Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

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Chapter 2
Moments and histories

Ch 2c

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Remiding Ch2b:

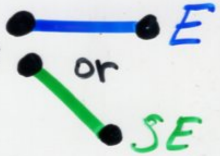
Restricted Laguerre histories

Combinatorial interpretation of moments
of orthogonal Sheffer polynomials

Definition restricted Laguerre history

$$h = (\omega_c ; p) \qquad p = (p_1, \ldots, p_n)$$

such that $p_i > 1$ for step ω_i =



In other words, during the insertion process $h \to \sigma$ the first open position $\sqcup$ is always kept at the beginning (of the sequence of values $1, 2, \ldots$ and $\sqcup$)

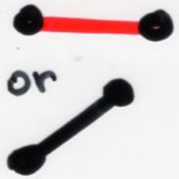
$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases} \qquad \begin{matrix} (k \geq 0) \\ \\ (k \geq 1) \end{matrix}$$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

$$\mu_n = n!$$

for a restricted Laguerre history, put a weight β for each choice

$p_i = 1$ with $w_i =$

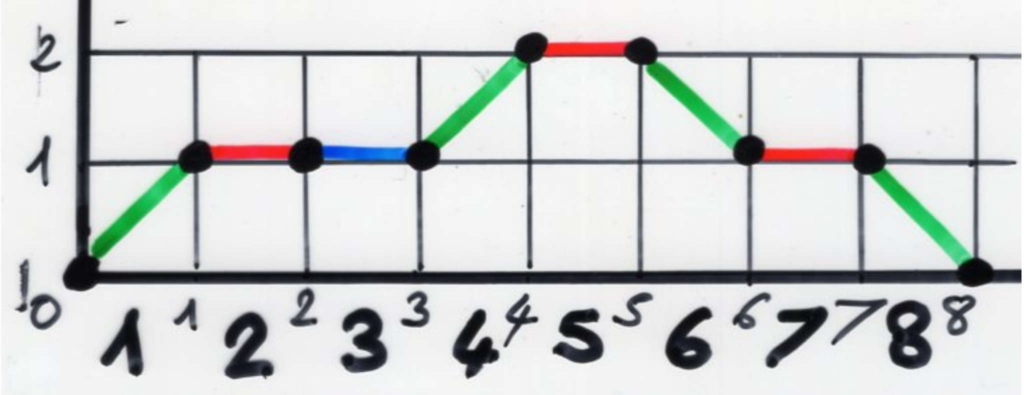


this is equivalent to say that the element i is a lr-min element of the corresponding permutation σ.

$$\begin{cases} a_k = k + \beta \\ b'_k = k + \beta \\ b''_k = k \\ c_k = k \end{cases}$$

$$\begin{cases} b_k = 2k + \beta \\ \lambda_k = (k - 1 + \beta)k \end{cases}$$

$$\mu_n = \beta(\beta+1) \cdots (\beta+n-1)$$



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word

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example

restricted
Laguerre
histories

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increasing
binary tree

1

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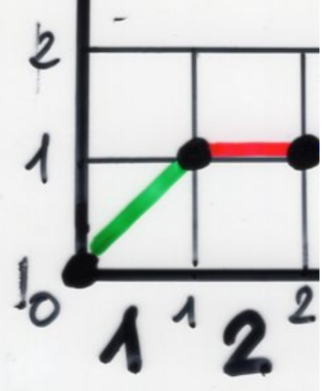

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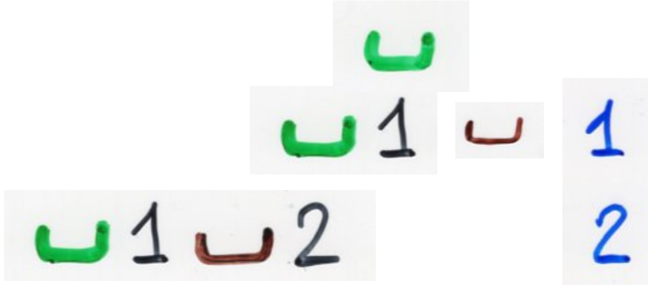

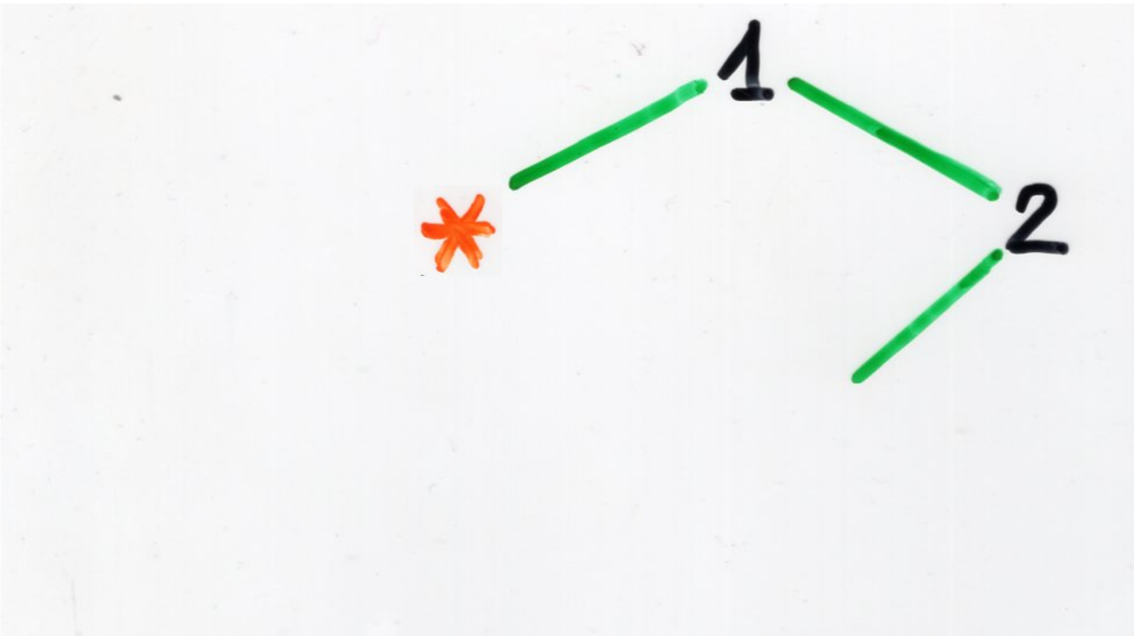

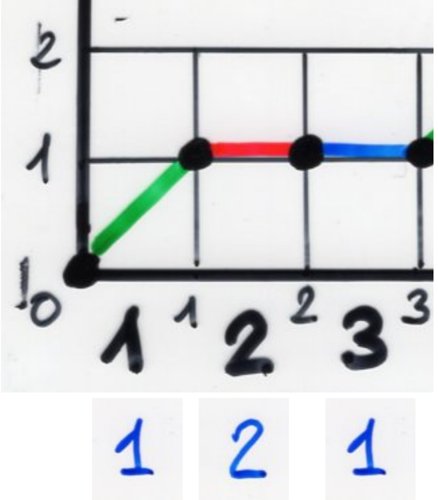
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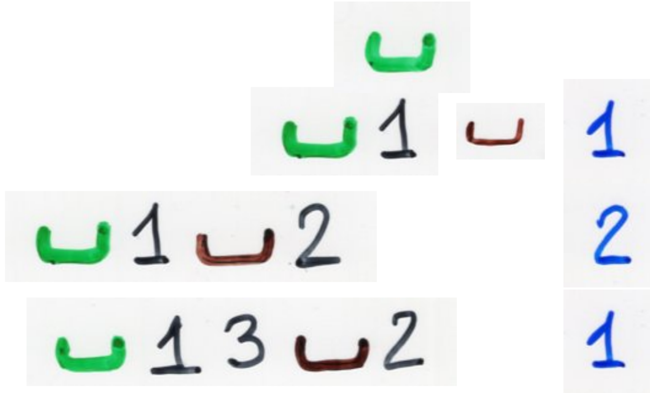
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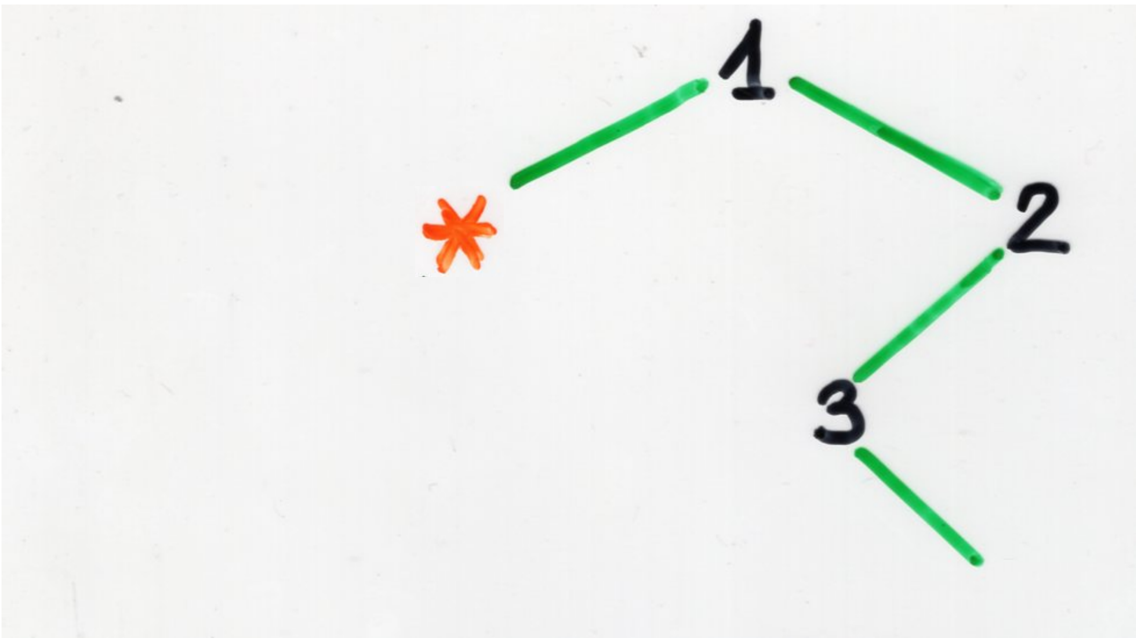
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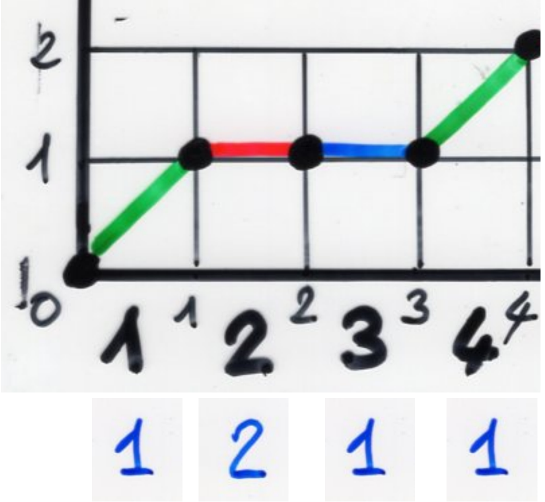

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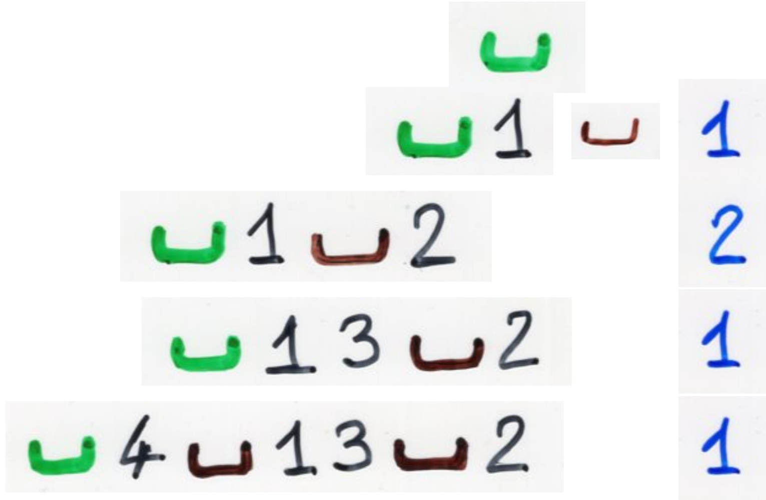

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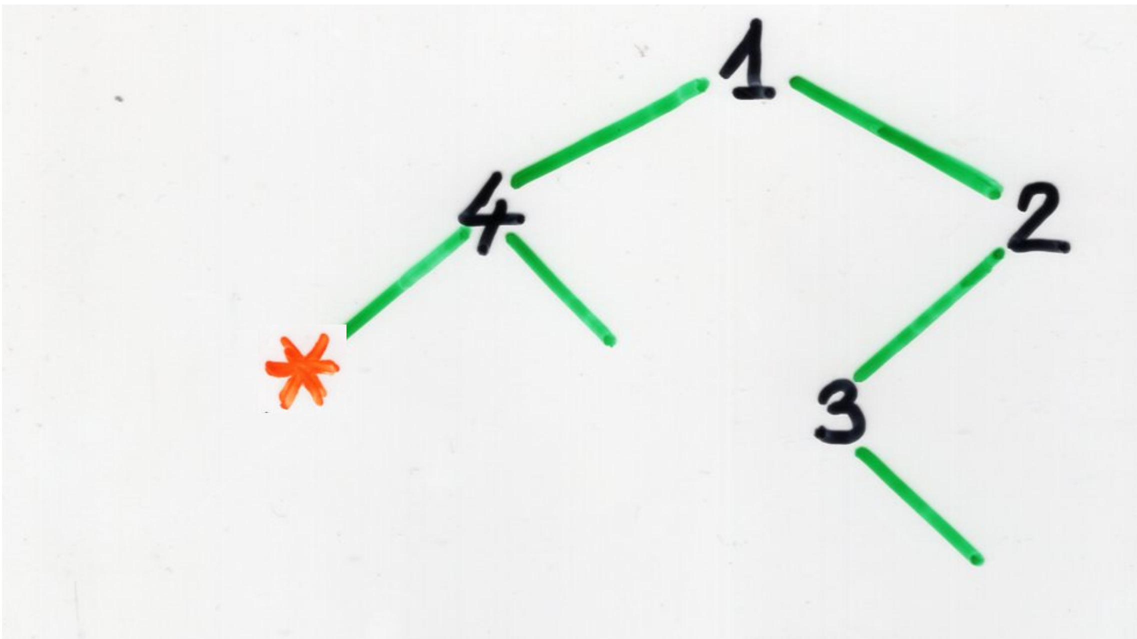

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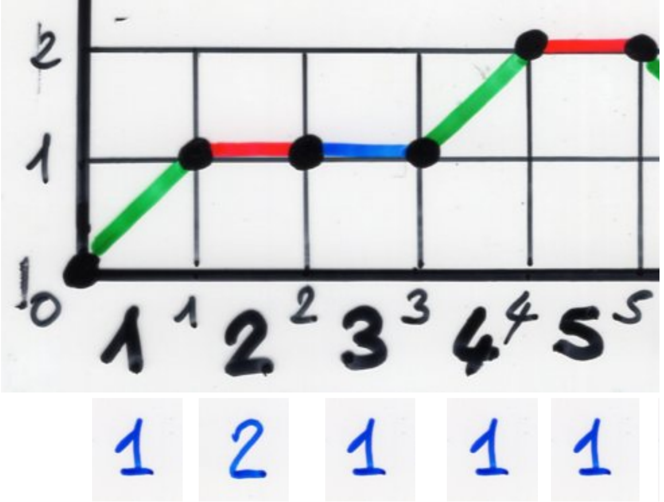
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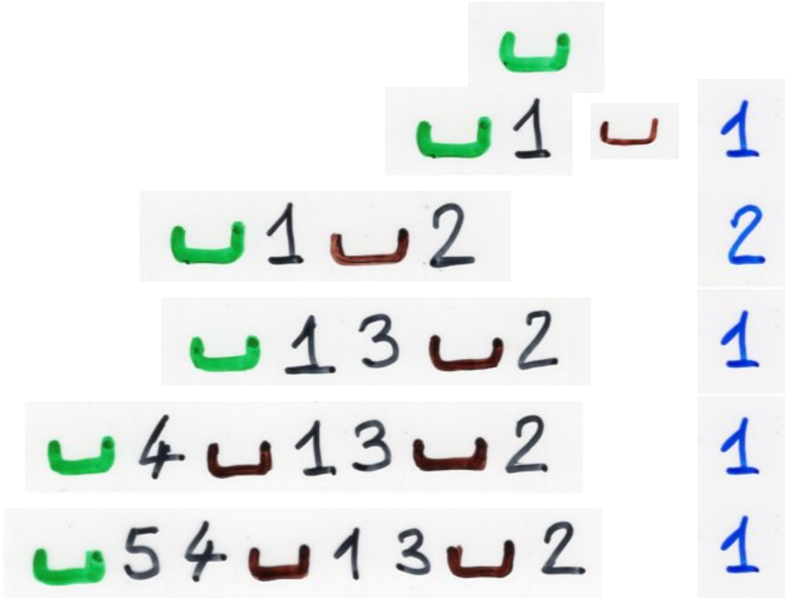
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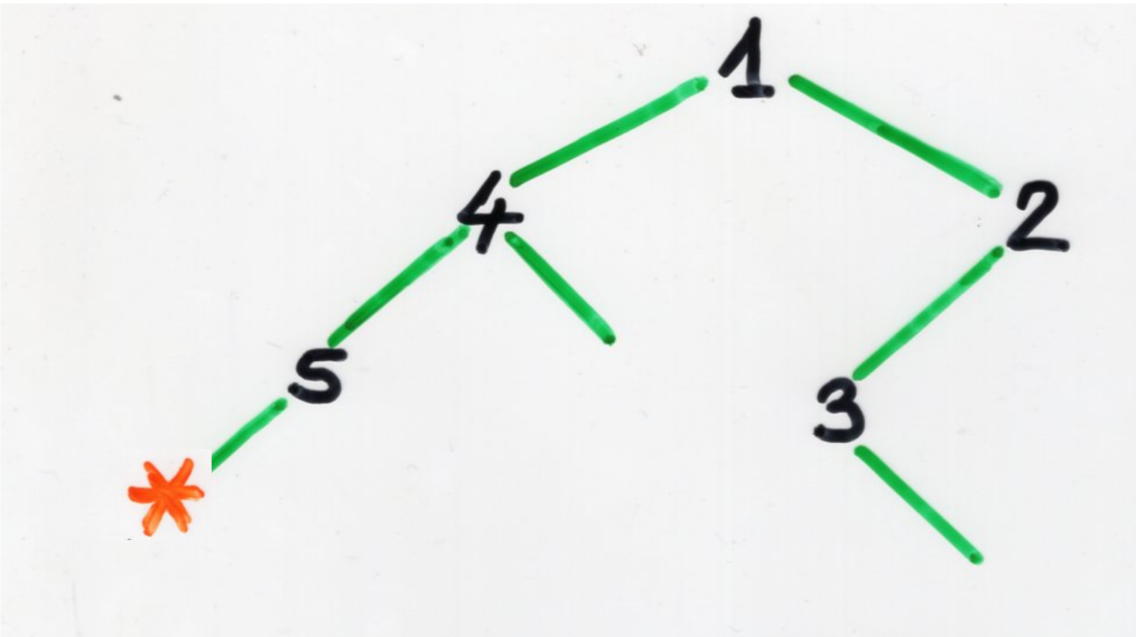
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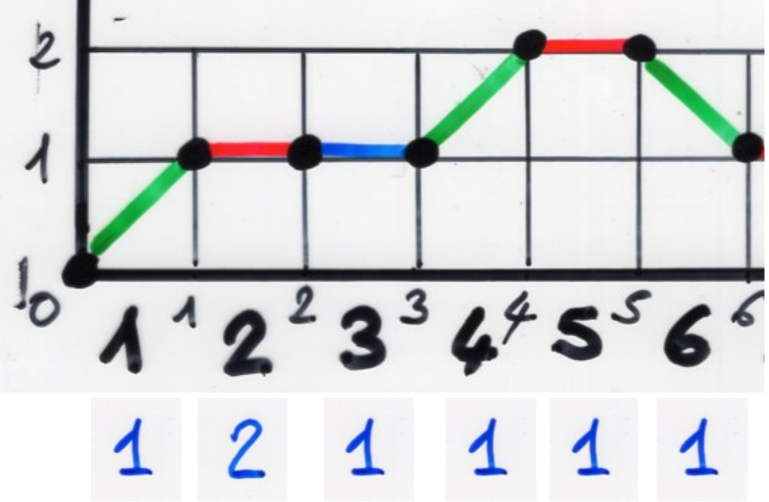

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⊔4⊔13⊔2 1
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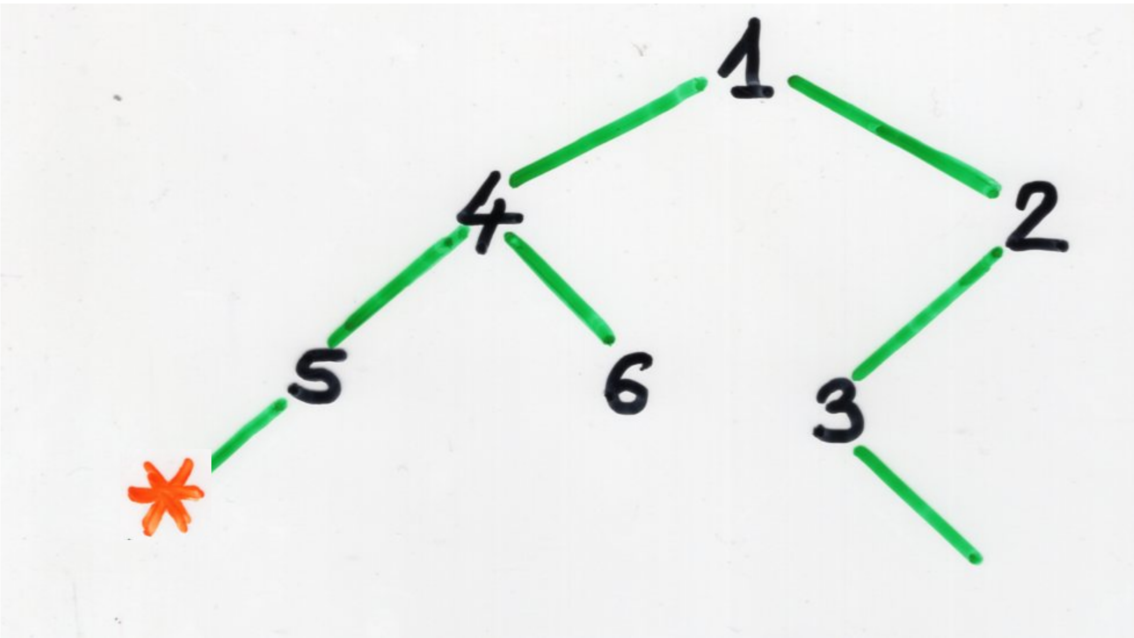

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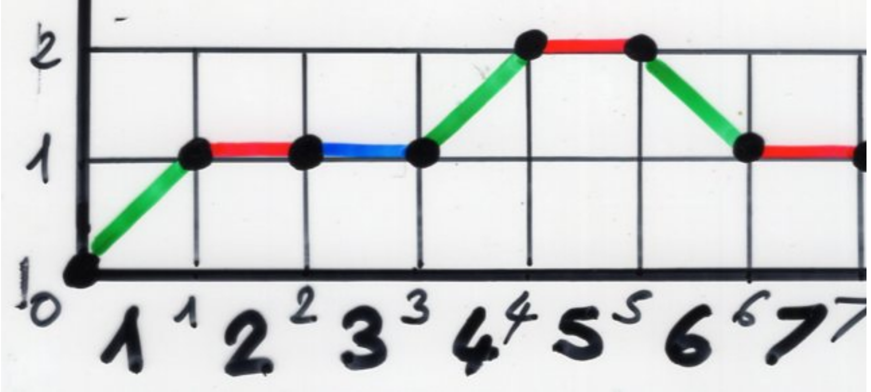
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5 4 6 1 3 2
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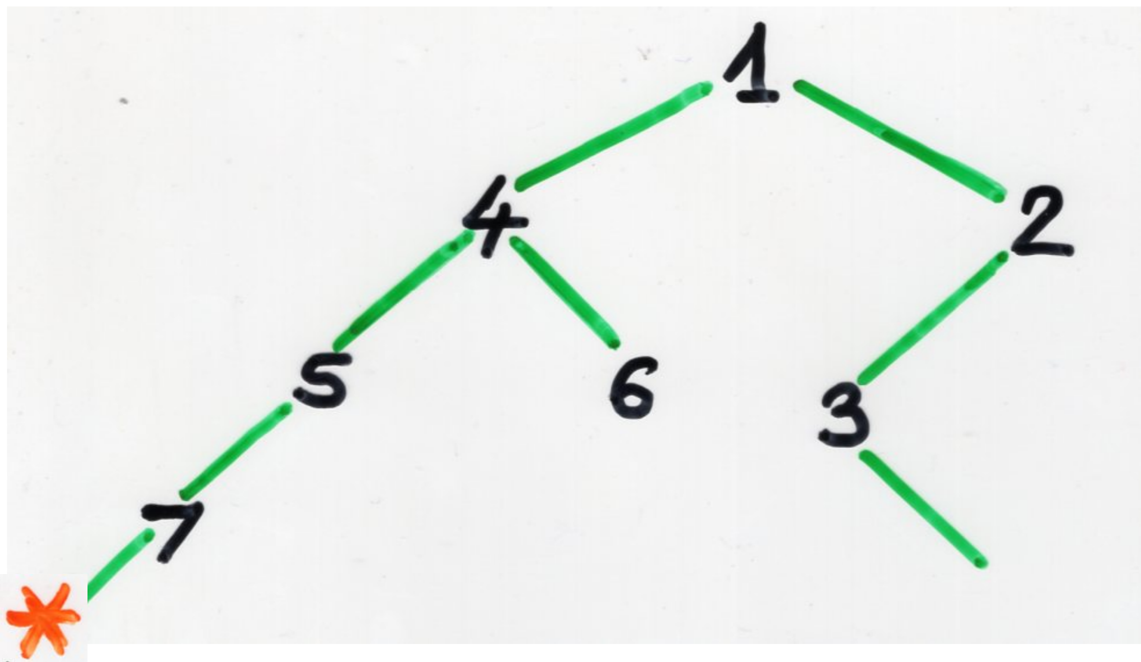
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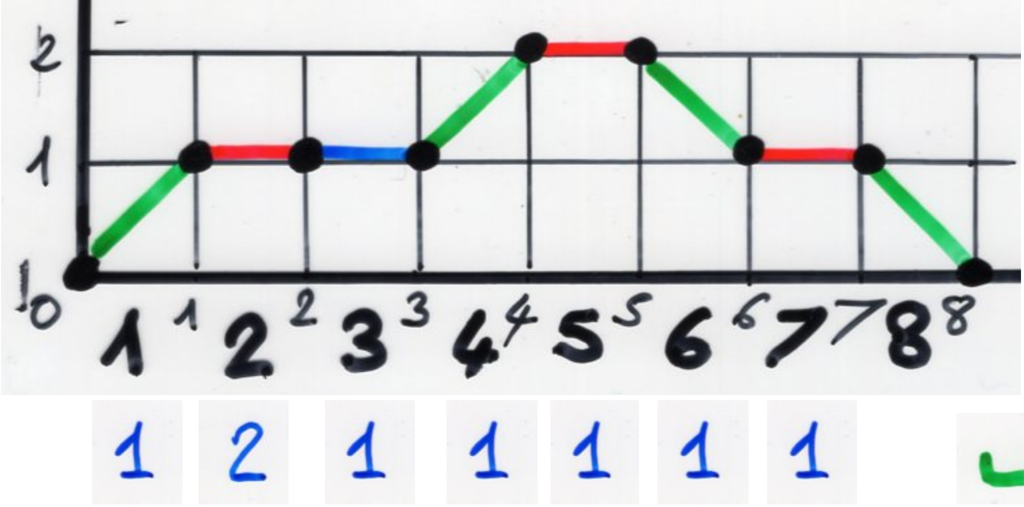
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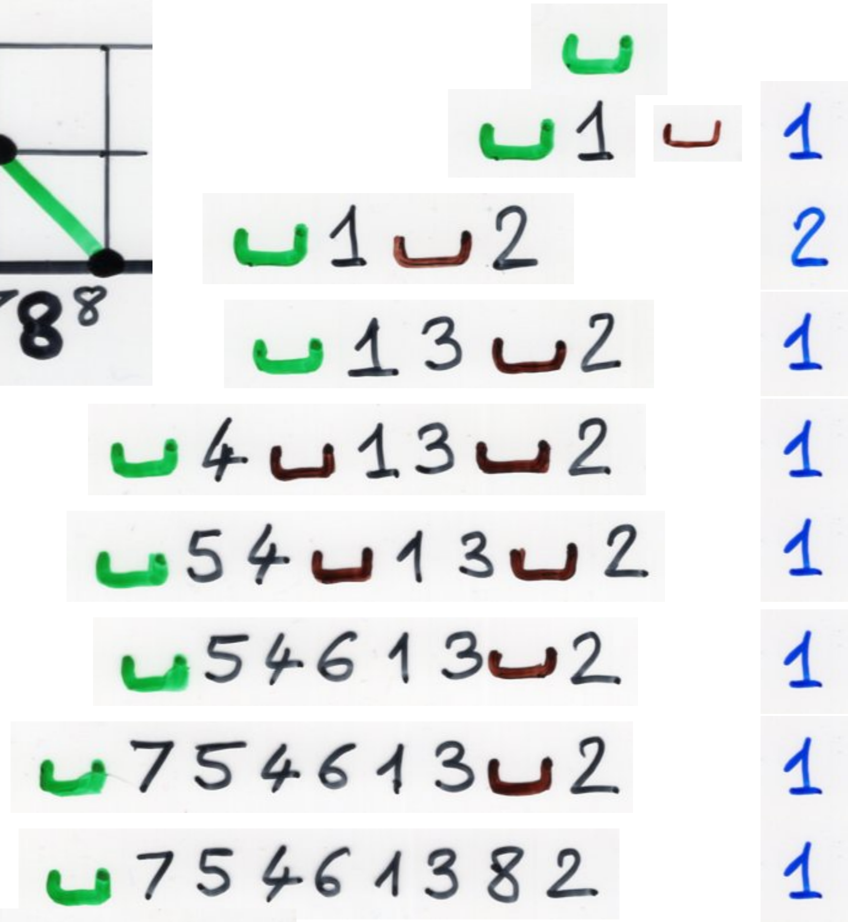
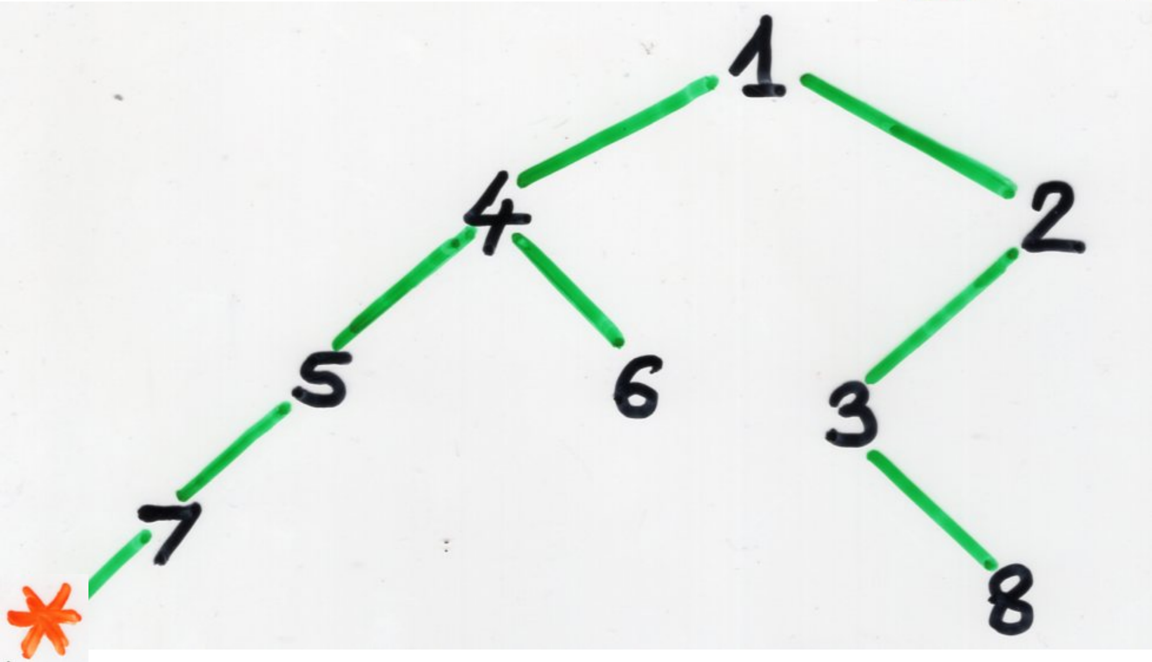

1 1
1 2 2
1 3 2 1
4 1 3 2 1
5 4 1 3 2 1
5 4 6 1 3 2 1
7 5 4 6 1 3 2 1
7 5 4 6 1 3 8 2 1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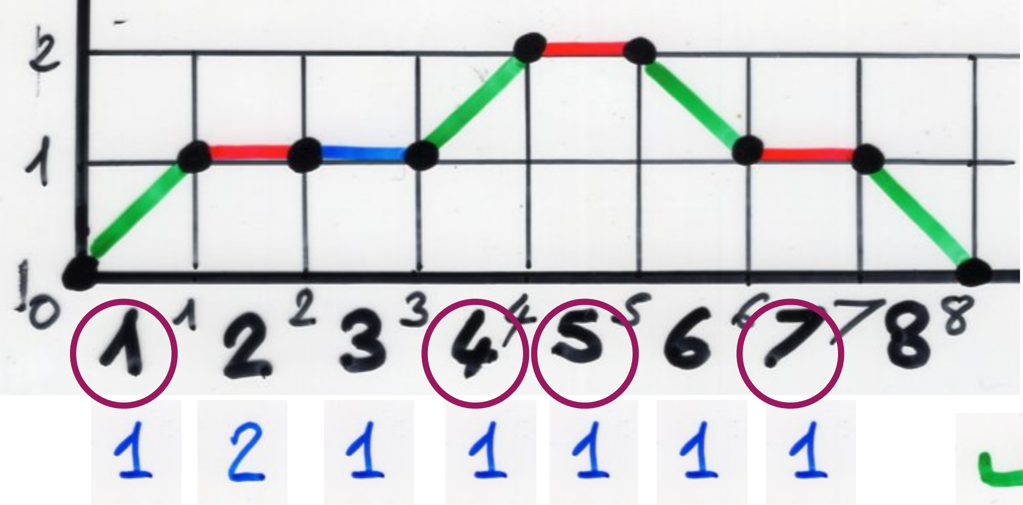
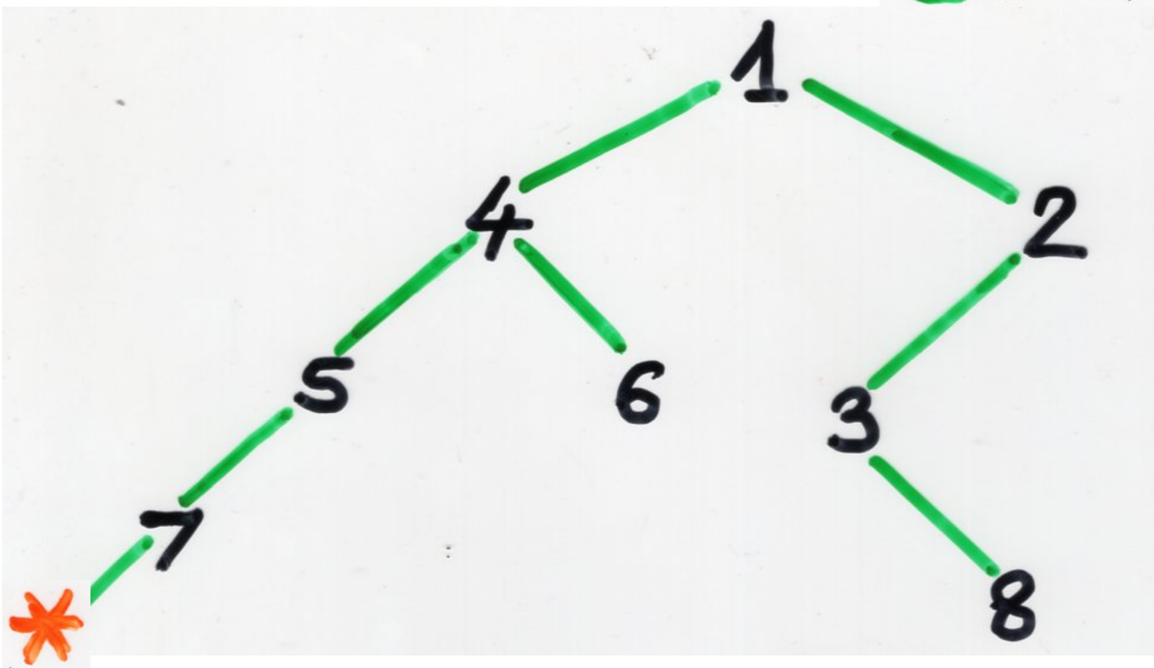




lr-min elements

$\sigma = 7 / 5 / 4\,6 / 1\,3\,8\,2$

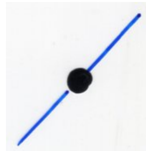
Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1}=0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\frac{(1+c)k+\beta c}{(1-c)}$	$\frac{ck(k-1+\beta)}{(1-c)^2}$	$=(1-c)^{\beta}\sum_{k\geq 0} k^n c^n \frac{(\beta)_k}{k!}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)}\left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

a — $v(\sigma)$ = number of valleys of σ

b — $p(\sigma)$ = number of peaks of σ

c — $dr(\sigma)$ = number of double rises of σ



d — $dd(\sigma)$ = number of double descents of σ

α — $f(\sigma)$ = number of lr-min elements which are a descent of σ

β — $s(\sigma)$ = number of lr-min elements of σ

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

a $v(\sigma)$ = number of valleys of σ

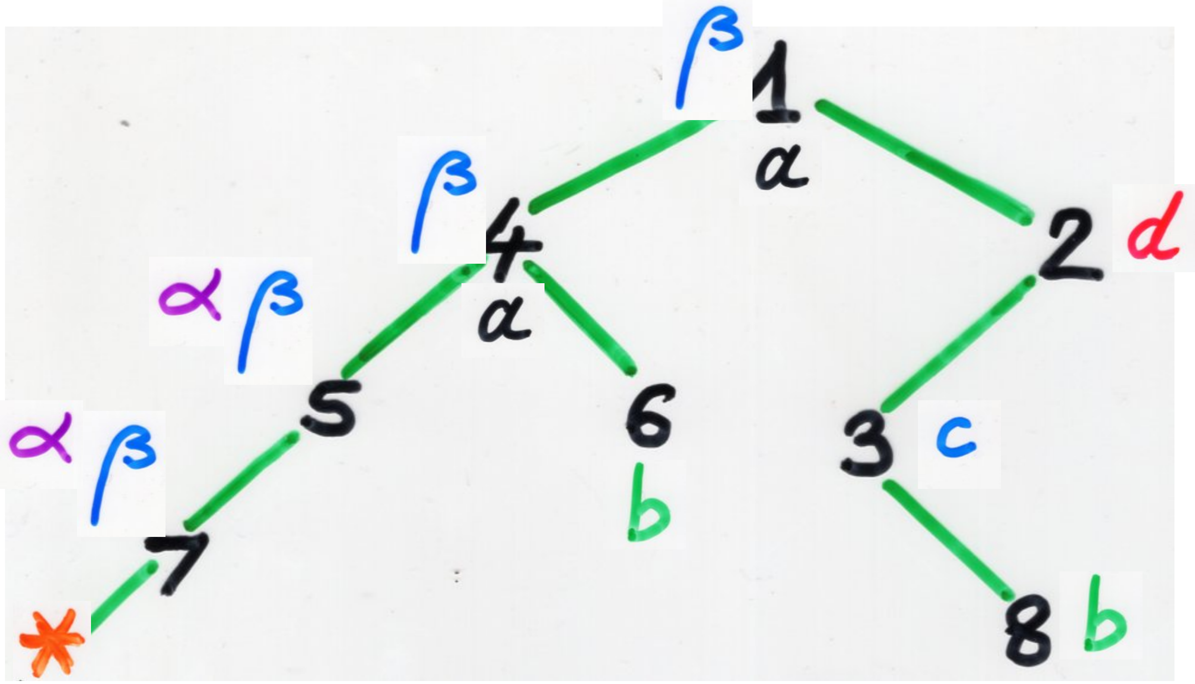
b $p(\sigma)$ = number of peaks of σ

c $dr(\sigma)$ = number of double rises of σ

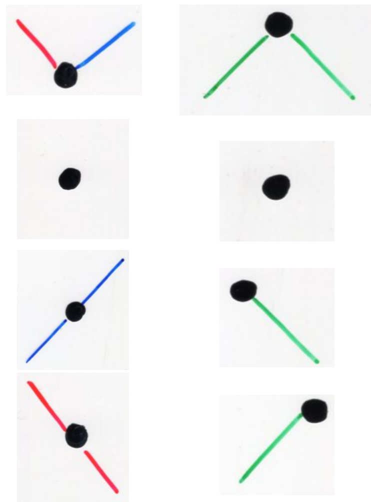
d $dd(\sigma)$ = number of double descents of σ

α $f(\sigma)$ = number of lr-min elements which are a descent of σ

β $s(\sigma)$ = number of lr-min elements of σ



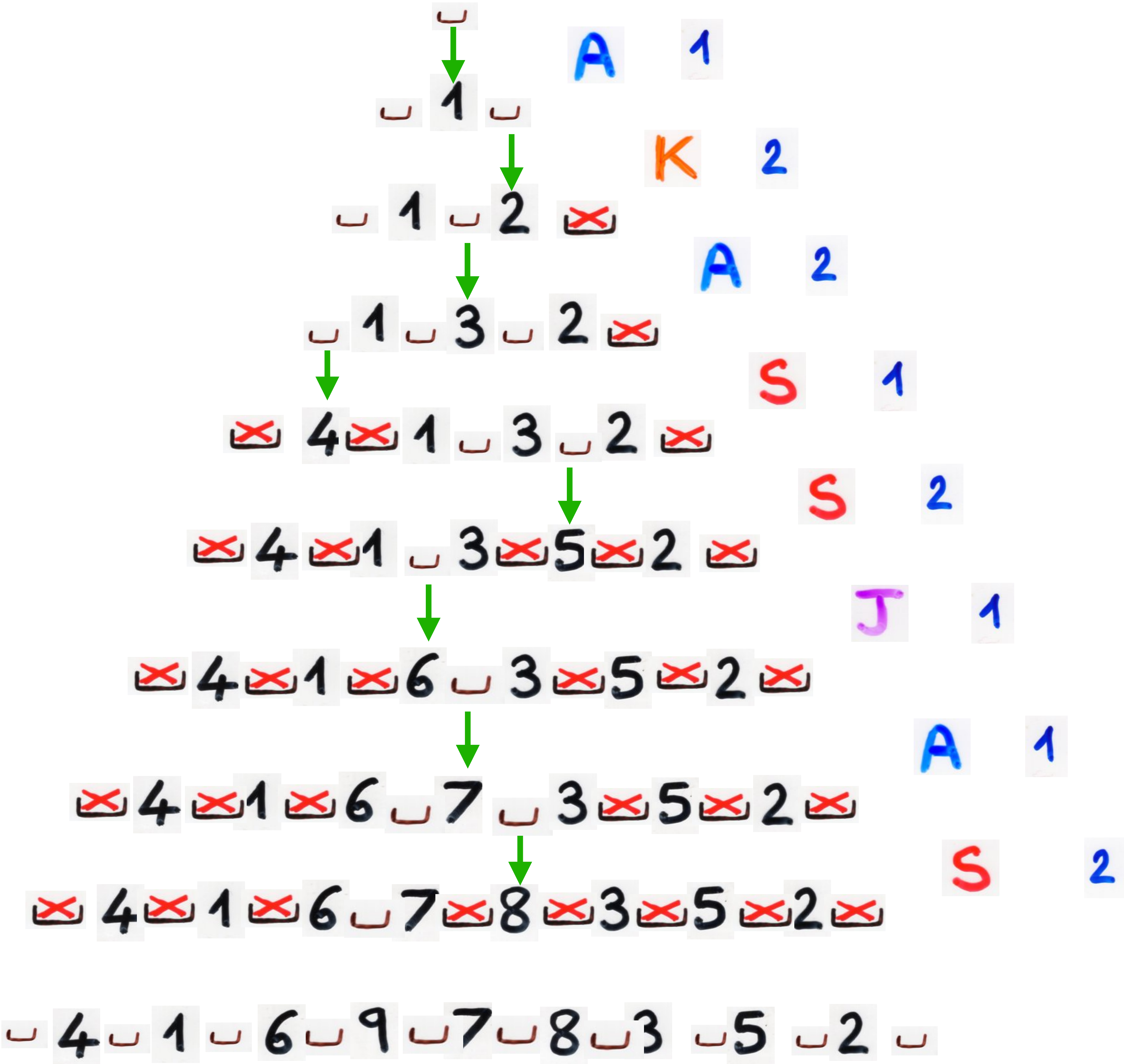
- **a** $v(\sigma)$ = number of valleys of σ
- **b** $p(\sigma)$ = number of peaks of σ
- **c** $dr(\sigma)$ = number of double rises of σ
- **d** $dd(\sigma)$ = number of double descents of σ
- **α** $f(\sigma)$ = number of lr-min elements which are a descent of σ
- **β** $s(\sigma)$ = number of lr-min elements of σ

