Course IMSc, Chennai, India

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Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 1
Paths and moments

Ch 1c

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

IMSc, Chennai
January 21, 2019

mirror website
www.imsc.res.in/~viennot

Reminding Ch 1b

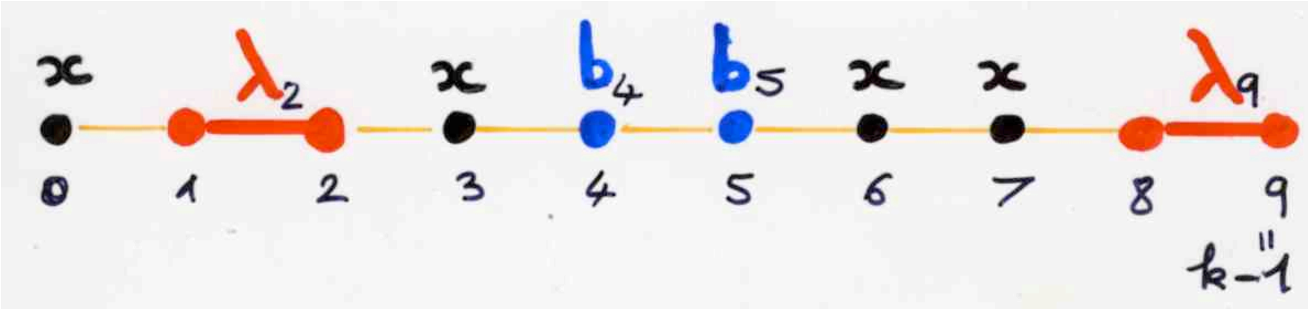
$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exist $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k \geq 1$

$$P_n(x) = \sum_{\substack{\alpha \\ \text{pavage of } [0,n-1]}} (-1)^{|\alpha|} v(\alpha)\, x^{ip(\alpha)}$$



$$v(\alpha) = b_4 b_5 \lambda_2 \lambda_9 \qquad (-1)^4 b_4 b_5 \lambda_2 \lambda_9 x^4$$

$ip(\alpha)$ = number of isolated points of α

$|\alpha|$ = number of pieces of the pavage α (monomers − dimers)

$$P_n(x) = \sum_{\substack{\alpha \\ \text{pavage of } [0, n-1]}} (-1)^{|\alpha|} v(\alpha) \, x^{ip(\alpha)}$$

(formal) Favard's Theorem

3-terms linear recurrence relation

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

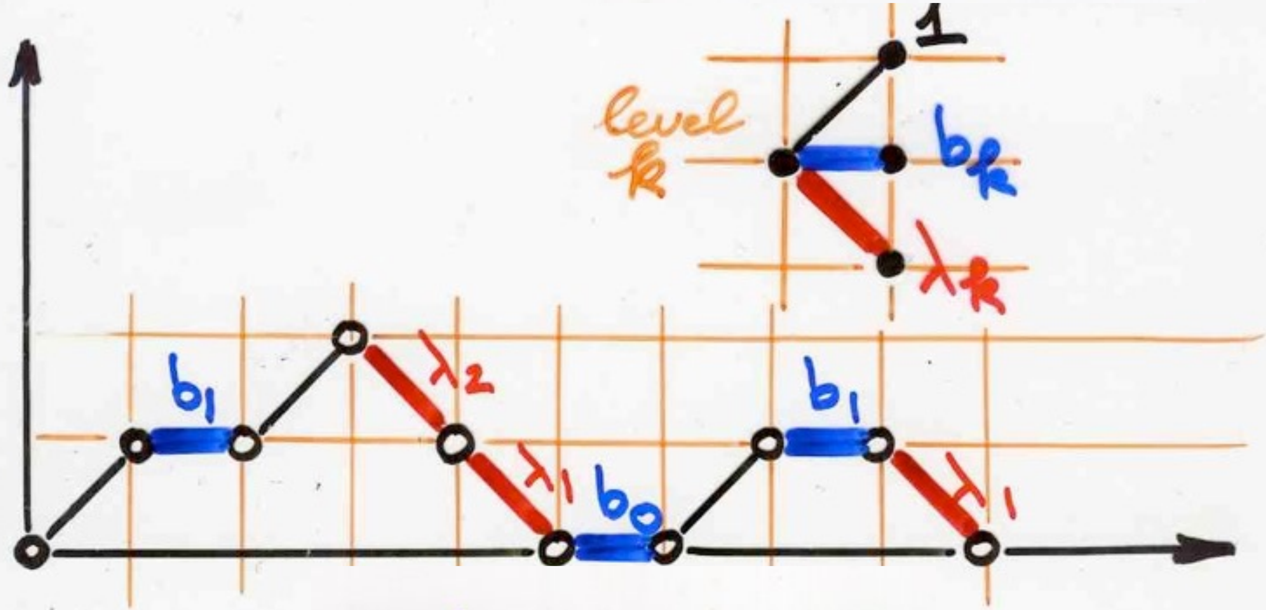
for every $k \geq 1$

$\Rightarrow$ orthogonality

$f(x^n) = \mu_n$ moments μ_n ?

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$ $b_k, \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

$$P_{k+1}(x) = (x - b_k)\,P_k(x) - \lambda_k\,P_{k-1}(x)$$
for every $k \geq 1$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

$$f(x^n) = \mu_n$$

Motzkin path
$|\omega| = n$

length

combinatorial proof

3-terms recurrence relation
implies orthogonaliy

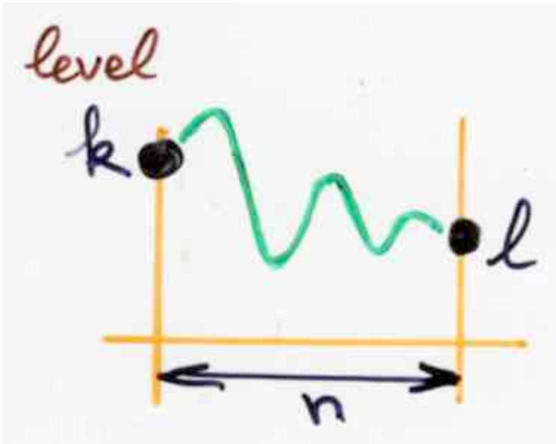
The main theorem

(main) **Theorem**

$$f(P_k P_l x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_l$$

ω "Motzkin path"

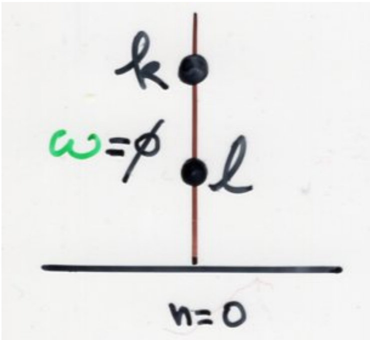
$|\omega| = n$ level $k \rightsquigarrow l$

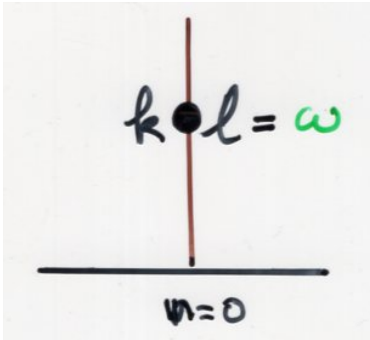


Corollary $\Rightarrow$ orthogonality

$n=0$

$$f(P_k P_l) = 0 \quad k \neq l$$
$$= \lambda_1 \cdots \lambda_l \quad k = l$$



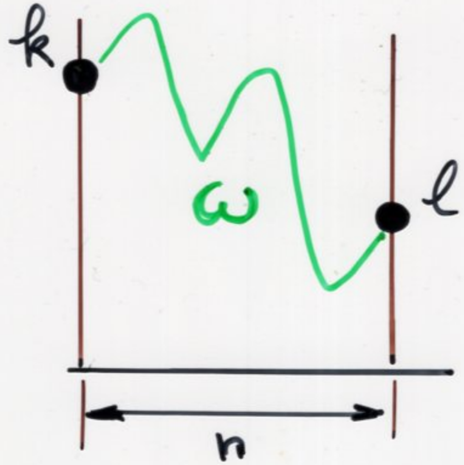


(main) **Theorem**

$$f(P_k P_\ell x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_\ell$$

"Motzkin path"

$|\omega| = n$ level $k \rightsquigarrow \ell$

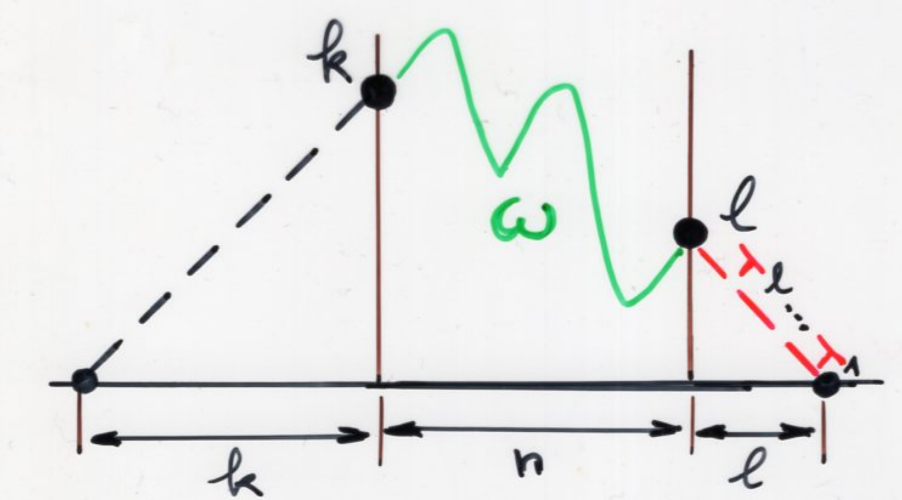


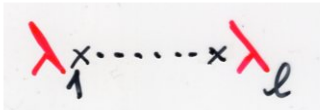
another formulation of the main **Theorem**

$$\mathfrak{f}(P_k P_l x^n) = \sum_{\substack{\omega \\ \text{Motzkin path level } 0 \to 0 \\ |\omega| = k+n+l}} v(\omega)$$

(i) first k steps are NE

(ii) last l steps are SE



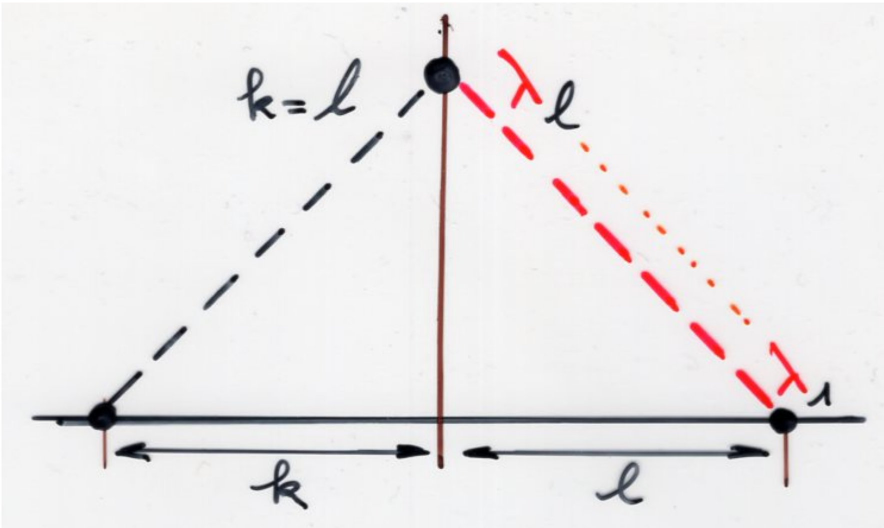


Corollary

$\Rightarrow$ orthogonality

$n=0$

$$f(P_k P_l) = 0 \quad k \neq l$$
$$= \lambda_1 \cdots \lambda_l \quad k = l$$



The « essence » of the fundamental sign-reversing involutions

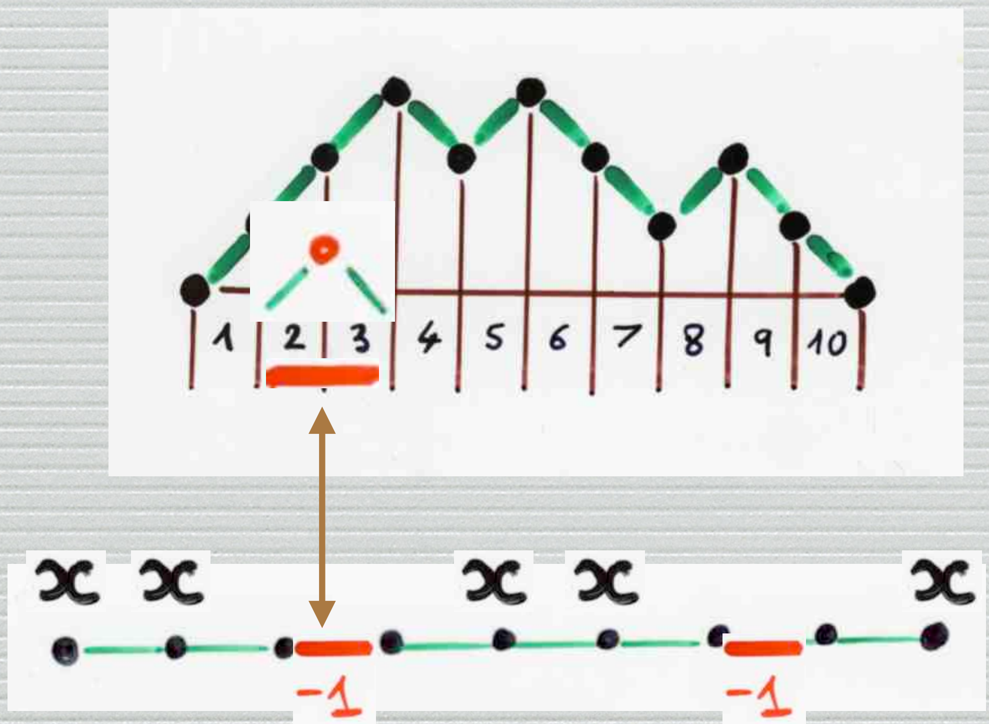
moments f (Tchebychev) 2nd kind

$f(x^n) = \mu_n$ moments

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$



$S_n(x)$

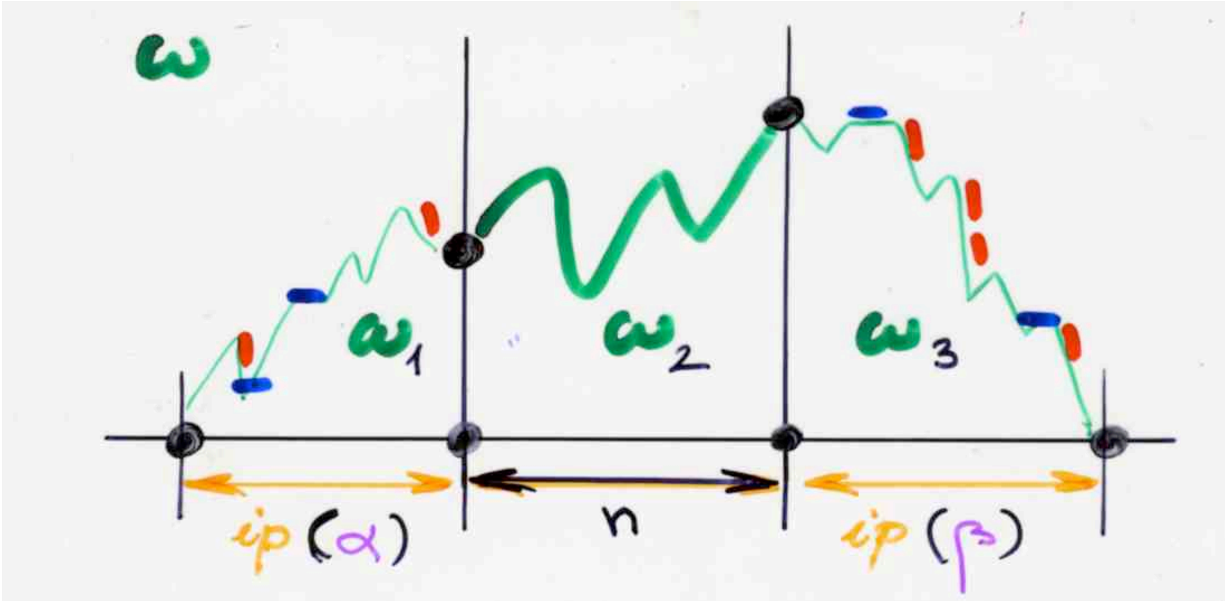
bijective proof

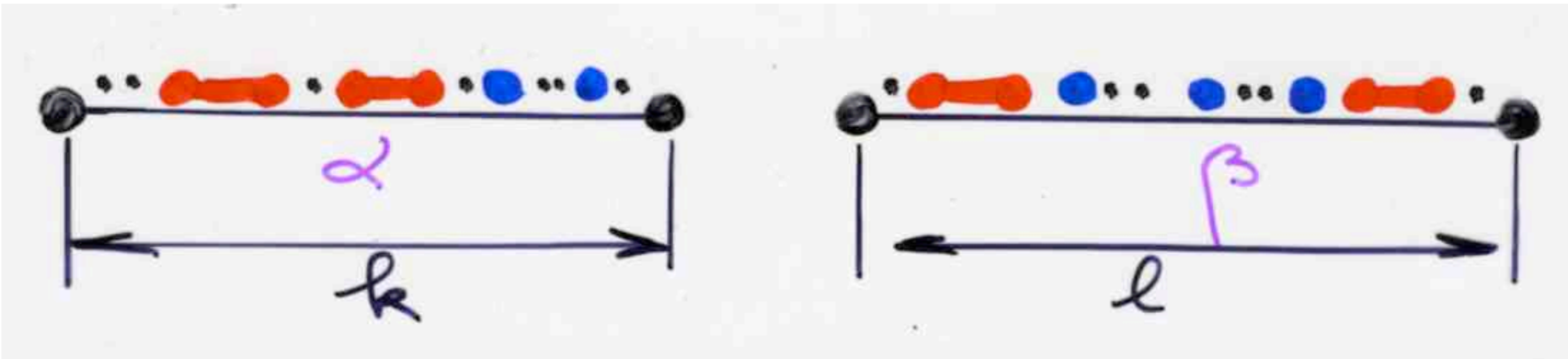
$$f(P_k P_l x^n) = \sum_{\alpha, \beta, \omega} (-1)^{|\alpha|+|\beta|} v(\alpha) v(\beta) v(\omega)$$

α pavage of $[0, k-1]$
β pavage of $[0, l-1]$
ω Motzkin path (level $0 \rightsquigarrow 0$)

$$|\omega| = ip(\alpha) + ip(\beta) + n$$

$$(\alpha, \beta, \omega) \in E_{n,k,l}$$

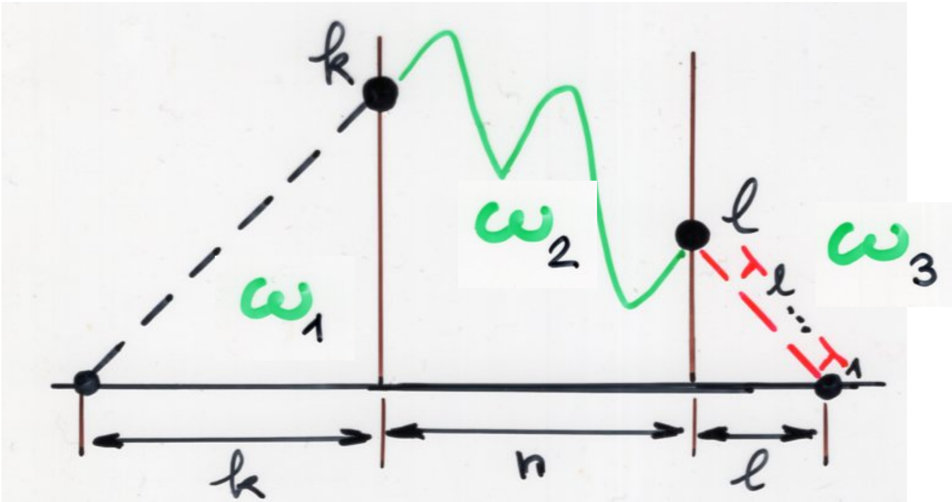


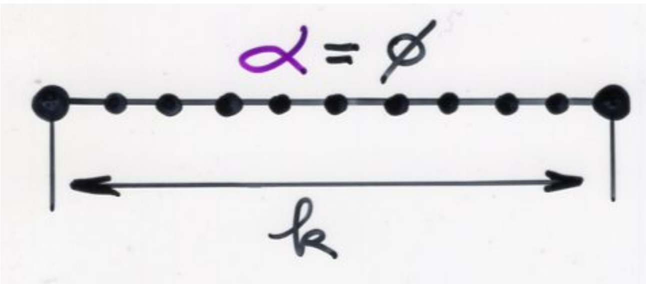


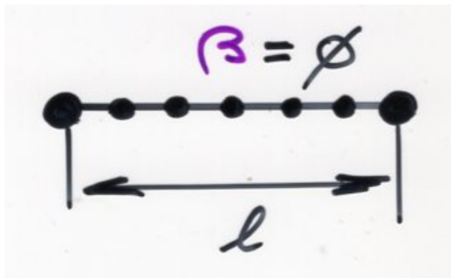
$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$

$F_{n,k,\ell} \subseteq E_{n,k,\ell}$

- α, β empty
- $\omega_1 =$ ($|\omega_1| = k$)
- $\omega_3 =$ ($|\omega_3| = \ell$)







$$\mathbf{F}_{n,k,\ell} \subseteq \mathbf{E}_{n,k,\ell} \begin{cases} - \ \alpha, \beta \ \text{empty} \\ - \ \omega_1 = \\ - \ \omega_3 = \end{cases}$$

$\omega_1 =$  $(|\omega_1| = k)$

$\omega_3 =$  $(|\omega_3| = \ell)$

construction of an **involution** Θ

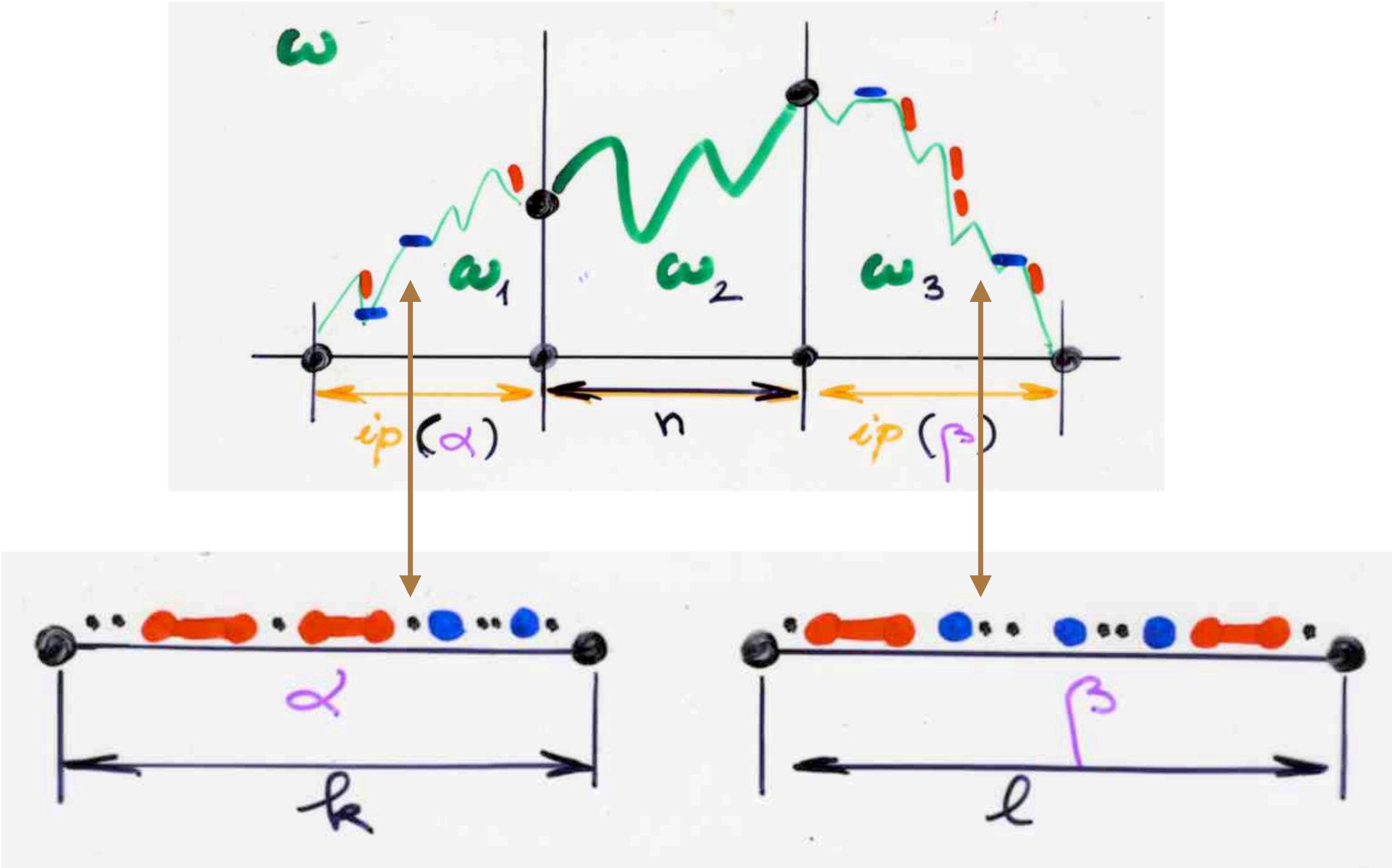
$$\mathbf{E}_{n,k,\ell} \setminus \mathbf{F}_{n,k,\ell} \longrightarrow \mathbf{E}_{n,k,\ell} \setminus \mathbf{F}_{n,k,\ell}$$

$$(\alpha, \beta, \omega) \xrightarrow{\Theta} (\alpha', \beta', \omega')$$

weight-preserving

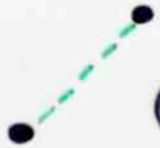
$$\begin{cases} v(\alpha')v(\beta')v(\omega') = v(\alpha)v(\beta)v(\omega) \\ (-1)^{|\alpha'| + |\beta'|} = -(-1)^{|\alpha| + |\beta|} \end{cases}$$

