Course IMSc, Chennai, India

January-March 2019

Combinatorial theory of orthogonal polynomials and continued fractions

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

mirror website
www.imsc.res.in/~viennot

Chapter 1
Paths and moments

Ch 1b

Xavier Viennot
CNRS, LaBRI, Bordeaux
www.viennot.org

IMSc, Chennai
January 14, 2019

mirror website
www.imsc.res.in/~viennot

Reminding Ch 1a

sequence $\{P_n(x)\}_{n\geq 0}$

4 examples

orthogonal polynomials

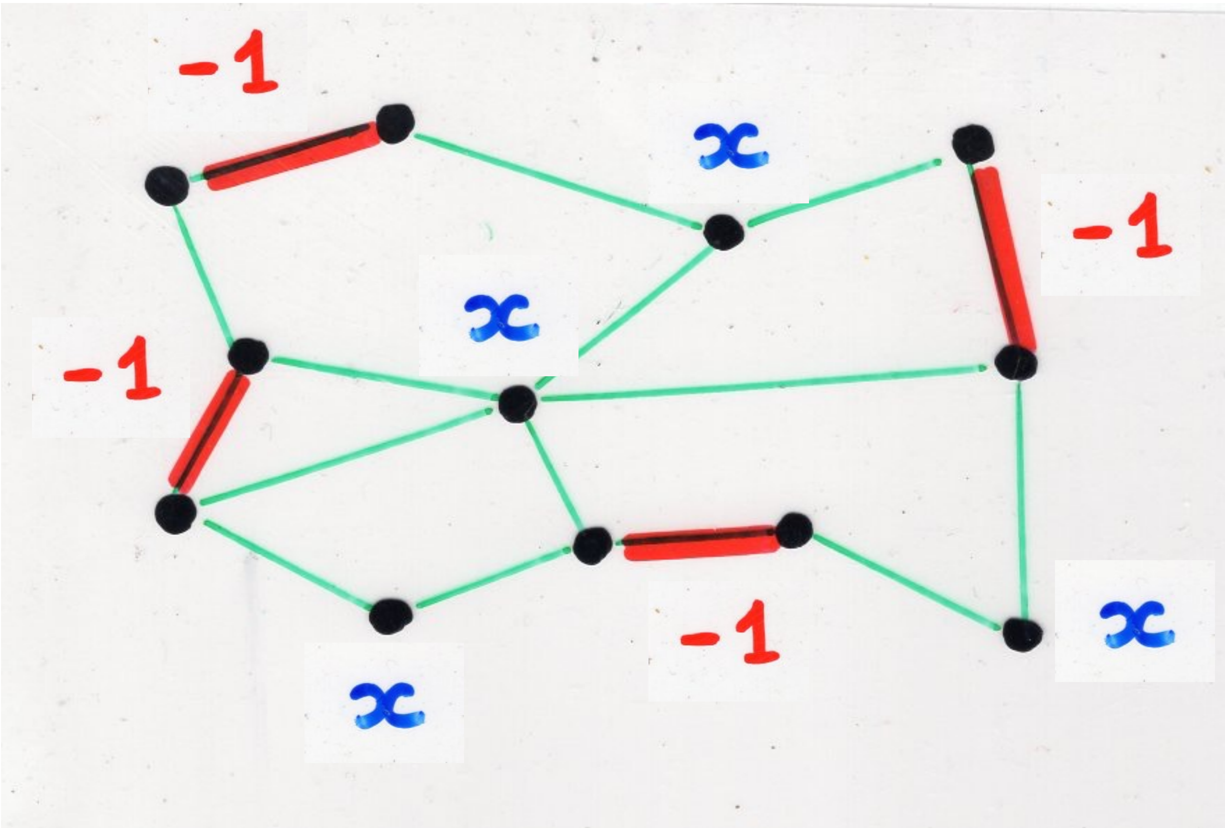
Tchebychev 1st kind $T_n(x)$
2nd kind $U_n(x)$

$$\cos(n\theta) = T_n(\cos\theta)$$

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

Hermite polynomial $H_n(x)$

Laguerre polynomial $L_n(x)$



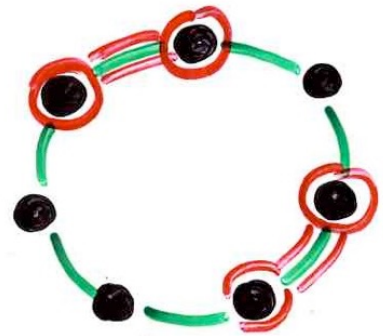
matching polynomial of a graph

monic polynomial

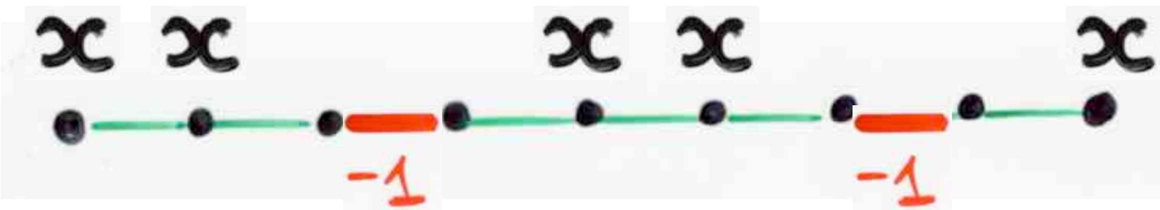
$P_k(x) = x^k + \ldots$

$\deg(P_k) = k$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

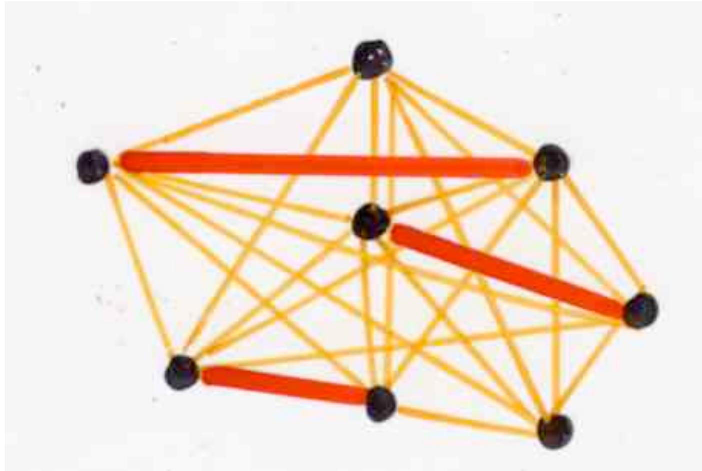


$$U_n(x) = S_n(2x)$$



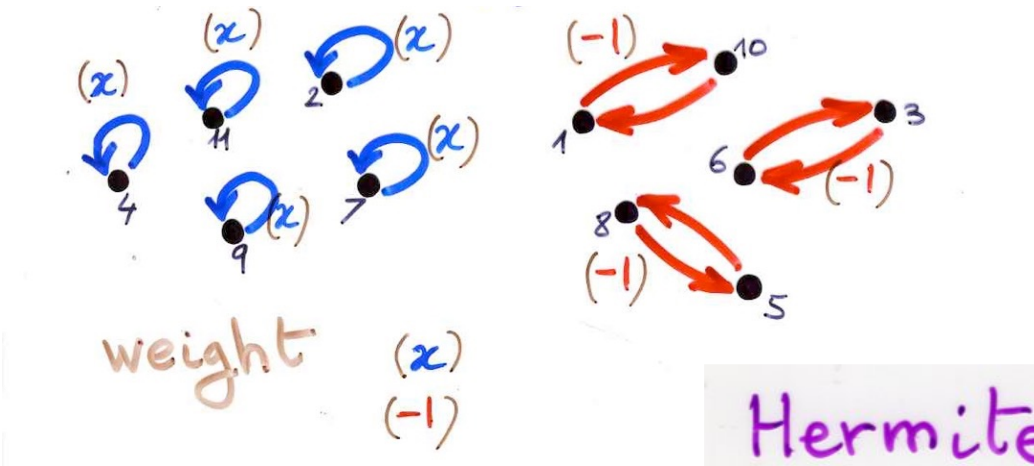
Hermite polynomial

$H_n(x)$



(combinatorial) Hermite polynomials

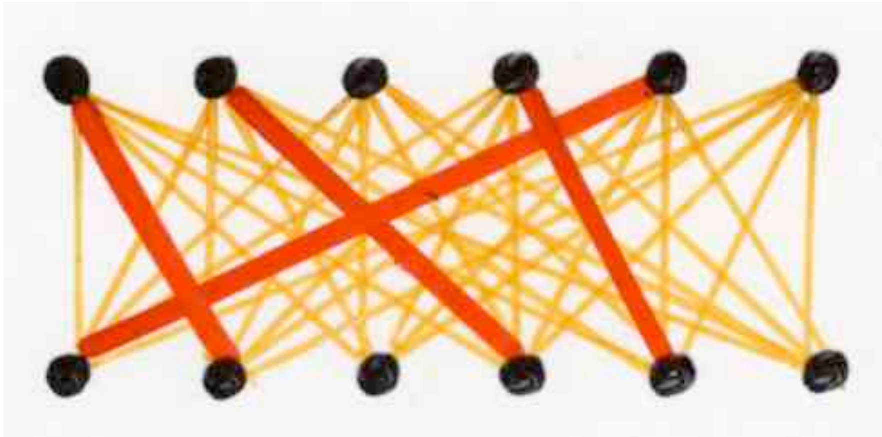
$2^{n/2} H_n(\sqrt{2}\, x)$



Hermite configuration

Laguerre polynomial

$L_n(x)$



usual Laguerre polynomials

$$\frac{(-1)^n}{n!} L_n(x)$$

$$L_n^{(\alpha)}(x)$$

$$\alpha = 0$$

Definition $\{P_n(x)\}_{n\geq 0}$ sequence of polynomials orthogonal iff $\exists$

$f : \mathbb{K}[x] \longrightarrow \mathbb{K}$ linear functional

(i) $\deg(P_n) = n$, for $n \geq 0$

degree

(ii) $f(P_k P_l) = 0$, for $k \neq l \geq 0$

(iii) $f(P_k^2) \neq 0$, for $k \geq 0$

$$f(x^n) = \mu_n$$

moments

moments of (Tchebychev) 1st kind / 2nd kind

$$\begin{cases} \mu_{2n} = \binom{2n}{n} \\ \mu_{2n+1} = 0 \end{cases}$$

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

$$U_n(x) = S_n(2x)$$

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Catalan number

moments of Hermite polynomial

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \times 3 \times \ldots \times (2n-1)$

(combinatorial) Hermite polynomials

number of involutions on $\{1, \ldots, 2n\}$ with no fixed points

moments Laguerre polynomials

$\mu_n = n!$

linearization coefficients

$$f\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right)$$

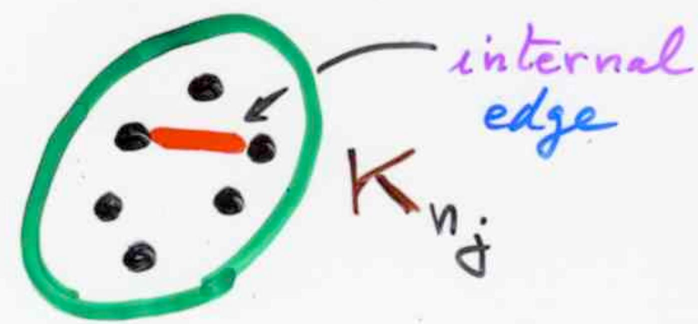
$$f\left(L_{n_1}(x)\,L_{n_2}(x)\cdots L_{n_k}(x)\right)$$

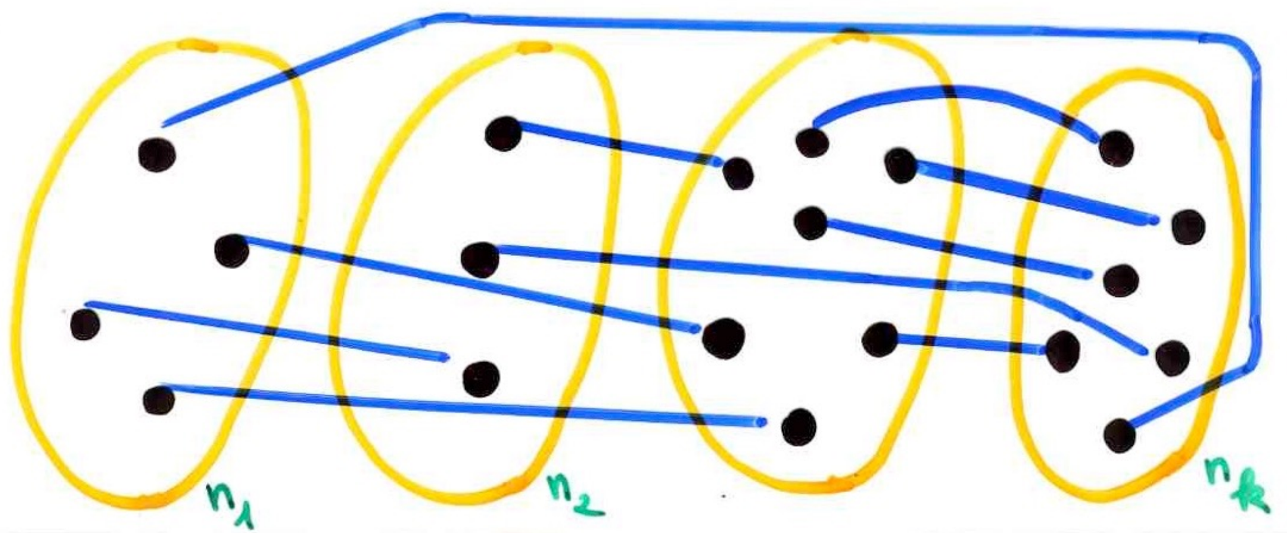
Proposition

positivity

$$\mathbf{f}\left(\mathbf{H}_{n_1}(x)\,\mathbf{H}_{n_2}(x)\cdots\mathbf{H}_{n_k}(x)\right) =$$

number of perfect matchings of the graph $K_{n_1} \oplus K_{n_2} \oplus \cdots \oplus K_{n_k}$ with no "internal" edges

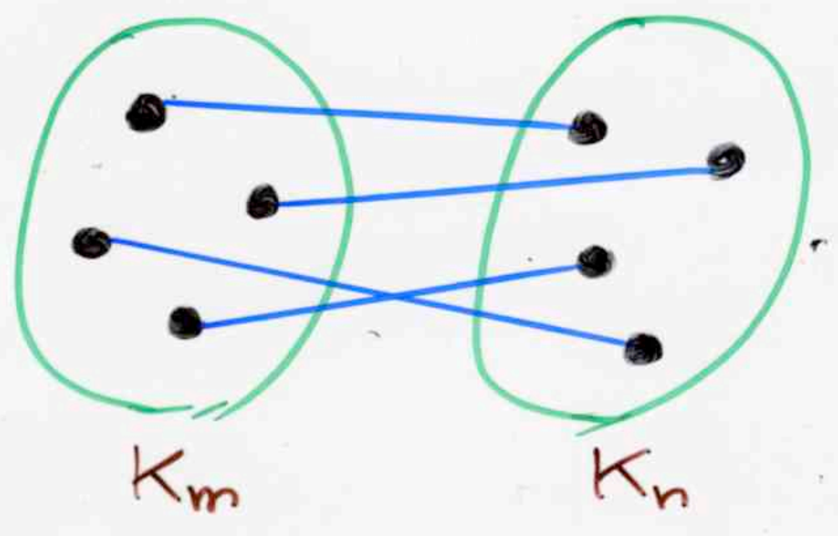




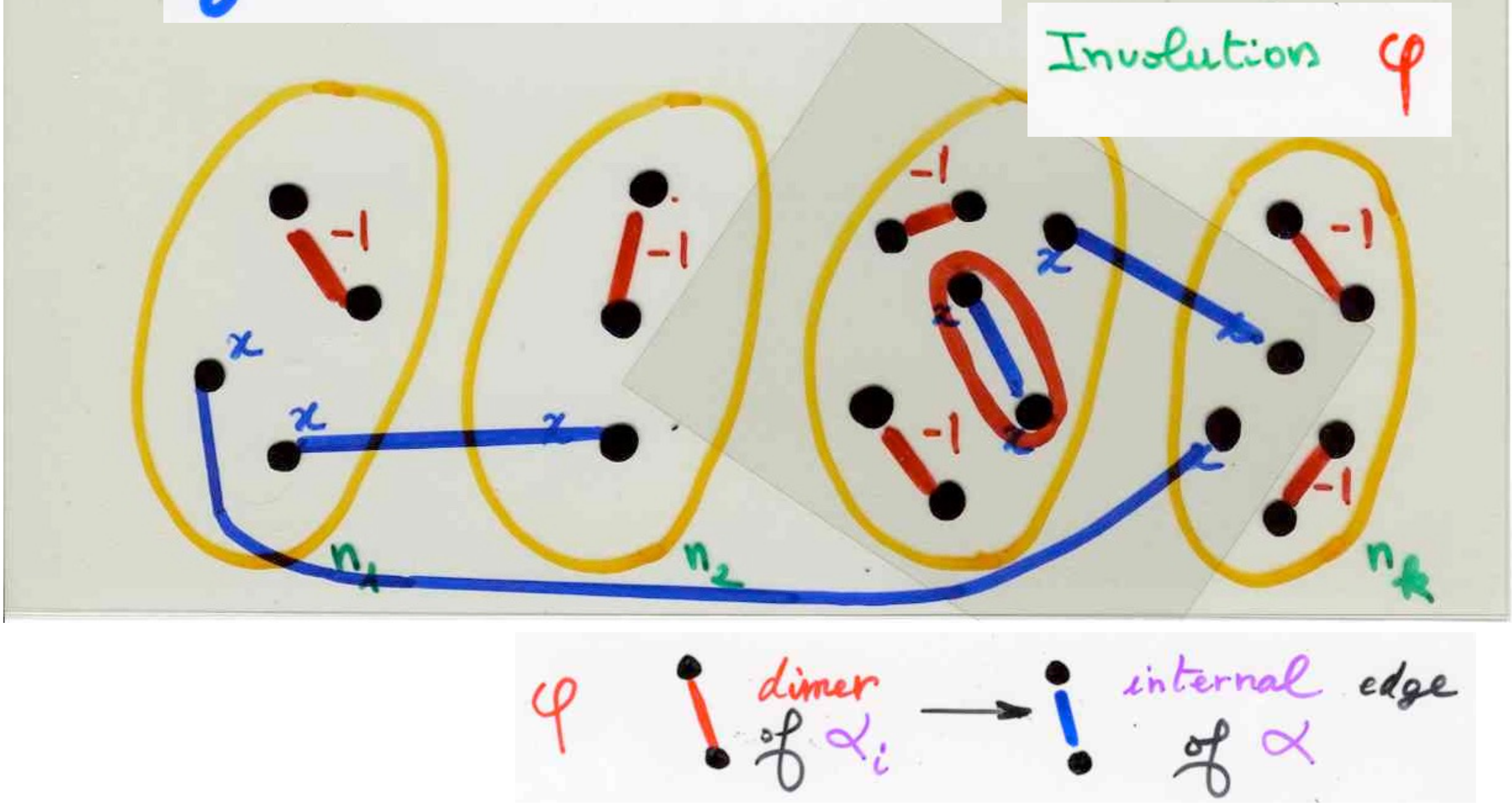
in particular:

orthogonality!

