Course IMSc, Chennai, India

January-March 2018

The cellular ansatz:
bijective combinatorics and quadratic algebra

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 5
Tableaux and orthogonal polynomials

Ch5c

The parameter « q »

IMSc, Chennai
12 March, 2018

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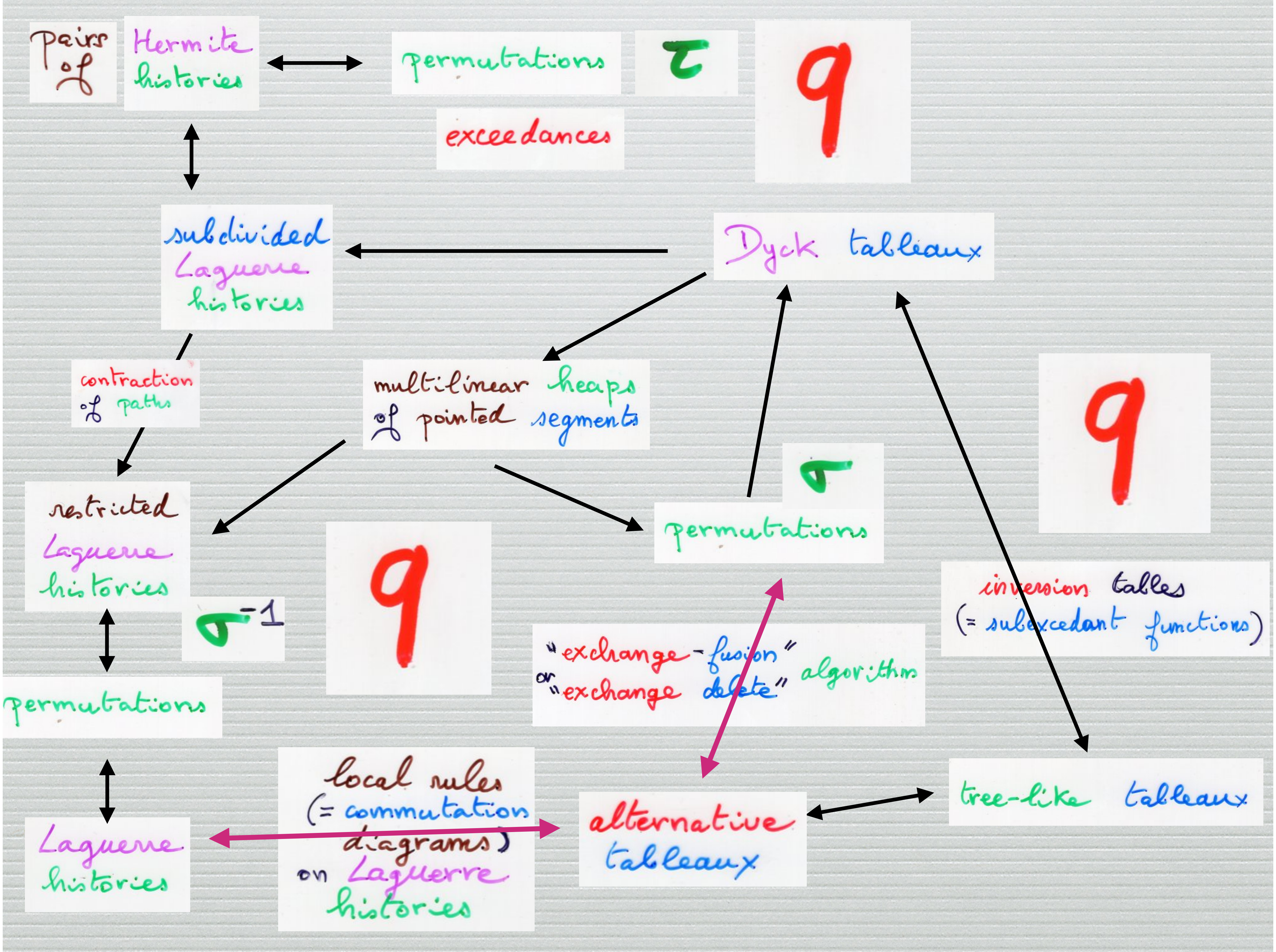

Pairs of Hermite histories
permutations
τ
exceedances
q
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
q
σ
permutations
restricted Laguerre histories
q
σ^-1
inversion tables
(= subexcedant functions)
"exchange-fusion" or "exchange delete" algorithms
permutations
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories


Pairs of Hermite histories
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Laguerre histories
local rules (= commutation diagrams) on Laguerre histories
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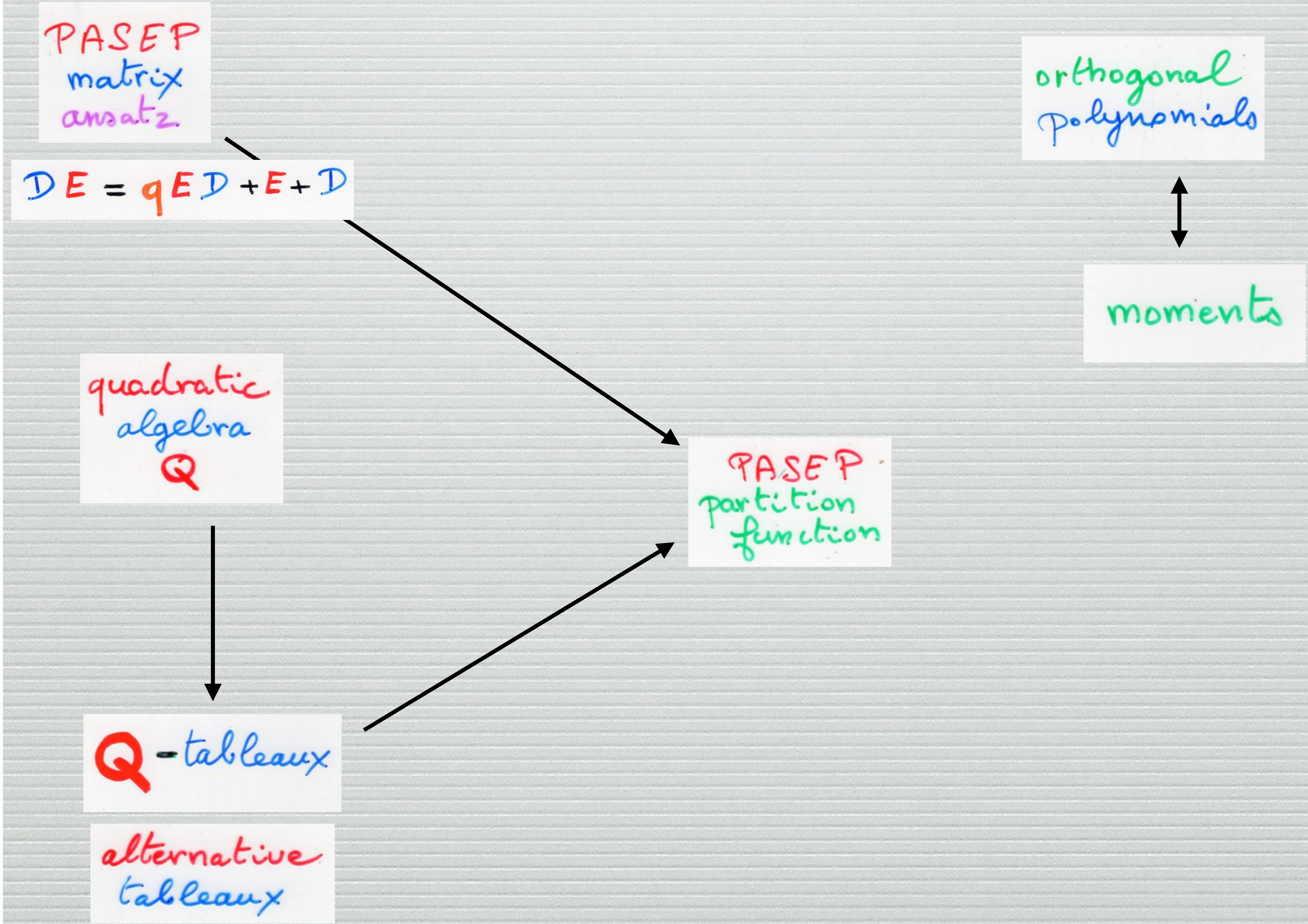
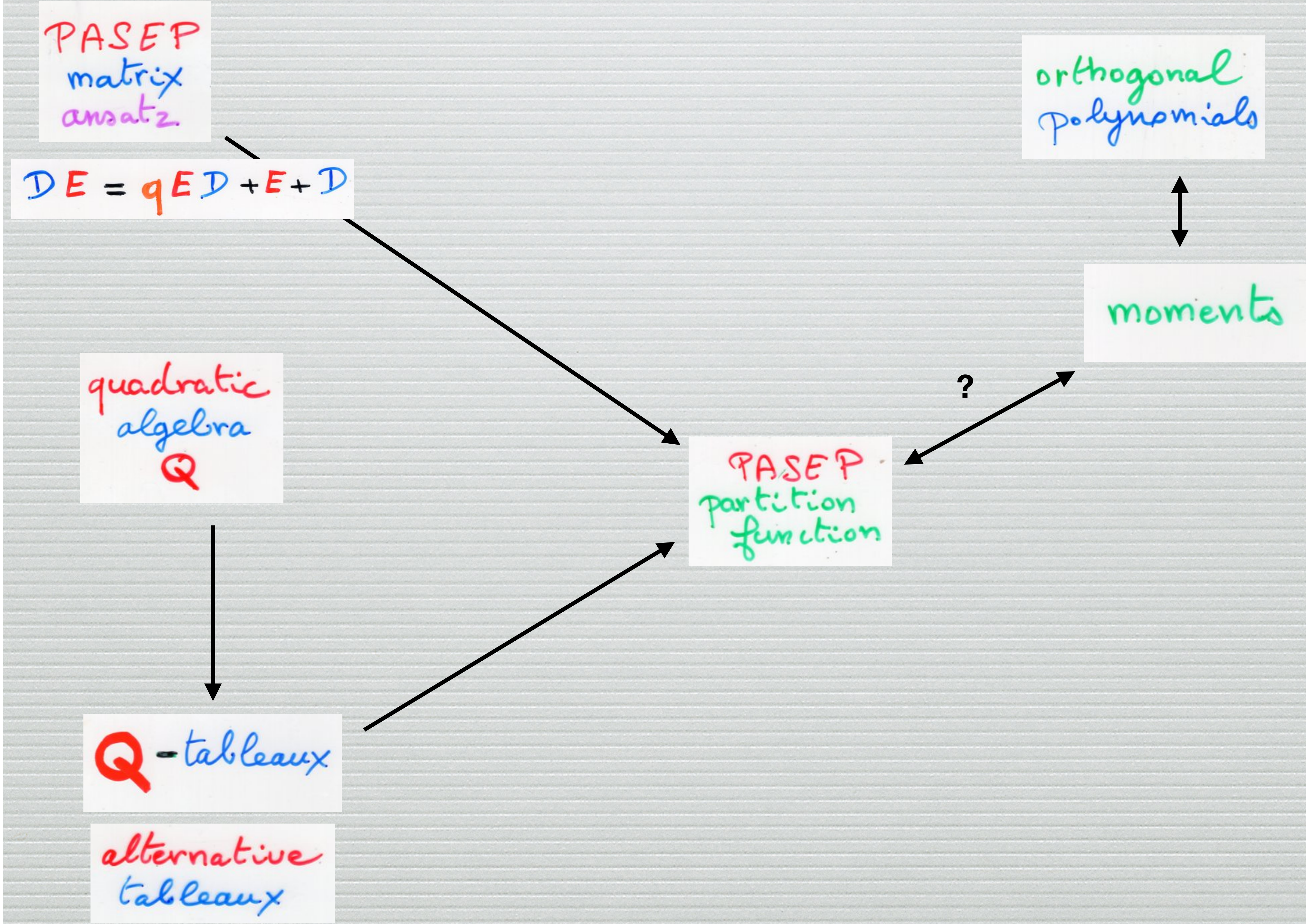
tree-like tableaux
inversion tables (= subexcedant functions)
"exchange-fusion" or "exchange delete" algorithm
Are you lost in the garden ?
Where are we ?
From where we come ?
Where are we going ?


orthogonal polynomials
moments
quadratic algebra Q
Q-tableaux


PASEP
matrix
ansatz
$DE = qED + E + D$
orthogonal
polynomials
moments
quadratic
algebra
Q
PASEP
partition
function
Q-tableaux
alternative
tableaux


PASEP
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DE = qED + E + D
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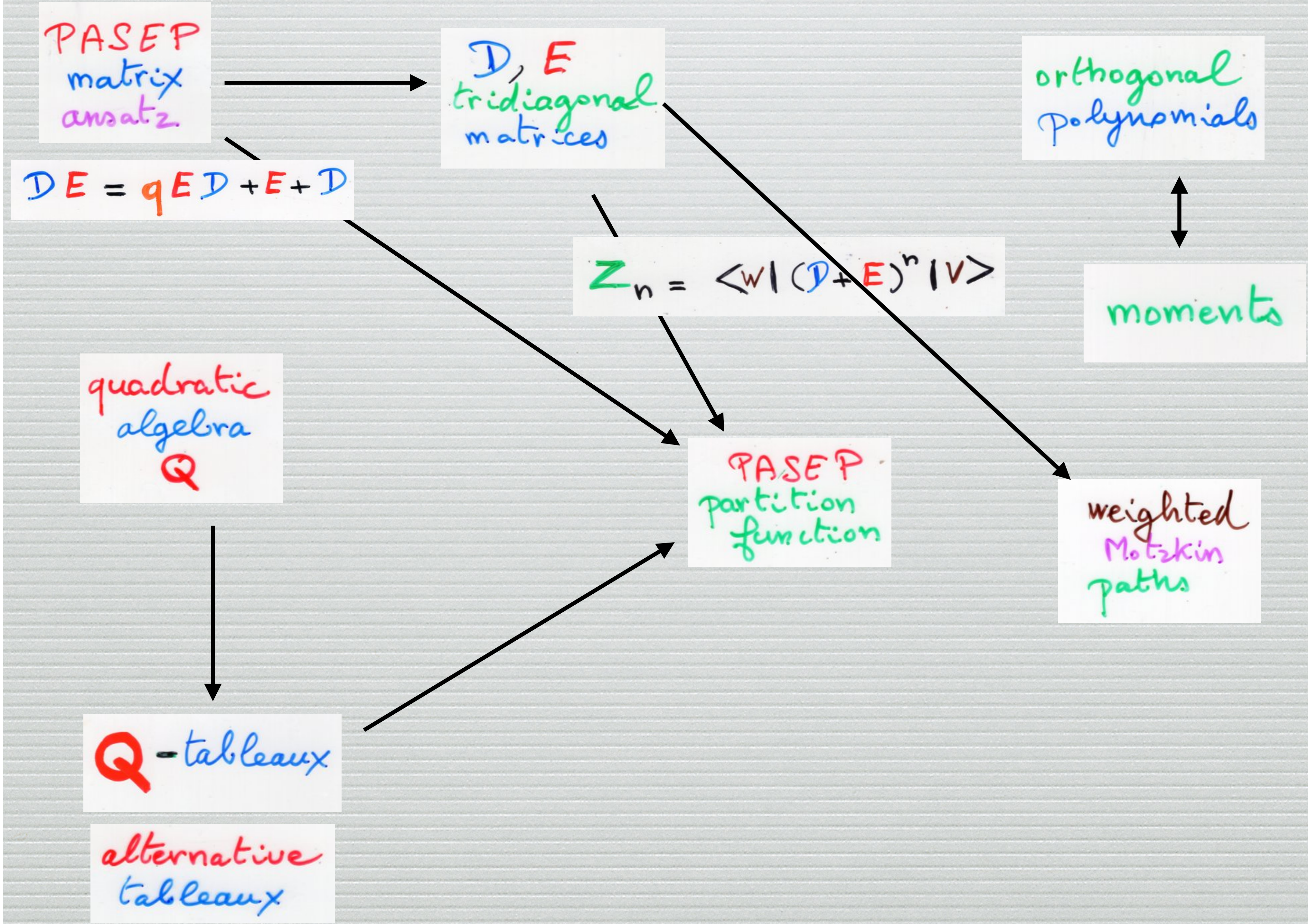
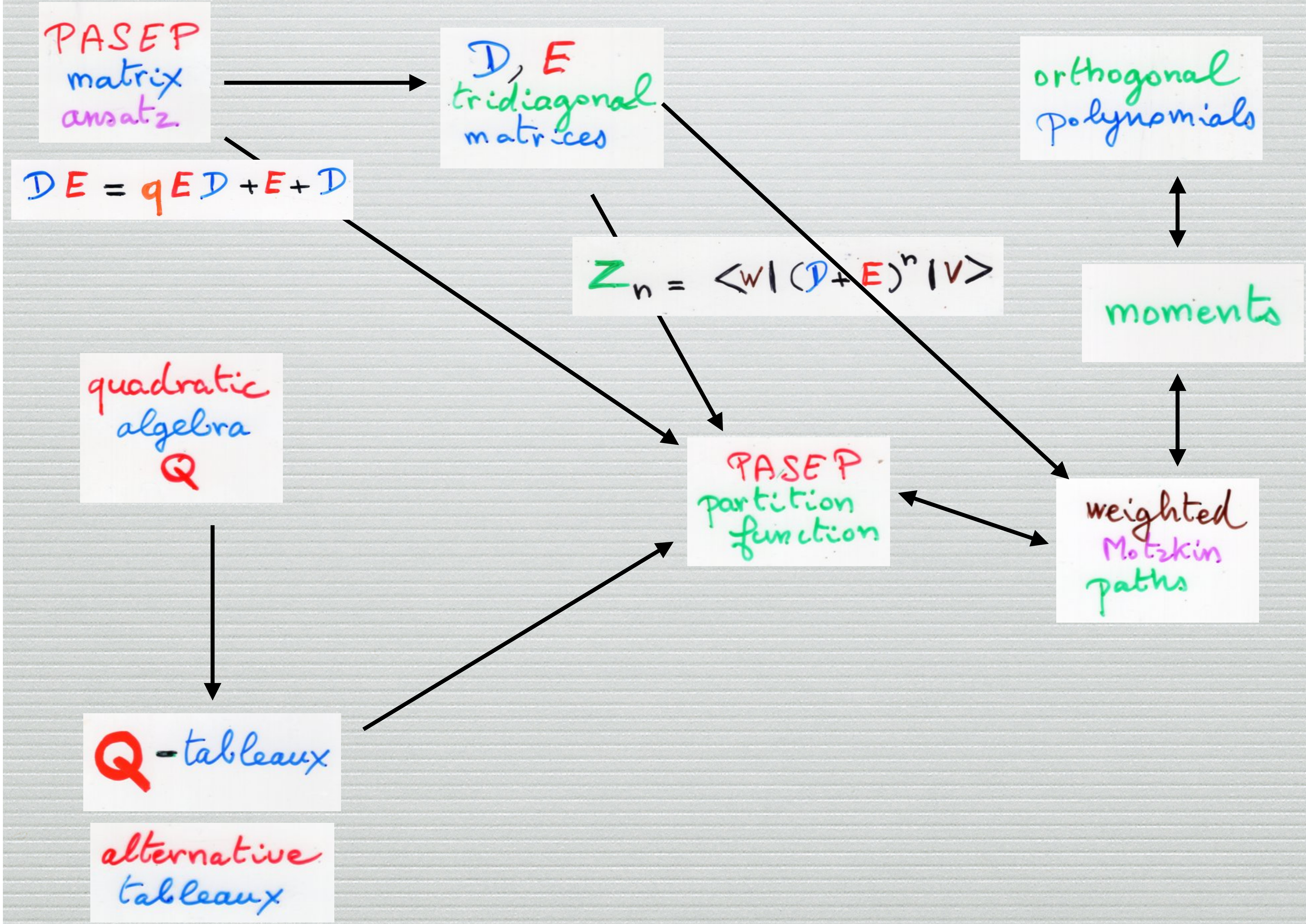

PASEP
matrix
ansatz
$DE = qED + E + D$
D, E
tridiagonal
matrices
orthogonal
polynomials
$Z_n = \langle W|(D+E)^n|V\rangle$
moments
quadratic
algebra
Q
PASEP
partition
function
weighted
Motzkin
paths
Q-tableaux
alternative
tableaux


PASEP
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DE = qED + E + D
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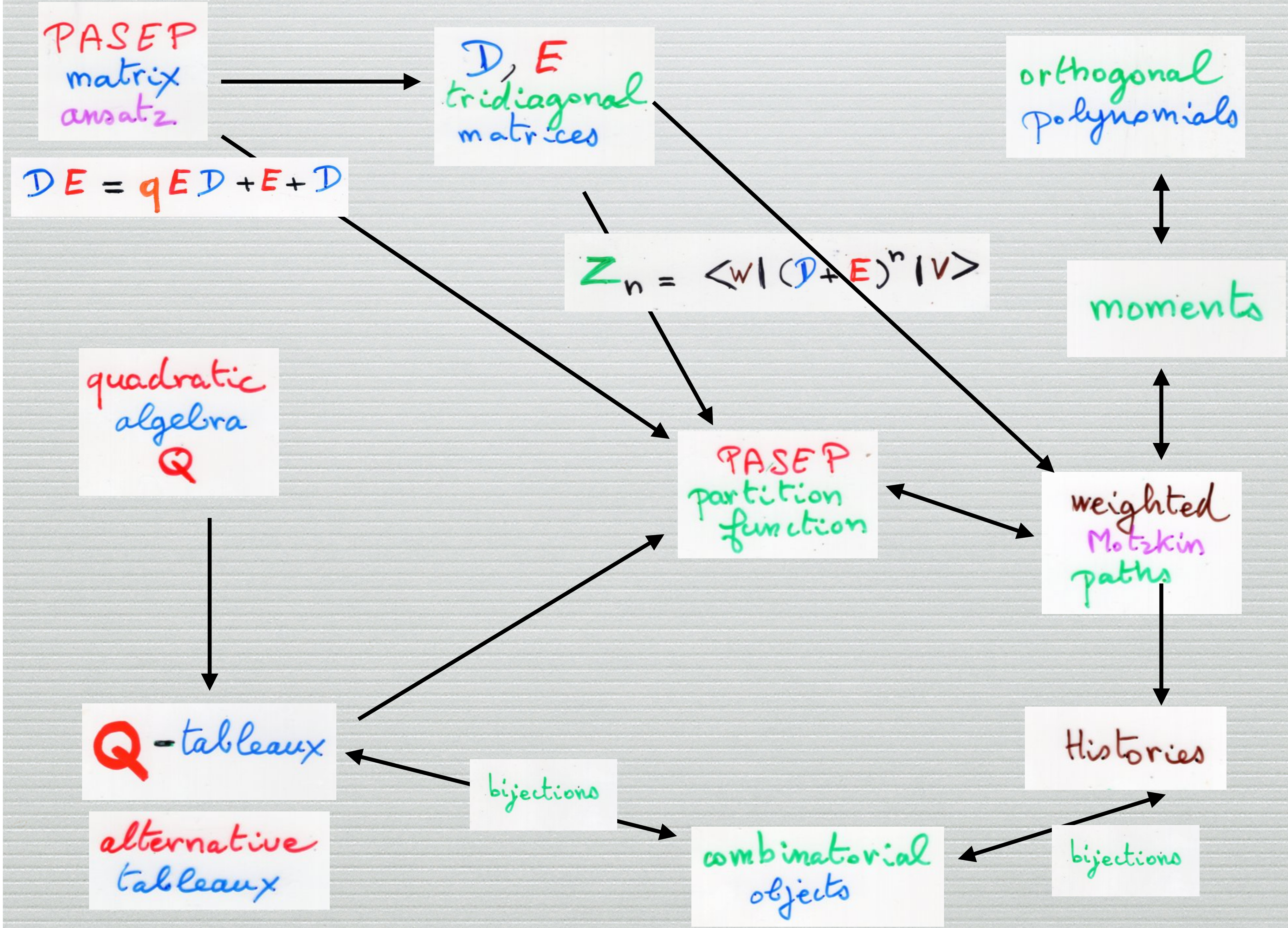
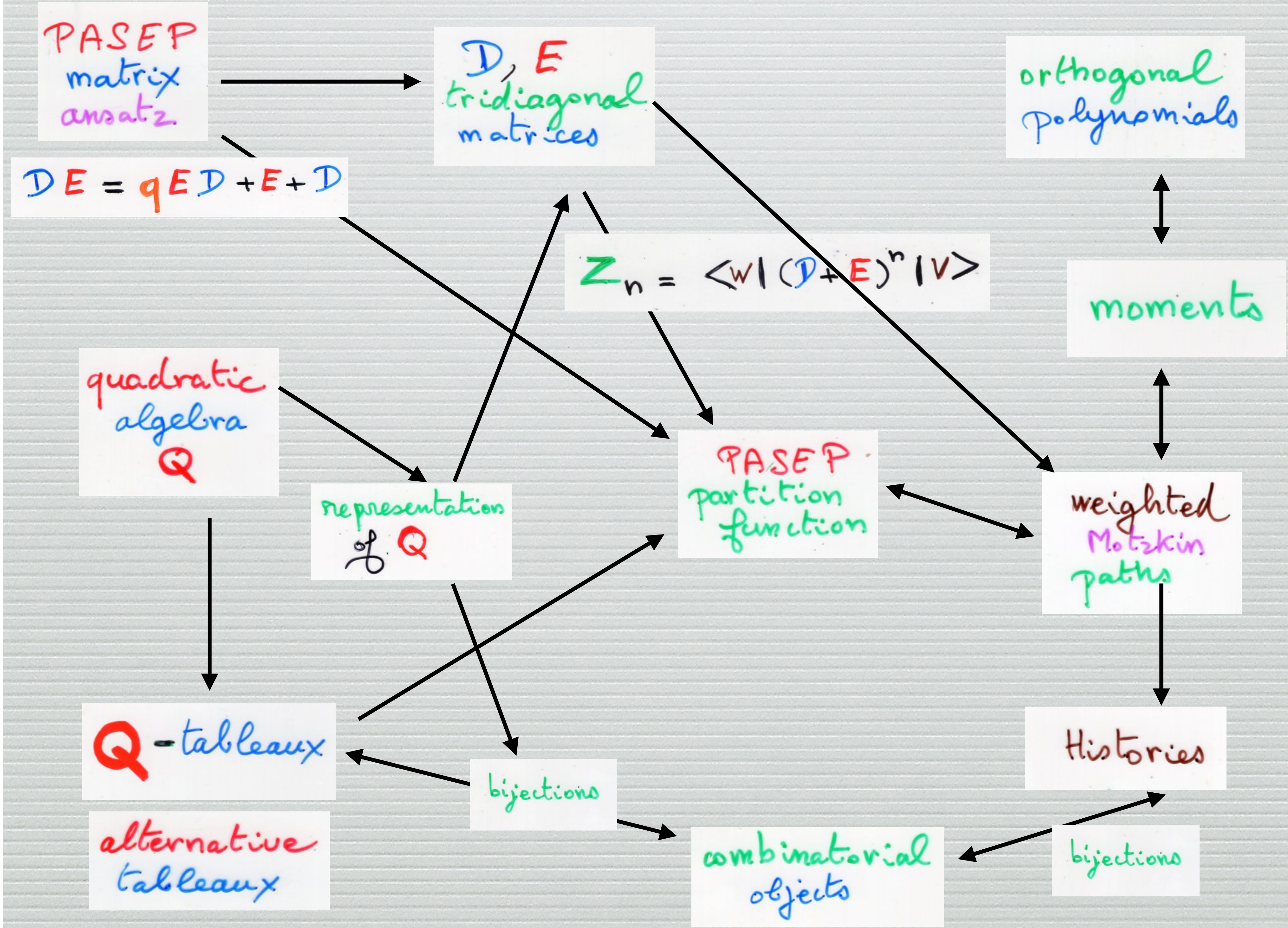

PASEP matrix ansatz
D, E tridiagonal matrices
orthogonal polynomials
DE = qED + E + D
Z_n = <W| (D+E)^n |V>
moments
quadratic algebra Q
PASEP partition function
weighted Motzkin paths
Q-tableaux
Histories
alternative tableaux
bijections
combinatorial objects
bijections


PASEP matrix ansatz
D, E tridiagonal matrices
orthogonal polynomials
DE = qED + E + D
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moments
quadratic algebra Q
representation of Q
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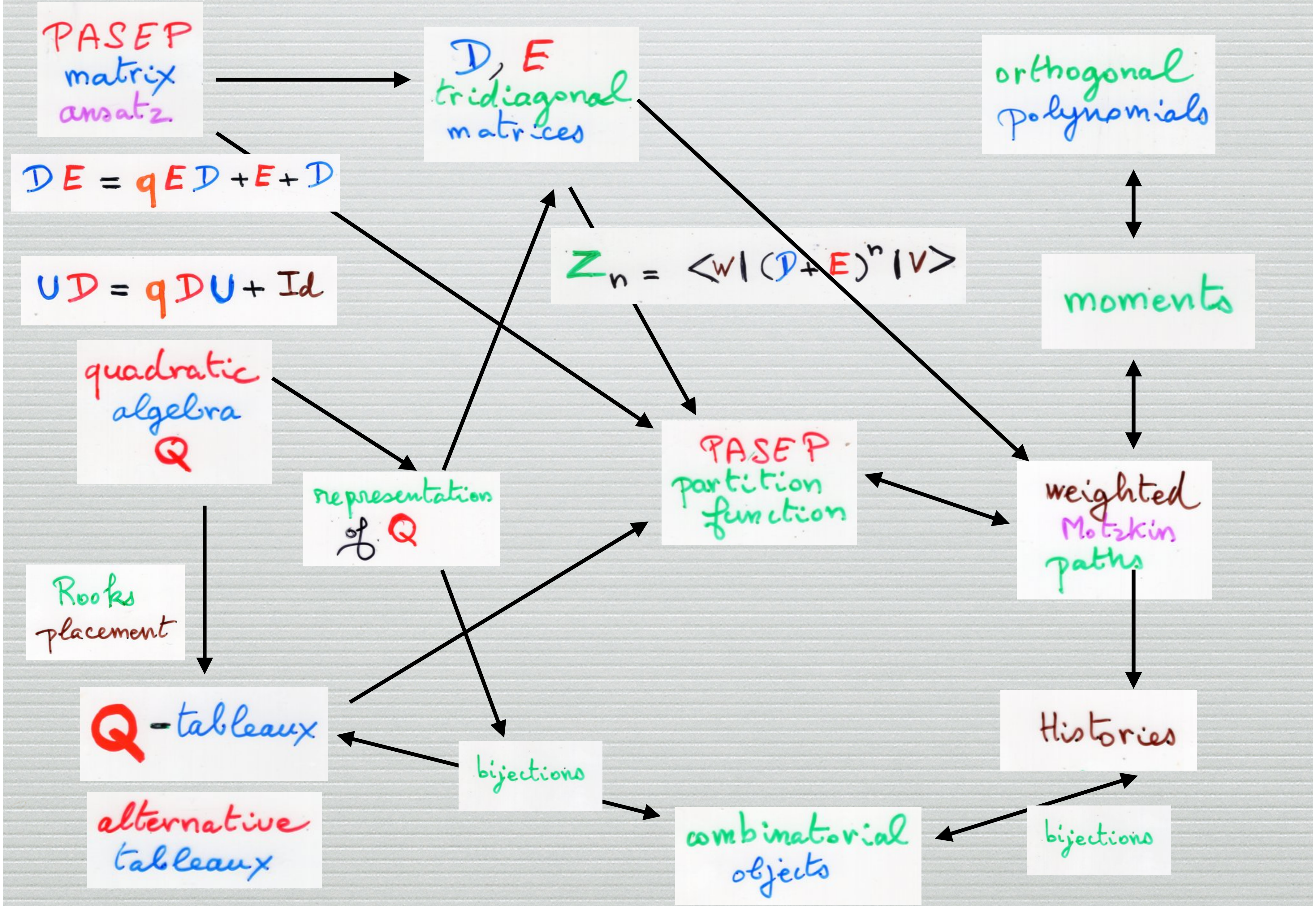
Histories


PASEP matrix ansatz
D, E tridiagonal matrices
orthogonal polynomials
DE = qED + E + D
UD = qDU + Id
Zn = <W| (D+E)^n |V>
moments
quadratic algebra Q
representation of Q
PASEP partition function
weighted Motzkin paths
Rooks placement
Q-tableaux
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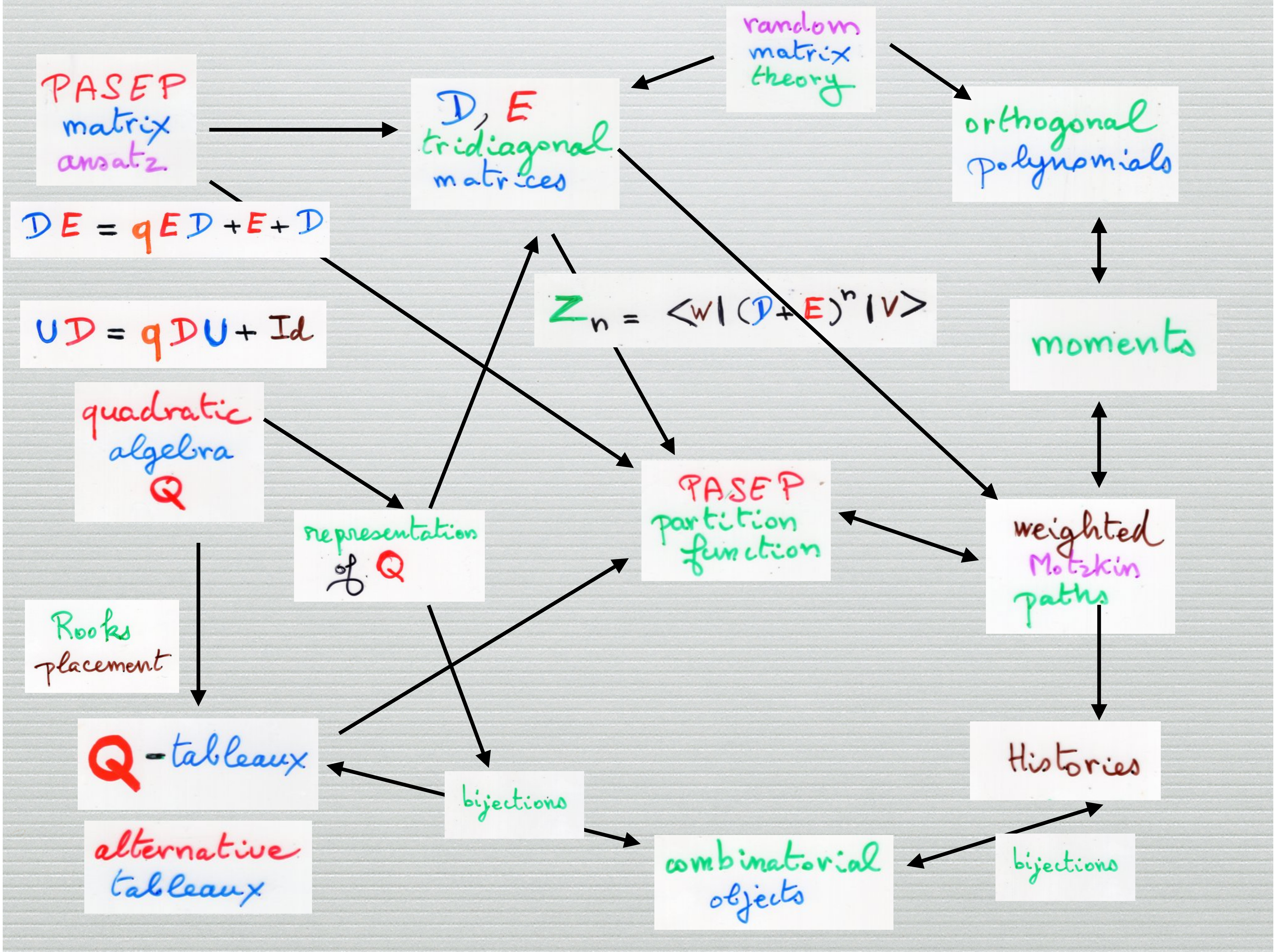
random matrix theory
PASEP matrix ansatz
D, E tridiagonal matrices
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DE = qED + E + D
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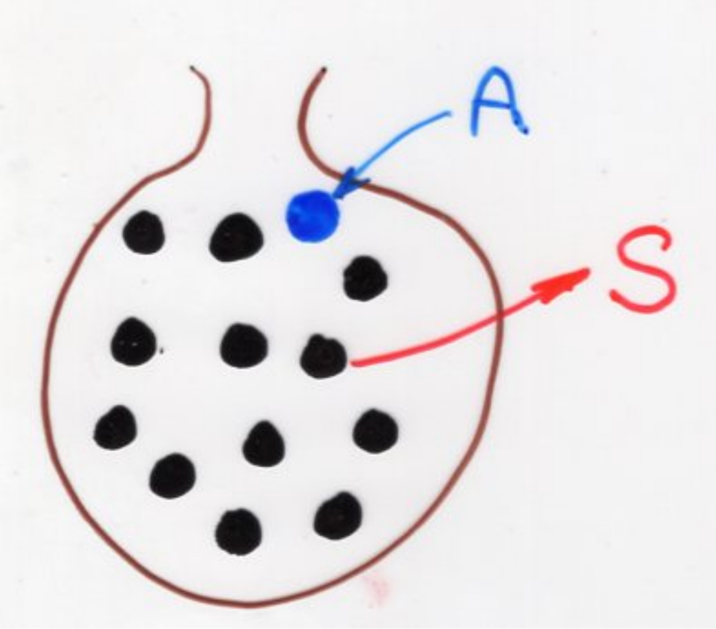
Polya urn

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix} \qquad S = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & k & \ddots \end{pmatrix}$$

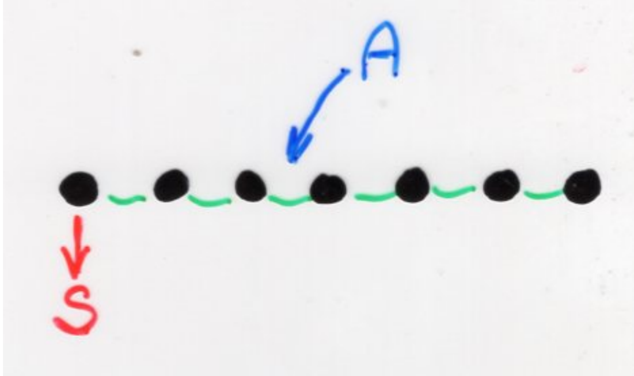
$$A\,|k\rangle = 1\,|(k+1)\rangle$$
$$S\,|k\rangle = k\,|(k-1)\rangle$$

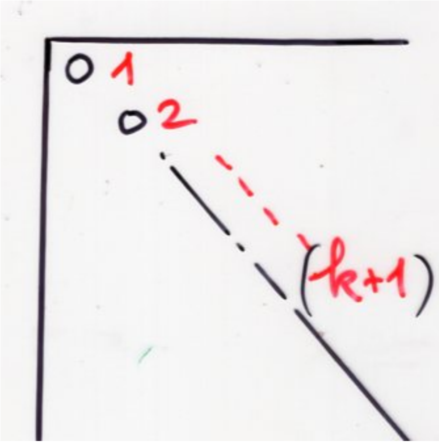


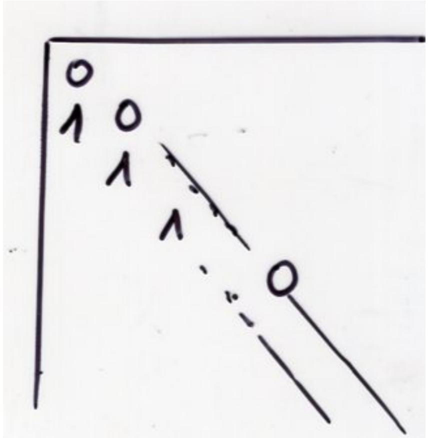
$$AS - SA = I$$

Priority queue

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$S\,|k\rangle = |(k-1)\rangle$$