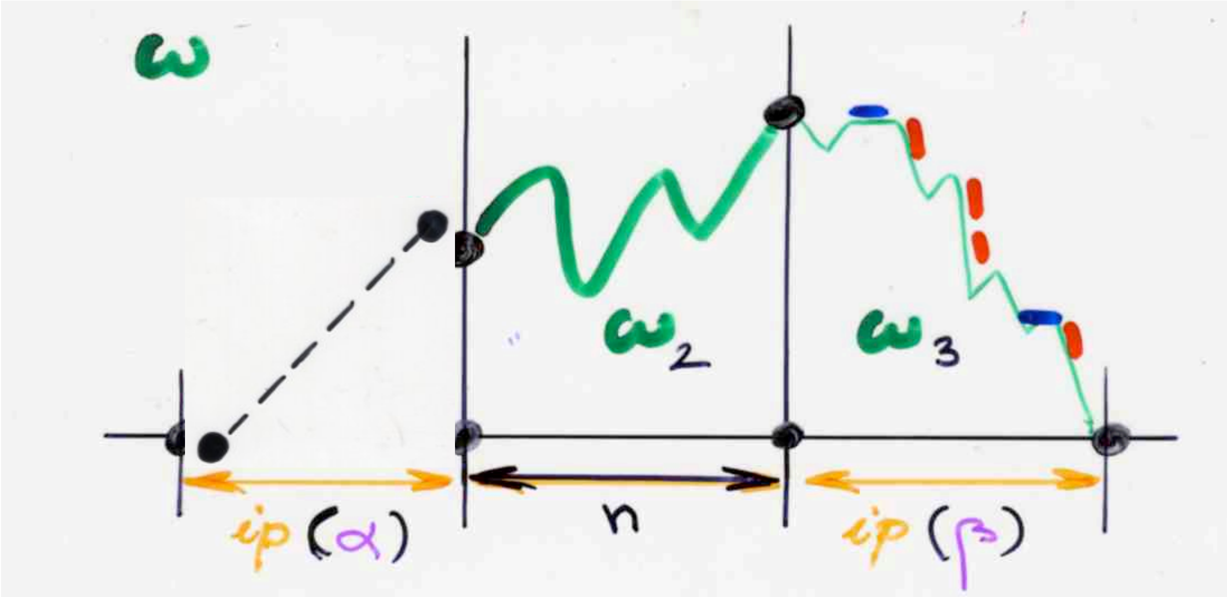
sign-reversing

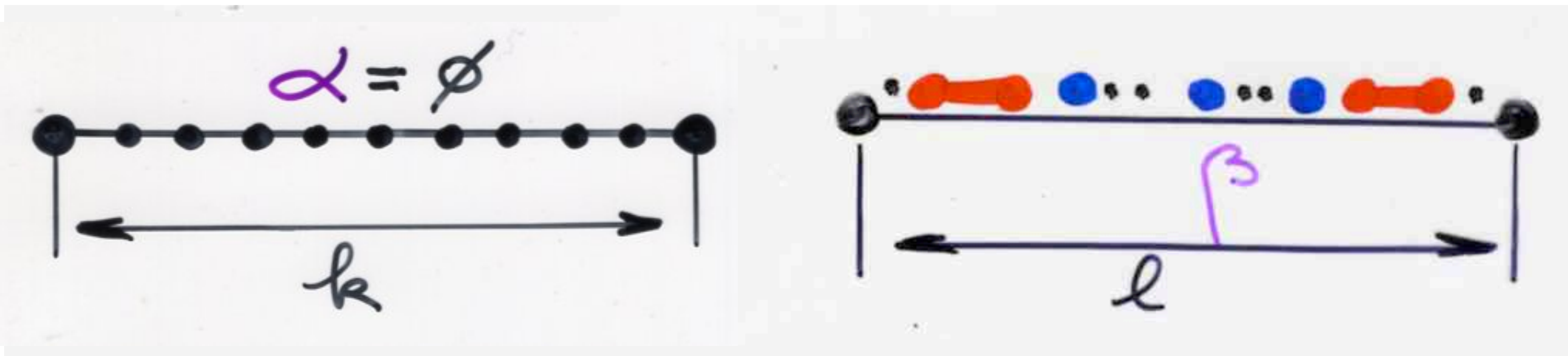


$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$

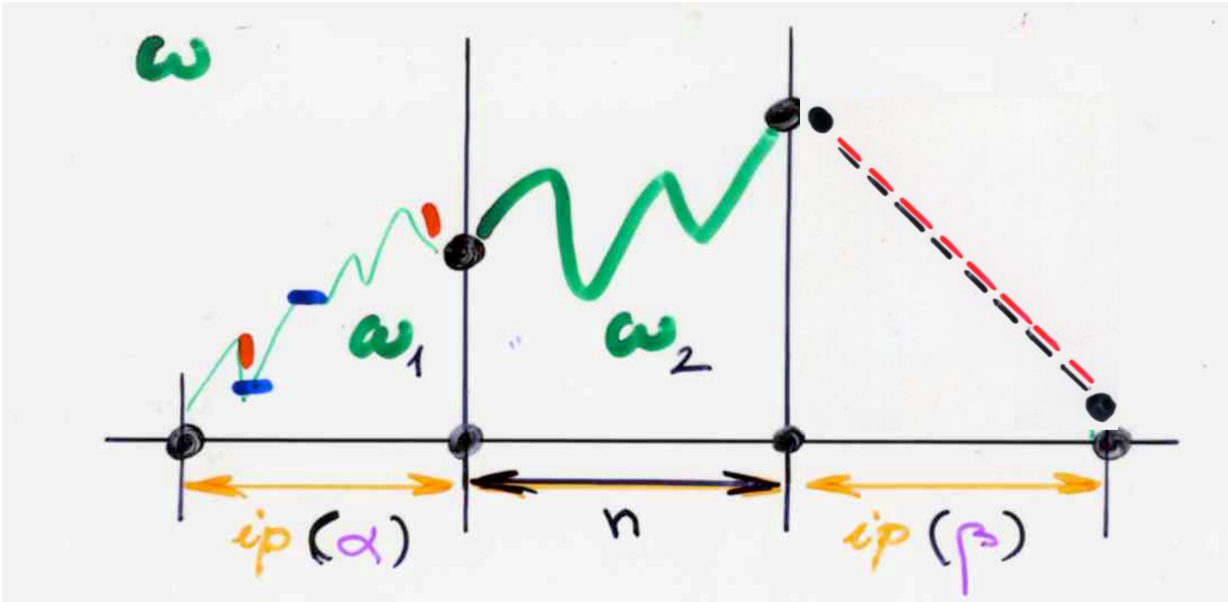
$\mathbf{L}_{n,k,\ell} \subseteq \mathbf{E}_{n,k,\ell}$

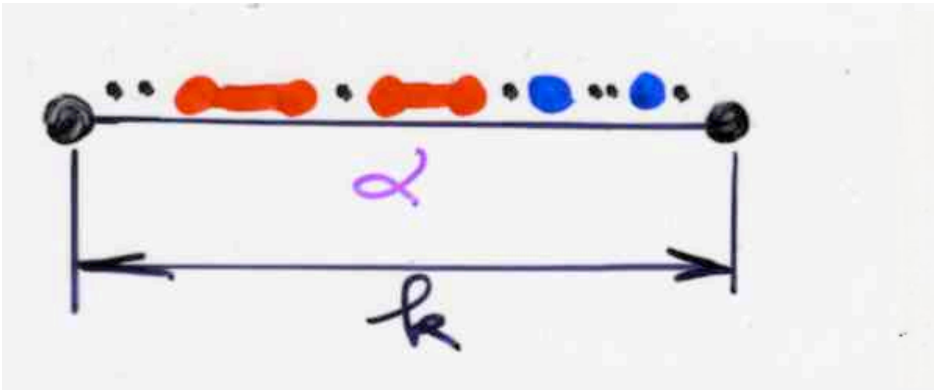
- α empty
- $\omega_1 =$  $(|\omega_1| = k)$

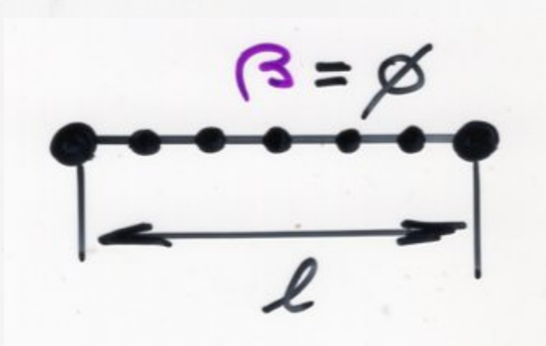




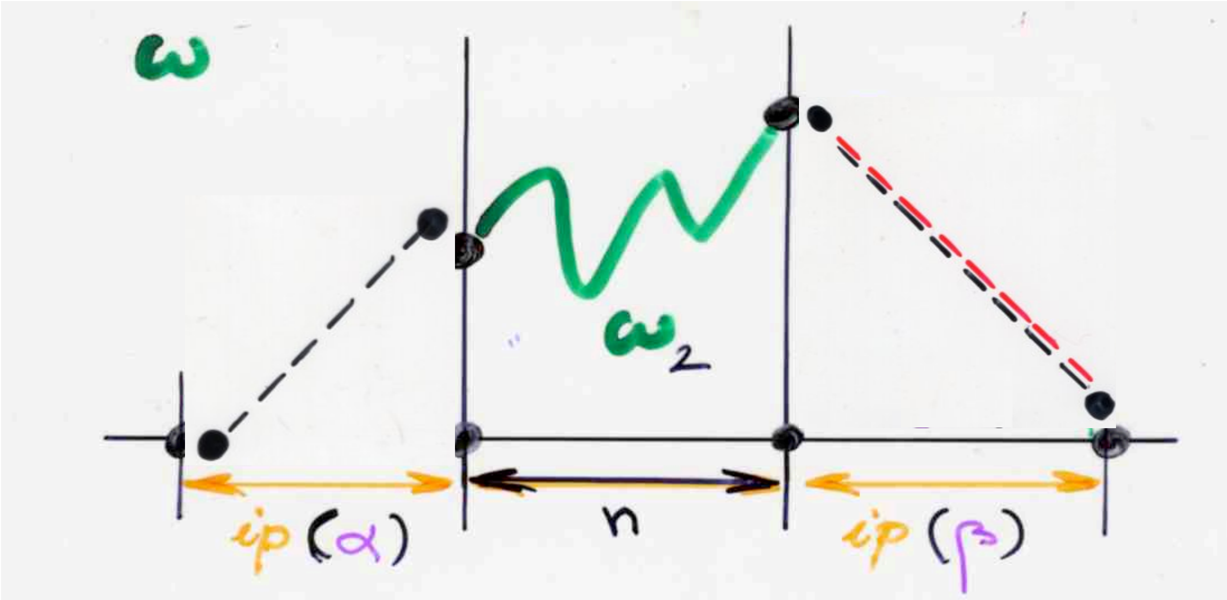
$$R_{n,k,\ell} \subseteq E_{n,k,\ell} \quad \begin{cases} - \beta \text{ empty} \\ - \omega_2 = \searrow \quad (|\omega_2| = \ell) \end{cases}$$

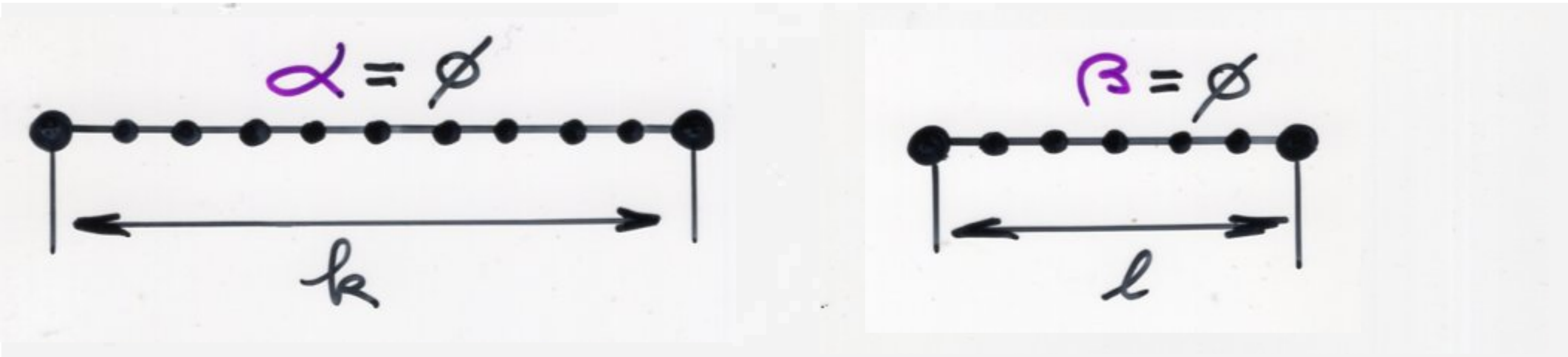


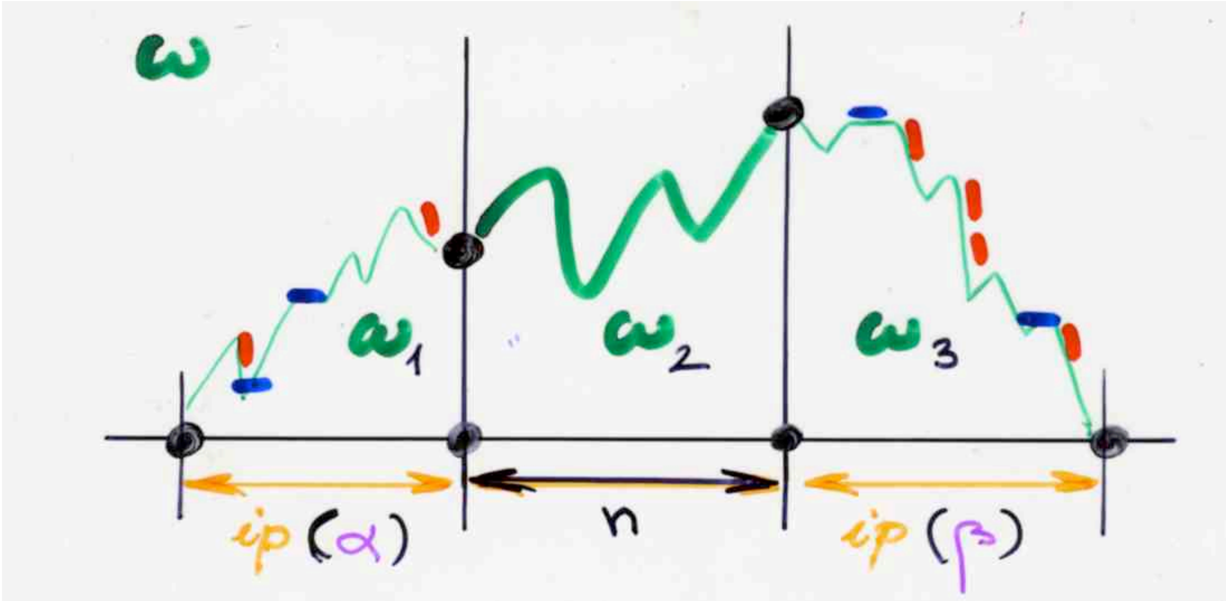


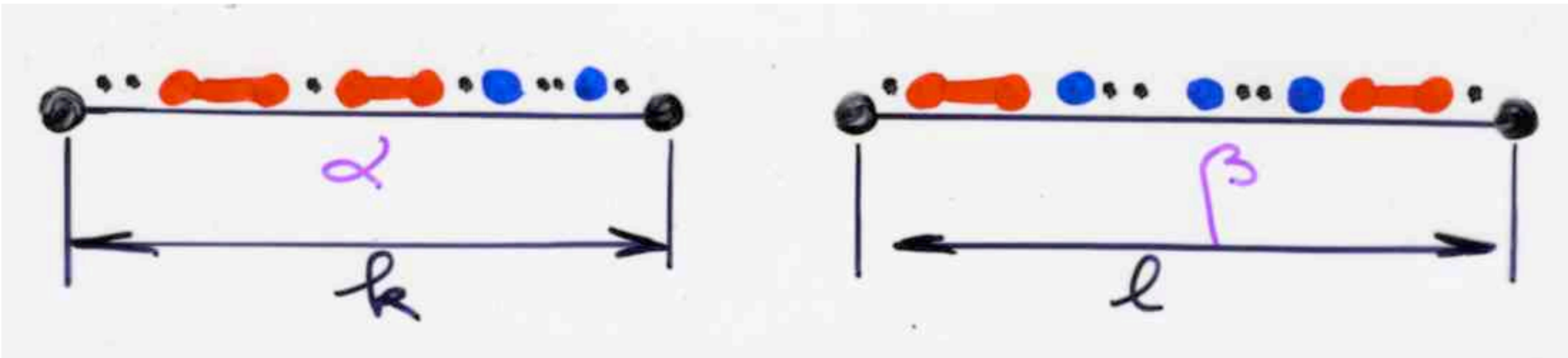


$$F_{n,k,l} = L_{n,k,l} \cap R_{n,k,l}$$

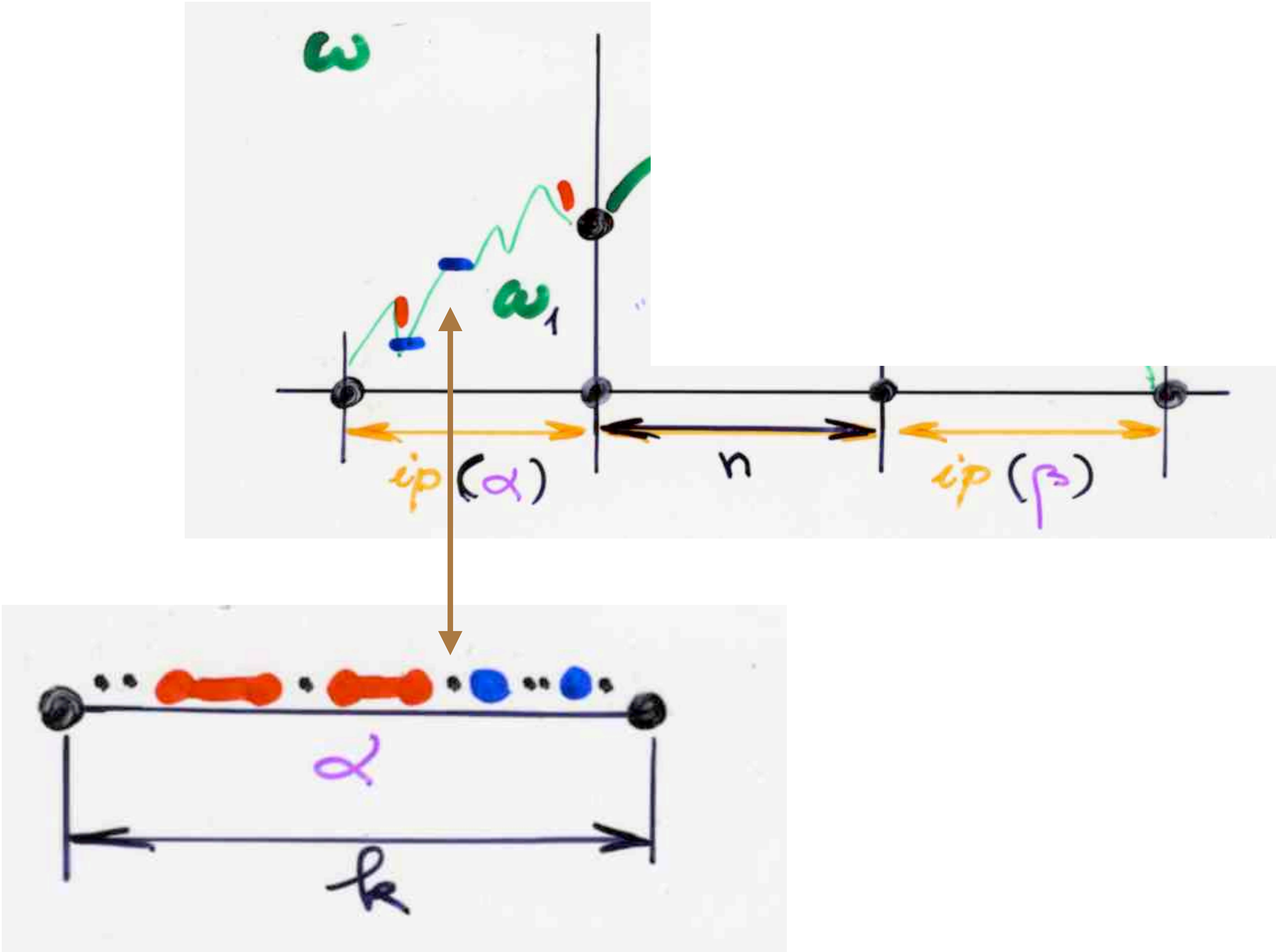








$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$



$$(\omega, \alpha, \beta) \in E_{n,k,\ell} \setminus L_{n,k,\ell}$$

$(\omega, \alpha, \beta) \in E_{n,k,\ell} \setminus L_{n,k,\ell}$

$h(\alpha)$ = smallest index i of $[0, k-1]$ "occupied" by a monomer or a dimer, if α is empty, the $h(\alpha) = \infty$

$h(\omega)$ = level (level of the starting point) of the first elementary step of ω_1 which is East or South-East

if no E or SE step in ω_1, then $h(\omega) = \infty$

this means, with $j = h(\omega)$

$h(\omega)$ and $h(\alpha)$ cannot be both ∞

thus we have 2 cases

$$\begin{cases} \text{(i)} & h(\alpha) \leq h(\omega) \\ \text{(ii)} & h(\alpha) > h(\omega) \end{cases}$$

first involution θ_1 on $E_{n,k,\ell} \setminus L_{n,k,\ell}$

(i) $h(\alpha) \leq h(\omega)$

delete from the pavage α
the left-most piece

i.e. monomer (i)
or dimer $(i, i+1)$

if $i = h(\alpha) \geq 0$

incorporate  resp. 
in the path ω_1

as a $(i+1)$ step resp. $(i+1, i+2)$ steps

equivalently: the level of first vertex
of  resp.  is i

$$(\alpha, \omega) \xrightarrow{\theta_1} (\alpha', \omega')$$

$$h(\omega') = h(\alpha)$$
$$h(\alpha') > h(\alpha)$$

we are in cas (ii)

the weight is preserved:

$$v(\alpha)v(\omega) = v(\alpha')v(\omega')$$

sign-reversing

(ii) $h(\alpha) > h(\omega)$

delete from the path ω_1
the $(i+1)^{th}$ step
resp. $(i, i+1)^{th}$ steps

and add the monomer (i)
resp. dimer $(i-1, i)$

to α

$(\alpha, \omega) \xrightarrow{\theta_1} (\alpha', \omega')$

$h(\omega') = h(\alpha) + 1$
$h(\alpha') > h(\alpha) + 1$

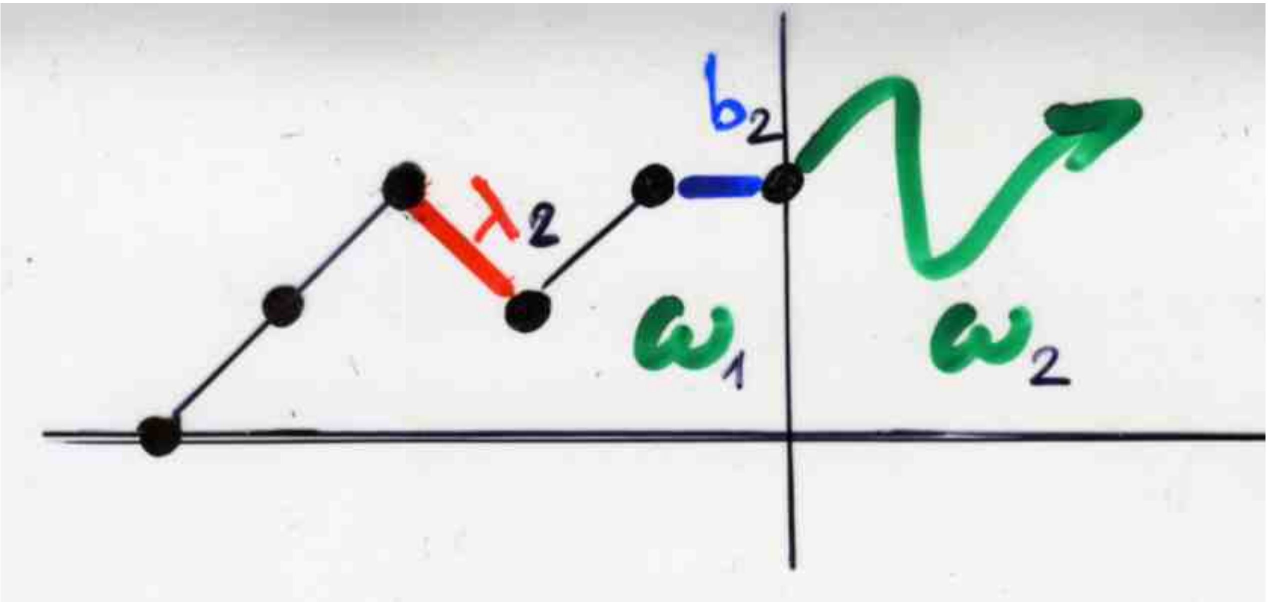
we are in cas (i)

the weight is preserved:
$v(\alpha)v(\omega) = v(\alpha')v(\omega')$

sign-reversing

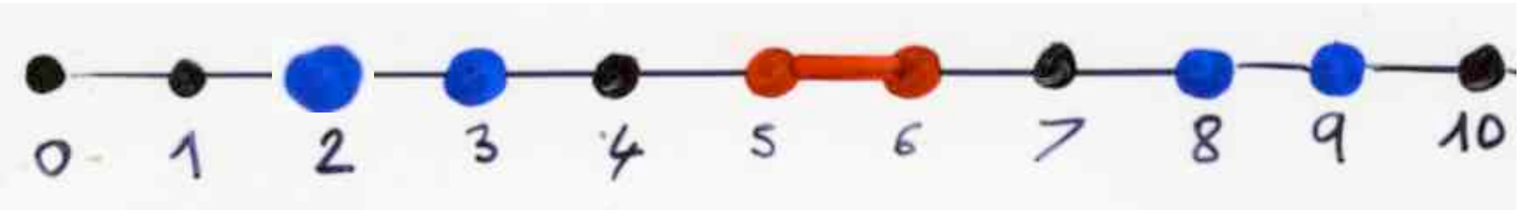
(i) $h(\alpha) \leq h(\omega)$

$h(\alpha) = h(\omega) = 2$



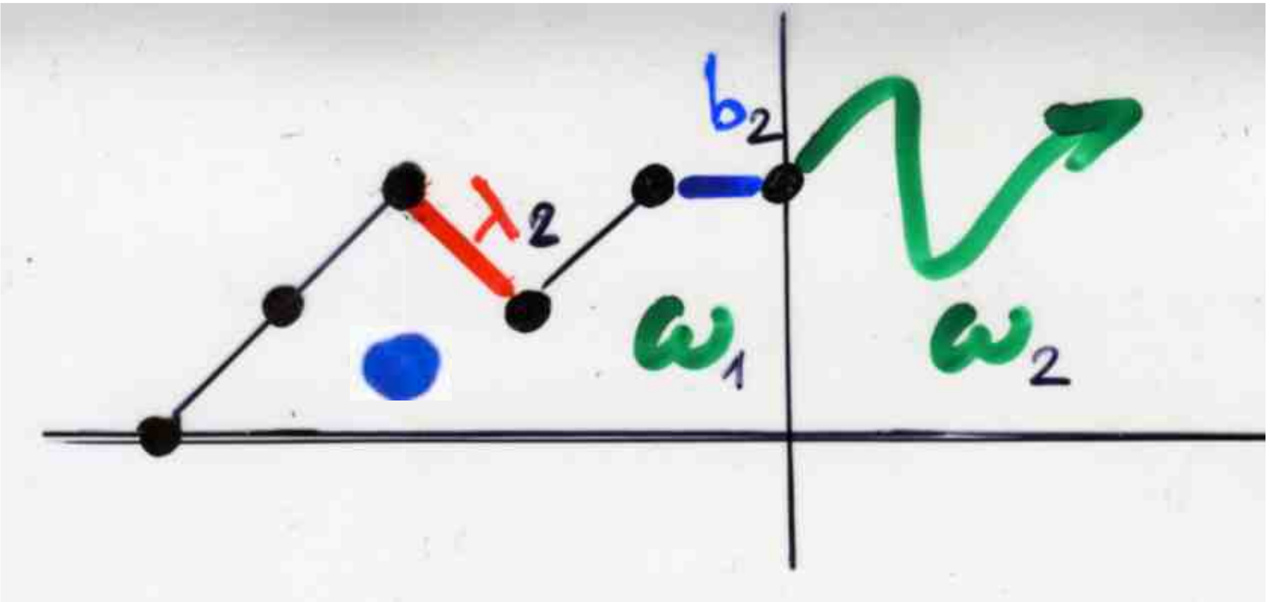
delete from the pavage α
the left-most piece

monomer (i)