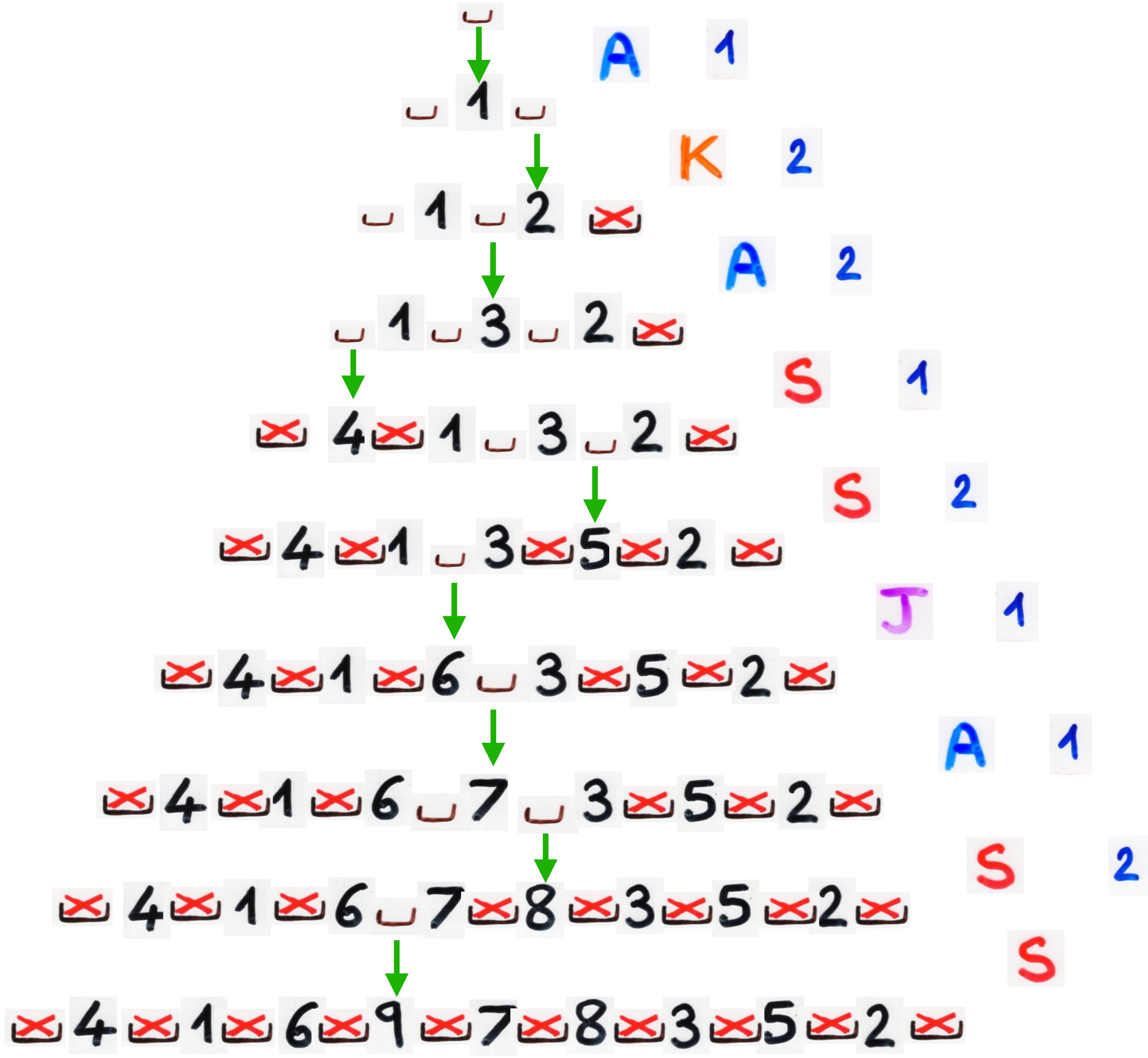


Closure of histories

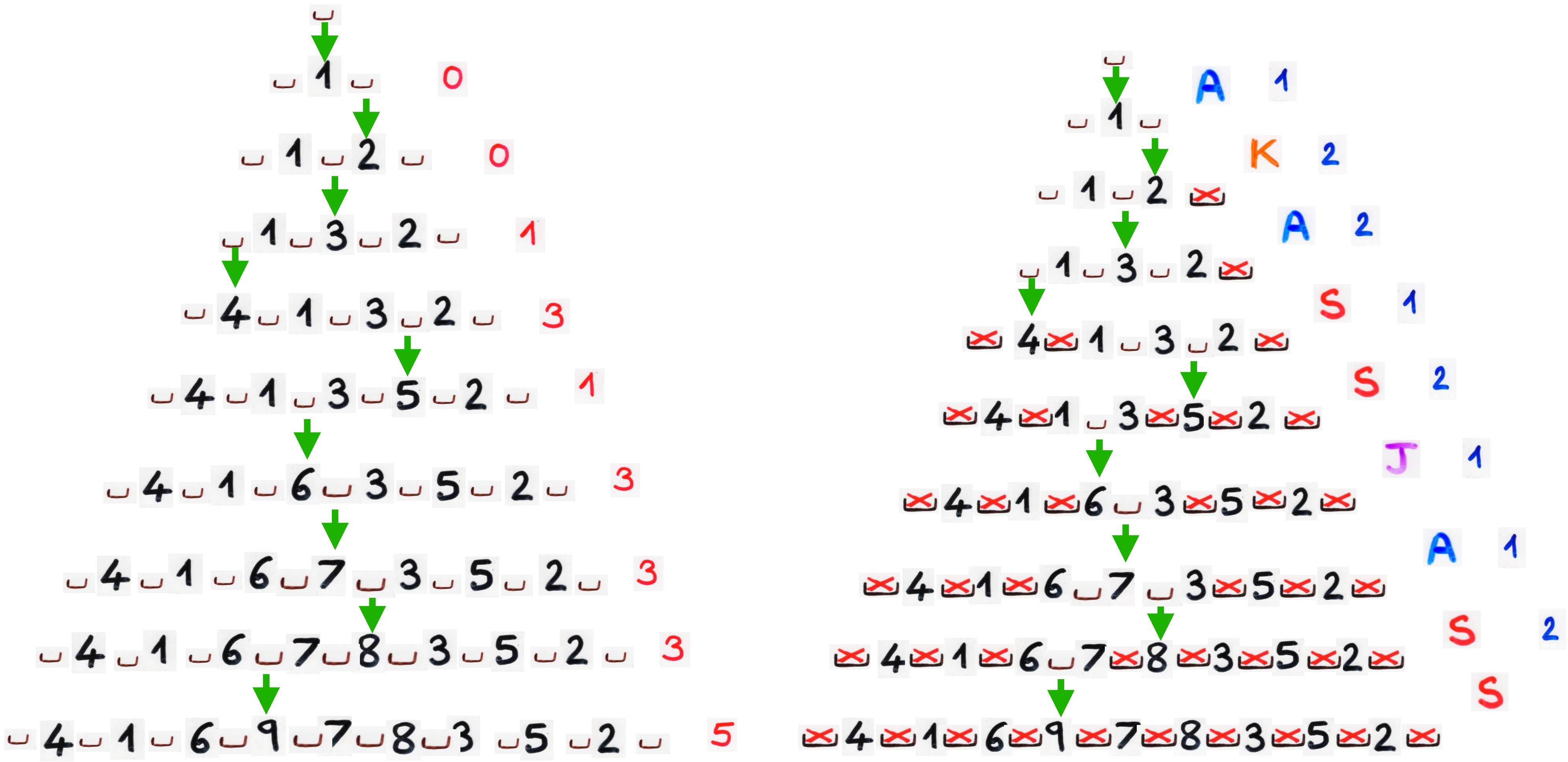
Open histories
Closed histories


1 0
1 2 0
1 3 2 1
4 1 3 2 3
4 1 3 5 2 1
4 1 6 3 5 2 3
4 1 6 7 3 5 2 3
4 1 6 7 8 3 5 2 3
4 1 6 9 7 8 3 5 2 5
inversion table


A 1
K 2
A 2
S 1
S 2
J 1
A 1
S 2
4 1 6 9 7 8 3 5 2



A 1
1
K 2
1 2
A 2
1 3 2
S 1
4 1 3 2
S 2
4 1 3 5 2
J 1
4 1 6 3 5 2
A 1
4 1 6 7 3 5 2
S 2
4 1 6 7 8 3 5 2
S
4 1 6 9 7 8 3 5 2



A 1
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1 1 1
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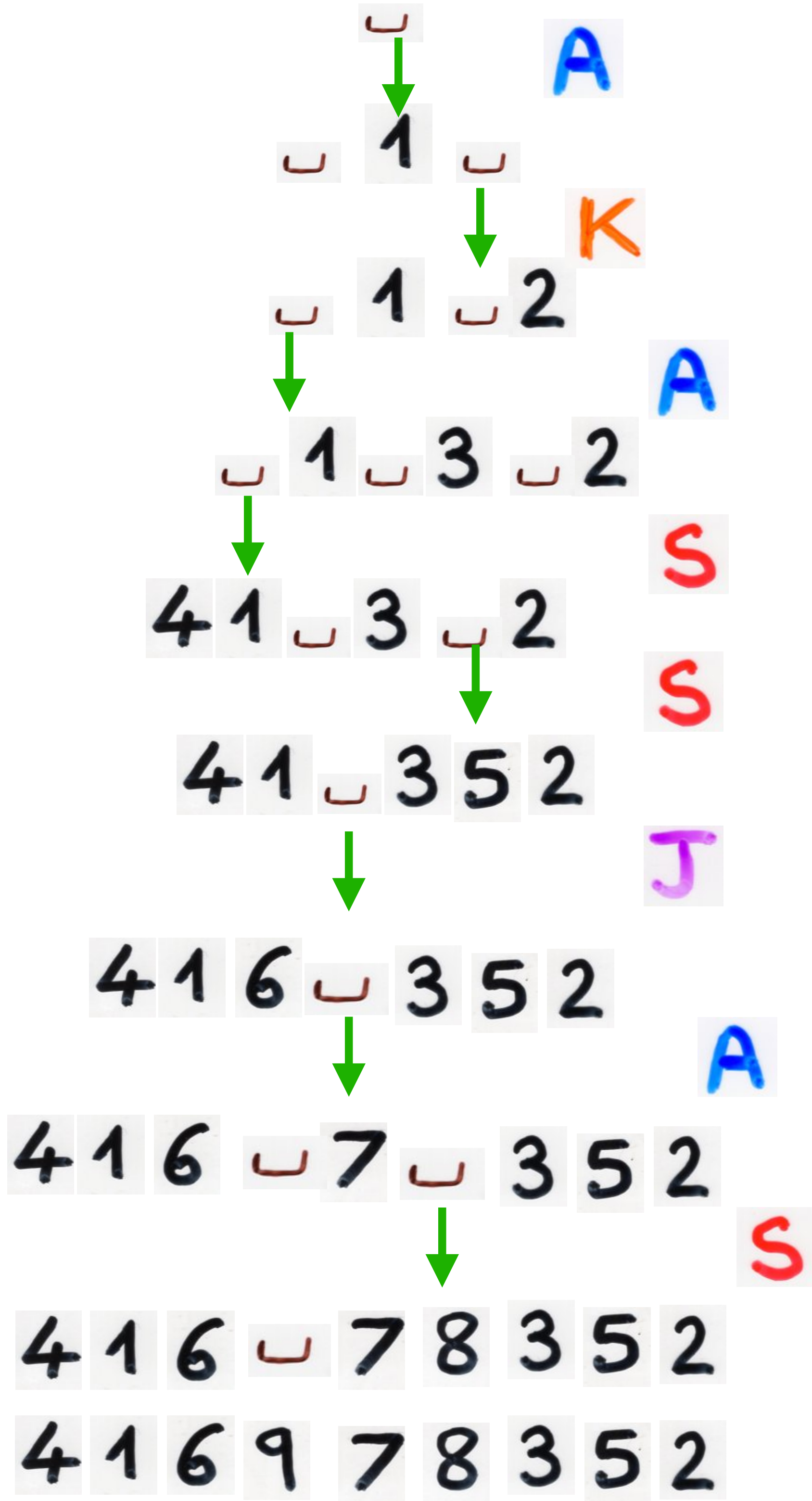
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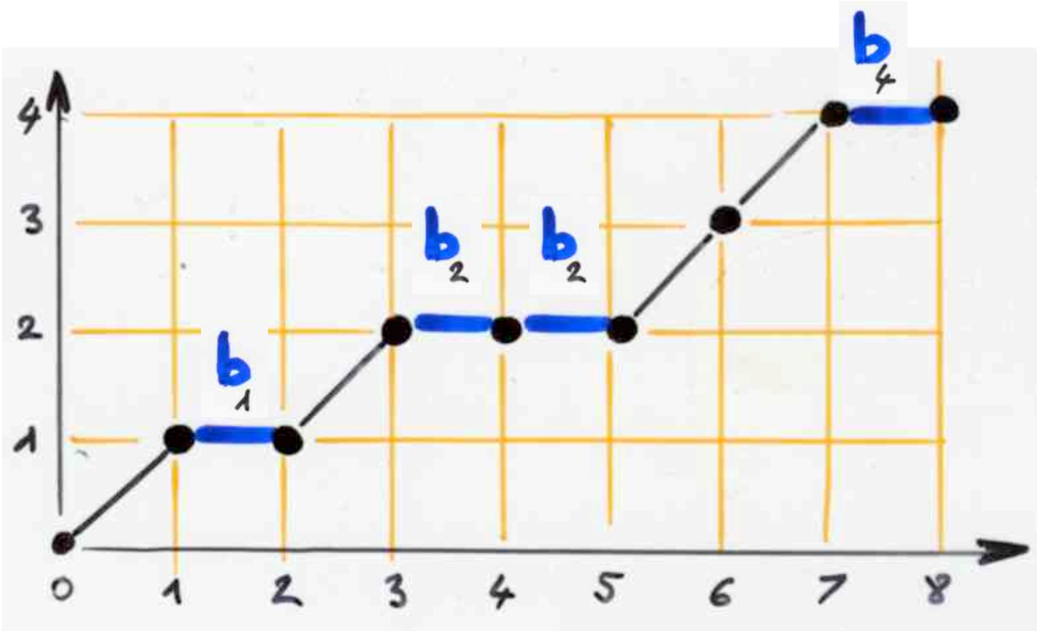
4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

Closure of the (open) history for set partitions

(Ch1d)

(Ch1d)

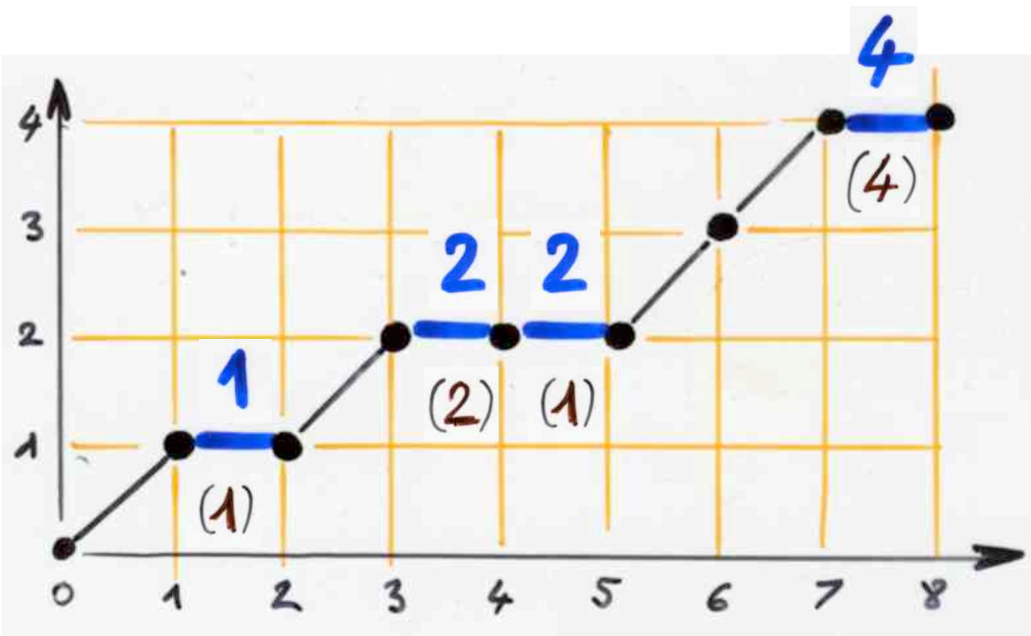


$\mu_{n,i} = S_{n,i}$

Stirling numbers

$\lambda_k = 0$ $\quad b_k = k$ $\quad k \geq 0$

= number of (set) partitions of $\{1, \dots, n\}$ into i blocks



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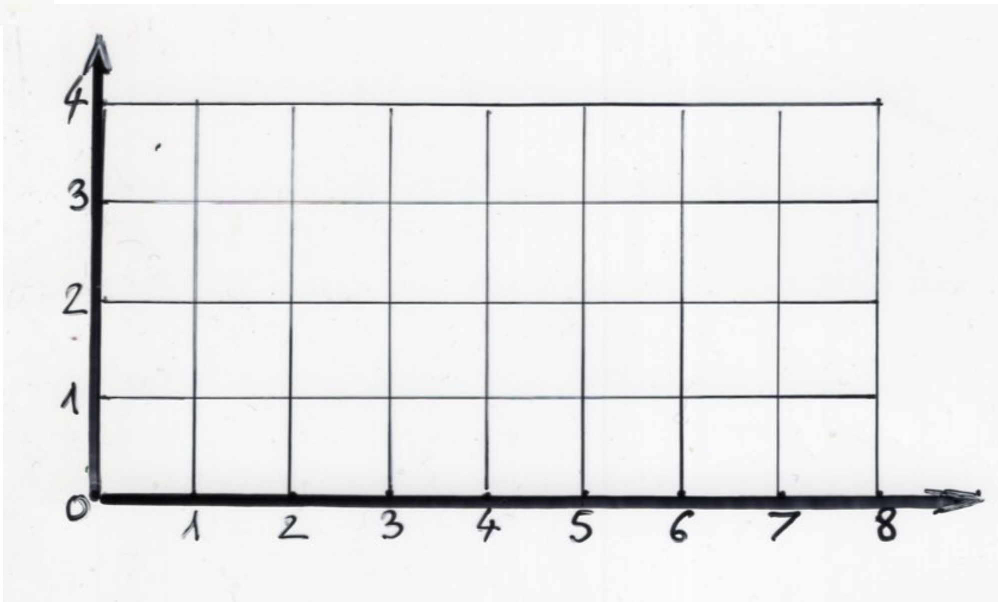
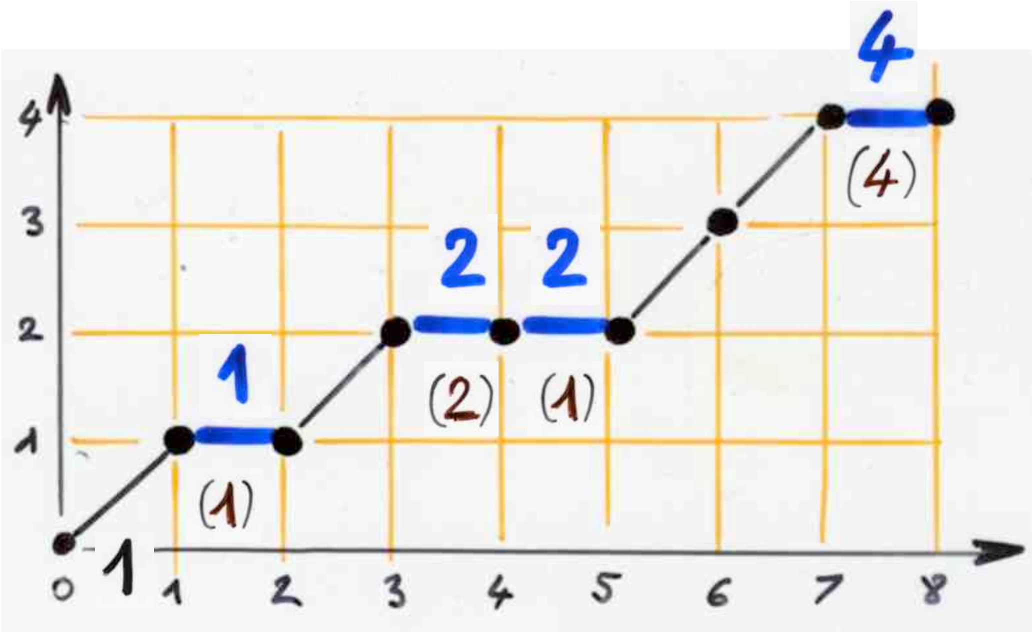
$\lambda_k = 0$ $b_k = k$ $k \geq 0$





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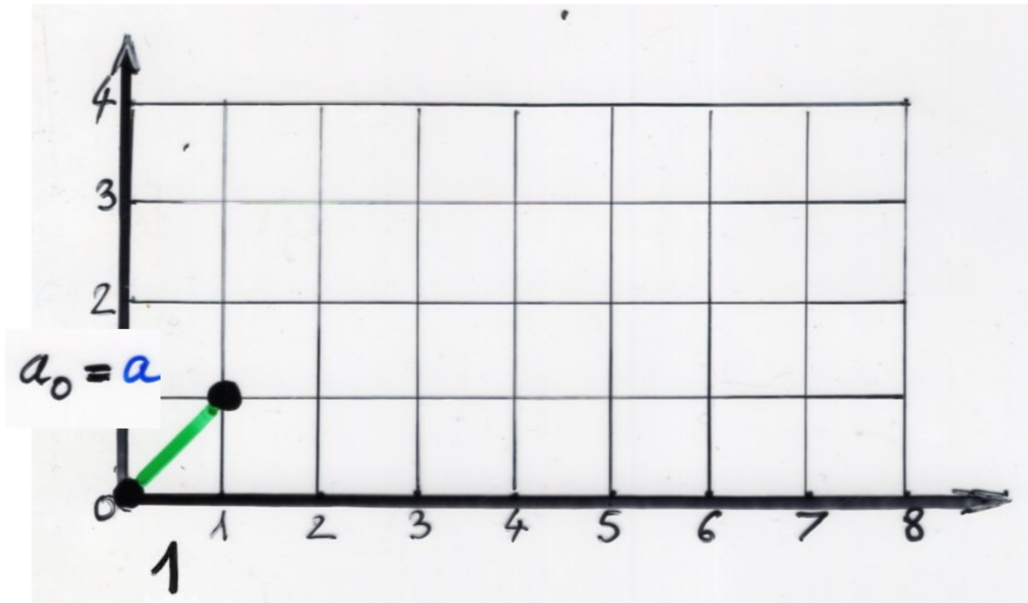

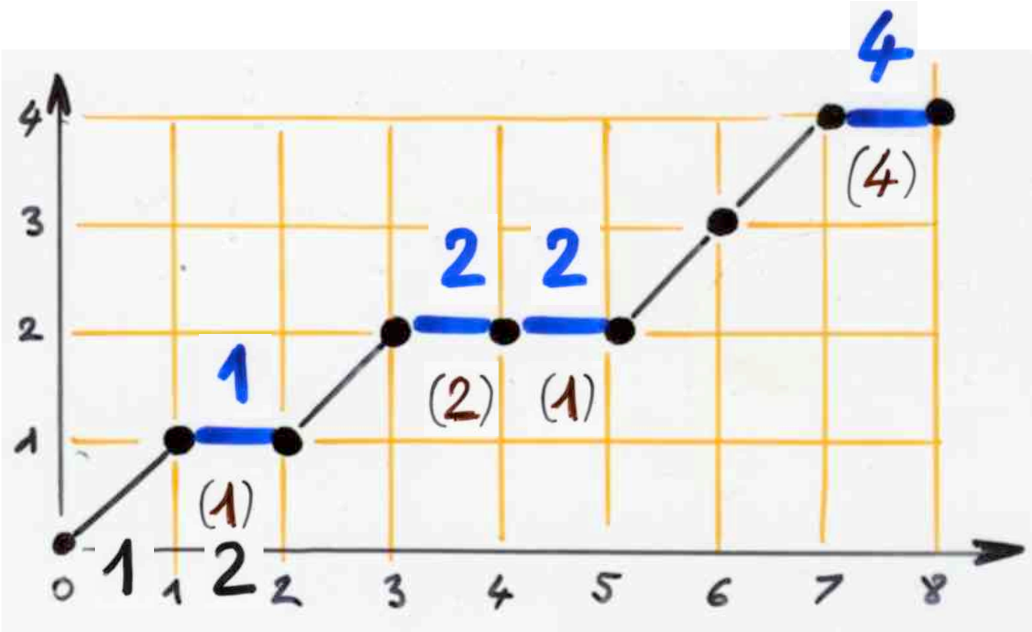

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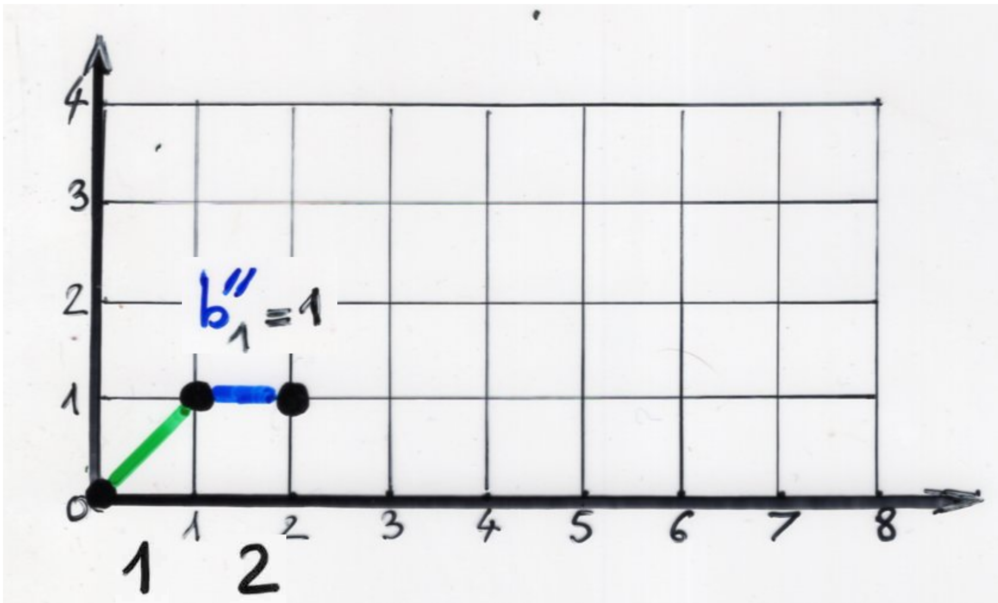

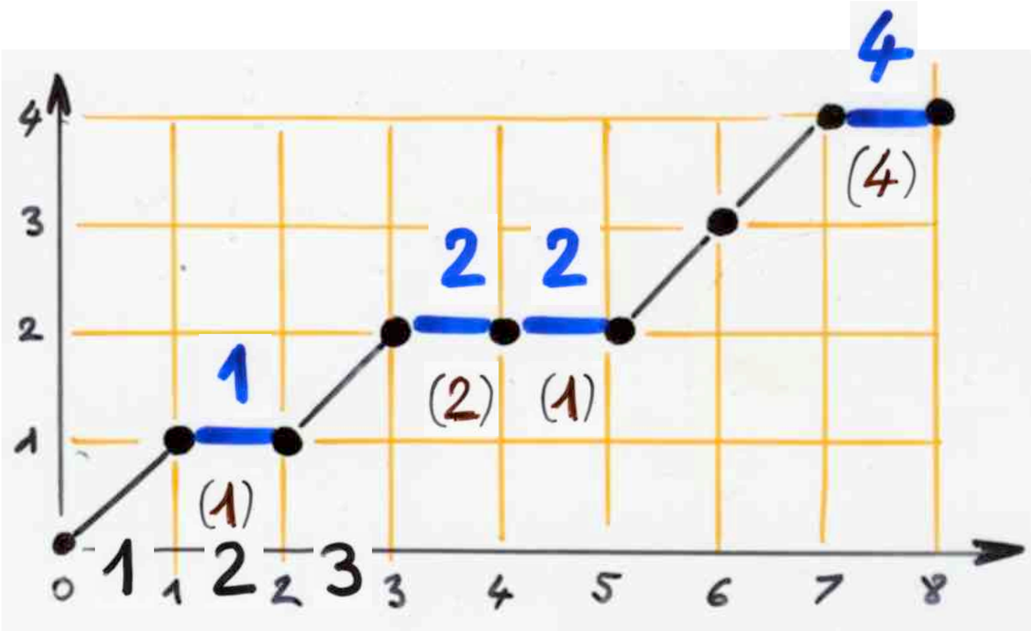

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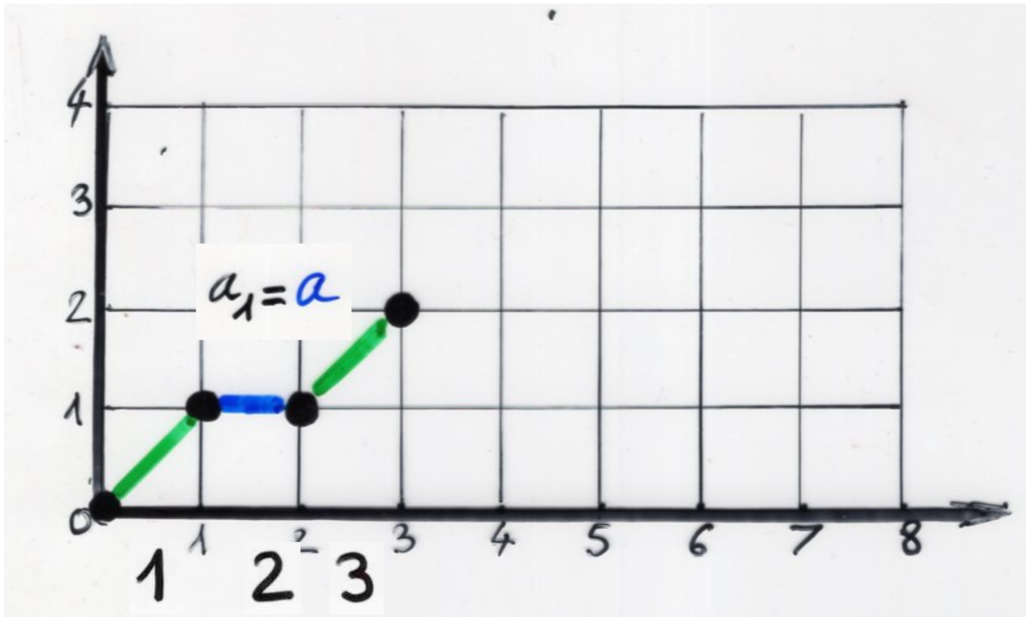

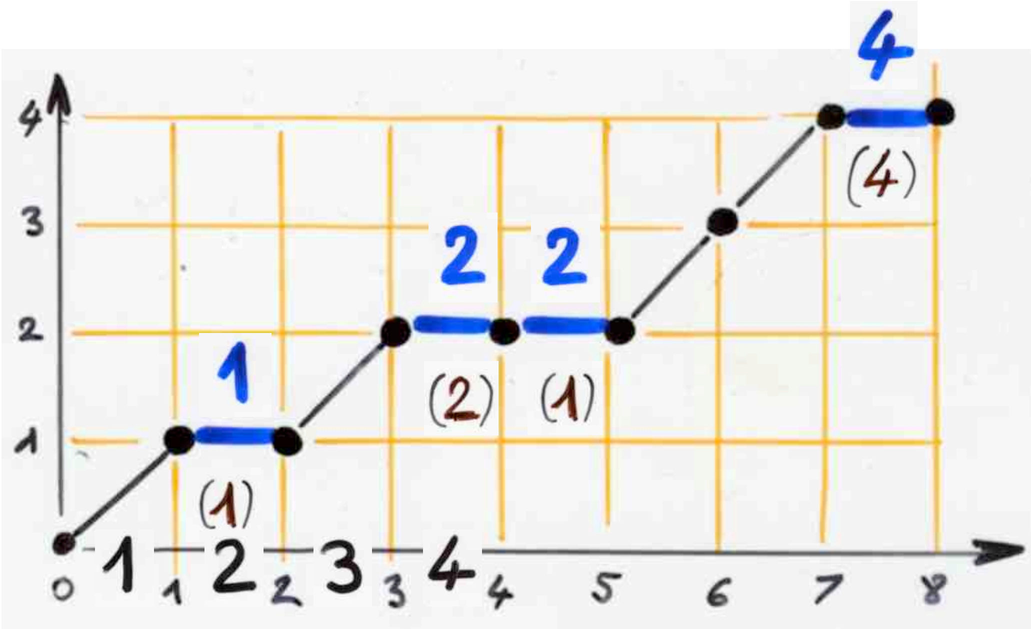

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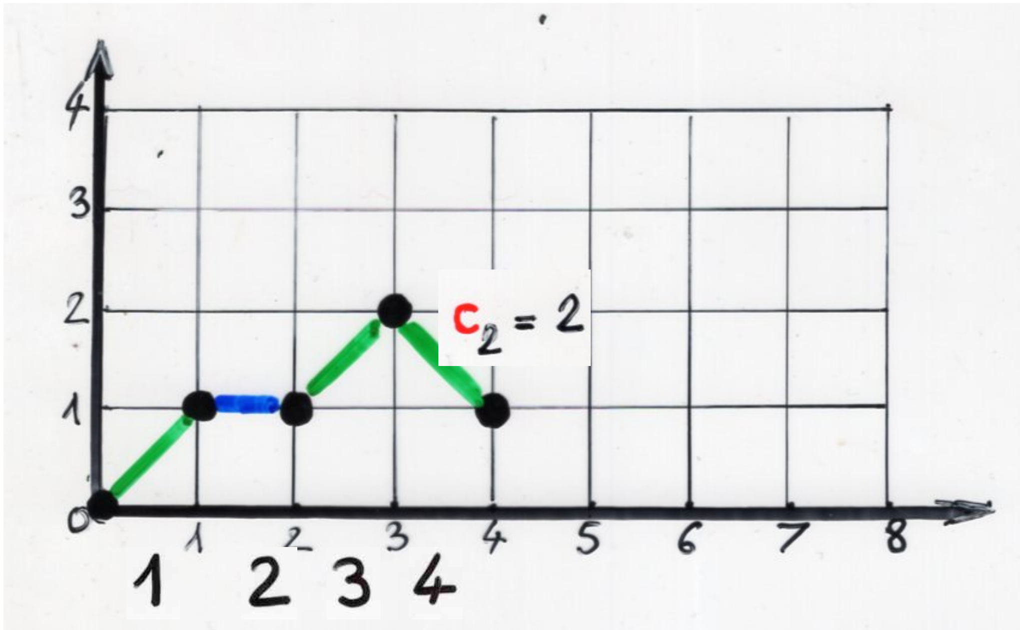

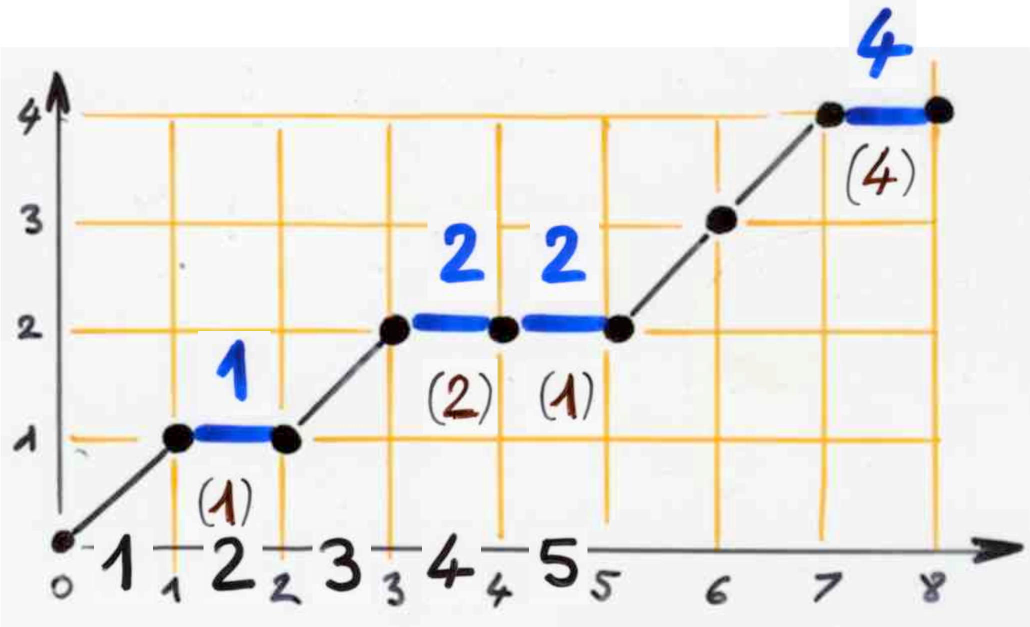

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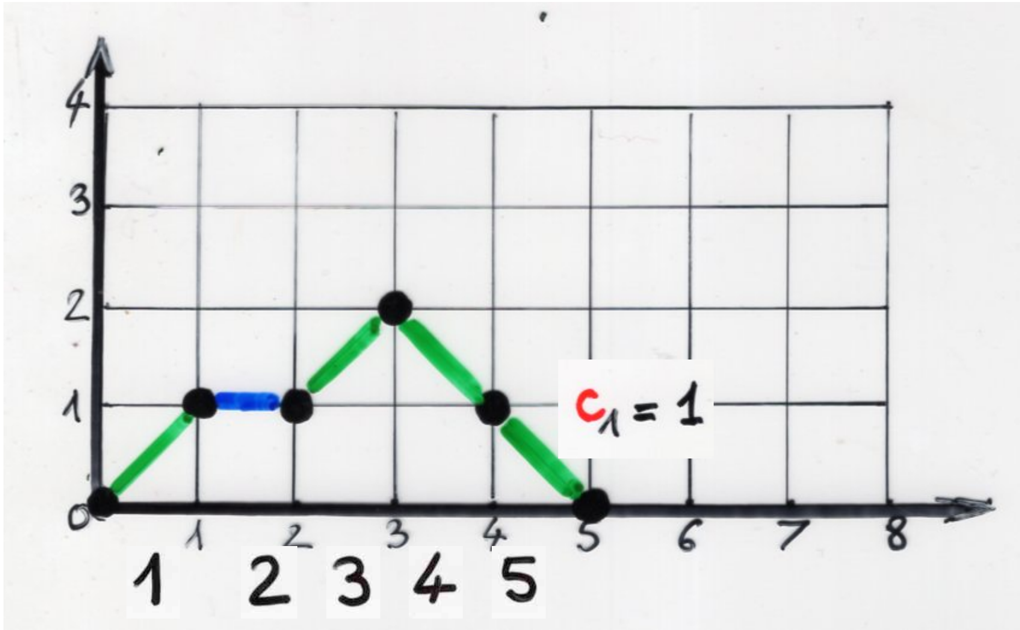

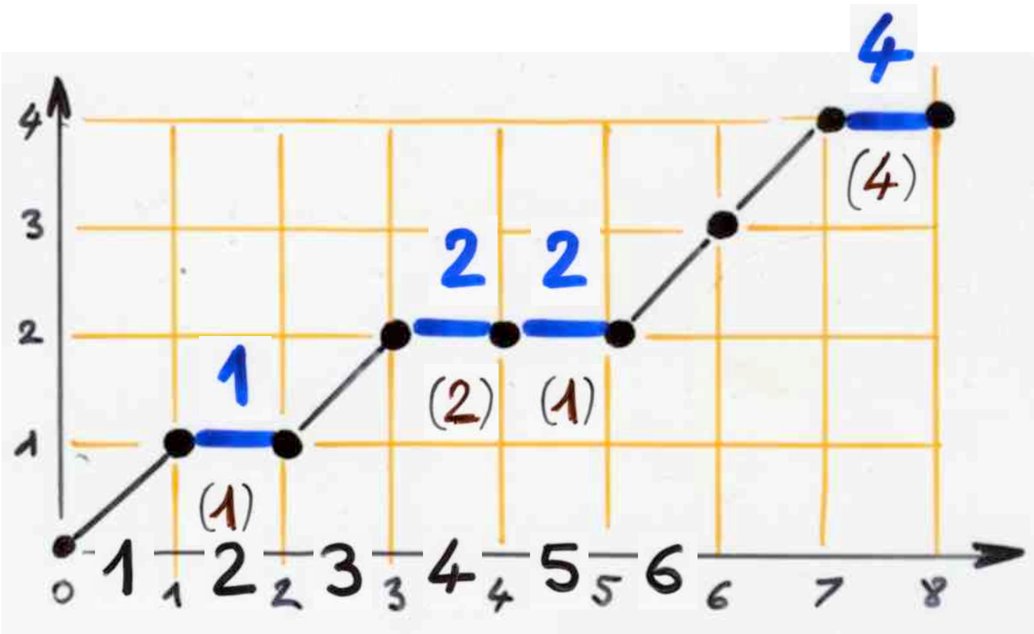

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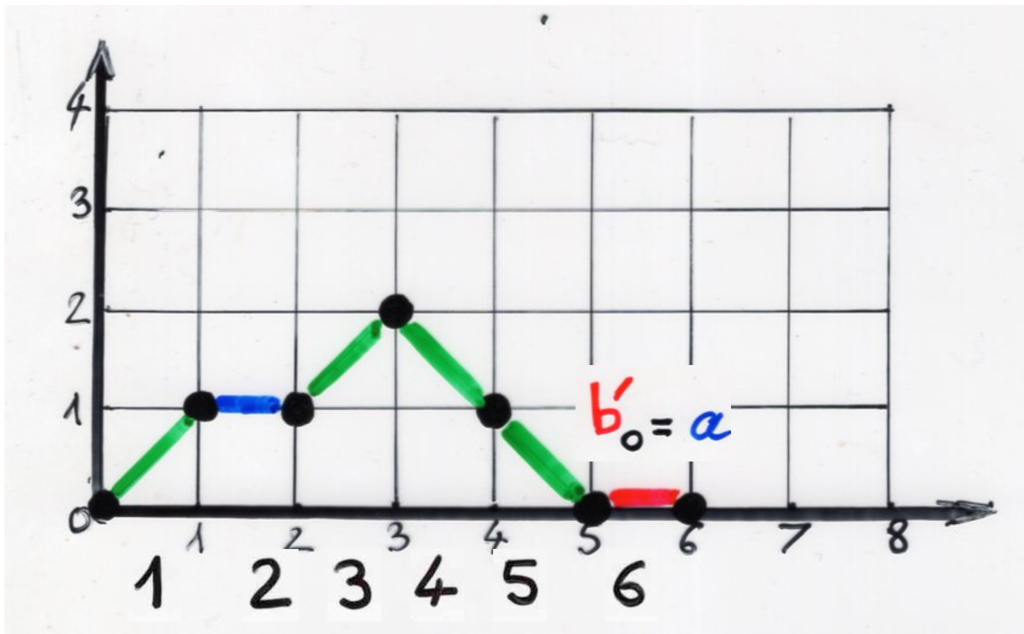

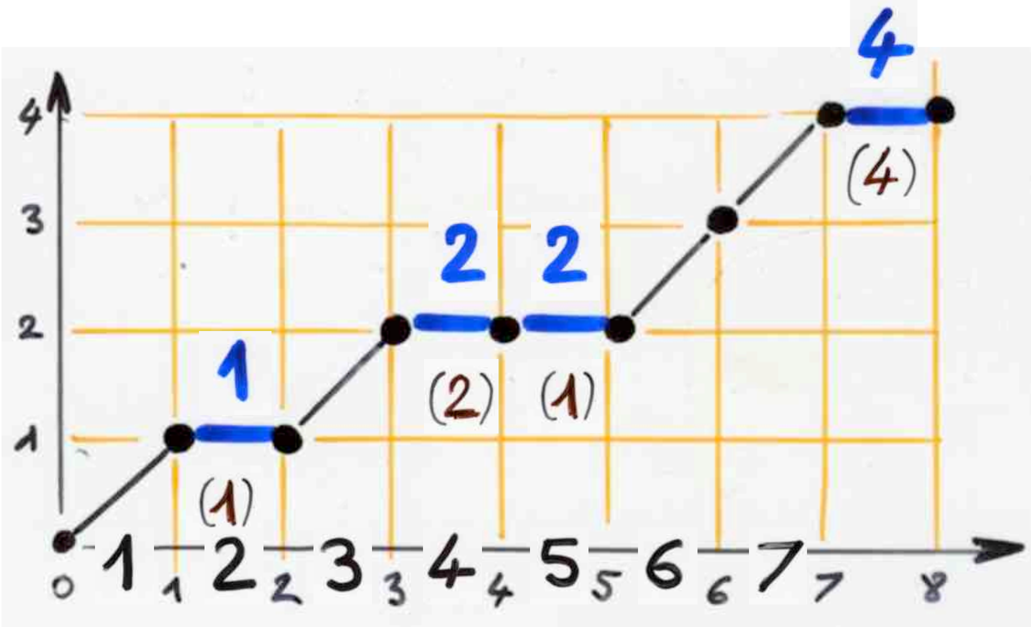

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[7, 8

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[6]

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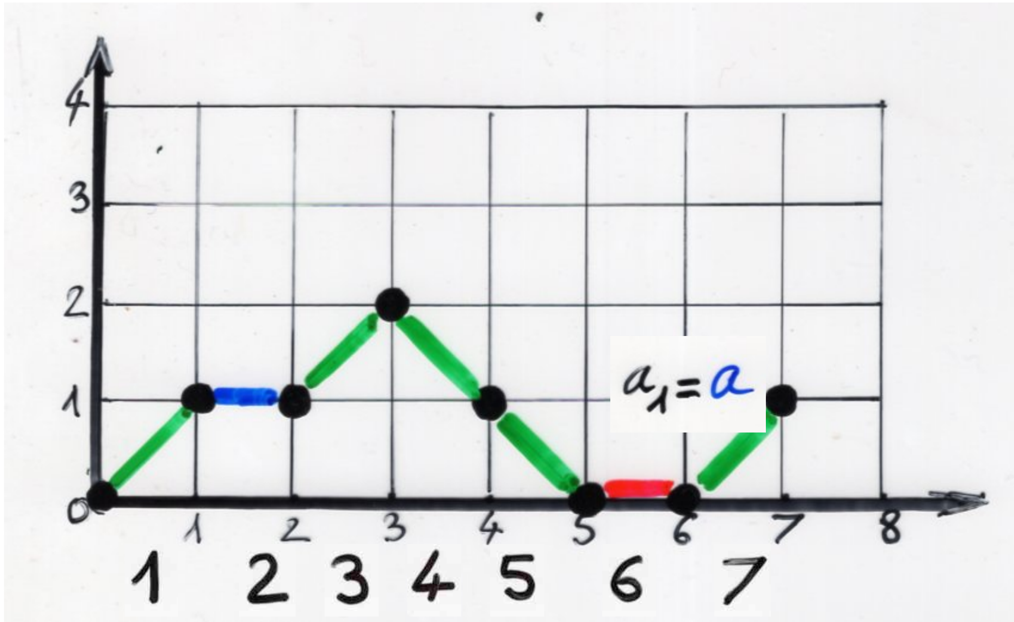