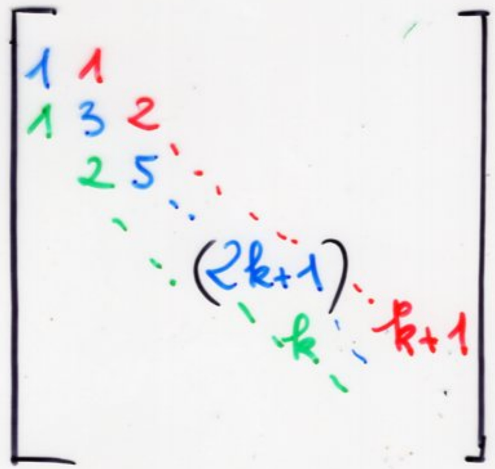
$A =$ 

$S =$ 

data structures

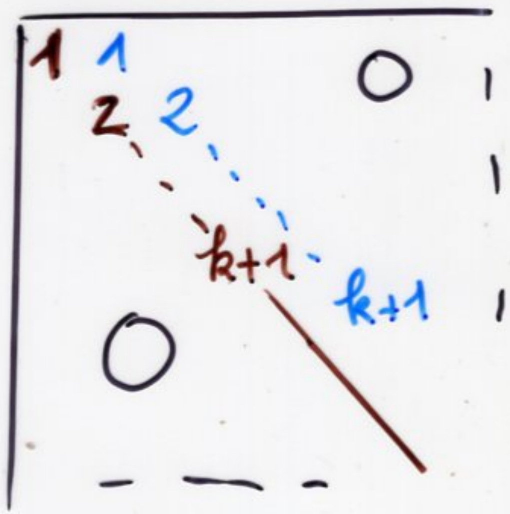
Computer Science

$$AS - SA = I$$

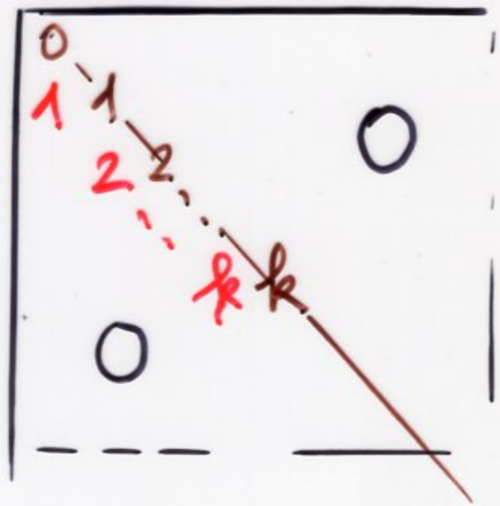


dictionnary data structure

$D =$



$E =$

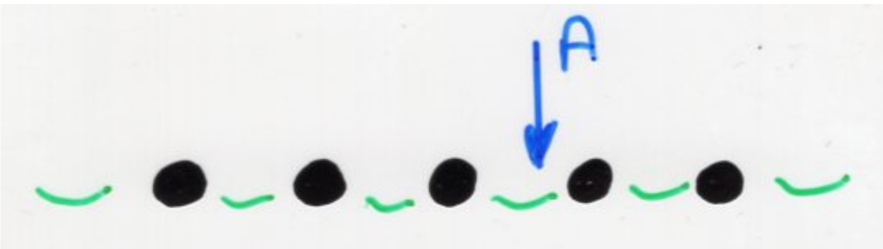


$$DE = qED + E + D$$

dictionnary data structure

$$A \, | k \rangle = (k+1) \, | (k+1) \rangle$$

$$S \, | k \rangle = k \, | (k-1) \rangle$$

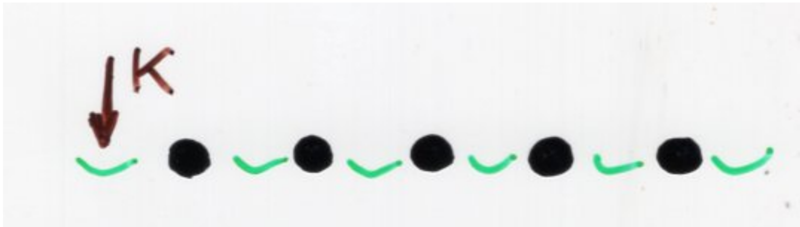


$$\begin{cases} D = A + K \\ E = S + J \end{cases}$$

$$DE = ED + E + D$$

$$J \, | k \rangle = k \, | k \rangle$$

$$K \, | k \rangle = (k+1) \, | k \rangle$$



Tridiagonal matrix

$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_k(x)$$

$$P_k(x) = \det(x I_k - A)$$

Where are we going ?

$$DE = qED + E + D$$

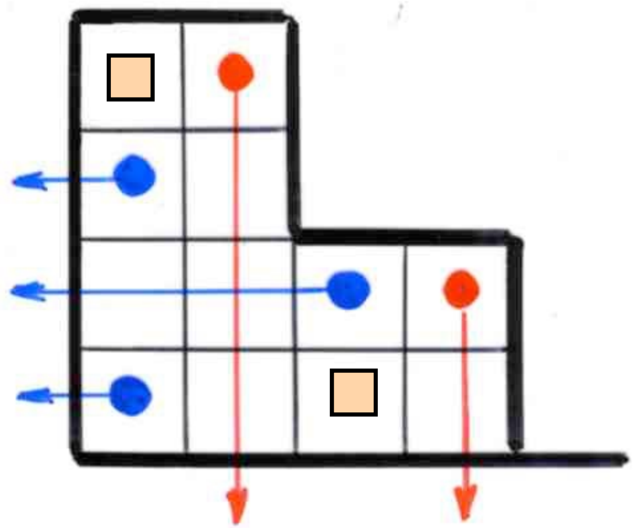
partition function $Z_n = \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$

alternative tableaux

$k(T)$ = nb of cells □

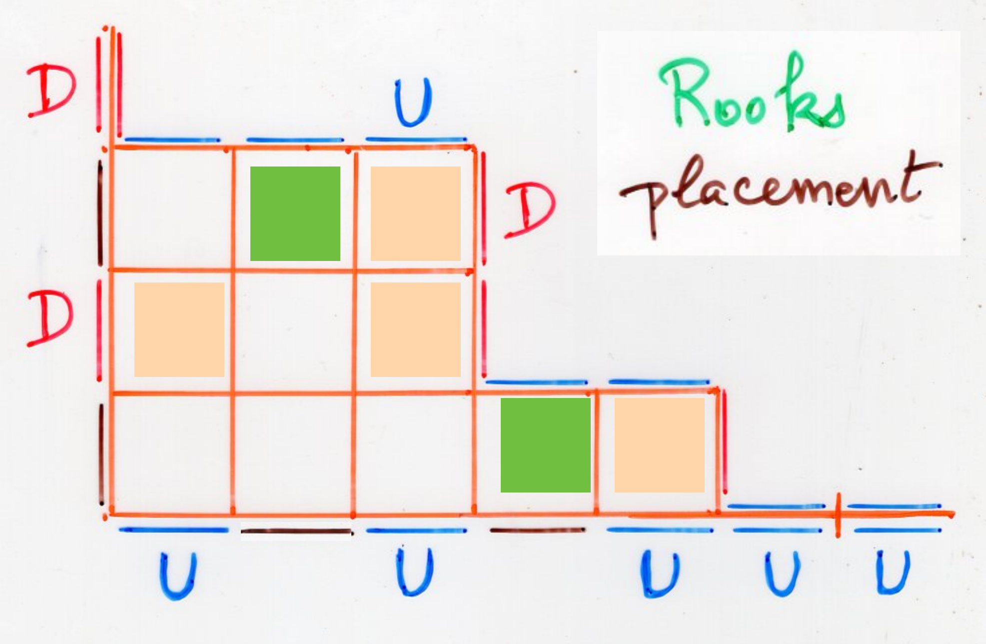
$i(T)$ = nb of rows without ● (blue)

$j(T)$ = nb of columns without ● (red)



$$UD = qDU + Id$$

$$\sum_{\substack{\text{rook placements } T \\ \text{"size } n\text{"}}} q^{k(T)} x^{i(T)} y^{j(T)}$$



$$\begin{cases} i(T) = \text{number of rows} \\ j(T) = \text{number of columns} \end{cases} \text{with no cell labeled}$$

$UD \rightarrow I_v I_h$

$q^{k(T)}$

$$D = \frac{1}{1-q}(1+d)$$

$$E = \frac{1}{1-q}(1+e)$$

$$de - qed = (1-q)\,Id$$

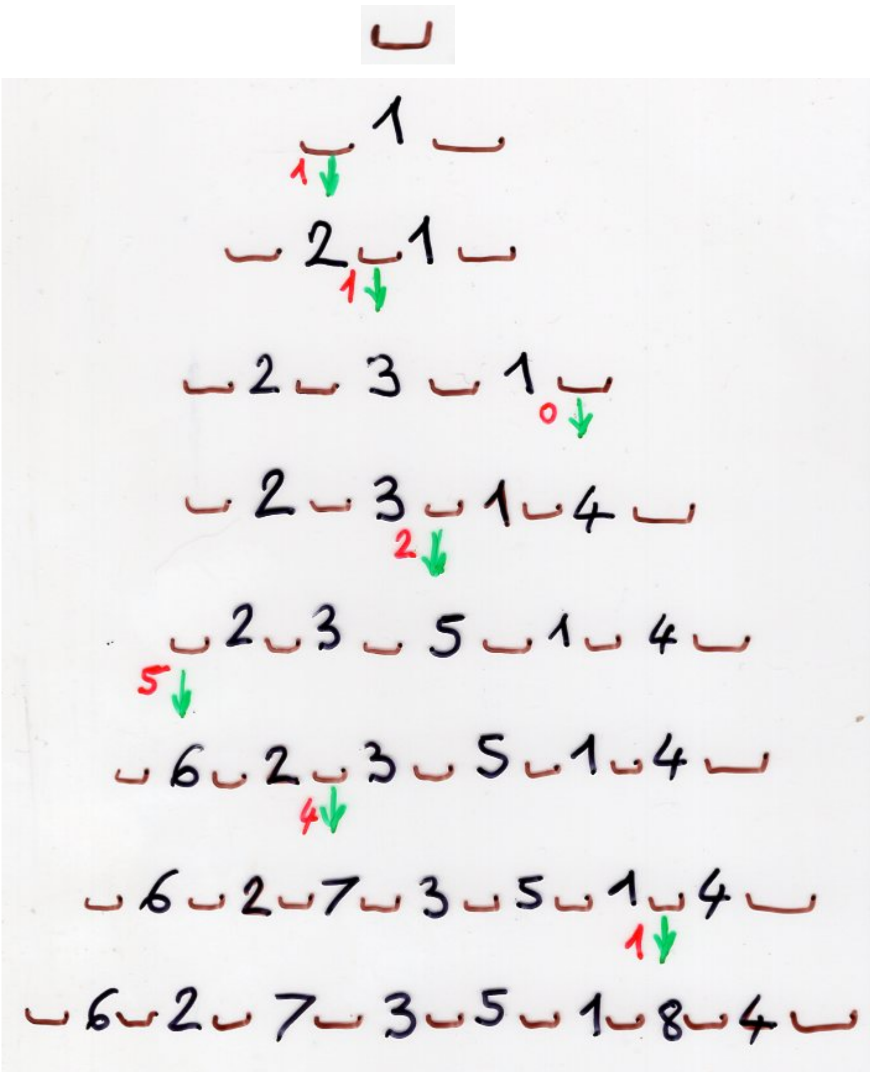
$$DE = qED + E + D$$

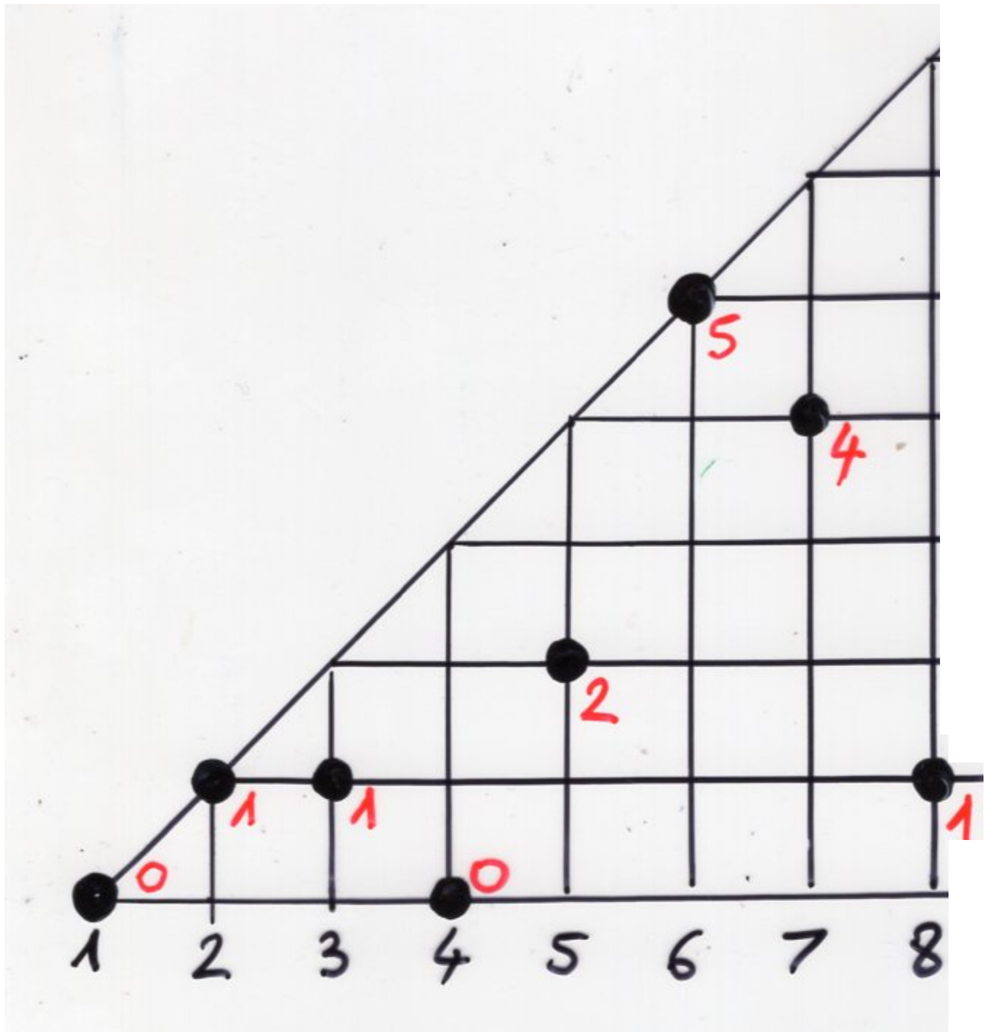
$$UD = qDU + Id$$

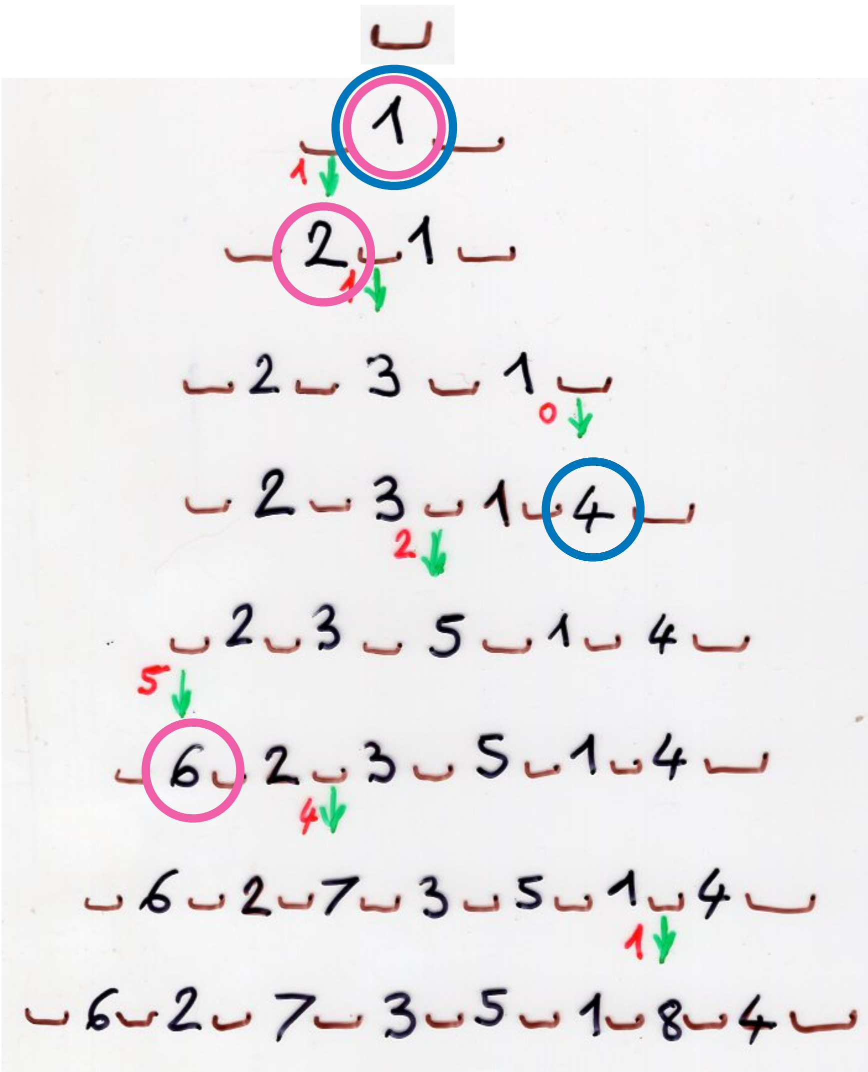
q-analogues : first step

Inversion tables

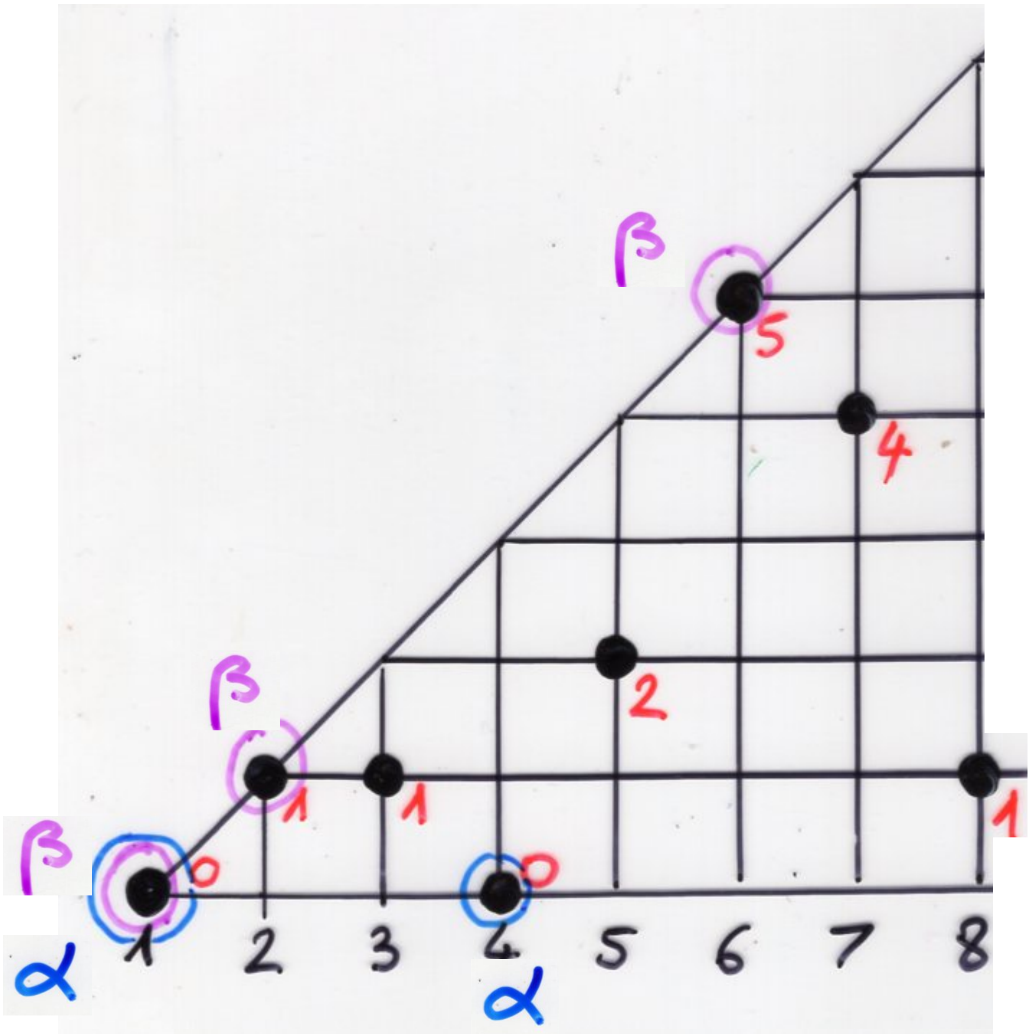
q-analogues of Hermite histories


1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4


1 2 3 4 5 6 7 8



1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4



β
β
β
α
α
1 2 3 4 5 6 7 8

distribution of permutations

3 parameters :
- number of inversions
- number of rl-min elements
- number of lr-min elements

$$[i\,;\alpha,\beta]_q = (\alpha + q + q^2 + \dots + q^{i-2} + \beta q^{i-1})$$

$$[1\,;\alpha,\beta]_q = \alpha\beta$$

$$[n\,;\alpha,\beta]!_q = \prod_{i=1}^{n-1} [i\,;\alpha,\beta]_q$$

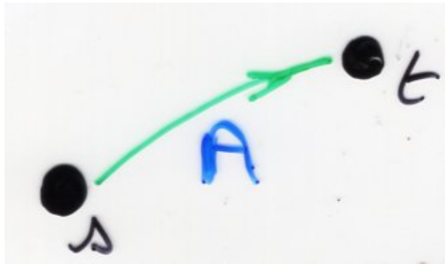
the philosophy of « histories »

and its q-analogues

history

operators

S states



weight

$V_A(s,t)$ = number of possibilities to apply A

$H = h_1 h_2 \cdots h_n$

sequence of operators

initial state s_0

$$s_i \xrightarrow{h_i} s_{i+1}$$

$P = (p_1, p_2, \ldots, p_n)$

$$0 \leq p_i < V_{h_i}(s_i, s_{i+1})$$

$H = h_1 h_2 \cdots h_n$

sequence of operators

initial state s_0

$P = (p_1, p_2, \ldots, p_n)$

$0 \leq p_i < V_{h_i}(s_i, t_i)$

q - weight

$V_q(H) = q^{\left(\sum_{i=1}^{n} p_i\right)}$

(α, β) - weight

$p_i = 0 \rightarrow \alpha$

$p_i = V_{h_i}(s_i, t_i) - 1 \rightarrow \beta$

$\alpha_{h_i} \quad \beta_{h_i}$

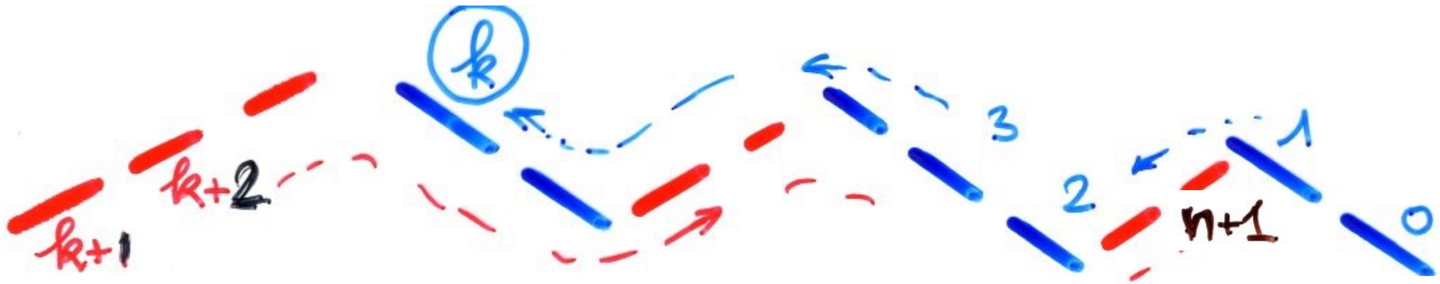
the maj index

$$\mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma(i) > \sigma(i+1)}} i$$

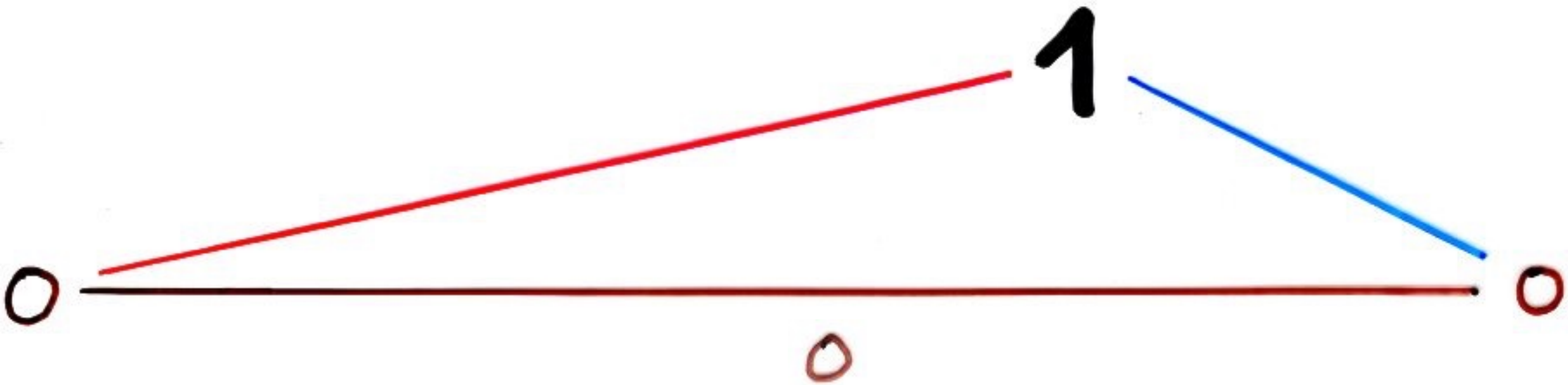
$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}$$

Mahonian
distribution

n+1
3
2
1
0



k
k+1
k+2
3
2
1
n+1
0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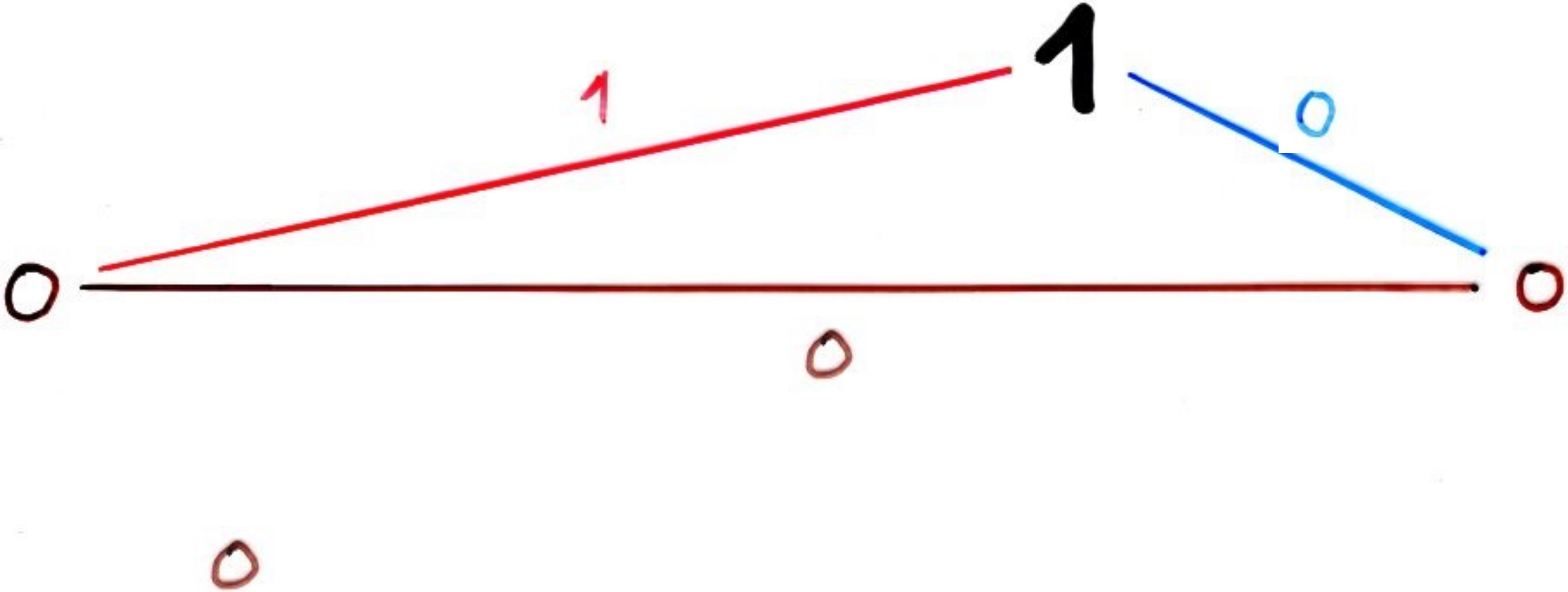
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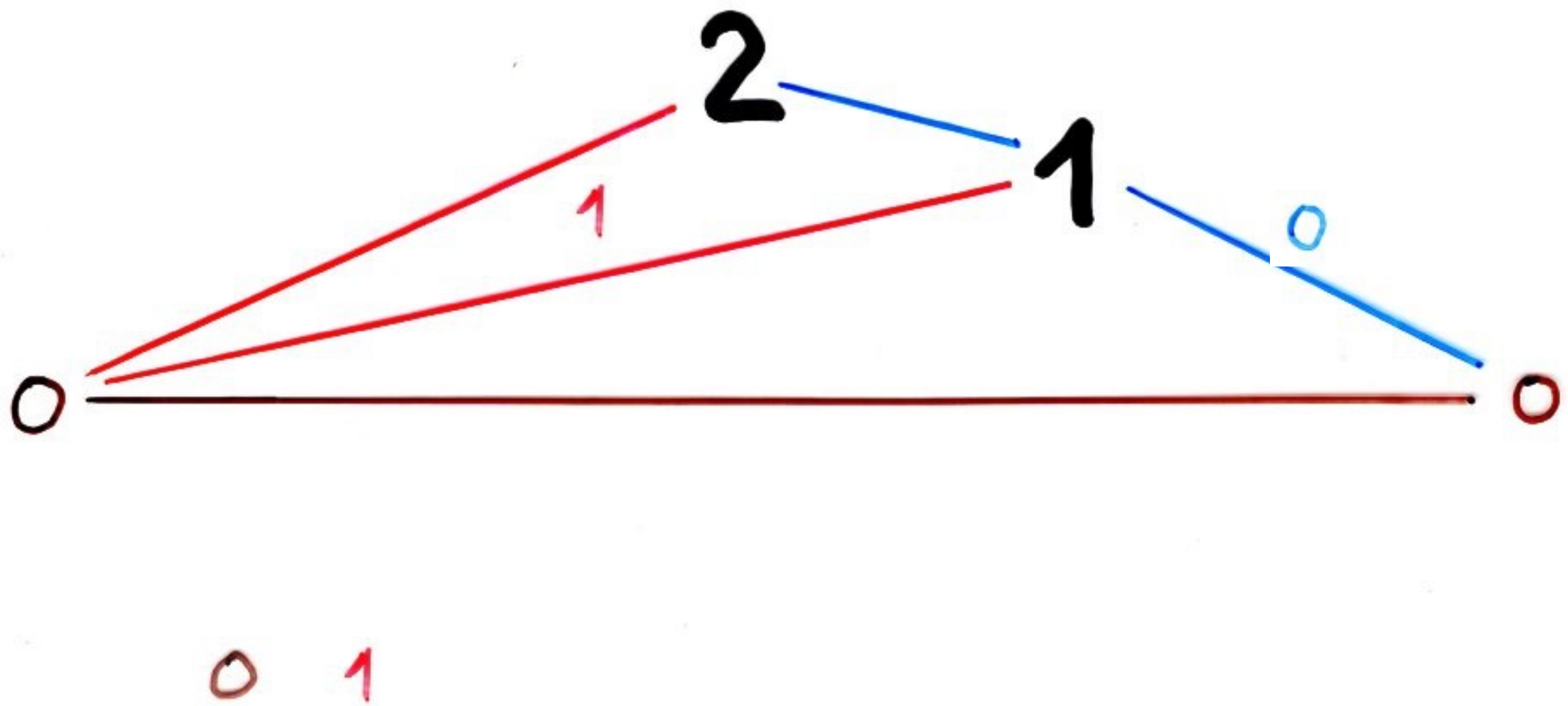
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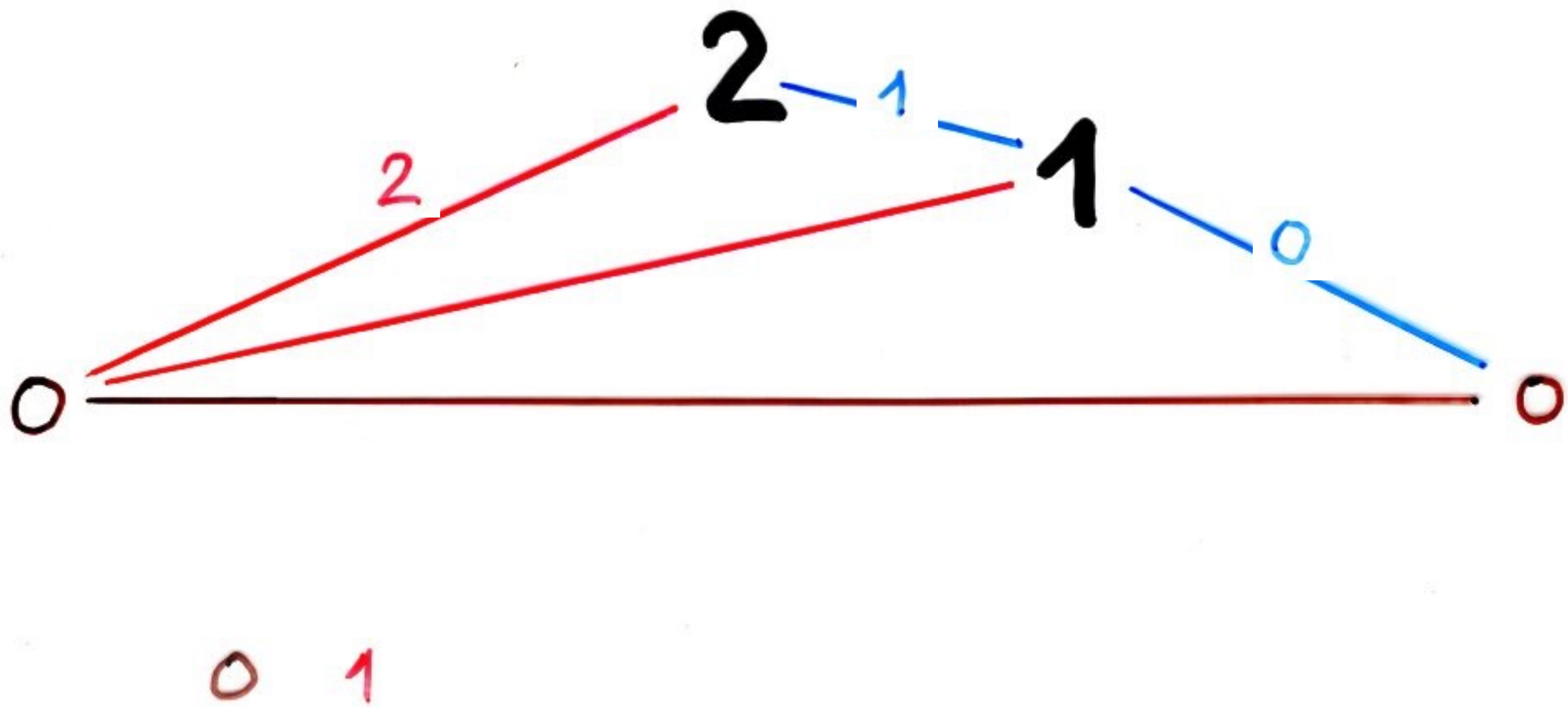
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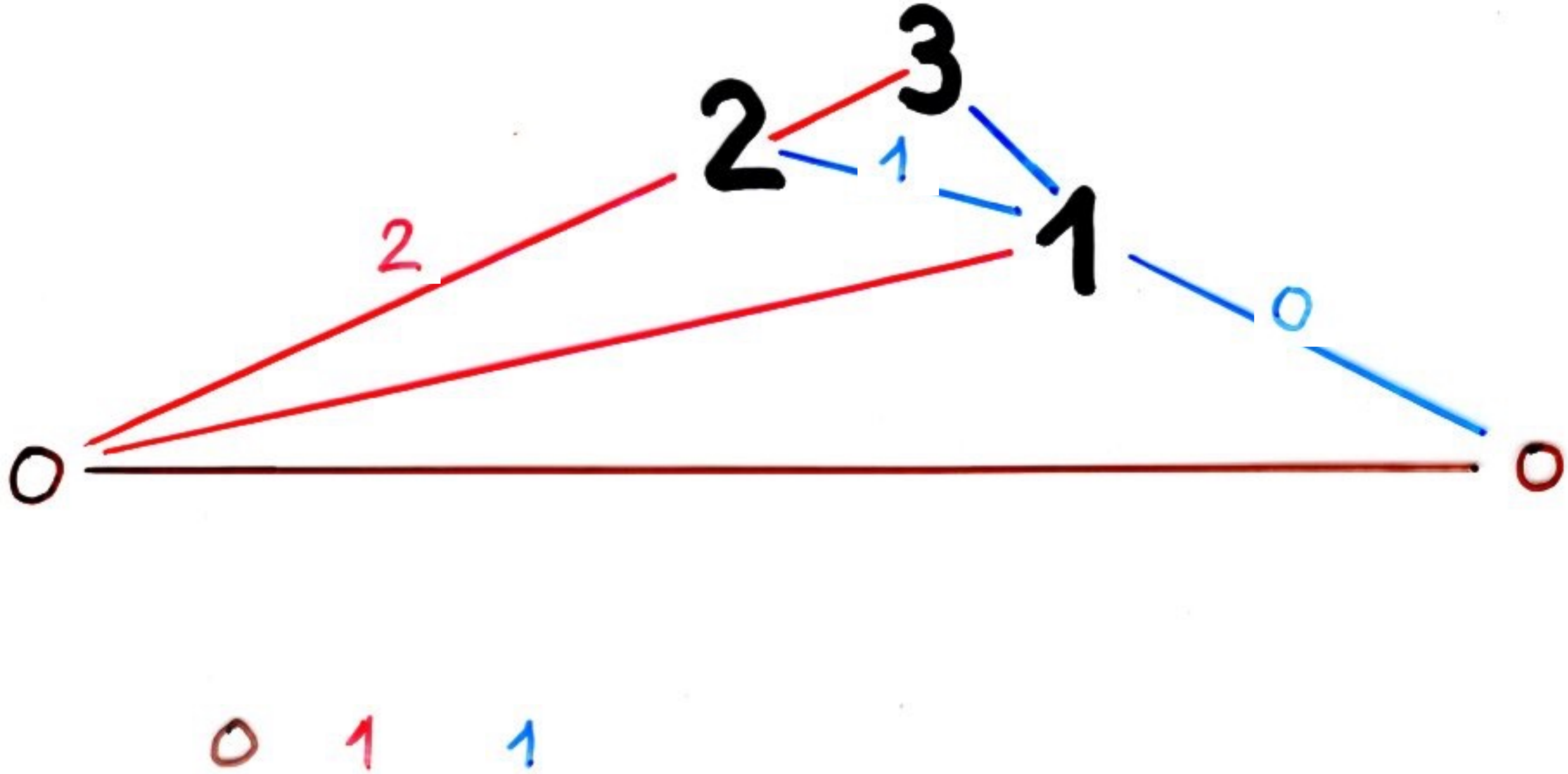