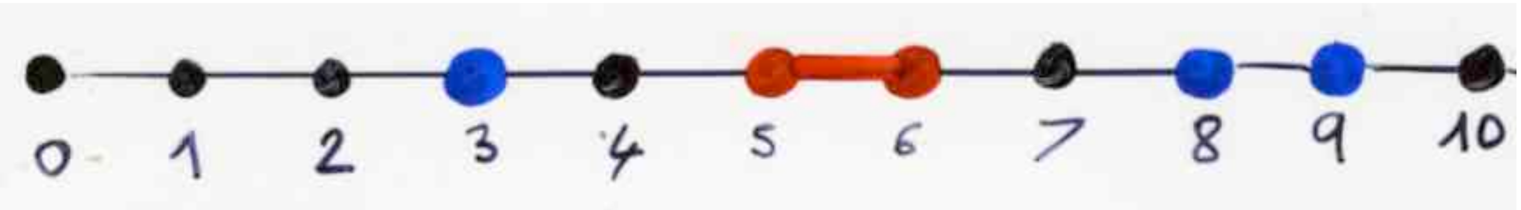
(i) $h(\alpha) \leq h(\omega)$

$h(\alpha) = h(\omega) = 2$



add ——
in the path ω_1
as a $(i+1)^{th}$ step

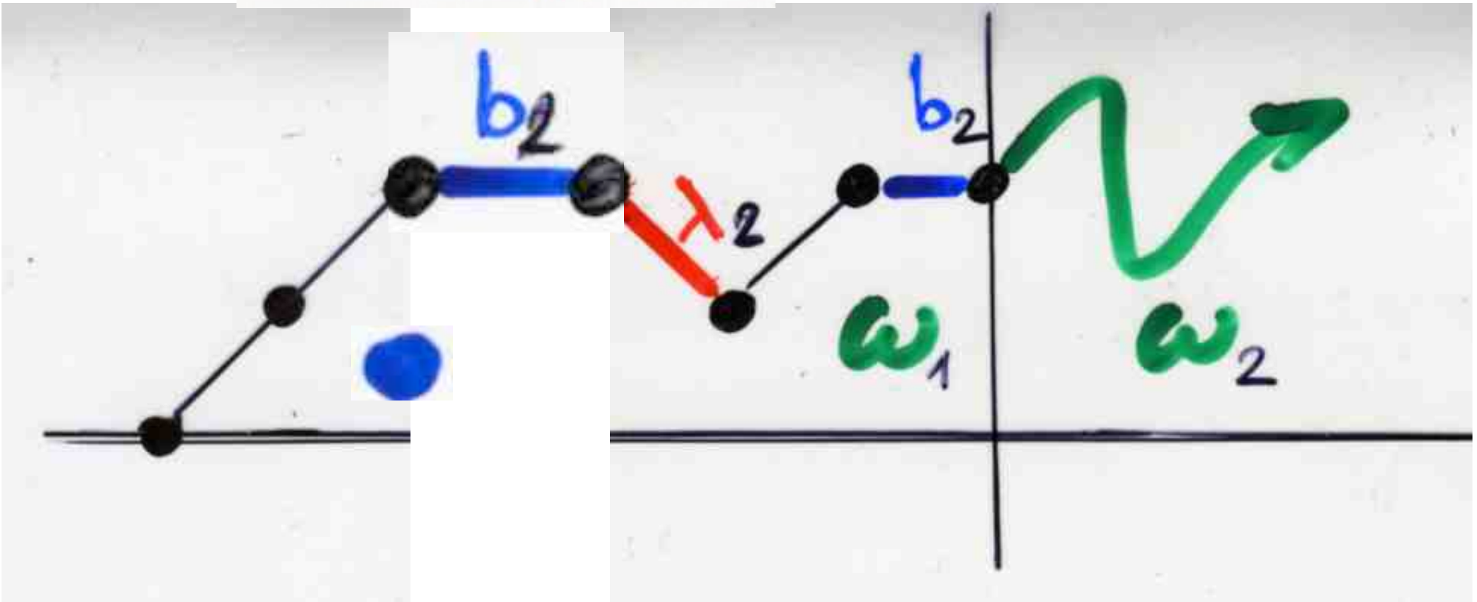
equivalently: the level of the first vertex of •—— is i

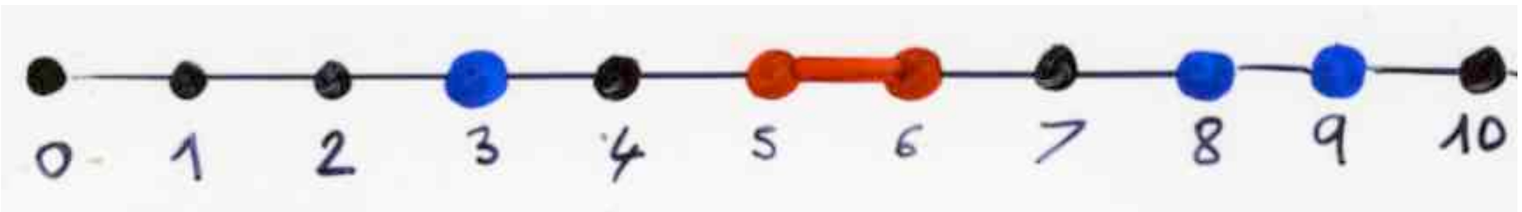


$$\begin{cases} h(\omega') = 2 \\ h(\alpha') = 3 \end{cases}$$

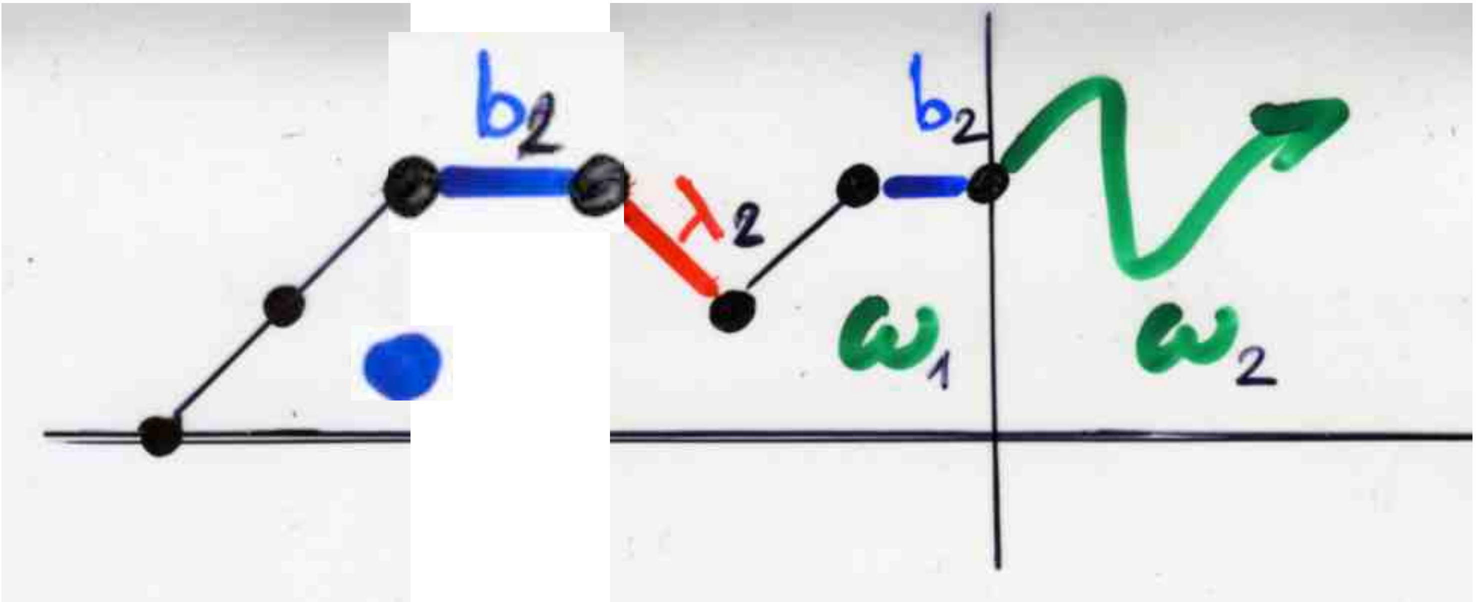
$$h(\omega') = h(\alpha) + 1$$
$$h(\alpha') > h(\alpha) + 1$$

(ii)

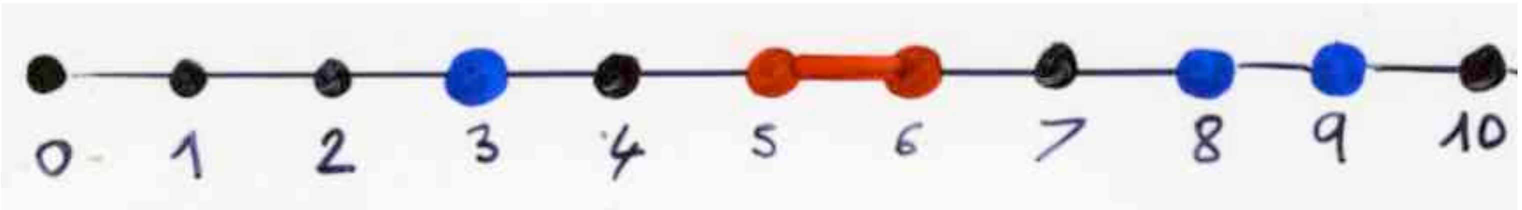




(ii) $h(\alpha) > h(\omega)$

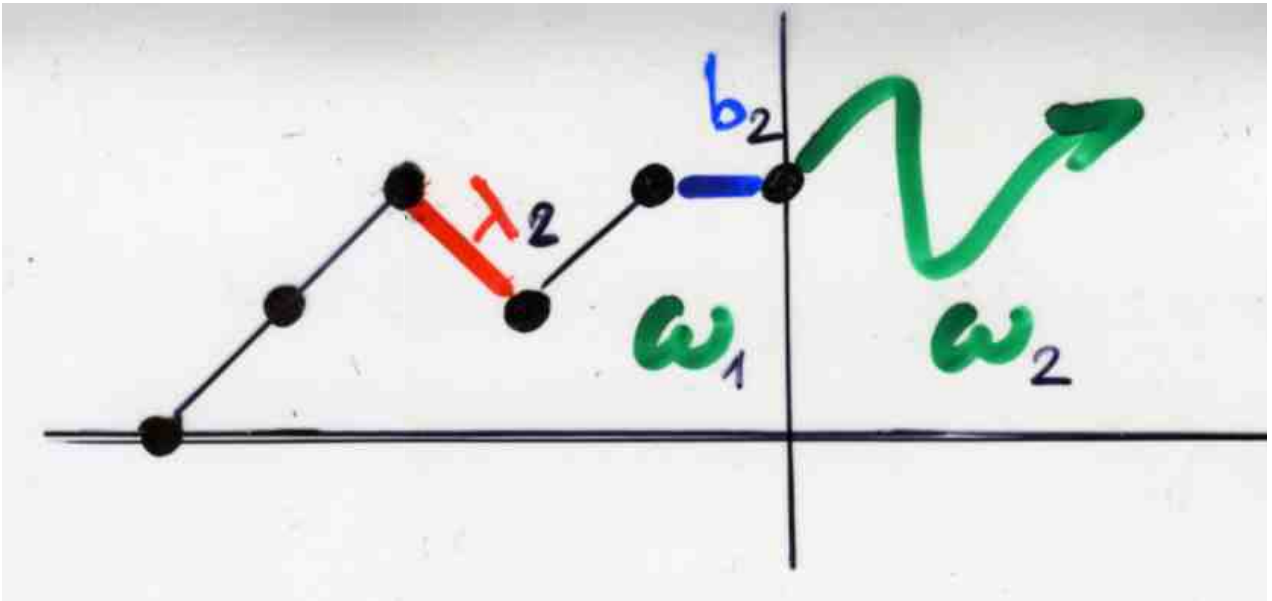


delete from the path ω_1 the $(i+1)^{th}$ step

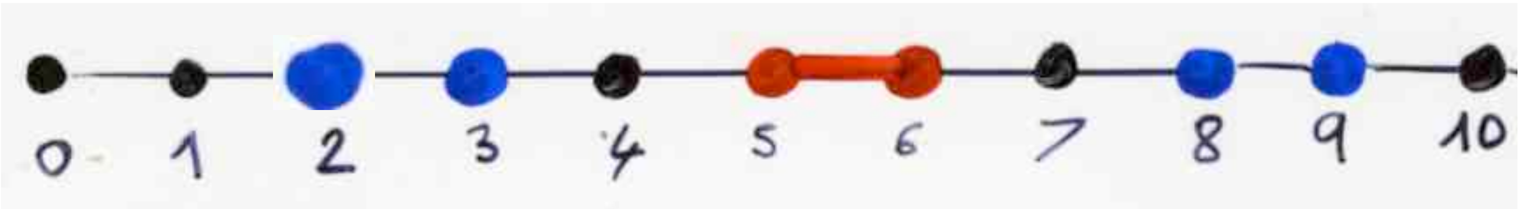


(i) $h(\alpha) \leq h(\omega)$

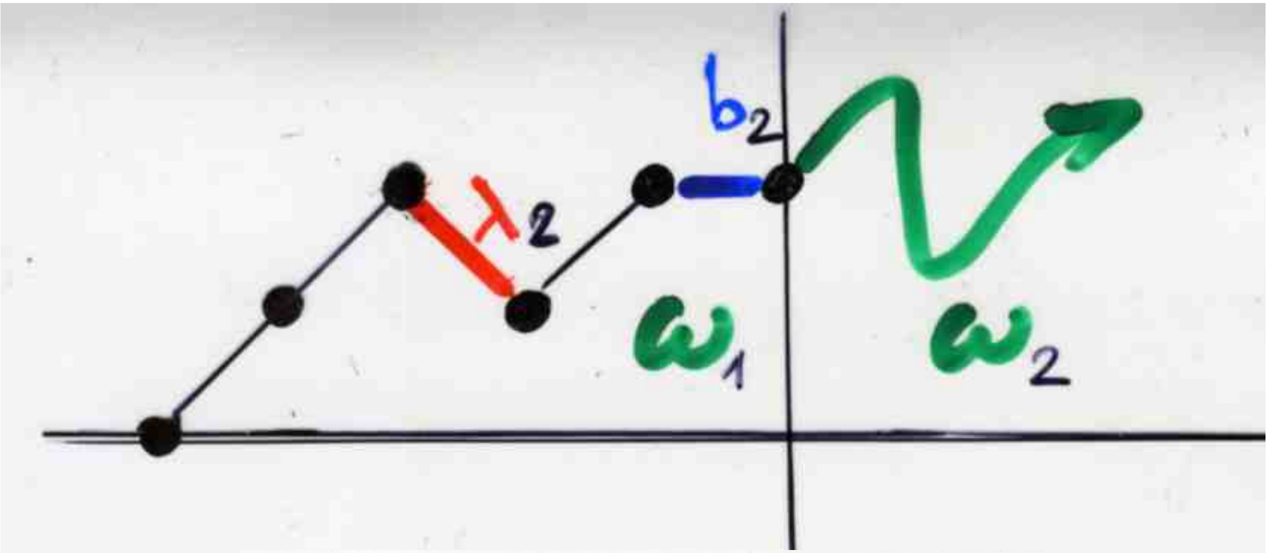
$h(\alpha) = h(\omega) = 2$



add the monomer (i) to the pavage α

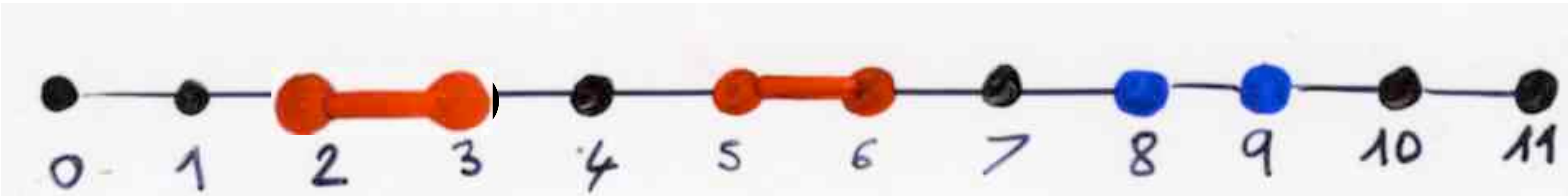


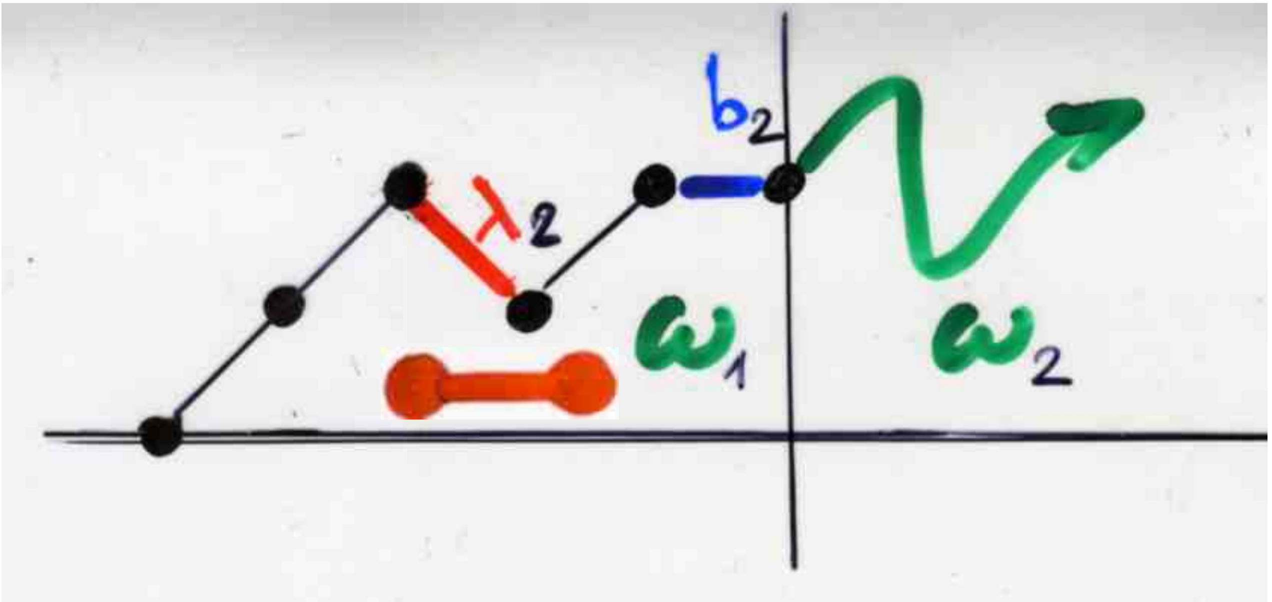
(i) $h(\alpha) \leq h(\omega)$ $h(\alpha) = h(\omega) = 2$



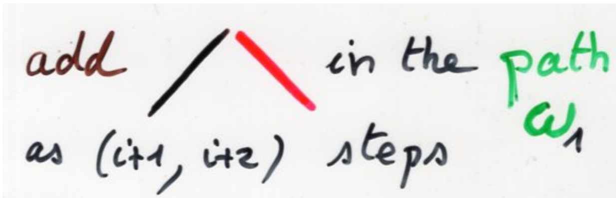
delete from the pavage α
the left-most piece

dimer $(i, i+1)$ $i = h(\alpha) \geq 0$

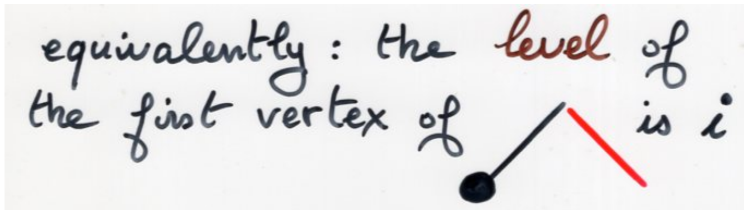




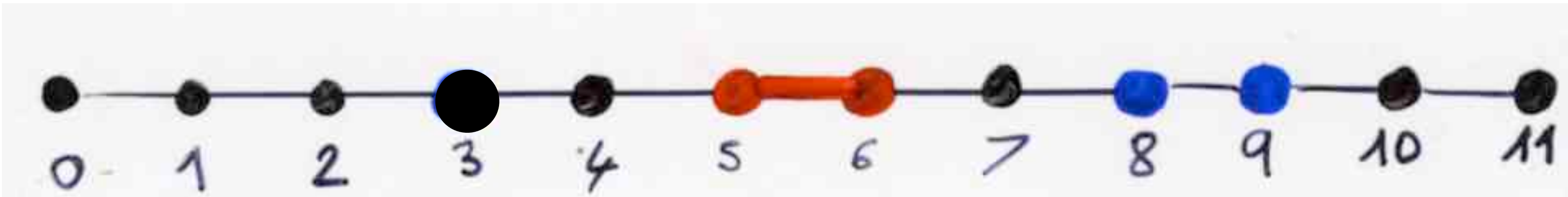
b_2
λ_2
ω_1
ω_2



add in the path ω_1
as (i+1, i+2) steps



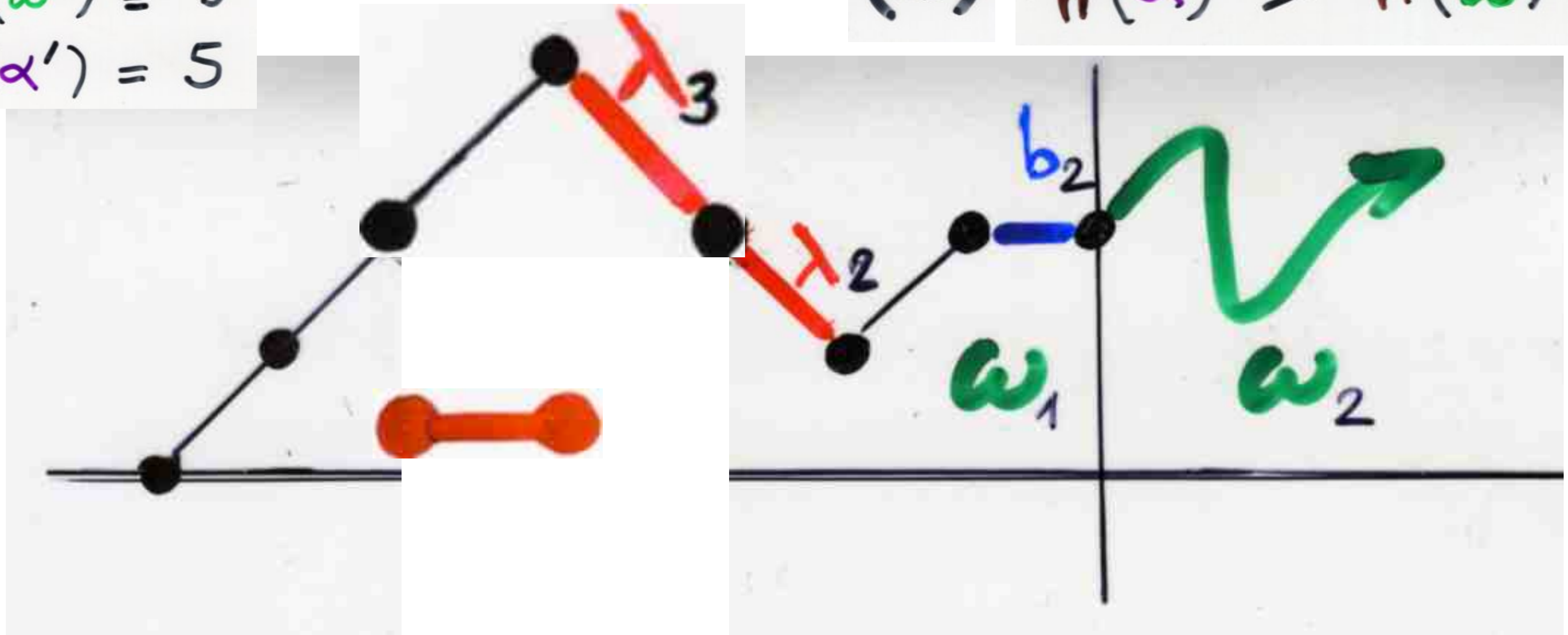
equivalently : the level of
the first vertex of is i



0 1 2 3 4 5 6 7 8 9 10 11

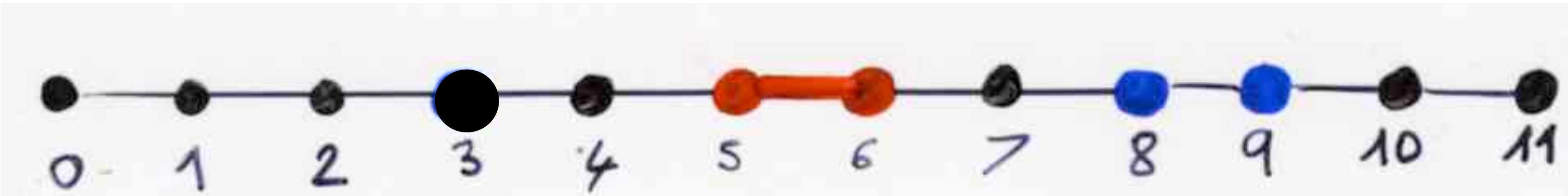
$$\begin{cases} h(\omega') = 3 \\ h(\alpha') = 5 \end{cases}$$

