Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 1
Paths and moments

Ch 1d

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

IMSc, Chennai
January 24, 2019

mirror website
www.imsc.res.in/~viennot

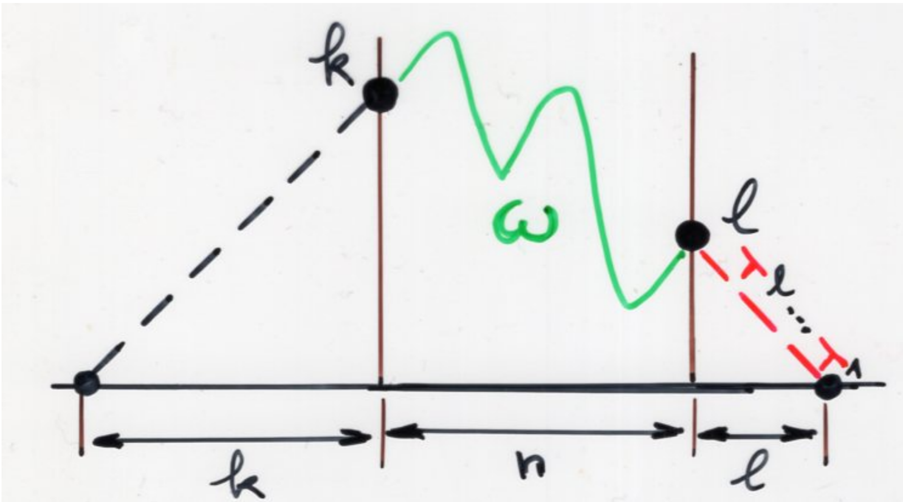
Reminding Ch 1c

another formulation of the main **Theorem**

$$f\left(P_k P_\ell x^n\right) = \sum_{\substack{\omega \\ \text{Motzkin path level } 0 \to 0 \\ |\omega| = k+n+\ell}} v(\omega)$$

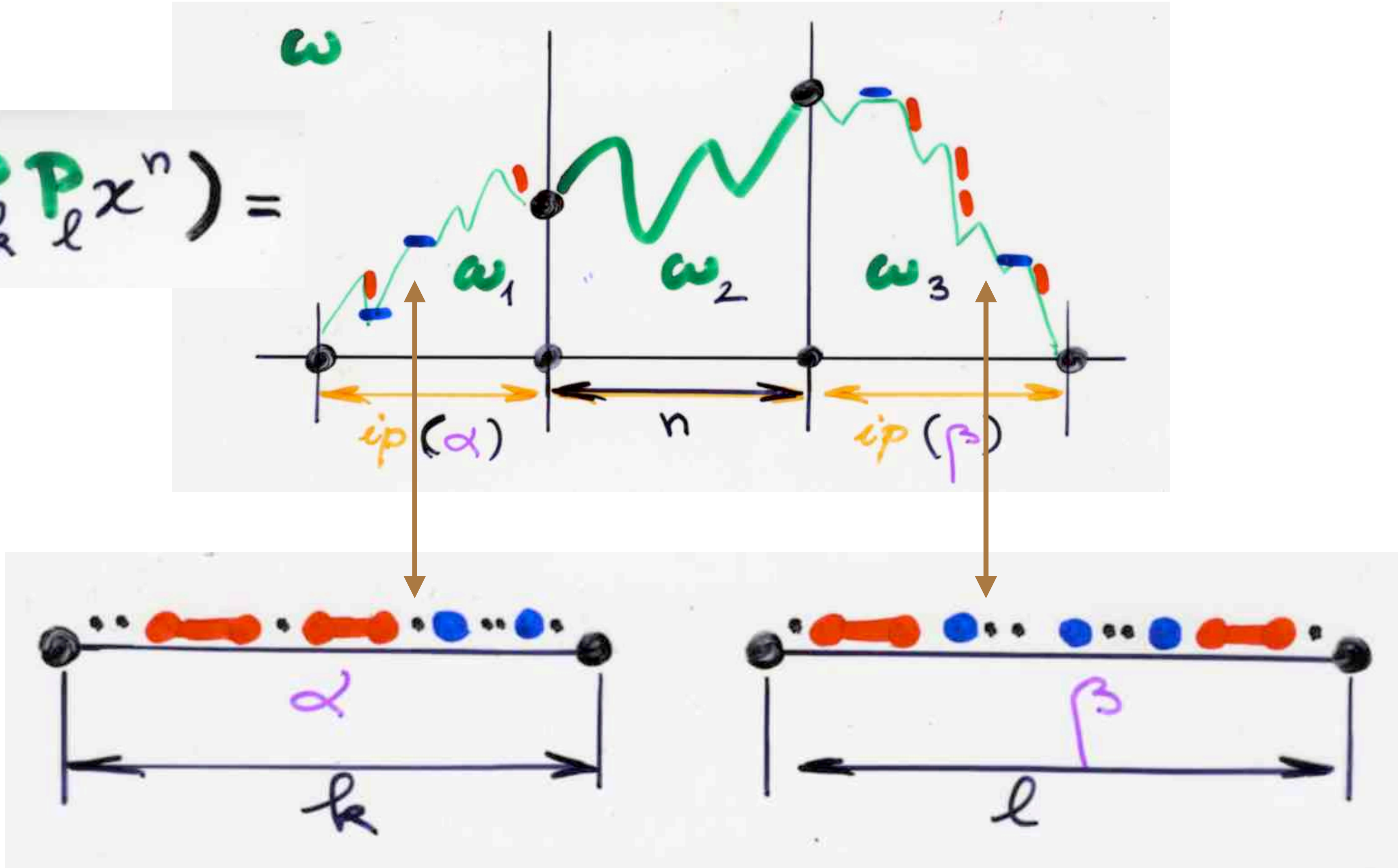
(i) first k steps are NE

(ii) last ℓ steps are SE



$\lambda_1 \times \cdots\cdots \times \lambda_\ell$

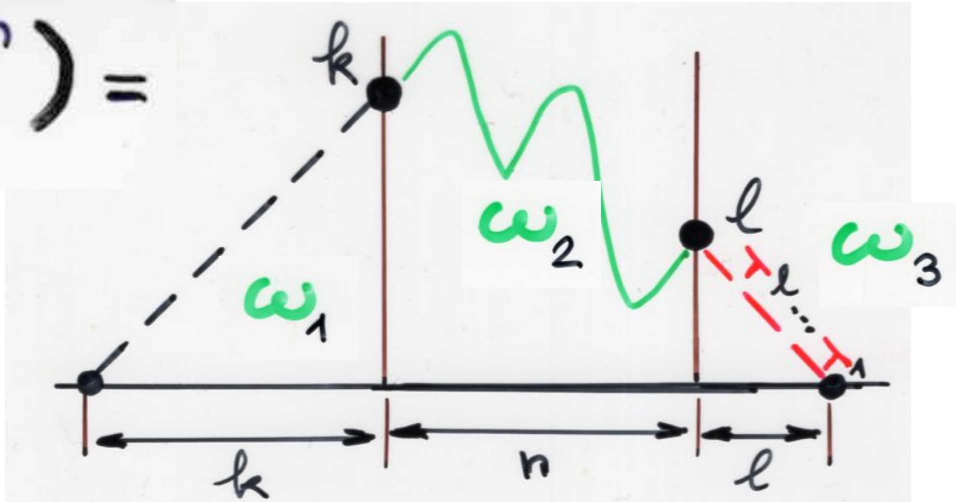
$$f(P_k P_\ell x^n) =$$

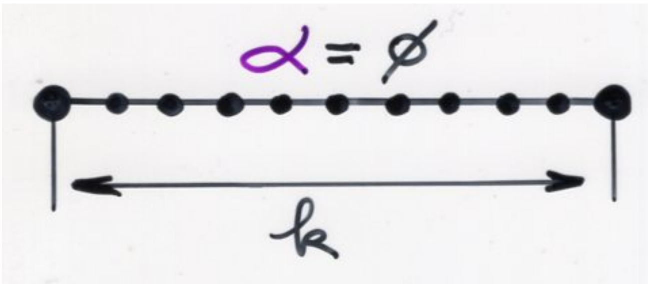


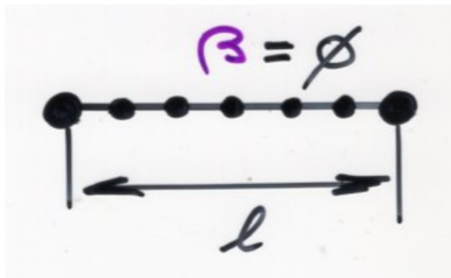
construction of an involution θ

$F_{n,k,\ell} \subseteq E_{n,k,\ell}$
- α, β empty
- $\omega_1 =$ (dashed segment) $(|\omega_1| = k)$
- $\omega_3 =$ (dashed segment) $(|\omega_3| = \ell)$

$f(P_k P_\ell x^n) =$







linearization coefficients

$$P_k(x)\, P_\ell(x) = \sum_n a_{k\ell}^{n}\, P_n(x)$$

$$a_{k\ell}^{n} = \frac{f(P_k P_n P_\ell)}{f(P_n^2)}$$

positivity

Proposition

Askey (1970)

$\lambda_{j+1} \geq \lambda_j \;, \; b_{j+1} \geq b_j$

If $\{\lambda_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 0}$ are increasing sequences
and $\lambda_j > 0$ for every $j \geq 1$,
then

$$a_{kl}^{n} \geq 0$$

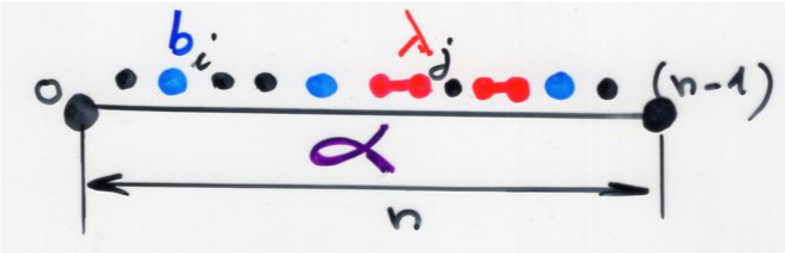
$$a_{kl}^{n} = \frac{f(P_k P_n P_l)}{f(P_n^2)}$$

combinatorial proof

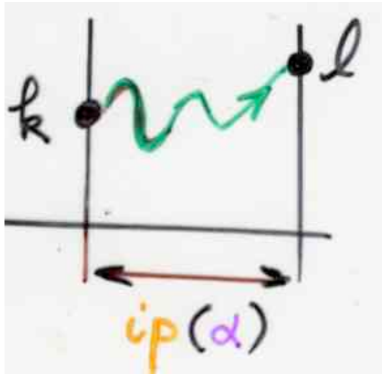
de Médicis, Stanton (1996)

$$f(P_k P_n P_\ell) = \lambda_1 \times \cdots \times \lambda_\ell \quad \sum_{(\alpha,\omega)\in M_{n,k,\ell}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)$$

$M_{n,k,\ell} = \{ (\alpha,\omega),$
- α pavage of $[0, n-1]$
- ω Motzkin path $k \to \ell$ level
- $|\omega| = ip(\alpha)$ $\}$



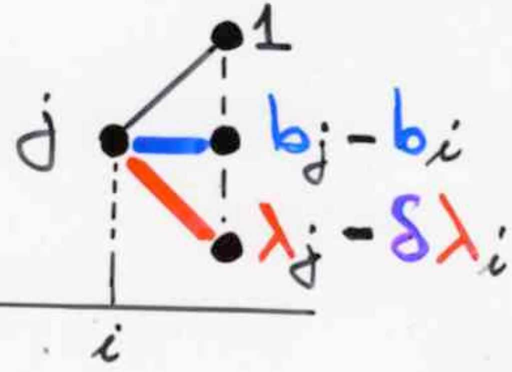
pavage α



- $x^{ip(\alpha)}$ number of isolated points of α

define a weight $\bar{v}$ on Motzkin paths

$\bar{v}$



$\delta = 1$

else $\delta = 0$

Proposition

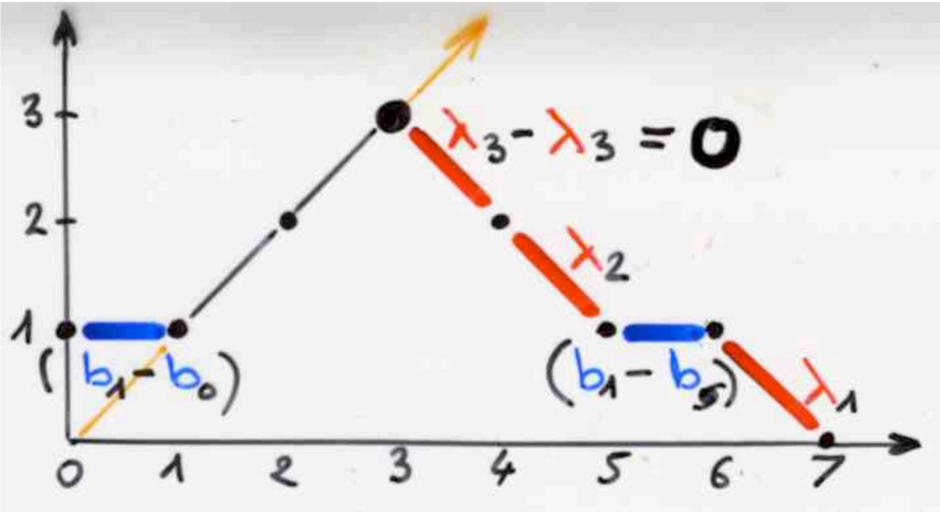
de Médicis, Stanton (1996)

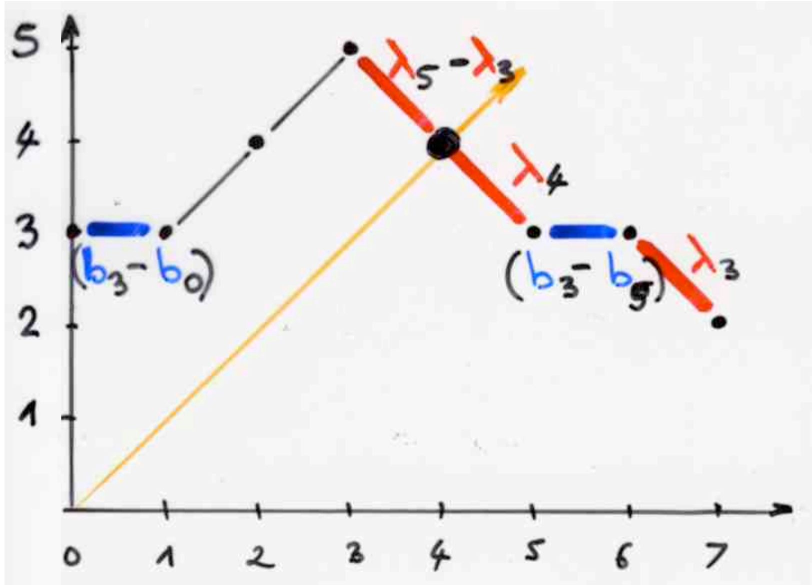
$$\sum_{(\alpha,\omega)\in M_{n,k,\ell}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega) = \sum_{\eta} \bar{v}(\eta)$$

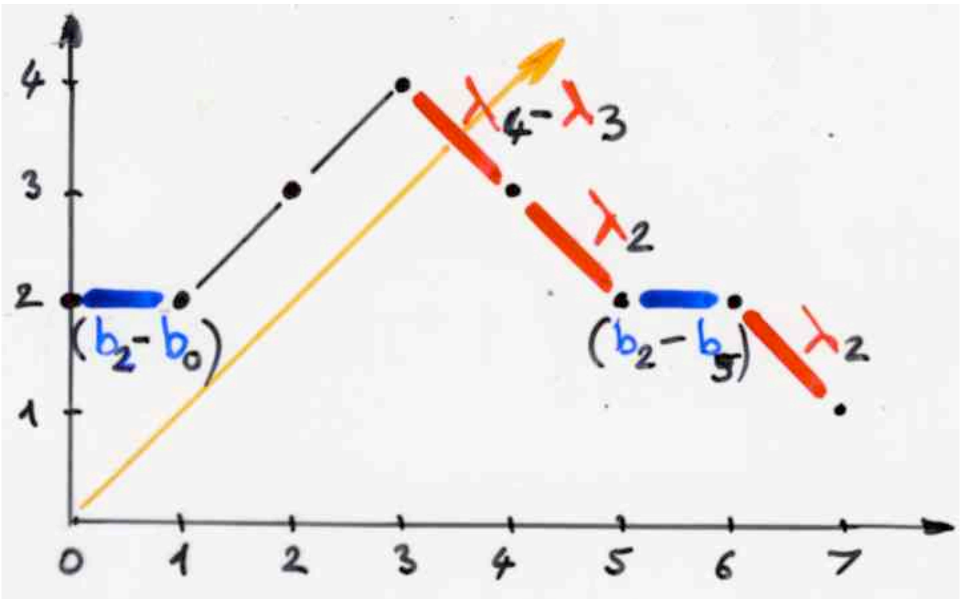
η Motzkin path

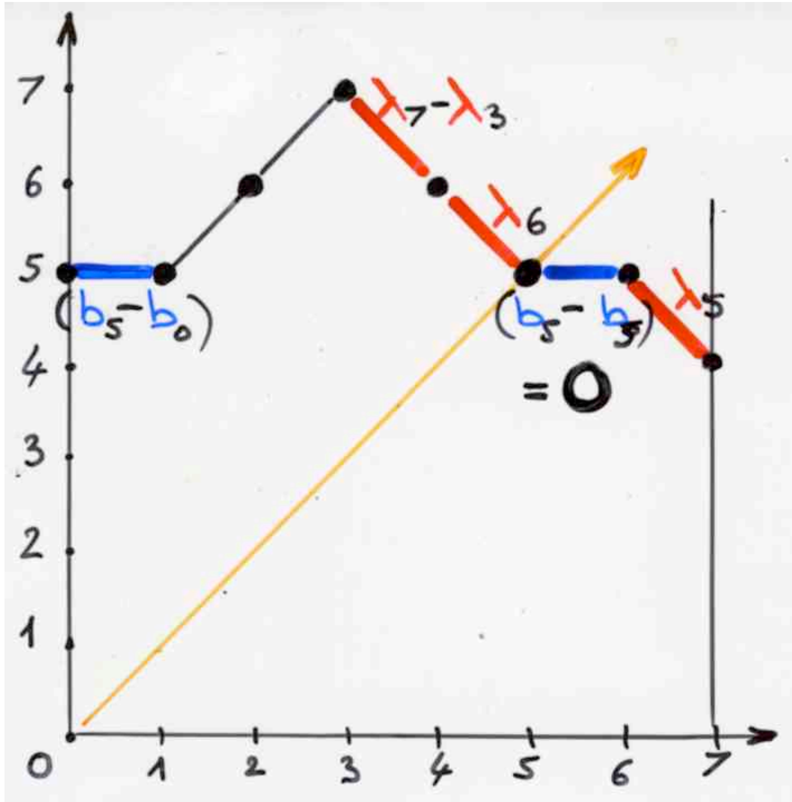
$|\eta| = n$

$\bar{V}(\eta)$

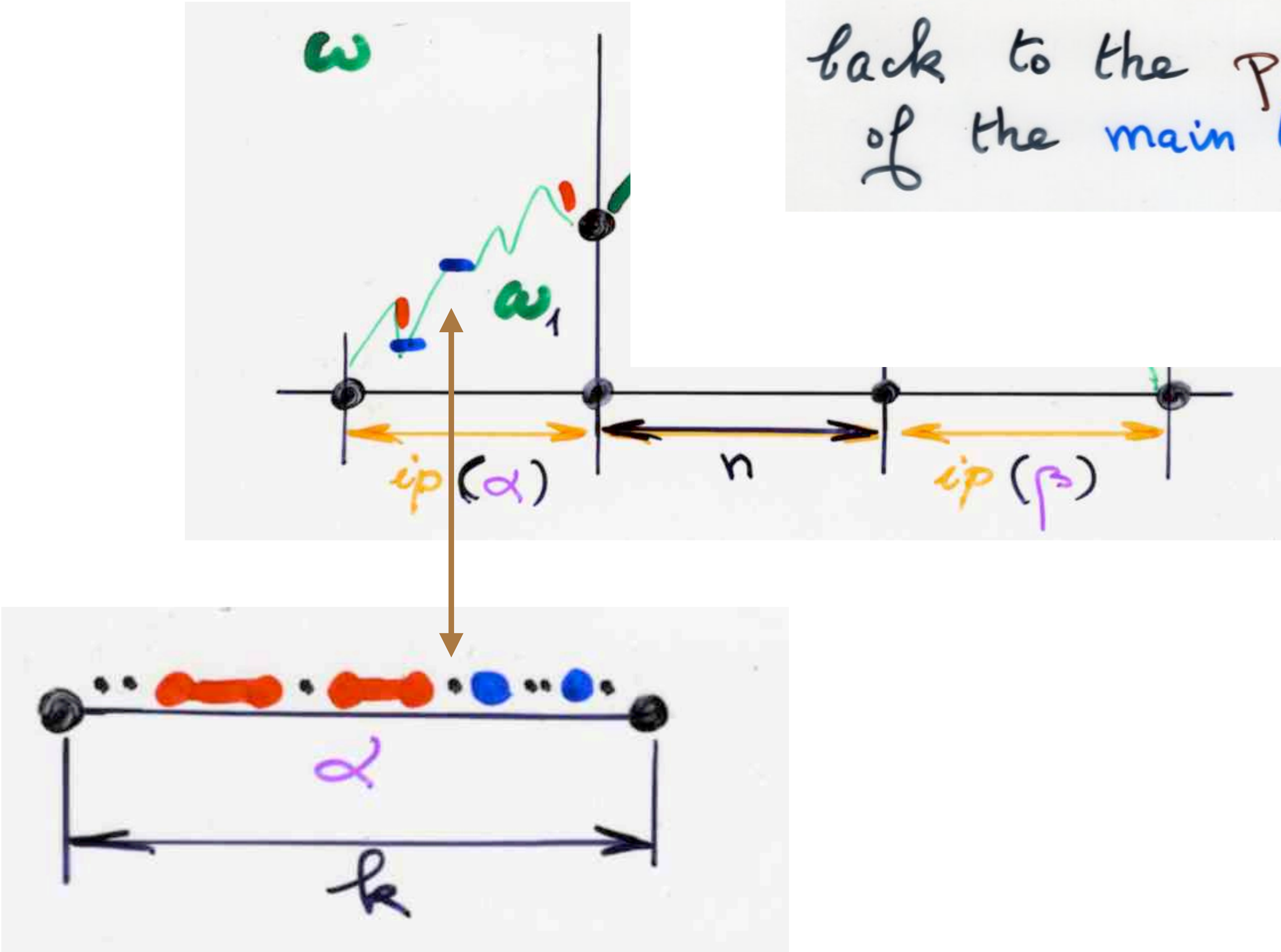

$\lambda_3 - \lambda_3 = 0$
λ_2
$(b_1 - b_0)$
$(b_1 - b_5)$
λ_1
0 1 2 3 4 5 6 7
1 2 3


$\lambda_5 - \lambda_3$
λ_4
$(b_3 - b_0)$
$(b_3 - b_5)$
λ_3
0 1 2 3 4 5 6 7
1 2 3 4 5


$\lambda_4 - \lambda_3$
λ_2
$(b_2 - b_0)$
$(b_2 - b_5)$
λ_2
0 1 2 3 4 5 6 7
1 2 3 4


$\lambda_7 - \lambda_3$
λ_6
$(b_5 - b_0)$
$(b_5 - b_5)$
$= 0$
λ_5
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7

back to the proof of the main theorem

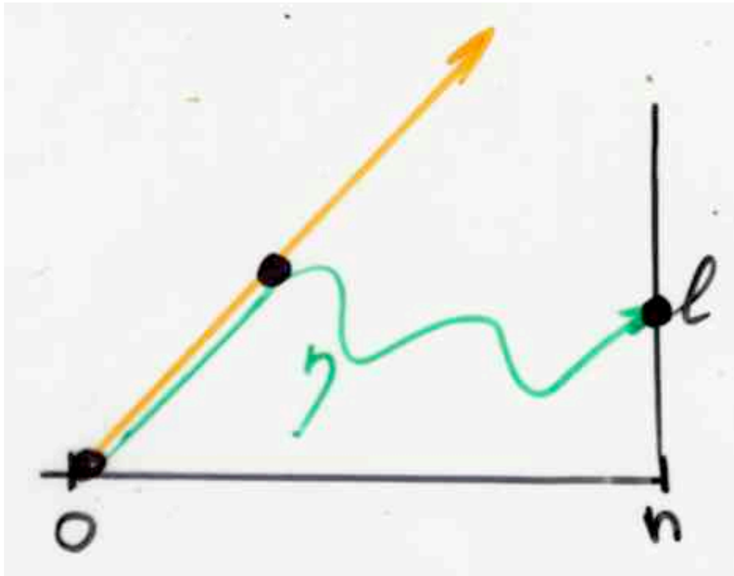


first involution θ_1 on $E_{n,k,\ell} \setminus L_{n,k,\ell}$

$k = 0$

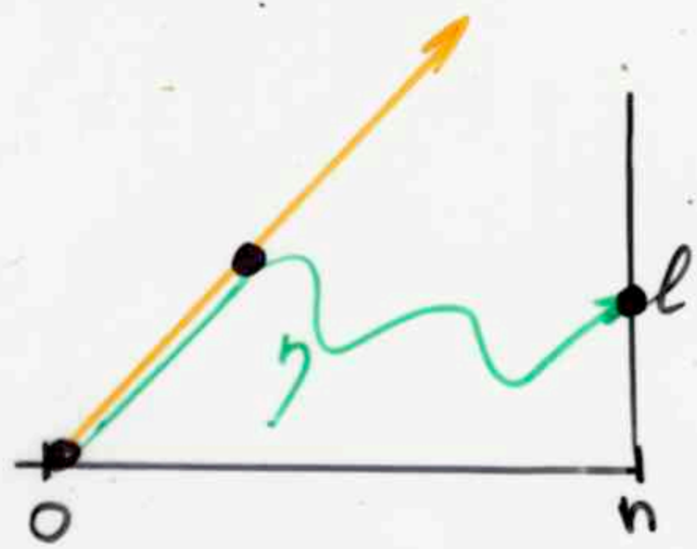
$M_{n,0,\ell} = E_{n,\ell}$

$$\sum_{(\alpha,\omega)\in E_{n,\ell}} (-1)^{|\alpha|} v(\alpha)\, v(\omega) = \sum_{|\eta|=n} \bar{v}(\eta)$$

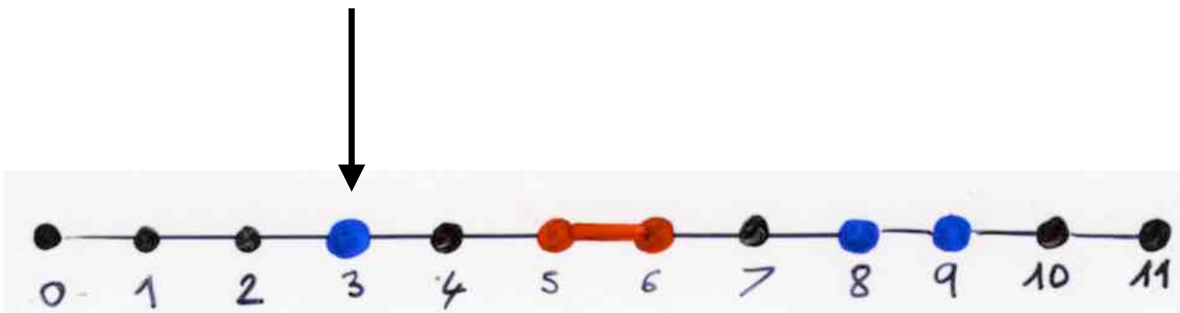


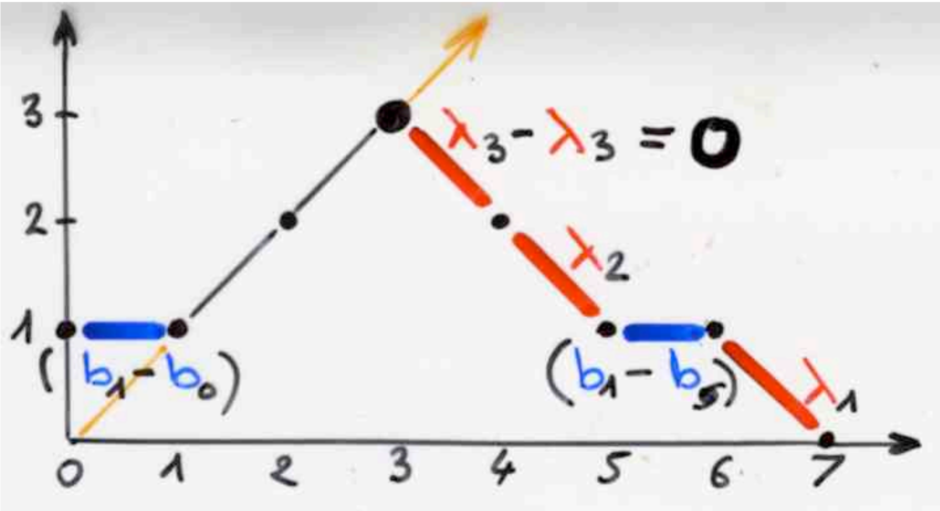
first involution θ_1 on $E_{n,k,\ell} \setminus L_{n,k,\ell}$

this means, with $\dot{j} = h(\omega)$

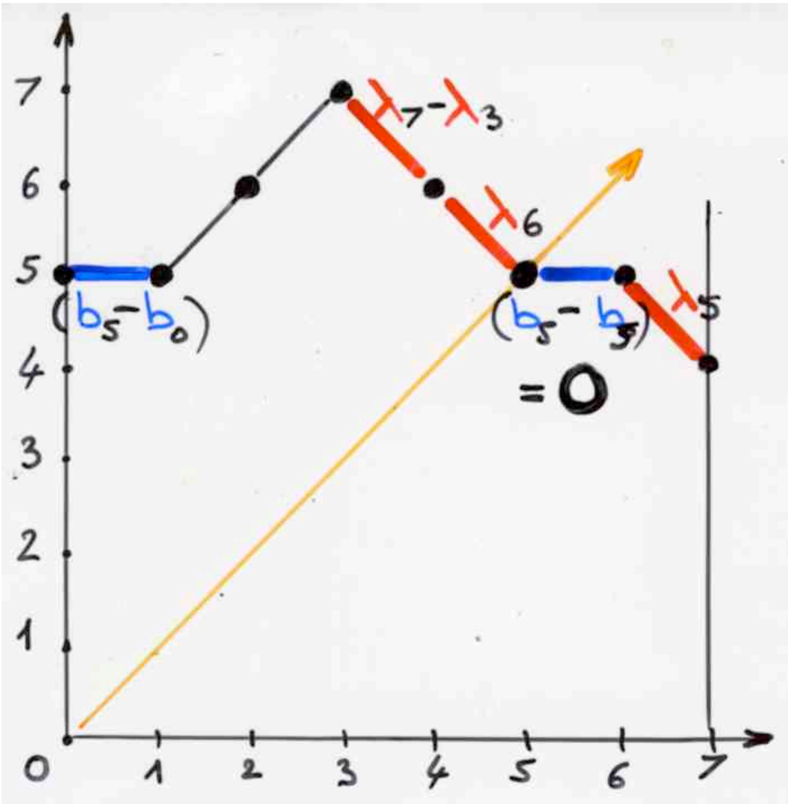


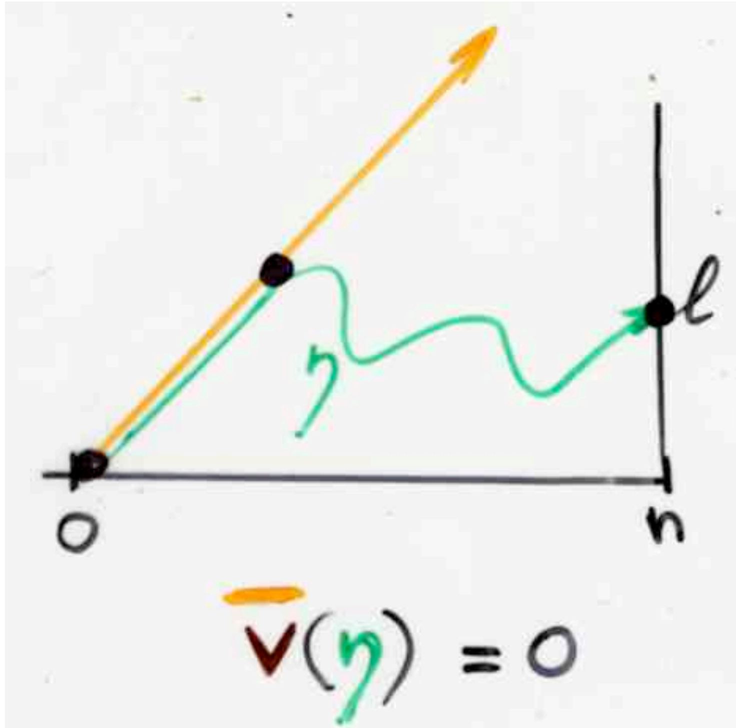
$\bar{V}(\eta) = 0$




$\lambda_3 - \lambda_3 = 0$
λ_2
λ_1
$(b_1 - b_0)$
$(b_1 - b_5)$

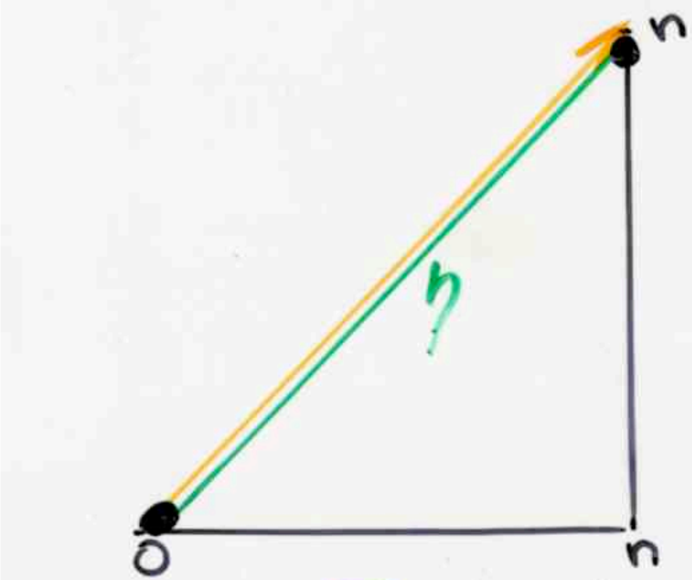
$\bar{V}(\eta)$


$\lambda_7 - \lambda_3$
λ_6
λ_5
$(b_5 - b_0)$
$(b_5 - b_5)$
$= 0$


η
ℓ
0
n
$\bar{V}(\eta) = 0$

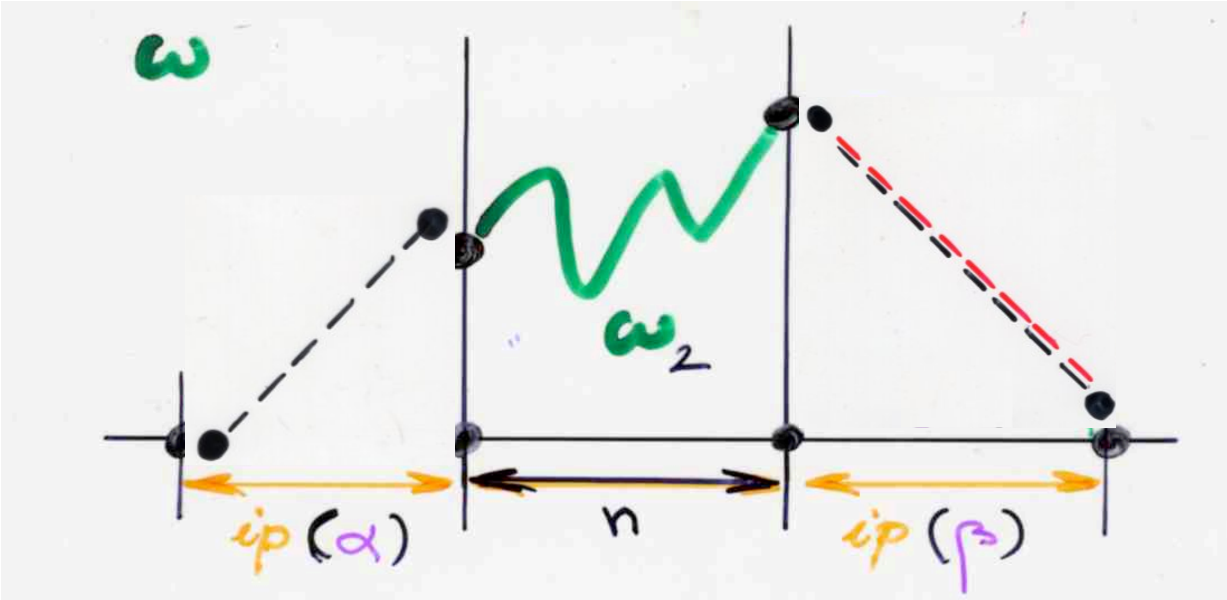
first involution Θ_1 on $E_{n,k,\ell} \setminus L_{n,k,\ell}$