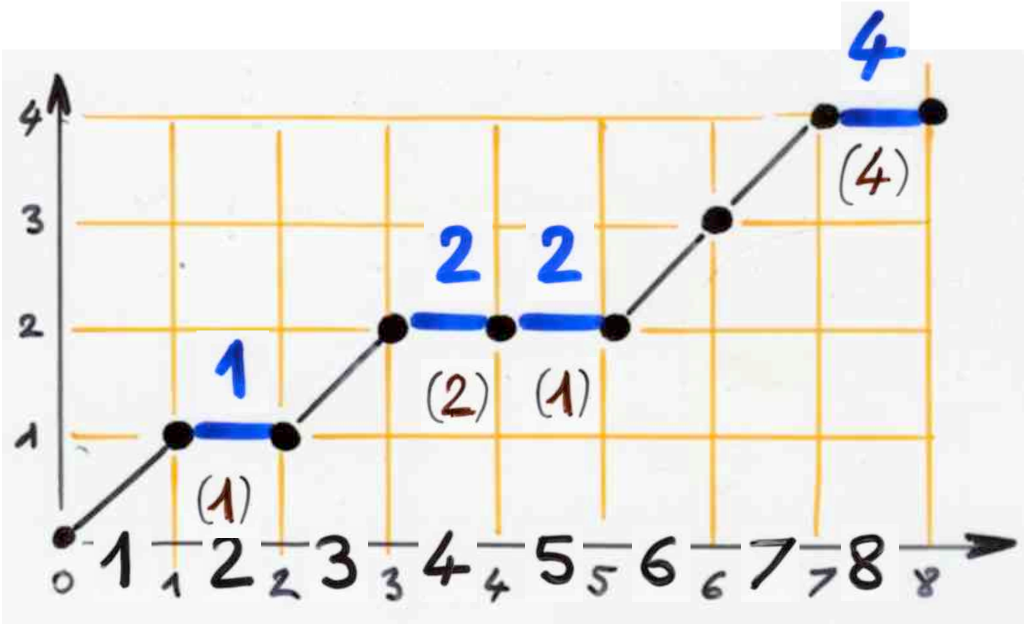

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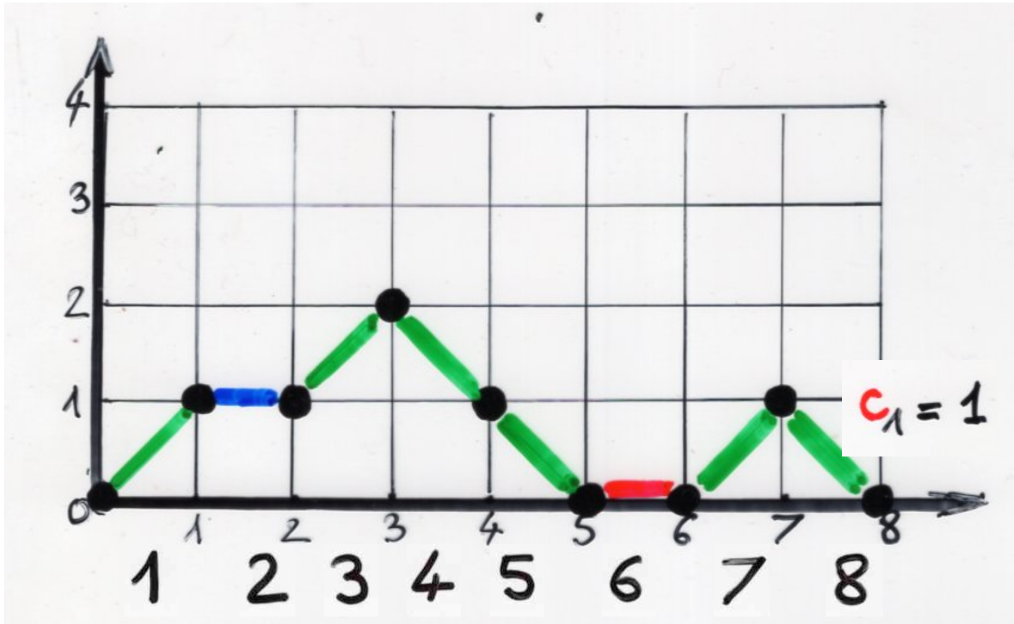
[7, 8]



"open" history

$\lambda_k = 0$ $b_k = k$ $k \geq 0$

Stirling numbers (2nd kind)

"closed" history

Charlier polynomials

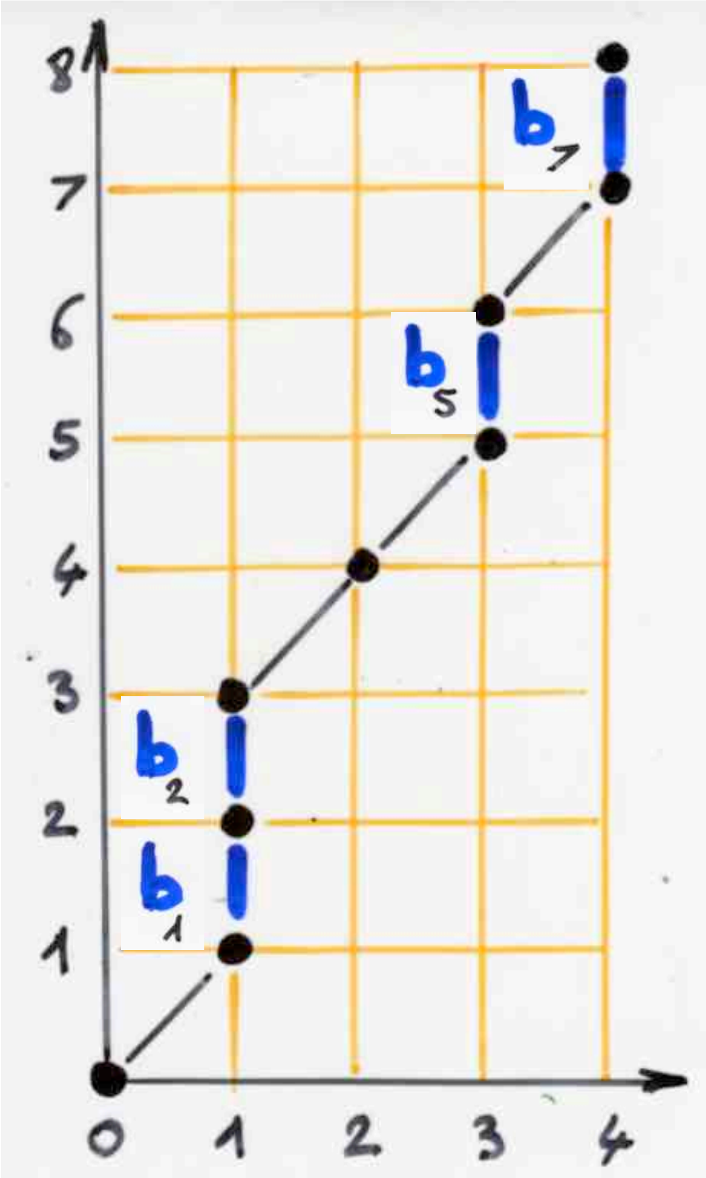
$\begin{cases} a_k = a \\ c_k = k \end{cases}$ $\begin{cases} b'_k = a \\ b''_k = k \end{cases}$ $\begin{cases} \lambda_k = ak \\ b_k = k + a \end{cases}$ $(k \geq 1)$ $(k \geq 0)$

$$\mu_n = \sum_{1 \leq k \leq n} S(n,k)\, a^k$$

moments

Closure of the (open) history for permutations

(Ch1d)

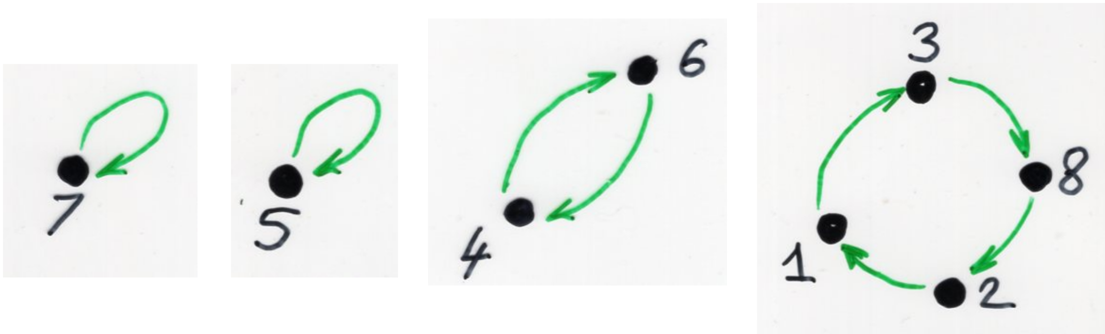


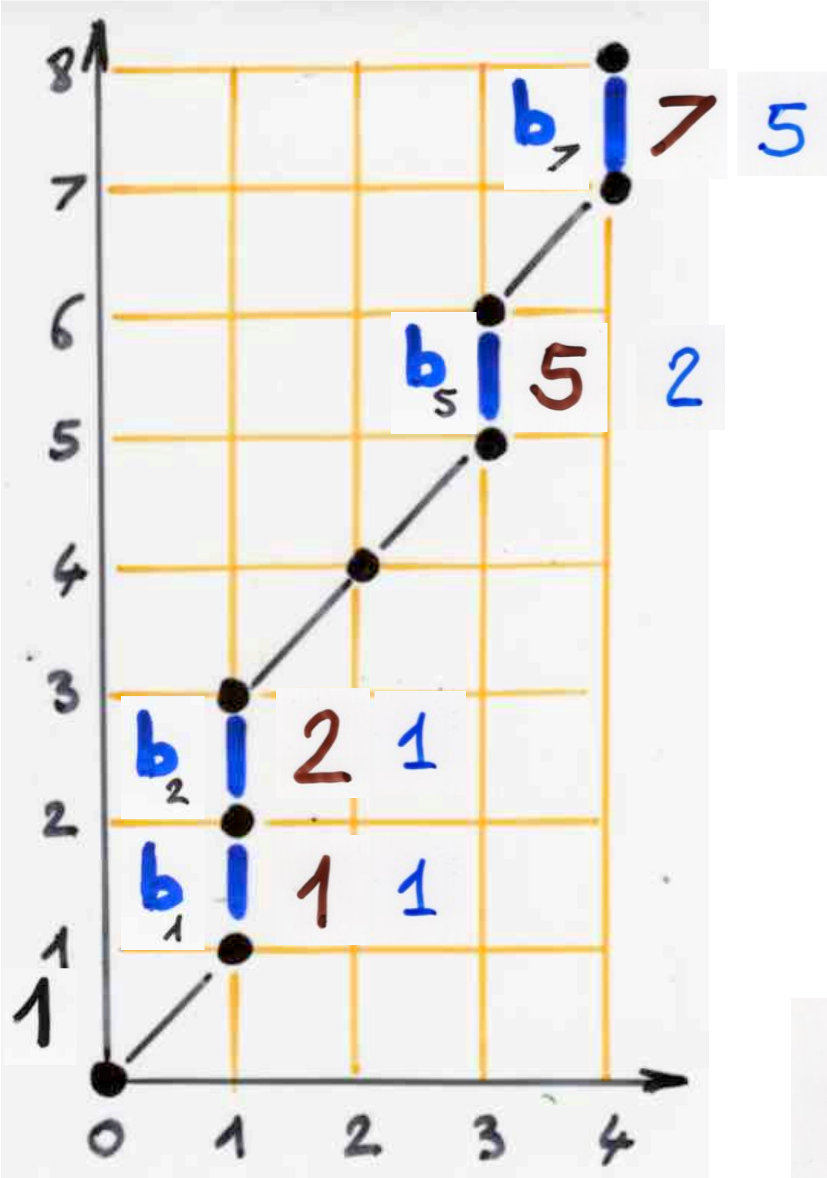
$\lambda_k = 0$ $b_k = k$ $k \geq 0$

$$P_{n,i} = (-1)^i s_{n,i}$$

Stirling numbers 1st kind

number of permutations of $\{1, \ldots, n\}$ having i cycles

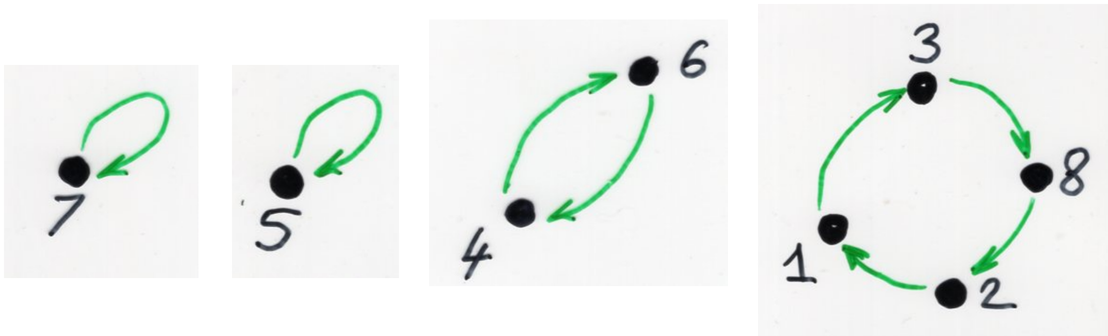




b1 1 1
b2 2 1
b5 5 2
b7 7 5
1

1
1

1



7
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4
6
3
8
2
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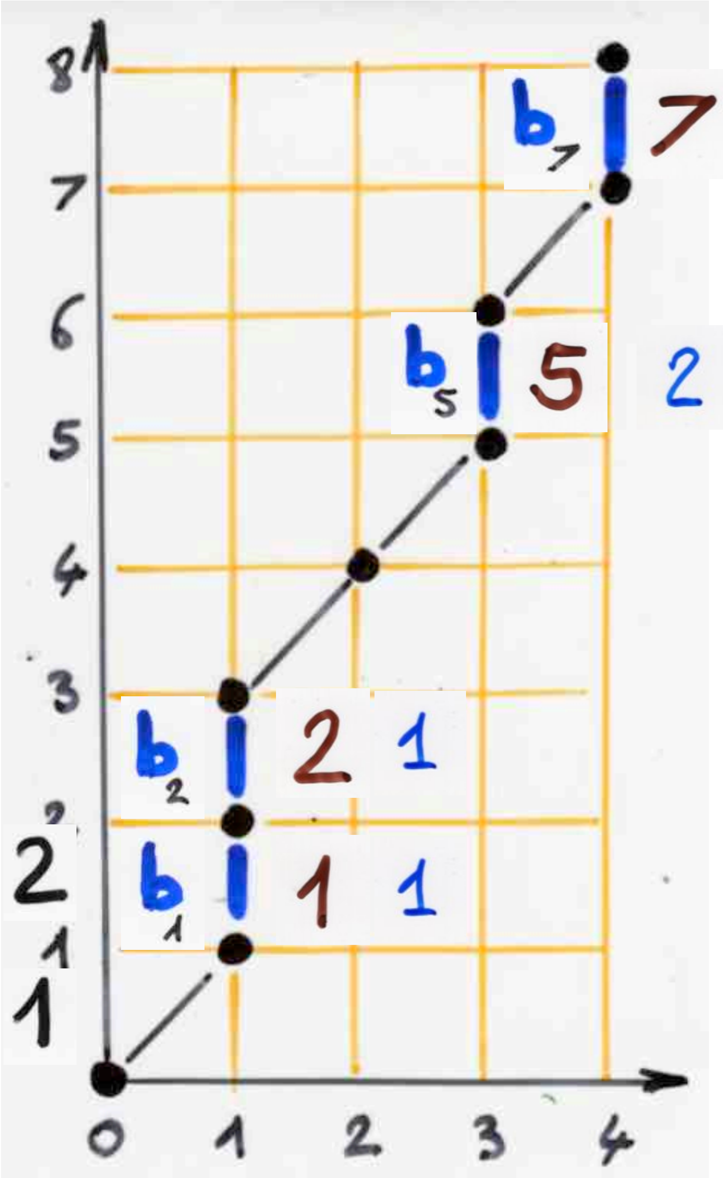

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1
0 1 2 3 4

2
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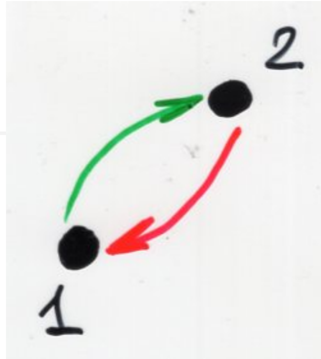
0
1
1
1



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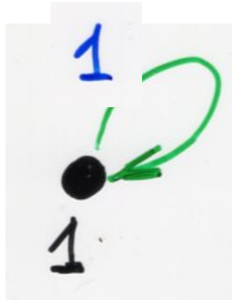

1
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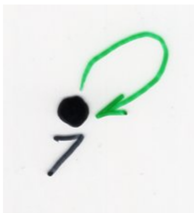

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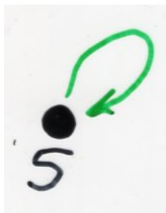

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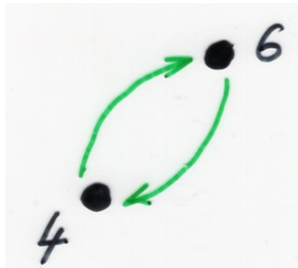

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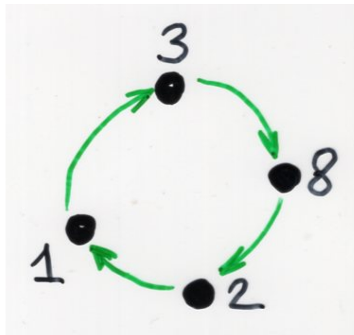

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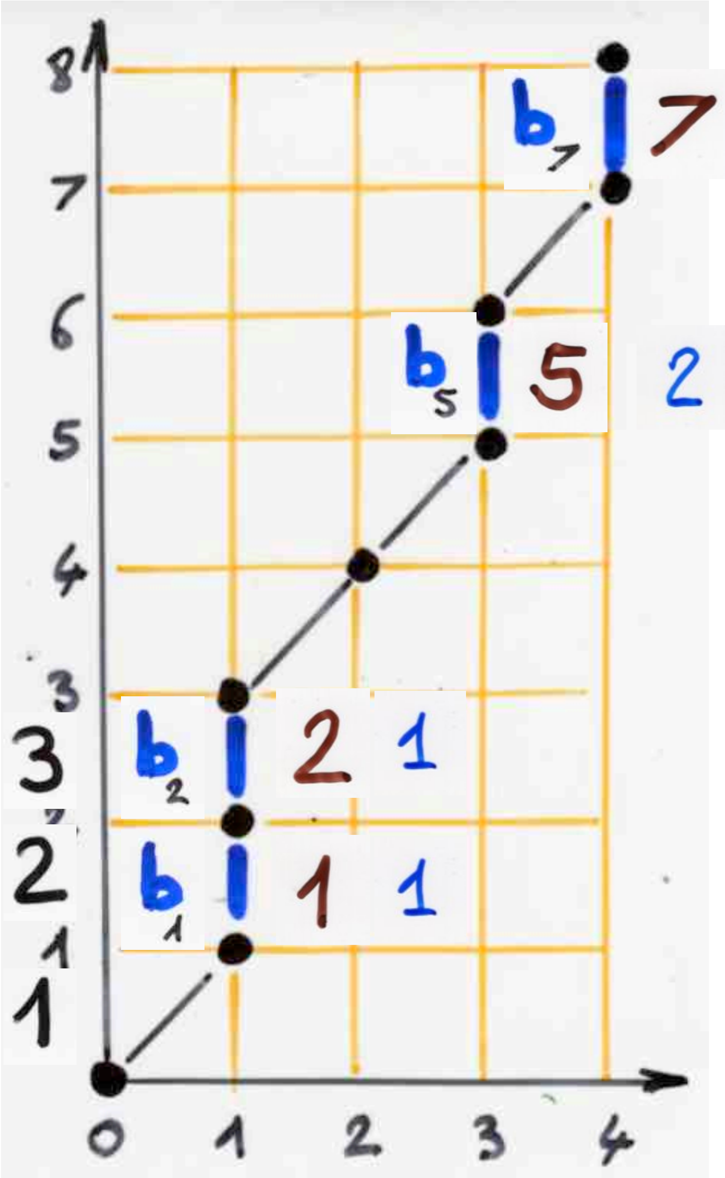
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b7 7
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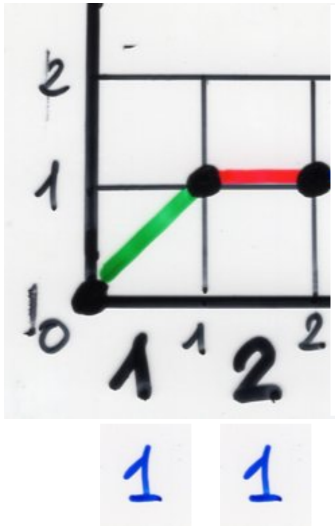
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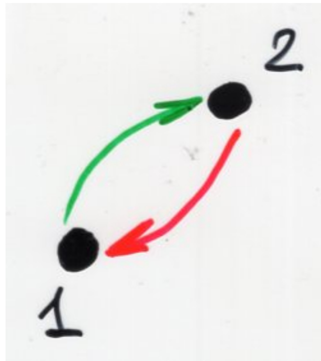
1 1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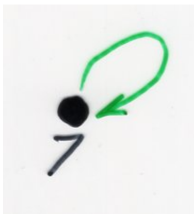
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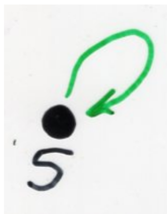
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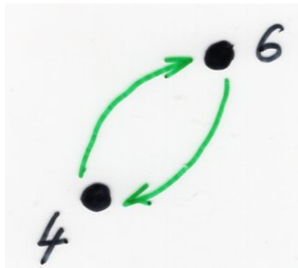
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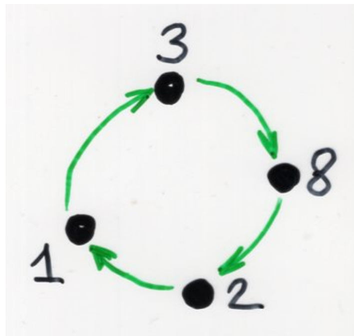
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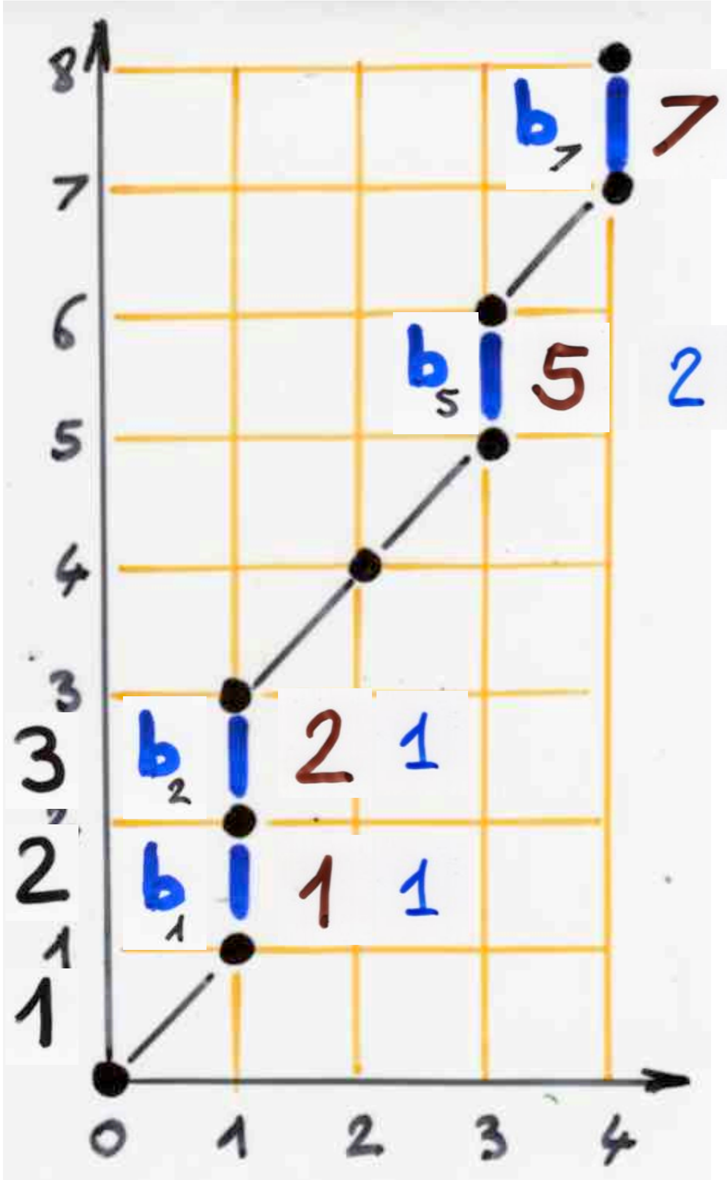
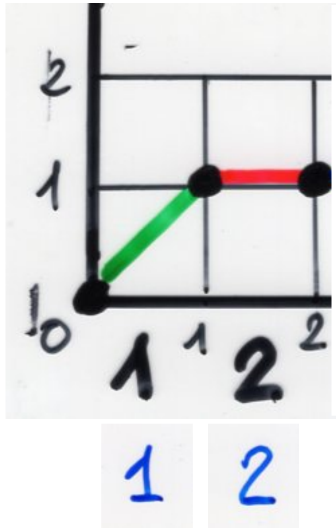
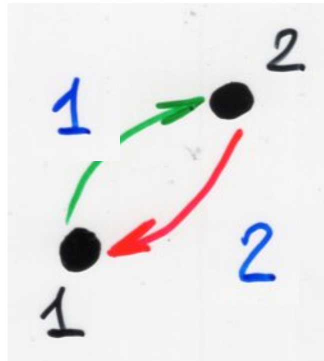
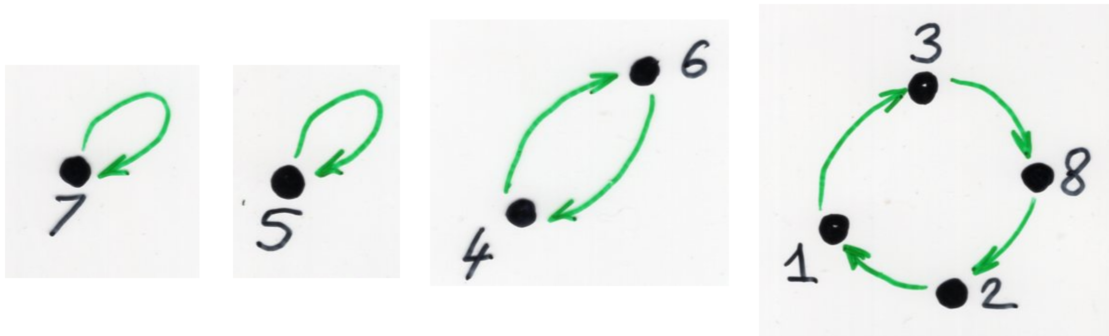
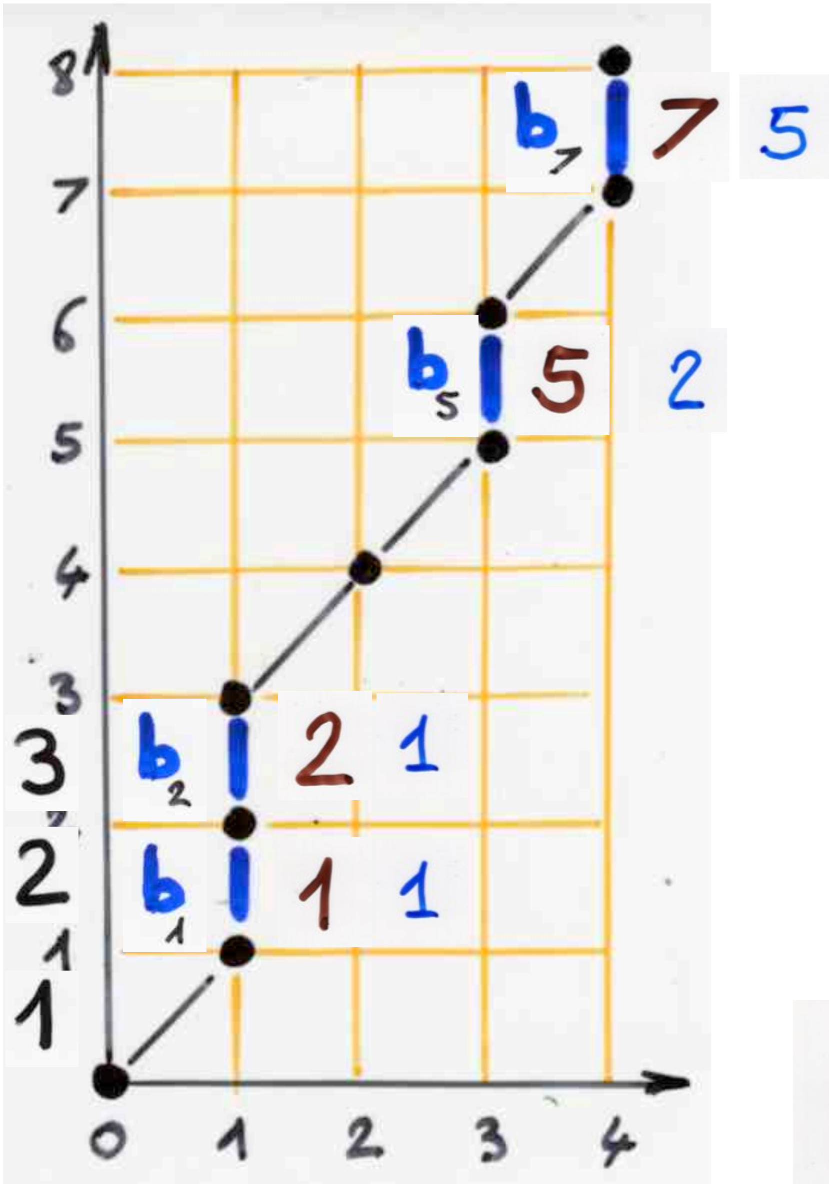


b1
b2
b5
b7

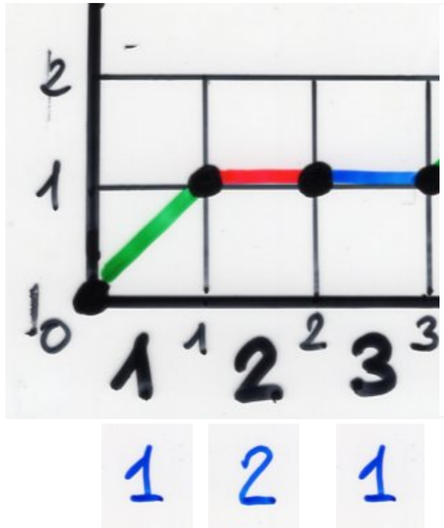




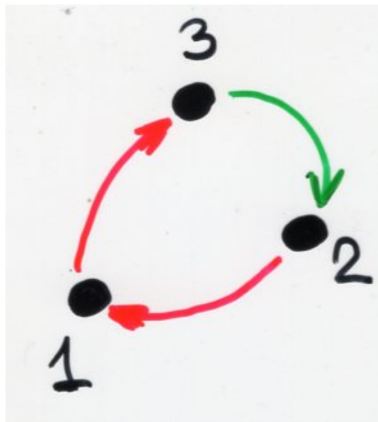


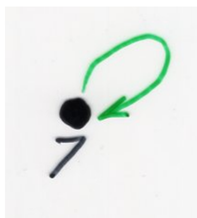


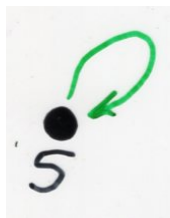
b_7 7 5
b_5 5 2
3 b_2 2 1
2 b_1 1 1
1

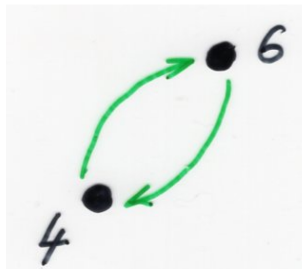


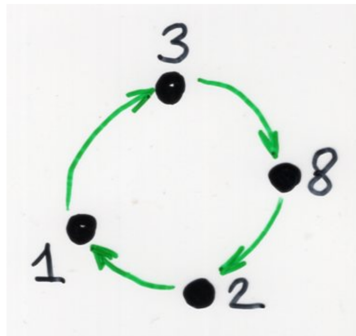
1 2 3
1 2 1

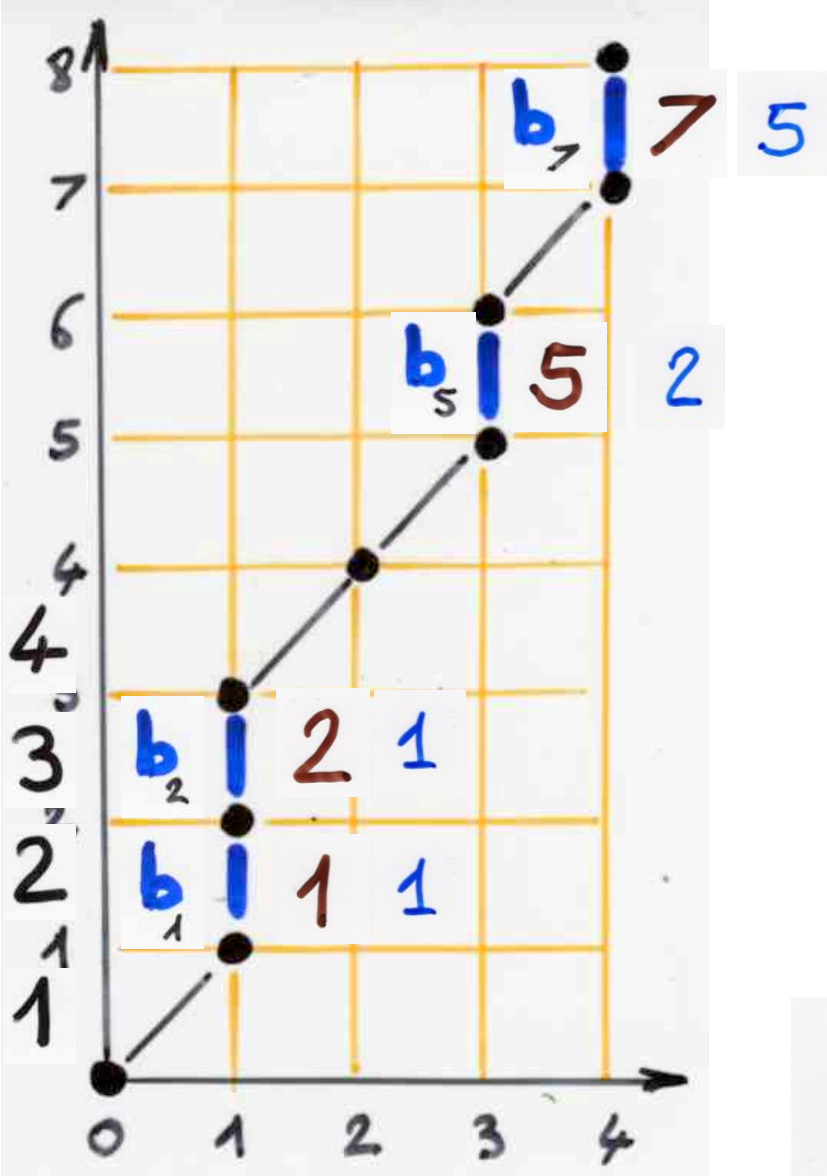




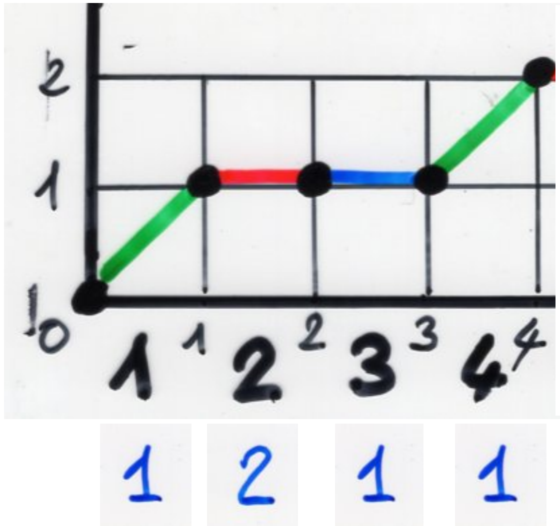




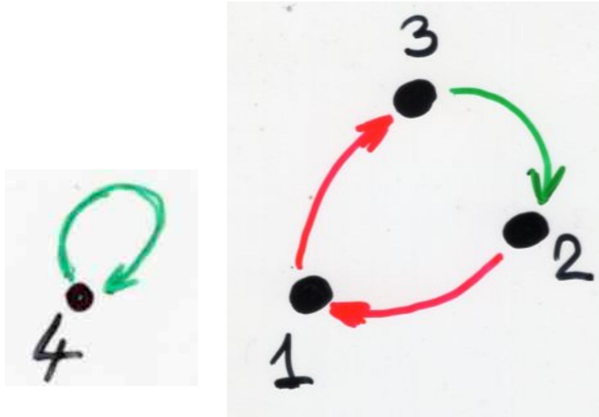


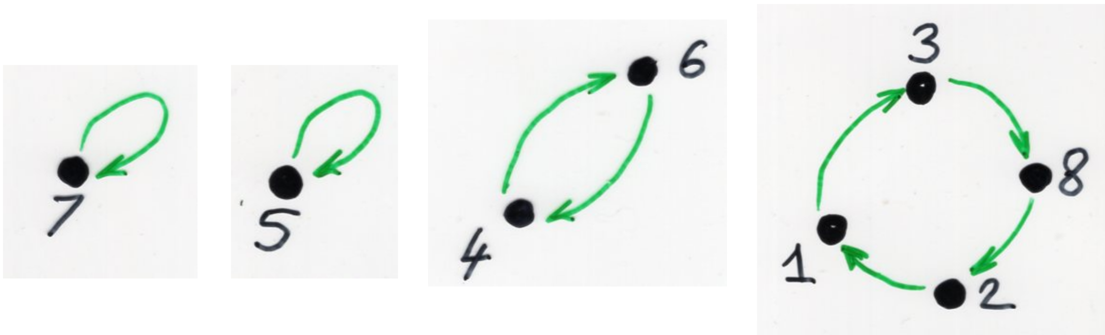


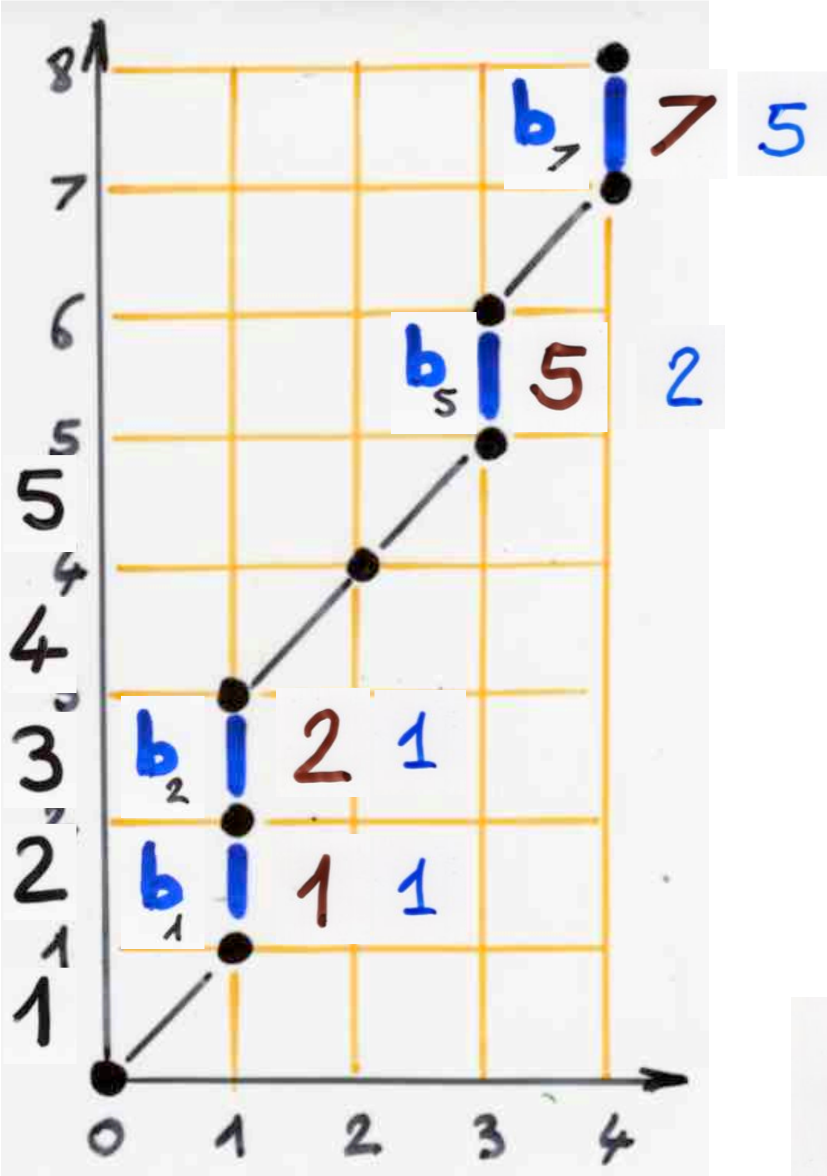
b1 1 1
b2 2 1
b5 5 2
b7 7 5



1 2 3 4
1 2 1 1







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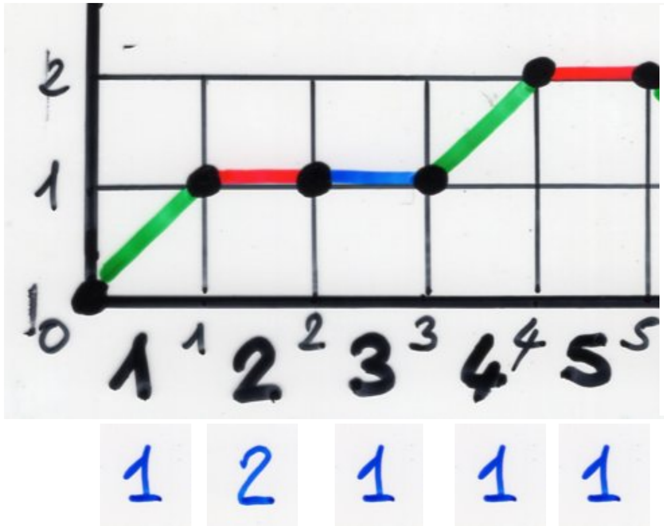
b5 5 2
b7 7 5



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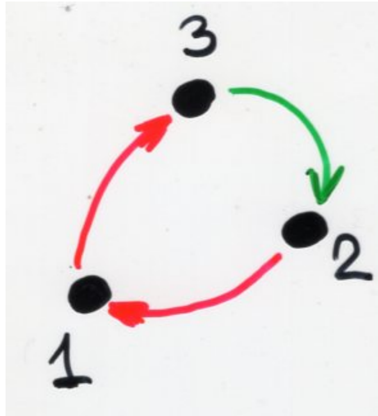
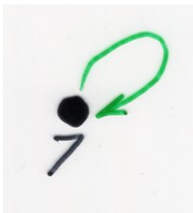
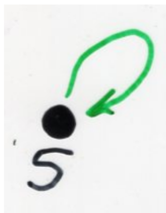
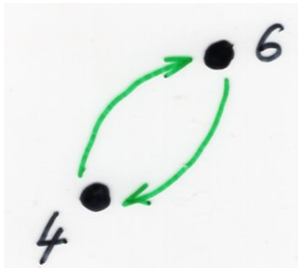
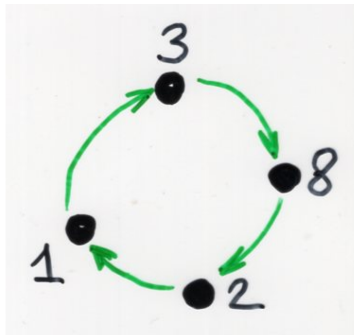
1 2 1 1 1

