Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

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mirror website

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Chapter 5
Orthogonal polynomials and exponential structures

Ch5a

IMSc, Chennai
February 25, 2019

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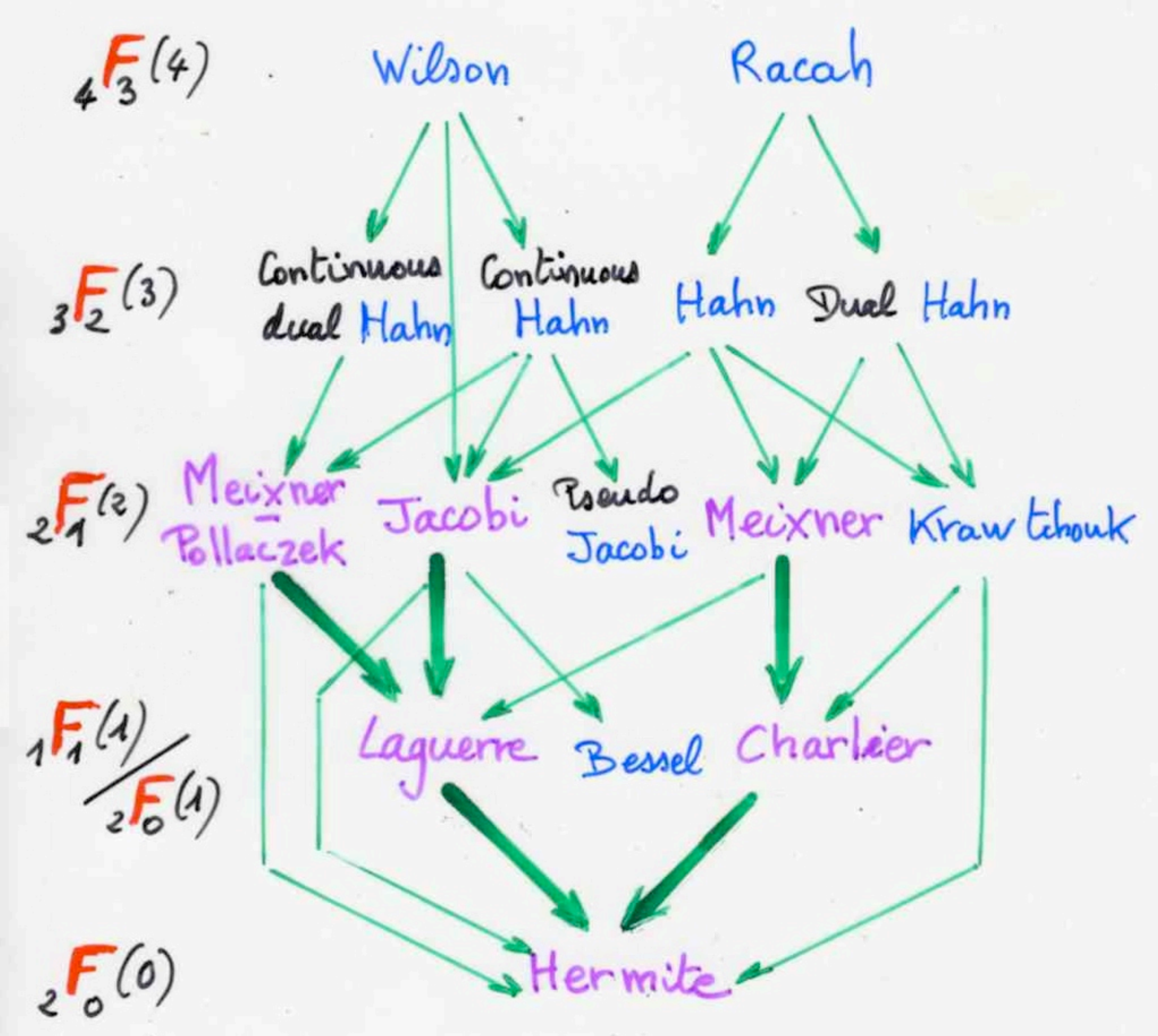
Askey scheme of hypergeometric orthogonal polynomials



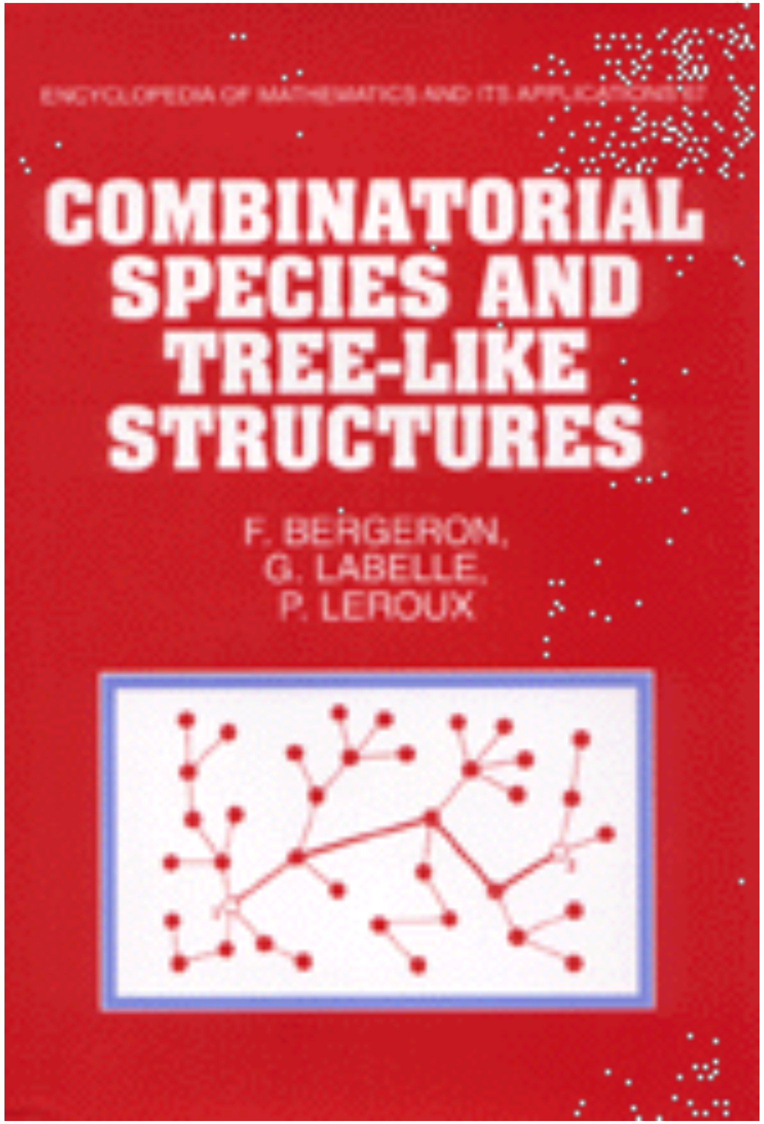
orthogonal Sheffer polynomials

Combinatorial model for exponential generating function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Species (combinatorial) structures

UQAM Montréal Québec



Encyclopedia of Mathematics and its Applications
Cambridge University Press (1977)

This lecture is dedicated to
My dear friend Pierre Leroux

San Diego, California, 1978-79

A. Garsia, X.V., R.Stanley's course, vol 1 and 2 (!), ...

Back to UQAM, Montréal, 1979, séminaire ...

1980 Groupe de recherche en Combinatoire de l'UQAM

Theory of combinatorial species, A.Joyal, G.Labelle,...

Exponential generating functions

Combinatorial interpretation of some families of orthogonal polynomials using species theory

1980-1990

P.Leroux, F. Bergeron,

visitors: V. Strehl, Y.N.Yeh, D.Foata, X.V., ...

1989, founder of the LACIM

Laboratoire de Combinatoire
et d'Informatique Mahtématique

UQAM, Montréal

One of the founder of the FPSAC colloques

(formal power series and algebraic combinatorics)

FPSAC' 88,Lille,90, Paris

……………………………

FPSAC' 91, Bordeaux

FPSAC' 92, Montréal

FPSAC' 93, Firenze

……………………………

FPSAC' 21, Bangalore

Société Mathématique de France (SMF)
Gazette des mathématiciens, Juillet 2008, n° 117

"Pierre Leroux (1942 - 2008)", X.V., p 59-66
"La carrière mathématique de Pierre Leroux",
Gilbert Labelle, p 67-74

http://smf.emath.fr/
⟶ "publications" "Gazette"

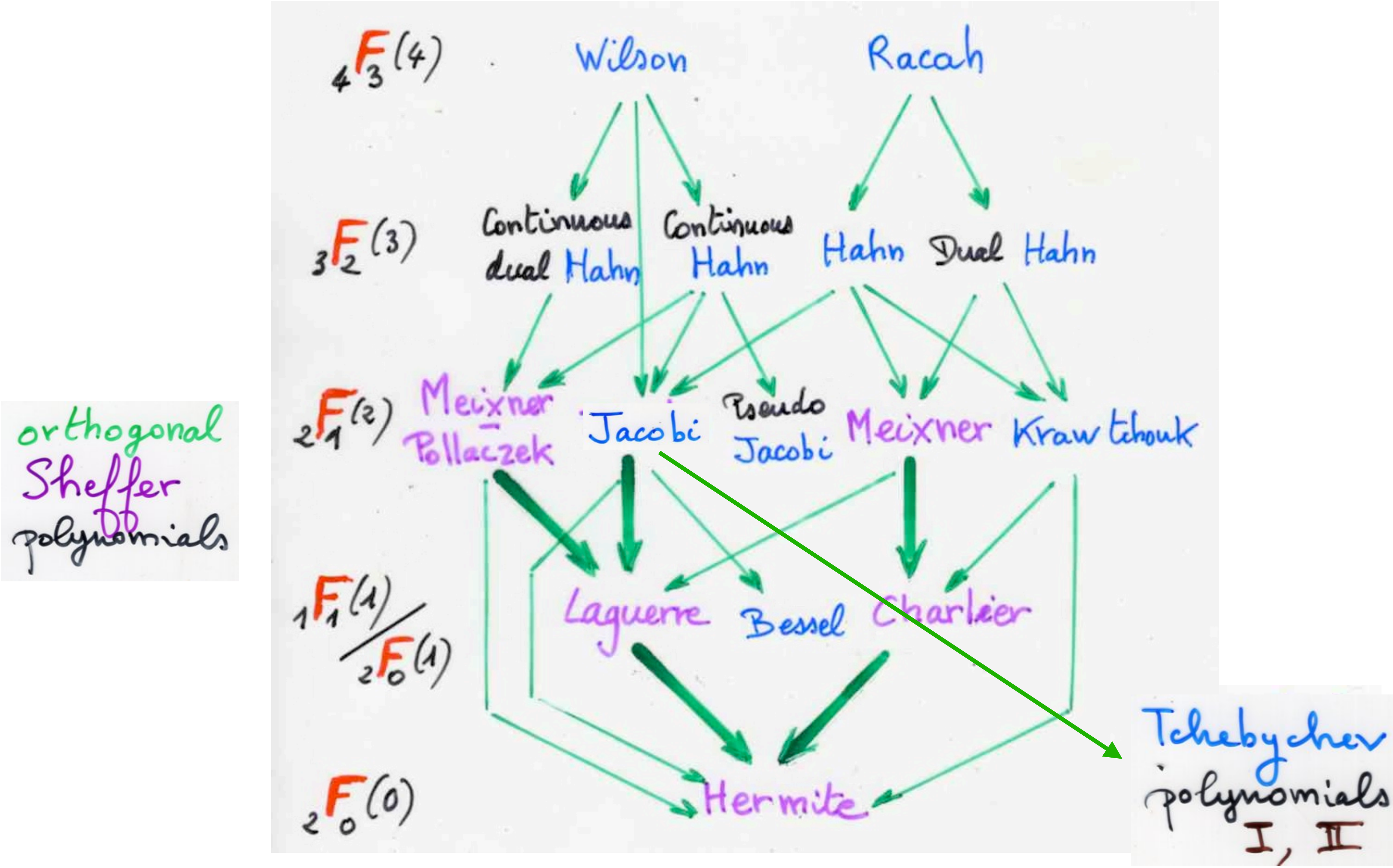

59th SLC
Bertinoro
Sept 2007

Hypergeometric series

and

orthogonal polynomials

Askey scheme of hypergeometric orthogonal polynomials



$\{u_n\}_{n\geq 0}$ geometric

$$\frac{u_{n+1}}{u_n} = c \quad \text{constant}$$

$$u_n = u_0\, c^n$$

$\{u_n\}_{n\geq 0}$ hypergeometric

$$\frac{u_{n+1}}{u_n} = R(n) \quad \text{rational function}$$

$$R(n) = \frac{(a_1+n)\cdots(a_r+n)}{(1+n)(b_1+n)\cdots(b_s+n)}\, z$$

$$(a)_k = \begin{cases} a(a+1)\cdots(a+k-1) & \text{if } k>0 \\ 1 & \text{if } k=0 \\ \dfrac{1}{(a-1)(a-2)\cdots(a+k)} & \text{if } k<0 \end{cases}$$

Pochhammer symbol
rising factorial

$$(1)_k = k!$$

$${}_rF_s\left[\begin{matrix} a_1, a_2, \ldots a_r \\ b_1, \ldots, b_s \end{matrix} ; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k}\, z^k$$

hypergeometric

Gauss (1812) hypergeometric series

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; z\right]$$

$\Gamma(n+1) = n!$

$a, b, c \in \mathbb{C}$

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

$\mathrm{Re}(c-a-b) > 0$

Gauss (1812)

${}_rF_s$ "generalized" hypergeometric series

$$ {}_2F_1\left[\begin{matrix}-n, b\\ c\end{matrix};1\right] = \frac{(c-b)_n}{(c)_n} $$

Vandermonde (1770)

Chu (1303)

$$ {}_2F_1\left[\begin{matrix}1-c-2n, -2n\\ c\end{matrix};-1\right] = (-1)^n \frac{(2n)!\,(c-1)!}{n!\,(c+n-1)!} $$

Kummer (1836)

$$ {}_3F_2\left[\begin{matrix}a, b, -n\\ c, 1+a+b-c-n\end{matrix};1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} $$

Pfaff-Saalschütz

(1797) (1890)

Orthogonal Sheffer polynomials

Sheffer polynomials

$$\sum_{n \geqslant 0} P_n(x) \frac{t^n}{n!} = g(t) \exp\left(x f(t)\right)$$

binomial type polynomials

$\{P_n(x)\}_{n\geq 0}$ orthogonal polynomials

Meixner (1934)

are Sheffer polynomials

$\Longleftrightarrow$ $\{P_n(x)\}_{n\geq 0}$ are one of the 5 possible types:

Hermite

Laguerre

Charlier

Meixner

Meixner-Pollaczek

$\{P_n(x)\}_{n\geqslant 0}$ orthogonal polynomials

are Sheffer polynomials

Meixner (1934)

positive-definite OPS

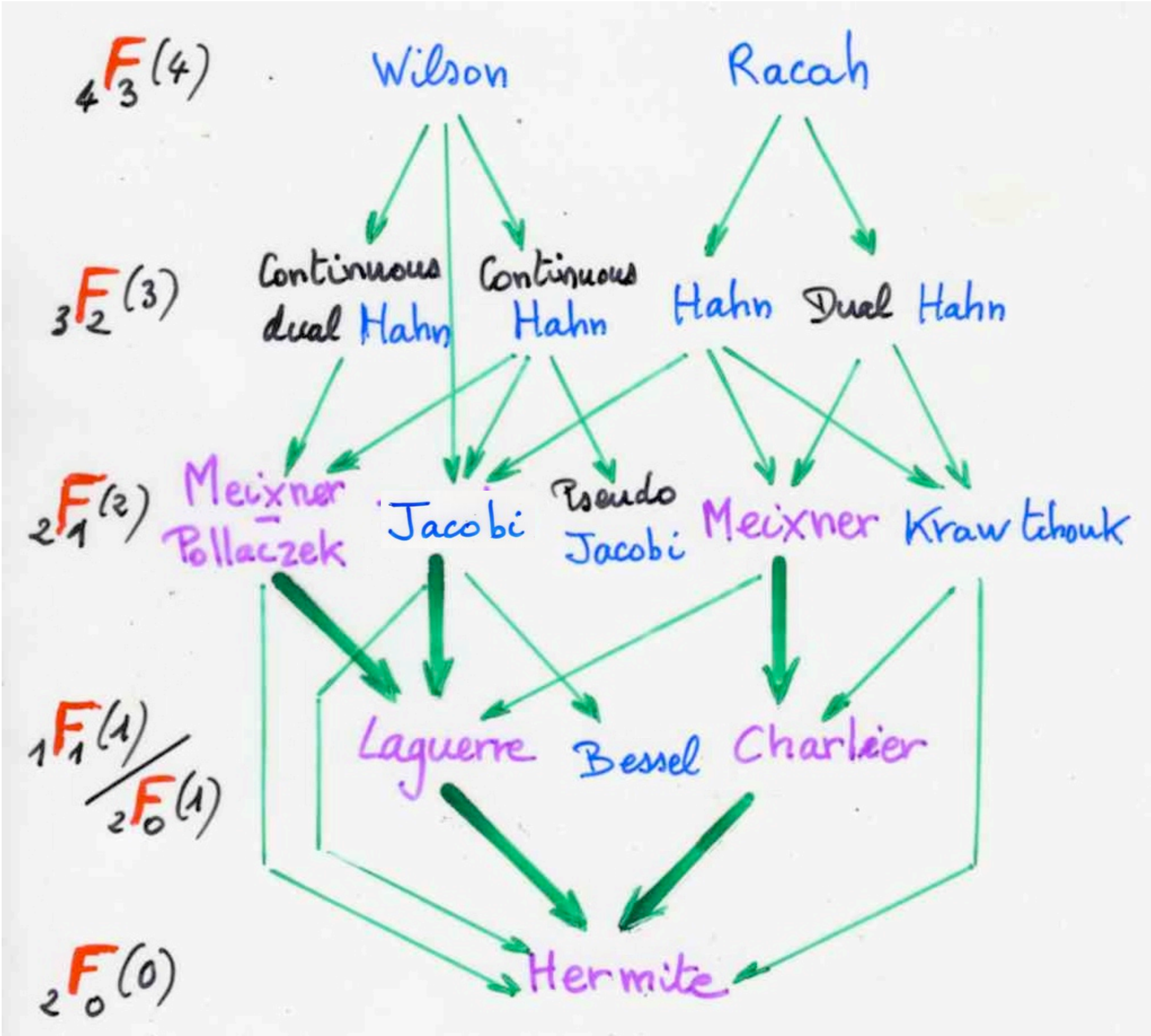
Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak+b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a,b,c,d \in \mathbb{R} \\ c \geqslant 0, \ c+d>0 \end{cases}$

(1) $a=0$, $c=0$	Hermite polynomials	$H_n(x)$
(2) $a\neq 0$, $a^2-4c=0$	Laguerre polynomials	$L_n^{(\alpha)}(x)$
(3) $a\neq 0$, $c=0$	Charlier polynomials	$C_n^{(a)}(x)$
(4) $a^2-4c>0$	Meixner polynomials	$M_n^{(\beta,c)}(x)$
(5) $a^2-4c<0$	Meixner - Pollaczek polynomials	$C_n^{(a)}(x)$

Askey scheme of hypergeometric orthogonal polynomials

orthogonal
Sheffer
polynomials



reminding

ABjC, Part 1 Chapter 3

(some ideas about)

species and
exponential generating functions

"naive" definition

U finite set

combinatorial structure

construction α

U underlying set with

α constructed on U,

supported by U

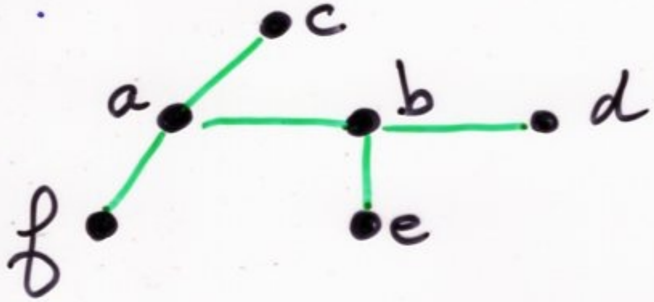
species F

structures of type F

set $F[U]$

F-structure $\alpha \in F[U]$

<u>example</u> Tree (= graph having no cycle)



Permutations,
(set) Partitions,
Graphs,
Endofunctions, ...