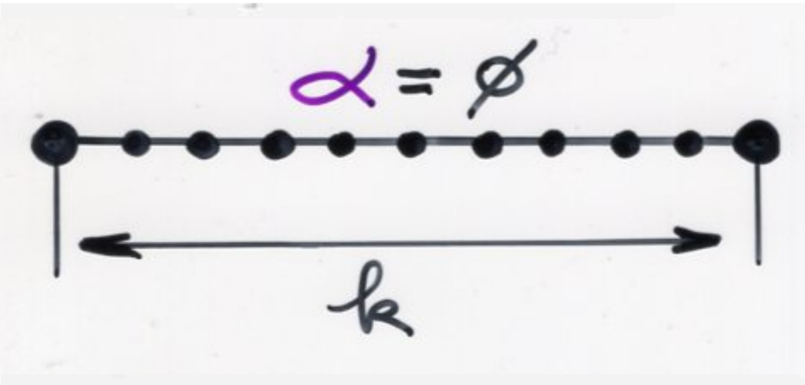
$h(\omega)$ and $h(\alpha)$ both ∞

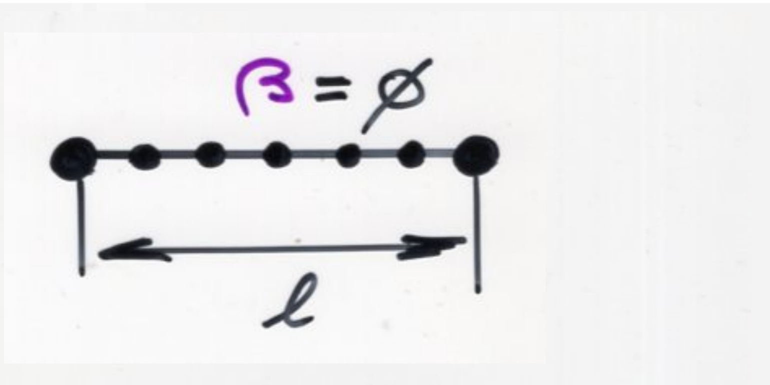


except $\bar{v}(\eta) = 1$

$$F_{n,k,\ell} = L_{n,k,\ell} \cap R_{n,k,\ell}$$



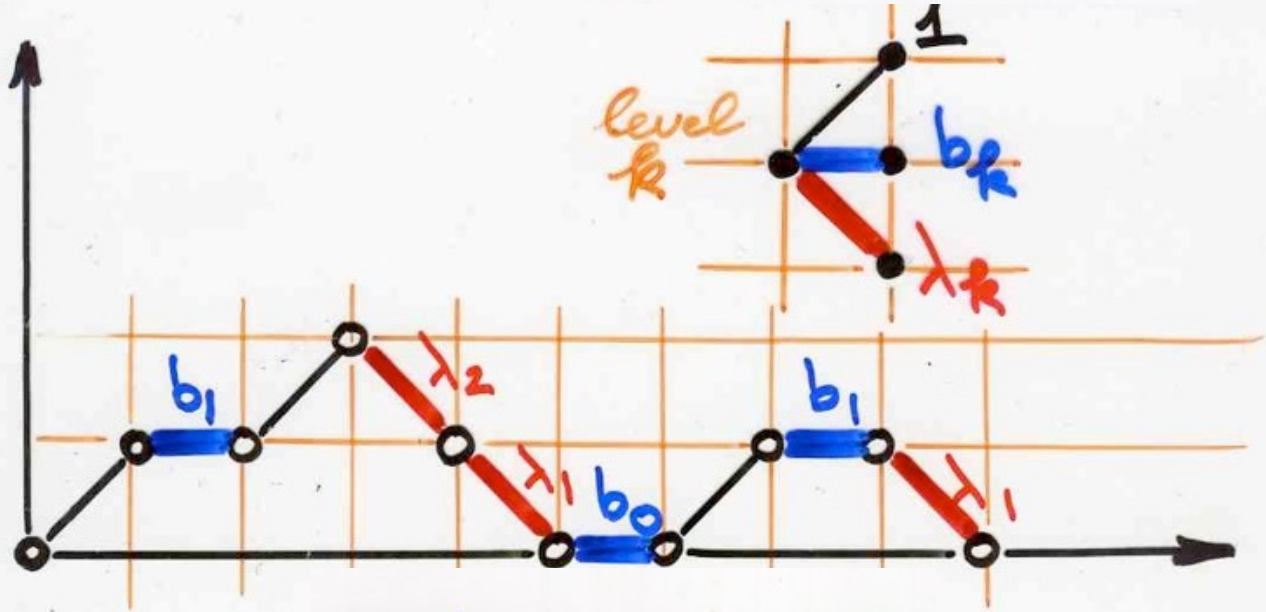




Favard paths

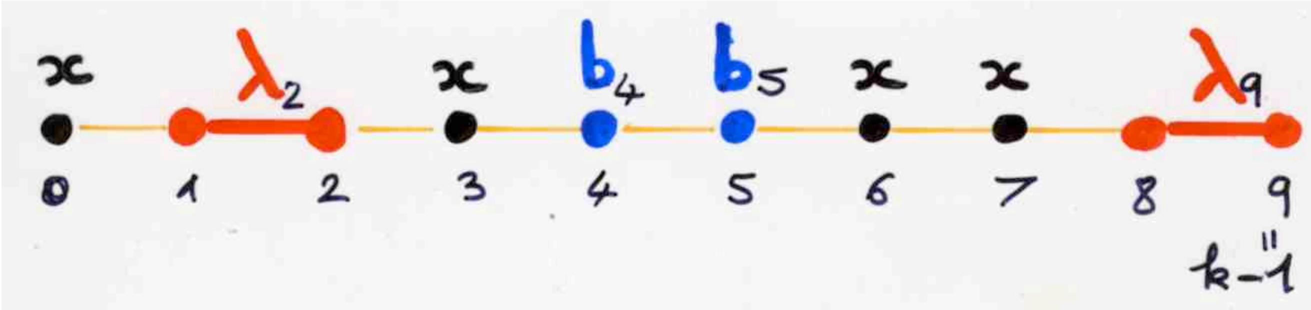
$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$ $b_k, \lambda_k \in \mathbb{K}$ ring

valuation v

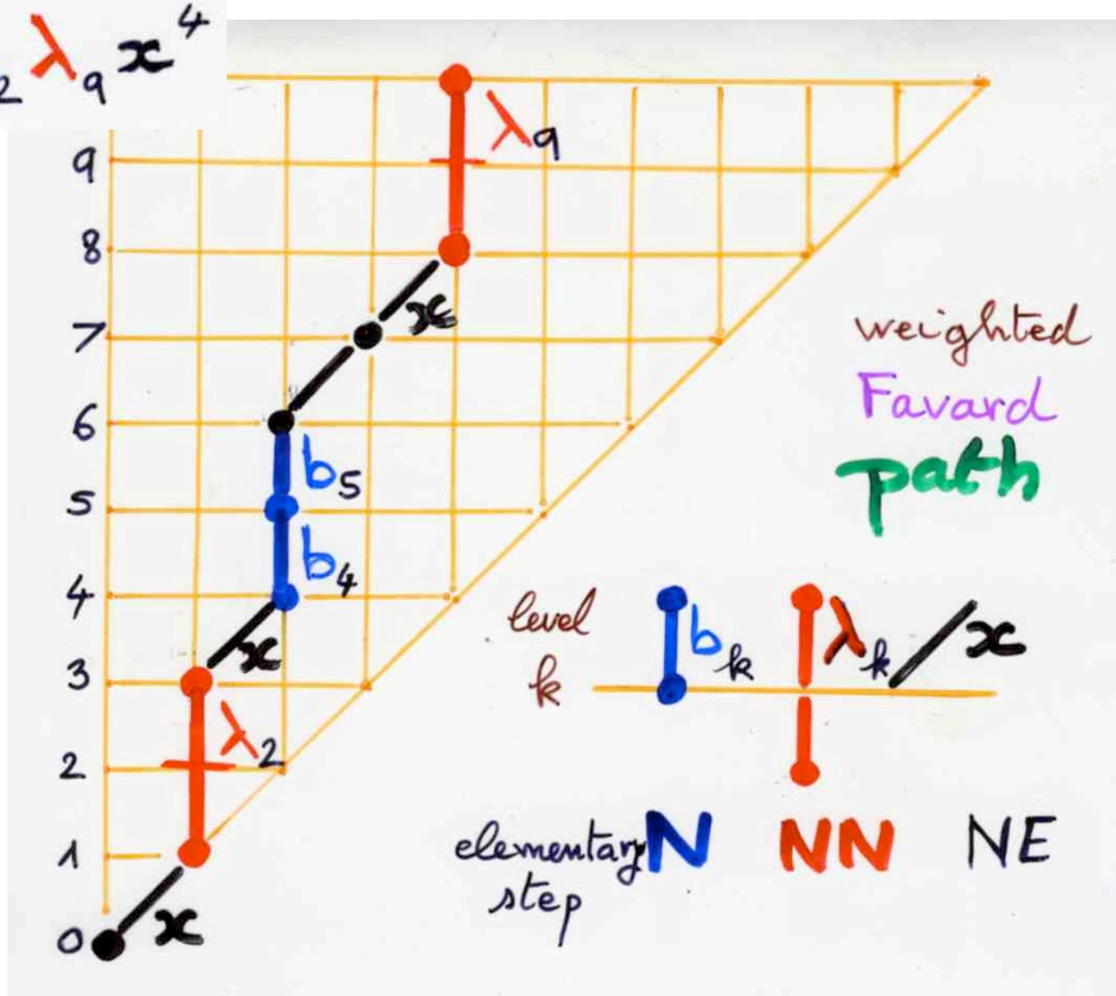


ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$



$$(-1)^4\, b_4 b_5 \lambda_2 \lambda_9\, x^4$$



$$v(\eta)$$

$$b_4 b_5 \lambda_2 \lambda_9$$

number of N, NN, NE, elementary steps of η

$$P_n(x) = \sum_{\substack{\eta \\ \text{Favard path } |\eta| = n}} (-1)^{N + NN(\eta)} v(\eta)\, x^{NE(\eta)}$$

$|\eta|$

the "length" of η is the number of steps where "NN" is counting for 2

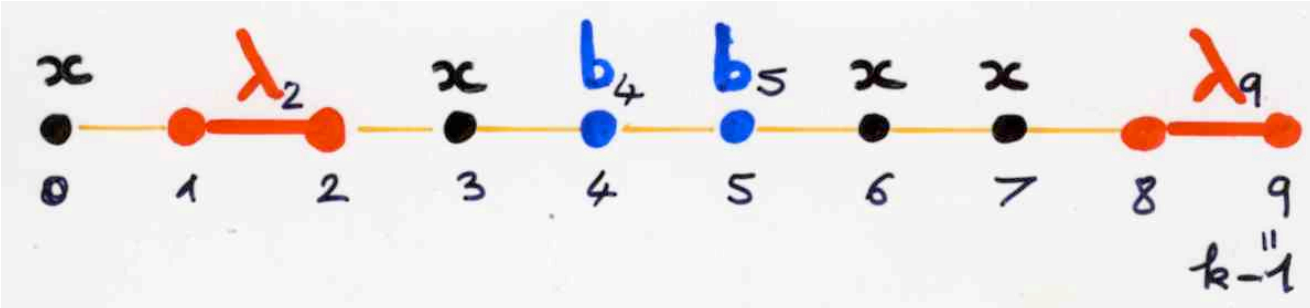
added after the video:

in other words $|\eta|$ is the level of the ending point.

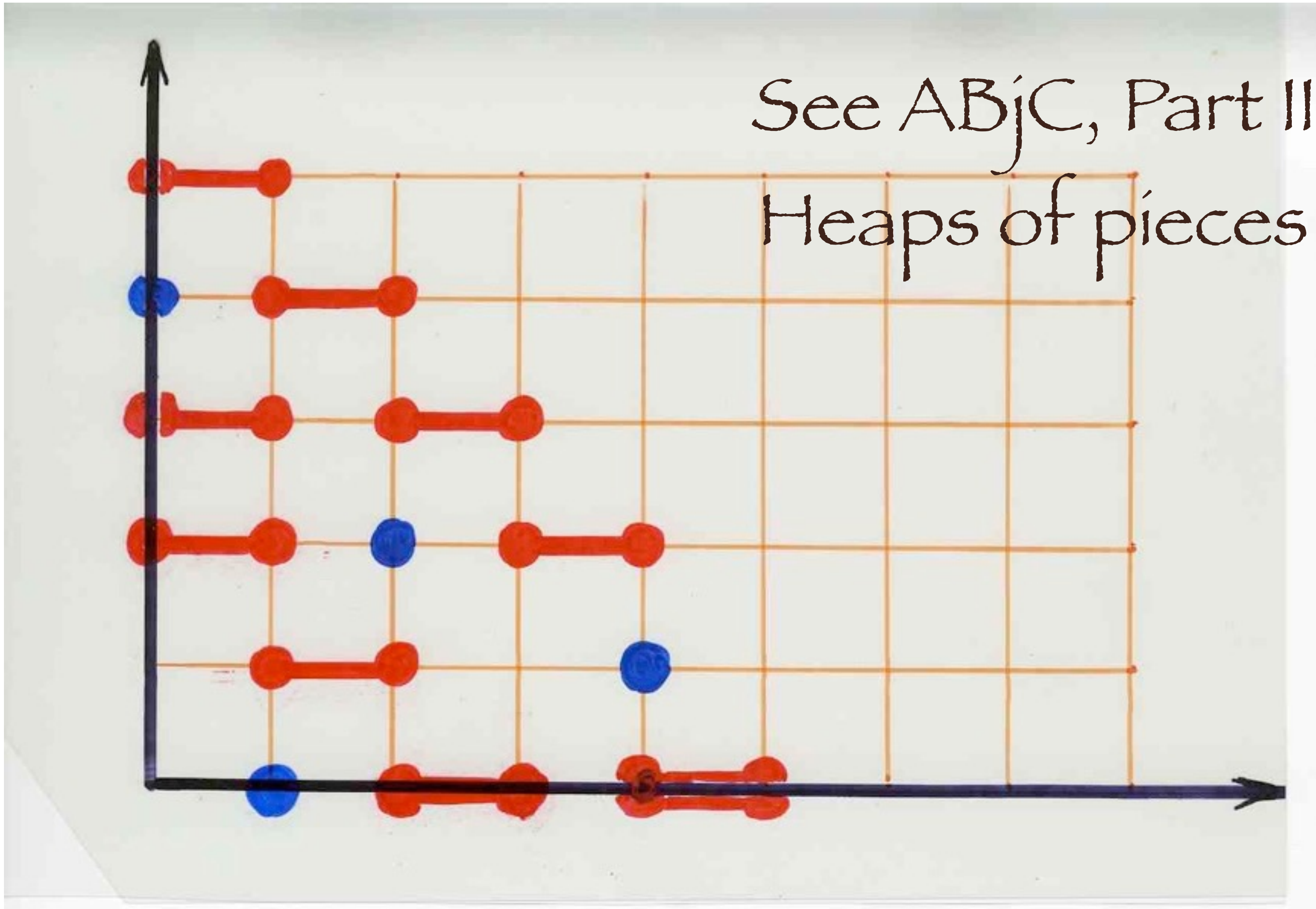
$$P_n(x) = \sum_{\substack{\alpha \\ \text{pavage of } [0, n-1]}} (-1)^{|\alpha|} v(\alpha)\, x^{ip(\alpha)}$$

Complements

other interpretation of the moments

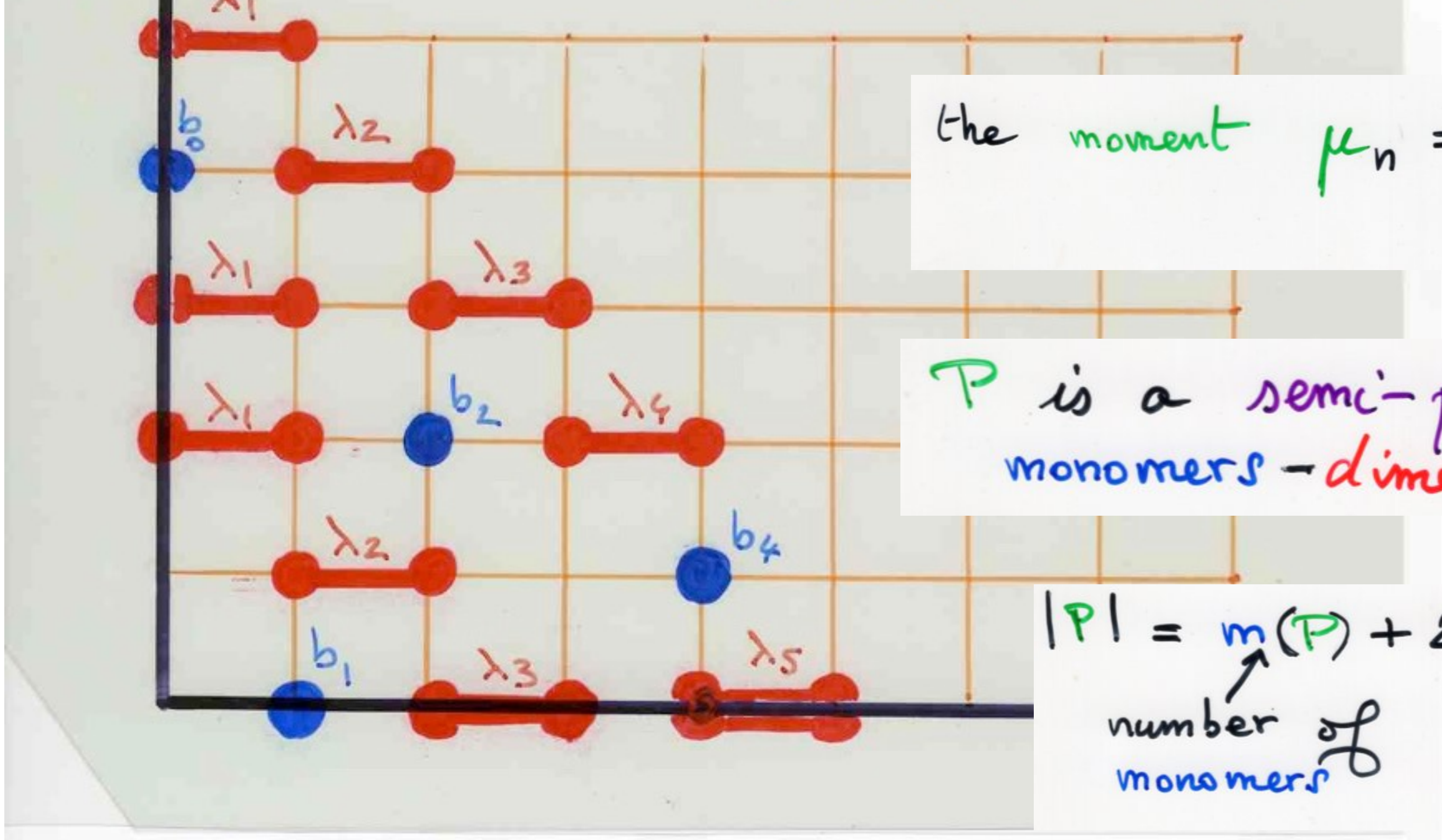


Pavage = trivial heap (→ Part II)
of monomers, dimers


See ABjC, Part II
Heaps of pieces

added after the video:

$$v(P) = b_0 b_1 b_2 b_4 \lambda_1^3 \lambda_2^2 \lambda_3^2 \lambda_4 \lambda_5$$



the moment $\mu_n = \sum_{|P|=n} v(P)$

P is a semi-pyramid of monomers - dimers on $\mathbb{N}$

$|P| = m(P) + 2d(P)$

$m(P)$: number of monomers

$d(P)$: number of dimers

Inverse polynomials

$$\{P_n(x)\}_{n\geq 0}$$

$$P_n \in \mathbb{K}[x]$$

monic

$$\deg(P_n) = n$$

$$P_n(x) = x^n + \dots$$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$\{b_k\}_{k\geq 0},\ \{\lambda_k\}_{k\geq 1}$$

$$b_k, \lambda_k \in \mathbb{K}$$

ring

Definition Inverse polynomials

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

$$Q_n(x) = \sum_{i=0}^{n} q_{n,i} x^i$$

inverse sequence $\{Q_n(x)\}_{n \geq 0}$

$$Q = (q_{n,i})_{i,n}$$

$$P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i} x^i$$

$$\begin{cases} p_{n,i} = 0, & i > n \\ q_{n,i} = 0, & i > n \end{cases}$$

triangular matrices

(1 on the diagonal)

$$Q = P^{-1}$$

Definition vertical polynomials

$$V_n(x) = \sum_{i=0}^{n} \mu_{n,i}\, x^i$$

$$\mu_{n,i} = \sum_{\substack{\omega \\ \text{"Motzkin" path} \\ |\omega|=n,\ 0 \to i}} v(\omega)$$

Motzkin path going from level 0 to level i

$$V = (\mu_{n,i})_{n,i \geq 0}$$

Proposition

$\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ $b_k, \lambda_k \in \mathbb{K}$

$\{V_n(x)\}_{n\geq 0}$ is the inverse sequence of $\{P_n(x)\}_{n\geq 0}$

defined by the 3-terms reccurence relation

in other words $V = Q = P^{-1}$

(classical) proof

(with $\lambda_k \neq 0$ for every $k \geq 1$)

$\mathbb{K}$ ring

integral domain

$a, b \neq 0 \Rightarrow ab \neq 0$

cancellation property

$a \neq 0$, $ab = ac \Rightarrow b = c$

$f(x^n) = \mu_n$

from the "main theorem"

$f(x^n P_k) = \lambda_1 \cdots \lambda_k \mu_{n,k}$

from the "main theorem"

$$f(x^n P_k) = \lambda_1 \cdots \lambda_k \mu_{n,k}$$

$$x^n = \sum_{i=0}^{n} q_{n,i} P_i(x)$$

Inverse polynomials

$$f(x^n P_k) = \sum_{i=0}^{n} q_{n,i} f(P_i P_k)$$

$$= q_{n,k} f(P_k P_k)$$

$$f(x^n P_k) = \lambda_1 \cdots \lambda_k q_{n,k}$$

Bijective proof

we have to prove :

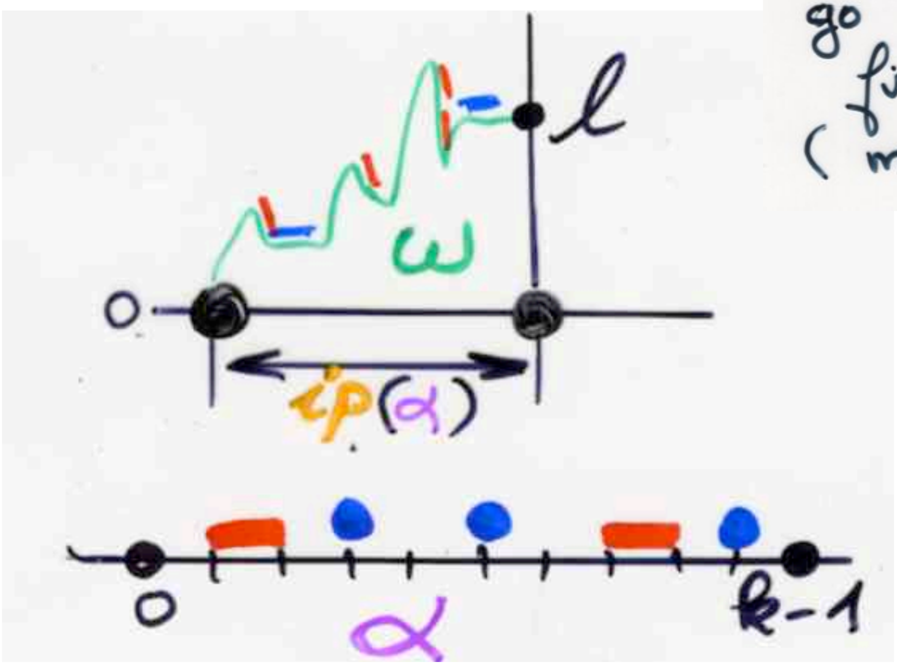
$$\sum_{0 \le i} p_{k,i}\,\mu_{i,l} = \delta_{k,l} \quad \text{for every } k,l \ge 0$$

$$p_{k,i} = \sum_{\substack{\alpha \text{ pavage of } [0,k-1] \\ ip(\alpha)=i}} (-1)^{|\alpha|}\, v(\alpha)$$

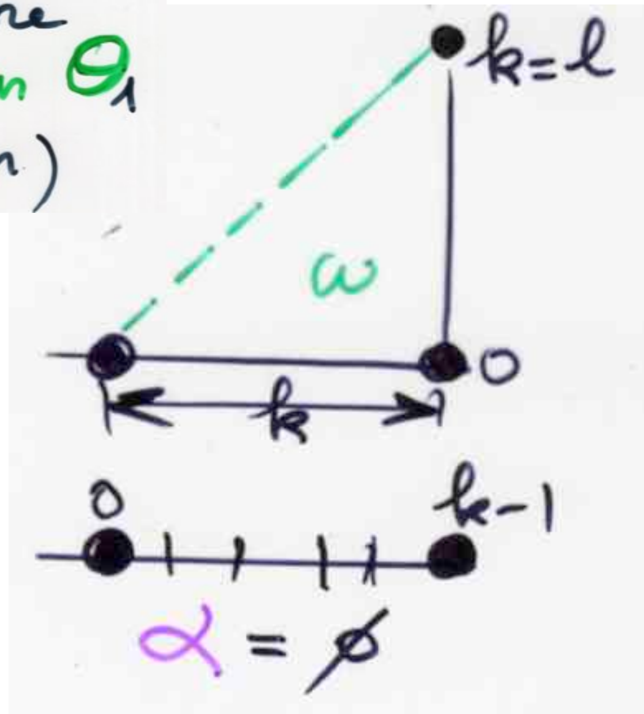
$$\mu_{i,l} = \sum_{\substack{\omega \text{ Motzkin path} \\ 0 \rightsquigarrow l \text{ level} \\ |\omega| = i}} v(\omega)$$

$$\sum_{0 \le i} p_{k,i}\,\mu_{i,l} = \sum_{(\alpha,\omega)\in E_{k,l}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)$$

$$E_{k,l} = \left\{ (\alpha,\omega),\ \begin{array}{l} -\ \alpha \text{ pavage of } [0,k-1] \\ -\ \omega \text{ Motzkin path} \\ \quad 0 \rightsquigarrow l,\ |\omega| = ip(\alpha) \end{array} \right\}$$



go back to the first involution θ_1
(main theorem)



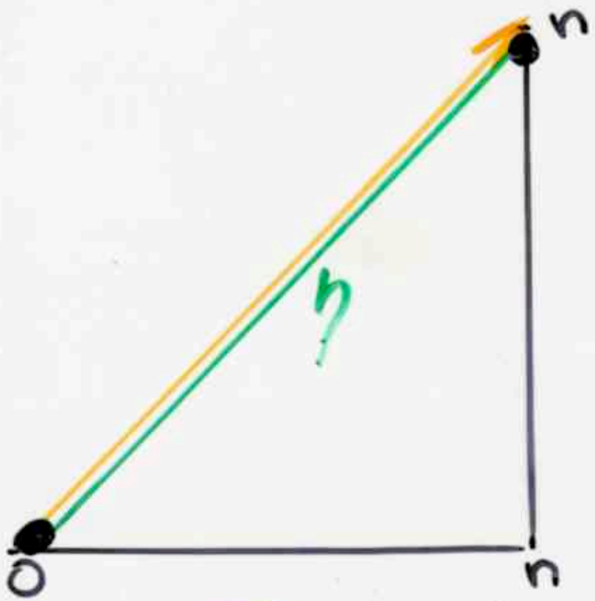
$$\sum_{(\alpha,\omega)\in E_{k,\ell}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega) = \delta_{k,\ell}$$

$$E_{k,\ell} = \left\{ (\alpha,\omega),\ \begin{array}{l} -\ \alpha \text{ pavage of } [0,k-1] \\ -\ \omega \text{ Motzkin path} \\ \quad 0 \to \ell,\ |\omega| = ip(\alpha) \end{array} \right\}$$

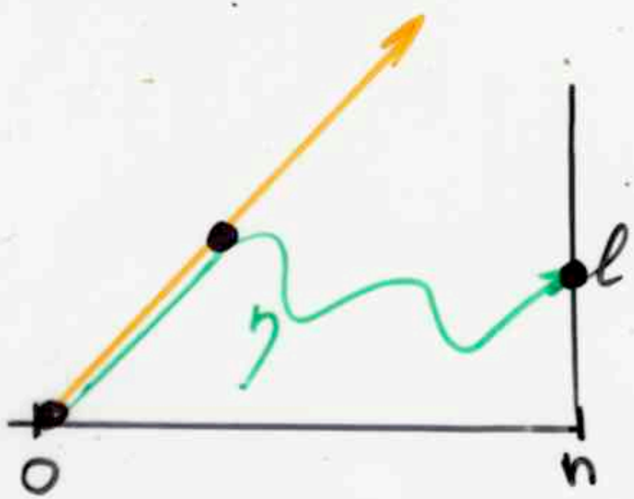
or apply de Médicis, Stanton's bijective methodology

$k = 0$ $\qquad M_{n,0,\ell} = E_{n,\ell}$

$$\sum_{(\alpha,\omega)\in E_{n,\ell}} (-1)^{|\alpha|} v(\alpha)v(\omega) = \sum_{|\eta|=n} \bar{v}(\eta)$$



except $\bar{v}(\eta) = 1$

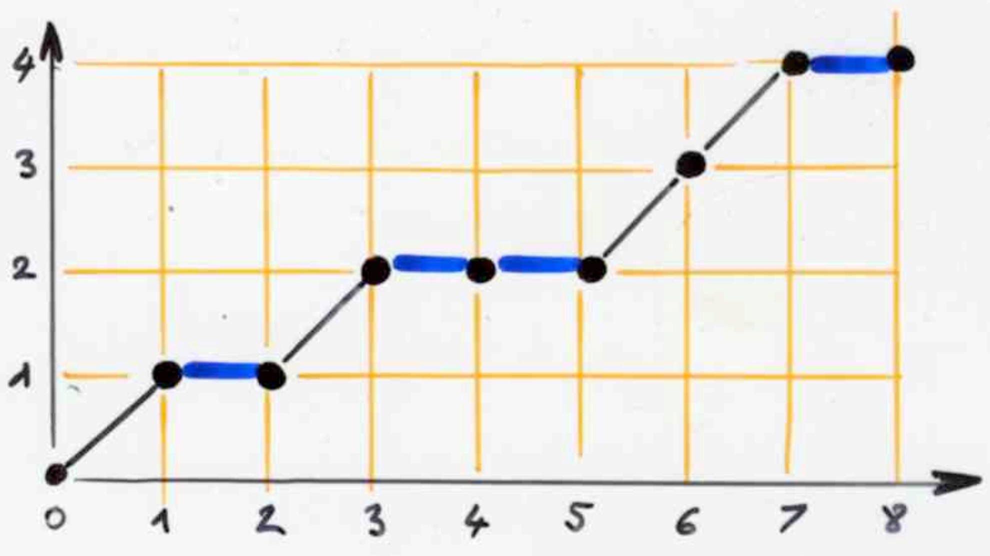


$\bar{v}(\eta) = 0$

Inverse relations: examples

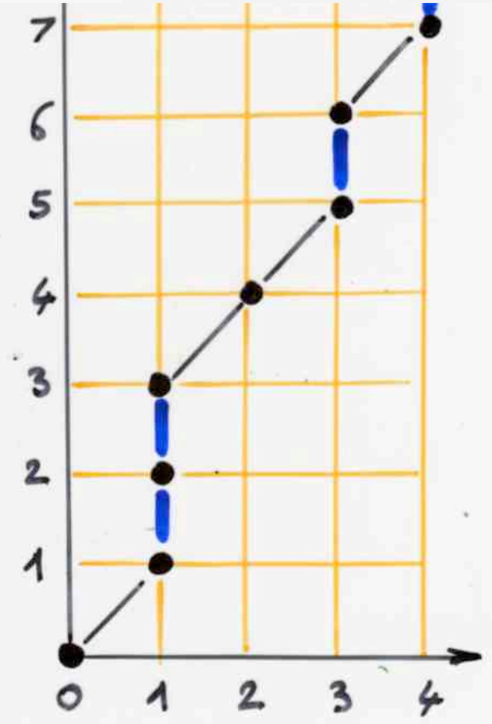
Tchebychev I and II

$$\begin{cases} b_k = 1 , & k \geq 0 \\ \lambda_k = 0 , & k \geq 1 \end{cases}$$



$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k$$

$$b_n = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} a_k$$



Riordan "Combinatorial identities"
(1968) Ch 2 Inverse relations I

$$T_n(x) = \cos(n\theta) \qquad x = \cos\theta$$

$$b_n(x) = 2\,T_n(x/2)$$

$$a_n = \sum_{k=0}^{m} \binom{n}{k} b_{n-2k} \qquad m = \left[\frac{n}{2}\right]$$

$$b_n = \sum_{k=0}^{m} (-1)^k \frac{n}{n-k} \binom{n-k}{k} a_{n-2k}$$

Riordan "Combinatorial identities" (1968) Ch 2 Inverse relations I

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad x = \cos\theta$$

$$b_n(x) = U_n(x/2)$$

$$a_n = \sum_{k=0}^{m} \left[\binom{n}{k} - \binom{n}{k-1} \right] b_{n-2k}$$

$$b_n = \sum_{k=0}^{m} (-1)^k \binom{n-k}{k} a_{n-2k}$$

Riordan "Combinatorial identities"
(1968) Ch 2 Inverse relations I

apply the inversion theorem

$$V = Q = P^{-1}$$

$\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 1}$ $b_k, \lambda_k \in \mathbb{K}$

$$P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i} x^i$$

$$V = (\mu_{n,i})_{n,i\geq 0}$$

$$\mu_{n,i} = \sum_{\omega} v(\omega)$$

ω "Motzkin" path

$|\omega| = n$, $0 \rightsquigarrow i$

$b_k = 0, \ k \geq 0$ $\lambda_k = 1, \ k \geq 1$

$S_n(x) = U_n(x/2)$ matching polynomial of the segment graph

$[0, n-1]$

$$S_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k}$$

$$Q_n(x) = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1} \right] x^{n-2k}$$

inverse polynomial

$\mu_{n,\, n-2k}$

ballot numbers

$\mu_{2n} = C_n$

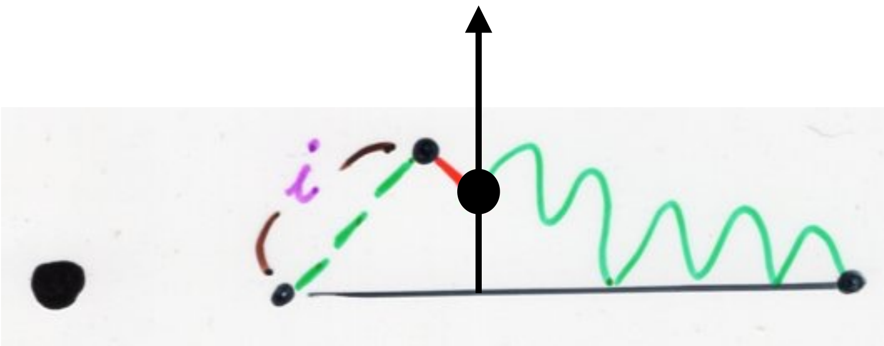
$\mu_{2n+1} = 0$

Catalan numbers

See ABjC, Part I, Ch2c

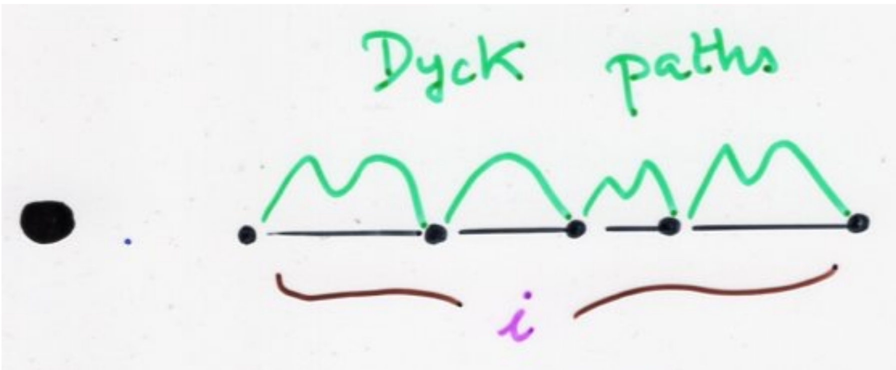
(α)- distribution

$$\frac{i}{2n-i}\binom{2n-i}{n}$$



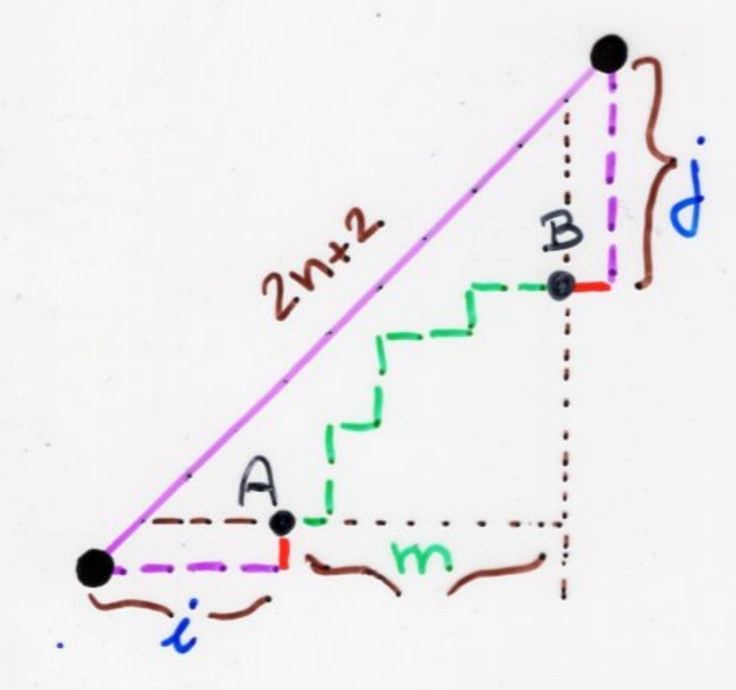
ballot
numbers

$\mu_{n,\,n-2k}$



For $j = 1$, we get the (α)-distribution of Catalan numbers :