$$\mathcal{f}\left(H_m(x)\, H_n(x) \right) = n!\, \delta_{m,n}$$



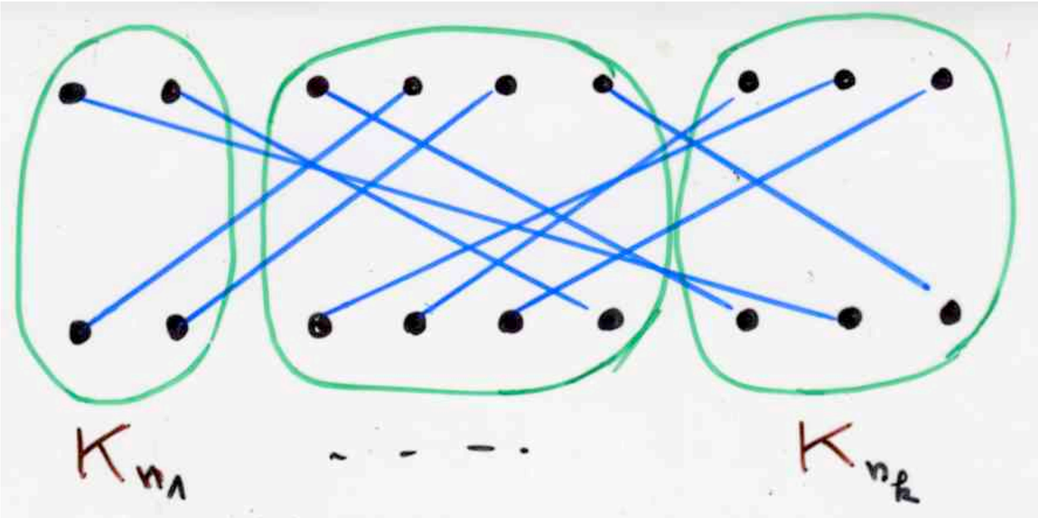
$$f\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right) =$$



Proposition **exercise**

$$f\left(L_{n_1}(x)\, L_{n_2}(x) \cdots L_{n_k}(x)\right) =$$

number of perfect matchings of the graph L $K_{n_1,n_1} \oplus \cdots \oplus K_{n_k,n_k}$ with no "internal" edges



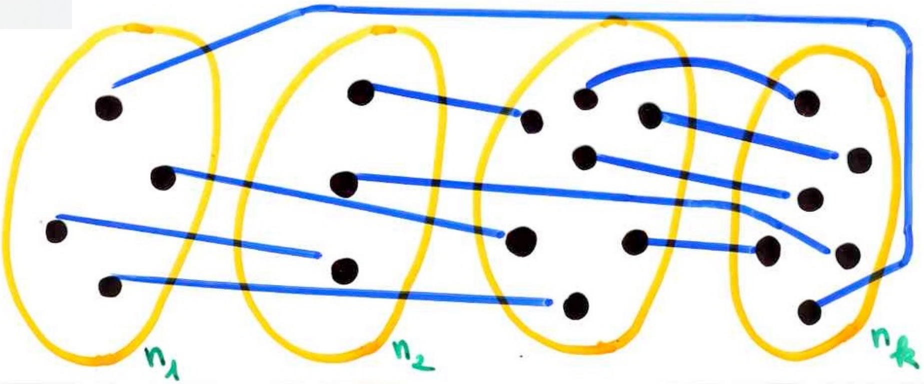
More about

the linearization coefficients

$$f\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right)$$

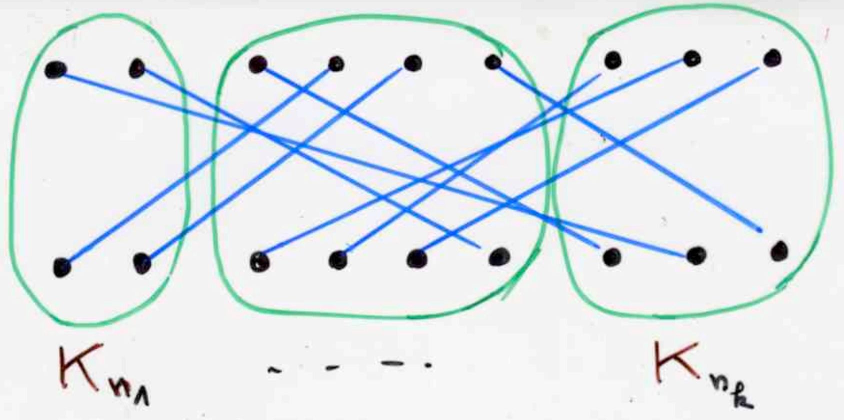
$$f\left(L_{n_1}(x)\,L_{n_2}(x)\cdots L_{n_k}(x)\right)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\prod_{i=1}^{k} H_{n_i}(x)\right)e^{-x^2/2}\,dx$$



$$\frac{1}{\Gamma(\alpha+1)}\int_{0}^{\infty}\left(\prod_{i=1}^{k} L_{n_i}^{(\alpha)}(x)\right)x^{\alpha}e^{-x}\,dx$$

here $\alpha = 0$

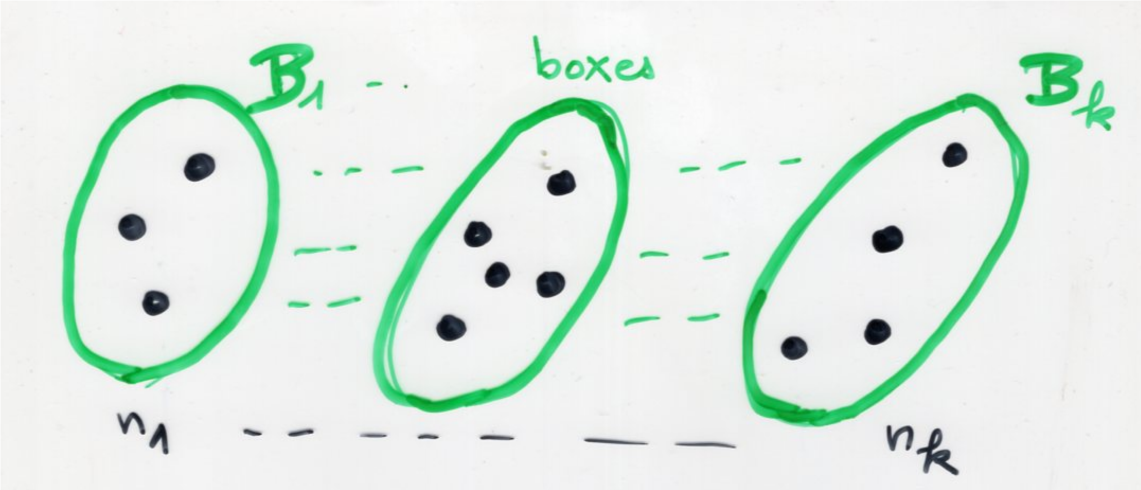


exercise

$$\int \left(H_\ell(x)\, H_m(x)\, H_n(x) \right) = \frac{n!\, s!\, m!}{(s-n)!\,(s-\ell)!\,(s-m)!}$$

$$s = \frac{n+\ell+m}{2}$$

$$\binom{\ell}{s-n}\binom{m}{s-\ell}\binom{n}{s-m} \quad (s-n)!\,(s-\ell)!\,(s-m)!$$



each ball ● in a box B_i
is moving to another box B_j
$i \neq j$

generalized
derangements

$n_1 = n_2 = \cdots = n_k$

derangements

permutations with no fixed points

d_n

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right)$$

$$\frac{d_n}{n!} \to \frac{1}{e}$$

Hermite polynomials $H_n(x)$

Azor, Gillis, Victor (1982)

Laguerre polynomial $L_n(x)$ $L_n^{(\alpha)}(x)$

Askey, Ismail, Rashed (1975)

$\alpha = 0$

Askey, Ismail (1976)

Foata, Zeilberger (1988)

Zeng (1988)(1990)(1992) ------

Lecture Notes (2016)

Kim, Zeng (2001)

Anshelevich (2005)

many others -----

5 Sheffer orthogonal polynomials

Complement

The power of bijective proof:

The Askey-Wilson integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\prod_{i=1}^{k} H_{n_i}(x)\right)e^{-x^2/2}\,dx$$

$$\frac{1}{\Gamma(\alpha+1)}\int_{0}^{\infty}\left(\prod_{i=1}^{k} L_{n_i}^{(\alpha)}(x)\right)x^{\alpha}e^{-x}\,dx$$

exponential structures

$$\sum_{n\geqslant 0} a_n \frac{t^n}{n!}$$

$$\sum_{n\geqslant 0} f(H_{n_1}\cdots H_{n_k})\,\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_k^{n_k}}{n_k!} =$$

elementary
symmetric
functions

$e_n(x_1, \dots x_k)$

$$\sum_{n\geqslant 0} f(H_{n_1} \cdots H_{n_k}) \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_k^{n_k}}{n_k!} = \exp\left(e_2(x_1, \dots, x_k)\right)$$

$f(L_{n_1} \cdots L_{n_k})$

$$\frac{1}{\left(1 - e_2 - 2e_3 - \dots - (k-1)e_k\right)^{(\alpha+1)}}$$

$L_n^{(\alpha)}(x)$

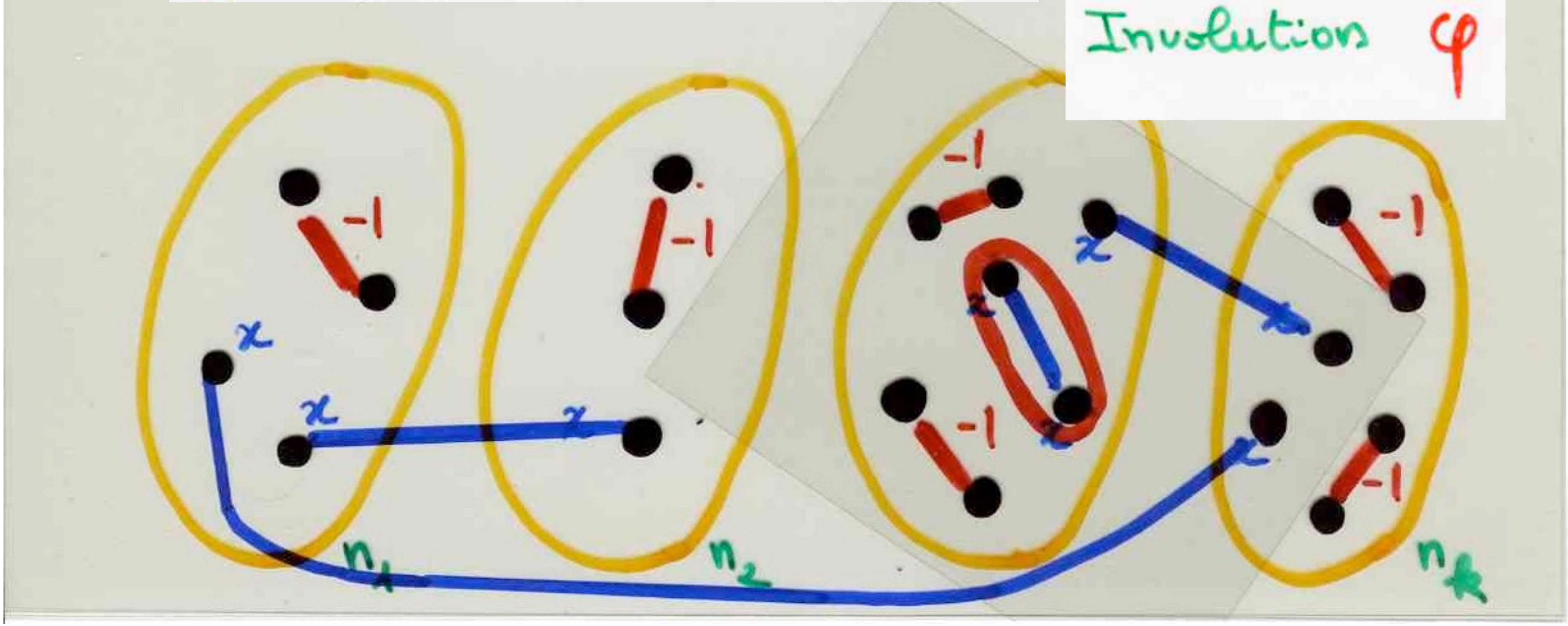
MacMahon Master theorem
β - extension

$\beta = \alpha + 1$

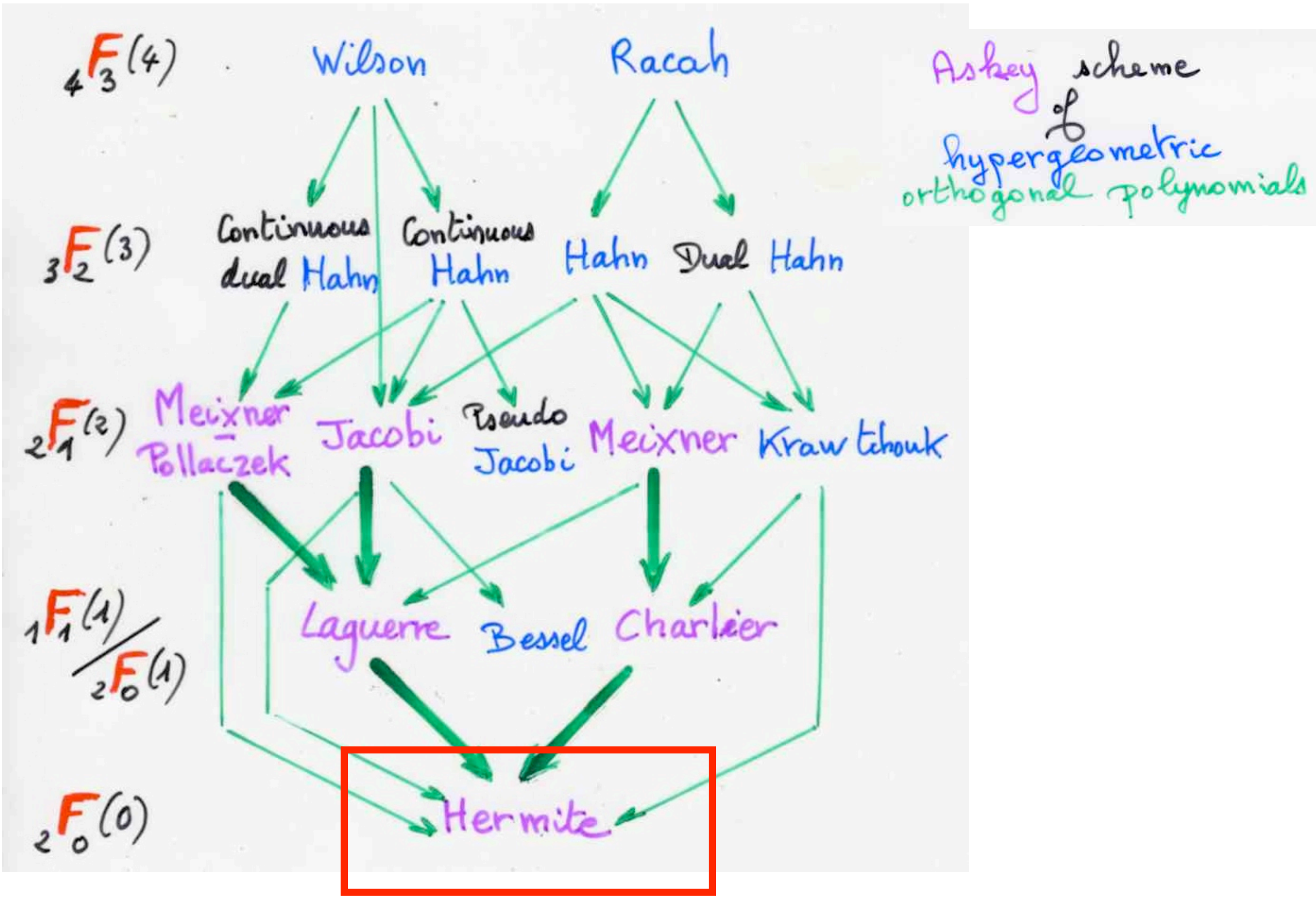
de Sainte-Catherine, X.V. (1985)