Transport of structures

$$U \xrightarrow{f} V$$

bijection

$$F[U] \xrightarrow{F[f]} F[V]$$

transport along f

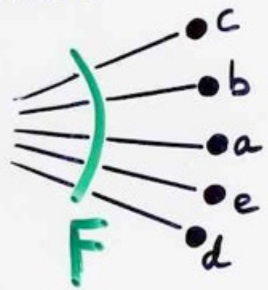
<u>example</u> trees $U = \{a, b, c\}$ $V = \{1, 2, 3\}$

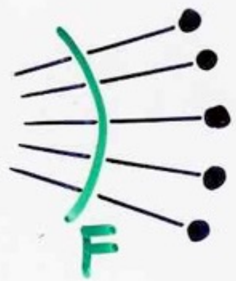
$$\begin{array}{ccc} U = \{a, b, c\} & & \{a - b - c, \; b - a - c, \; a - c - b\} \\ f \downarrow \quad \downarrow \downarrow \downarrow & & F[f] \downarrow \\ V = \{1, 2, 3\} & & \{1 - 2 - 3, \; 2 - 1 - 3, \; 1 - 3 - 2\} \end{array}$$

coherent transport

$$\mathbf{F}[f \circ g] = \mathbf{F}[f] \circ \mathbf{F}[g] \qquad \mathbf{F}[Id_U] = Id_{\mathbf{F}[U]}$$

Convention.





enumeration $a_n = \left| F[\{1, 2, \ldots, n\}] \right|$

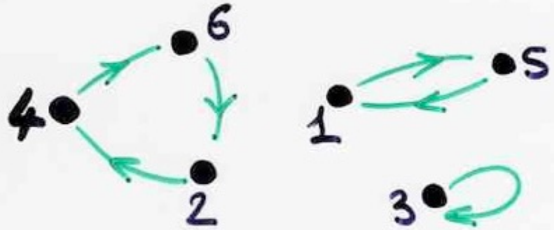
Definition generating function of the species F

$$F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Examples

Permutations S

$a_n = n!$ $\quad S(t) = \frac{1}{1-t}$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}$$

$$\sigma = 543612$$

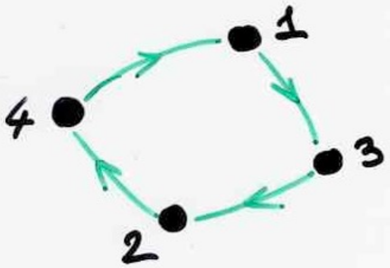
Total order L

$a_n = n!$ $\quad L(t) = \frac{1}{1-t}$

Cycle C

$a_n = (n-1)!$

$$C(t) = \sum_{n\geq 1} \frac{t^n}{n} = \log(1-t)^{-1}$$



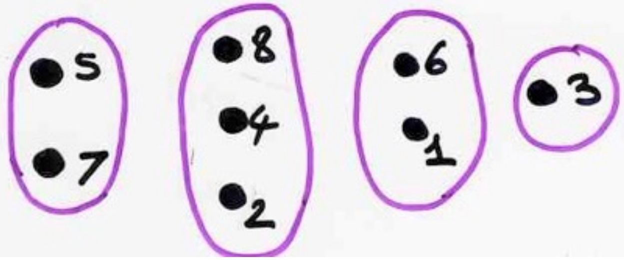
circular permutations

set E (ensemble) $E[t] = e^t$

uniform species

"Ensemble"
e
Euler

8. Partition B



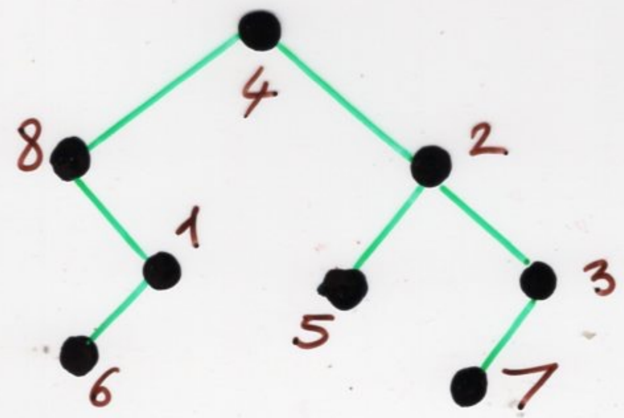
Bell number

$a_n = B_n$
nombre de Bell

$$B(t) = \exp(e^t - 1)$$

(labeled) binary tree

$$a_n = n!\, C_n$$

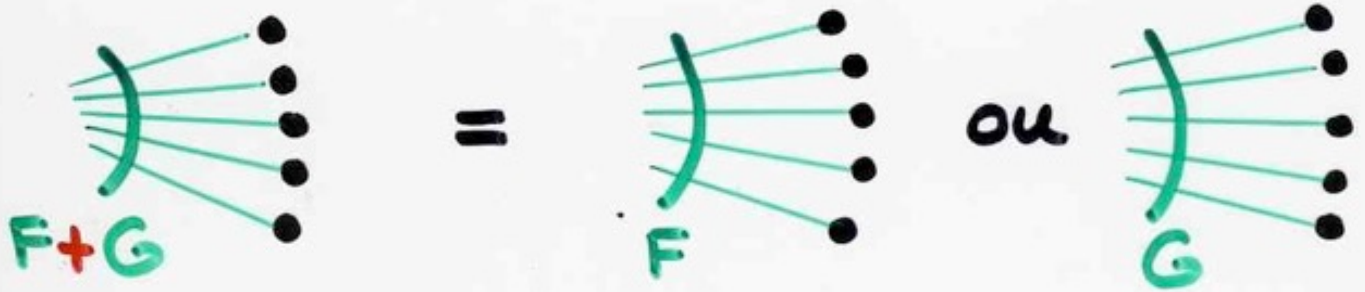


$$y = \sum_{n \geqslant 0} C_n t^n$$

$$= \sum_{n \geqslant 0} (n!\, C_n) \frac{t^n}{n!}$$

Def. sum

$$(F+G)[U] = F[U] + G[U] \quad \text{(disjoint union)}$$



Prop.

$$(F+G)[t] = F[t] + G[t]$$

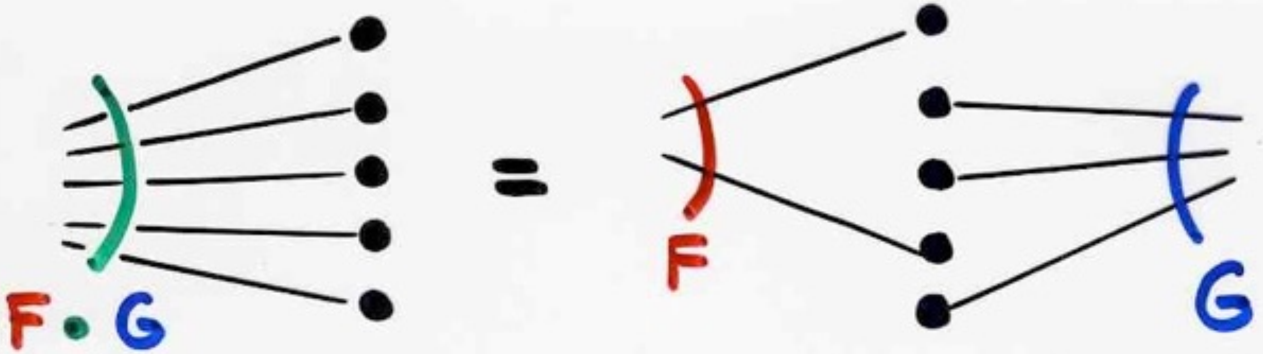
$$c_n = a_n + b_n$$

ex.

$$E = EP + EI$$

$$e^t = \mathrm{ch}\,t + \mathrm{sh}\,t$$

Déf. product $F \cdot G$



$\gamma \in F\cdot G[U]$ $\quad \gamma = (U_1, U_2, \alpha, \beta)$ $\quad \{U_1, U_2\}$ partition of U

$\alpha \in F[U_1] \quad \beta \in G[U_2]$

Prop.

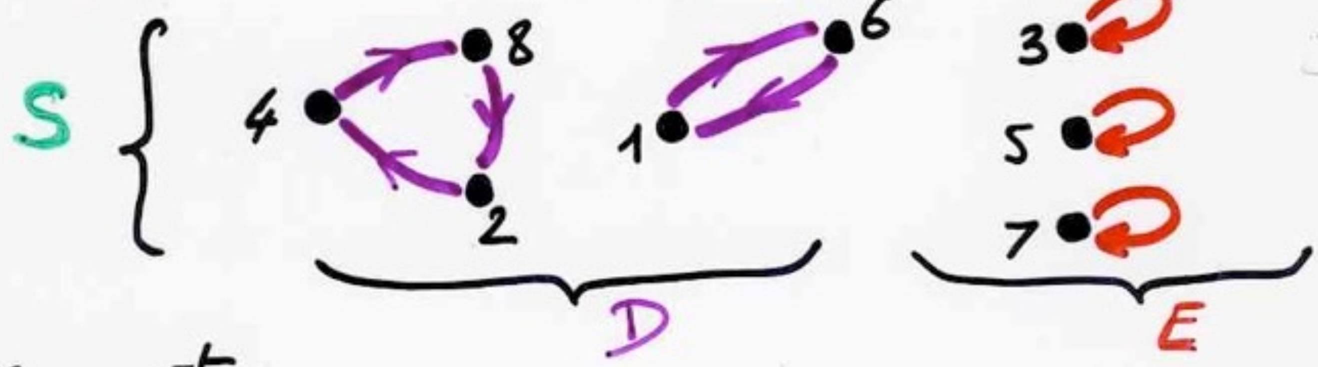
$$F\cdot G[t] = F[t]\, G[t]$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k\, b_{n-k}$$

ex. Dérangements D

$$S = D \cdot E$$ set

permutation   ensemble



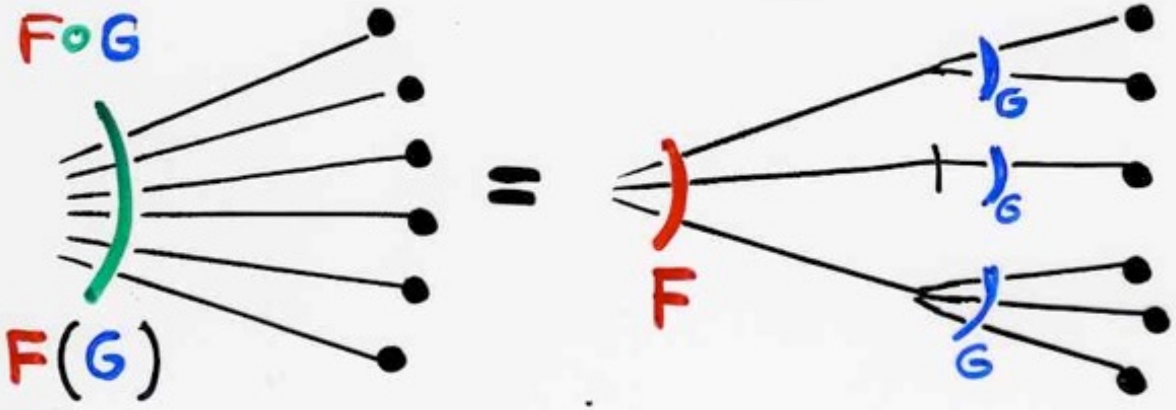
$$D[t] = \frac{e^{-t}}{1-t}$$

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{(-1)^n}{n!}\right)$$

$$p_n = \frac{d_n}{n!} \longrightarrow \frac{1}{e}$$

Def. F $\quad G \quad G[\varnothing] = \varnothing$

substitution of G into F



$\gamma \in F(G)[U]$

$$\gamma \begin{cases} \bullet \ \text{partition } \{U_1, \ldots, U_k\} \text{ de } U \\ \quad\ \text{classes} \neq \varnothing \\ \bullet \ \beta_i \in G[U_i] \ , \quad i = 1, \ldots, k \\ \bullet \ \alpha \in F[U/\equiv] \end{cases}$$

F- "assemblée" of G-structures

ex.- permutation = assemblee of cycles

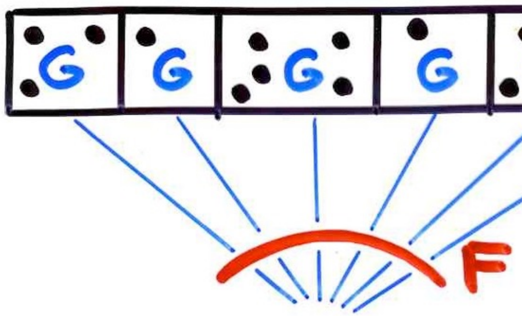
Prop. $(F \circ G)(t) = F(G(t))$

$$c_n = \sum_{\substack{k=0 \\ n_1+\dots+n_k=n \\ n_1,\dots,n_k \geq 1}}^{n} \frac{n!}{k!\, n_1! \dots n_k!}\, a_k\, b_{n_1} \dots b_{n_k}$$

Cor $F = E$

$$(E \circ G)(t) = \exp(G(t))$$

"assemblée" of G-structures E^G

$H =$ 

$$H = F(G)$$

$$F = E$$

species set

$$e^t$$

$$H = \exp(G)$$

"assemblée" of G-structures

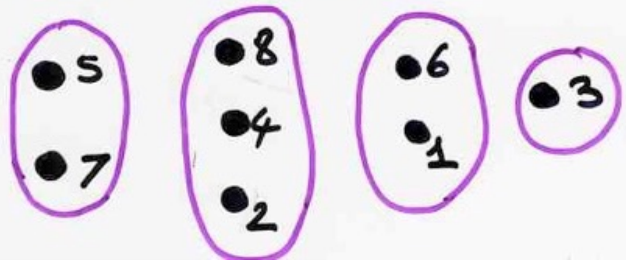
$$h(t) = \exp(g(t))$$

$B = \exp(E^{+})$

non-empty set

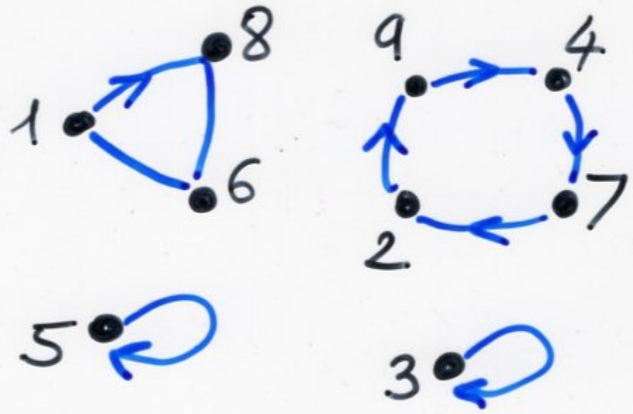
Bell number

Partition B

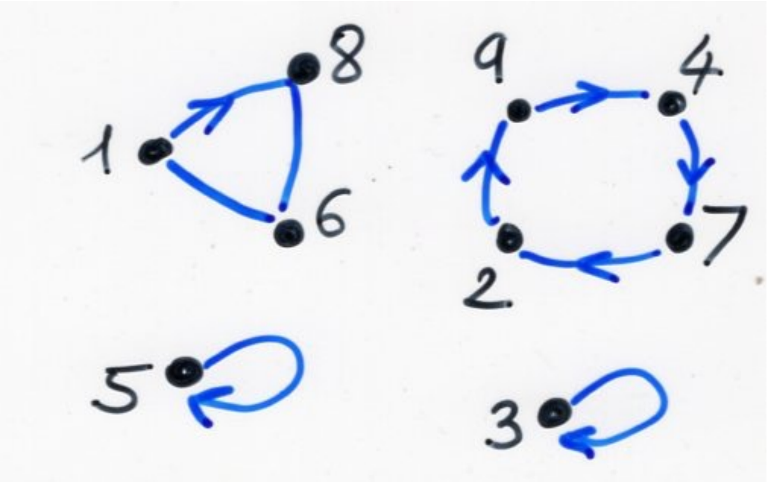


$a_n = B_n$
nombre de Bell

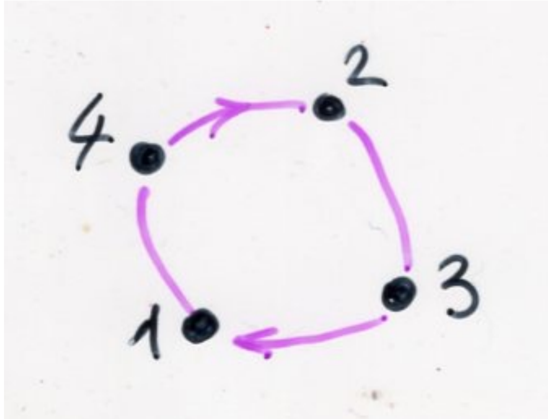
$B(t) = \exp(e^t - 1)$



permutation = exp (cycle)



cyclic
permutation

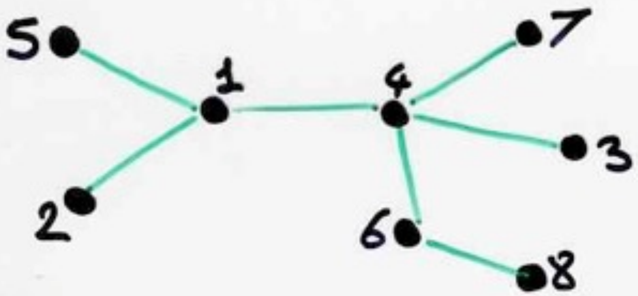


$$\sum_{n\geqslant 0} n!\,\frac{t^n}{n!} = \frac{1}{1-t}$$

$$\sum_{n\geqslant 1} (n-1)!\,\frac{t^n}{n!} = \sum_{n\geqslant 1} \frac{t^n}{n}$$

$$= \log \frac{1}{1-t}$$

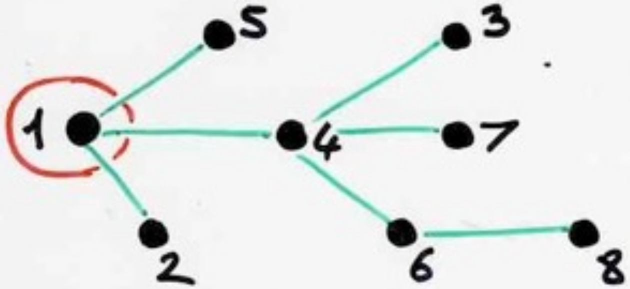
12. (Cayley) tree a



$$a_n = n^{n-2}$$

13. Arborescence A

pointed vertex (root)