$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$



$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

$b_k = 0, \quad k \geq 0$

$$\begin{cases} \lambda_k = 1, & k \geq 2 \\ \lambda_1 = 2 \end{cases}$$

$C_n(x)$

matching polynomial of Γ_n, the "cycle graph" $|\gamma| = n$

$$C_n(x) = 2\, T_n(x/2)$$

$$C_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}$$

$$Q_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{k} x^{n-2k}$$

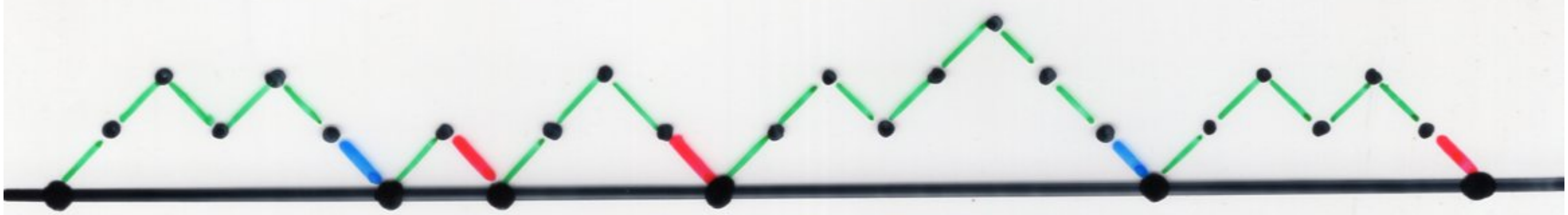
$\mu_{n,\, n-2k}$

inverse polynomial

$\mu_{2n} = \binom{2n}{n}, \quad \mu_{2n+1} = 0$

$b_k = 0 , \quad k \geq 0$

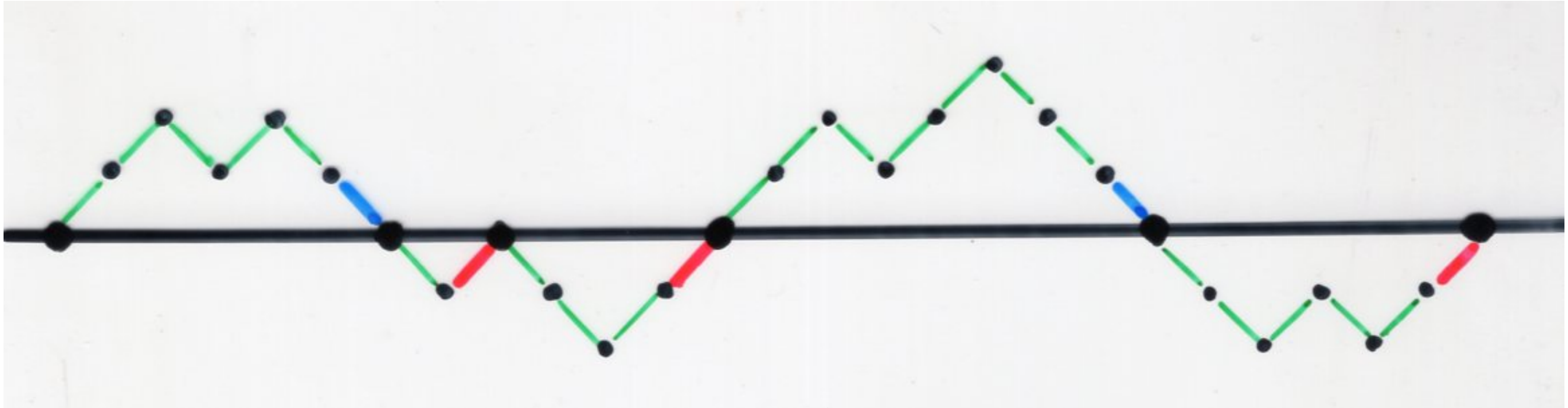
$$\begin{cases} \lambda_k = 1, & k \geq 2 \\ \lambda_1 = 2 \end{cases}$$



$\mu_{2n} = \binom{2n}{n} , \quad \mu_{2n+1} = 0$

$b_k = 0, \quad k \geq 0$

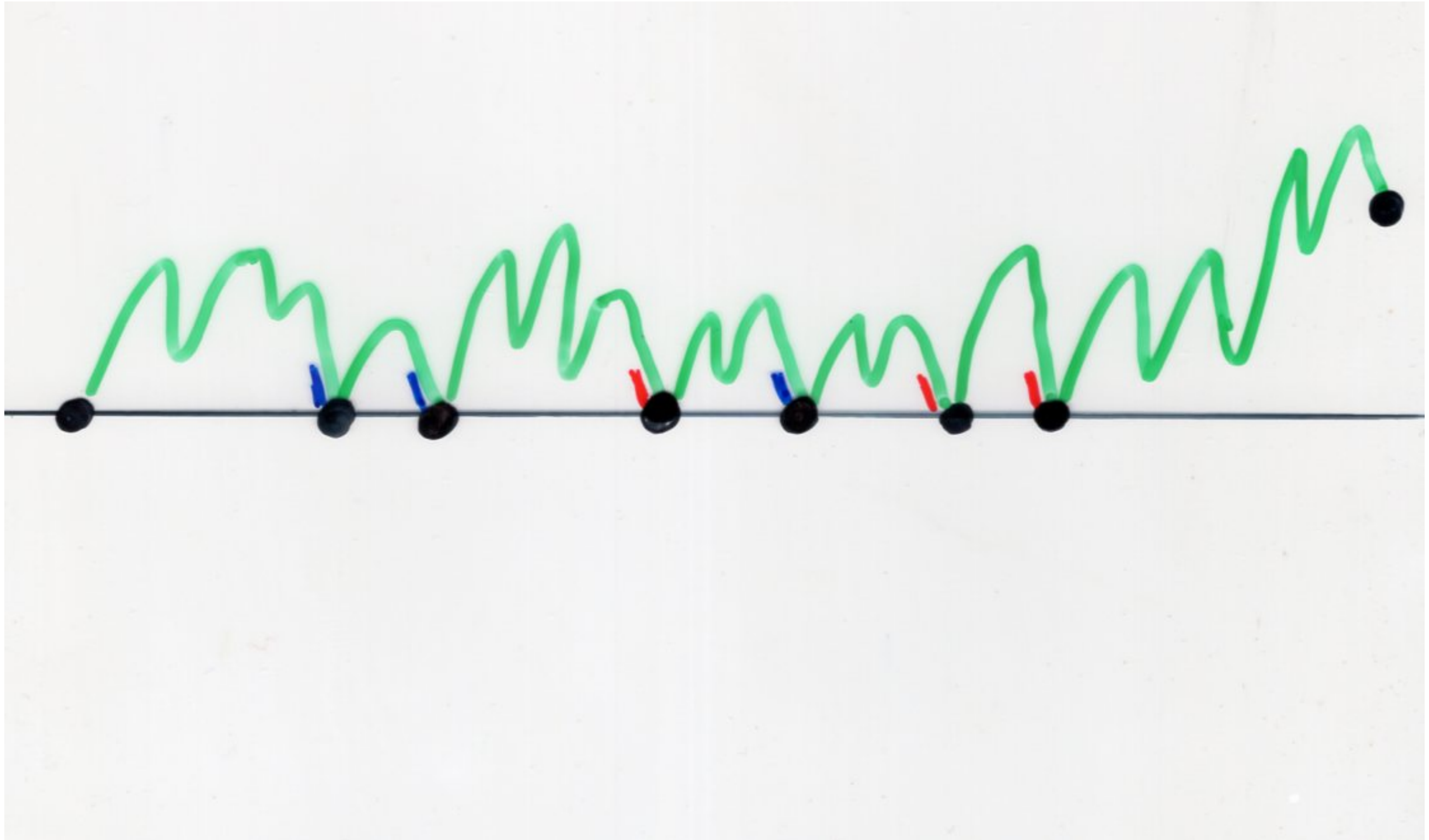
$$\begin{cases} \lambda_k = 1, & k \geq 2 \\ \lambda_1 = 2 \end{cases}$$



$\mu_{2n} = \binom{2n}{n}, \quad \mu_{2n+1} = 0$

$b_k = 0, \quad k \geq 0$

$$\begin{cases} \lambda_k = 1, & k \geq 2 \\ \lambda_1 = 2 \end{cases}$$

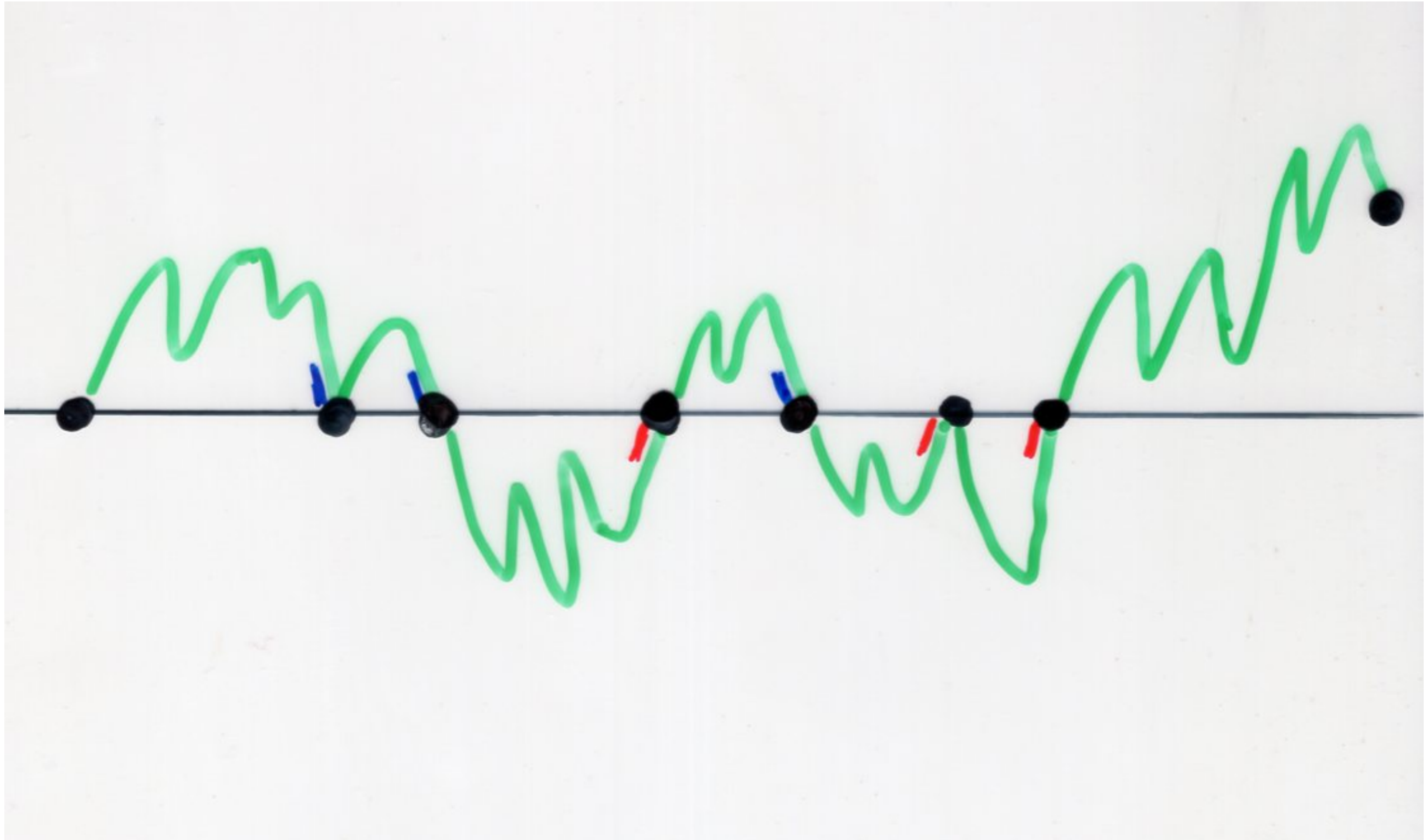


inverse polynomial

$$Q_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{k} x^{n-2k}$$

$b_k = 0, \quad k \geq 0$

$$\begin{cases} \lambda_k = 1, & k \geq 2 \\ \lambda_1 = 2 \end{cases}$$



inverse polynomial

$$Q_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{k} x^{n-2k}$$

Inverse relations: examples

Stirling numbers I and II

$\lambda_k = 0$ $b_k = k$ $k \geq 0$

$\mu_{n,i} = s_{n,i}$ Stirling numbers 1st kind

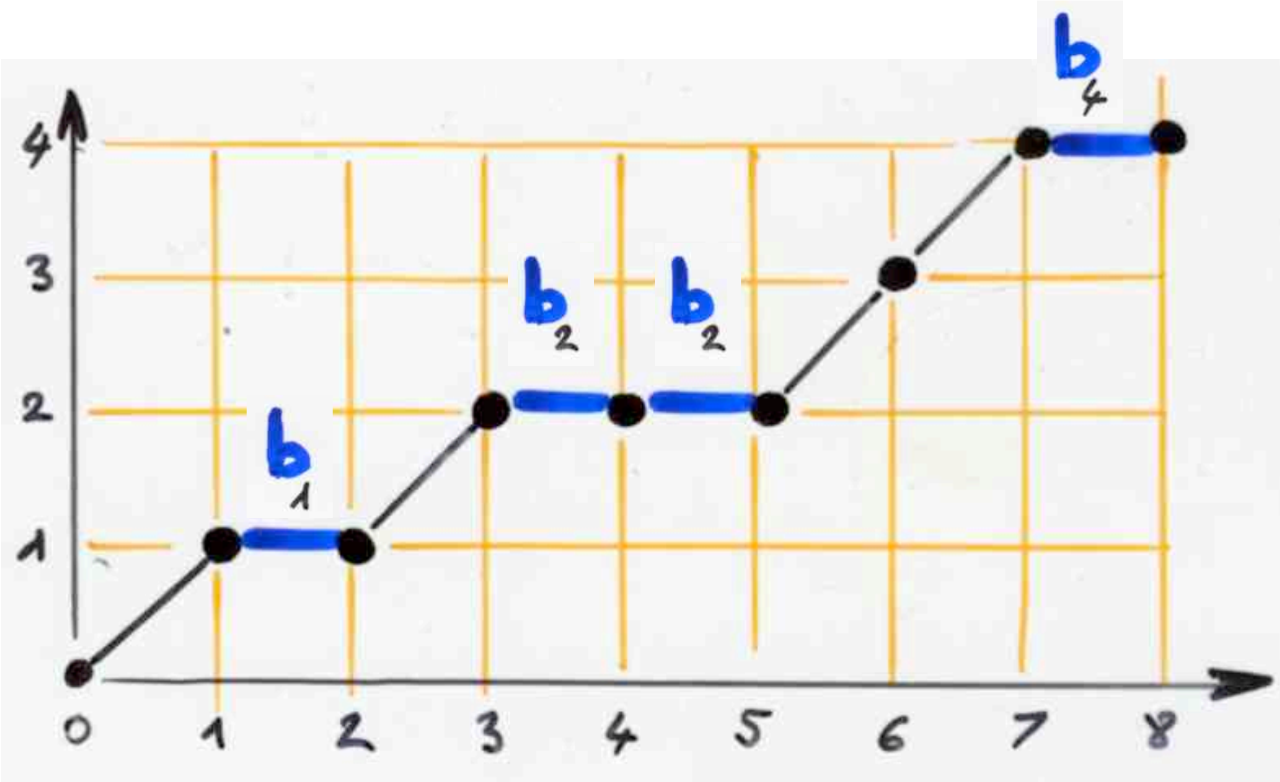
$=$ number of (set) partitions of $\{1, \ldots, n\}$ into i blocks

