Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 2
Moments and histories

Ch 2a

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

IMSc, Chennai
January 28, 2019

mirror website
www.imsc.res.in/~viennot

From Chapter 1 ...

Paths and moments

4 examples orthogonal polynomials

Tchebychev 1st kind $T_n(x)$

2nd kind $U_n(x)$

Hermite polynomial $H_n(x)$

Laguerre polynomial $L_n(x)$

combinatorial
interpretation

- coefficients
  of polynomials
- moments
- linearization
  coefficients

"direct" proof
of orthogonality

moments of (Tchebychev) 1st kind
2nd kind

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

$$U_n(x) = S_n(2x)$$

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Catalan number

Hermite polynomial

(combinatorial) Hermite polynomials

$H_n(x)$

Laguerre polynomial

$L_n(x)$

$L_n^{(\alpha)}(x)$ $\quad \alpha = 0$

moments of Hermite polynomial

$\mu_{2n+1} = 0$

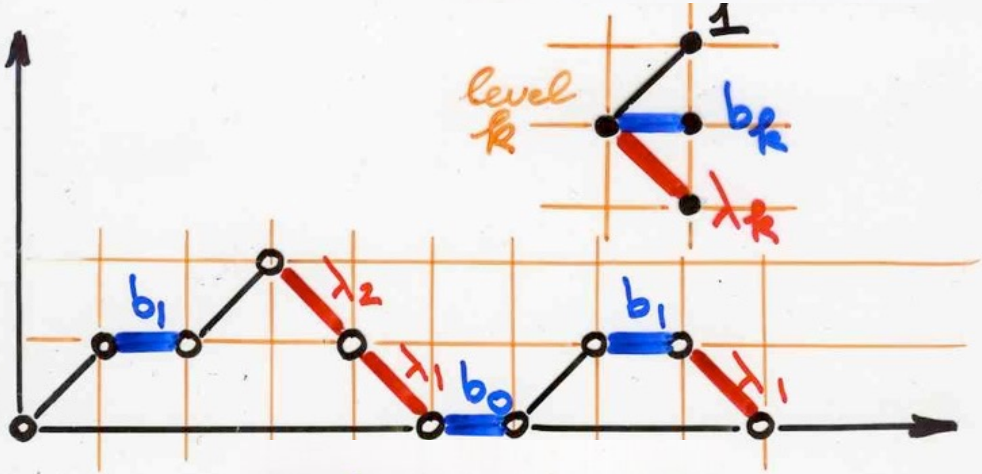
$\mu_{2n} = 1 \times 3 \times \ldots \times (2n-1)$

moments Laguerre polynomials

$\mu_n = n!$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path $|\omega| = n$

$$f(x^n) = \mu_n$$

moments

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

Tchebychev 1st kind $T_n(x)$

2nd kind $U_n(x)$

$\lambda_n = 1$ $(n \geqslant 1)$

$\begin{cases} \lambda_1 = 2 \\ \lambda_n = 1 \quad (n \geqslant 2) \end{cases}$

Hermite polynomial

$H_n(x)$

$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

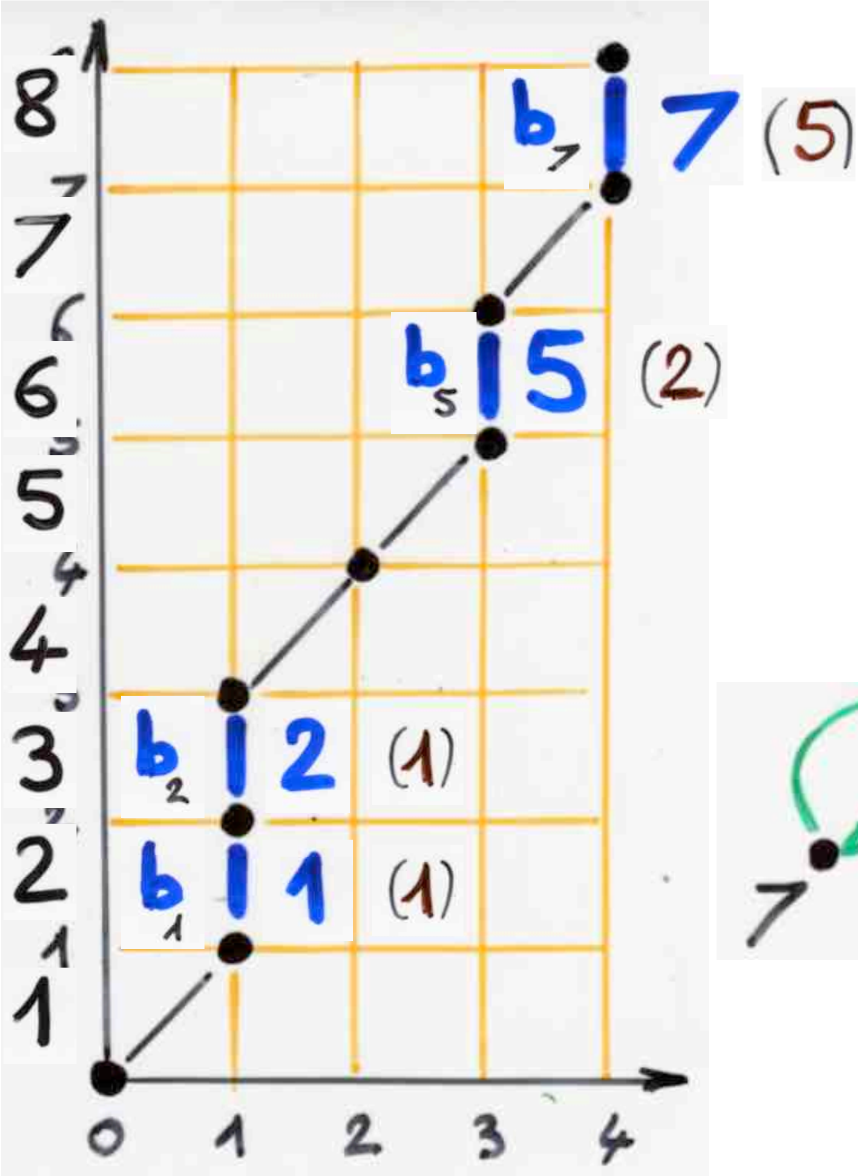
Laguerre polynomial

$L_n(x)$

$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$

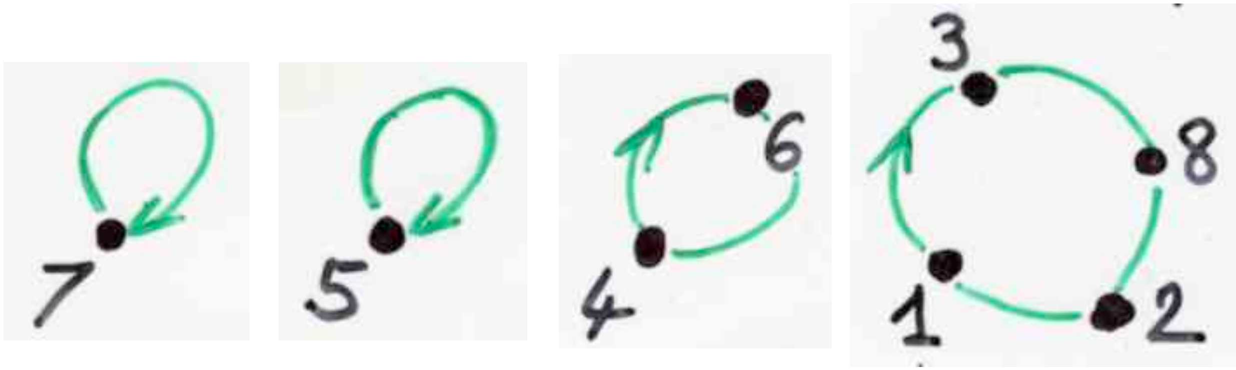
First steps

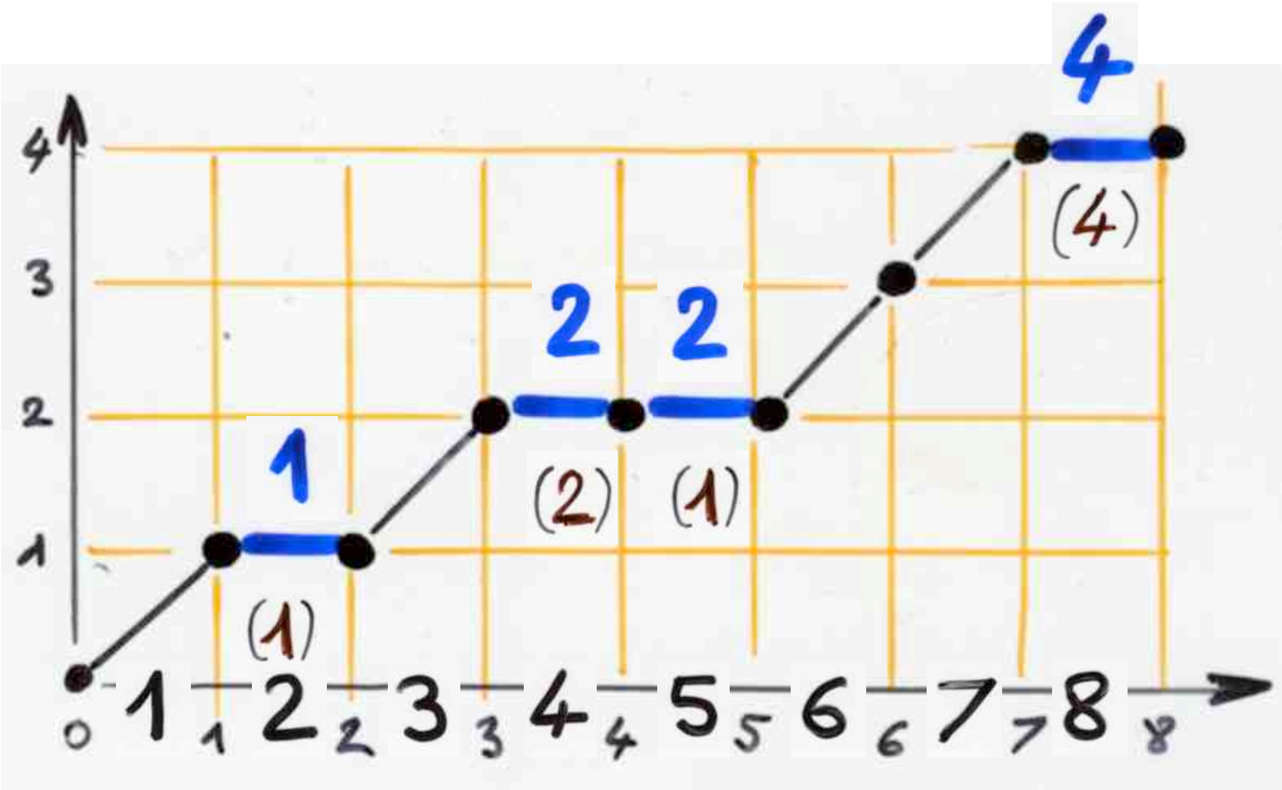
with the notion of histories



Stirling numbers

number of permutations of $\{1,\dots,n\}$ having i cycles





Stirling numbers

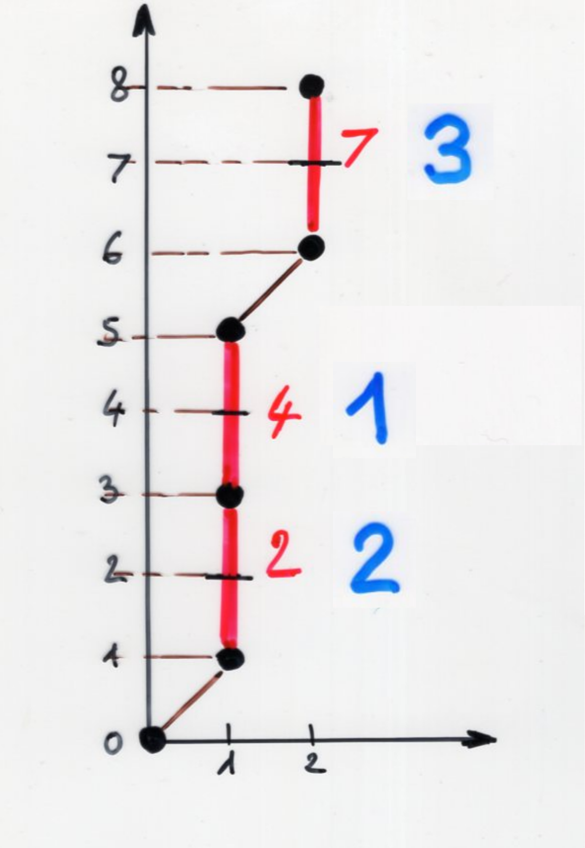
number of (set) partitions of $\{1, \dots, n\}$ into i blocks