0
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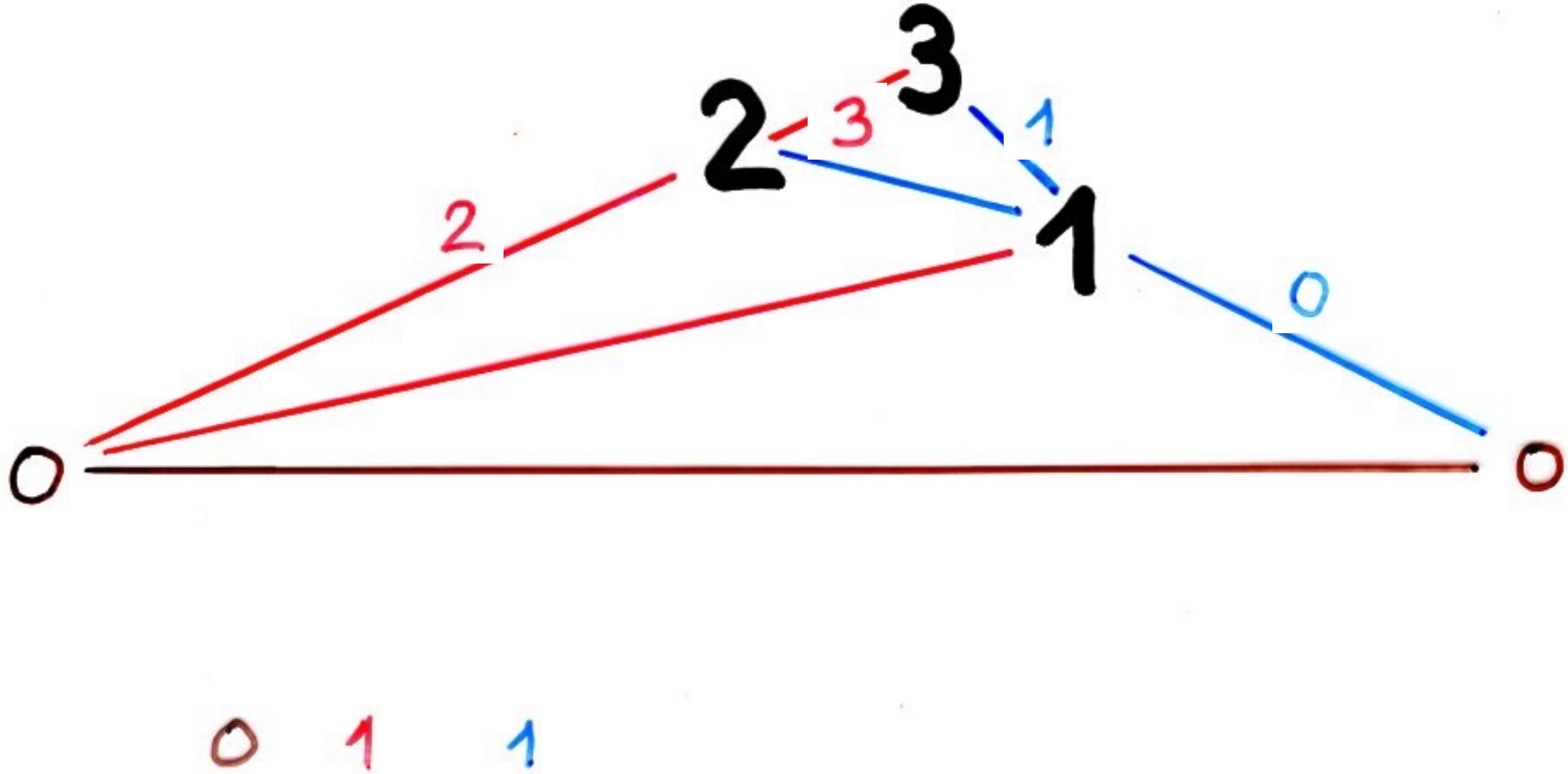
0 1 1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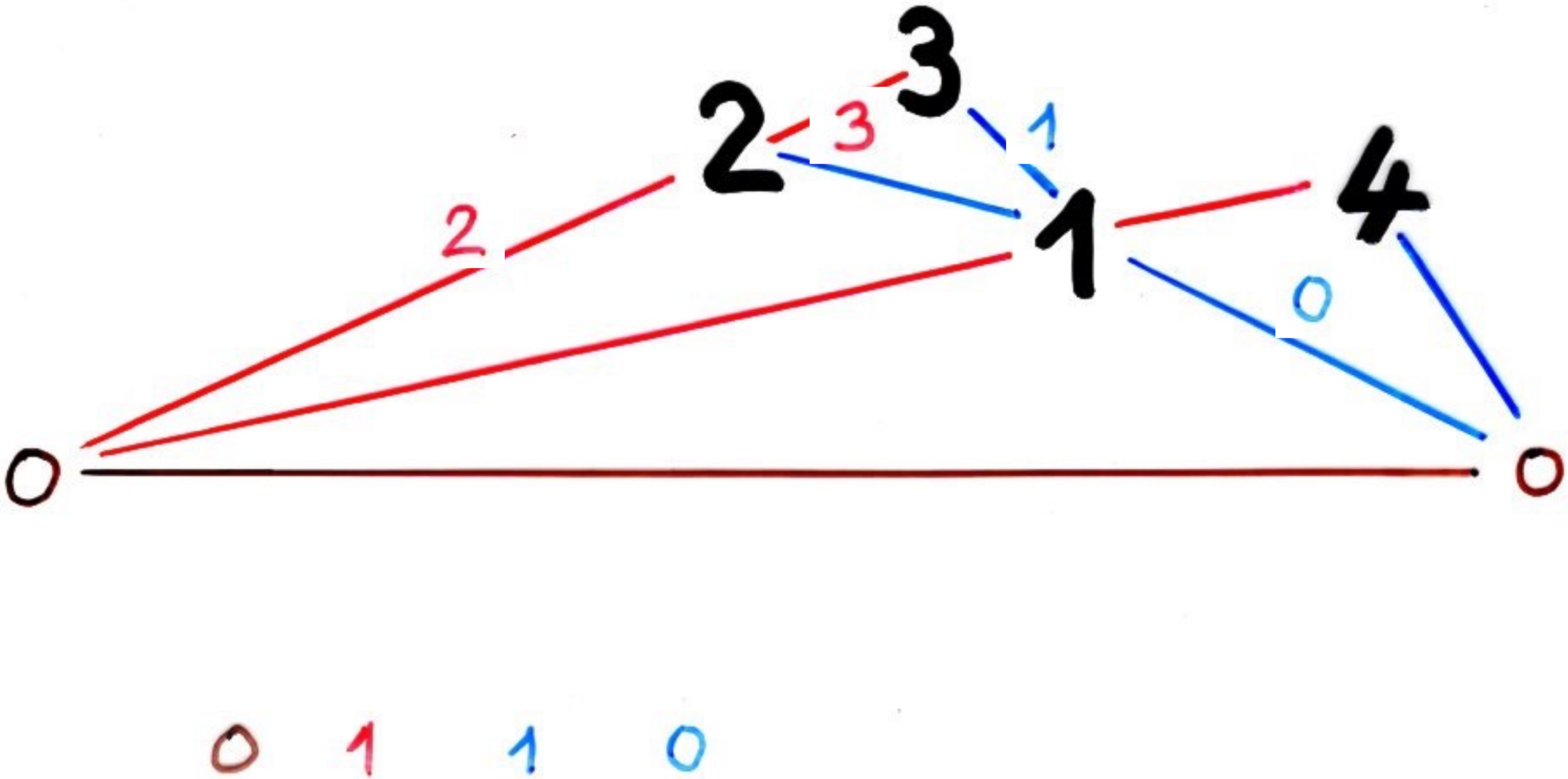

3
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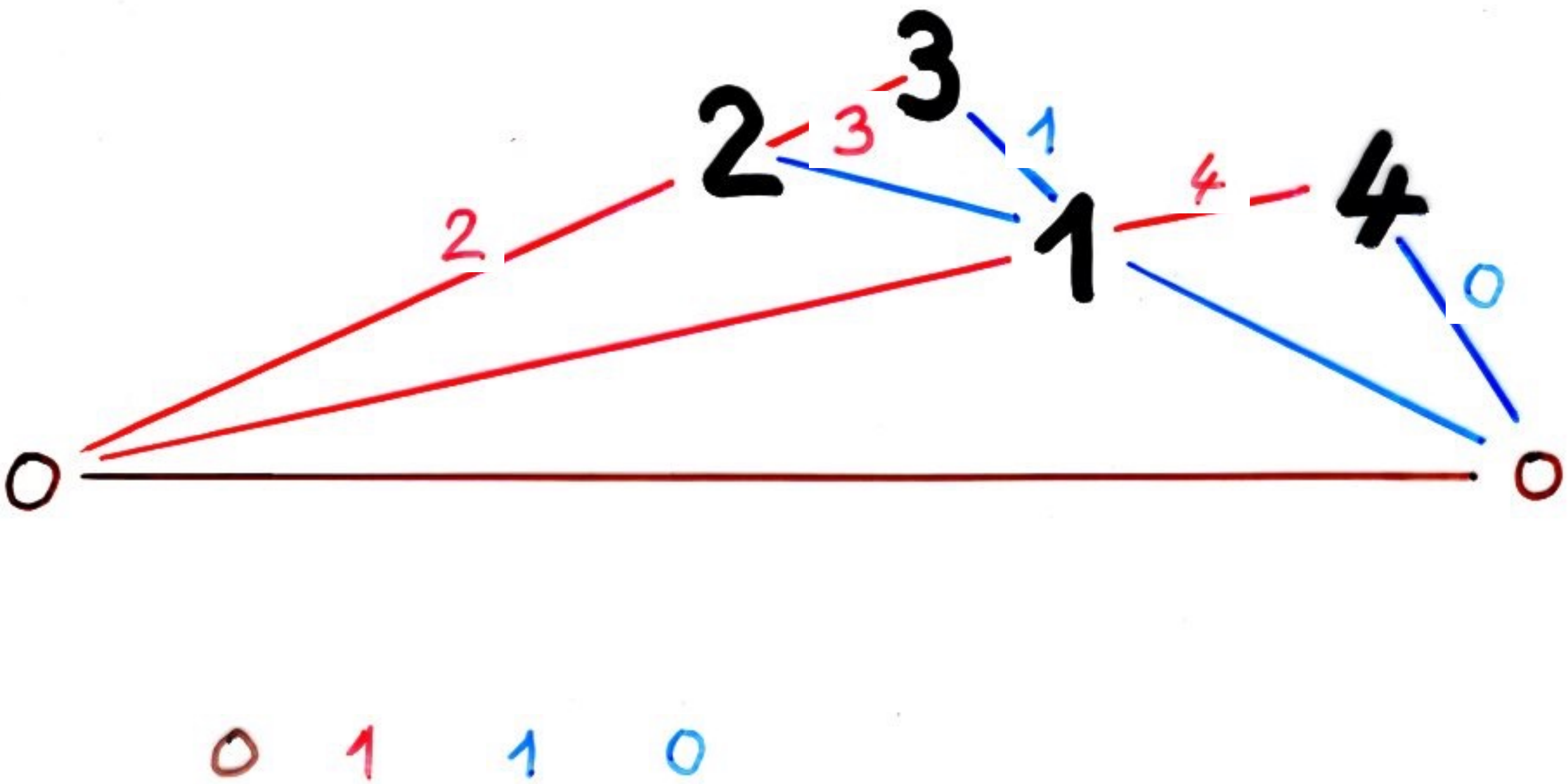
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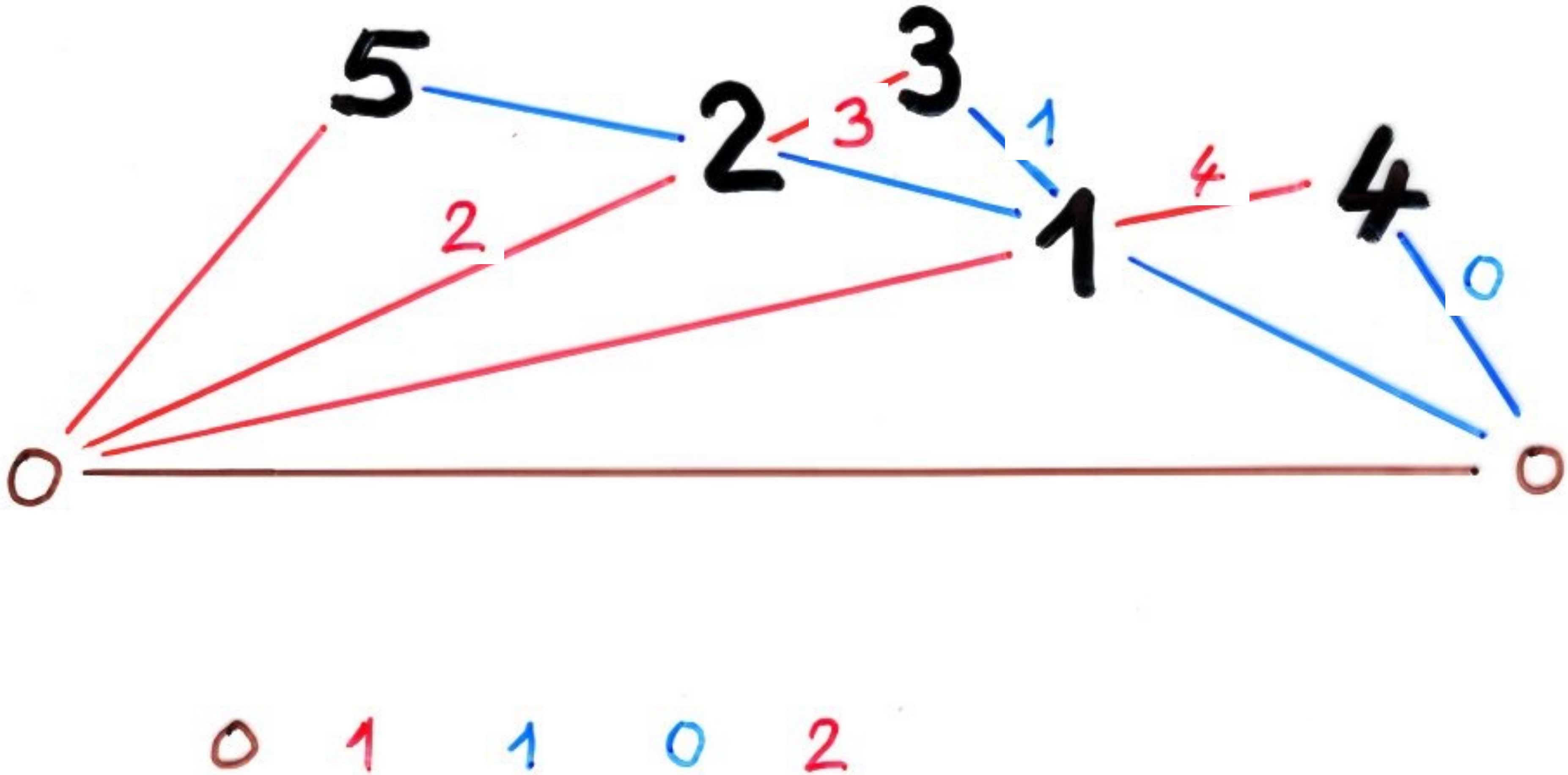
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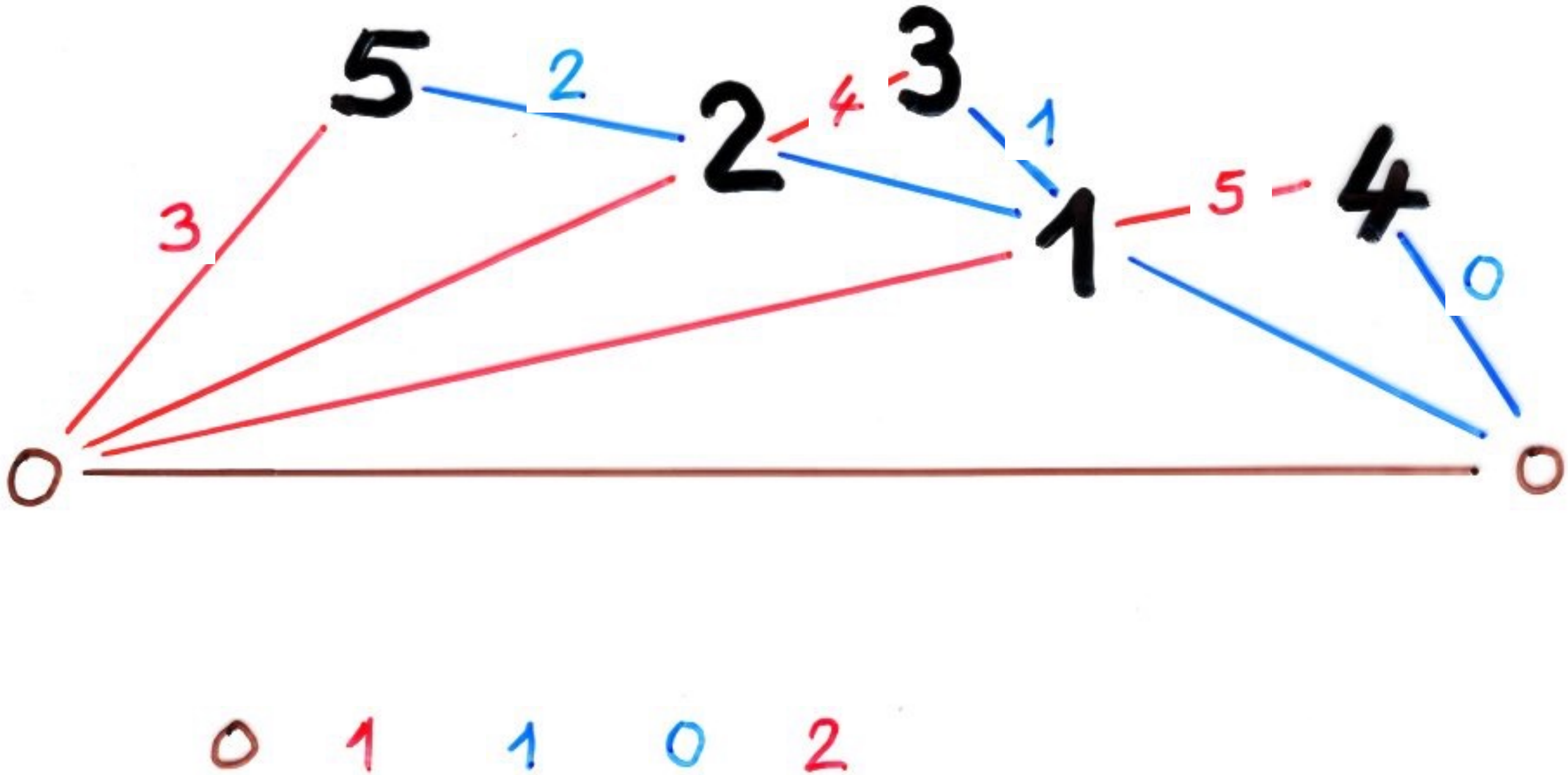
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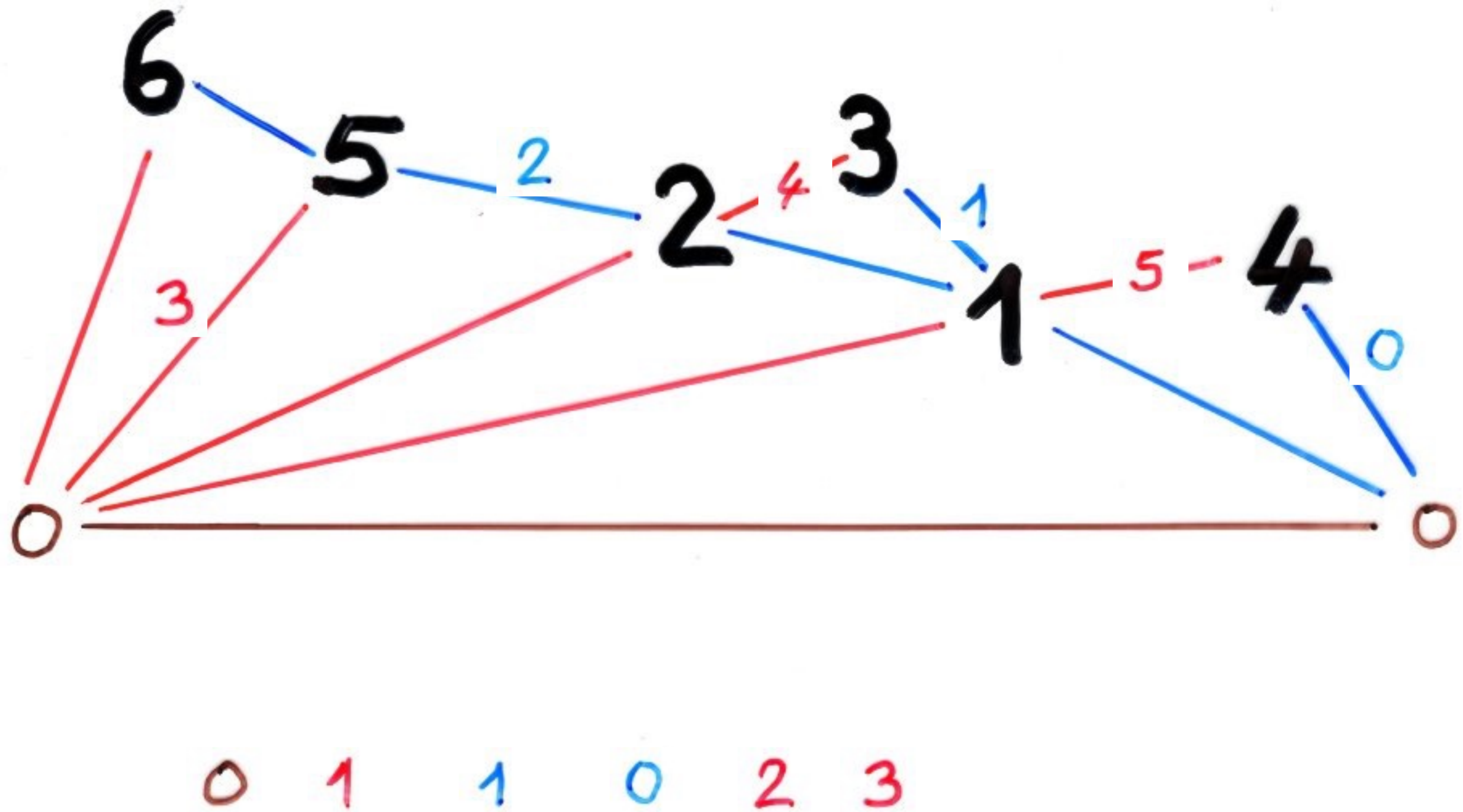

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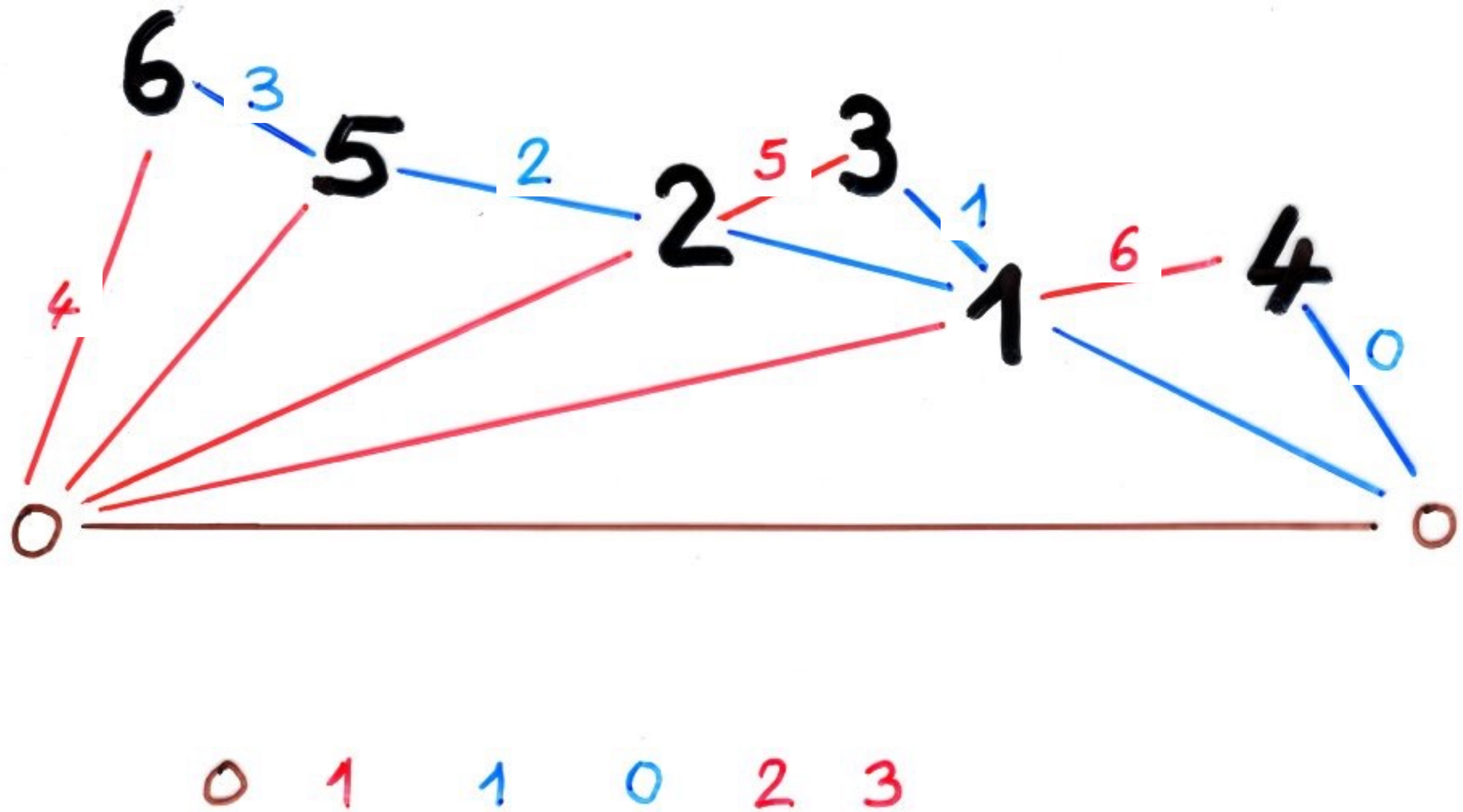
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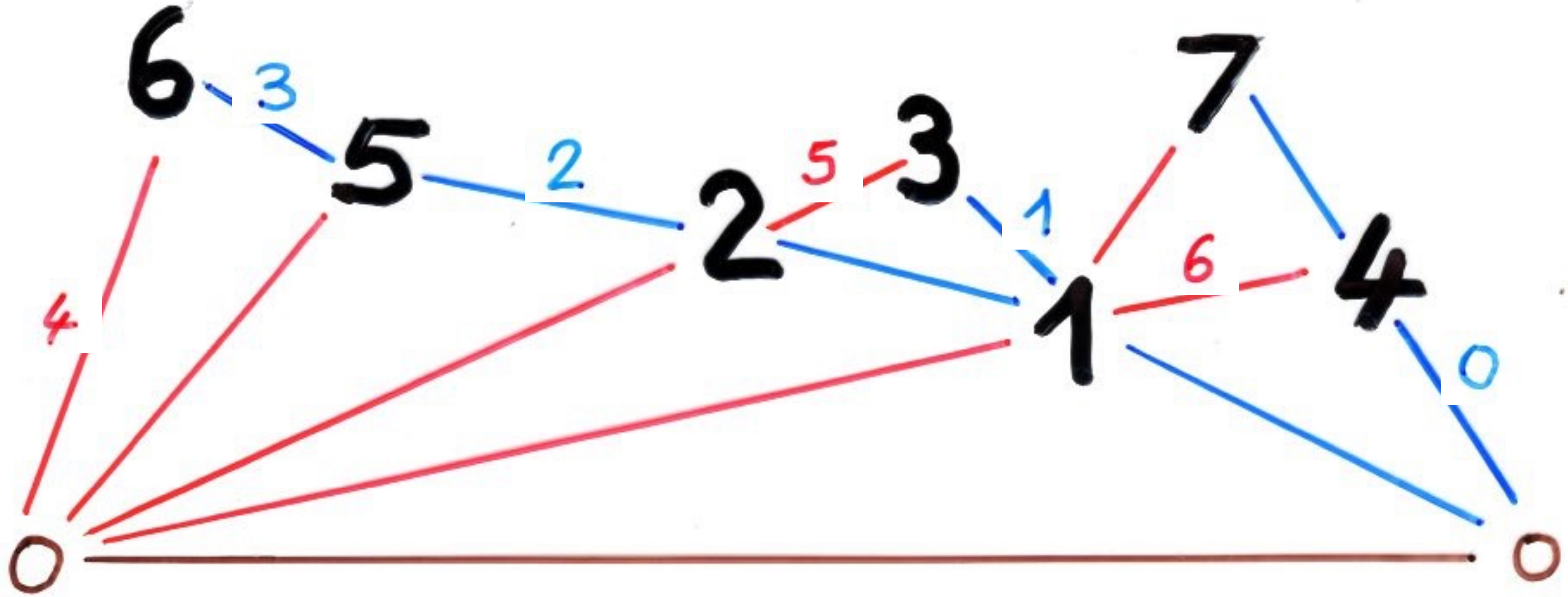

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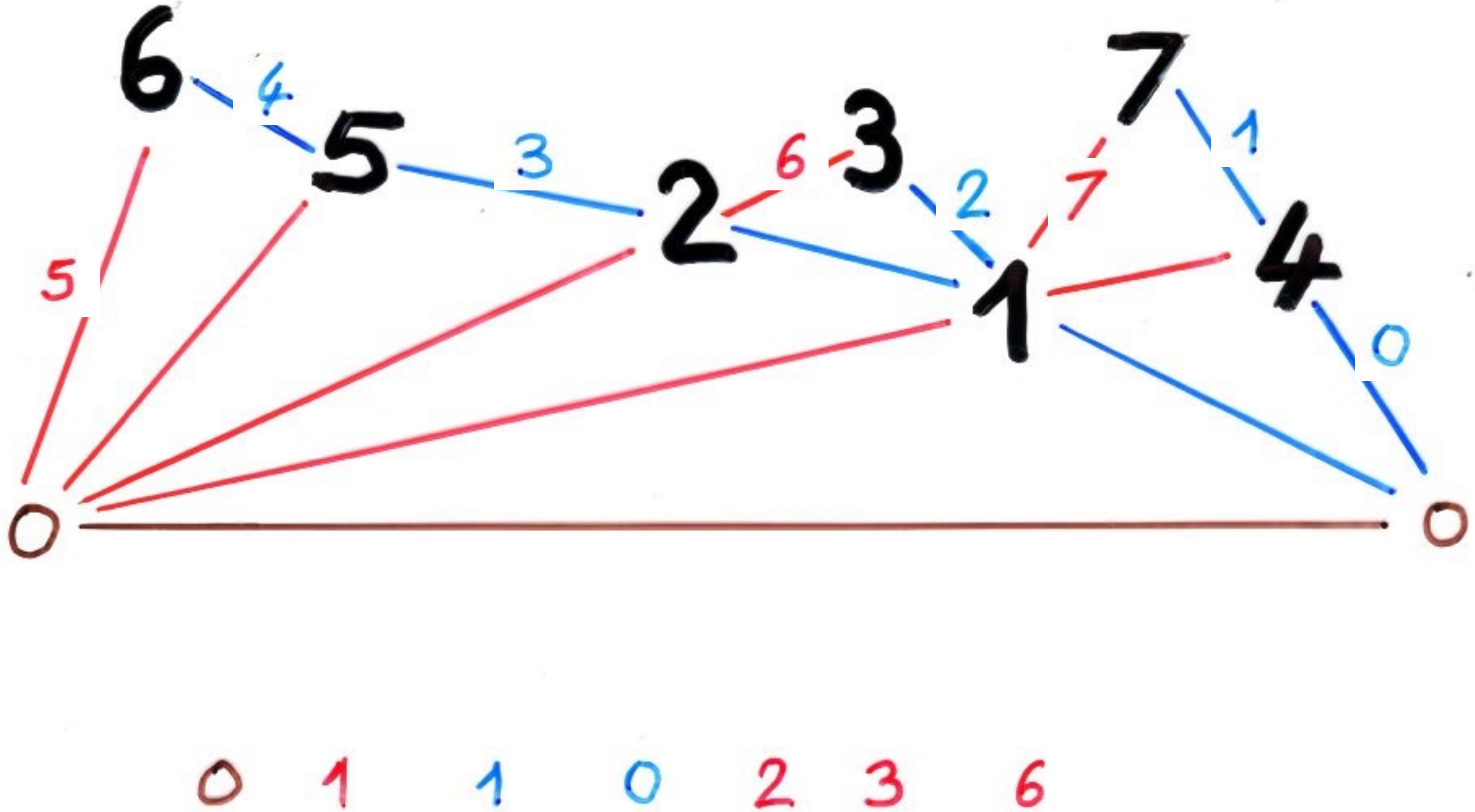
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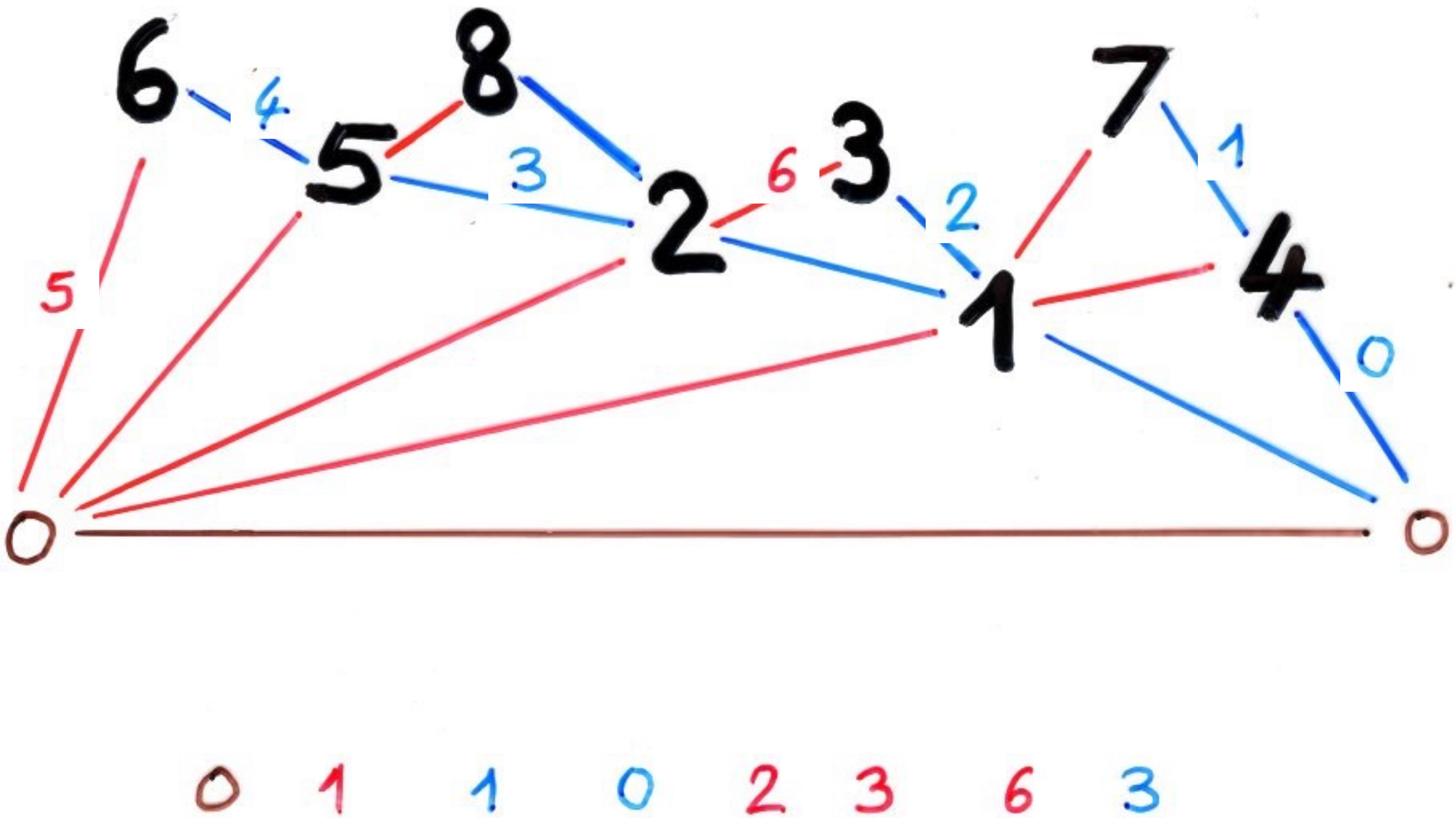
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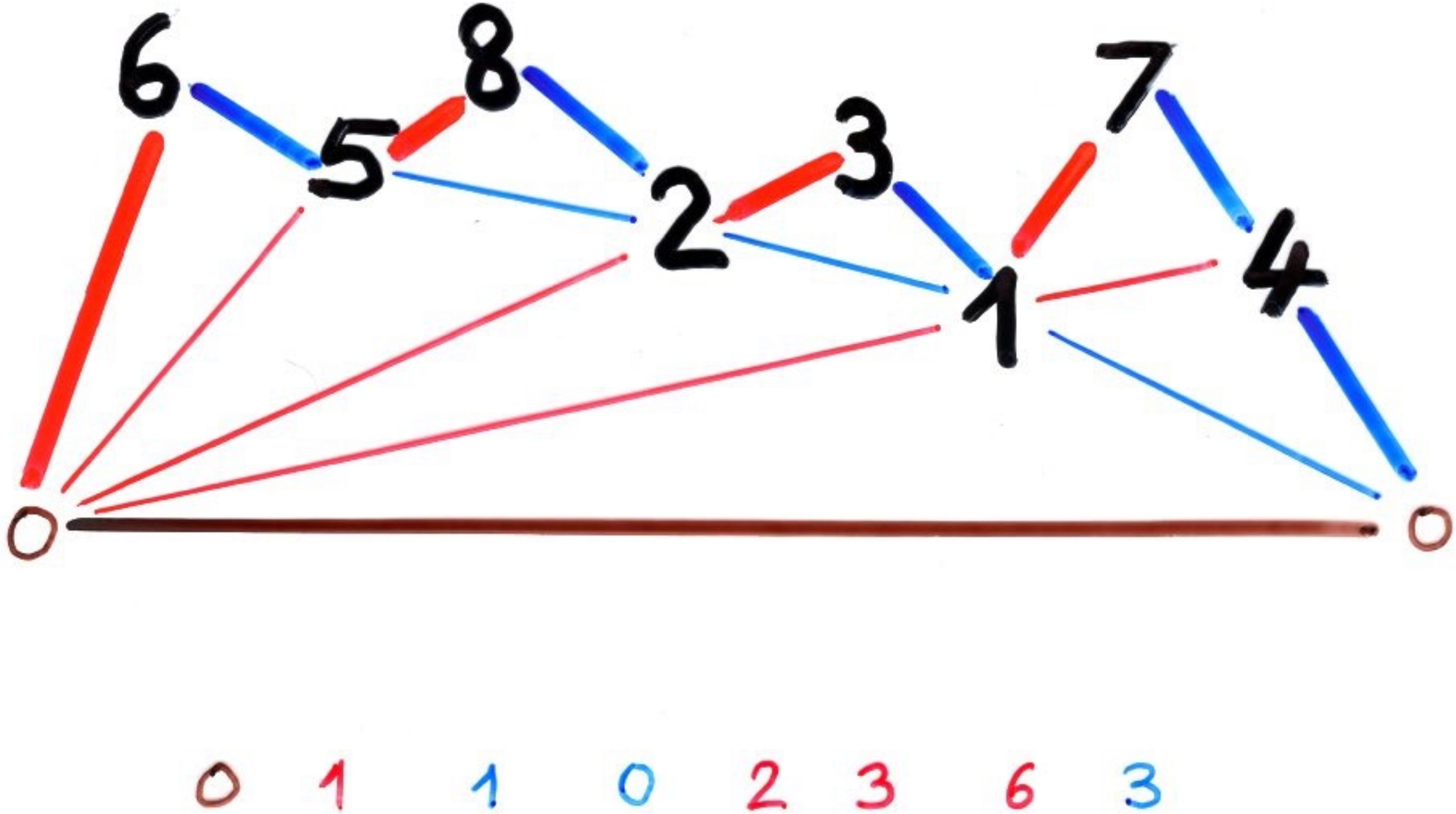
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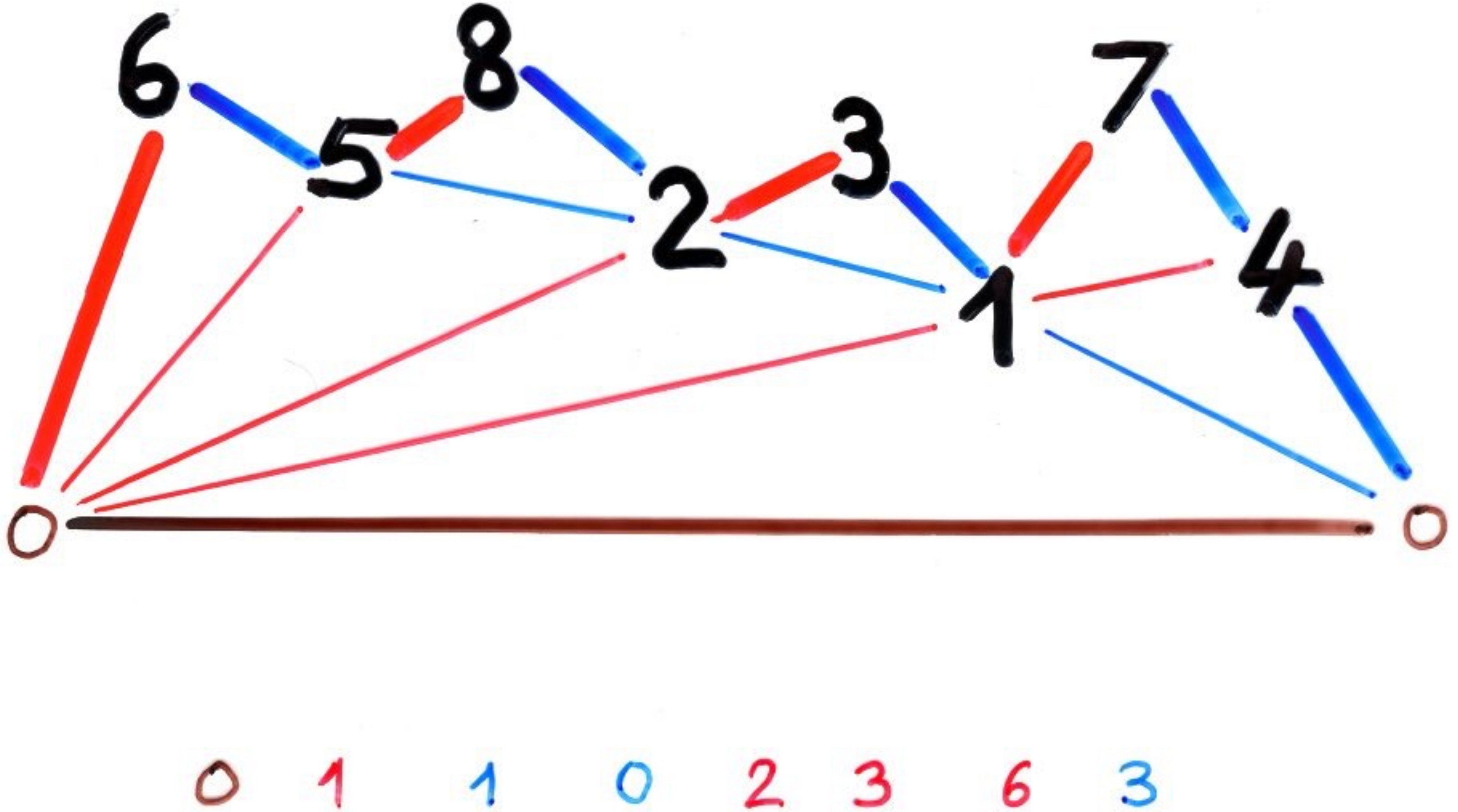
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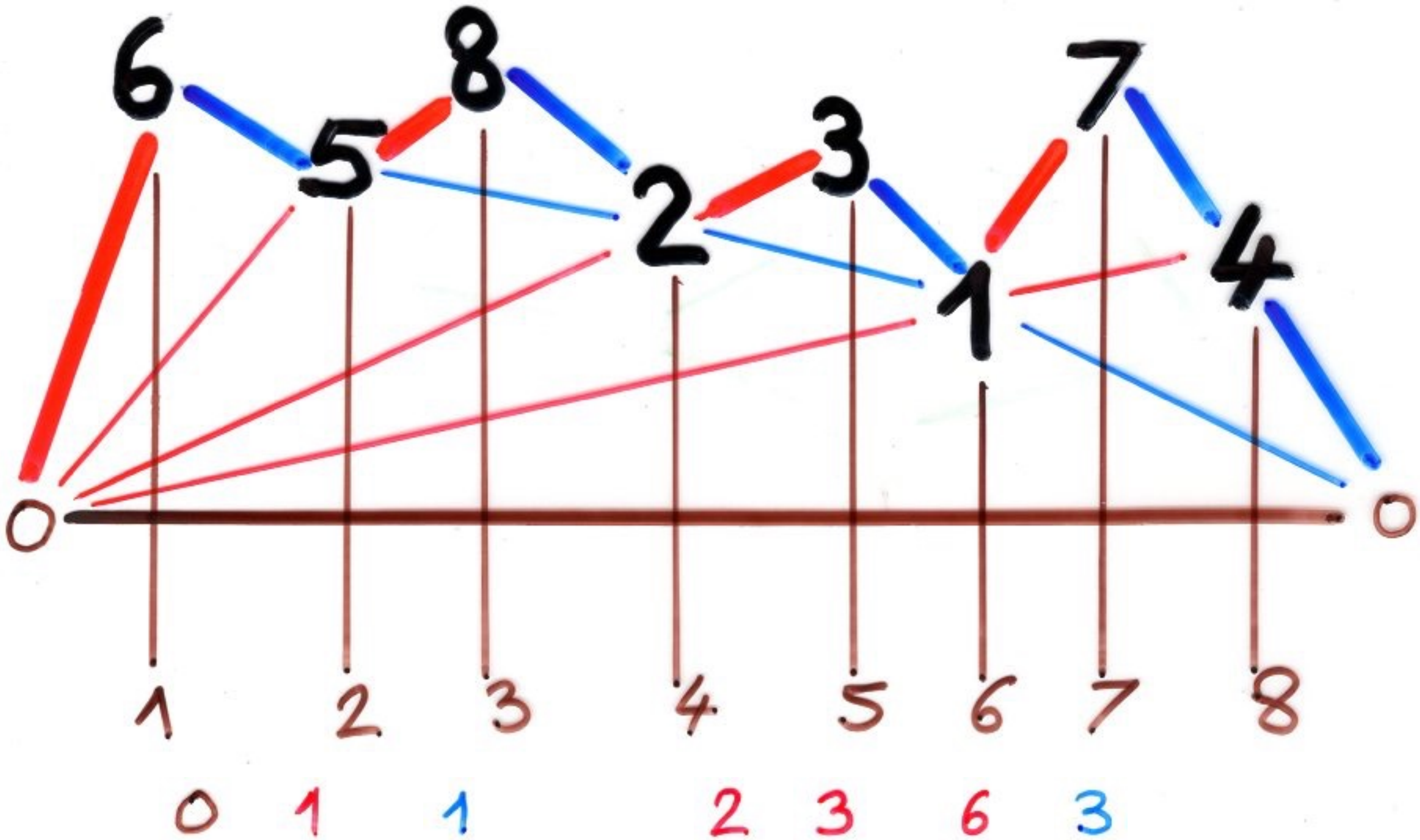
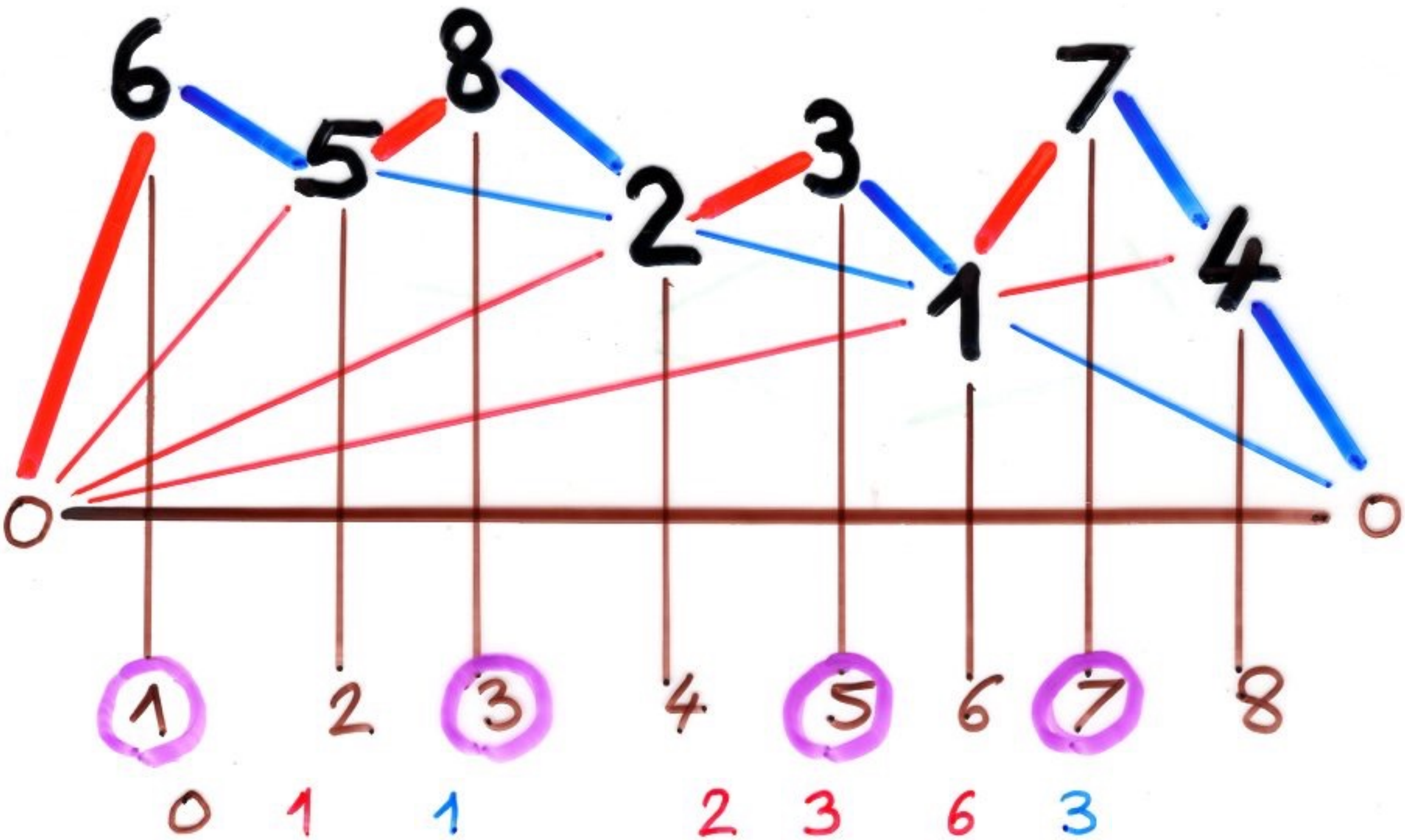
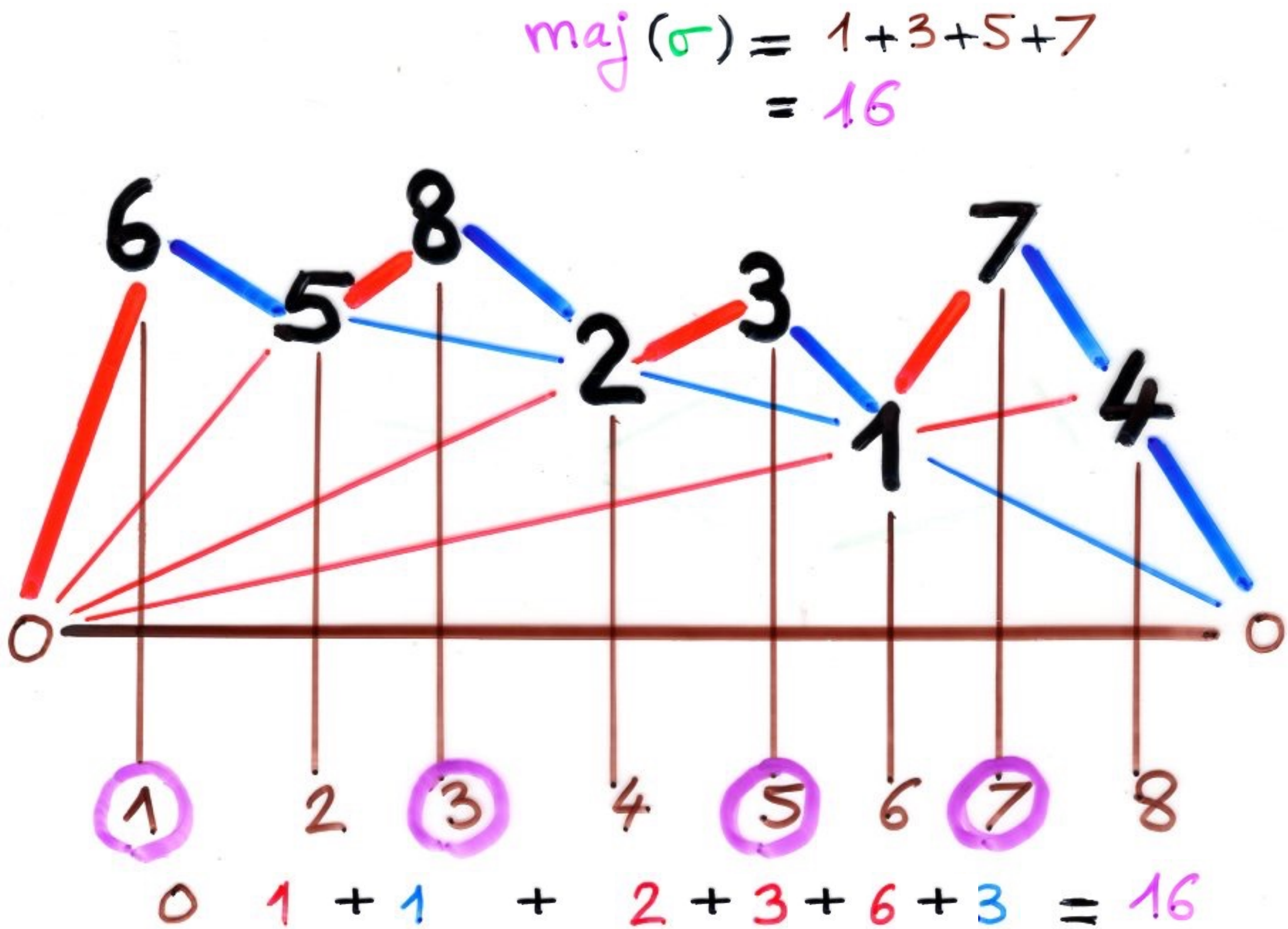