$\rho_{n,i} = (-1)^i S_{n,i}$ Stirling numbers 2nd kind

$=$ number of permutations of $\{1, \ldots, n\}$ having i cycles

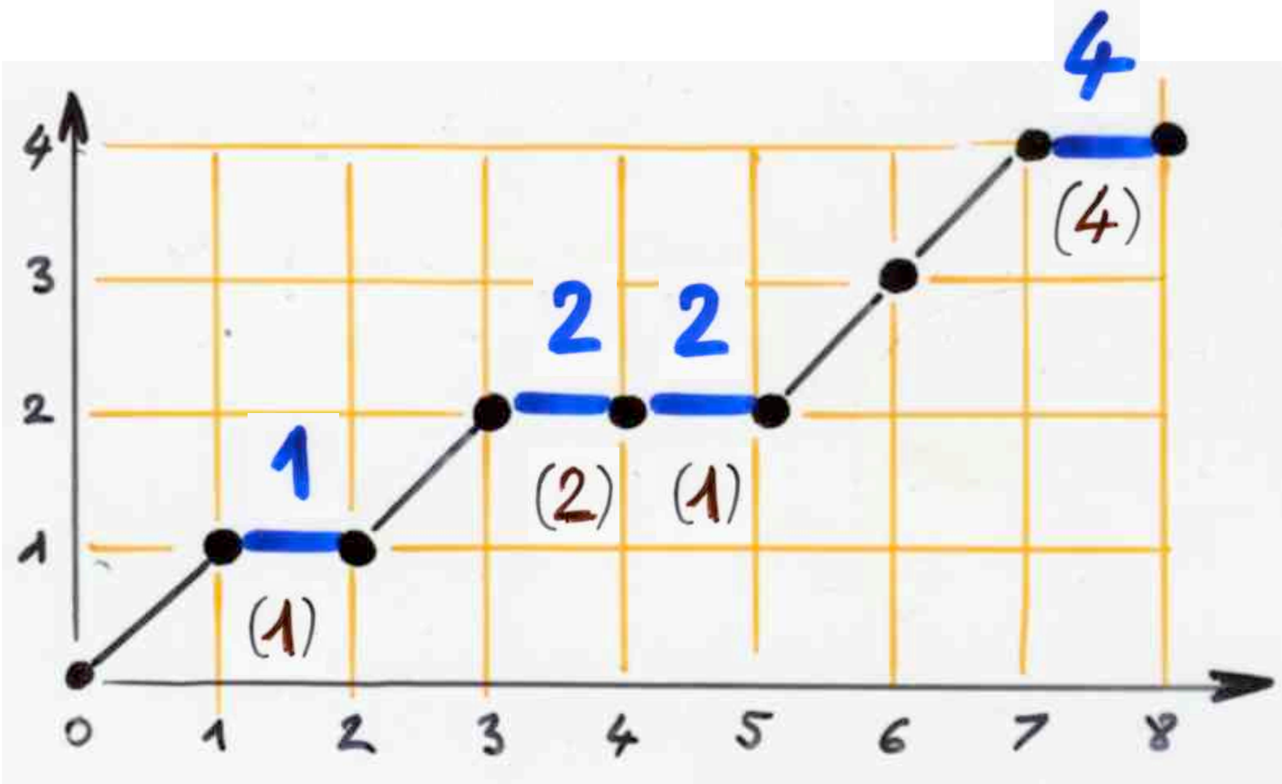
$\mu_{n,i} = s_{n,i}$

(idea of)
history



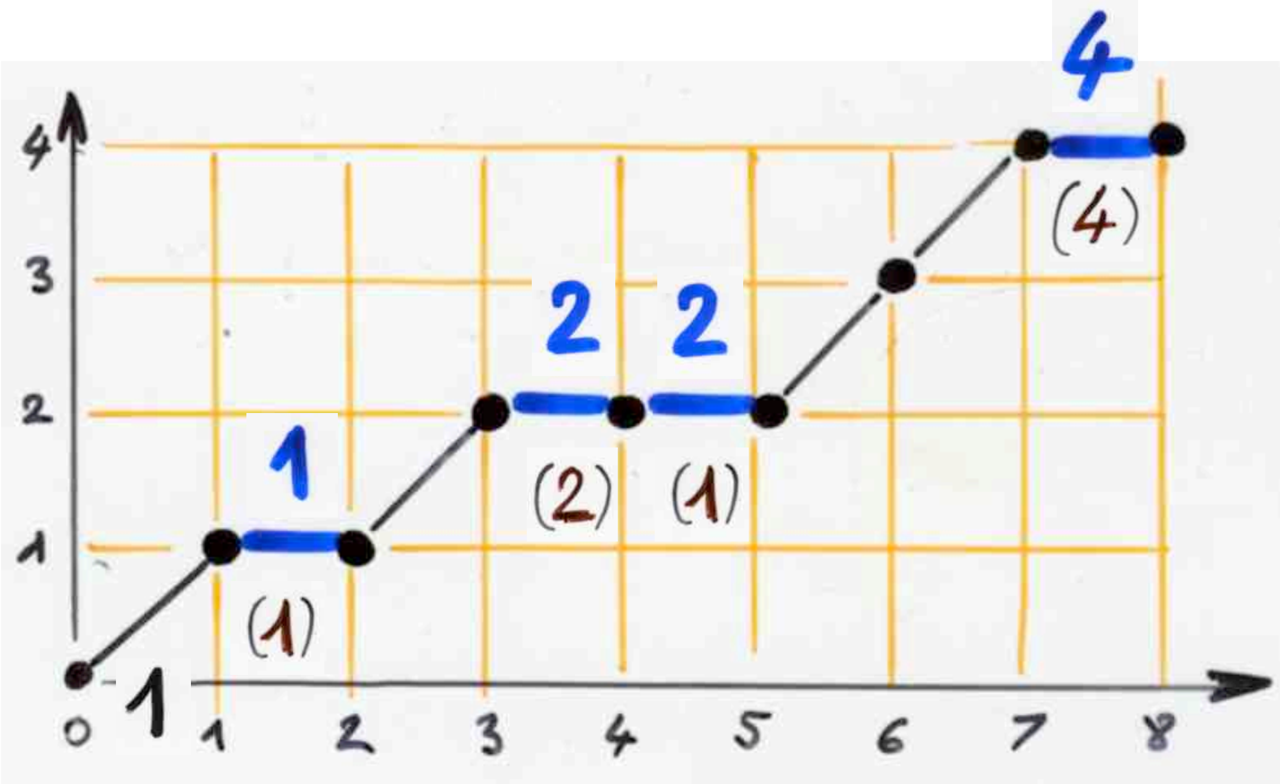
$$\mu_{n,i} = s_{n,i}$$

(idea of)
history



$\mu_{n,i} = s_{n,i}$

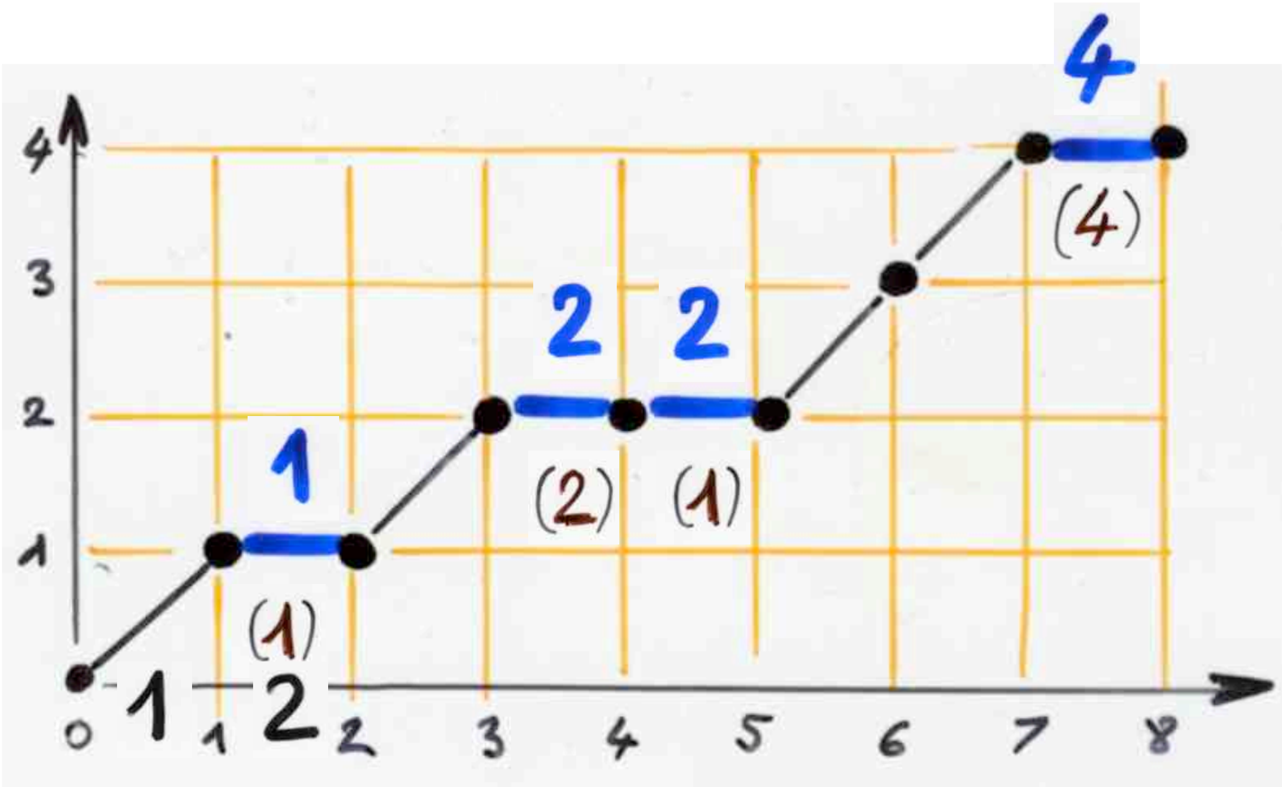
(idea of)
history



[1

$$\mu_{n,i} = s_{n,i}$$

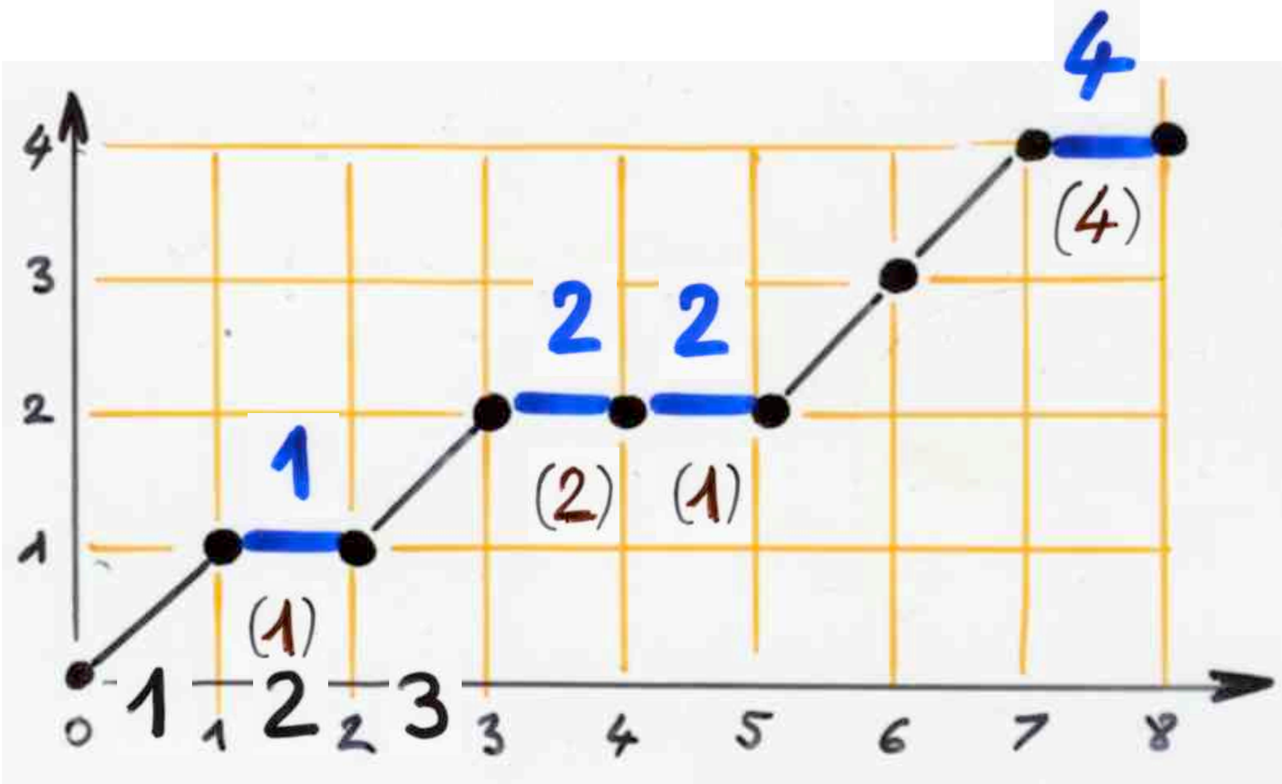
(idea of)
history



[1, 2

$\mu_{n,i} = s_{n,i}$

(idea of)
history

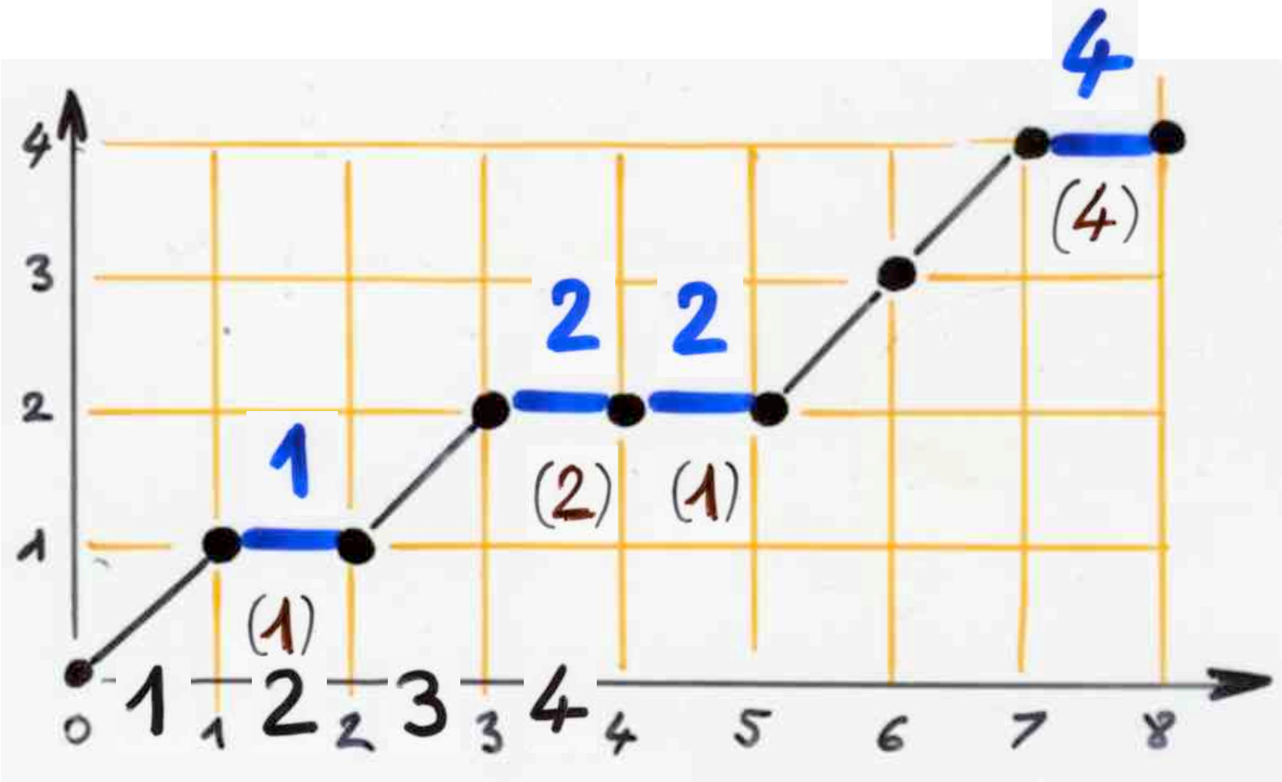


[1 , 2

[3

$\mu_{n,i} = s_{n,i}$

(idea of) history

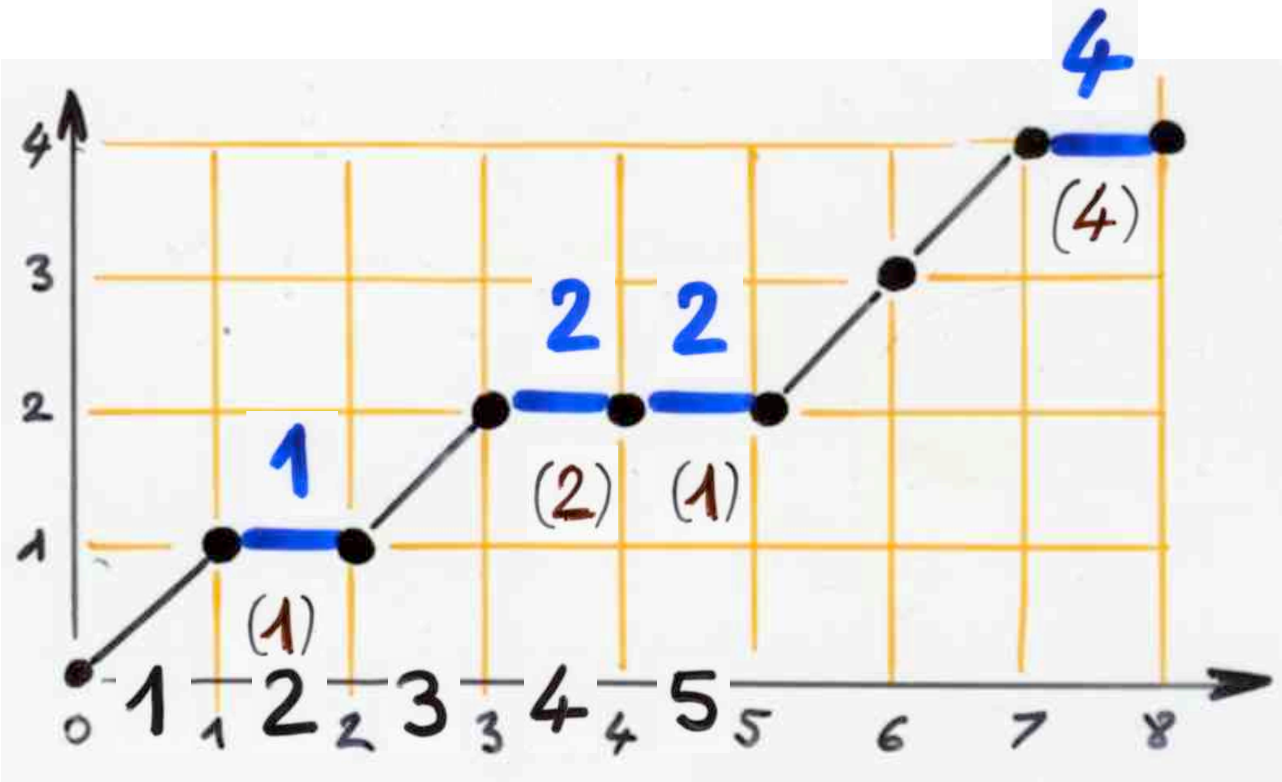


[1 , 2

[3 , 4

$\mu_{n,i} = s_{n,i}$

(idea of)
history

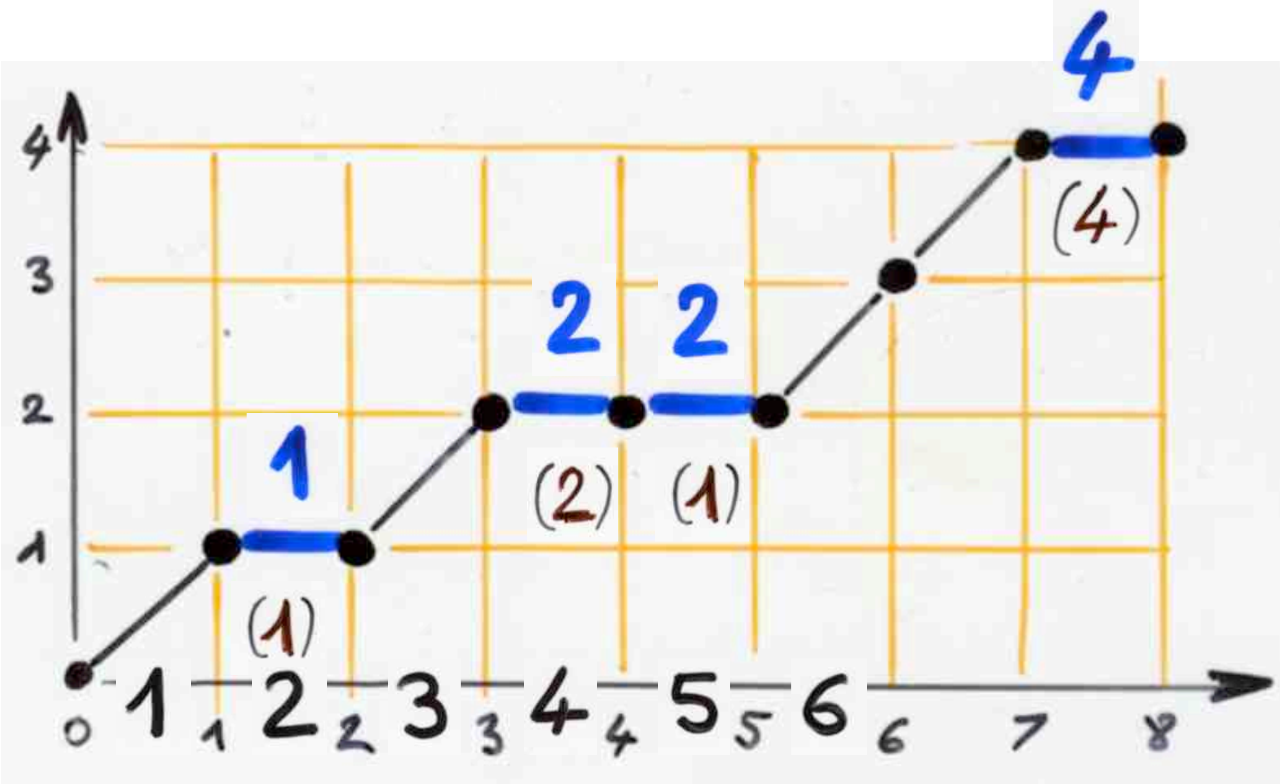


[1 , 2 , 5

[3 , 4

$\mu_{n,i} = s_{n,i}$

(idea of)
history



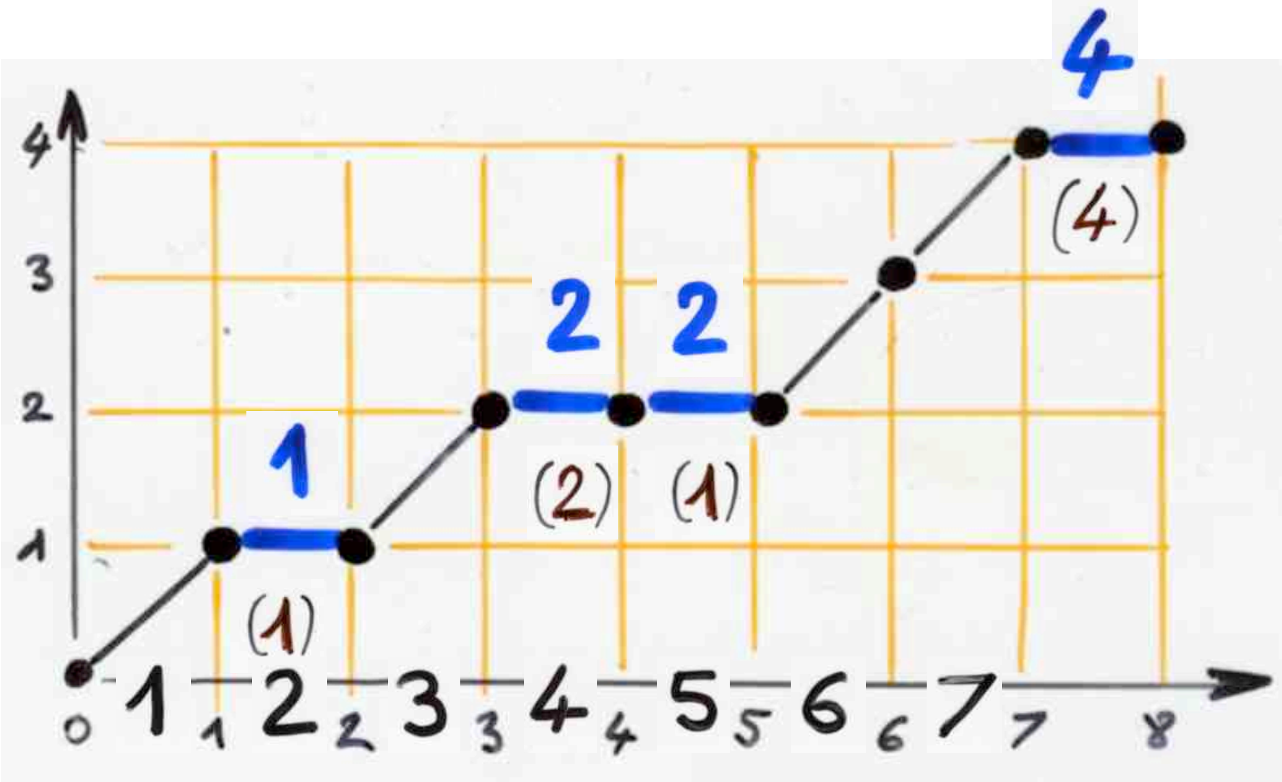
[1 , 2 , 5

[3 , 4

[6

$$\mu_{n,i} = s_{n,i}$$

(idea of)
history



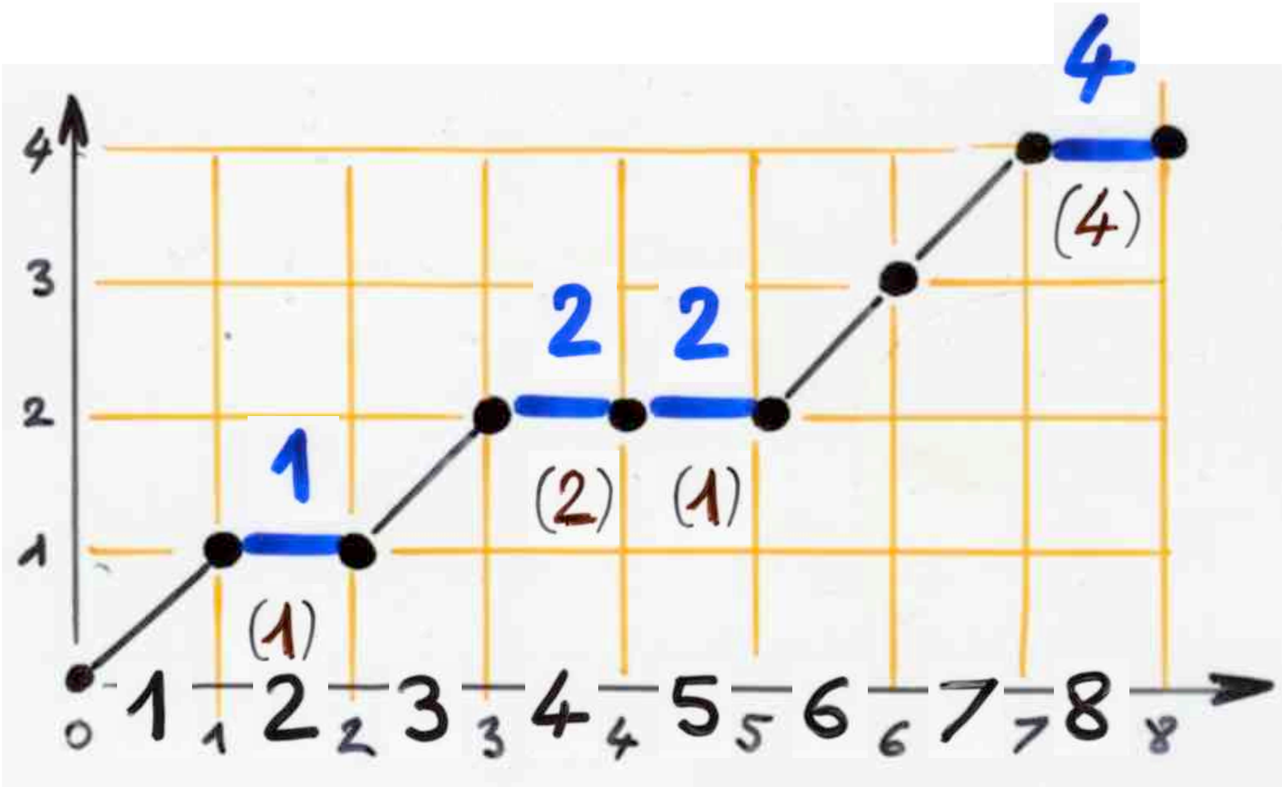
[1 , 2 , 5

[3 , 4

[6

[7

$$\mu_{n,i} = s_{n,i}$$



(idea of)
history

[1 , 2 , 5

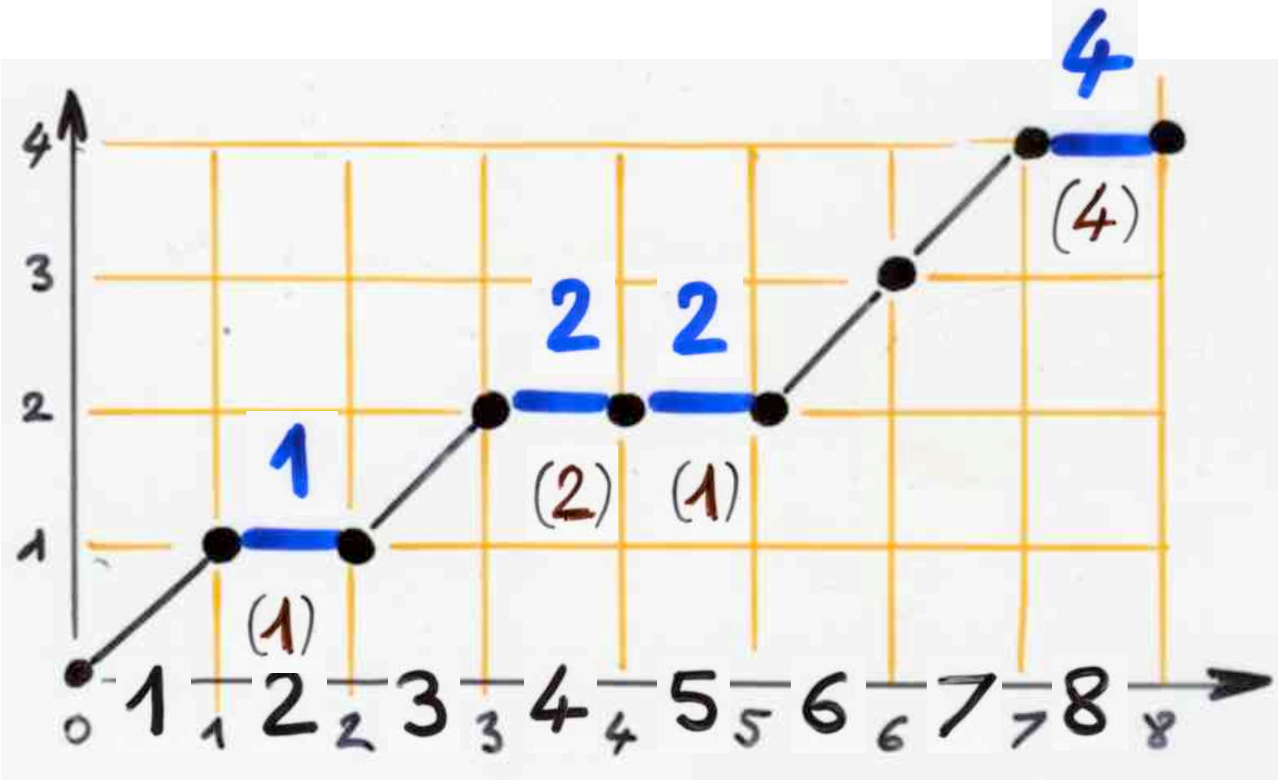
[3 , 4

[6

[7, 8

$\mu_{n,i} = s_{n,i}$

(idea of)
history



[1 , 2 , 5

[3 , 4

[6 ,

[7, 8

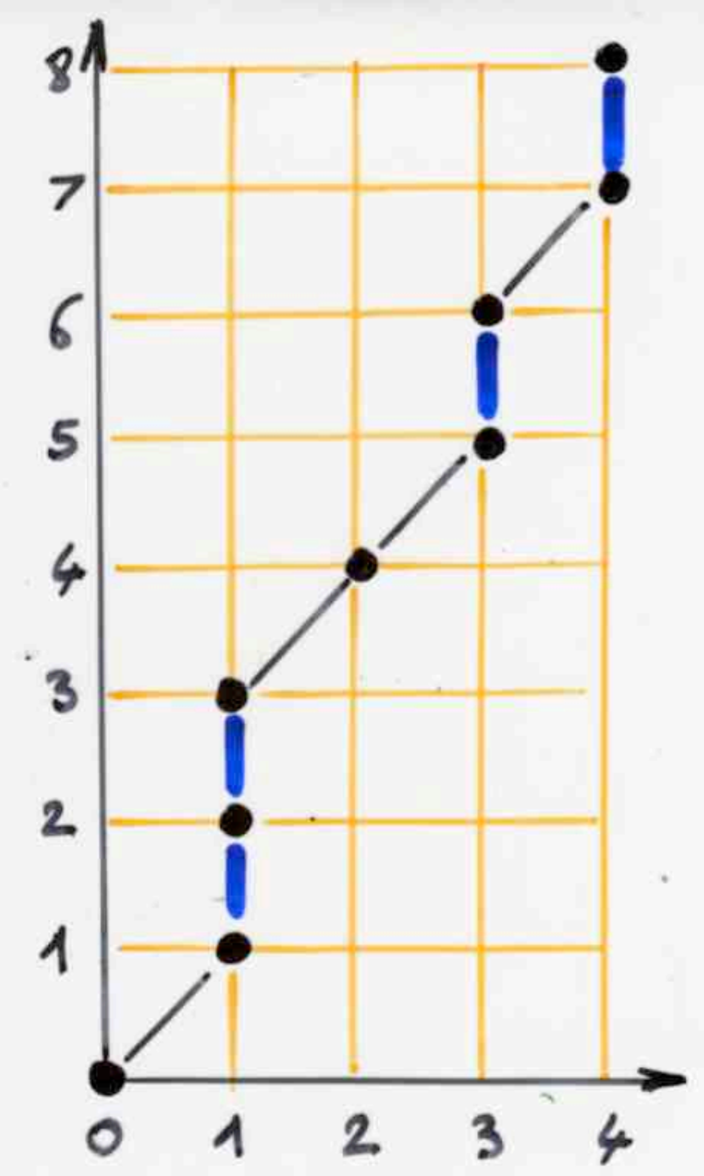
$\lambda_k = 0$ $b_k = k$ $k \geq 0$

$\mu_{n,i} = s_{n,i}$ Stirling numbers 1st kind

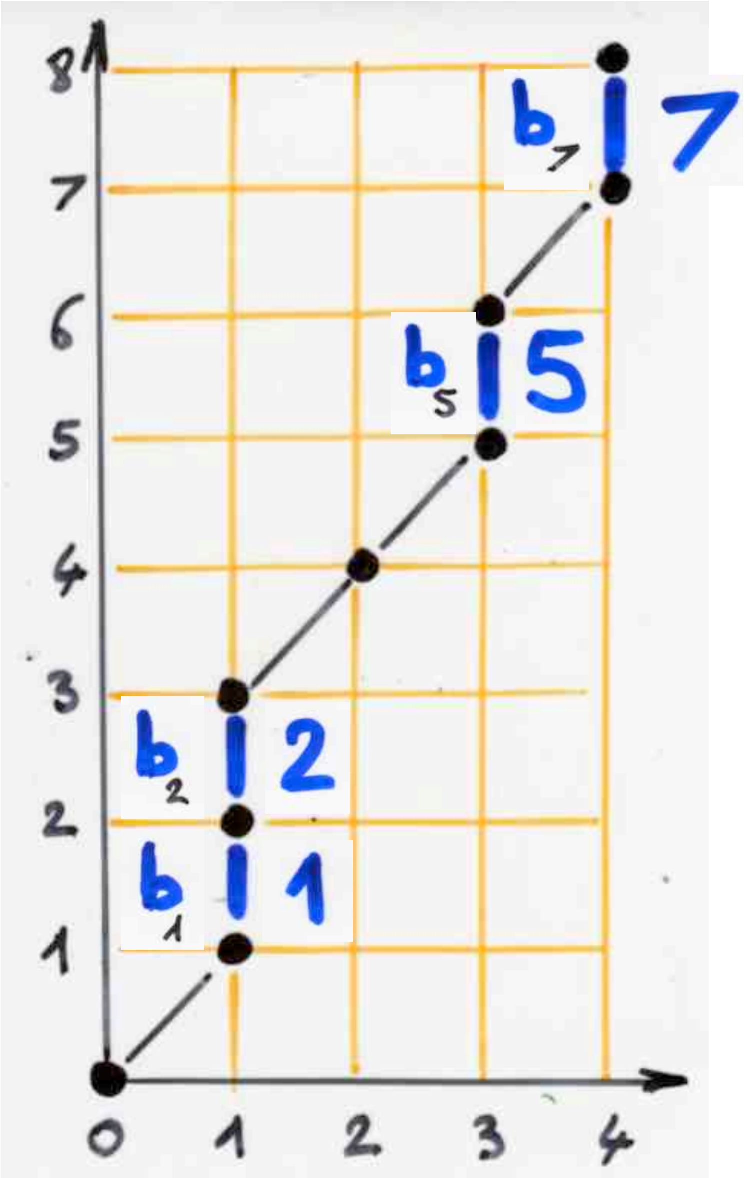
$=$ number of (set) partitions of $\{1, \dots, n\}$ into i blocks