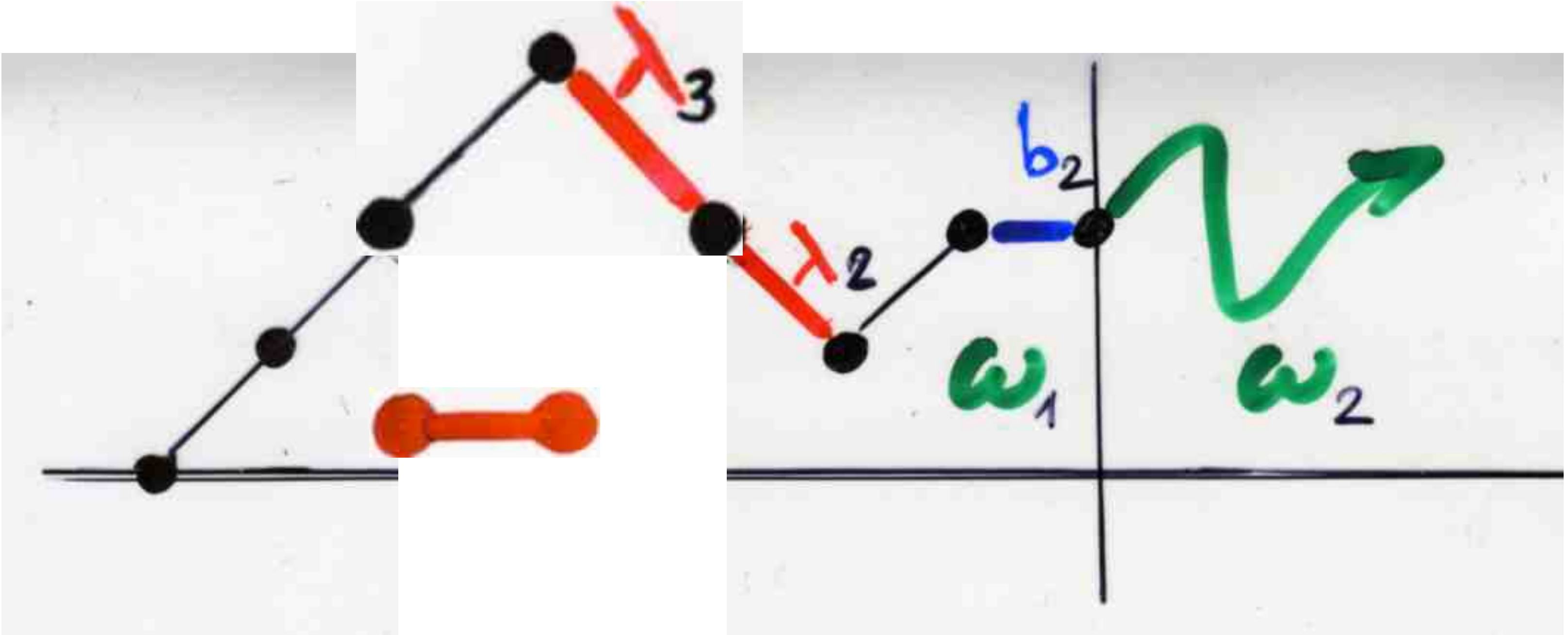
(ii) $h(\alpha) > h(\omega)$



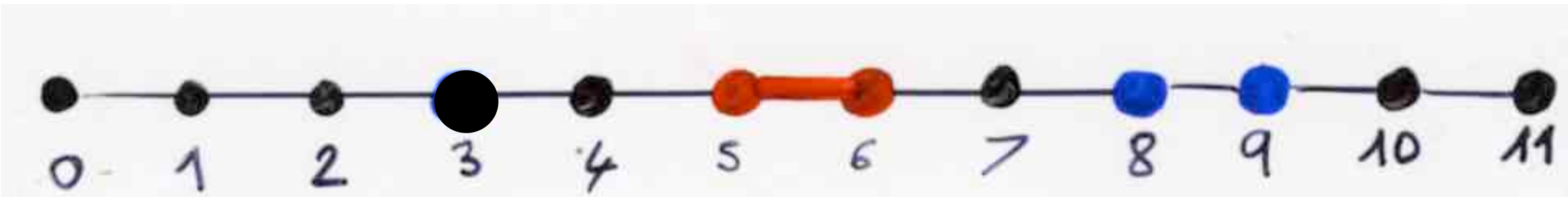
add in the path ω_1 as $(i+1, i+2)$ steps

equivalently : the level of the first vertex of is i



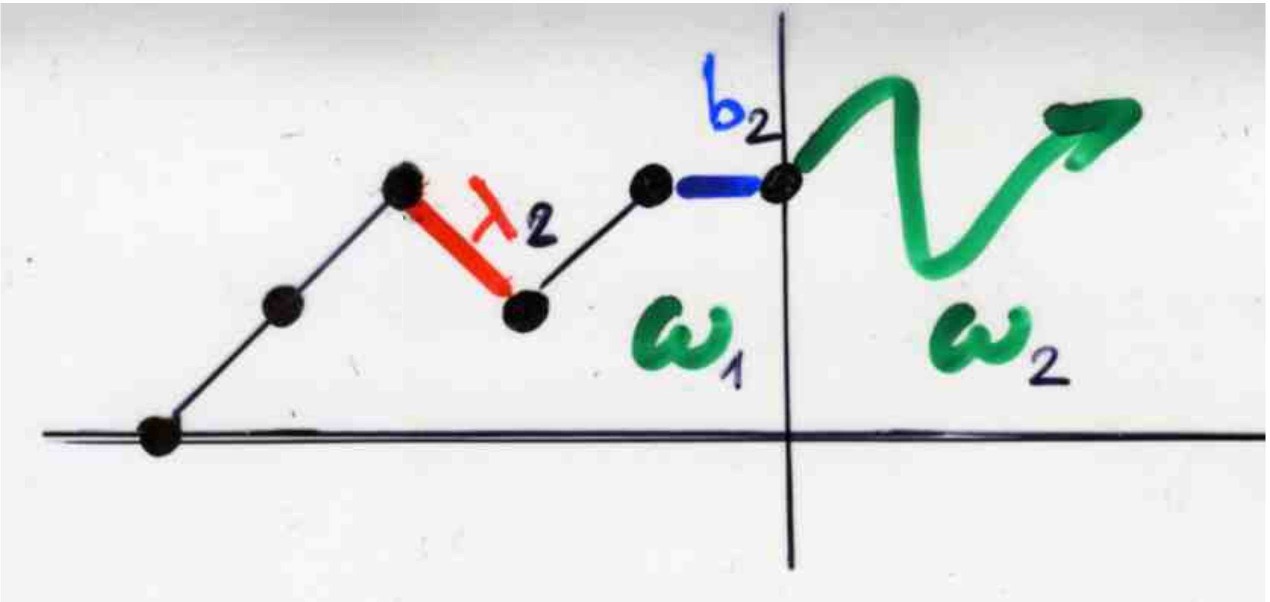


delete from the path ω_1
the $(i, i+1)$ steps

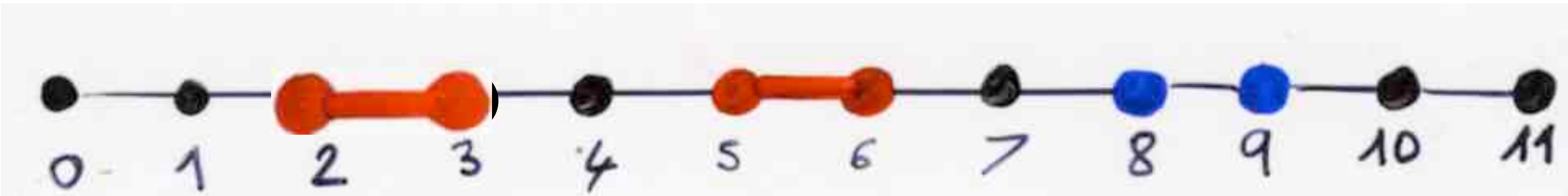


(i) $h(\alpha) \leq h(\omega)$

$h(\alpha) = h(\omega) = 2$

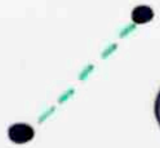


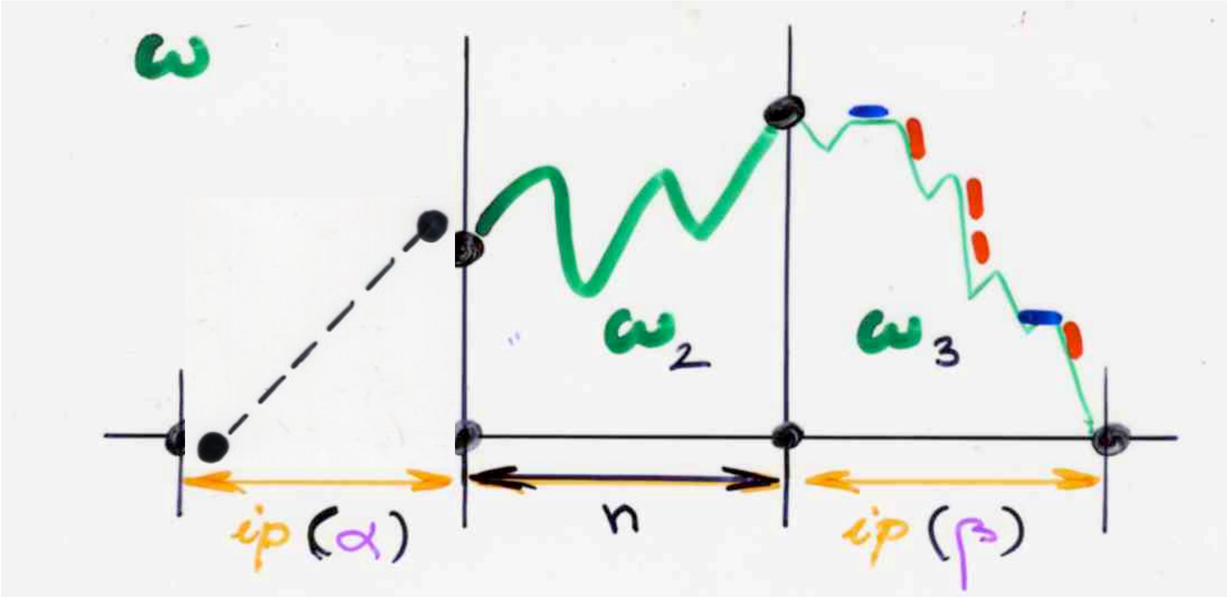
add the dimer $(i-1, i)$
to the pavage α

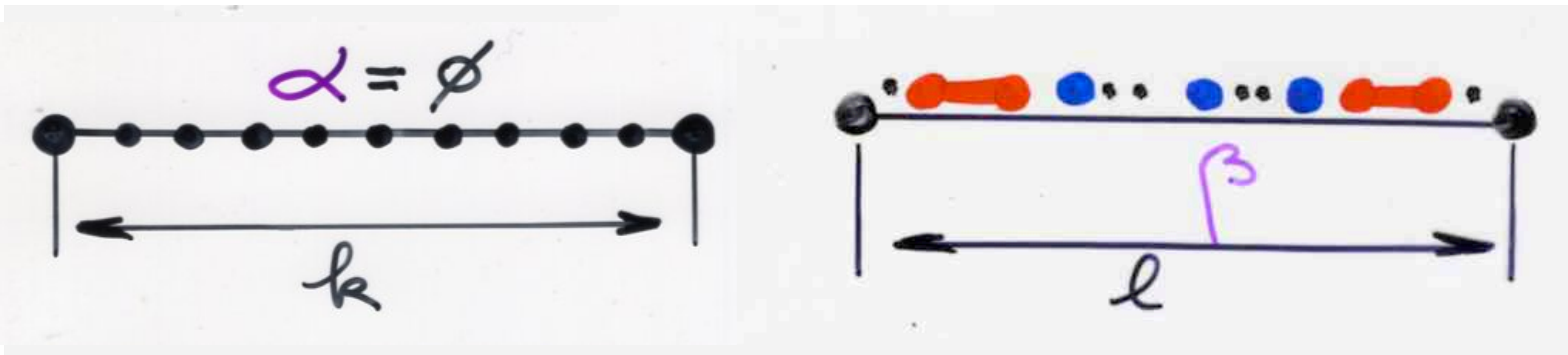


$$\sum_{(\alpha,\beta,\omega)\in E_{n,k,\ell}} (-1)^{|\alpha|+|\beta|}\, v(\alpha)\, v(\beta)\, v(\omega)$$

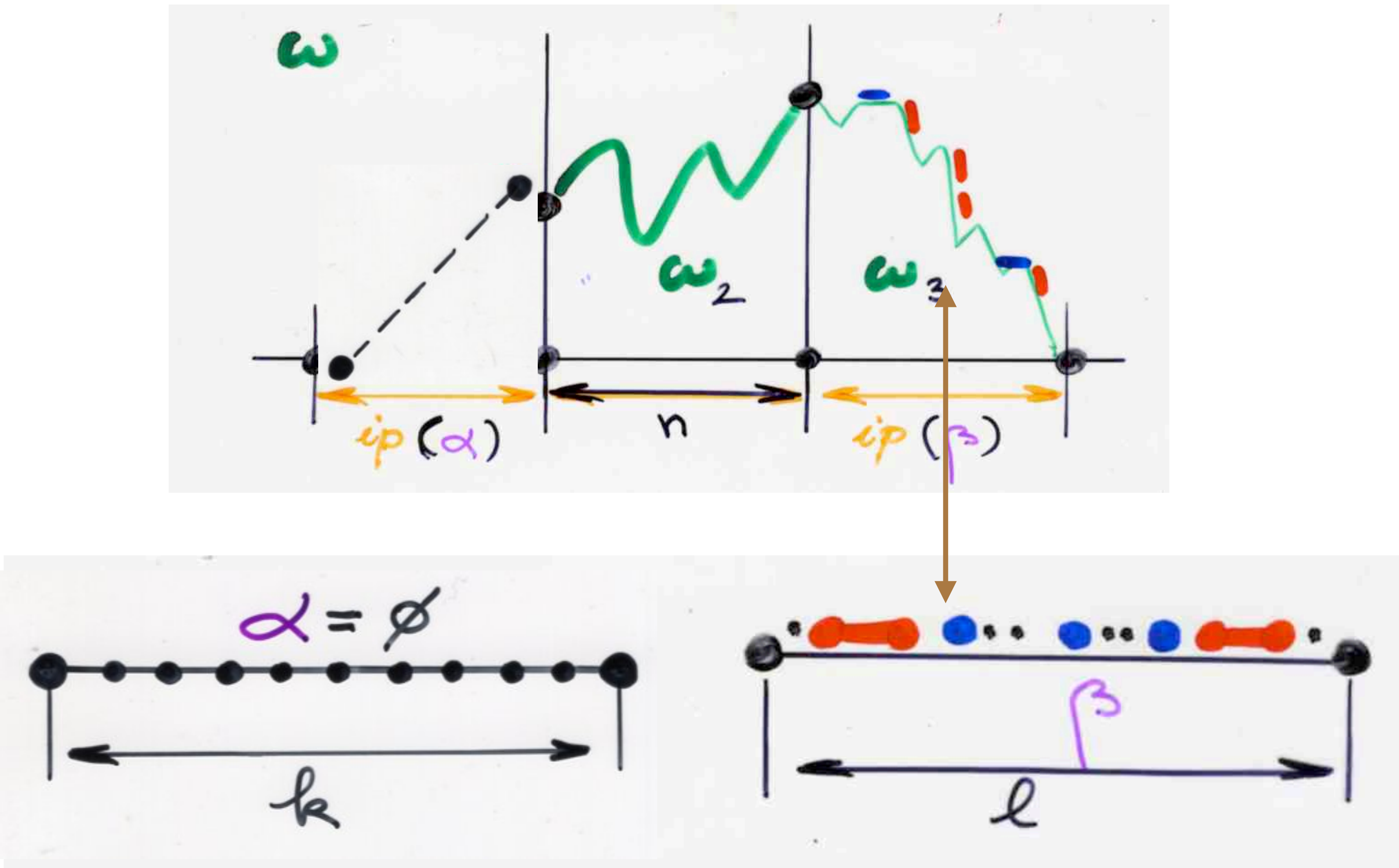
$$= \sum_{(\alpha,\beta,\omega)\in L_{n,k,\ell}} (-1)^{|\alpha|+|\beta|}\, v(\alpha)\, v(\beta)\, v(\omega)$$

$L_{n,k,\ell} \subseteq E_{n,k,\ell}$ $\begin{cases} - \ \alpha \text{ empty} \\ - \ \omega_1 = \end{cases}$  $(|\omega_1| = k)$

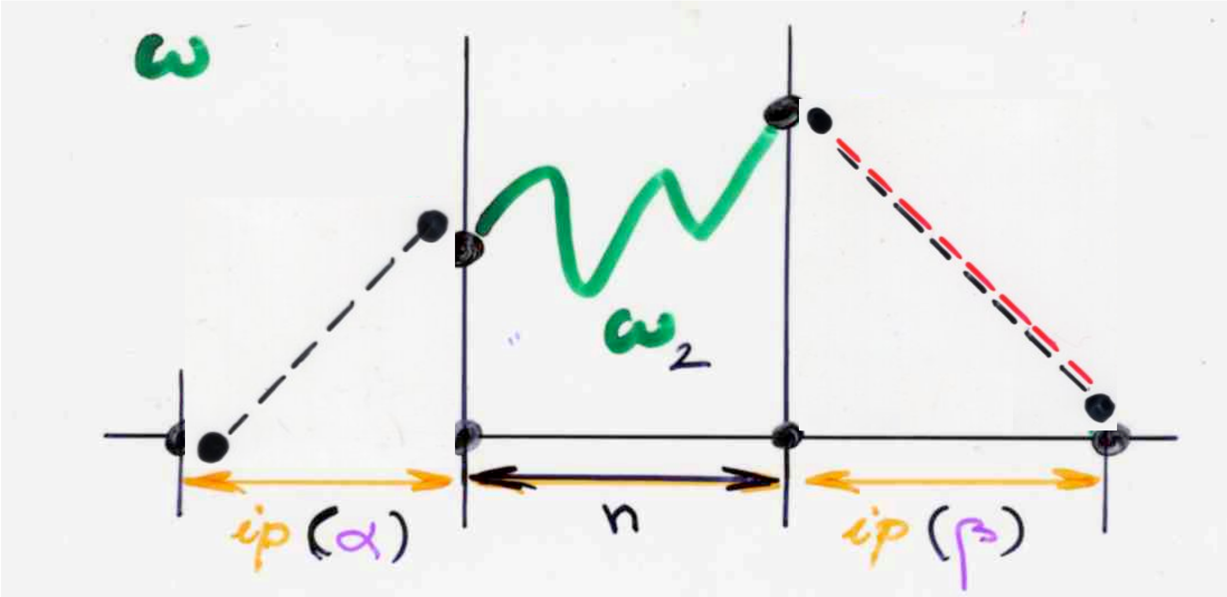


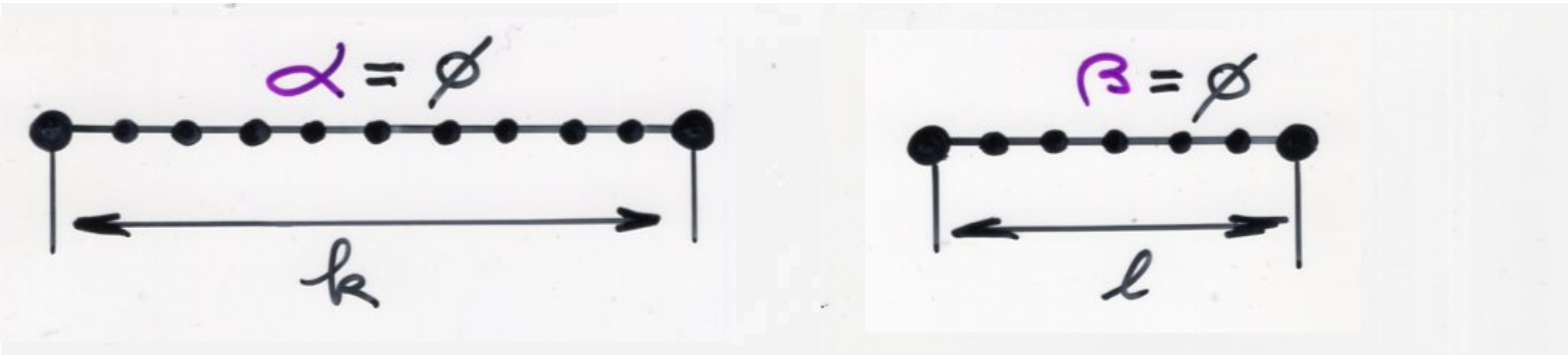


second involution θ_2 on $L_{n,k,\ell} \setminus R_{n,k,\ell}$



$$L_{n,k,\ell} \cap R_{n,k,\ell} = F_{n,k,\ell}$$





$$\sum_{(\alpha,\beta,\omega)\in E_{n,k,\ell}} (-1)^{|\alpha|+|\beta|}\, v(\alpha)\, v(\beta)\, v(\omega)$$

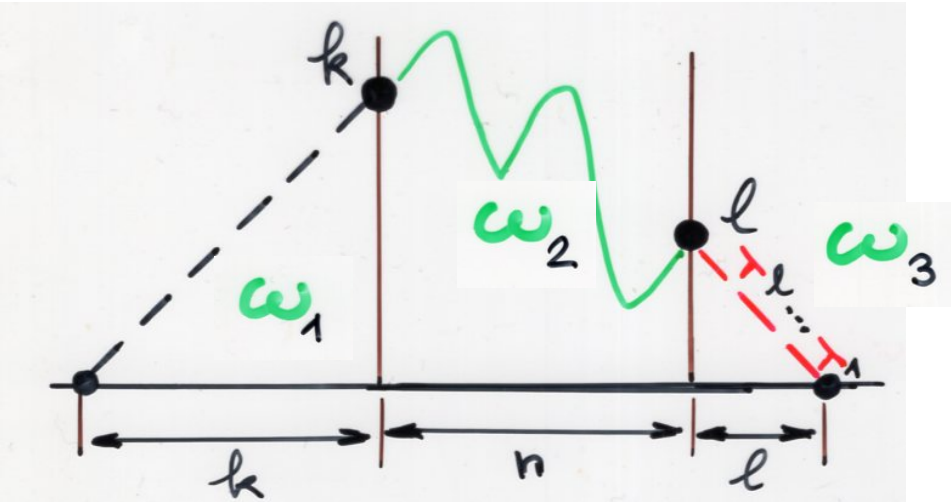
$$= \sum_{(\alpha,\beta,\omega)\in L_{n,k,\ell}} (-1)^{|\alpha|+|\beta|}\, v(\alpha)\, v(\beta)\, v(\omega)$$

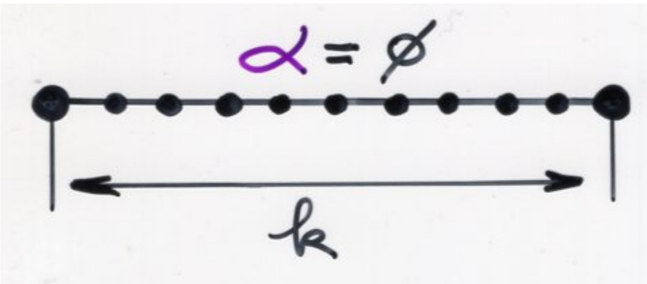
$$= \sum_{(\alpha,\beta,\omega)\in F_{n,k,\ell} = L_{n,k,\ell}\cap R_{n,k,\ell}} (-1)^{|\alpha|+|\beta|}\, v(\alpha)\, v(\beta)\, v(\omega)$$

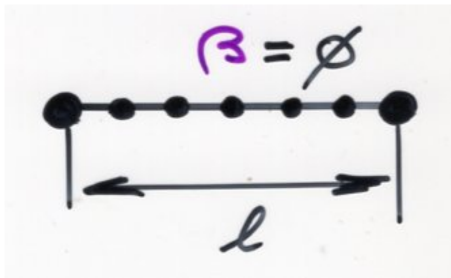
$F_{n,k,\ell} \subseteq E_{n,k,\ell}$

- α , β empty
- $\omega_1 =$  $(|\omega_1| = k)$
- $\omega_3 =$  $(|\omega_3| = \ell)$

$F_{n,k,\ell} \subseteq E_{n,k,\ell}$
- α, β empty
- $\omega_1 =$ ($|\omega_1| = k$)
- $\omega_3 =$ ($|\omega_3| = \ell$)







(main) **Theorem**

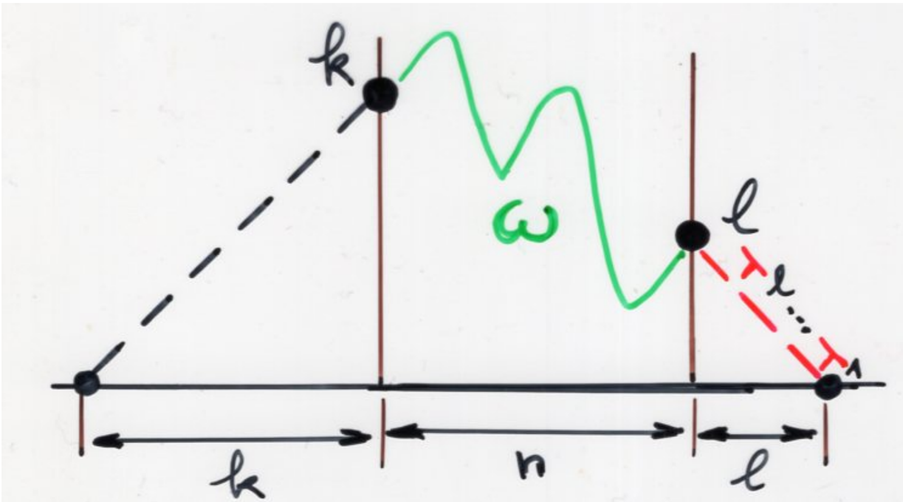
$$f\left(P_k P_\ell x^n\right) = \lambda_1 \times \cdots \times \lambda_\ell \sum_{\omega} v(\omega)$$

ω Motzkin path level $0 \to 0$

$|\omega| = k+n+\ell$

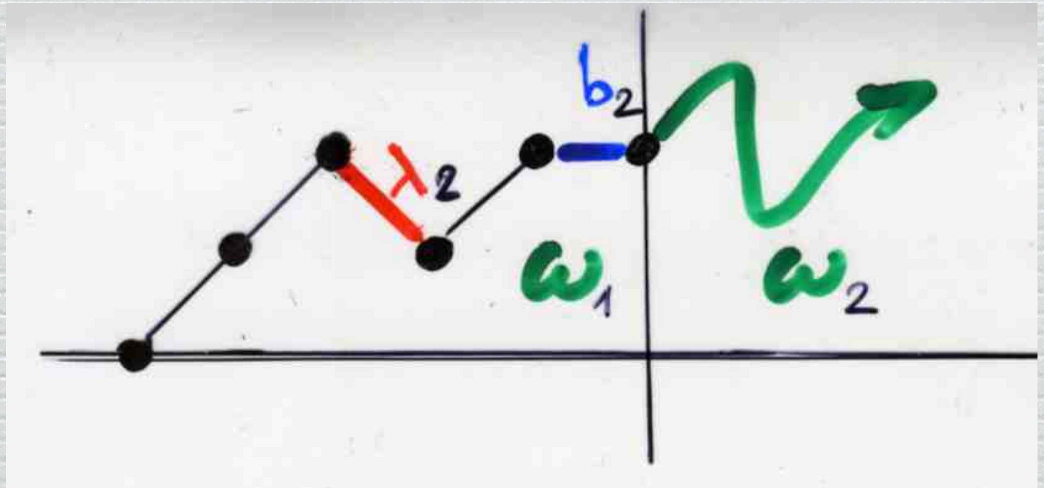
(i) first k steps are NE

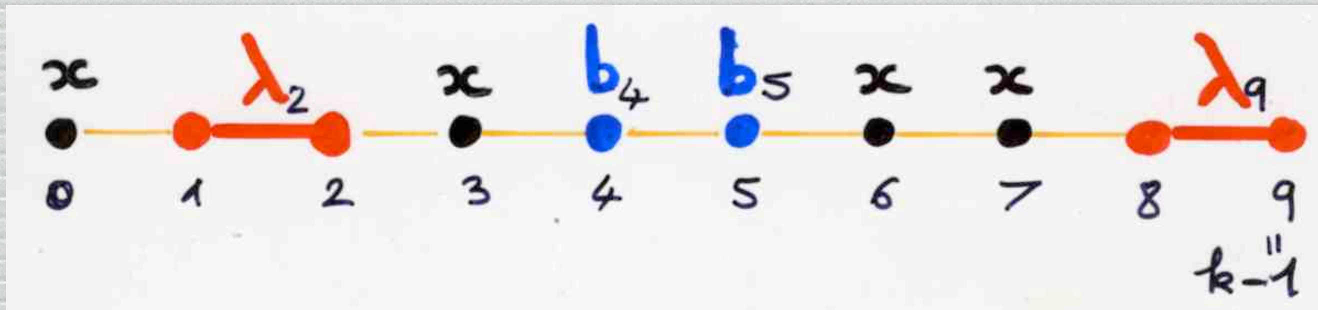
(ii) last ℓ steps are SE



□

The « essence » of the fundamental sign-reversing involutions





3 bijective proofs :