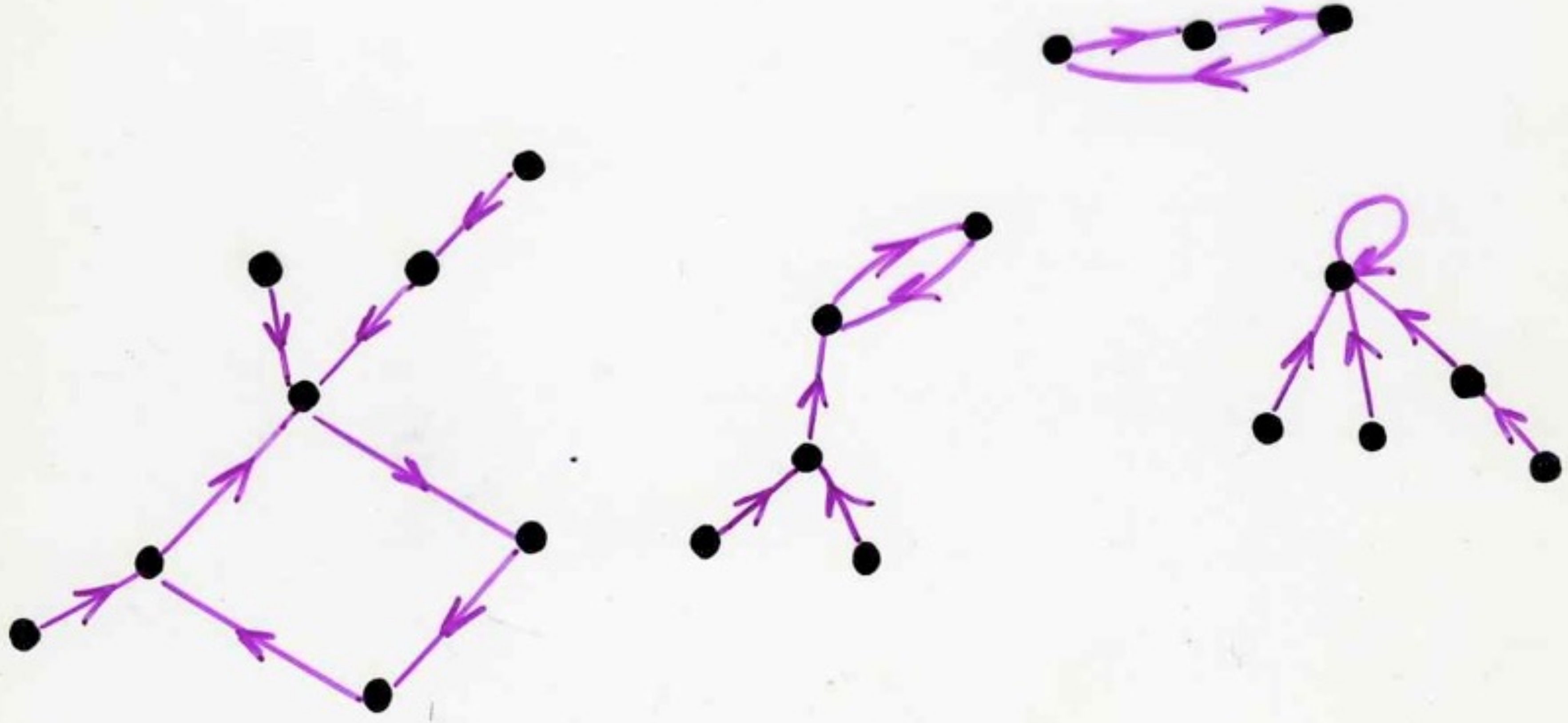


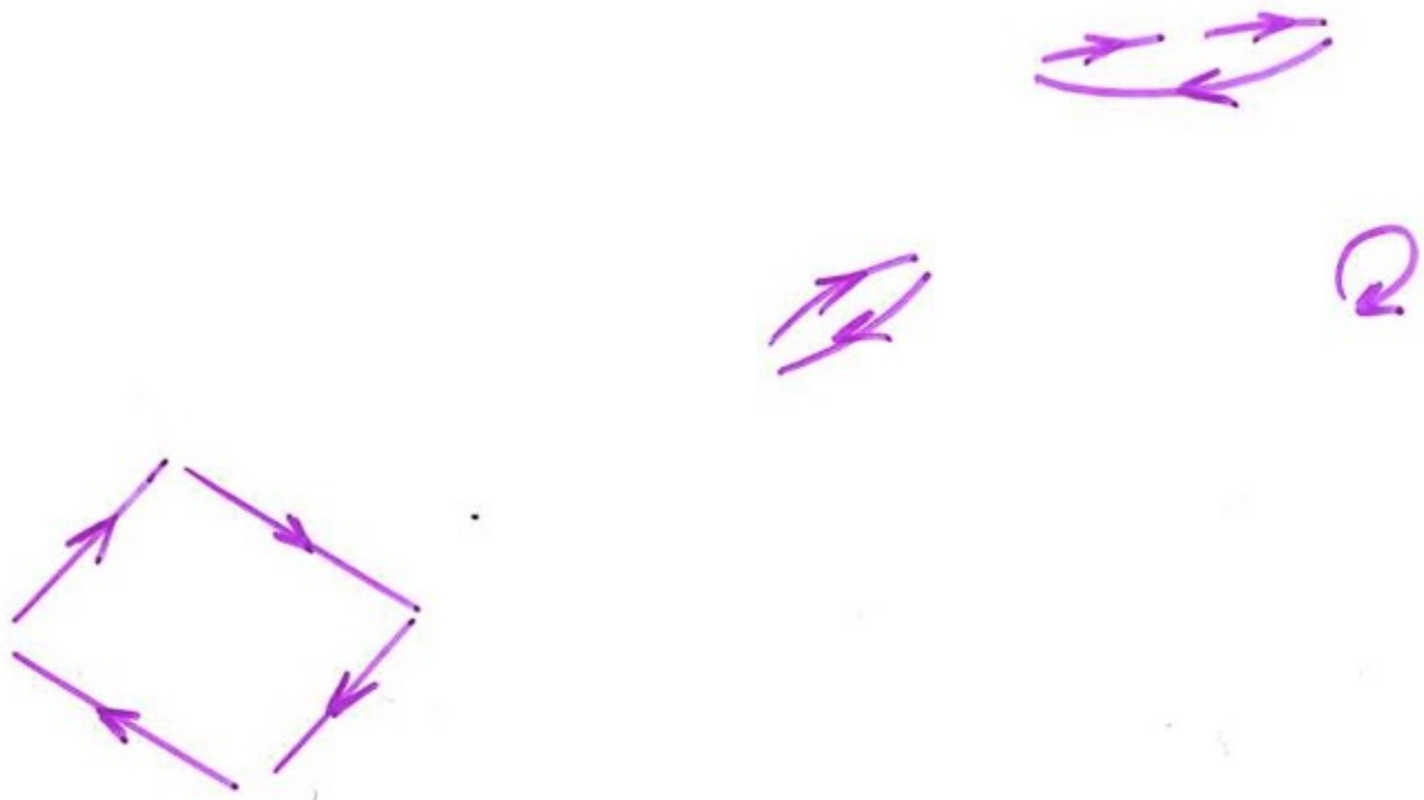
$$a_n = n^{n-1}$$

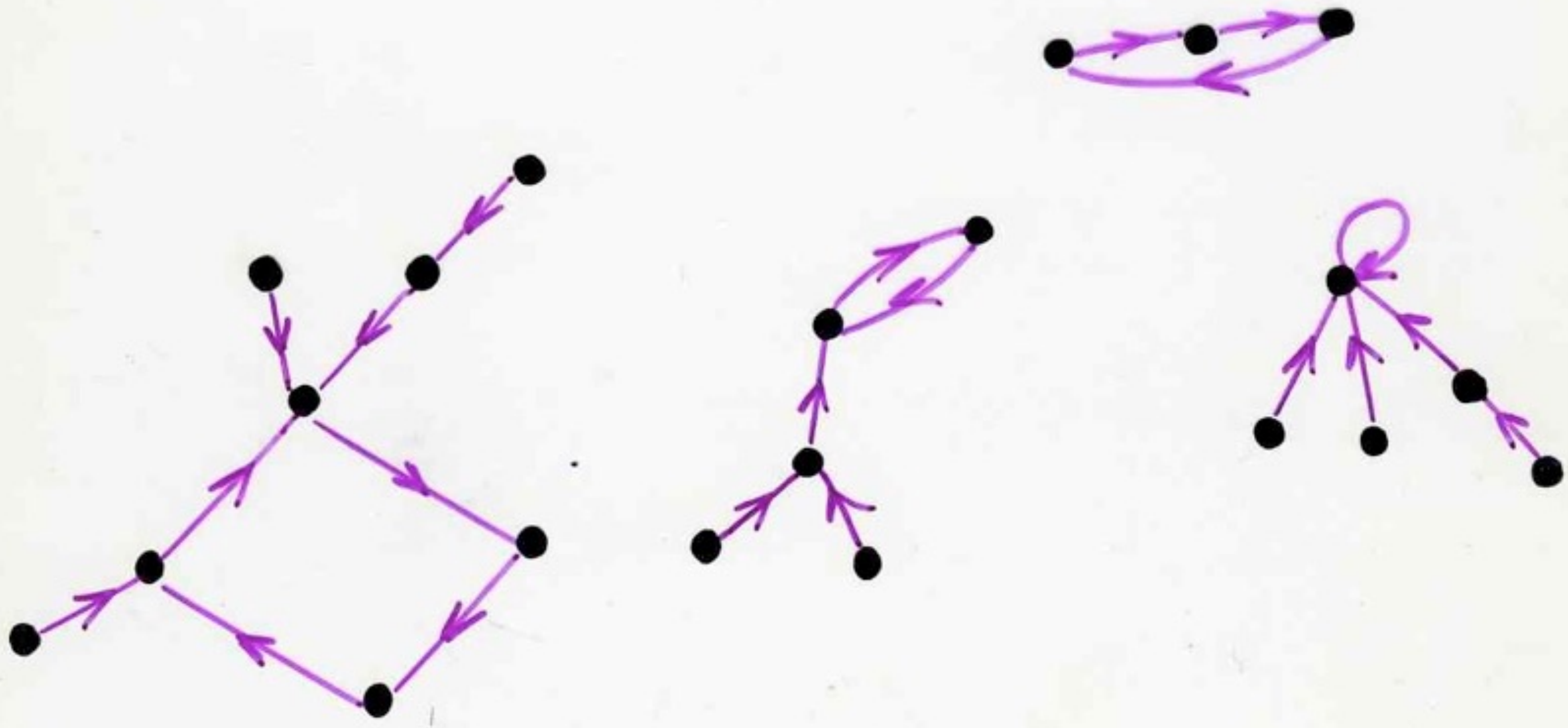
ex. Endofonctions

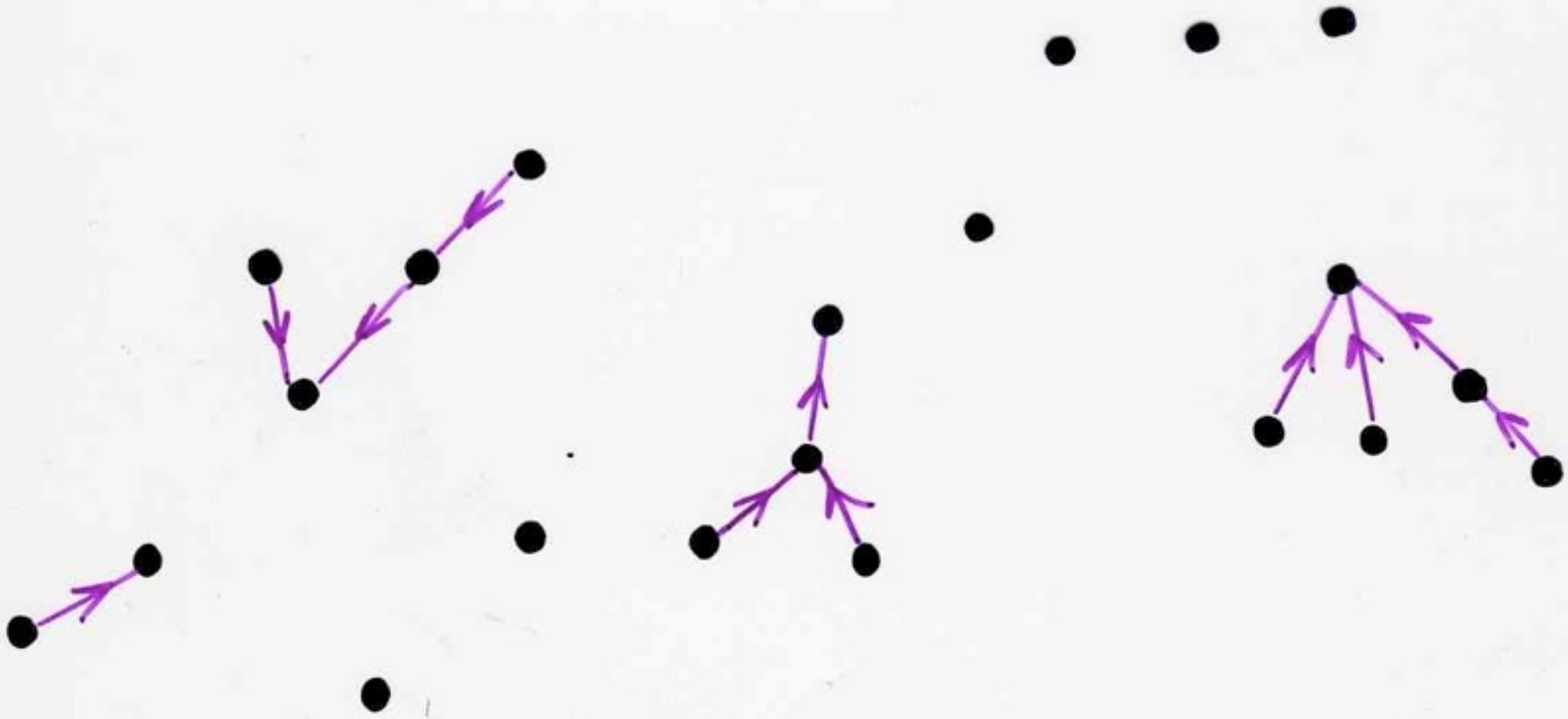
$$\mathrm{End} = S \circ A$$

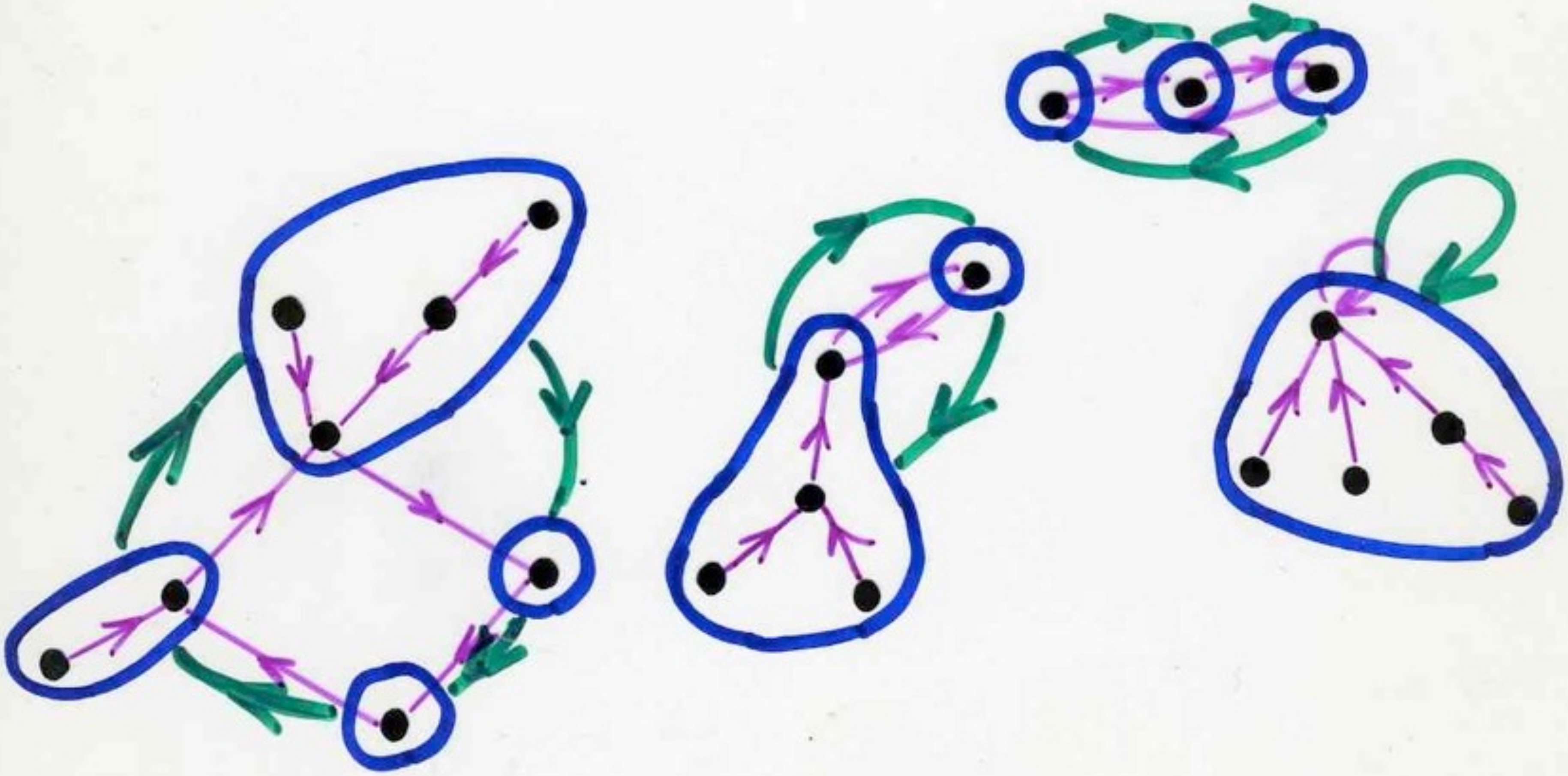
endofonctions   permutations   arborescence

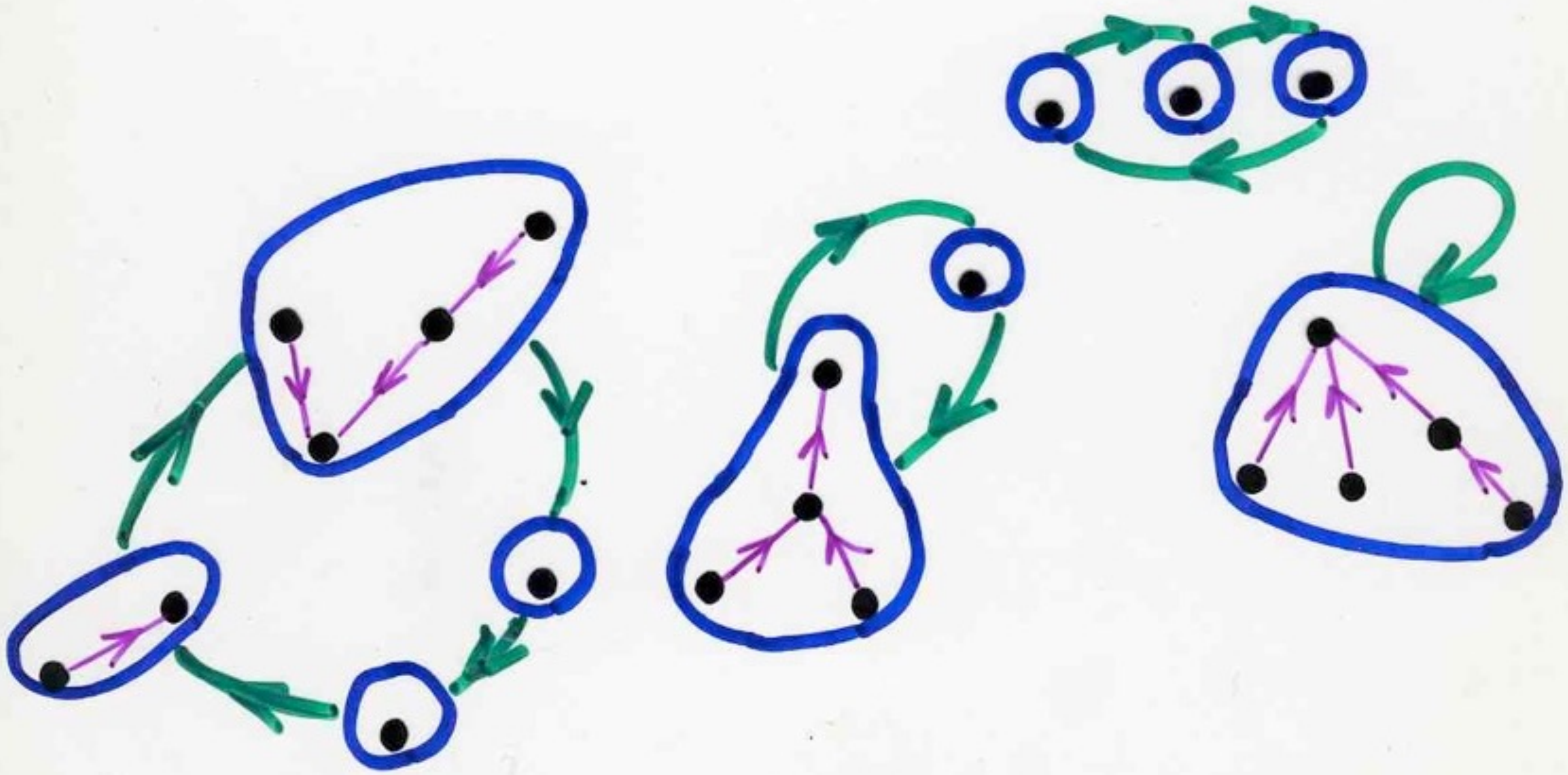












$\mathbb{K}$ commutative ring

<u>Definition</u> weighted species $\mathbf{F}_v$

$\alpha \in \mathbf{F}[U] \longrightarrow v(\alpha) \in \mathbb{K}$

weight

(or valuation)

of the $\mathbf{F}$-structure α

$f : U \longrightarrow V$ $\quad \alpha \in \mathbf{F}[U] \xrightarrow{\mathbf{F}[f]} \beta \in \mathbf{F}[V]$

$v(\alpha) = v(\beta)$

Definition generating power series $\mathbf{F}_v(t)$

$$\mathbf{F}_v(t) = \sum_{n \geq 0} P_n \frac{t^n}{n!}$$

$$P_n \in \mathbb{K}$$

$$P_n = \sum_{\substack{\alpha \in \mathbf{F}[U] \\ \text{with } |U| = n}} v(\alpha)$$

Combinatorial interpretation of Hermite polynomials

Hermite polynomial

$H_n(x)$

$$H_n(x) = (2x)^n \, {}_2F_0\left[\begin{matrix} -n/2, \; -(n-1)/2 \\ - \end{matrix} ; -\frac{1}{x^2}\right]$$

"physicists" Hermite polynomial $H_n(x)$

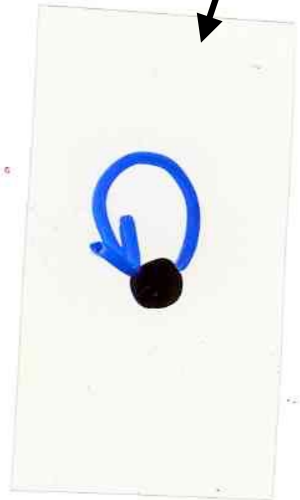
$$\sum_{n \geq 0} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

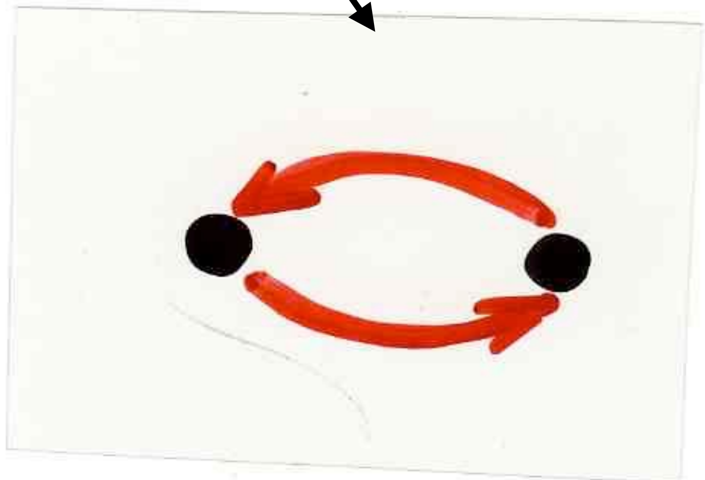
exp ( (x) + (-1) )