$P_{n,i} = (-1)^i S_{n,i}$ Stirling numbers 2nd kind

$=$ number of permutations of $\{1, \dots, n\}$ having i cycles

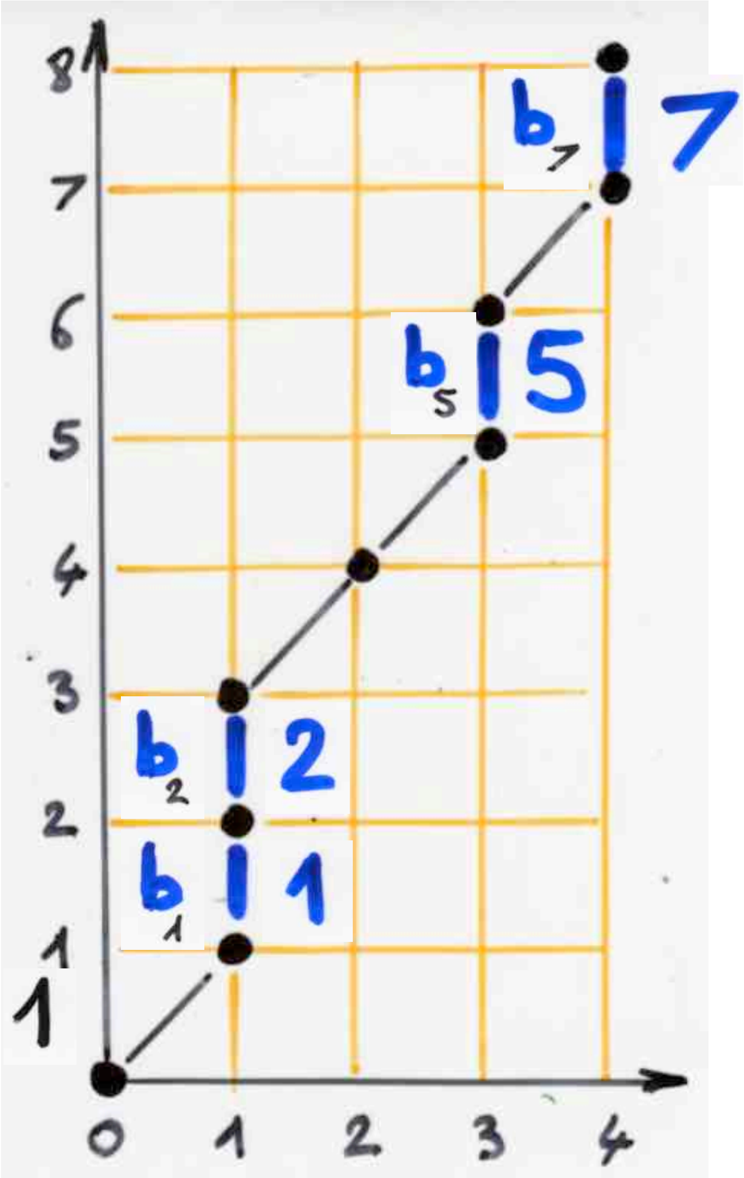


(idea of)
history

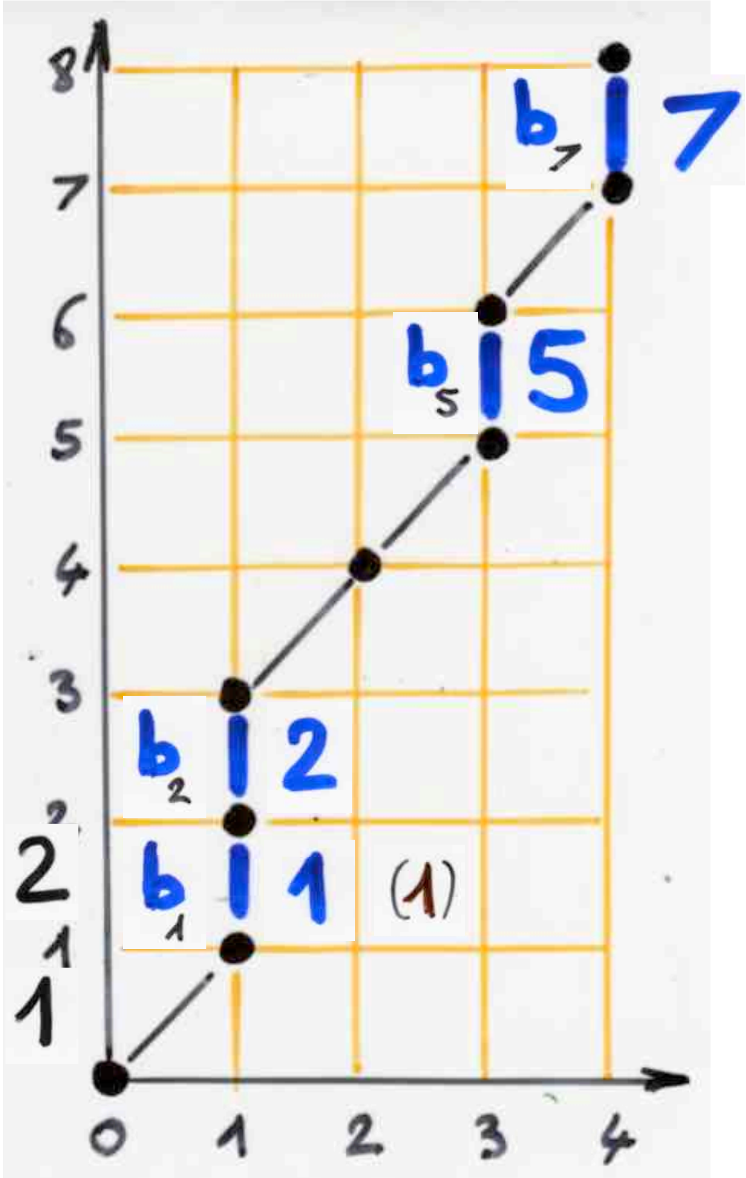


(idea of)
history

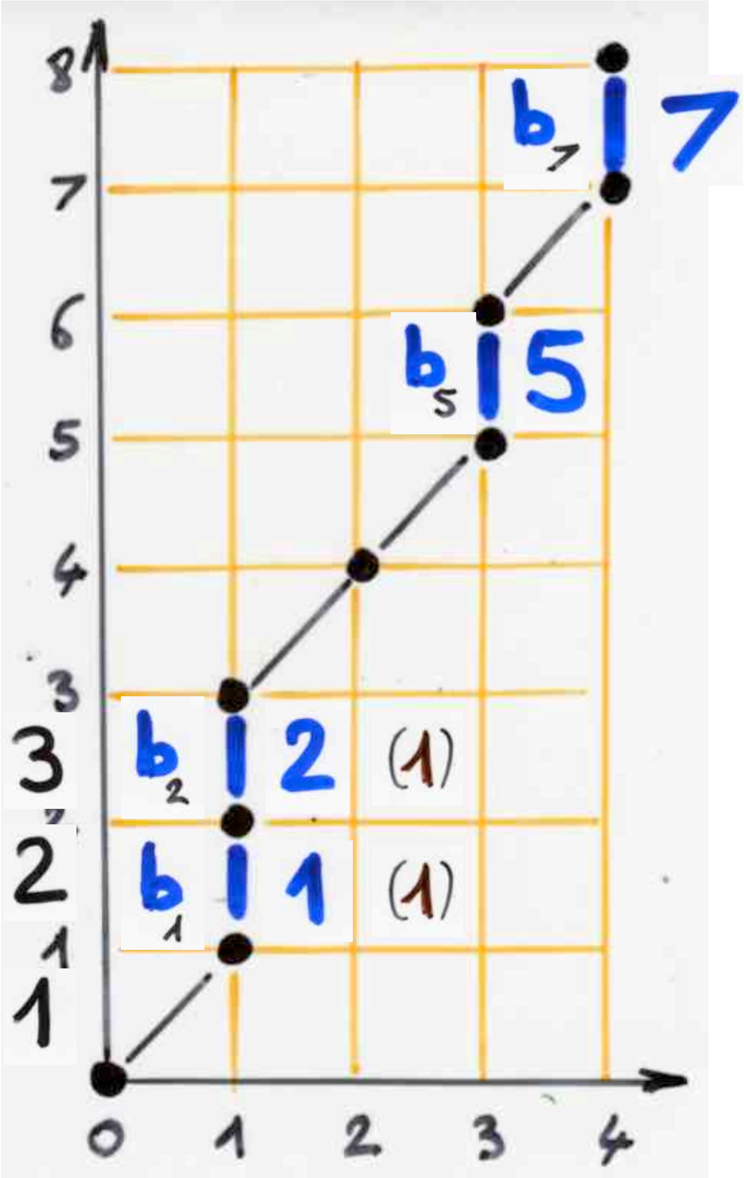
(idea of)
history

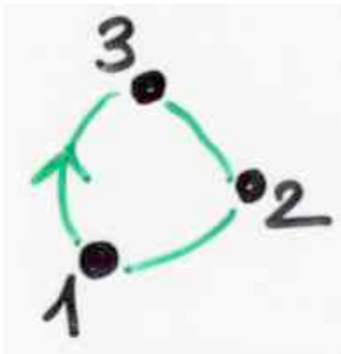


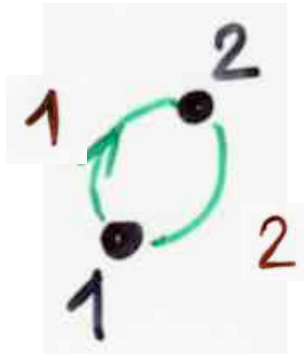
(idea of)
history

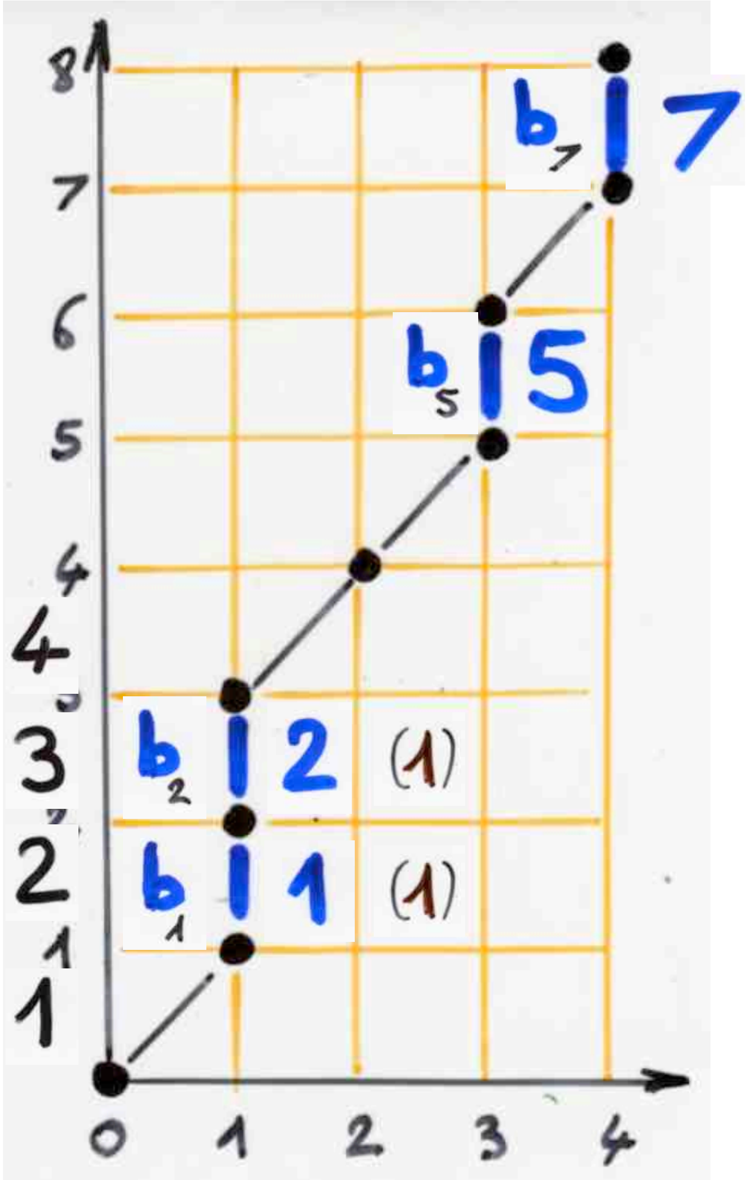


(idea of)
history

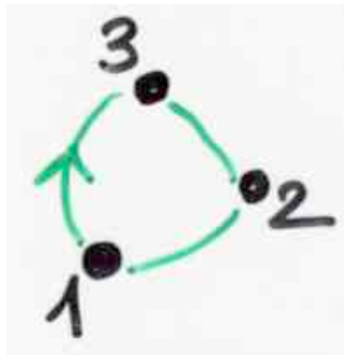




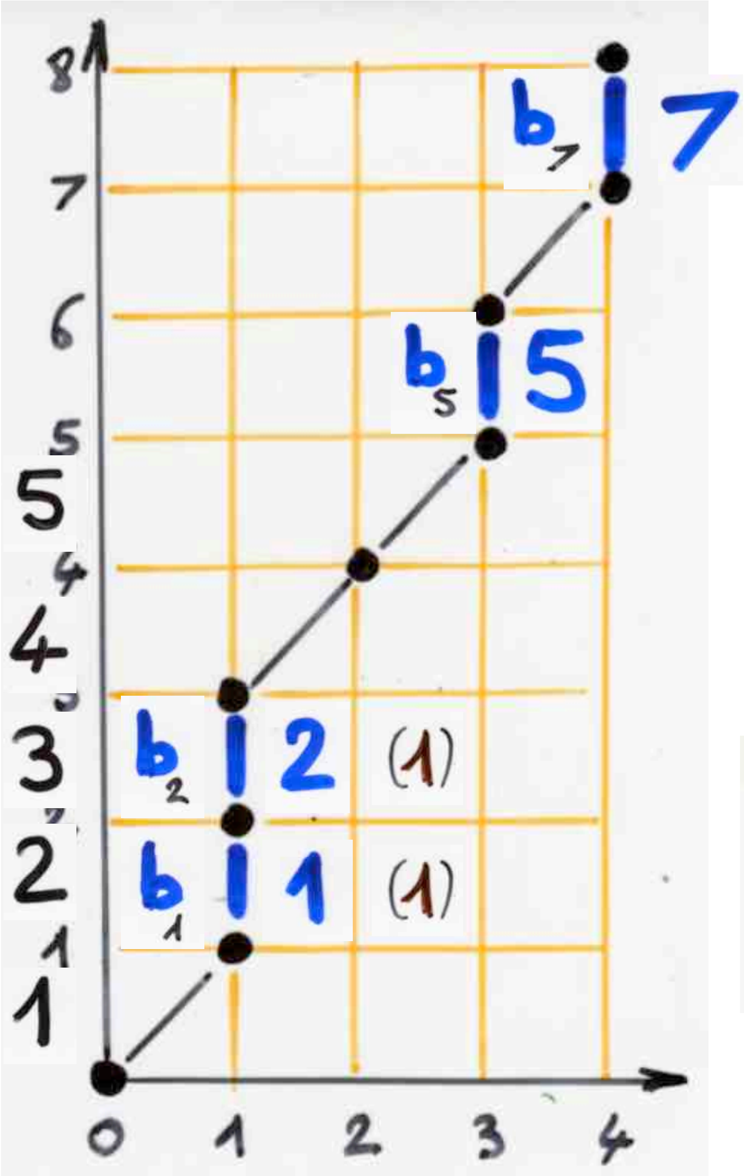


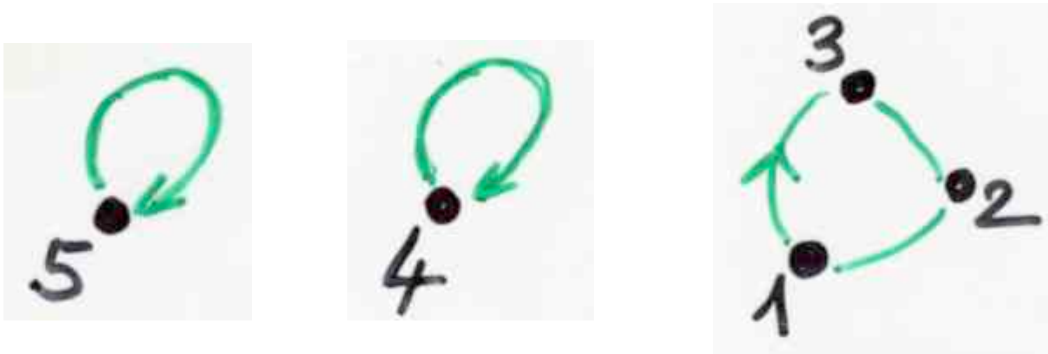


(idea of)
history

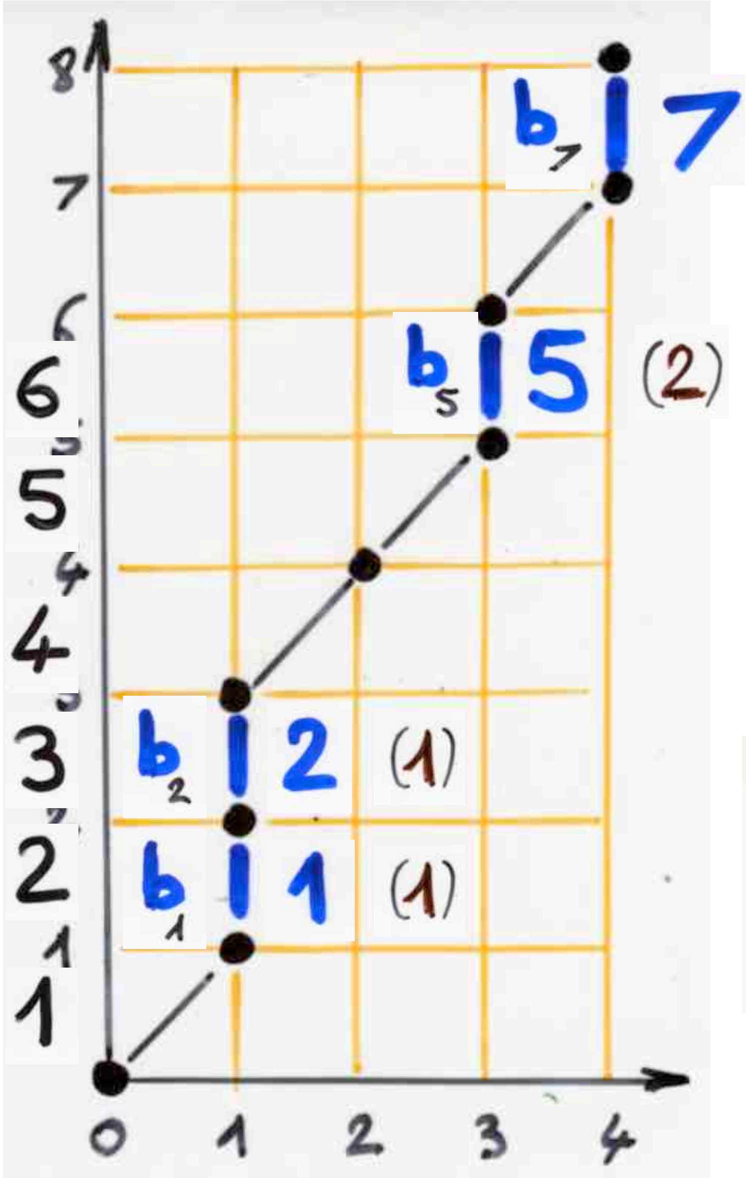


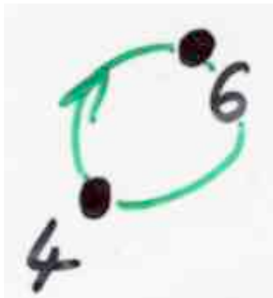
(idea of)
history

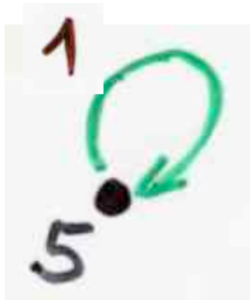


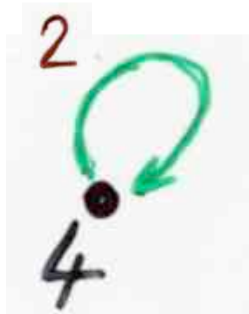


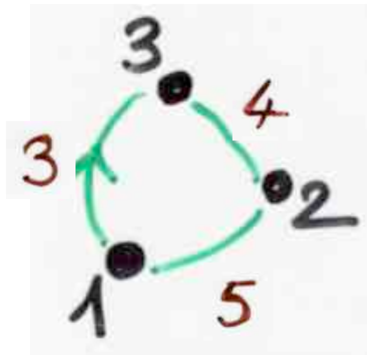
(idea of)
history



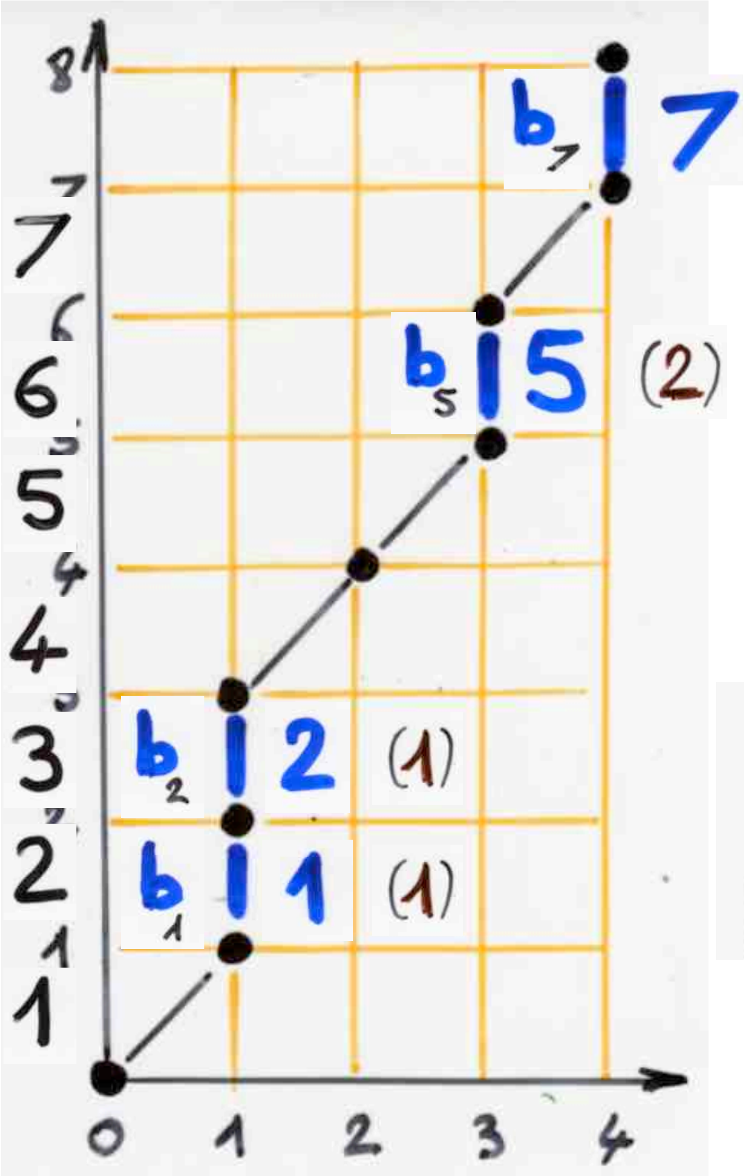


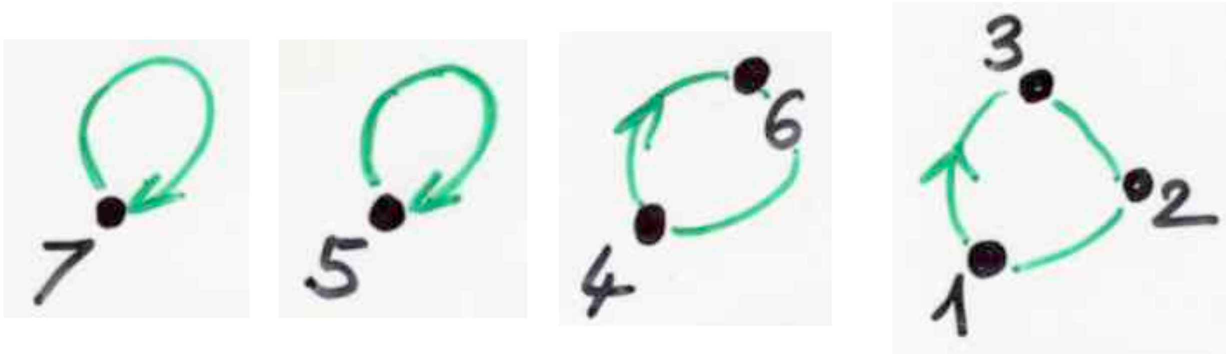




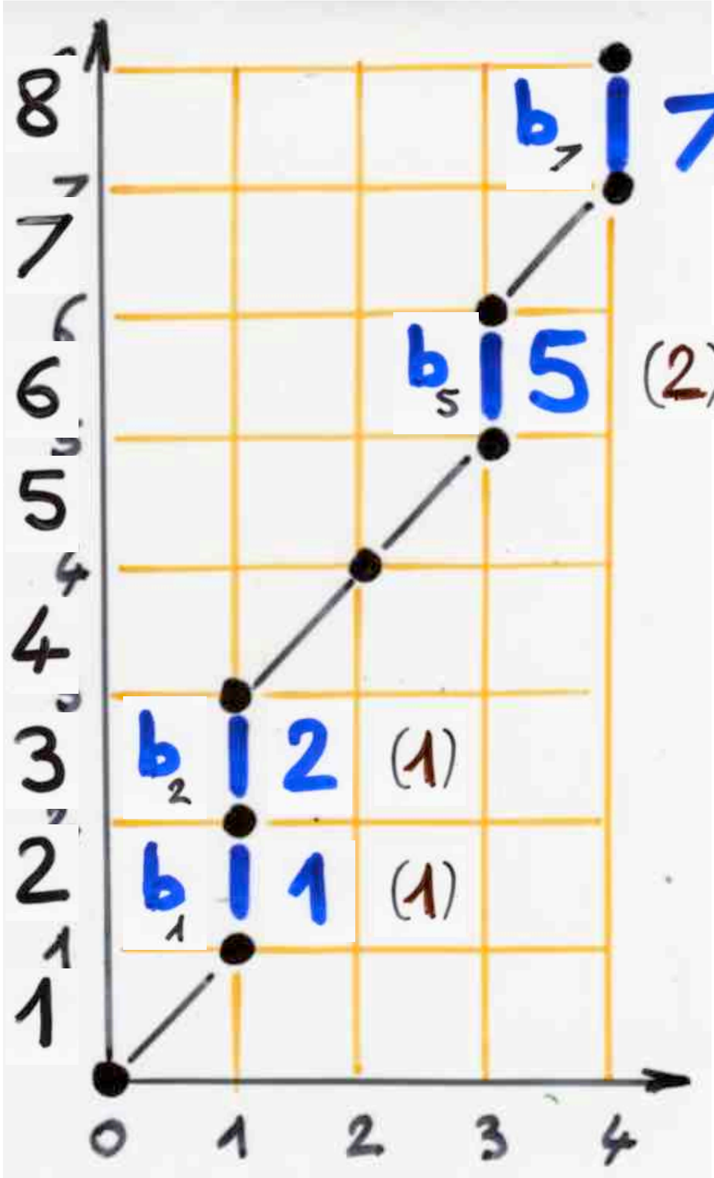


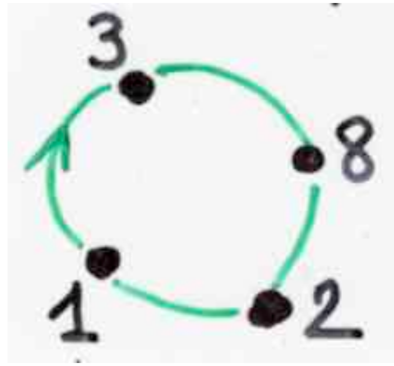
(idea of)
history

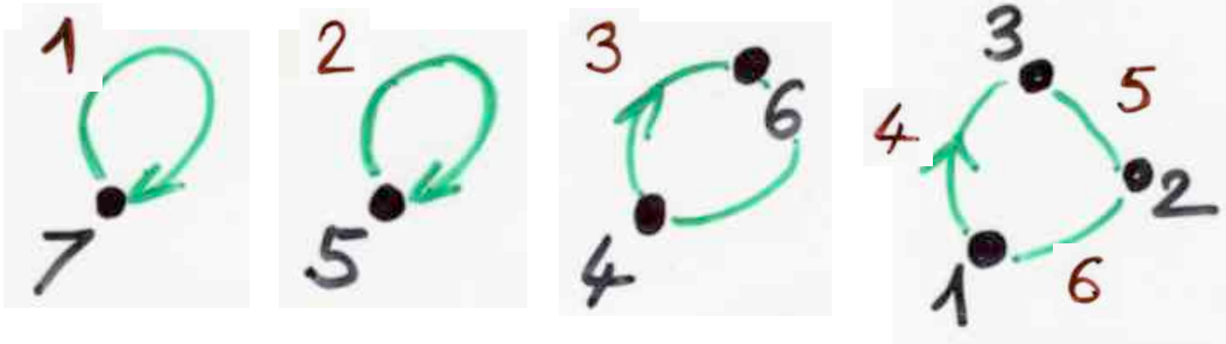




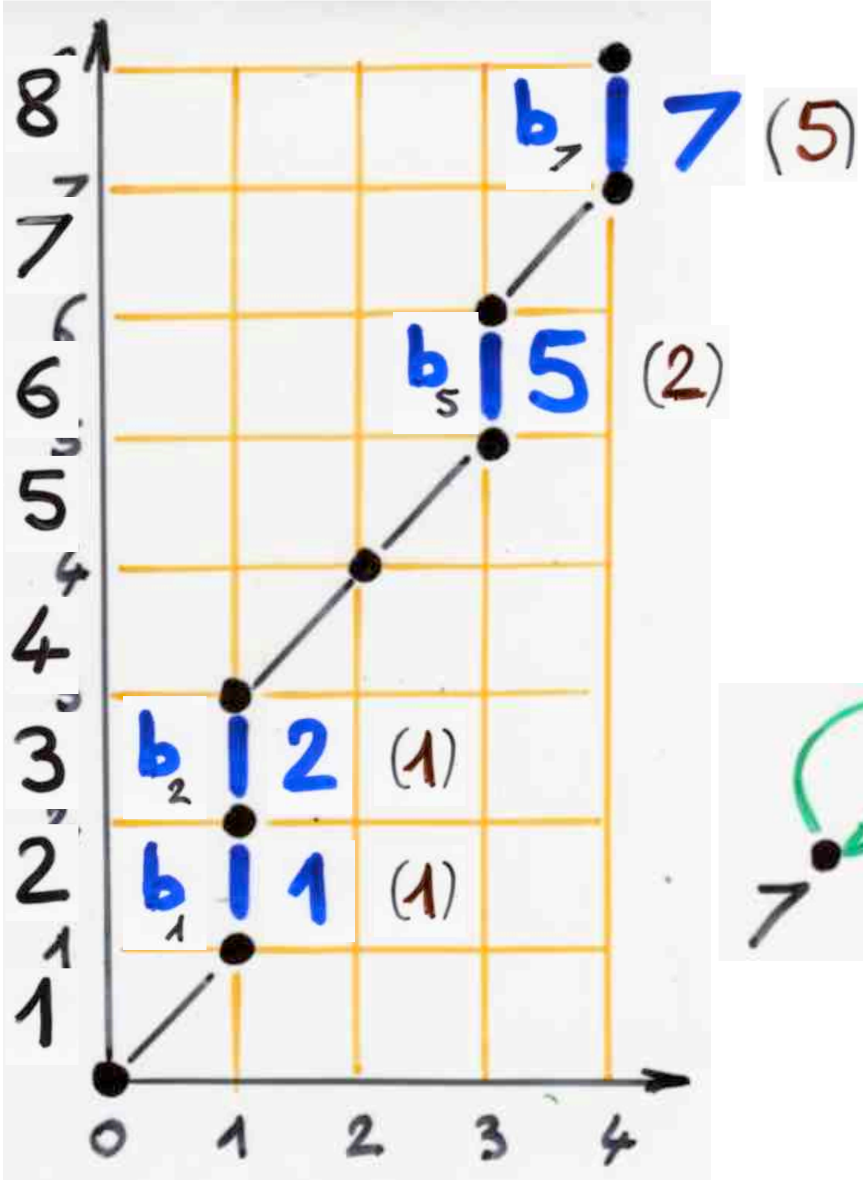
(idea of)
history

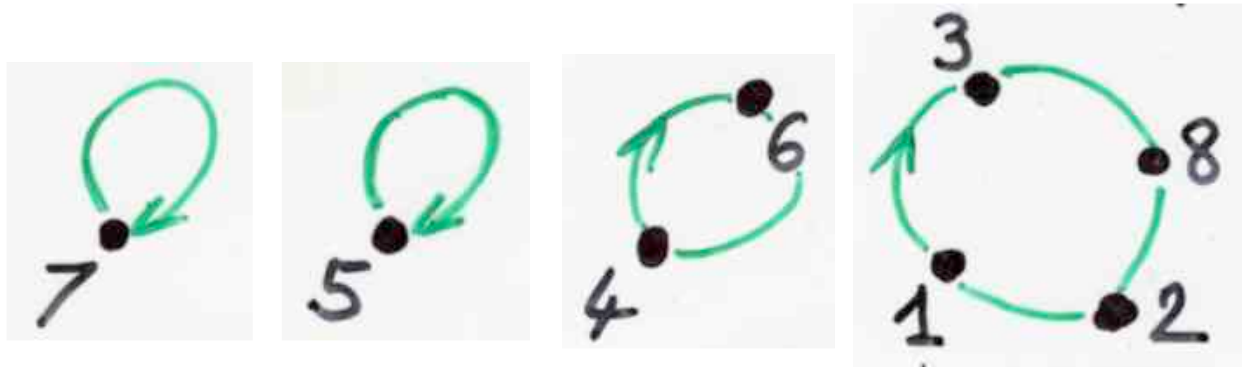






(idea of) history





Inverse relations: examples

Symmetric functions

symmetric polynomials $\mathbb{K}[x_1, .., x_n]$

$$P(x_1, .., x_n)$$

$\sigma \in \mathfrak{S}_n$

$$P(x_{\sigma(1)}, .., x_{\sigma(n)}) = P(x_1, .., x_n)$$

→ complements Ch4c
plactic monoid, product of Schur functions

See ABjC, Part I, Ch4c

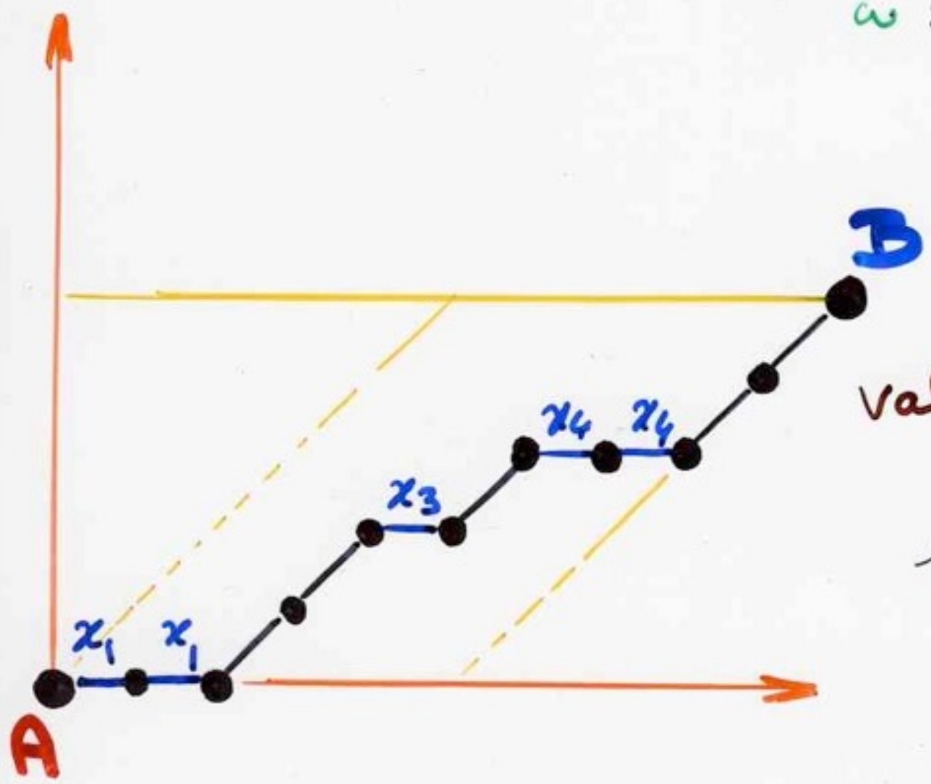
Definition Homogeneous (or complete) symmetric functions

$$h_p(x_1, \dots, x_m) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_m) \\ \text{compositions of } p \\ (\alpha_i \geq 0,\ \alpha_1+\dots+\alpha_m=p)}} x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

Lemma $h_p(x_1, \dots, x_m) = \sum_{\omega} v(\omega)$

ω Motzkin path

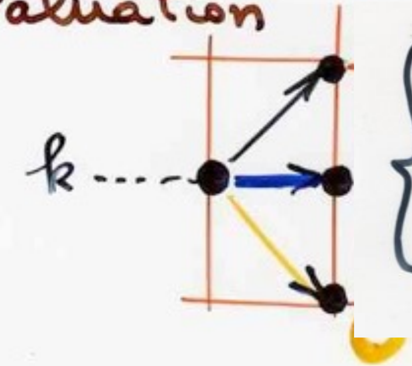
$\omega : A \rightsquigarrow B$



$A = (0,0)$

$B = (p+m-1, m-1)$

valuation

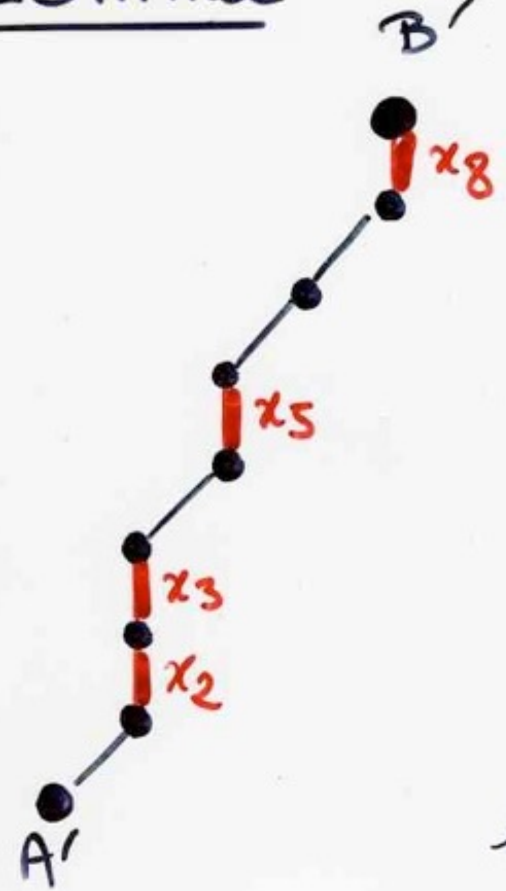


$$\begin{cases} \lambda_k = 0 \\ b_k = x_{k+1} \end{cases}$$

Definition symmetric elementary function

$$e_p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le m} x_{i_1} \cdots x_{i_p}$$

Lemma



$$e_p = \sum_{\omega} v(\omega)$$

"Favard" path

$\omega : A' \rightsquigarrow B'$

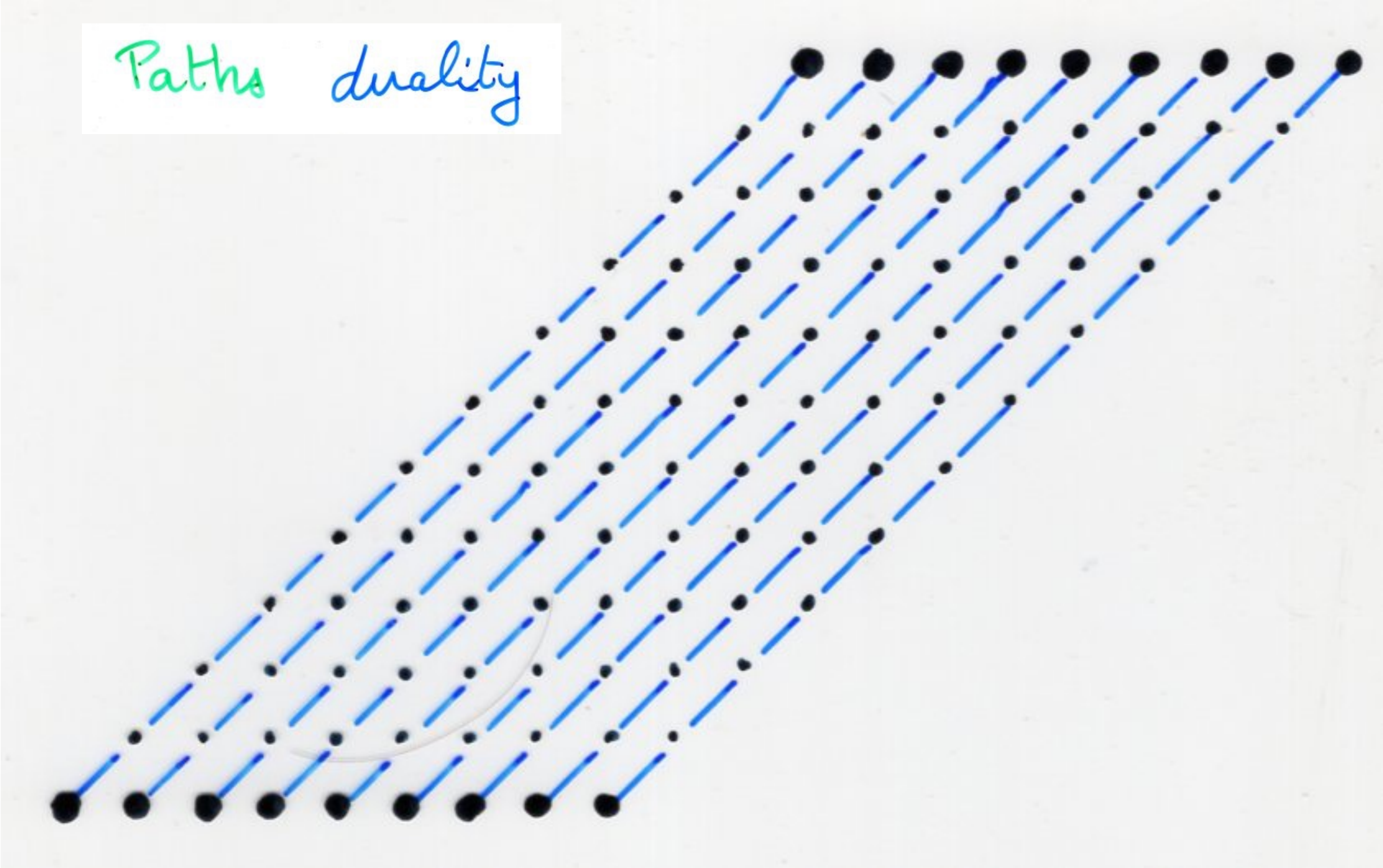
$A' = (0, 0)$

$B' = (m, m-p)$

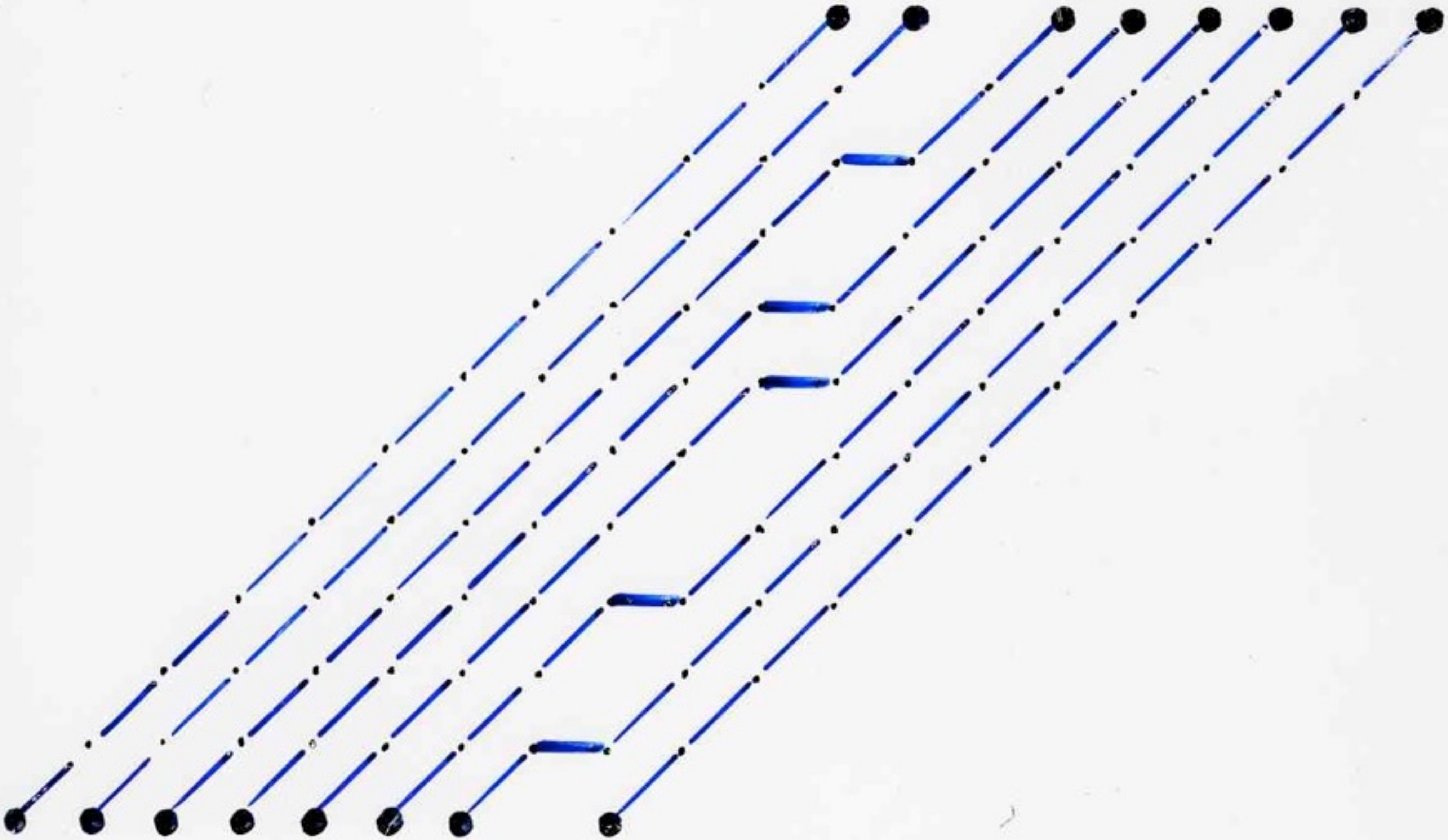
valuation :

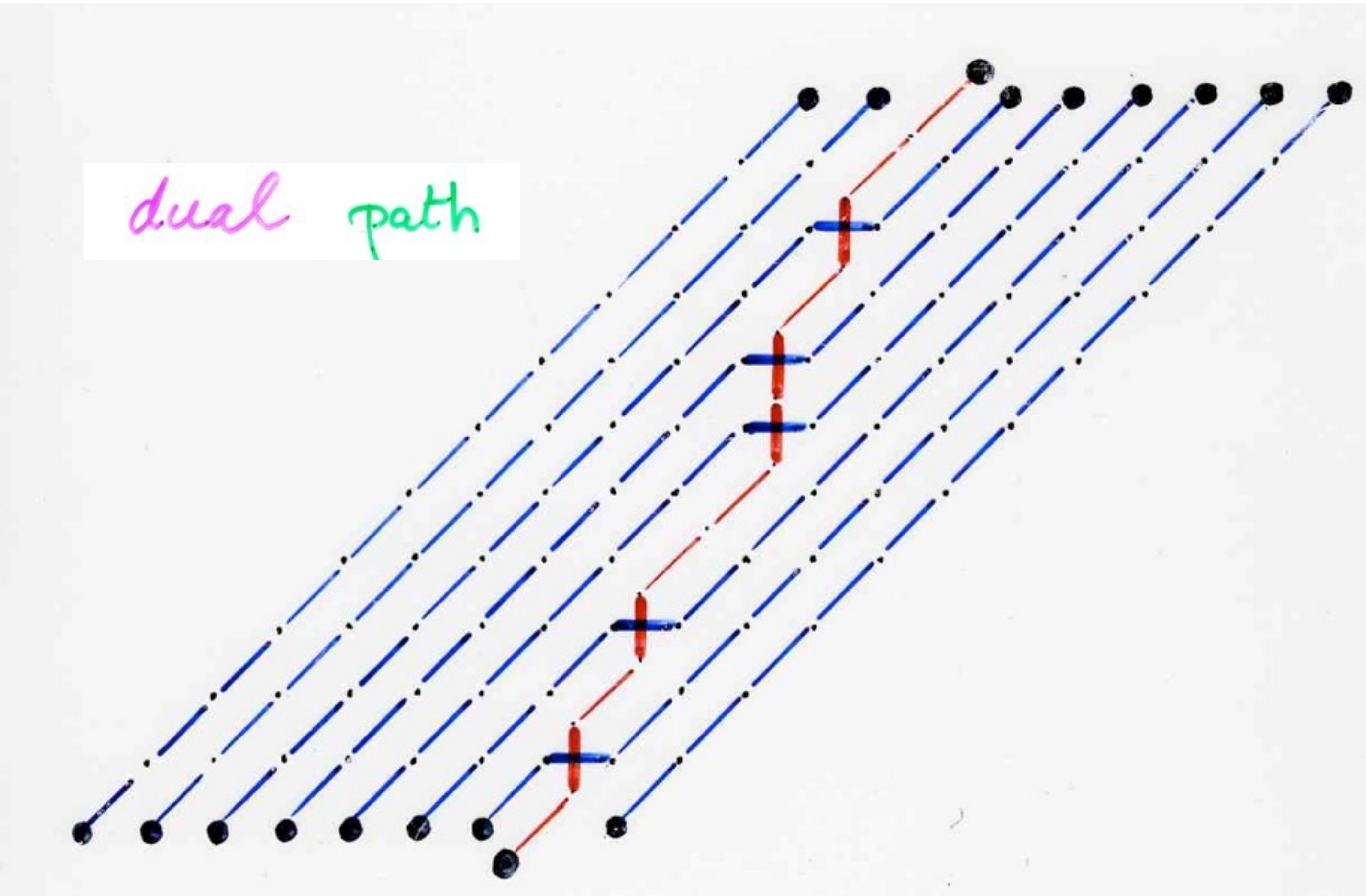
level k ---
$$\begin{cases} \lambda_k = 0 \\ b_k = x_{k+1} \end{cases}$$

