\,\mathcal{R}^{-1}\left[1-(X-Y)t+\mathcal{R}\right]^{-\alpha}\left[1-(Y-X)t+\mathcal{R}\right]^{-\beta}$$

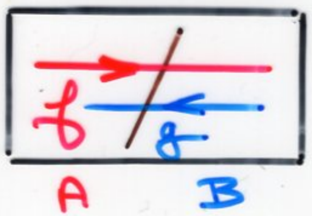
$$\mathcal{R} = \left[1-2(X+Y)t+(X-Y)^2t^2\right]^{1/2}$$

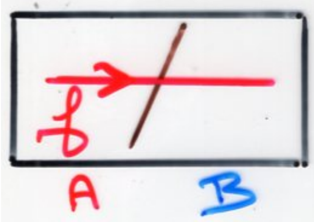
limit formula

<u>example</u>

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

Jacobi → Laguerre





J. Labelle, Y.N. Yeh (1989)

Meixner Polynomials

Meixner

$$M_n(x;\beta,c) = (\beta)_n \, {}_2F_1\left[\begin{matrix}-n, -x \\ \beta\end{matrix}; 1-\frac{1}{c}\right]$$

$$\sum_{i+j=n} \binom{n}{i,j} (-x)_i (\beta+i)_j (c^{-1}-1)^i$$

$$M_n(x;\beta,c) = (\beta+x)_n \, {}_2F_1\left[\begin{matrix}-n, -x \\ 1-\beta-n-x\end{matrix}; c^{-1}\right]$$

$$\sum_{i+j=n} \binom{n}{i,j} (-x)_i (\beta+x)_j \, c^{-i}$$

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta}$$

Meixner configurations

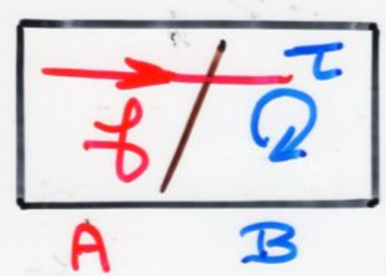
Foata, J. Labelle (1983)

endofunction $\Psi : E \to E$ (A, B)

- $f = \Psi/_A$ injective map
- $\tau = \Psi/_B$ permutation

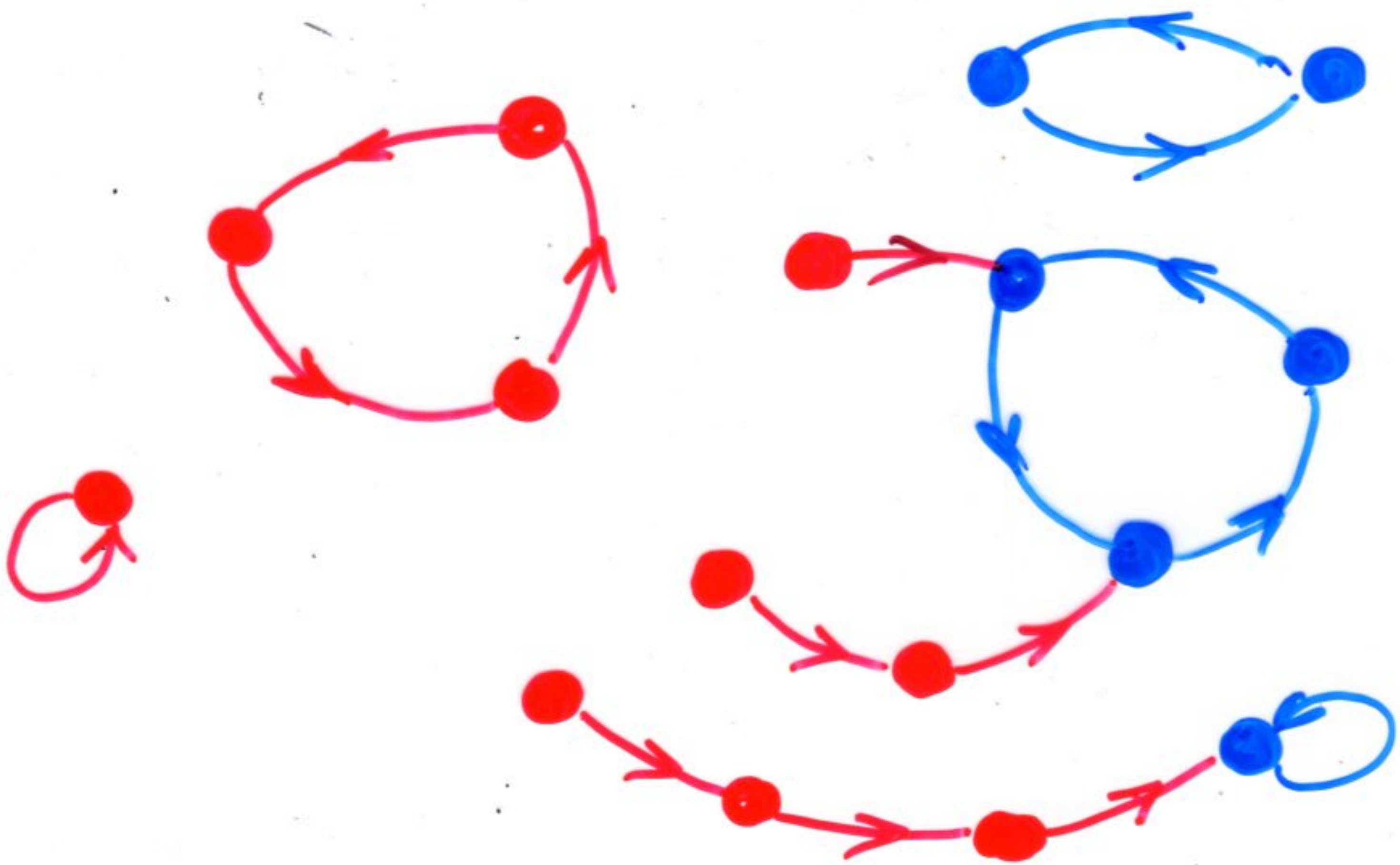
Meixner configurations

$$\mathbf{M}[A, B] = \mathbf{L}[A, B] \times \mathbf{S}[B]$$

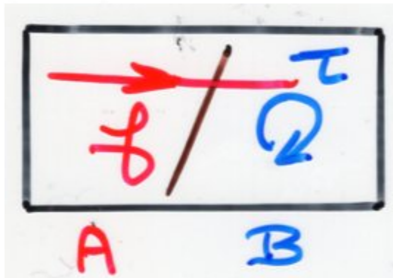


(A, B)