$$f\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right) =$$

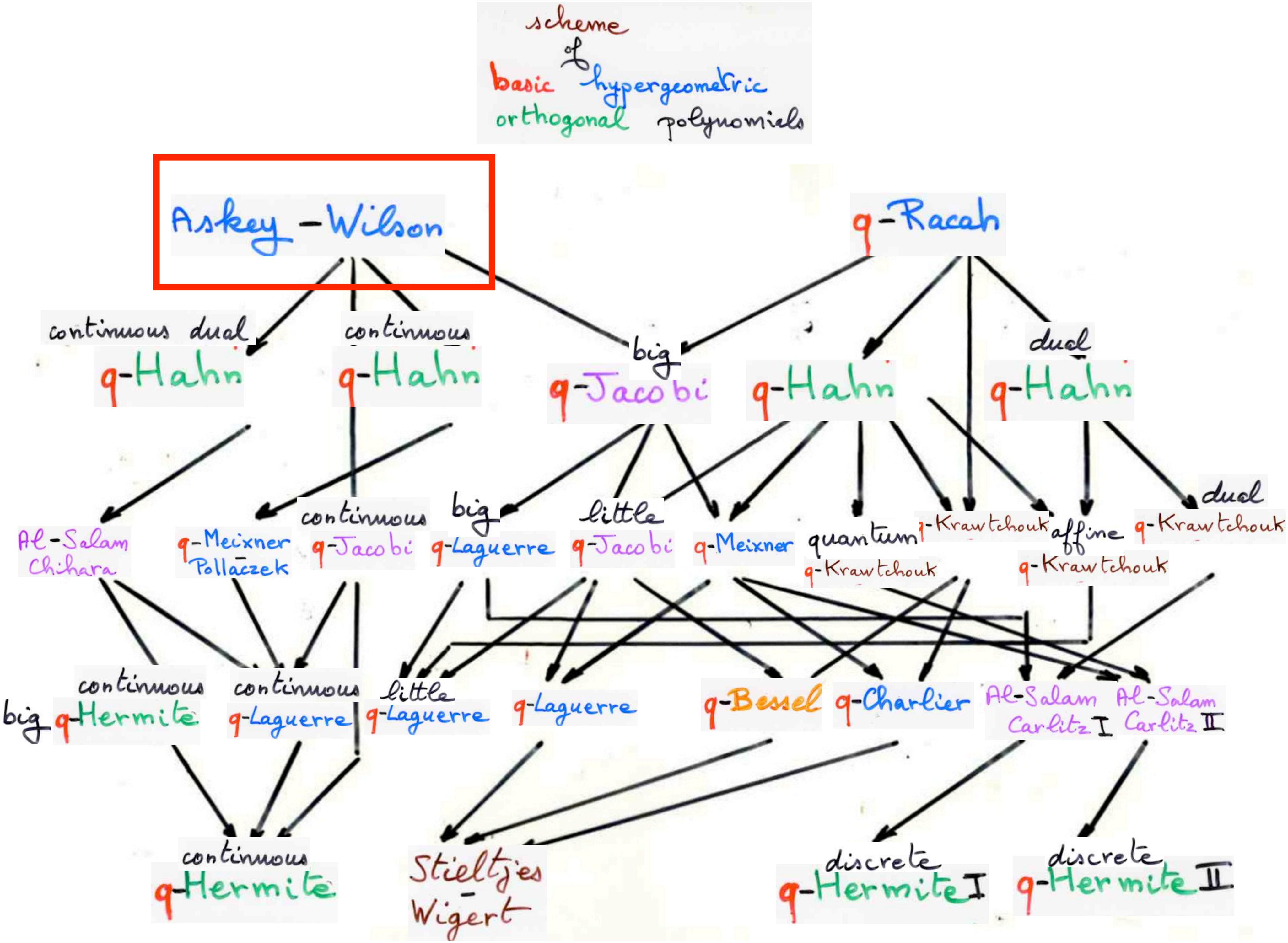
Involution φ



φ dimer of α_i → internal edge of α



$$f\left(H_{n_1}(x)\,H_{n_2}(x)\cdots H_{n_k}(x)\right) =$$


scheme of basic hypergeometric orthogonal polynomials
Askey-Wilson
q-Racah
continuous dual q-Hahn
continuous q-Hahn
big q-Jacobi
q-Hahn
dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
continuous q-Jacobi
big q-Laguerre
little q-Jacobi
q-Meixner
quantum q-Krawtchouk
q-Krawtchouk
affine q-Krawtchouk
dual q-Krawtchouk
big q-Hermite
continuous q-Laguerre
little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
continuous q-Hermite
Stieltjes-Wigert
discrete q-Hermite I
discrete q-Hermite II

Askey-Wilson polynomials

$$P_n(x) = P_n(x; a, b, c, d \mid q)$$

$$P_n(x) = a^{-n} (ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k}$$

$$(a_1, a_2, \ldots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k)$$

${}_4\phi_3$ basic hypergeometric function

Askey-Wilson polynomials

$$\int_0^{\pi} p_n(\cos\theta, a, b, c, d, q)\, p_m(\cos\theta, a, b, c, d, q)\, w(\cos\theta, a, b, c, d, q)\, d\theta = 0 \quad n \neq m$$

$$w(\cos\theta, a, b, c, d \,|\, q) =$$

$$\frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$$

The Askey-Wilson integral

$$\frac{(q)_\infty}{2\pi}\int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta = \frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

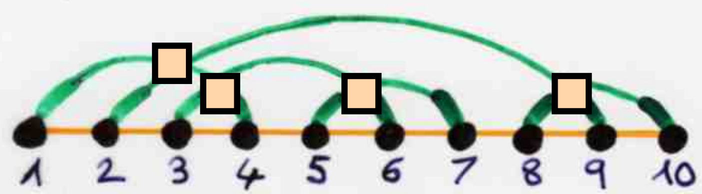
integral of the product
of 4 q-Hermite polynomials

q-analogue of Hermite polynomials

$$H_k(x;q)$$

$$H_{k+1}(x) = x H_k(x) - k H_{k-1}(x)$$

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$



$$(a)_\infty = \prod_{i \geq 0} (1 - a q^i)$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi H_k(\cos\theta | q) H_l(\cos\theta | q) (e^{2i\theta})_\infty (e^{-2i\theta})_\infty = (q)_k \delta_{kl}$$

The Askey-Wilson integral

integral of the product
of 4 q-Hermite polynomials

Ismail, Stanton, X.V. (1987)

$$\mathfrak{f}\left(\mathbf{H}_{n_1}(x)\,\mathbf{H}_{n_2}(x)\cdots\mathbf{H}_{n_k}(x)\right)$$

linearization coefficients and orthogonality

exemple:
Tchebychev 2nd kind

moments $\mathcal{f}$
(Tchebychev)

2nd kind

$\mathcal{f}(x^n) = \mu_n$

moments

$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan
number

Dyck paths

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

path on X

$$\omega = (s_0, \dots, s_i, s_{i+1}, \dots, s_n)$$

$s_i \in X \quad i = 0, \dots, n$

ω goes from s_0 to s_n

notation $\quad s_0 \overset{\omega}{\rightsquigarrow} s_n$

s_0 starting vertex

s_n ending vertex

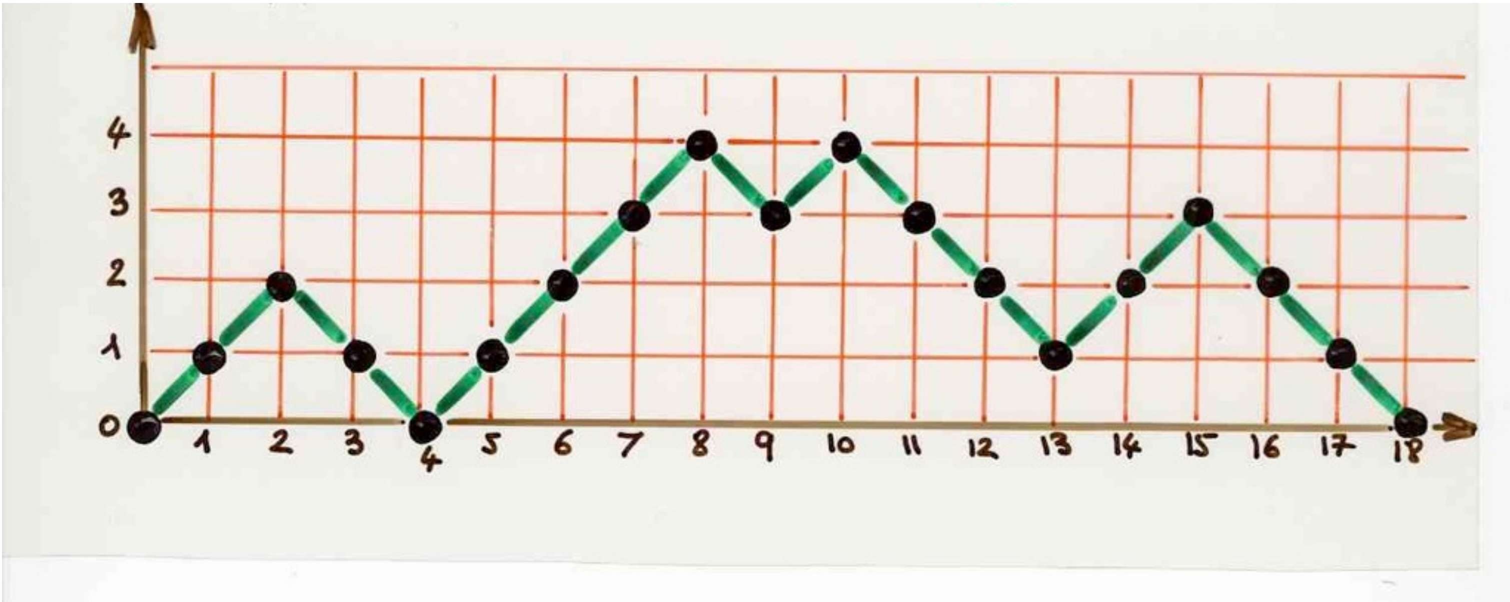
(s_i, s_{i+1}) elementary step

length $|\omega| = n$

(number of elementary steps)

$n+1$ vertices

Dyck paths



$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

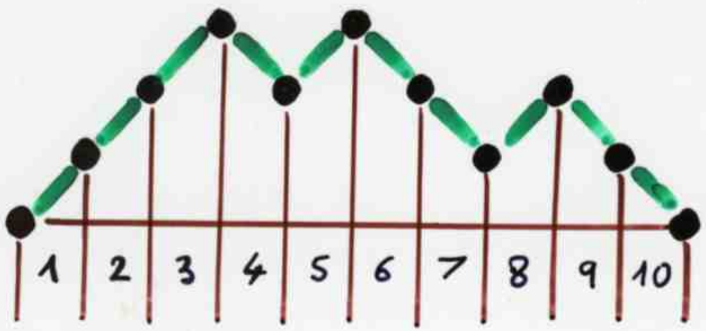
$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

Catalan number

The « essence » of the fundamental sign-reversing involutions

moments f (Tchebychev) 2nd kind

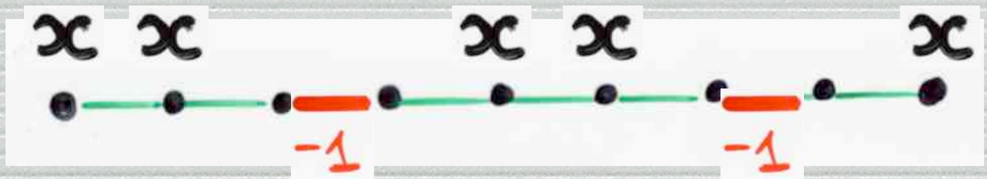
$f(x^n) = \mu_n$ moments



$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$



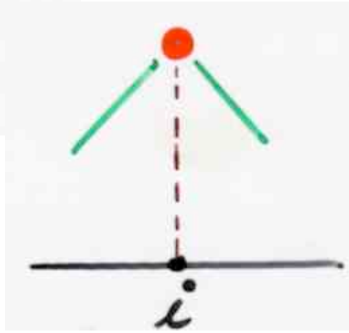
$S_n(x)$

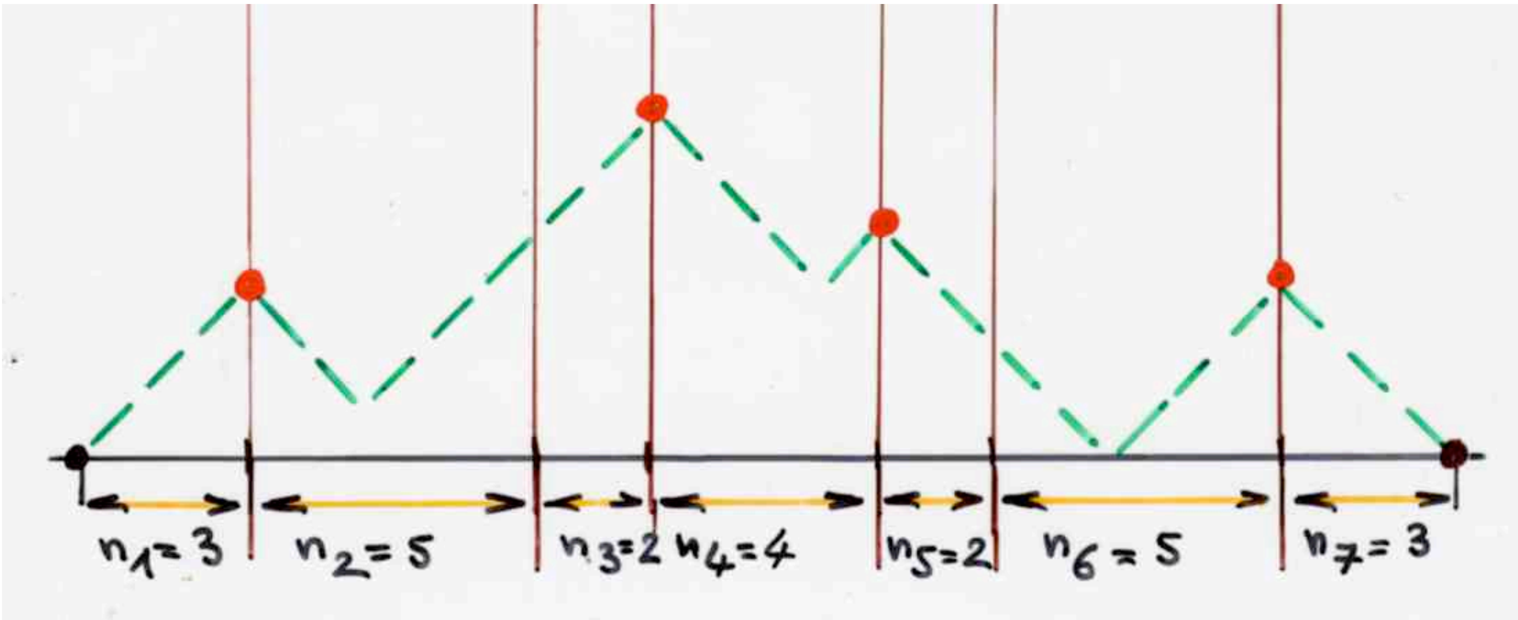
Proposition

$$f\left(S_{n_1}(x)\cdots S_{n_k}(x)\right) =$$

number of Dyck paths ω of length $|\omega| = n_1 + \cdots + n_k$ such that the abscissas i of the peaks of ω are in the set

$$\{n_1,\ n_1+n_2,\ \ldots,\ n_1+n_2+\cdots+n_{k-1}\}$$



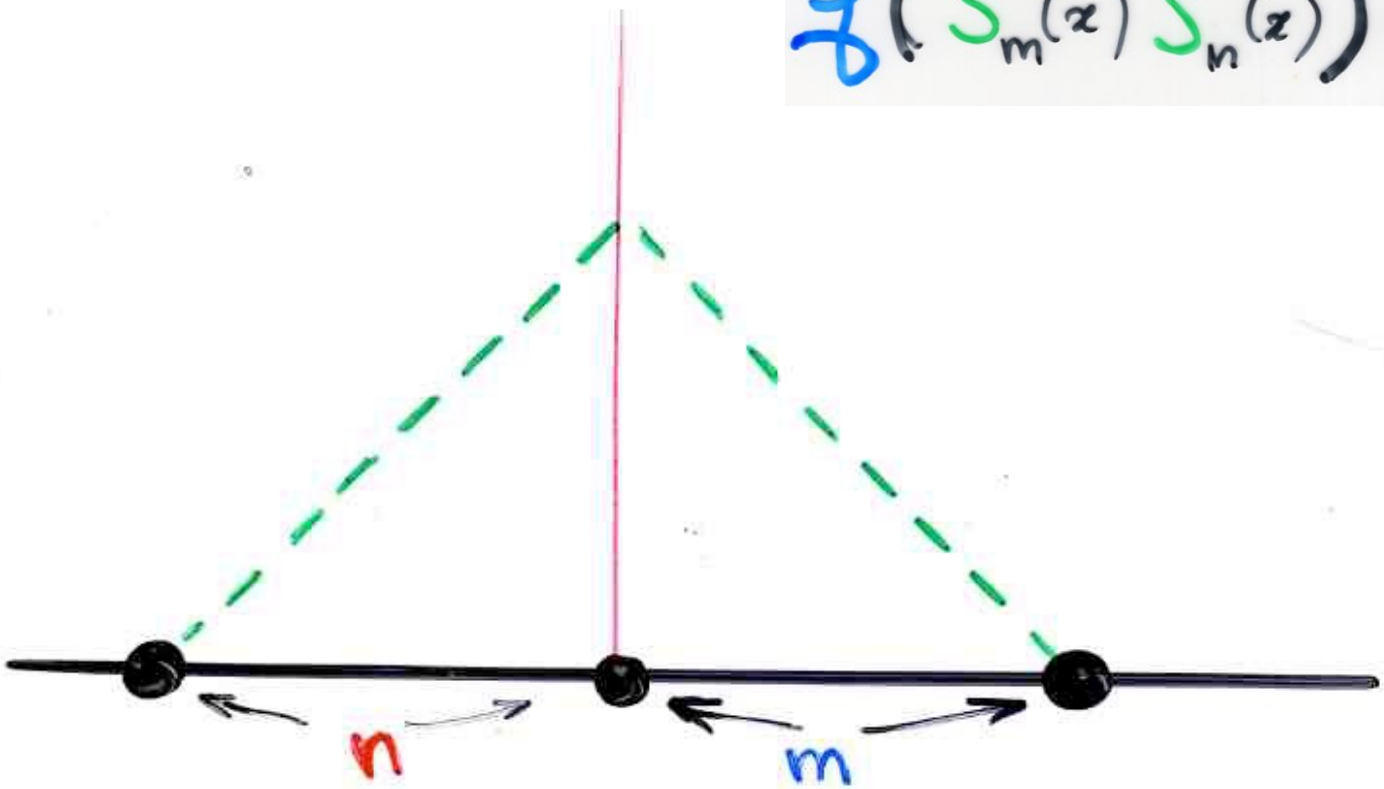


in particular:

Corollary

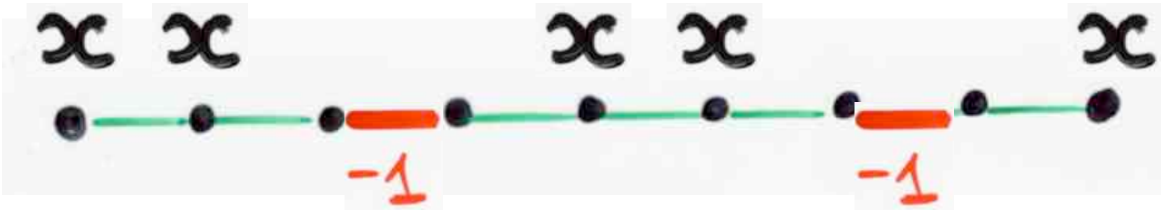
orthogonality!