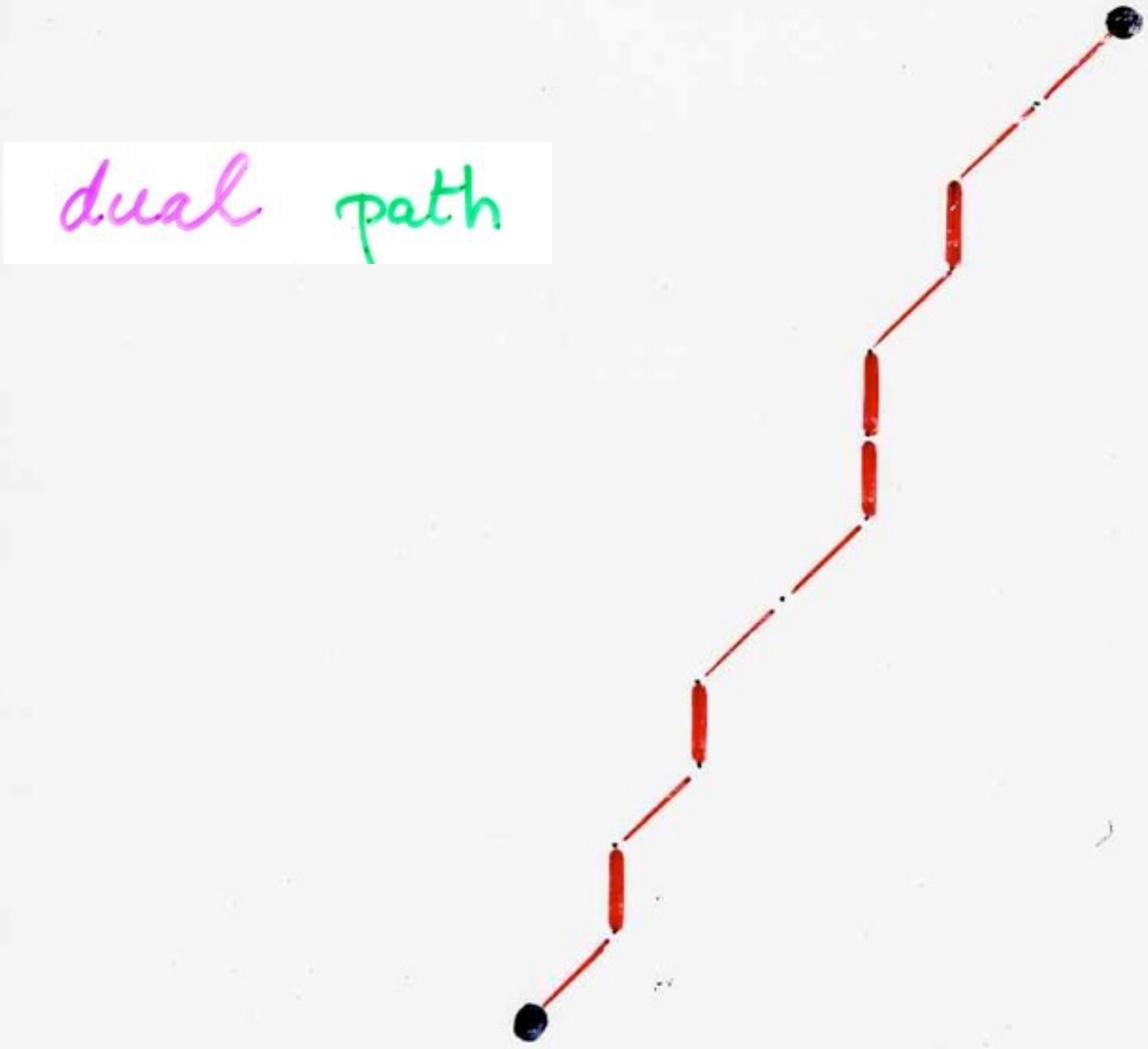
Duality of paths

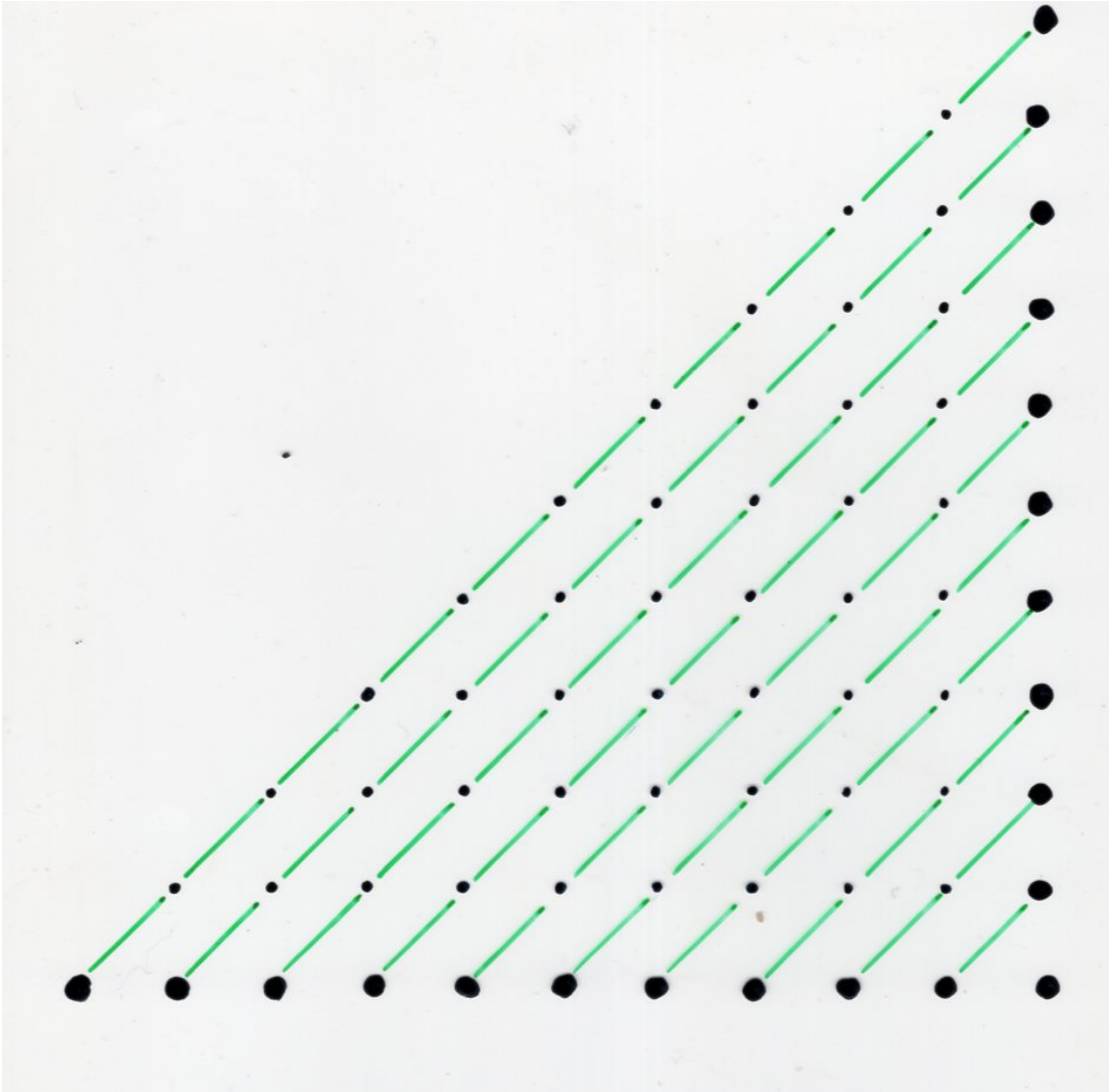


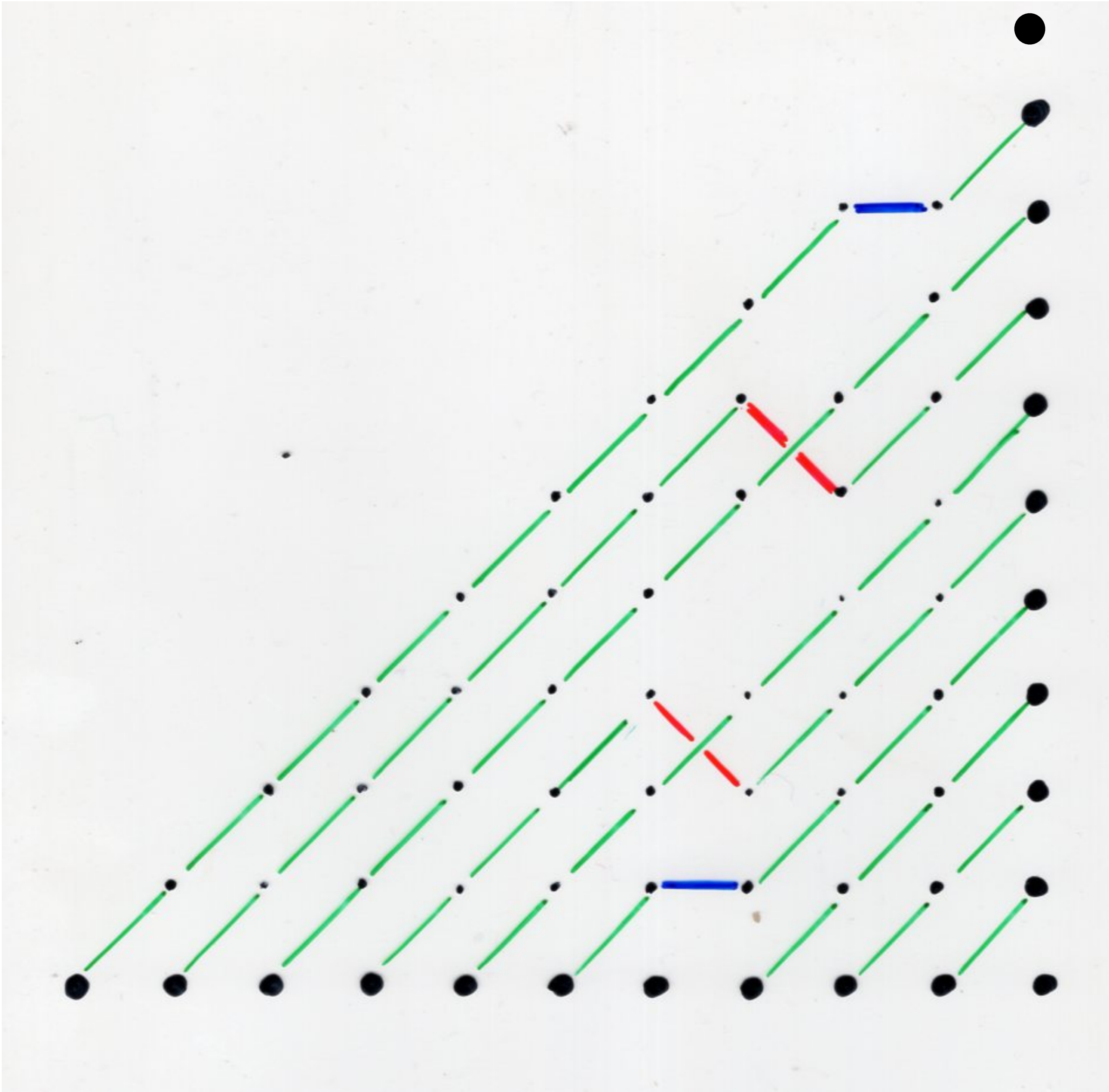
Paths duality

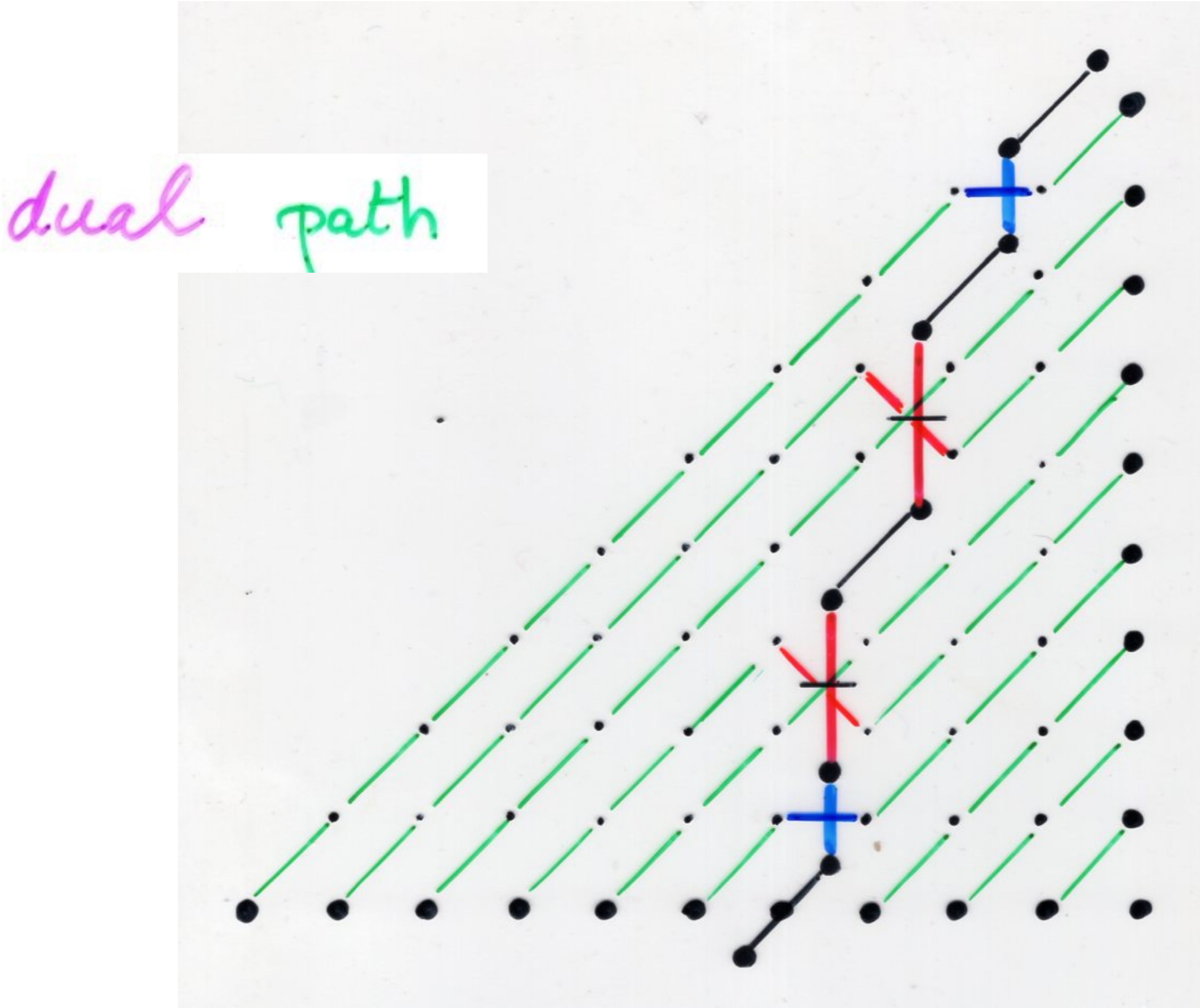


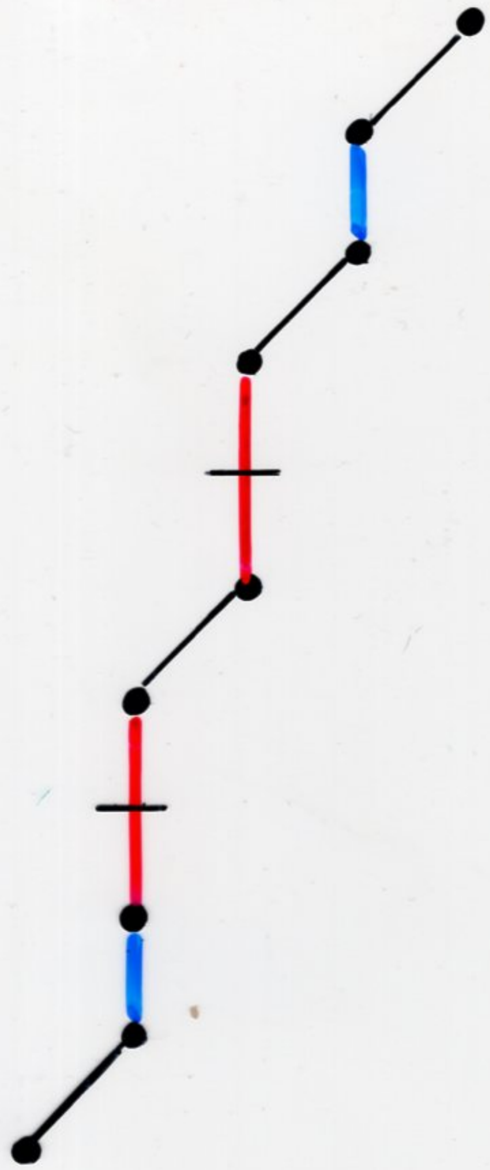

dual path


dual path

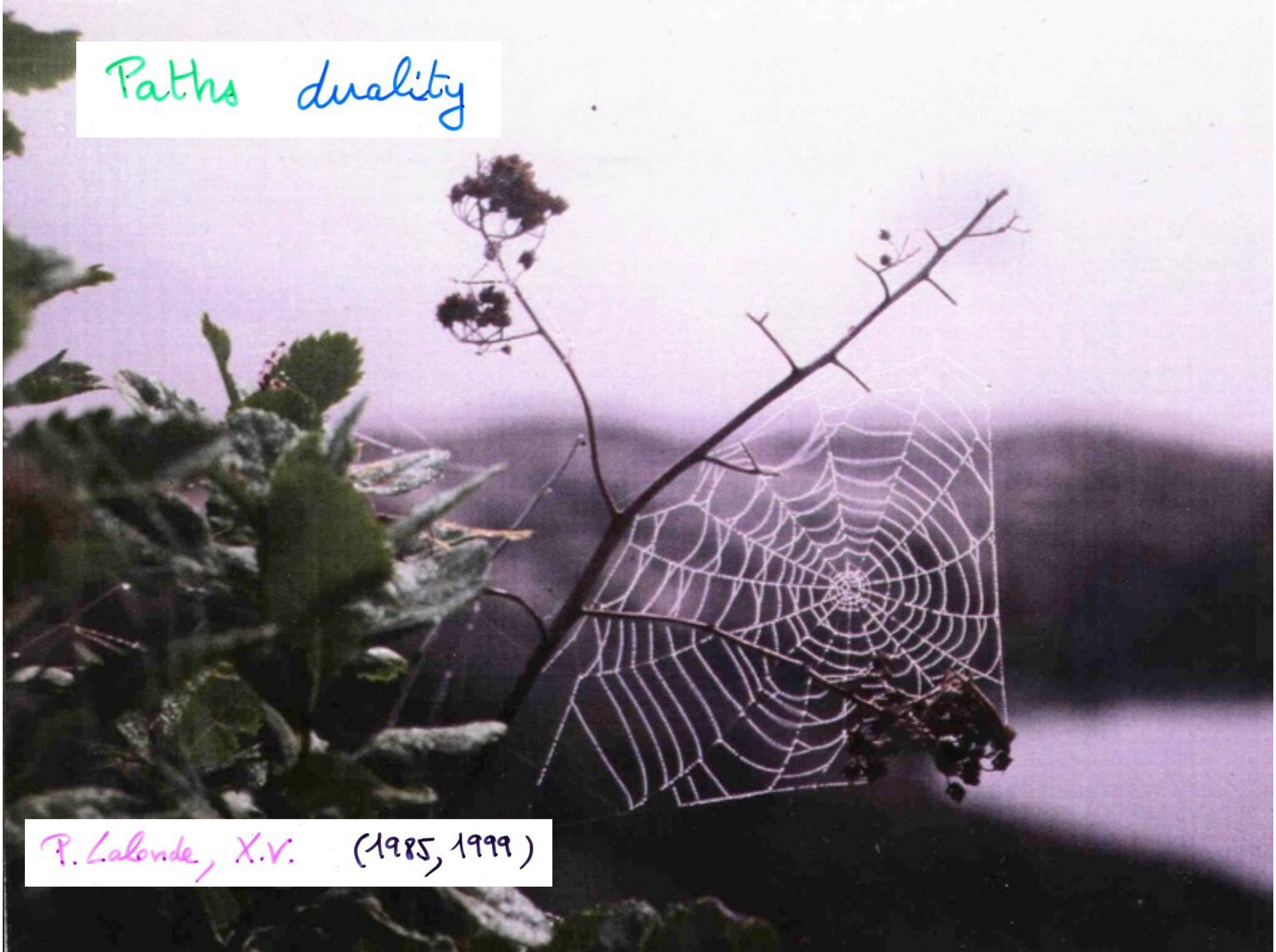





dual path



Paths duality



P. Lalonde, X.V. (1985, 1999)

Inverse relations: examples

Hermite polynomials
and
two kinds of Hermite histories

Hermite histories II

the inversion theorem

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$V = Q = P^{-1}$$

$H_n(x)$

$$P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i} x^i$$

$$V = (\mu_{n,i})_{n,i \geq 0}$$

$$\mu_{n,i} = \sum_{\omega} v(\omega)$$

ω "Motzkin" path

$|\omega| = n$, $0 \to i$

orthogonal Sheffer polynomials

→ Riordan arrays

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

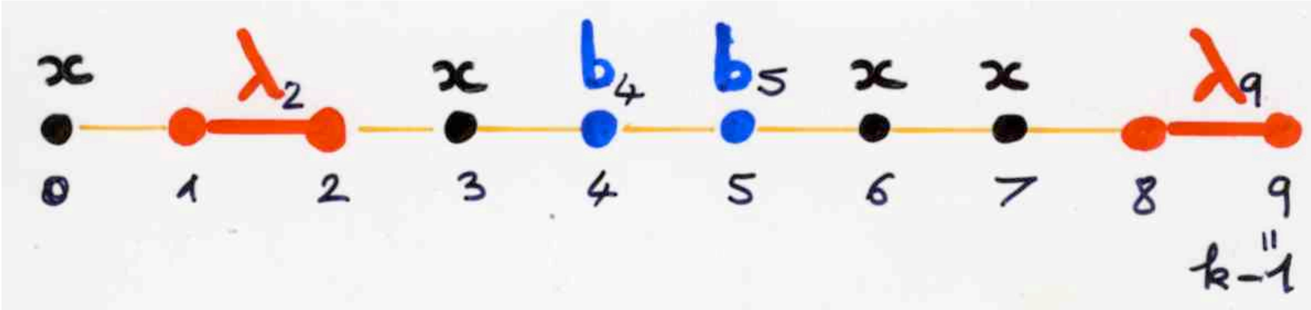
number of N, NN, NE, elementary steps of η

$$H_n(x) = \sum_{\eta} (-1)^{N+NN(\eta)} v(\eta)\, x^{NE(\eta)}$$

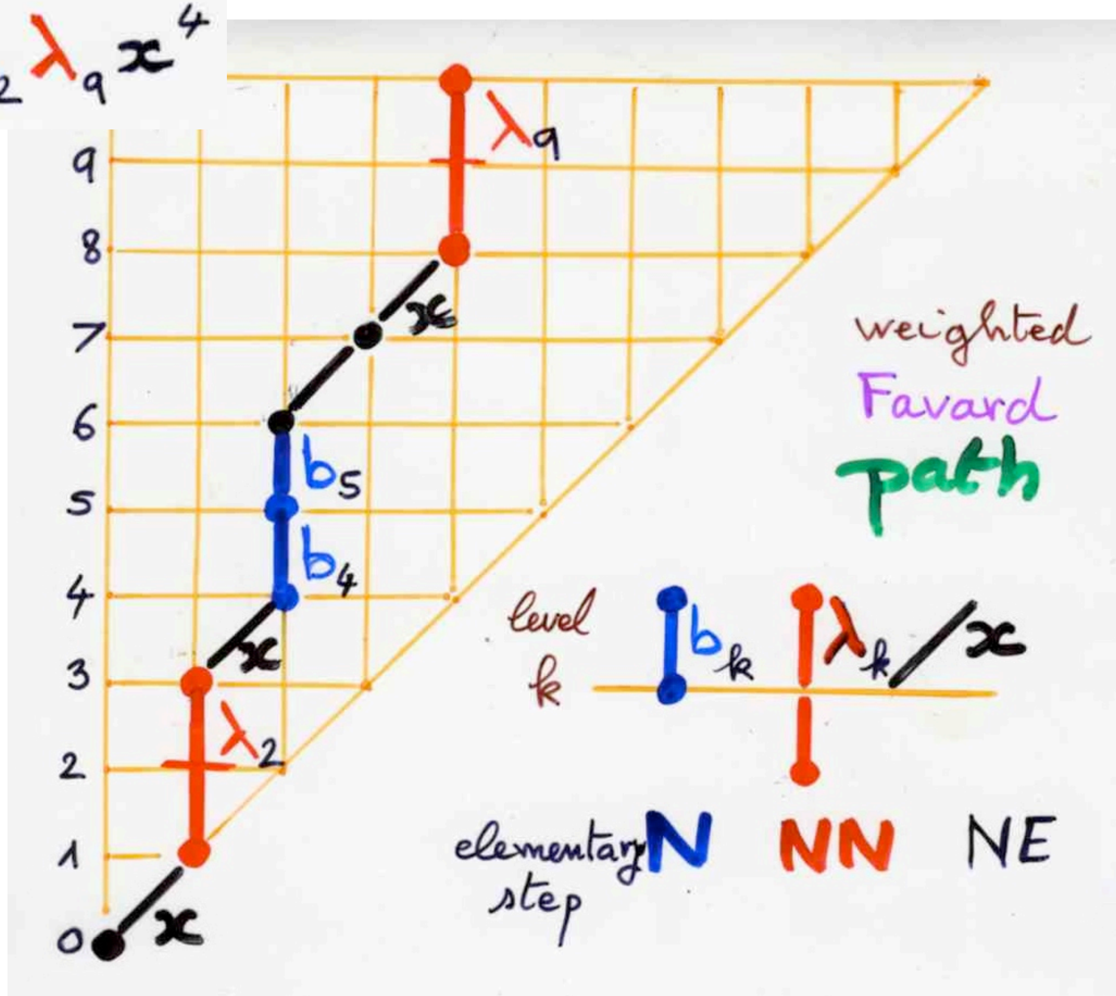
Favard path $|\eta| = n$

$$H_n(x) = \sum_{\alpha} (-1)^{|\alpha|} v(\alpha)\, x^{ip(\alpha)}$$

pavage of $[0, n-1]$

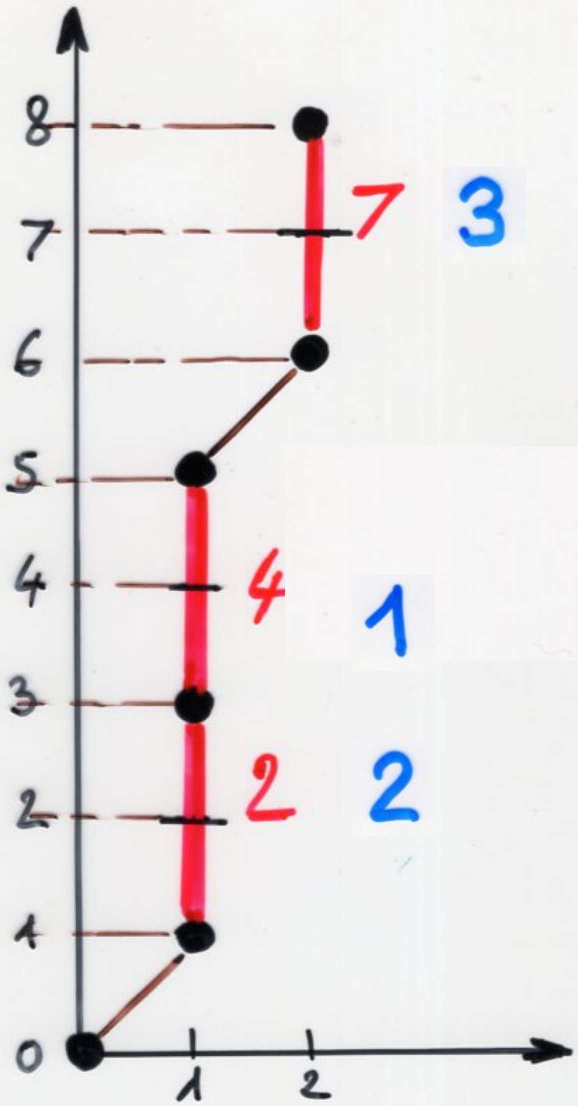


$$(-1)^4\, b_4 b_5 \lambda_2 \lambda_9\, x^4$$



$$v(\eta)$$

$$b_4\, b_5\, \lambda_2\, \lambda_9$$

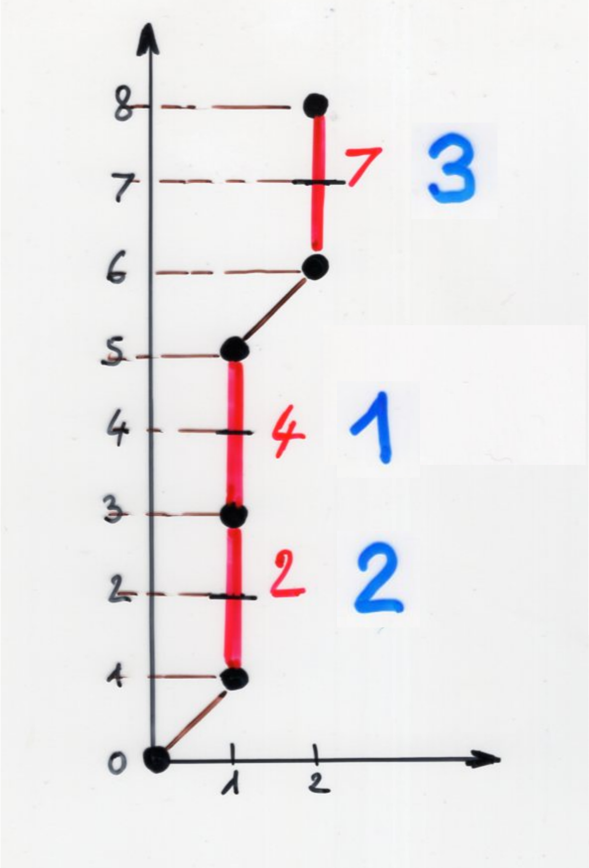


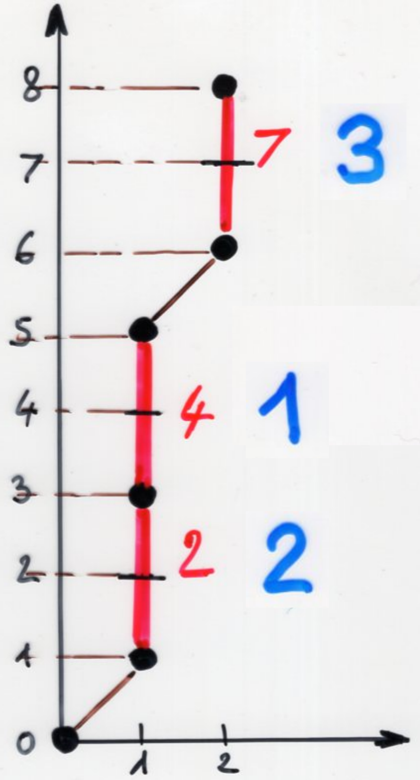
Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