Meixner configurations

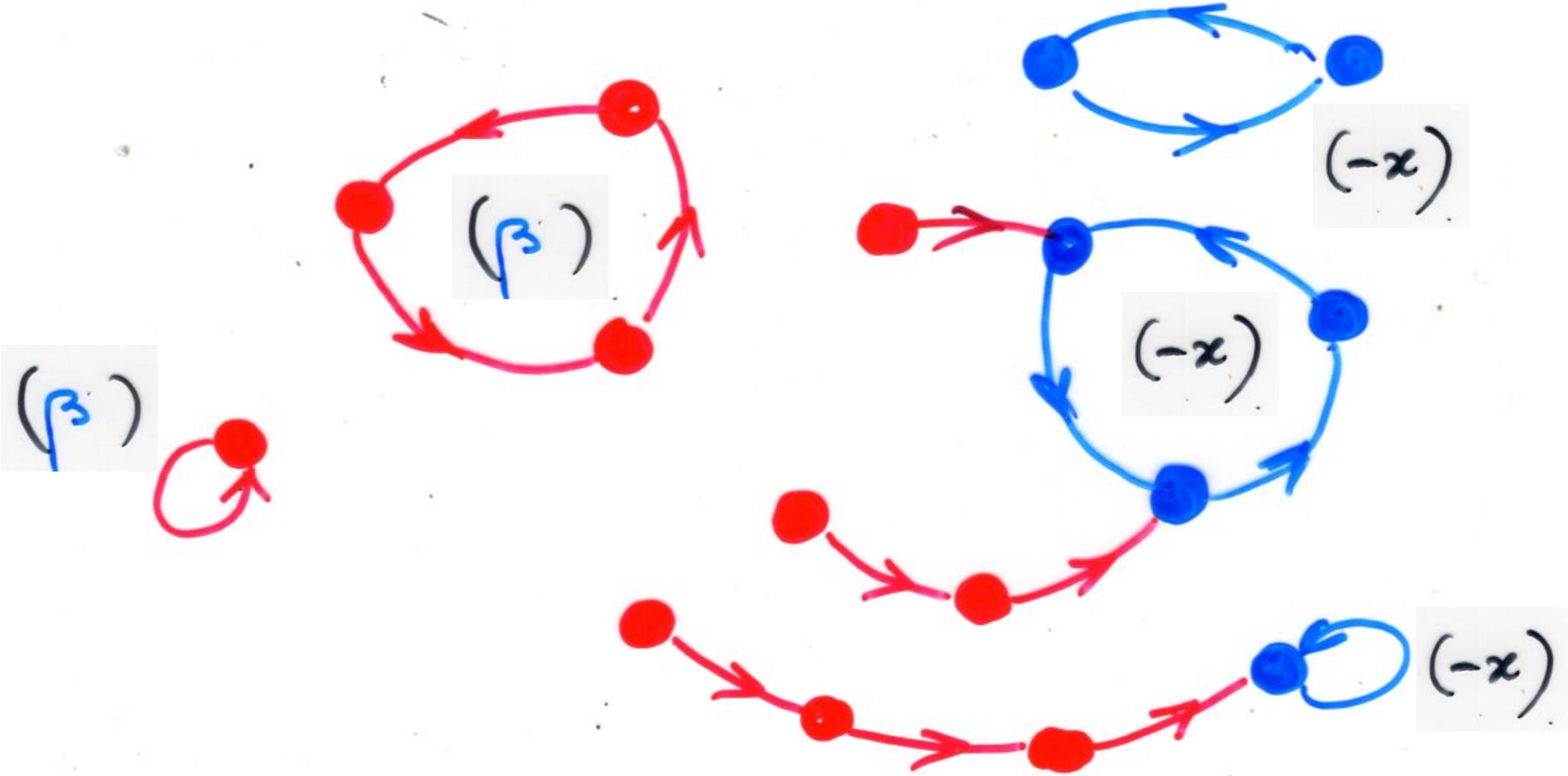


$$M[A, B] = L[A, B] \times S[B]$$

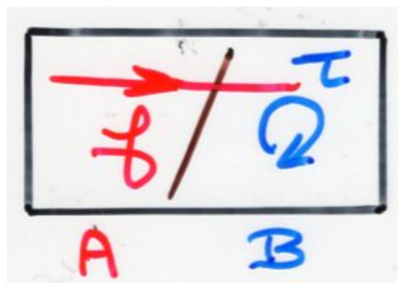


$$w(f,\tau) = \beta^{cyc(f)} (-x)^{cyc(\tau)} (c^{-1}-1)^{|B|}$$

$(c^{-1}-1)$

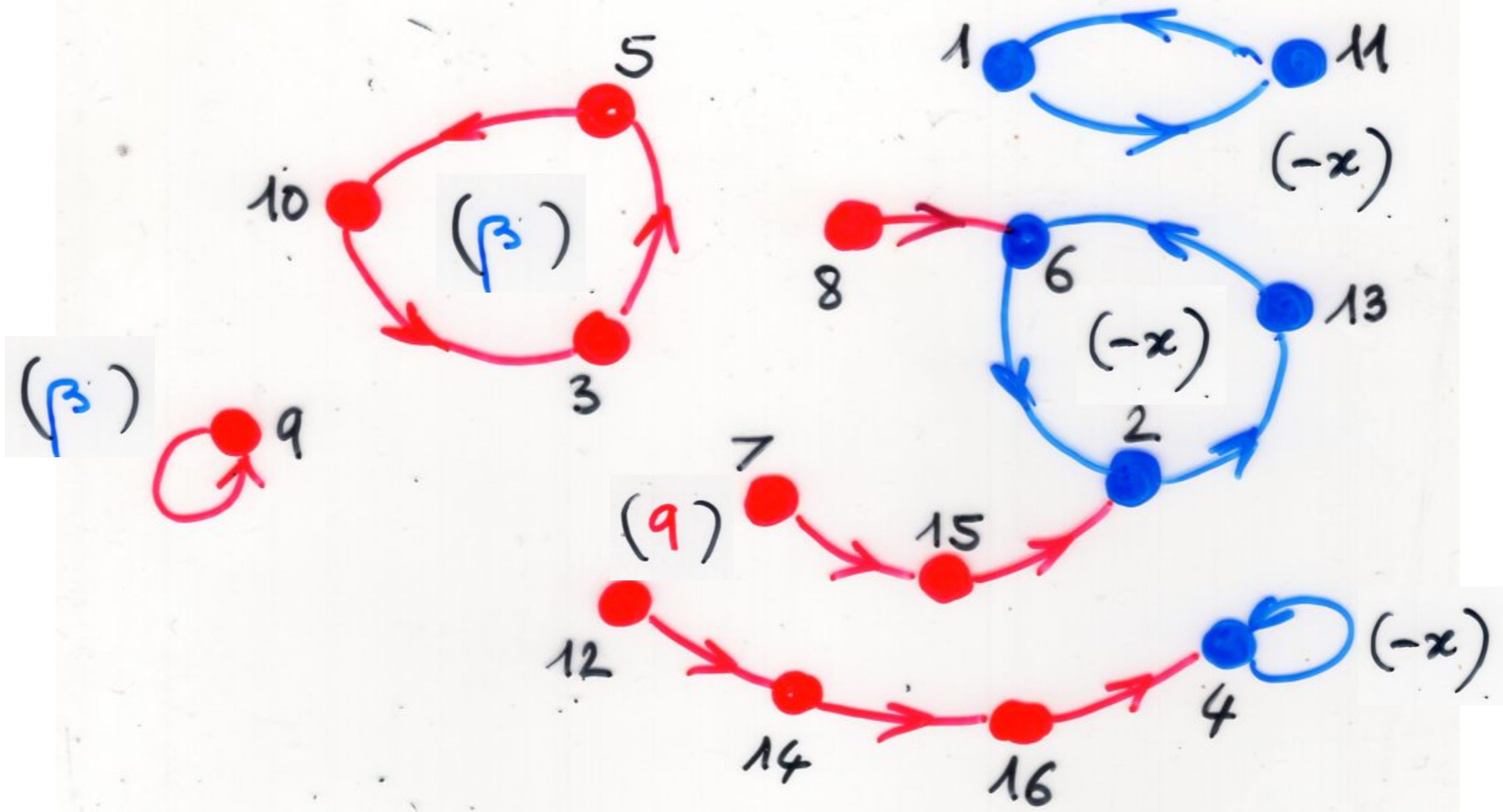


$$M[A,B] = L[A,B] \times S[B]$$



$$w(f,\tau) = \beta^{cyc(f)}(-x)^{cyc(\tau)}(c^{-1}-1)^{|B|}$$

$(c^{-1}-1)$



$$w(f,\tau) = \beta^{\mathrm{cyc}(f)}(-x)^{\mathrm{cyc}(\tau)}(c^{-1}-1)^{|B|}$$

<u>Proposition</u>

$$M_n(x;\beta\ c) = \sum_{(f,\tau)\in M[A,B]} w(f,\tau)$$

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^{x}(1-t)^{-x-\beta}$$

$$\frac{\gamma t}{1-t}$$

exponential generating function for non empty lists (=paths) with weight γ

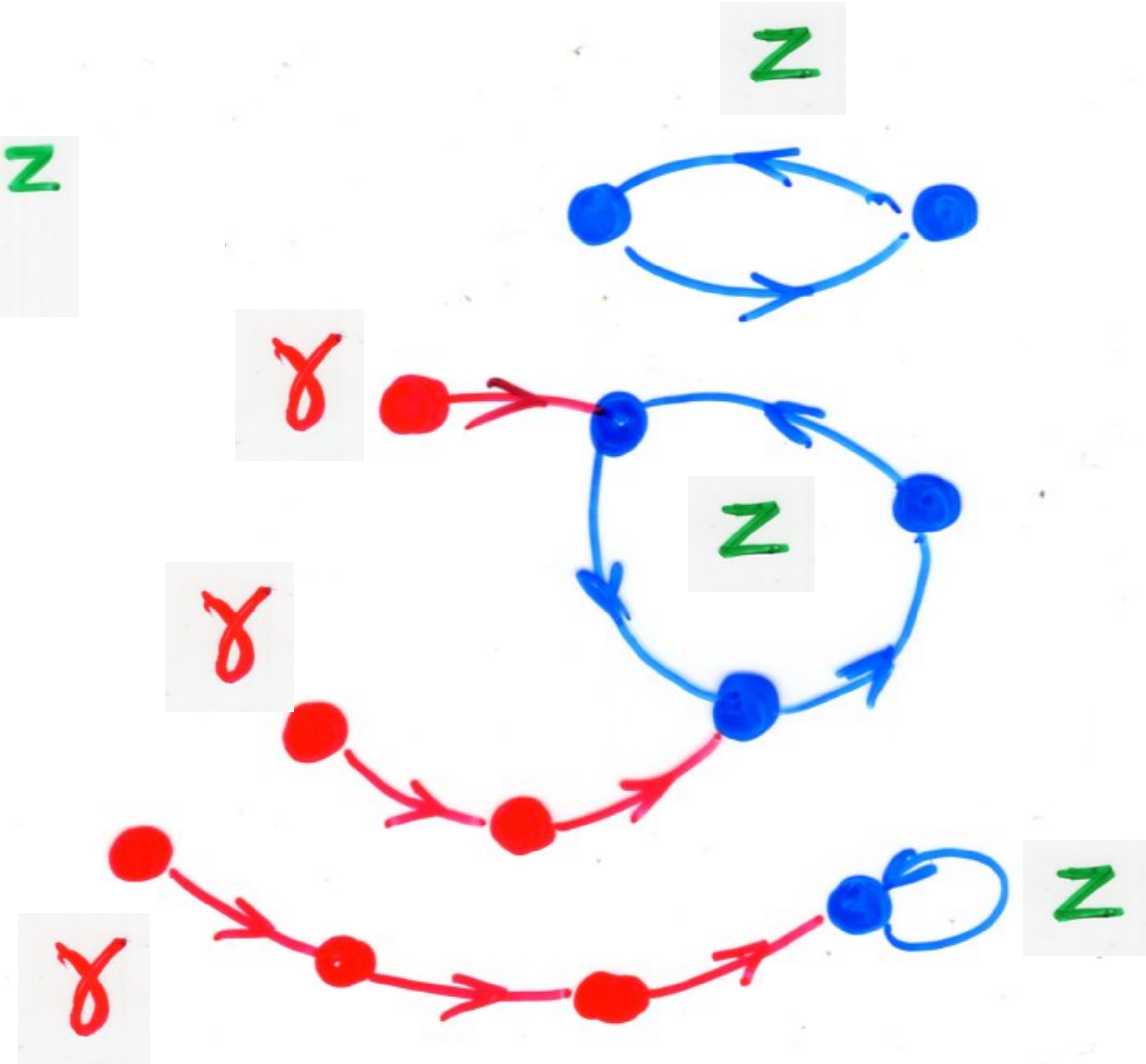
$$\left[1-\gamma\frac{t}{1-t}\right]^{-z}$$

exp. g. f: permutations (non empty lists weight γ)

species $\quad S \circ L^1$

weight z for each cycle of the permutation

$$\left[1 - \gamma \frac{t}{1-t}\right]^{-z}$$



exp. g.f. Meixner configurations

$$\left(\frac{1}{1-t}\right)^{\beta}\left[1-\frac{\gamma t}{1-t}\right]^{-z}$$

$$(1-t)^{-\beta+z}\,(1-(1+\gamma)t)^{-z}$$

$$z=-x \qquad \gamma=(c^{-1}-1)$$

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^{x}(1-t)^{-x-\beta} \qquad \square$$