[1 , 2 , 5

[3 , 4

[6

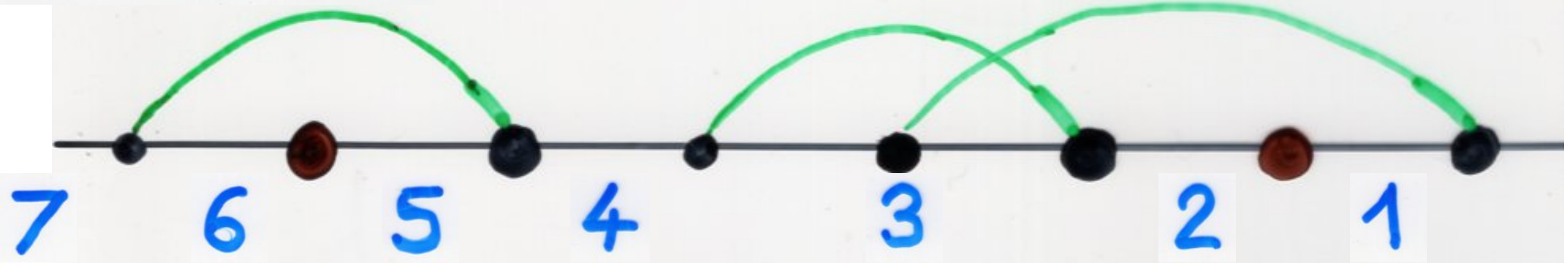
[7 , 8



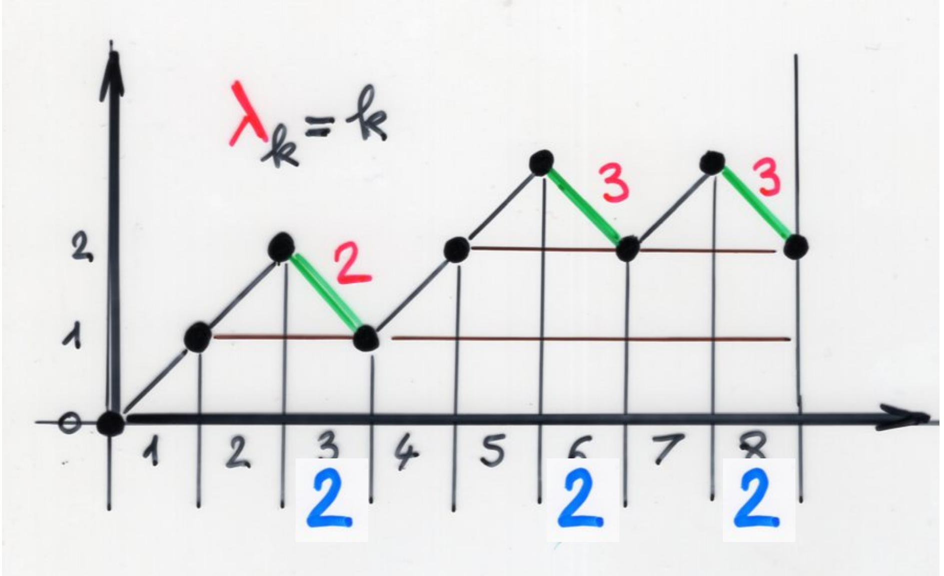
Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$H_n(x) = \sum_{0 \leq 2k < n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!} x^{n-2k}$$



$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{\mathrm{fix}(\sigma)}$$



moments

$$\mu_{2n+1} = 0$$

$$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

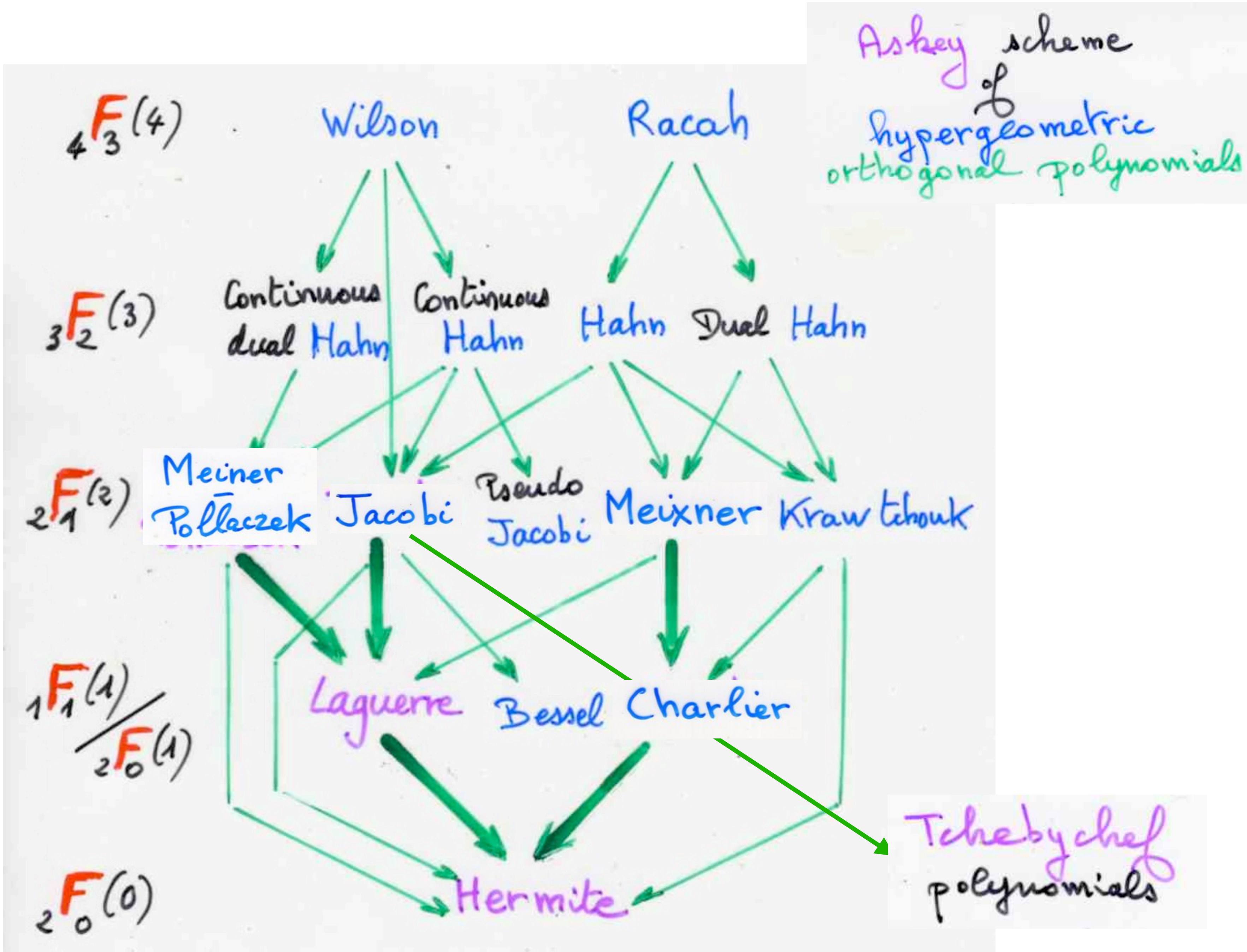
Hermite polynomials

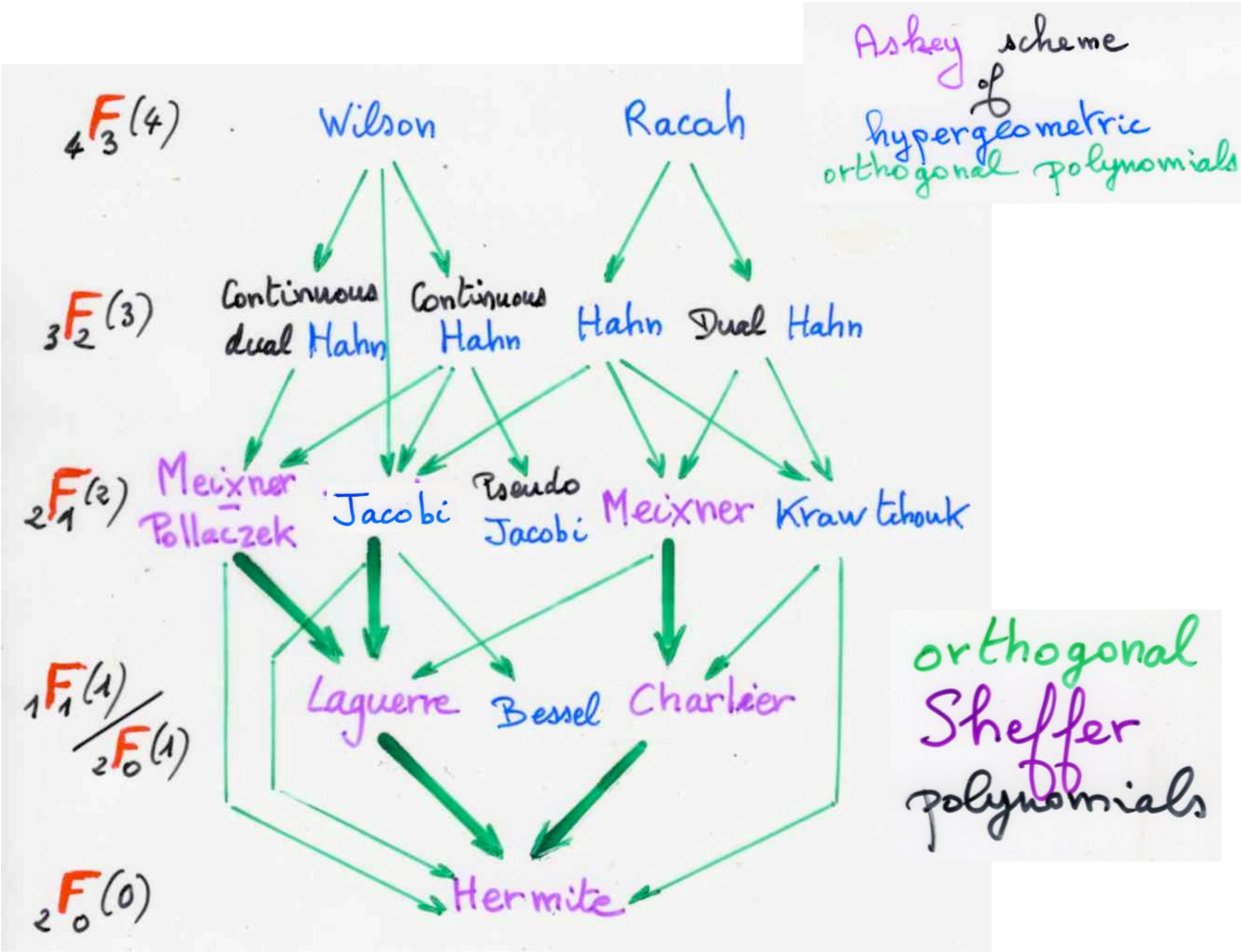
$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

Hermite polynomials again

!

Orthogonal Sheffer polynomials


Askey scheme of hypergeometric orthogonal polynomials
${}_4F_3(4)$
Wilson
Racah
${}_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
${}_2F_1(2)$
Meiner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
${}_1F_1(1)$ / ${}_2F_0(1)$
Laguerre
Bessel
Charlier
${}_2F_0(0)$
Hermite
Tchebychef polynomials



Askey scheme of hypergeometric orthogonal polynomials
$_4F_3(4)$
Wilson
Racah
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
$_1F_1(1)$ / $_2F_0(1)$
Laguerre
Bessel
Charlier
$_2F_0(0)$
Hermite
orthogonal Sheffer polynomials

Sheffer polynomials

$$\sum_{n \geqslant 0} P_n(x) \frac{t^n}{n!} = g(t) \exp\big(x f(t)\big)$$

Rota
umbral calculus

delta operator Q

$$D x^n = n x^{(n-1)}$$

$\{P_n(x)\}_{n\geq 0}$ orthogonal polynomials

Meixner (1934)

are Sheffer polynomials

$\Longleftrightarrow$ $\{P_n(x)\}_{n\geq 0}$ are one of the 5 possible types:

Hermite

Laguerre

Charlier

Meixner

Meixner-Pollaczek

$\{P_n(x)\}_{n\geq 0}$ orthogonal polynomials are Sheffer polynomials

Meixner (1934) $\Longleftrightarrow$

positive-definite OPS

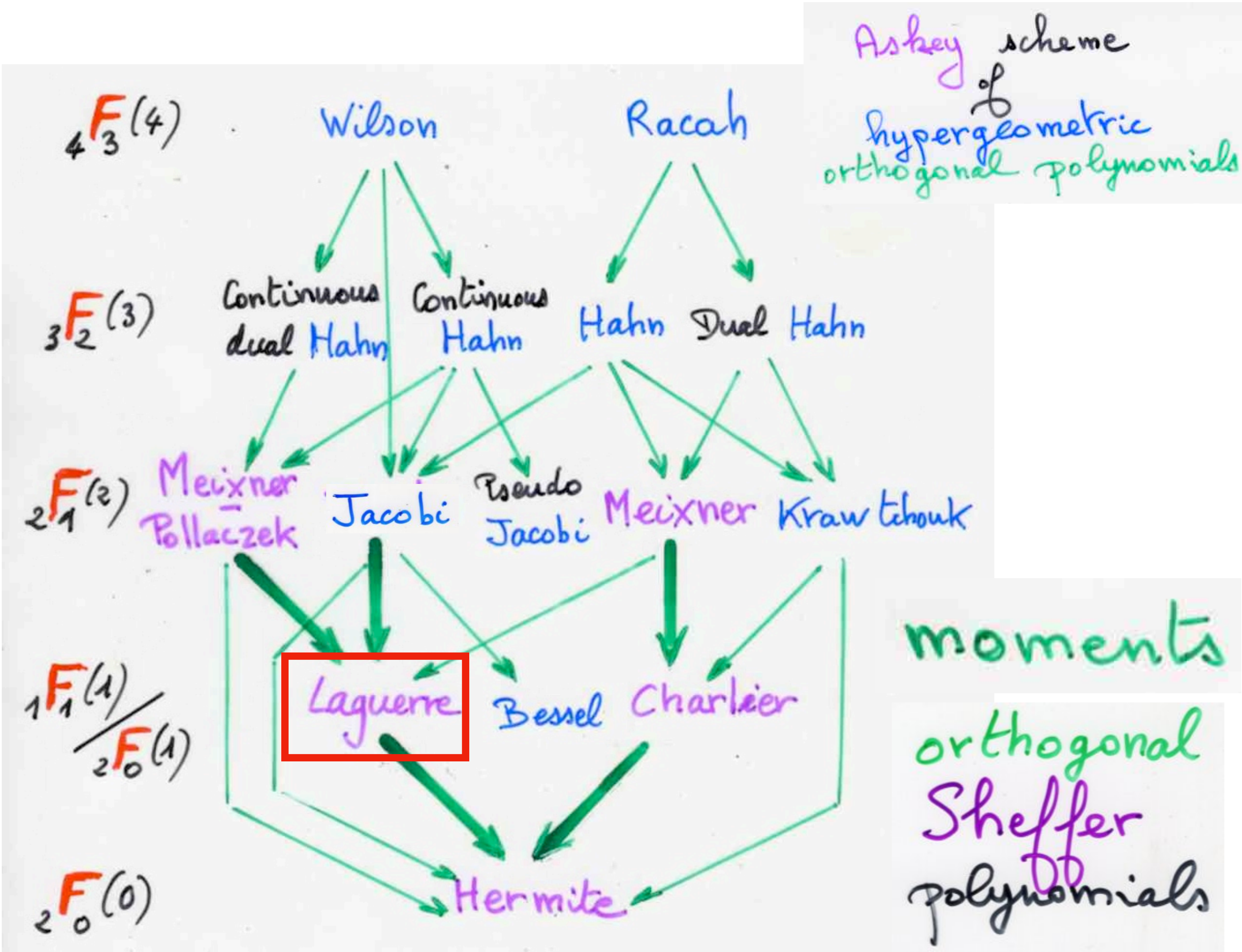
Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck + d) \end{cases}$

with $\begin{cases} a, b, c, d \in \mathbb{R} \\ c \geq 0, \quad c + d > 0 \end{cases}$

positive-definite OPS

Sheffer type $\Leftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$

(1) $a=0, \; c=0$	Hermite polynomials	$H_n(x)$
(2) $a\neq 0, \; a^2-4c=0$	Laguerre polynomials	$L_n^{(\alpha)}(x)$
(3) $a\neq 0, \; c=0$	Charlier polynomials	$C_n^{(a)}(x)$
(4) $a^2-4c>0$	Meixner polynomials	$M_n^{(\beta,c)}(x)$
(5) $a^2-4c<0$	Meixner-Pollaczek polynomials	$C_n^{(a)}(x)$