6 5 8 2 3 1 7 4
0 0
1 2 3 4 5 6 7 8
0 1 1 2 3 6 3

$$\mathrm{maj}(\sigma) = 1+3+5+7 = 16$$



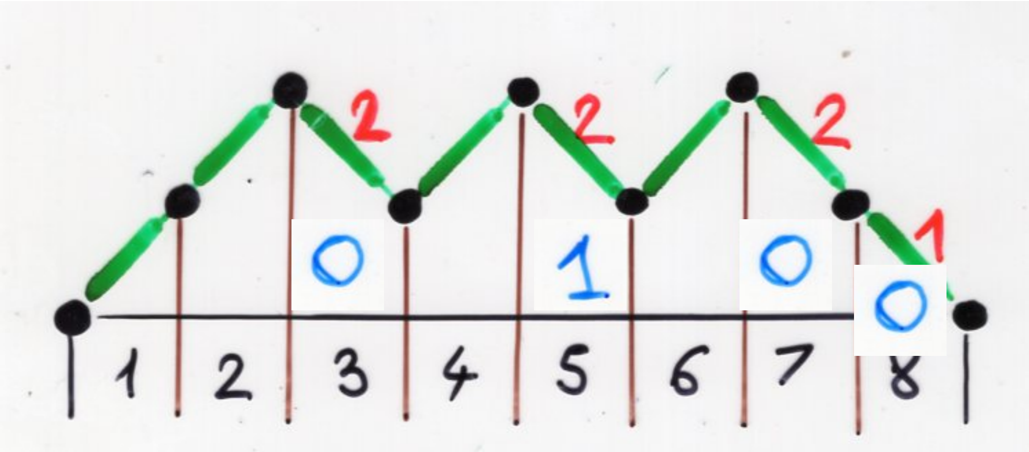


maj(σ) = 1+3+5+7
= 16
6 5 8 2 3 1 7 4
0 0
1 2 3 4 5 6 7 8
0 1 + 1 + 2 + 3 + 6 + 3 = 16

q-analog of Hermite histories

$v(\omega_i)$

$\lambda_k = k$

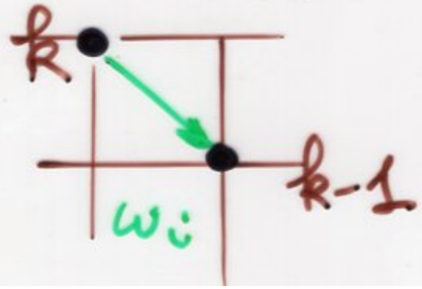


$\omega = (\omega_1, \omega_2, \dots, \omega_{2n})$

Dyck path

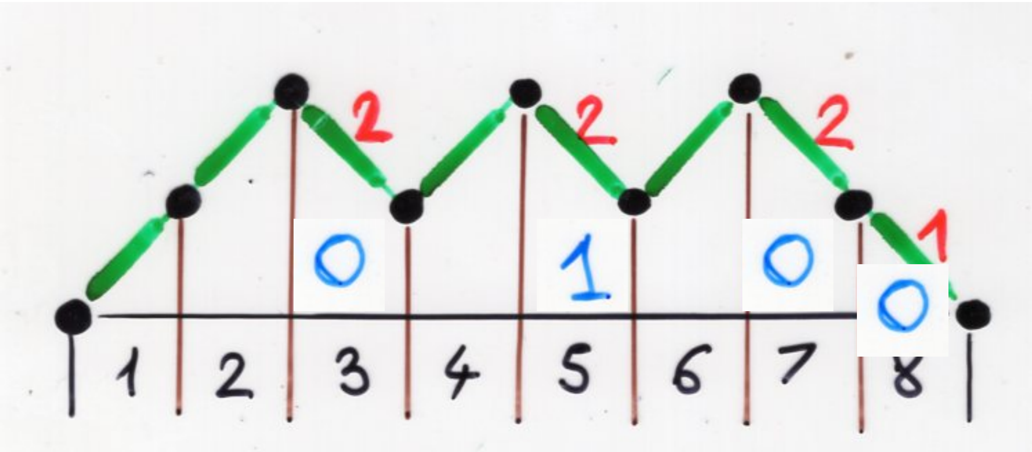
$v(\omega_i) = k$

if



choice function p_i

$0 \leq p_i < v(\omega_i)$

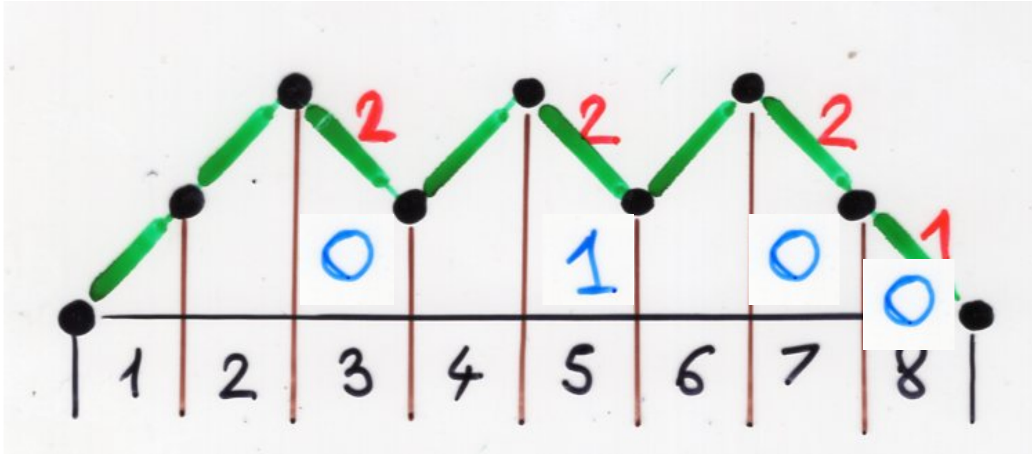


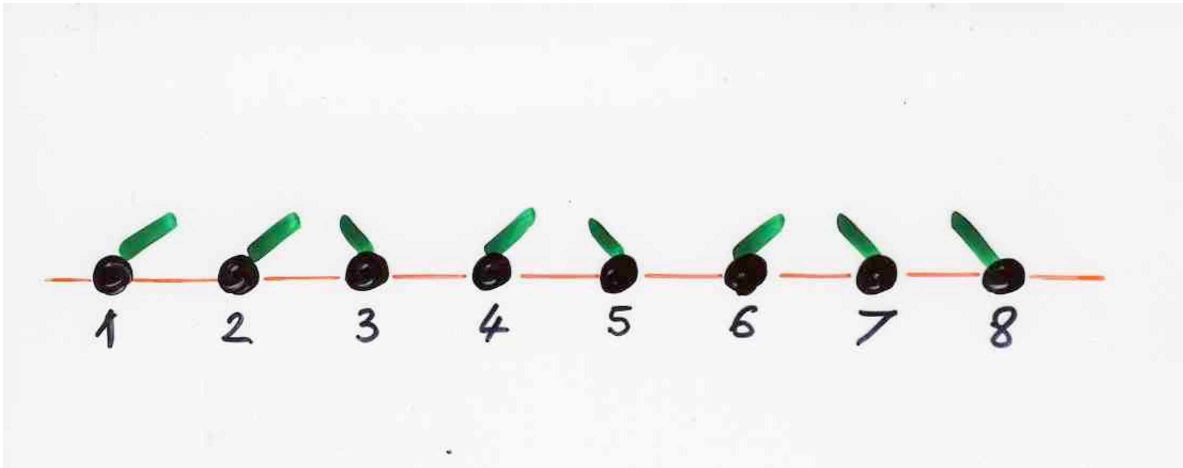
ω
Dyck path

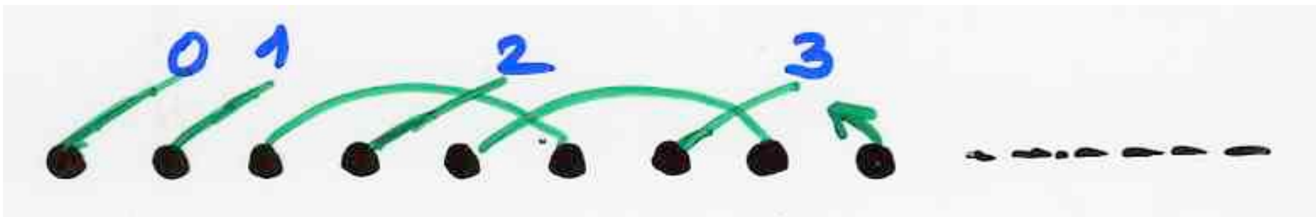
$$\lambda_k = [k]_q$$

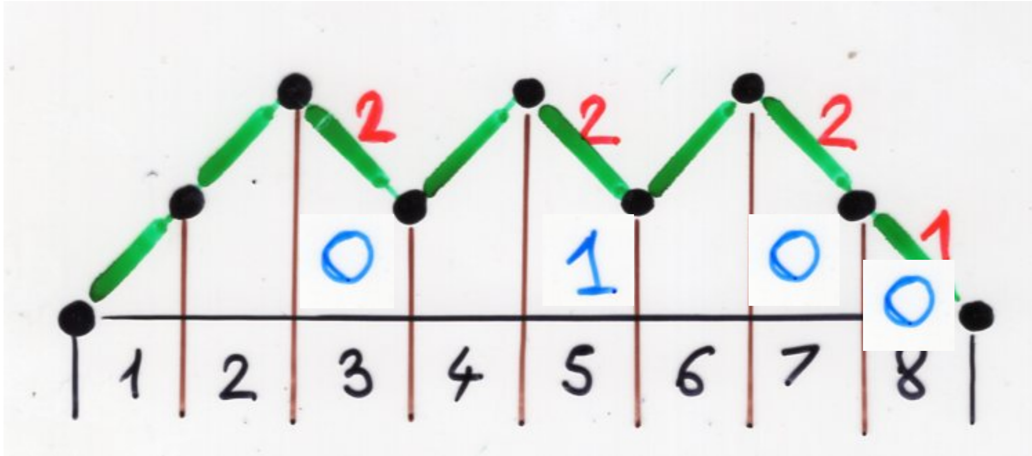
$$v_q(\omega) = \sum_h v_q(h)$$

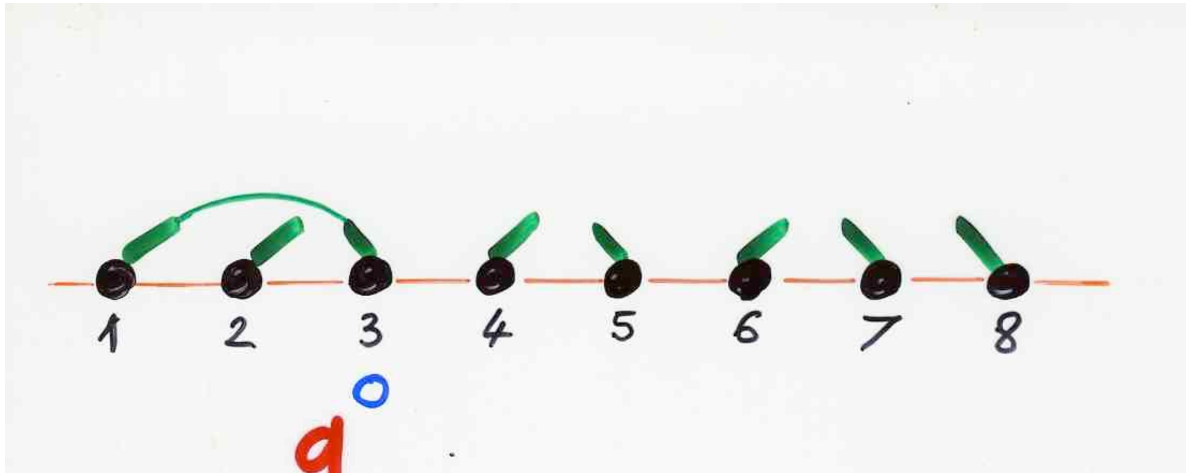
Hermite history related to ω


2
2
2
1
0
1
0
0
1 2 3 4 5 6 7 8

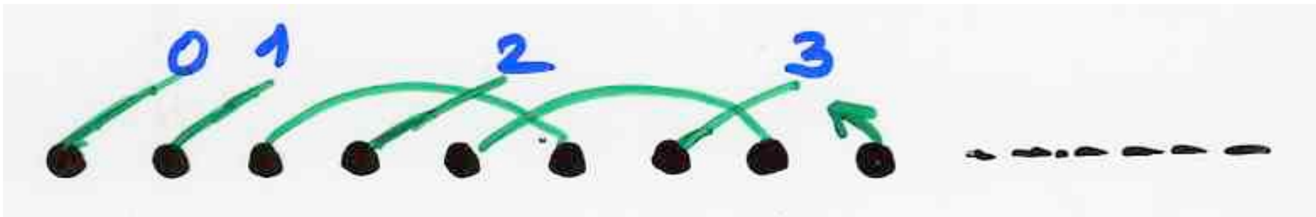

1 2 3 4 5 6 7 8

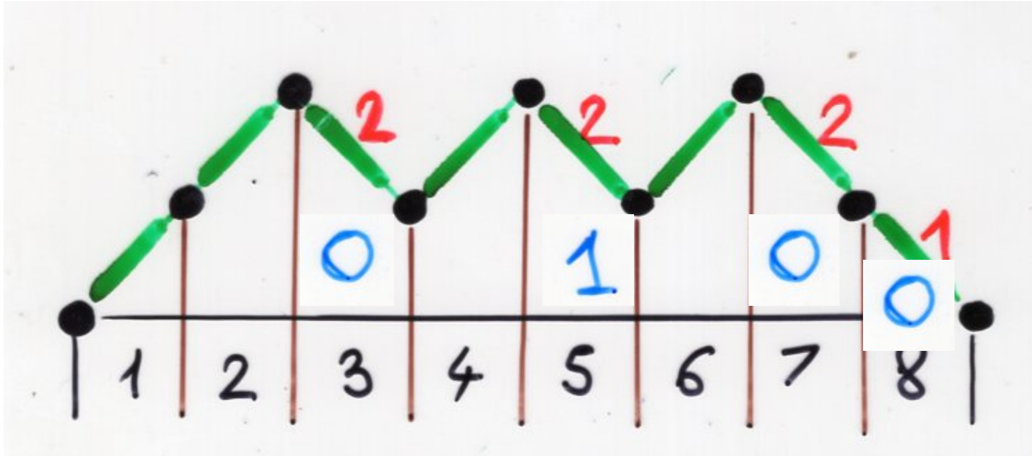

0 1 2 3

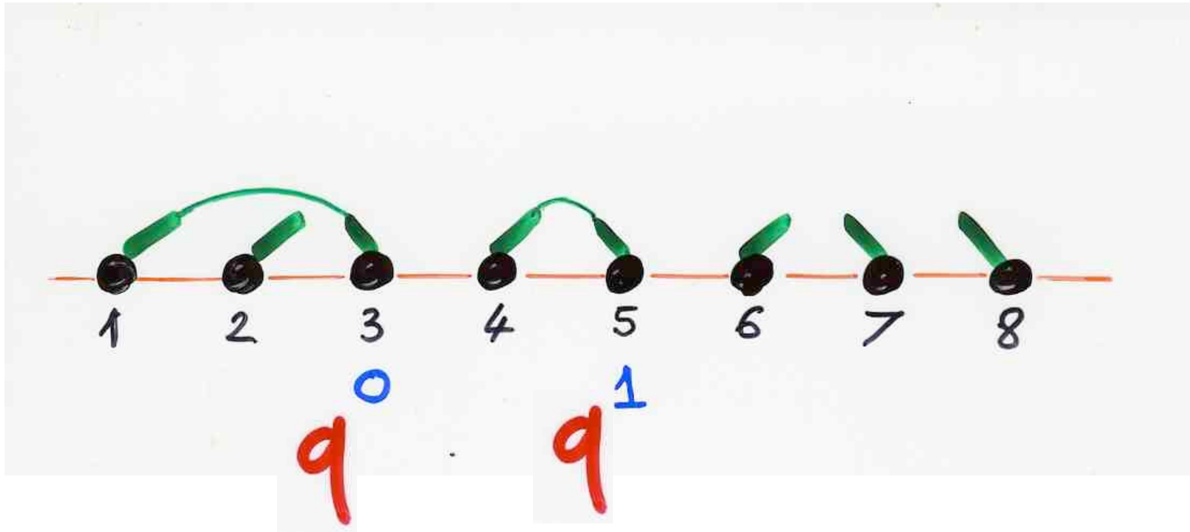


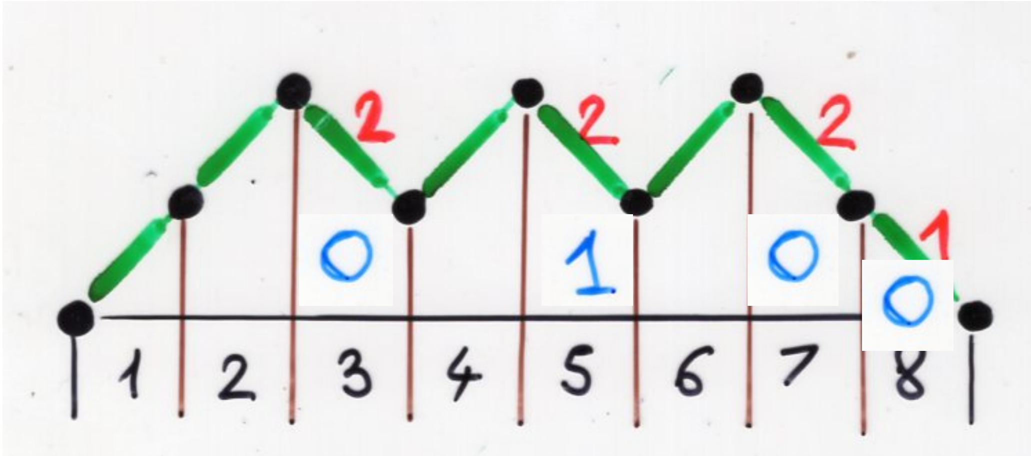


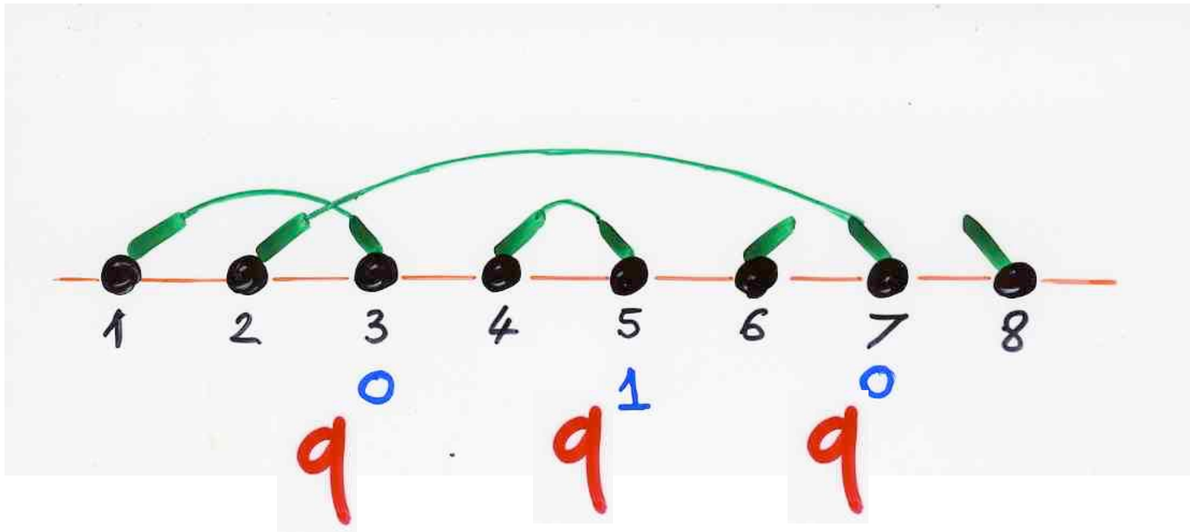
9^0

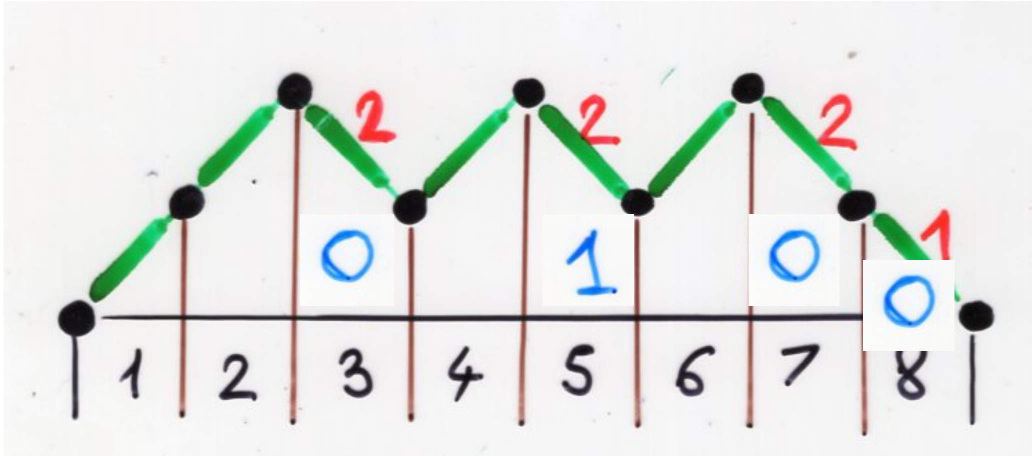


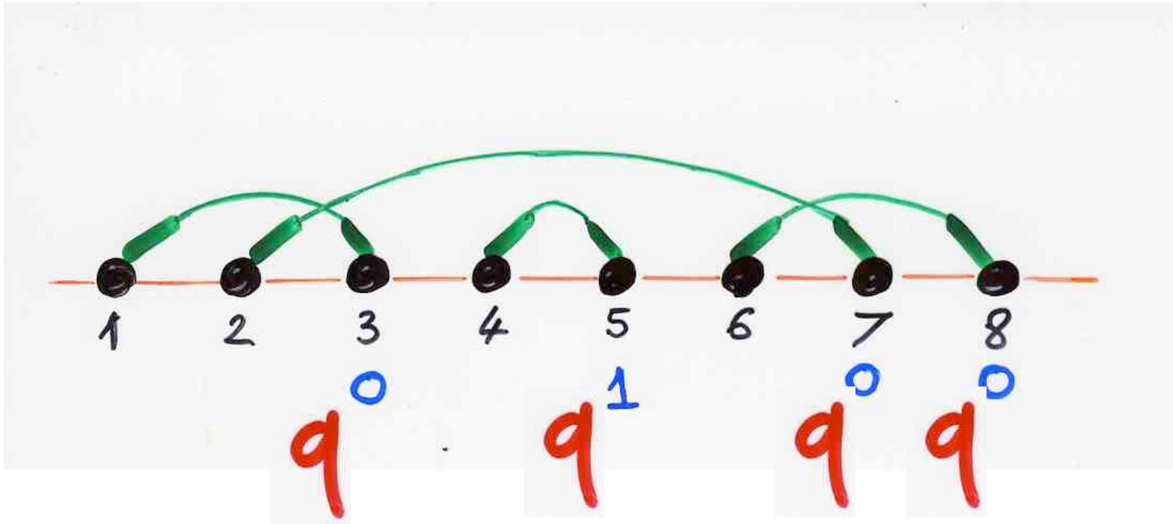

2
2
2
1
0
1
0
0
1 2 3 4 5 6 7 8


1 2 3 4 5 6 7 8
q⁰ q¹

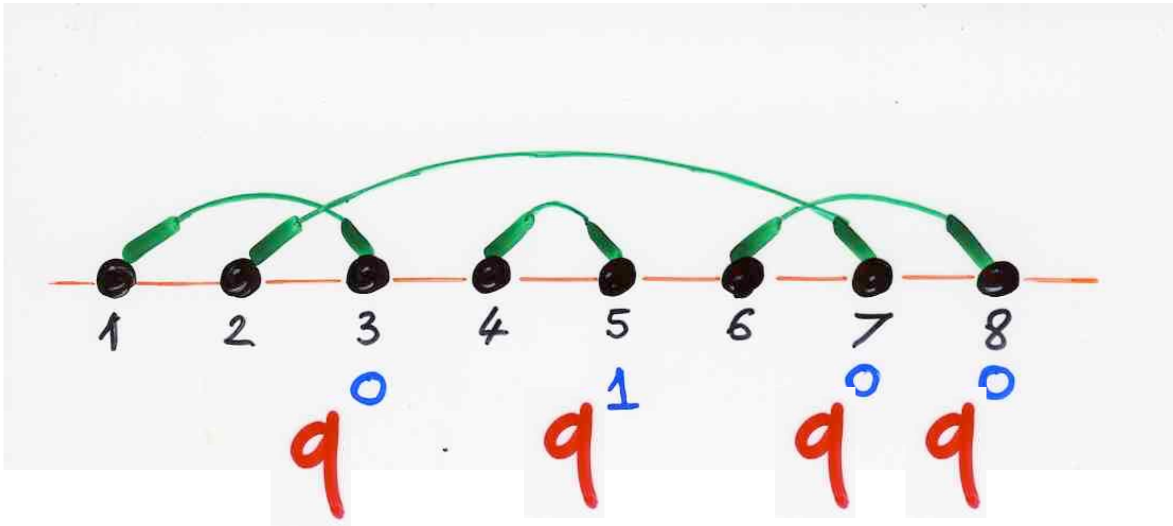

2
2
2
1
0
1
0
0
1
2
3
4
5
6
7
8

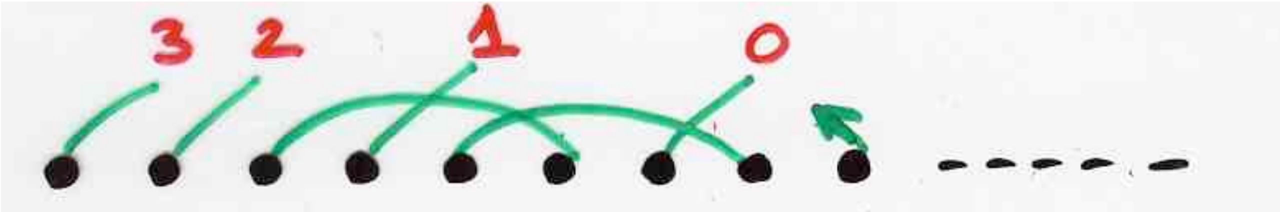

1
2
3
4
5
6
7
8
q^0
q^1
q^0


2
2
2
1
0
1
0
0
1
2
3
4
5
6
7
8

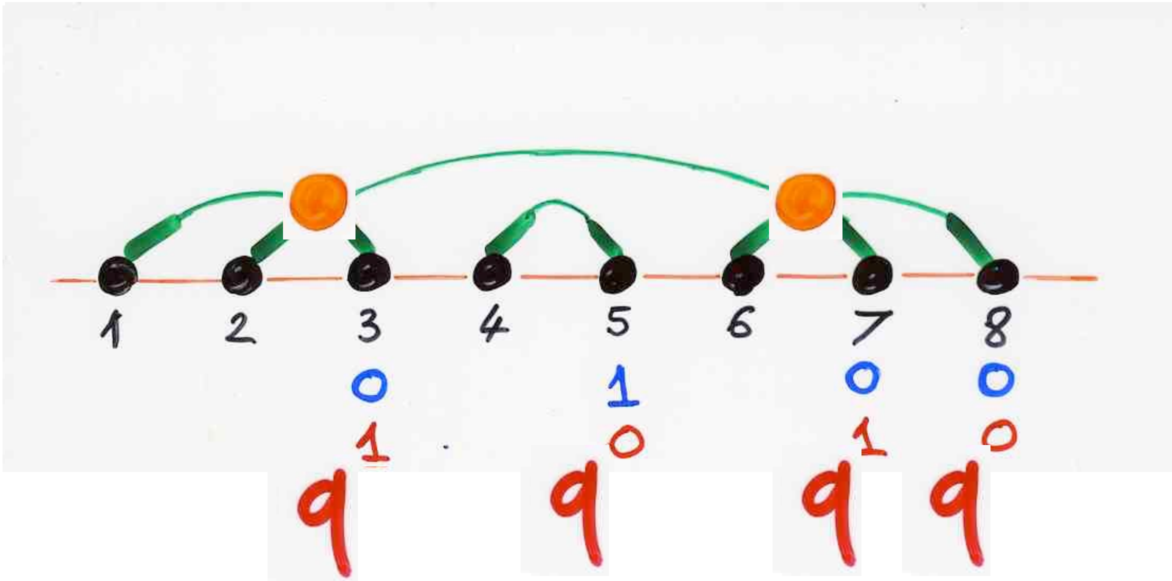

1
2
3
4
5
6
7
8
0
1
0
0
q
q
q
q


nesting


1 2 3 4 5 6 7 8
q^0
q^1
q^0
q^0

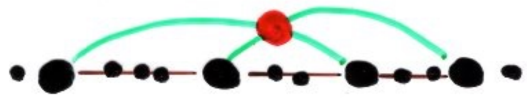

3 2 1 0


crossing

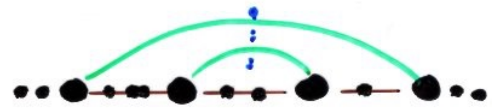

1 2 3 4 5 6 7 8
0 1 0 0
q1 q0 q1 q0

$$\sum_{\substack{\text{chord diagrams } I\\ [1,2n]}} q^{cr(I)} = \sum_{\substack{I\\ \text{chord diagrams}\\ [1,2n]}} q^{nest(I)} = \sum_{\substack{h\\ \text{Hermite histories}\\ |h|=2n}} q^{sum(h)}$$