Meixner polynomials

Limit formula

limit formula

$$\lim_{c \to 1} M_n\left(\frac{cx}{c-1}; \beta, c\right) = L_n^{(\beta-1)}(x)$$

$$M_n\left(\frac{cx}{c-1};\beta,c\right) \qquad \sum_{(f,\tau)\in M[A,B]} w\left(\beta,\frac{-cx}{c-1},c^{-1}-1\right)$$

$$w\left(\beta,\frac{-cx}{c-1},c^{-1}-1\right)$$

$$=\beta^{cyc(f)}(-x)^{cyc(\tau)}(c^{-1}-1)^{|B|-cyc(\tau)}$$

when $c\to 1$

only Meixner configurations with $|B|=cyc(\tau)$ will "survive" (i.e give a non-zero contribution)

(f, τ) with $|B| = cyc(\tau)$
is isomorphic to a Laguerre configution

$$w(f) = \beta^{cyc(f)} (-x)^{|B|}$$

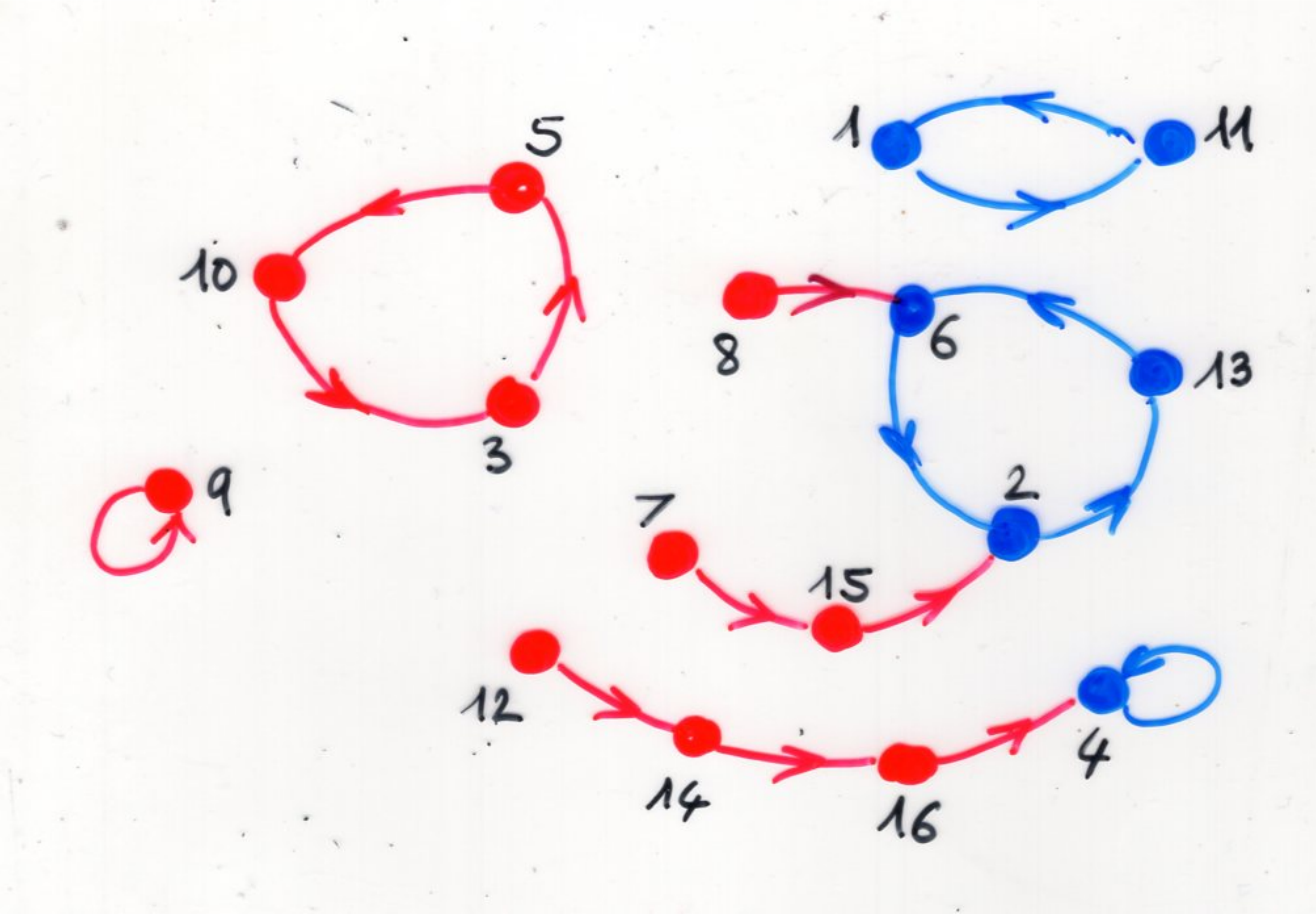
$$L_n^{(\beta-1)}(x) = \sum_{f \in \mathbb{L}[A,B]} w(f)$$