$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

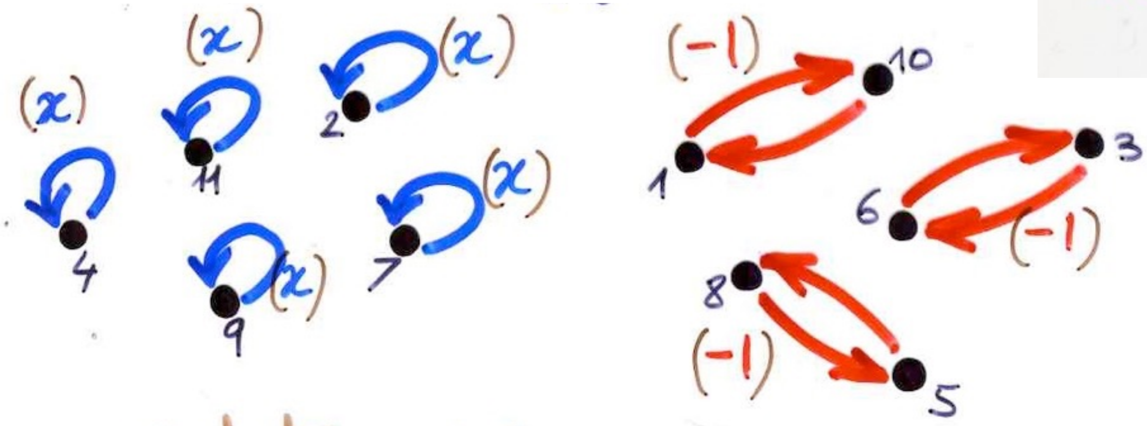
(combinatorial) Hermite polynomials

binomial type polynomials





Hermite configuration



weight (x)
(-1)

(combinatorial)
Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{fix(\sigma)}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 2 & 6 & 4 & 8 & 3 & 7 & 5 & 9 & 1 & 11 \end{pmatrix}$$

$$H_n(x) = \sum_{0 \le 2k \le n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!} x^{n-2k}$$

(combinatorial)
Hermite polynomials

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$

Mehler identity
for Hermite polynomials

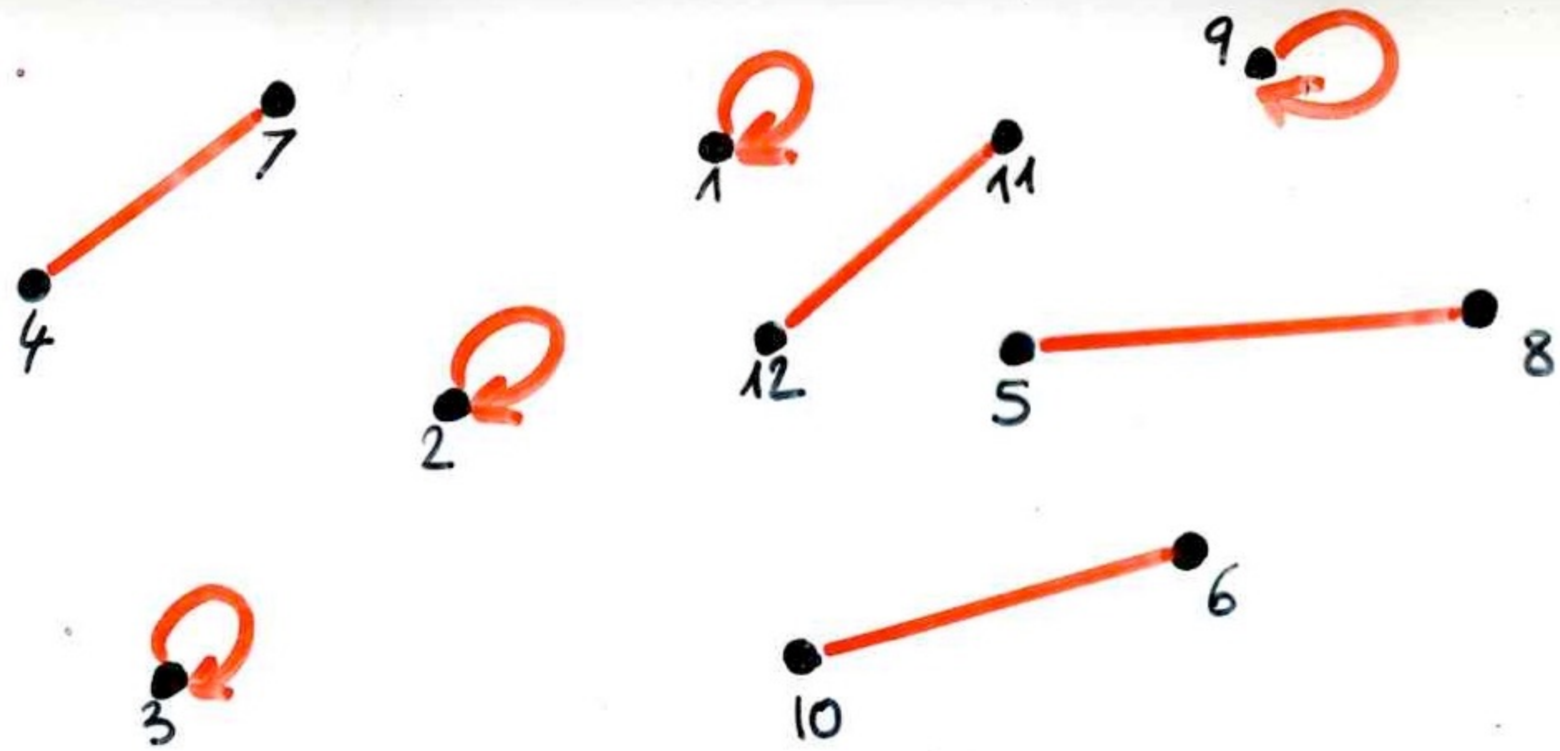
Combinatorial proof
of formulae

Mehler identity

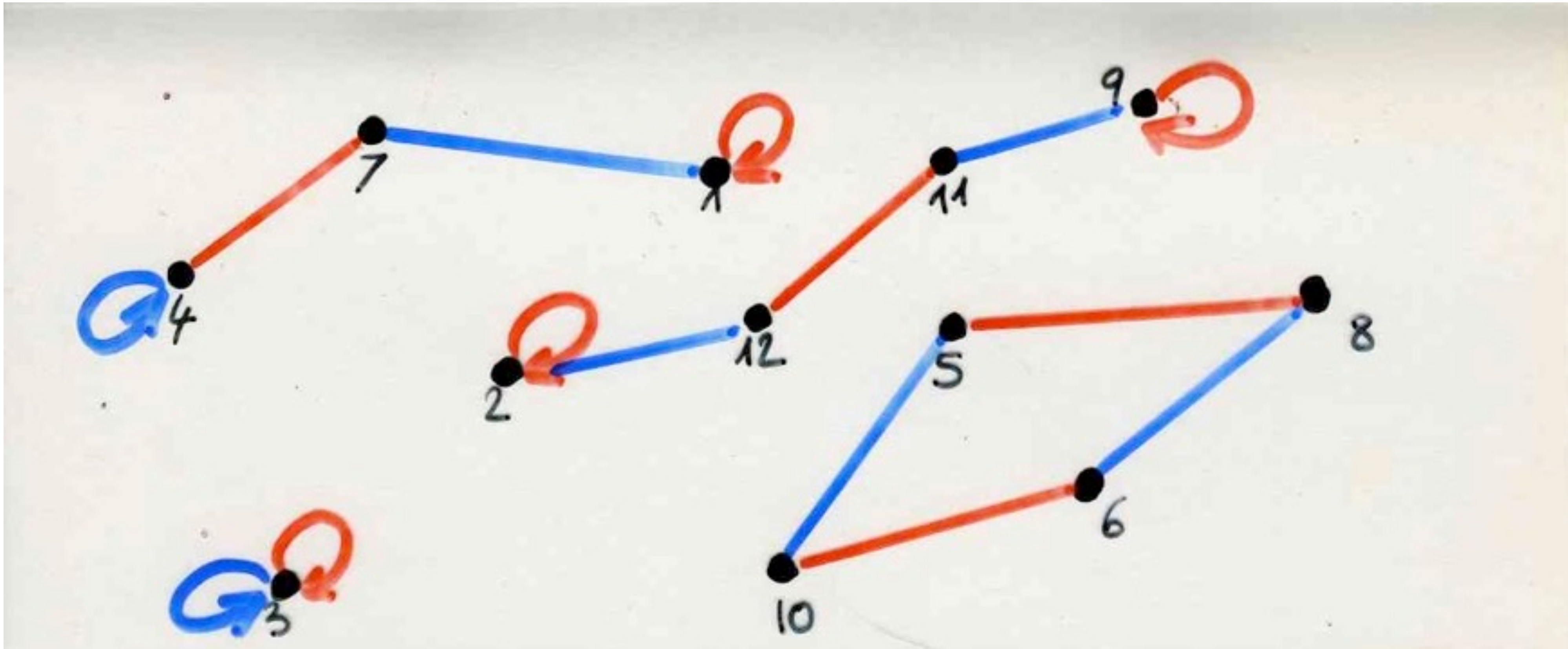
$$\sum_{n\geq 0} H_n(x)H_n(y)\frac{t^n}{n!}$$

$$= (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

ABjC, Part I, Ch3b, p.26



$$\sum_{n \geq 0} H_n(x) \qquad \frac{t^n}{n!}$$



$$\sum_{n \geq 0} H_n(x) H_n(y) \frac{t^n}{n!}$$

$$(1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

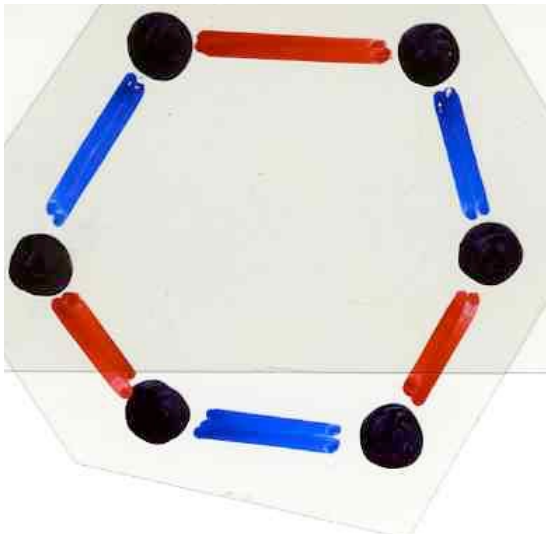
$$\frac{4xyt}{(1-4t^2)}$$

$$\frac{-4x^2t^2}{(1-4t^2)}$$

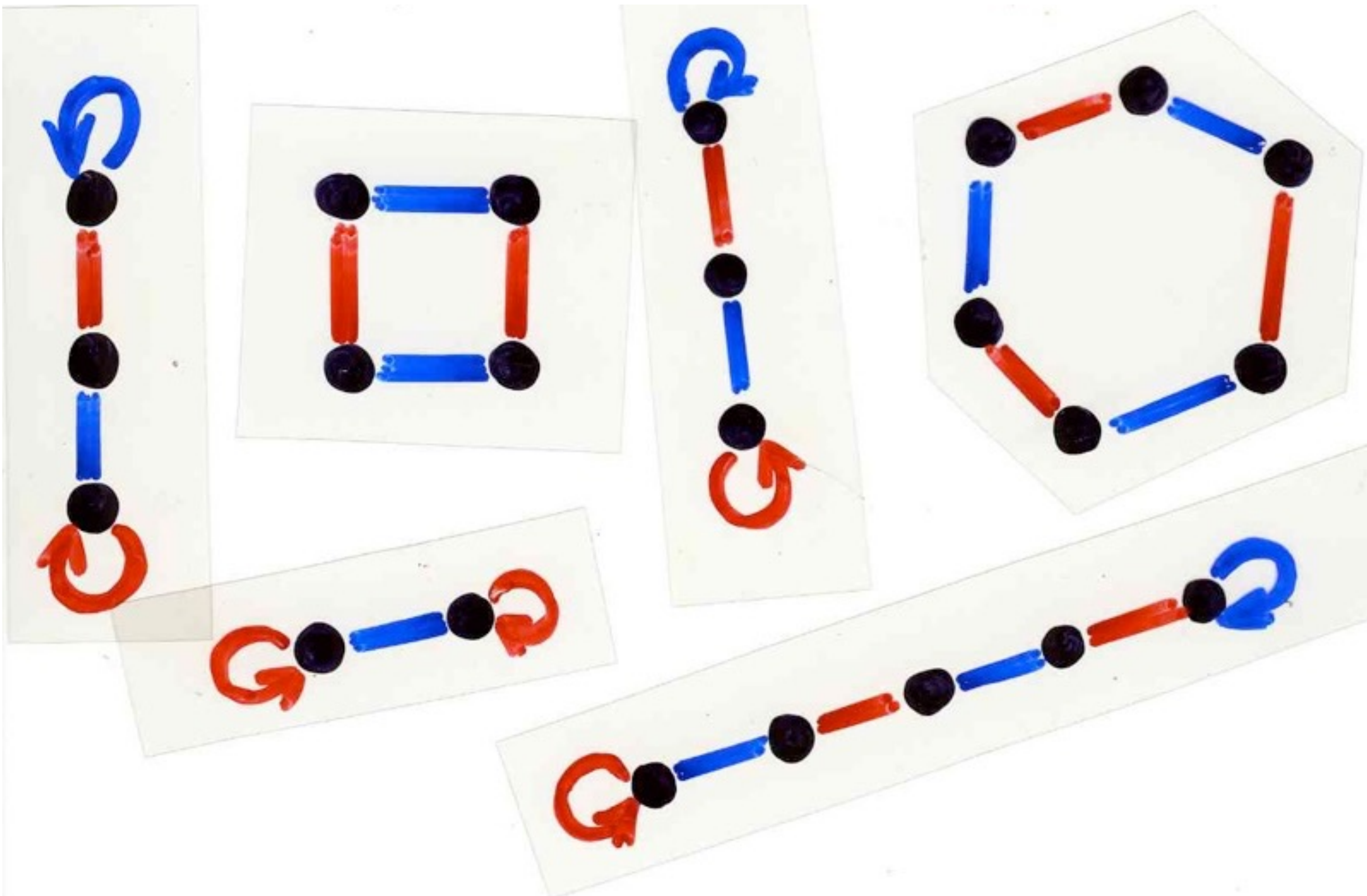
$$\frac{-4y^2t^2}{(1-4t^2)}$$

$$= (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2} \log \frac{1}{(1-4t^2)}\right]$$

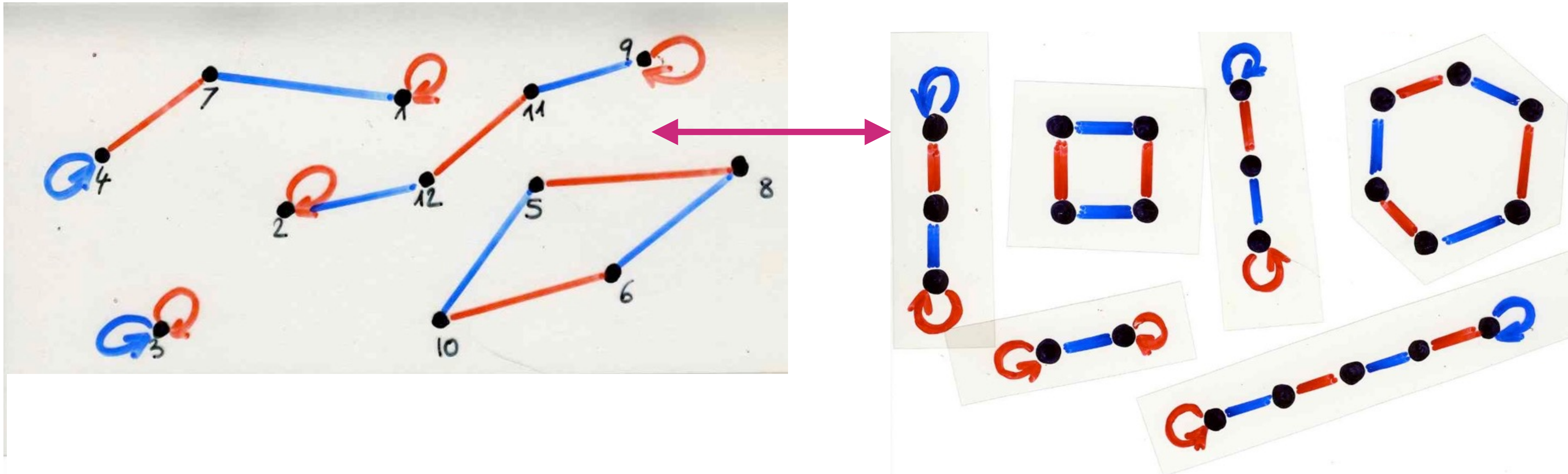


$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}\right] \qquad \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



$$\sum_{n\geq 0} \mathbf{H}_n(x)\,\mathbf{H}_n(y)\,\frac{t^n}{n!}$$

$$= (1-4t^2)^{-1/2}\,\exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$



Laguerre polynomials

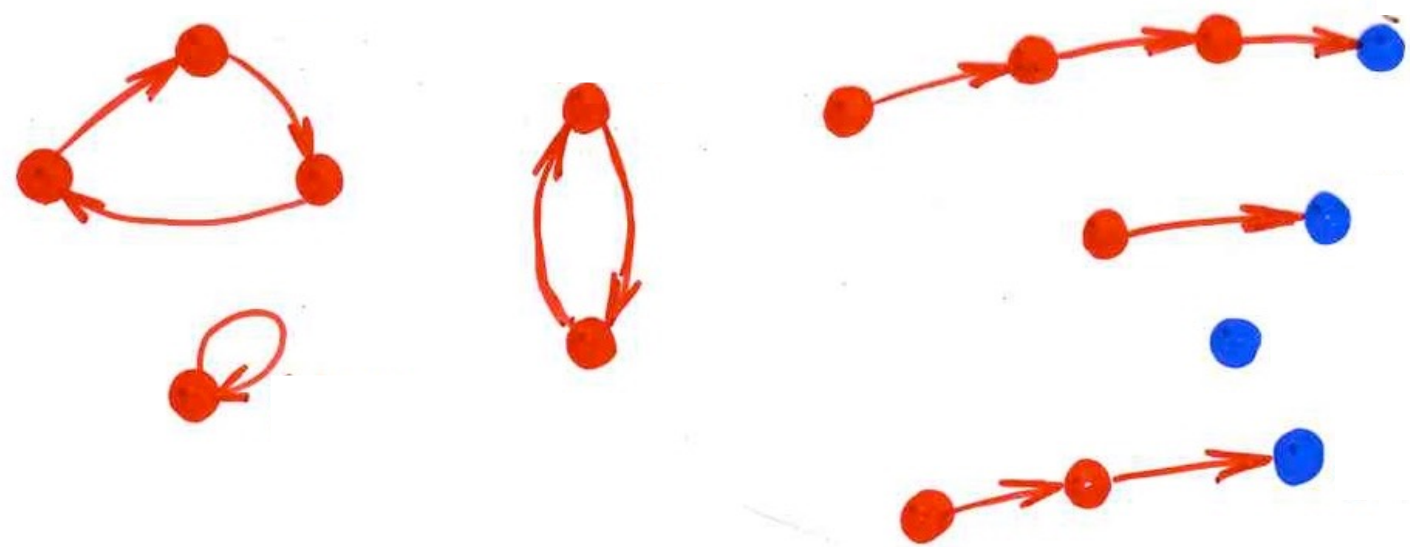
Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\middle| x\right) \qquad (\alpha > -1)$$

$\tilde{L}_n^{(\alpha)}(x)$ monic (combinatorial) Laguerre polynomial

$$\sum_{n\geq 0} \tilde{L}_n^{(\alpha)}(x)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

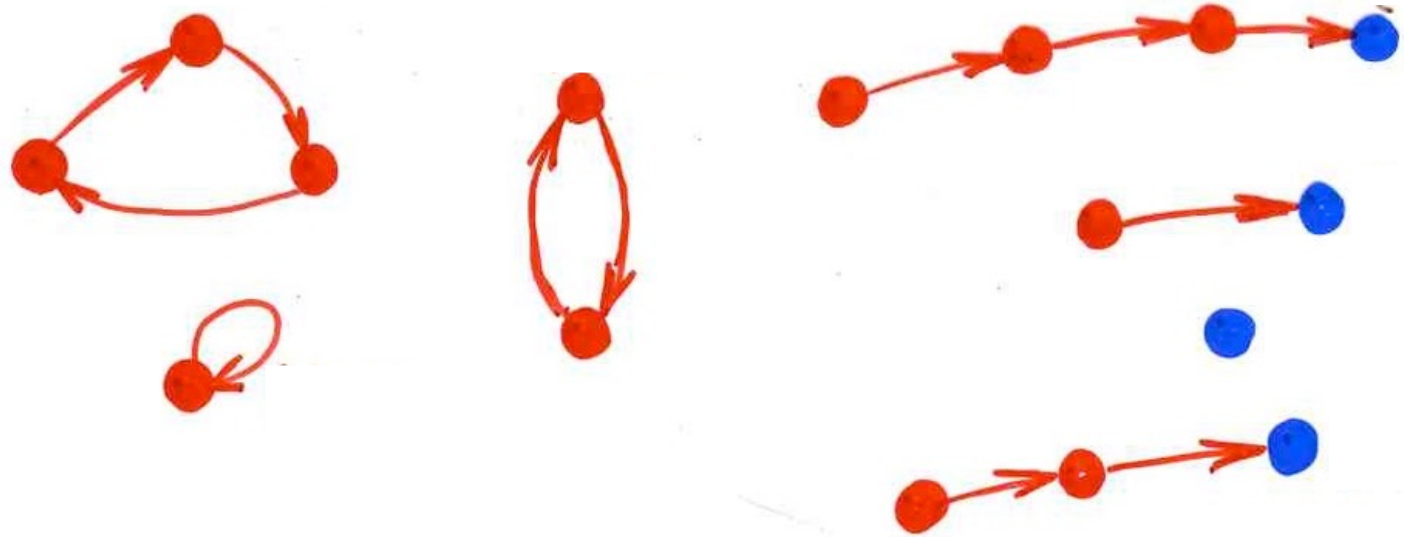
$$\sum_{n\geq 0} \tilde{L}_n^{(\alpha)}(x)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$