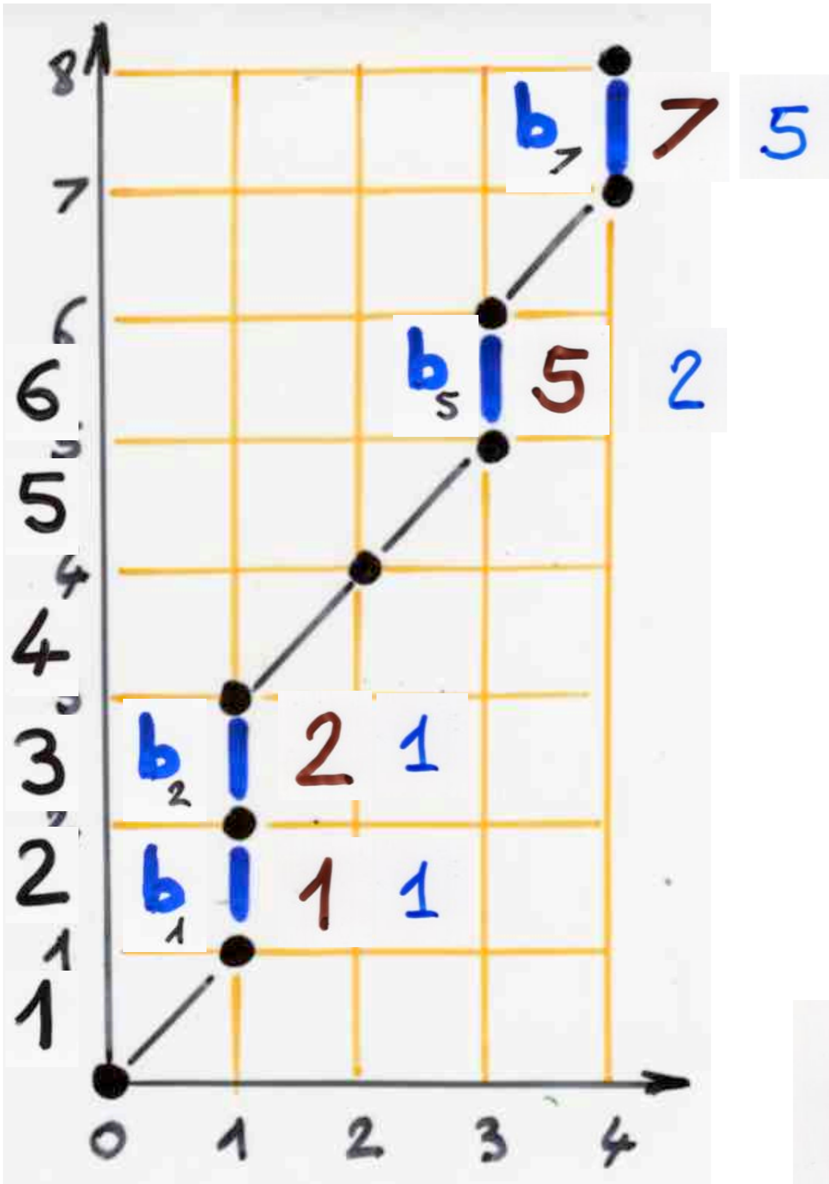












b1 1 1
b2 2 1
b5 5 2
b7 7 5
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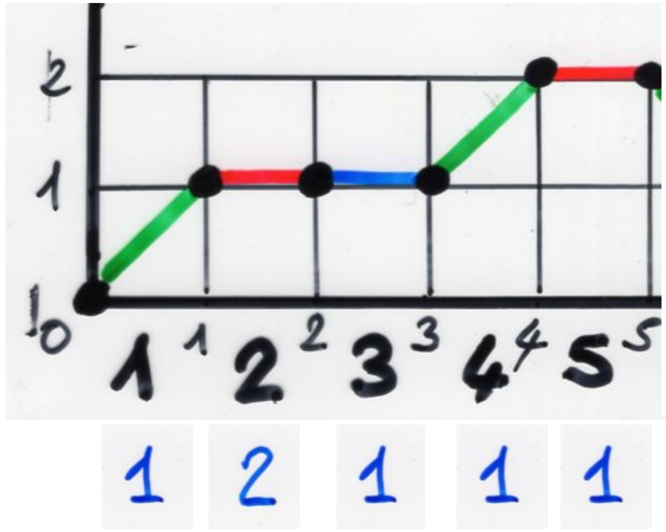
1 2 3 4 5 6



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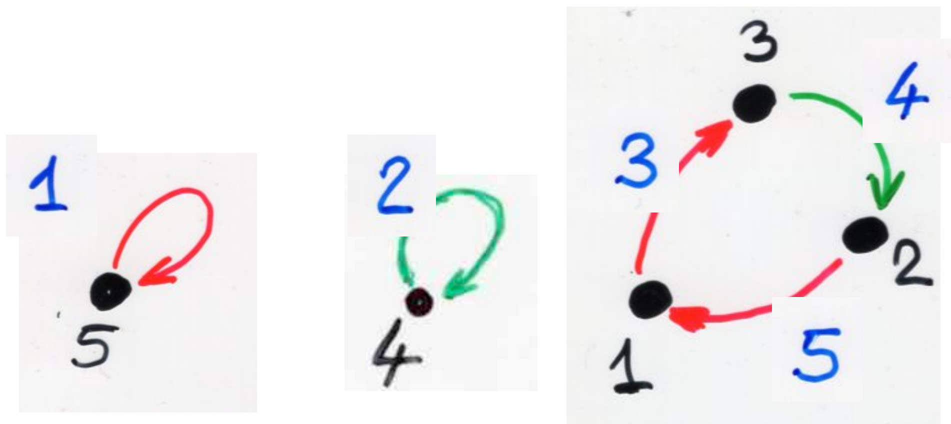

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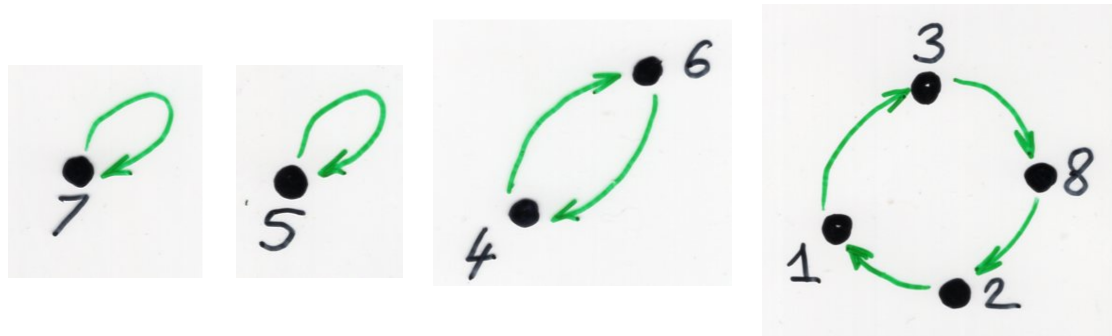

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3
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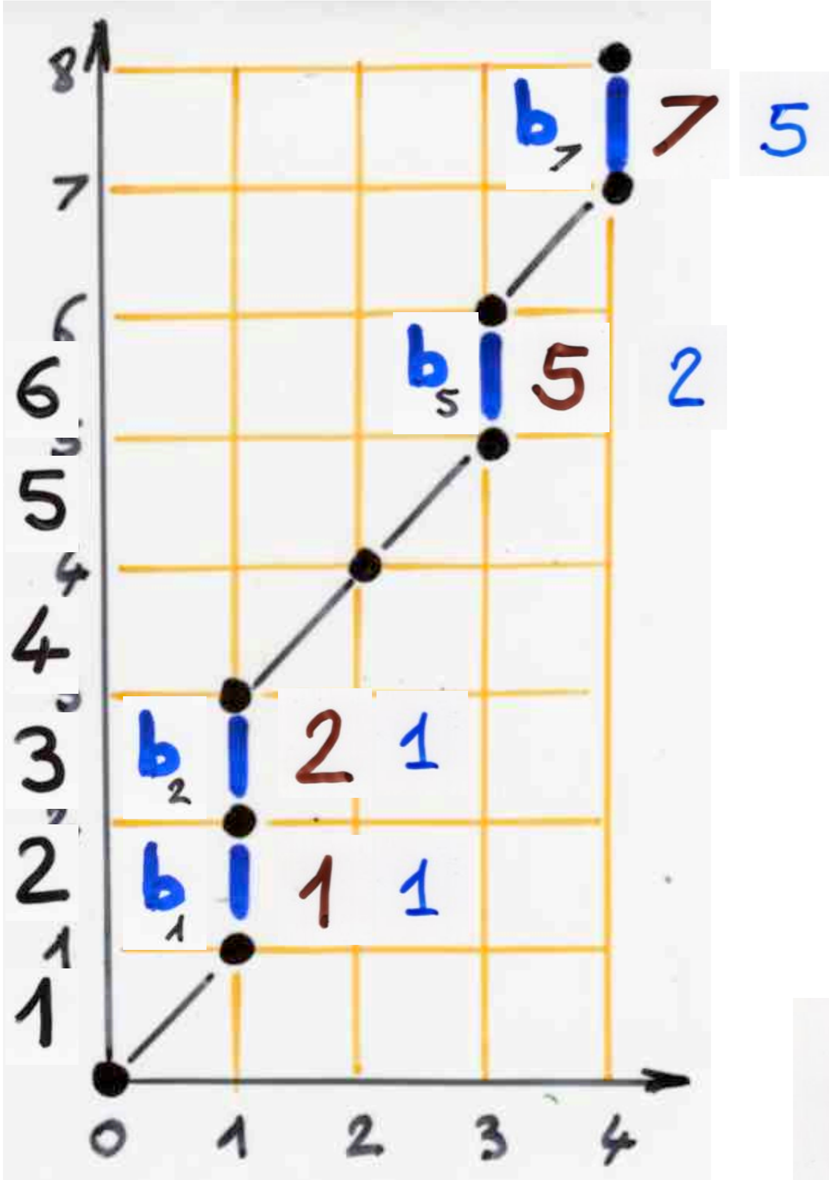

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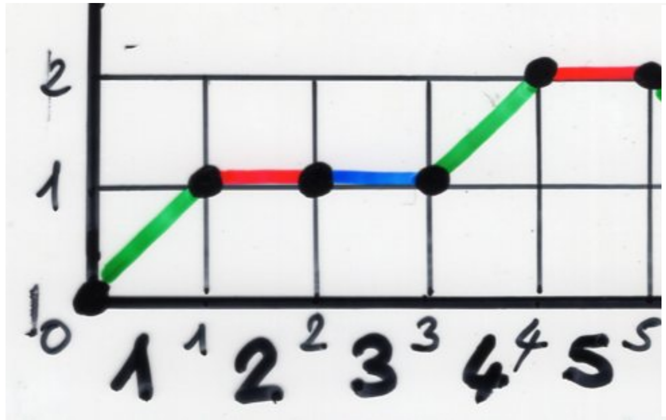
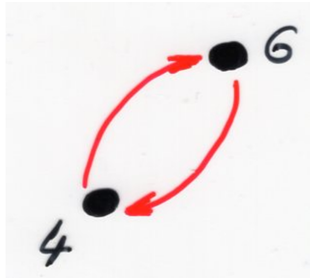
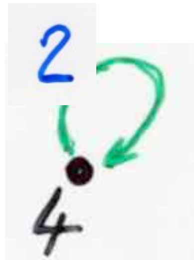
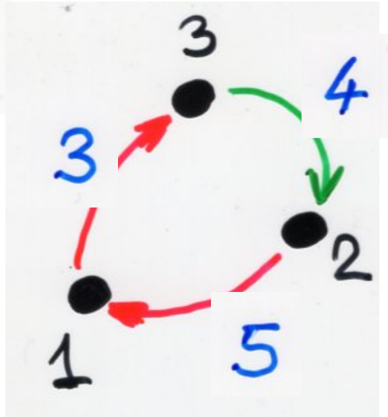
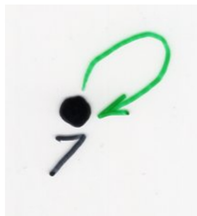
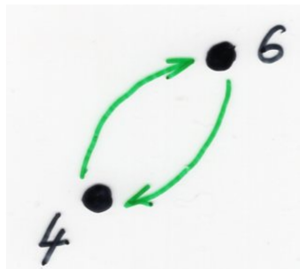


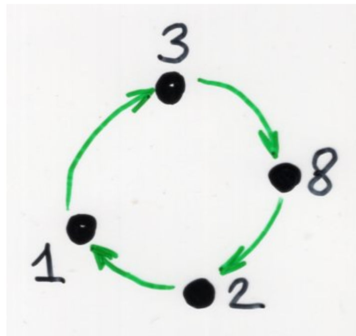


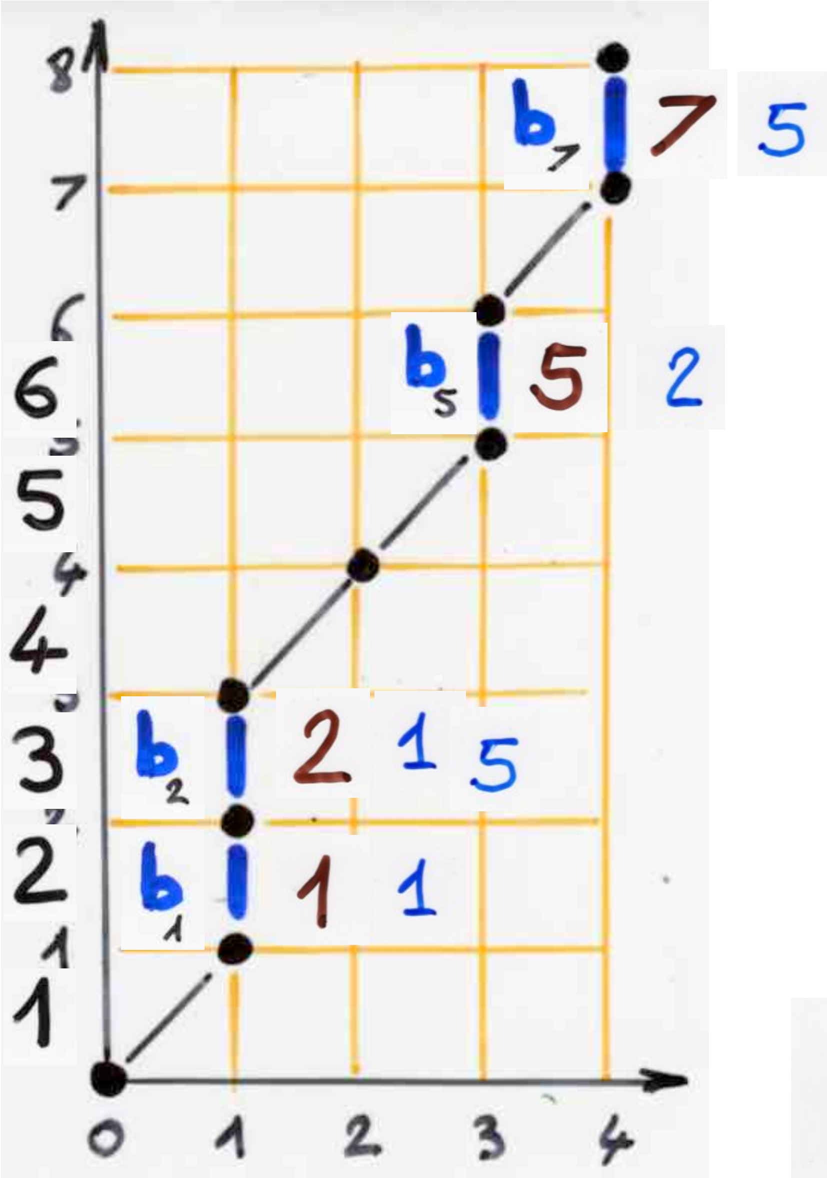




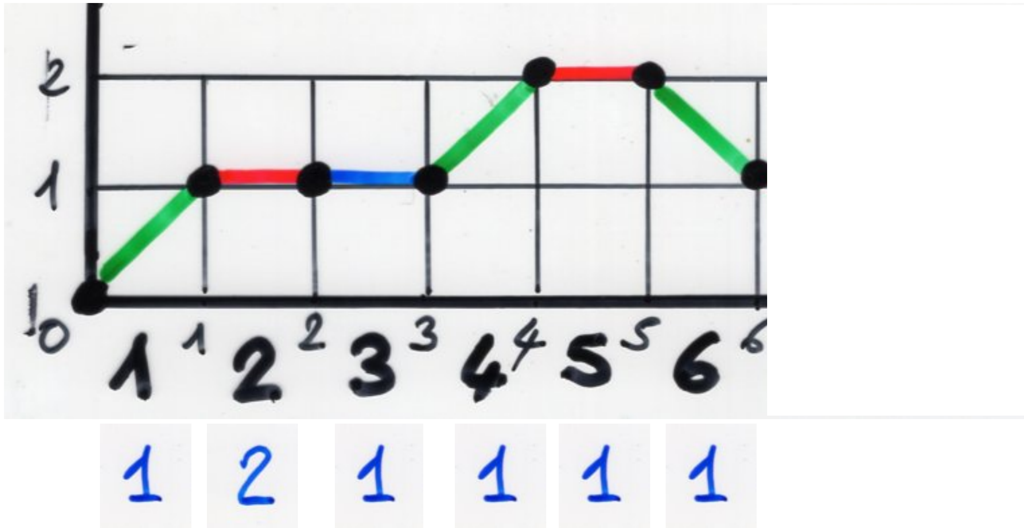








b₇ 7 5
b₅ 5 2
b₂ 2 1 5
b₁ 1 1



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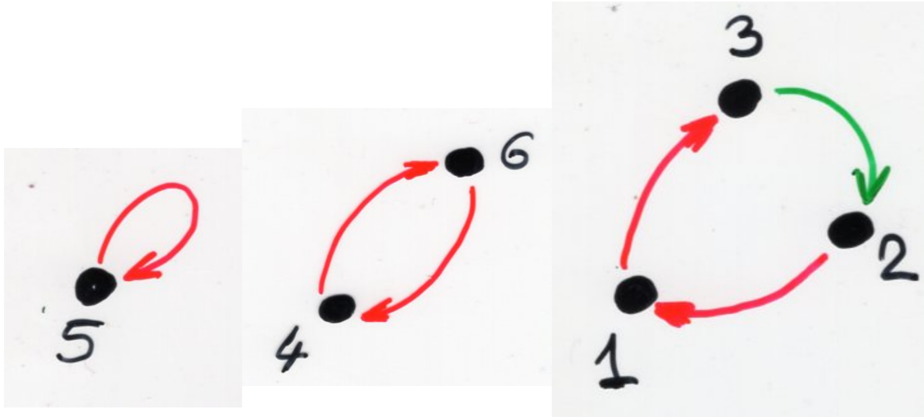

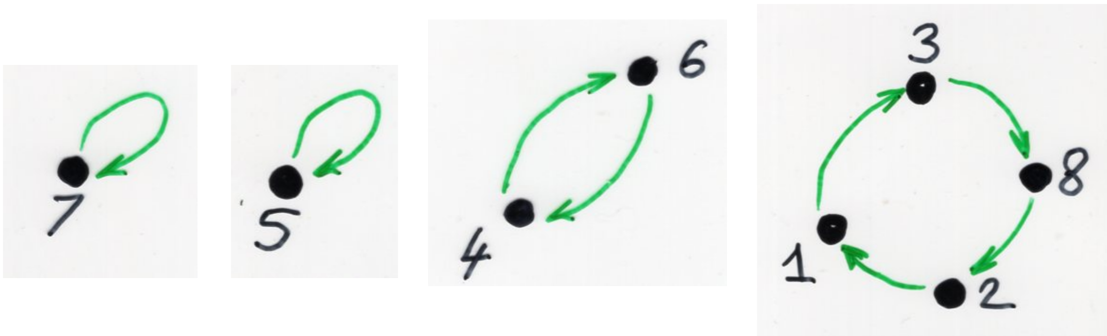

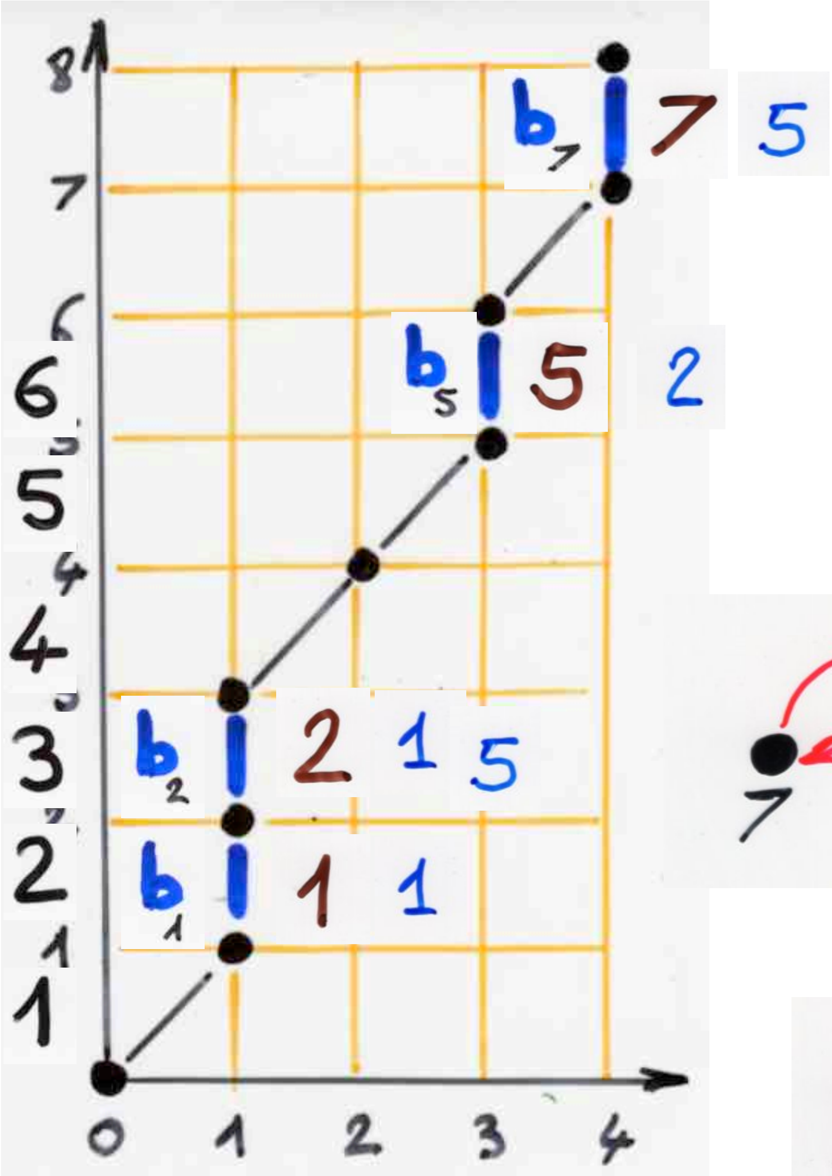

b_1 1 1
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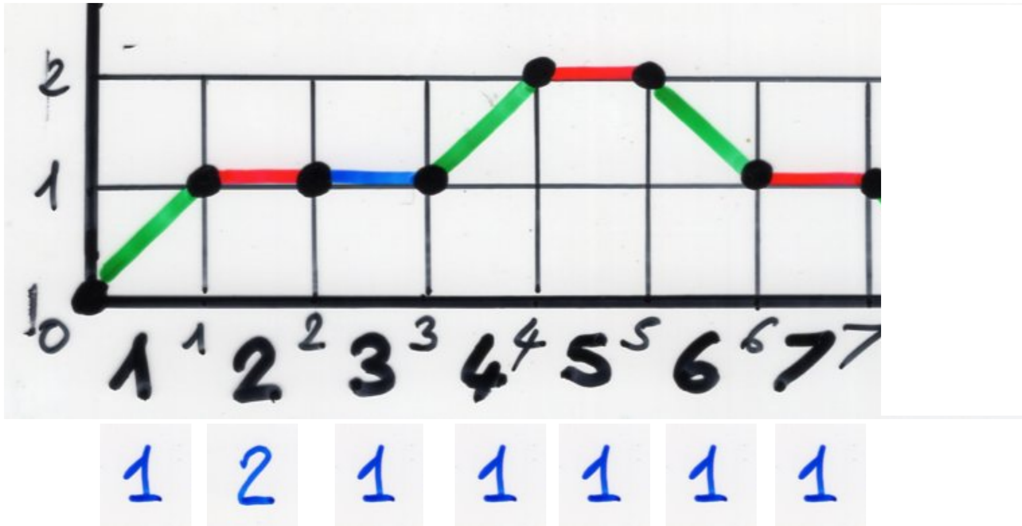

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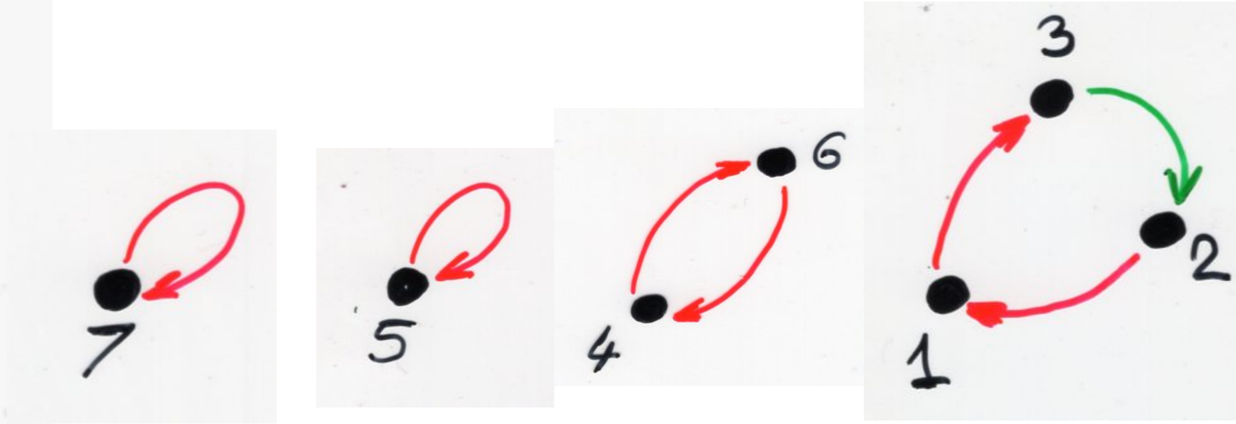
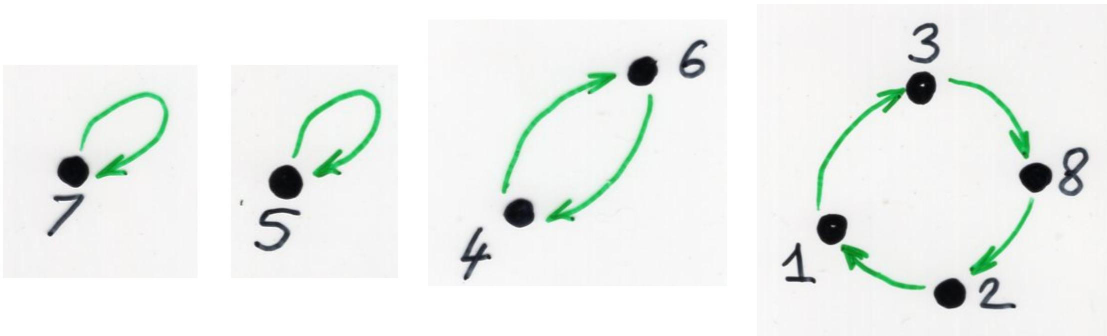
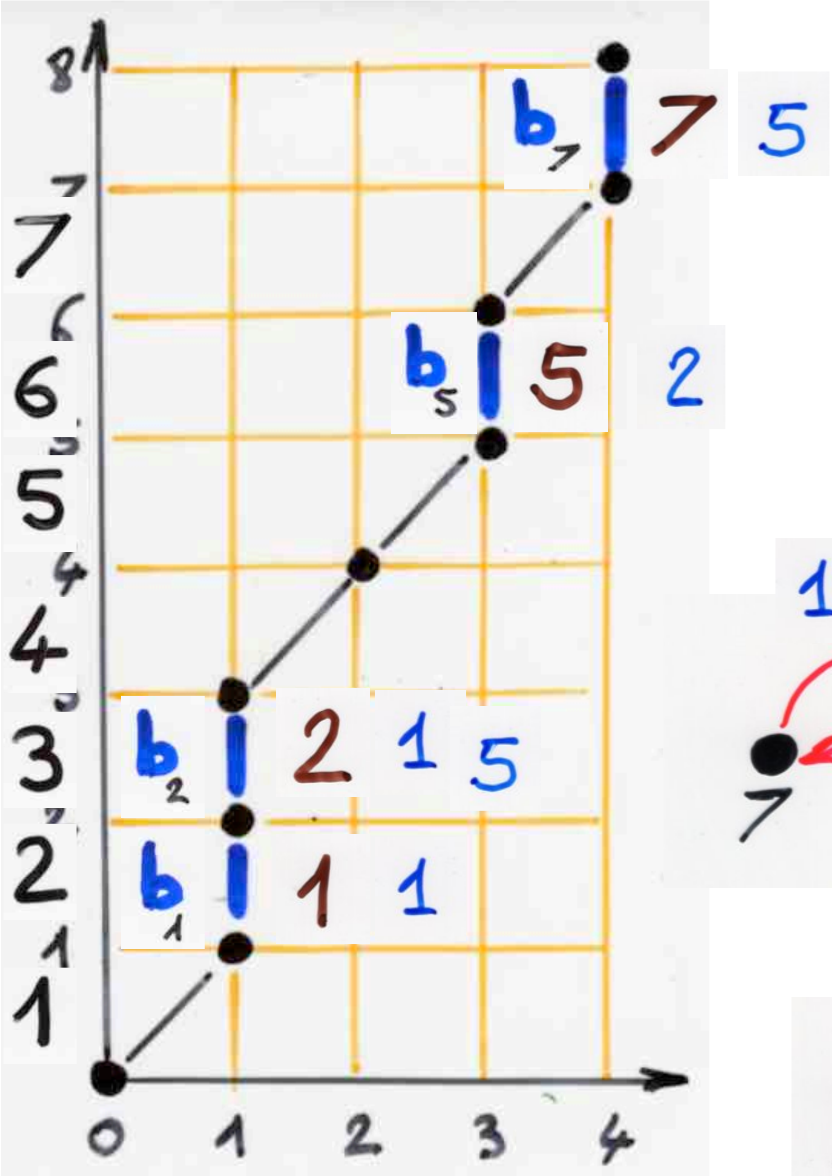
1 2 1 1 1 1 1



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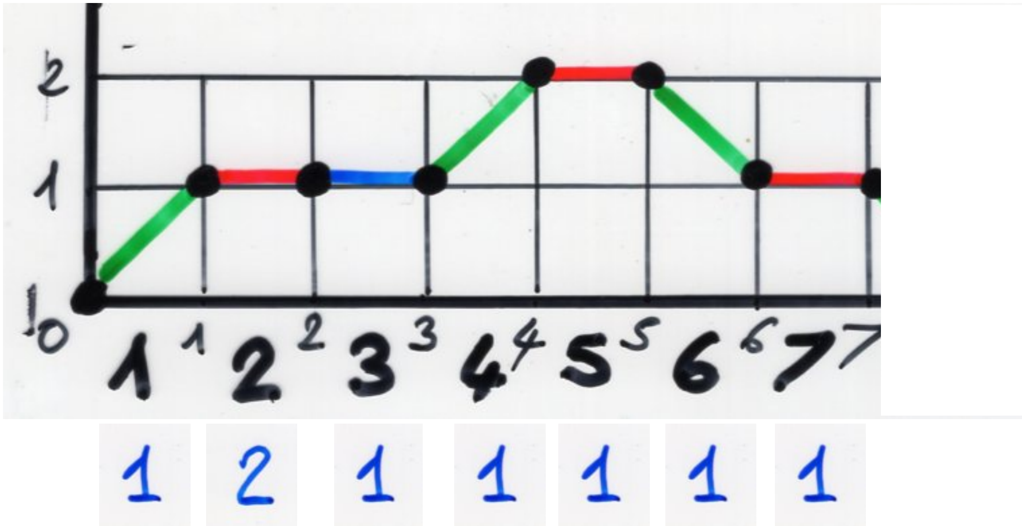
b5 5 2
b7 7 5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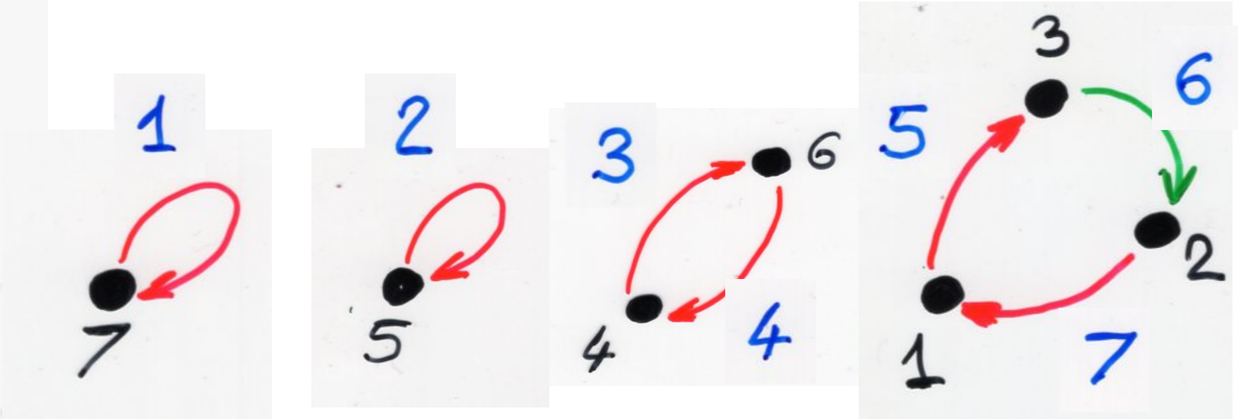
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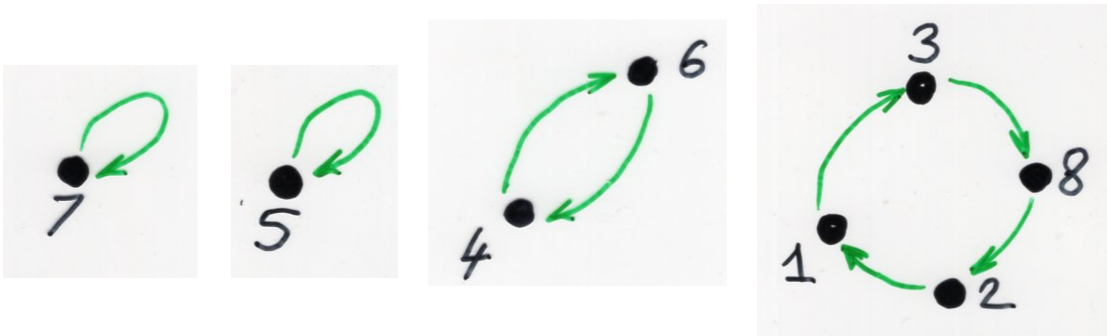

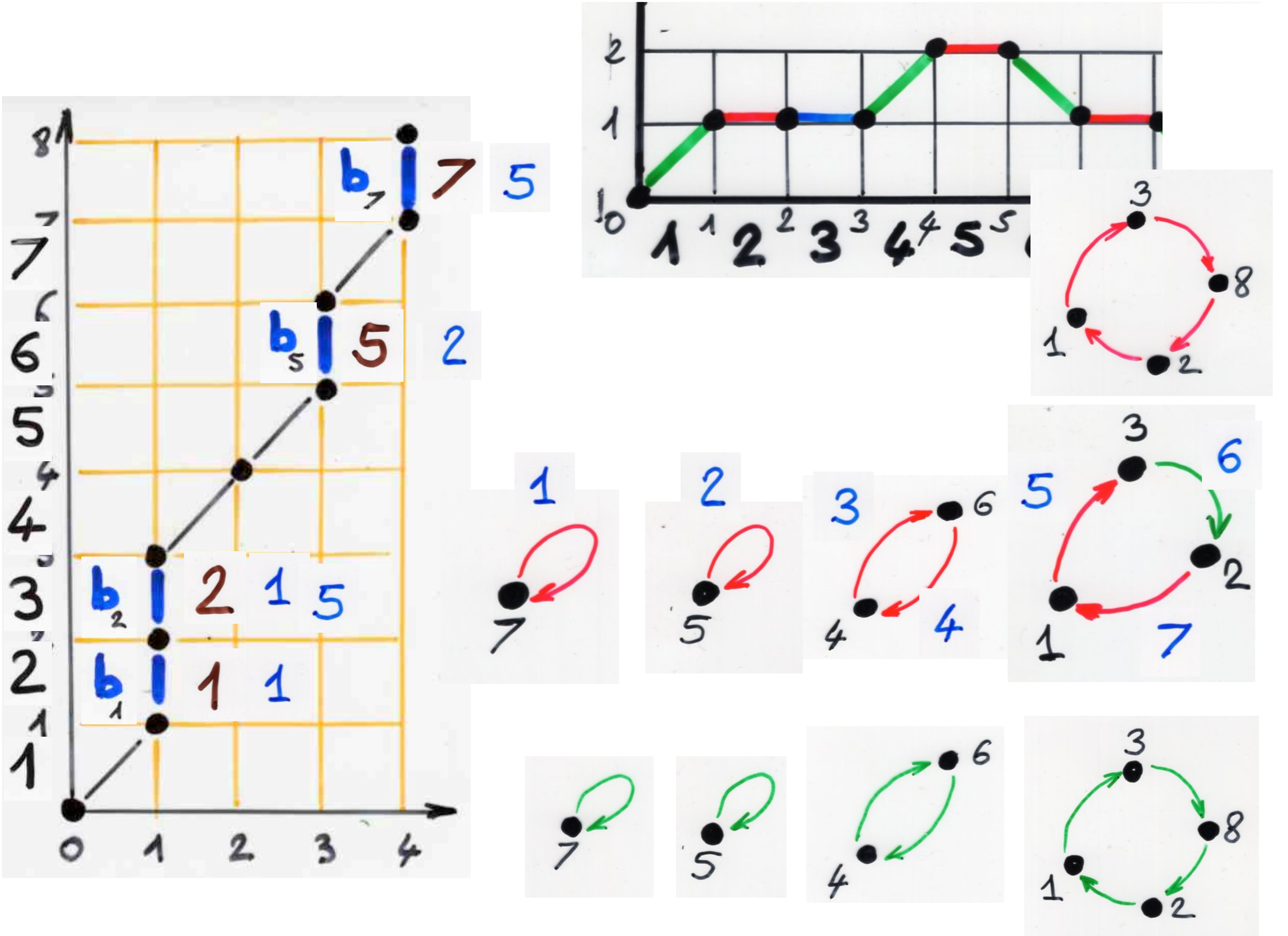

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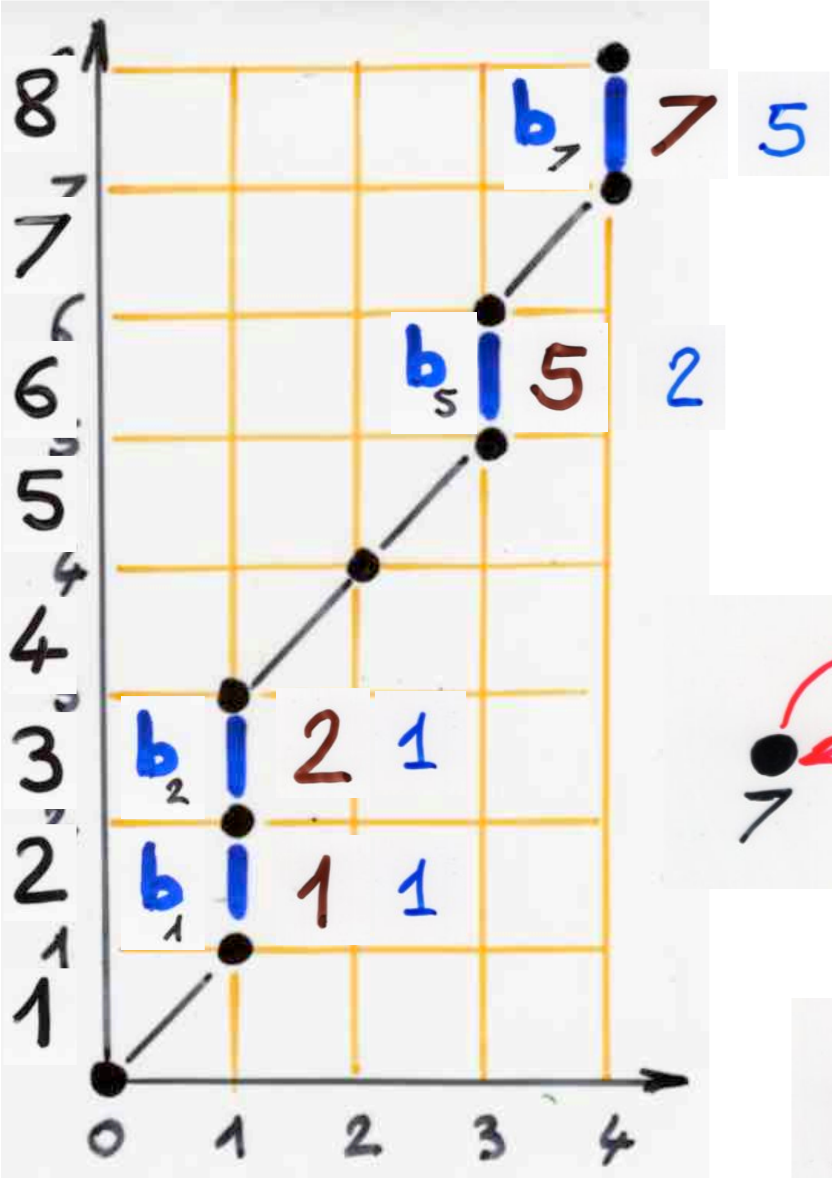

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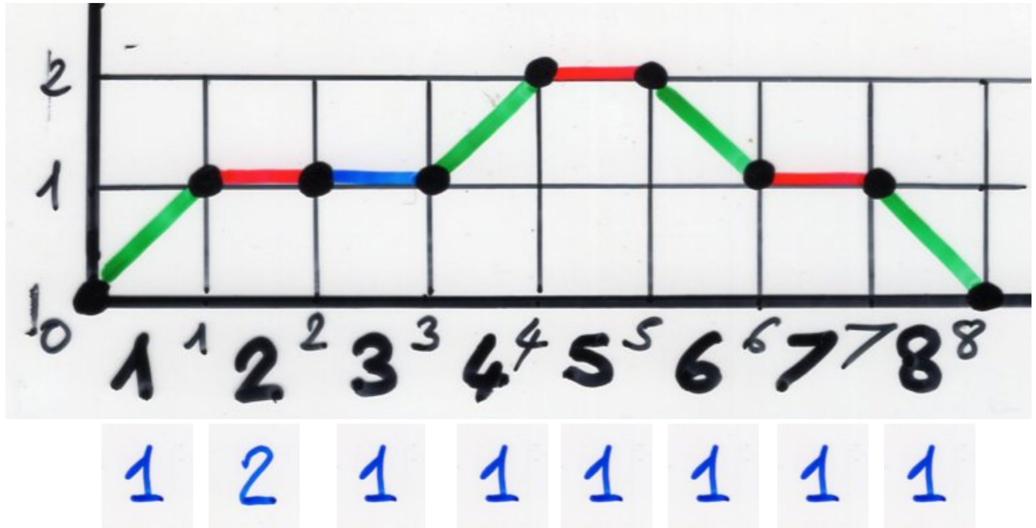
b2 2 1
b5 5 2
b7 7 5



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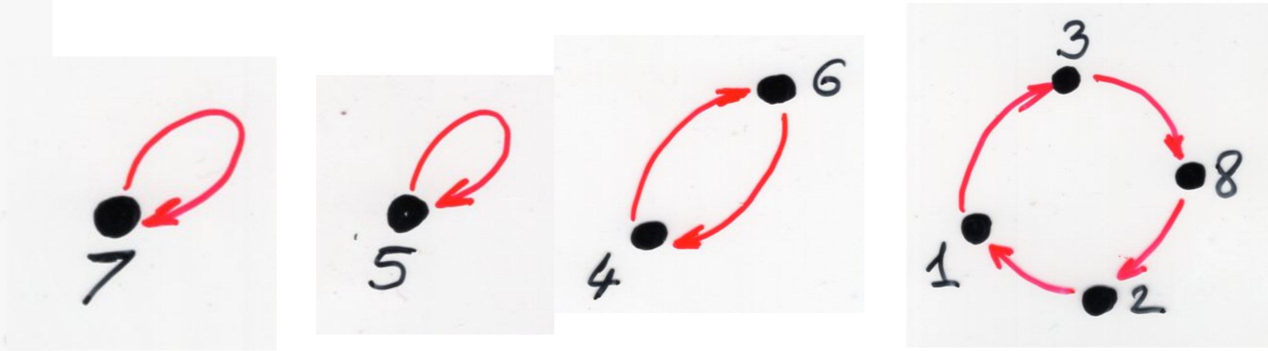

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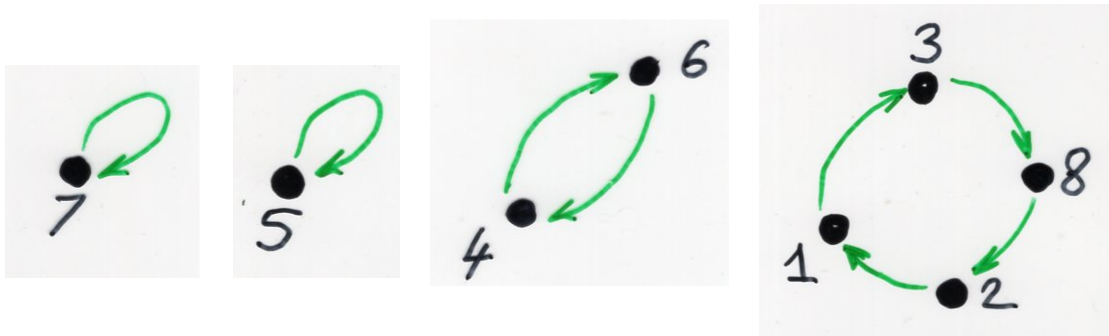
4 6
3 8 2 1



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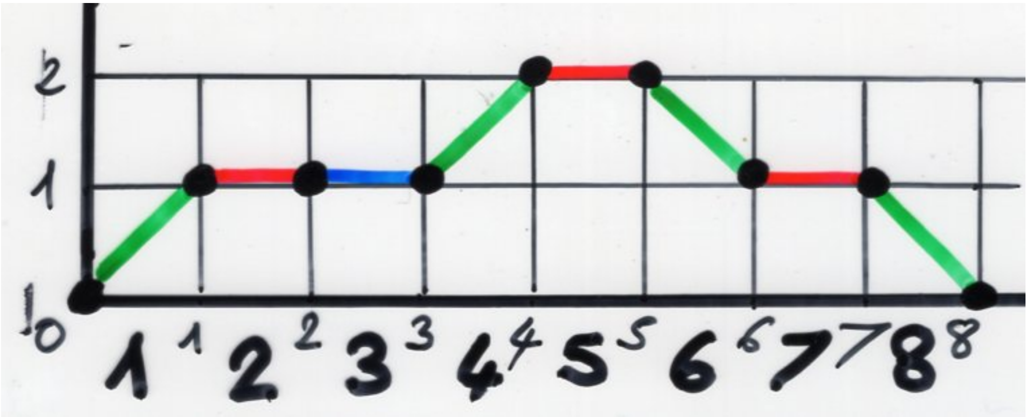
4 6
3 8 2 1