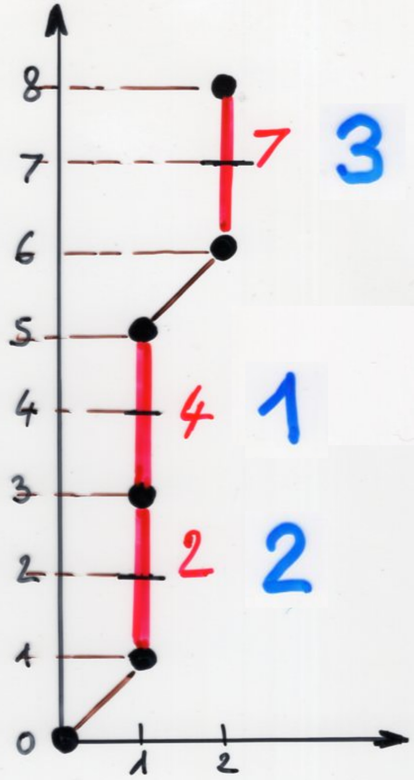
$H_n(x)$

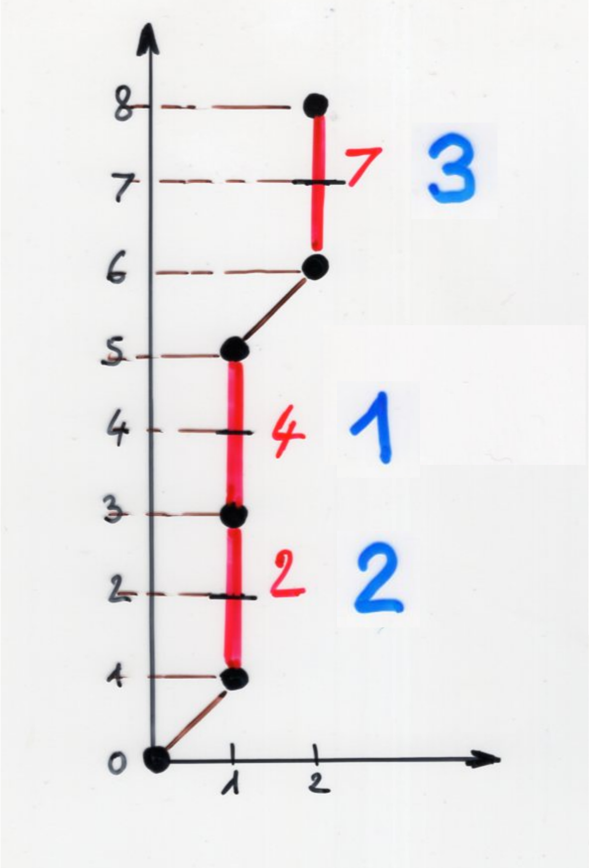
(combinatorial) Hermite polynomials

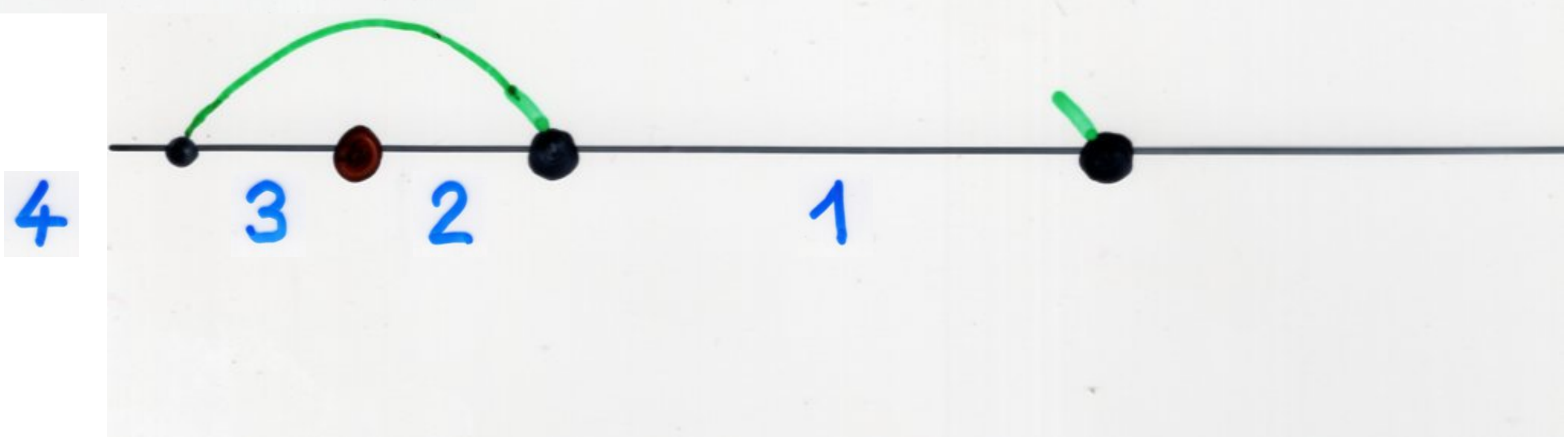

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

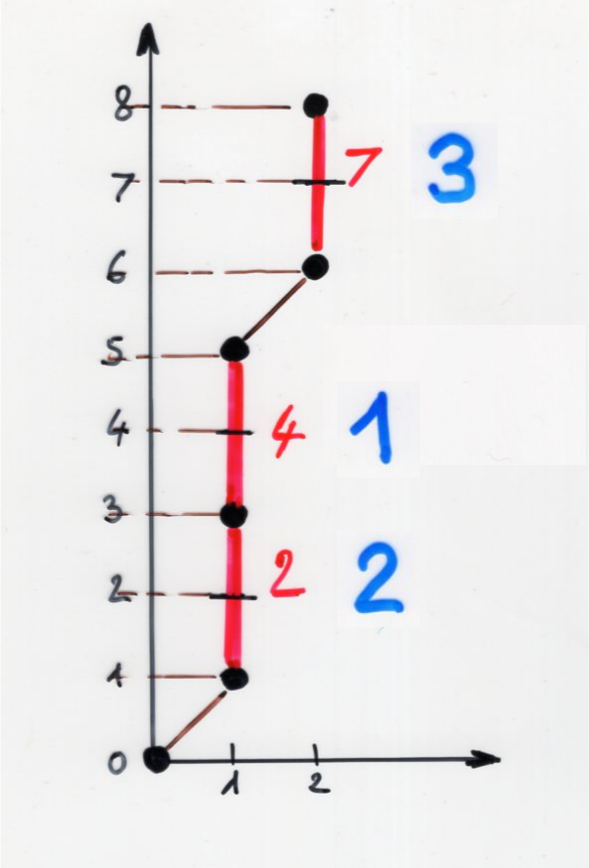

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

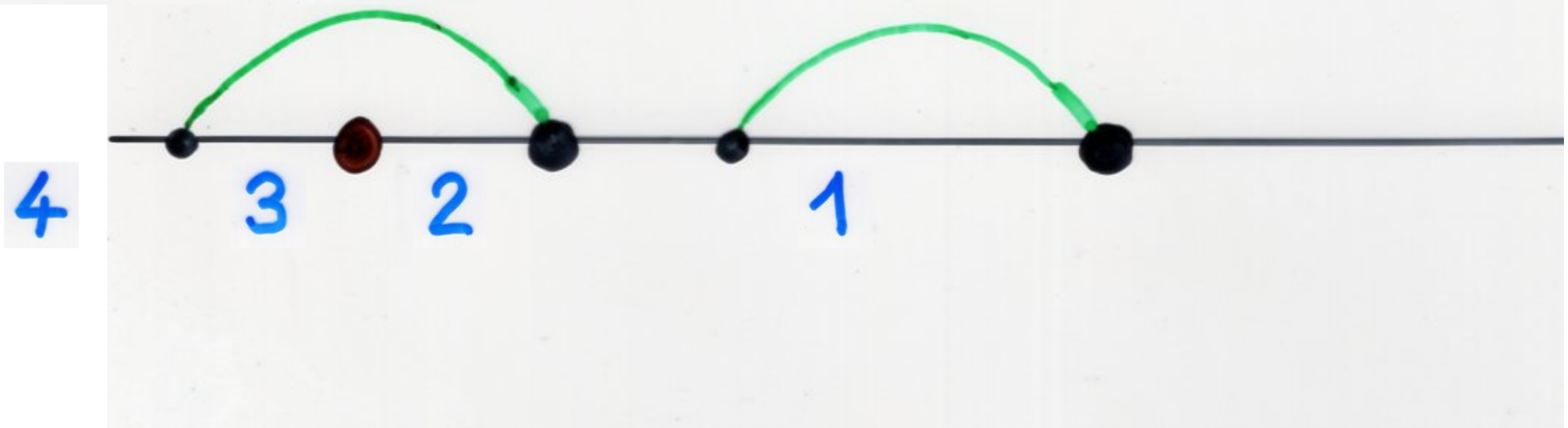

2
1

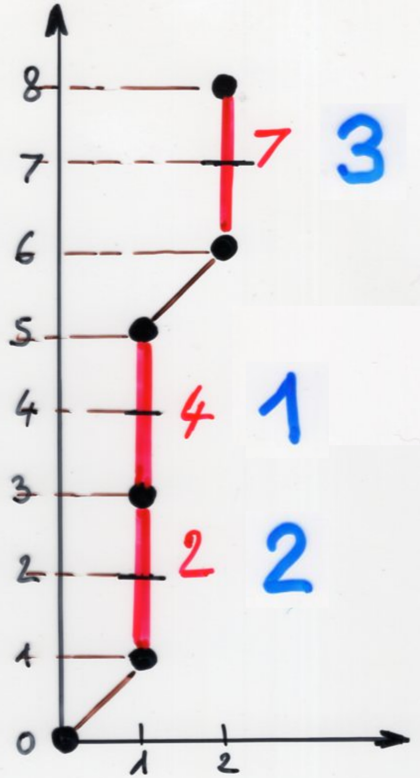

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

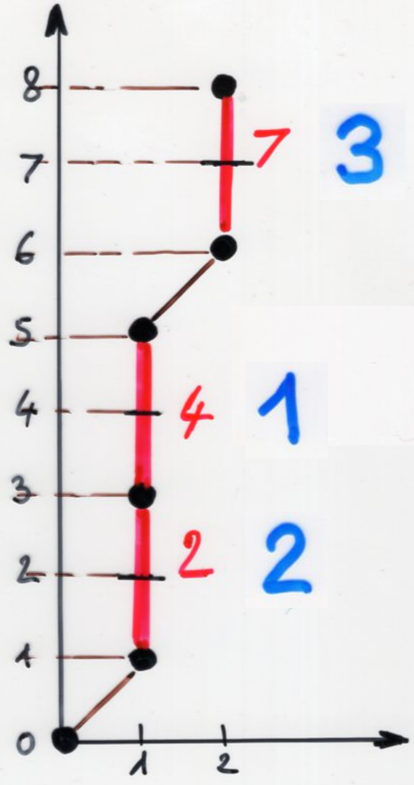

8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

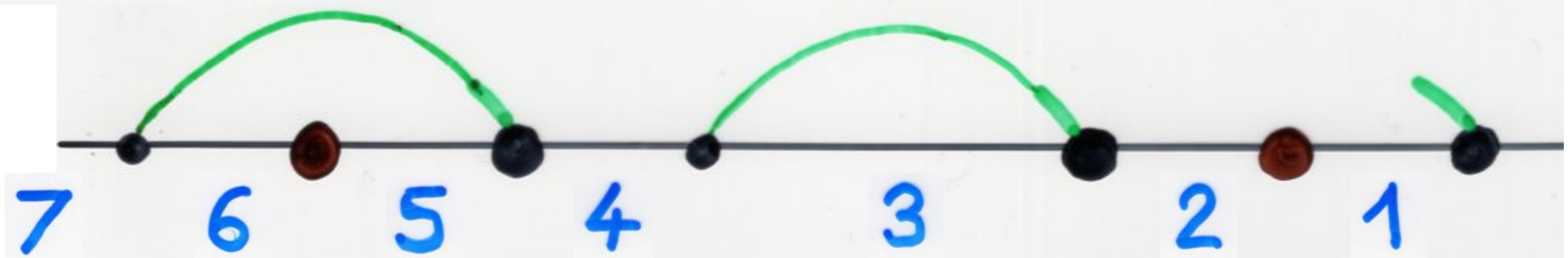

4
3
2
1

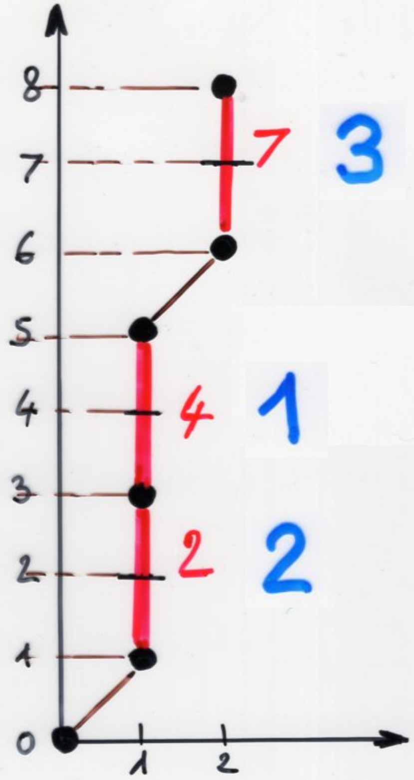

8
7
6
5
4
3
2
1
0
1
2
7
4
2
3
1
2


4
3
2
1


8
7
6
5
4
3
2
1
0
1
2
7
4
2
3
1
2


8
7
6
5
4
3
2
1
0
1
2
7
3
4
1
2
2

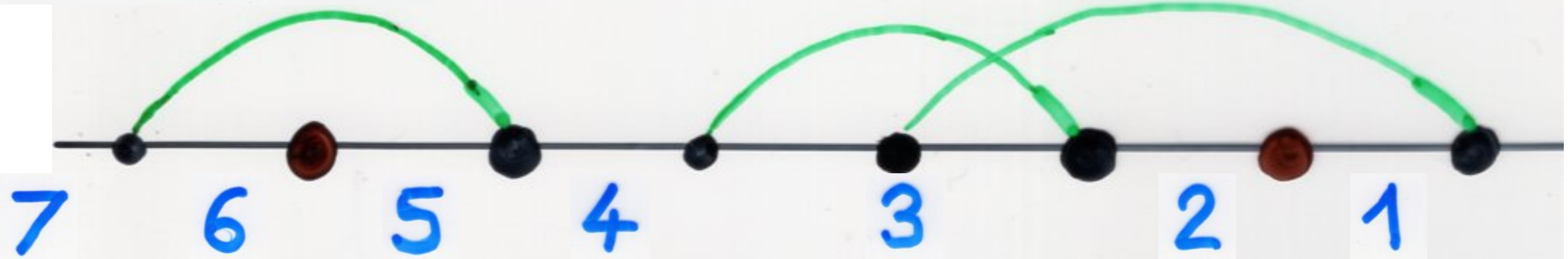

7
6
5
4
3
2
1



Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$H_n(x) = \sum_{0 \leq 2k < n} (-1)^k \frac{n!}{2^k k! (n-2k)!} x^{n-2k}$$



$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{fix(\sigma)}$$

Inverse relations: examples

Hermite polynomials
and
two kinds of Hermite histories

Hermite histories I

the inversion theorem

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$V = Q = P^{-1}$$

$$P = (p_{n,i})_{i,n}$$

$$P_n(x) = \sum_{i=0}^{n} p_{n,i} x^i$$

$$V = (\mu_{n,i})_{n,i \geq 0}$$

$$\mu_{n,i} = \sum_{\omega} v(\omega)$$

ω "Motzkin" path

$|\omega| = n$, $0 \to i$

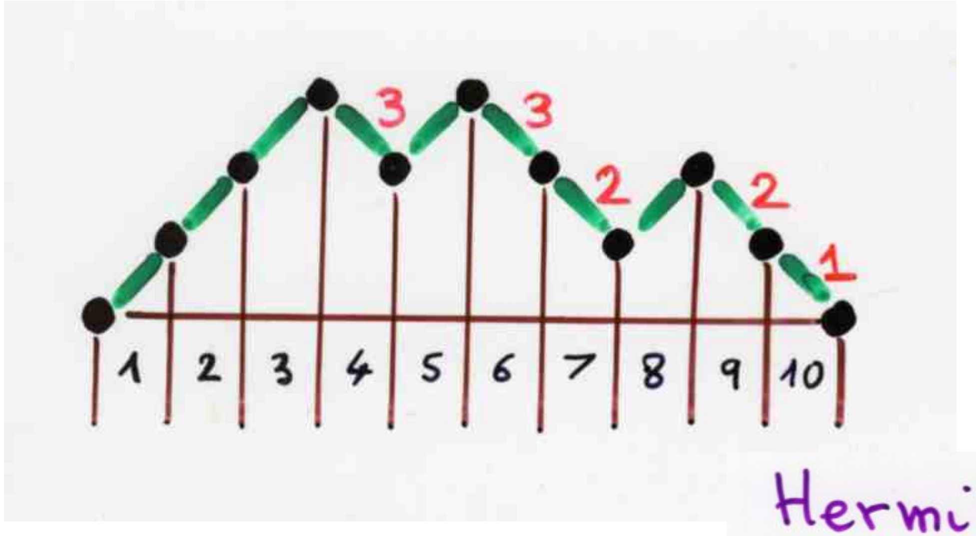
Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

moments

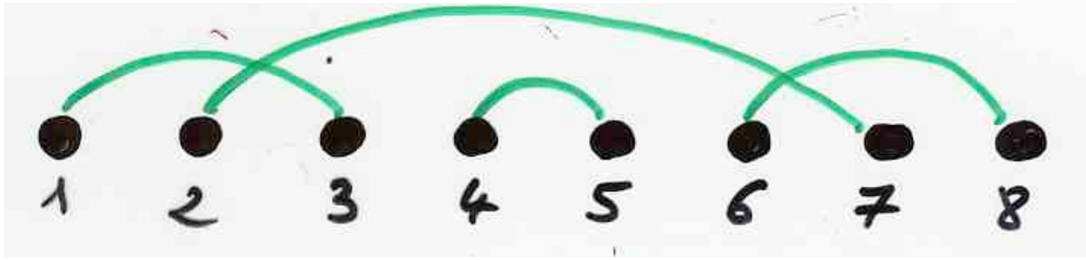
$$\mu_{2n,0} = 1 \times 3 \times \dots \times (2n-1)$$

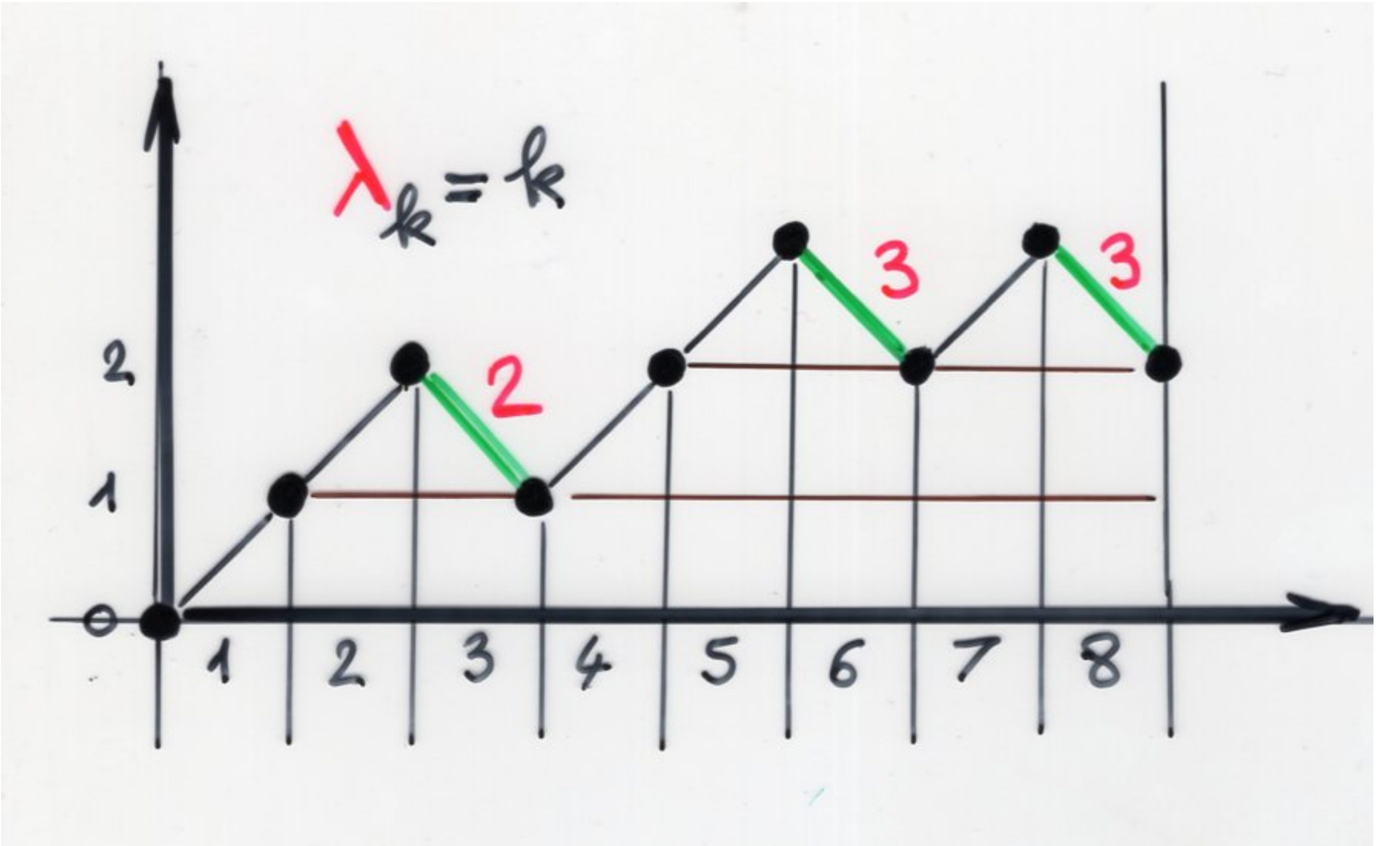
involutions on $\{1, \dots, 2n\}$ with no fixed points



Hermite histories

$$1 \leq i \leq \lambda_k$$





extension for $0 \leq i \leq 2n$

$\mu_{2n,i}$

$1 \leq i \leq \lambda_k$

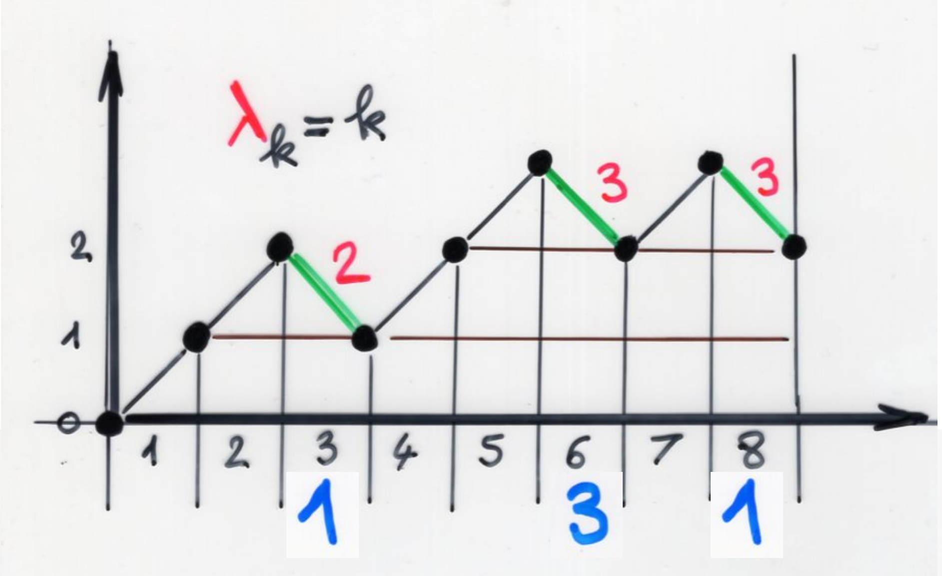
$$\mu_{n,i} = \sum_{\substack{\omega \\ \text{"Motzkin" path} \\ |\omega|=n,\ 0 \to i}} v(\omega)$$

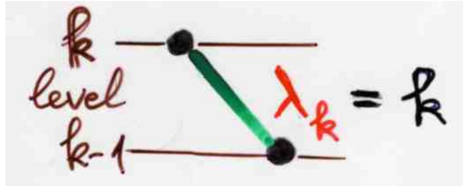
Motzkin path going from level 0 to level i

$$V_n(x) = \sum_{i=0}^{n} \mu_{n,i}\, x^i$$

$$V = (\mu_{n,i})_{n,i \geq 0}$$

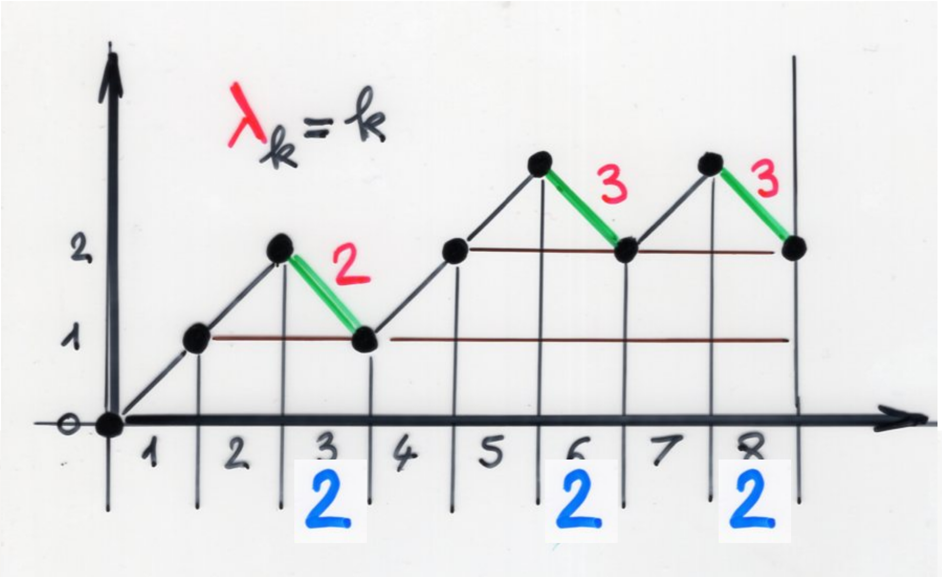
$$V = Q = P^{-1}$$



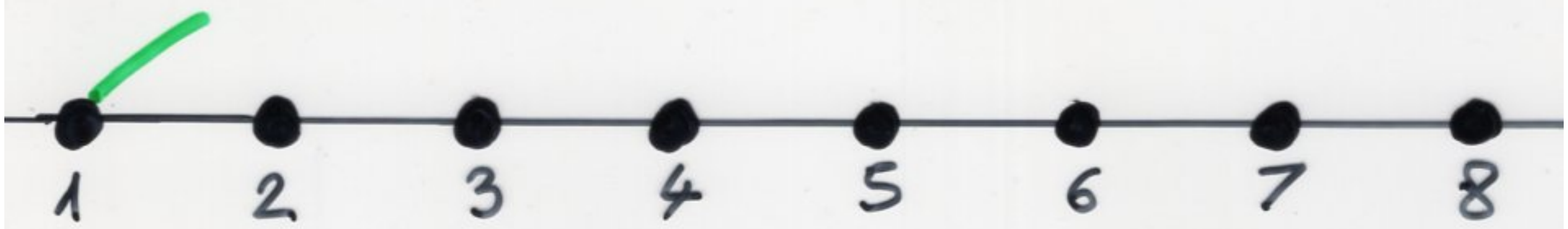


Hermite history

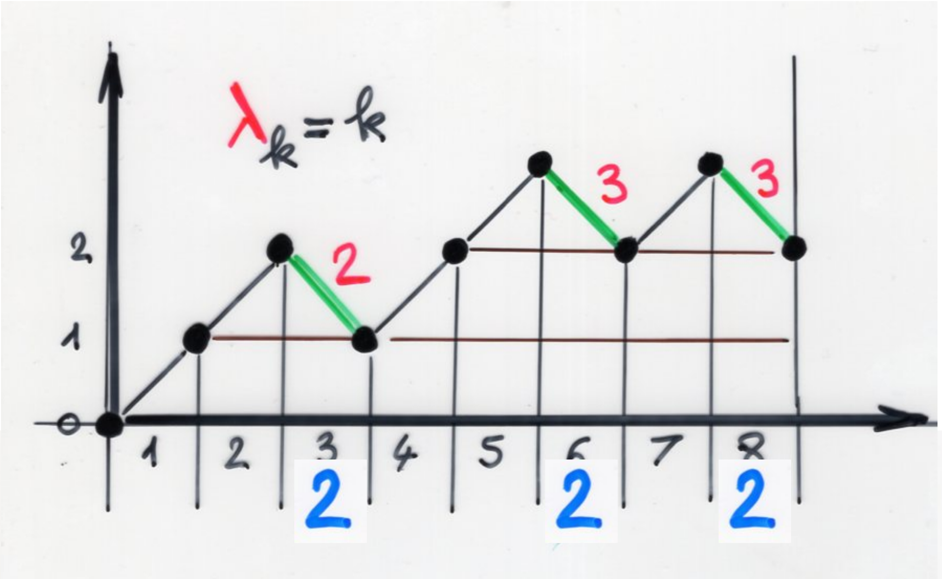
$$1 \leq i \leq \lambda_k$$

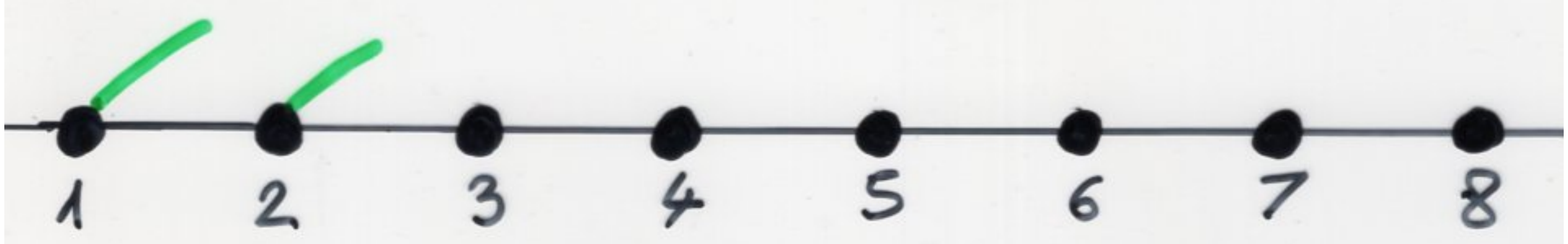


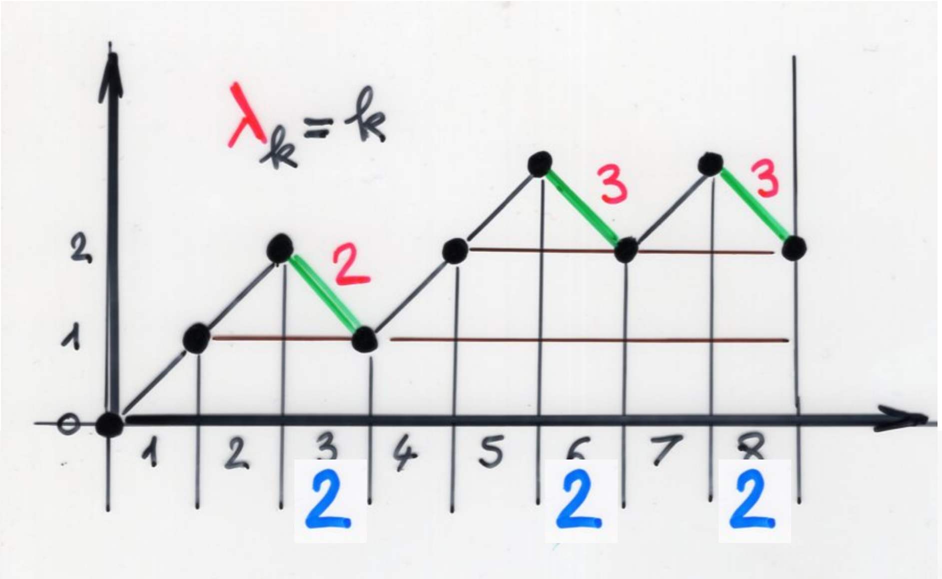
$\lambda_k = k$
2
3
3
0
1
2
1
2
3
4
5
6
7
8
2
2
2



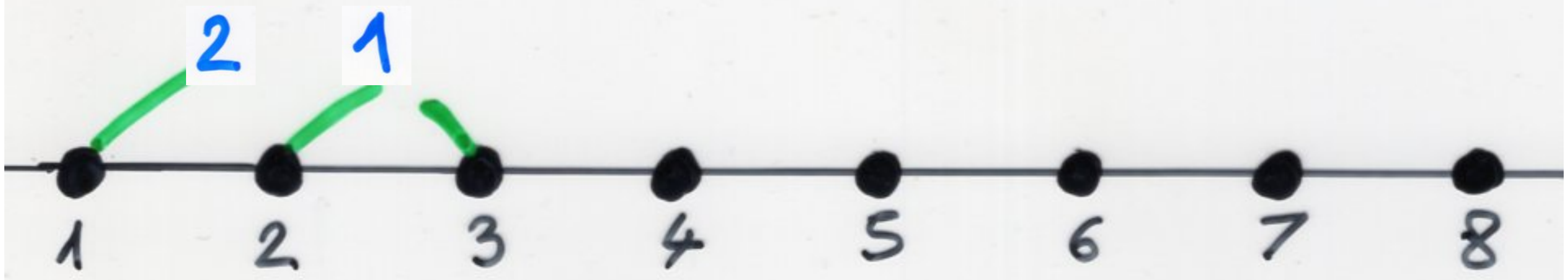
1
2
3
4
5
6
7
8


λ_k = k
2
3
3
0
1
2
1
2
3
4
5
6
7
8
2
2
2

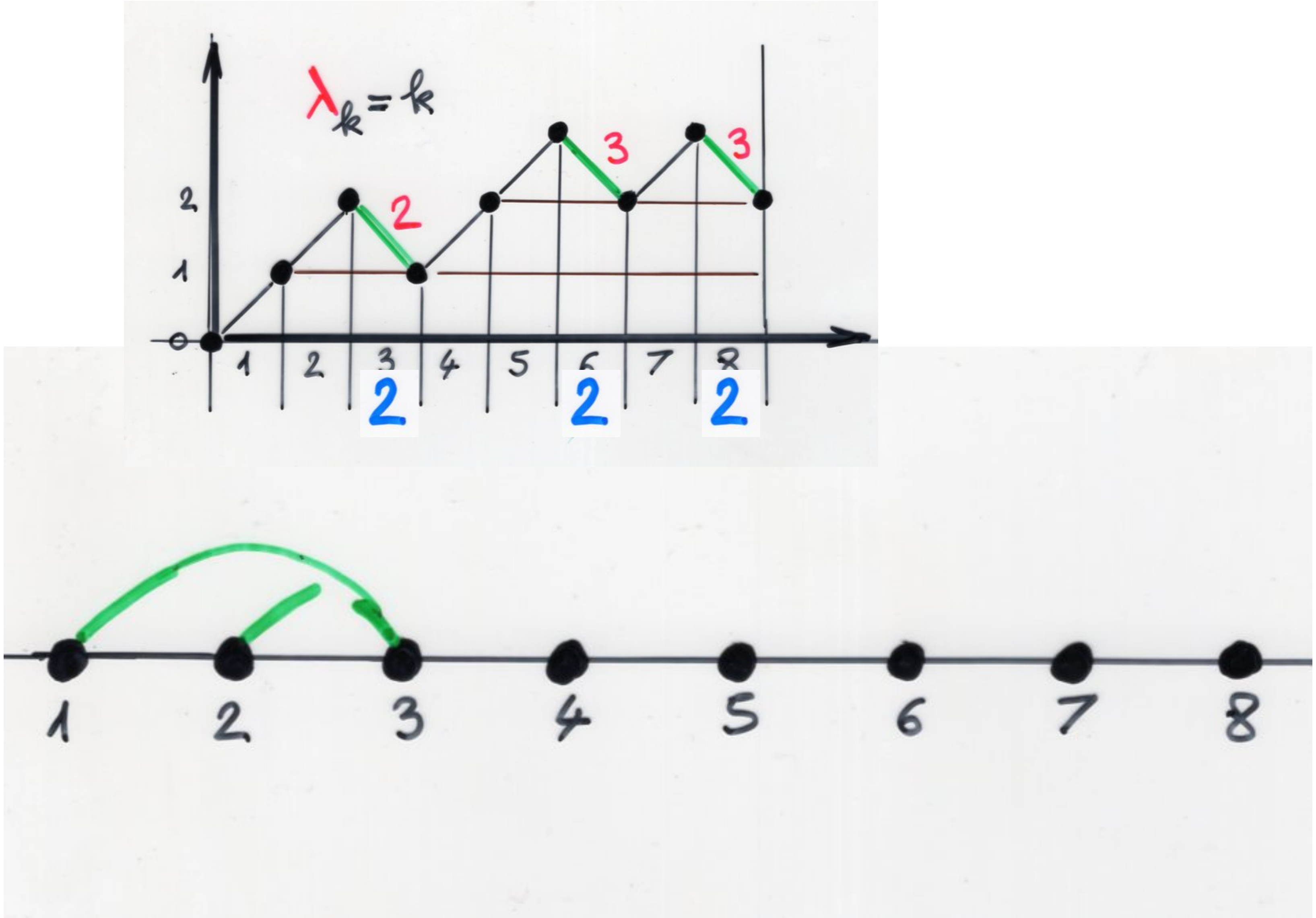

1
2
3
4
5
6
7
8



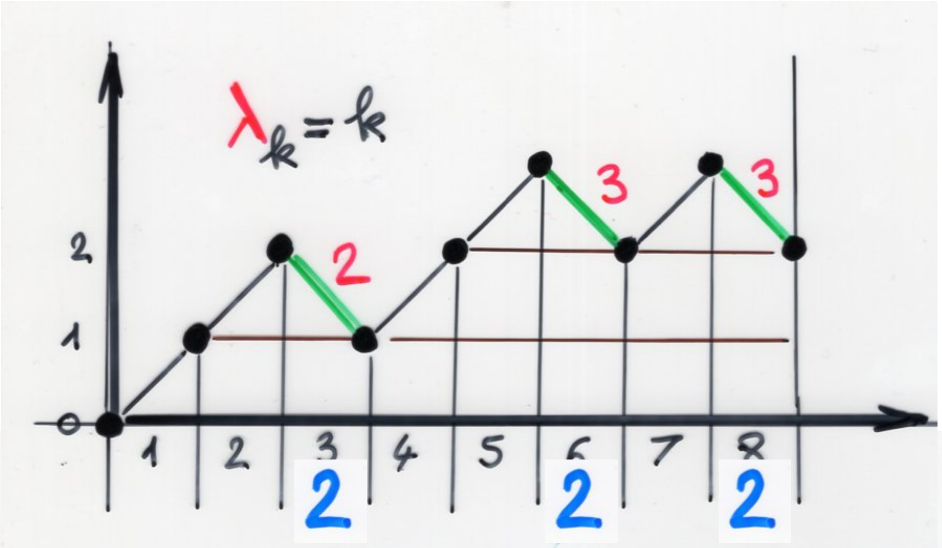
$\lambda_k = k$
2
3
3
0
1
2
1 2 3 4 5 6 7 8
2
2
2



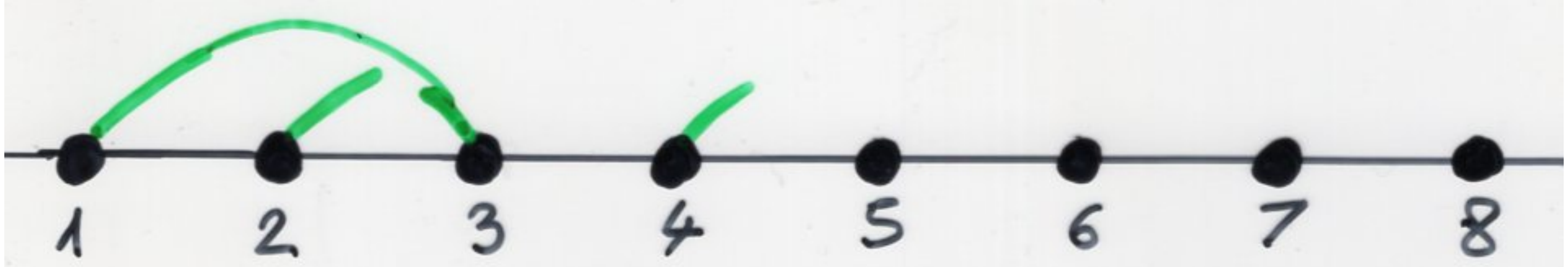
2
1
1 2 3 4 5 6 7 8



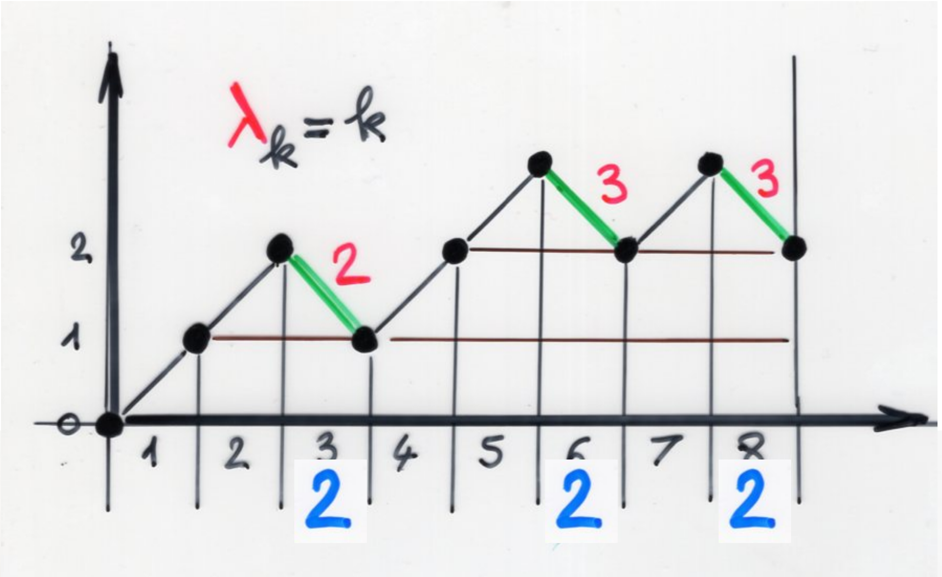
λk = k
2
3
3
2
2
2
0 1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8



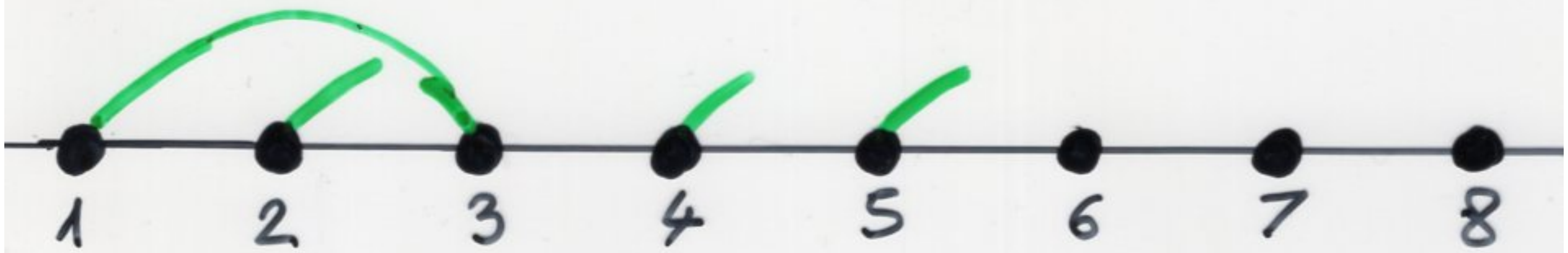
$\lambda_k = k$
2
1
0
1 2 3 4 5 6 7 8
2
3
3
2
2
2



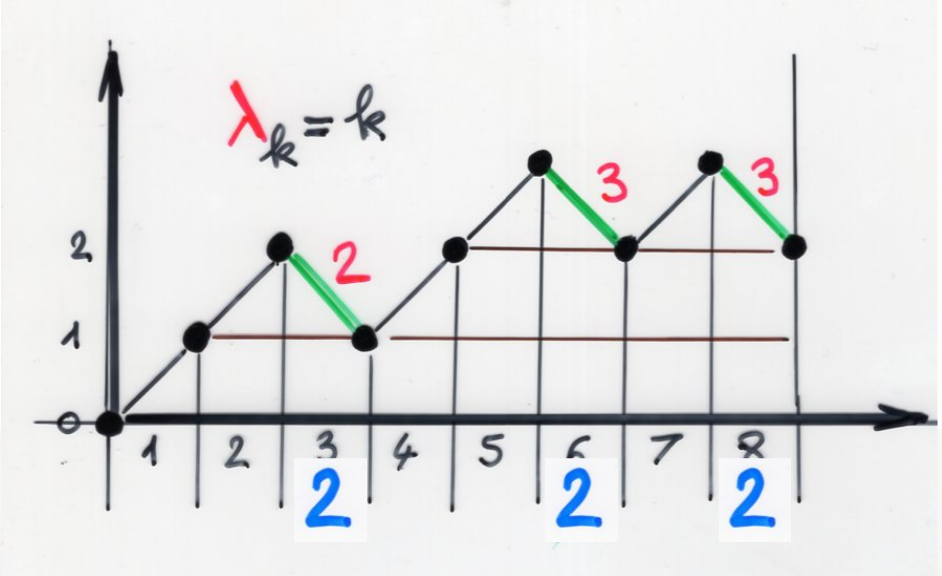
1 2 3 4 5 6 7 8



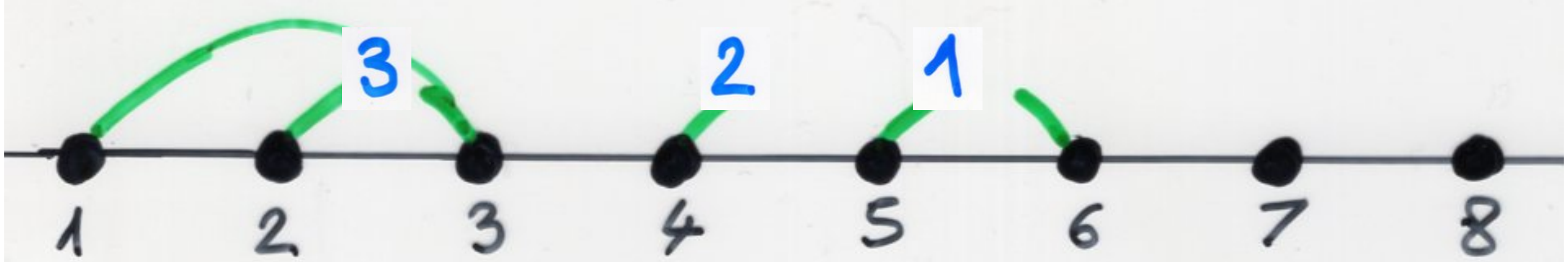
$\lambda_k = k$
0
1
2
1 2 3 4 5 6 7 8
2
3
3
2
2
2