Askey scheme of hypergeometric orthogonal polynomials
${}_4F_3(4)$
Wilson
Racah
${}_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
${}_2F_1(2)$
Meixner Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
${}_1F_1(1)/{}_2F_0(1)$
Laguerre
Bessel
Charlier
${}_2F_0(0)$
Hermite
moments
orthogonal
Sheffer
polynomials

Laguerre polynomials

Laguerre polynomials

$$\int_0^\infty L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, e^{-x} x^{\alpha}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{mn}$$

Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\,\middle|\, x\right) \qquad (\alpha > -1)$$

$$n!\, L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-n)_k}{k!}(k+\alpha+1)_{n-k}\, x^k$$

$\tilde{L}_n^{(\alpha)}(x)$ monic (combinatorial) Laguerre polynomial

$$\sum_{n\geq 0} \tilde{L}_n^{(\alpha)}(x)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{-xt}{1-t}\right)$$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$b_k = 2k + \alpha + 1$$

monic Laguerre (combinatorial) polynomial

$$\lambda_k = k(k+\alpha)$$

$$\mu_n = (\alpha+1)(\alpha+2)\cdots(\alpha+n)$$

$$L_n^{(\alpha)}(x) \qquad \alpha = 0$$

$$\mu_n = n!$$

$$L_n^{(1)}(x)$$

$$b_k = (2k+2)$$

$$\lambda_k = k(k+1)$$

moments

$$\mu_n = (n+1)!$$

$$\mu_n = (n+1)!$$

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

Laguerre histories

$$\mu_n = n!$$

$$b_k = (2k+1)$$
$$\lambda_k = k^2$$

restricted Laguerre histories

bijection

$$h = (\omega_c ; \underbrace{(p_1, \dots, p_n)}_{p}) \quad |\omega| = n \qquad \longleftrightarrow \qquad \text{permutations } \sigma \in \mathfrak{S}_{n+1}$$

Laguerre histories

$(n+1)!$

ABjC, Part I, Ch4
ABjC, Part III, Ch5

J. Françon, X.V. (1979)

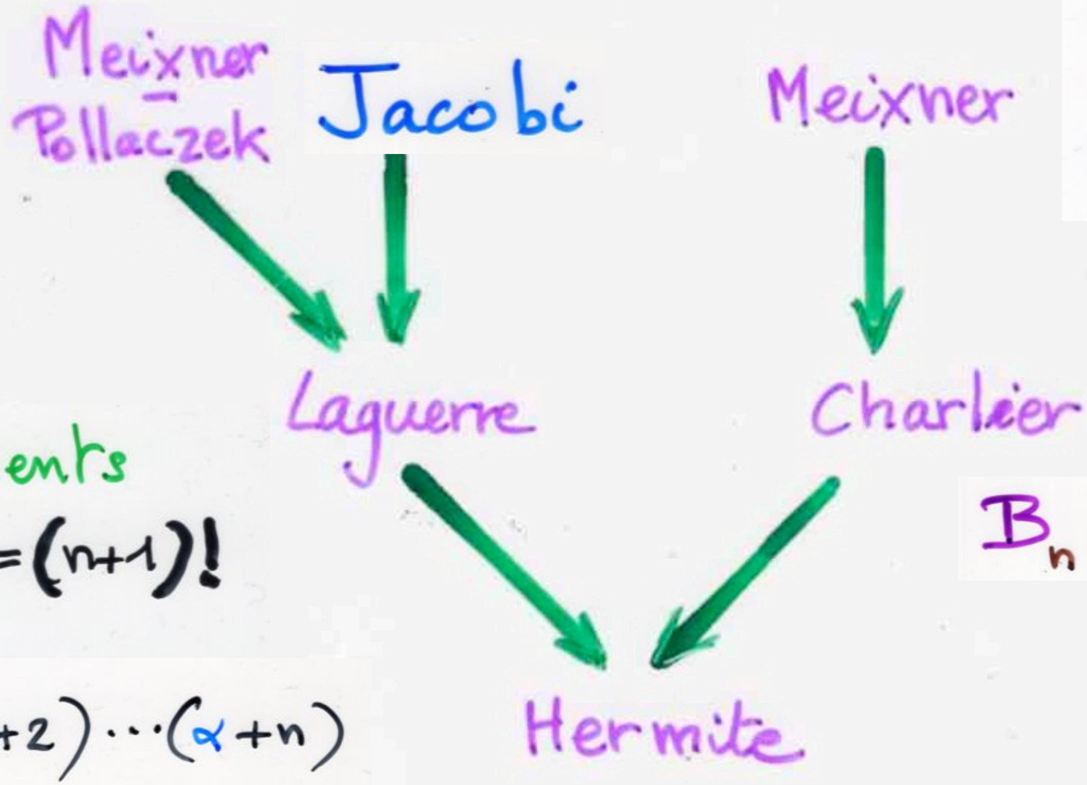
$$
\begin{tikzcd}
\text{Meixner-Pollaczek} \arrow[dr] & \text{Jacobi} \arrow[d] & & \text{Meixner} \arrow[d] \\
& \text{Laguerre} \arrow[dr] & & \text{Charlier} \arrow[dl] \\
& & \text{Hermite} &
\end{tikzcd}
$$

moments

$$\mu_n = (n+1)!$$

E_{2n}

secant number



number of ordered partitions

moments

$\mu_n = (n+1)!$

B_n Bell number

number of partitions

$(\alpha+1)(\alpha+2)\cdots(\alpha+n)$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \times 3 \times \dots \times (2n-1)$

Sheffer orthogonal polynomials	b_k	λ_k	moments μ_n
Laguerre $L_n^{(\alpha)}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)_n = (\alpha+1)\cdots(\alpha+n)$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1\times 3\times\ldots\times(2n-1)$ $\mu_{2n+1}=0$
Charlier $C_n^{(a)}(x)$	$k+a$	ak	$\sum_{k=1}^{n} S_{n,k}\, a^k$
Meixner $m_n(\beta,c;x)$	$\frac{(1+c)k+\beta c}{(1-c)}$	$\frac{ck(k-1+\beta)}{(1-c)^2}$	$=(1-c)^\beta \sum_{k\geq 0} k^n c^n \frac{(\beta)_k}{k!}$
Meixner Pollaczek $P_n(\delta,\eta;x)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)}\left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$

permutations

classic

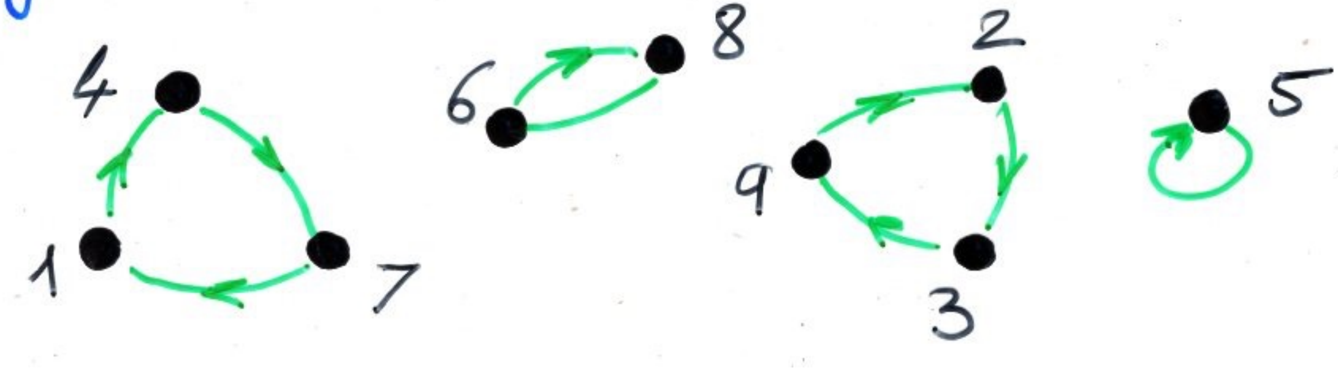
permutations very classic

different representations

as a bijection $\{1, 2, \dots, n\} \xrightarrow{\sigma} \{1, 2, \dots, n\}$

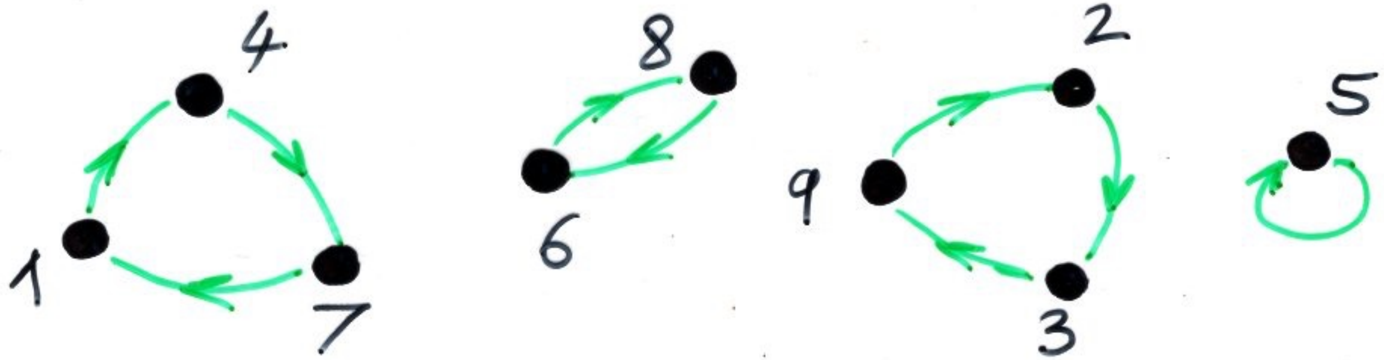
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

cycles notation



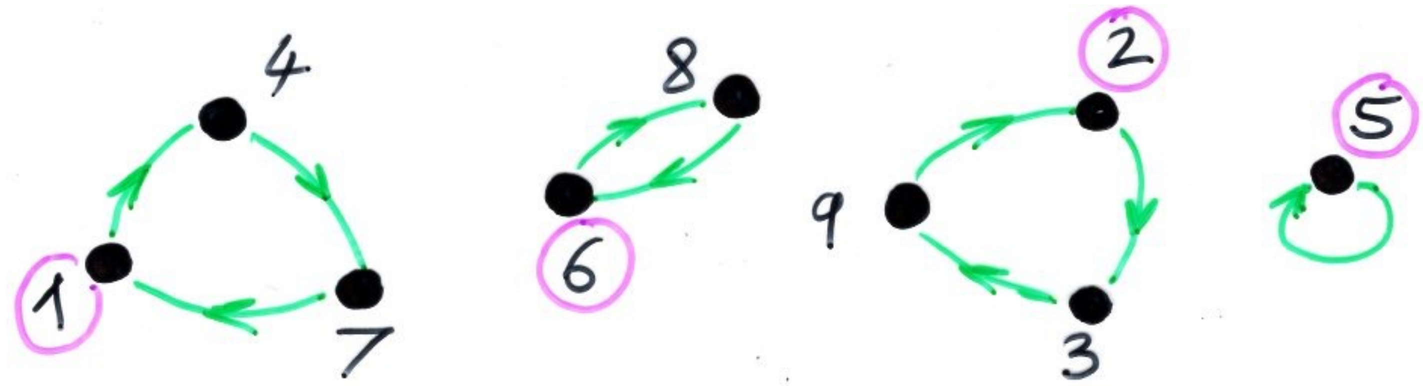
a classical bijection very classic !

$$\sigma \underset{\text{notation}}{\text{cycles}} \xrightarrow{f} \text{word} \quad \tau = f(\sigma)$$



a classical bijection   very classic !