$\square$

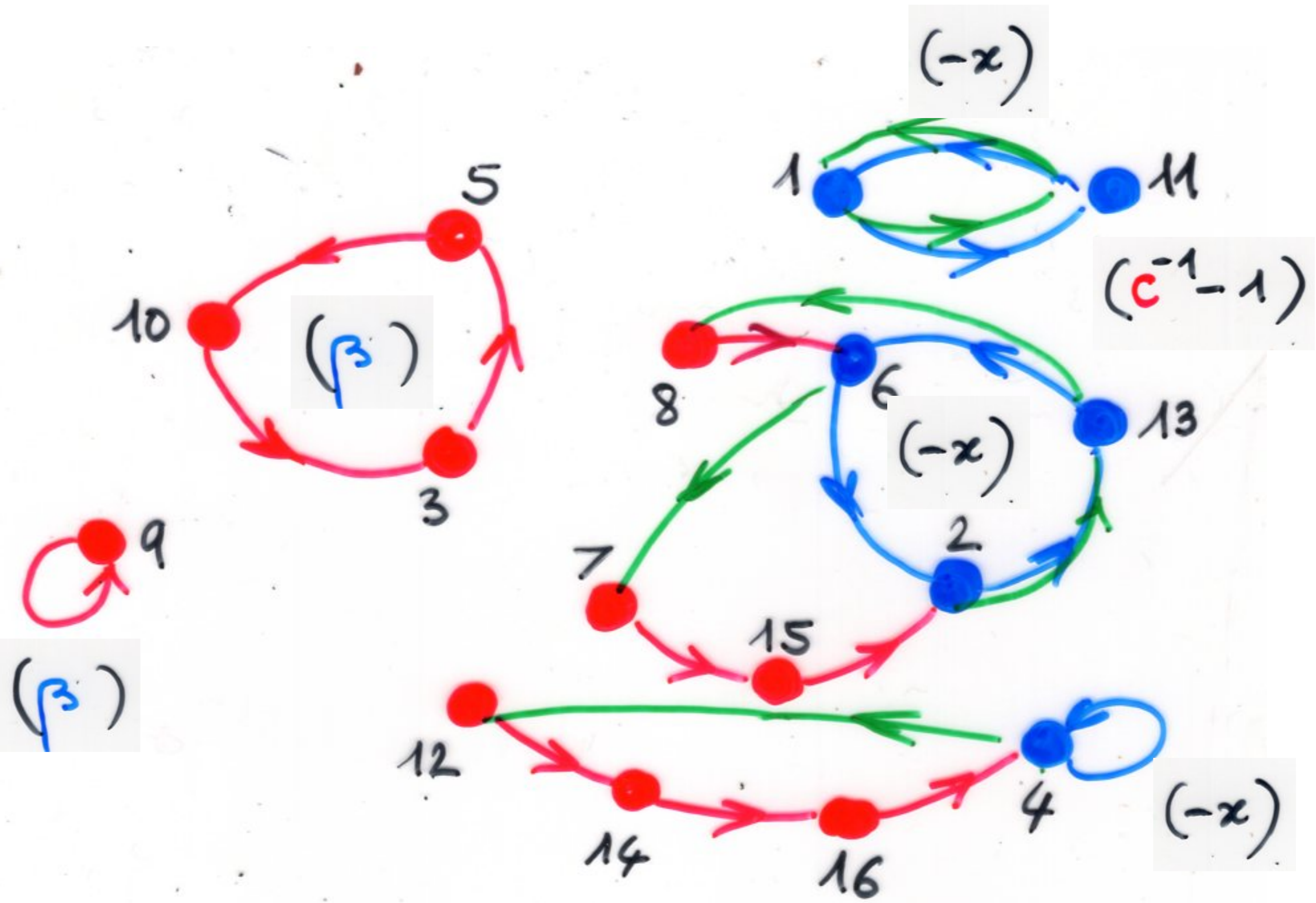
Meixner polynomials

Interpretation with colored permutations

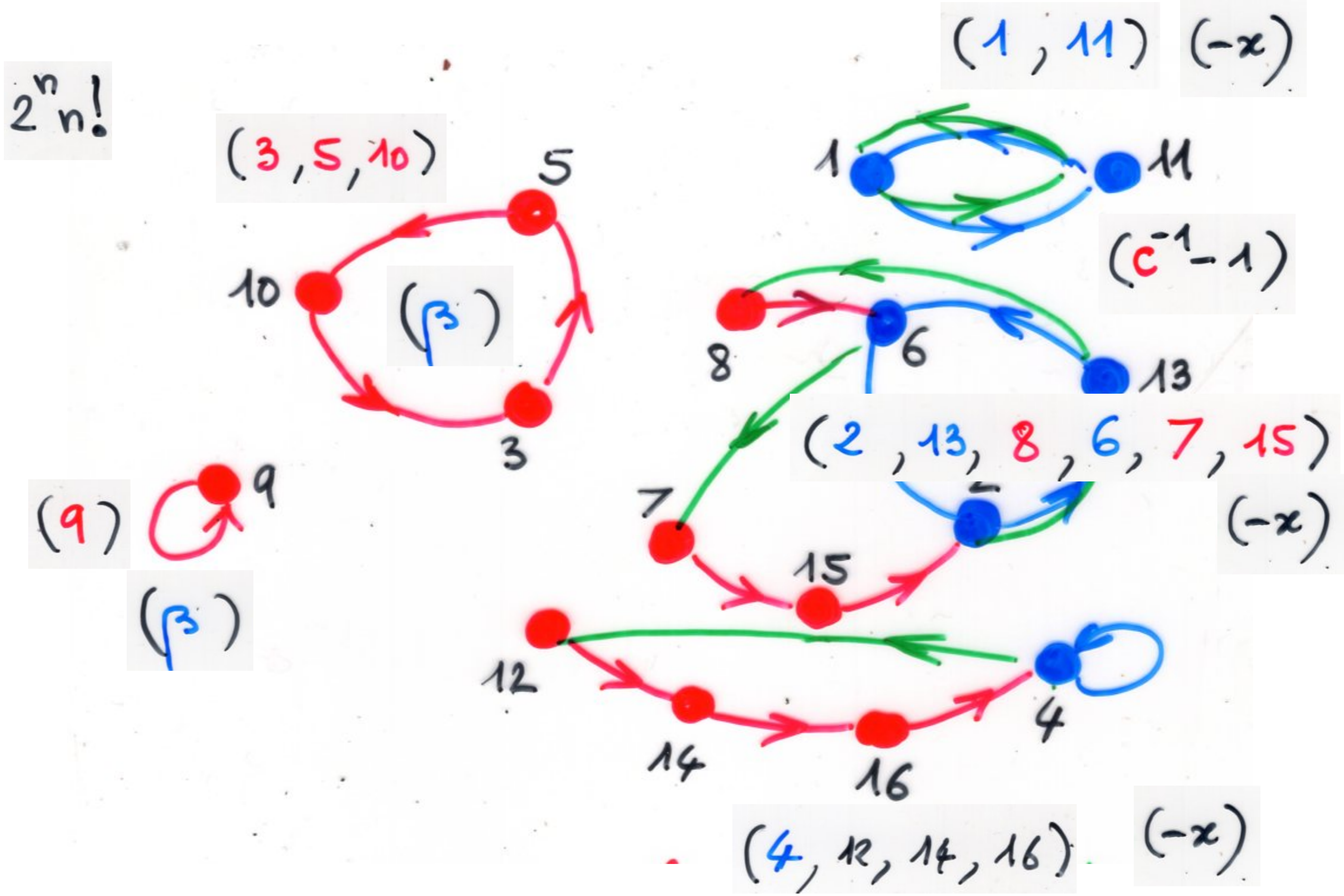
Meixner configuration → colored permutation



Meixner configuration → colored permutation



Meixner configuration → colored permutation



colored permutation

$(1, 11)$

$(3, 5, 10)$

$(2, 13, 8, 6, 7, 15)$

(9)

$(4, 12, 14, 16)$

$$w(\sigma_c) = \beta^{cyc(\sigma_c)} (-x)^{cyc(\sigma_c)} (c^{-1} - 1)^{|B|}$$

$cyc(\sigma_c)$ = number red cycles of σ_c

$cyc(\sigma_c)$ = number of non red of σ_c

Proposition

$$M_n(x; \beta\ c) = \sum_{\sigma_c \in \mathfrak{S}_n^c} w(\sigma_c)$$

Meixner polynomials

A third interpretation

Kreweras polynomials

"partially underlined" permutations

i ; $\underline{i}$

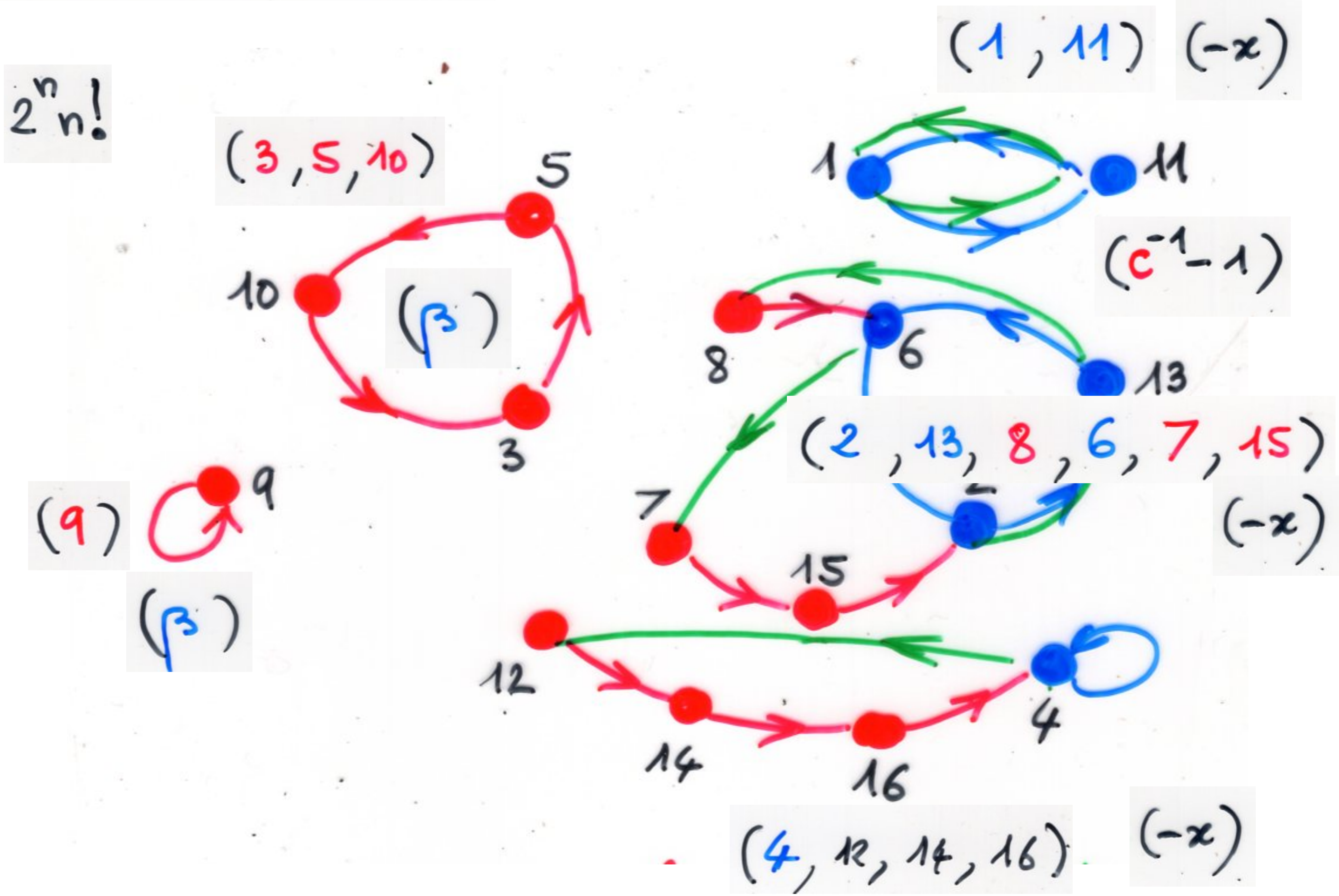
$1 \leq i \leq n$

2-colored permutation

σ_c or $\underline{\sigma}$

$w(\sigma_c)$

Meixner configuration → colored permutation



total order on $E = A \cup B$

any $j \in B$ is $<$ any $i \in A$

$(4, 12, 14, 16)$ $(2, 13, 8, 6, 7, 15)$ $(1, 11)$

(9) $(3, 5, 10)$

$\downarrow$

$\underline{\sigma} = 4, 12, 14, 16, 2, 13, 8, 6, 7, 15, 1, 11, 9, 3, 5, 10$

$$w(\underline{\sigma}) = \beta^{lr(\underline{\sigma})} (-x)^{lr(\underline{\sigma})} (c^{-1} - 1)^{|B|}$$

$\underline{\sigma} = 4, 12, 14, 16, 2, 13, 8, 6, 7, 15, 1, 11, 9, 3, 5, 10$

$$w(\underline{\sigma}) = \beta^{lr(\underline{\sigma})} (-x)^{lr(\underline{\sigma})} (c^{-1} - 1)^{|B|}$$

$lr(\sigma_c)$ red left-to-right minimum elements

$lr(\sigma_c)$ blue left-to-right minimum elements

$$\underline{\sigma} = 4, 12, 14, 16 \mid 2, 13, 8, 6, 7, 15 \mid 1, 11, \mid 9 \mid 3, 5, 10$$

$$(4, 12, 14, 16) \quad (2, 13, 8, 6, 7, 15) \quad (1, 11)$$

$$(9) \quad (3, 5, 10)$$

$\beta = 1 \, , \, c = \frac{1}{2}$

Kreweras polynomials

$$K_n(x) = \sum_{\sigma \in \mathfrak{S}_n} (-x)^{lr(\sigma)} = M_n(x; 1, \tfrac{1}{2})$$

$\sigma = 4, 12, 14, 16, 2, 13, 8, 6, 7, 15, 1, 11, 9, 3, 5, 10$

$lr(\sigma_c)$ blue left-to-right minimum elements

$\beta = 1$, $c = \frac{1}{2}$

$$\begin{cases} \tilde{b}_k = 3k+1 \\ \tilde{\lambda}_k = 2k^2 \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} 2^{d(\sigma)}$$

= number of ordered partitions of $\{1, 2, \dots, n\}$

<u>exercise</u> direct proof by constructing a bijection between ordered partitions and some histories associated to weighted colored Motzkin paths with weight $\tilde{b}_k = 3k+1$, $\tilde{\lambda} = 2k^2$

$c = \frac{1}{2}$ $\quad \tilde{b}_k = 3k + \beta$

$\tilde{\lambda}_k = 2k(k+\beta-1)$

parameter β : number of blocks ?