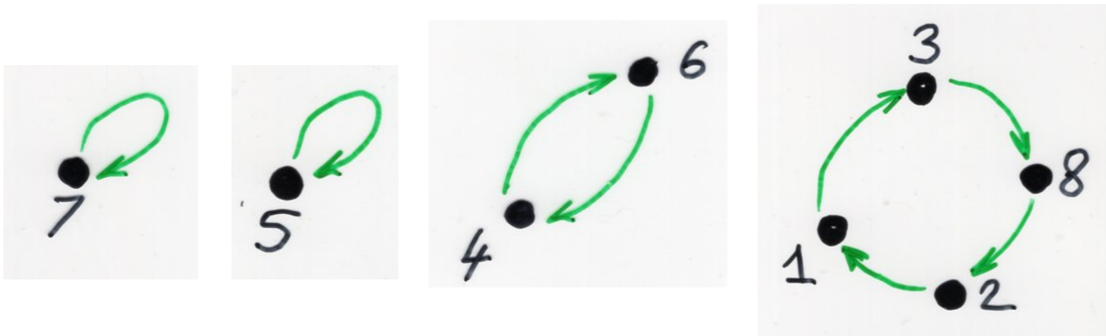


1 2 1 1 1 1 1 1

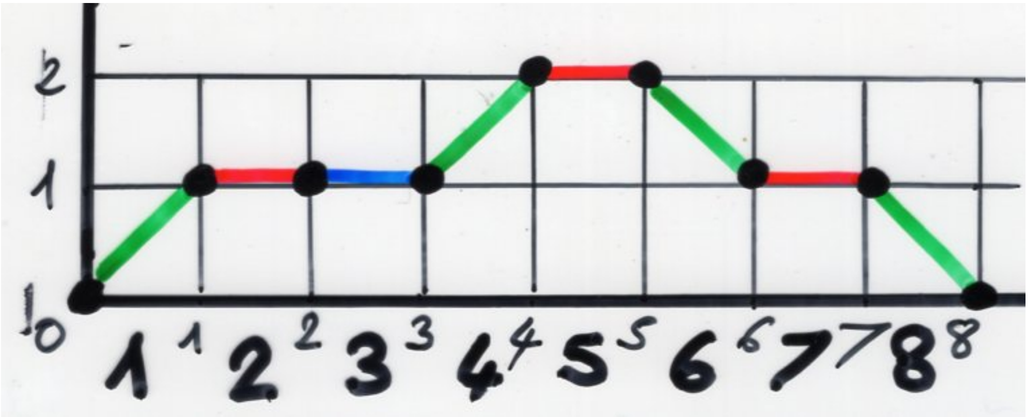
restricted Laguerre histories

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases} \quad \begin{matrix} (k \geq 0) \\ \\ (k \geq 1) \end{matrix}$$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$



$$\mu_n = n!$$

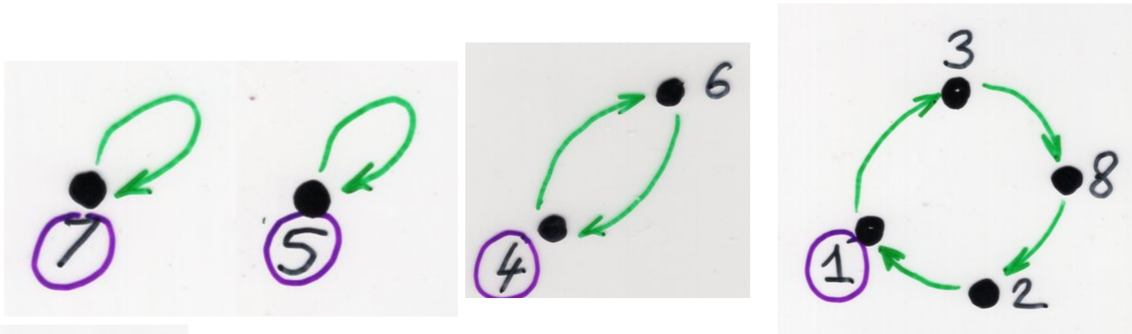


1 2 1 1 1 1 1 1

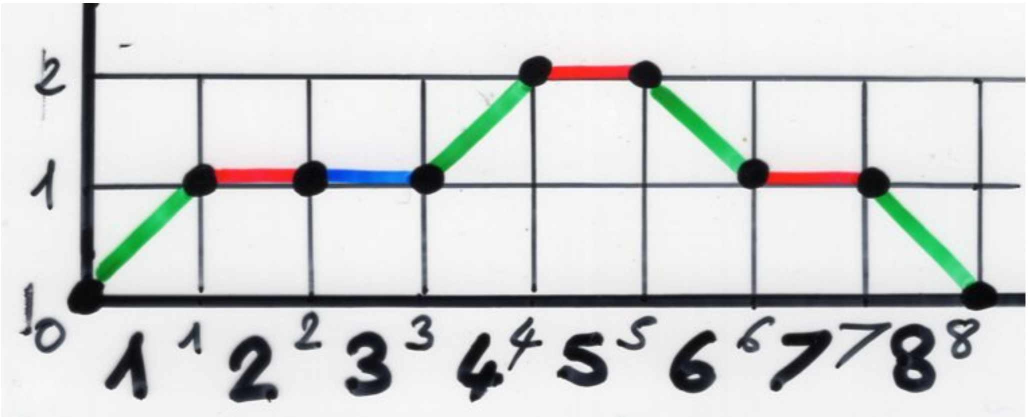
restricted Laguerre histories

$$\begin{cases} a_k = k + \beta \\ b'_k = k + \beta \\ b''_k = k \\ c_k = k \end{cases}$$

$$\begin{cases} b_k = 2k + \beta \\ \lambda_k = (k-1+\beta)k \end{cases}$$



$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$



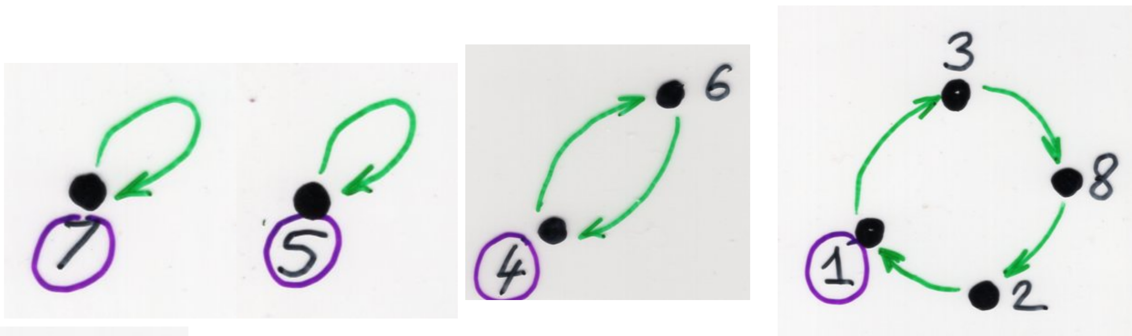
1 2 1 1 1 1 1 1

restricted Laguerre histories

$$\begin{cases} a_k = k + \beta \\ b'_k = k + \beta \\ b''_k = k \\ c_k = k \end{cases}$$

$$\begin{cases} b_k = 2k + \beta \\ \lambda_k = (k - 1 + \beta)k \end{cases}$$

σ = (7) / (5) / (4) 6 / (1) 3 8 2



$$\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

a — $v(\sigma)$ = number of valleys of σ

b — $p(\sigma)$ = number of peaks of σ

c — $dr(\sigma)$ = number of double rises of σ

d — $dd(\sigma)$ = number of double descents of σ

α — $f(\sigma)$ = number of lr-min elements which are a descent of σ

β — $s(\sigma)$ = number of lr-min elements of σ

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

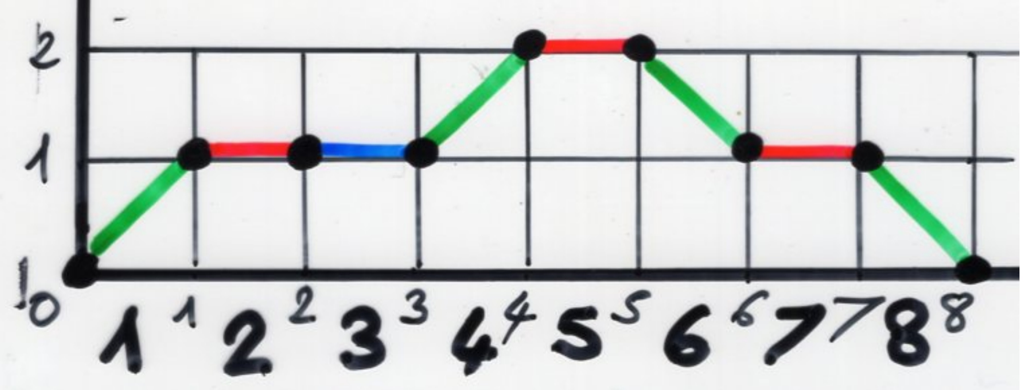
$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

a	cycle valley	$\sigma^{-1}(x) > x < \sigma(x)$	$cv(\sigma)$
b	cycle peak	$\sigma^{-1}(x) < x > \sigma(x)$	$cp(\sigma)$
c	cycle double rise	$\sigma^{-1}(x) < x < \sigma(x)$	$cdr(\sigma)$
d	cycle double descent	$\sigma^{-1}(x) > x > \sigma(x)$	$cdd(\sigma)$
α	fixed point	$\sigma(x) = x$	
β	number of cycles		$cyc(\sigma)$

Restricted Laguerre histories for

pemutations (cycles notation)

pemutations (word notation)



1 2 1 1 1 1 1

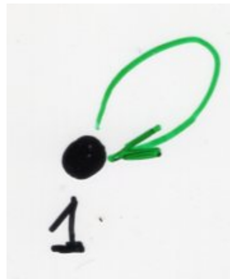
1

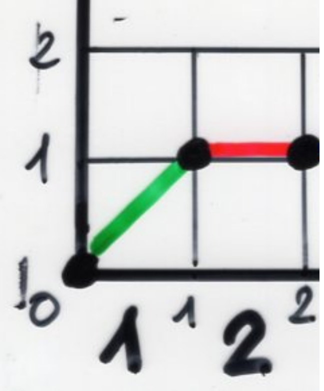
␣

␣ 1

1

*



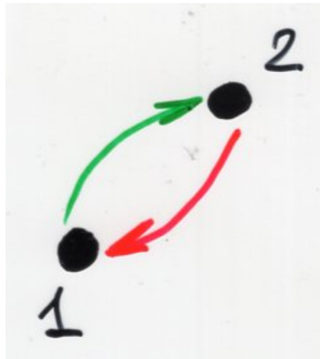


1 2

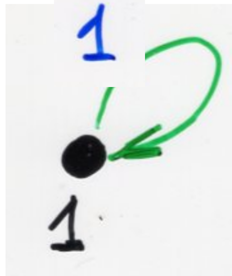
$\sqcup$

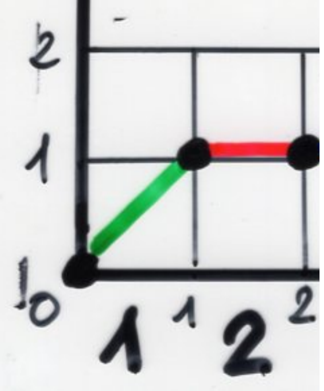
$\sqcup 1$ 1

$\sqcup 1 \sqcup 2$ 2



1

* 



1 2

⊔

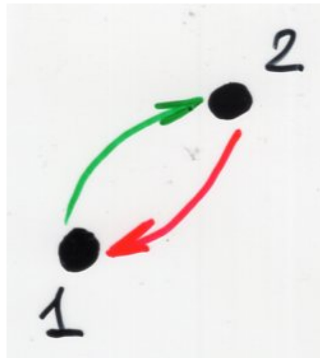
⊔1

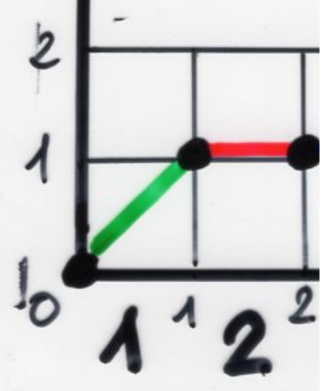
⊔1⊔2

1

2

*





1 2

␣

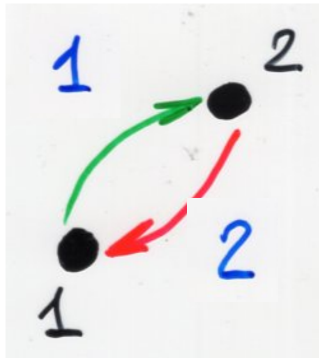
␣1

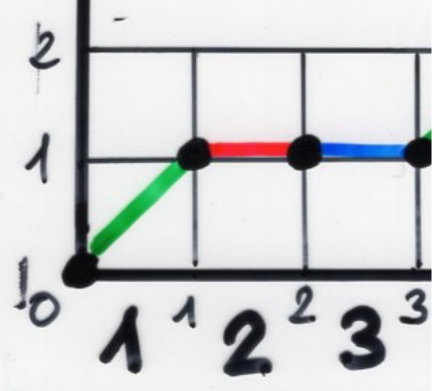
␣1␣2

1

2

*

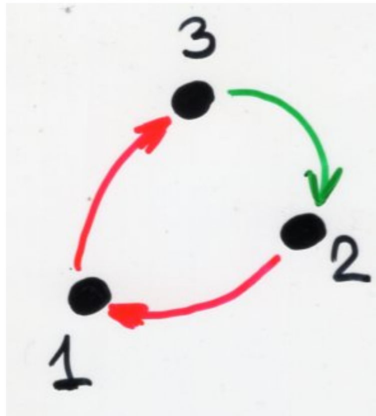


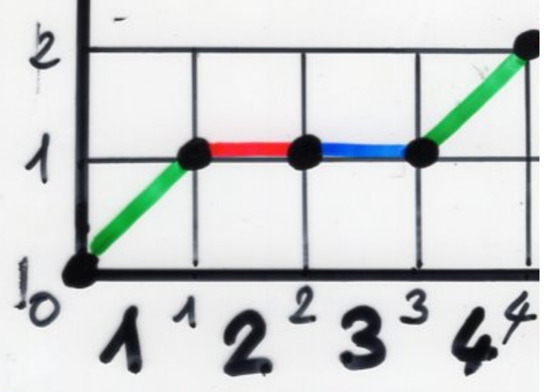


1 2 1

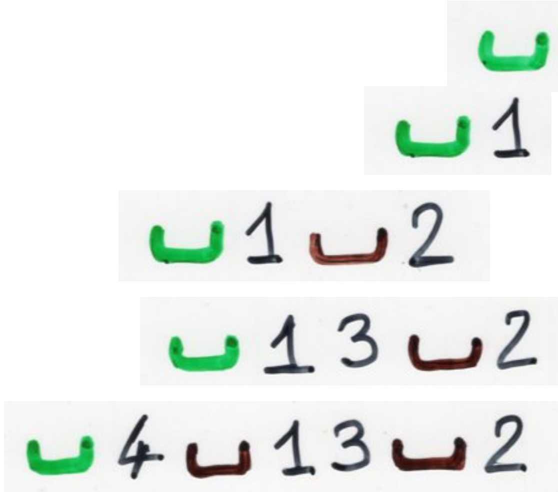
␣	
␣ 1	1
␣ 1 ␣ 2	2
␣ 1 3 ␣ 2	1

*

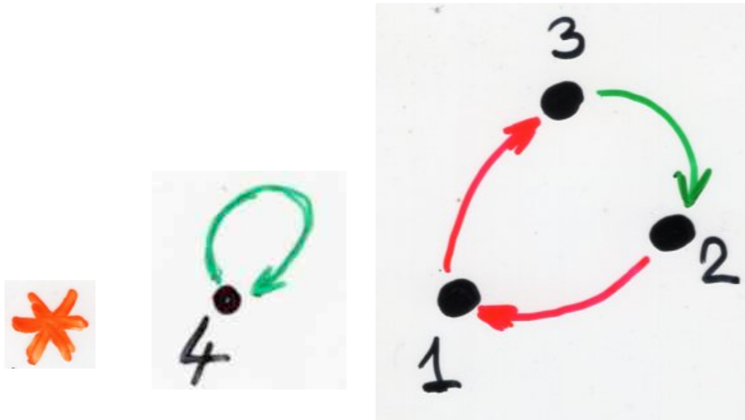


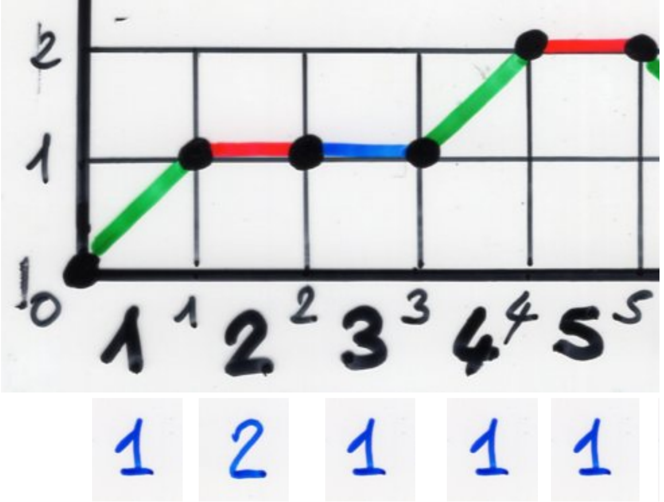


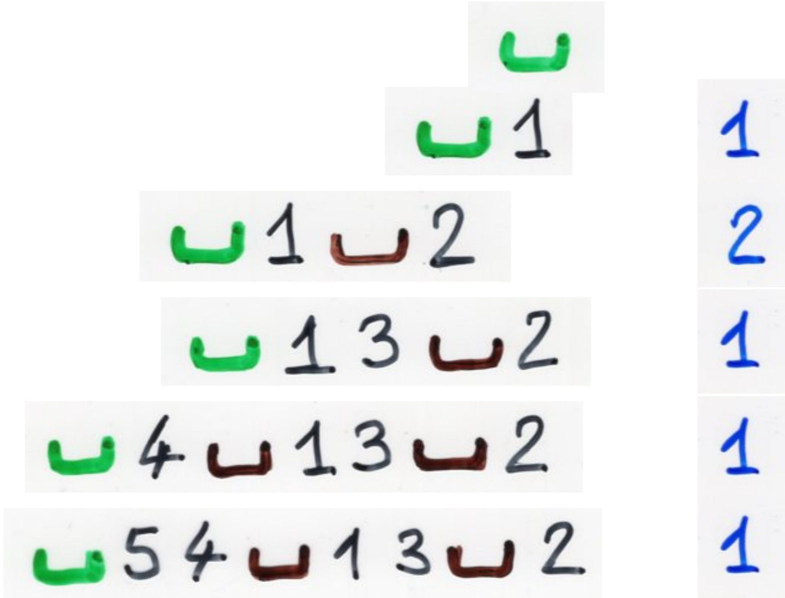
1 2 1 1

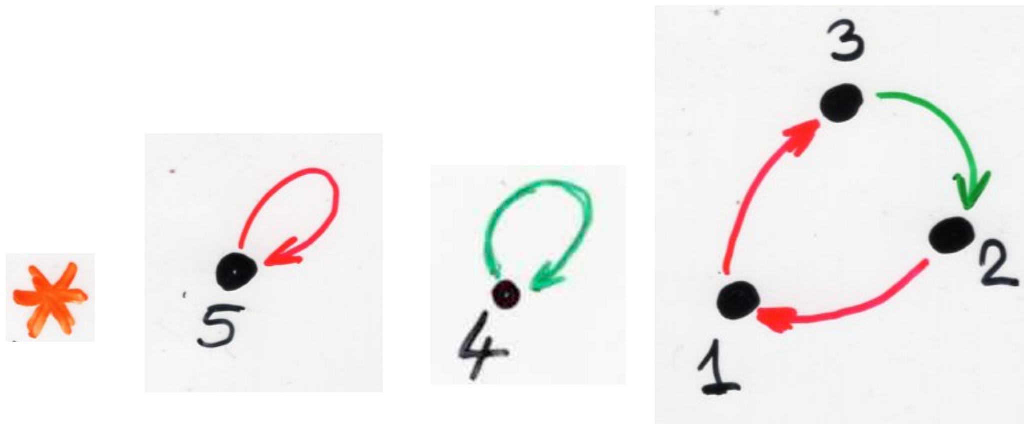


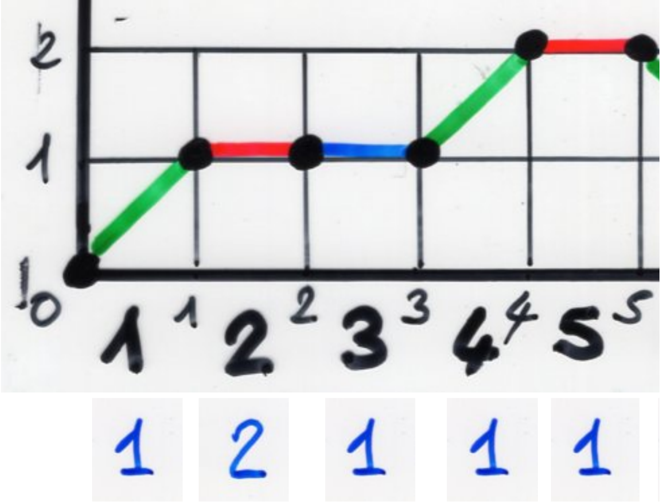
1
2
1
1



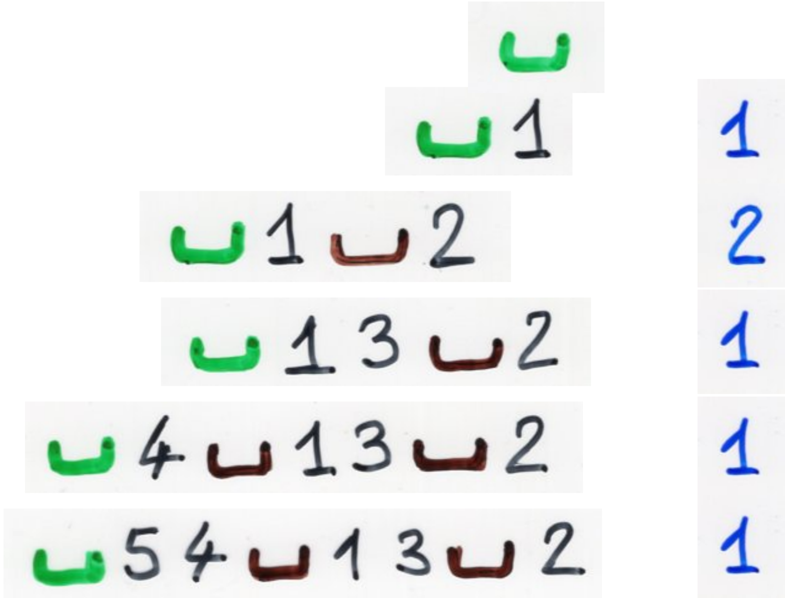

1 2 3 4 5
1 2 1 1 1


1
2
1
1
1

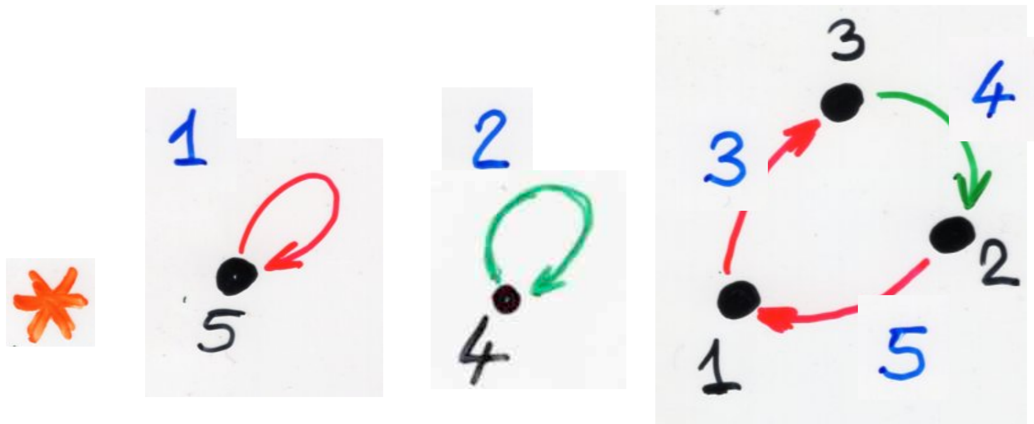

5
4
3
2
1



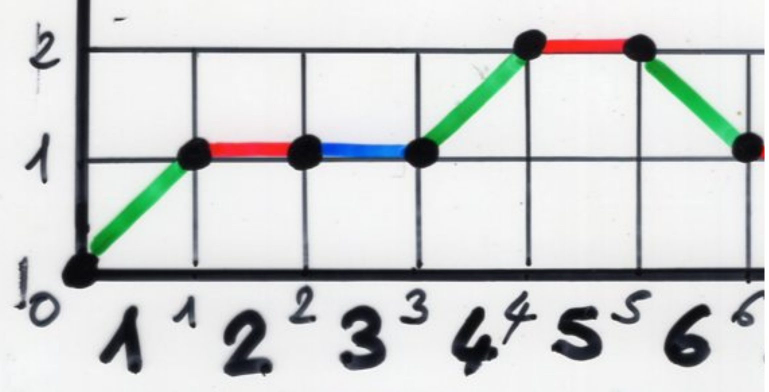
2
1
0
1 2 3 4 5
1 2 1 1 1



1
1 2
1 3 2
4 1 3 2
5 4 1 3 2
1
2
1
1
1

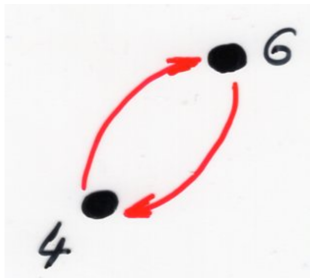


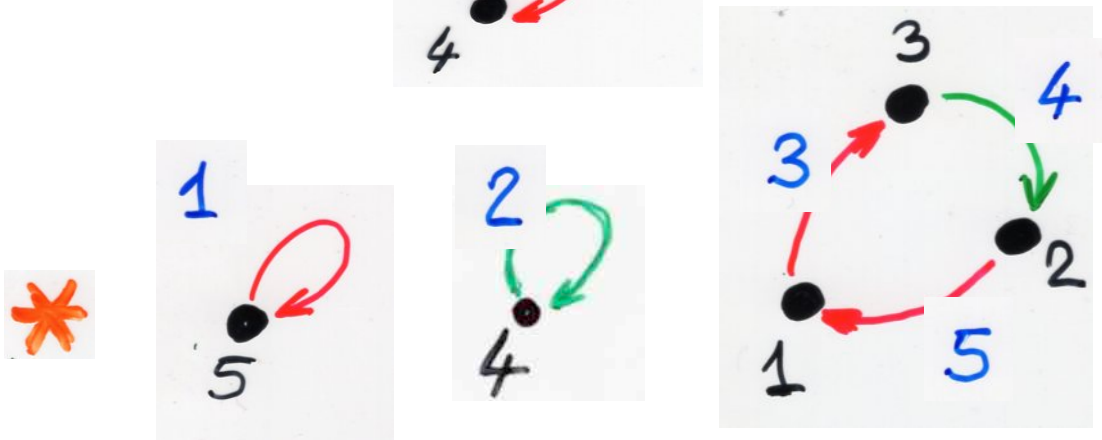
1
5
2
4
3
4
3
2
1
5

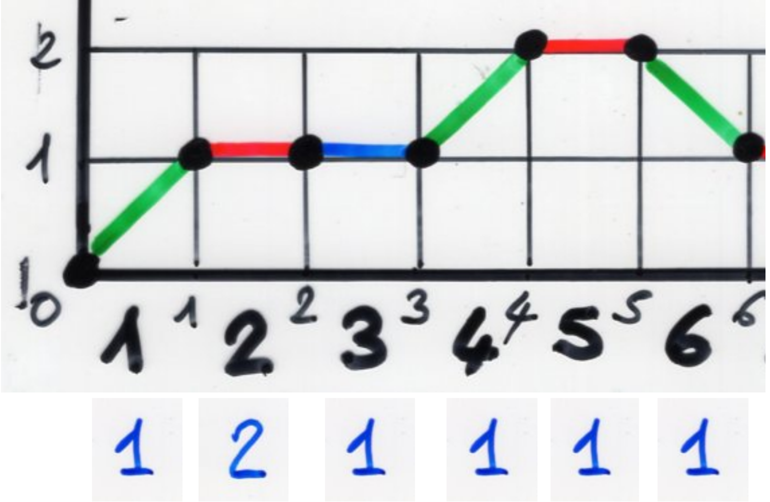


1 2 1 1 1 1

⊔	
⊔1	1
⊔1⊔2	2
⊔13⊔2	1
⊔4⊔13⊔2	1
⊔54⊔13⊔2	1
⊔54613⊔2	1

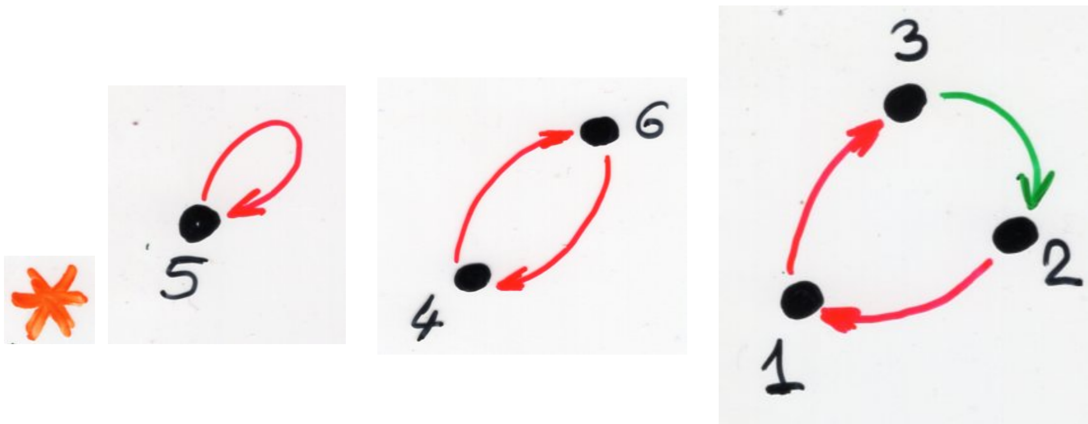




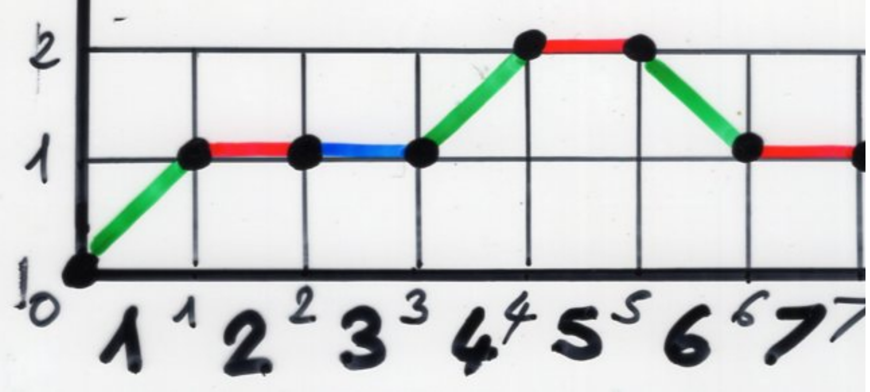


2
1
0
1 2 3 4 5 6
1 2 1 1 1 1

⊔
⊔1 1
⊔1⊔2 2
⊔13⊔2 1
⊔4⊔13⊔2 1
⊔54⊔13⊔2 1
⊔54613⊔2 1



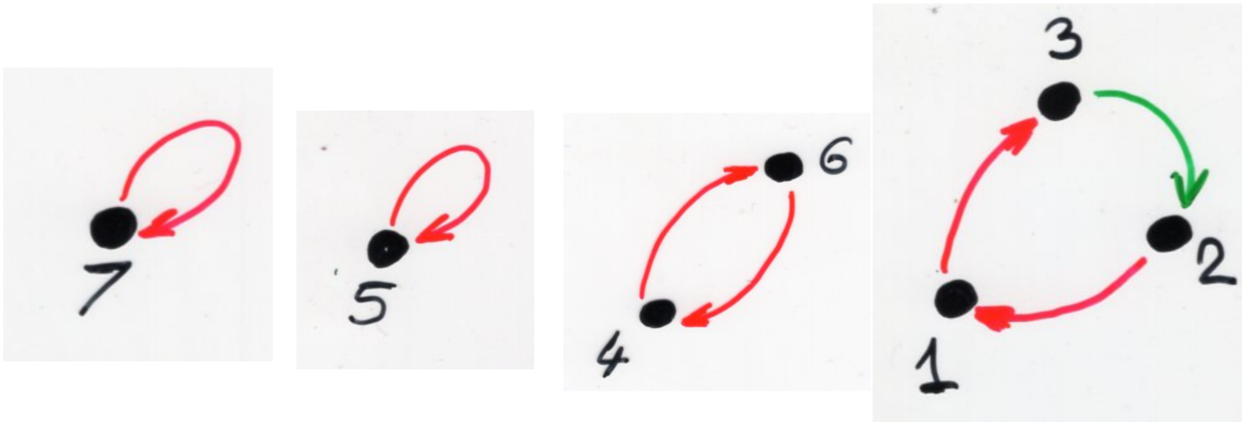
5
4
6
3
2
1

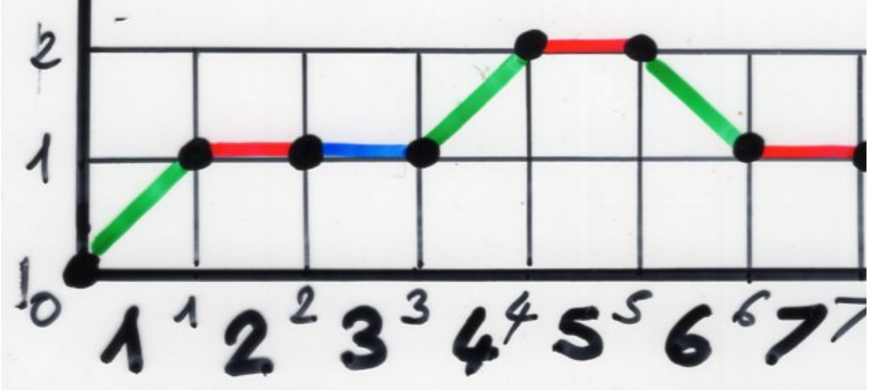


1	2	1	1	1	1	1

⊔	
⊔1	1
⊔1⊔2	2
⊔13⊔2	1
⊔4⊔13⊔2	1
⊔54⊔13⊔2	1
⊔54613⊔2	1
⊔754613⊔2	1

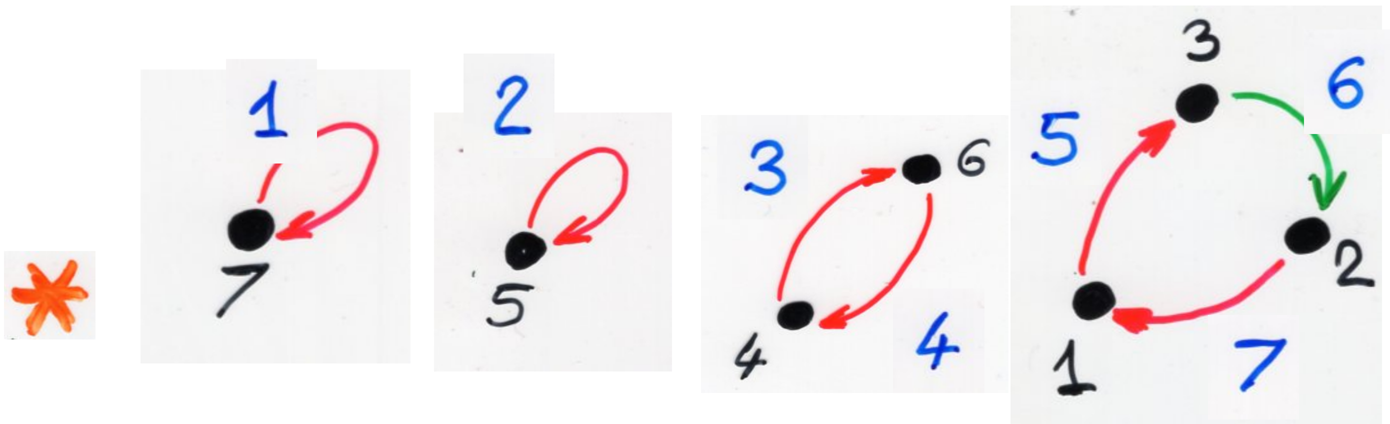
*

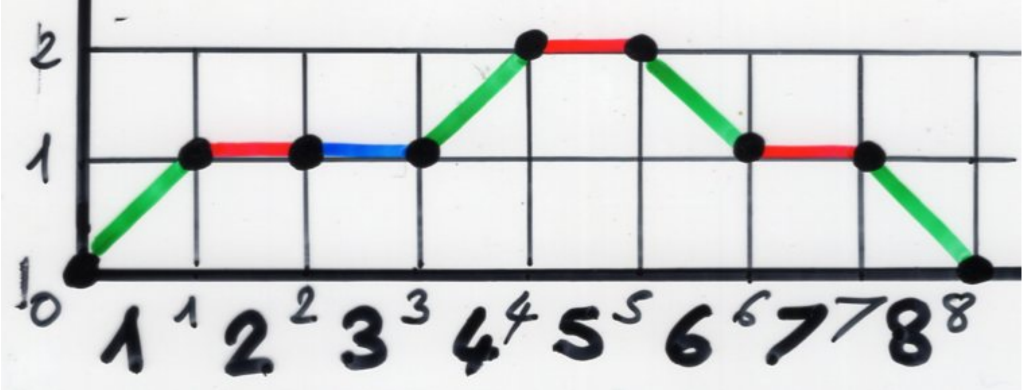




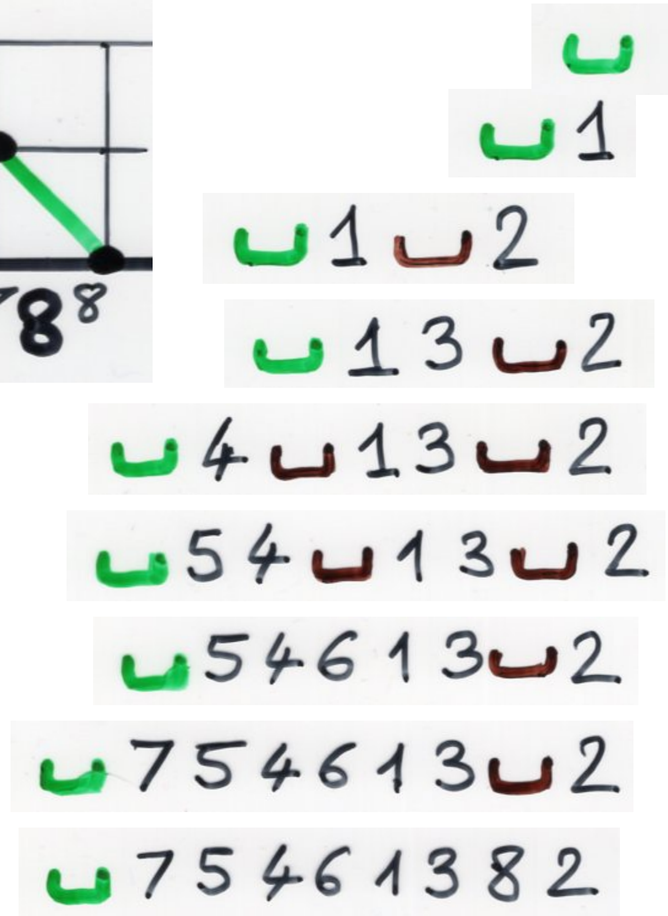
1 2 1 1 1 1 1

⊔	
⊔1	1
⊔1⊔2	2
⊔13⊔2	1
⊔4⊔13⊔2	1
⊔54⊔13⊔2	1
⊔54613⊔2	1
⊔754613⊔2	1





1 2 1 1 1 1 1



1
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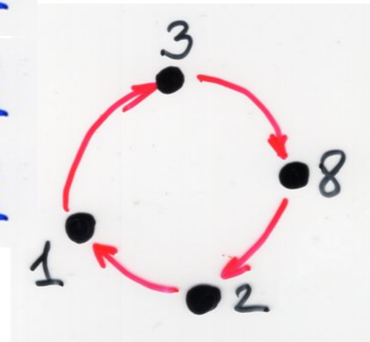
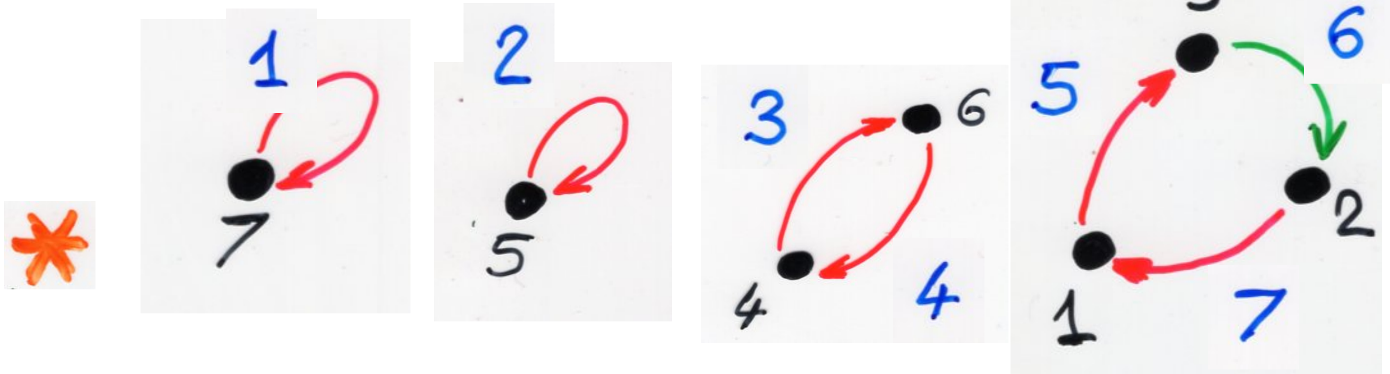
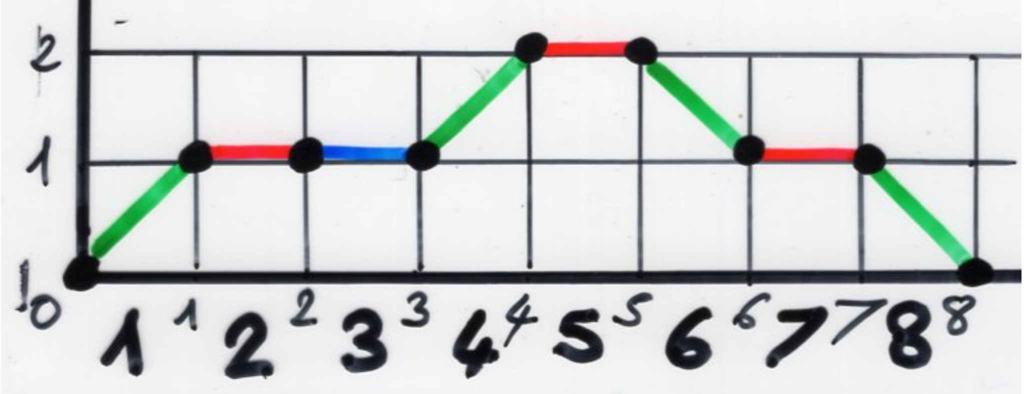
1







1 2 1 1 1 1 1

⊔	
⊔ 1	1
⊔ 1 ⊔ 2	2
⊔ 1 3 ⊔ 2	1
⊔ 4 ⊔ 1 3 ⊔ 2	1
⊔ 5 4 ⊔ 1 3 ⊔ 2	1
⊔ 5 4 6 1 3 ⊔ 2	1
⊔ 7 5 4 6 1 3 ⊔ 2	1
⊔ 7 5 4 6 1 3 8 2	1

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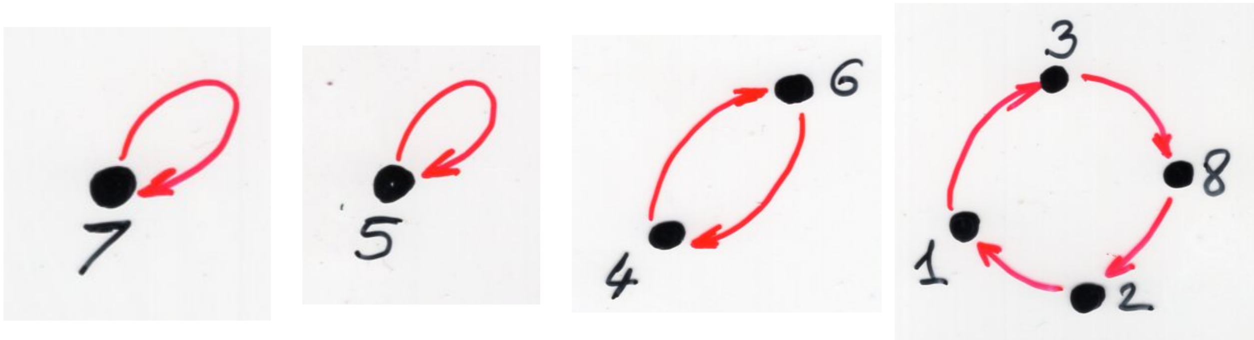
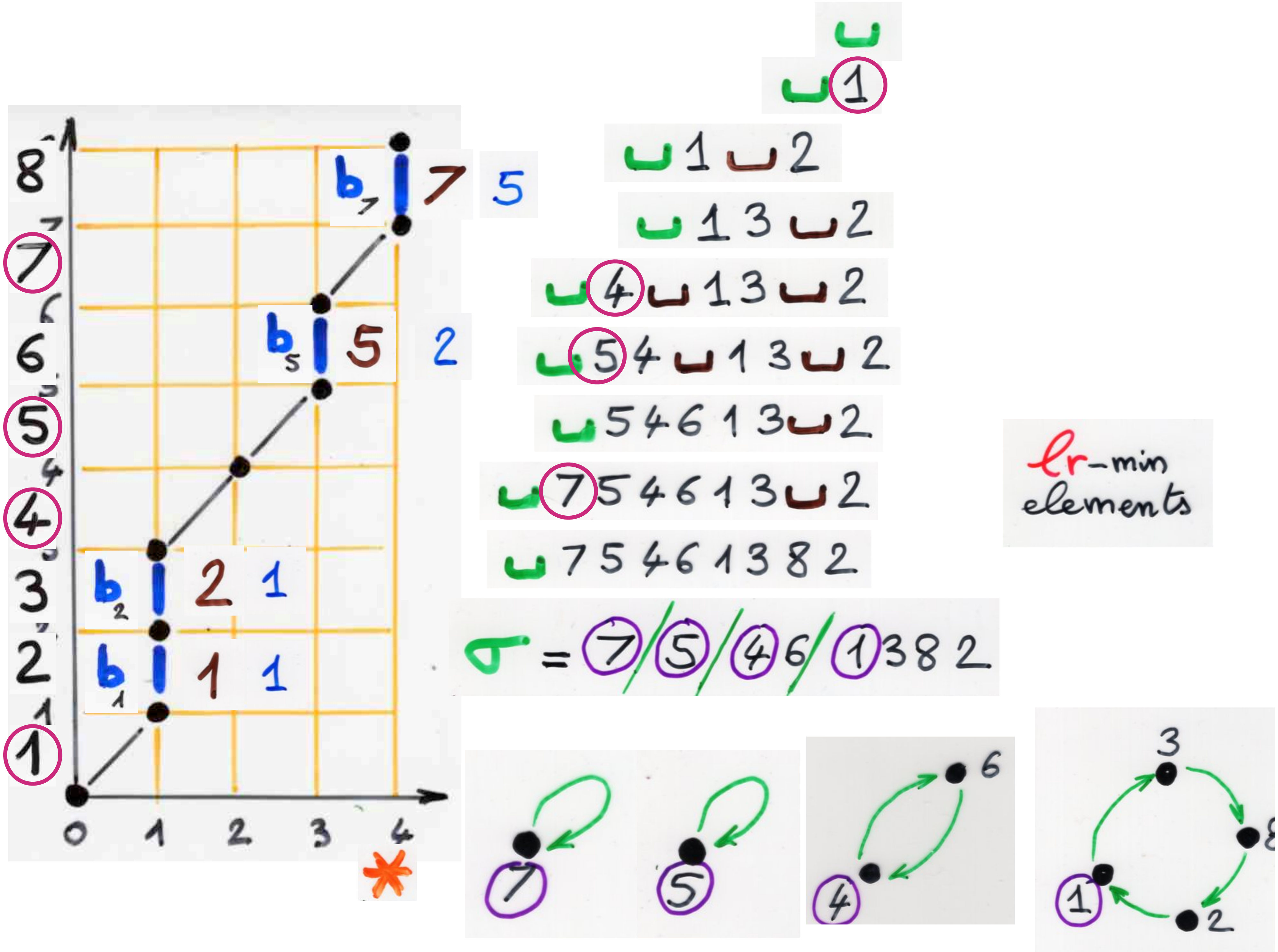




⊔
⊔ 1
⊔ 1 ⊔ 2
⊔ 1 3 ⊔ 2
⊔ 4 ⊔ 1 3 ⊔ 2
⊔ 5 4 ⊔ 1 3 ⊔ 2
⊔ 5 4 6 1 3 ⊔ 2
⊔ 7 5 4 6 1 3 ⊔ 2
⊔ 7 5 4 6 1 3 8 2
lr-min elements
σ = 7 / 5 / 4 6 / 1 3 8 2
b1 1 1
b2 2 1
b5 5 2
b7 7 5
0 1 2 3 4
1 2 3 4 5 6 7 8

Back to an exercise of Ch2b

$\beta = 1, \quad c = \frac{1}{2}$

$$\begin{cases} \tilde{b}_k = 3k+1 \\ \tilde{\lambda}_k = 2k^2 \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} 2^{d(\sigma)}$$

= number of ordered partitions of $\{1, 2, \ldots, n\}$

<u>exercise</u> direct proof by constructing a bijection between ordered partitions and some histories associated to weighted colored Motzkin paths with weight $\tilde{b}_k = 3k+1$, $\tilde{\lambda} = 2k^2$

$c = \frac{1}{2}$ $\quad \tilde{b}_k = 3k + \beta$

$\tilde{\lambda}_k = 2k(k + \beta - 1)$

parameter β : number of blocks ?