Laguerre configuration $f \in \mathbf{L}[A, B]$

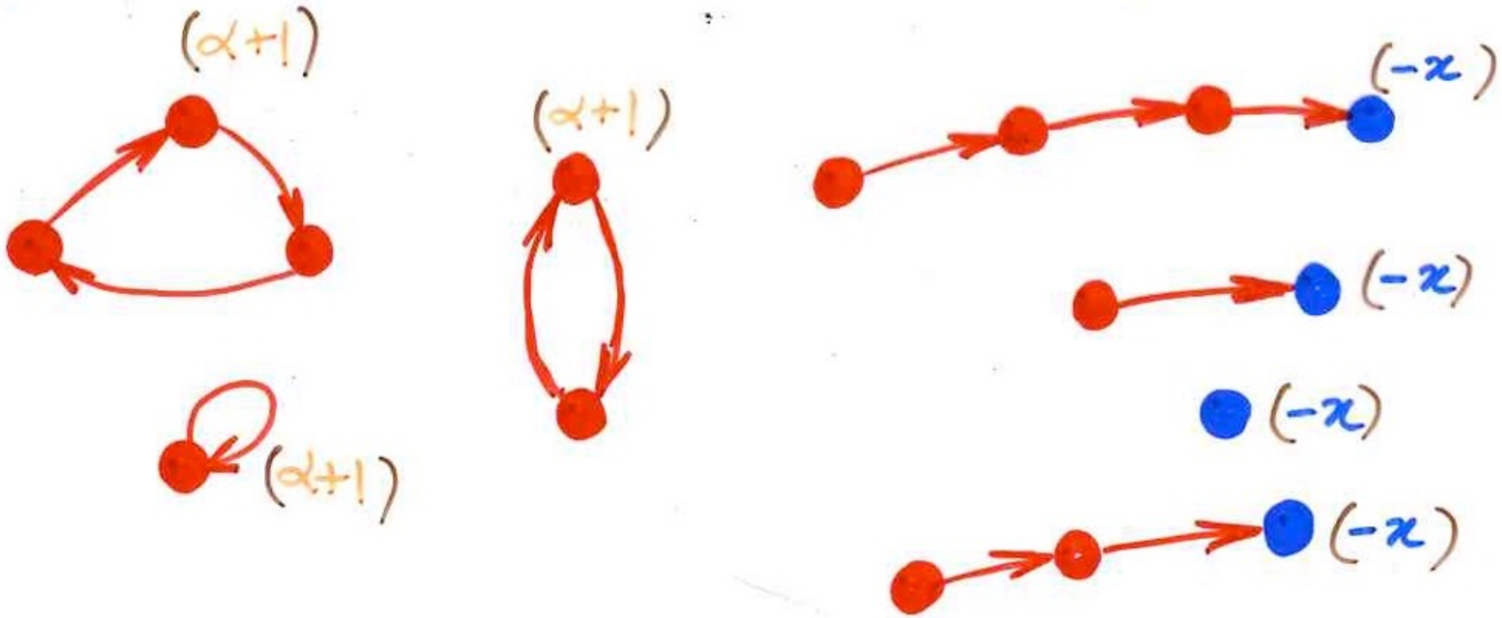
$E = A \cup B$
$A \cap B = \emptyset$

injective map
$f : A \to E$



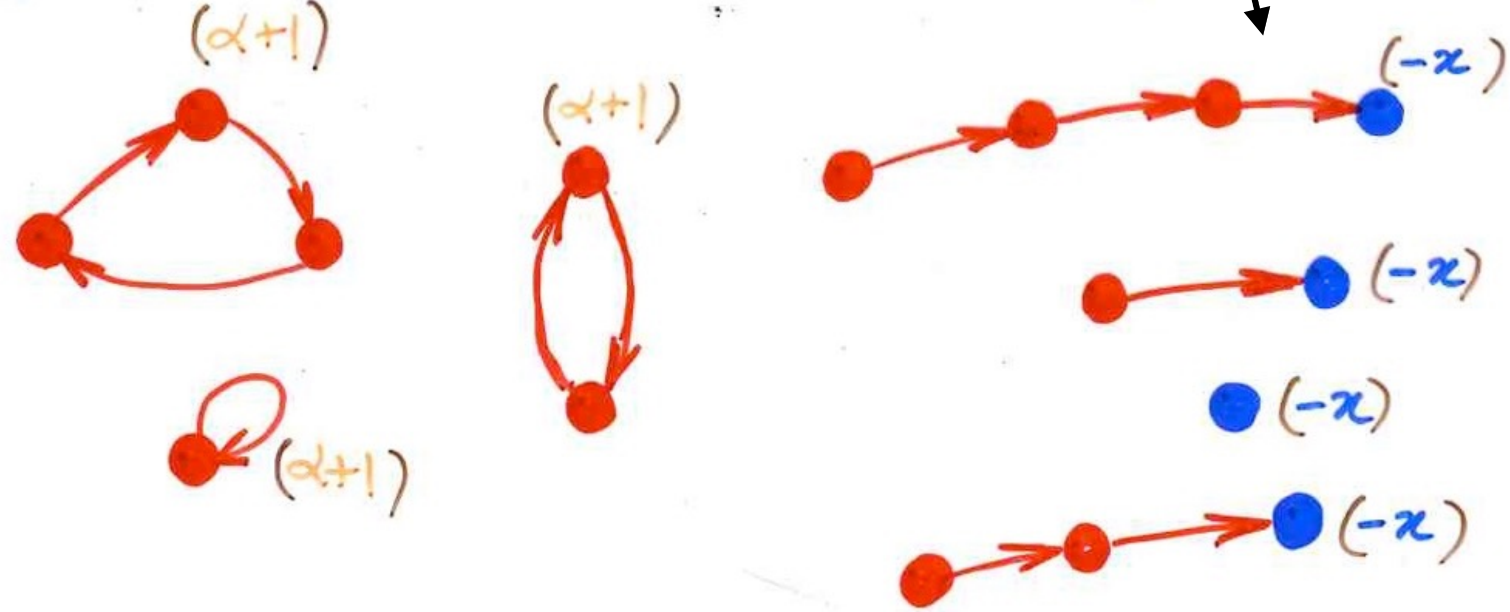
$f \in \mathbf{L}[A,B]$ $\qquad$ $w(f) = (\alpha+1)^{\mathrm{cyc}(f)}(-x)^{|B|}$

Proposition $\qquad$ $\tilde{L}_n^{(\alpha)}(x) = \sum_{f \in \mathbf{L}[A,B]} w(f)$



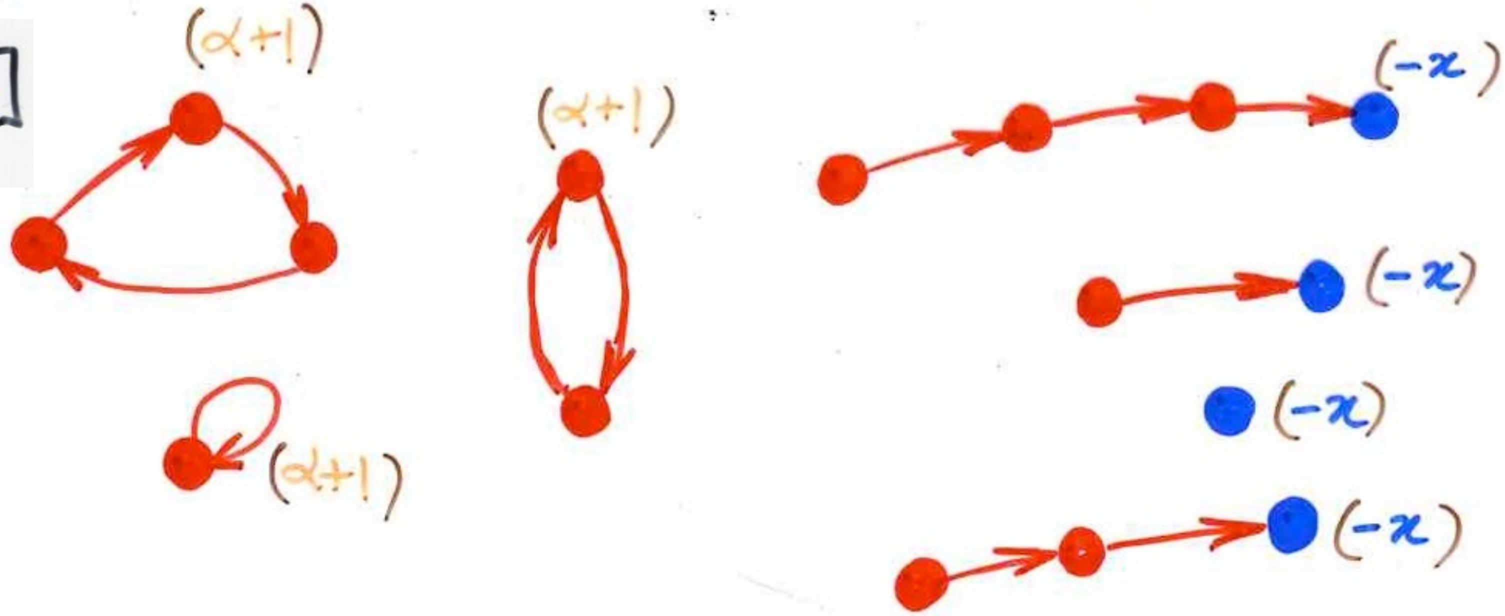
$$\sum_{n \geq 0} \tilde{L}_n^{(\alpha)}(x)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

$$\exp\left((\alpha+1)\log\frac{1}{(1-t)}\right) = \exp\left(\log\frac{1}{(1-t)^{(\alpha+1)}}\right)$$



$$\tilde{L}_n^{(\alpha)}(x) = \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i \, (-x)^j$$

$f \in L[A, B]$



Charlier polynomials

Charlier polynomials

$$C_n^{(a)}(x) = {}_2F_0\left[\begin{matrix}-n, -x \\ -\end{matrix}; a^{-1}\right]$$

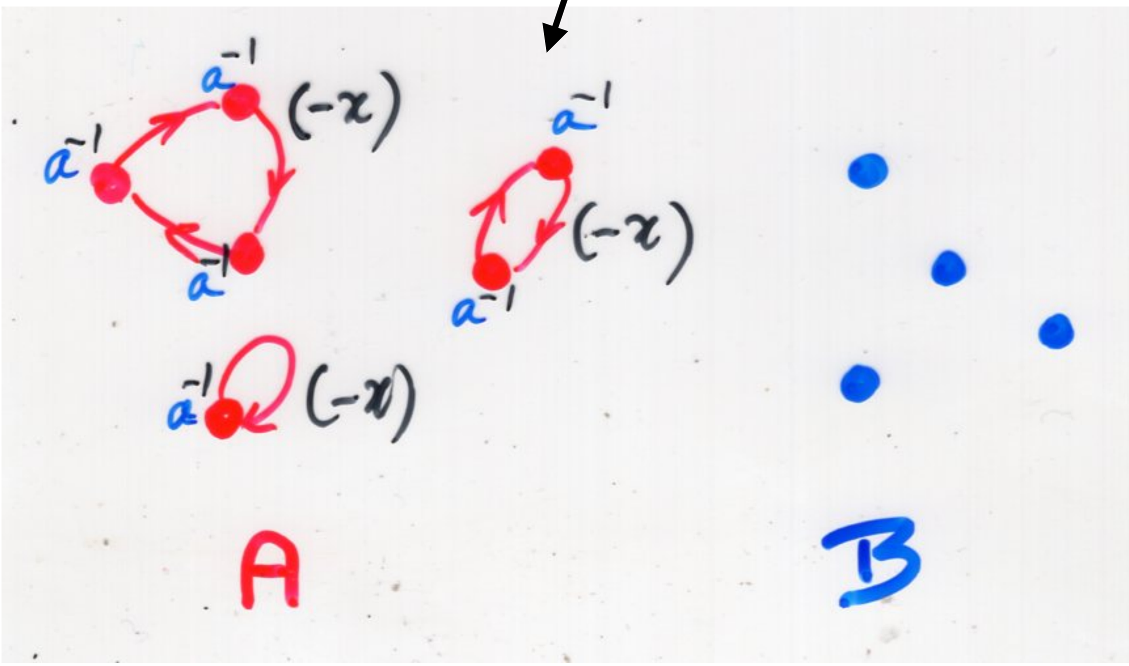
$$\sum_{n \geq 0} C_n^{(a)}(x) \frac{t^n}{n!} = e^t (1 - t/a)^x$$

$$\frac{1}{(1 - t/a)^{-x}}$$

$$\sum_{n\geq 0} C_n^{(a)}(x)\,\frac{t^n}{n!} = e^t\,\frac{1}{(1-t/a)^{-x}}$$

$$E = A \cup B$$
$$A \cap B = \emptyset$$

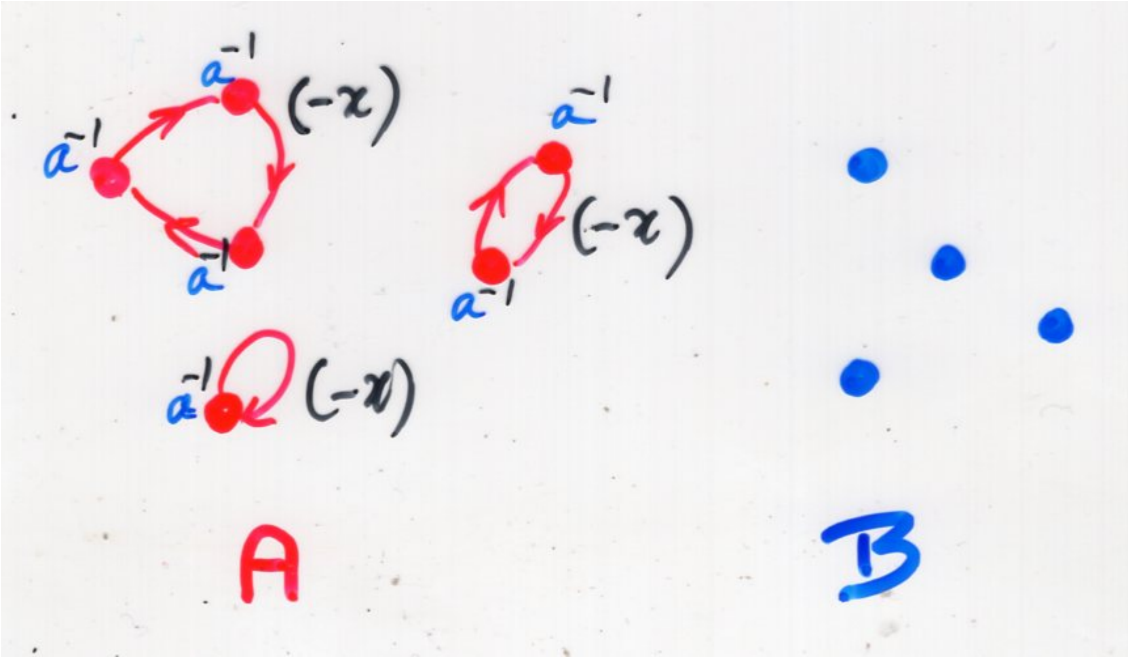
Charlier configurations



$(\sigma, 1_B) \in \mathcal{C}[A, B]$

Charlier configurations

$$w(\sigma, 1_B) = (-x)^{cyc(\sigma)} (a^{-1})^{|A|}$$

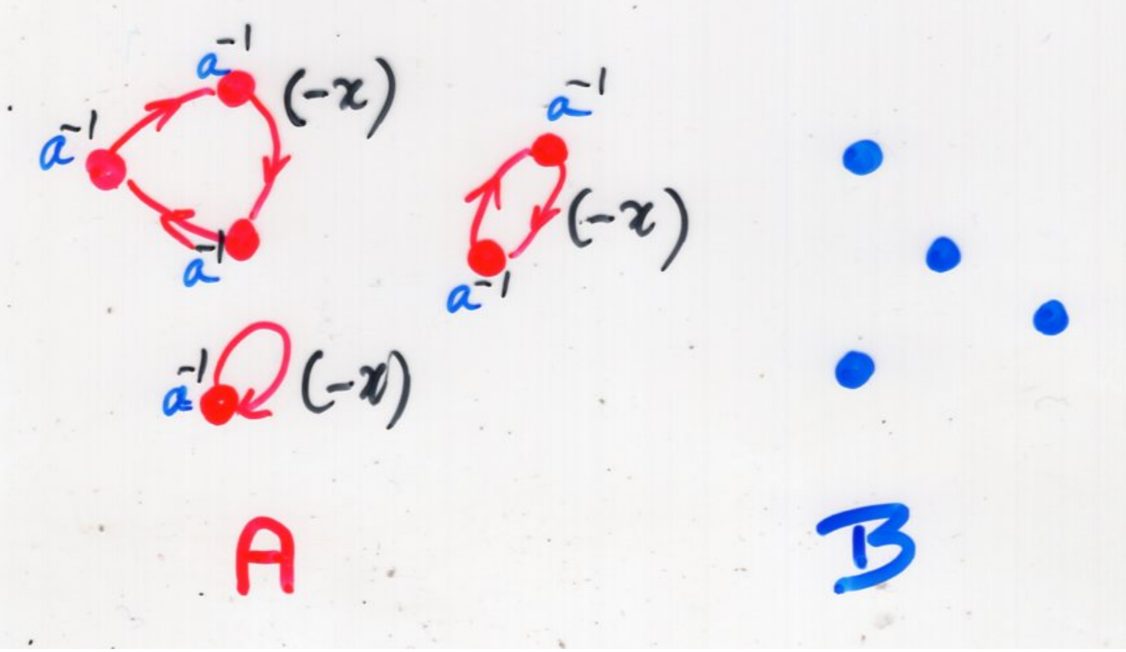


Charlier polynomials

Proposition

$$C_n^{(a)}(x) = \sum_{(\sigma, 1) \in C[A, B]} w(\sigma, 1_B)$$

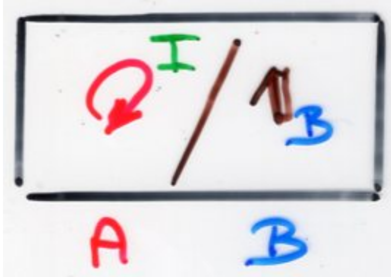
$$= \sum_{n \geq i \geq 0} \binom{n}{i} (a^{-1})^i (-x)_i$$



(A, B)

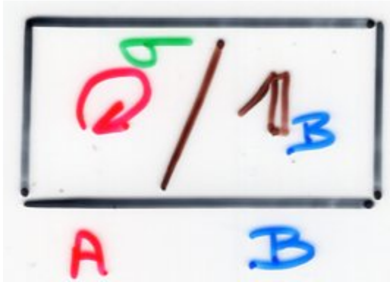
Hermite configurations

$$\mathbf{H}[A, B] = \mathbf{I}[A] \times \{\mathbb{1}_B\}$$



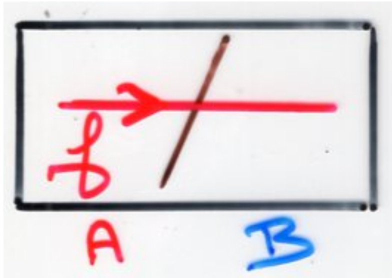
Charlier configurations

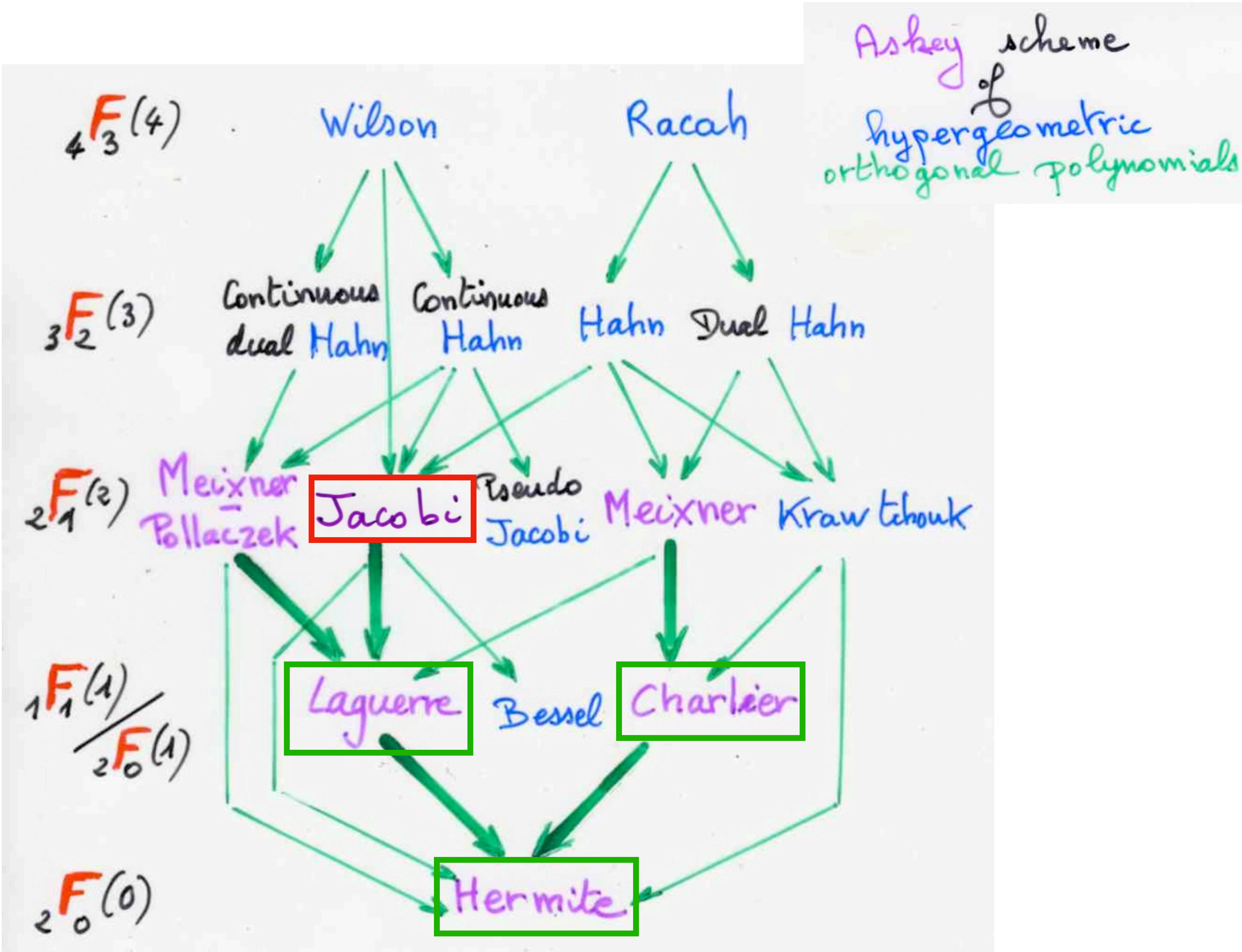
$$\mathbf{C}[A, B] = \mathbf{S}[A] \times \{\mathbb{1}_B\}$$