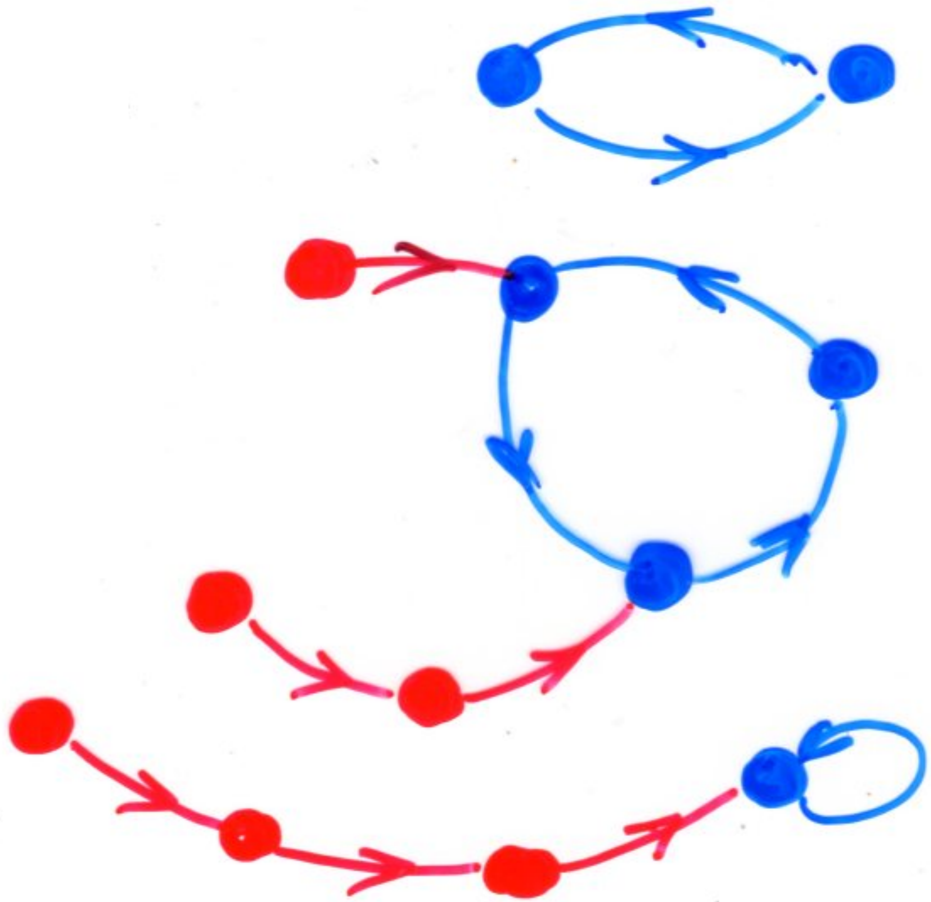
Octopus

F. Bergeron (1990)

species

$S \circ L^{1}$

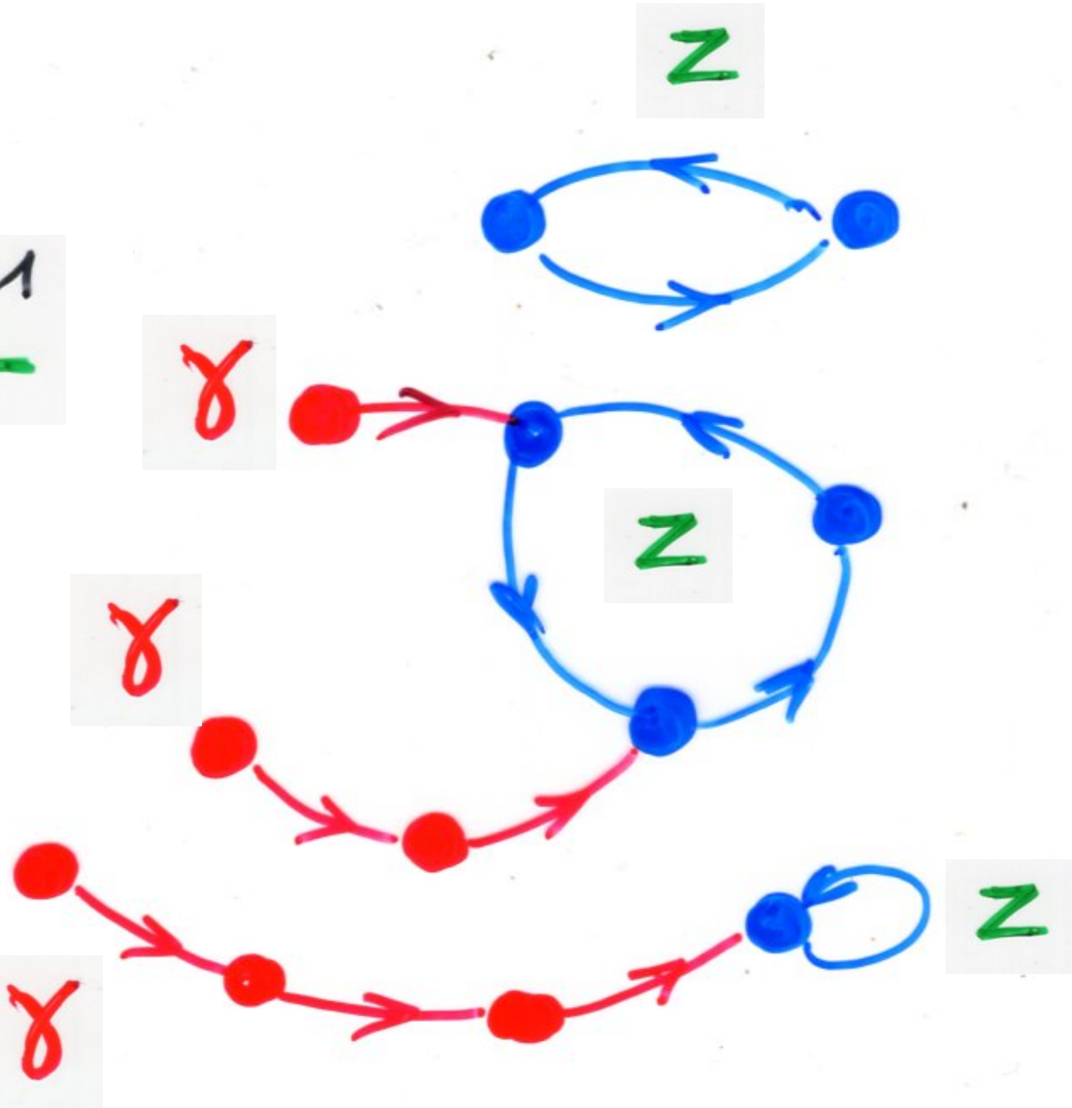
chain



species

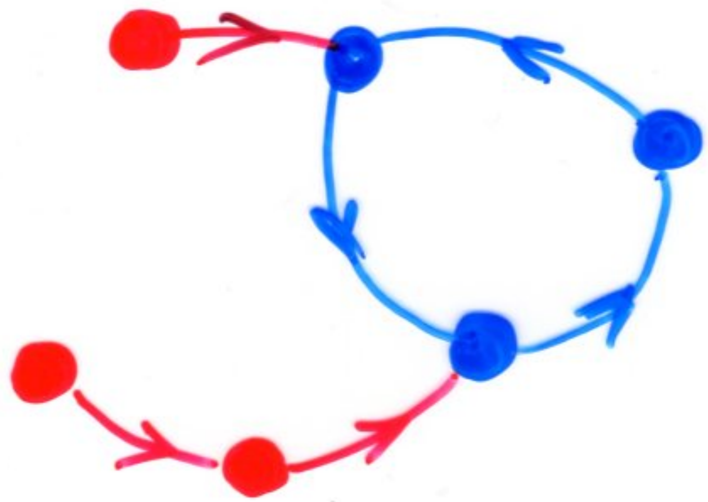
$$S \circ L^1$$

$$\left[1 - \gamma \frac{t}{1-t}\right]^{-z}$$



species $\mathcal{O}$

octopus

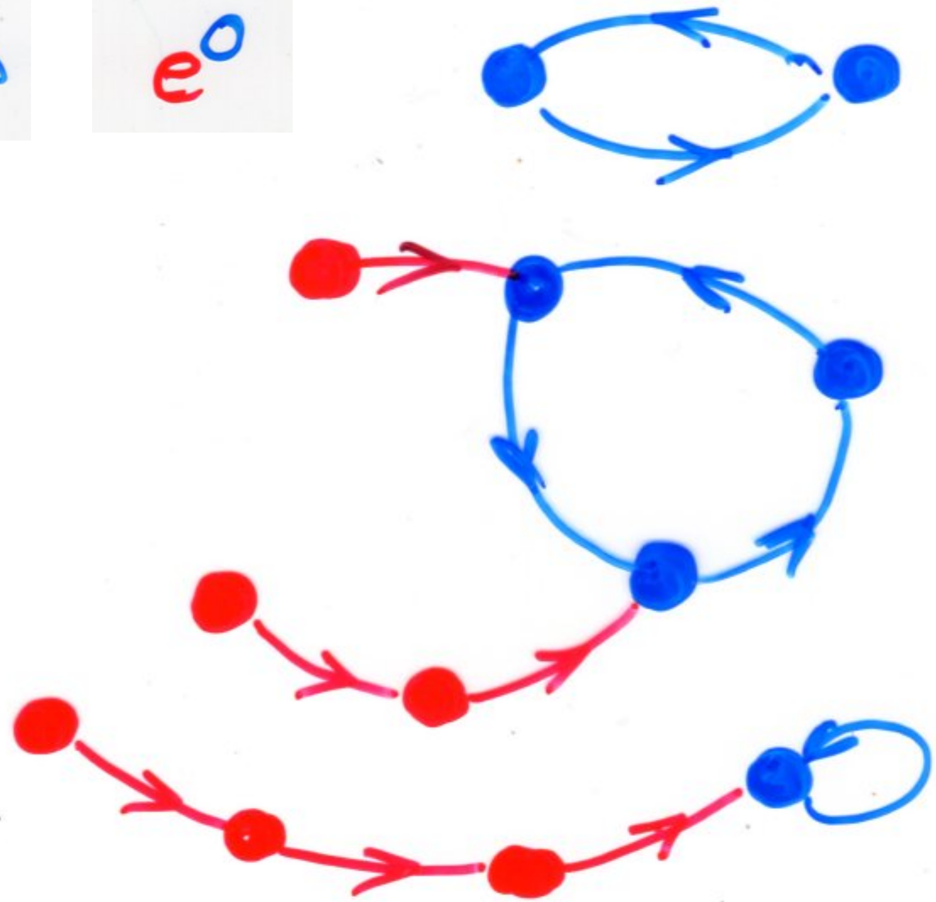


$$-\log\left[1-\frac{t}{1-t}\right]$$

$\log = \ln$

"assemblée" of octopus e^0

$S \circ L^1$



$\{c_i\}_{i\geq 1}$ $\{\tau_i\}_{i\geq 1}$

$C(\Omega)$ cycle of Ω

Ω octopus

$$w(\Omega) = c_k \prod_{j\in C(\Omega)} \tau_{\ell(j)}$$

$k = |C(\Omega)|$

$\ell(j) =$ length of $T(j)$

tentacle fixed on j

exp. generating function:

$$O_w(t) = \sum_{n\geq 1} \sum_{\Omega\in O[n]} w(\Omega)\frac{t^n}{n!} = \sum_{k\geq 1} \frac{c_k}{k}\Big(\sum_{i\geq 1} \tau_i t^i\Big)^k$$