$$\oint \left(S_m(x)\, S_n(x) \right) = \delta_{mn}$$



$$U_n(x) = S_n(2x)$$

$$S_n(x) = \sum_{\substack{\alpha \text{ matching} \\ \text{of } [0,\, n-1]}} (-1)^{|\alpha|} \, x^{n-2|\alpha|}$$



$|\alpha|$ = number of dimers of α

$ip(\alpha)$ = number of isolated points of α $= n - 2|\alpha|$

$$f\left(S_{n_1}(x)\cdots S_{n_k}(x)\right) = \sum (-1)^{|\alpha_1|+\cdots+|\alpha_k|}$$

$(\alpha_1, \dots, \alpha_k ; \omega)$

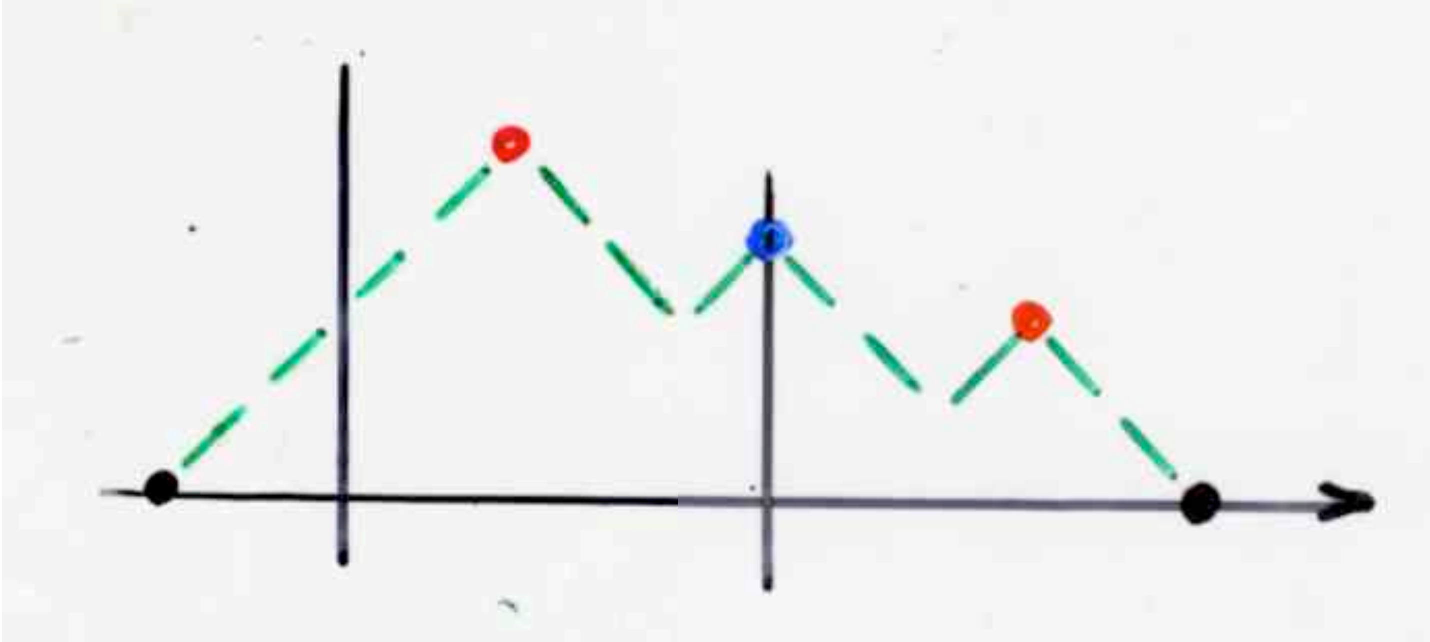
α_i matching of $[0, i-1]$

ω Dyck path

$|\omega| = ip(\alpha_1) + \dots + ip(\alpha_k)$

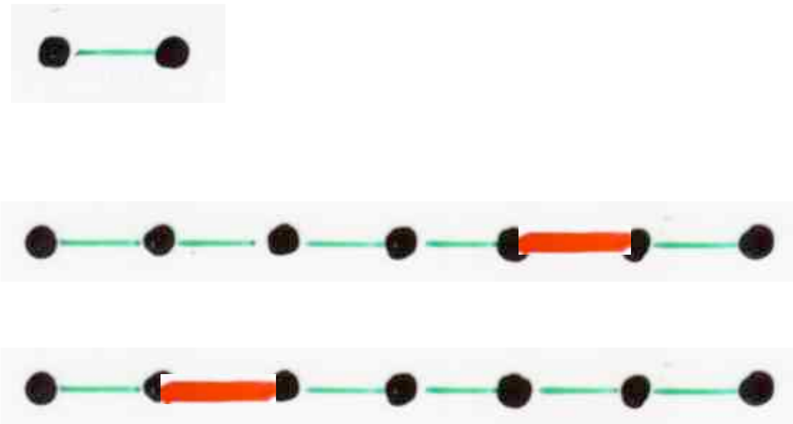
$E_{n_1, \dots, n_k}$

$E_{n_1,\dots,n_k}$



ω Dyck path

$|\omega| = ip(\alpha_1) + \dots + ip(\alpha_k)$



$n_1=2,\ n_2=7,\ n_3=7$

$(\alpha_1, \dots, \alpha_k ; \omega)$

α_i matching of $[0, i-1]$

$E_{n_1,\dots,n_k}$

$(\alpha_1,\dots,\alpha_k;\omega)$

α_i matching of $[0,i-1]$

ω Dyck path

$|\omega| = ip(\alpha_1)+\dots+ip(\alpha_k)$

$$F_{n_1,\dots,n_k} = \{(\alpha_1,\dots,\alpha_k;\omega)\} \subseteq E_{n_1,\dots,n_k}$$

α_j empty matching of $[0,j-1]$

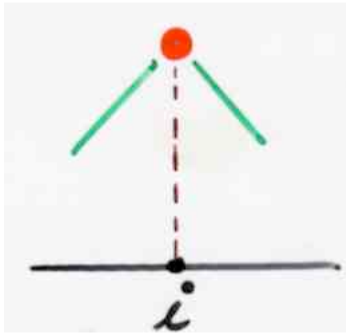
ω Dyck path

$|\omega| = n_1+\dots+n_k$

$j=1,\dots,k$

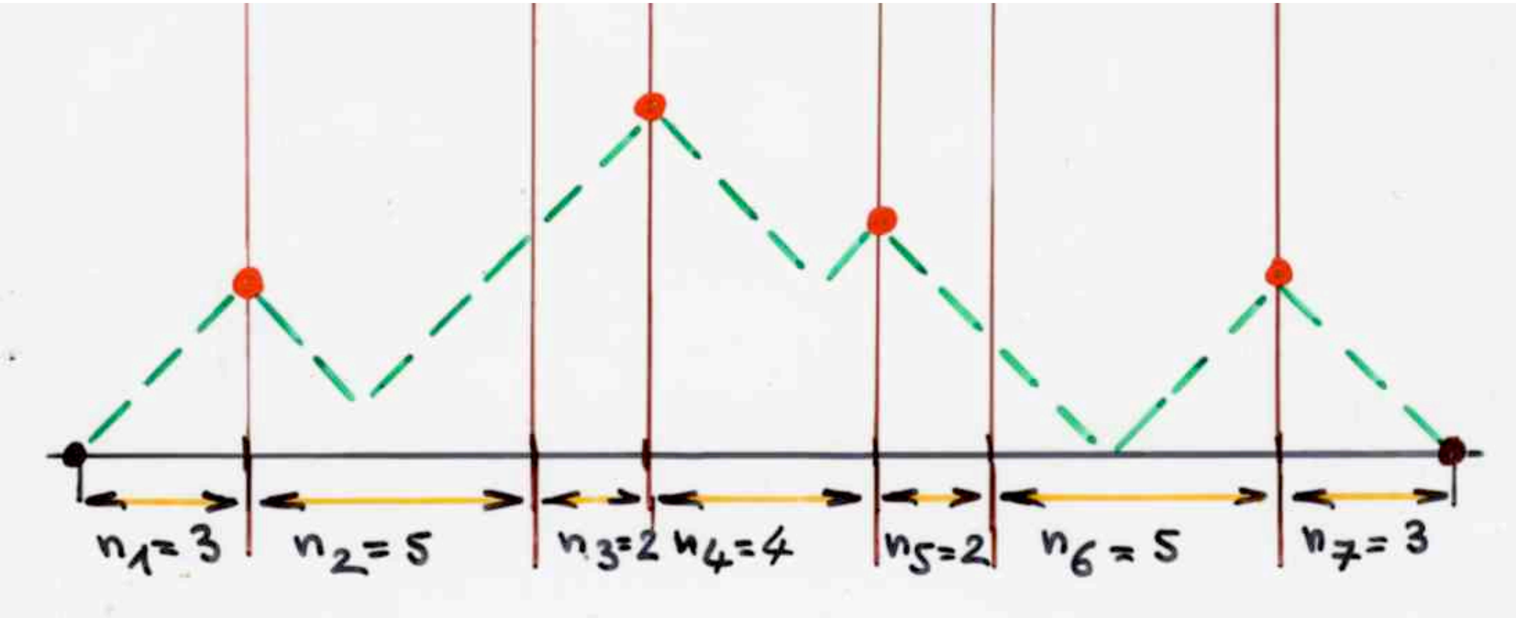
such that the abscissas i of the peaks of ω are in the set

$$\{n_1, n_1+n_2, \dots, n_1+n_2+\dots+n_{k-1}\}$$



construction of an involution φ

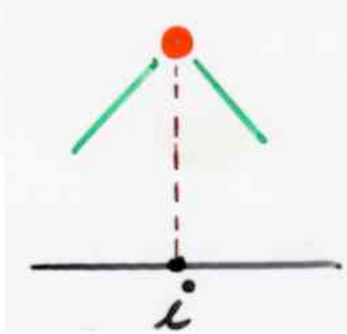
$$E_{n_1,\dots,n_k} \setminus F_{n_1,\dots,n_k} \longrightarrow E_{n_1,\dots,n_k} \setminus F_{n_1,\dots,n_k}$$



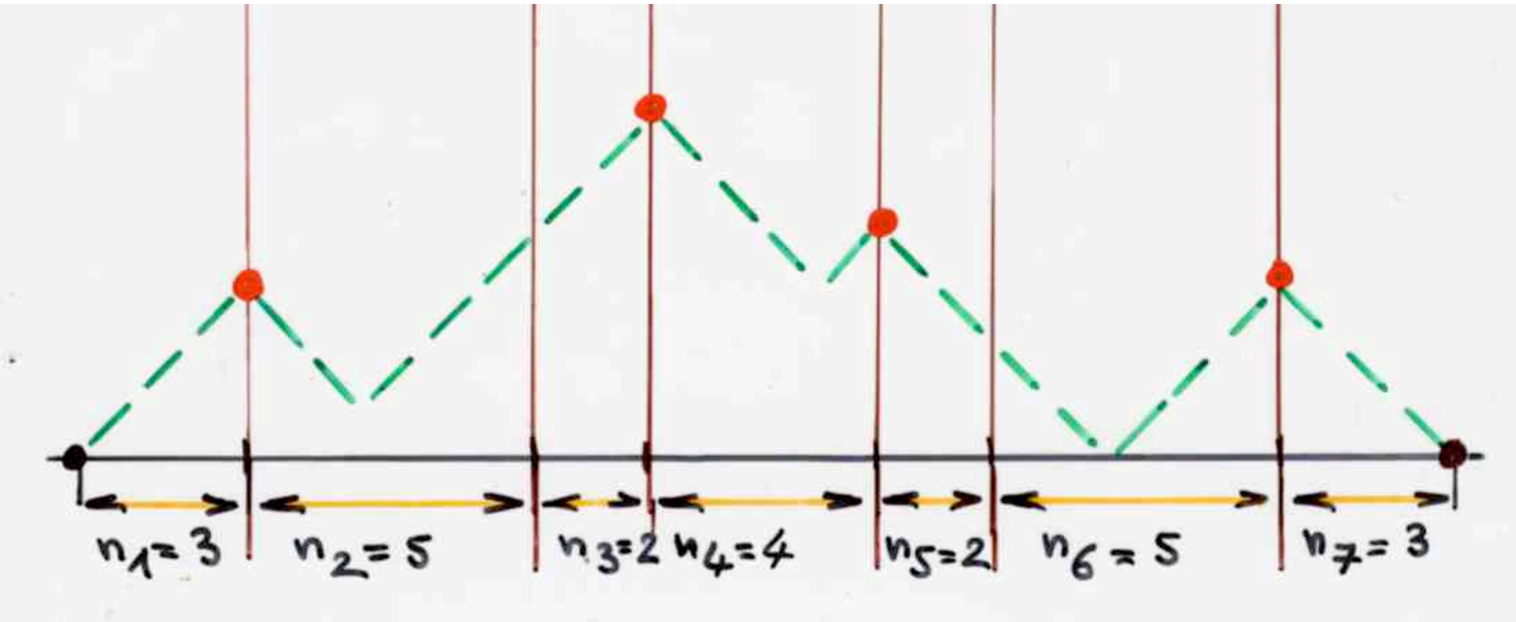
$$F_{n_1,\dots,n_k} = \{(\alpha_1, \dots, \alpha_k; \omega)\} \subseteq E_{n_1,\dots,n_k}$$

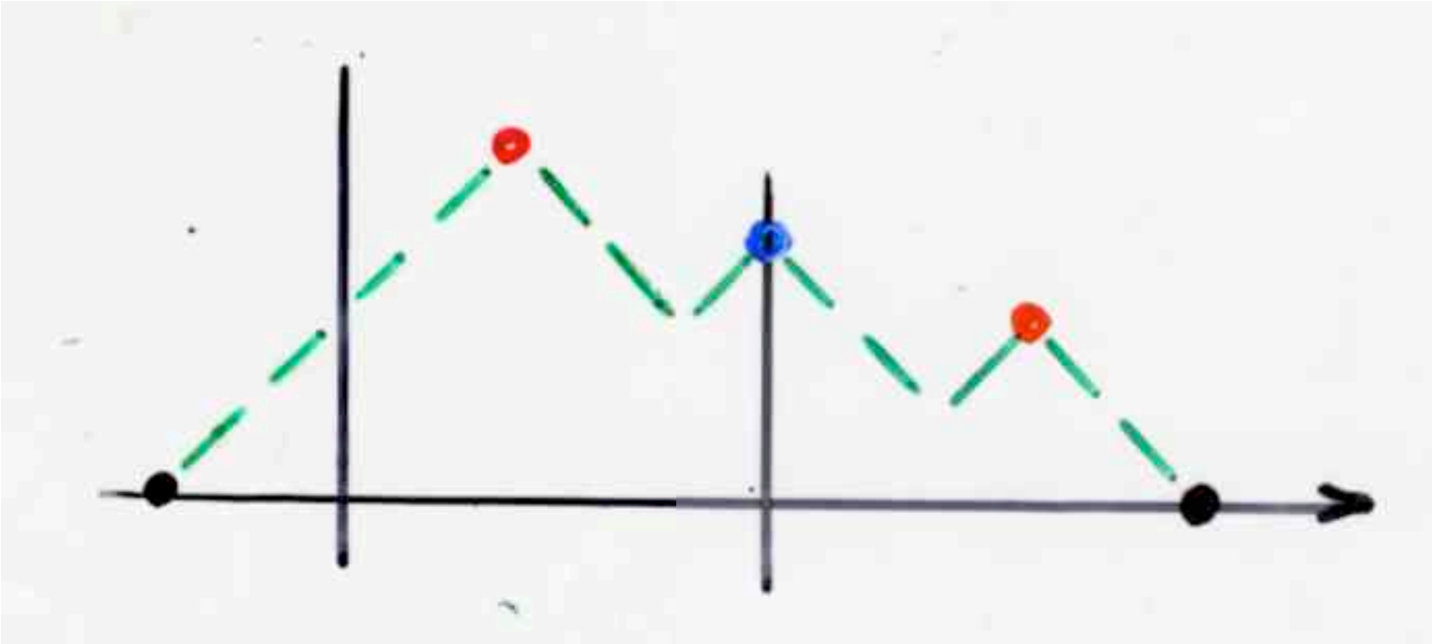
Proposition $f\left(S_{n_1}(x) \cdots S_{n_k}(x)\right)$

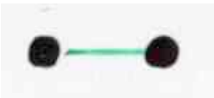
ω Dyck path $|\omega| = n_1 + \dots + n_k$ such that the abscissas i of the peaks of ω are in the set



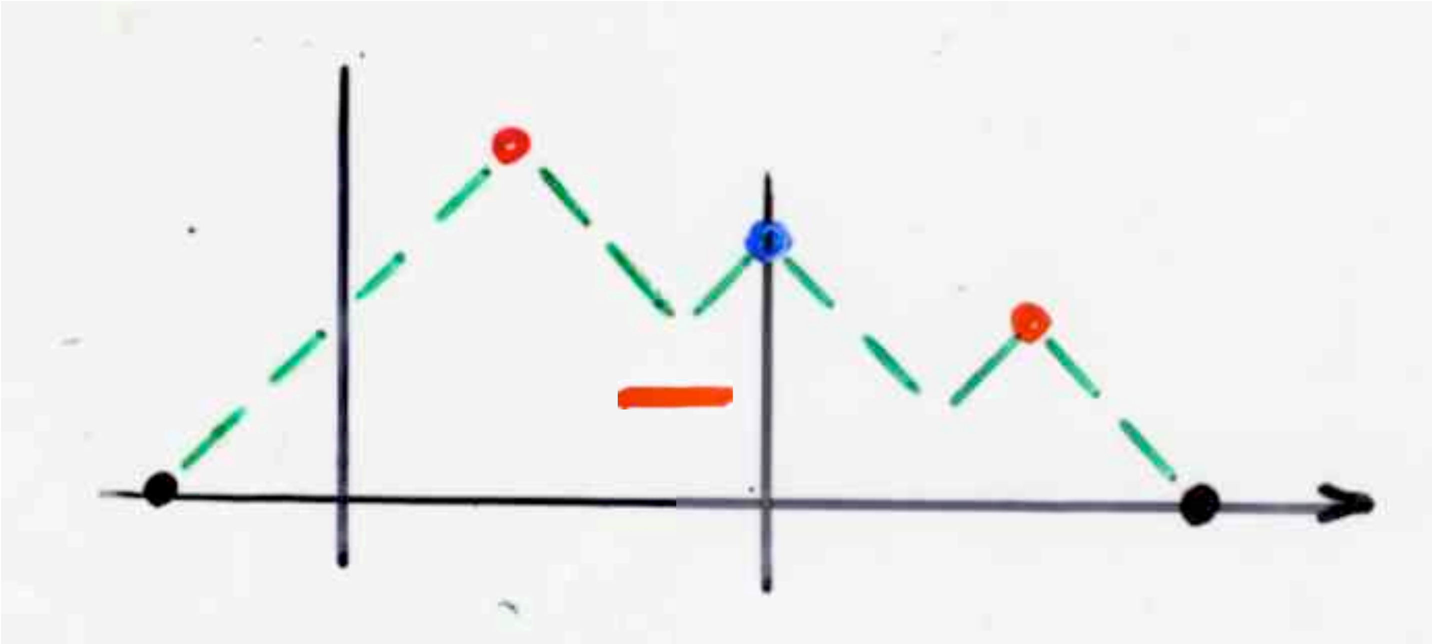
$$\{n_1, n_1+n_2, \dots, n_1+n_2+\dots+n_{k-1}\}$$