σ cycles notation $\xrightarrow{f}$ word $\tau = f(\sigma)$



$\tau = / 6 \; 8 / 5 / 2 \; 3 \; 9 / 1 \; 4 \; 7$

$\tau = 6\ 8\ 5\ 2\ 3\ 9\ 1\ 4\ 7$

$w = x_1 x_2 \ldots x_n$

word with distinct letters

lr-min

left to right minimum element

$x_i = \min(x_1, x_2, \ldots, x_i)$

$$\tau = 6\ 8\ 5\ 2\ 3\ 9\ 1\ 4\ 7$$

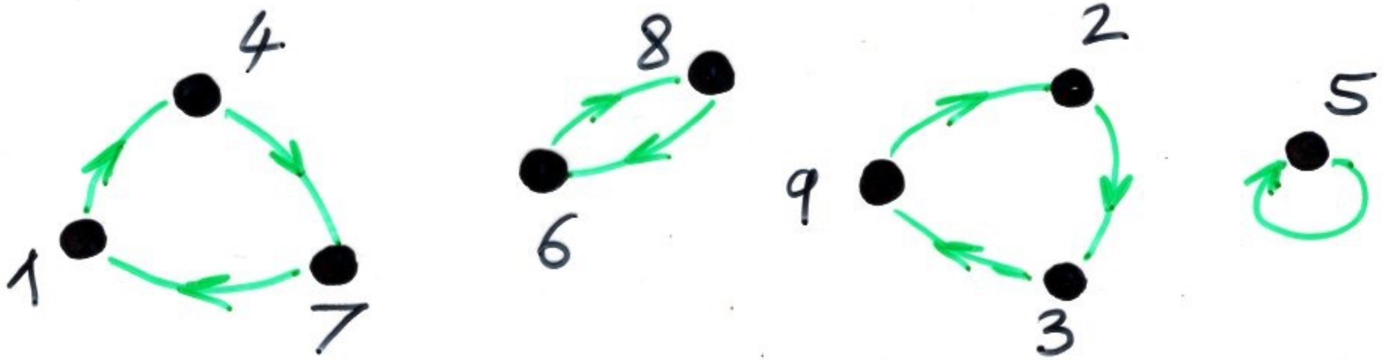
$w = x_1 x_2 \ldots x_n$

word with distinct letters

lr-min

left to right minimum element

$$x_i = \min(x_1, x_2, \ldots, x_n)$$



Foata (1965)

"transformation fondamentale"

rise $\sigma(i) < \sigma(i+1)$

descent $\sigma(i) > \sigma(i+1)$

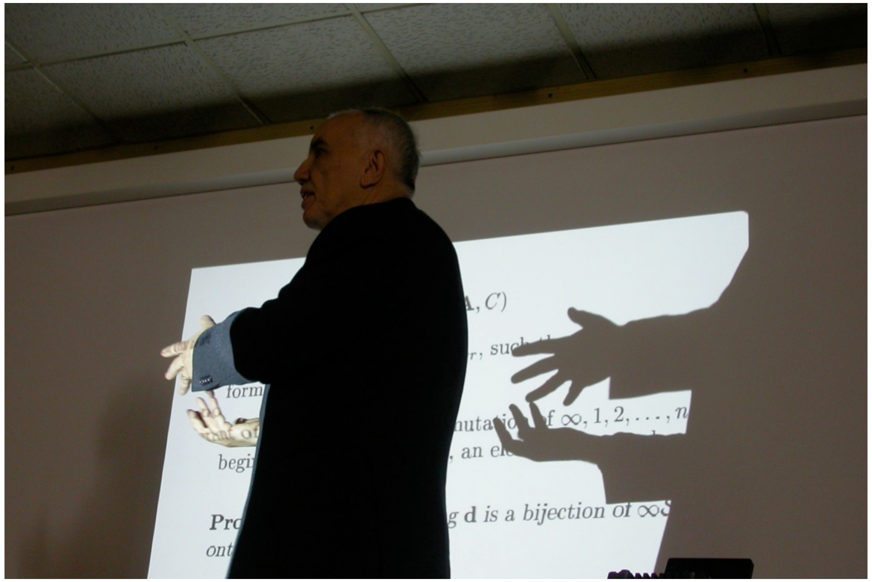
$$4 - 7 - 1 - 9 - 2 - 3 - 5 - 8 - 6$$

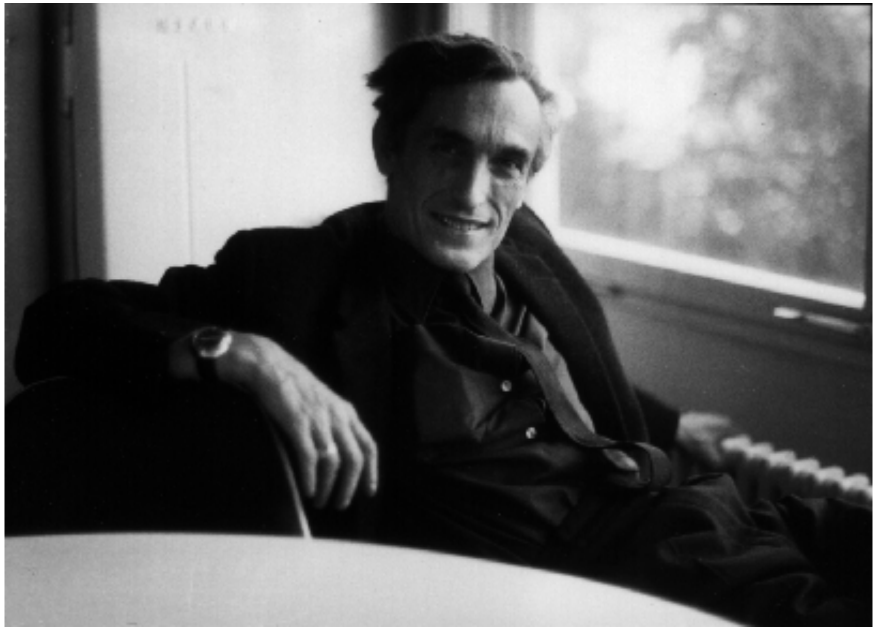
$a_{n,k}$ = number of $\sigma \in \mathfrak{S}_n$ having k rises

$$A_n(x) = \sum_k a_{n,k}\, x^k$$

Euler (1755)

eulerian polynomials





D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens"
(1970)

excedance

$i < \sigma(i)$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

excedance

$i < \sigma(i)$

$\mathrm{exc}(\sigma)$ = number of excedances

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{rise}(\sigma)}$$

$\mathrm{rise}(\sigma)$ = number of rises of σ

eulerian distribution

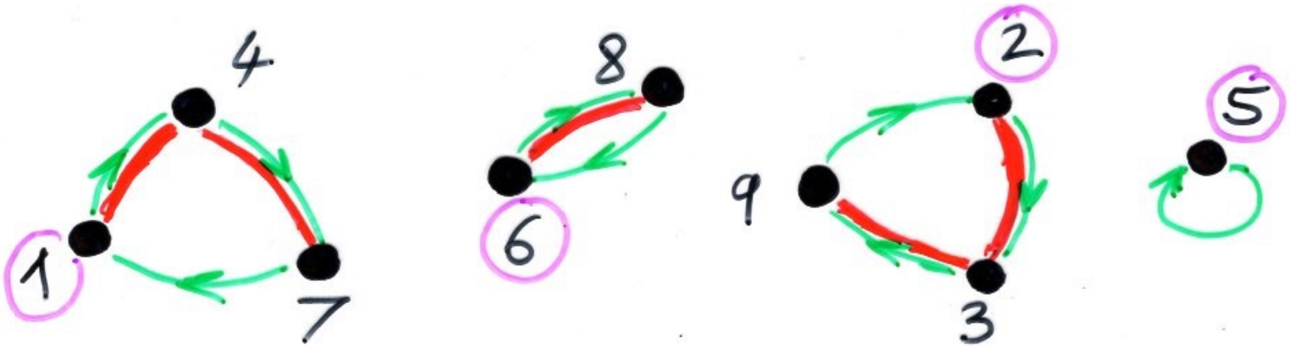
$A_n(x)$ eulerian polynomial

excedance

$i < \sigma(i)$

$\mathrm{exc}(\sigma)$ = number of excedances

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$



$$\tau = /\, 6 - 8 \,/\, 5 \,/\, 2 - 3 - 9 \,/\, 1 - 4 - 7$$

up-down sequence of a permutation

4 7 1 9 2 3 5 8 6

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$

Definition

sub-excedante functions

$$f : [1, n] \longrightarrow [0, n-1]$$

pour tout $1 \leqslant i \leqslant n$, $0 \leqslant f(i) < i$

$\mathcal{F}_n$ set of sub-excedante functions

$$|\mathcal{F}_n| = n!$$

$\sigma \in \mathfrak{S}_n \longrightarrow f \in \mathcal{F}_n$

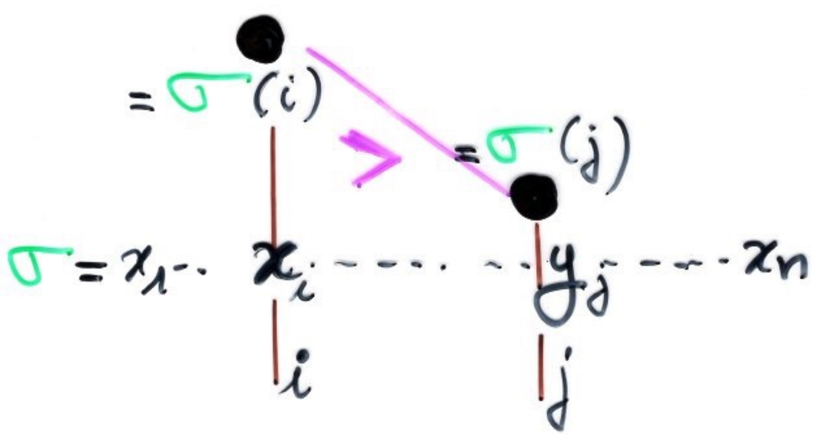
Inversion table

$$\sigma = \frac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

$1 \leqslant x \leqslant n \qquad x = \sigma(i)$

$f(x) =$ number of j, $i < j \leqslant n$ with $\sigma(j) < \sigma(i)$



inversion of σ

$(i, j) \qquad 1 \leqslant i < j \leqslant n$

$\sigma(i) > \sigma(j)$

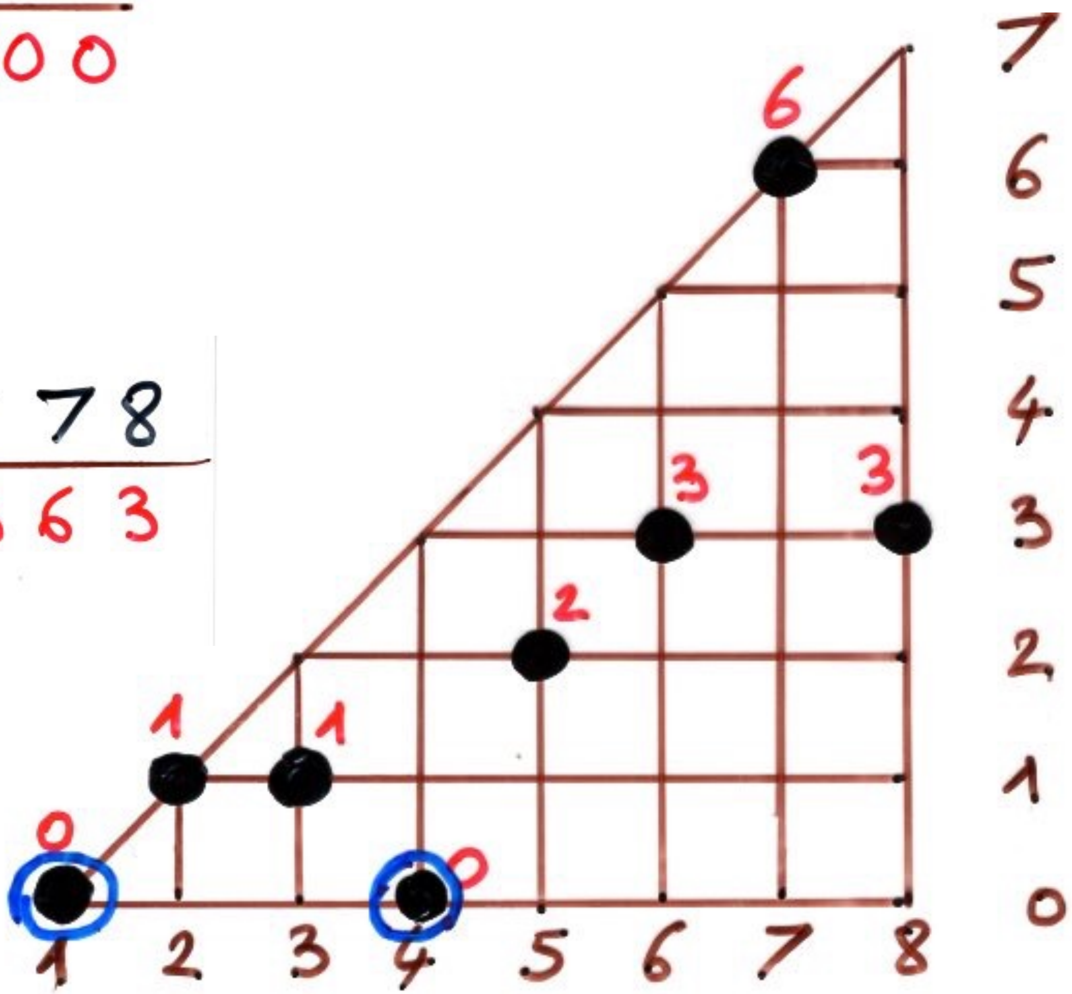
$inv(\sigma) =$ number of inversions

Inversion table

$\sigma = \dfrac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$

rl- min

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

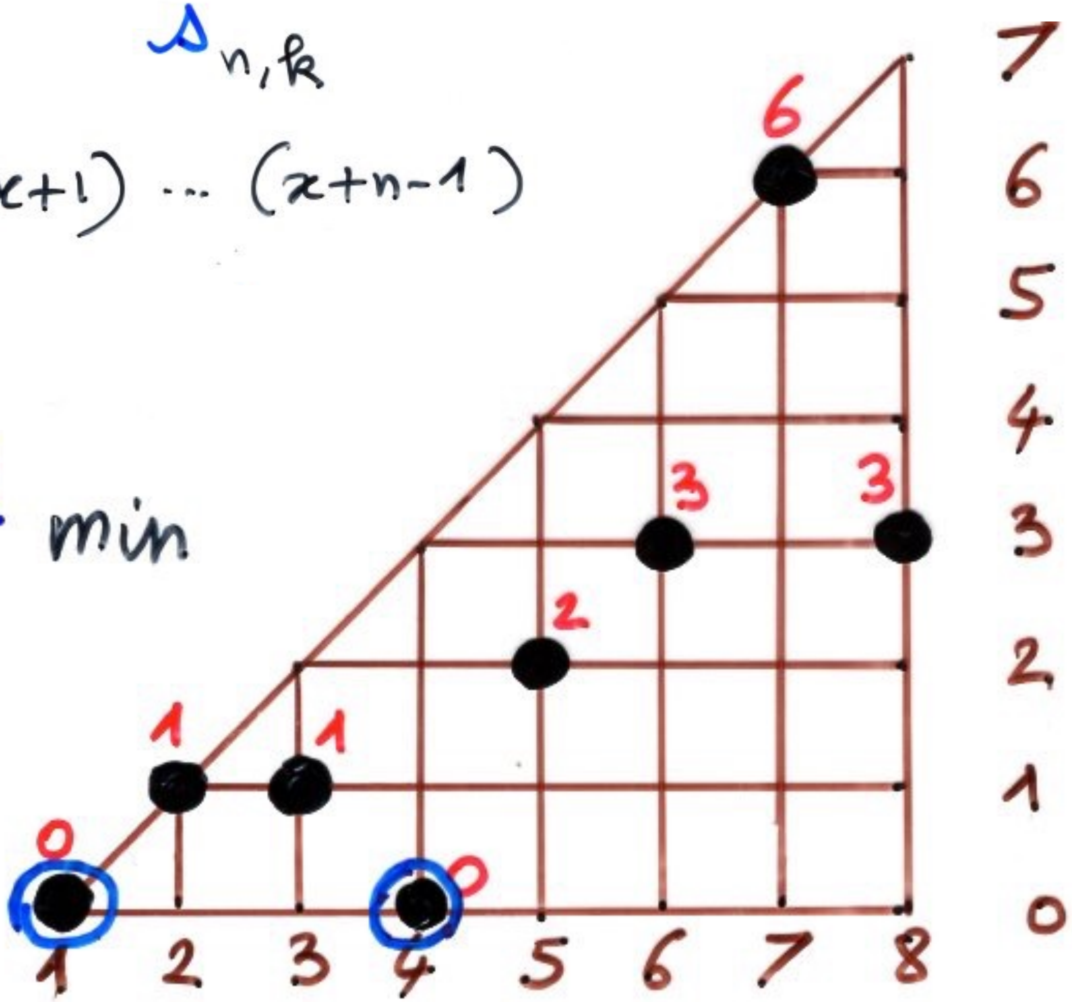


Inversion table

Stirling numbers $s_{n,k}$

$$\sum_{1 \le k \le n} s_{n,k}\, x^k = x(x+1)\cdots(x+n-1)$$

rl- min



number of inversions

$\mathrm{inv}(\sigma) = 19$

$0+0+1+3+1+3+3+3+5$

$q^{0+0+1+3+1+3+3+3+5}$

q^i

␣	
␣ 1 ␣	0
␣ 1 ␣ 2 ␣	0
␣ 1 ␣ 3 ␣ 2 ␣	1
␣ 4 ␣ 1 ␣ 3 ␣ 2 ␣	3
␣ 4 ␣ 1 ␣ 3 ␣ 5 ␣ 2 ␣	1
␣ 4 ␣ 1 ␣ 6 ␣ 3 ␣ 5 ␣ 2 ␣	3
␣ 4 ␣ 1 ␣ 6 ␣ 7 ␣ 3 ␣ 5 ␣ 2 ␣	3
␣ 4 ␣ 1 ␣ 6 ␣ 7 ␣ 8 ␣ 3 ␣ 5 ␣ 2 ␣	3
␣ 4 ␣ 1 ␣ 6 ␣ 9 ␣ 7 ␣ 8 ␣ 3 ␣ 5 ␣ 2 ␣	5