The origin of the notion of « histories »

computer science

Data structures
and histories

example:

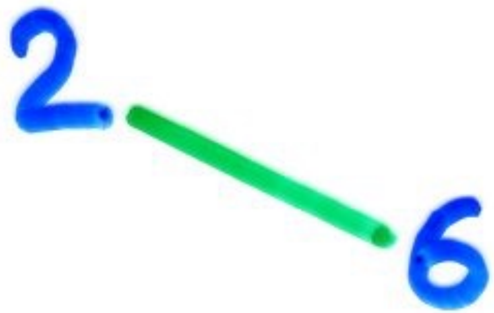
binary search trees

analysis of algorithms

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree

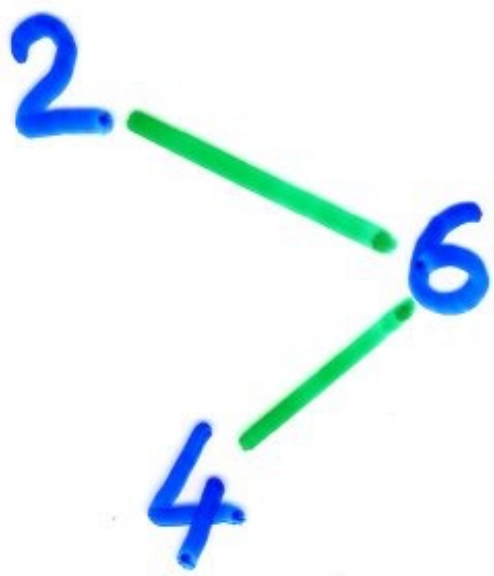
2

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



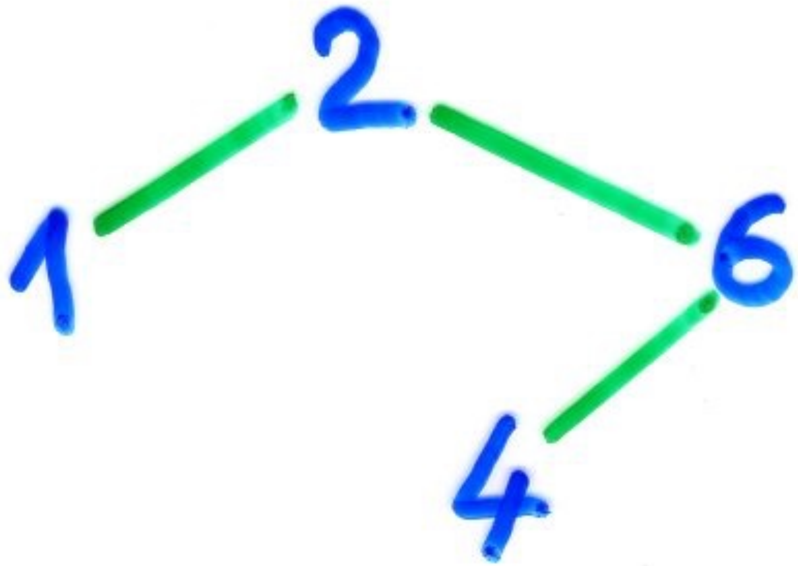
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$

binary search tree



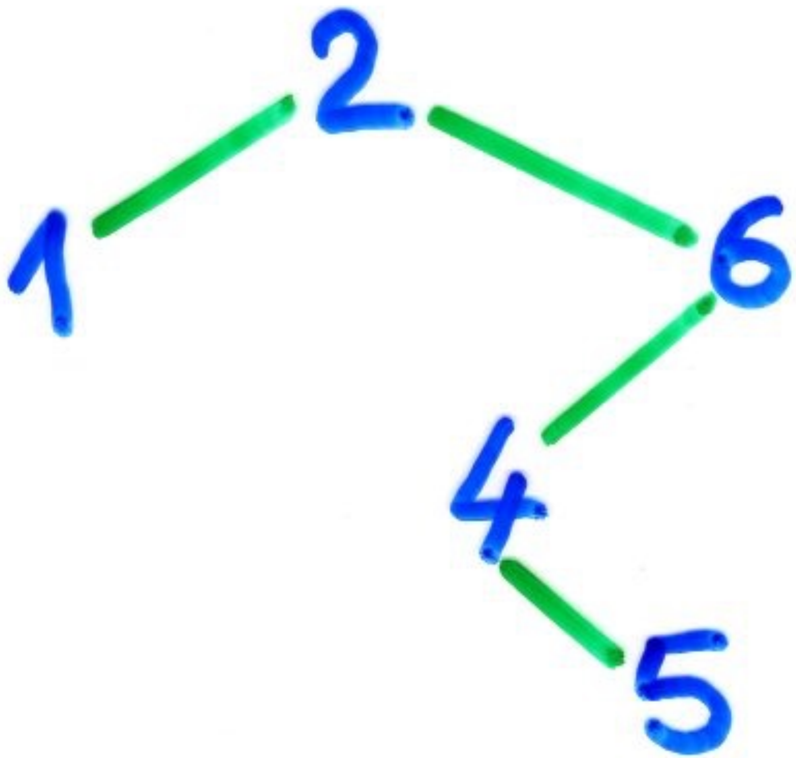
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$

binary search tree

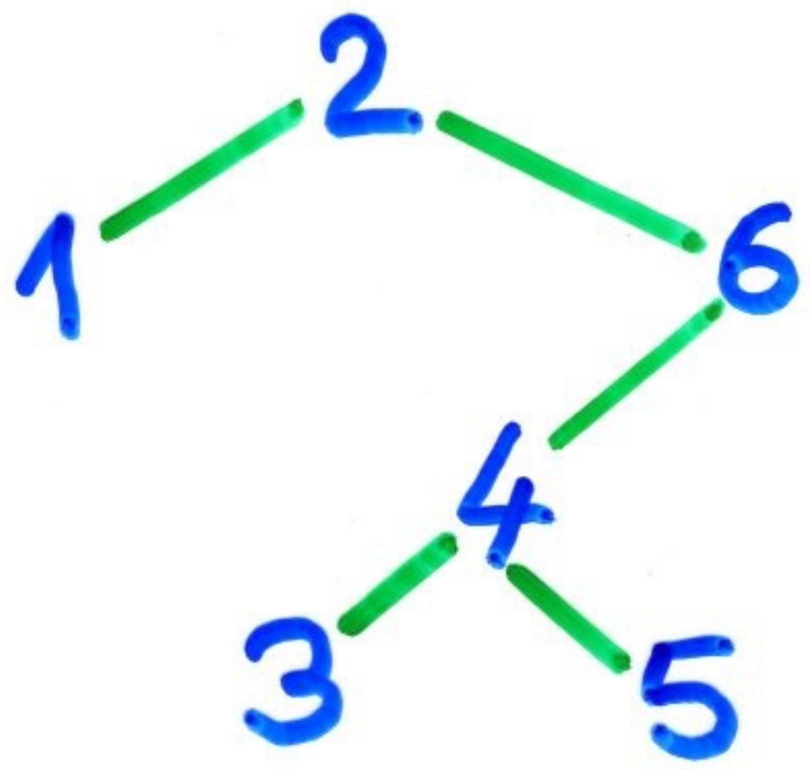


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$

binary search tree



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$ binary search tree



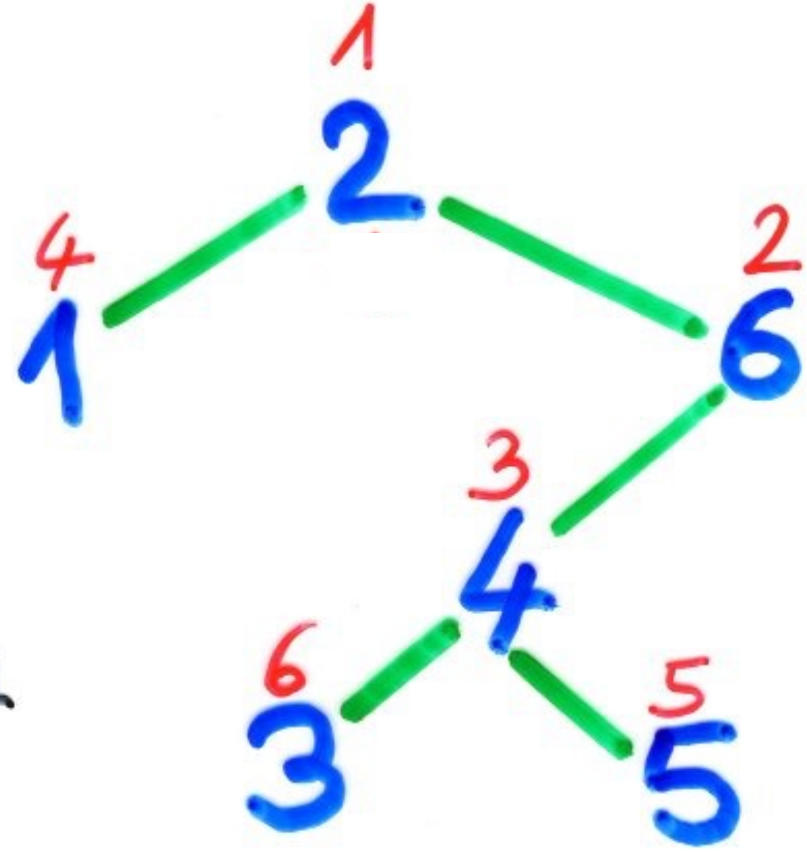
$\pi(B) = 1\ 2\ 3\ 4\ 5\ 6$

= identity permutation

<u>example</u>: Insertion in a random binary search tree

$2(H_n - 1)$ average cost

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$$



$$\tau = \sigma^{-1}$$

$$\tau = 4\ 1\ 6\ 3\ 5\ 2$$

Data structures and histories

integrated cost

sequence of primitive operations

J. Françon (1978)

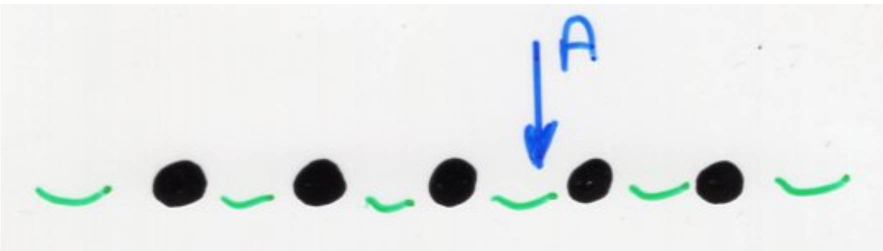
dictionnary data structure

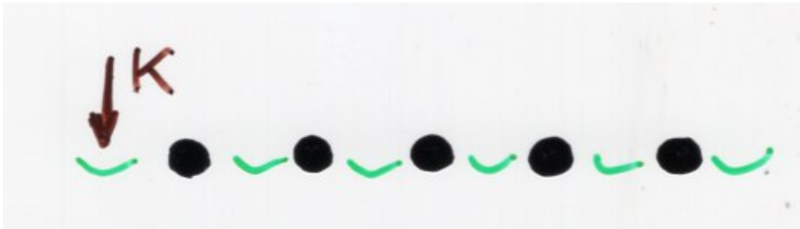
add or delete any element

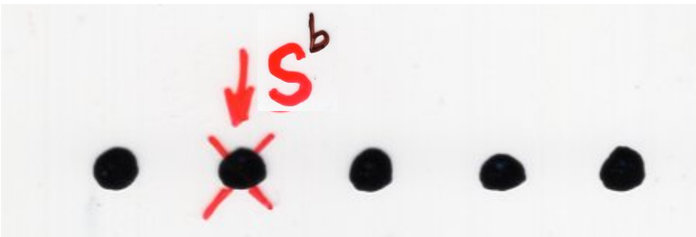
ask questions

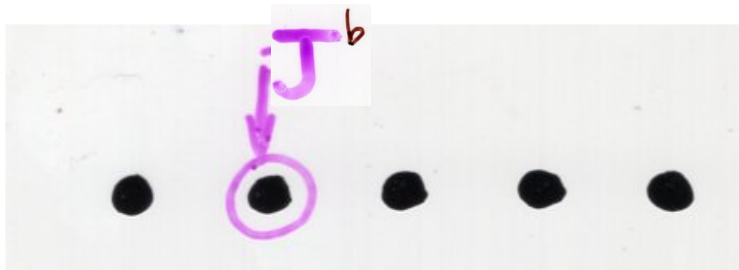
J^b positive

K negative

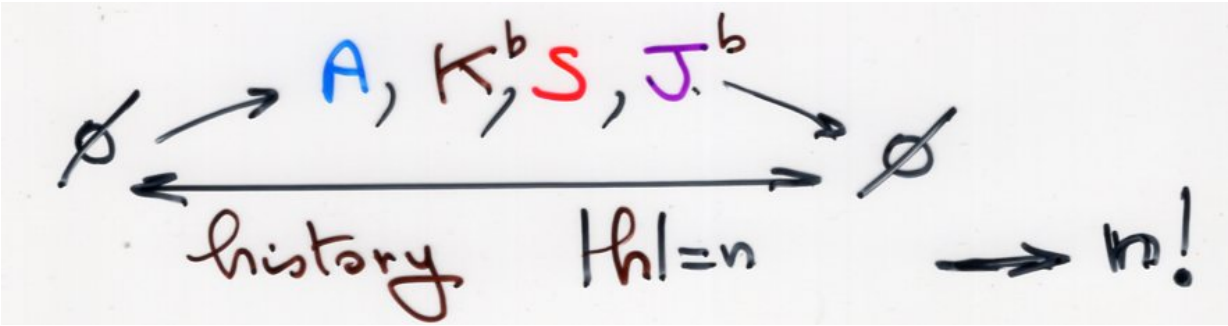






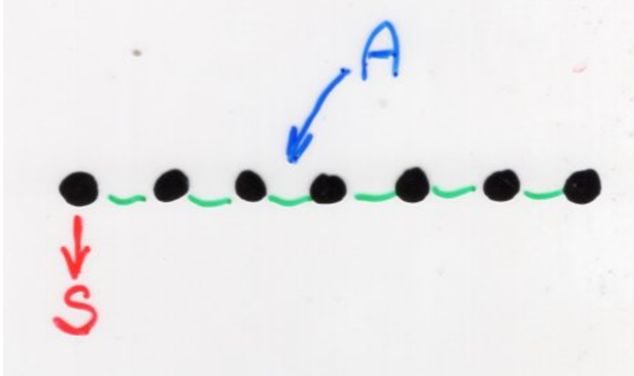


Françon (1978) "histoires de fichiers"



Priority queue

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$S\,|k\rangle = |(k-1)\rangle$$



Computation of the integrated cost
of a data structure
for a random sequence
of primitive operation
knowing the average cost
of a single primitive operation
(under certain conditions)

Françon, Flajolet, Vuillemin (1980, ...)

24

17

10

8

24

17

← 12

10

8

Representation of an history

for the data structure « dictionnary »

$$A|k\rangle = (k+1)|(k+1)\rangle$$

$$J^b|k\rangle = k|k\rangle$$

$$K|k\rangle = (k+1)|k\rangle$$

$$S^b|k\rangle = k|(k-1)\rangle$$

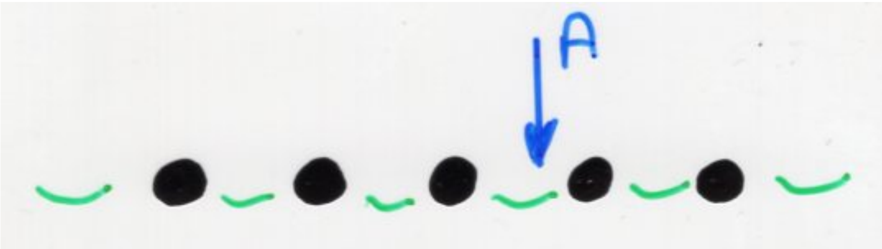
dictionnary data structure

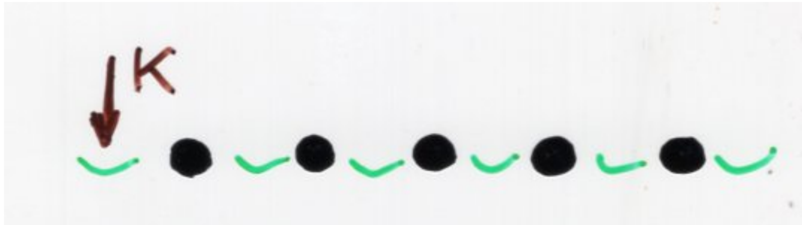
add or delete any element

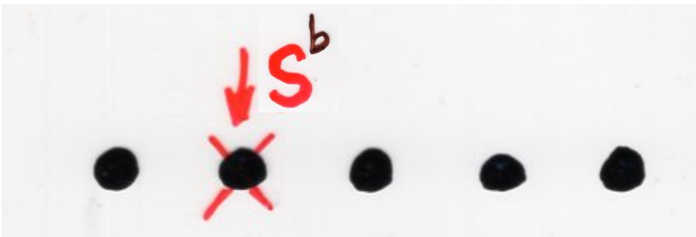
ask questions

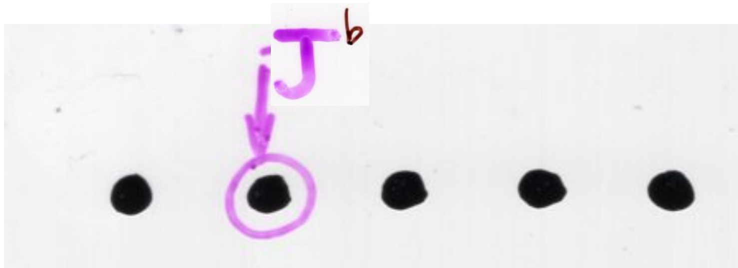
J^b positive

K negative

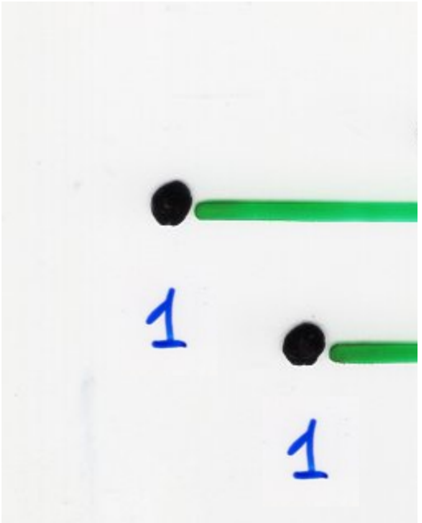




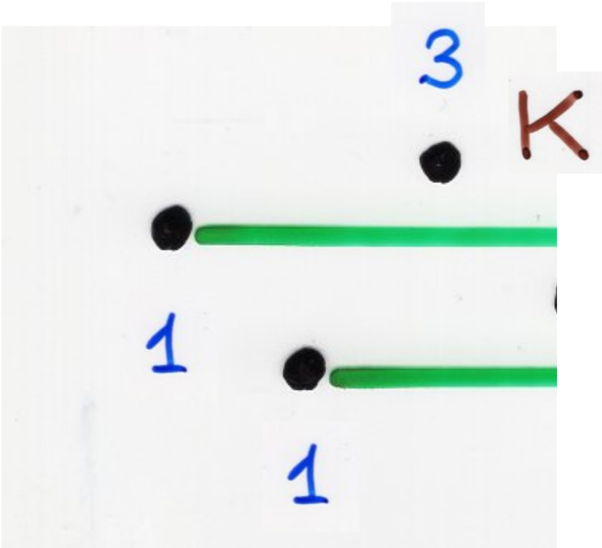


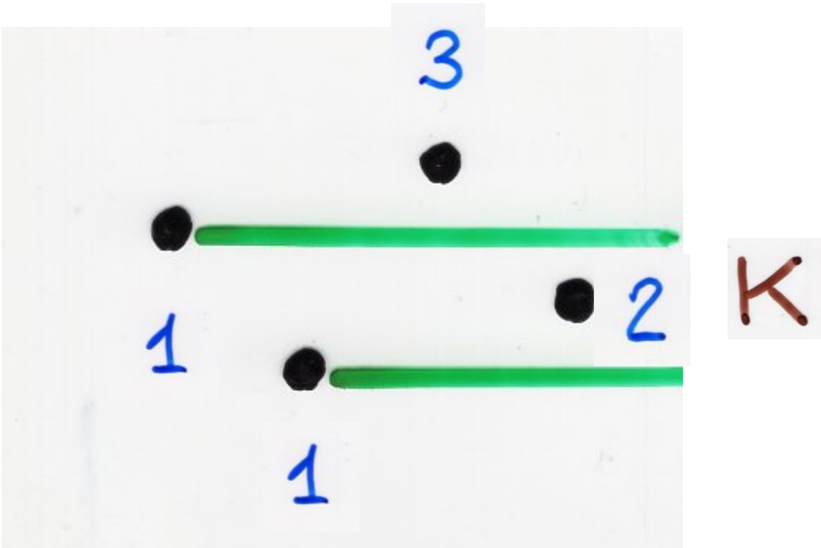


A



A


3
K
1
1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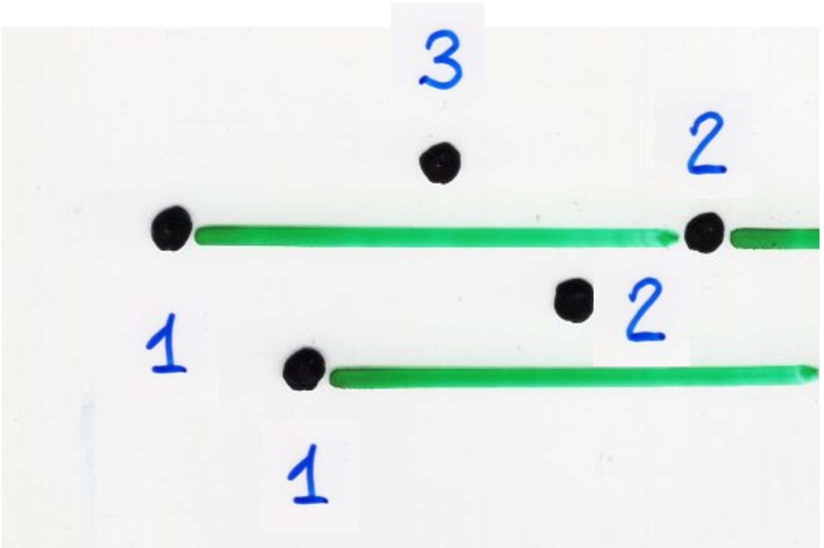
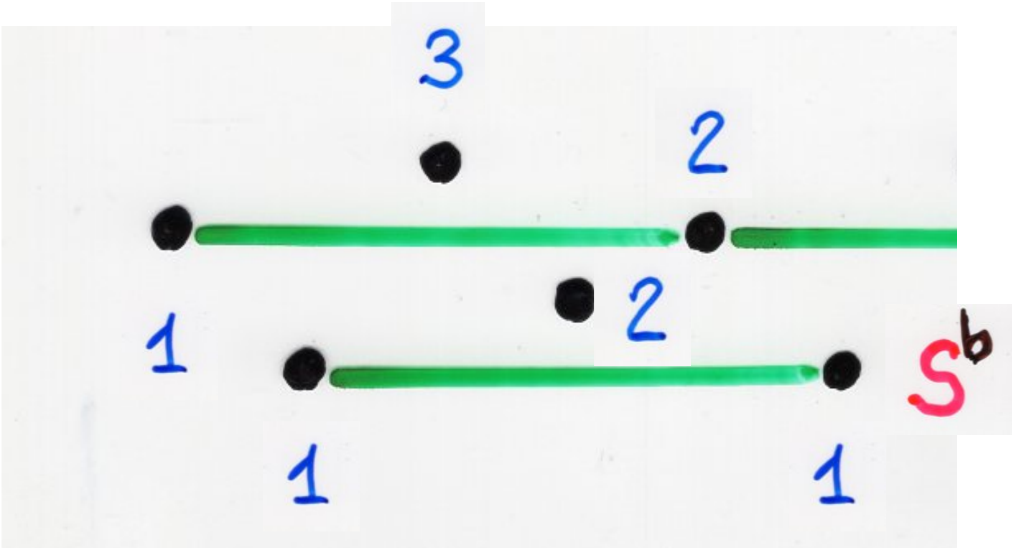
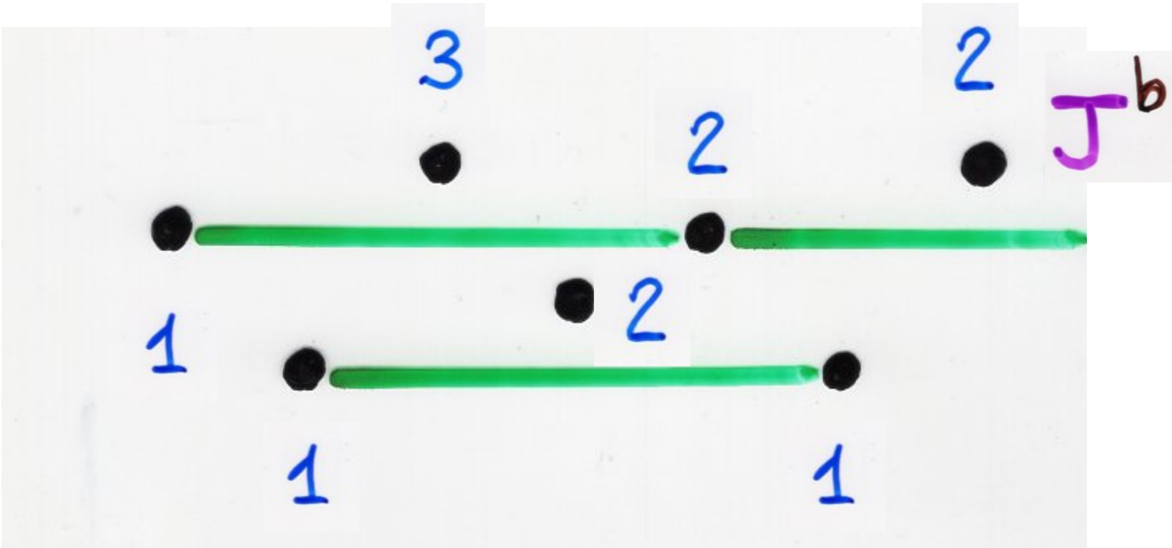
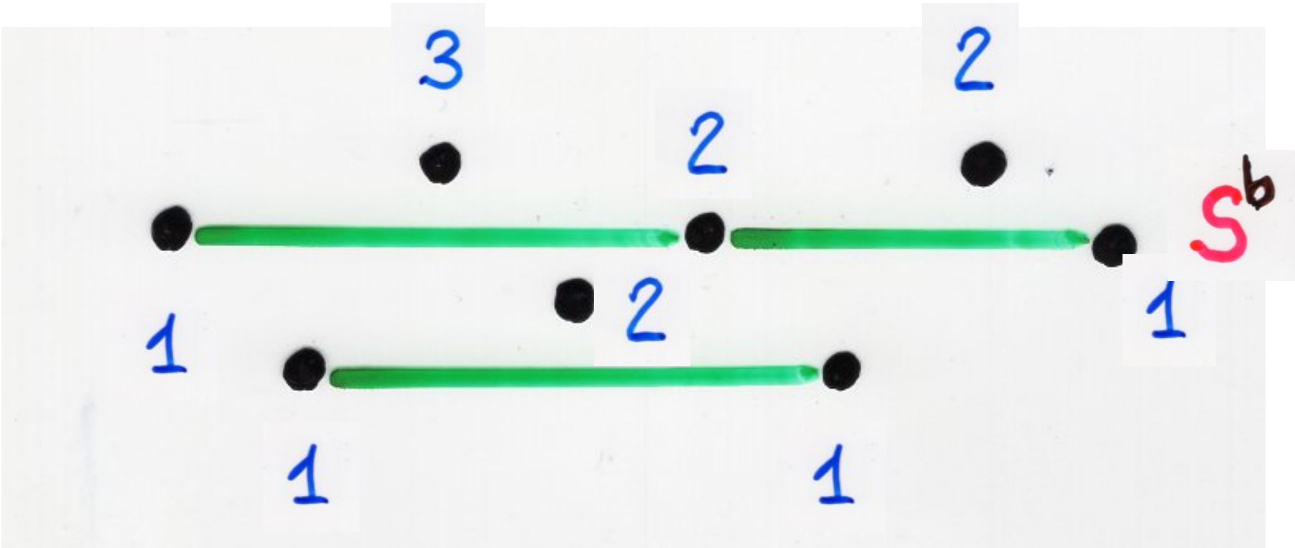
1
2
K
1



J^b


3
2
2
1
1
1
S^b


3
2
2
J^b
2
1
1
1



3
2
2
S^b
2
1
1
1
1

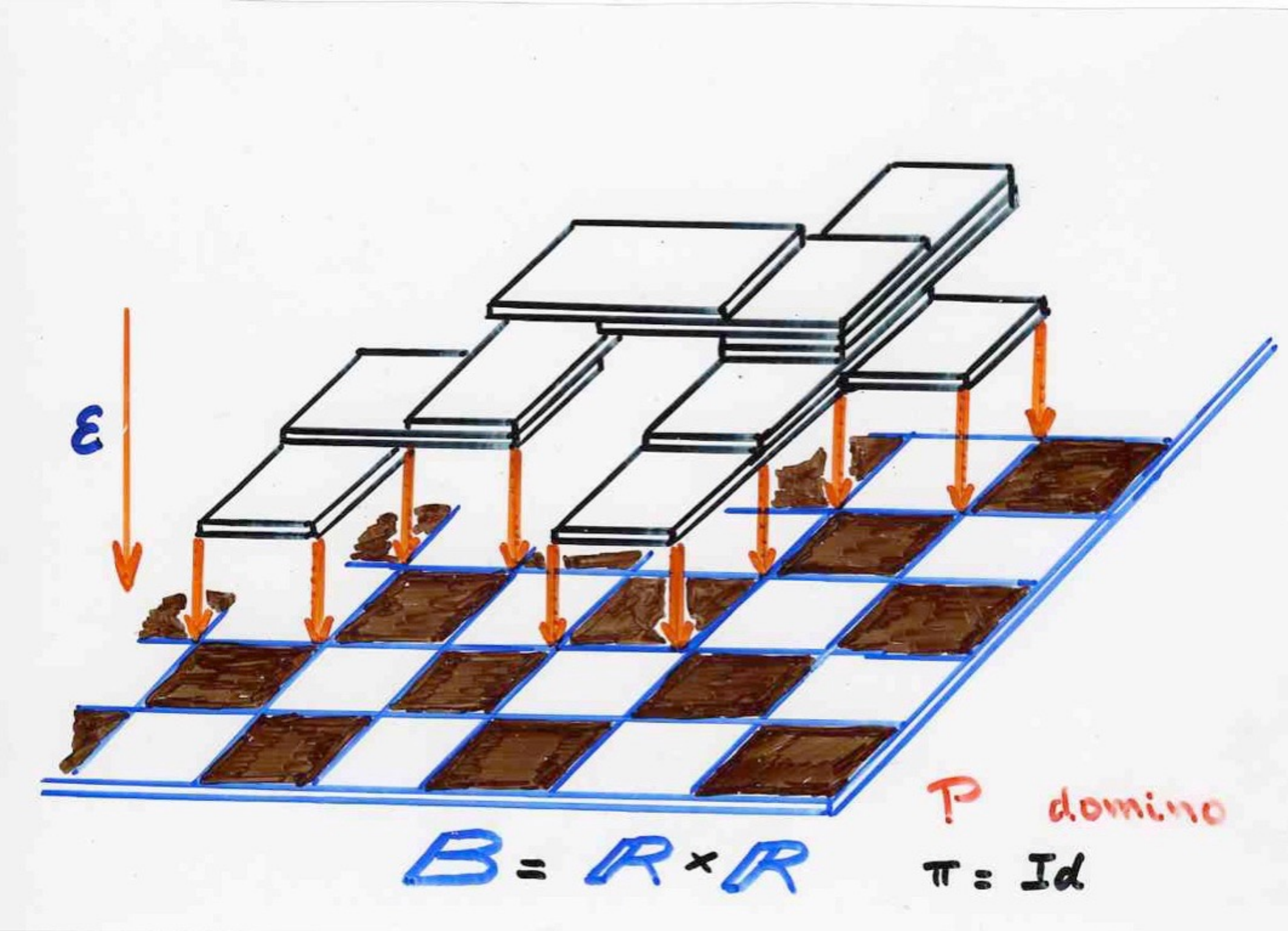
Laguerre heaps of segments

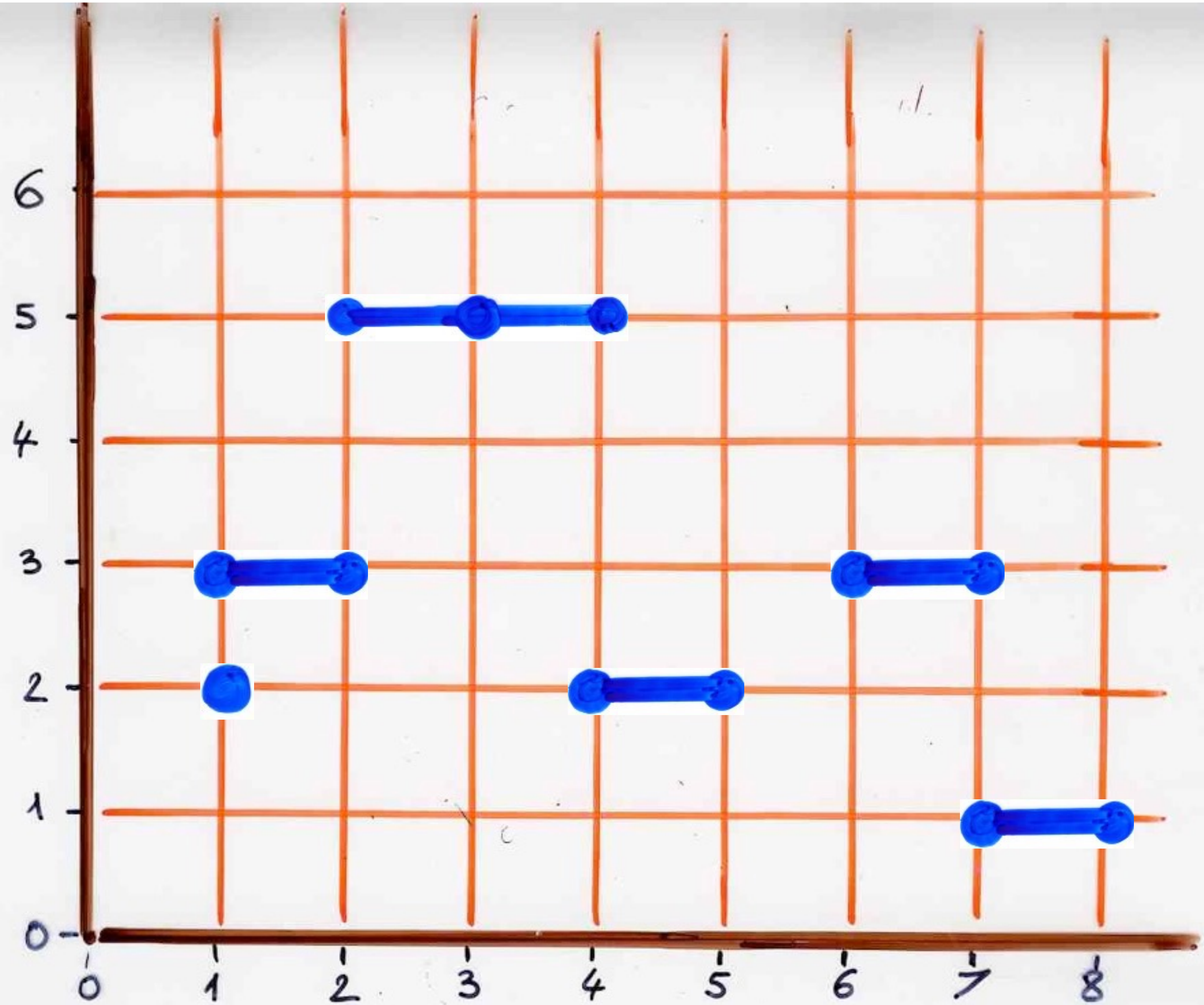
Reminding the notion of heaps of pieces

ABjC, part II

Introduction
Heaps

From BJC 2, Ch 1a





6
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0
1
2
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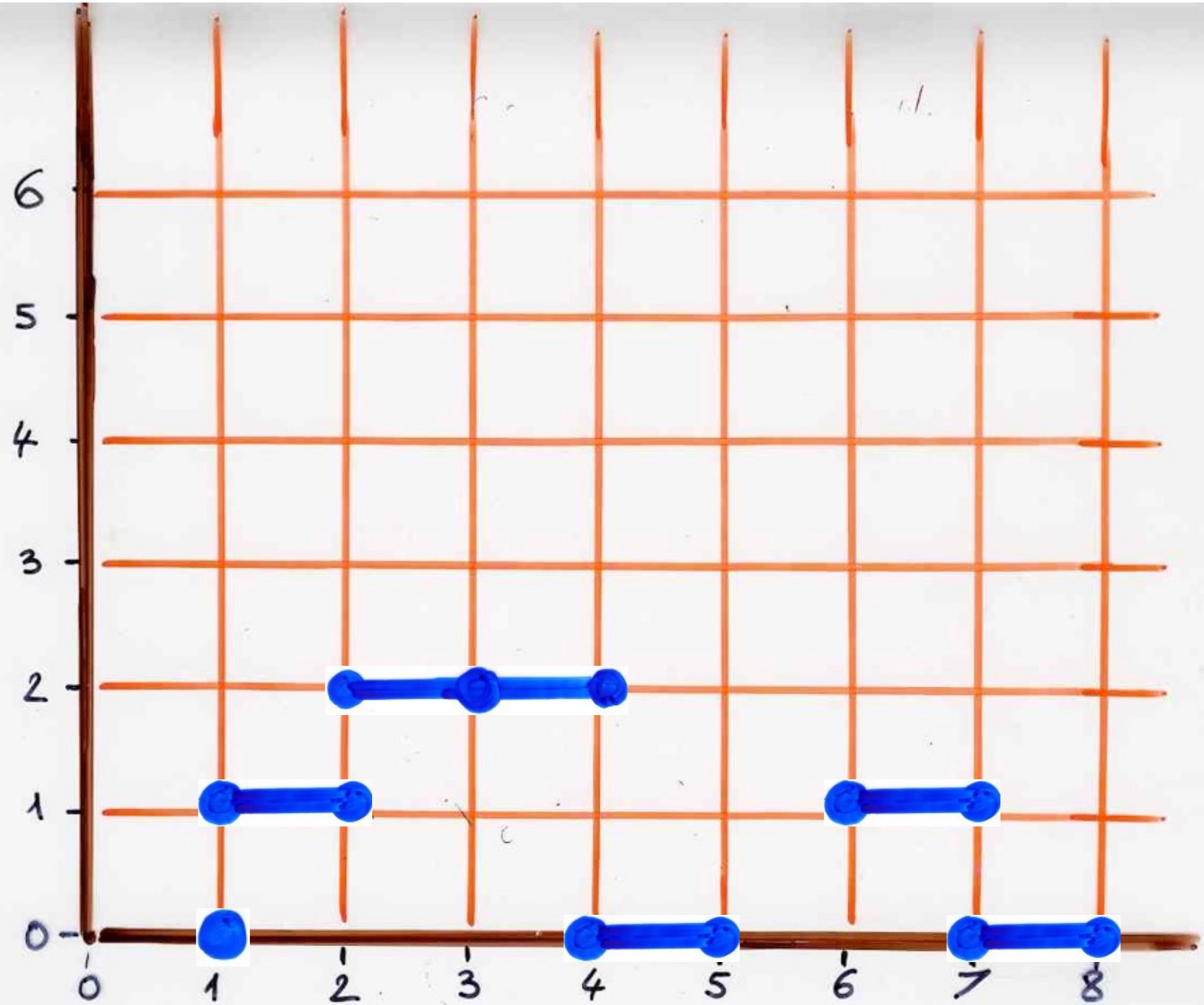
4
5
6
7
8



6
5
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0
0
1
2
3
4
5
6
7
8

heap definition

- $\mathcal{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathcal{P}$ { symmetric, reflexive }
  (dependency relation)
- heap E, finite set of pairs
  (α, i) $\alpha \in \mathcal{P}$, $i \in \mathbb{N}$ (called pieces)
  α ← projection, i ← level

(i)

(ii)

heap — definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P {symmetric, reflexive}
  (dependency relation)
- heap E, finite set of pairs
  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)
  α: projection, i: level