Laguerre configurations

$$\mathbf{L}[A, B] = \left\{ \begin{array}{l} \text{injective map } f \\ \text{from } A \text{ to } A+B \end{array} \right\}$$





Askey scheme of hypergeometric orthogonal polynomials
${}_4F_3(4)$
Wilson
Racah
${}_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
${}_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
${}_1F_1(1)$ / ${}_2F_0(1)$
Laguerre
Bessel
Charlier
${}_2F_0(0)$
Hermite

Jacobi polynomials

Carl Jacobi
1804 - 1851

$$n!\, P_n^{(\alpha,\beta)}(x) = (\alpha+1)_n \; {}_2F_1\left[\begin{matrix}-n,\; n+\alpha+\beta+1\\ \alpha+1\end{matrix}\; ; \; \frac{(1-x)}{2}\right]$$

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) =$$

$$\frac{(-1)^n}{2^n\, n!}\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{0\le j\le n}\binom{n+\alpha}{n-j}\binom{n+\beta}{j}\left(\frac{x-1}{2}\right)\left(\frac{x+1}{2}\right)^{n-j}$$

Proposition

$$\sum_{n\geq 0} P_n^{(\alpha,\beta)}(x)\, t^n = 2^{\alpha+\beta} R^{-1} (1-t+R)^{-\alpha} (1+t+R)^{-\beta}$$

$$R = (1 - 2xt + t^2)^{1/2}$$

$$\mathrm{P}_n^{(\alpha,\beta)}(x) = \sum_{0\leq j\leq n} \binom{n+\alpha}{n-j}\binom{n+\beta}{j}\left(\frac{x-1}{2}\right)\left(\frac{x+1}{2}\right)^{n-j}$$

$$n!\,\mathrm{P}_n^{(\alpha,\beta)}(x) = \mathcal{P}_n^{(\alpha,\beta)}\left(\frac{x+1}{2},\frac{x-1}{2}\right)$$

$$n!\,\mathrm{P}_n^{(\alpha,\beta)}\left(\frac{X+Y}{X-Y}\right)(X-Y)^n = \mathcal{P}_n^{(\alpha,\beta)}(X,Y)$$

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = \sum_{\substack{i,j\geq 0\\ i+j=n}} \binom{n}{i}(\alpha+1+j)_i\,(\beta+1+i)_j\; X^i Y^j$$

Proposition

$$R = (1 - 2xt + t^2)^{1/2}$$

$$\sum_{n \geqslant 0} P_n^{(\alpha,\beta)}(x)\, t^n = 2^{\alpha+\beta} R^{-1} (1 - t + R)^{-\alpha} (1 + t + R)^{-\beta}$$

$$\mathcal{R} = \left[1 - 2(X+Y)t + (X-Y)^2 t^2\right]^{1/2}$$

$$\sum_{n \geqslant 0} \mathcal{P}_n^{(\alpha,\beta)}(X,Y)\, \frac{t^n}{n!} = 2^{\alpha+\beta} \mathcal{R}^{-1} \left[1 - (X-Y)t + \mathcal{R}\right]^{-\alpha} \left[1 - (Y-X)t + \mathcal{R}\right]^{-\beta}$$

Foata, Leroux (1983)

Jacobi endofunctions Φ

(A, B)

$E = A \cup B$

$A \cap B = \emptyset$

- ordered partition (A, B) of E
- endofunction $\Phi : E \to E$

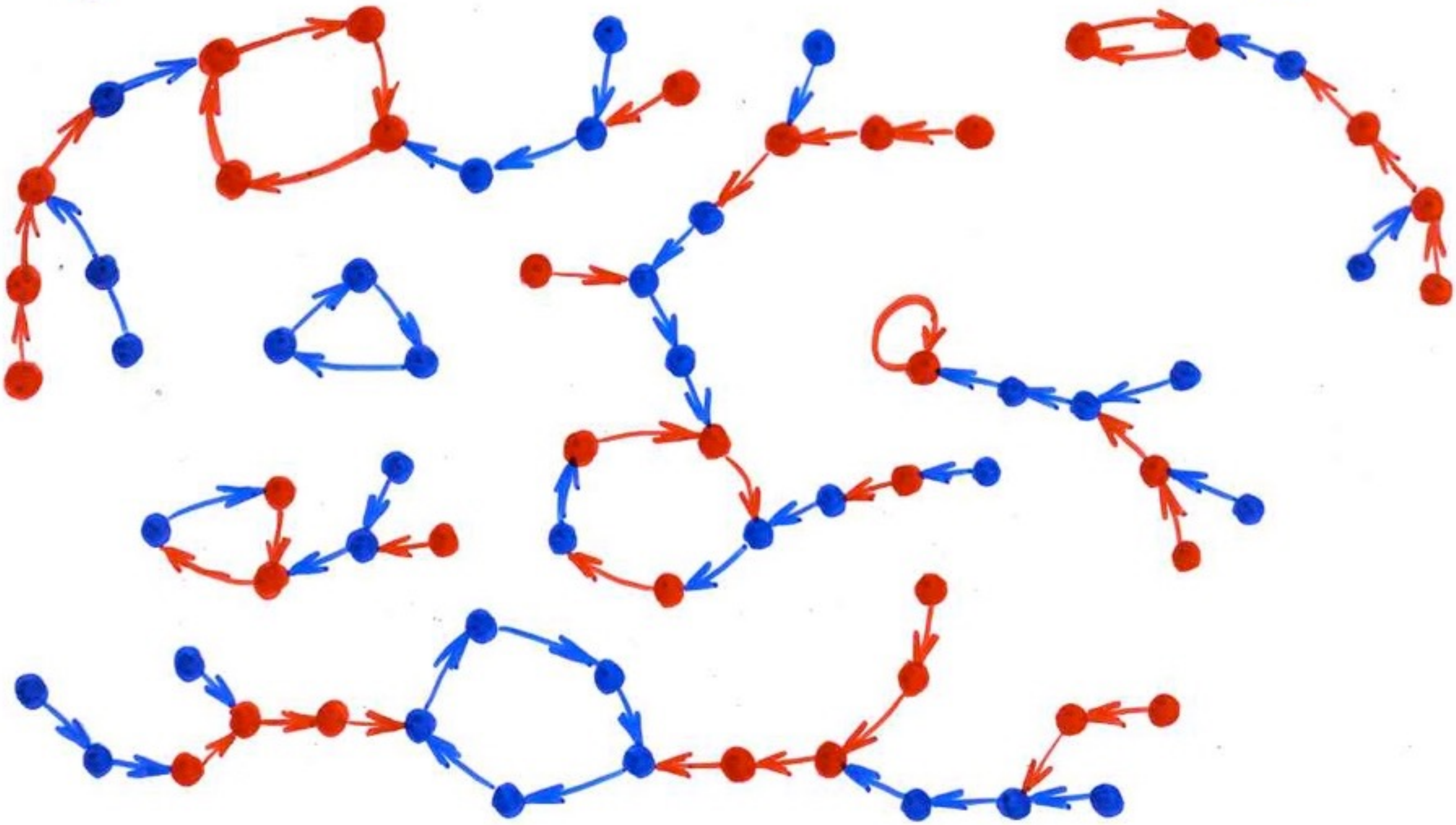
such that the restrictions

$f = \Phi|_A : A \to E$ and $g = \Phi|_B : B \to E$

are injective

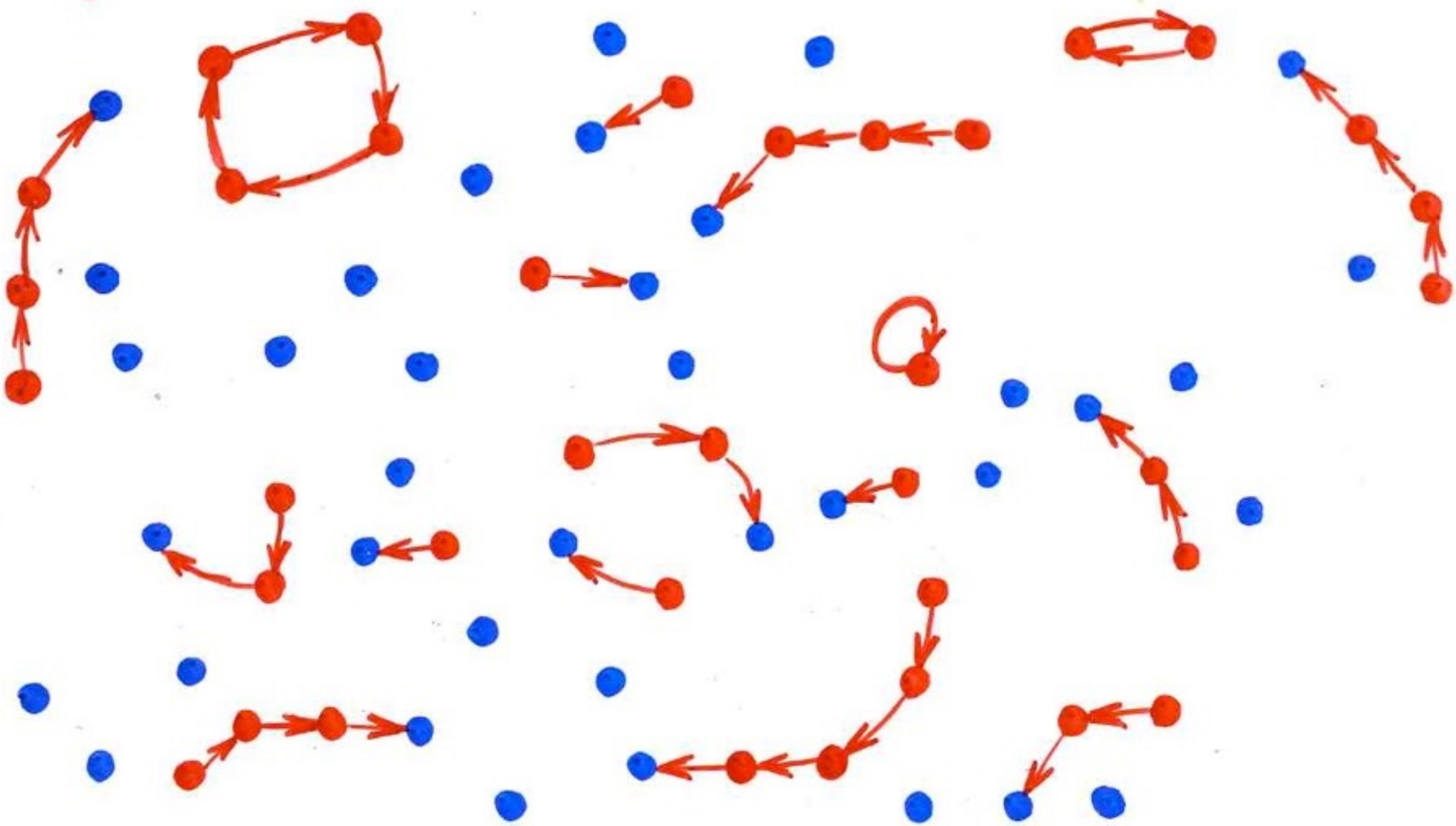
(A, B)

$f : A \to A + B \qquad A + B \leftarrow B : g$

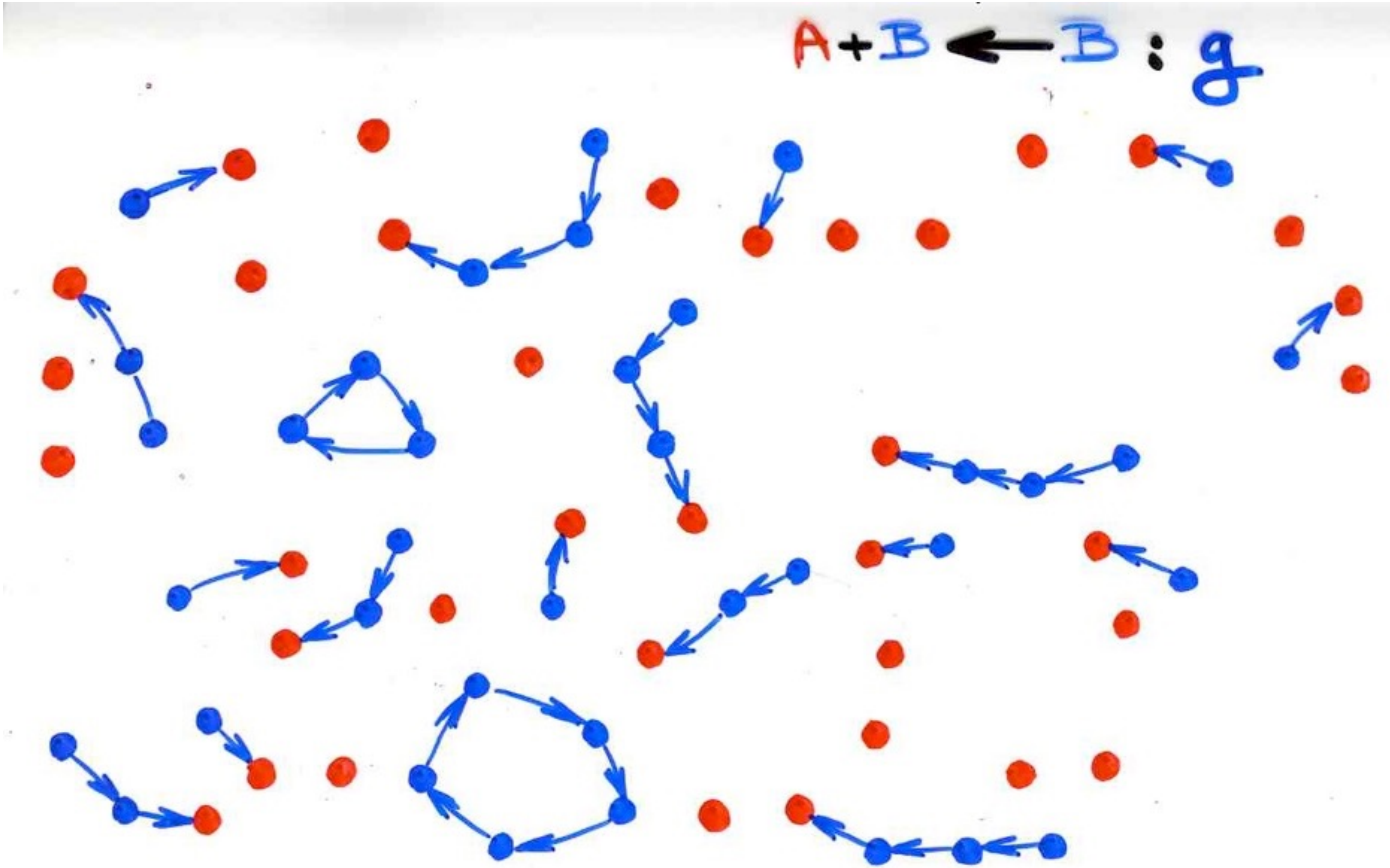


(A, B)

$f : A \to A + B$

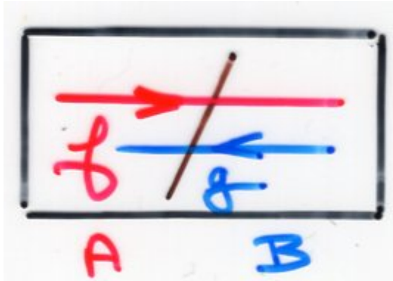


(A, B)

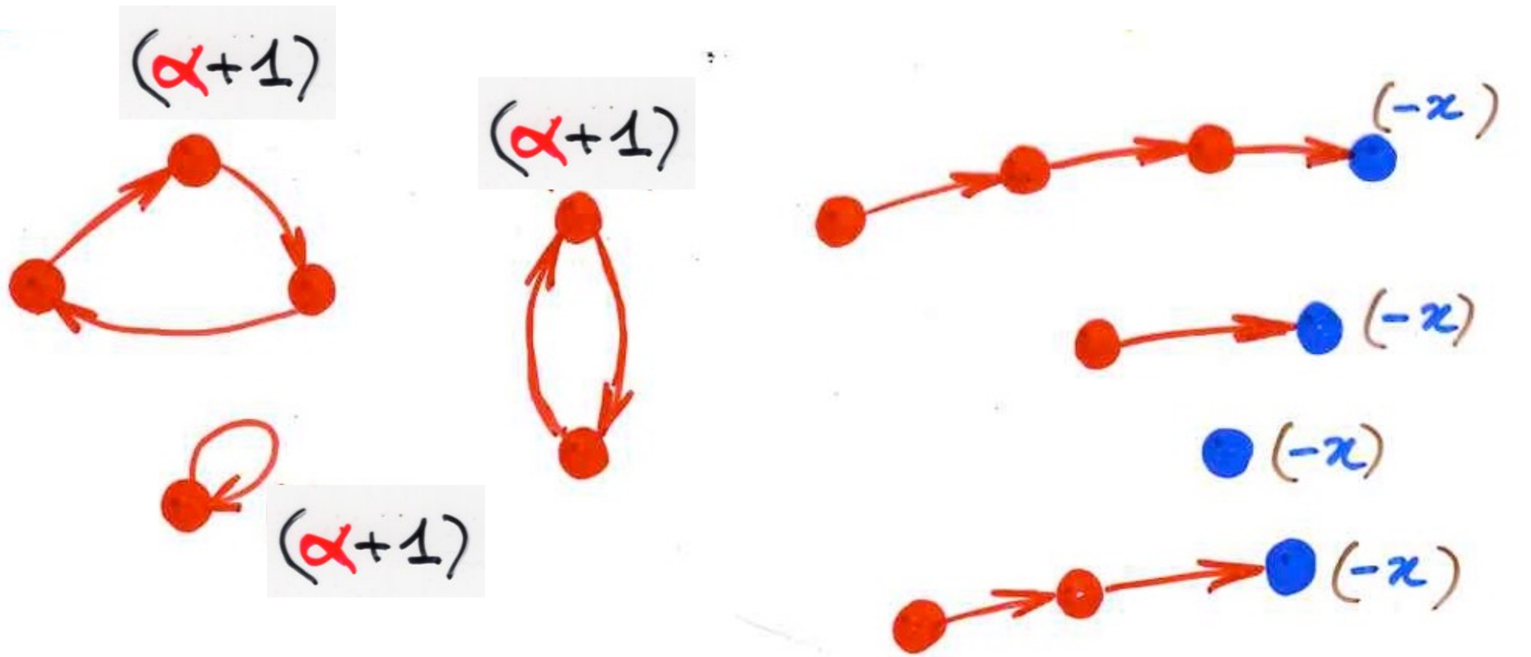


Jacobi configurations

$$\mathbf{J}[A,B] = \mathbf{L}[A,B] \times \mathbf{L}[B,A]$$

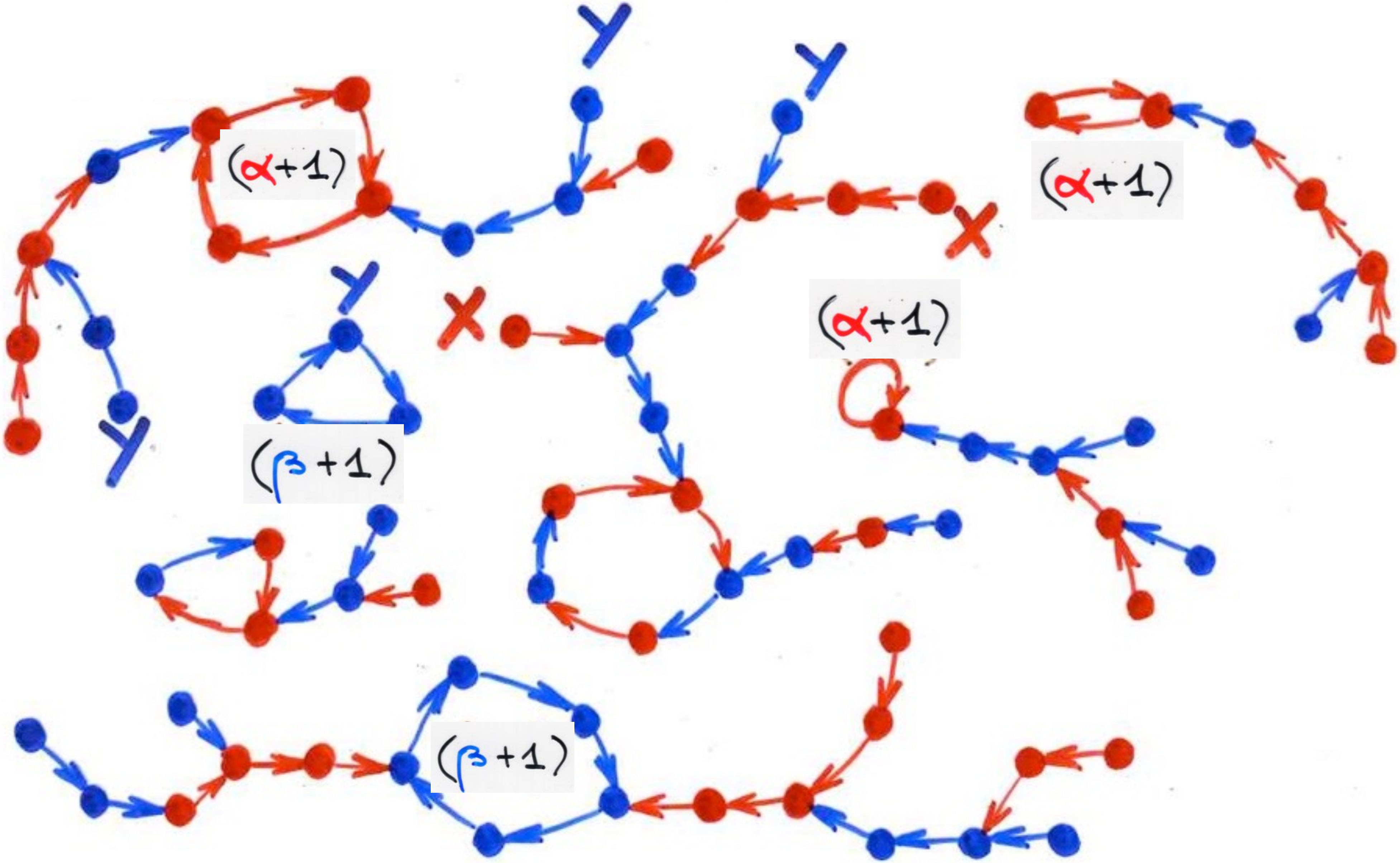


Laguerre configuration



$$(f, g) \in \mathbf{L}[A, B]$$

$$w(f, g) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)} X^{|A|} Y^{|B|}$$