Laguerre histories
and
moment of Laguerre polynomials

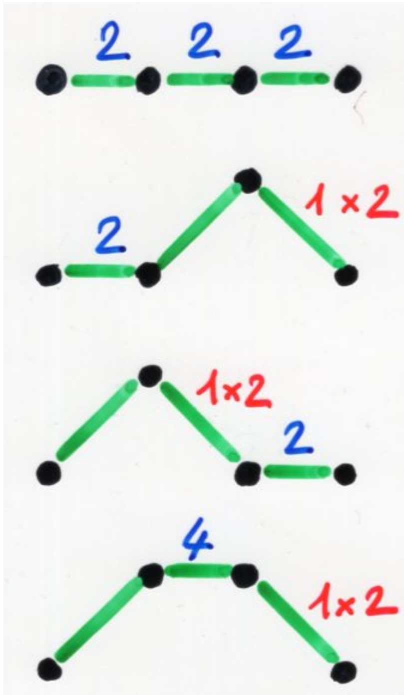
Laguerre polynomials

$L^{(1)}_n(x)$

moments

$\mu_n = (n+1)!$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$



8

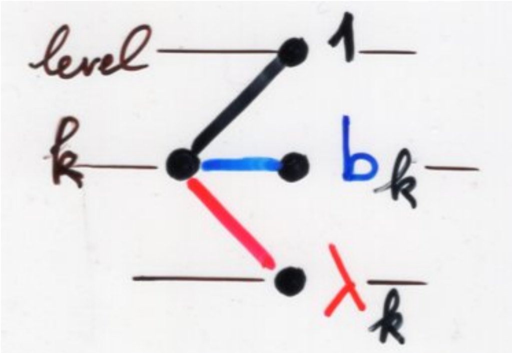
4

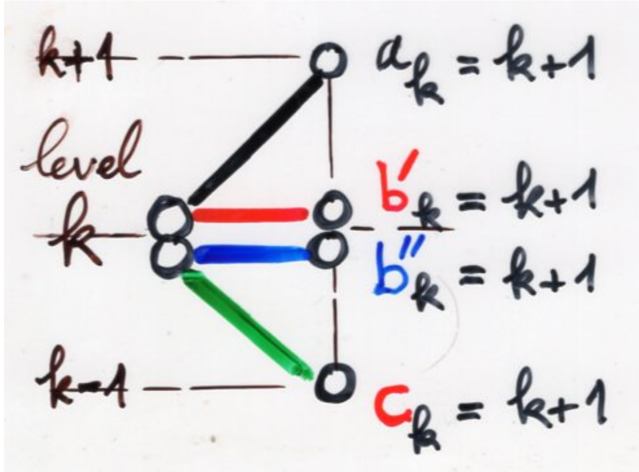
4

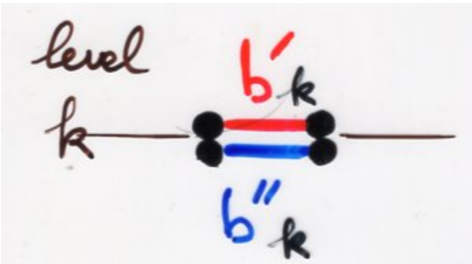
8

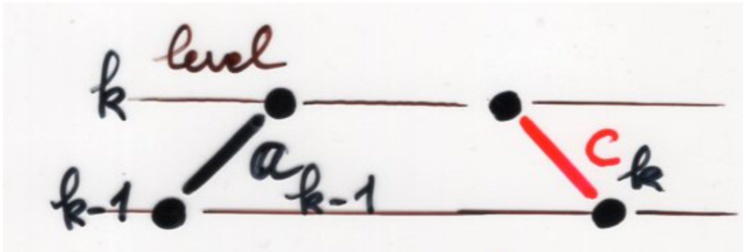
$4! =$ 24

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$



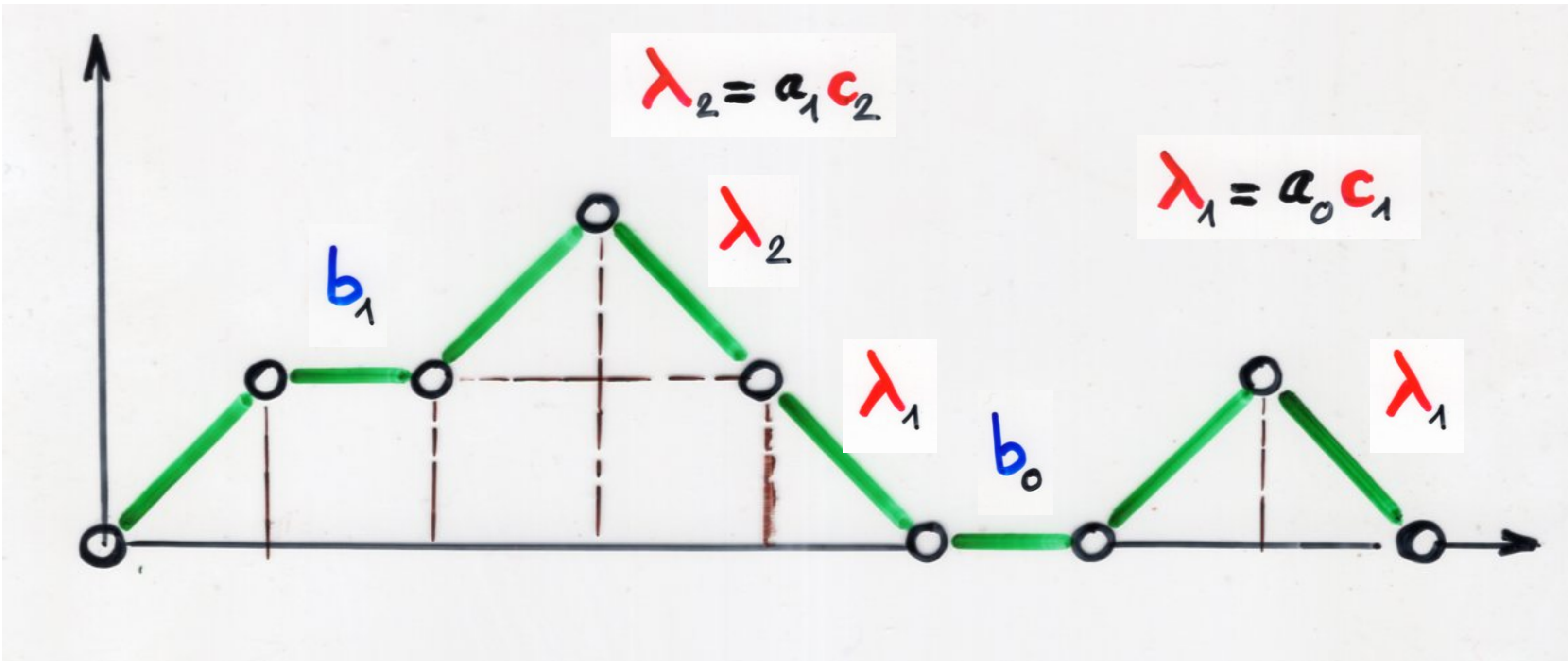


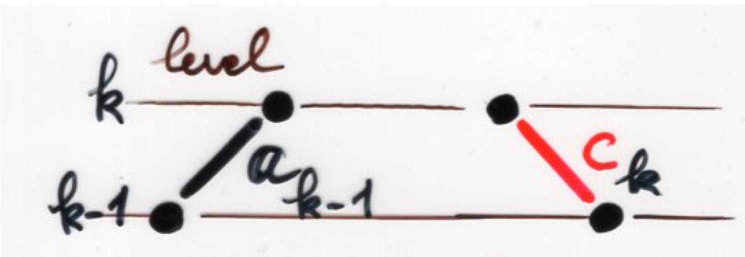




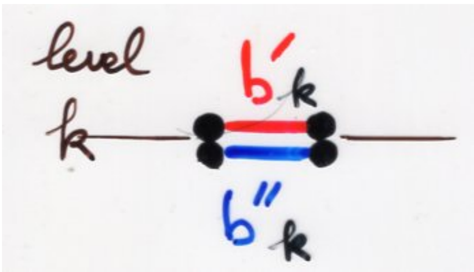
$$b_k = b'_k + b''_k$$

$$a_{k-1}\, c_k = \lambda_k$$

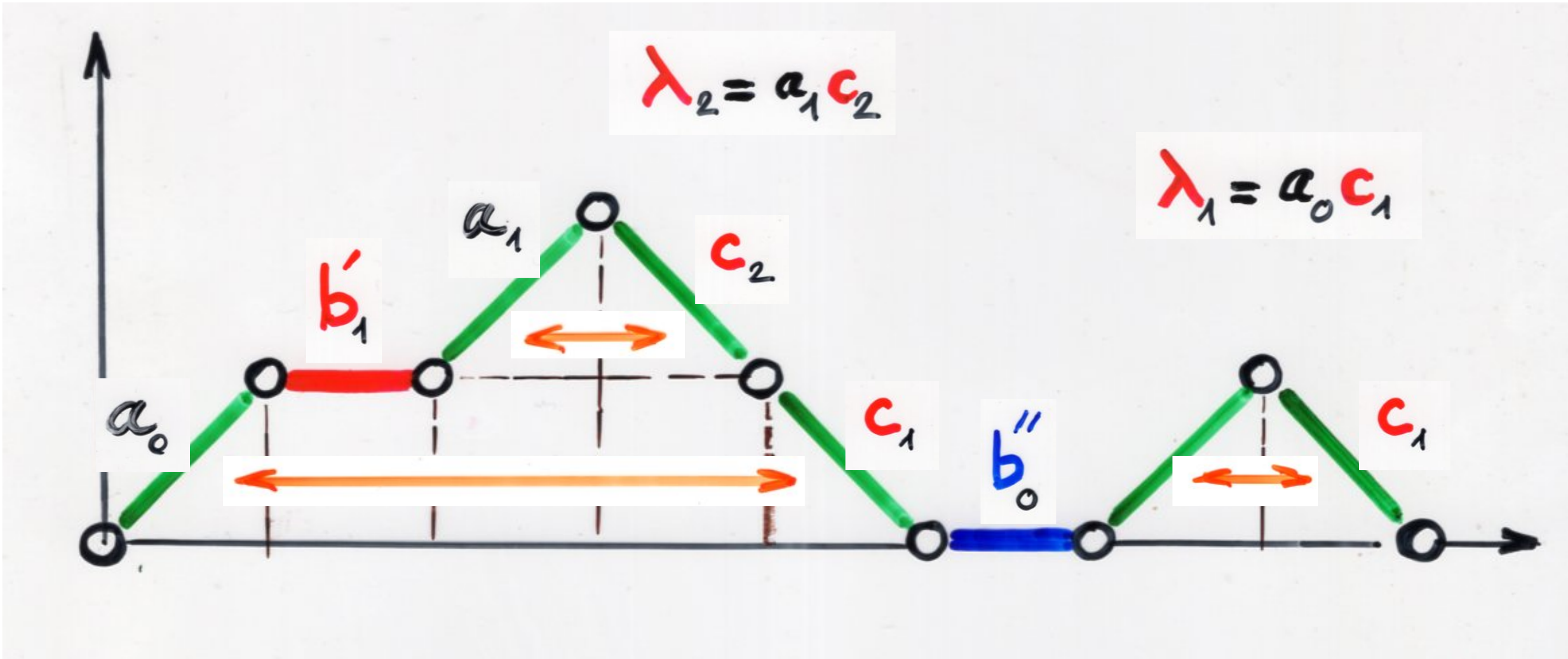


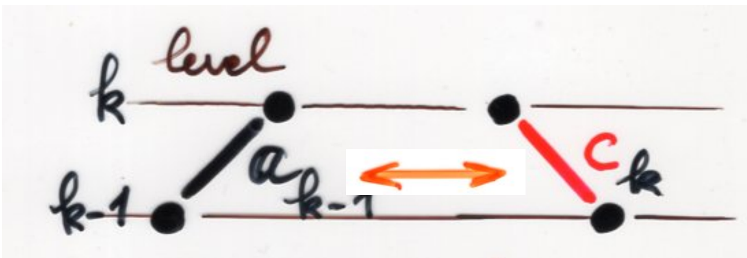


$$a_{k-1}\, c_k = \lambda_k$$



$$b_k = b'_k + b''_k$$





$$a_{k-1} c_k = \lambda_k$$

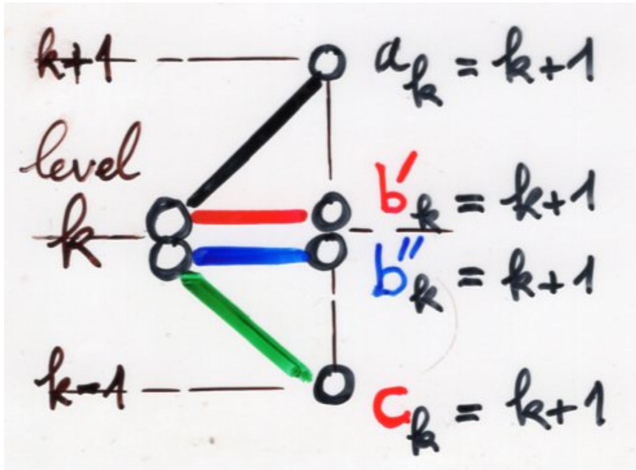
$$\sum_{\substack{|\omega|=n \\ \text{Motzkin} \\ \text{path}}} v(\omega) = \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin} \\ \text{path}}} v^{*}(\omega) = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases} \qquad \begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$

$$v^{*}(\omega)$$



positive-definite OPS

Sheffer type $\Longleftrightarrow \begin{cases} b_k = ak + b \\ \lambda_k = k(ck+d) \end{cases}$

with $\begin{cases} a, b, c, d \in \mathbb{R} \\ c \geq 0, \quad c+d > 0 \end{cases}$