Interpretation of Gegenbauer polynomials

Gegenbauer polynomials

ultraspherical polynomials

$$C_n^{(\lambda)}(x)$$

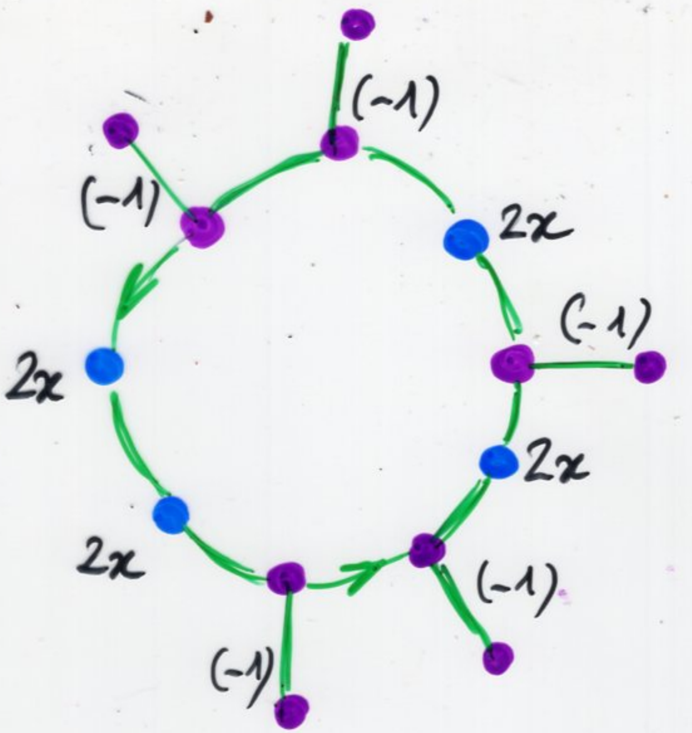
$$\sum_{n\geq 0} C_n^{(\lambda)}(x)\, t^n = (1-2xt+t^2)^{-\lambda}$$

Gegenbauer octopus

tentacle length ≤ 2

weight $\lambda\,(2x)^a\,(-1)^b$

$$\begin{cases} a = \text{number of tentacles length } 1 \\ b = \quad '' \qquad '' \quad \text{length } 2 \end{cases}$$



assemblée of octopus Ω

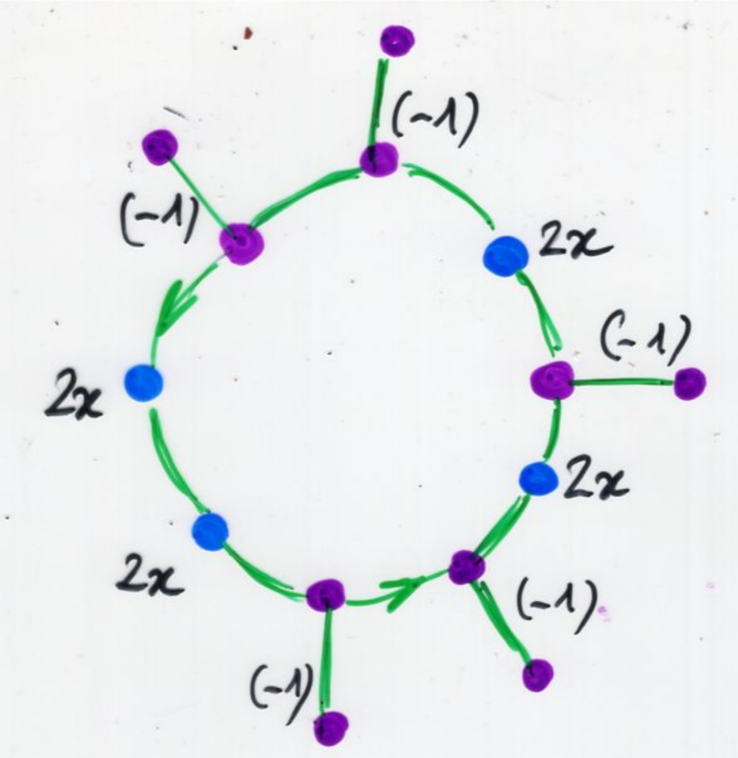
Tchebychev II

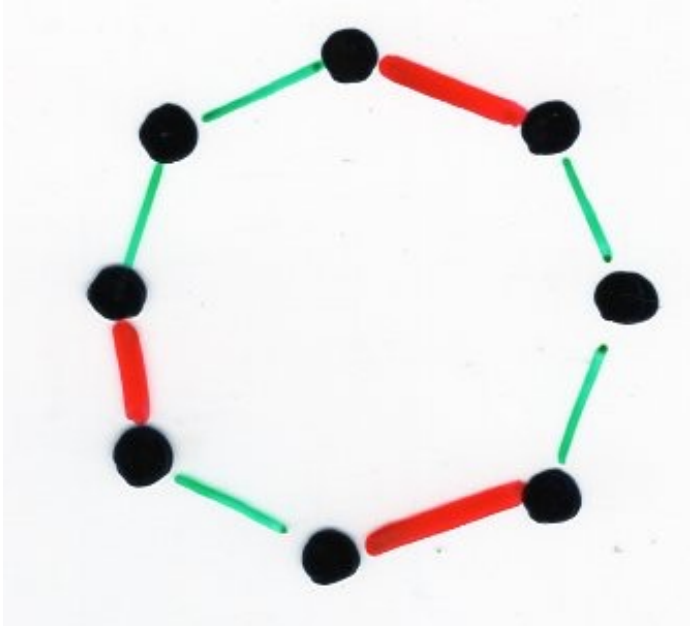
$$U_n(x) = C_n^{(1)}(x)$$

$$\sum_{n\geq 0} U_n(x)\, t^n = \frac{1}{1-2xt+t^2}$$

Gegenbauer octopus

$$T_n(x) = \frac{1}{2} C_n(2x)$$





Interpretation of Meixner-Pollaczeck polynomials

Meixner-Pollaczeck

$$\sum_{n \geq 0} P_n(x;\eta,\delta)\frac{t^n}{n!} = \left[(1+\delta t)^2 + t^2\right]^{-\eta/2} \exp\left[x \arctan\left(\frac{t}{1+\delta t}\right)\right]$$

$$\delta \in \mathbb{R}, \ \eta > 0$$

$$\left[(1+\delta t)^2+t^2\right]^{-\eta/2}\exp\left[x \arctan\left(\frac{t}{1+\delta t}\right)\right]$$

(i) Gegenbauer octopus

weight $(-\eta/2)(2\delta)^a(-(1+\delta^2))^b$

(ii) octopus with cycle odd length $2k+1$

weight $x(-1)^k(-\eta/2)^{n-2k-1}$

n number of vertices of Ω

Pairs of permutations

Pair of permutations

$$\mathbf{T}[A, B] = \mathbf{S}[A] \times \mathbf{S}[B]$$

$$(\sigma, \tau) \in \mathbf{T}[A, B]$$

J. Labelle, Y. N. Yeh (1989)

$$w(\sigma, \tau) = u^{cyc(\sigma)} v^{cyc(\tau)} r^{|A|} s^{|B|}$$

$$T(t) = \sum_{n \geq 0} T_n \frac{t^n}{n!}$$

$$T(t) = (1 - rt)^{-u} (1 - st)^{-v}$$

$$T_n(r, s; u, v)$$

$$T_n = \sum_{i+j=n} \binom{n}{i} (u)_i \, (v)_j \, r^i s^j$$

$$T[A,B] = S[A] \times S[B]$$

$$w(\sigma,\tau) = u^{cyc(\sigma)} v^{cyc(\tau)} r^{|A|} s^{|B|}$$

$$T_n(r,s;u,v)$$

$$T_n(c^{-1},1;-x,\beta+x) = M_n(x;\beta,c)$$

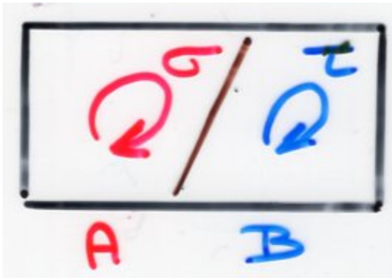
Interpretation of Meixner-Pollaczeck polynomials