$(f, g) \in \mathbf{L}[A, B]$

$$w(f, g) = (\alpha+1)^{cyc(f)} (\beta+1)^{cyc(g)} \quad x^{|A|} y^{|B|}$$

<u>theorem</u> $|E| = n$ (A, B)

$$\mathcal{P}_n^{(\alpha, \beta)}(x, y) = \sum_{(f, g) \in \mathbf{J}[A, B] = \mathbf{L}[A, B] \times \mathbf{L}[B, A]} w(f, g)$$

(A, B) $\qquad |A| = i,\ |B| = j$

$$f : A \to E\ ,\ g : B \to E$$

$$w(f) = (\alpha+1)^{\mathrm{cyc}(f)}\, x^i$$

$$w(g) = (\beta+1)^{\mathrm{cyc}(g)}\, y^j$$

$$\sum_{(f,g)\in \mathbf{J}[A,B]} w(f,g) = \sum_{f\in \mathbf{L}[A,B]} w(f) \cdot \sum_{g\in \mathbf{L}[B,A]} w(g)$$

$$\sum_{(f,g)\in \mathbf{J}[A,B]} w(f,g) = \sum_{f\in \mathbf{L}[A,B]} w(f) \cdot \sum_{g\in \mathbf{L}[B,A]} w(g)$$

$$(\alpha+1+j)\,X^{i} \qquad\qquad (\beta+1+i)_j\, Y^{j}$$

$$\sum_{(f,g)\in \mathbf{J}[A,B]} w(f,g) = (\alpha+1+j)_i\,(\beta+1+i)_j\; X^{i} Y^{j}$$

(A, B) $\quad |A| = i,\ |B| = j$

$$\sum_{(f,g) \in \mathcal{J}[A,B]} w(f,g) = (\alpha+1+j)_i\,(\beta+1+i)_j\ X^i Y^j$$

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = \sum_{\substack{i,j \geq 0 \\ i+j=n}} \binom{n}{i} (\alpha+1+j)_i\,(\beta+1+i)_j\ X^i Y^j$$

<u>theorem</u>

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = \sum_{(f,g) \in \mathcal{J}[A,B]} w(f,g)$$

$$\sum_{n \geq 0} \mathcal{P}_n^{(\alpha,\beta)}(x,y)\frac{t^n}{n!} = \sum_{n\geq 0} \sum_{(f,g)\in \mathcal{J}[A,B]} w(f,g)\frac{t^n}{n!}$$

$\Phi_w(t)$

type a
type b
type m

$\Phi_w(t)$

$\Phi_v(t)$

$v(f,g) = x^{|A|}\,y^{|B|}$

$\alpha = \beta = 0$

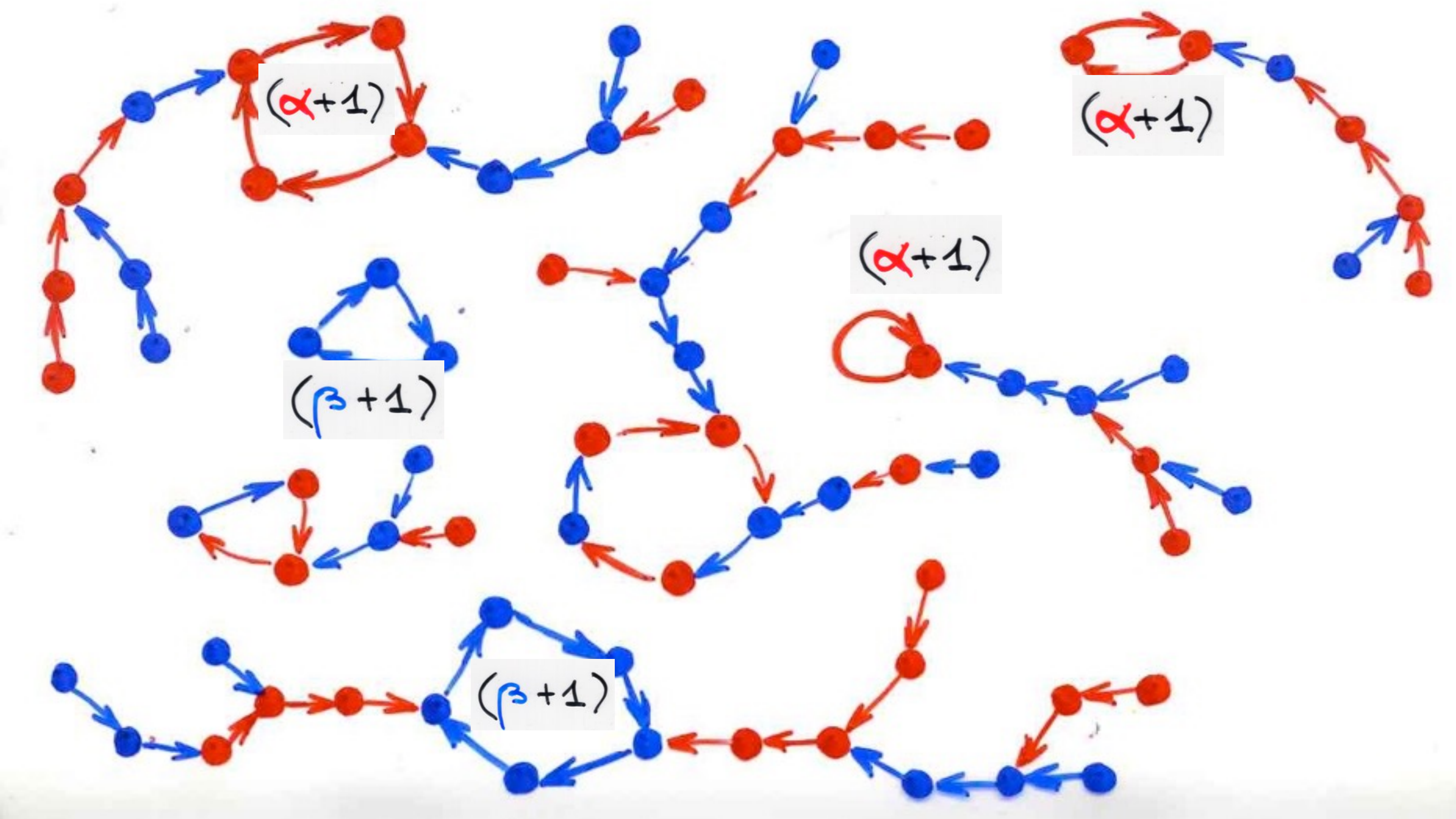
Lemme

$$\Phi_v(t) = \Phi_a(t)\,\Phi_b(t)\,\Phi_m(t)$$

$$\Phi_w(t) = \Phi_\alpha(t)\,\Phi_\beta(t)\,\Phi_m(t)$$

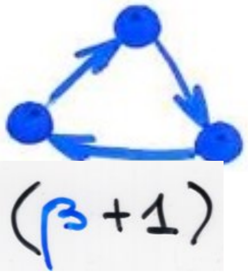
$$\phi_v(t) = \phi_a(t)\,\phi_b(t)\,\phi_m(t)$$

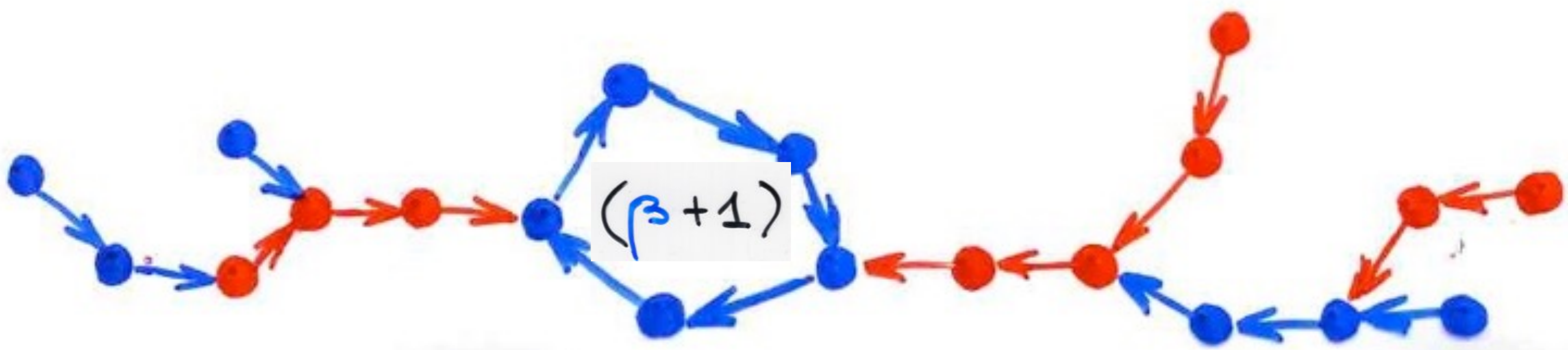
$$\phi_w(t) = \phi_\alpha(t)\,\phi_\beta(t)\,\phi_m(t)$$



$$\phi_v(t) = \phi_a(t)\,\phi_b(t)\,\phi_m(t)$$

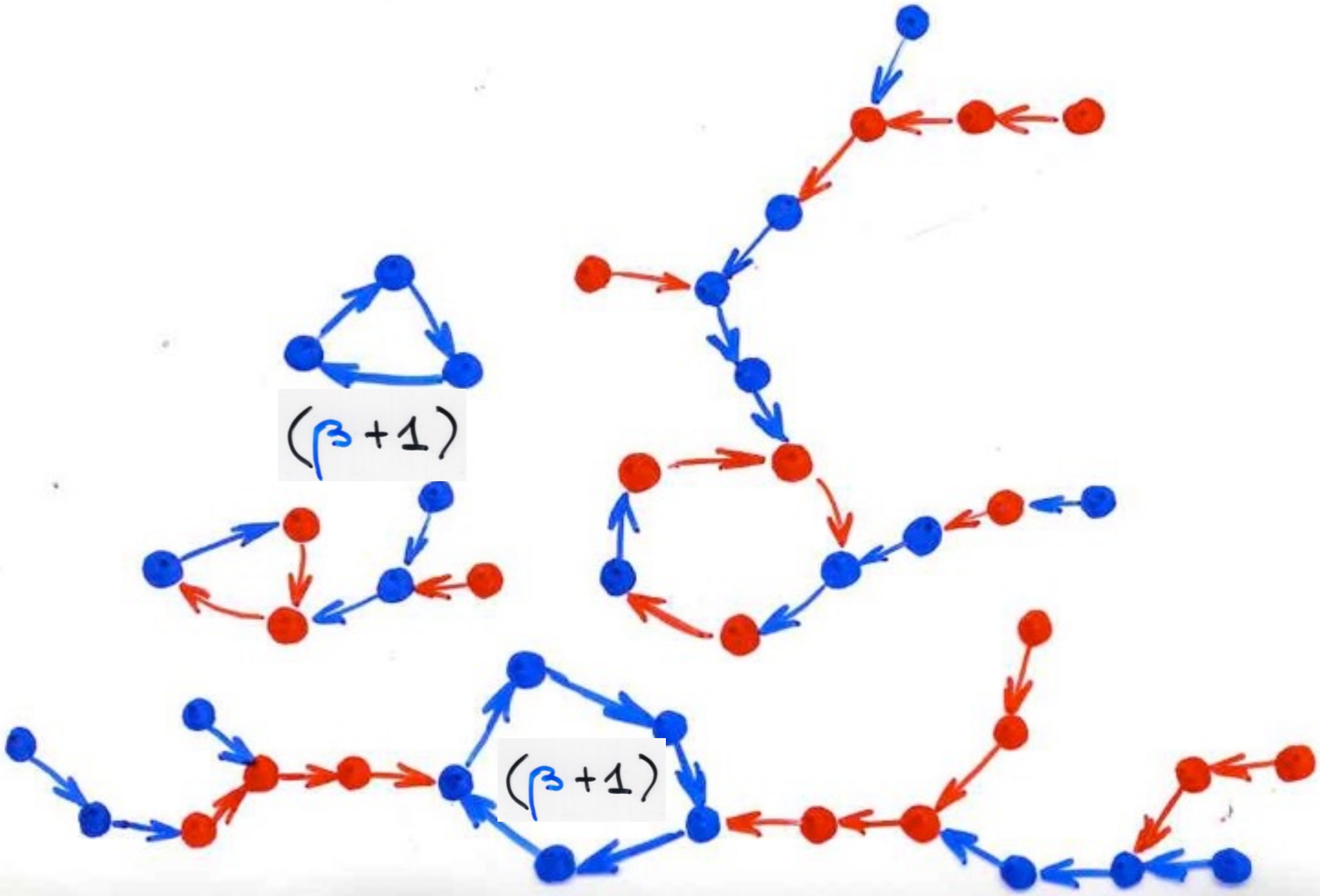
$$\phi_w(t) = \phi_\alpha(t)\,\phi_\beta(t)\,\phi_m(t)$$





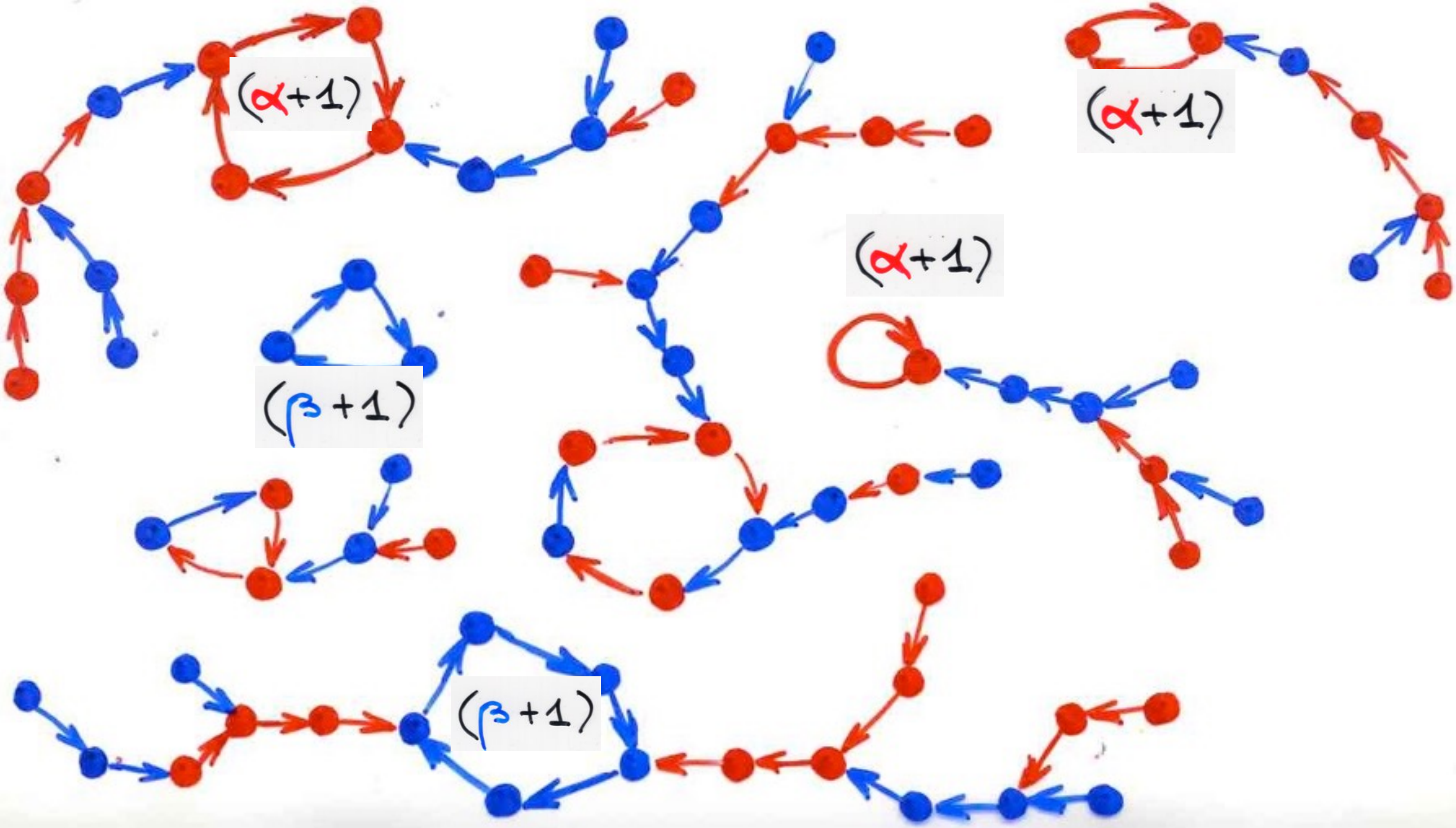
$$\phi_v(t) = \phi_a(t)\,\phi_b(t)\,\phi_m(t)$$

$$\phi_w(t) = \phi_\alpha(t)\,\phi_\beta(t)\,\phi_m(t)$$



$$\phi_v(t) = \phi_a(t)\,\phi_b(t)\,\phi_m(t)$$

$$\phi_w(t) = \phi_\alpha(t)\,\phi_\beta(t)\,\phi_m(t)$$



Lemme

$$\phi_\alpha(t) = \left(\phi_a(t)\right)^{\alpha+1}$$

$$\phi_\beta(t) = \left(\phi_b(t)\right)^{\beta+1}$$

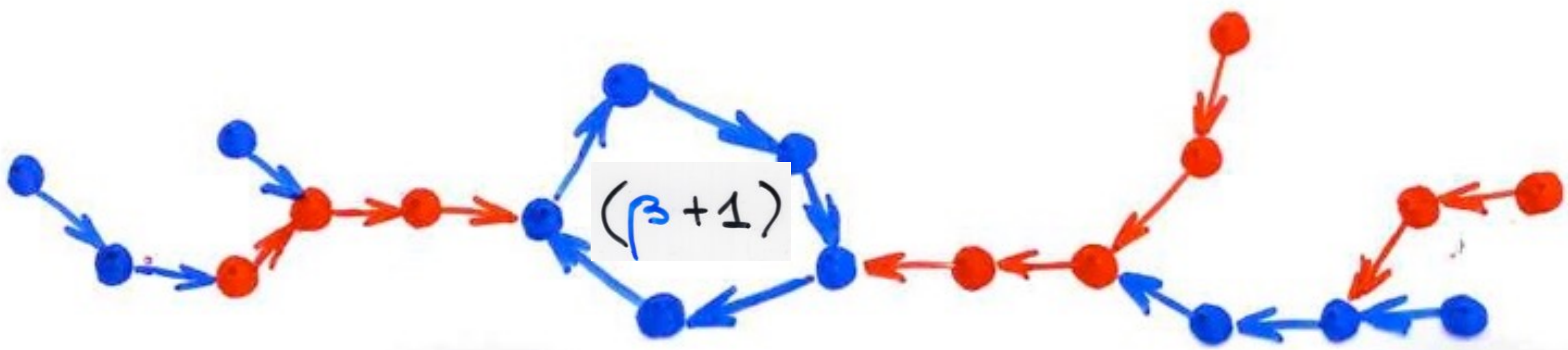
$(f,g) \in L[A,B]$ type a k connected componants