Hermite

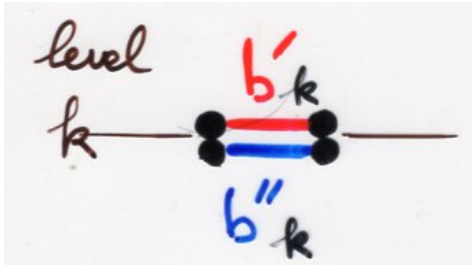
Laguerre

Charlier

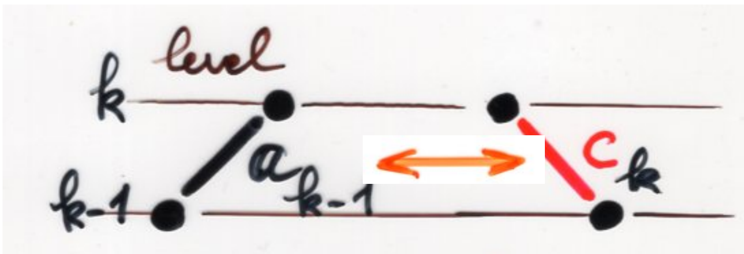
Meixner

Meixner-Pollaczek

$$b_k = b'_k + b''_k$$



$$a_{k-1} c_k = \lambda_k$$



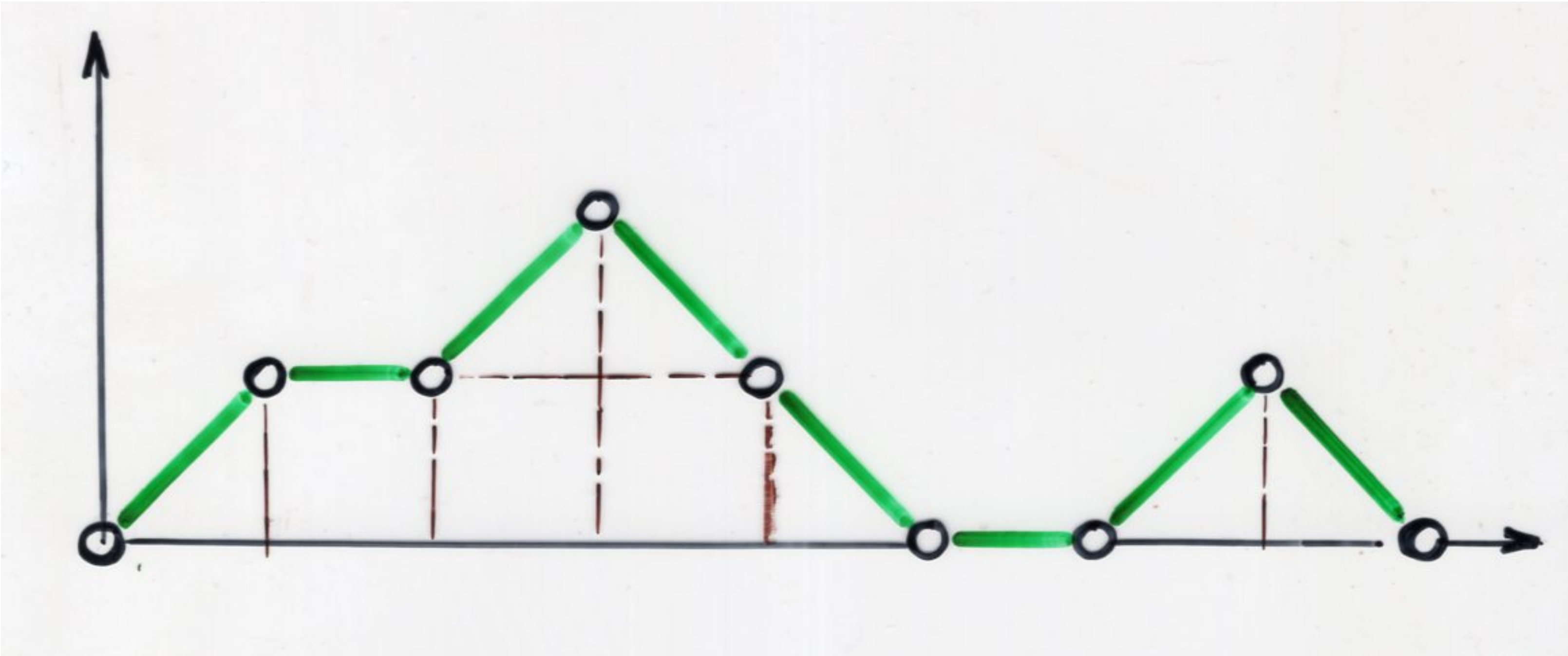
Laguerre histories

definition

Laguerre history

$h = (\omega_c, p)$

Motzkin path

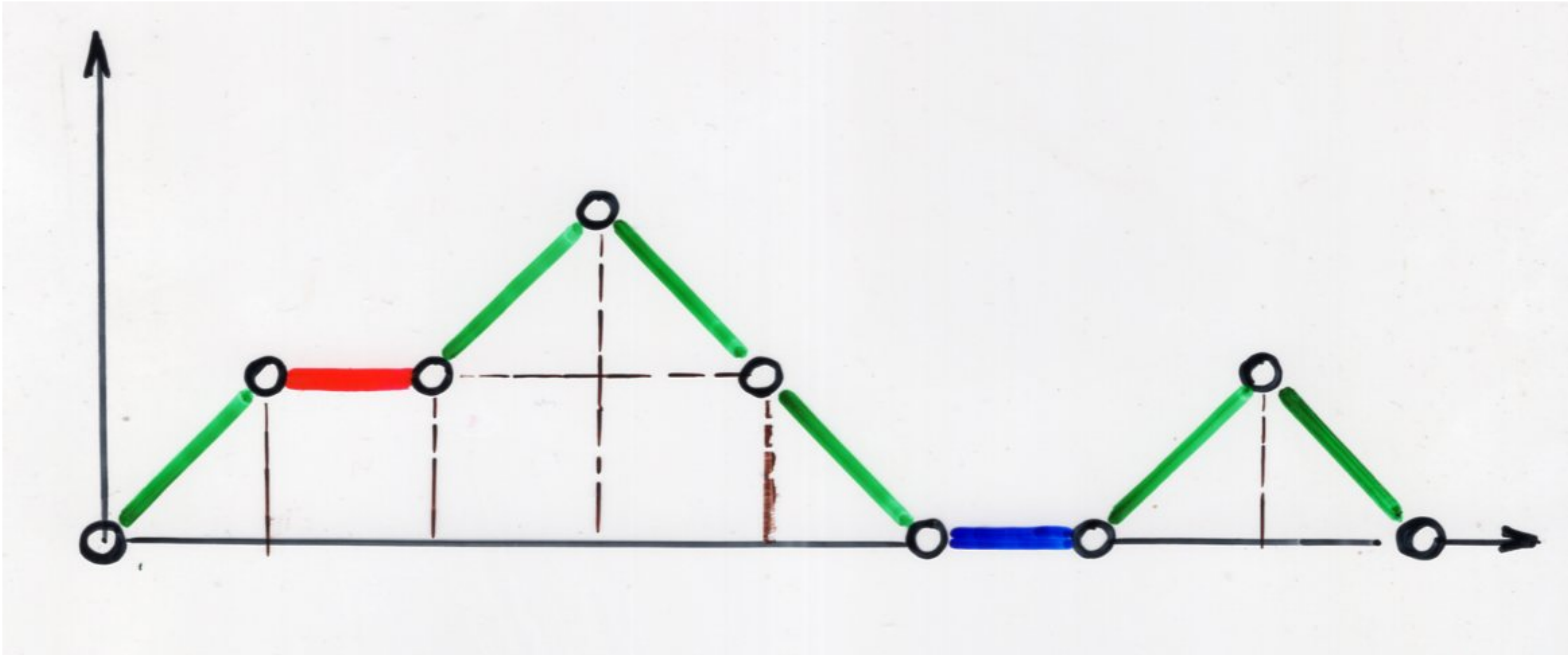


Laguerre history

$$h = (\omega_c, p)$$

Motzkin path

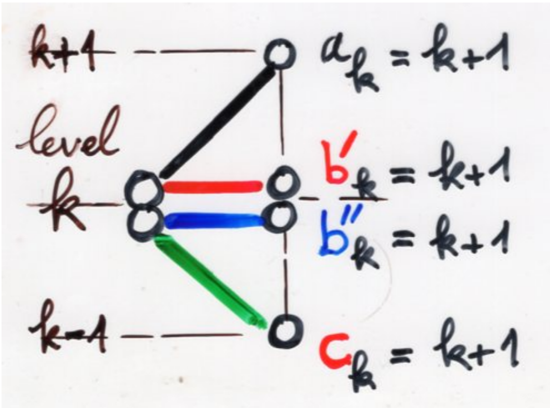
2 colors East steps



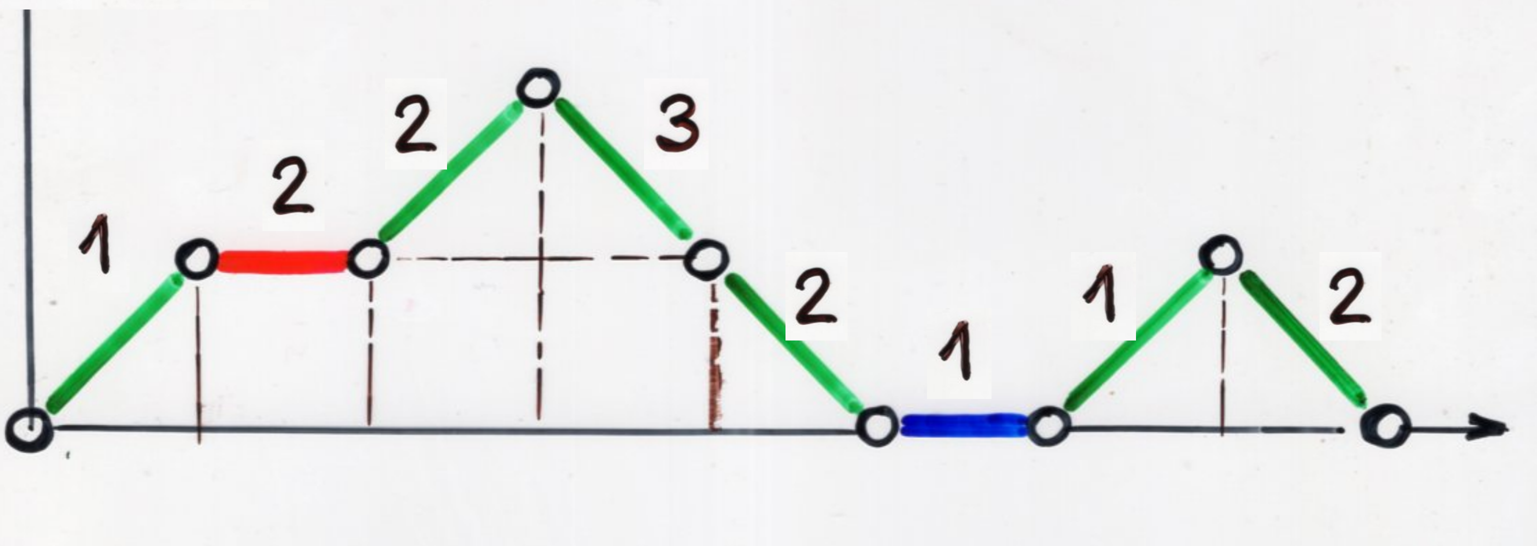
Laguerre history $h = (\omega_c, p)$

Motzkin path

2 colors East steps



$v^*(\omega)$



Laguerre history

$h = (\omega_c, P)$

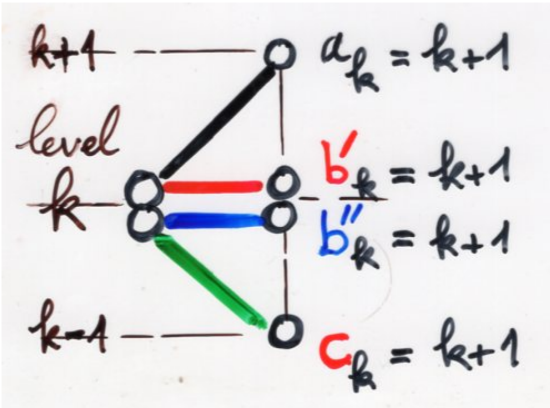
Motzkin path

2 colors East steps

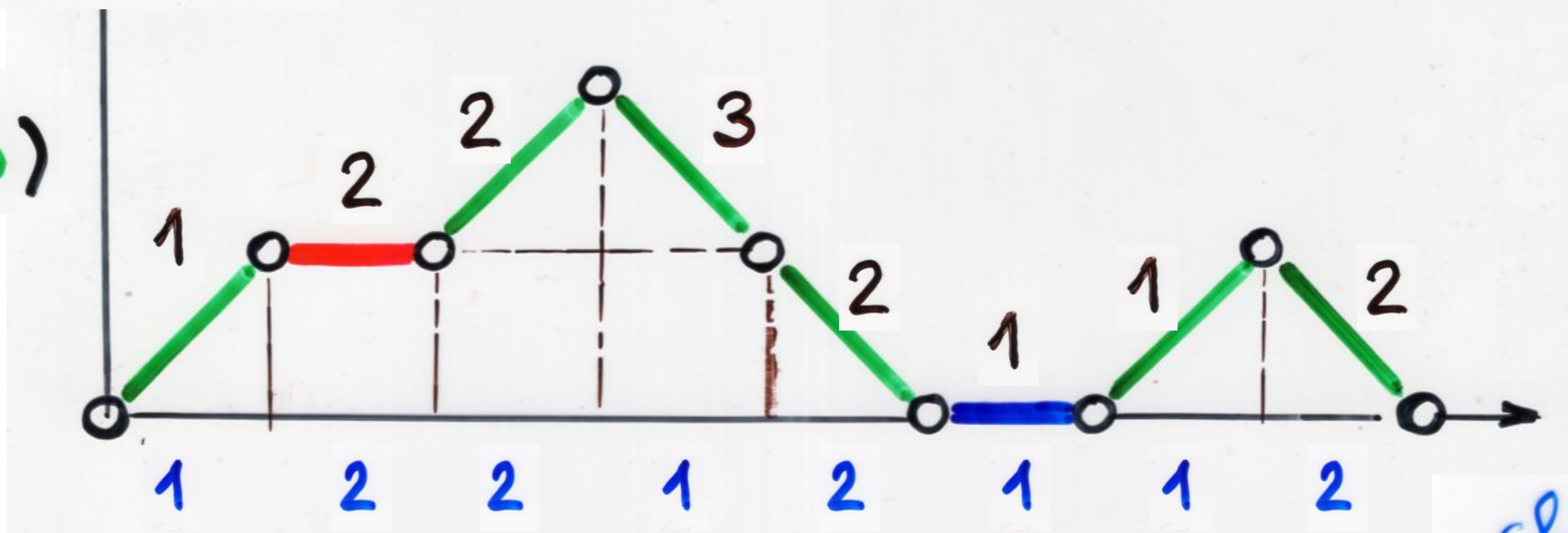
$P = (p_1, \dots, p_n)$

$1 \leq p_i \leq v(\omega_i)$

$\omega = (\omega_1 \cdots \omega_n)$



$v^*(\omega)$



choice function

bijection

$$h = (\omega_c ; \underbrace{(p_1, \dots, p_n)}_{P}) \longleftrightarrow \text{permutations } \sigma \in \mathfrak{S}_{n+1}$$

$|\omega| = n$

Laguerre histories

$(n+1)!$

$|h| = |\omega|$

length of the history

J. Françon, X.V. (1979)

bijection

Laguerre histories $\longrightarrow$ permutations

description with words

The FV bijection

(Françon-XV 1978)