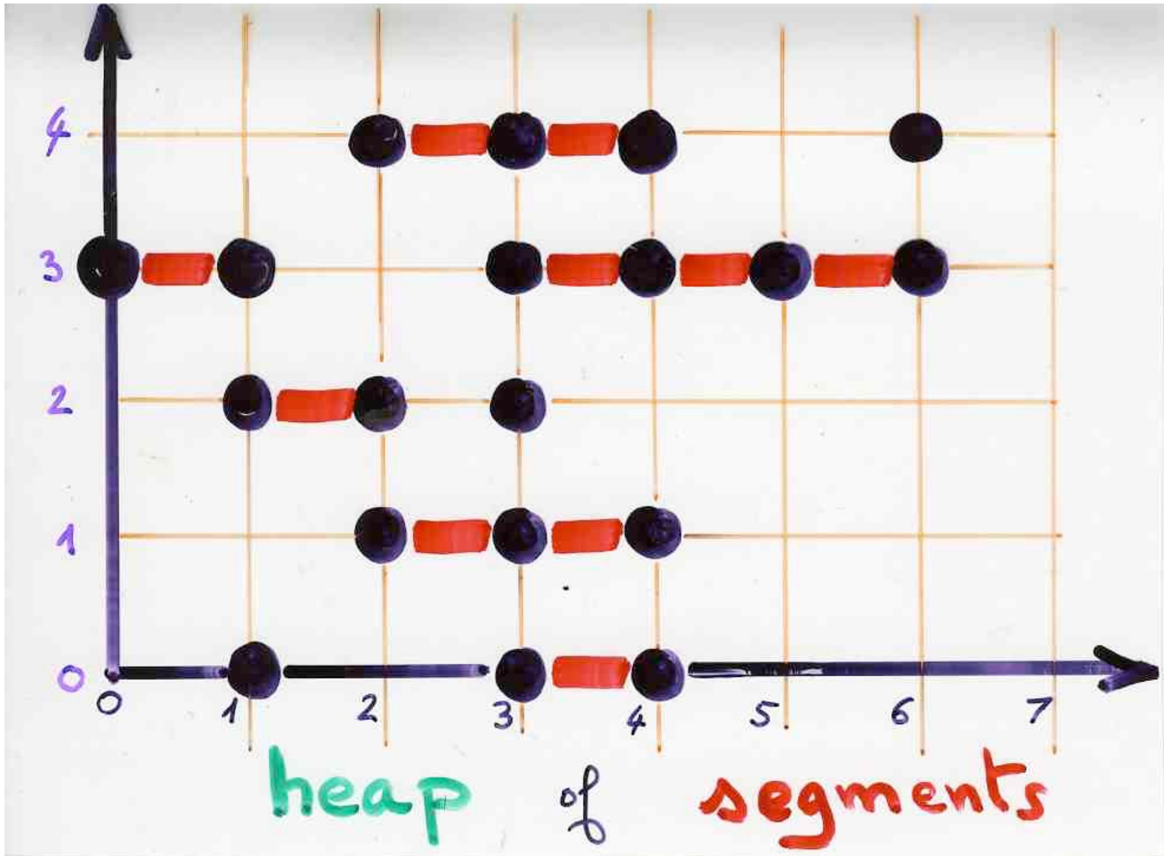
(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \,\mathcal{C}\, \beta \Rightarrow i \neq j$

(ii) $(\alpha, i) \in E$, $i > 0 \Rightarrow \exists \beta \in P$, $\alpha \,\mathcal{C}\, \beta$, $(\beta, i-1) \in E$

ex: heap of segments over $\mathbb{N}$

$P = \{ [a,b] = \{a, a+1, \dots, b\} , 0 \leq a \leq b \}$

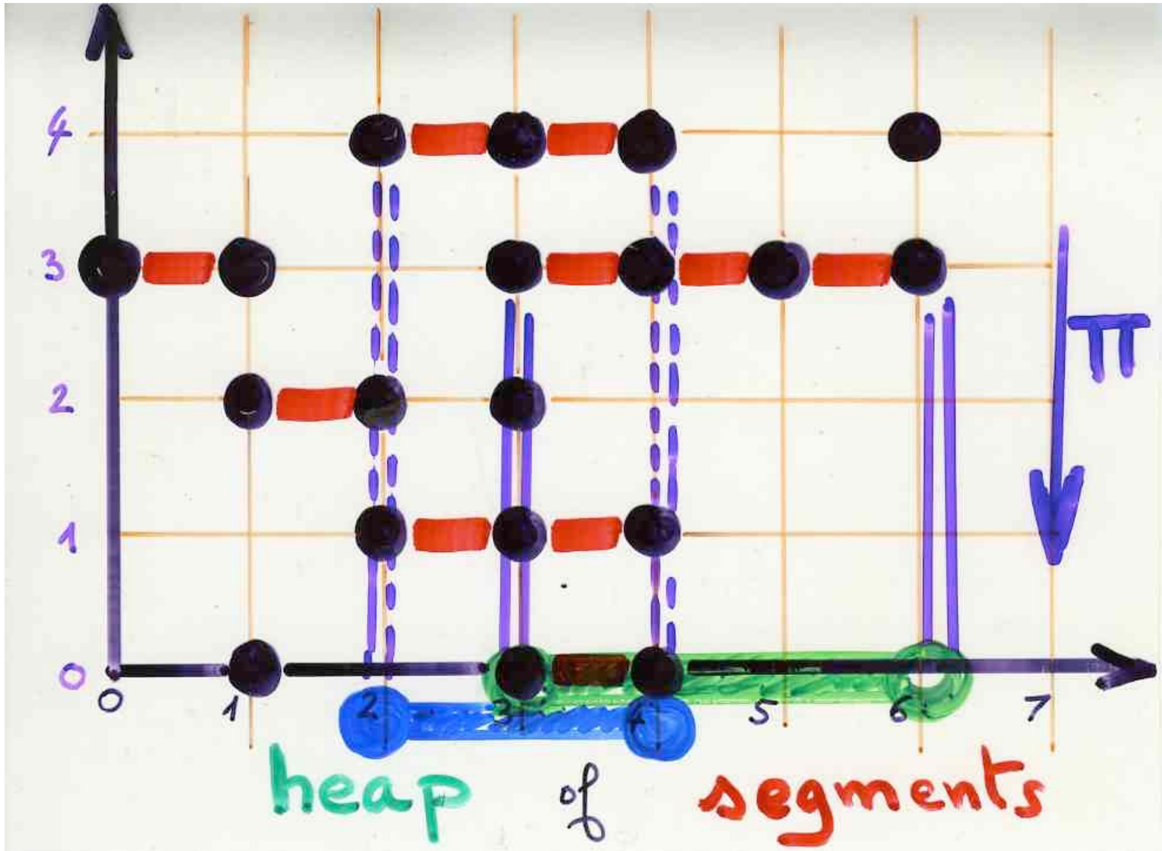
$\mathcal{C}$ $\quad [a,b] \, \mathcal{C} \, [c,d] \Leftrightarrow [a,b] \cap [c,d] \neq \emptyset$



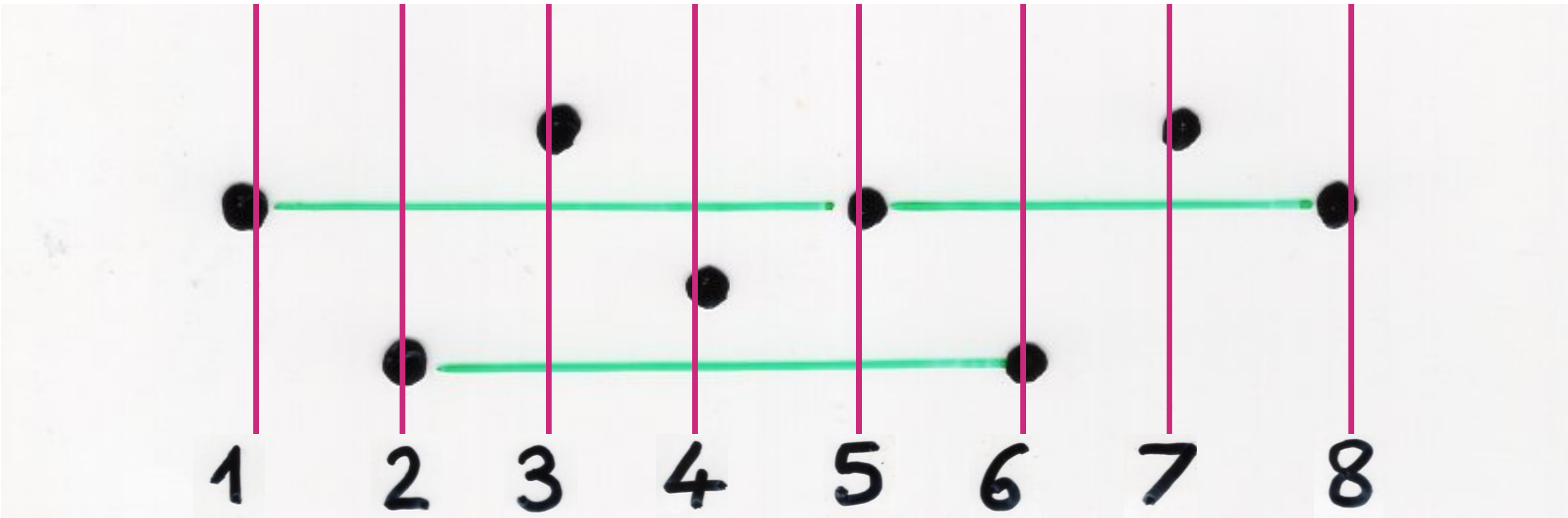
ex: heap of segments over $\mathbb{N}$

$P = \{ [a, b] = \{a, a+1, \ldots, b\} , 0 \leq a \leq b \}$

$\mathcal{C}$ $\quad [a, b] \,\mathcal{C}\, [c, d] \iff [a, b] \cap [c, d] \neq \emptyset$



Definition Laguerre heap on [1, n]



Definition Laguerre heap on $[1, n]$

- basic piece: pointed segments

  segment $[a, b] = \{a, a+1, \ldots, b\}$

  $0 \leq a \leq b$

- pointed: choice of points $a \leq j \leq b$ including a and b

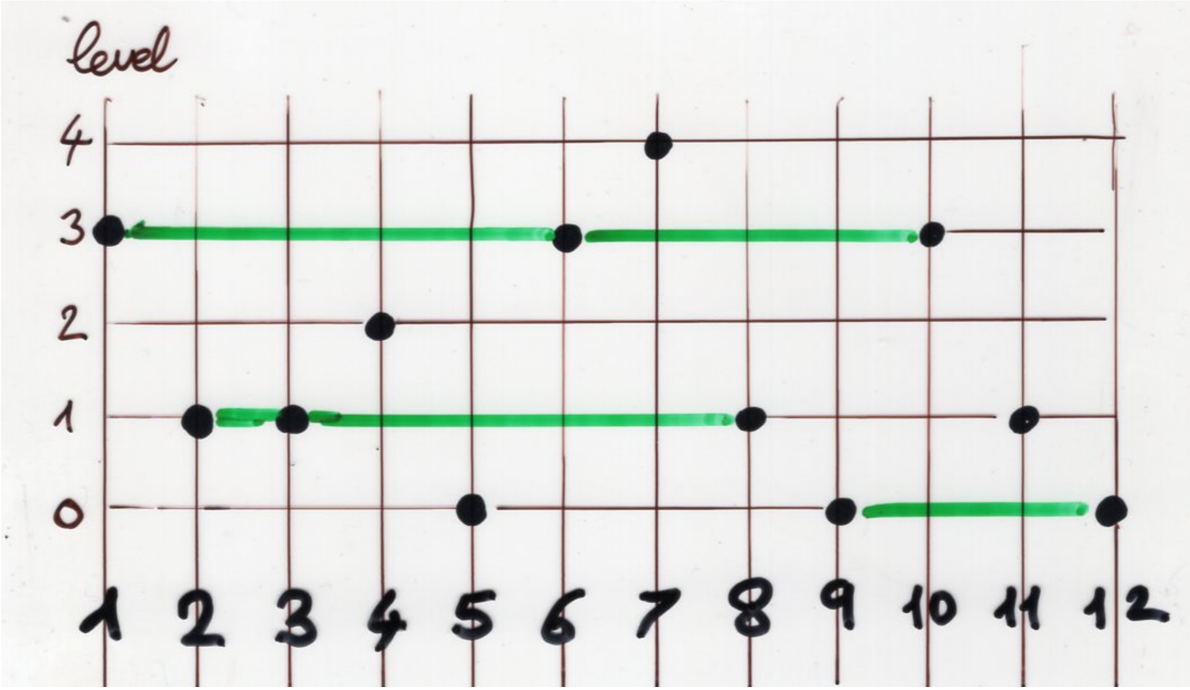
- dependency relation

  $[a, b] \cap [c, d] \neq \emptyset$

  (same as for segments)

- multilinear:

for each j, $1 \leq j \leq n$, there exist one and only one pointed segment of the heap such that j is one of the pointed element of that segment



level
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12

Bijection

(restricted) Laguerre histories

↕

Laguerre heaps

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

$$J^b\,|k\rangle = k\,|k\rangle$$

$$K\,|k\rangle = (k+1)\,|k\rangle$$

$$S^b\,|k\rangle = k\,|(k-1)\rangle$$

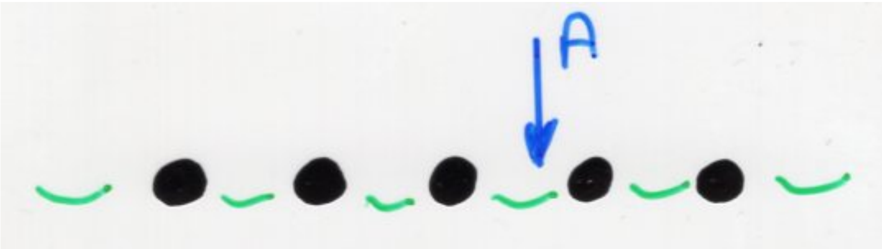
dictionnary data structure

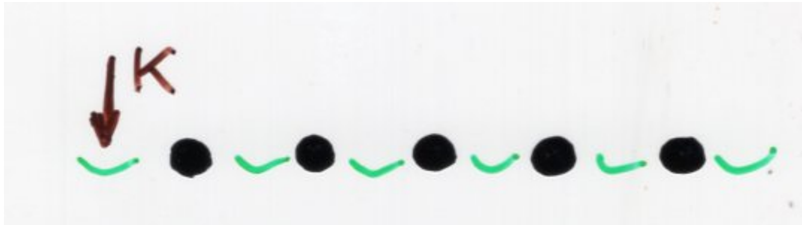
add or delete any element

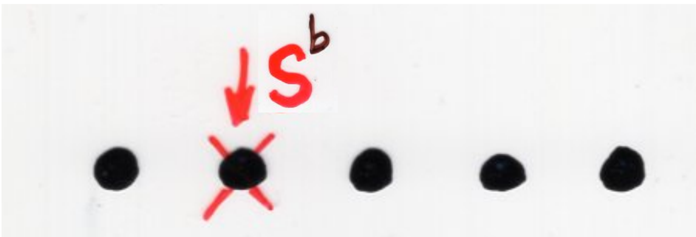
ask questions

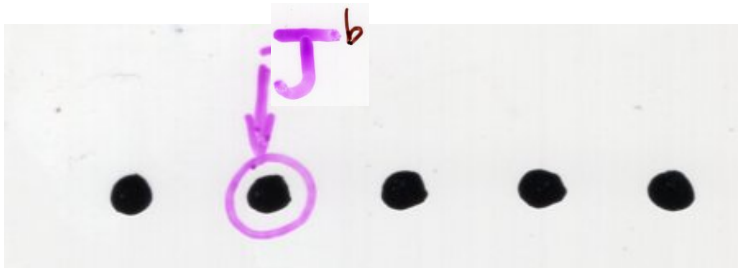
J^b positive

K negative









$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

$$\kappa_n = n!$$

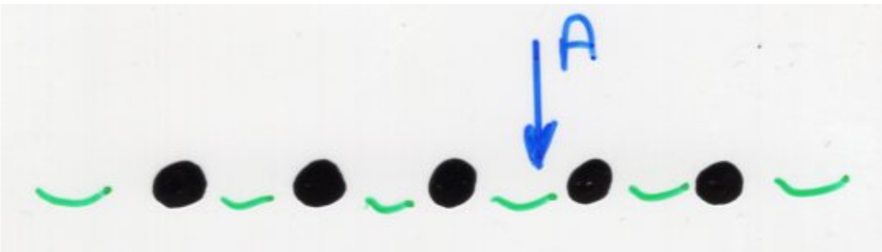
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

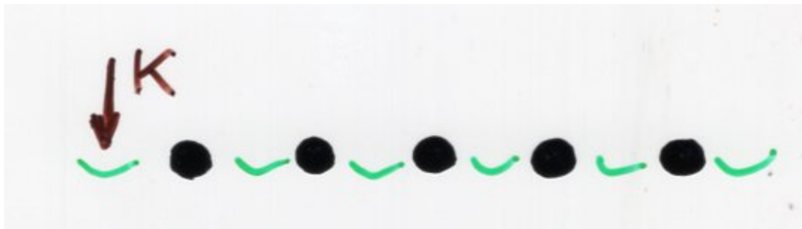
$$K\,|k\rangle = (k+1)\,|k\rangle$$

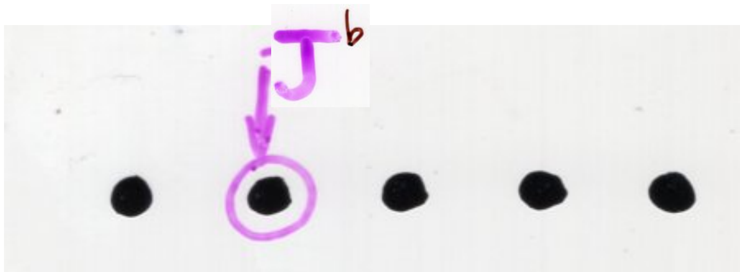
$$J^b\,|k\rangle = k\,|k\rangle$$

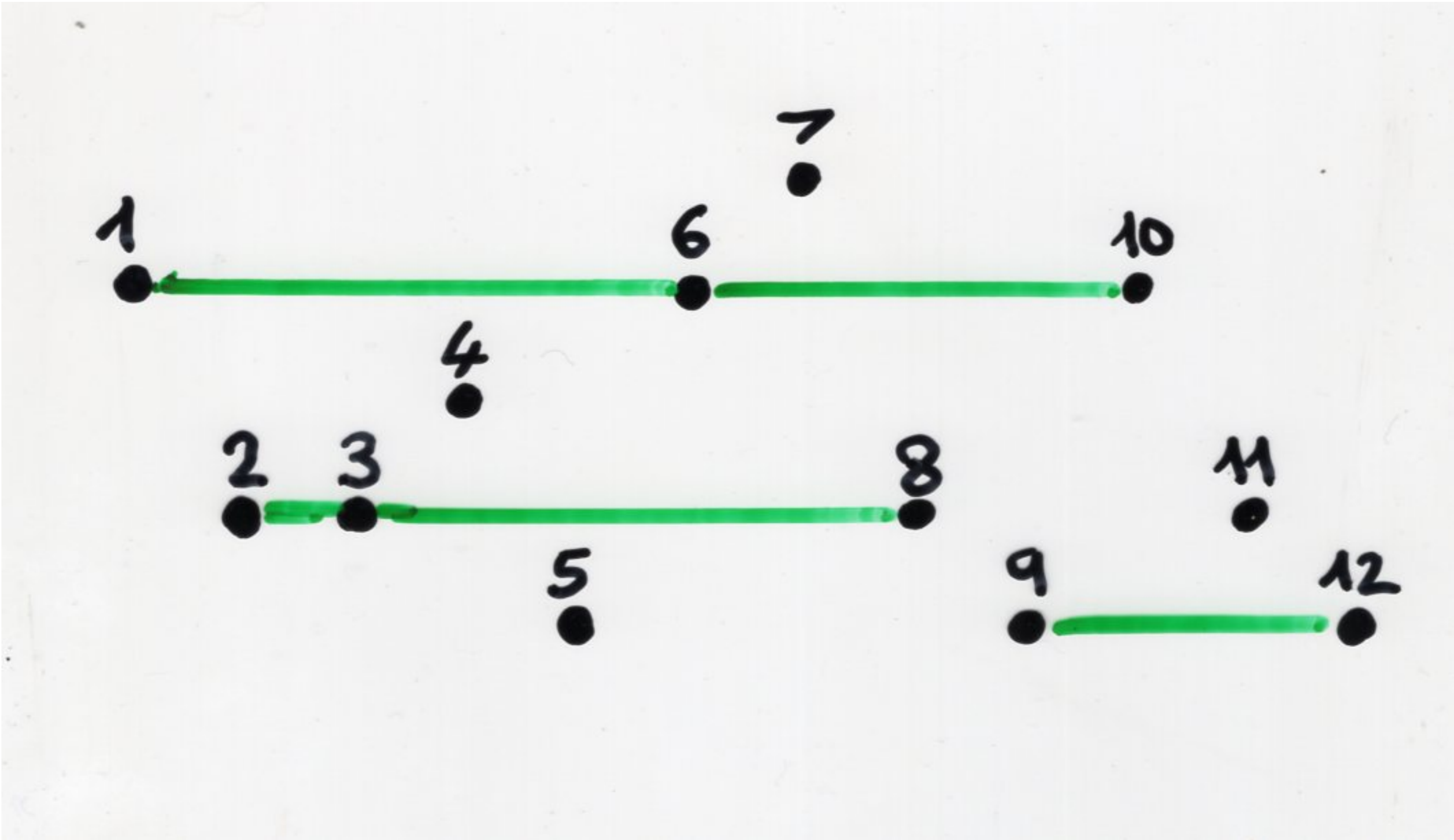
$$S^b\,|k\rangle = k\,|(k-1)\rangle$$

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases}$$



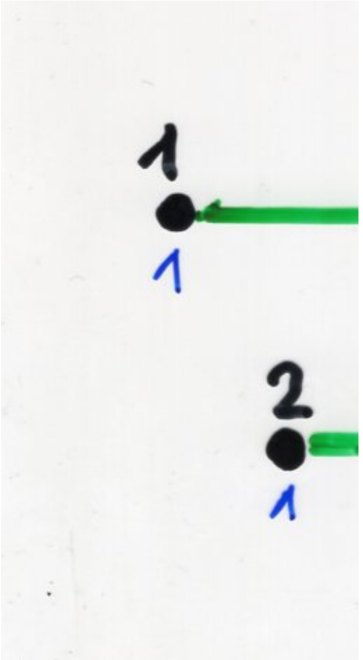


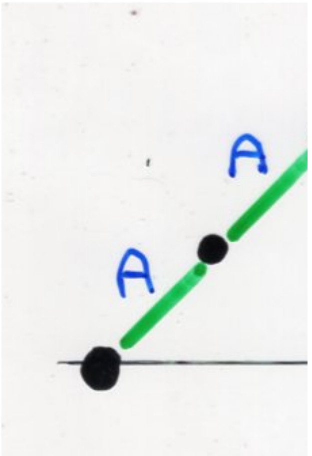


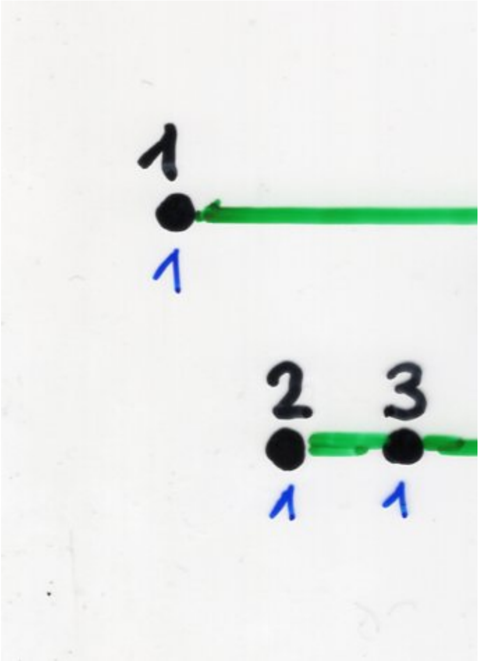

7
1
6
10
4
2
3
8
11
5
9
12

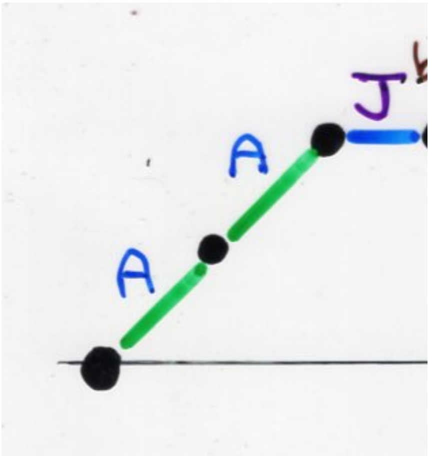

1
1

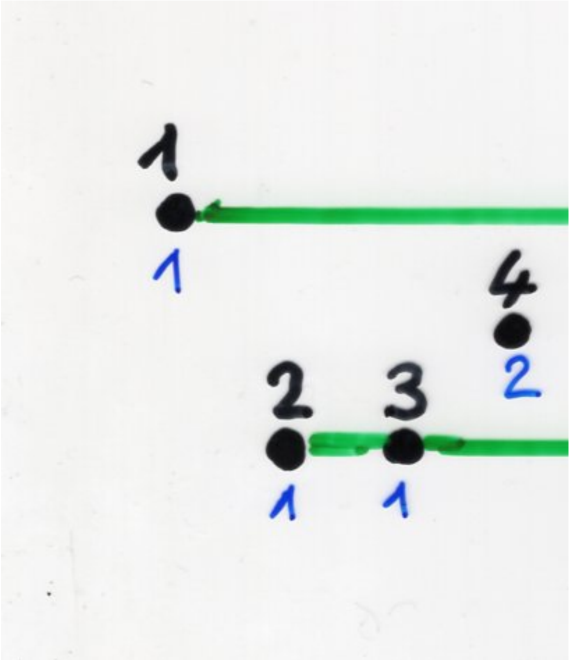

A

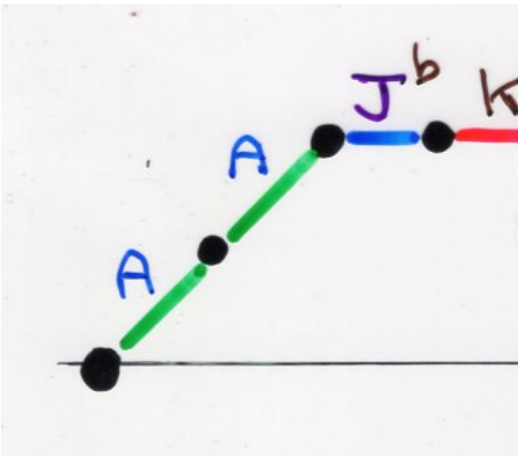

1
1
2
1

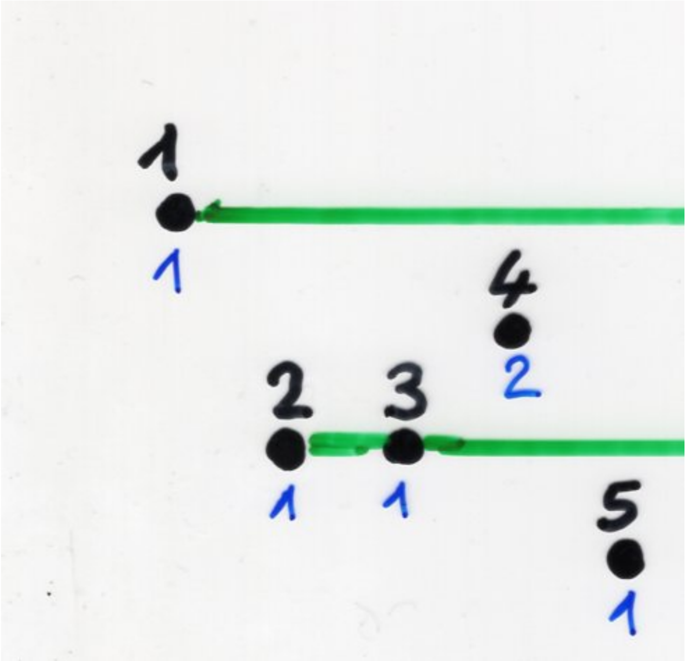

A
A


1
1
2
3
1
1


J
A
A


1
1
4
2
2
3
1
1


A
A
J b



1
1
4
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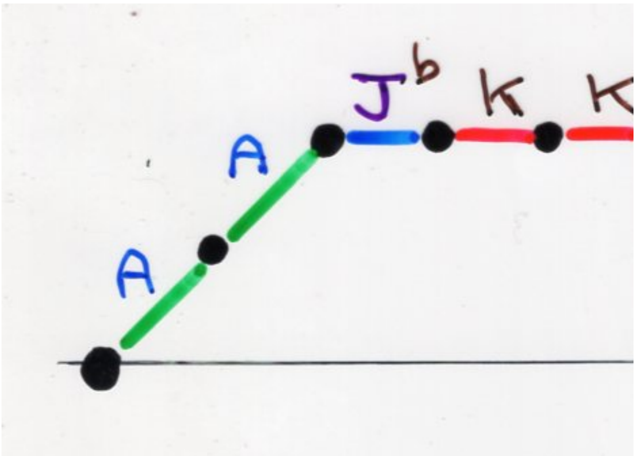
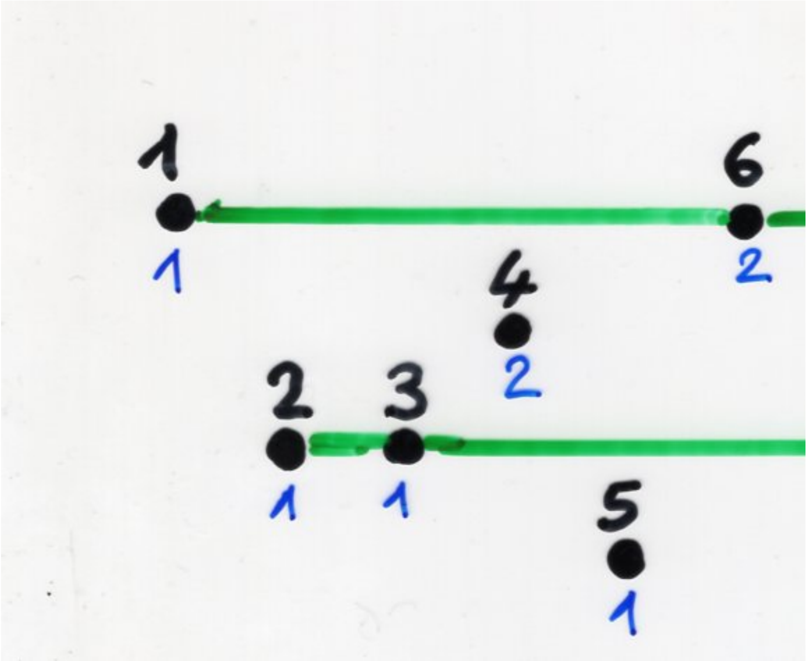
3
1
1
5
1



A
A
J^b
K
K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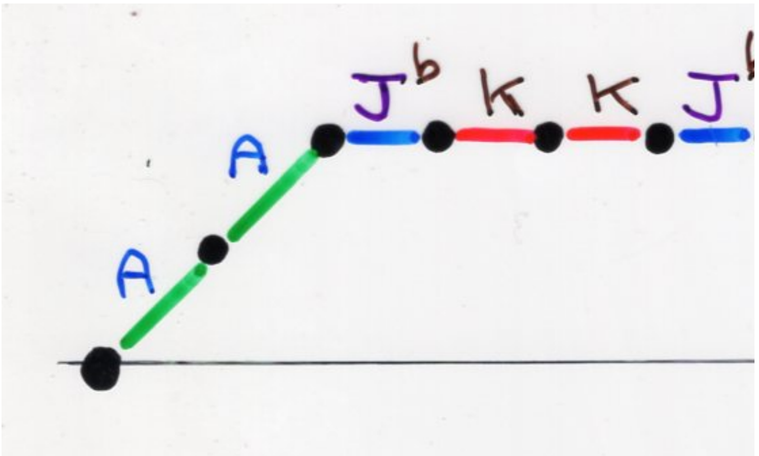
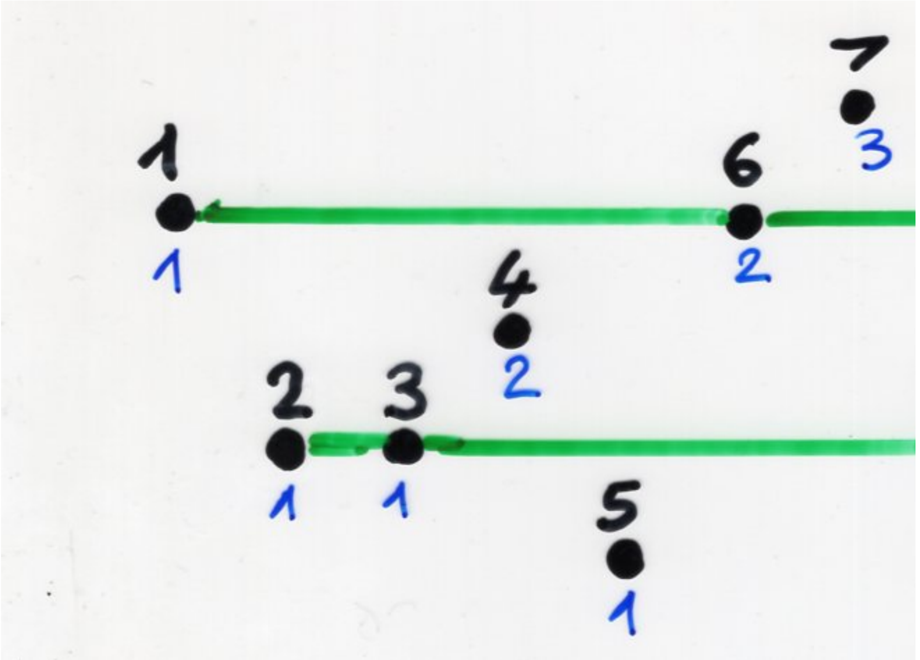

A
A
J^b
K
K
J



1
1
6
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7
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4
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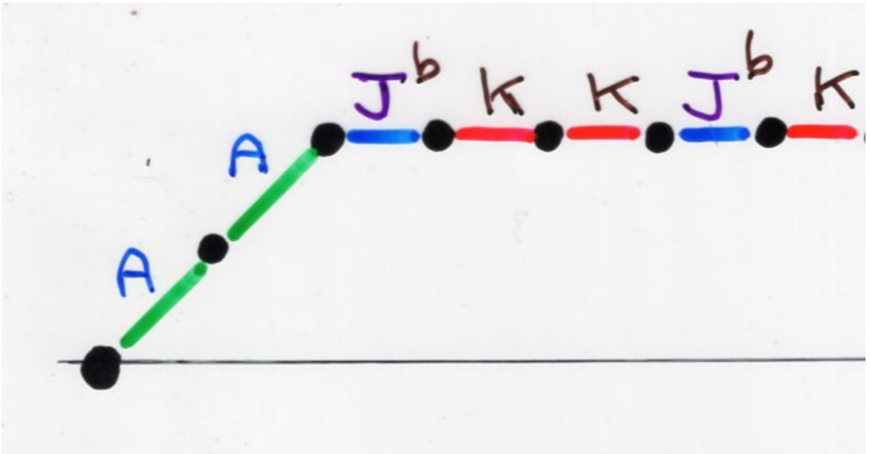
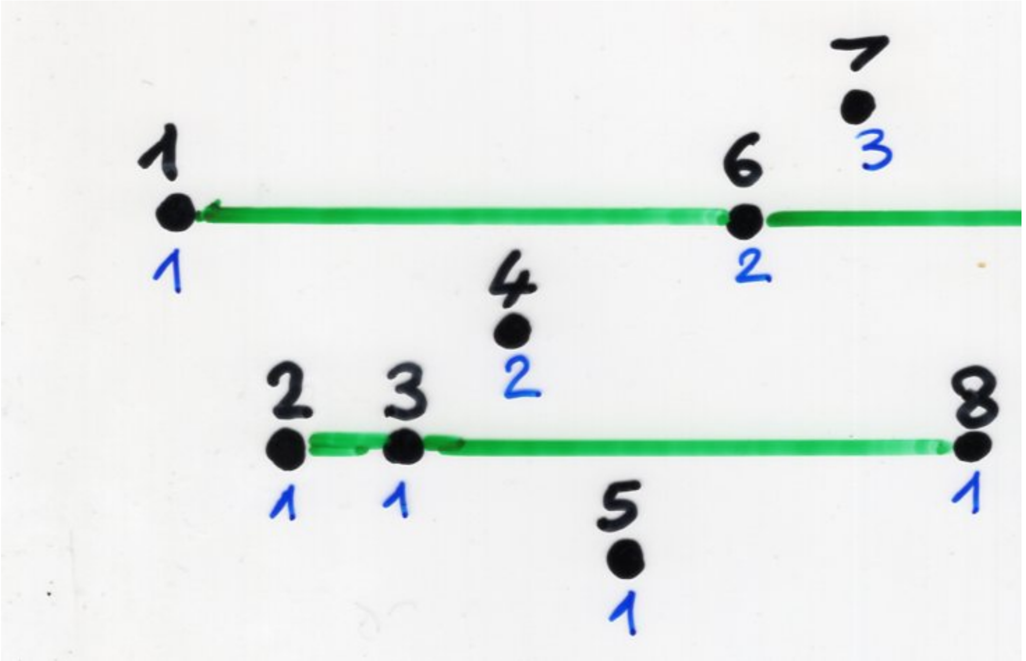
3
1
5
1



A
A
J b
K
K
J b
K



1
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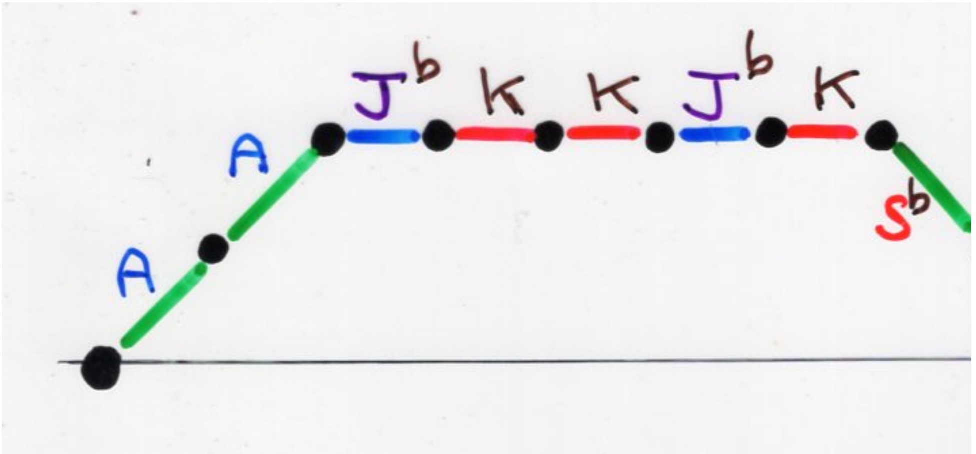
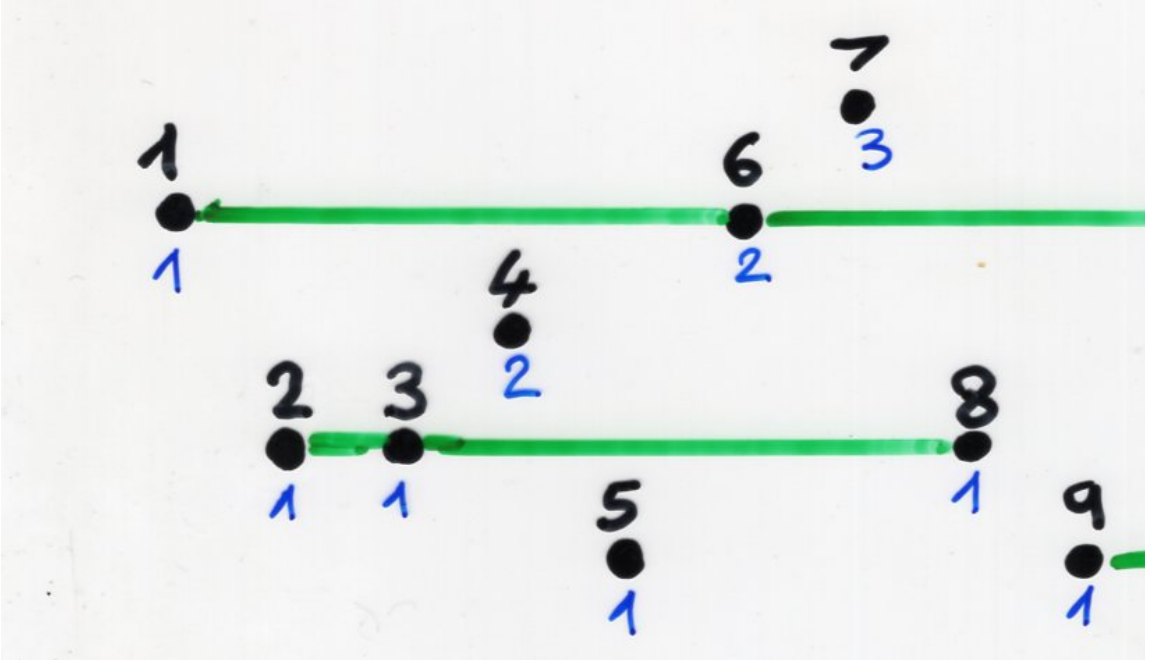
3
8
1
1
1
5
1



A
A
J b
K
K
J b
K
S b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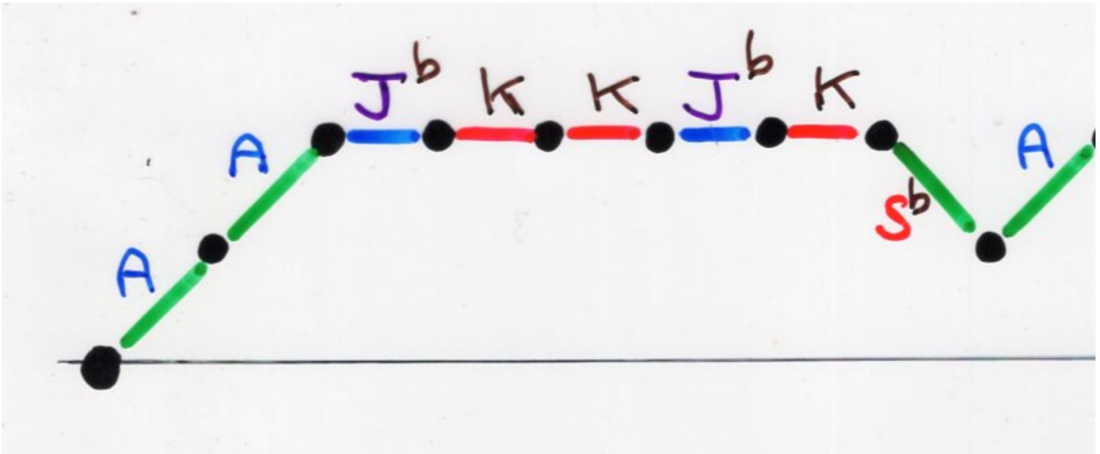
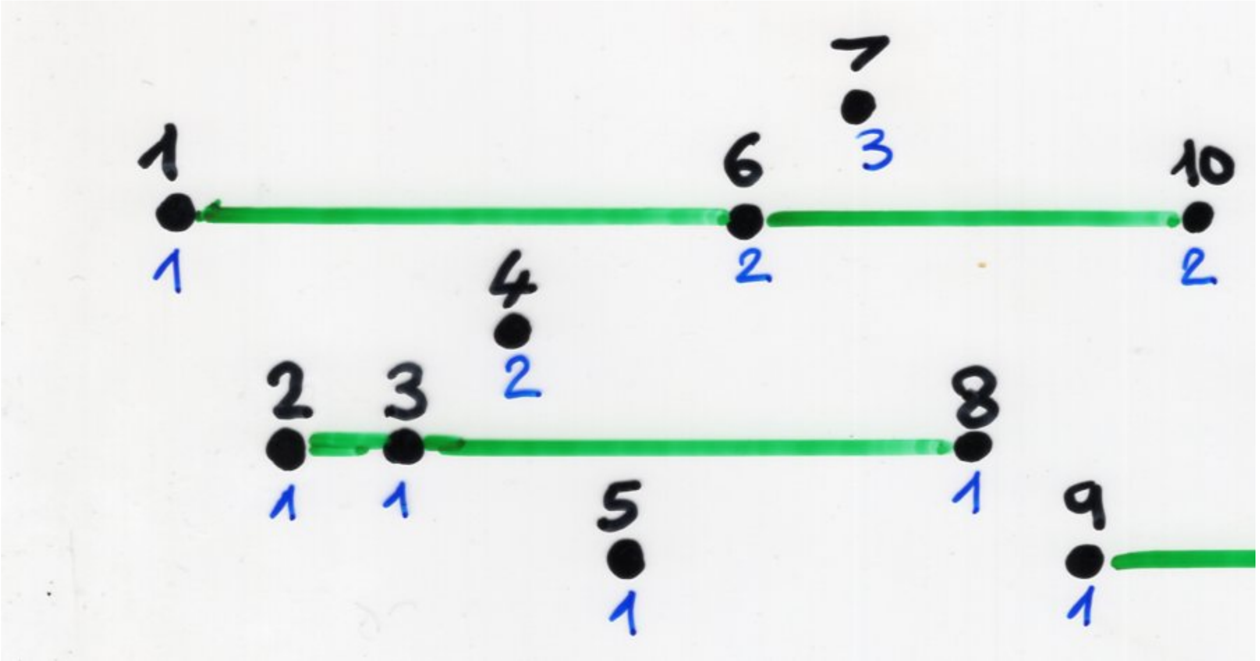
1