- 3-term recurrence ⇒ orthogonality
  (Favard theorem)
- inverse polynomials
- positivity of some linearization coefficients

same ("essence" of) bijection

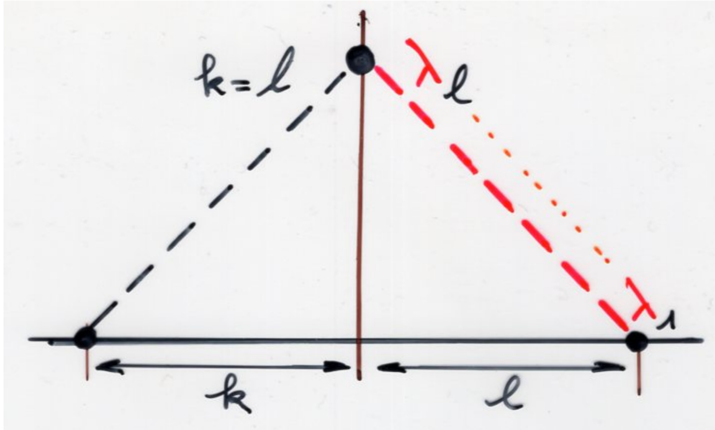
- 3 bijective proofs Ch 1
- convergents of continued fractions and orthogonal polynomial
- Ramanujan algorithm

Corollary

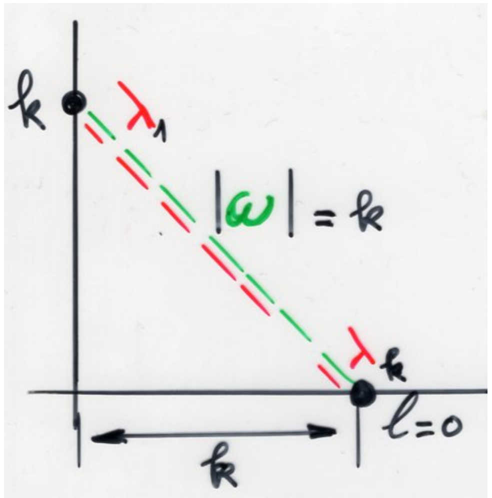
$\mathbb{K}$ field

$$\lambda_k = \frac{f(P_k^2)}{f(P_{k-1}^2)}$$

$$= \frac{f(x^k P_k)}{f(x^{k-1} P_{k-1})}$$



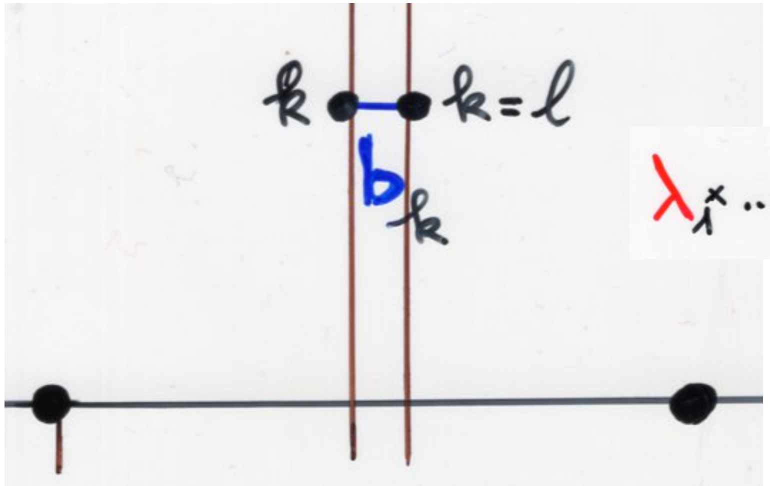
$\lambda_1 \times \ldots \times \lambda_k$



Corollary

$\mathbb{K}$ field

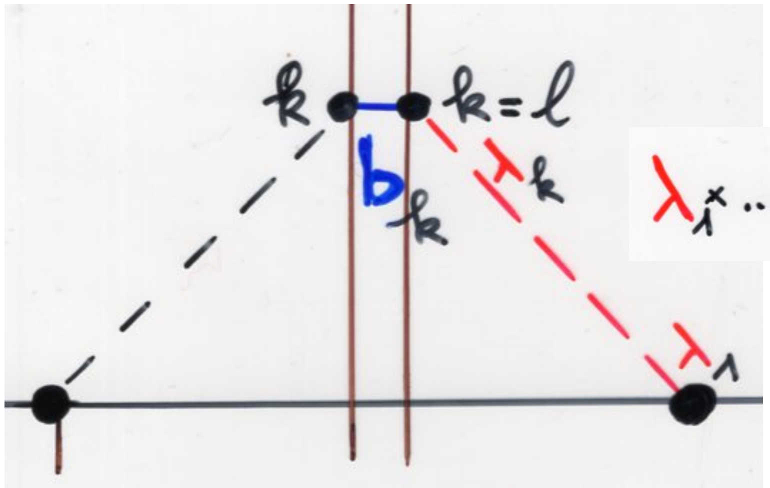
$$b_k = \frac{f(x P_k^2)}{f(P_k^2)}$$



$$\lambda_1 \times \cdots \times \lambda_k$$

Corollary $\mathbb{K}$ field

$$b_k = \frac{f(xP_k^2)}{f(P_k^2)}$$



$$\lambda_1 \times \cdots \times \lambda_k$$

A bijective proof for the

Positivity of linearization coefficients

<u>Lemma</u>

$$P_k(x)\, P_\ell(x) = \sum_n a_{k\ell}^{n} P_n(x)$$

$$a_{k\ell}^{n} = \frac{\mathcal{f}(P_k P_n P_\ell)}{\mathcal{f}(P_n^2)}$$

Proposition

Askey (1970)

$\lambda_{j+1} \geq \lambda_j$, $b_{j+1} \geq b_j$

If $\{\lambda_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 0}$ are increasing sequences

and $\lambda_j > 0$ for every $j \geq 1$,

then

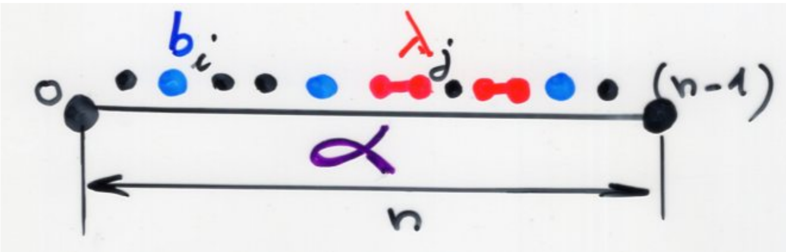
$$a_{kl}^{n} \geq 0$$

combinatorial proof

de Médicis, Stanton (1996)

$$f(P_k P_n P_\ell) = \sum_{\substack{\alpha \\ \text{pavage} \\ [0,\, n-1]}} f(P_k\, x^{ip(\alpha)}\, P_\ell)$$

pavage α



$\bullet \quad x^{ip(\alpha)}$

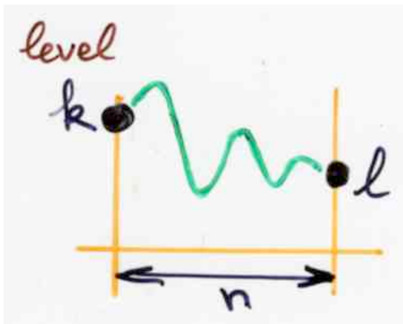
number of isolated points of α

from the main theorem:

$$f(P_k P_l x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_l$$

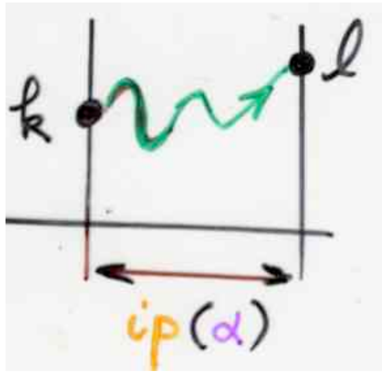
ω "Motzkin path"

$|\omega| = n$ level $k \rightsquigarrow l$

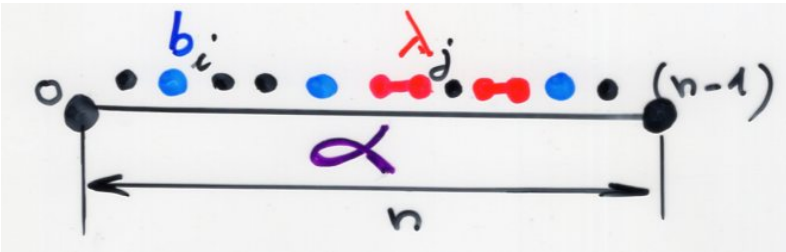


$$f(P_k P_n P_\ell) = \lambda_1 \cdots \lambda_\ell \sum_{(\alpha,\omega)\in M_{n,k,\ell}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)$$

$M_{n,k,\ell} = \{ (\alpha, \omega),$
- α pavage of $[0, n-1]$
- ω Motzkin path $k \rightsquigarrow \ell$ level
- $|\omega| = ip(\alpha)$ $\}$



pavage α

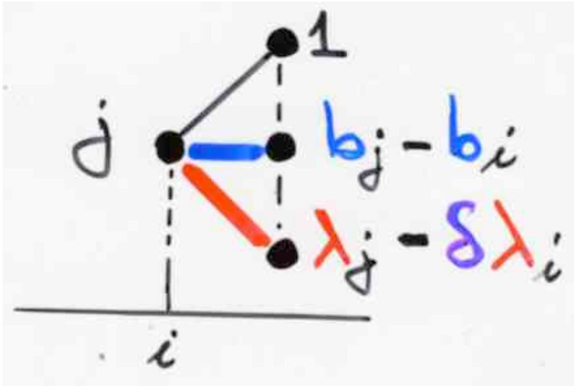


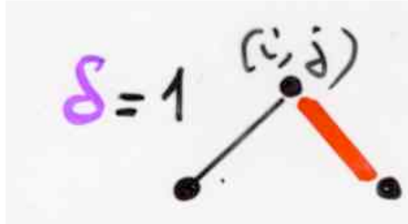
- $x^{ip(\alpha)}$

number of isolated points of α

define a weight $\bar{v}$ on Motzkin paths

$\bar{v}$





else $\delta = 0$

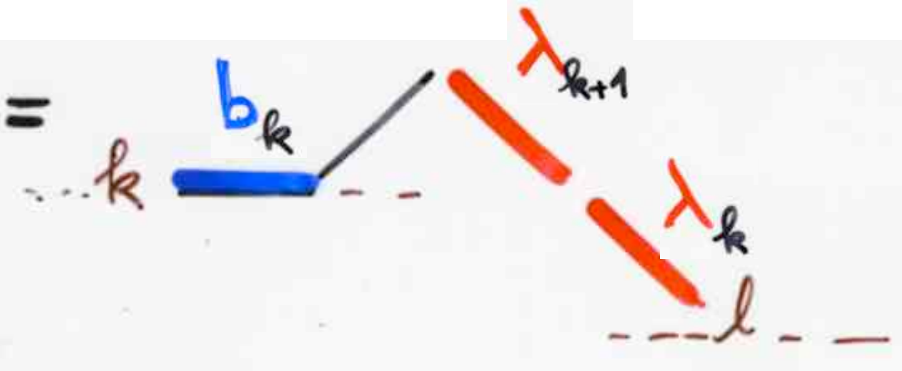
Proposition de Médicis, Stanton (1996)

$$\sum_{(\alpha,\omega)\in M_{n,k,l}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega) = \sum_{\eta} \bar{v}(\eta)$$

η Motzkin path $k \to l$ level

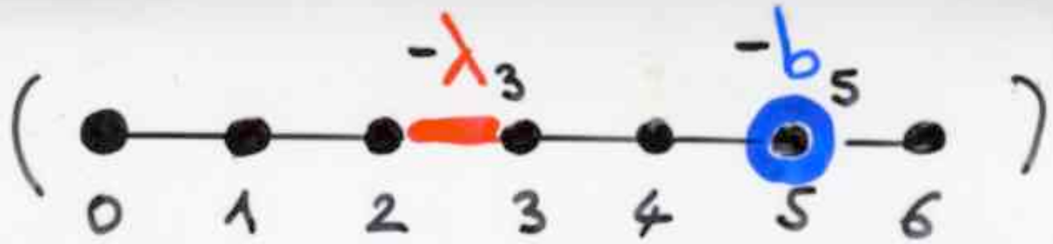
$|\eta| = n$

$\omega =$



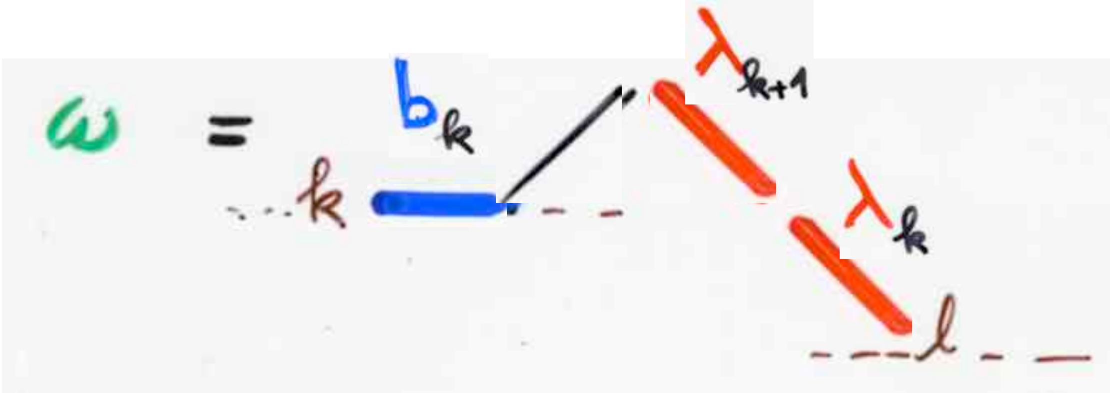
define a map ψ

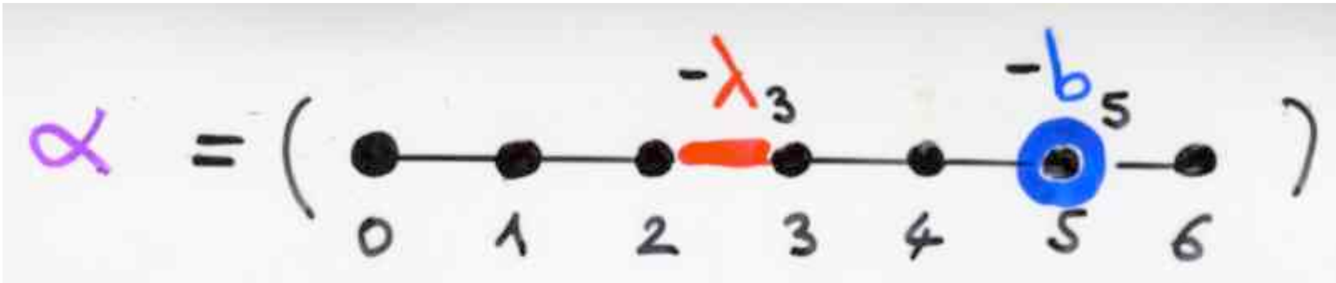
$\alpha =$

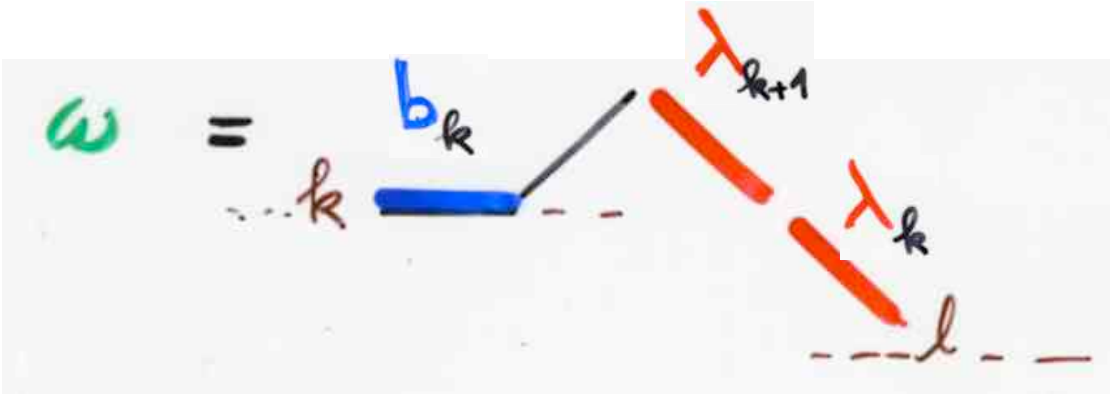


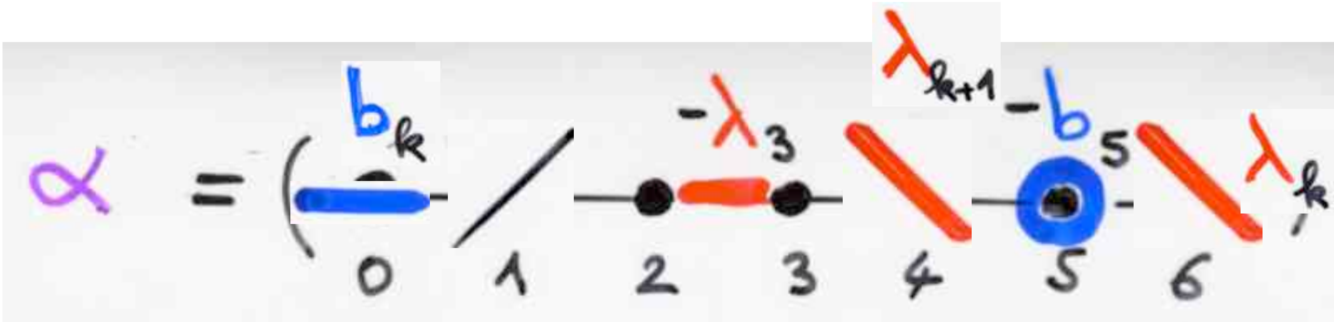
$$(\alpha, \omega) \in M_{n,k,\ell} \xrightarrow{\psi} \eta$$

Motzkin path

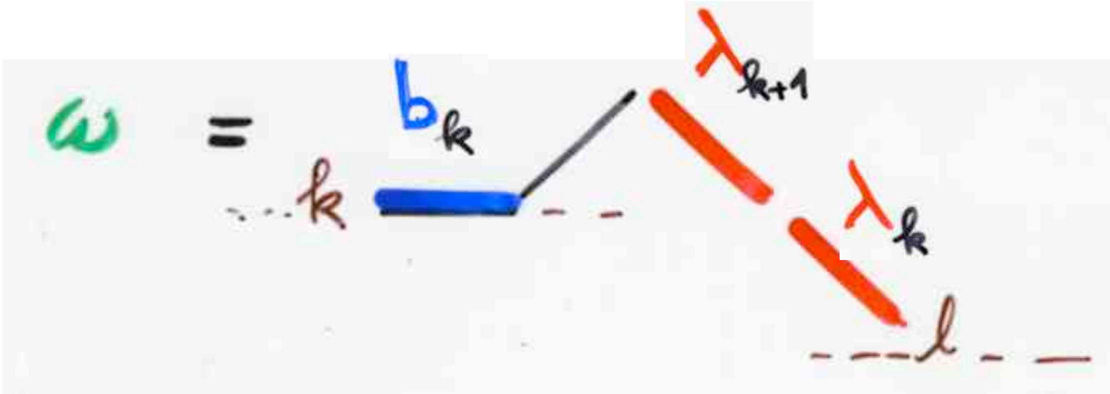

ω =
b k
k
λ k+1
λ k
ℓ

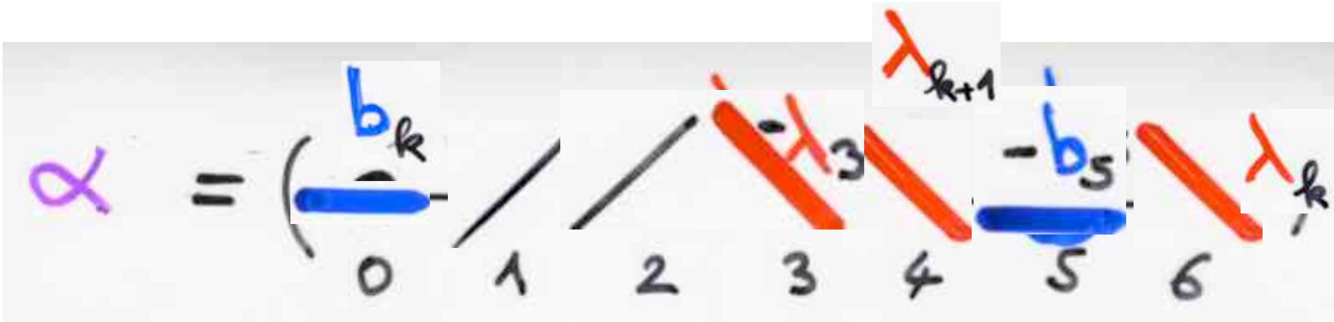

α =
-λ 3
-b 5
0 1 2 3 4 5 6

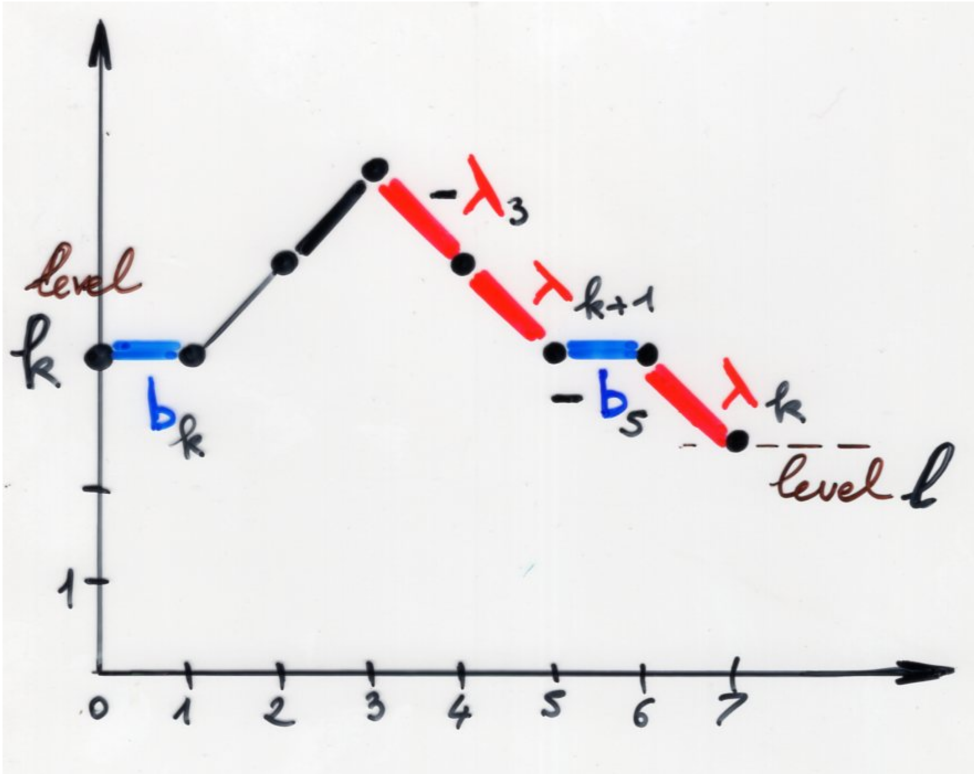

ω =
b_k
k
λ_{k+1}
λ_k
ℓ


α =
b_k
-λ_3
λ_{k+1}
-b_5
λ_k
0 1 2 3 4 5 6

ψ





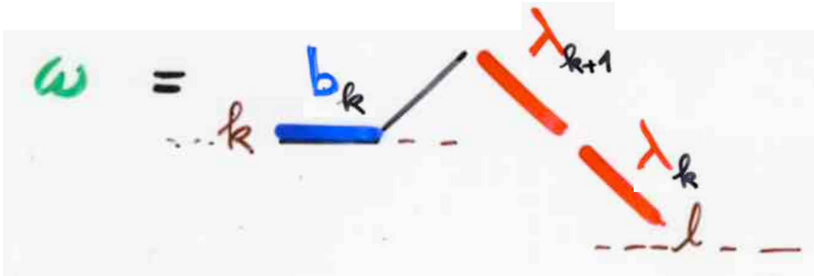


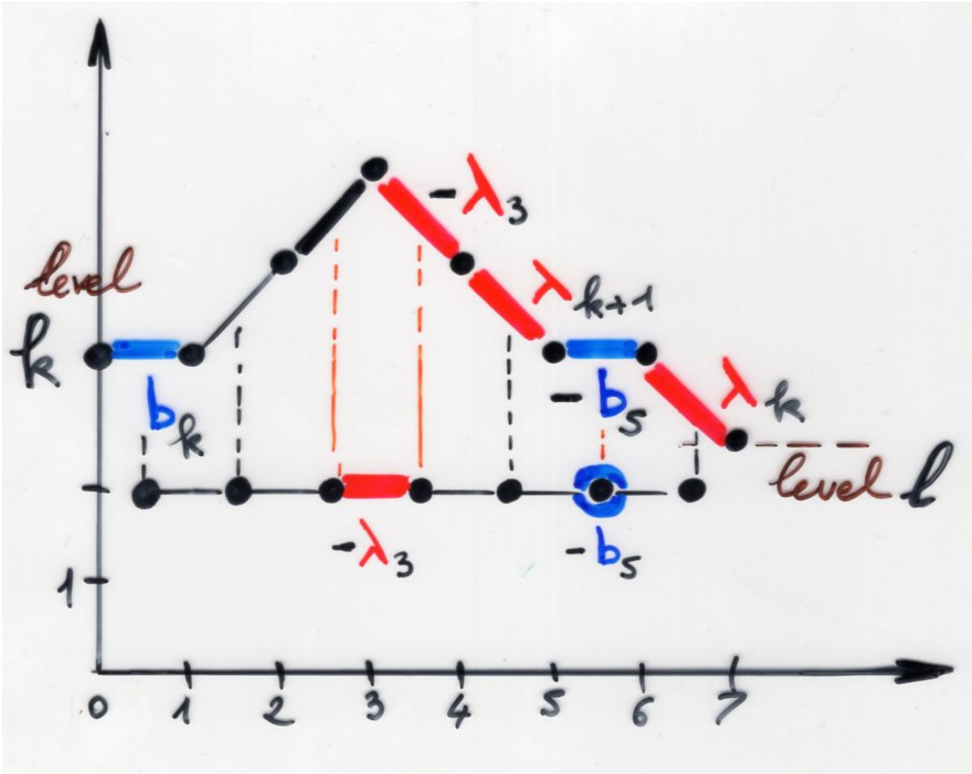
$$(\alpha, \omega) \in M_{n,k,\ell} \xrightarrow{\ \Psi\ } \eta$$