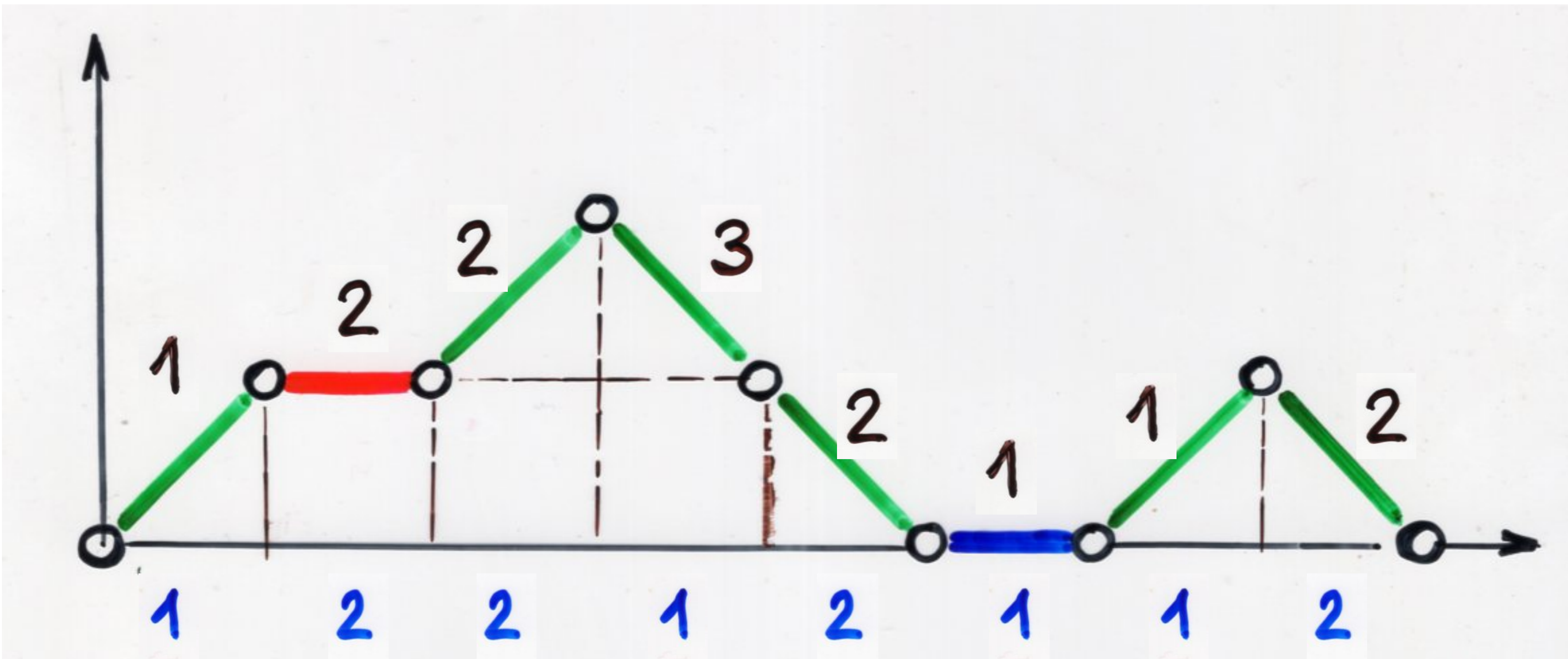
$\omega = (\omega_1 \cdots \omega_n)$

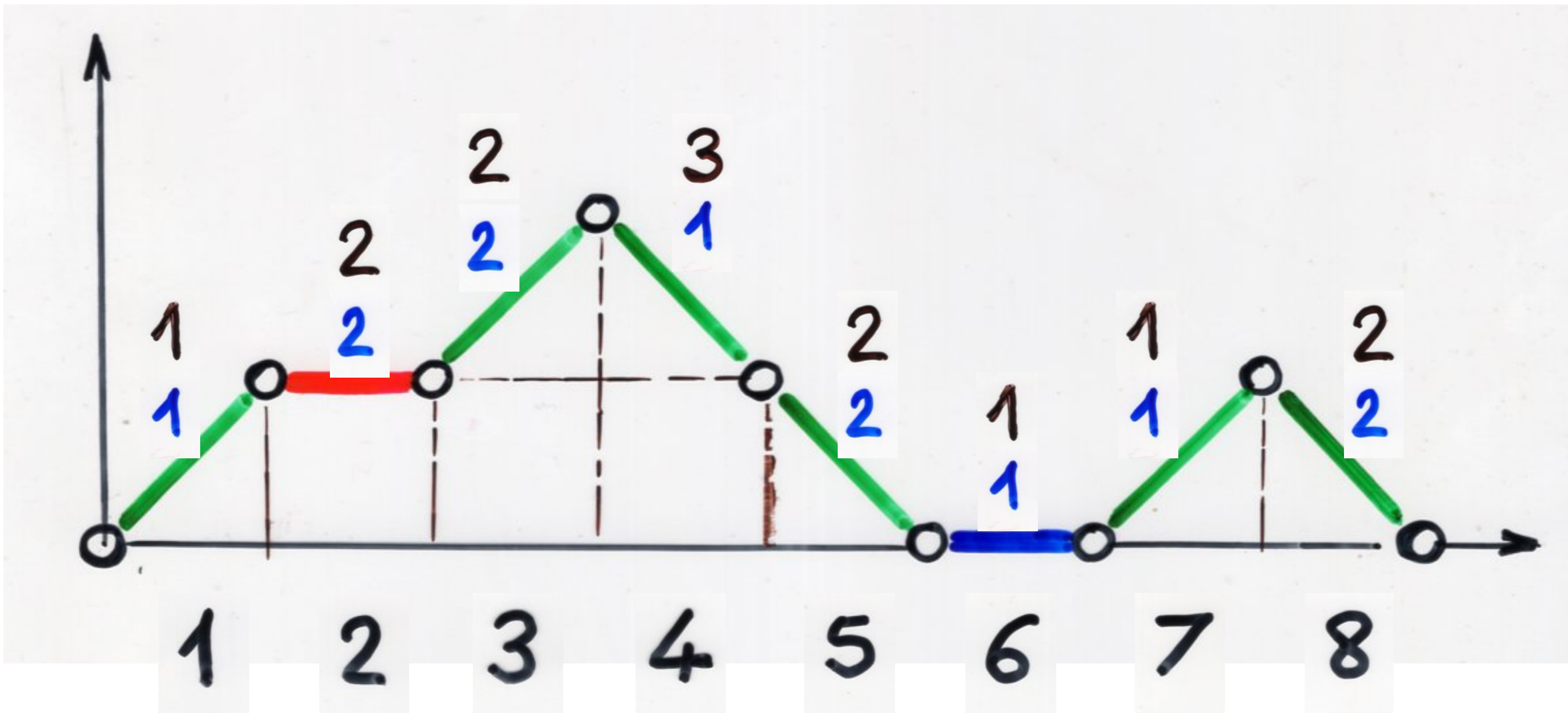
sequence of primitive operations

choice function

$1 \leq p_i \leq v(\omega_i)$

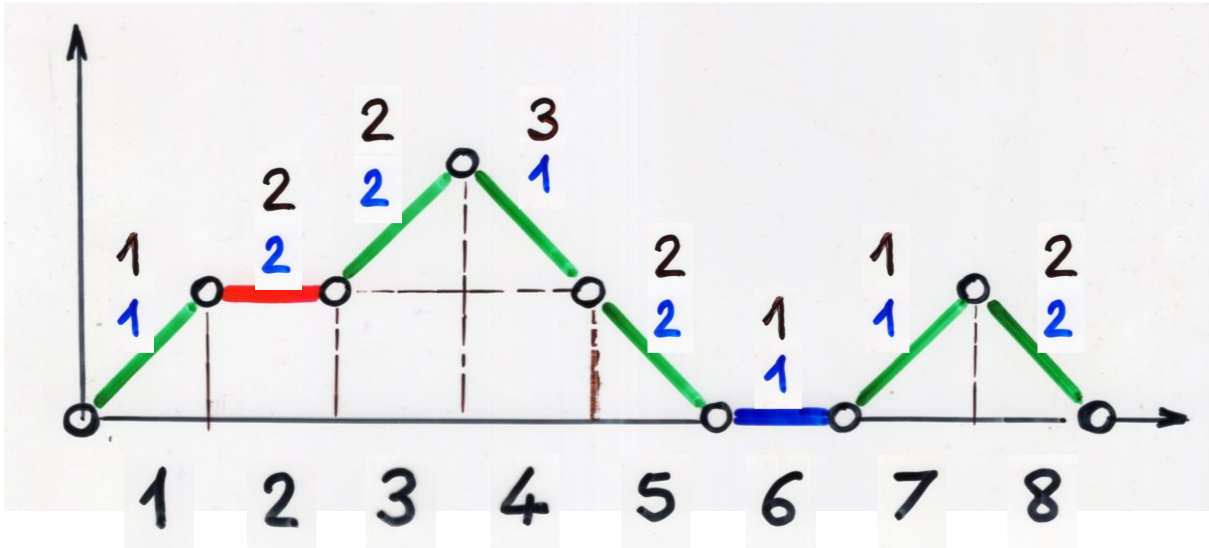
number of possibilities

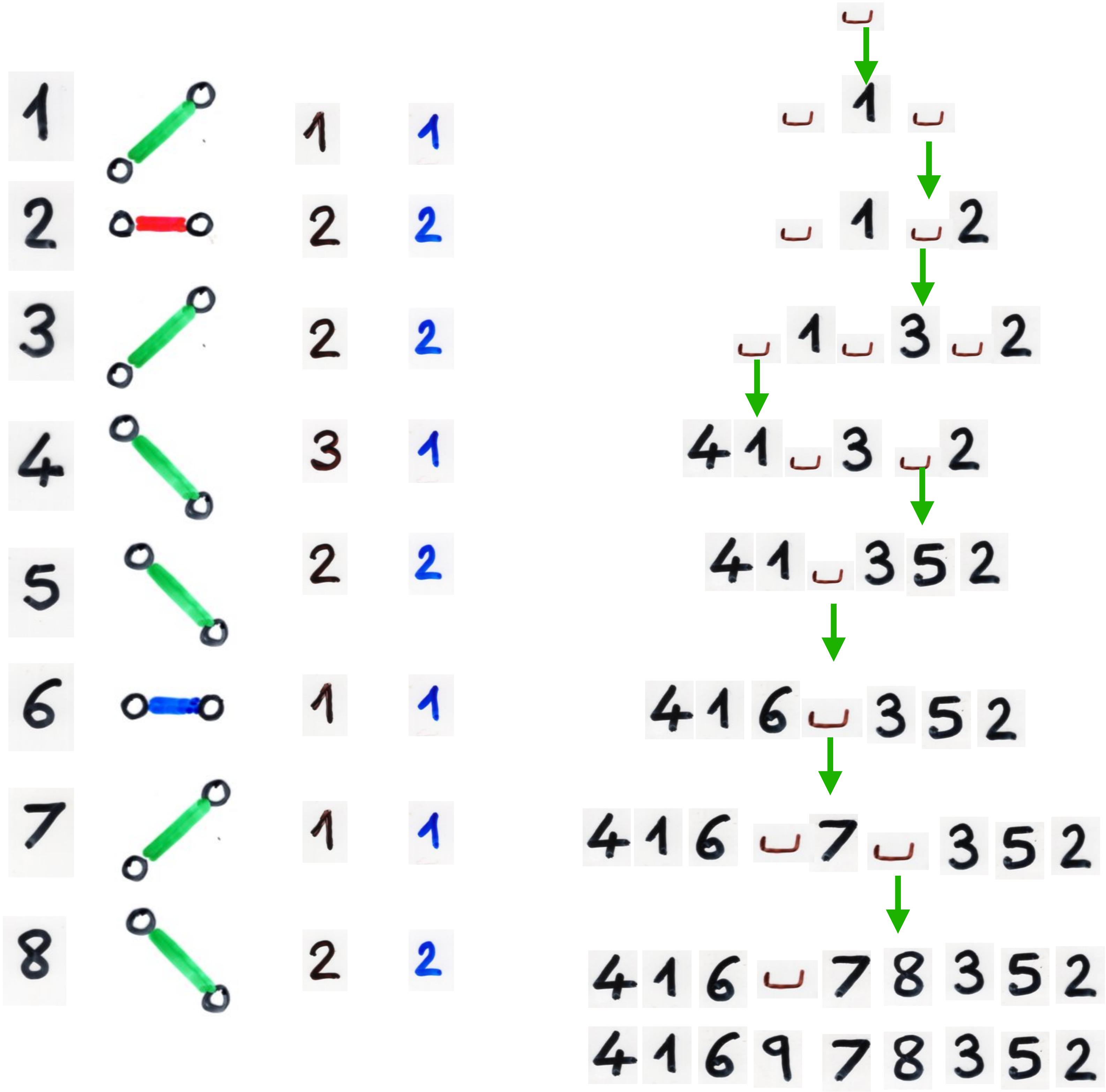




1
1
2
2
2
2
3
1
2
2
1
1
1
1
2
2
1 2 3 4 5 6 7 8

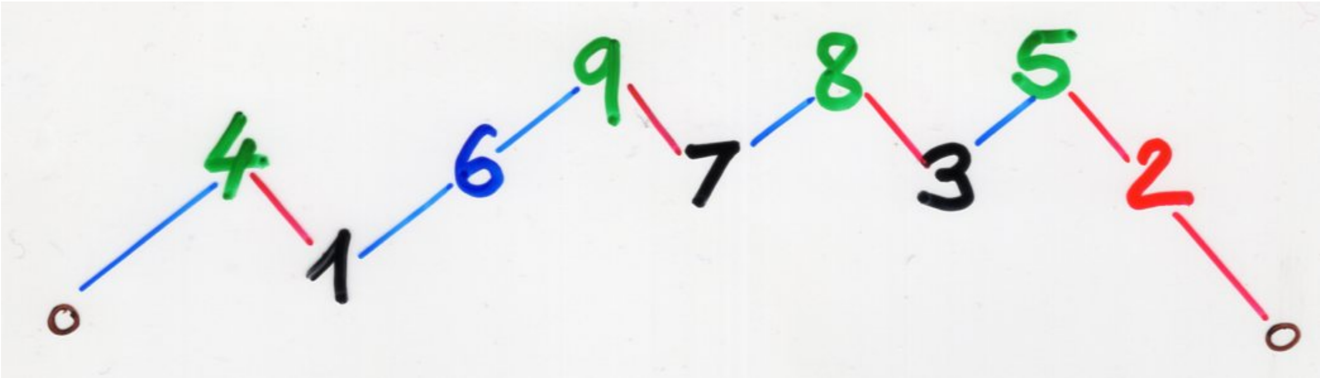
1	1	1
2	2	2
3	2	2
4	3	1
5	2	2
6	1	1
7	1	1
8	2	2




1 1 1
2 2 2
3 2 2
4 3 1
5 2 2
6 1 1
7 1 1
8 2 2
␣
␣ 1 ␣
␣ 1 ␣ 2
␣ 1 ␣ 3 ␣ 2
4 1 ␣ 3 ␣ 2
4 1 ␣ 3 5 2
4 1 6 ␣ 3 5 2
4 1 6 ␣ 7 ␣ 3 5 2
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

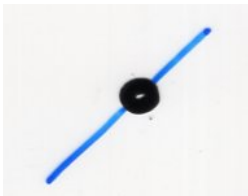
reciprocal bijection

permutations $\longrightarrow$ Laguerre histories

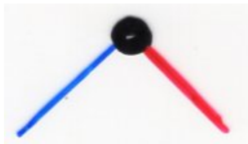


$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$

valley (through)

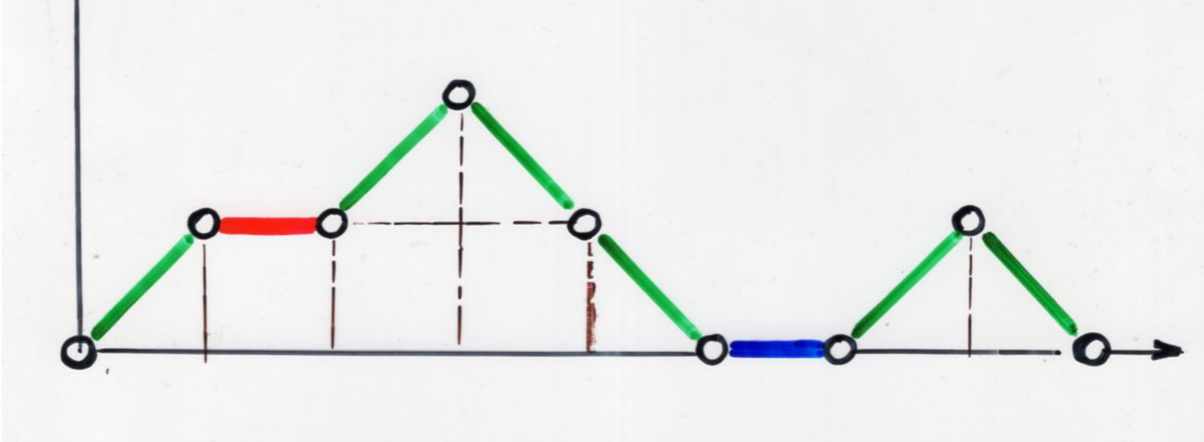


double rise

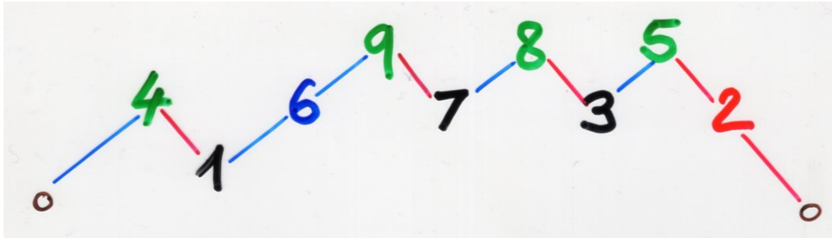


peak

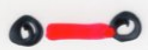
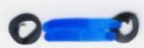
double descent



permutation σ



ω_c

	valleys	1, 3, 7
	peaks	4, 5, 8, 9
	double descents	2
	double rise	6

2- colored Motzkin path

Definition $\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

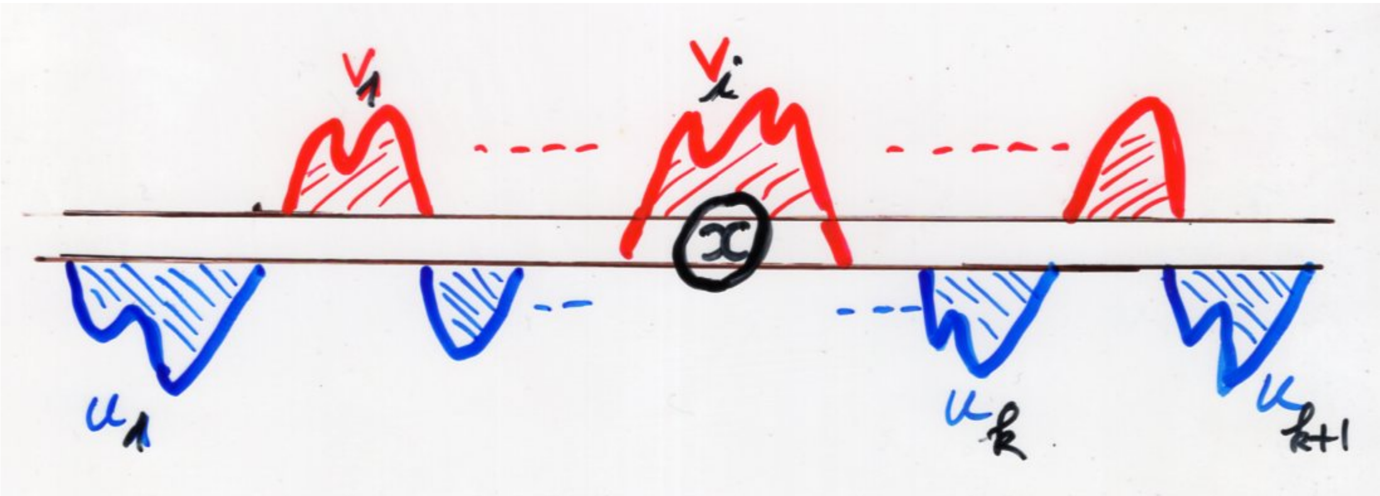
x- decomposition

- $\sigma = u_1 v_1 \cdots u_k v_k u_{k+1}$
- letters $(u_i) < x$
- letters $(v_j) \geq x$
- words $v_1, u_2, \ldots, u_k, v_k$ non-empty

example $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$ $x = 3$

$$\underbrace{}_{u_1}\overbrace{4}^{v_1}\underbrace{1}_{u_2}\overbrace{6\quad 9\quad 7\quad 8\quad 3\quad 5}^{v_2}\underbrace{2}_{u_3}$$