$cr(I)$ = number of crossings



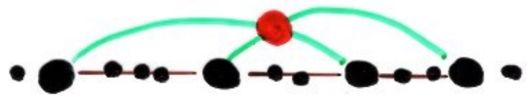
$nest(I)$ = number of nestings

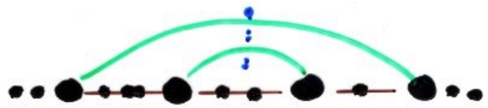


$$sum(h) = \sum_i p_i$$

exercise

$$\sum_{\substack{I \\ \text{chord diagrams} \\ [1,2n]}} q^{cr(I)}\, t^{nest(I)}$$





(q,t)-polynomial
symmetric in q and t

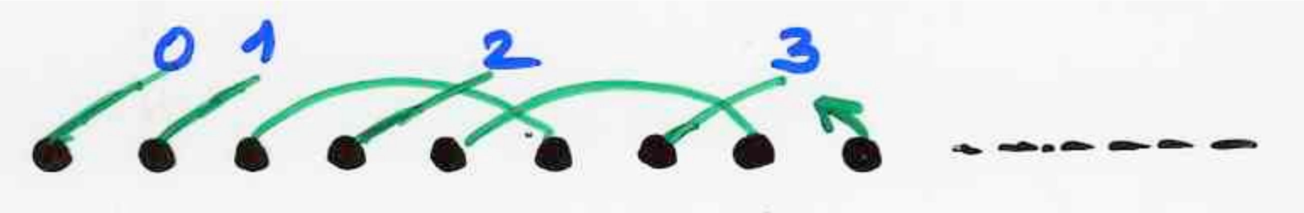
rook placements

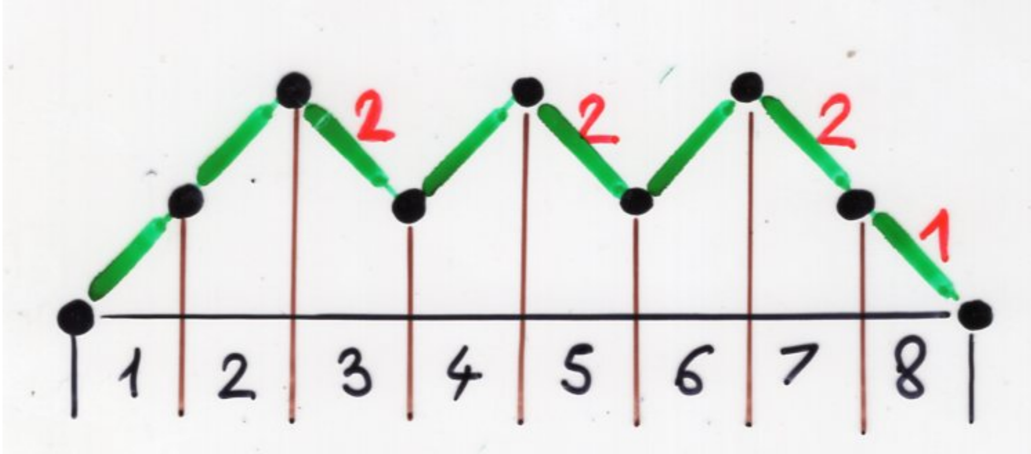
(with no empty rows or columns)

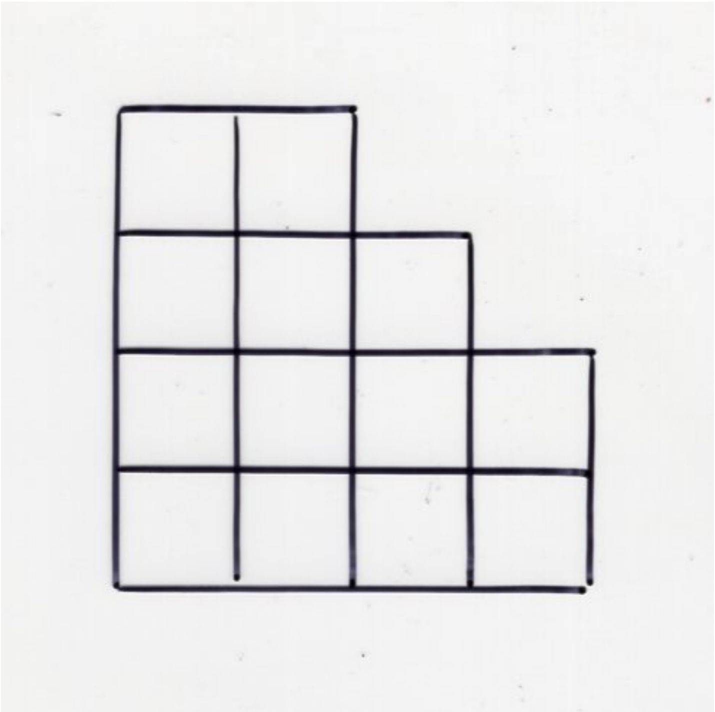
quadratic algebra Q

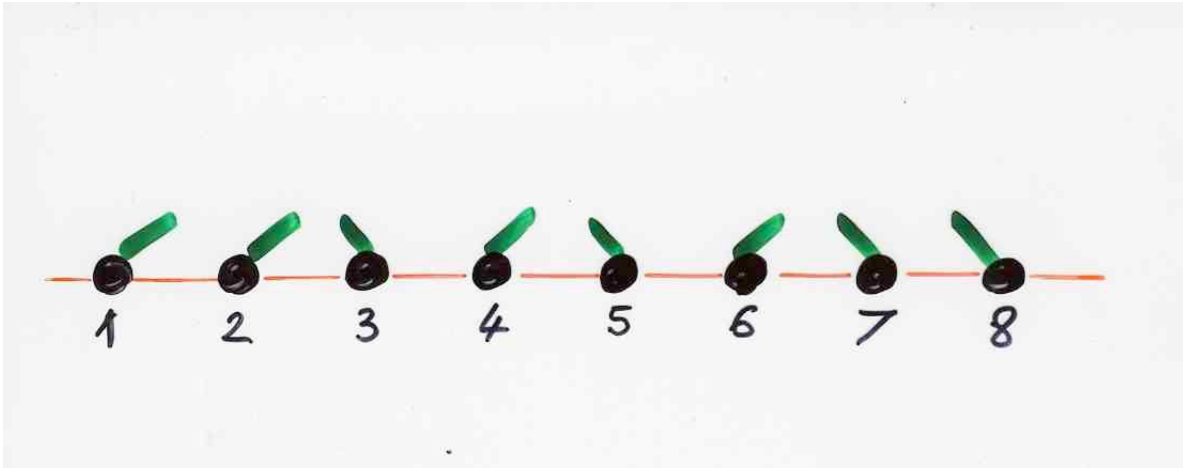
Q-tableaux

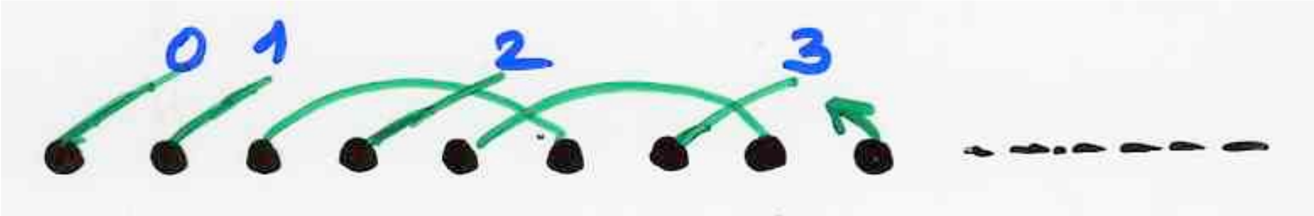
$UD = DU + Id$

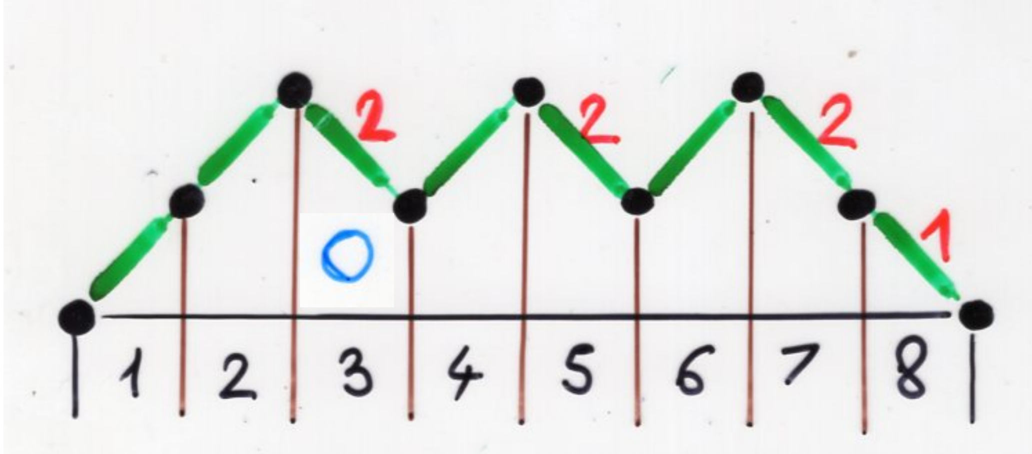

0
1
2
3

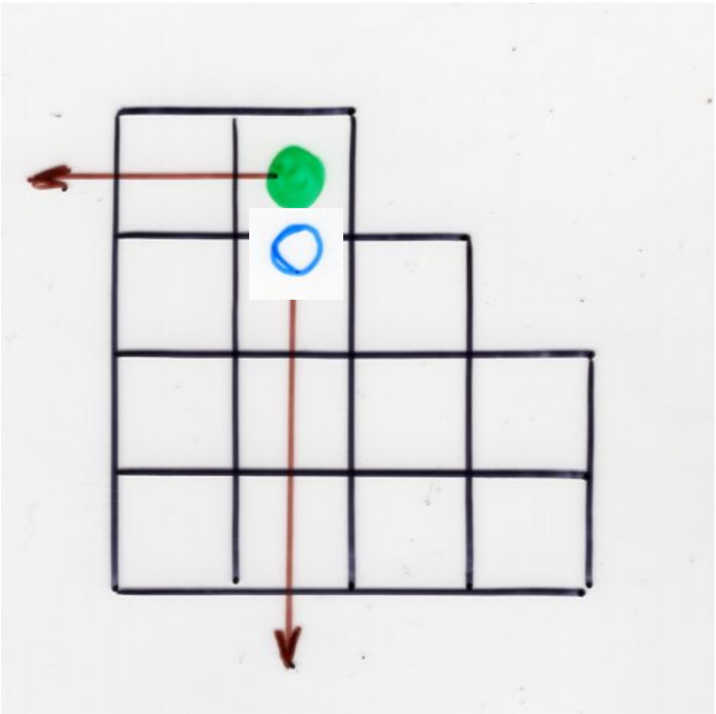

2
2
2
1
1 2 3 4 5 6 7 8

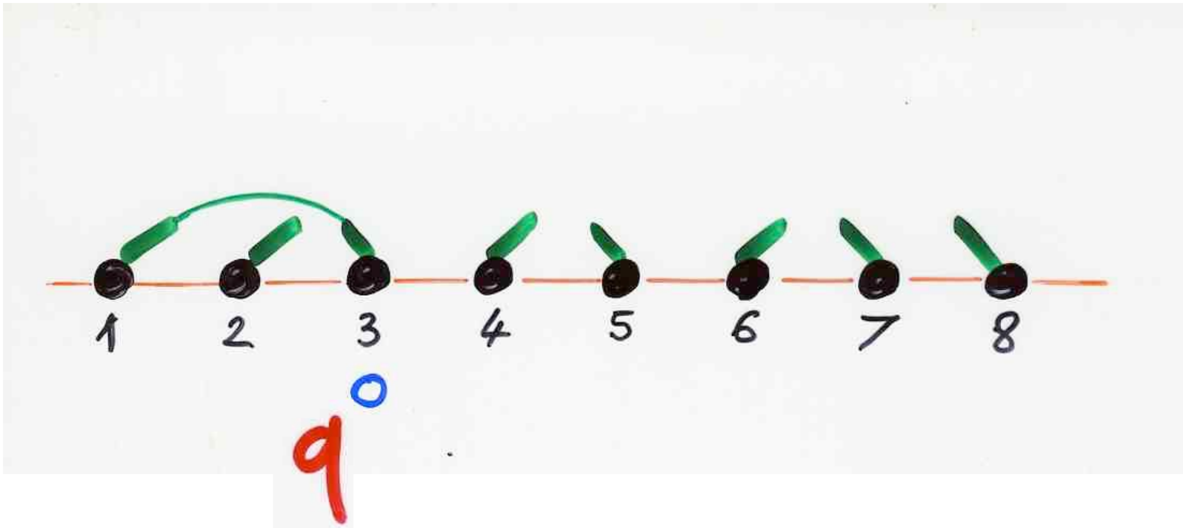


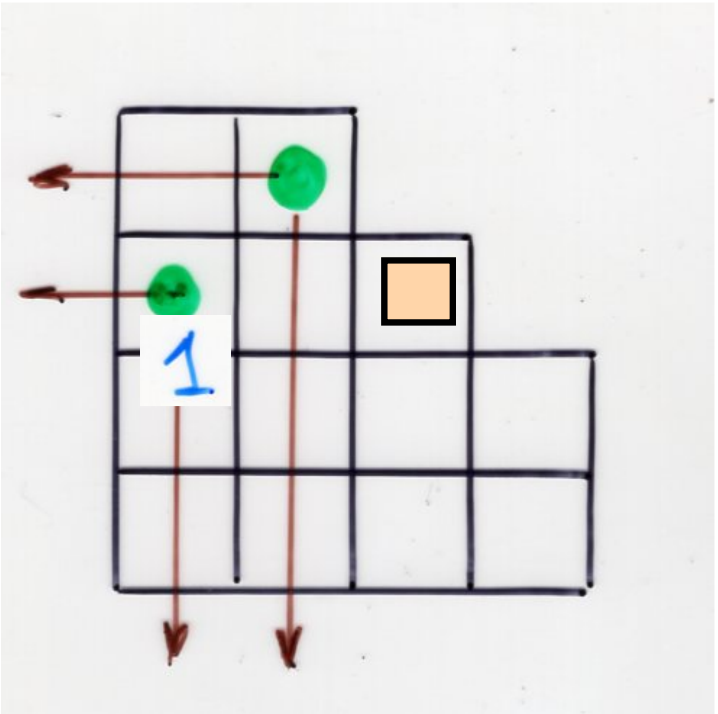

1 2 3 4 5 6 7 8

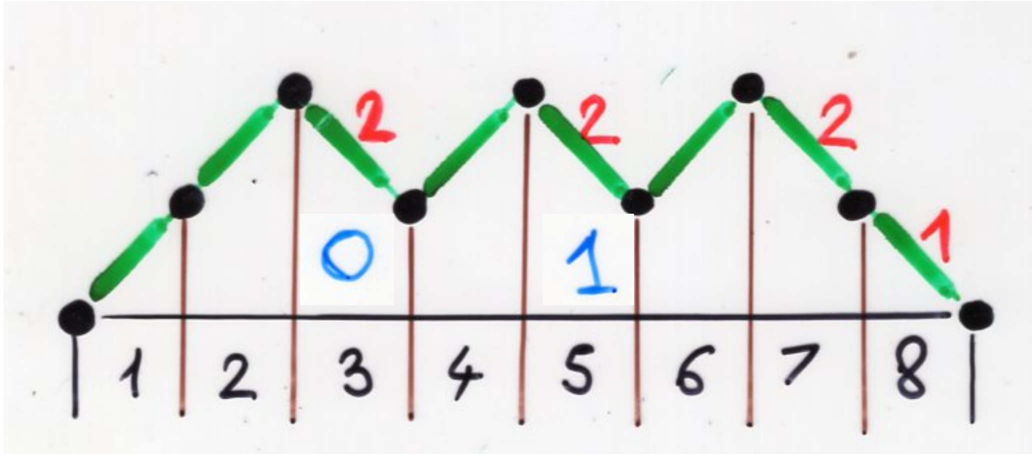

0
1
2
3

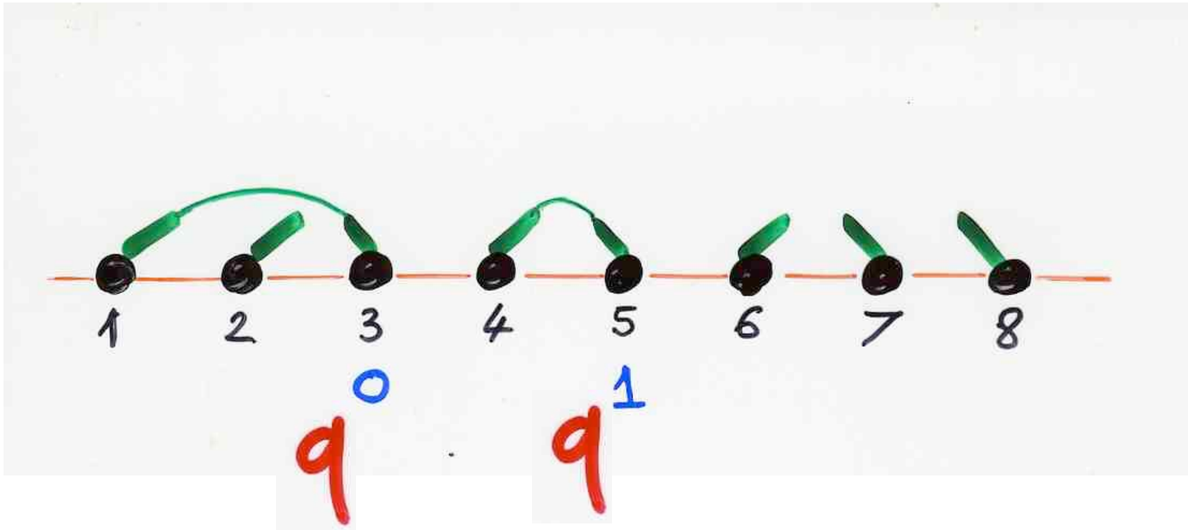

2
2
2
1
0
1 2 3 4 5 6 7 8

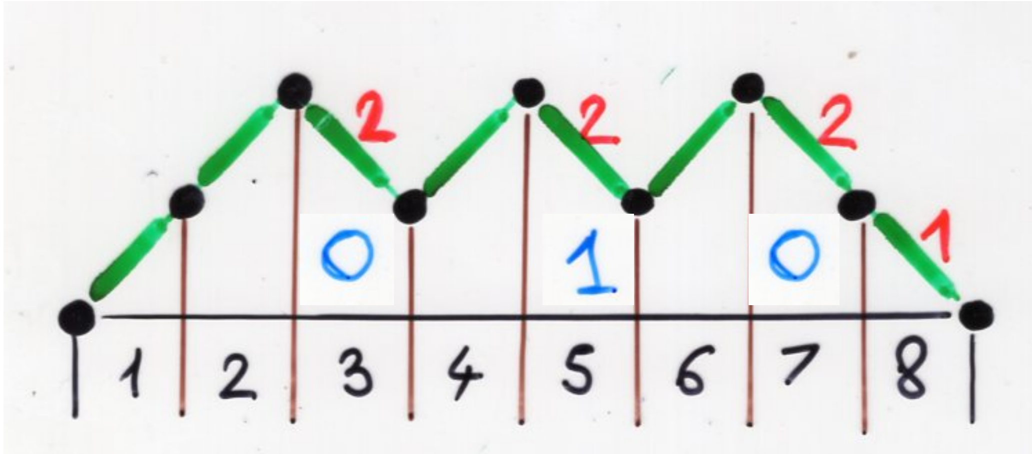

0

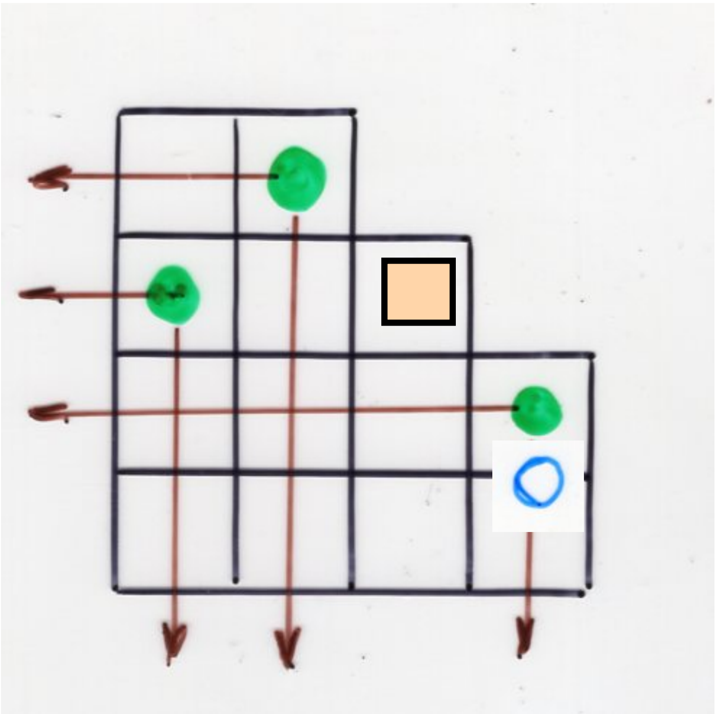

1 2 3 4 5 6 7 8
9°

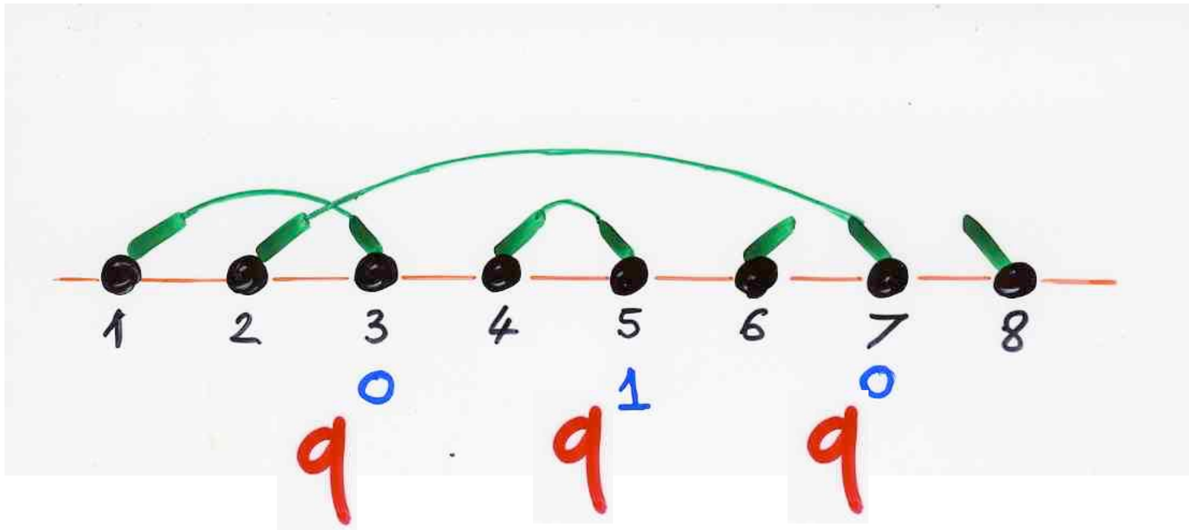

1

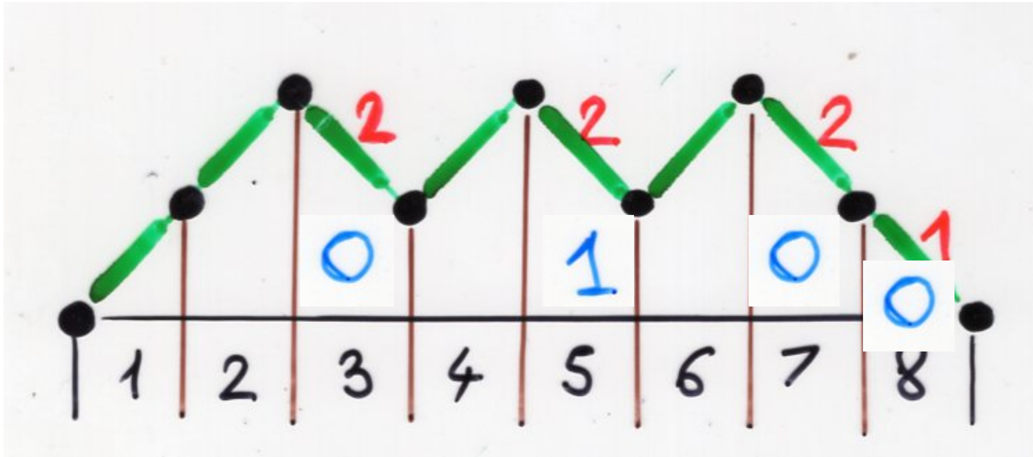

2
2
2
1
0
1
1 2 3 4 5 6 7 8

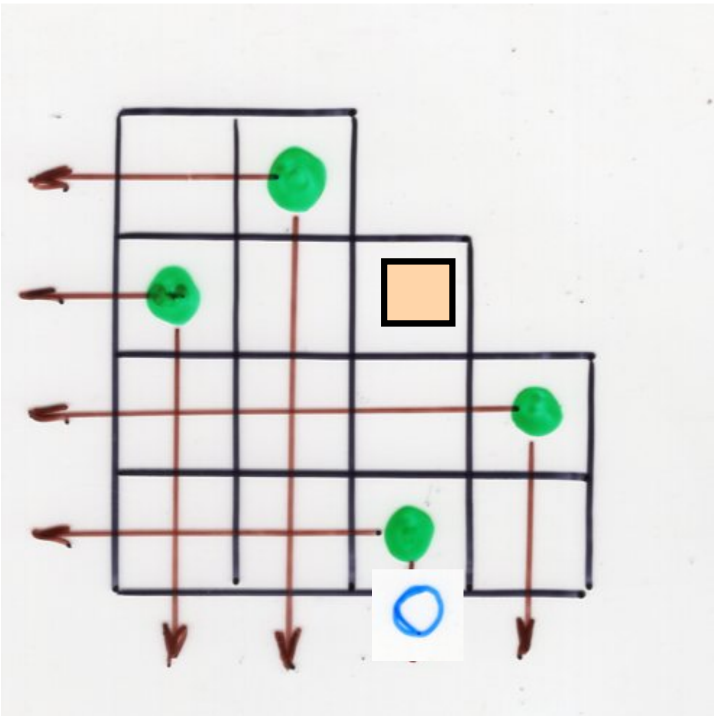

1 2 3 4 5 6 7 8
q⁰
q¹

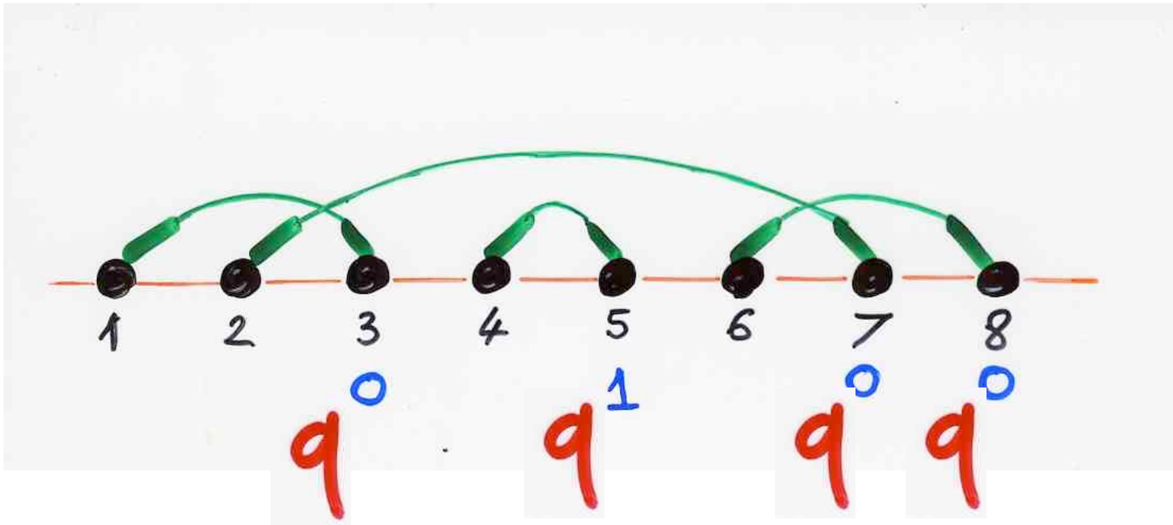

2
2
2
1
0
1
0
1 2 3 4 5 6 7 8

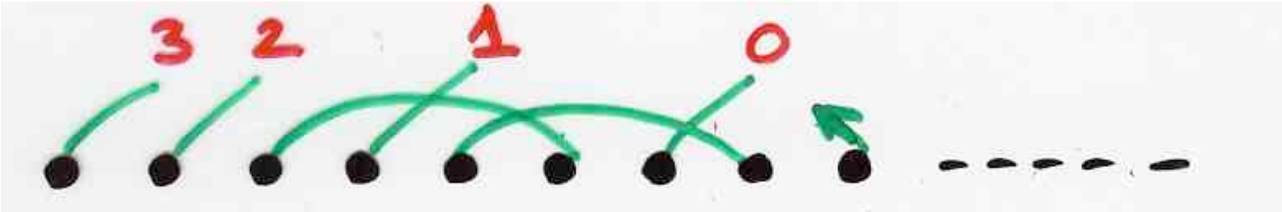

0

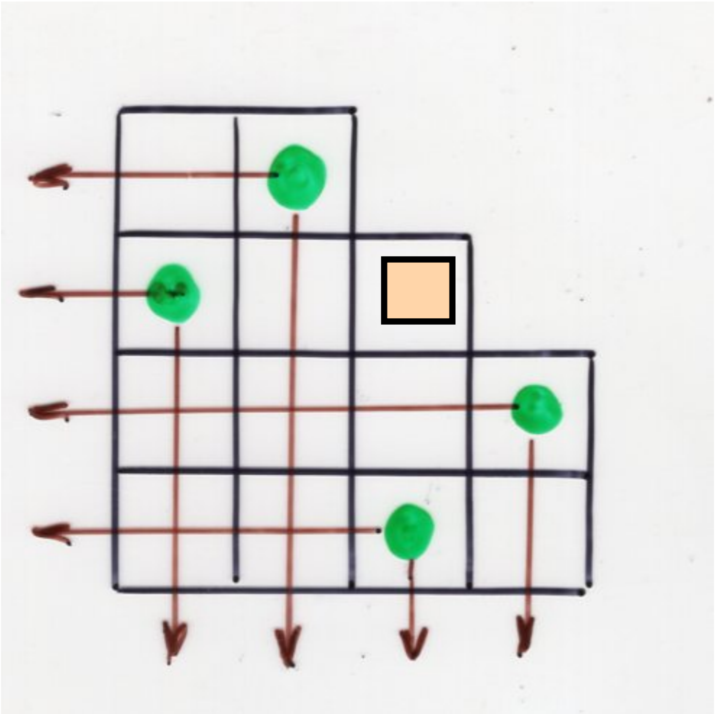

1 2 3 4 5 6 7 8
q^0 q^1 q^0

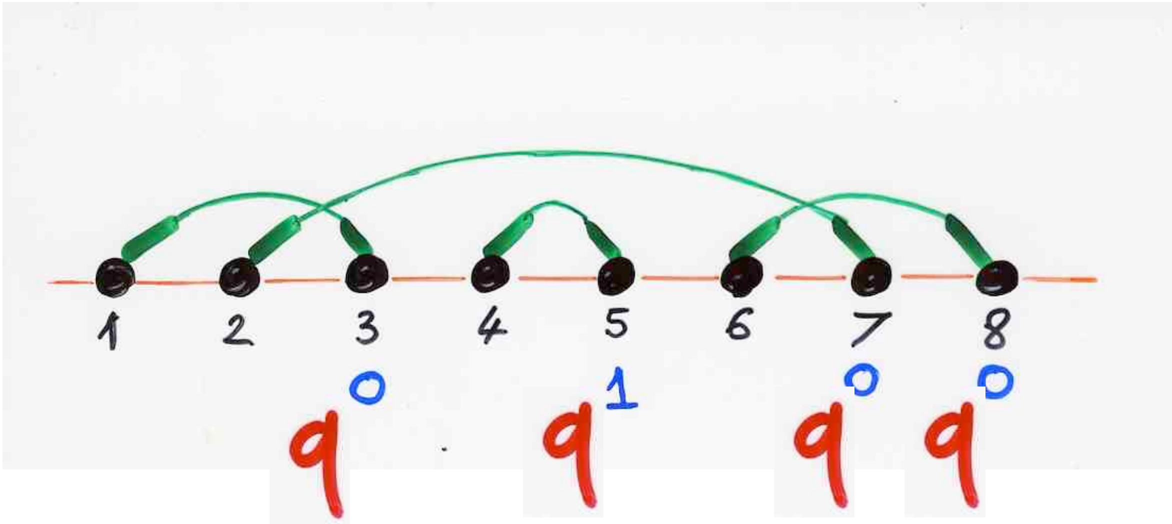

2
2
2
1
0
1
0
0
1 2 3 4 5 6 7 8

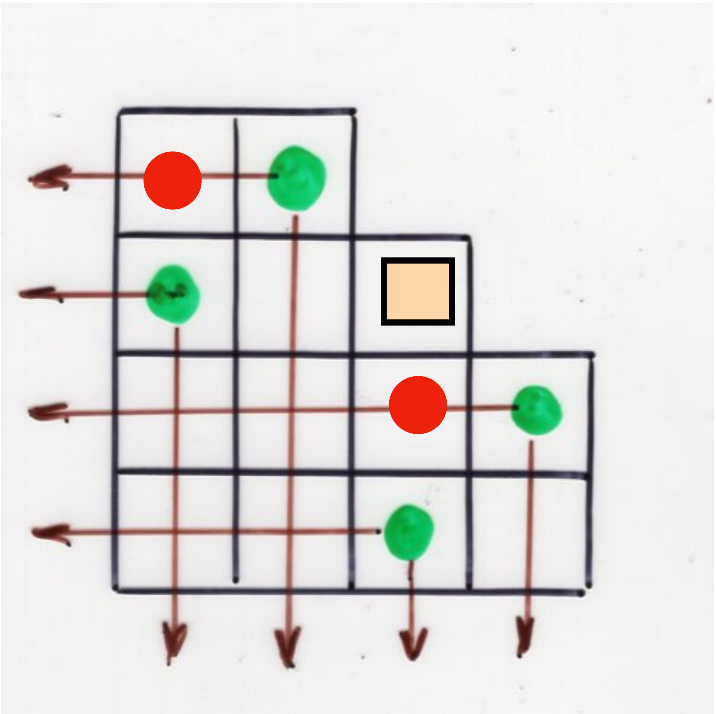

0


1 2 3 4 5 6 7 8
0 1 0 0
q q q q

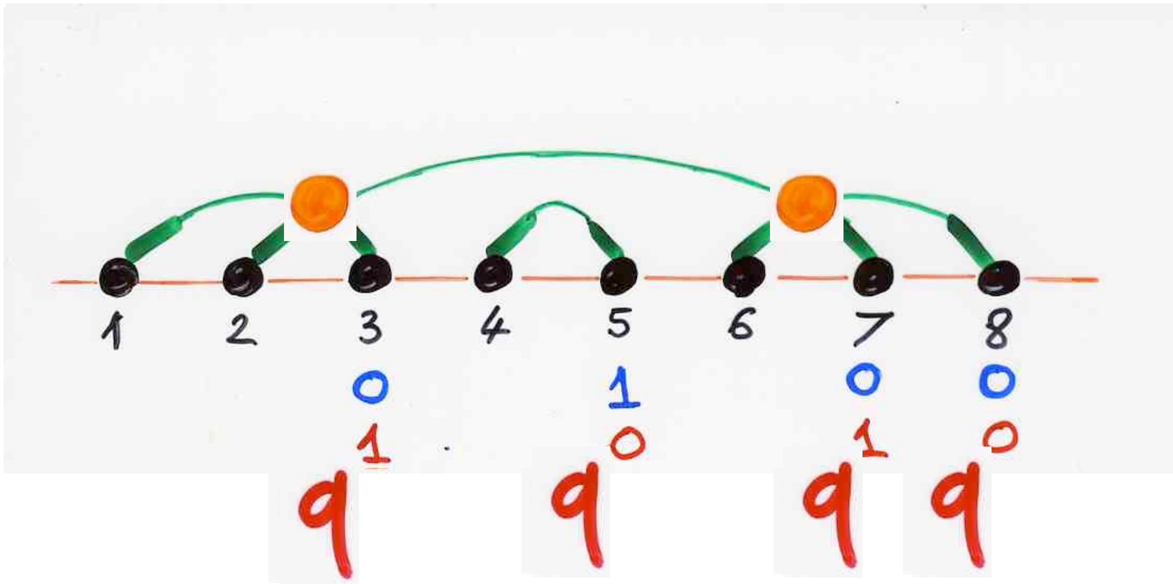

3
2
1
0




1
2
3
4
5
6
7
8
q^0
q^1
q^0
q^0

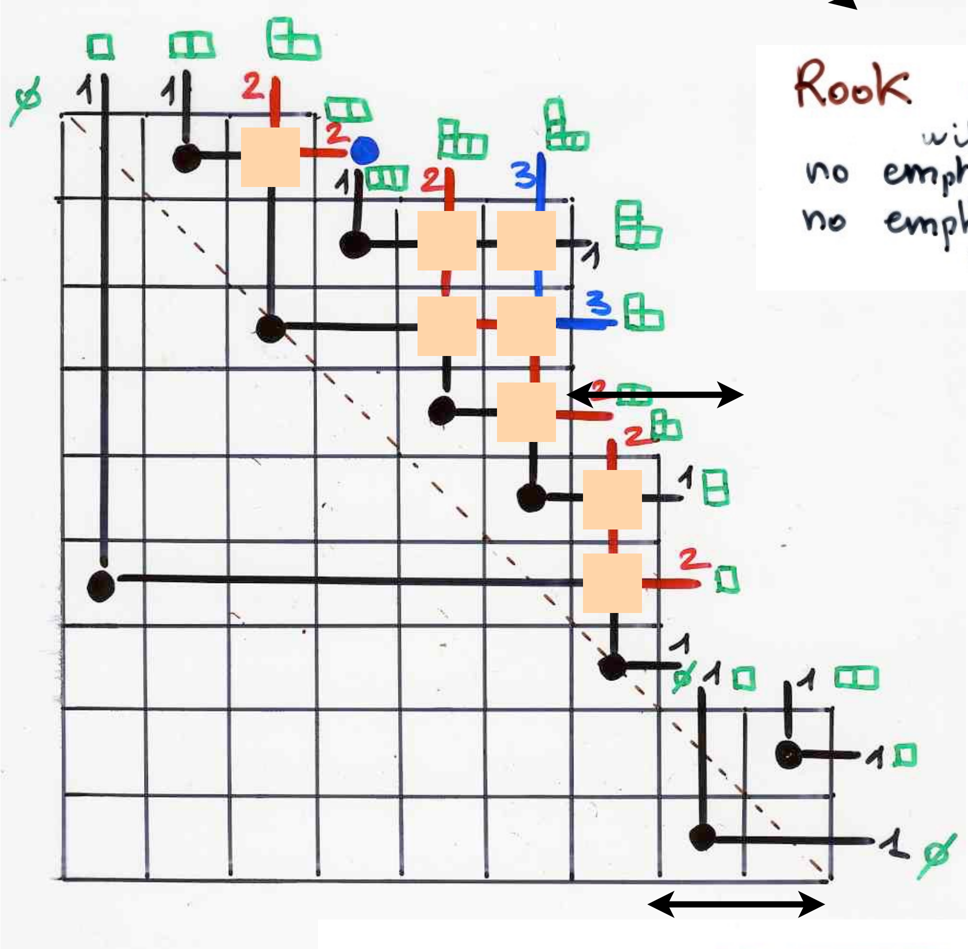


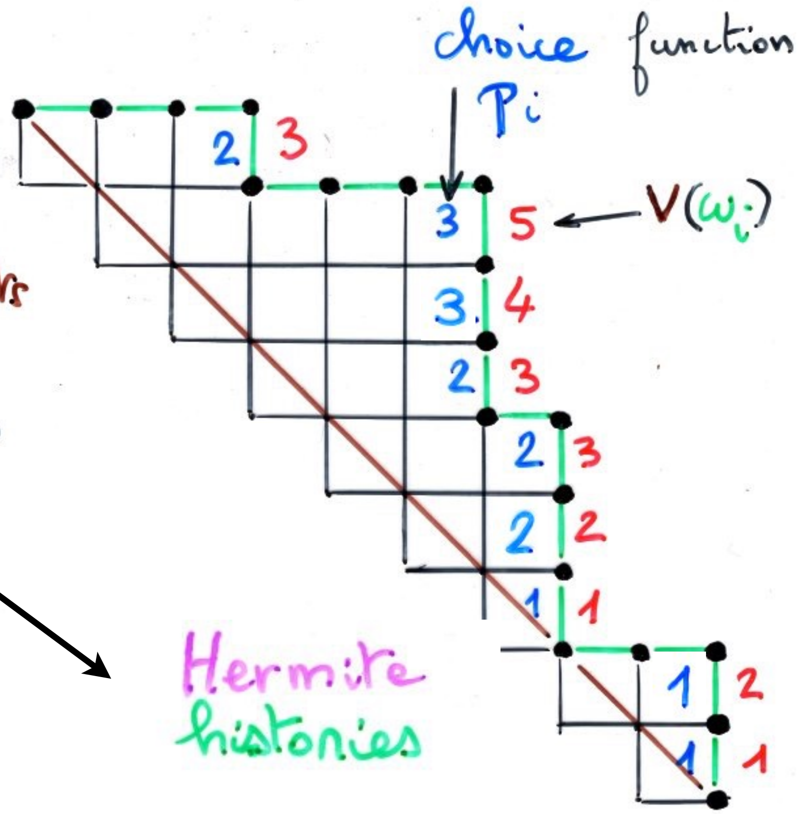

crossing


1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0
q q q q

sequences of oscillating tableaux starting and ending at ∅

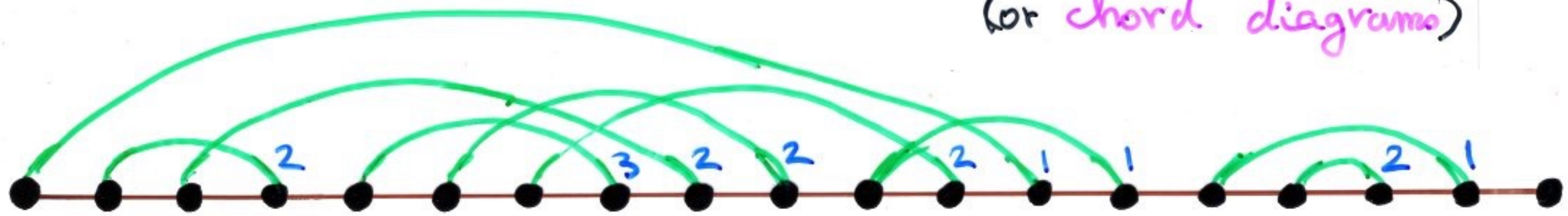
Rook placements with no empty row no empty column