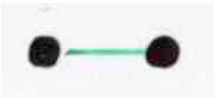




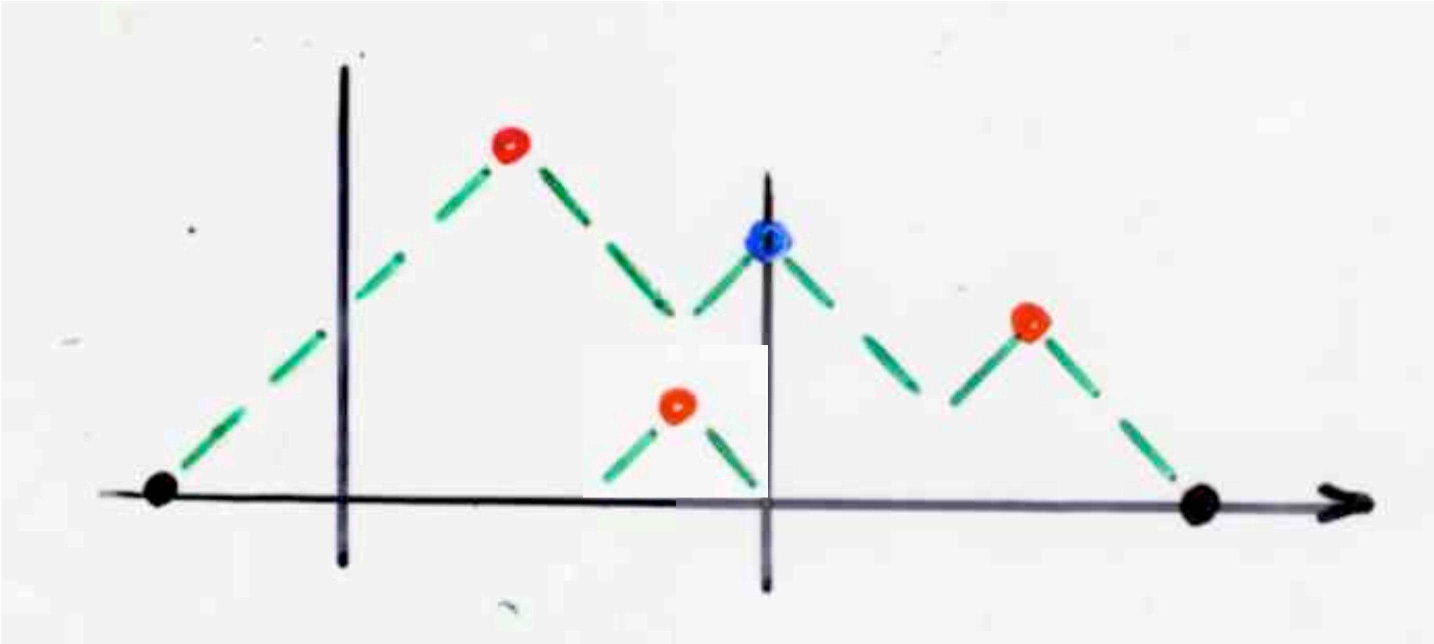


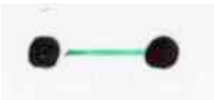
$n_1 = 2,\ n_2 = 7,\ n_3 = 7$



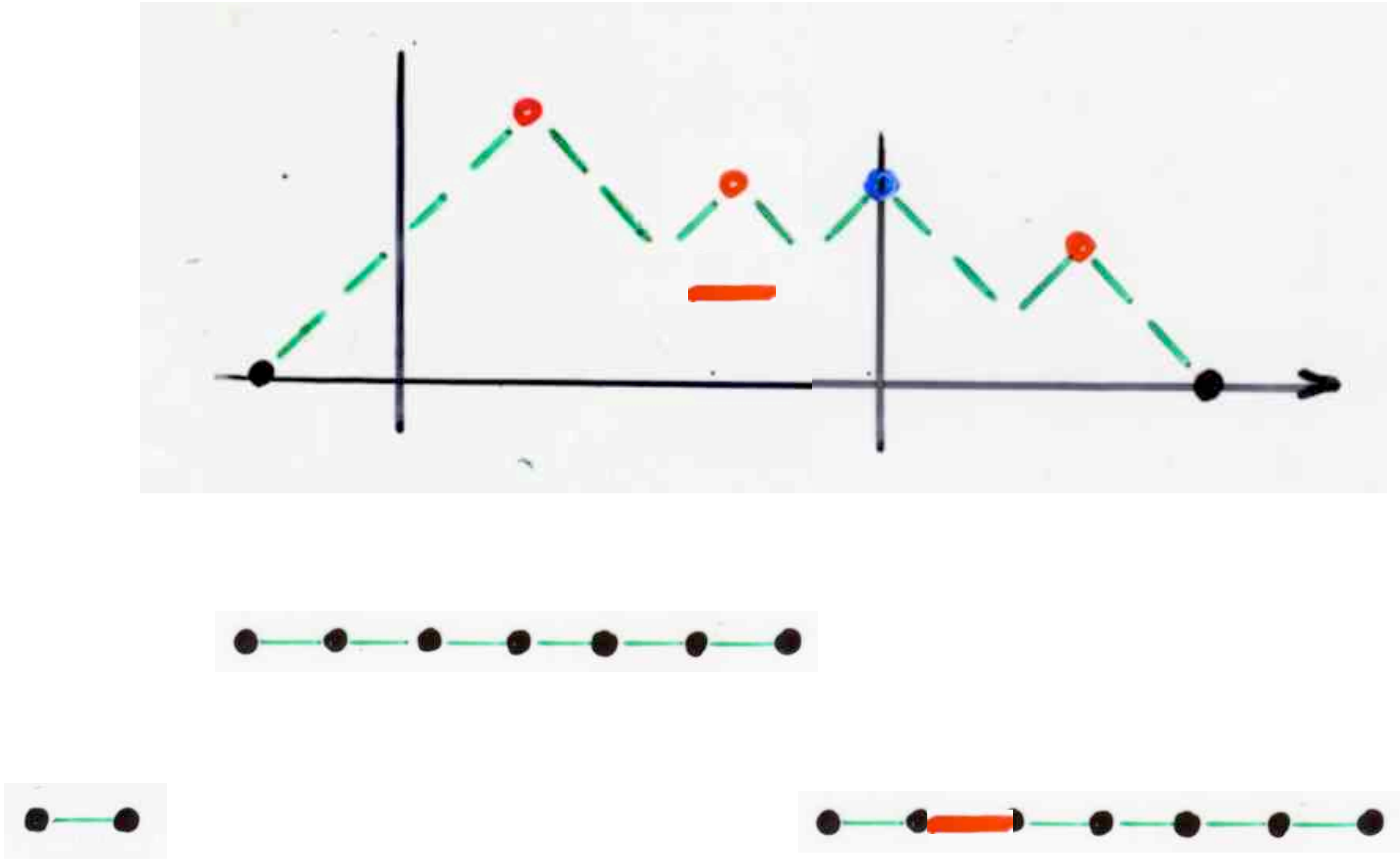


$n_1 = 2, \; n_2 = 7, \; n_3 = 7$

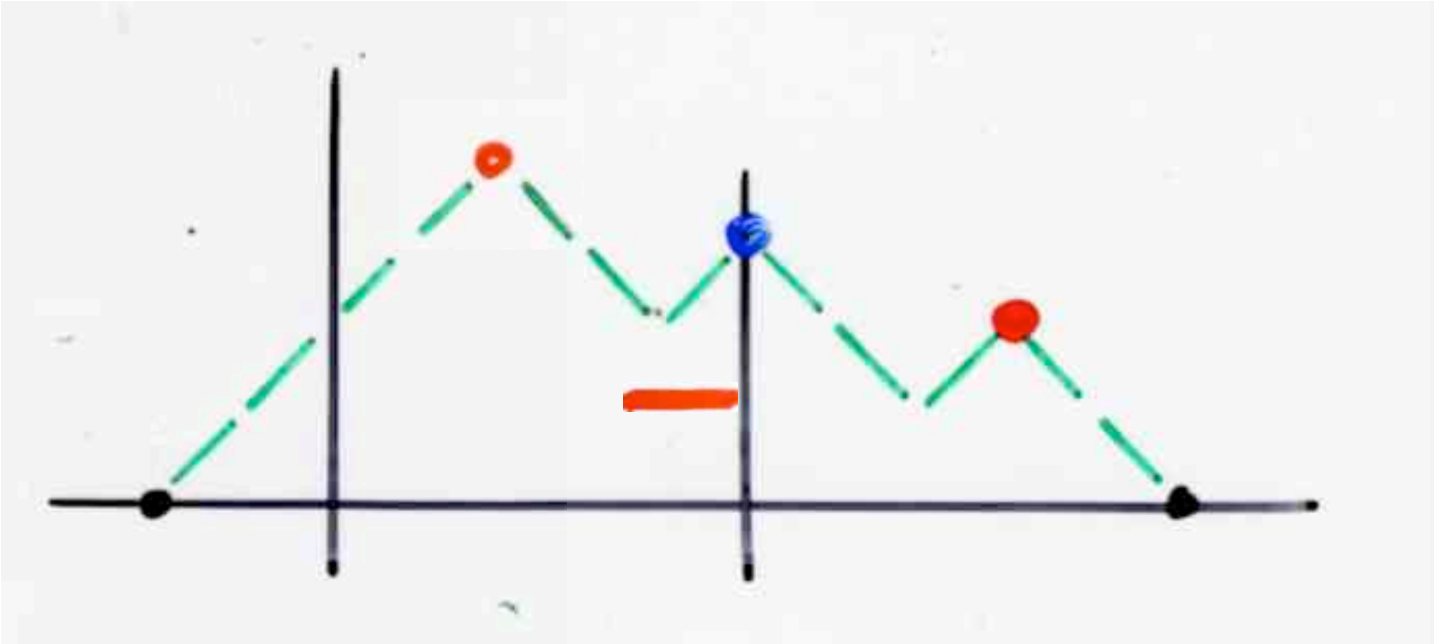


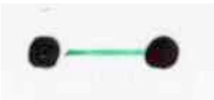


$n_1=2,\ n_2=7,\ n_3=7$

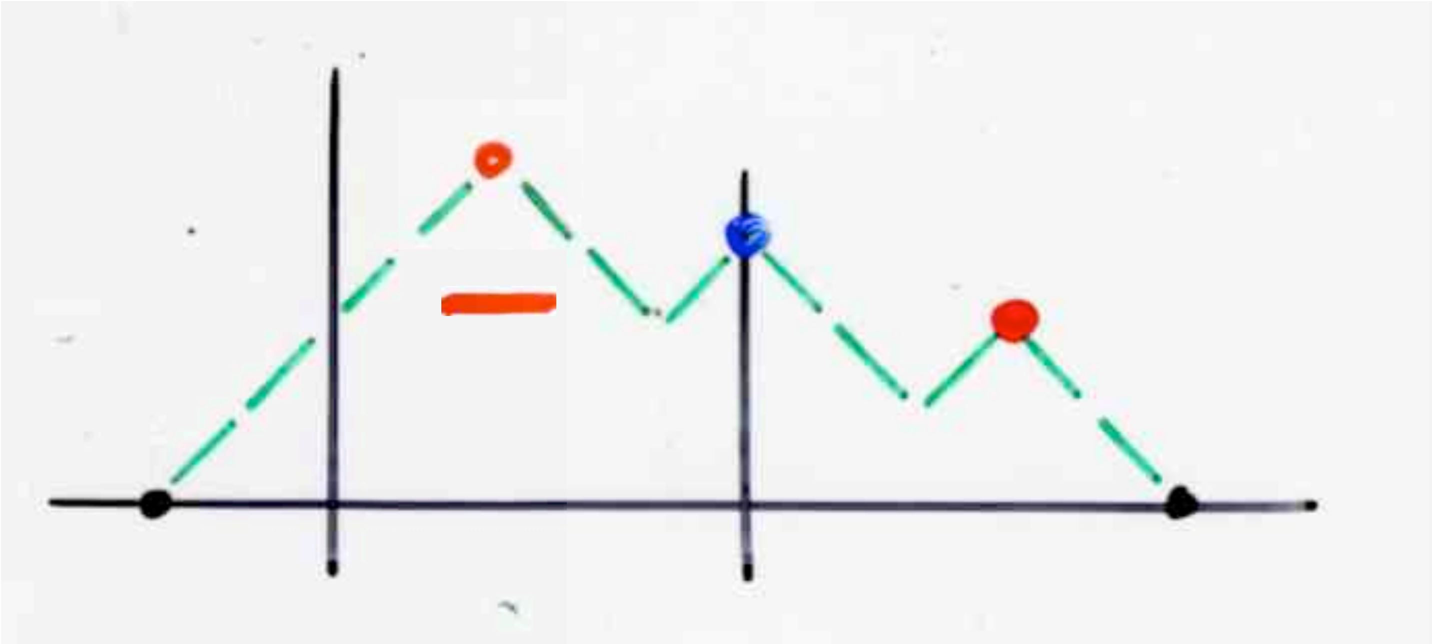


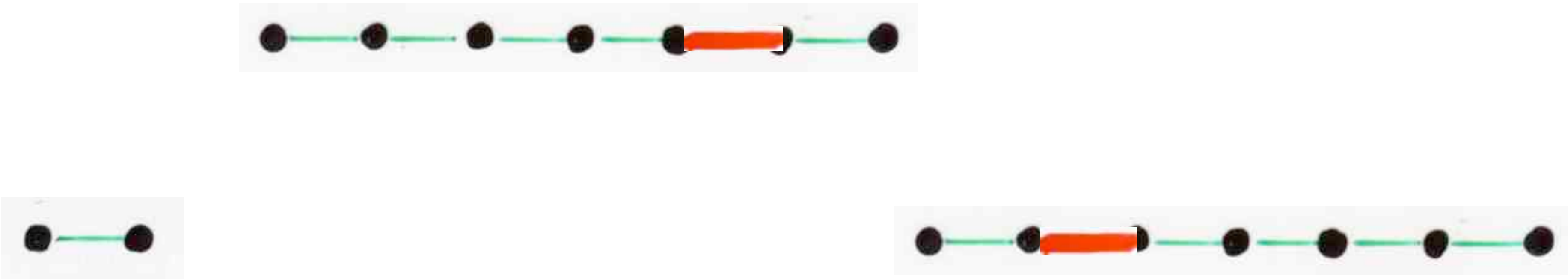
$n_1=2,\ n_2=7,\ n_3=7$



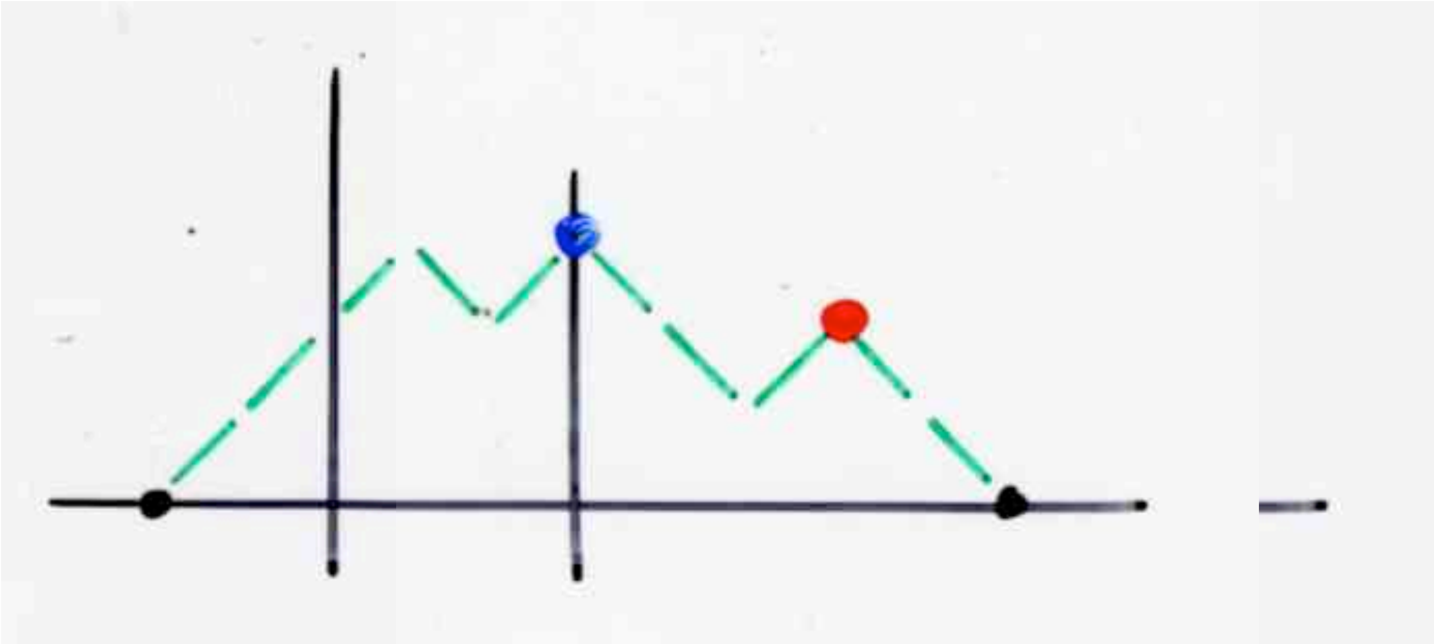


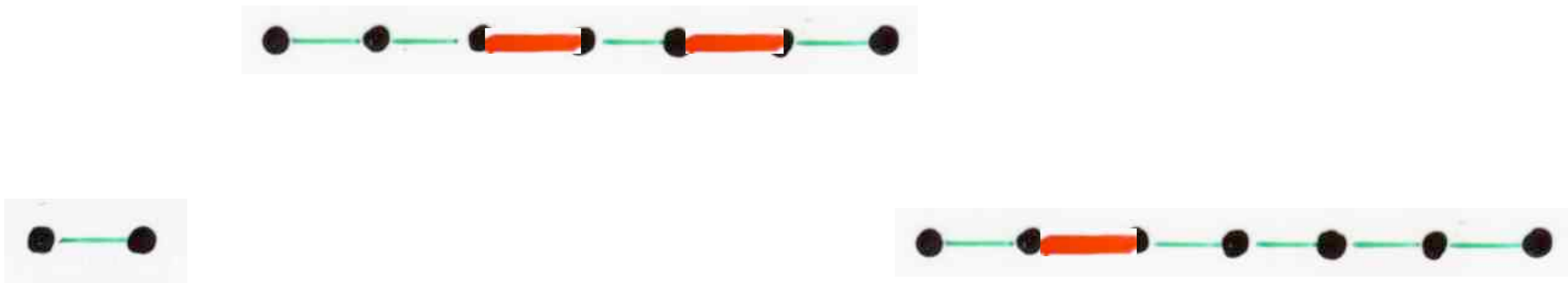
$n_1 = 2, \; n_2 = 7, \; n_3 = 7$



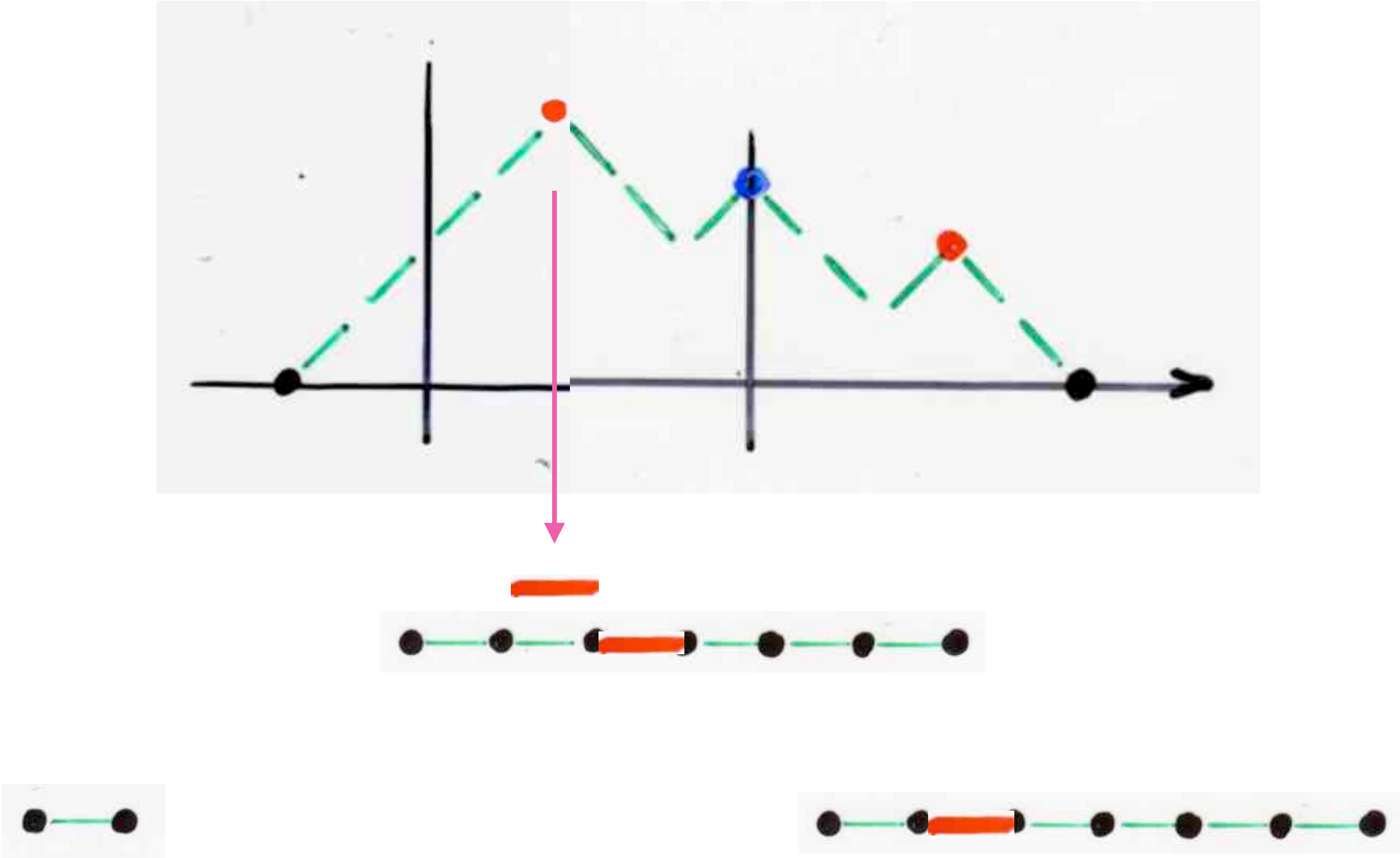


$n_1=2,\ n_2=7,\ n_3=7$

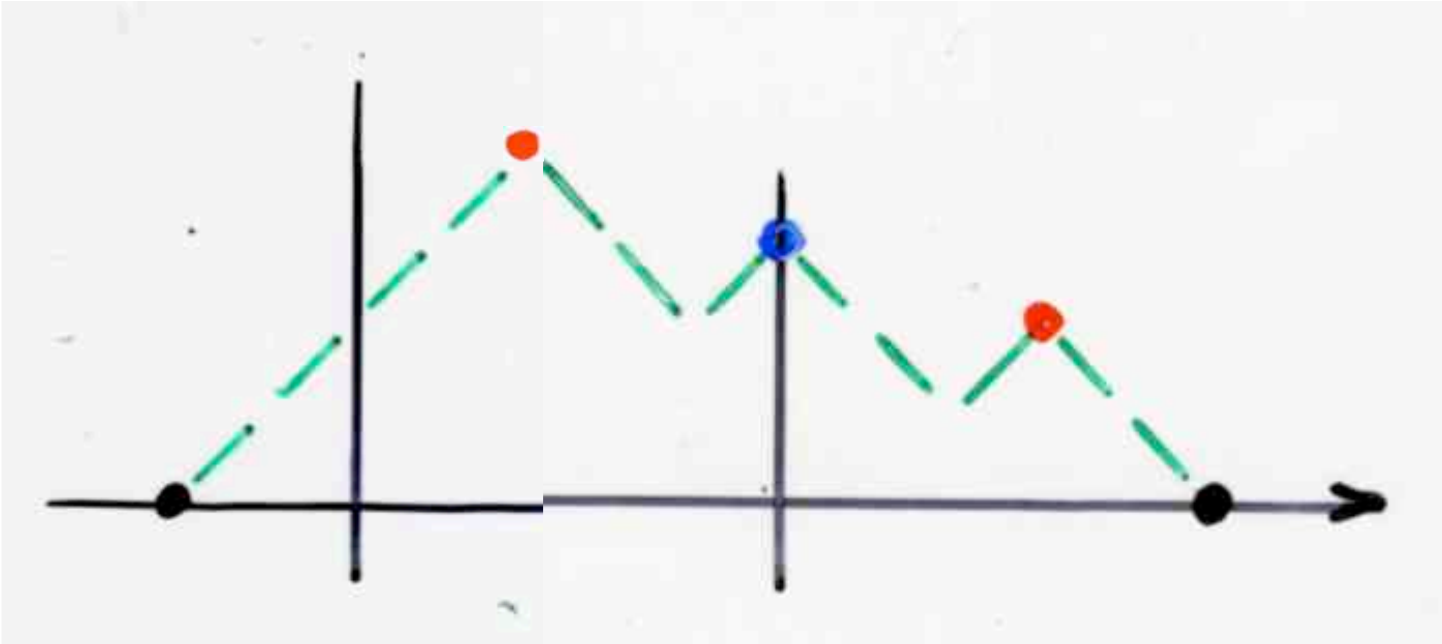


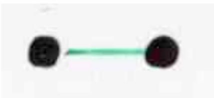