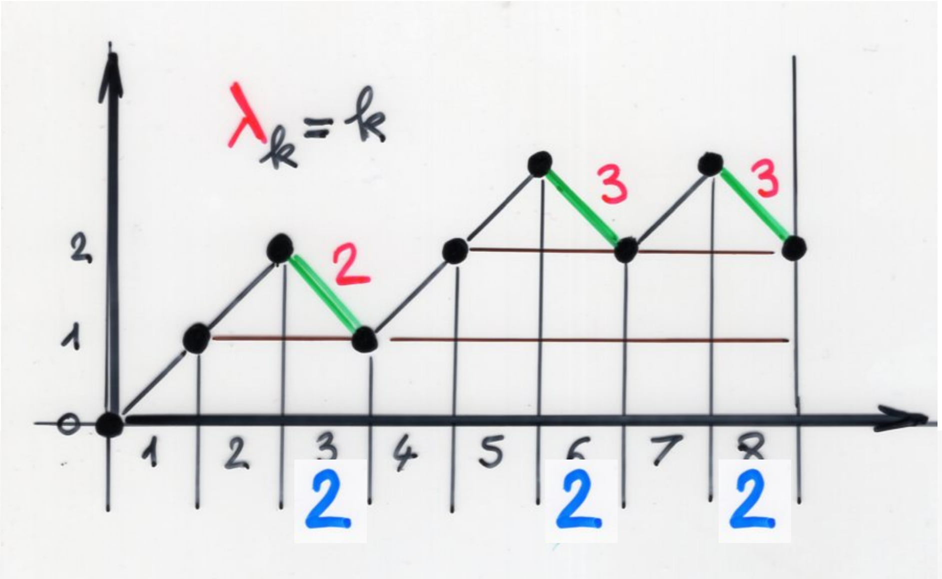
1 2 3 4 5 6 7 8



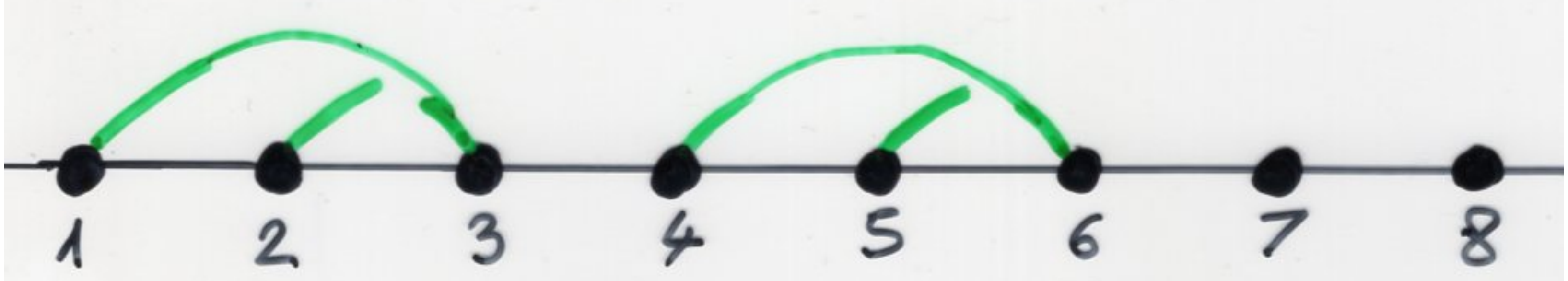
$\lambda_k = k$
2
3
3
0
1
2
1
2
3
4
5
6
7
8
2
2
2



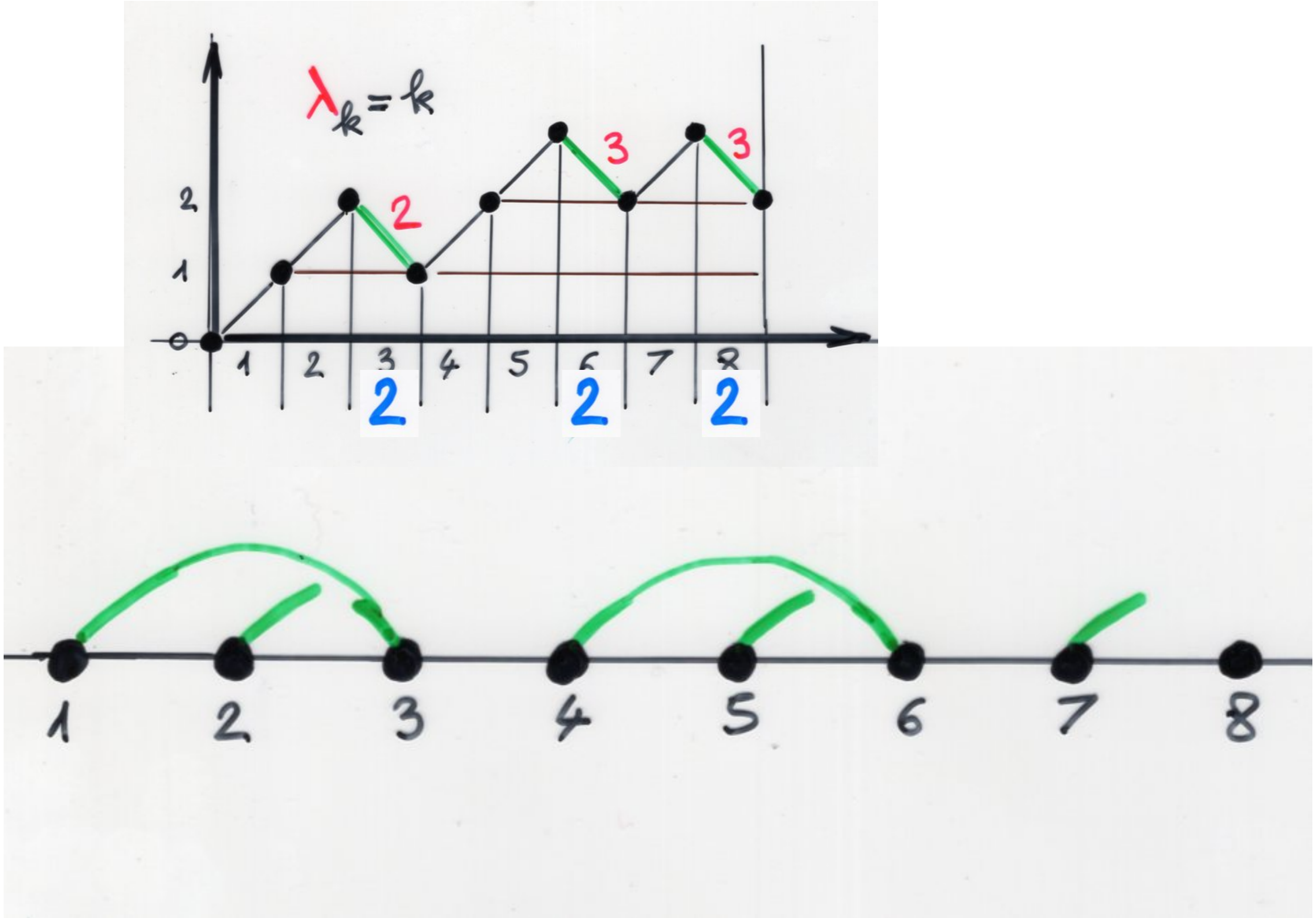
3
2
1
1
2
3
4
5
6
7
8



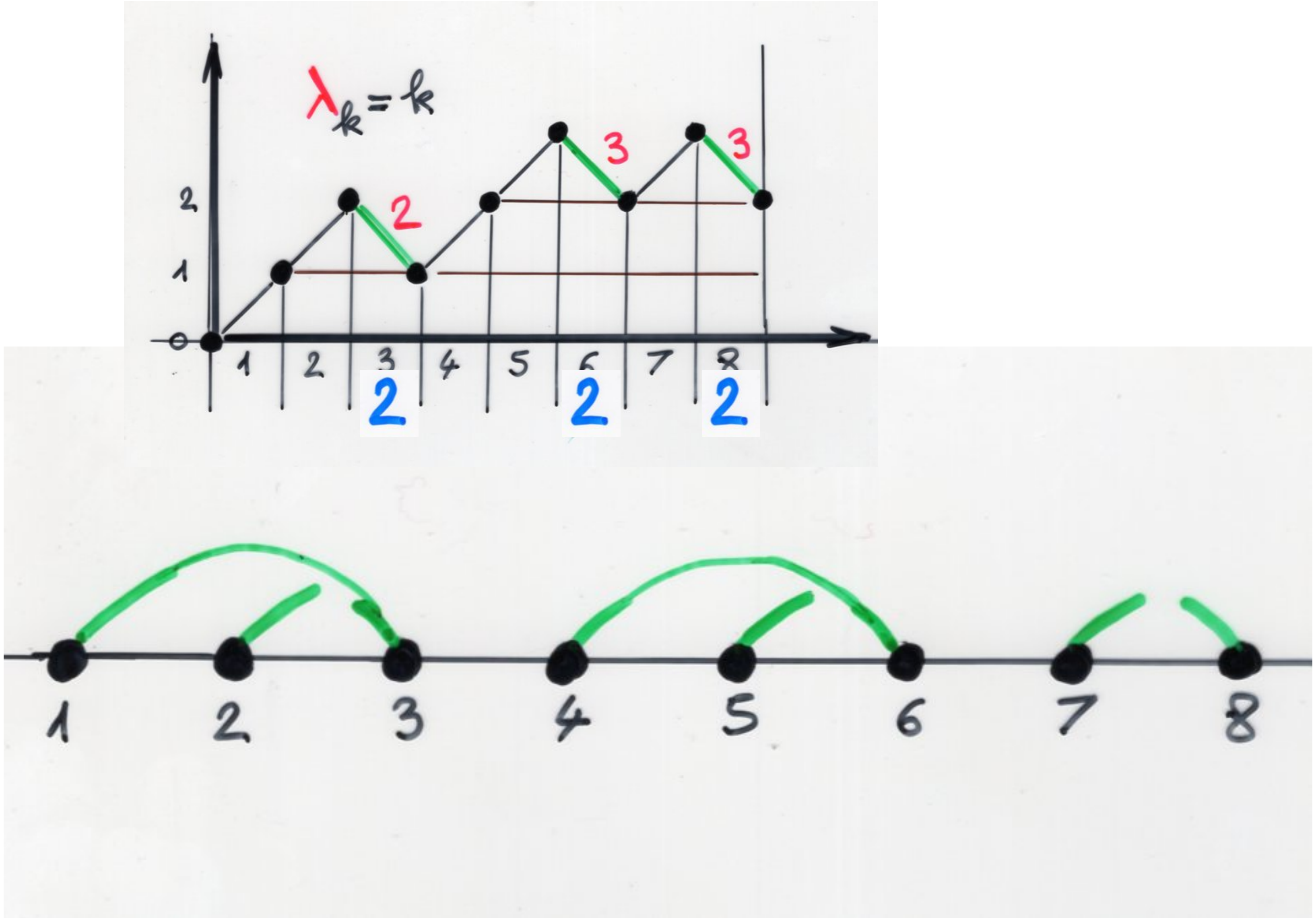
$\lambda_k = k$
2
3
3
0
1
2
1 2 3 4 5 6 7 8
2
2
2

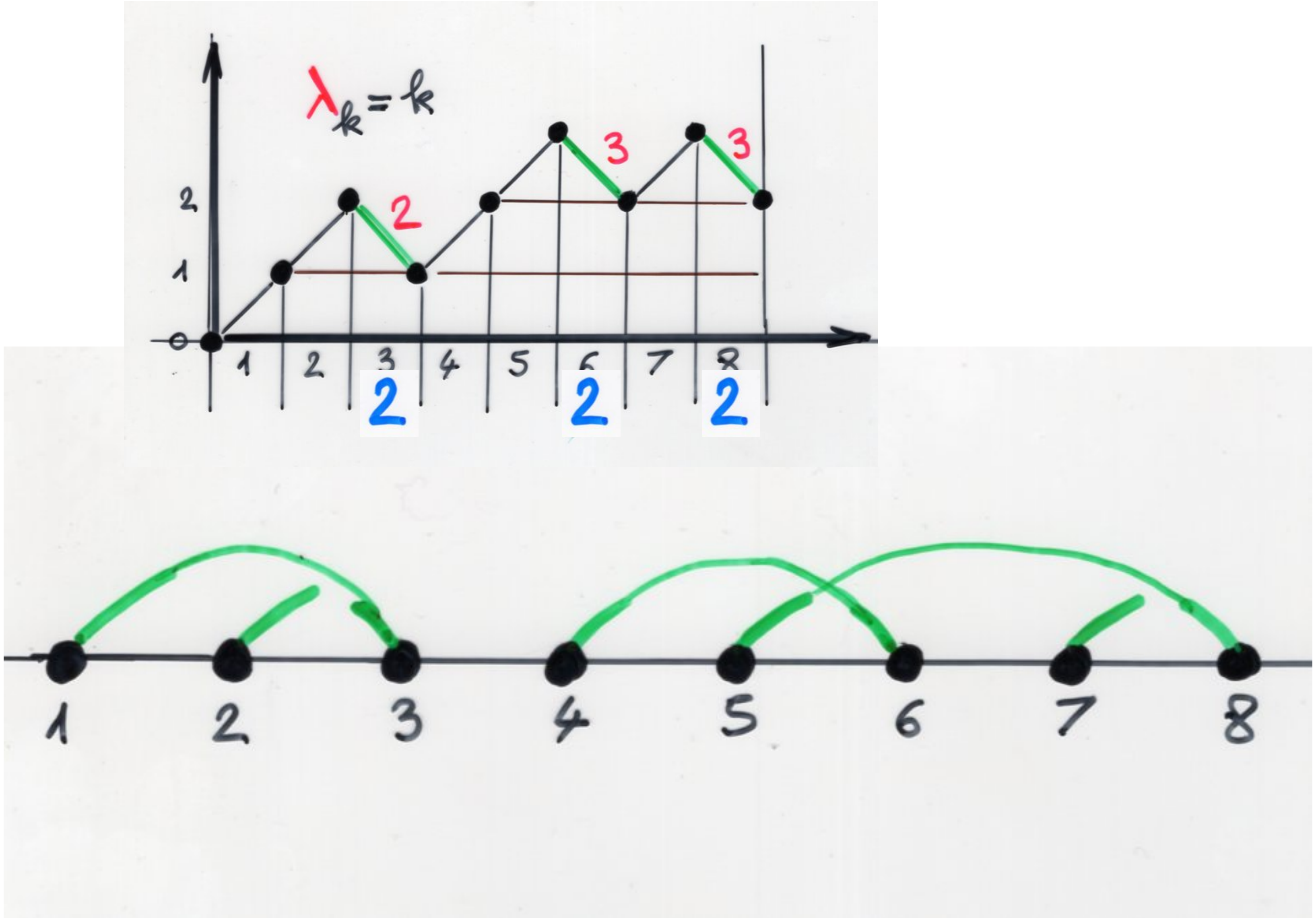


1 2 3 4 5 6 7 8

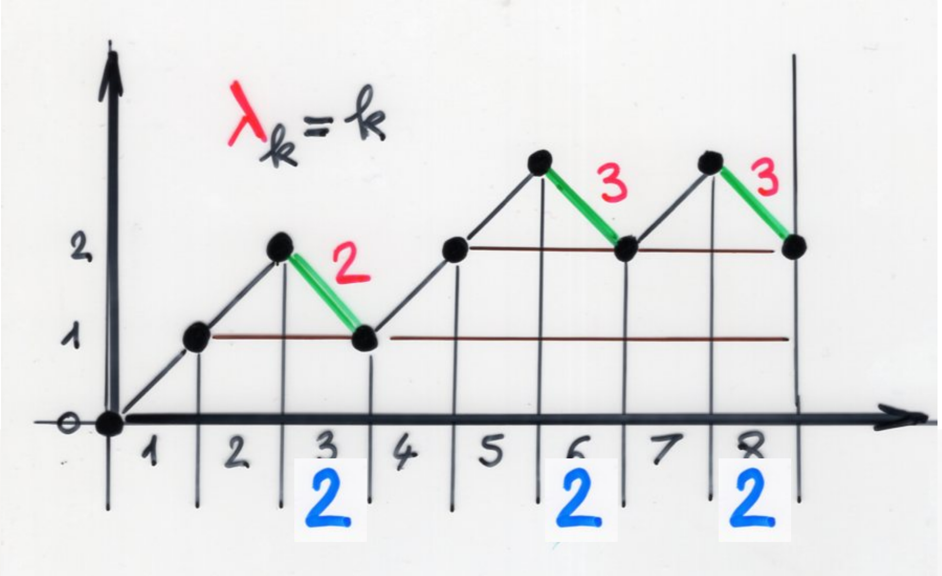


λk = k
0
1
2
1 2 3 4 5 6 7 8
2
3
3
2 2 2
1 2 3 4 5 6 7 8


λ_k = k
0
1
2
1 2 3 4 5 6 7 8
2
3
3
2 2 2
1 2 3 4 5 6 7 8



$\lambda_k = k$
2
1
0
1 2 3 4 5 6 7 8
2 3 3
2 2 2
1 2 3 4 5 6 7 8



Hermite polynomials again !

$$H_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{involution}}} (-1)^{d(\sigma)} x^{fix(\sigma)}$$

the inversion theorem

$H_n(x)$

Hermite polynomials

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$$V = Q = P^{-1}$$

$$P = (p_{n,i})_{i,n}$$

$$p_{n,i} = (-1)^{\frac{(n-i)}{2}} \mu_{n,i}$$

$$V = (\mu_{n,i})_{n,i \geqslant 0}$$

Complements

some remarks about
q-Hermite polynomials

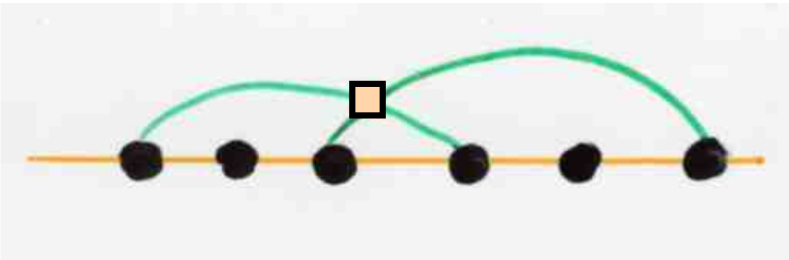
q-analogue of Hermite polynomials

$H_k(x;q)$

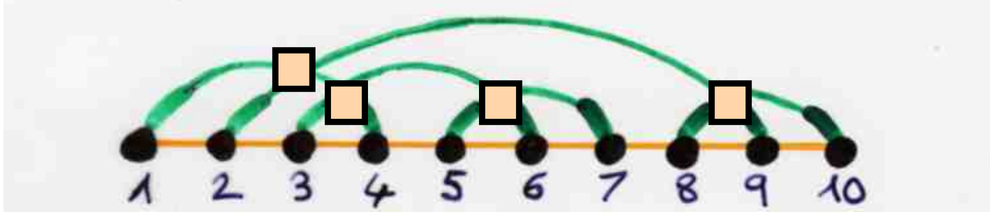
$\lambda_k = [k]_q$

$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$

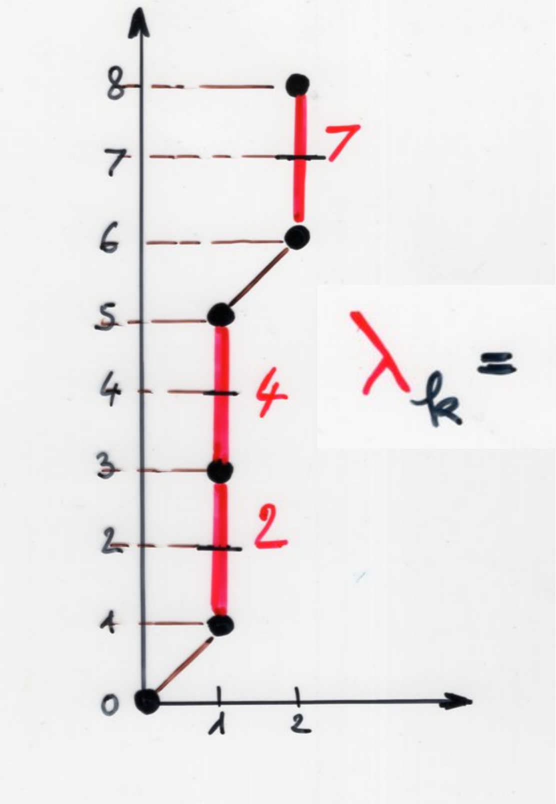
q-Hermite I (continuous)



crossing

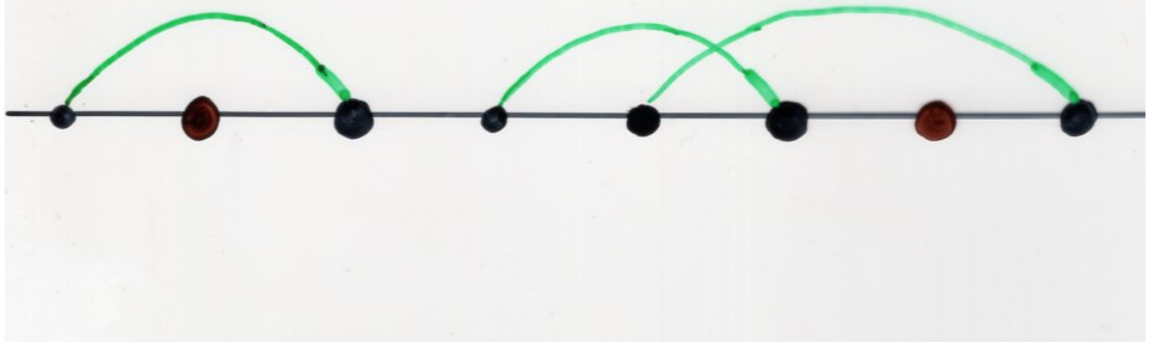


moments



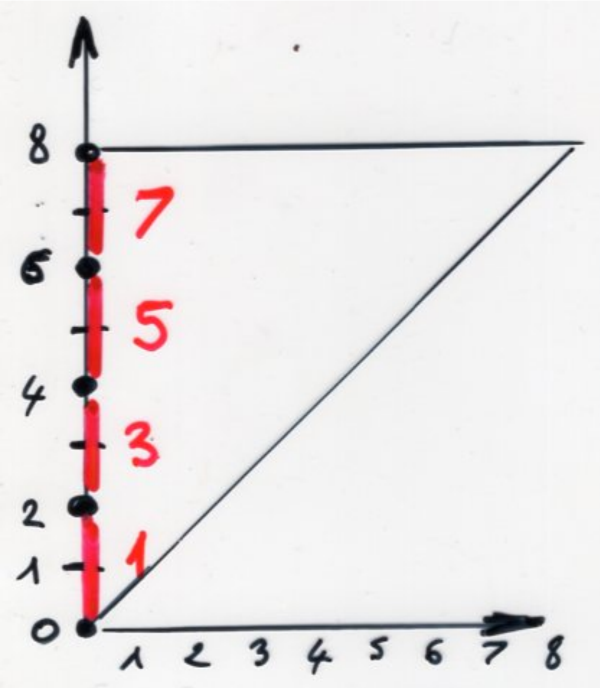
$\lambda_k = [k]_q$

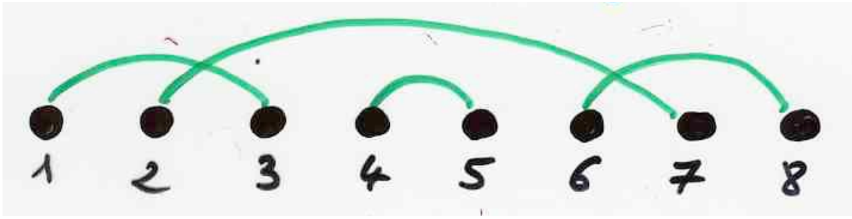
$H_k(x;q)$



$[1]_q \cdot [3]_q \cdots [2n-1]_q$

moments





q-Hermite II
(discrete I)

$\lambda_k = q^{k-1}[k]_q$

Complements

« beta-analogue » of Tchebychev 2nd kind

"β-analog" of Tchebychev 2nd kind

Corrections after the video:

moments

$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

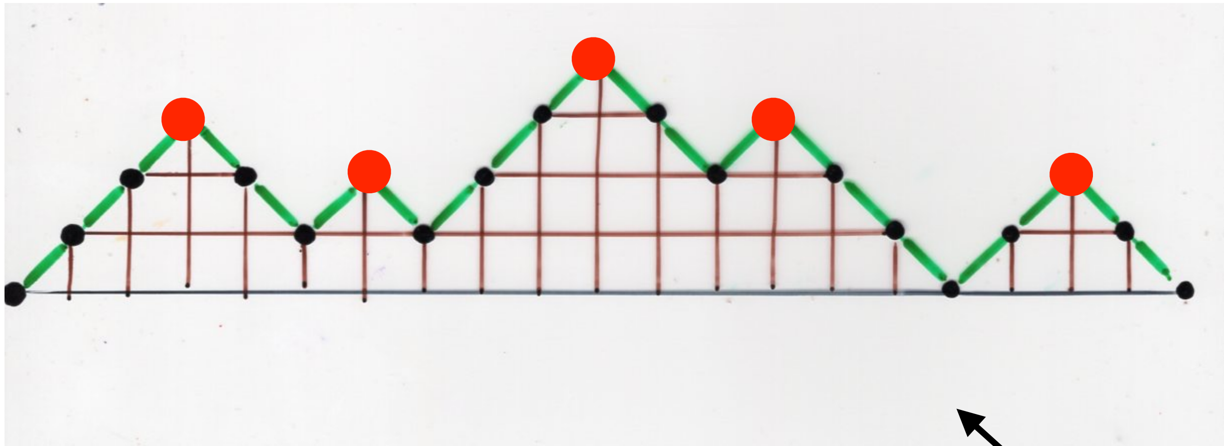
↑

number of Dyck paths ω, $|\omega| = 2n$ having k peaks

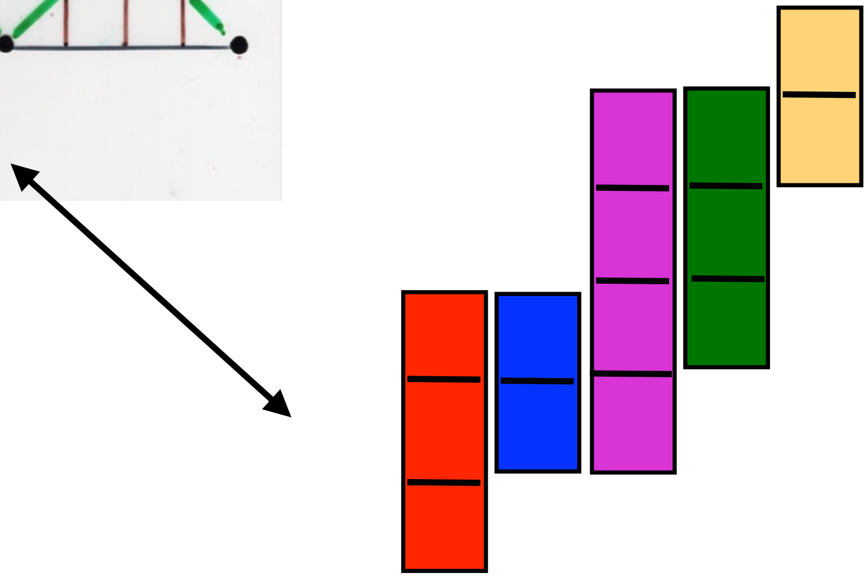
Proposition

$$\lambda_k = 1 \quad k \text{ even}$$
$$= \beta \quad k \text{ odd}$$

(β)- distribution on Catalan numbers

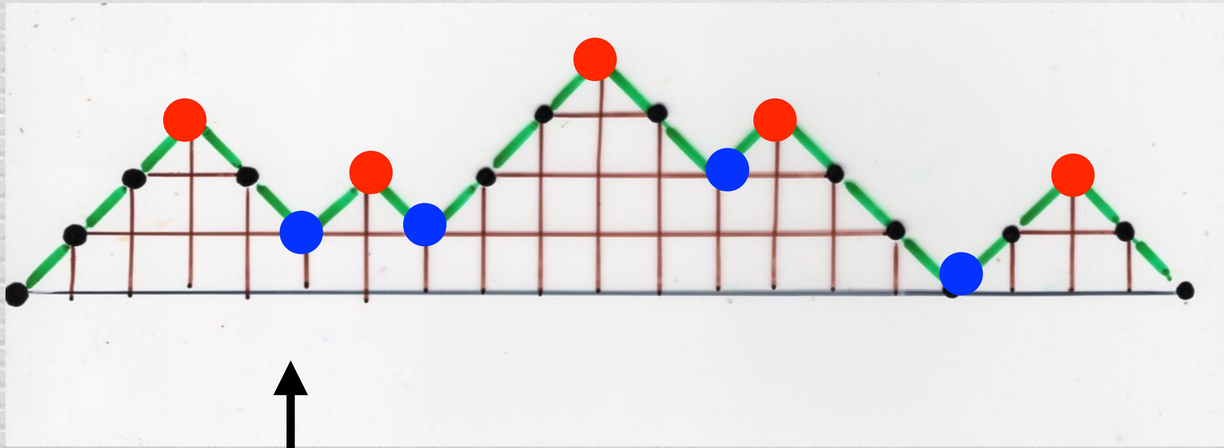


number of peaks in Dyck paths

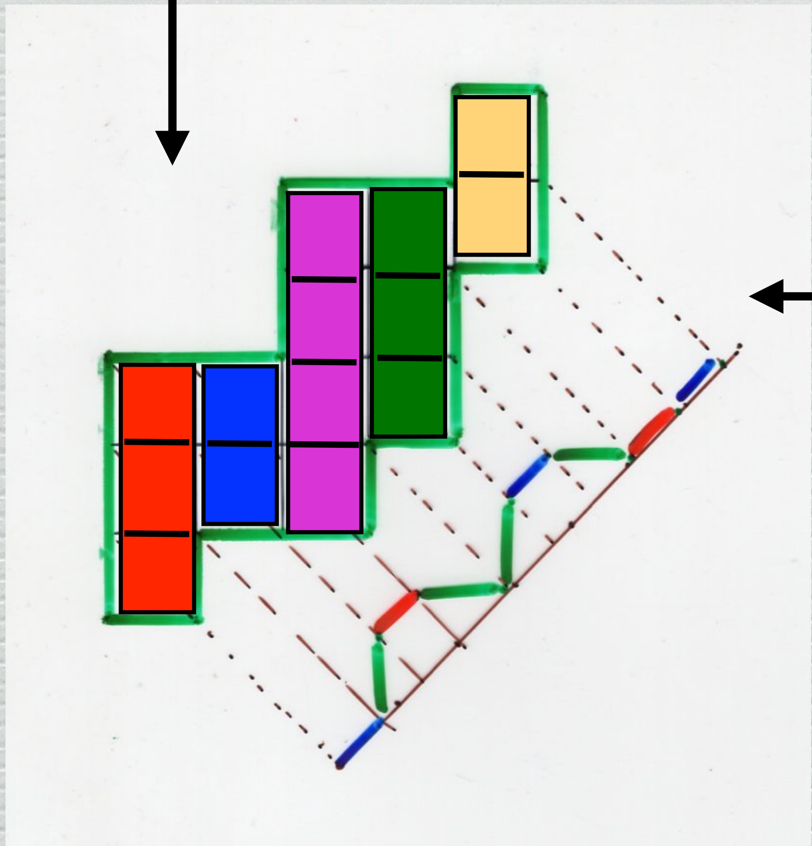


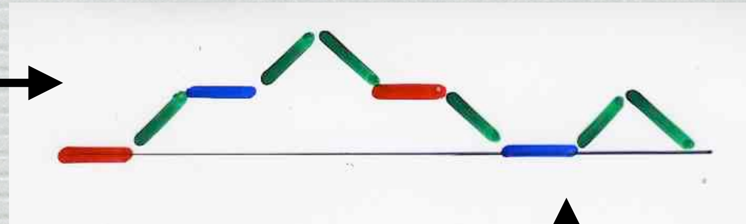
number of columns in staircase polygons

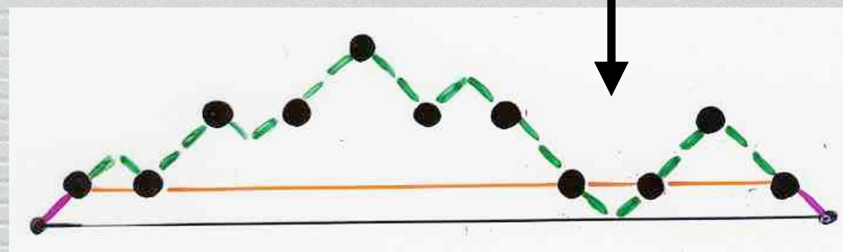
See ABjC, Part I, Ch2a

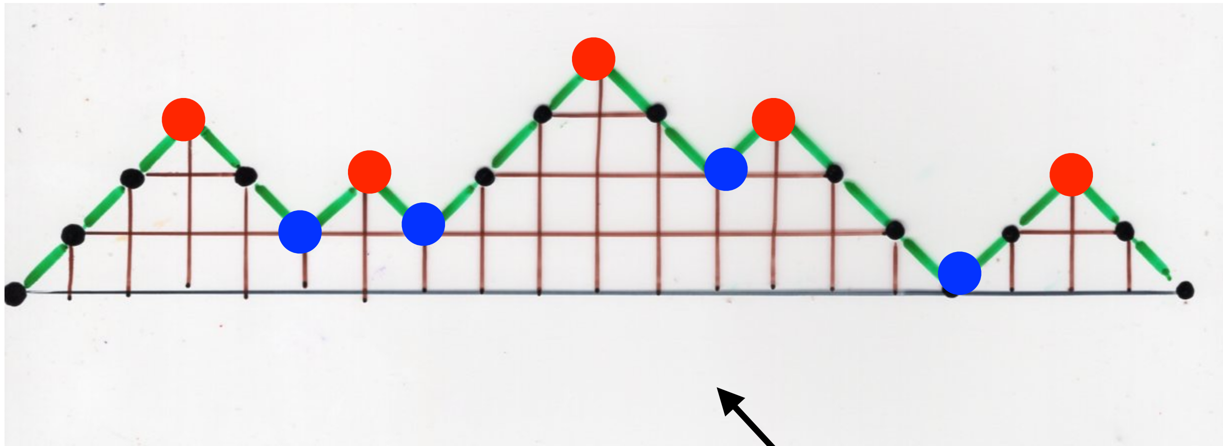


(β)- distribution $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$







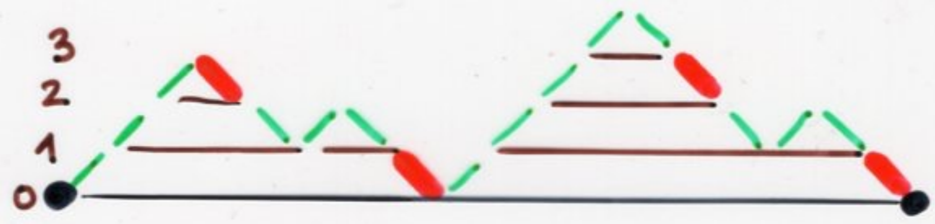


moments

number of Dyck paths having k peaks

ω, $|\omega| = 2n$

$$\mu_{2n}(\beta) = \sum_{1 \leq k \leq n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$



● Dyck path
number of SE steps
from odd to even level

Proposition

$$\lambda_k = 1 \quad k \text{ even}$$
$$= \beta \quad k \text{ odd}$$