Hermite histories

involutions on 2n with no fixed points (or chord diagrams)



rook placements

quadratic algebra Q

Q-tableaux

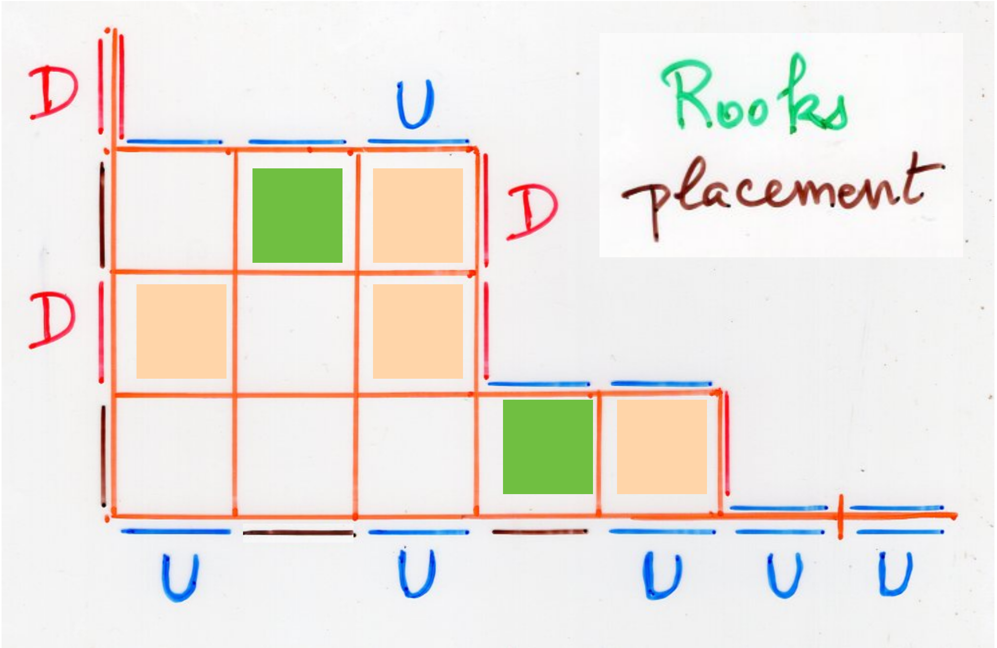
$$UD = qDU + Id$$

$$UD = qDU + Id$$

Lemma Every word w with letters U and D can be written in a unique way

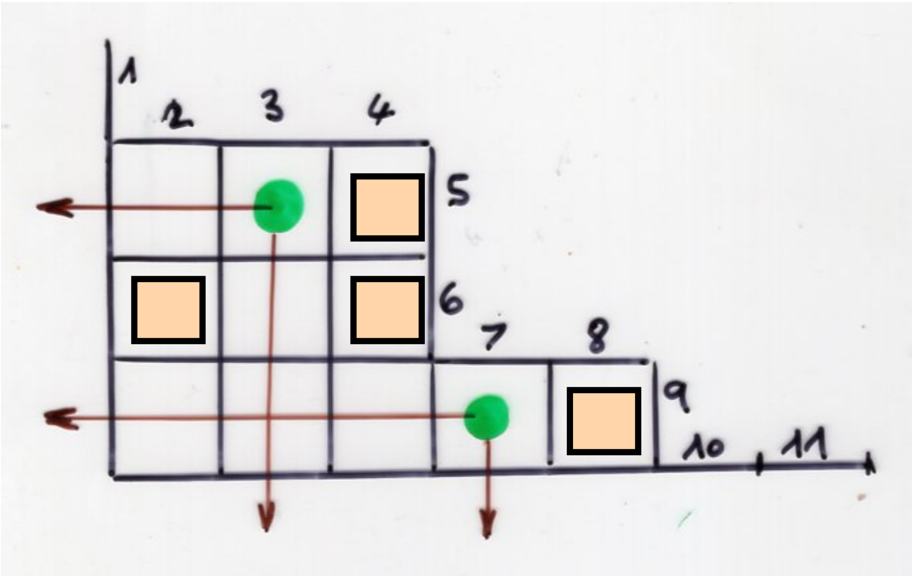
$$w = \sum_{i,j \geq 0} c_{ij}(w)\, D^i U^j$$

$$\sum_{\substack{\text{rook placements } T \\ \text{"size } n\text{"}}} q^{k(T)}\, x^{i(T)}\, y^{j(T)}$$

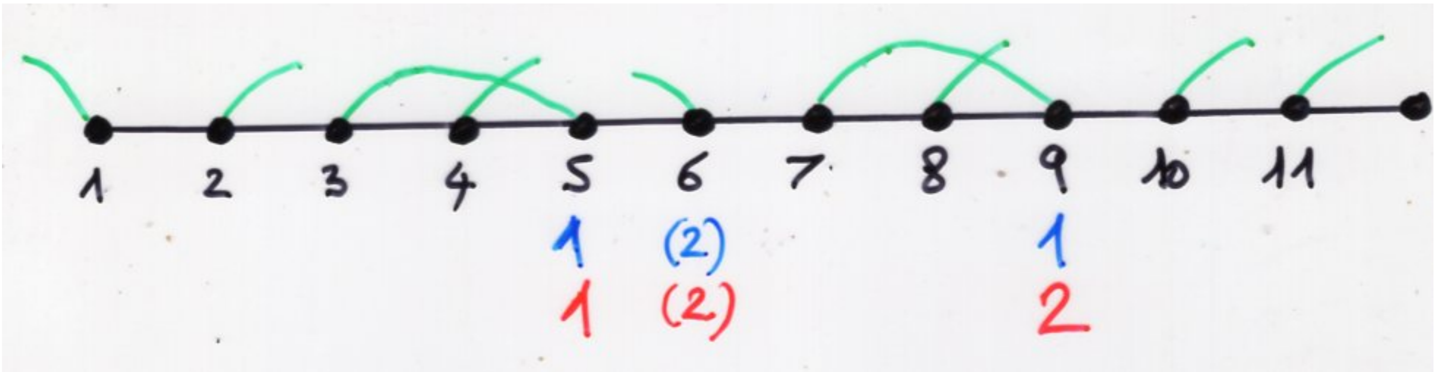


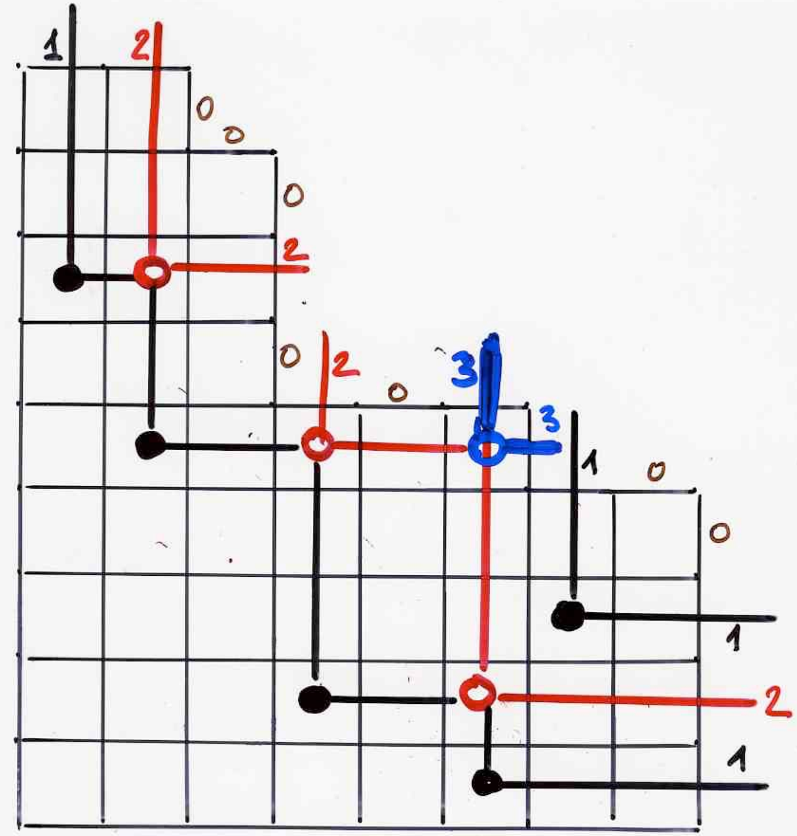
$\begin{cases} i(T) = \text{number of rows} \\ j(T) = \text{number of columns} \end{cases}$ with no cell labeled $UD \rightarrow I_v\, I_h$

Josuat-Vergès (2011)



involutions on 2n
with 2-colored fixed points
↕
Rook placements





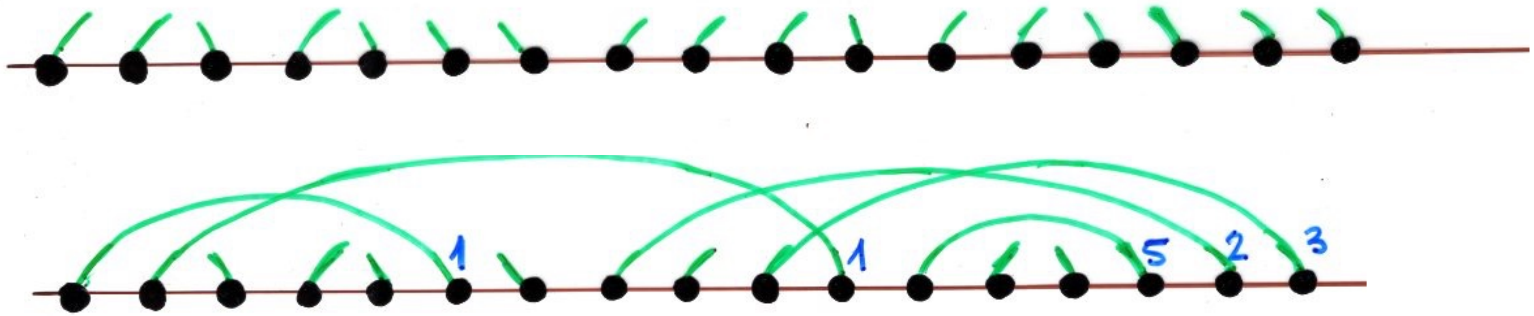
involutions on 2n
with 2-colored fixed points

↕

Rook placements

↕

sequence of 2n
2-colored vacillating tableaux
starting and ending at ∅



The parameter « q »

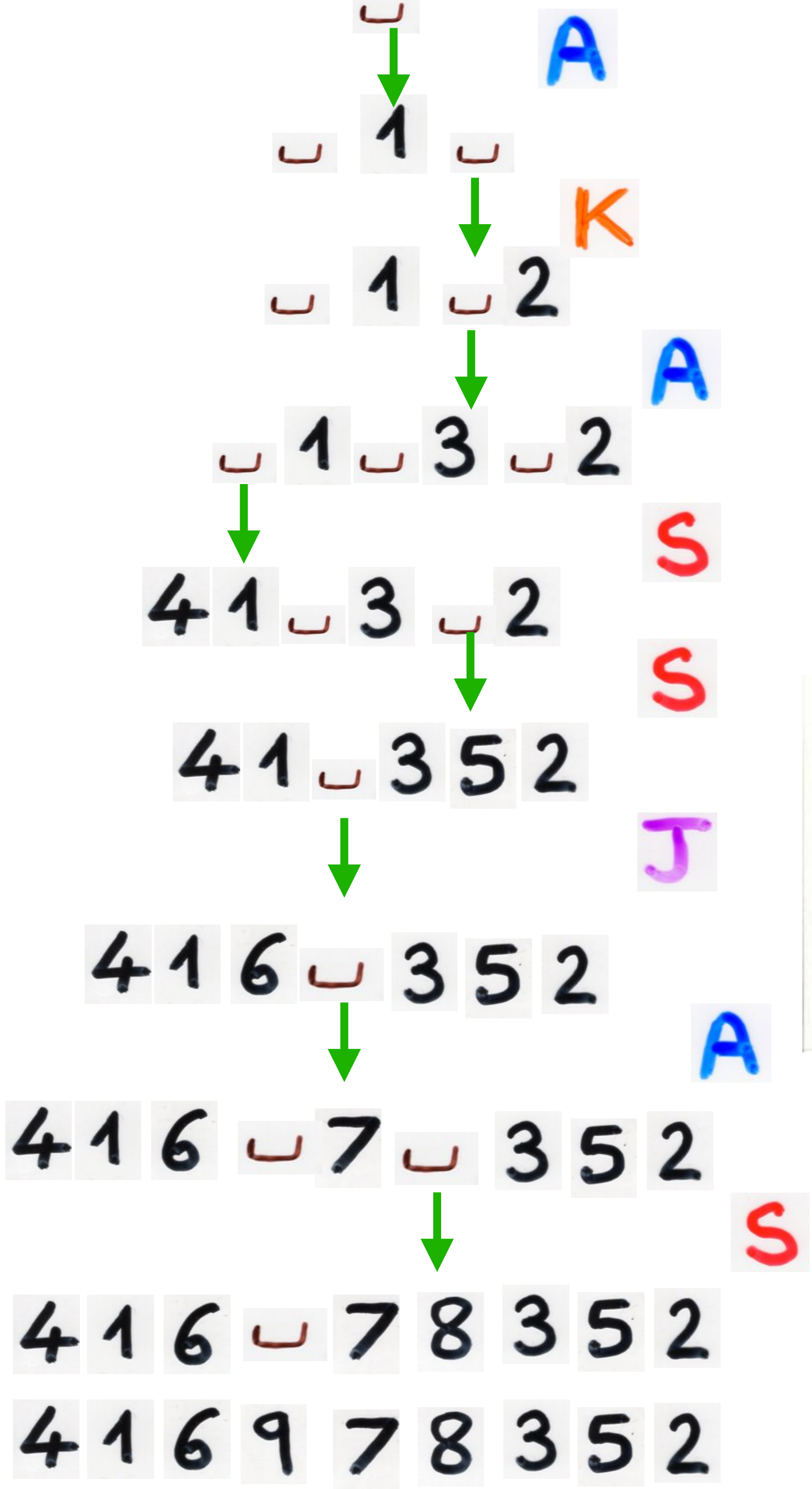
of the PASEP

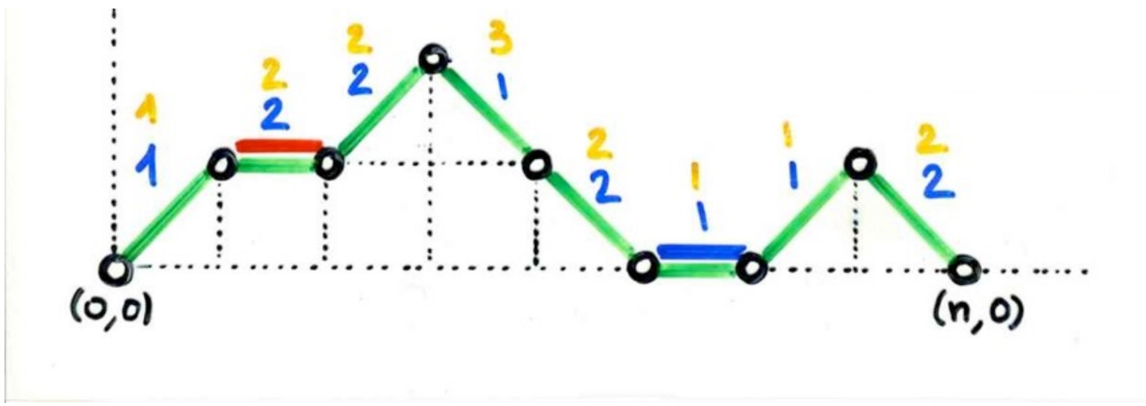
q-Laguerre Laguerre histories

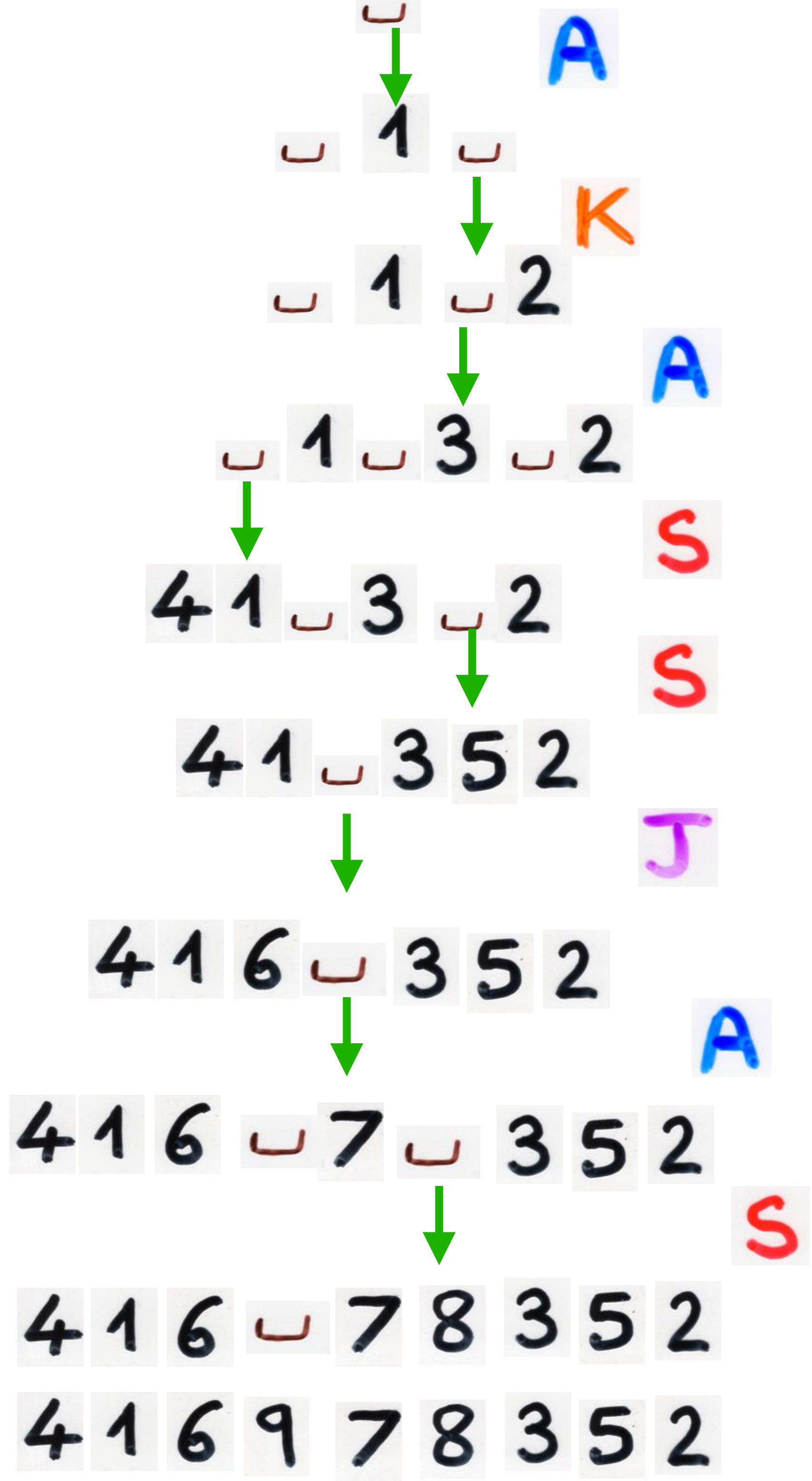
q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$


A
K
A
S
S
J
A
S
4 1 6 9 7 8 3 5 2


(0,0)
(n,0)



"q-analogue" of Laguerre histories

choice function

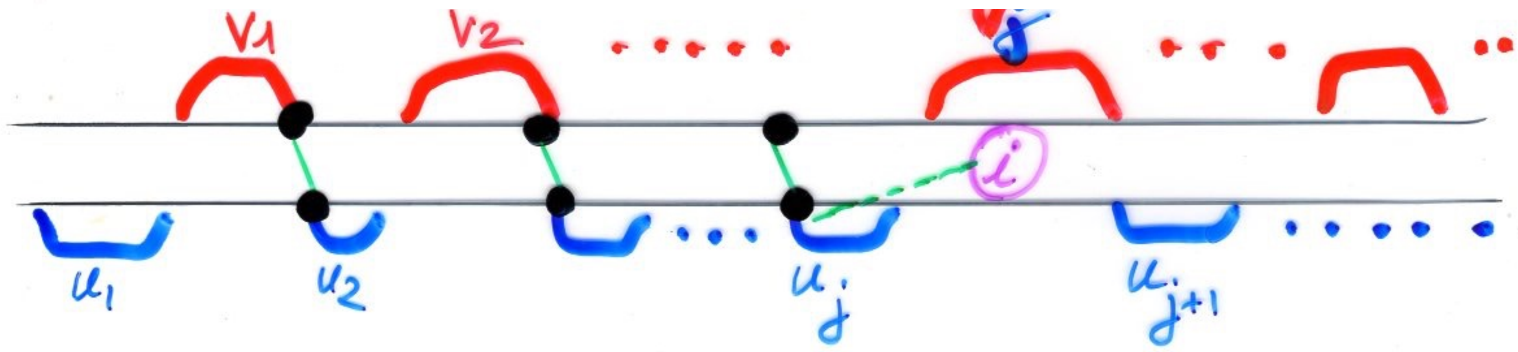
$i =$	1	2	3	4	5	6	7	8
$p_i =$	1	2	2	1	2	1	1	2
$p_i - 1 =$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories q^4

weighted q-Laguerre histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern $(31-2)$



q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

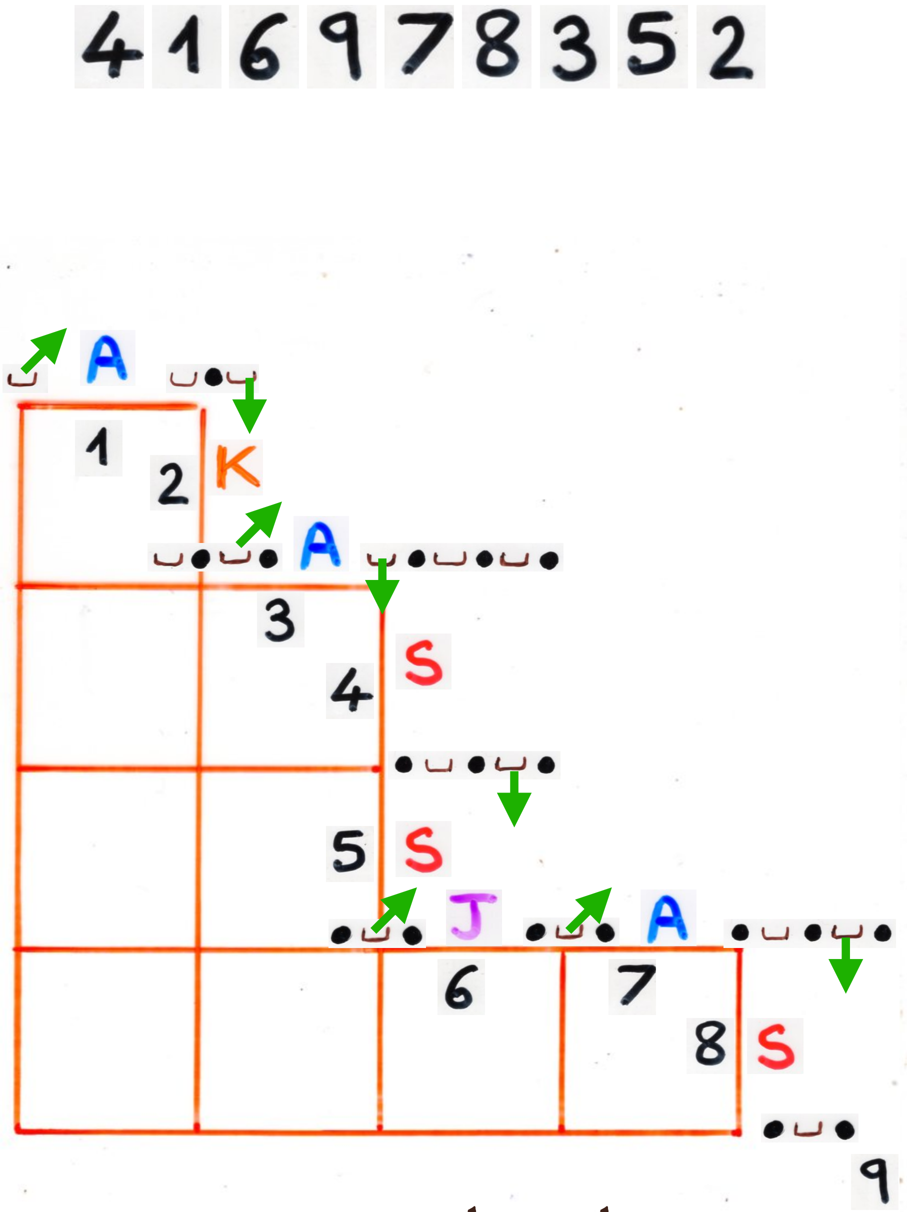
$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q-Laguerre restricted histories

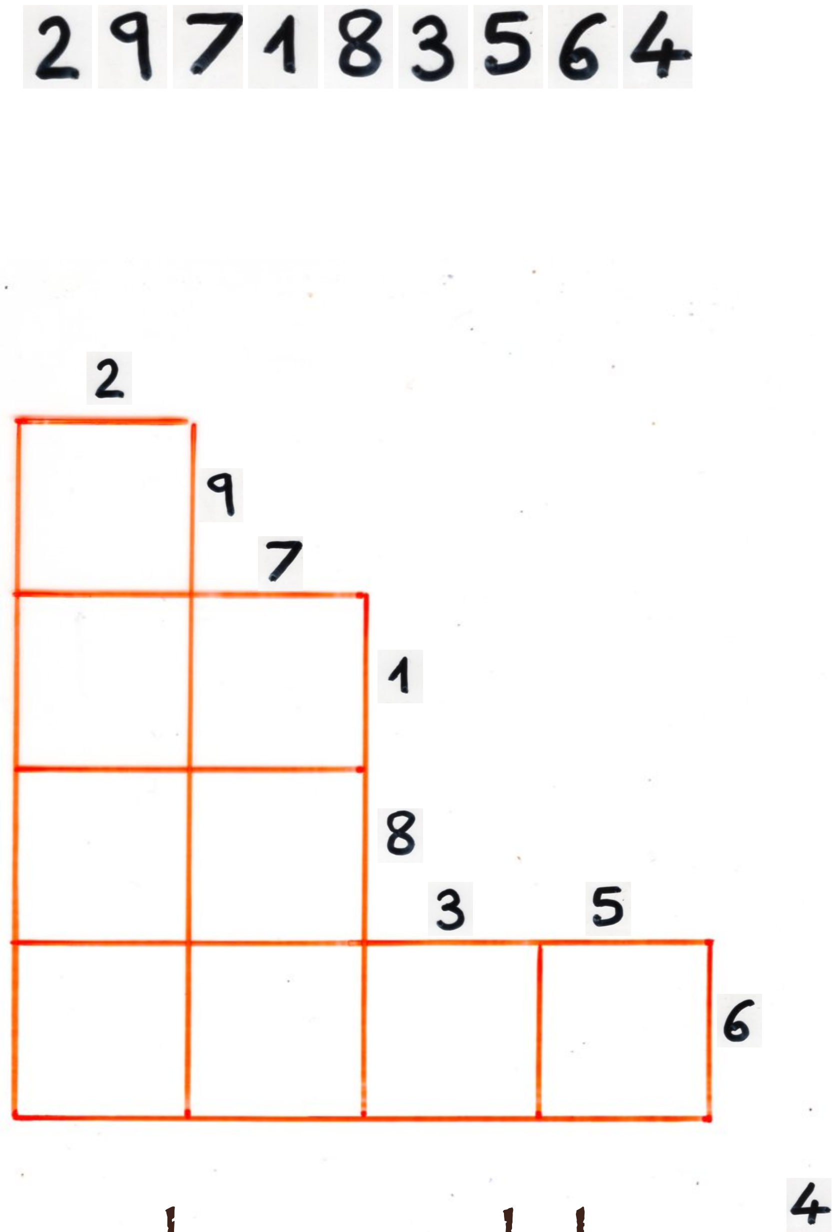
The parameter « q »