Meixner-Pollaczeck

$$\sum_{n\geqslant 0} \mathbf{P}_n(x;\eta,\delta)\frac{t^n}{n!} = \left[(1+\delta t)^2+t^2\right]^{-\eta/2}\exp\left[x \arctan\left(\frac{t}{1+\delta t}\right)\right]$$

$$\delta\in\mathbb{R},\ \eta>0$$

$$\mathbf{T}[A,B] = \mathbf{S}[A]\times\mathbf{S}[B]$$



$$\sum_{n\geq 0} \mathcal{P}_n^{\lambda}(x;\varphi)\, t^n = \frac{(1-te^{i\varphi})^{-\lambda-ix}}{(1-te^{i\varphi})^{\lambda+ix}}$$

$$n!\,\mathcal{P}_n^{\lambda}(x;\varphi) = (2\lambda)_n\, e^{in\varphi}\; {}_2F_1\left[\begin{matrix}-n,\ \lambda+ix\\ 2\lambda\end{matrix};\ 1-e^{-2i\varphi}\right]$$

$$T_n(r,s;u,v)$$

$$n!\,\mathcal{P}_n^{\lambda}(x;\varphi) = T_n(e^{i\varphi}, e^{-i\varphi}; \lambda-ix, \lambda+ix)$$

$$= T_n(2i\sin\varphi, e^{i\varphi}; \lambda+ix, 2\lambda)$$

Interpretation of Krawtchouk polynomials

Krawtchouk polynomials

$$K_n(x; p, N) = {}_2F_1\left[\begin{matrix}-n, -x \\ -N\end{matrix}; \frac{1}{p}\right]$$

for $0 \leqslant n \leqslant N$, $0 < p < 1$

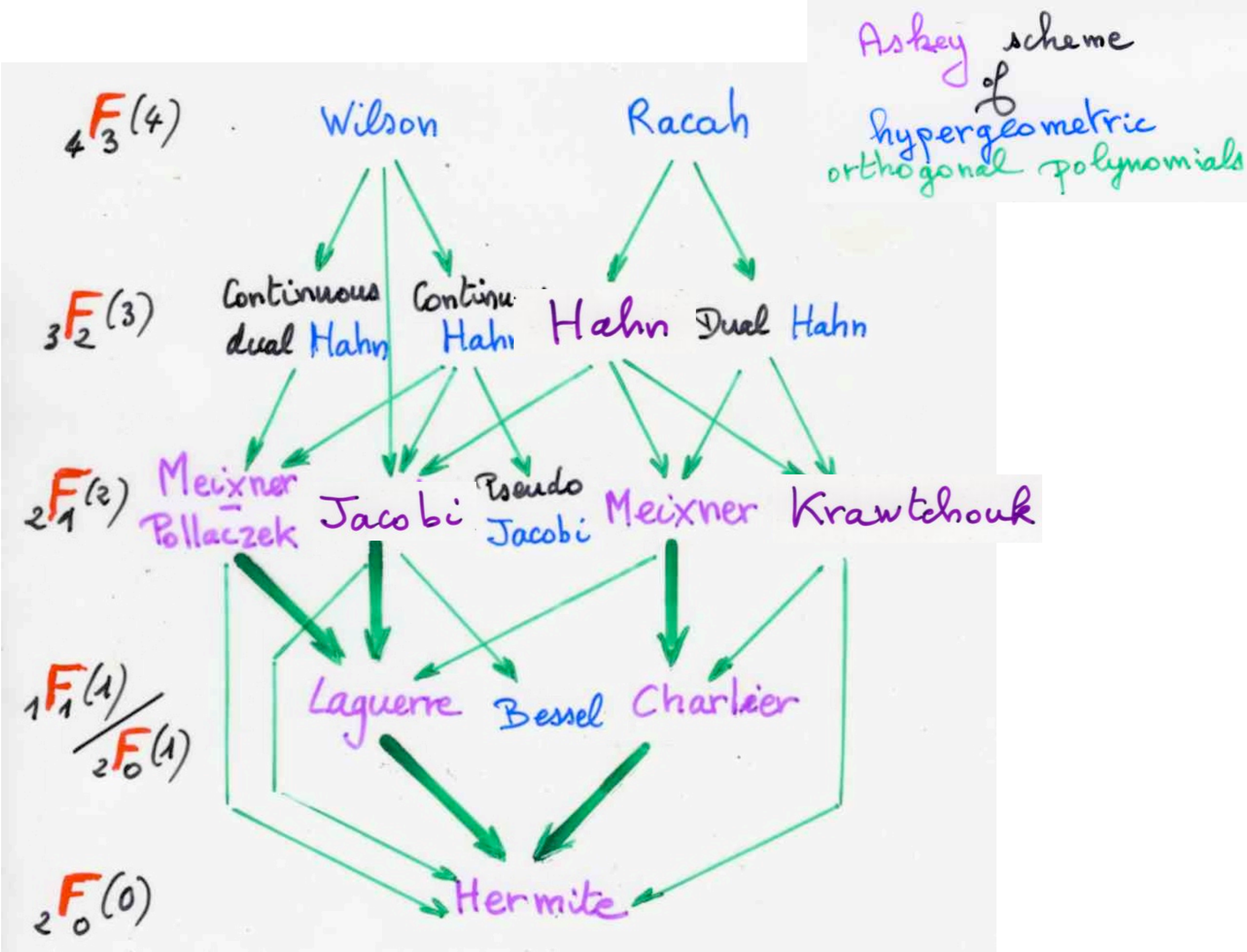
Krawtchouk configurations

$$K[A, B] = S[A] \times S[B]$$

$$T_n(-q p^{-1}, 1; -x, x-N) = (-N)_n \, K_n(x; p; N)$$

$$T_n(r, s; u, v)$$

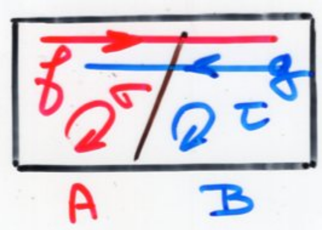
Interpretation of Hahn polynomials


Askey scheme of hypergeometric orthogonal polynomials
$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continu Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
$_1F_1(1)$ / $_2F_0(1)$
Laguerre
Bessel
Charlier
$_2F_0(0)$
Hermite

Hahn

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x\\ \alpha+1,\ -N\end{matrix};1\right] \qquad 0\le n\le N$$

Hahn configurations

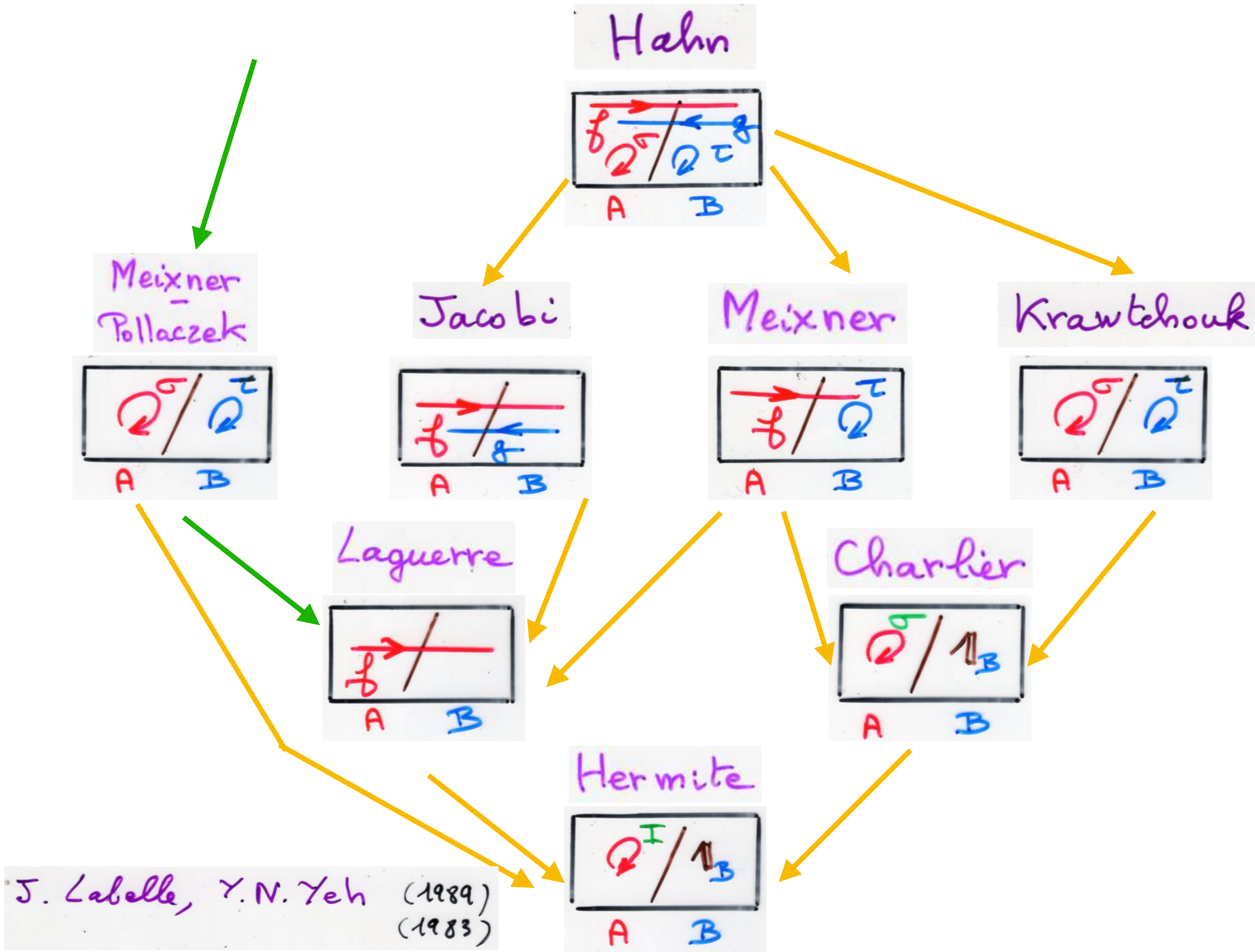


$$Q[A,B] = L[A,B]\times S[A]\times L[B,A]\times S[B]$$

$$w(f,\sigma,g,\tau) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)}(x-N)^{cyc(\sigma)}(-x)^{cyc(\tau)}(-1)^{|B|}$$

$$(\alpha+1)_n(-N)_n Q_n(x;\alpha,\beta,N) = \sum_{(f,\sigma,g,\tau)\in Q[A,B]} w(f,\sigma,g,\tau)$$