Motzkin path

$|\eta| = n$

$\eta : k \rightsquigarrow \ell$



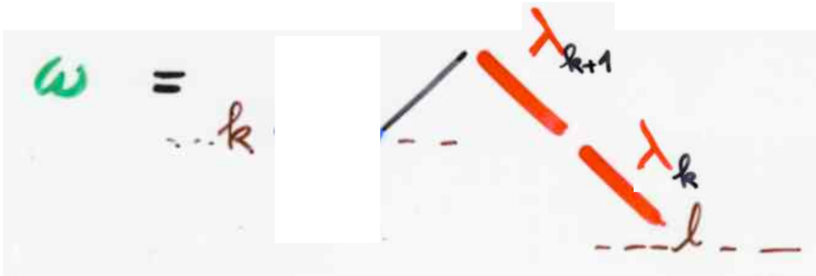


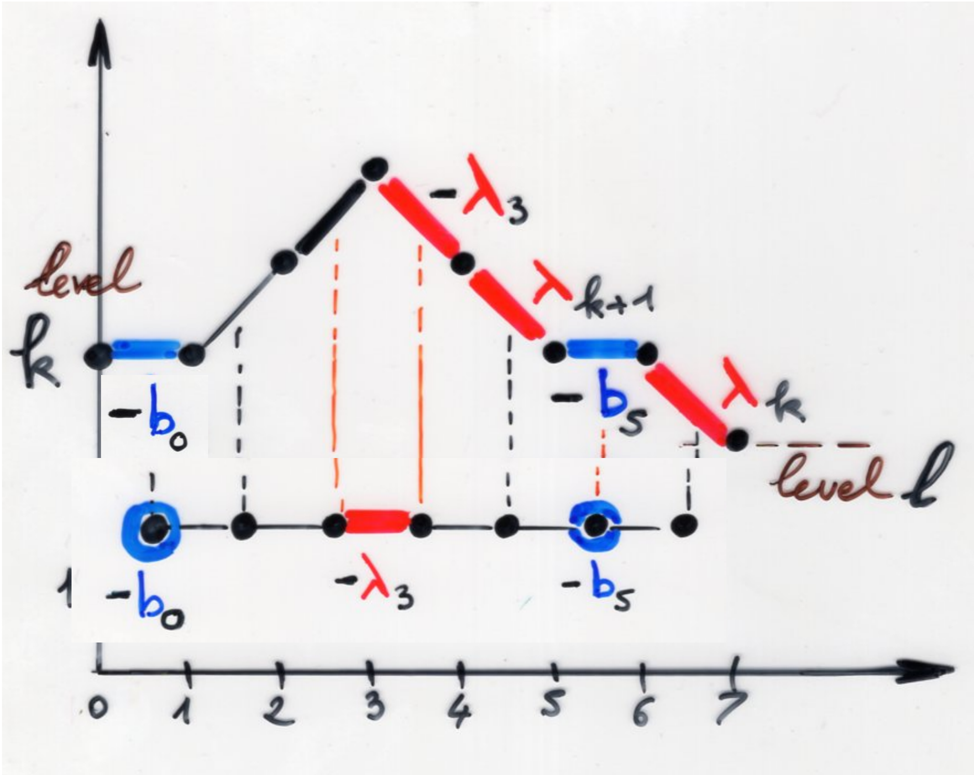
$$(\alpha, \omega) \in M_{n,k,\ell} \xrightarrow{\Psi} \eta$$

Motzkin path

$|\eta| = n$

$\eta : k \rightsquigarrow \ell$


ω =
k
λ_{k+1}
λ_k
ℓ


level
k
-b_0
-λ_3
λ_{k+1}
-b_5
λ_k
level ℓ
-b_0
-λ_3
-b_5
0
1
2
3
4
5
6
7


(α, ω) ∈ M_{n,k,ℓ}
ψ
η

Lemma

- η is a Motzkin path
  $|\eta| = n$ and $\eta : k \rightsquigarrow \ell$ level

- $\psi^{-1}(\eta) = M_{n,k,\ell}$

$$M_{n,k,\ell} = \left\{ (\alpha, \omega), \begin{array}{l} \bullet\ \alpha \text{ pavage of } [0, n-1] \\ \bullet\ \omega \text{ Motzkin path } k \rightsquigarrow \ell \text{ level} \\ \bullet\ |\omega| = ip(\alpha) \end{array} \right\}$$

Definition

marked path η

(η, E)

$E = (\varepsilon_1, \dots, \varepsilon_r)$

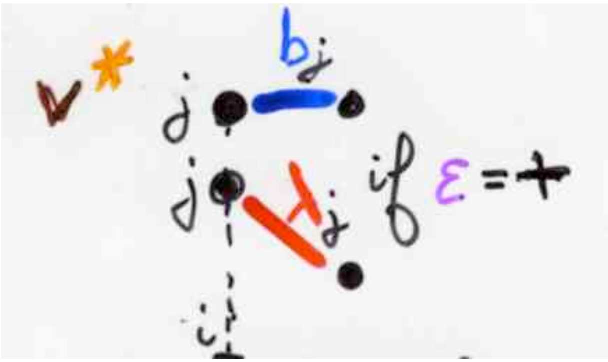
$\varepsilon_i \in \{+, -\}$

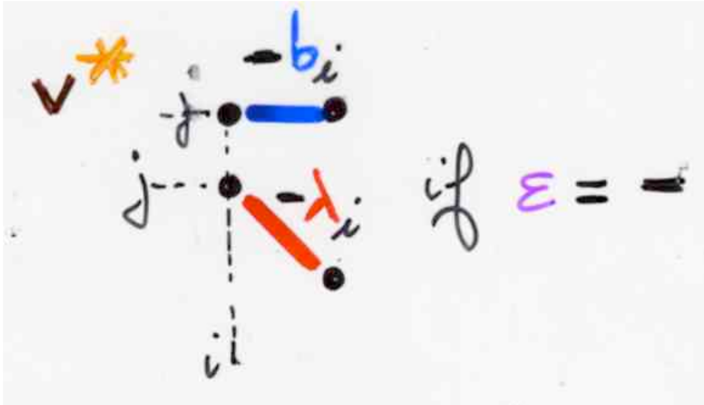
label (or mark) on the i^{th} step of η of the form  or 

bijection:

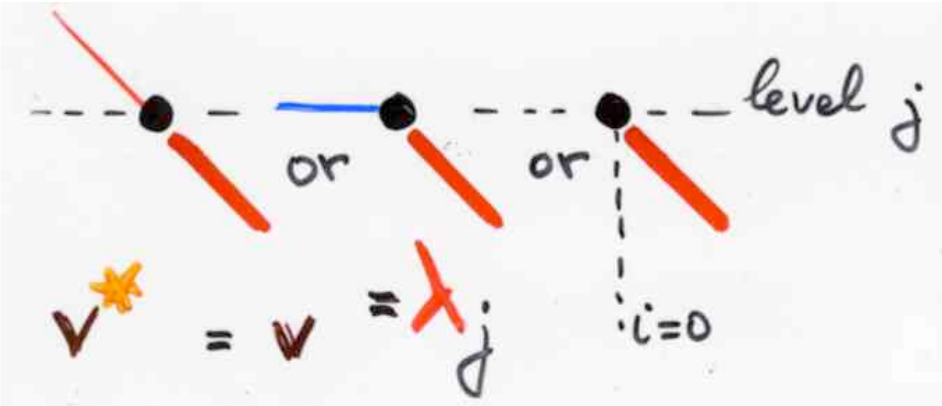
$$(\alpha, \omega) \in \psi^{-1}(\eta) \longleftrightarrow \text{marked path } (\eta, E)$$

$$(-1)^{|\alpha|} v(\alpha) v(\omega) = v^{*}(\eta, E)$$





starting point (i, j)

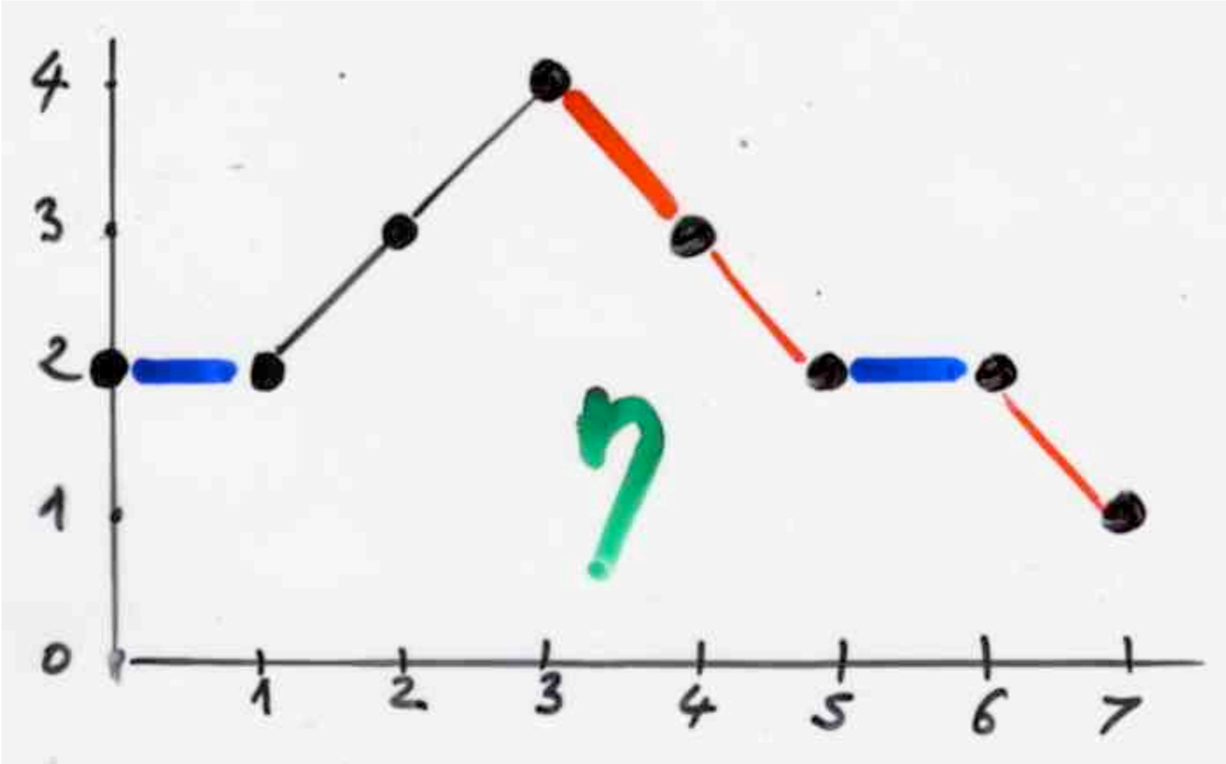


$$(-1)^{|\alpha|} v(\alpha) v(\omega) = v^{*}(\eta, E)$$

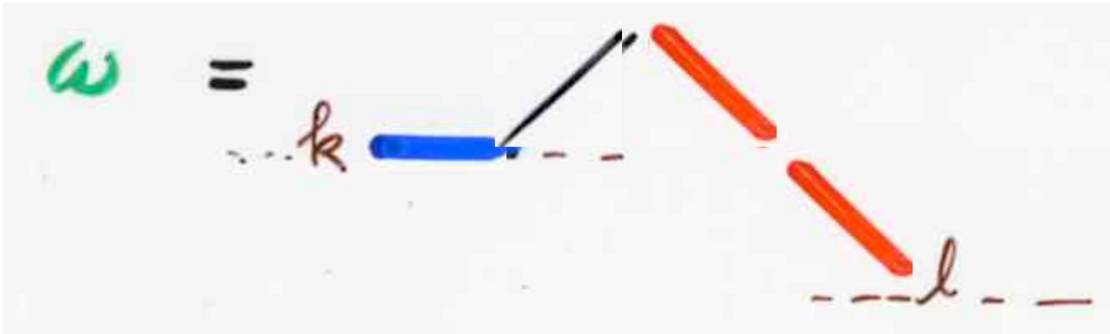
$$\sum_{(\alpha,\omega)\in\psi^{-1}(\eta)} (-1)^{|\alpha|} v(\alpha) v(\omega) = \sum_{E=(\varepsilon_1,\dots,\varepsilon_r)} v^{*}(\eta, E)$$

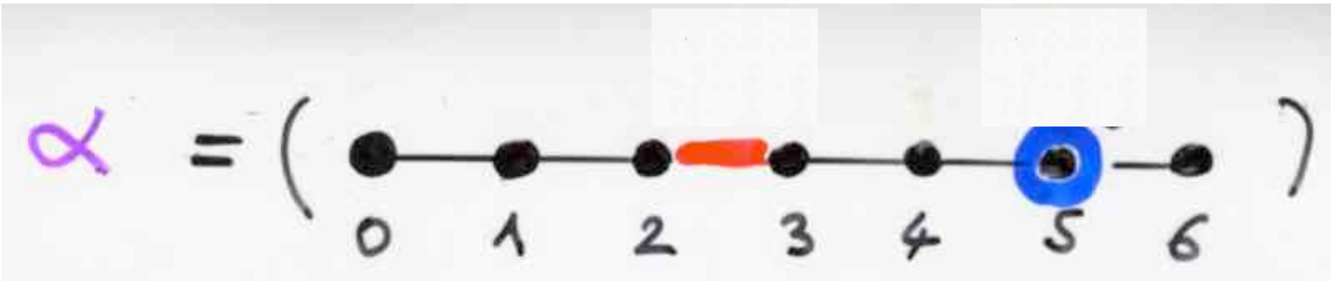
$$= \bar{v}(\eta)$$

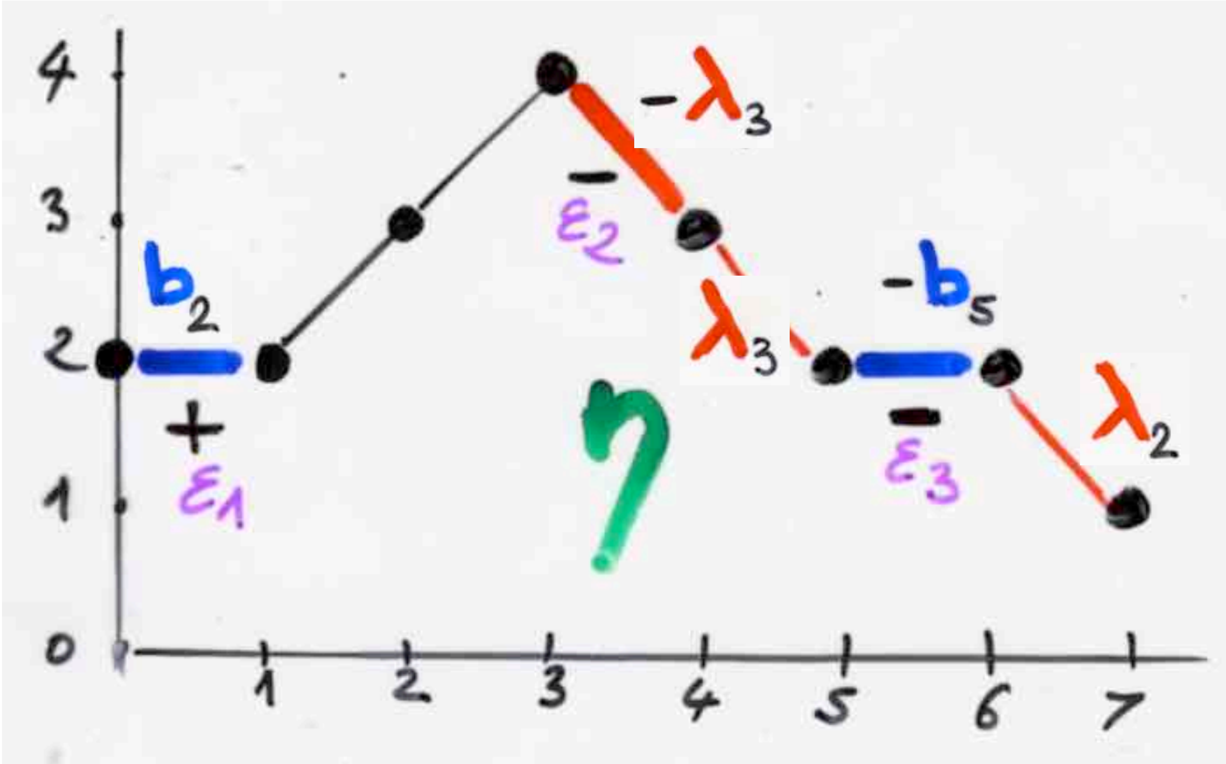
$$\sum_{\substack{(\alpha,\omega)\in M_{n,k,\ell} \\ \psi(\alpha,\omega)=\eta}} (-1)^{|\alpha|} v(\alpha) v(\omega) = \bar{v}(\eta)$$



$E = (+, -, -)$

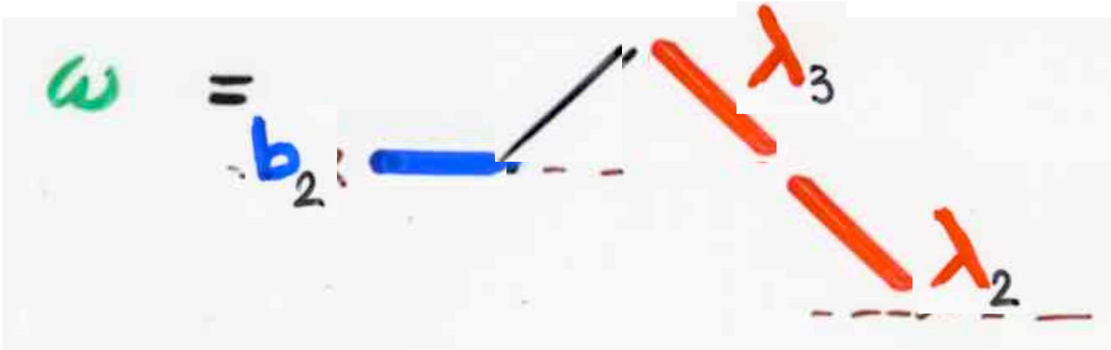




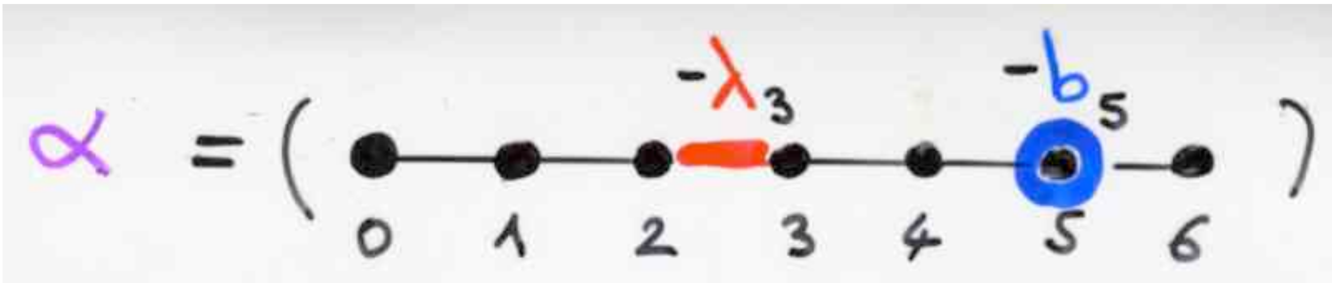


4
3
2
1
0
1
2
3
4
5
6
7
b_2
+
ε_1
$-\lambda_3$
−
ε_2
λ_3
η
$-b_5$
−
ε_3
λ_2

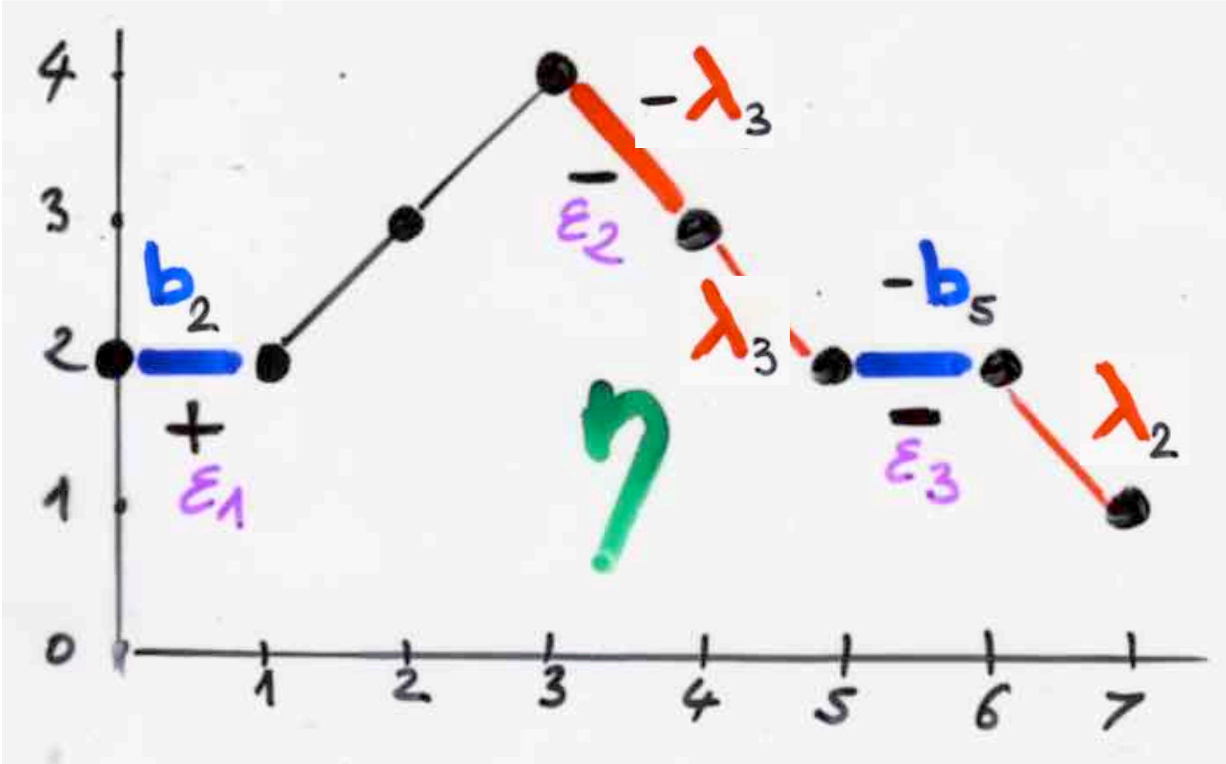
$E = (+, -, -)$



ω =
b_2
λ_3
λ_2



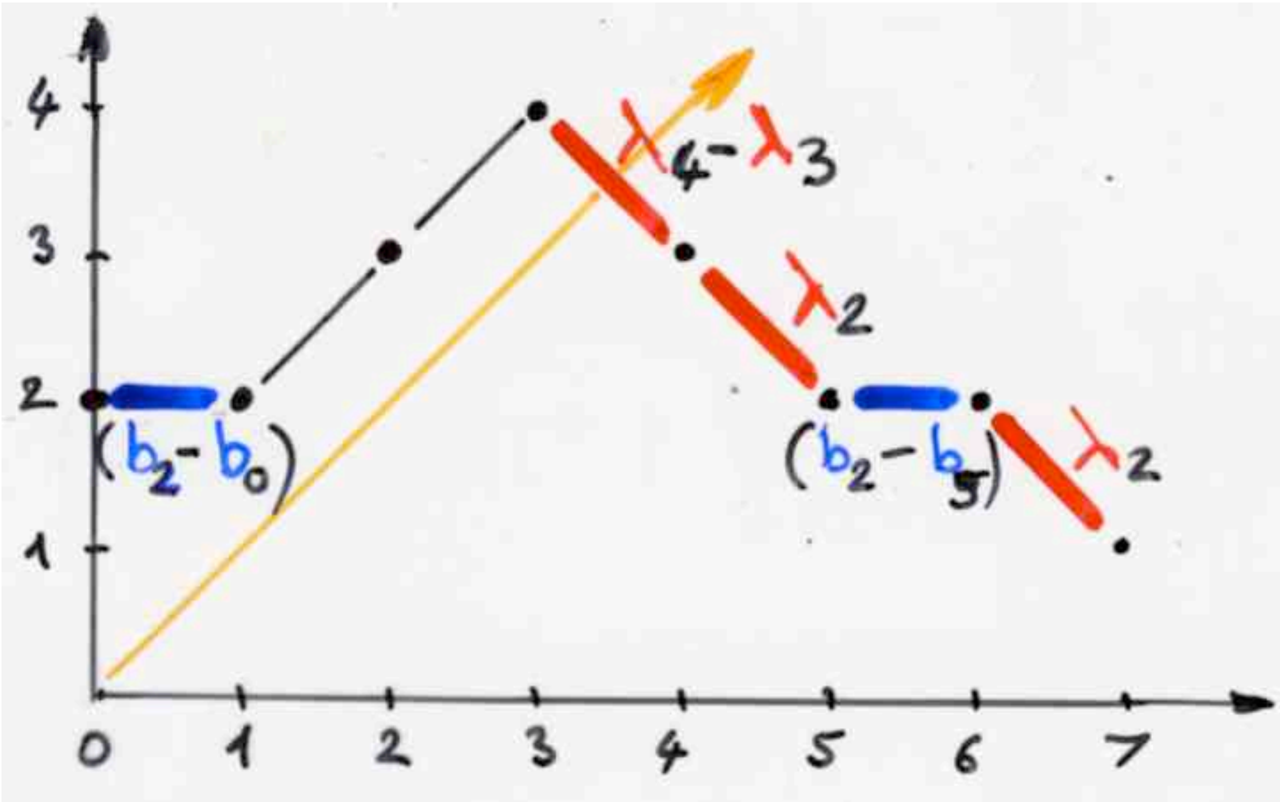
α = (
)
$-\lambda_3$
$-b_5$
0
1
2
3
4
5
6



$E = (+,-,-)$

$$\sum_{E=(\varepsilon_1,\dots,\varepsilon_r)} v^{*}(\eta, E) = \bar{v}(\eta)$$

$$(b_2 - b_0)(\lambda_4 - \lambda_3)\,\lambda_3\,(b_2 - b_5)\,\lambda_2$$

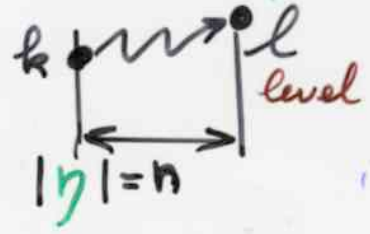


$$(b_2 - b_0)(\lambda_4 - \lambda_3)\,\lambda_3\,(b_2 - b_5)\,\lambda_2 \quad = \bar{v}(\eta)$$

$\lambda_1 \times \cdots \times \lambda_\ell$

$$\sum_{(\alpha,\omega)\in M_{n,k,\ell}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega) = \sum_{\eta} \sum_{\substack{(\alpha,\omega)\in M_{n,k,\ell} \\ \psi(\alpha,\omega)=\eta}} (-1)^{|\alpha|}\, v(\alpha)\, v(\omega)$$