A
A
J^b
K
K
J^b
K
S^b
A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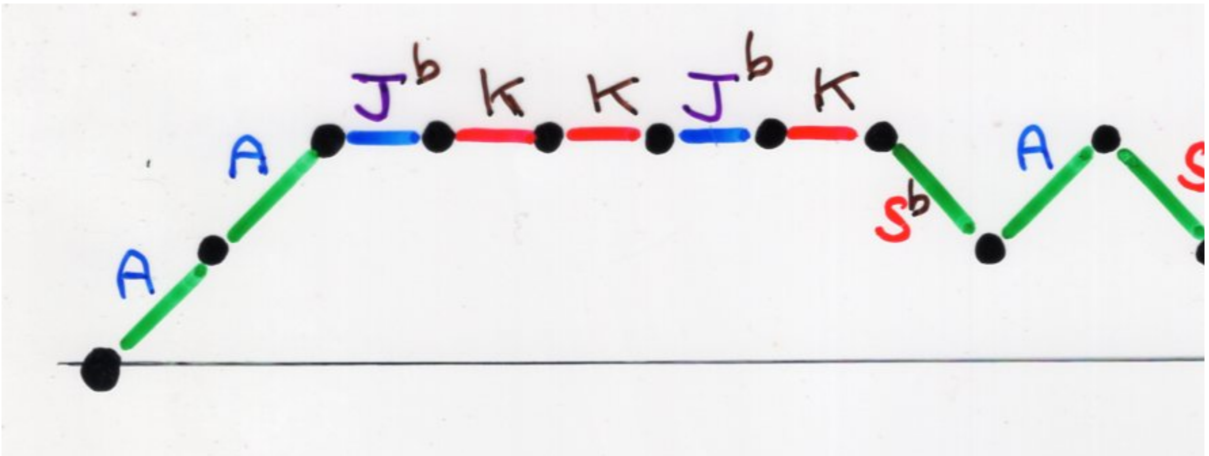
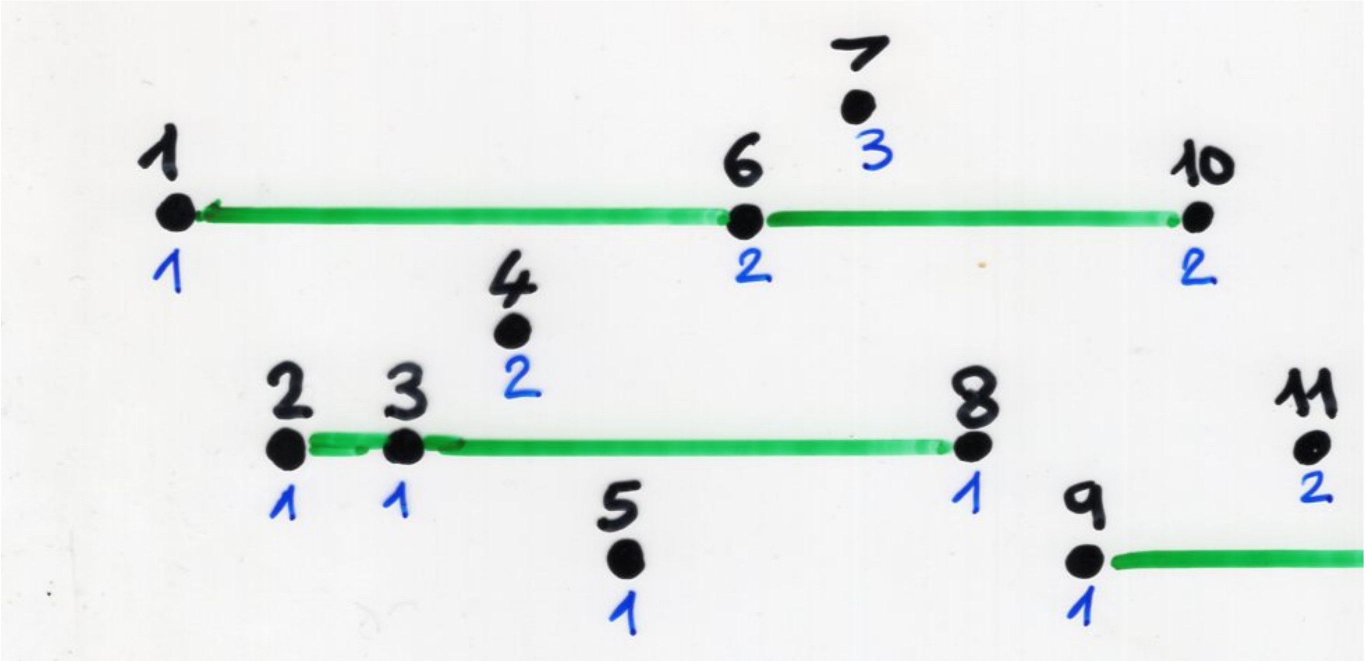

J^b
K
K
J^b
K
A
A
S^b
A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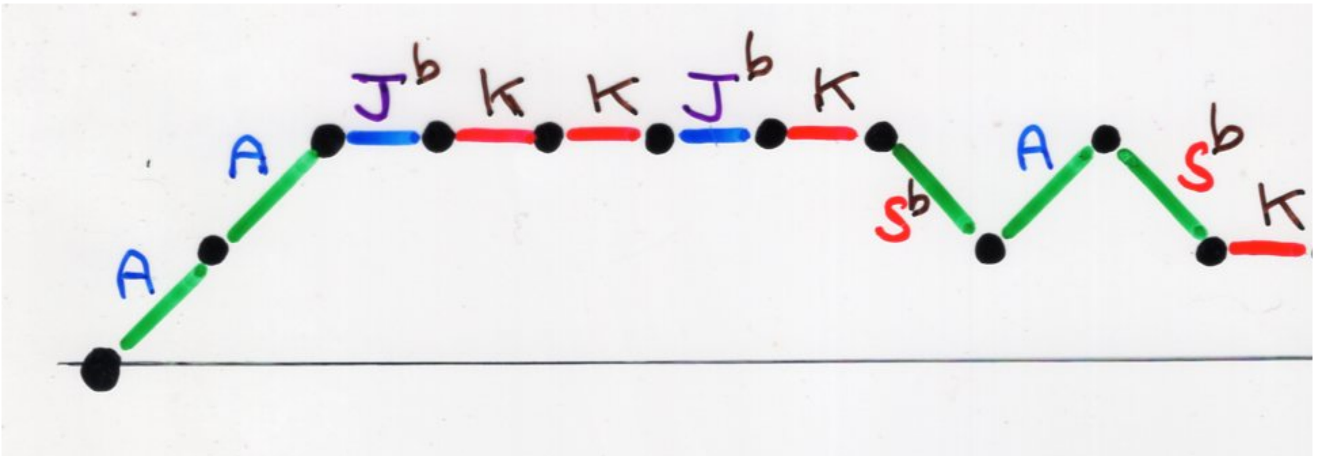
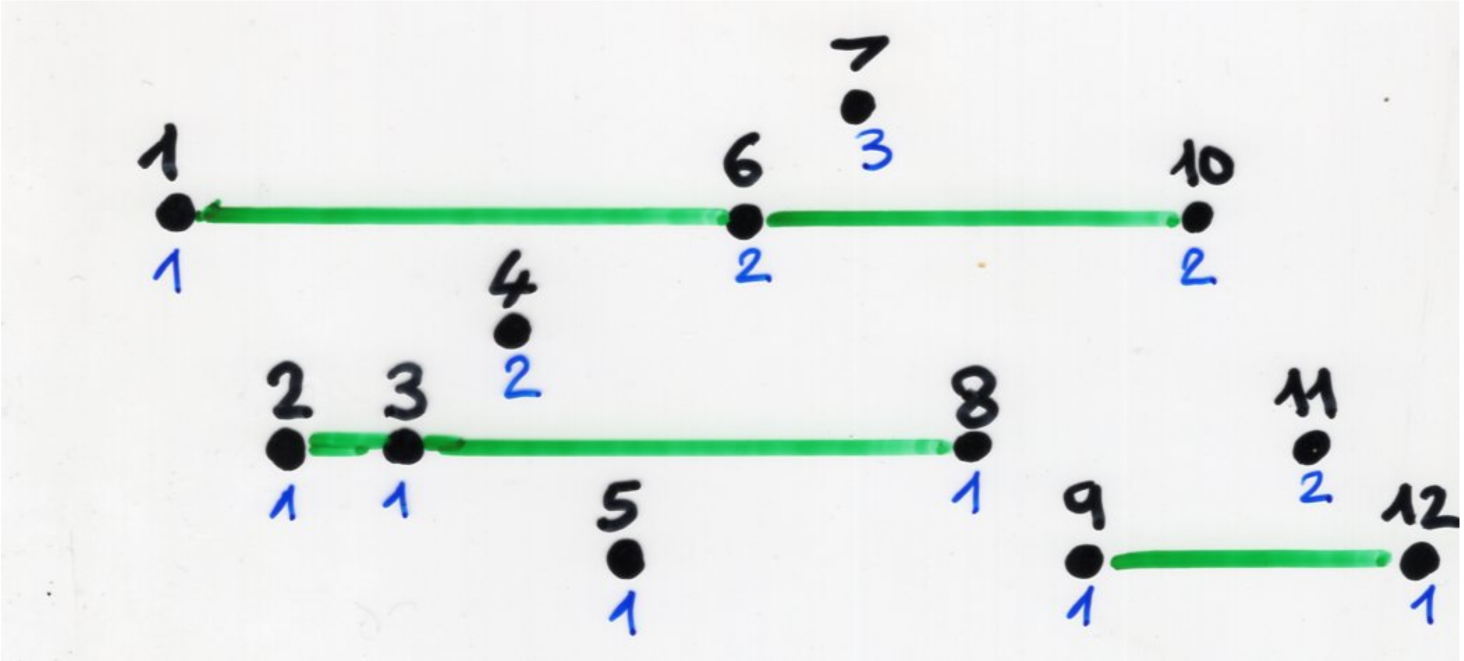

A
A
J b
K
K
J b
K
S b
A
S b
K



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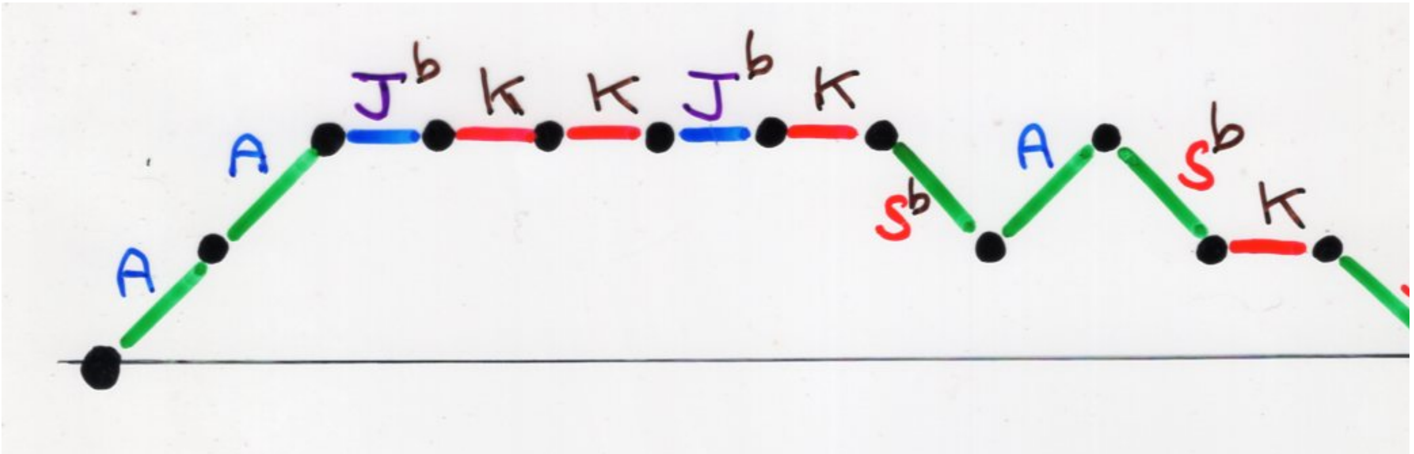
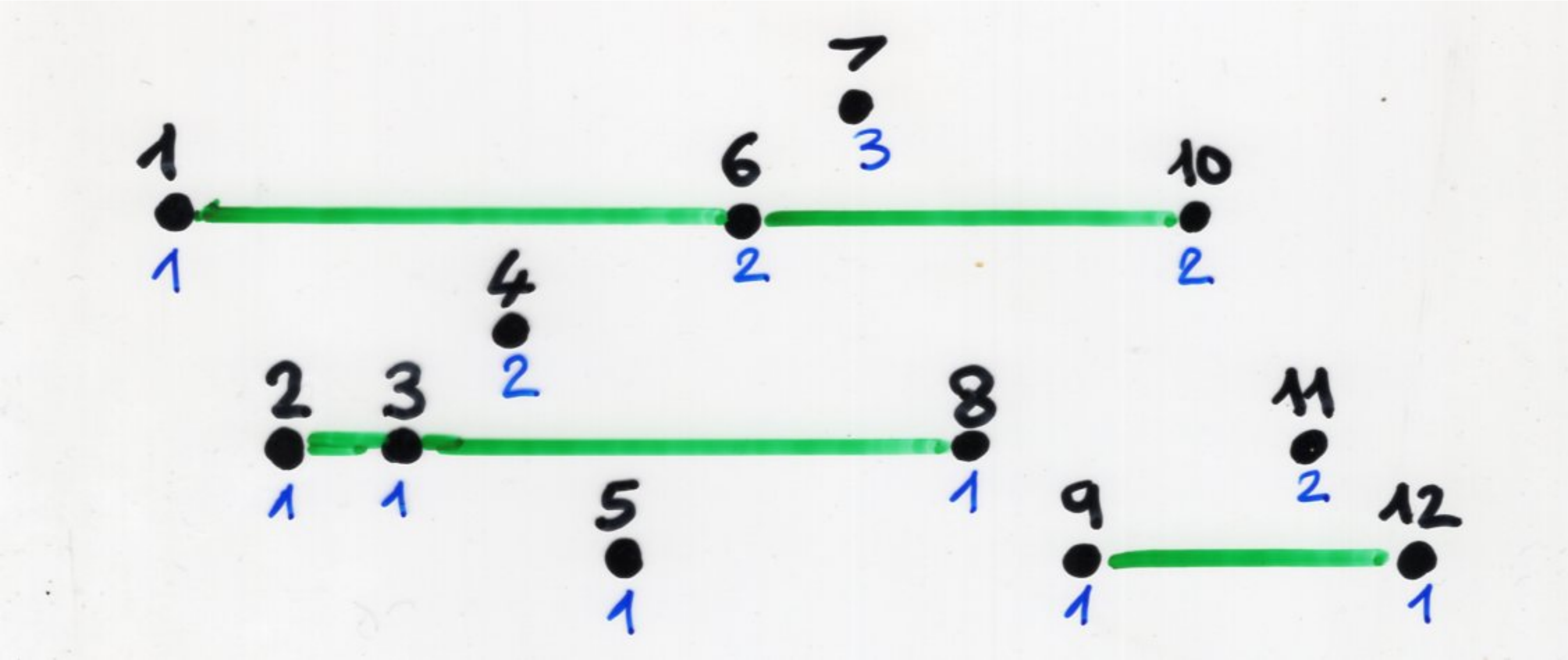
12
1



A
A
J b
K
K
J b
K
S b
A
S b
K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3
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8
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5
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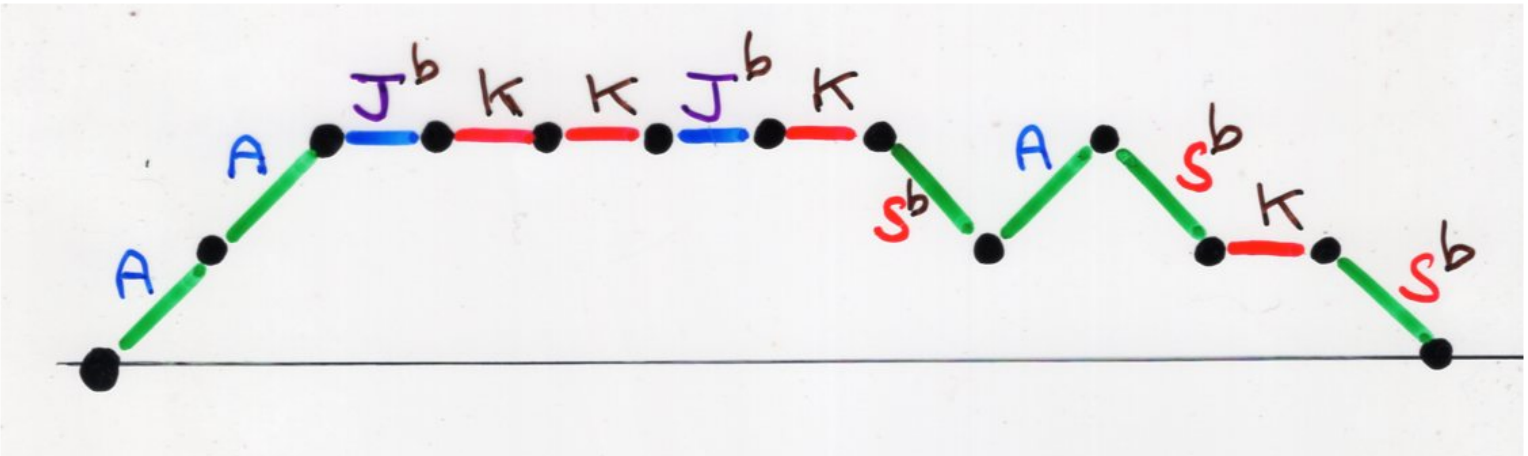
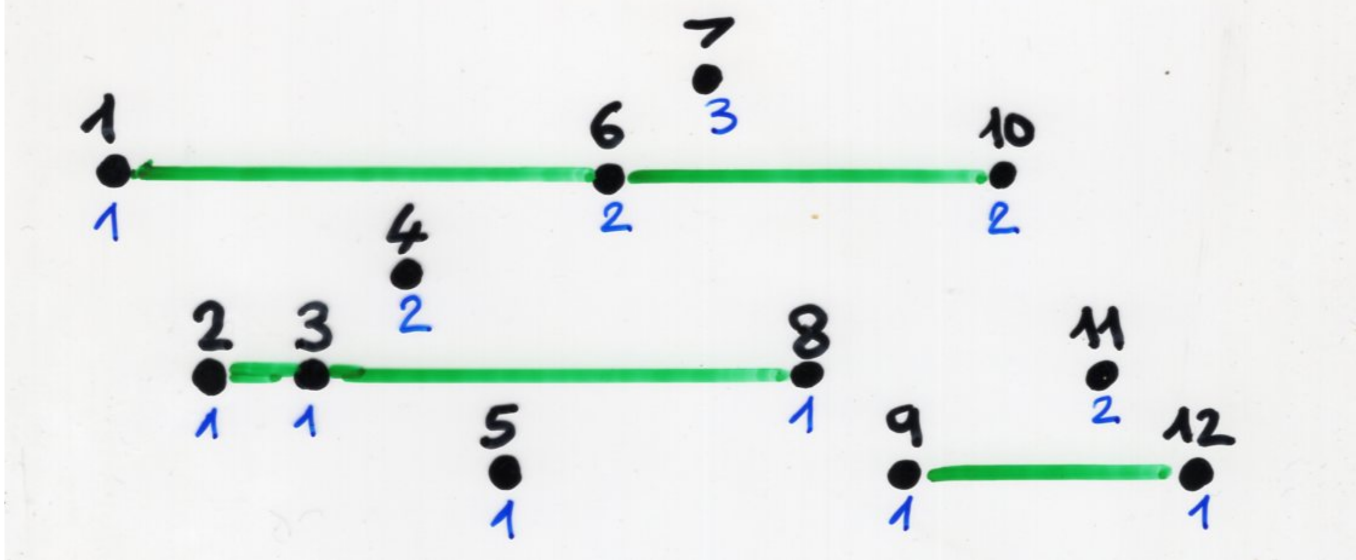
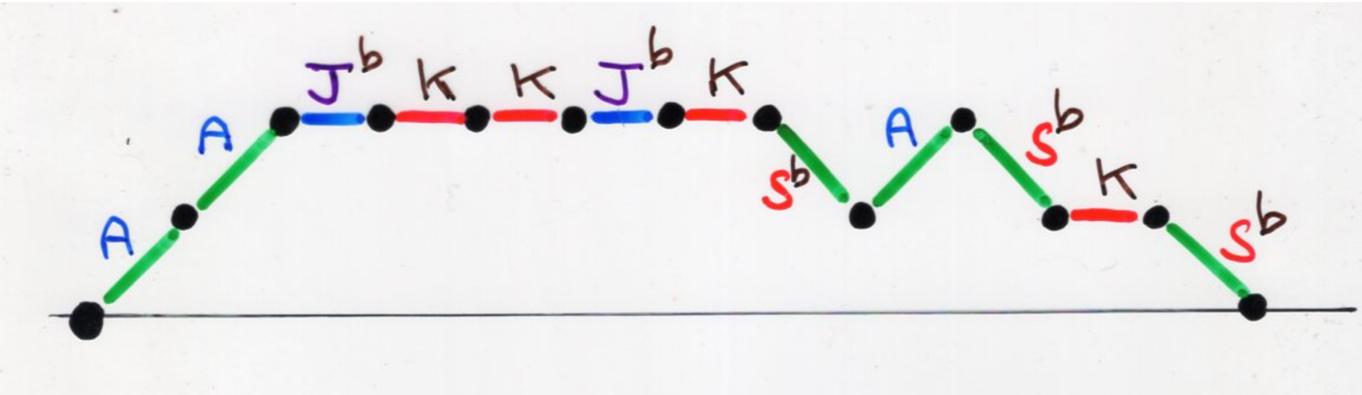
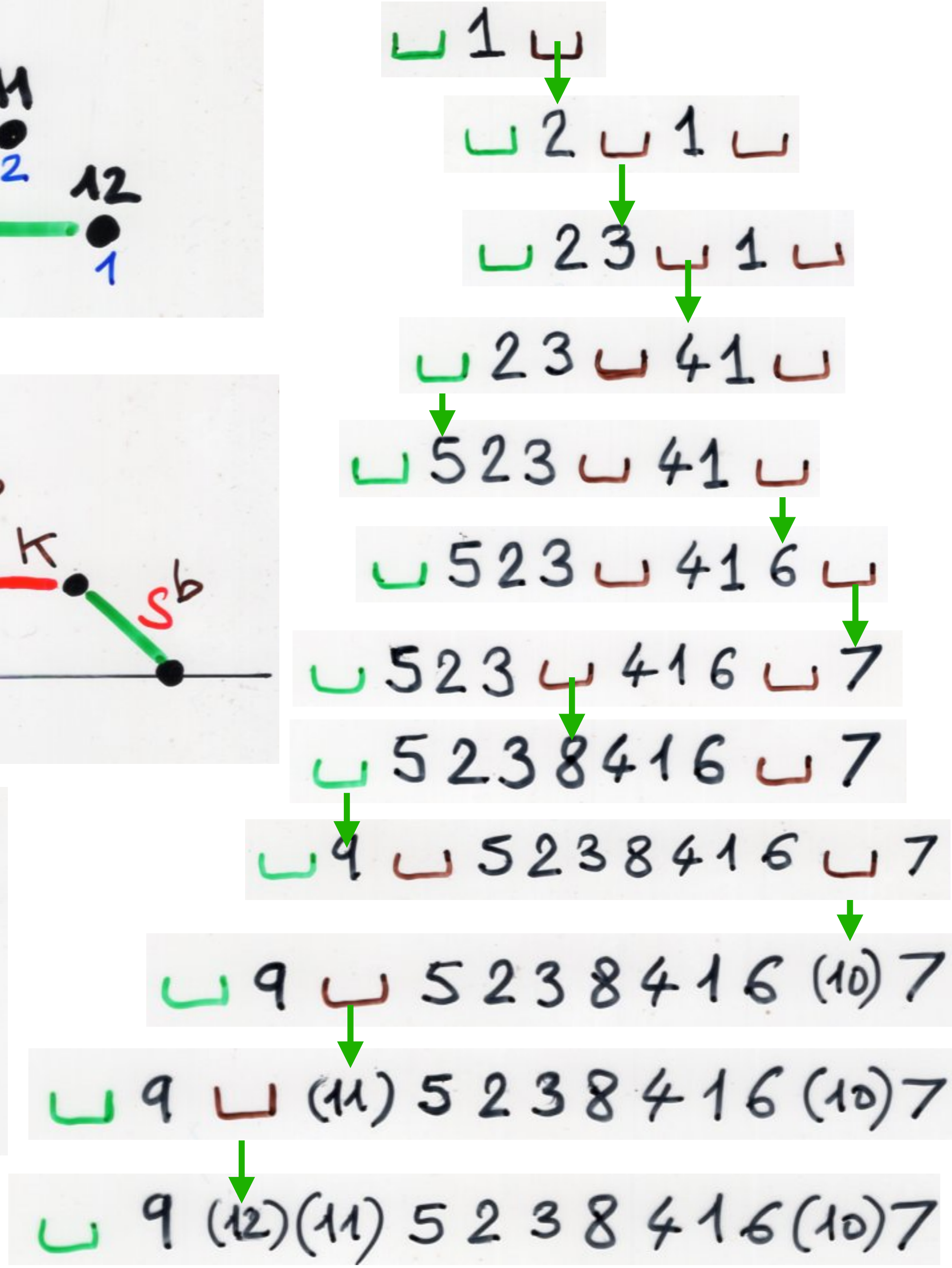
11
2
9
1
12
1


A
A
J b
K
K
J b
K
S b
A
S b
K
S b







$$A|k\rangle = (k+1)\,|(k+1)\rangle$$

$$K|k\rangle = (k+1)\,|k\rangle$$

$$J^b|k\rangle = k\,|k\rangle$$

$$S^b|k\rangle = k\,|(k-1)\rangle$$

$$\begin{cases} a_k = k+1 \\ b'_k = k+1 \\ b''_k = k \\ c_k = k \end{cases}$$

$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)\,ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)}\, b^{p(\sigma)}\, c^{dr(\sigma)}\, d^{dd(\sigma)}\, \alpha^{f(\sigma)}\, \beta^{s(\sigma)}$$

a — $v(\sigma)$ = number of valleys of σ  A

b — $p(\sigma)$ = number of peaks of σ  S^b

c — $dr(\sigma)$ = number of double rises of σ  J^b

d — $dd(\sigma)$ = number of double descents of σ  K

α — $f(\sigma)$ = number of lr-min elements which are a descent of σ

β — $s(\sigma)$ = number of lr-min elements of σ

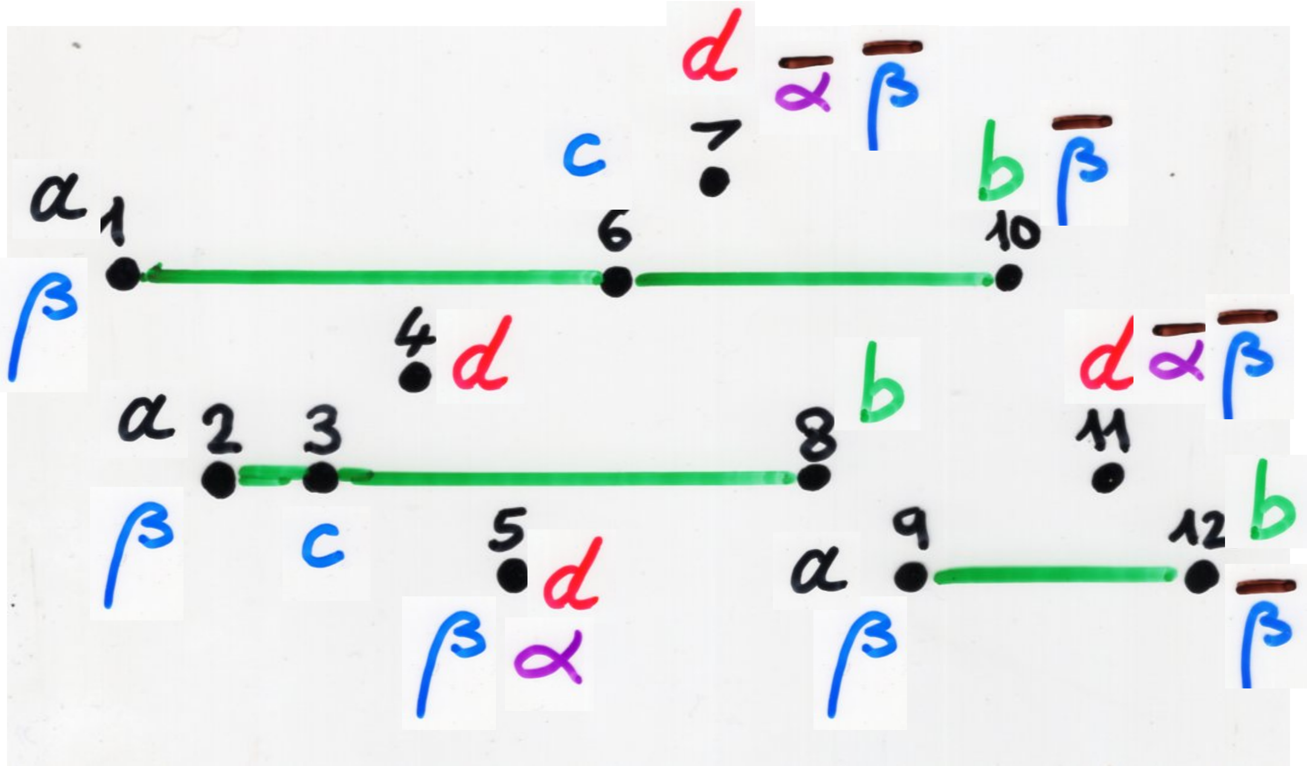
$$\begin{cases} b_k = (\alpha\beta + k(c+d)) \\ \lambda_k = k(k-1+\beta)ab \end{cases}$$

$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{v(\sigma)} b^{p(\sigma)} c^{dr(\sigma)} d^{dd(\sigma)} \alpha^{f(\sigma)} \beta^{s(\sigma)}$$

a	cycle valley	$\sigma^{-1}(x) > x < \sigma(x)$	$cv(\sigma)$	A
b	cycle peak	$\sigma^{-1}(x) < x > \sigma(x)$	$cp(\sigma)$	S^b
c	cycle double rise	$\sigma^{-1}(x) < x < \sigma(x)$	$cdr(\sigma)$	J^b
d	cycle double descent	$\sigma^{-1}(x) > x > \sigma(x)$	$cdd(\sigma)$	K
α	fixed point	$\sigma(x) = x$		
β	number of cycles		$cyc(\sigma)$	

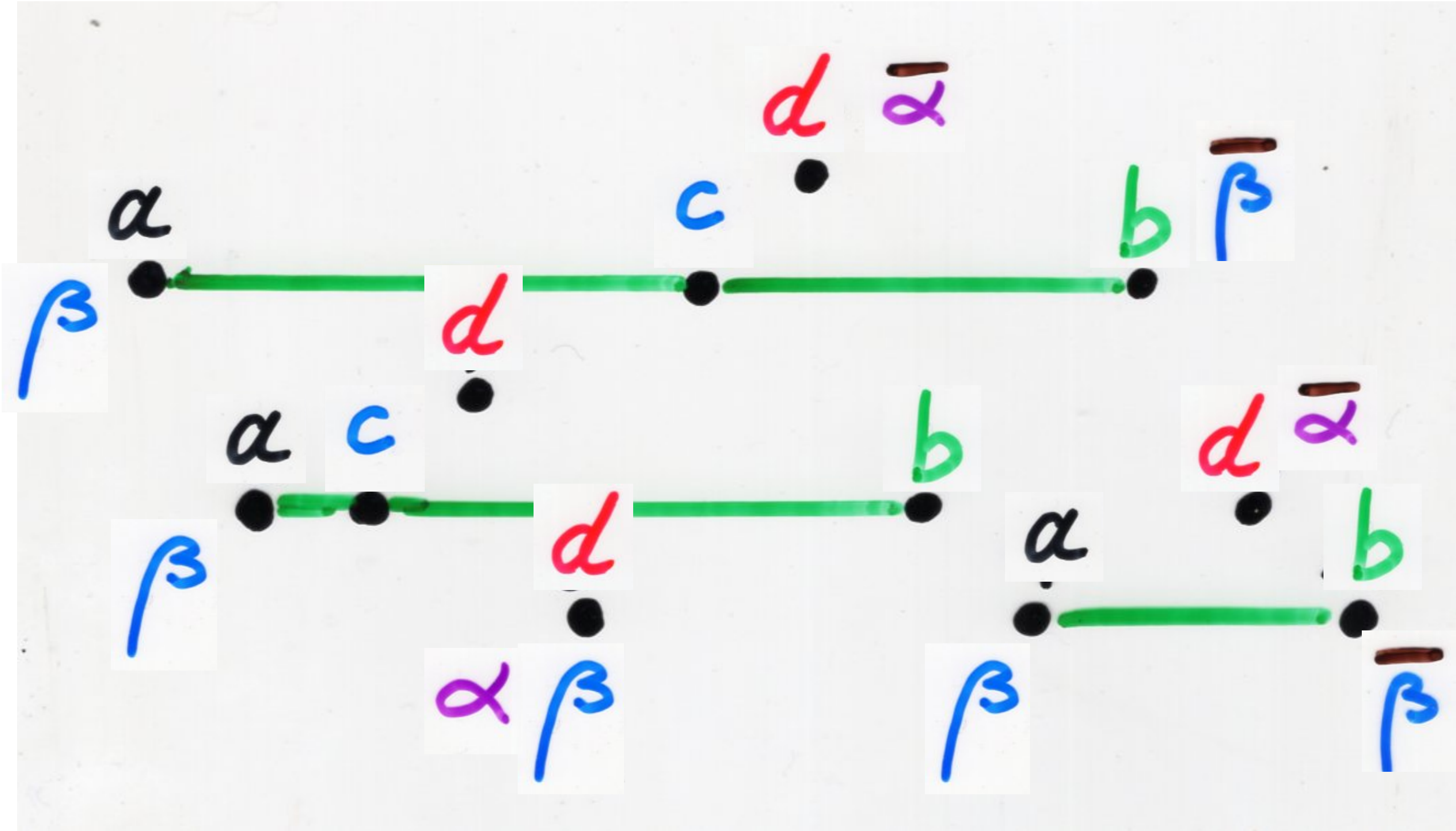
$$\mu_n = \sum_{\sigma \in \mathfrak{S}_n} a^{(\ )} b^{(\ )} c^{(\ )} d^{(\ )} \alpha^{(\ )} \beta^{(\ )} \bar{\alpha}^{(\ )} \bar{\beta}^{(\ )}$$

a	A
b	S^b
c	J^b
d	K
α	$\bar{\alpha}$
β	$\bar{\beta}$

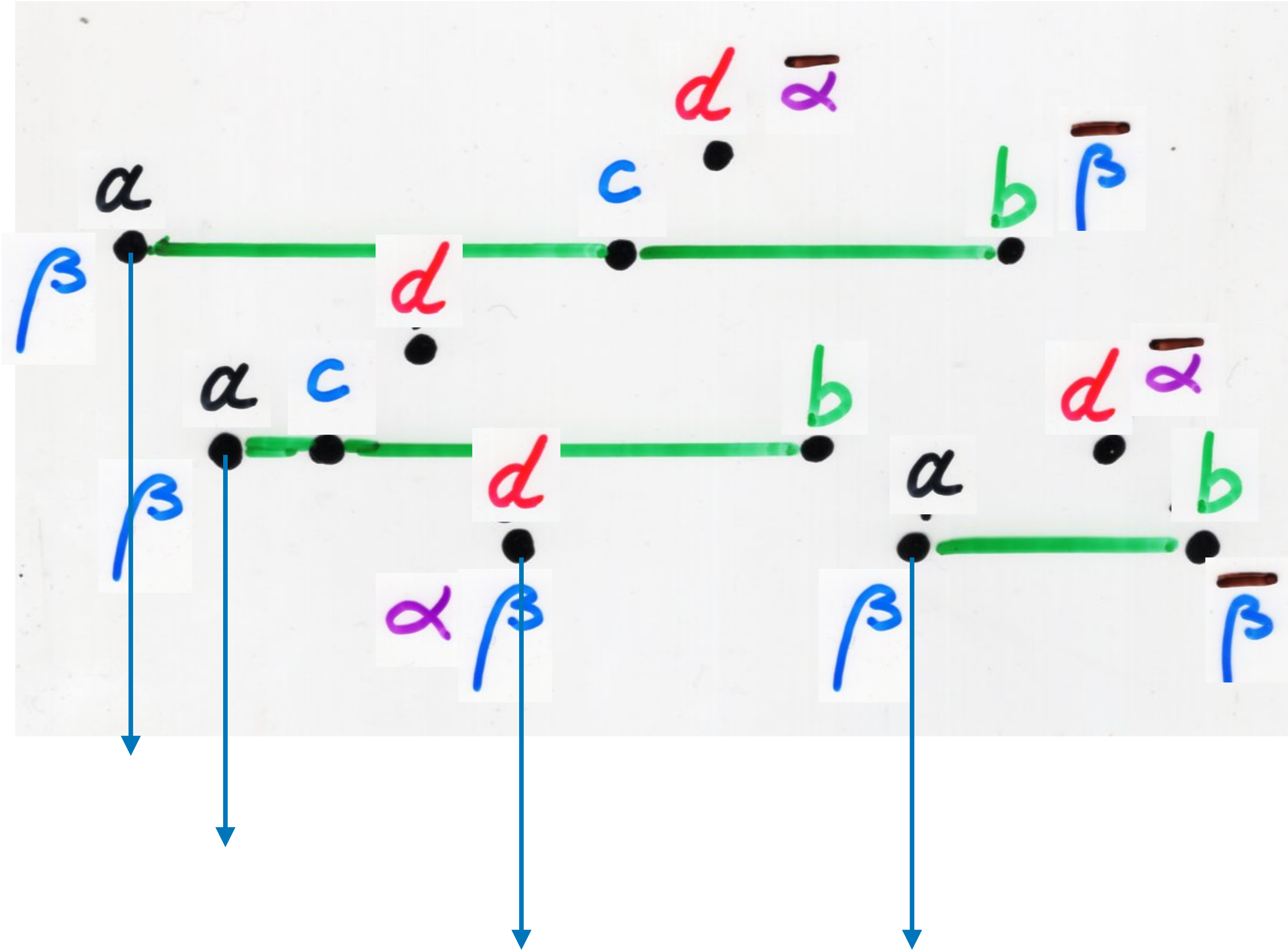


$\sqcup$ 9 (12)(11) 5 2 3 8 4 1 6 (10) 7

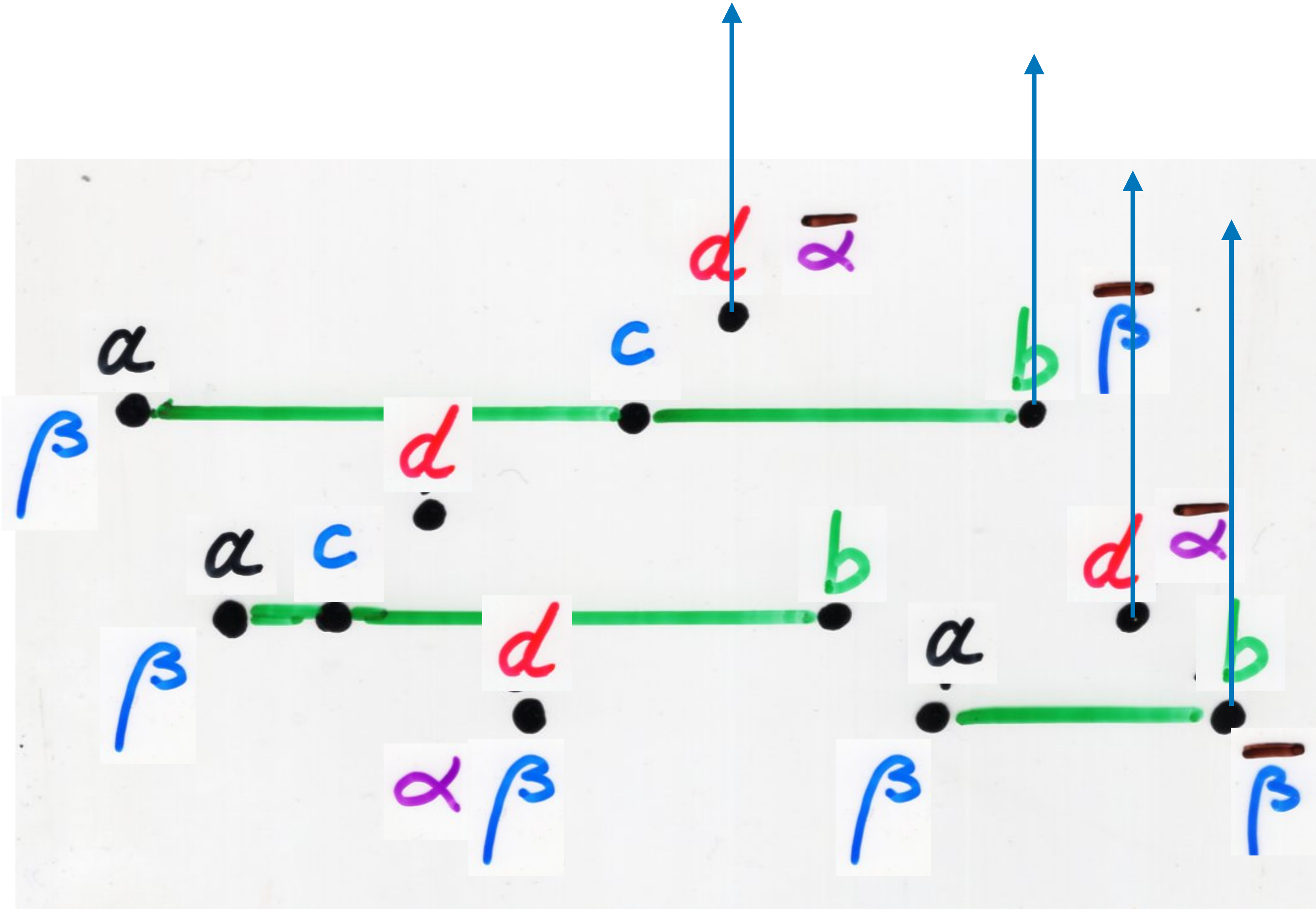
rl-max



d
$\bar{\alpha}$
a
c
b
$\bar{\beta}$
β
d
a
c
b
d
$\bar{\alpha}$
β
d
a
b
α
β
β
$\bar{\beta}$



a
β
c
d
ᾱ
b
β̄
a
c
d
b
β
d
ᾱ
a
b
α
β
β
β̄



d
α
a
c
b
β
d
a
c
b
d
α
β
a
b
d
α
β
β
β

Conclusion: relation with the PASEP

ABjC, Part III

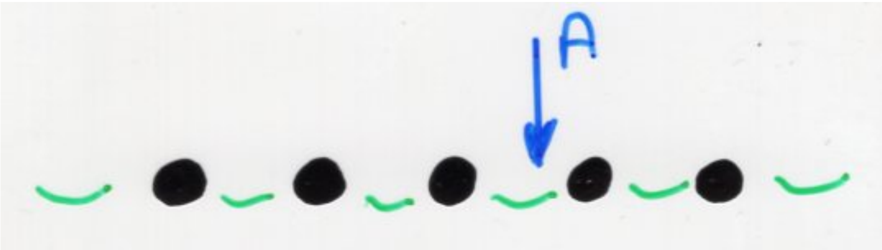
$$A|k\rangle = (k+1)|(k+1)\rangle \qquad J^b|k\rangle = k|k\rangle$$

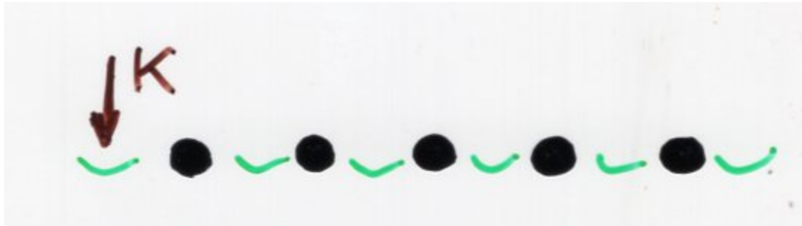
$$K|k\rangle = (k+1)|k\rangle \qquad S^b|k\rangle = k|(k-1)\rangle$$

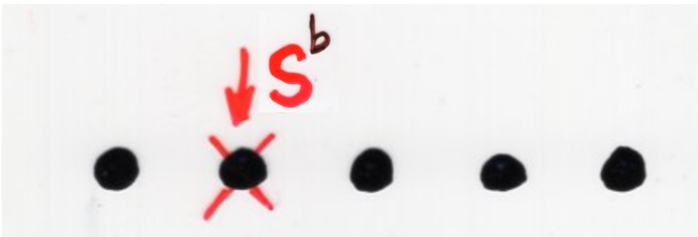
dictionnary data structure

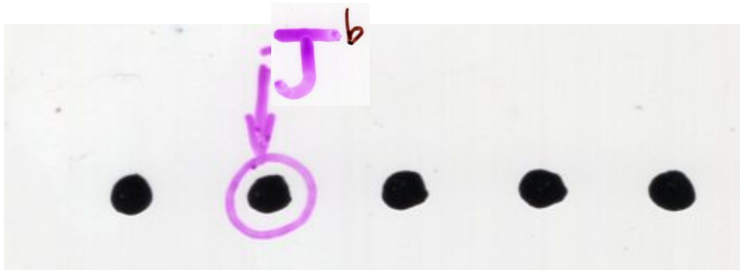
add or delete any element

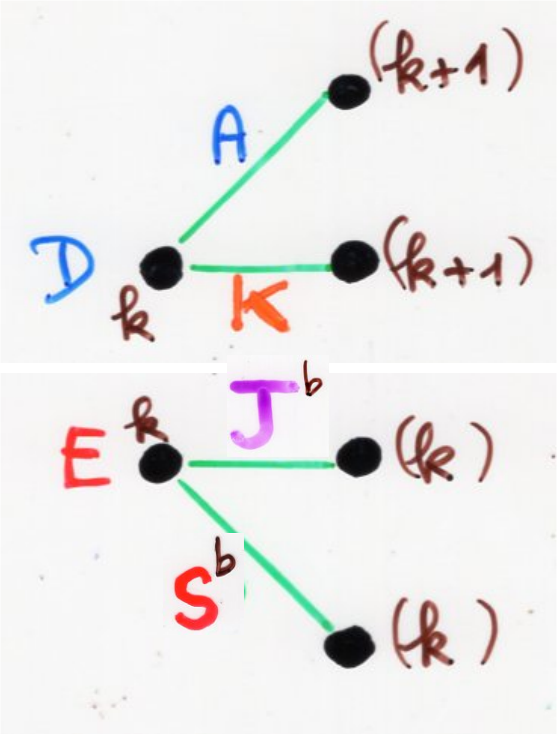
ask	questions
J^b	positive
K	negative











D, E "restricted"

$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

$$b_k = (k+1)$$

$$\lambda_k = k^2$$

$$\mu_n = n!$$

restricted
Laguerre
histories

PASEP
algebra

D, E "large"

$$\langle k | A = (k+1) \langle (k+1)|$$
$$\langle k | K = (k+1) \langle k|$$
$$\langle k | J = (k+1) \langle k|$$
$$\langle k | S = (k+1) \langle (k-1)|$$

$$D = A + K$$
$$E = S + J$$

$$DE = ED + E + D$$

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$

Laguerre histories

PASEP algebra

$$DE = qED + E + D$$

PASEP
algebra

ABjC, Part III

No class Thursday 7

Next class: Monday 11 February