- $\sigma = u_1 v_1 \cdots u_k v_k u_{k+1}$
- letters $(u_i) < x$
- letters $(v_j) \geqslant x$
- words $v_1, u_2, \ldots, u_k, v_k$



reciprocal bijection

$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow \left(\omega_c \,;\, (p_1, \dots, p_n)\right)$$

$$\omega_c = \omega_1 \cdots \omega_n$$

(i) ω_i is i^{th} step

$\Longleftrightarrow$ i

valley

peak

double descent

double rise

(ii) $p_i = j \Longleftrightarrow$ i is a letter of the word v_j in the i-decomposition of

$$\sigma = u_1 v_1 \cdots v_j \cdots u_k v_k u_{k+1}$$

example

$P_i = j \iff$ i is a letter of the word v_j in the i-decomposition of

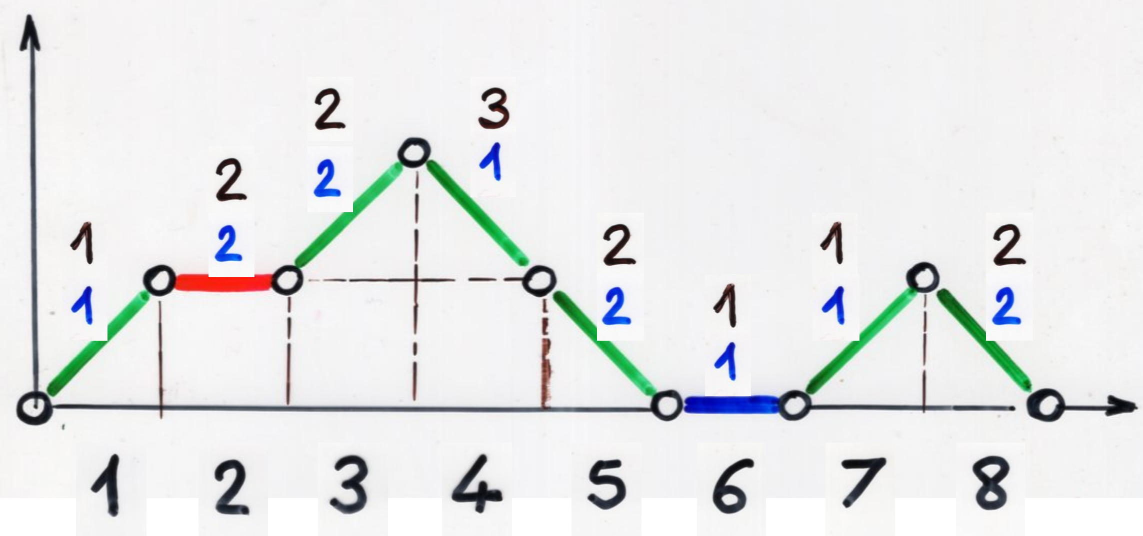
$$\sigma = u_1 v_1 \cdots v_j \cdots u_k v_k u_{k+1}$$

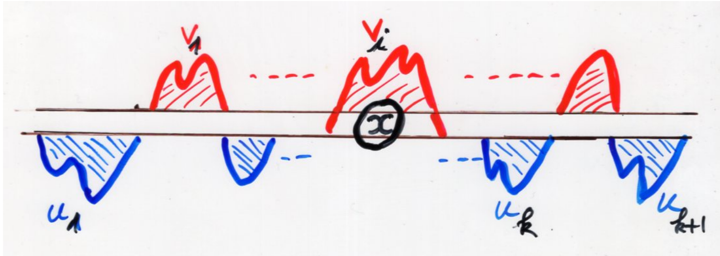
$$\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

$$\underbrace{}_{u_1}\overbrace{4}^{v_1}\underbrace{1}_{u_2}\overbrace{6\;9\;7\;8\;3\;5}^{v_2}\underbrace{2}_{u_3}$$

$i = 3$

$j = 2$





1		1	1
2		2	2
3		2	2
4		3	1
5		2	2
6		1	1
7		1	1
8		2	2

$\sqcup$

$\sqcup\ 1\ \sqcup$

$\sqcup\ 1\ \sqcup\ 2$

$\sqcup\ 1\ \sqcup\ 3\ \sqcup\ 2$

$4\ 1\ \sqcup\ 3\ \sqcup\ 2$

$4\ 1\ \sqcup\ 3\ 5\ 2$

$\sqcup\sqcup = \sqcup$ $\qquad 4\ 1\ 6\ \sqcup\ 3\ 5\ 2$

$4\ 1\ 6\ \sqcup\ 7\ \sqcup\ 3\ 5\ 2$

$4\ 1\ 6\ \sqcup\ 7\ 8\ 3\ 5\ 2$

$4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$

Lemma

$P_i = j$ is also defined by :

$j = 1 +$ number of triples (a, b, i) having the patter $(31\text{-}2)$, that is:

$$a = \sigma(k),\quad b = \sigma(k+1),\quad i = \sigma(\ell)$$

with $k < k+1 < \ell$ and $b < i < a$