$n_1 = 2, \; n_2 = 7, \; n_3 = 7$

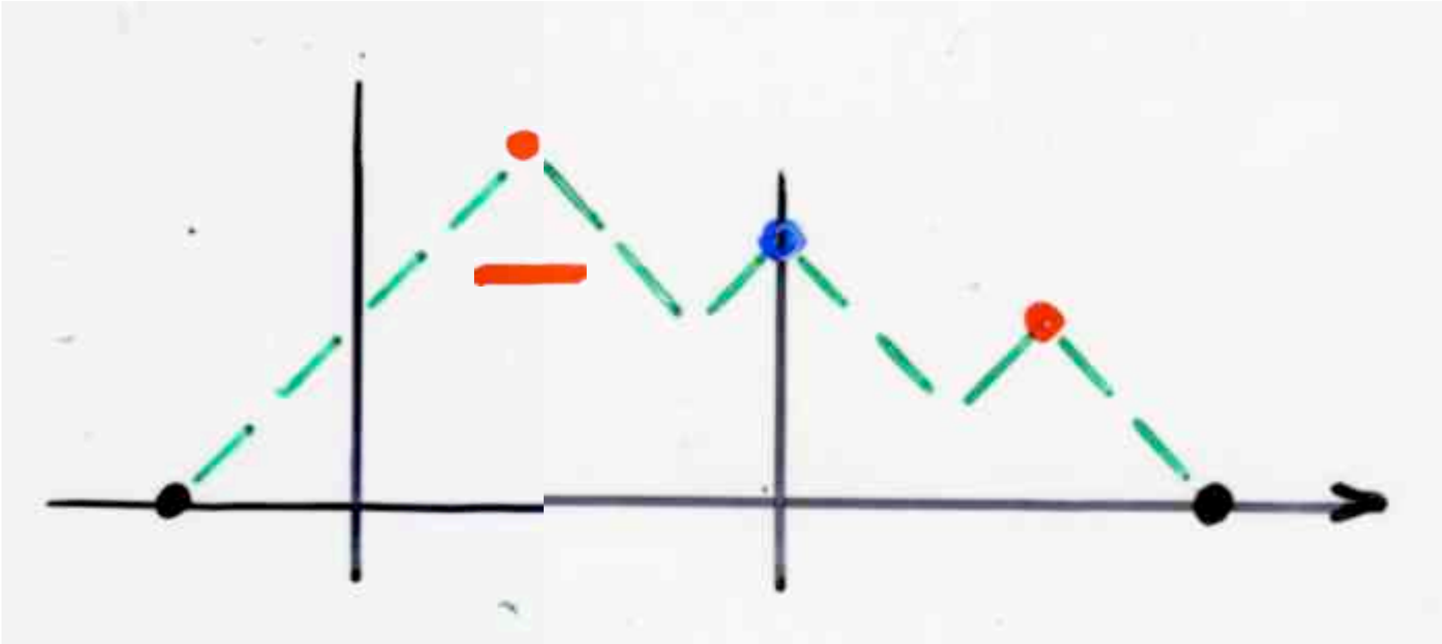


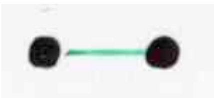
$n_1=2, \; n_2=7, \; n_3=7$



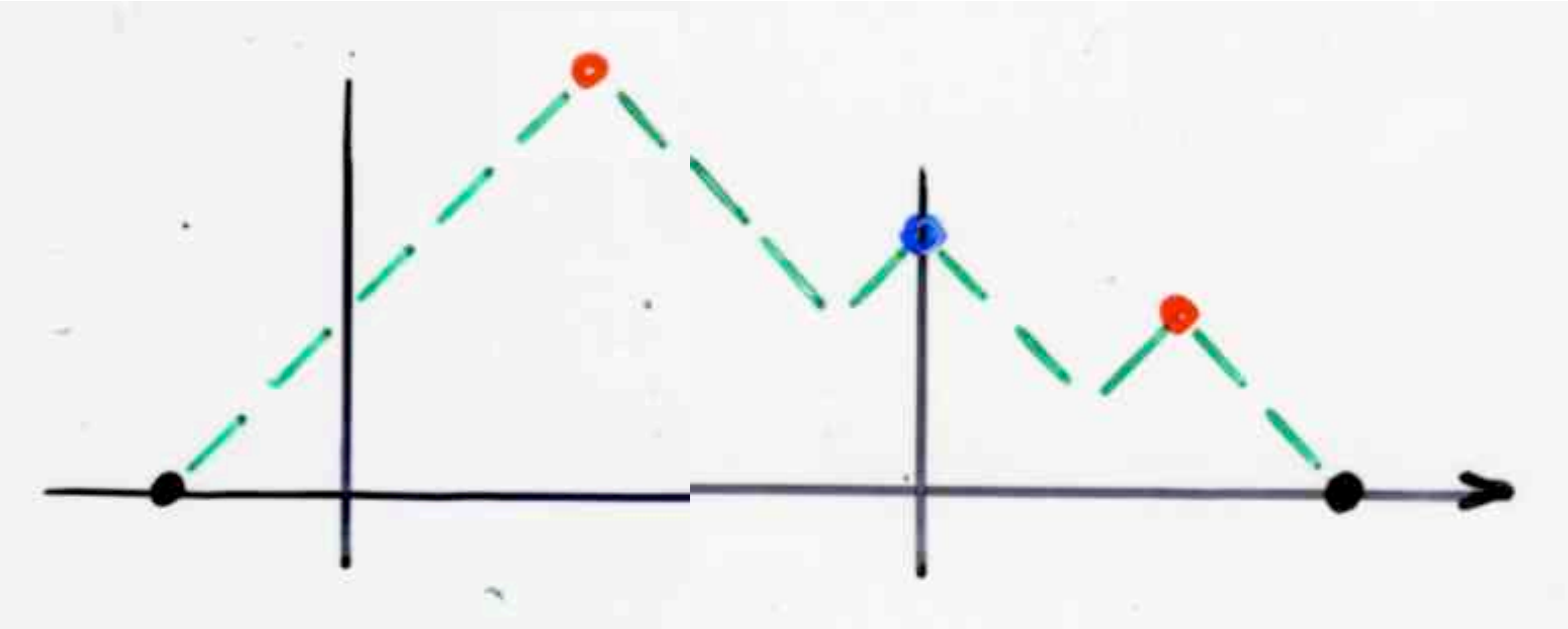


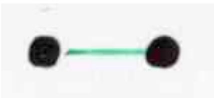
$n_1=2,\ n_2=7,\ n_3=7$





$n_1 = 2, \; n_2 = 7, \; n_3 = 7$





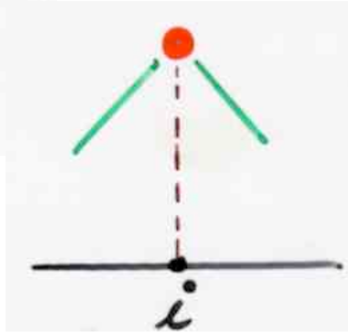
$n_1=2,\ n_2=7,\ n_3=7$

$$f\left(S_{n_1}(x)\cdots S_{n_k}(x)\right) \approx$$

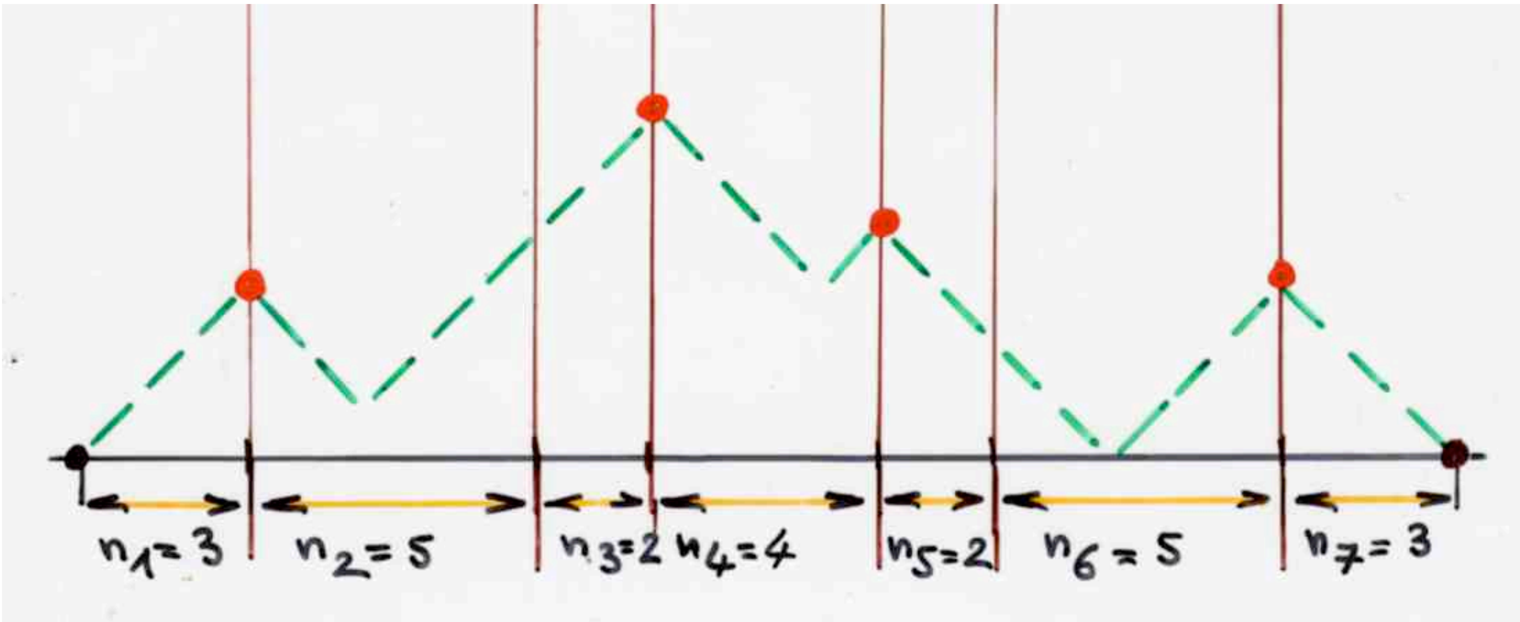
ω Dyck path

$|\omega| = n_1 + \dots + n_k$

such that the abscissas i of the peaks of ω are in the set

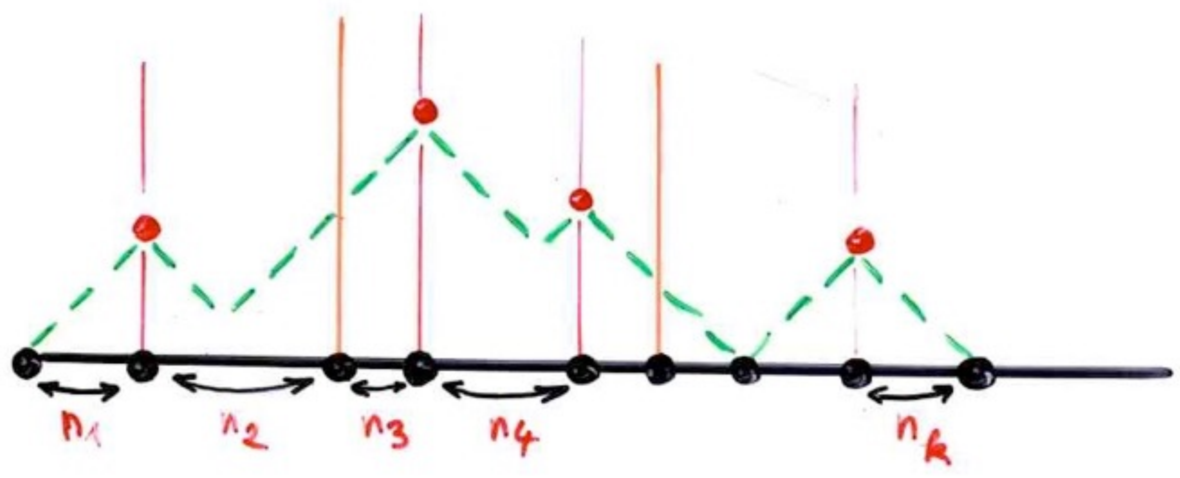


$$\{n_1,\ n_1+n_2,\ \dots,\ n_1+n_2+\dots+n_{k-1}\}$$



□

$$\frac{2}{\pi}\int_{-1}^{+1} U_{n_1}(x)\,U_{n_2}(x)\ldots U_{n_k}(x)\,(1-x^2)^{1/2}\,dx =$$