of the PASEP

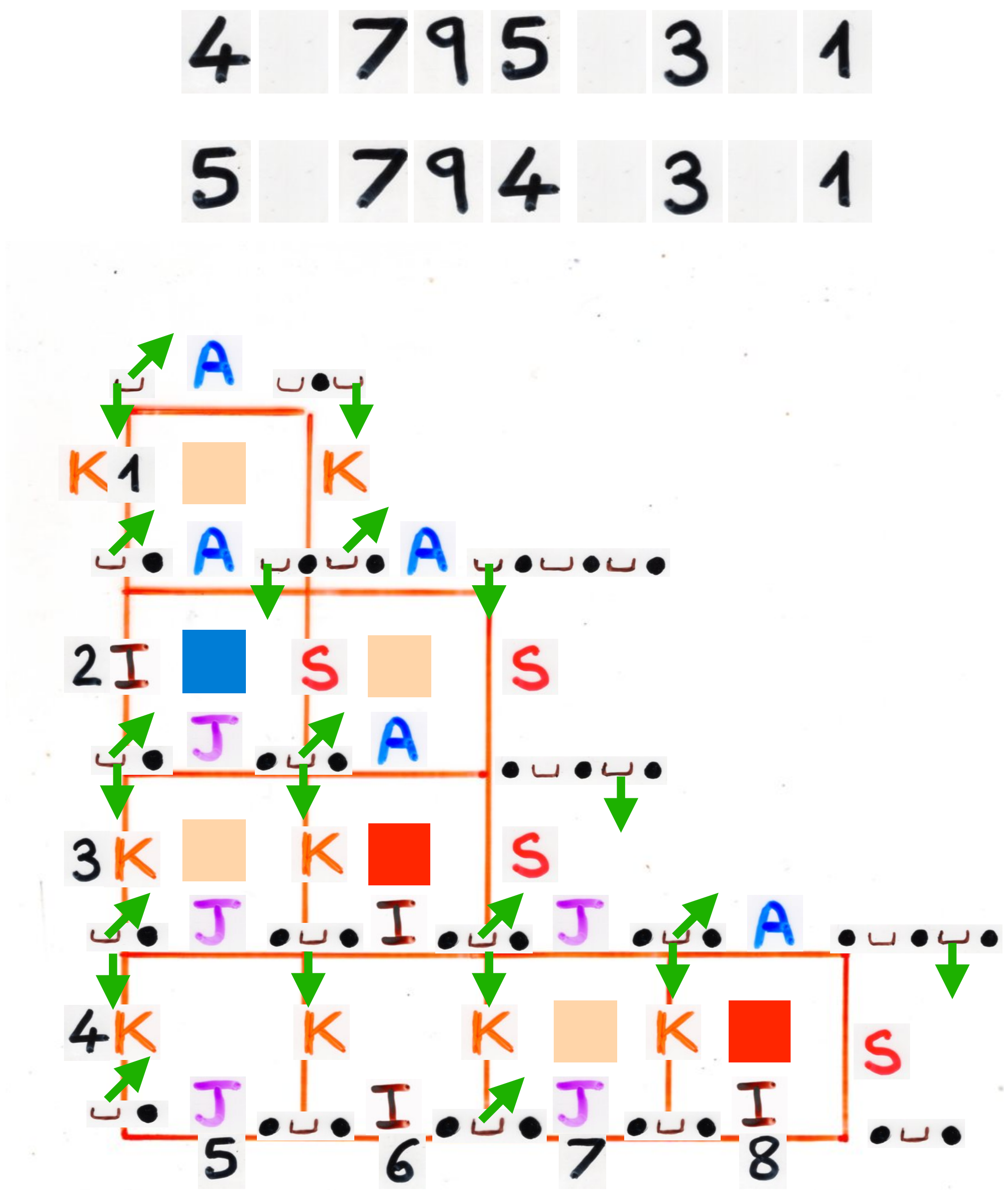
The « essence » of the index 32-1



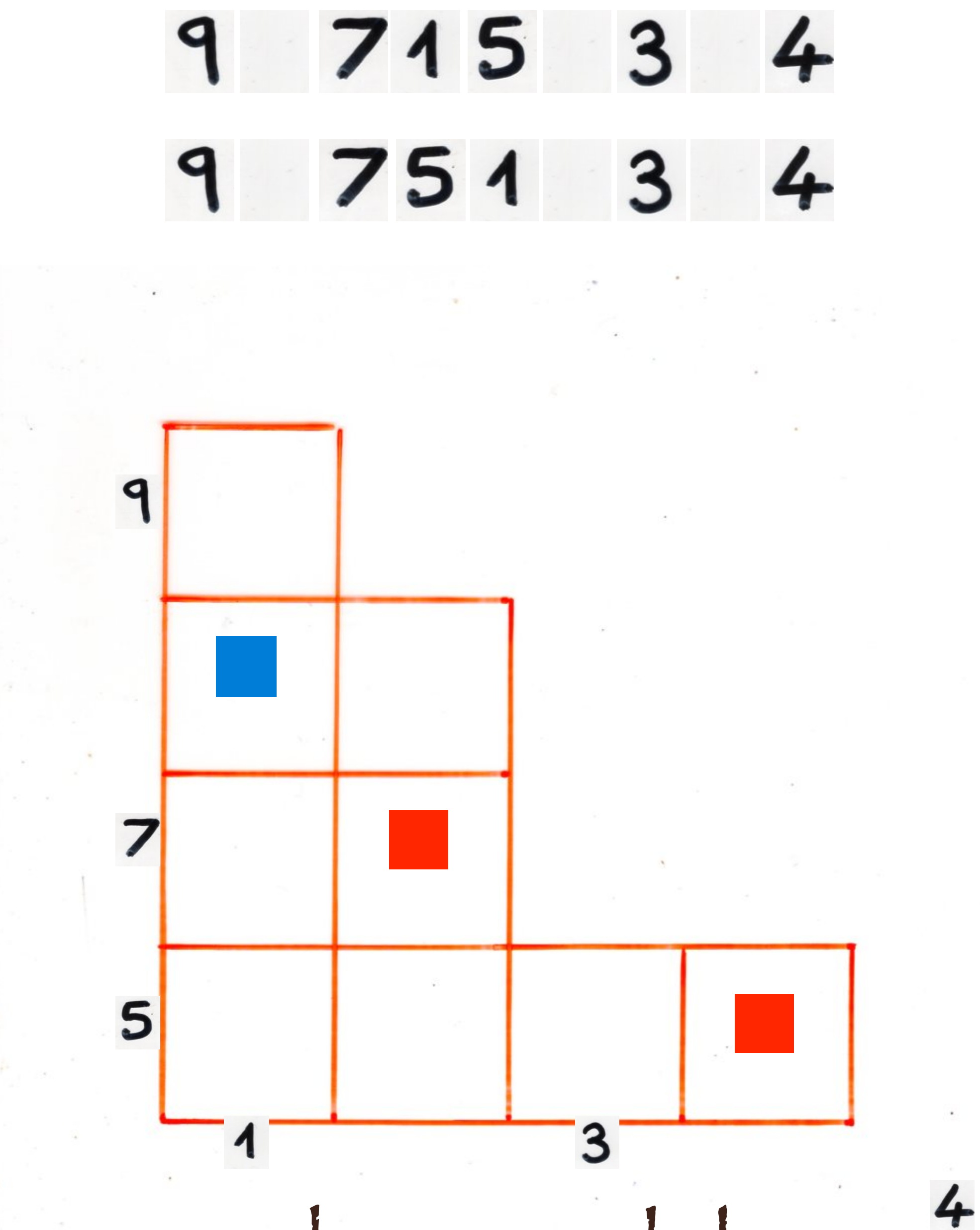
Local rules commutation diagrams



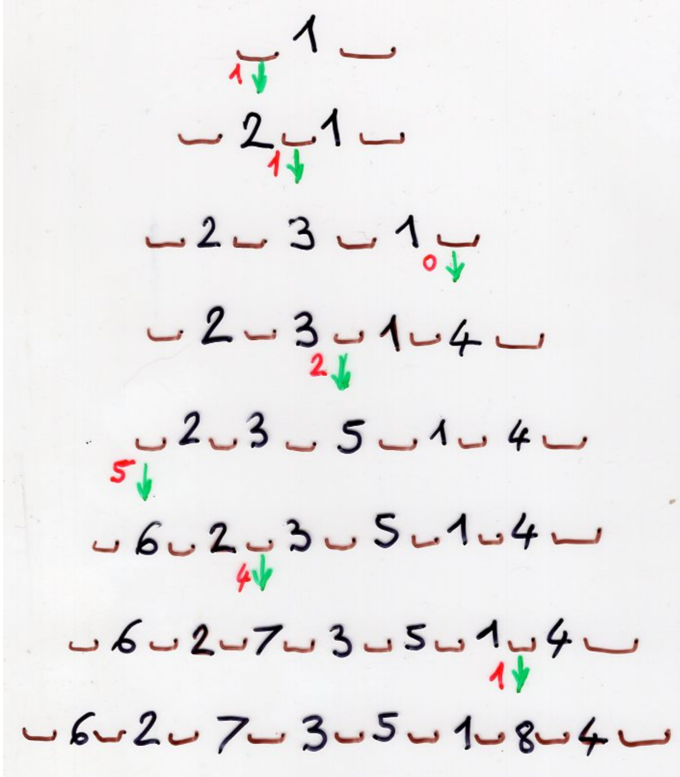
Exchange-delete Algorithm

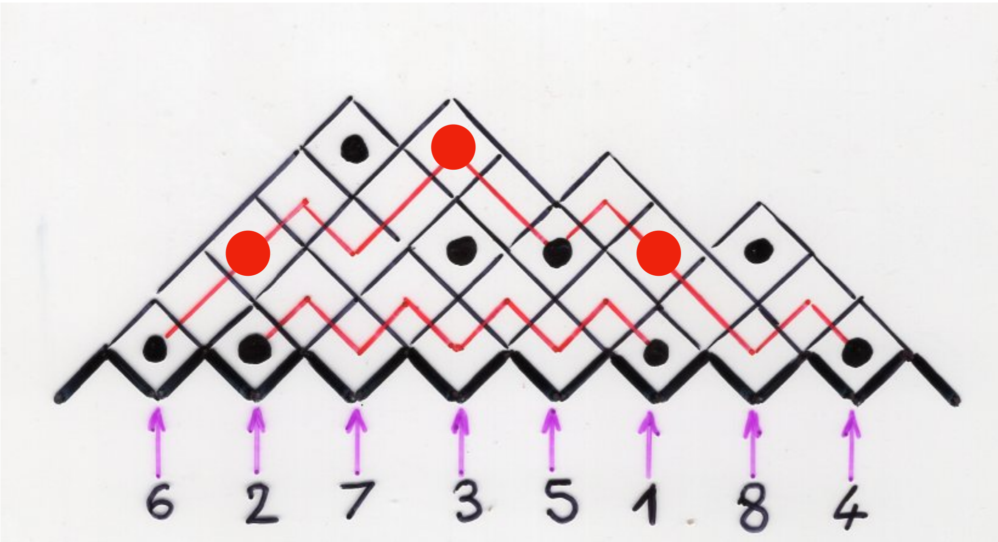


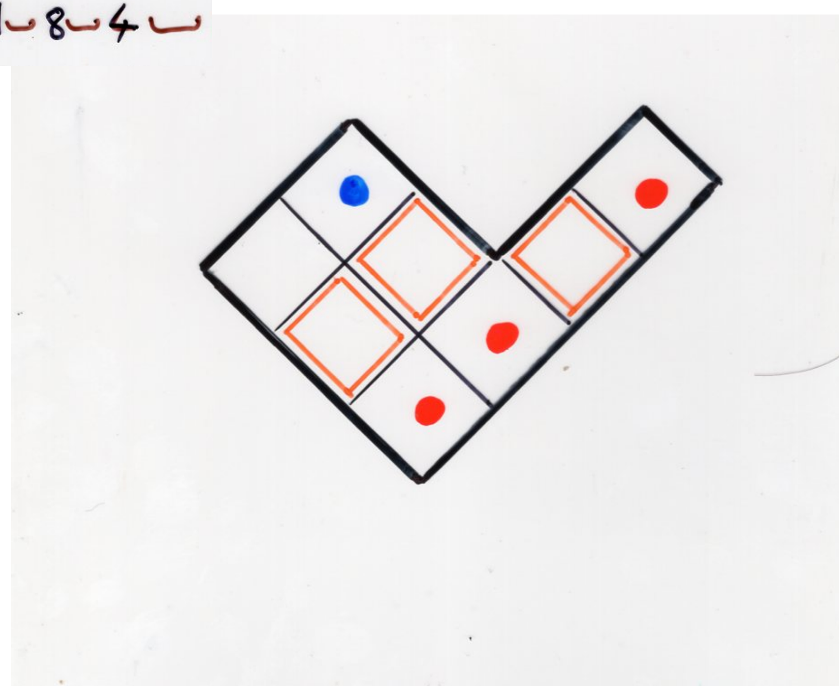
Local rules commutation diagrams



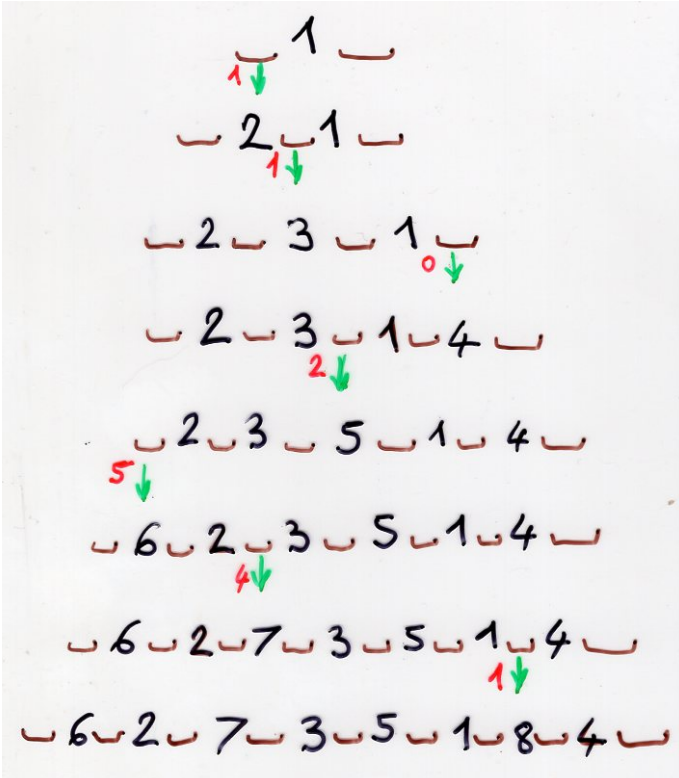
Exchange-delete Algorithm

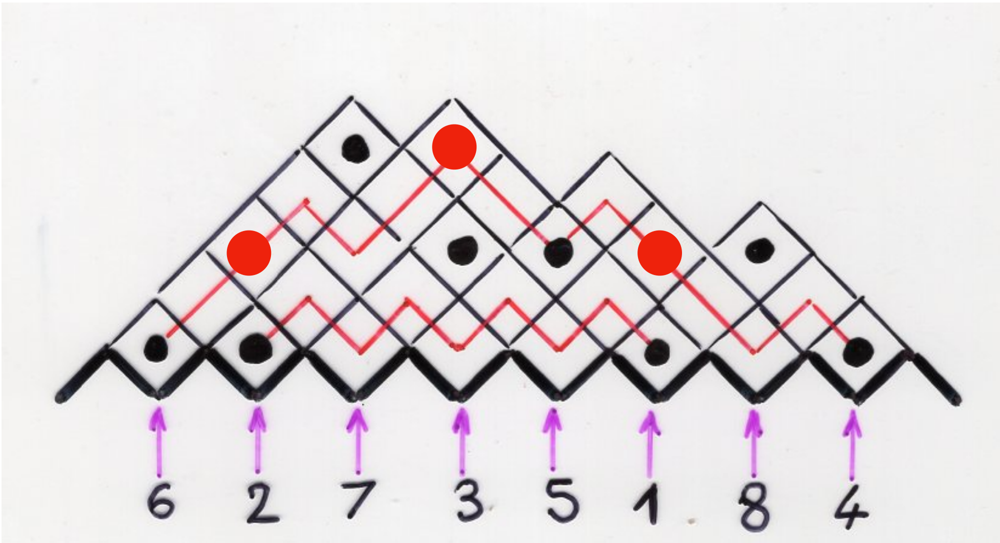


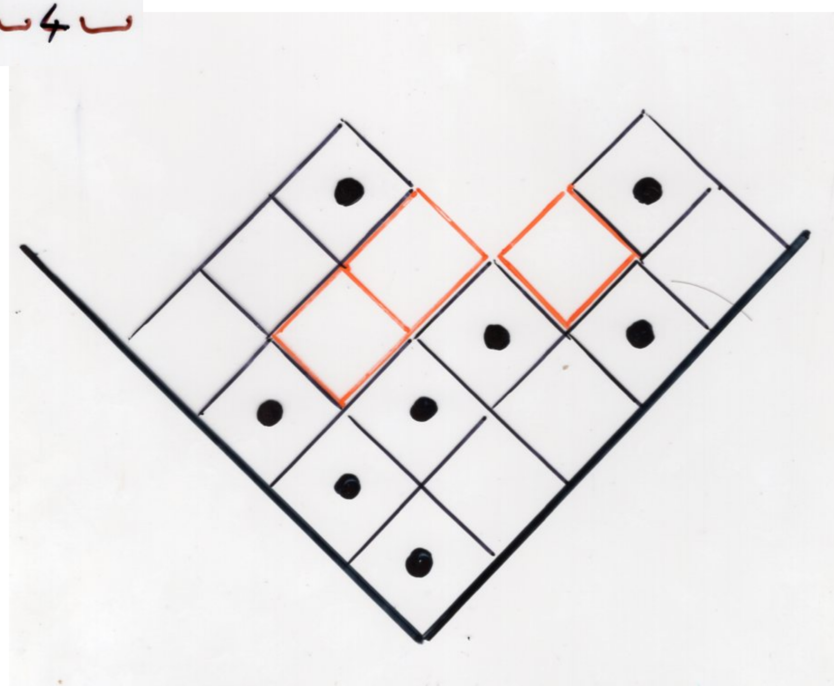




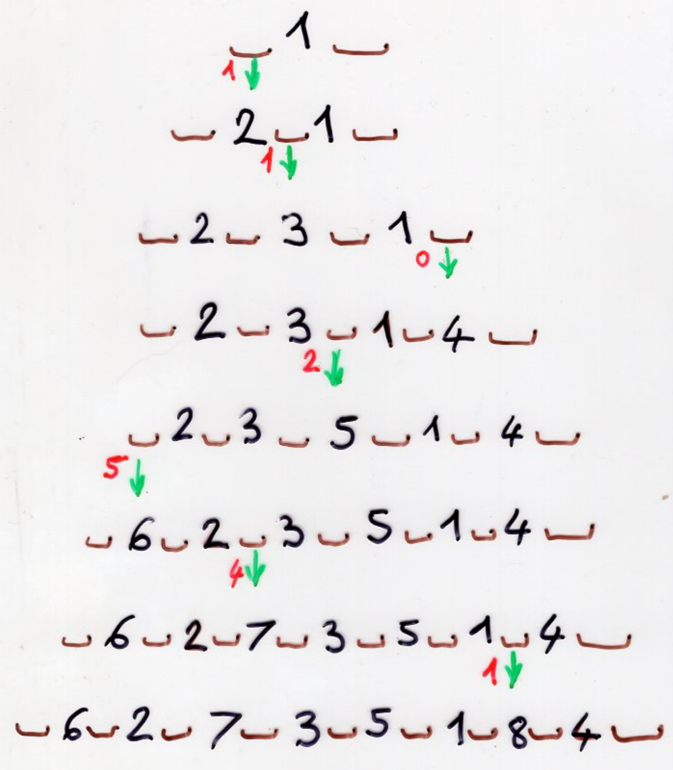
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



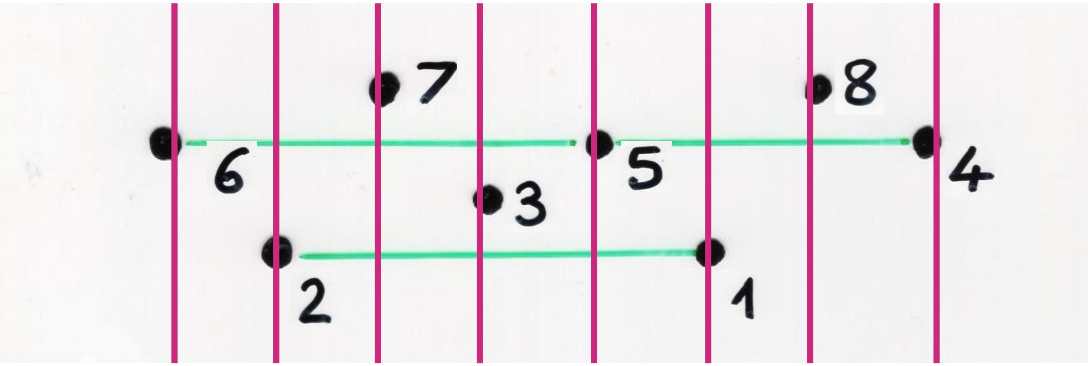


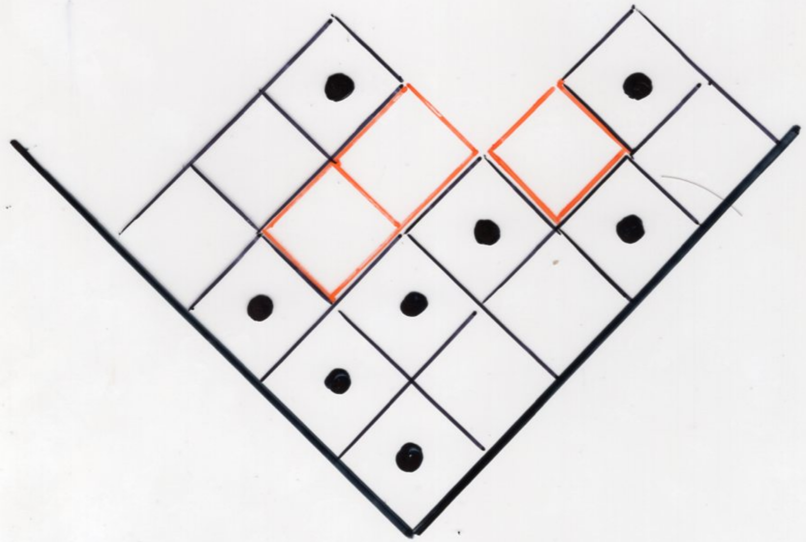


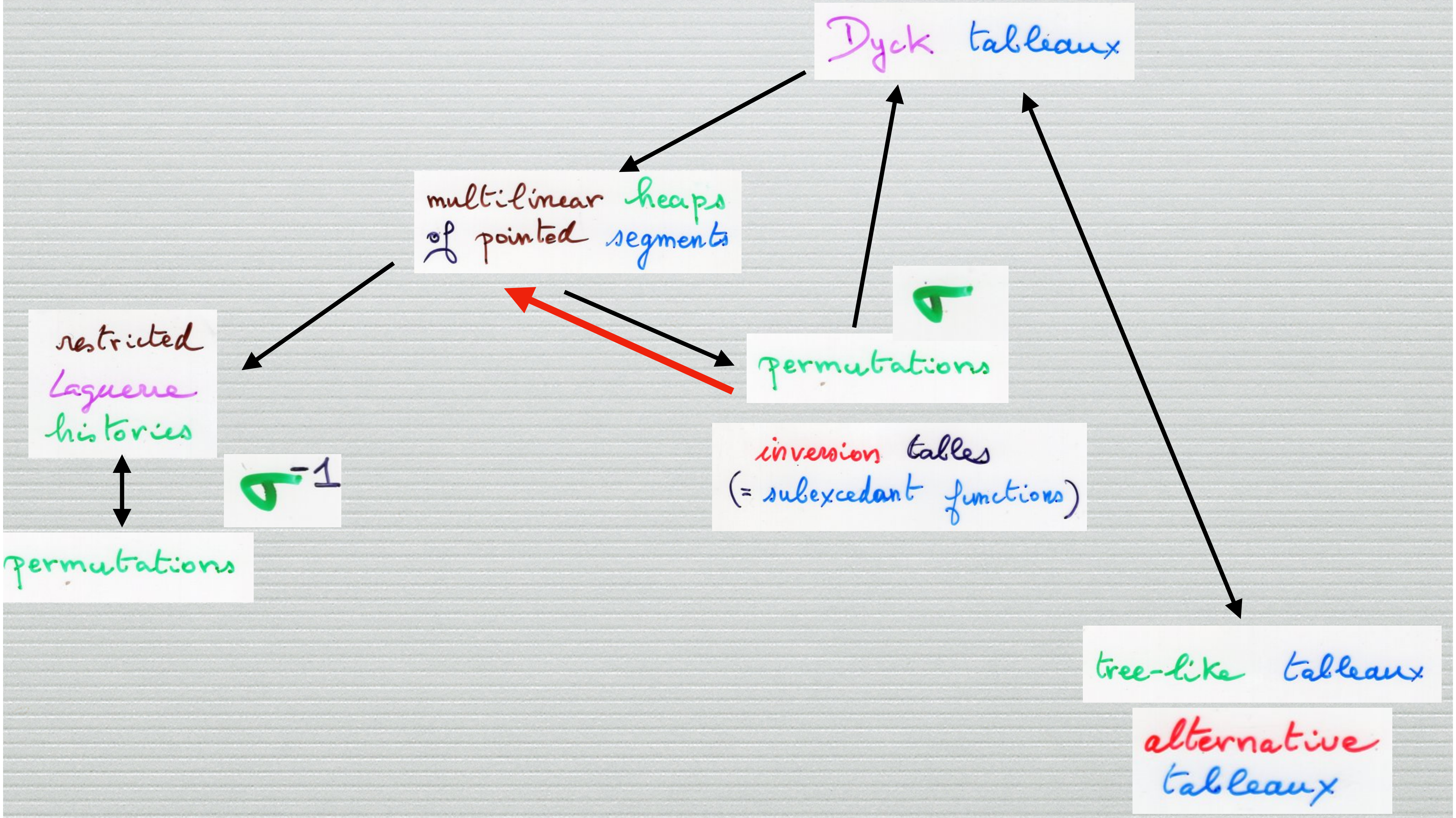
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$

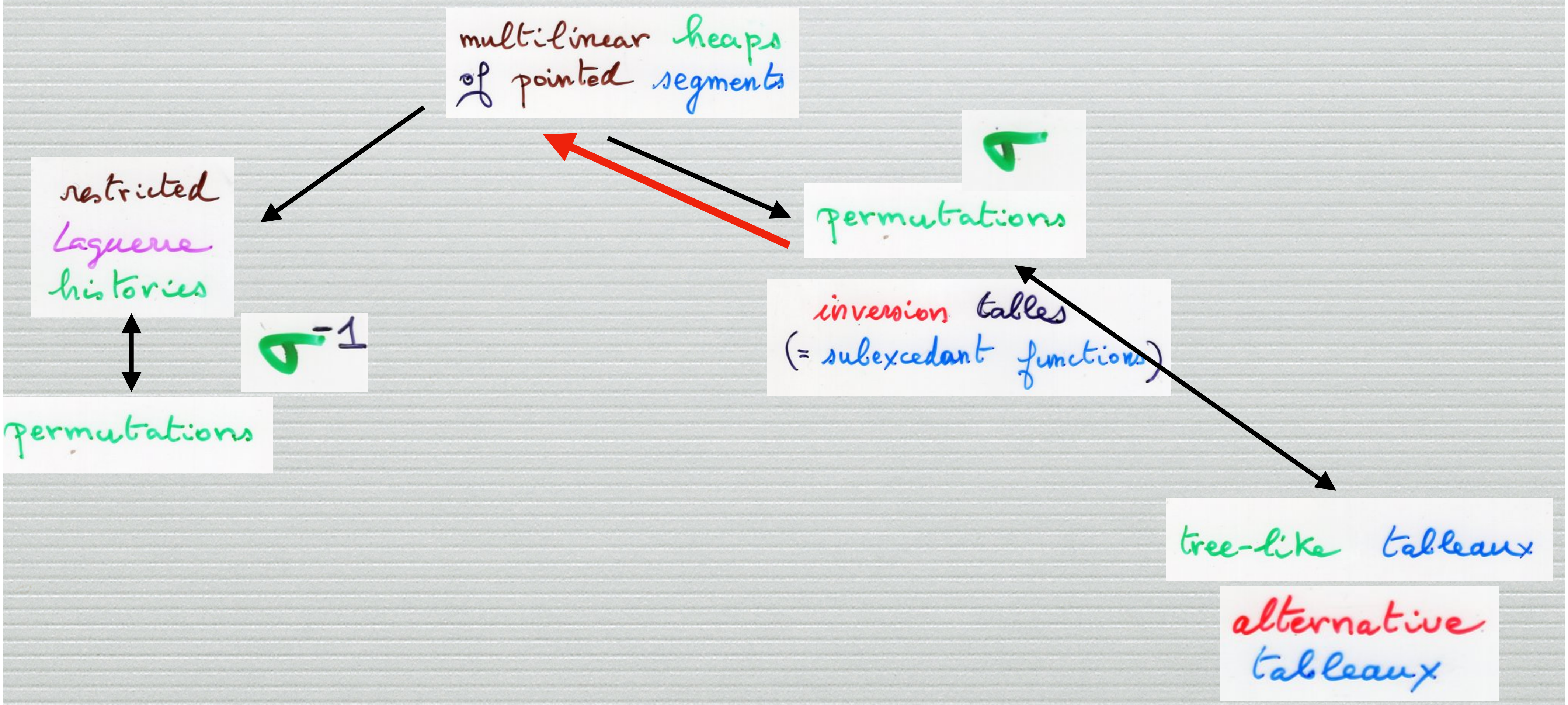


σ = 6 2 7 3 5 1 8 4

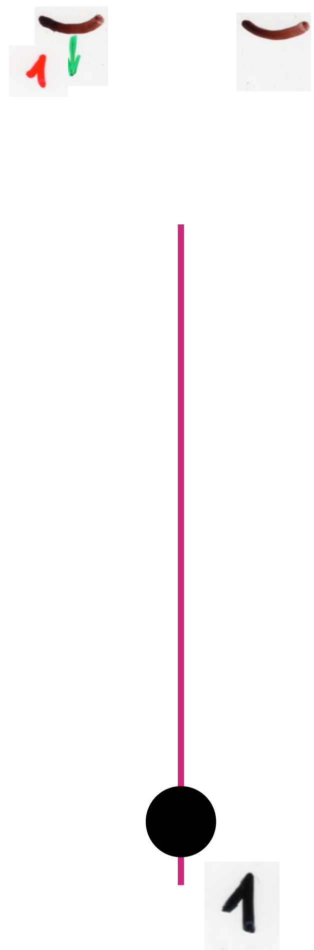


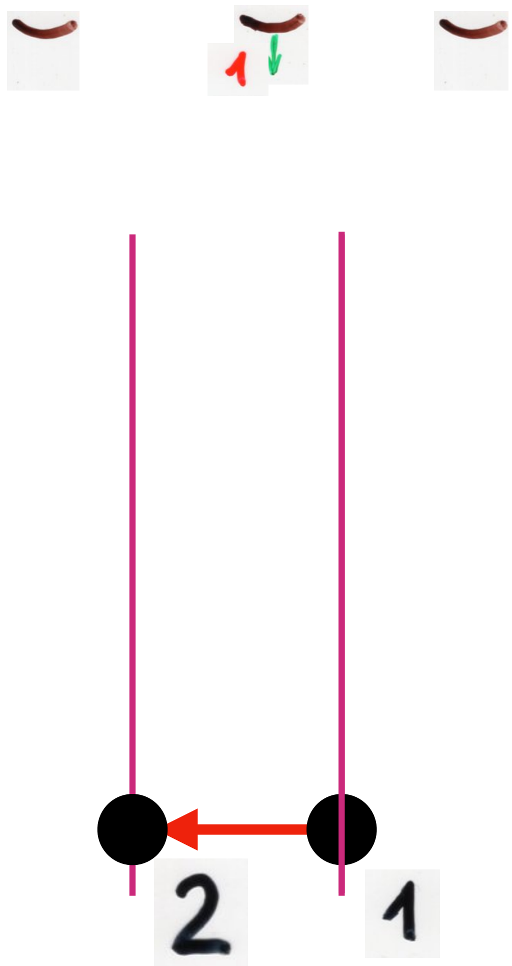


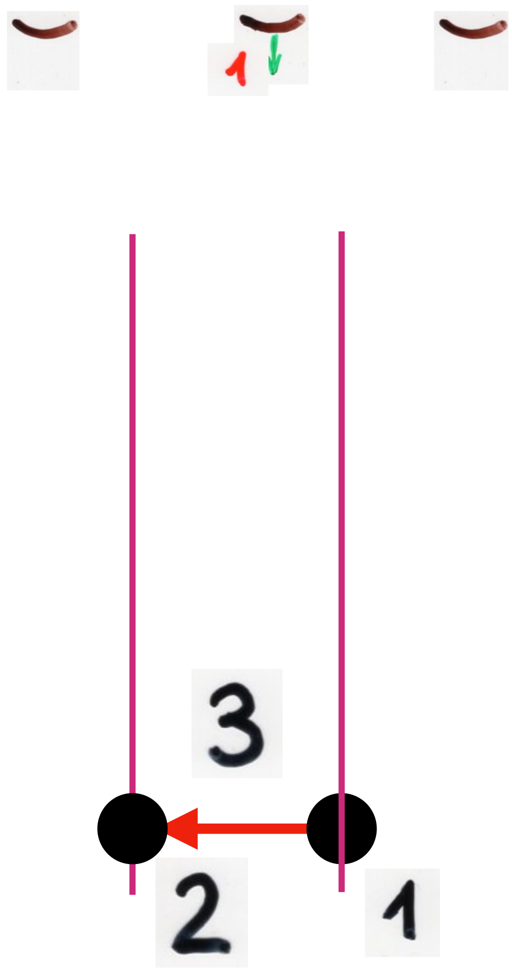

Dyck tableaux
multilinear heaps of pointed segments
restricted Laguerre histories
permutations
σ
inversion tables
(= subexcedant functions)
σ⁻¹
permutations
tree-like tableaux
alternative tableaux

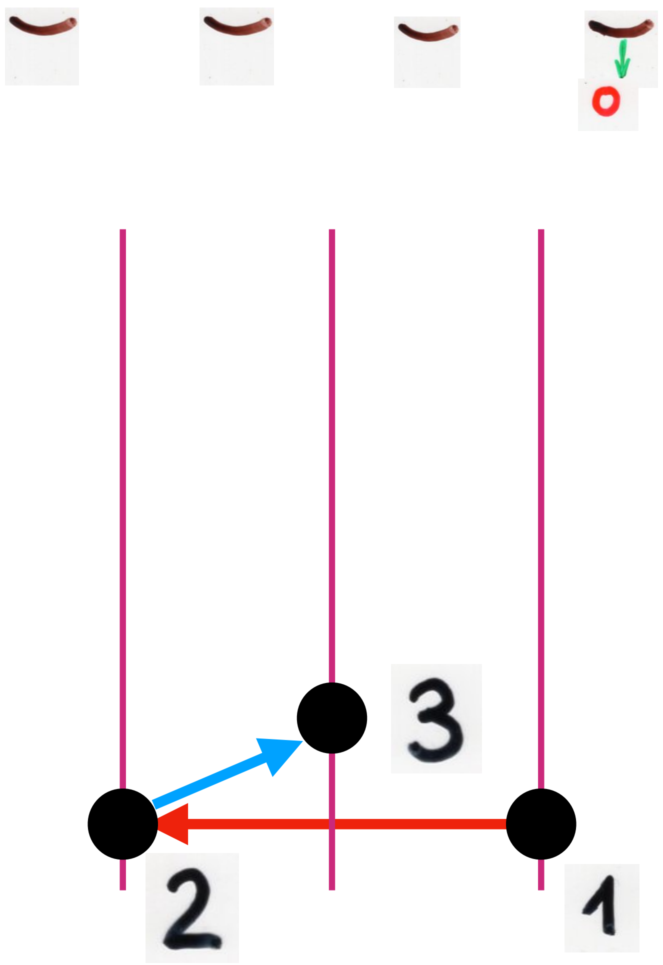


multilinear heaps of pointed segments
restricted Laguerre histories
σ^{-1}
permutations
σ
permutations
inversion tables
(= subexcedant functions)
tree-like tableaux
alternative tableaux