$$w(f,g) = (\alpha+1)^k \, w(f_1,g_1) \cdots (f_k,g_k)$$

$(f, g) \in L[A, B]$ type a k connected componants

$$w(f, g) = (\alpha + 1)^k \, w(f_1, g_1) \cdots (f_k, g_k)$$

$(\beta + 1)$



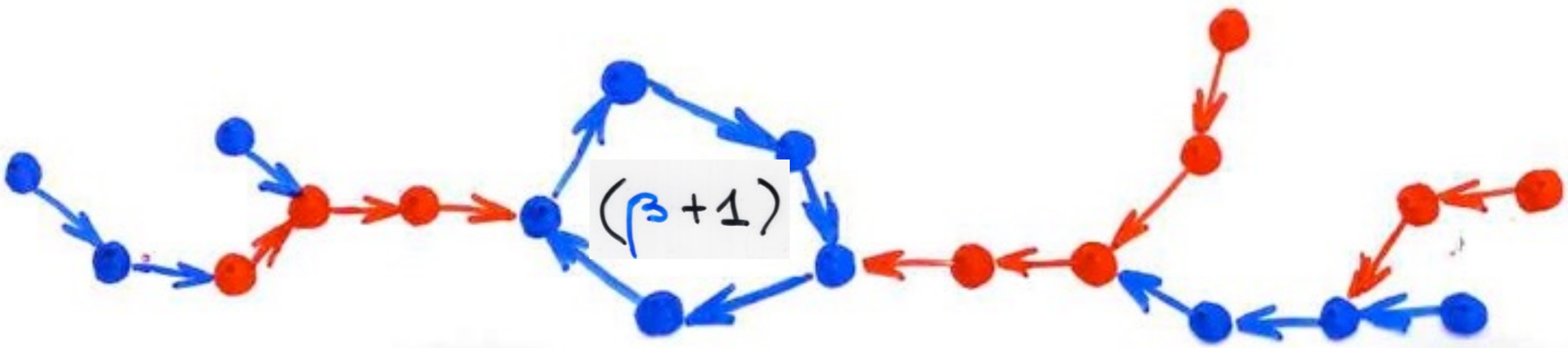
Lemme

$$\Phi_{\alpha}(t) = \exp \Phi_{\alpha, c}(t)$$

$$= \exp\left((\alpha+1)\,\Phi_{a, c}(t)\right)$$

$$= \left(\Phi_{a}(t)\right)^{(\alpha+1)}$$

$(\beta+1)$



we want to prove

$$\sum_{n\geq 0} \mathcal{P}_n^{(\alpha,\beta)}(x,y)\frac{t^n}{n!} = 2^{\alpha+\beta}\mathcal{R}^{-1}\left[1-(x-y)t+\mathcal{R}\right]^{-\alpha}\left[1-(y-x)t+\mathcal{R}\right]^{-\beta}$$

$$\phi_w(t) = \phi_\alpha(t)\ \phi_\beta(t)\ \phi_m(t)$$

$$\phi_\alpha(t) = \left(\phi_a(t)\right)^{\alpha+1}$$

$$\phi_\beta(t) = \left(\phi_b(t)\right)^{\beta+1}$$

It is sufficient to prove:

$$\phi_m = \mathcal{R}^{-1}$$

$$\phi_a(t) = 2\left[1-(x-y)t+\mathcal{R}\right]^{-1}$$

$$\phi_b(t) = 2\left[1-(y-x)t+\mathcal{R}\right]^{-1}$$

$C_a(t)$ $C_b(t)$

exponential generating function for contraction type a, b

connected Jacobi configuration type a with unique cycle of length 1

Lemme

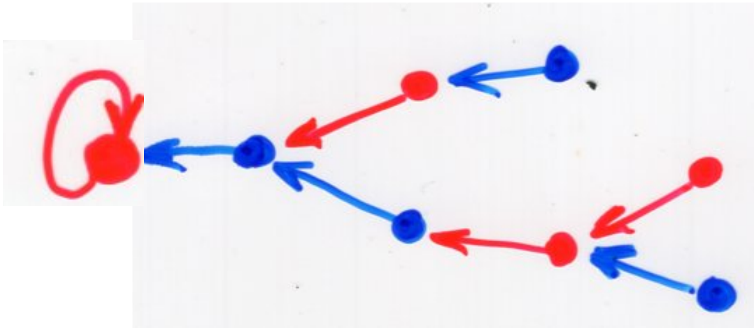
$$\phi_a(t) = \left(1 - C_a(t)\right)^{-1}$$

$$\phi_b(t) = \left(1 - C_b(t)\right)^{-1}$$

$$\phi_m(t) = \left[1 - \left(C_a(t) + C_b(t)\right)\right]^{-1}$$

$C_a(t)$ $C_b(t)$

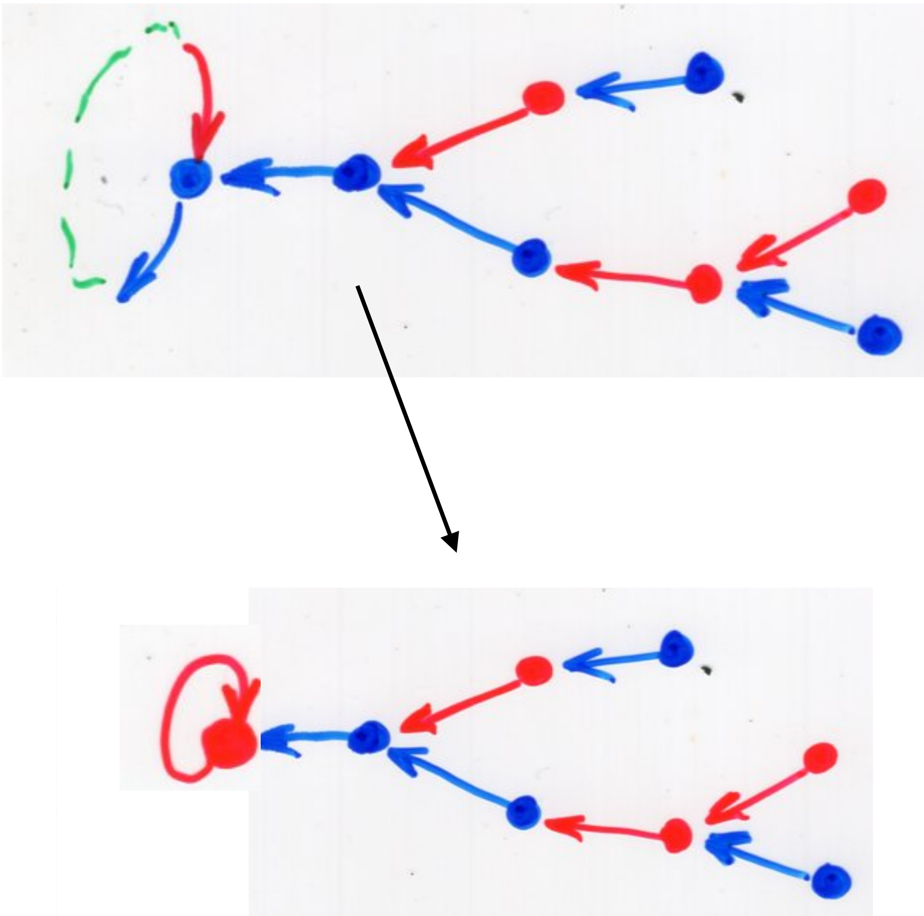
connected Jacobi configuration
type a with unique cycle of length 1



$$\phi_a(t) = \left(1 - C_a(t)\right)^{-1}$$

$$\phi_b(t) = \left(1 - C_b(t)\right)^{-1}$$

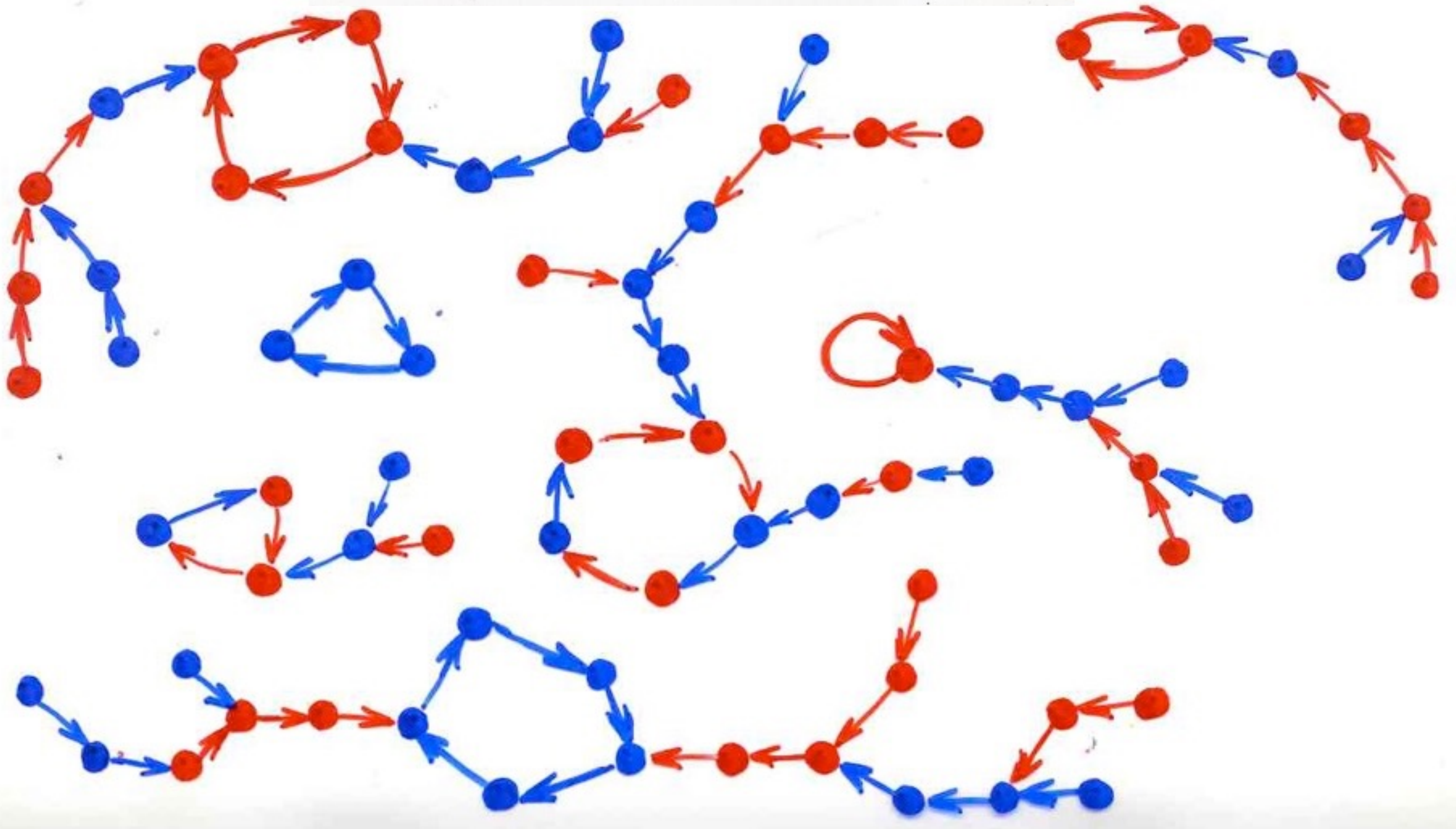
$$\phi_m(t) = \left[1 - \left(C_a(t) + C_b(t)\right)\right]^{-1}$$



$$\phi_a(t) = \left(1 - C_a(t)\right)^{-1}$$

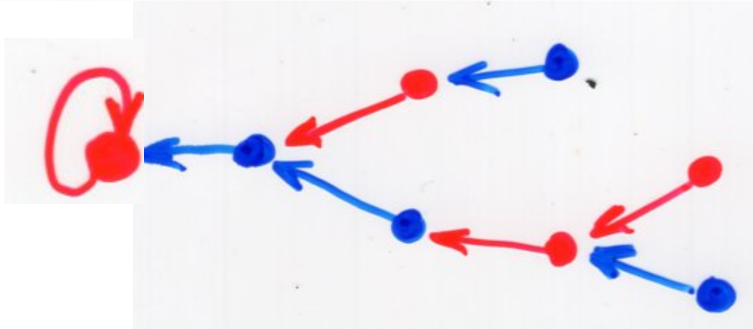
$$\phi_b(t) = \left(1 - C_b(t)\right)^{-1}$$

$$\phi_m(t) = \left[1 - \left(C_a(t) + C_b(t)\right)\right]^{-1}$$

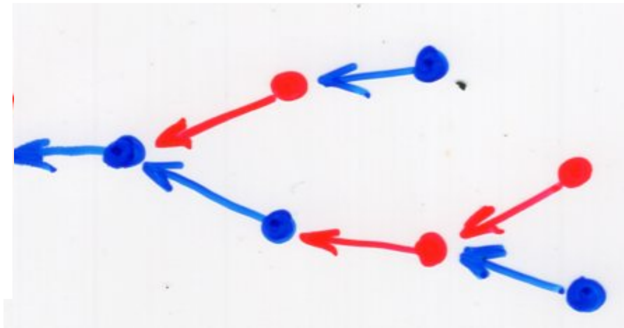


$C_a(t)$ $C_b(t)$

connected Jacobi configuration
type a with unique cycle of length 1

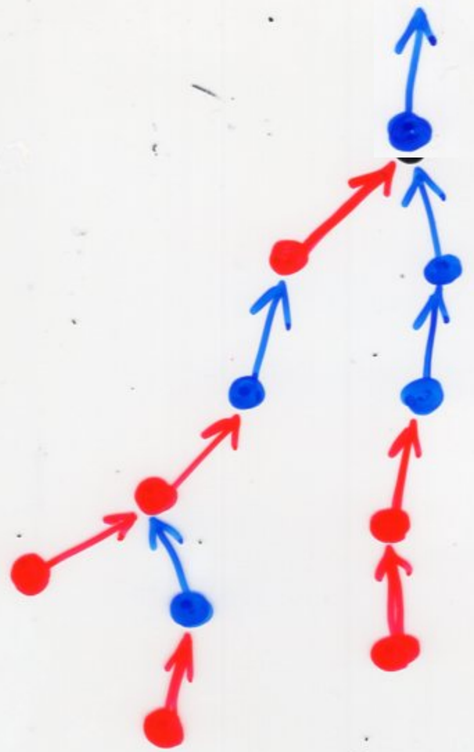


Jacobi arborescence ψ "unlabeled"



Jacobi arborescence ψ "unlabeled"

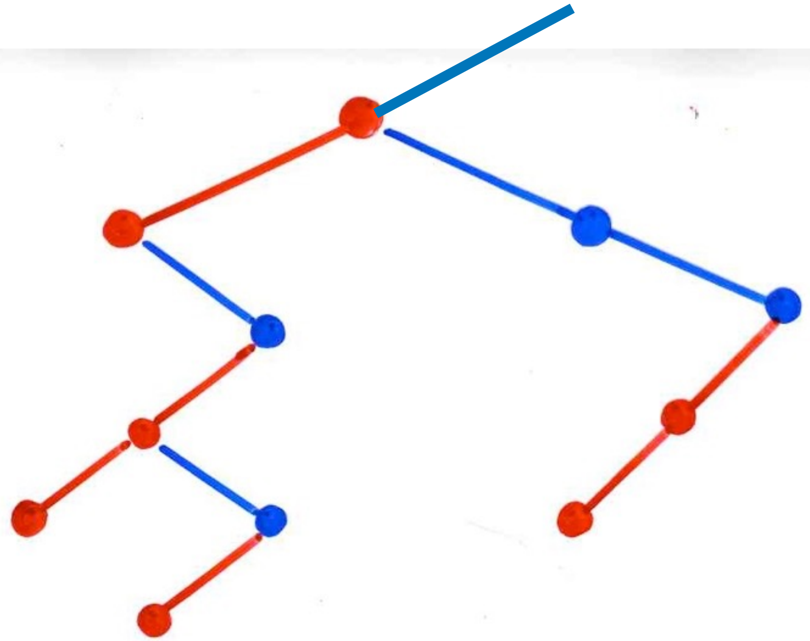
$A(t)$ ordinary generating function
Jacobi arborescence, v weight

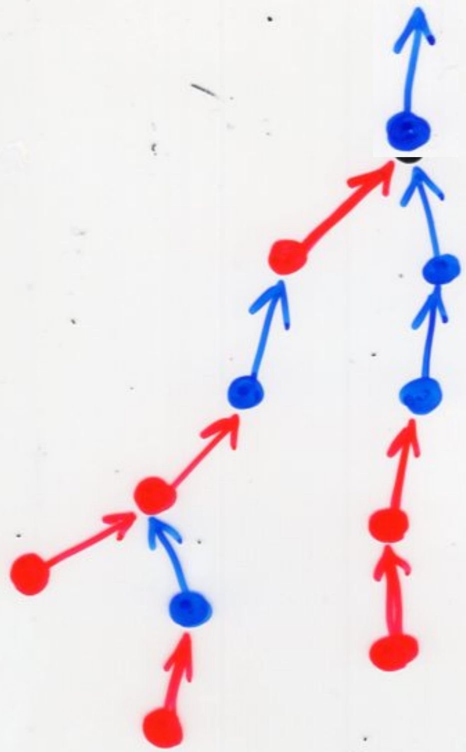


$$y = \sum_{n \geq 0} c_n t^n$$

$$= \sum_{n \geq 0} (n! c_n) \frac{t^n}{n!}$$

"labeled" binary tree





Jacobi arborescence ψ "unlabeled"

$A(t)$ ordinary generating function
Jacobi arborescence, $\vee$ weight

$$C_a(t) = X t (1 + Y A(t))$$

$$C_b(t) = Y t (1 + X A(t))$$

$$C_a(t) = X t \left(1 + Y A(t)\right)$$

$$C_b(t) = Y t \left(1 + X A(t)\right)$$

$$A(t) = t \left(1 + X A(t)\right)\left(1 + Y A(t)\right)$$

$$X Y t \left(A(t)\right)^2 + \left((X+Y)t - 1\right) A(t) + t = 0$$

$$A(t) = \left(1 - (X+Y)t - R\right) \Big/ 2XYt$$

$$R = \left[1 - 2(X+Y)t + (X-Y)^2 t^2\right]^{1/2}$$

$$A(t) = \left(1-(X+Y)t-\mathcal{R}\right)/2XYt$$

$$\mathcal{R} = \left[1-2(X+Y)t+(X-Y)^2t^2\right]^{1/2}$$

$$C_a(t) = Xt(1+YA(t))$$

$$C_b(t) = Yt(1+XA(t))$$

$$\phi_a(t) = \left(1-C_a(t)\right)^{-1}$$

$$\phi_b(t) = \left(1-C_b(t)\right)^{-1}$$

$$\phi_m(t) = \left[1-(C_a(t)+C_b(t))\right]^{-1}$$

We have proved

$$\phi_m = \mathcal{R}^{-1}$$

$$\phi_a(t) = 2\left[1-(x-y)t+\mathcal{R}\right]^{-1}$$

$$\phi_b(t) = 2\left[1-(y-x)t+\mathcal{R}\right]^{-1}$$

It was sufficient to prove these 3 identities

$\square$

Jacobi polynomials

$$\left[X(1+\alpha+\delta_Y)+Y(1+\beta+\delta_X)\right]\left[D_X+D_Y\right]\mathcal{L}_n^{(\alpha,\beta)} =$$

$$(1+\alpha+\beta+n)\, n\, \mathcal{L}_n^{(\alpha,\beta)}$$

Leroux, Strehl (1985)

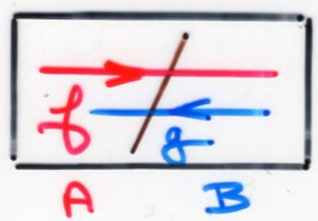
Limit formula

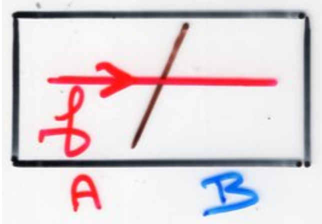
limit formula

example

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

Jacobi $\longrightarrow$ Laguerre





J. Labelle, Y.N. Yeh (1989)

$$P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = \sum_{(f,g)\in J[A,B]} \overline{w}(f,g)$$

$$\overline{w}(f,g) = (\alpha+1)^{cyc(f)}(\beta+1)^{cyc(g)}(1-x\beta^{-1})^{|A|} \quad (-x\beta^{-1})^{|B|}$$

$$\lim_{\beta\to\infty} \overline{w}(f,g) = 0 \qquad \text{unless } \ cyc(g) = |B|$$

i.e. $g = 1_B$

$$\lim_{\beta \to \infty} w(f, 1_B) = (\alpha+1)^{\mathrm{cyc}(f)} (-x)^{|B|}$$

$$= w(f)$$

$f \in \mathbf{L}[A, B]$

Laguerre configuration

$\square$