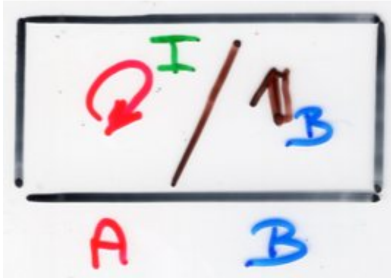
$$Q[A,B] = L[A,B] \times S[A] \times L[B,A] \times S[B]$$


Hahn
f
g
σ
τ
A
B
Meixner-Pollaczek
Jacobi
Meixner
Krawtchouk
Laguerre
Charlier
Hermite
I
1B
J. Labelle, Y.N. Yeh (1989) (1983)

(A, B)

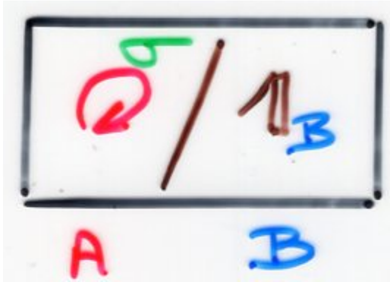
Hermite configurations

$$\mathbf{H}[A,B] = \mathbf{I}[A] \times \{1_B\}$$



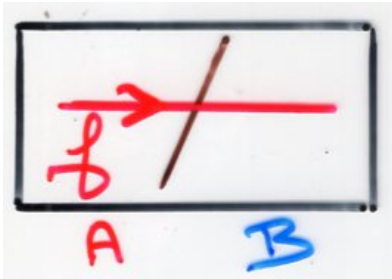
Charlier configurations

$$\mathbf{C}[A,B] = \mathbf{S}[A] \times \{1_B\}$$



Laguerre configurations

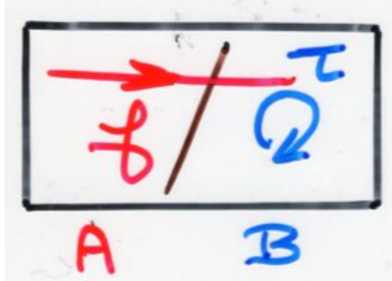
$$\mathbf{L}[A,B] = \left\{ \begin{array}{l} \text{injective map } f \\ \text{from } A \text{ to } A+B \end{array} \right\}$$



(A, B)

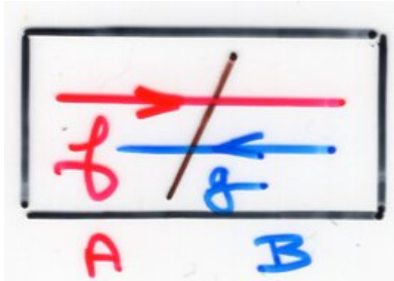
Meixner configurations

$$M[A,B] = L[A,B] \times S[B]$$



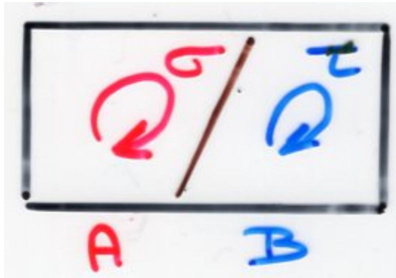
Jacobi configurations

$$J[A,B] = L[A,B] \times L[B,A]$$



Krawtchouk configurations

$$K[A,B] = S[A] \times S[B]$$

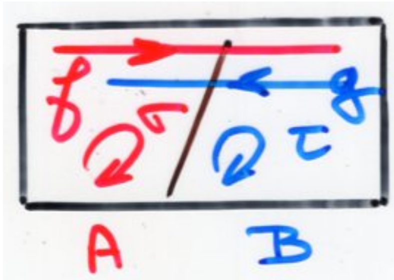


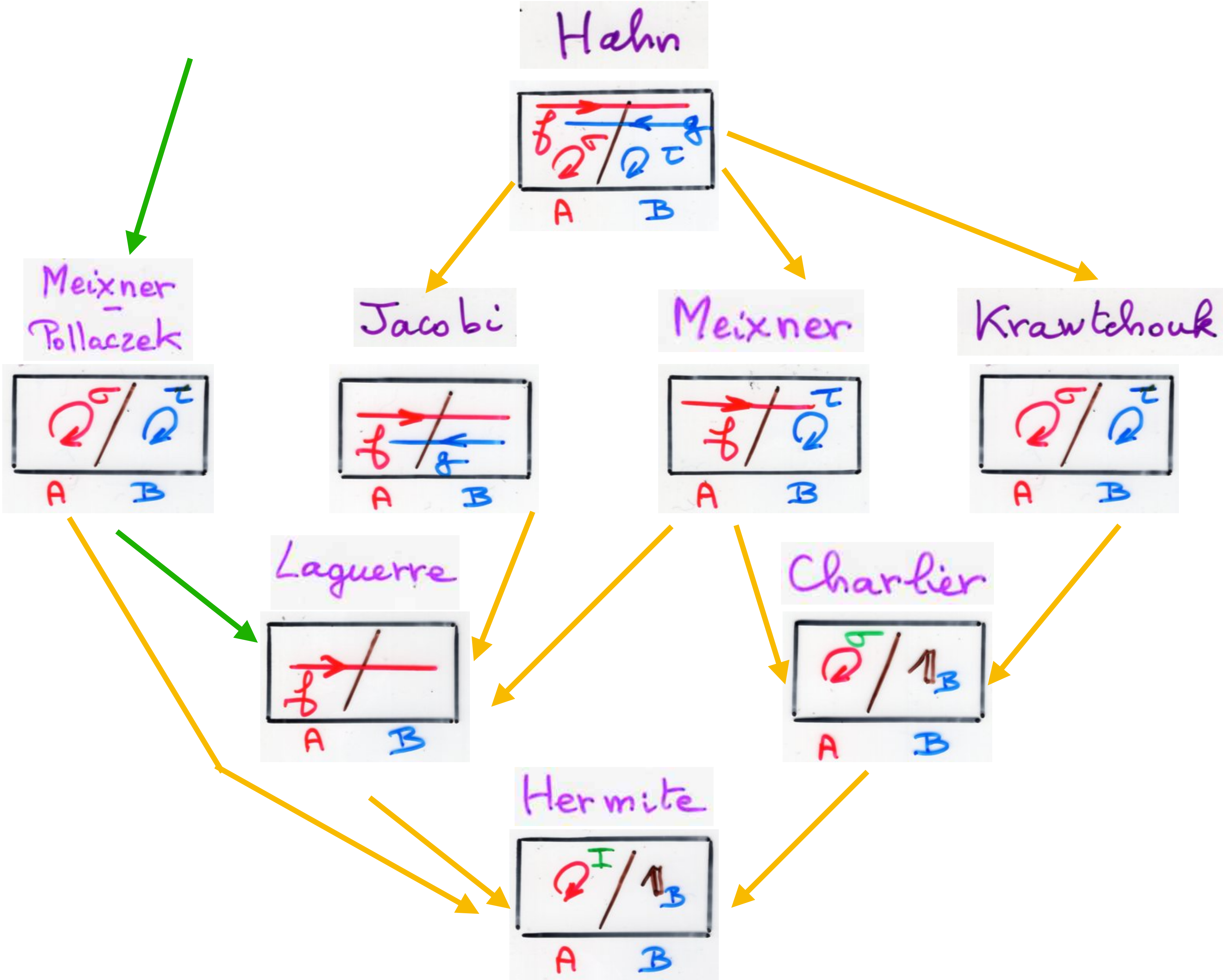
Hahn configurations

$$\mathbf{Q}[A,B] = \mathbf{L}[A,B] \times \mathbf{S}[A] \times \mathbf{L}[B,A] \times \mathbf{S}[B]$$

$$= \mathbf{M}[A,B] \times \mathbf{M}[B,A]$$

$$= \mathbf{J}[A,B] \times \mathbf{K}[A,B]$$





Hahn
f
g
σ
τ
A
B
Meixner-Pollaczek
σ
τ
A
B
Jacobi
f
g
A
B
Meixner
f
τ
A
B
Krawtchouk
σ
τ
A
B
Laguerre
f
A
B
Charlier
σ
1_B
A
B
Hermite
I
1_B
A
B

Sheffer polynomials

(Ch 5c)

Sheffer polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\exp\left(x f(t)\right)$$

binomial type polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \exp\left(x f(t)\right)$$

delta operator Q

$$Q = \sum_{k \geq 0} \frac{a_k}{k!} D^k$$

$$D x^n = n x^{(n-1)}$$

binomial type polynomials

Rota
umbral calculus

$$Q(B_n) = B_{n-1}$$

$$\sum_{n \geq 0} B_n(x) \frac{t^n}{n!} = \exp\left(x q^{<-1>}(t)\right)$$

$$t = q(u) \qquad u = q^{<-1>}(t)$$

Sheffer polynomials

S , Q delta operators

$S = s(D)$

$b_0 \neq 0$

$Q = q(D)$

$a_1 \neq 0$

$$s(t) = \sum_{k \geq 0} b_k \frac{t^k}{k!}$$

$$q(t) = \sum_{k \geq 1} a_k \frac{t^k}{k!}$$

$\{P_n(x)\}_{n \geq 0}$ Sheffer polynomials

$B_n = S(P_n)$ binomial type polynomials

Sheffer polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\exp\left(x f(t)\right)$$

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = \frac{1}{s\left(q^{<-1>}(t)\right)}\exp\left(x\, q^{<-1>}(t)\right)$$

$$\sum_{n\geq 0} Q_n(x)\frac{t^n}{n!} = s(t)\exp\left(x\, q(t)\right)$$

Inverse polynomials

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

See Ch 1d

$$Q_n(x) = \sum_{i=0}^{n} q_{n,i} x^i$$

inverse sequence

$$\{Q_n(x)\}_{n \geq 0}$$

combinatorial interpretation of the operator Q and S for the 5 classes of Sheffer orthogonal polynomials with:

Laguerre histories

$\left\{ \begin{array}{ll} \text{restricted} & S \\ \text{large} \rightarrow & Q \end{array} \right.$

duality

orthogonal polynomial ⟷ moments μ_n