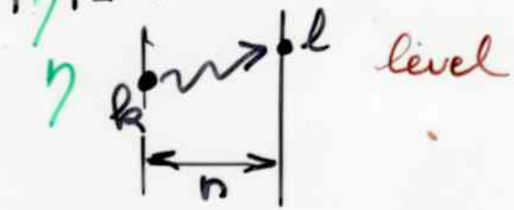
η Motzkin path



$$f(P_k P_n P_\ell) = \sum_{|\eta|=n} \bar{v}(\eta)$$

η Motzkin path

$|\eta| = n$



$\square$

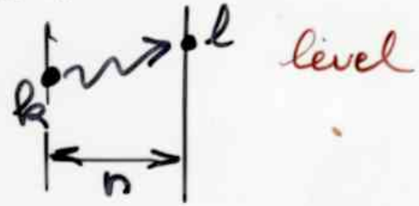
$$a_{k\ell}^{n} = \frac{f(P_k P_n P_\ell)}{f(P_n^2)} \longleftarrow \lambda_1 \times \cdots \times \lambda_n$$

$$f(P_k P_n P_\ell) = \lambda_1 \times \cdots \times \lambda_\ell \sum_{|\eta|=n} \bar{v}(\eta)$$

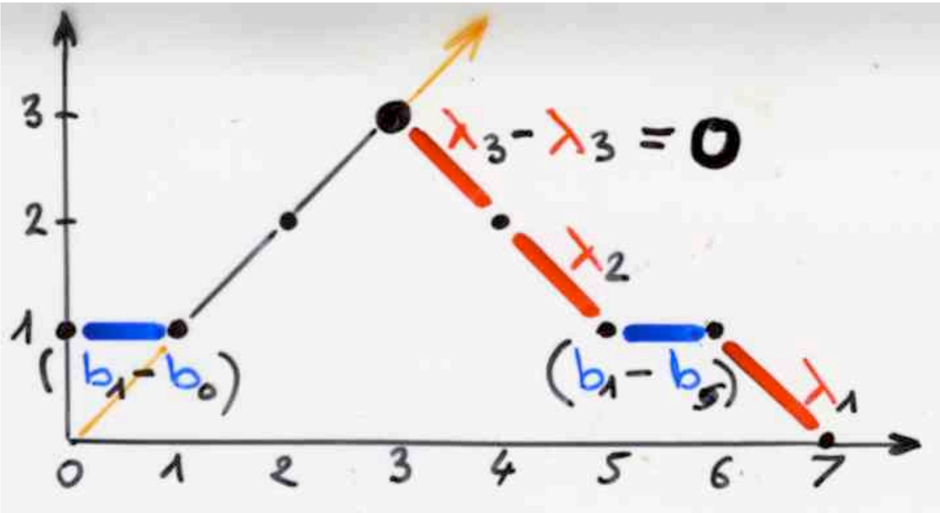
η Motzkin path

$|\eta| = n$

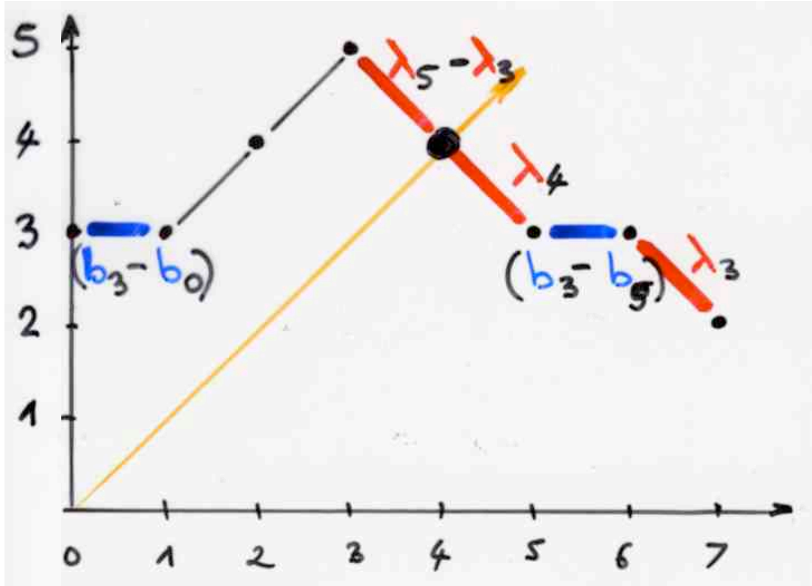
η



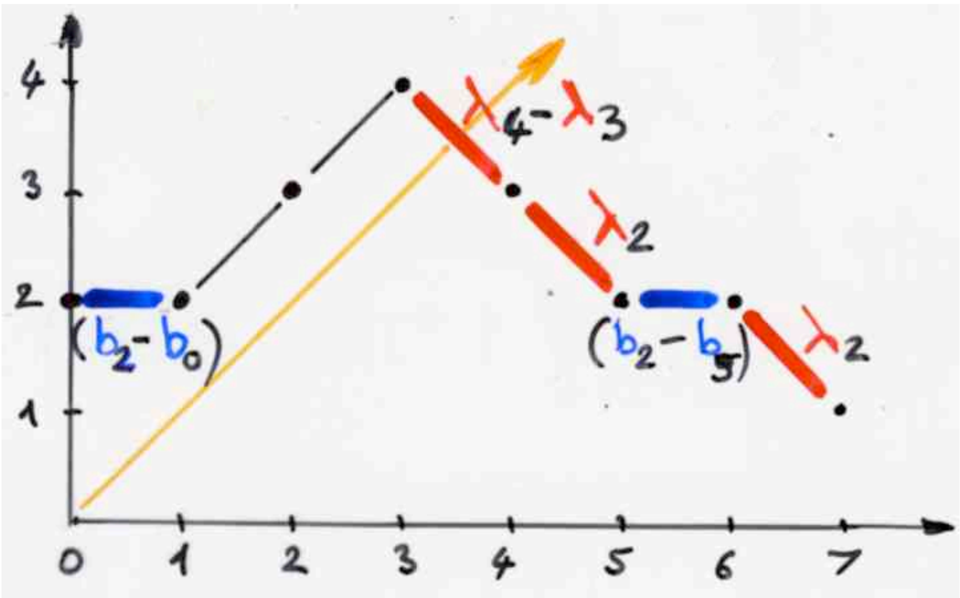
$\bar{V}(\eta)$



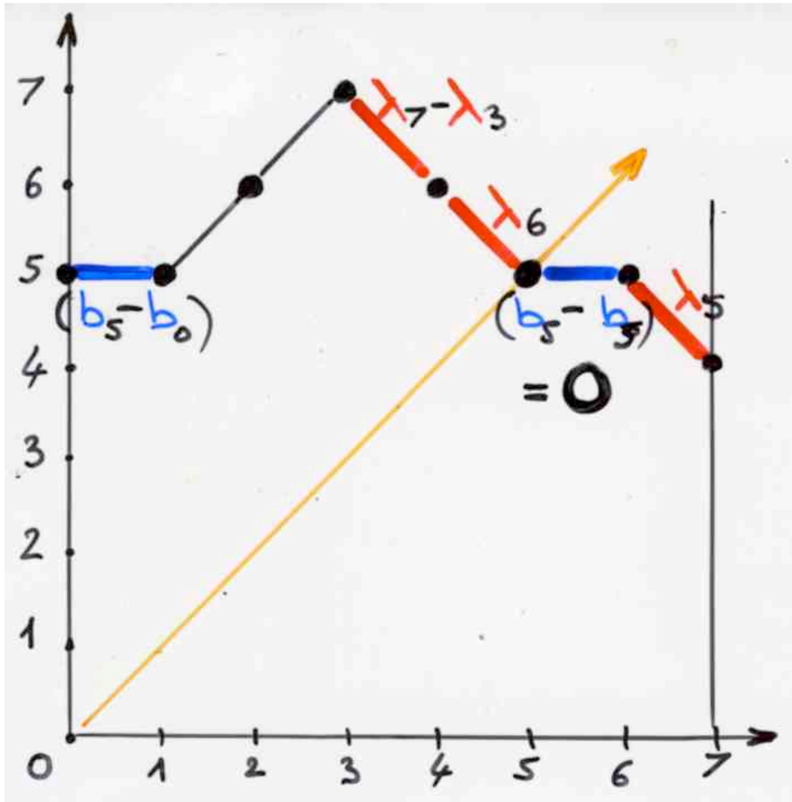
λ3 - λ3 = 0
λ2
λ1
(b1 - b0)
(b1 - b5)
0 1 2 3 4 5 6 7
1 2 3



λ5 - λ3
λ4
λ3
(b3 - b0)
(b3 - b5)
0 1 2 3 4 5 6 7
1 2 3 4 5



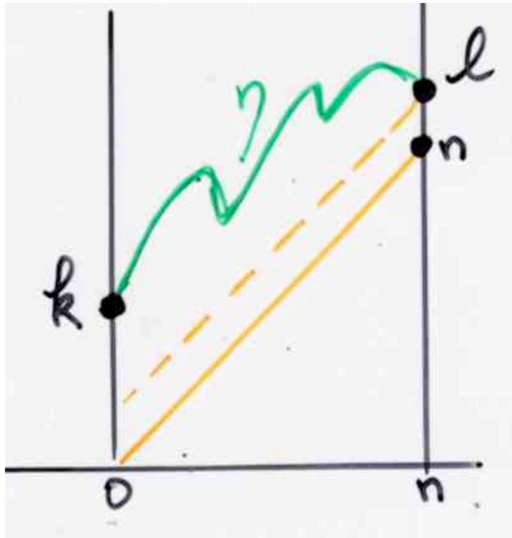
λ4 - λ3
λ2
λ2
(b2 - b0)
(b2 - b5)
0 1 2 3 4 5 6 7
1 2 3 4



λ7 - λ3
λ6
λ5
(b5 - b0)
(b5 - b5)
= 0
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7

If $n \leq \ell$, then all the vertices of the path η are above the diagonal Δ

$\Rightarrow$ all labels $b_j - b_i$ and $\lambda_j - \lambda_i$ satisfy $j \geq i$



Corollary

Askey (1970)

If $\{\lambda_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 0}$ are increasing sequences

and $\lambda_j > 0$ for every $j \geq 1$,

then

$$a_{kl}^{n} \geq 0$$

$\lambda_{j+1} \geq \lambda_j$, $b_{j+1} \geq b_j$

combinatorial proof

de Médicis, Stanton (1996)

Corollary

Askey (1970)

If $\{\lambda_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 0}$ are increasing sequences and $\lambda_j > 0$ for every $j \geq 1$,

then

$$a_{kl}^{n} \geq 0$$

$\lambda_{j+1} \geq \lambda_j$, $b_{j+1} \geq b_j$

$$a_{kl}^{n} = \frac{f(P_k P_n P_l)}{f(P_n^2)}$$

de Médicis, Stanton (1996)

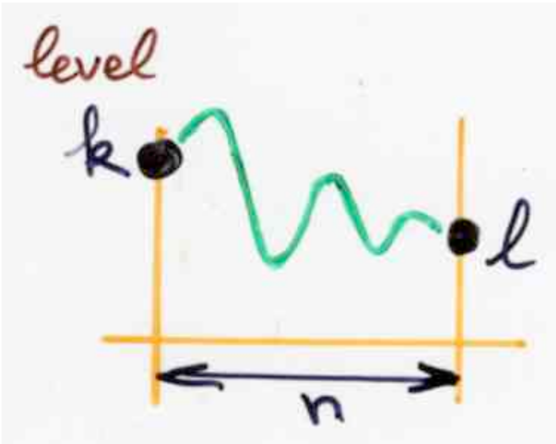
Back to the proof of the main theorem

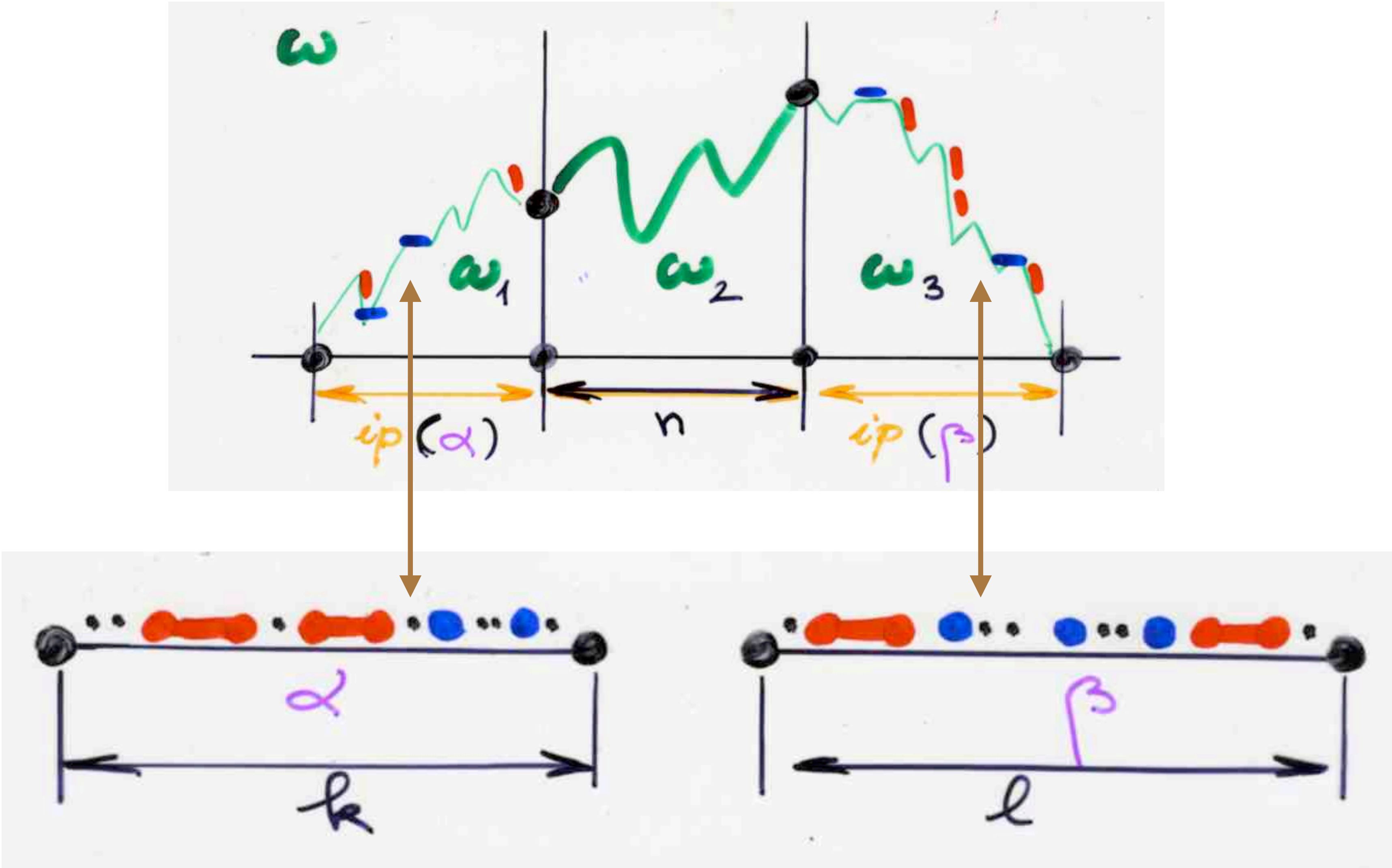
(main) **Theorem**

$$f(P_k P_l x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_l$$

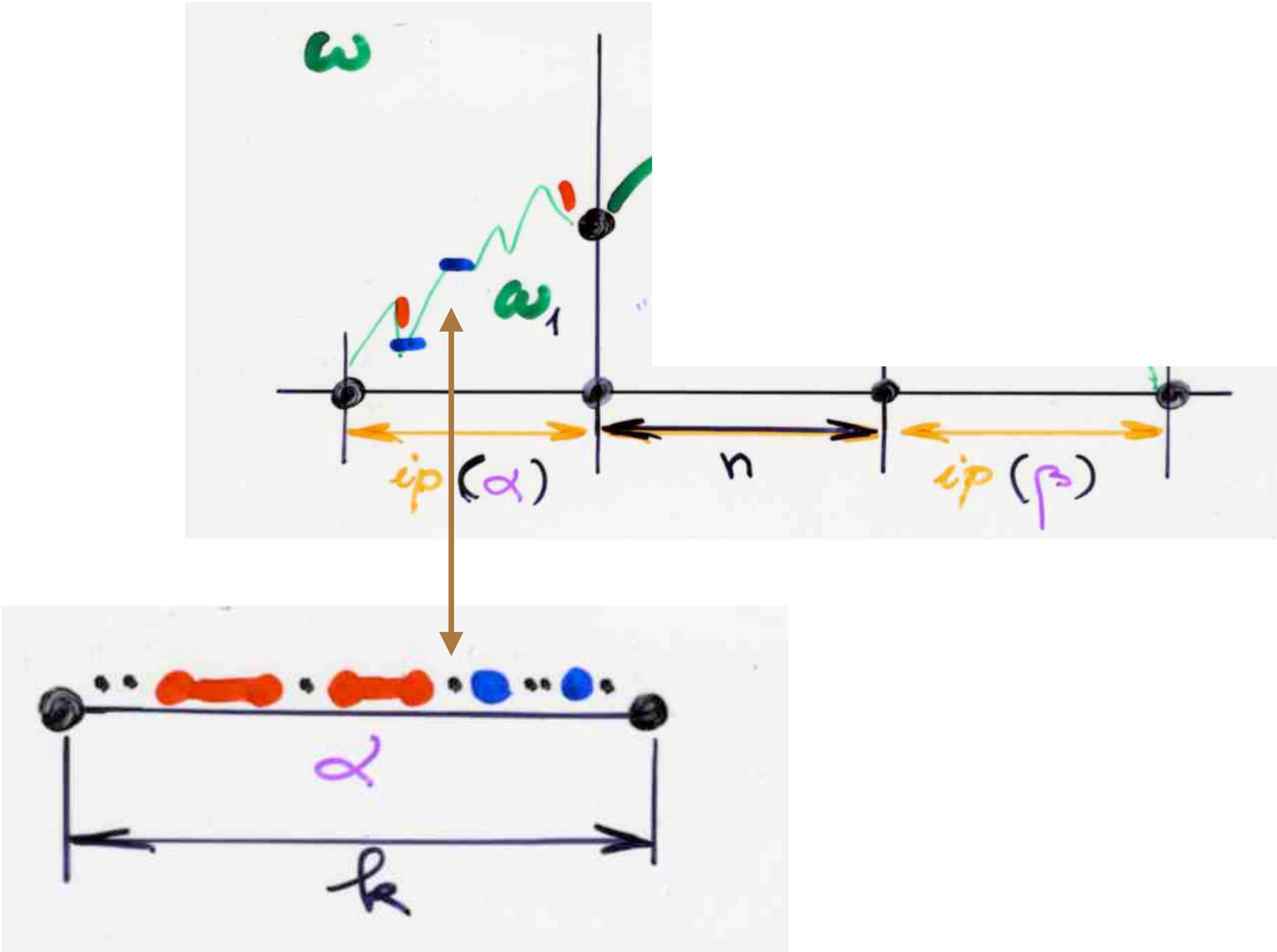
"Motzkin path" ω

$|\omega| = n$ level $k \rightsquigarrow l$





$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$

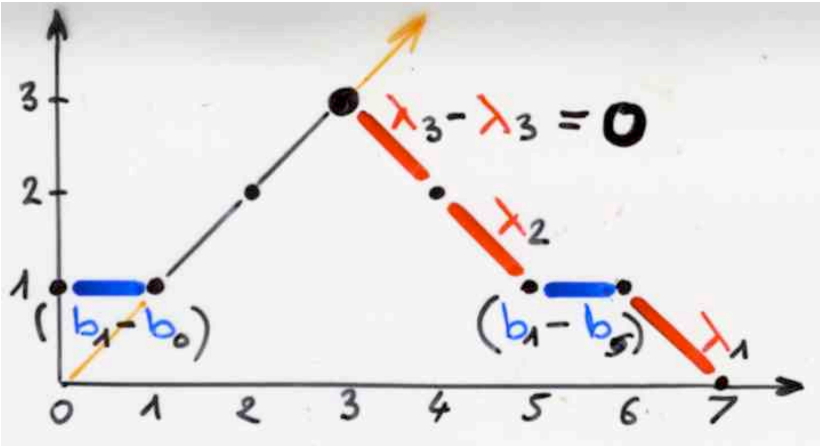


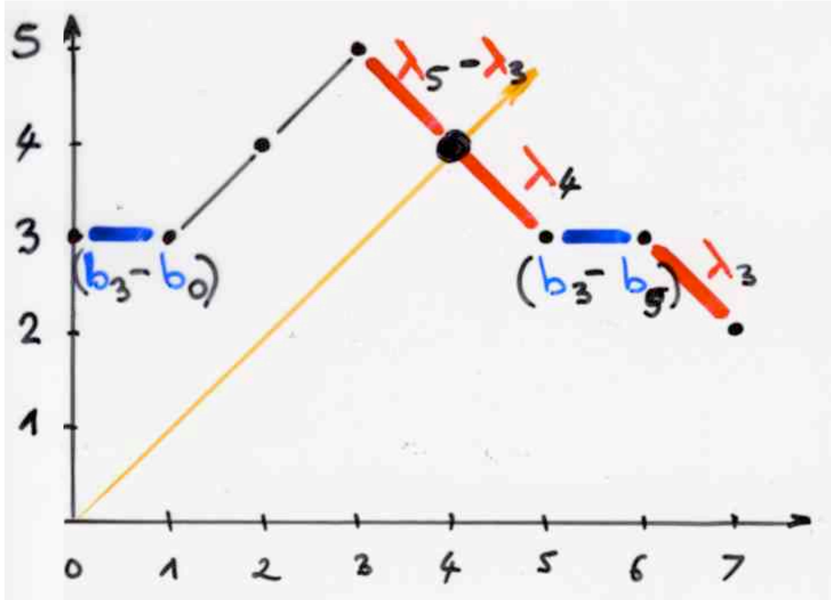
$$(\omega, \alpha, \beta) \in E_{n,k,\ell} \setminus L_{n,k,\ell}$$

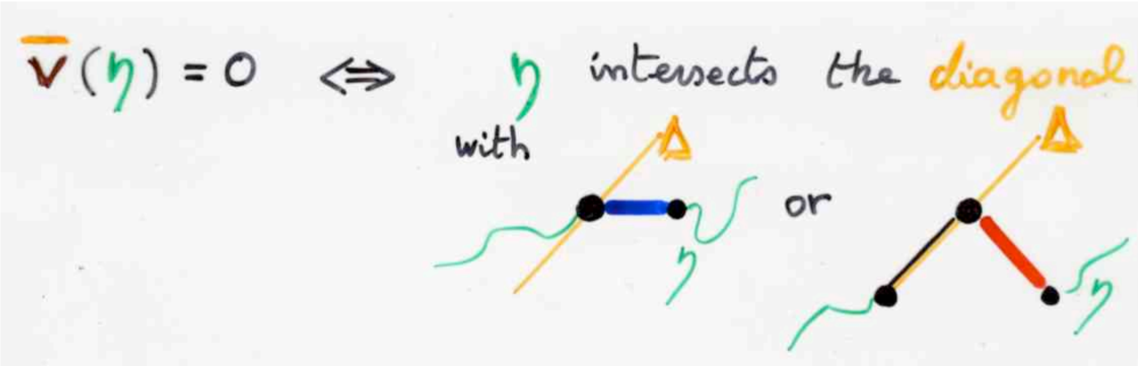
$k = 0$

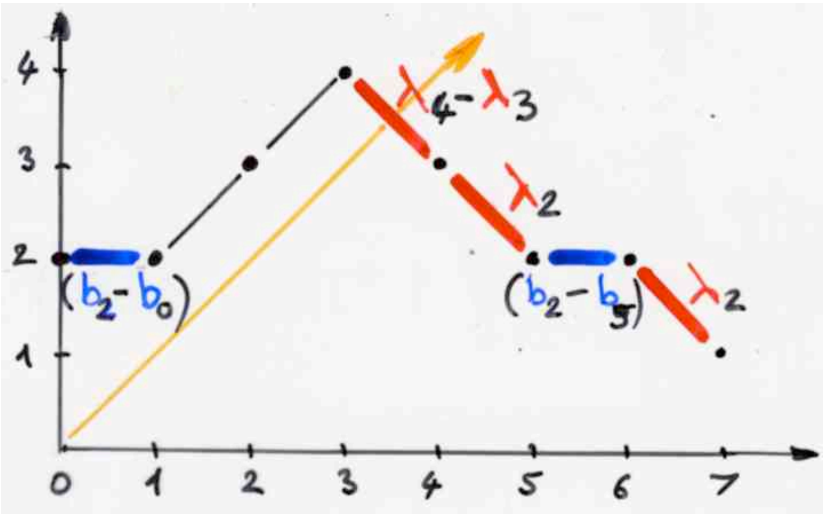
$M_{n,0,\ell} = E_{n,\ell}$

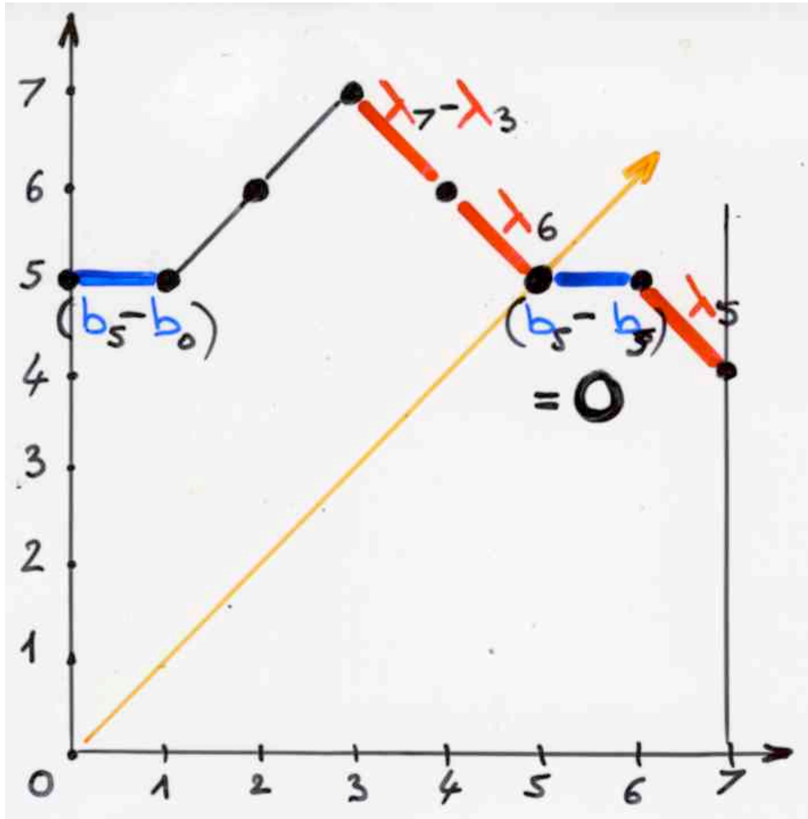
$$\sum_{(\alpha,\omega) \in E_{n,\ell}} (-1)^{|\alpha|} v(\alpha) v(\omega) = \sum_{|\eta| = n} \bar{v}(\eta)$$


λ3 - λ3 = 0
λ2
λ1
(b1 - b0)
(b1 - b5)