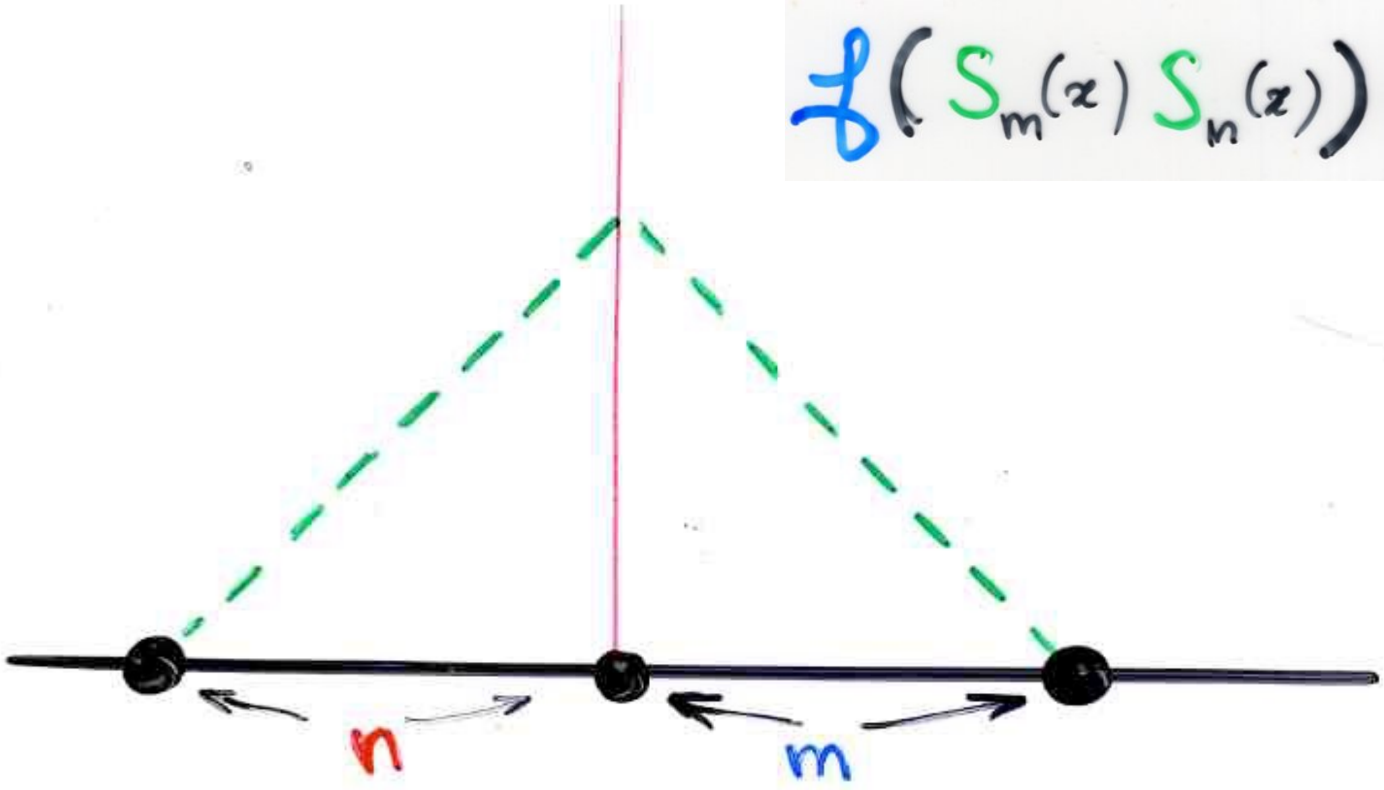


in particular:

Corollary

orthogonality!

$$\oint \left(S_m(x)\, S_n(x) \right) = \delta_{mn}$$



$\square$

orthogonal polynomials
some elementary lemma

$\mathbb{K}$ ring

field $\mathbb{R}$, $\mathbb{C}$

or $\mathbb{Q}[\alpha, \beta, \dots]$

$\mathbb{K}[x]$

polynomials in x

$\{P_n(x)\}_{n \geq 0}$

sequence of polynomials

$P_n(x) \in \mathbb{K}[x]$.

$f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$f(x^n) = \mu_n$

moments

(i) $\deg(P_n) = n$, for $n \geq 0$

degree

(ii) $f(P_k P_l) = 0$, for $k \neq l \geq 0$

(iii) $f(P_k^2) \neq 0$, for $k \geq 0$

$$f \longleftrightarrow \{P_n(x)\}_{n\geq 0}$$

unicity (up to a multiplicative factor)

<u>Lemma</u> $\{P_n(x)\}_{n\geq 0}$ orthogonal for f and g,

then $f = cg$, $c \in \mathbb{K}$, $c \neq 0$

Lemma

f linear functional on $\mathbb{K}[x]$
$\{P_k(x)\}_{k\geq 0}$ sequence of polynomials of $\mathbb{K}[x]$
the following conditions are equivalent

(i) $\{P_k(x)\}_{k\geq 0}$ orthogonal for f

(ii) $f(QP_k)=0$ for every $Q\in\mathbb{K}[x]$ with $\deg(Q)<k$
and $f(QP_k)\neq 0$ if $\deg(Q)=k$

(iii) $f(x^{\ell}P_k) = c_k\,\delta_{k\ell}$ with $c_k\neq 0$
($\delta_{k\ell}$: Kronecker symbol)

Lemma $\{P_k(x)\}_{k\geq 0}$ orthogonal for $\mathcal{f}$

$Q(x)$ polynomial degree n

$$Q(x) = \sum_{k=0}^{n} c_k P_k(x)$$

then $$c_k = \frac{\mathcal{f}(Q P_k)}{\mathcal{f}(P_k^2)}$$

Corollary

If $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are orthogonal for f, then $\exists\ \{c_n\}_{n\geq 0}$

$$P_n(x) = c_n Q_n(x) \quad \text{for } n \geq 0$$

$$c_n \in \mathbb{K}, \quad c_n \neq 0$$

$$f \longleftrightarrow \{P_n(x)\}_{n\geq 0}$$

unicity (up to a multiplicative factor)

Lemma

$$P_k(x)\,P_\ell(x) = \sum_n a_{k\ell}^{n}\,P_n(x)$$

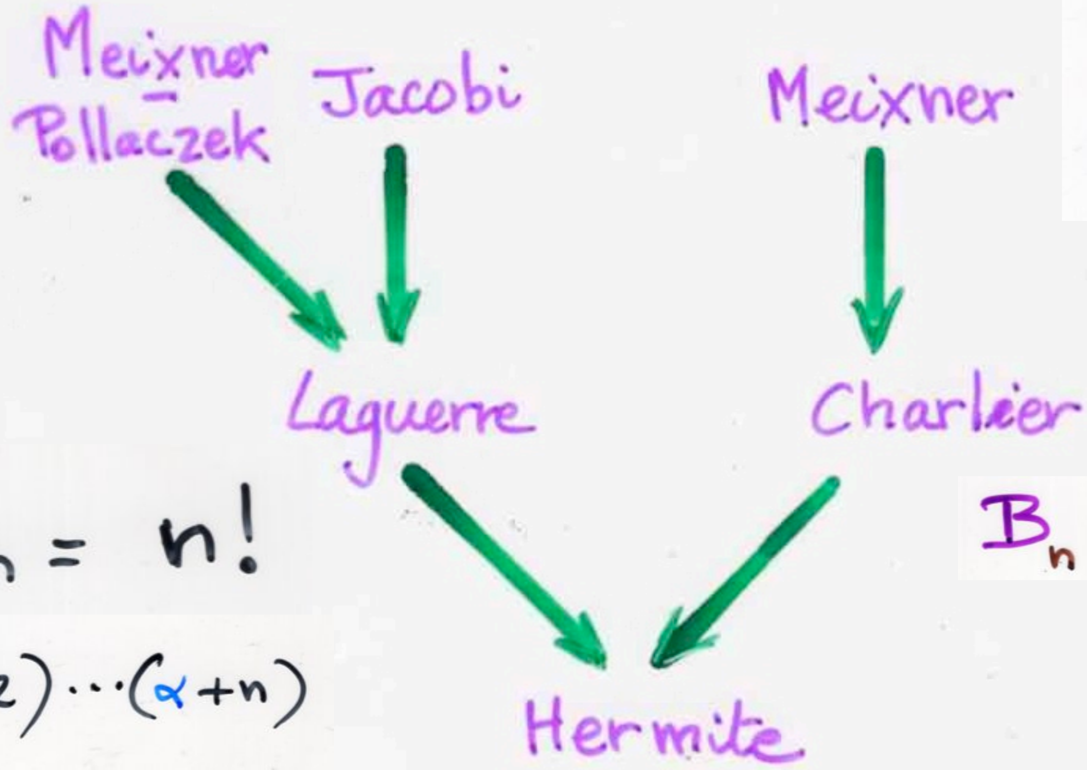
$$a_{k\ell}^{n} = \frac{\mathcal{L}(P_k P_n P_\ell)}{\mathcal{L}(P_n^2)}$$

moments of
some classical orthogonal polynomials

E_{2n}

secant number

number of ordered partitions



$\mu_n = n!$

$(\alpha+1)(\alpha+2)\cdots(\alpha+n)$

B_n Bell number

number of partitions

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1,2,\ldots,2n\}$

Favard's theorem

3-terms linear recurrence relation and pavages

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exit $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)\,P_k(x) - \lambda_k\,P_{k-1}(x)$$

for every $k\geq 1$

$$T_n(x) = \frac{1}{2} C_n(2x)$$

$$C_{n+1}(x) = x\,C_n(x) - \lambda_n C_{n-1}(x)$$

$$\begin{cases} \lambda_1 = 2 \\ \lambda_n = 1 \quad (n \geqslant 2) \end{cases}$$

$$U_n(x) = S_n(2x)$$

$$S_{n+1}(x) = x\,S_n(x) - S_{n-1}(x)$$

Hermite polynomial

$H_n(x)$

$$\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

Laguerre polynomial

$L_n(x)$

$$\begin{cases} b_k = (2k+1) \\ \lambda_k = k^2 \end{cases}$$

$L_n^{(\alpha)}(x)$ $\quad \alpha = 0$

$\{P_n(x)\}_{n\geq 0}$ sequence of monic orthogonal polynomials

There exit $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coefficients in $\mathbb{K}$ such that

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geq 1$

Proof

Proof

$$P_{k+1}(x) - x P_k(x) = \underbrace{\sum^{k} c_i P_i(x)}_{\text{polynomial } \deg \leqslant k}$$

for every polynomial $Q(x)$, $\deg Q < k$

$$f\big(P_k(x)\, Q(x)\big) = 0$$

$$P_{k+1}(x)\,P_j(x) - x\,P_k(x)\,P_j(x) = \sum_{i=0}^{k} c_i\,P_i(x)\,P_j(x)$$

$$f\left(P_{k+1}P_j\right) - f\left(P_k \times \underbrace{x P_j}_{\deg < k}\right) = c_j\, f\left(P_j\, P_j\right)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0 \qquad\qquad \neq 0$$

for $j < k-1$

$$\Rightarrow c_j = 0$$

(formal) Favard's Theorem

3-terms linear recurrence relation

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

for every $k \geq 1$

$\Rightarrow$ orthogonality

$$\{P_n(x)\}_{n\geq 0}$$

sequence of polynomials

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$\{b_k\}_{k\geq 0}, \ \{\lambda_k\}_{k\geq 1}$$

$$b_k, \lambda_k \in \mathbb{K}$$

ring

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$P_0 = 1$$

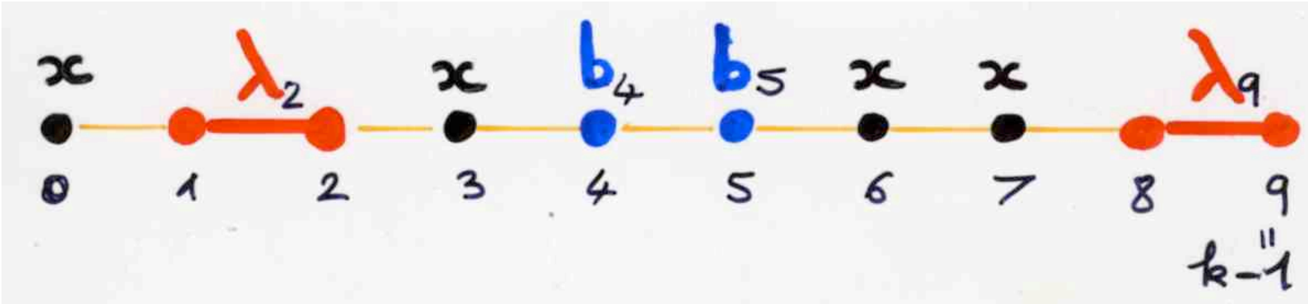
$$P_1 = (x - b_0)$$

$$\{P_n(x)\}_{n \geq 0}$$

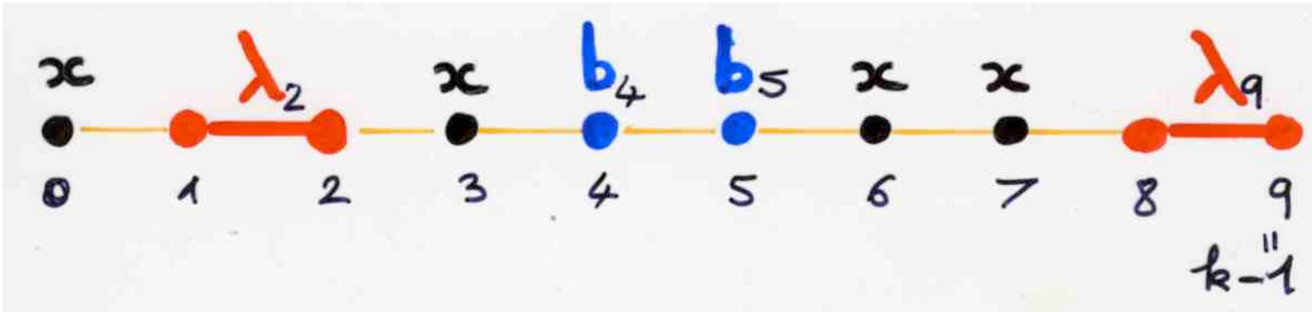
sequence of polynomials

"pavage"

monomer, dimer



$$v(\alpha) = b_4 b_5 \lambda_2 \lambda_9$$



$$v(\alpha) = b_4 \, b_5 \, \lambda_2 \, \lambda_9$$

$ip(\alpha)$ = number of isolated points of α

$|\alpha|$ = number of pieces of the pavage α (monomers - dimers)

$$(-1)^4 \, b_4 \, b_5 \, \lambda_2 \, \lambda_9 \, x^4$$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$\{P_n(x)\}_{n \geq 0}$$

sequence of polynomials

$$P_0 = 1$$

$$P_1 = (x - b_0)$$

$$P_n(x) = \sum_{\alpha} (-1)^{|\alpha|} v(\alpha)\, x^{ip(\alpha)}$$

α pavage of $[0, n-1]$

Moments
and
weighted Motzkin paths

$\{b_k\}_{k \geq 0}$ $\{\lambda_k\}_{k \geq 1}$

$b_k, \lambda_k \in \mathbb{K}$

ring

μ_n ?