λ5 - λ3
λ4
λ3
(b3 - b0)
(b3 - b5)

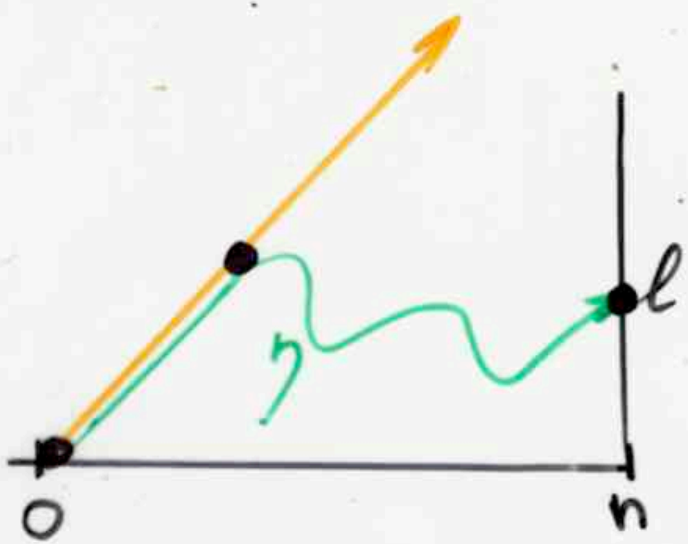

V̄(η) = 0 ⇔ η intersects the diagonal
with
Δ
or
η


λ4 - λ3
λ2
λ2
(b2 - b0)
(b2 - b5)

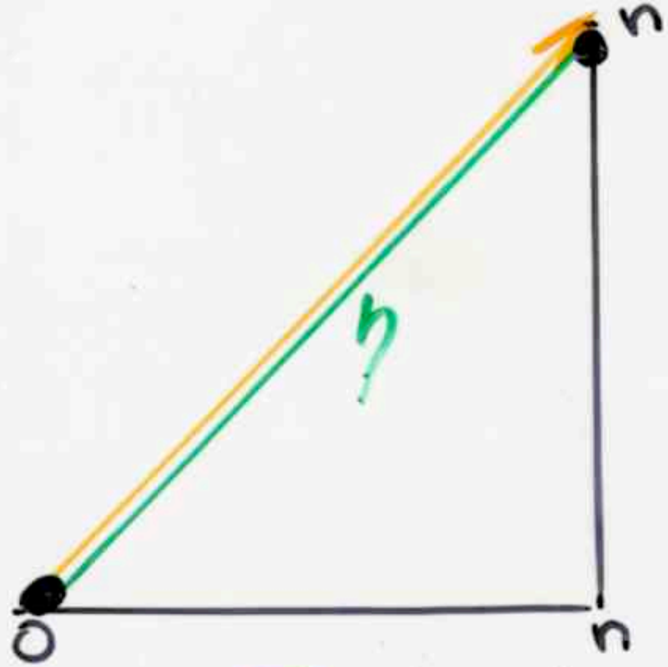

λ7 - λ3
λ6
λ5
(b5 - b0)
(b5 - b5)
= 0

$k = 0$ $\qquad\qquad M_{n,0,\ell} = E_{n,\ell}$

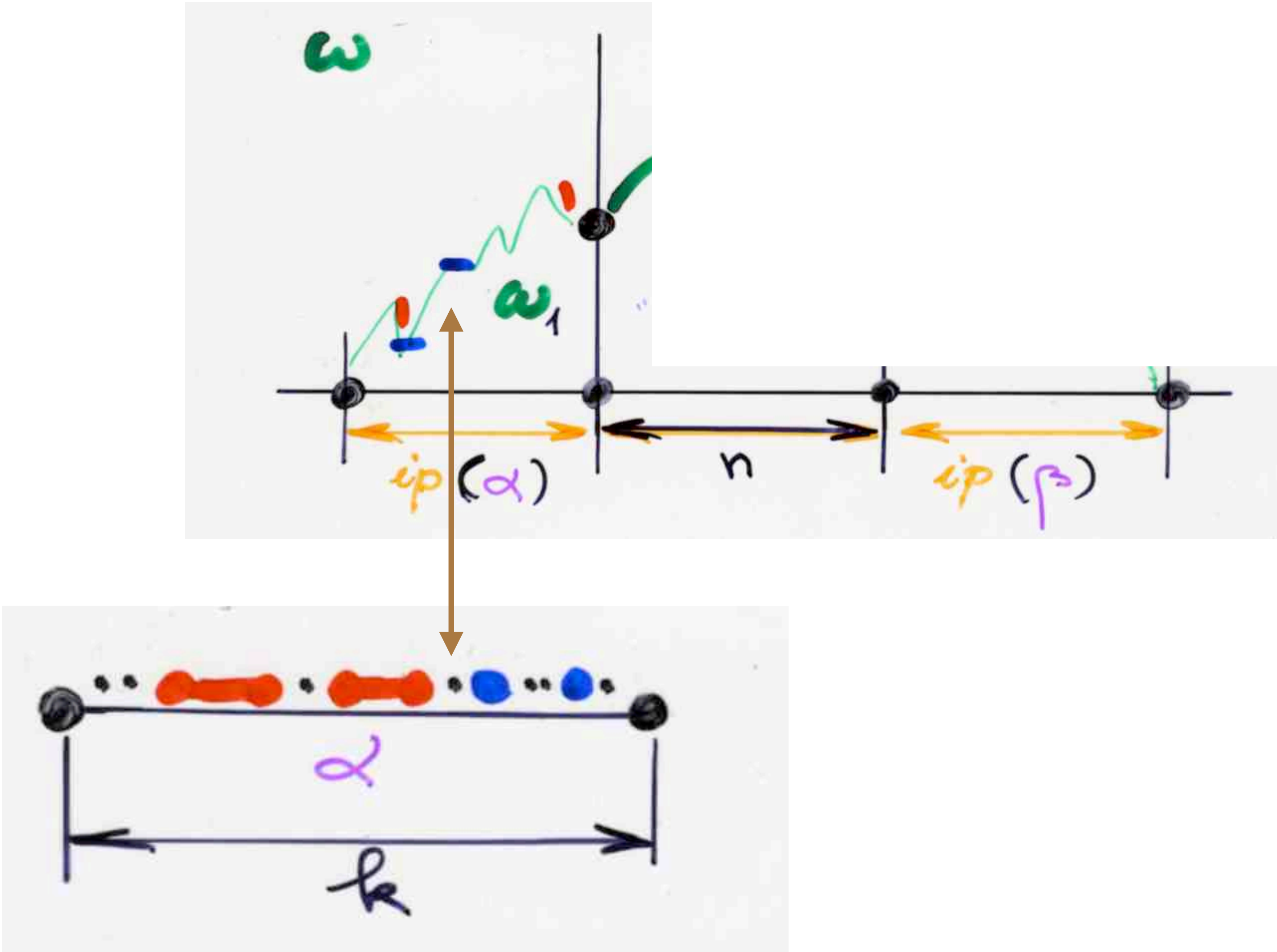
$$\sum_{(\alpha,\omega)\in E_{n,\ell}} (-1)^{|\alpha|} v(\alpha)\, v(\omega) = \sum_{|\eta|=n} \bar{v}(\eta)$$



$\bar{v}(\eta) = 0$



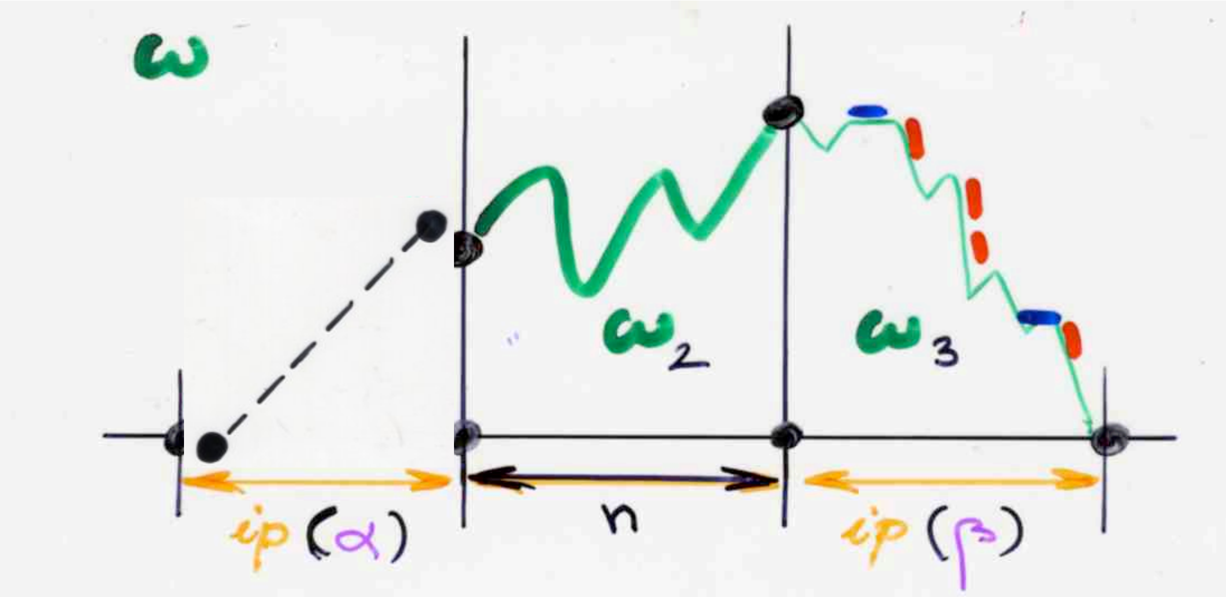
except $\qquad \bar{v}(\eta) = 1$

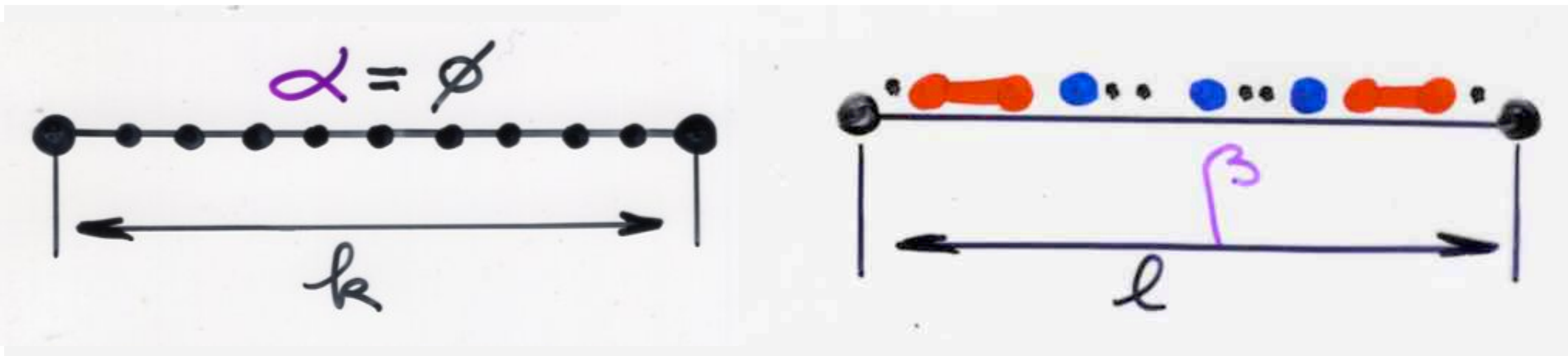


$$(\omega, \alpha, \beta) \in E_{n,k,\ell} \setminus L_{n,k,\ell}$$

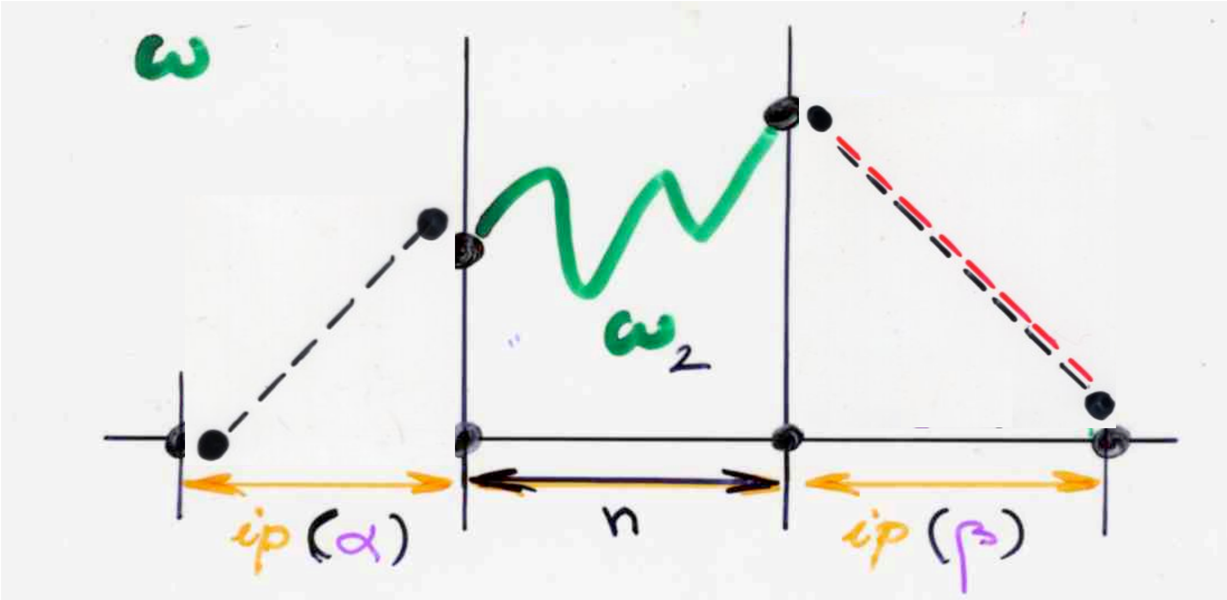
$L_{n,k,\ell} \subseteq E_{n,k,\ell}$

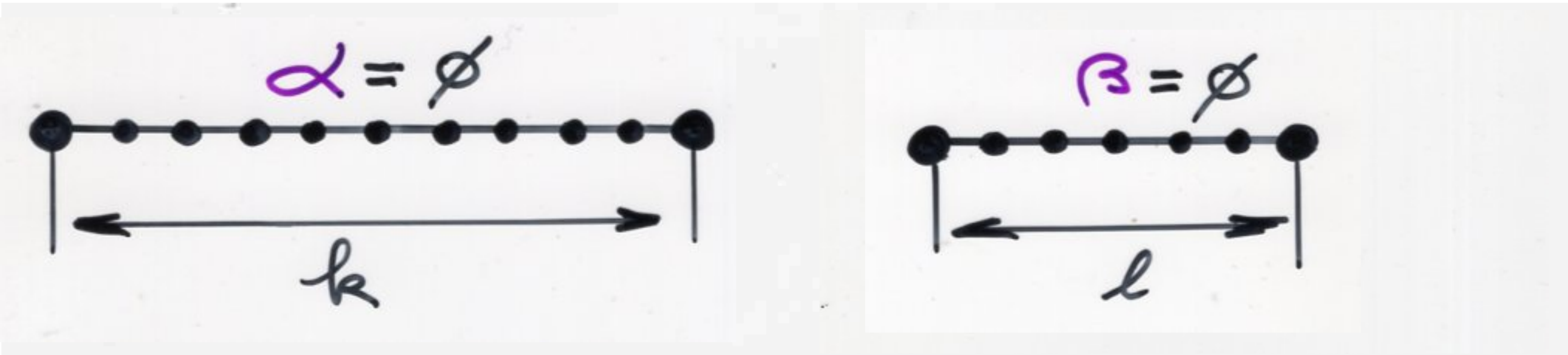
- α empty
- $\omega_1 =$  $(|\omega_1| = k)$





$$F_{n,k,\ell} = L_{n,k,\ell} \cap R_{n,k,\ell}$$

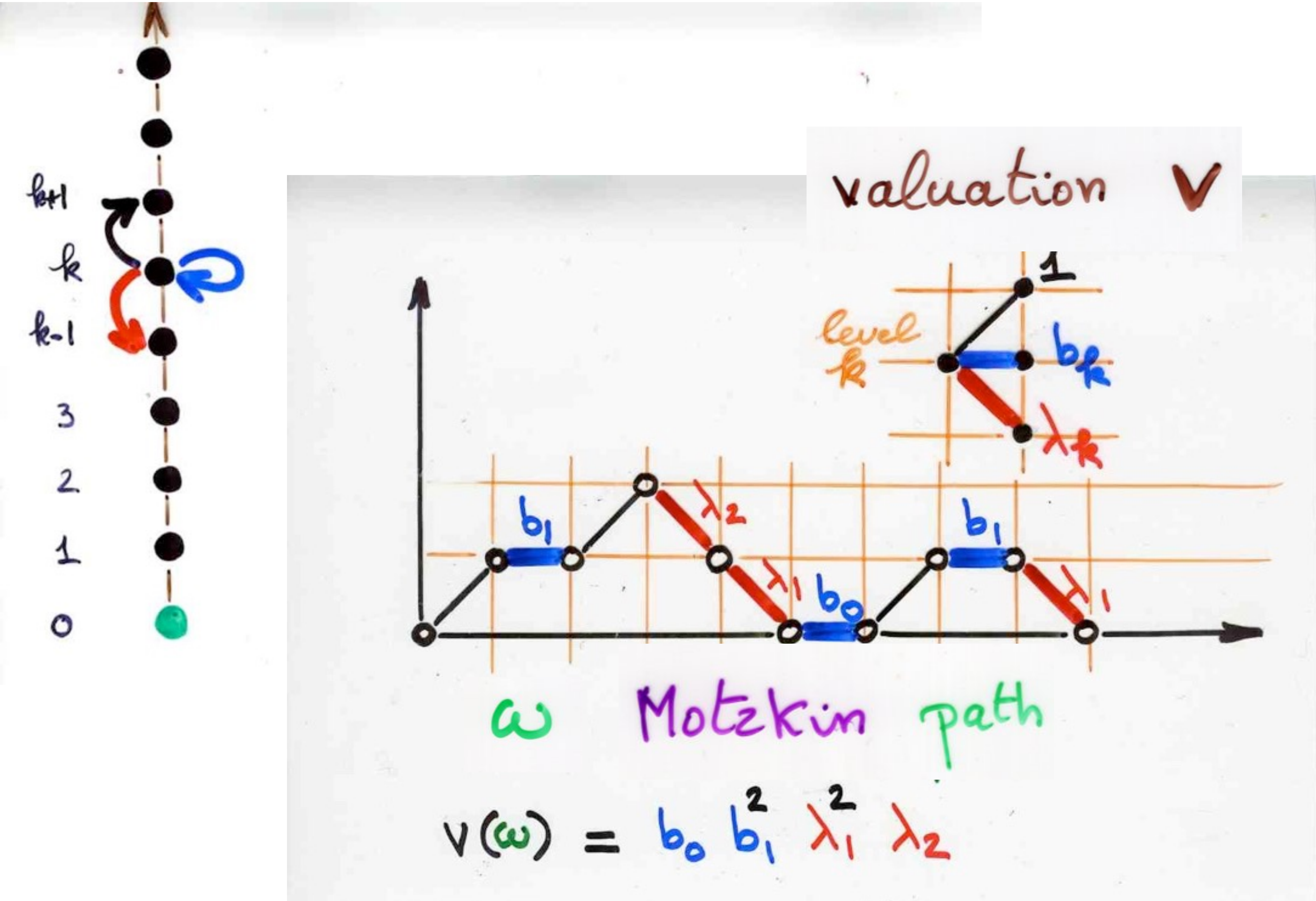




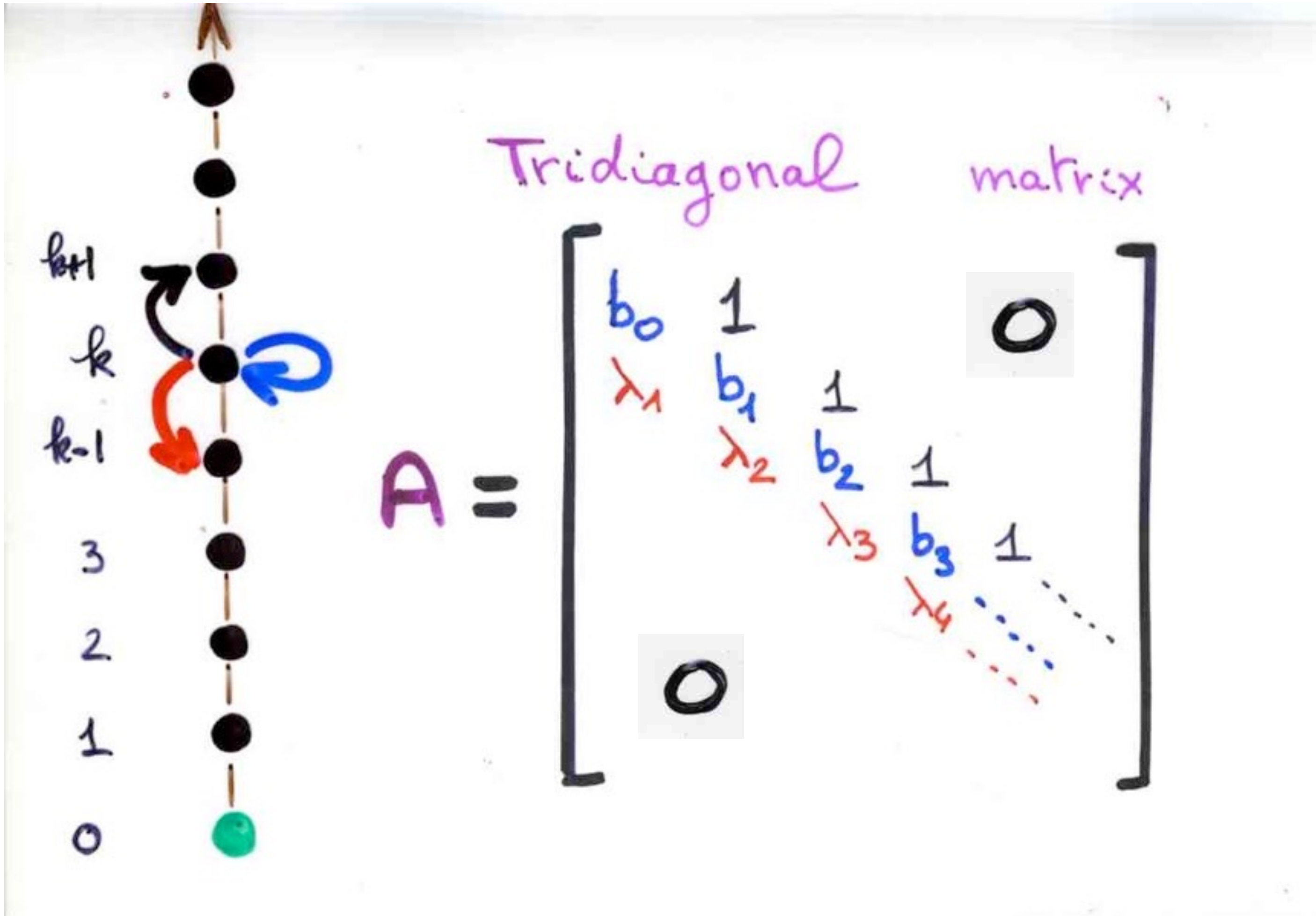
Tridiagonal matrices

Tridiagonal matrix

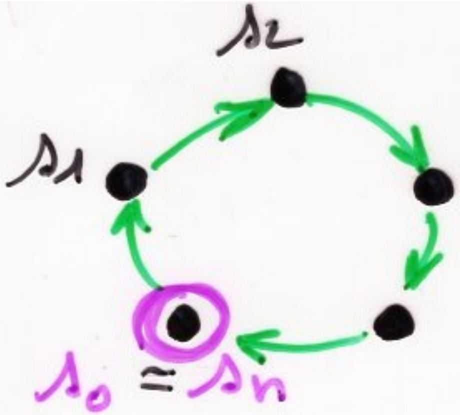
$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & \mathbf{0} & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & \mathbf{0} & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

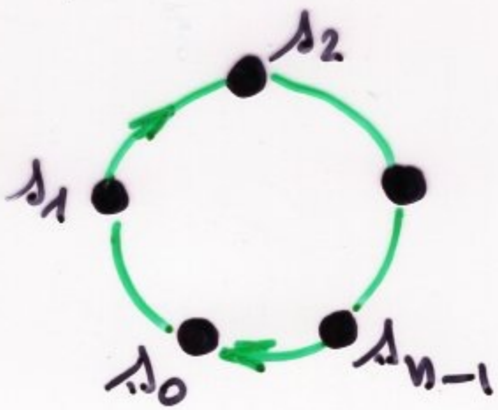


valuation v
k+1
k
k-1
3
2
1
0
level k
1
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
v(ω) = b_0 b_1^2 λ_1^2 λ_2


Tridiagonal matrix
k+1
k
k-1
3
2
1
0
A =
b0 1
λ1 b1 1
λ2 b2 1
λ3 b3 1
λ4
0
0

elementary circuit $\omega = (s_0, \dots, s_n)$
with $s_0 = s_n$, all vertices are disjoint
except $s_0 = s_n$.

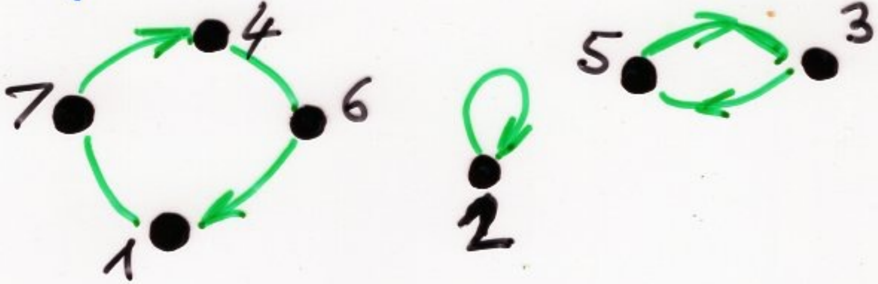




Cycle = elementary circuit up to a circular permutation of the vertices

Cycles

of a permutation σ



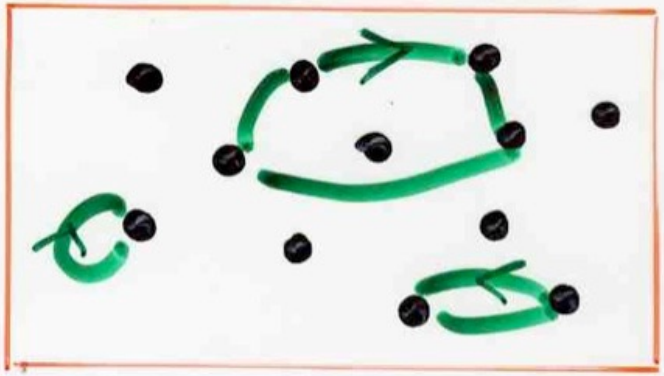
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\mathrm{inv}(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

permutations of $\mathfrak{S}_n$

$$\det(I_n - A) = \sum_{\{\gamma_1, \ldots, \gamma_r\}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$

2 by 2 disjoint cycles



$$A_n = \begin{bmatrix} b_0 & 1 & & & 0 \\ \lambda_1 & b_1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \lambda_{n-1} & b_{n-1} \end{bmatrix}$$

$$P_n(x) \doteq \det(I_n x - A_n)$$

characteristic polynomial of the matrix A_n