Path (or walk)

$$\omega = (s_0, s_1, \ldots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

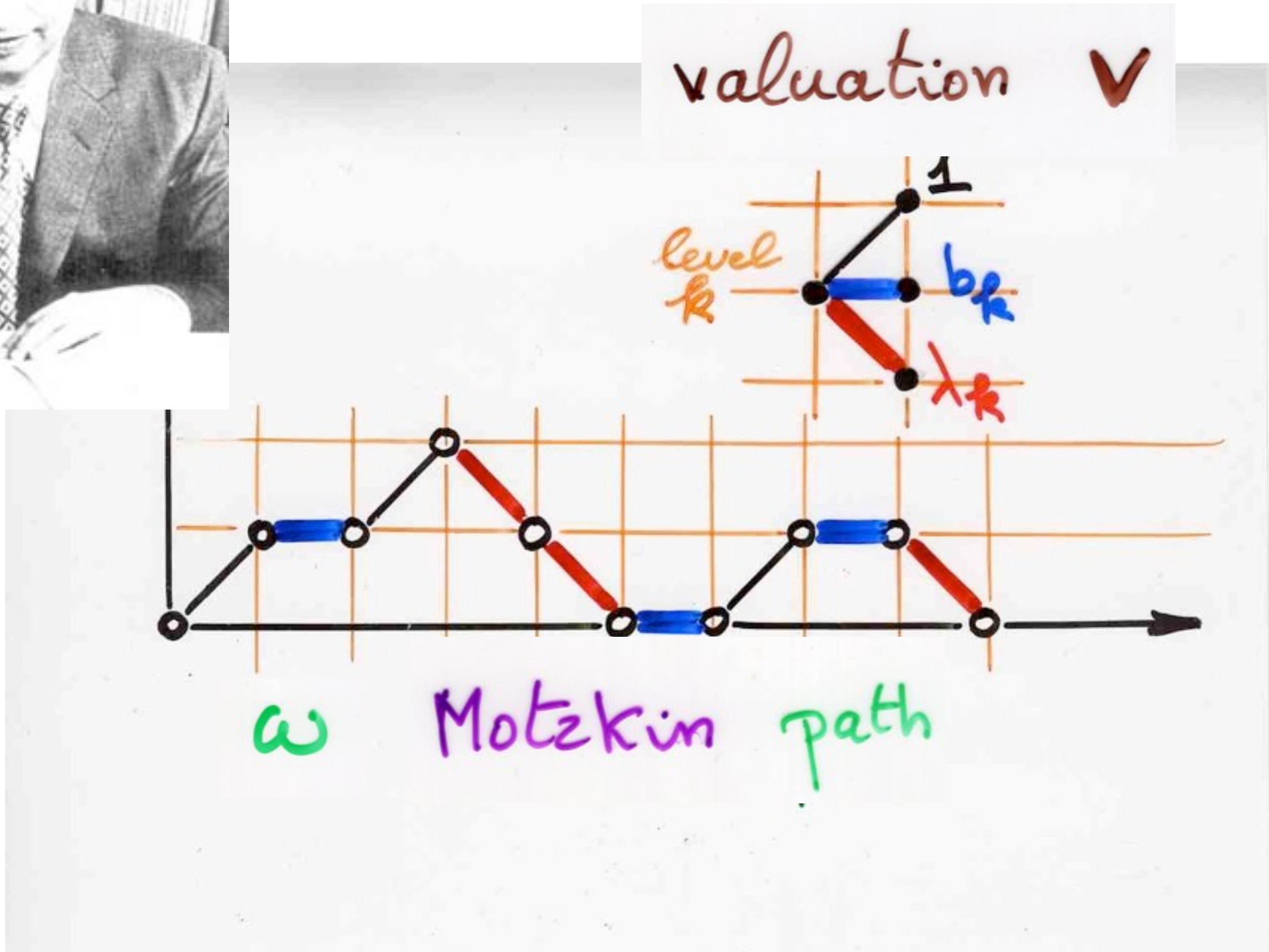
length n

(s_i, s_{i+1}) elementary step

valuation (weight)

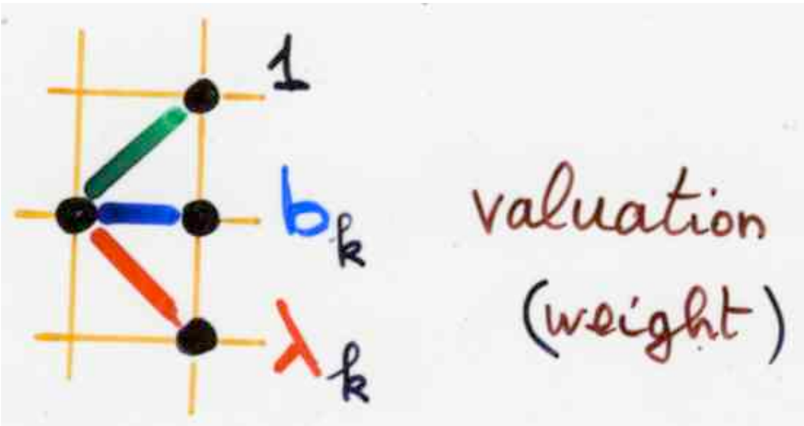
$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

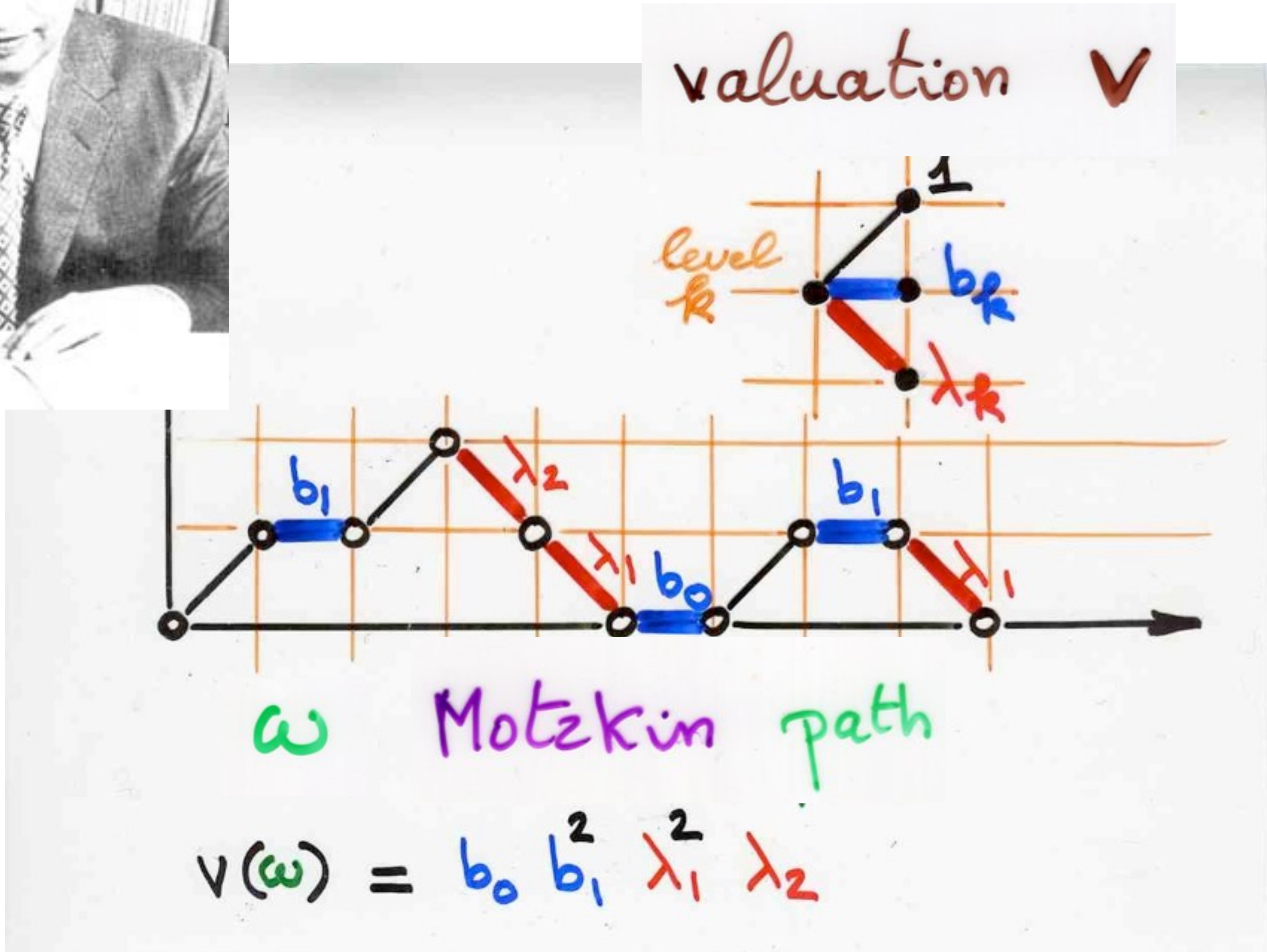
$$v : S \times S \to \mathbb{K}[x]$$


valuation v
1
level k
b_k
λ_k
ω Motzkin path

$\{b_k\}_{k \geq 0}$, $\{\lambda_k\}_{k \geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring





valuation v
1
level k
b_k
λ_k
b_1
λ_2
λ_1
b_0
b_1
λ_1
ω Motzkin path
$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$

$$P_{k+1}(x) = (x - b_k)\, P_k(x) - \lambda_k\, P_{k-1}(x)$$

for every $k \geqslant 1$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

Motzkin path

$|\omega| = n$

length

$$f(x^n) = \mu_n$$

combinatorial proof

3-terms recurrence relation
implies orthogonaliy

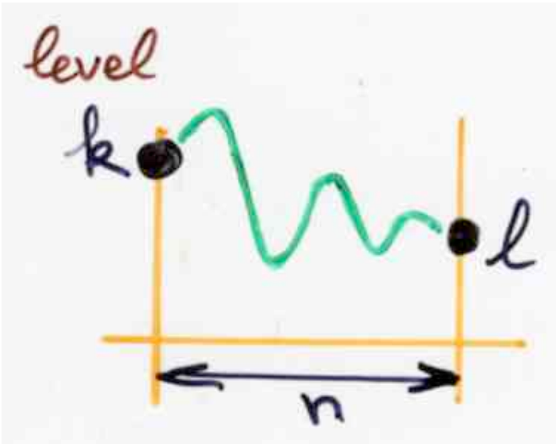
The main theorem

(main) **Theorem**

$$f(P_k P_l x^n) = \sum_{\omega} v(\omega)\, \lambda_1 \cdots \lambda_l$$

"Motzkin path" ω

$|\omega| = n$ level $k \rightsquigarrow l$

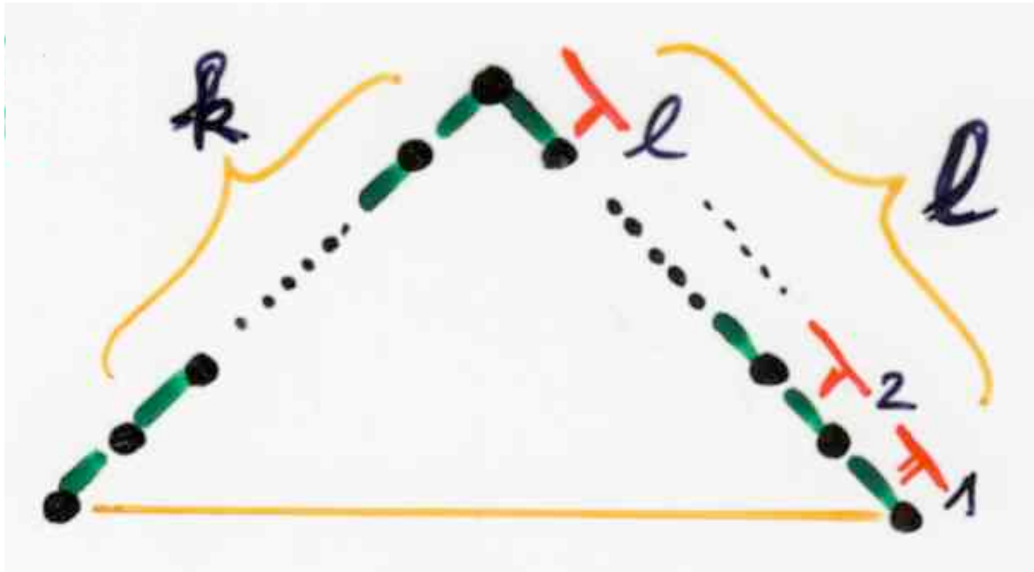


Corollary

$\Rightarrow$ orthogonality

$n=0$

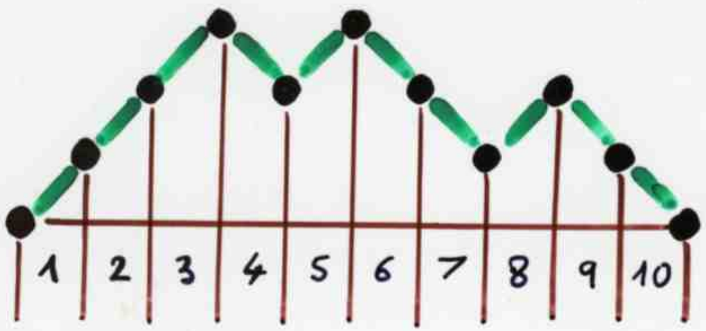
$$f(P_k P_\ell) = 0 \quad k \neq \ell$$
$$= \lambda_1 \cdots \lambda_\ell \quad k = \ell$$



The « essence » of the fundamental sign-reversing involutions

moments $\mathfrak{f}$ (Tchebychev) 2nd kind

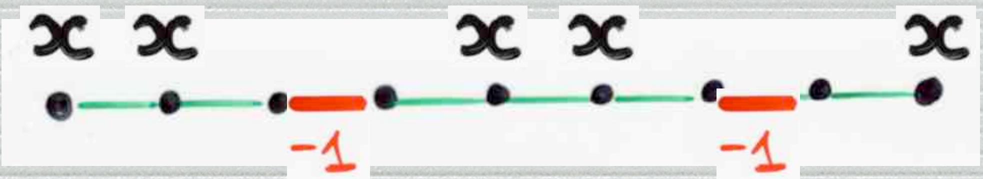
$\mathfrak{f}(x^n) = \mu_n$ moments



$$\begin{cases} \mu_{2n} = C_n \\ \mu_{2n+1} = 0 \end{cases}$$

Catalan number

$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$



$S_n(x)$

bijective proof

$$f(P_k P_l x^n) = \sum_{\alpha, \beta, \omega} (-1)^{|\alpha|+|\beta|} v(\alpha) v(\beta) v(\omega)$$

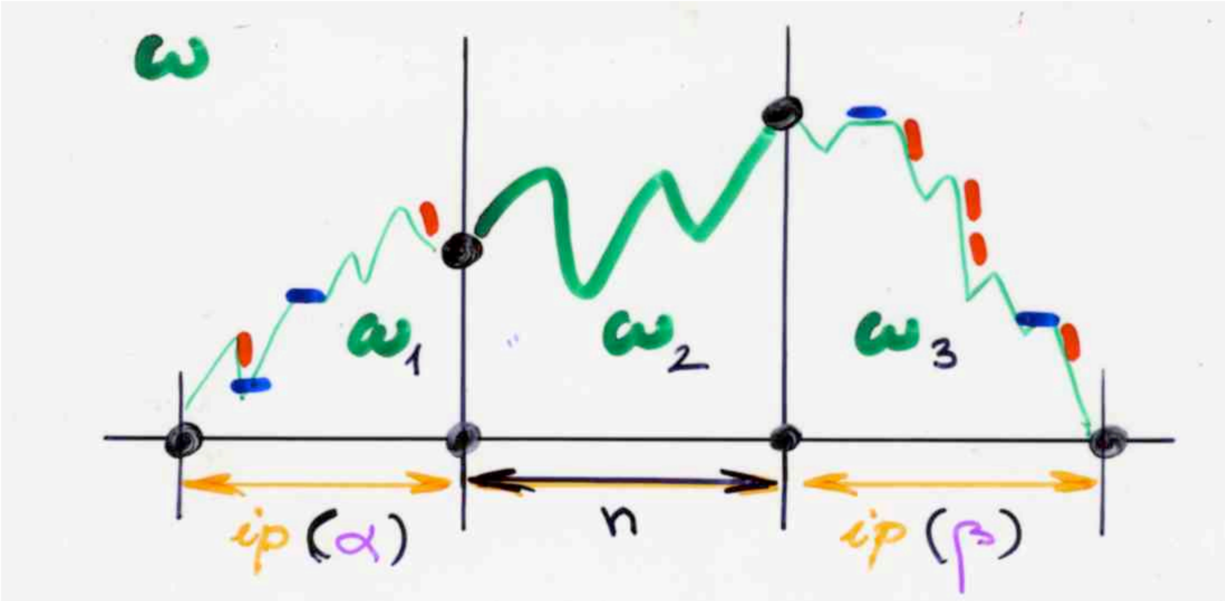
α pavage of $[0, k-1]$

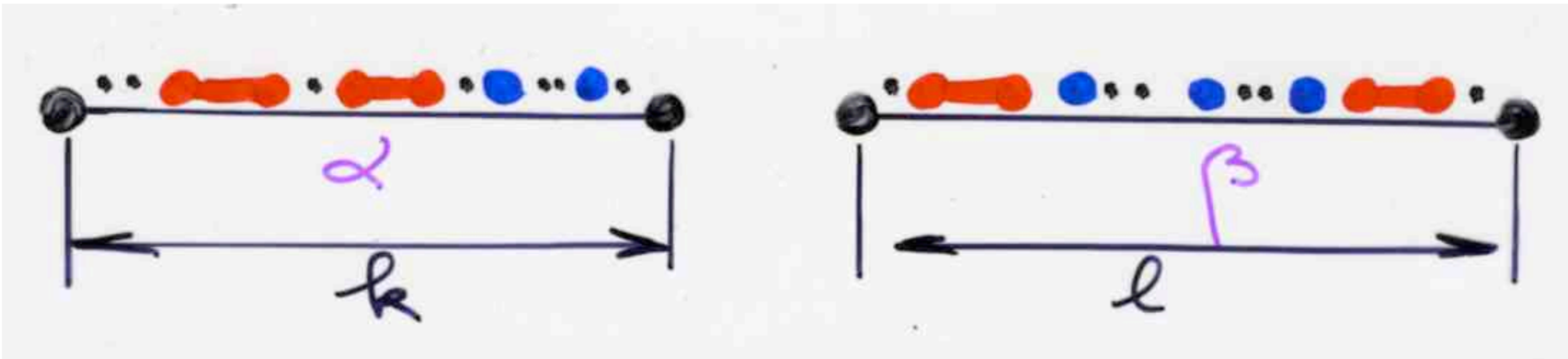
β pavage of $[0, l-1]$

ω Motzkin path (level $0 \rightsquigarrow 0$)

$$|\omega| = ip(\alpha) + ip(\beta) + n$$

$$(\alpha, \beta, \omega) \in E_{n,k,l}$$





$$(\alpha, \beta, \omega) \in E_{n,k,\ell}$$

Proof of the main theorem:

next lecture (Ch1c)