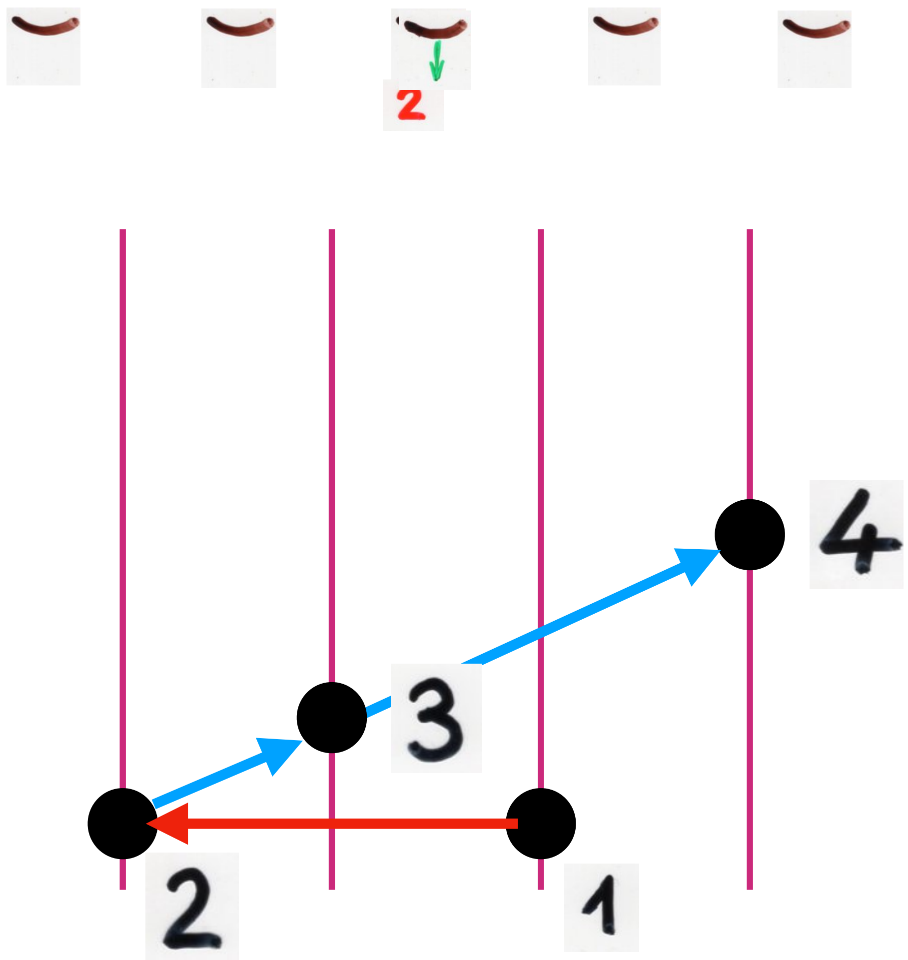

1
1

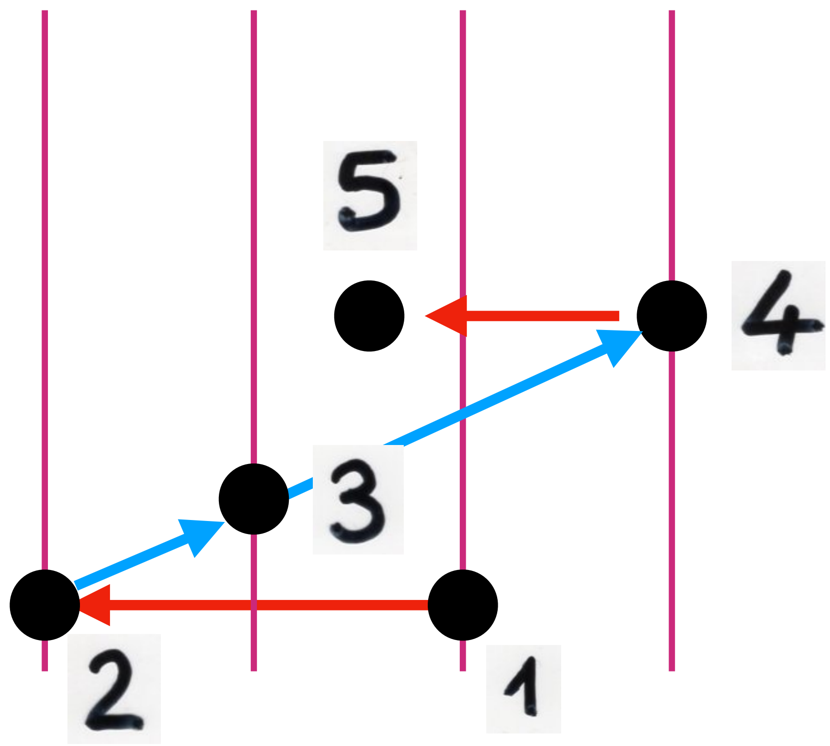

1
2
1

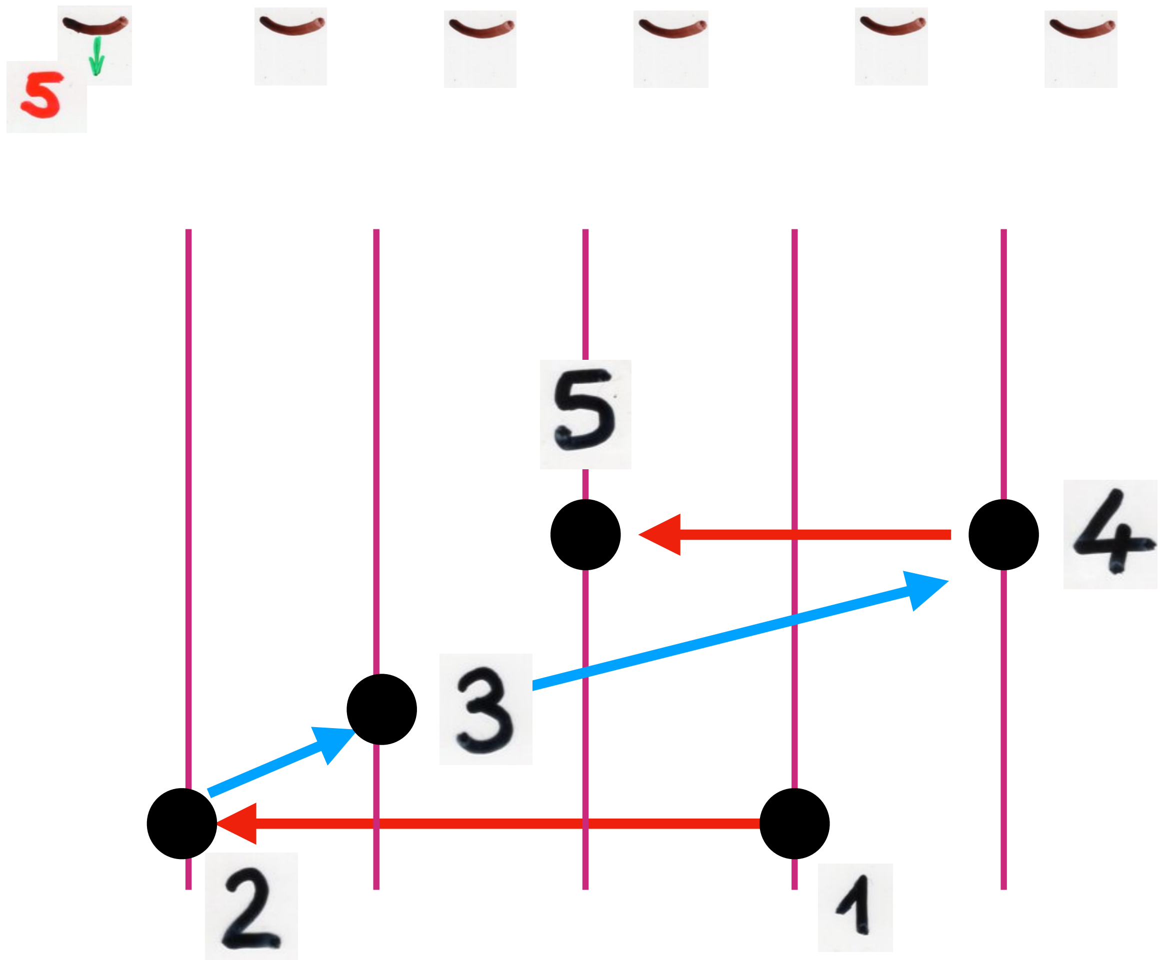

1
3
2
1

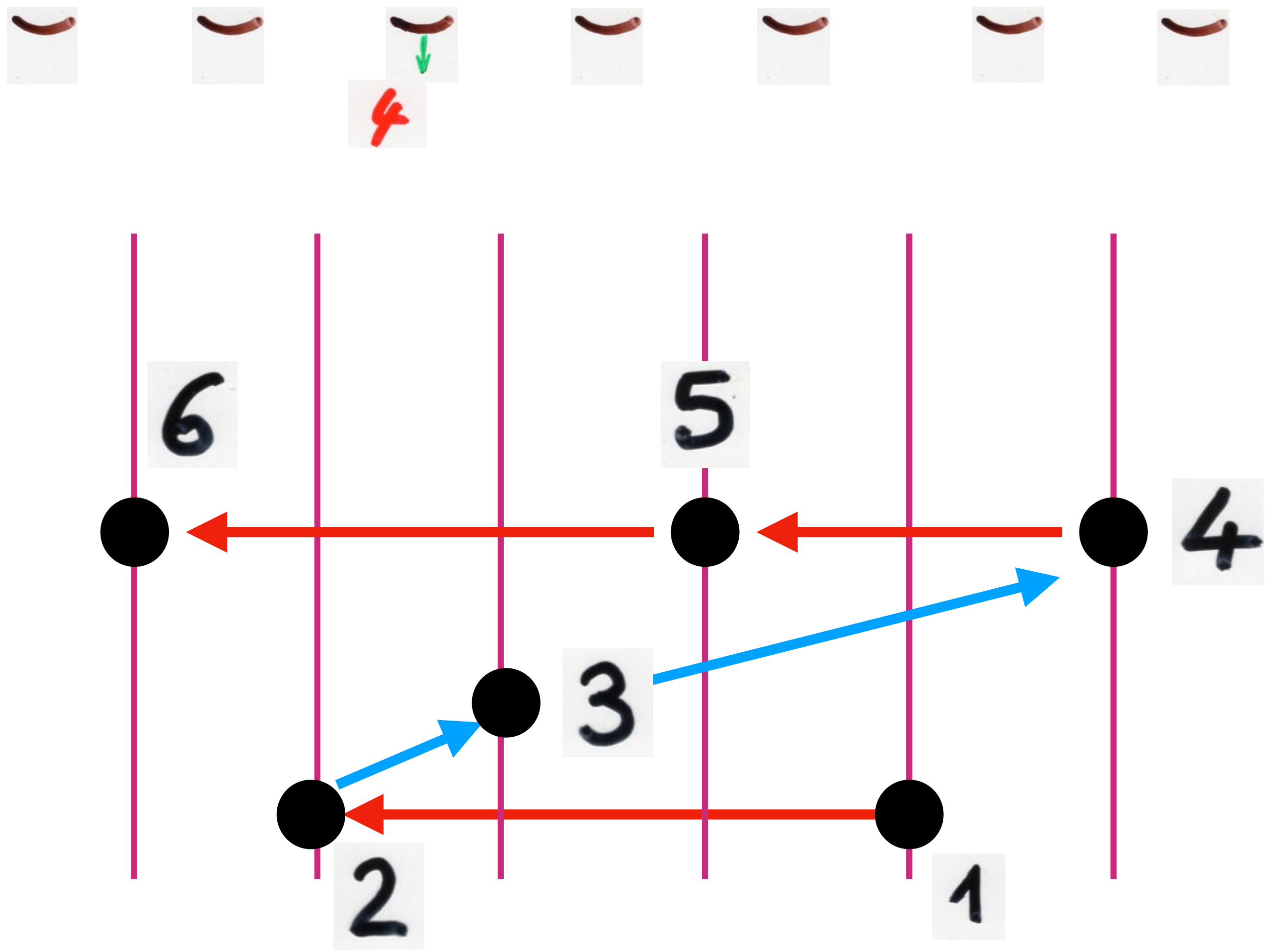


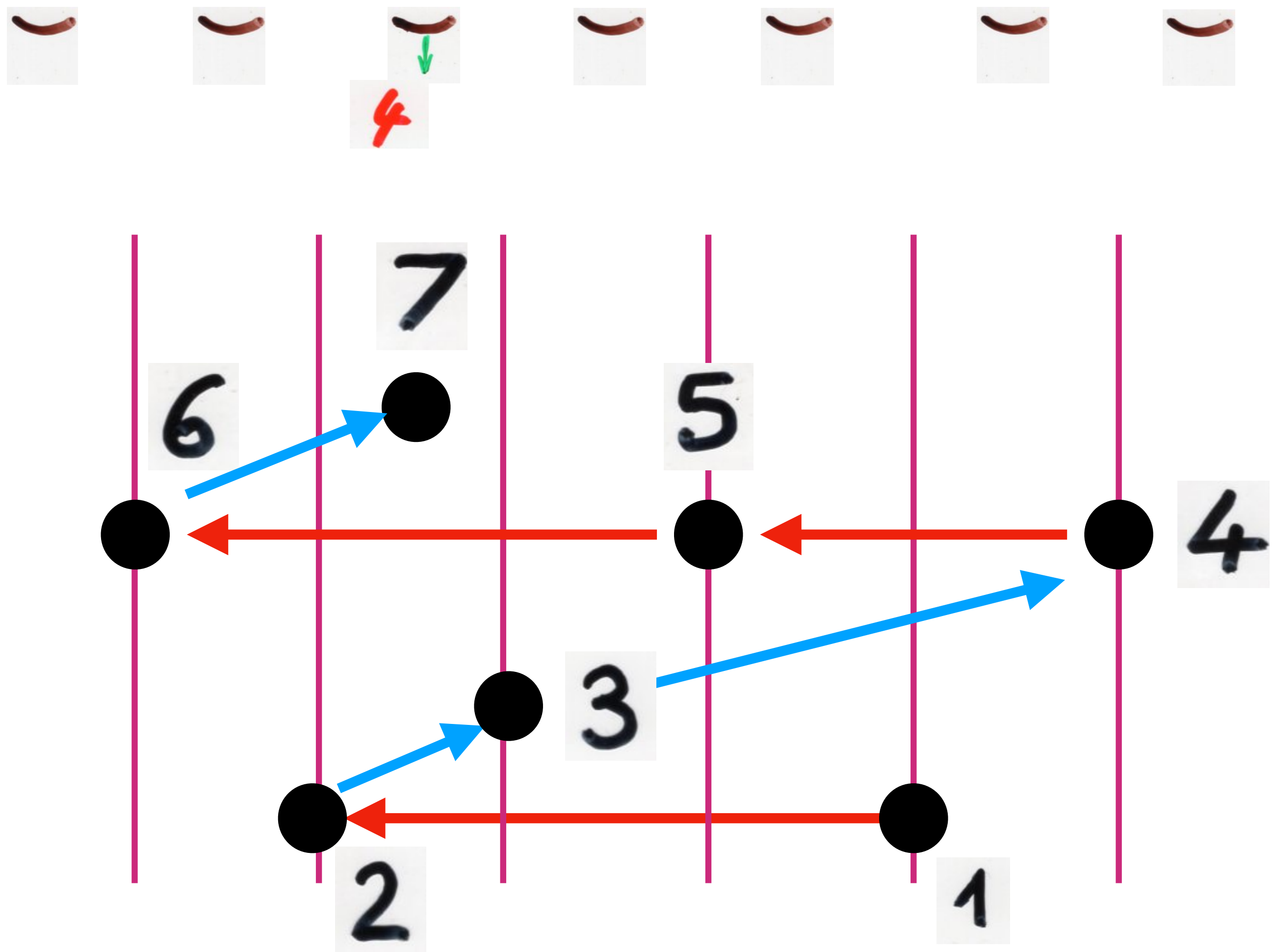
0
3
2
1

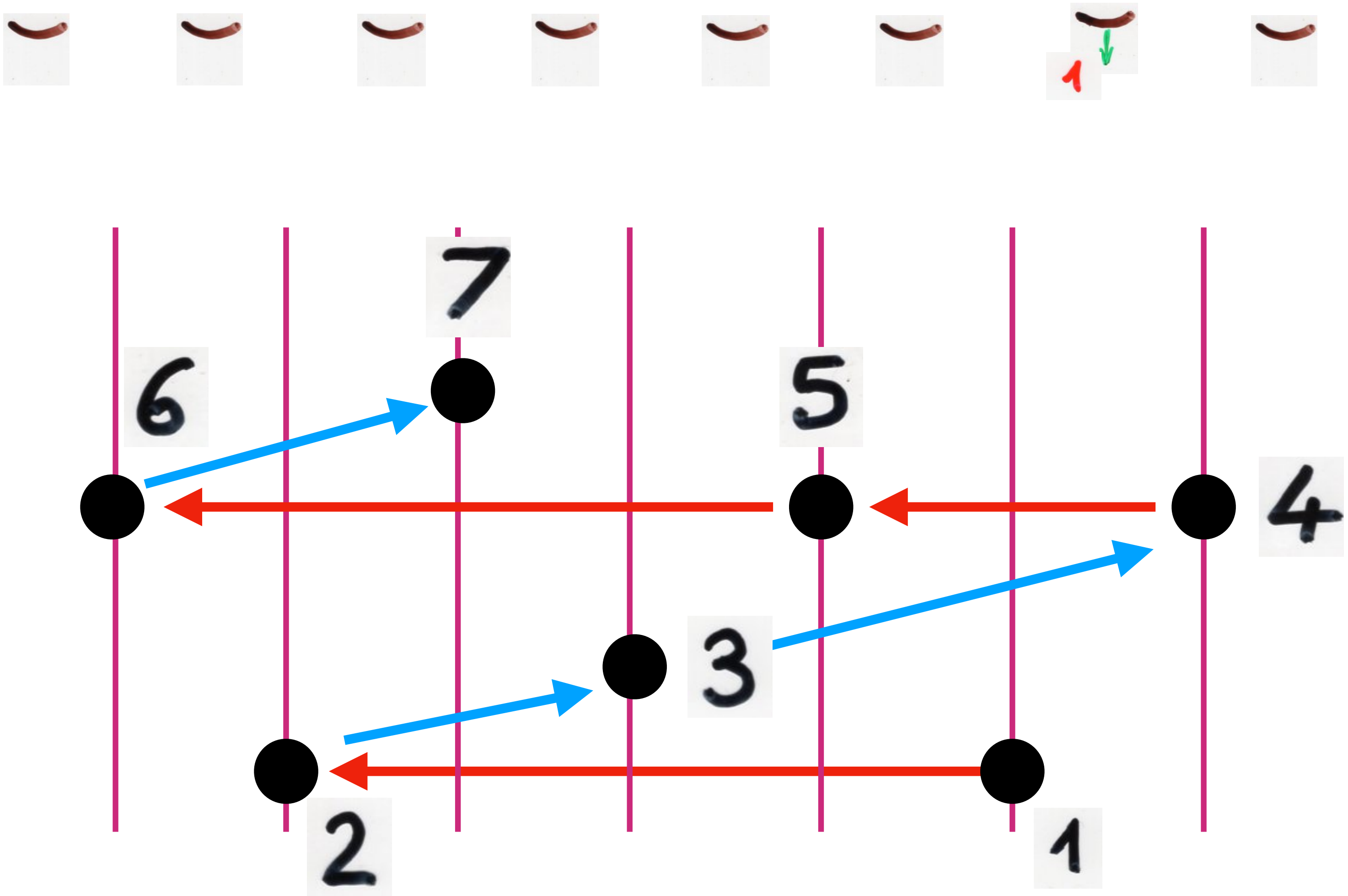

2
4
3
2
1

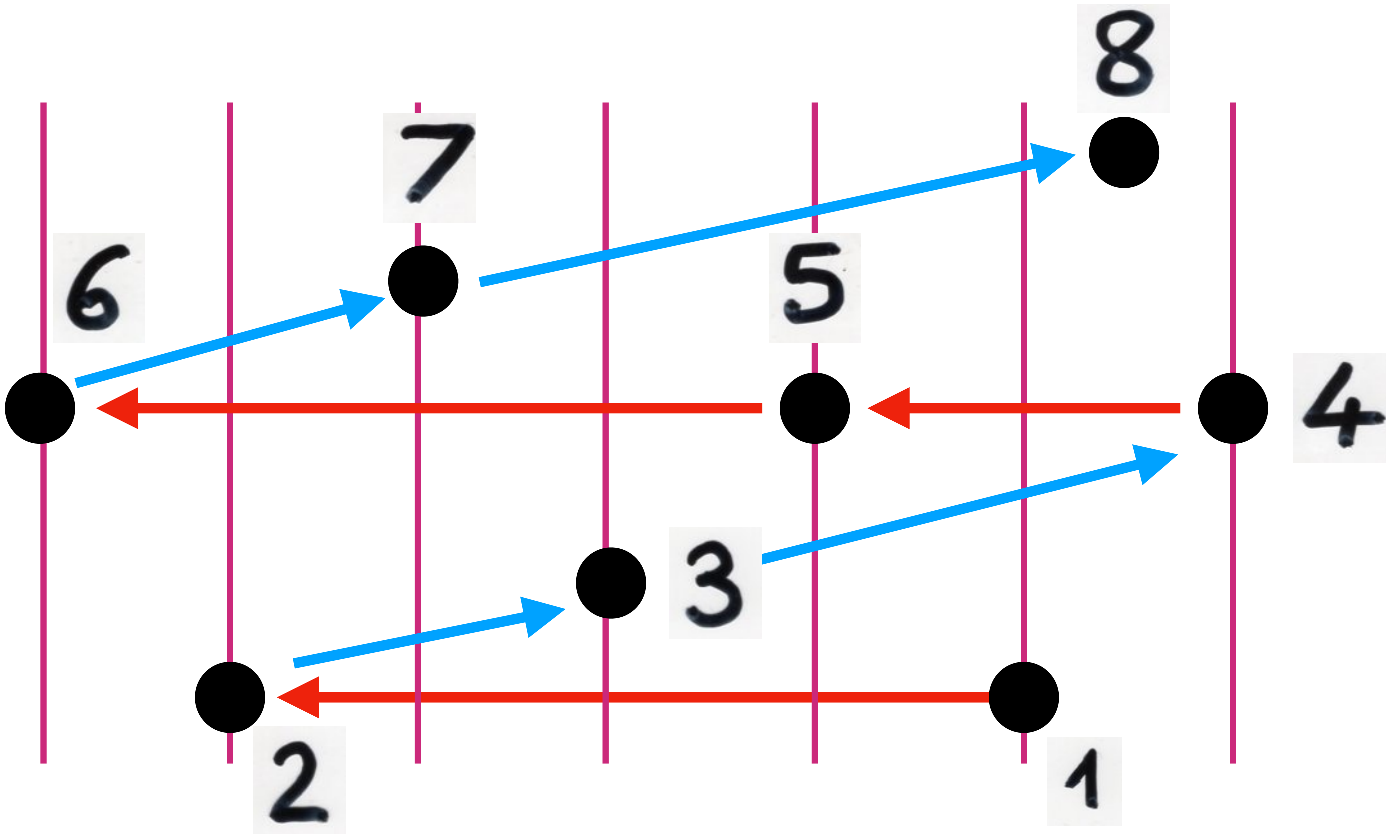

5
4
3
2
1


5
5
4
3
2
1

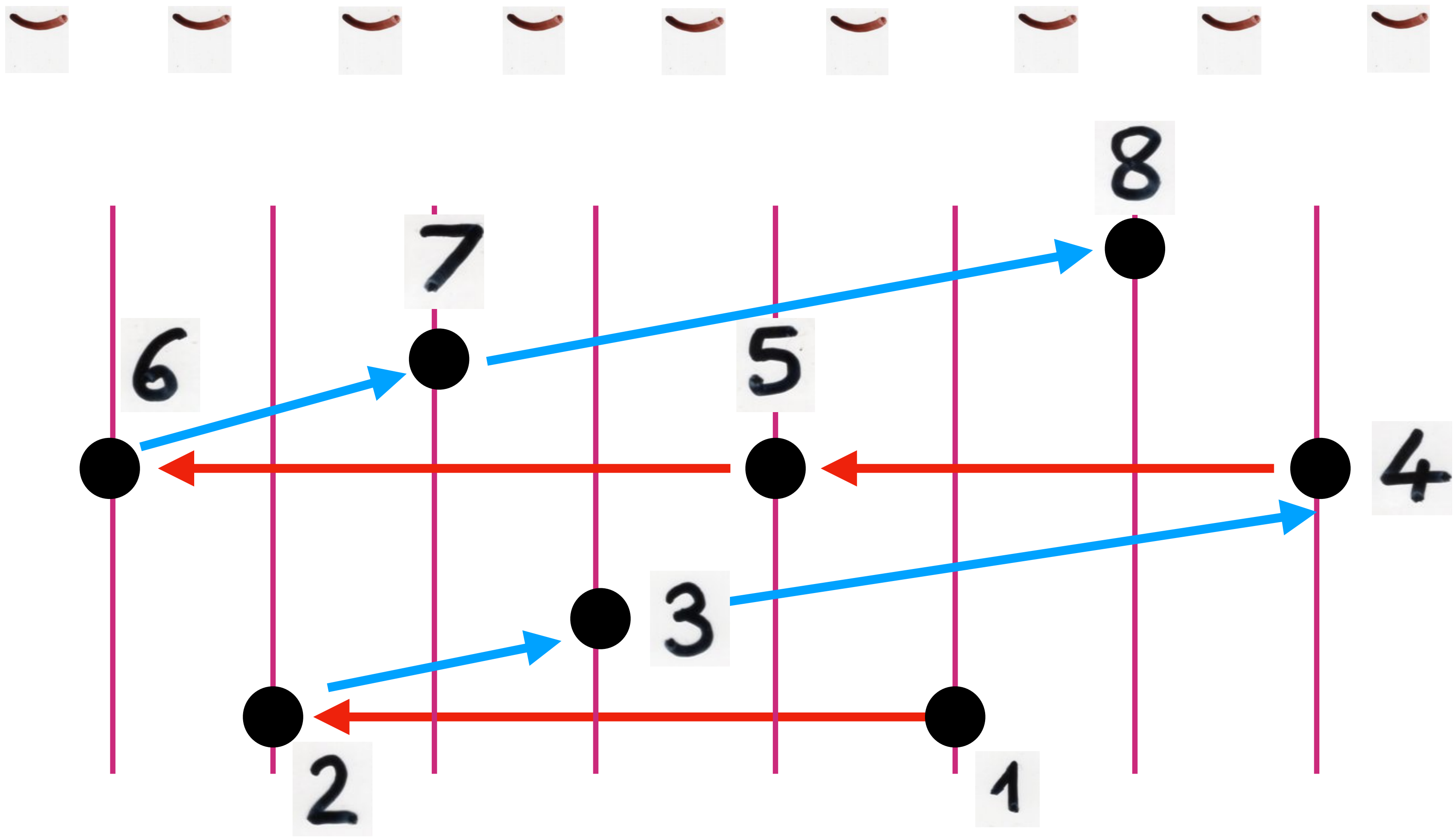

4
6
5
4
3
2
1


4
7
6
5
4
3
2
1

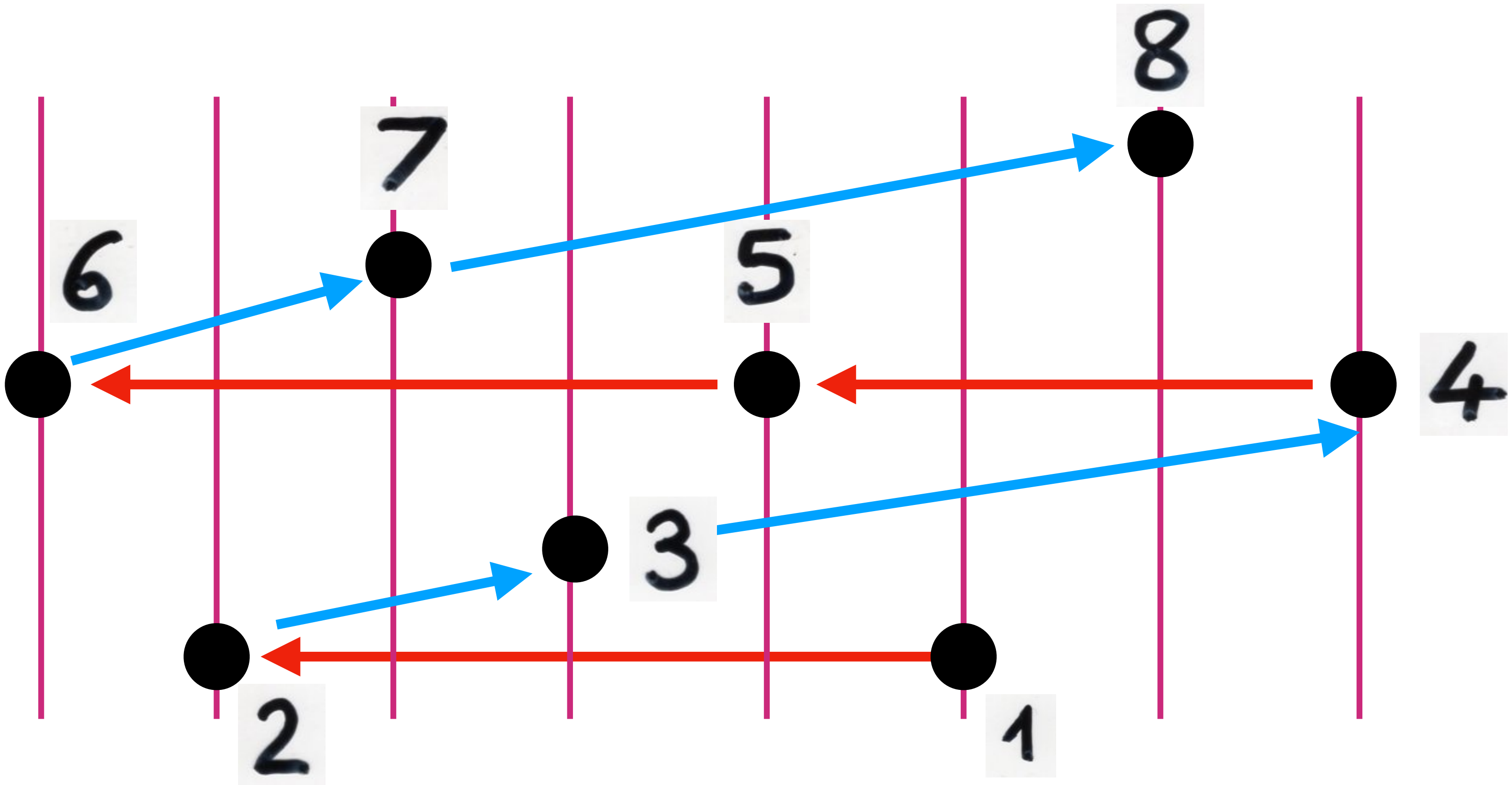

1
7
6
5
4
3
2
1

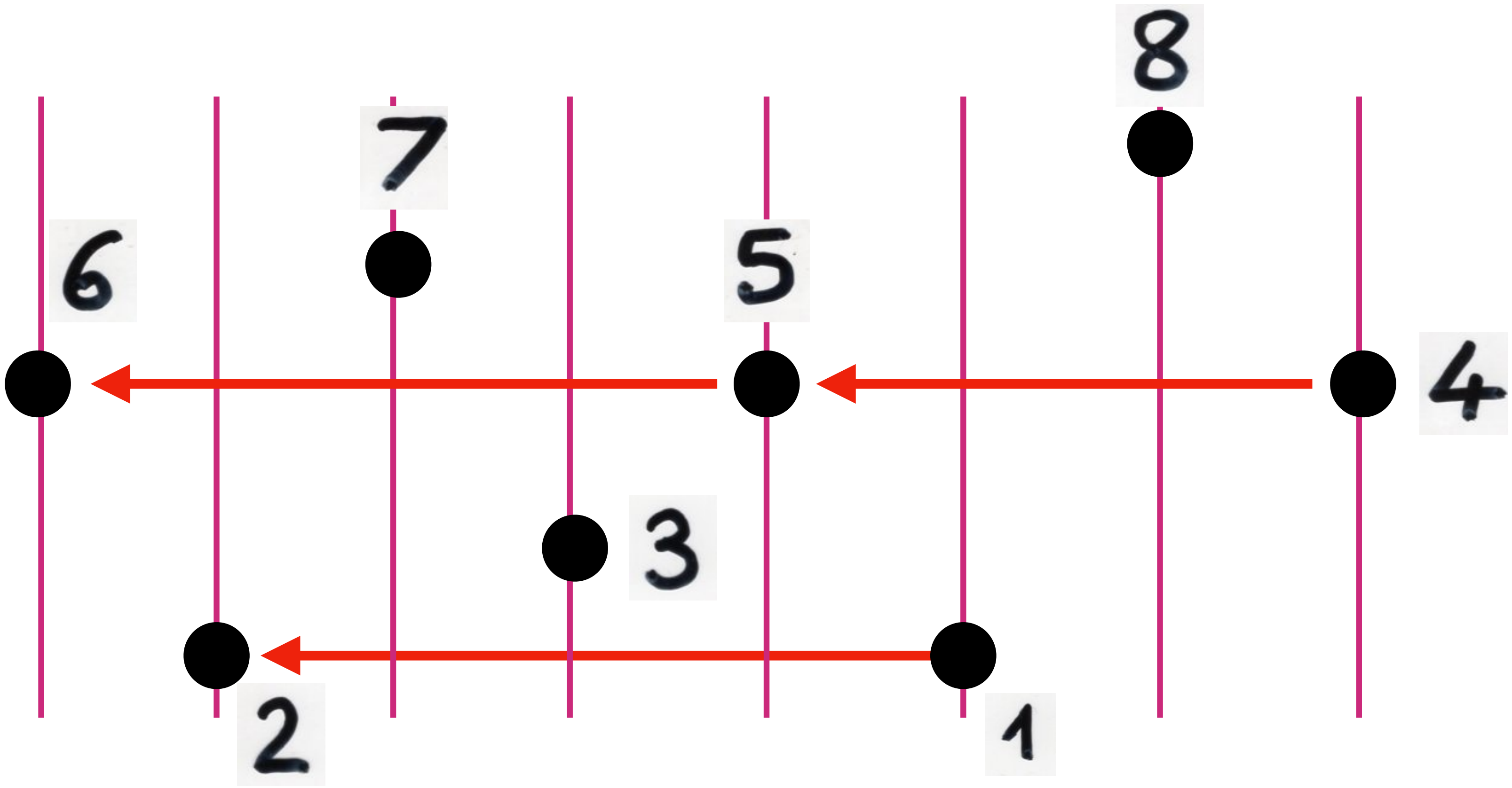


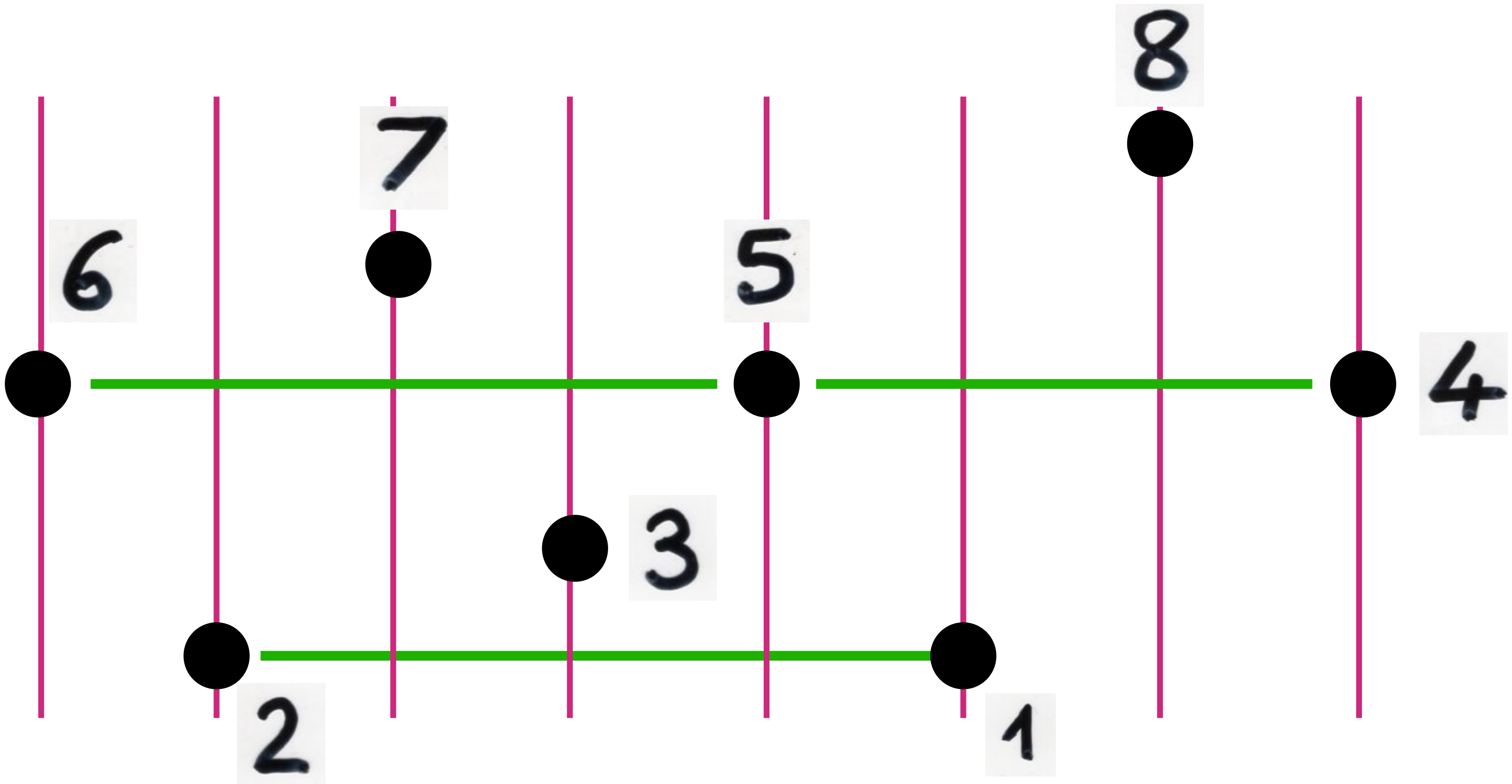
8
7
6
5
4
3
2
1



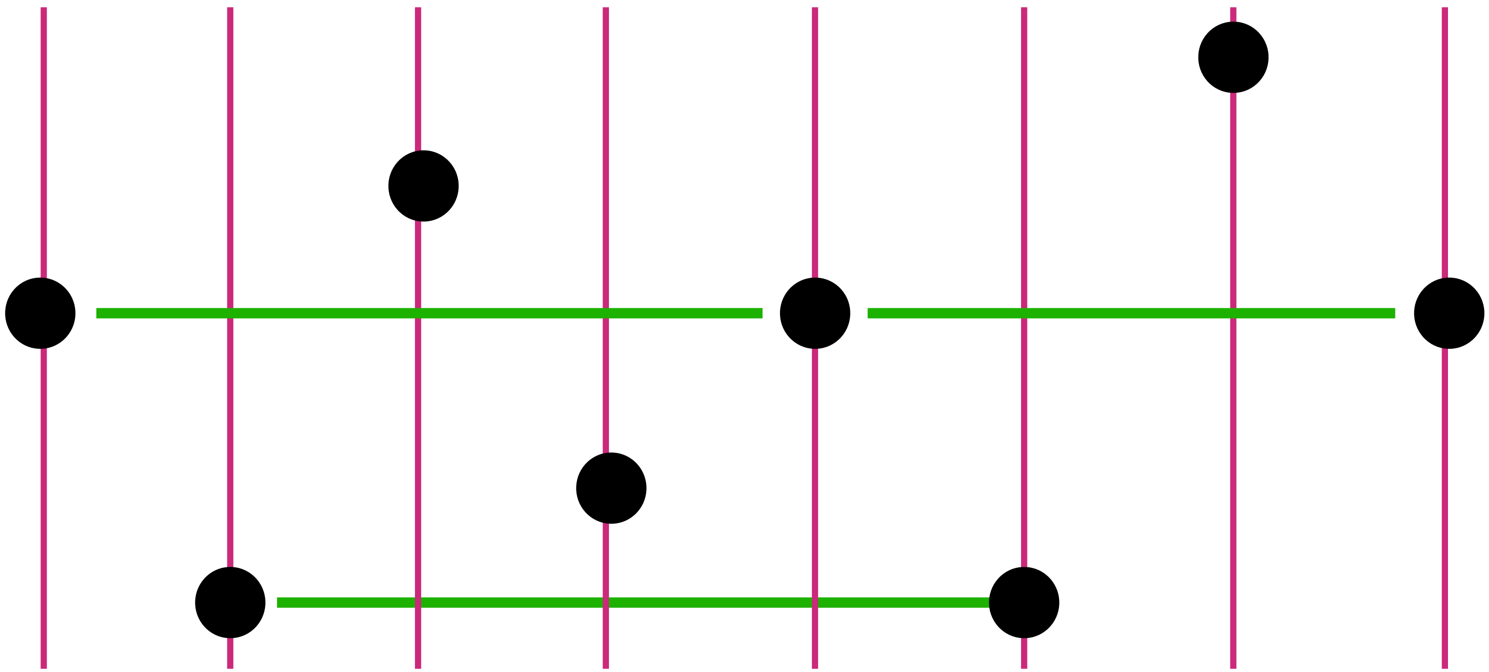
8
7
6
5
4
3
2
1


8
7
6
5
4
3
2
1


8
7
6
5
4
3
2
1


8
7
6
5
4
3
2
1

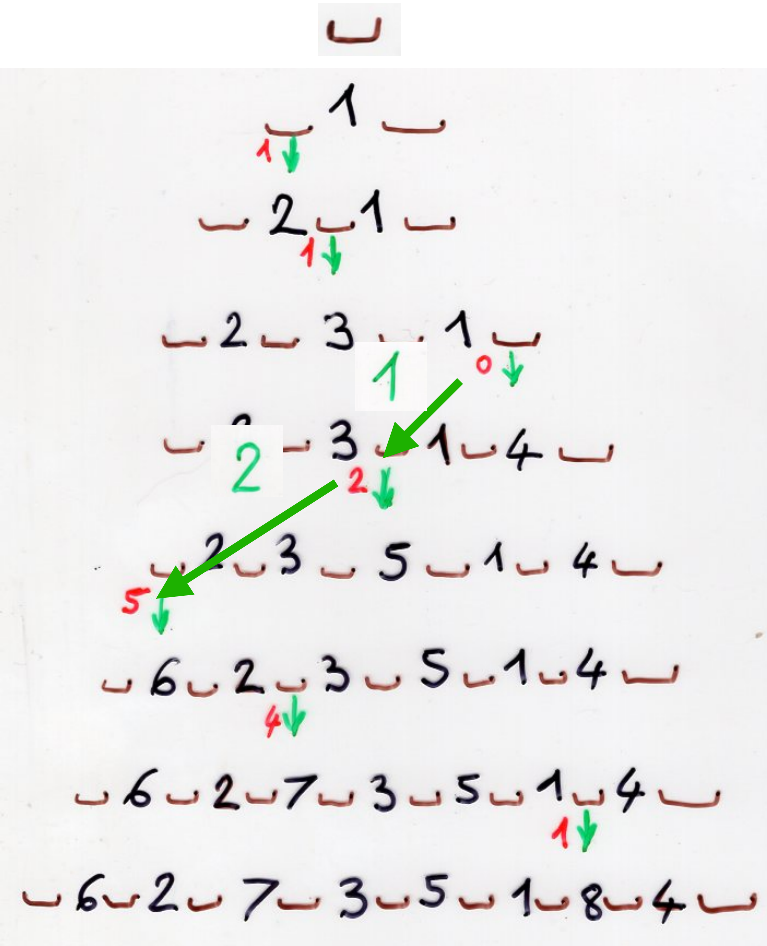
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

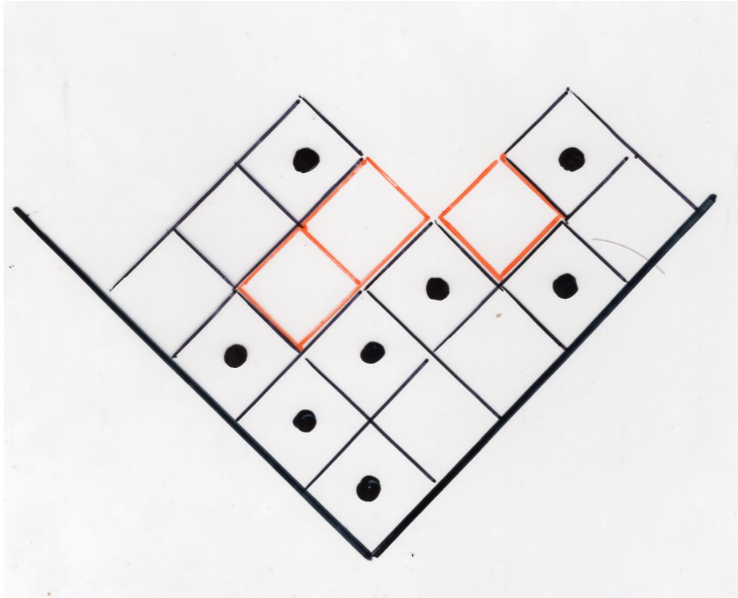


bijection

$f \longrightarrow T$

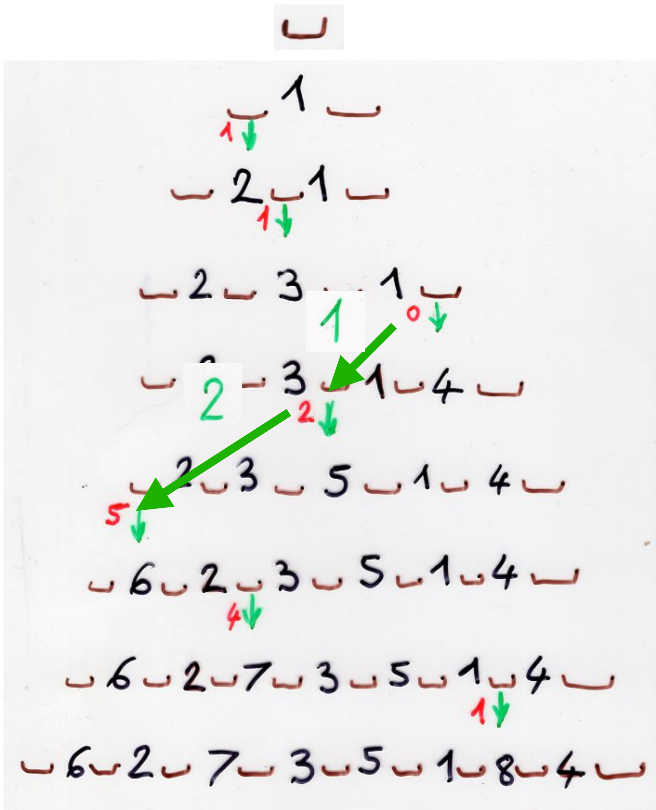
number of crossings $cr(T)$

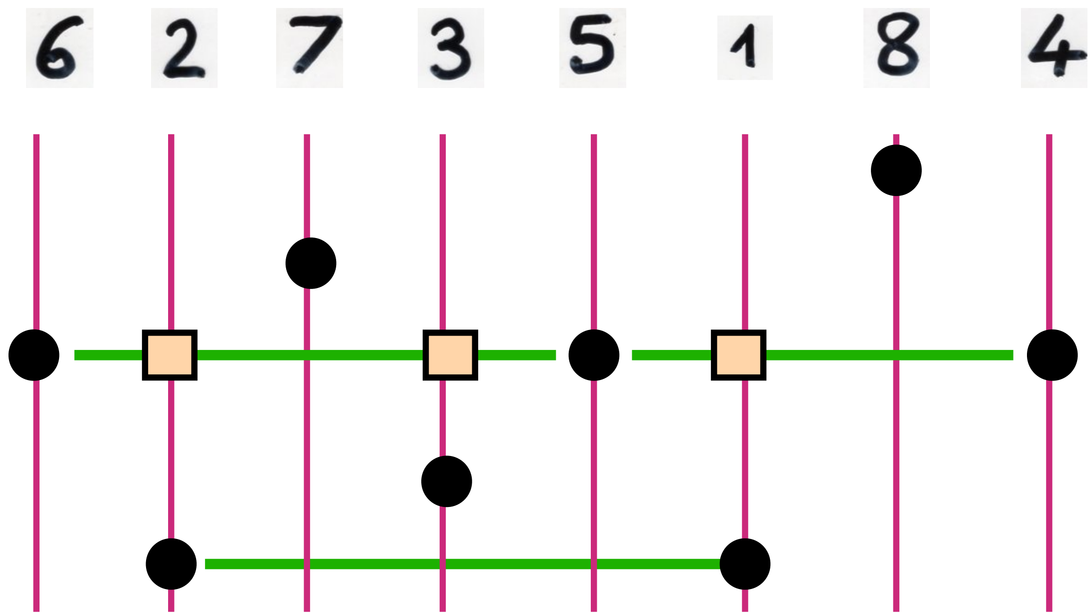


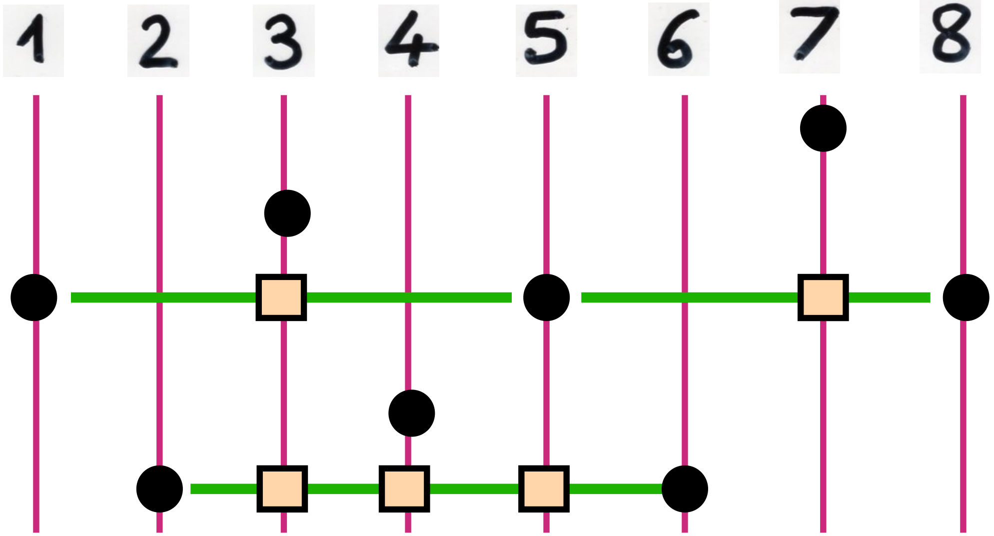


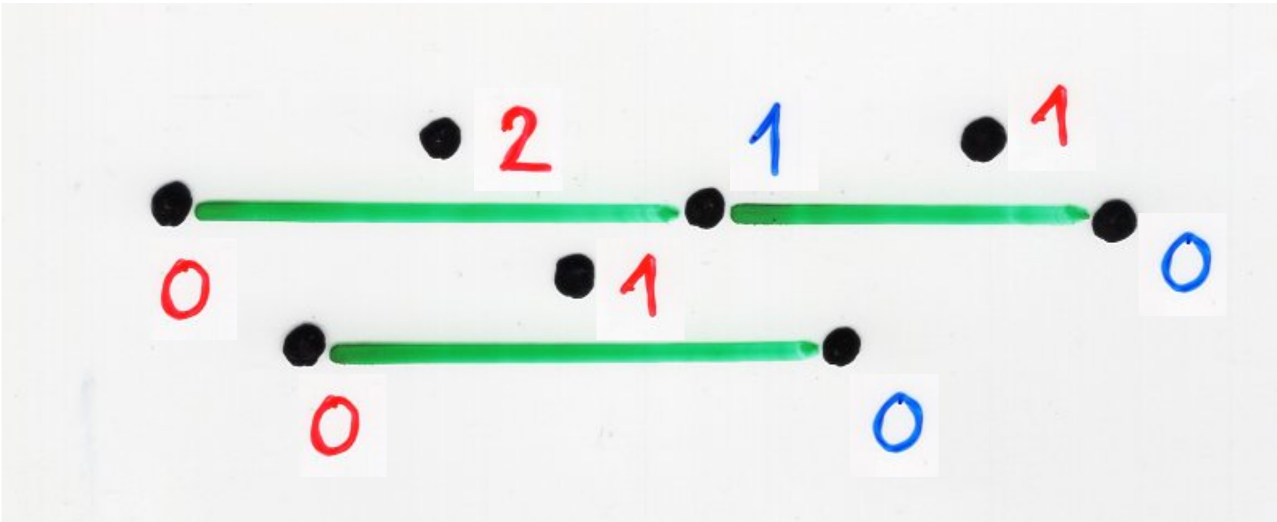
= sum of the length of all rim-hooks added in the insertion algorithm

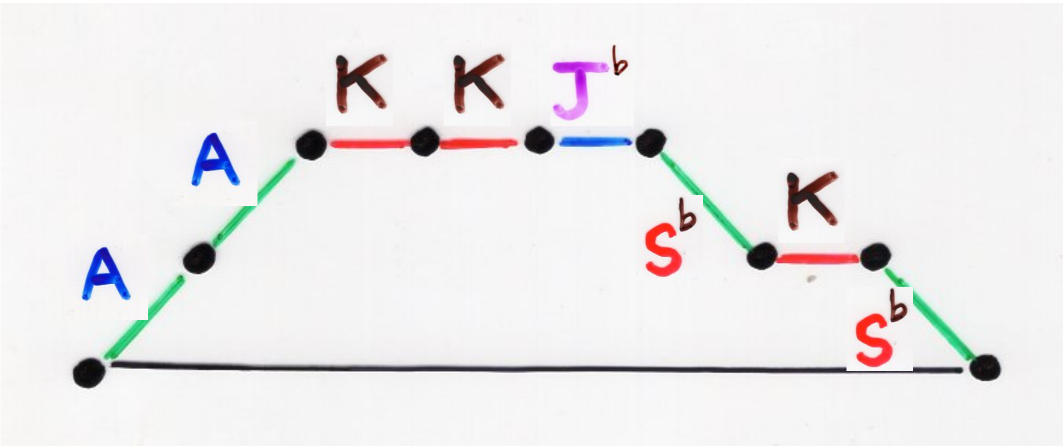
$$= \sum_{1 \le i \le (n-1)} \max\left[\left(f(i+1) - f(i) - 1\right), 0\right]$$

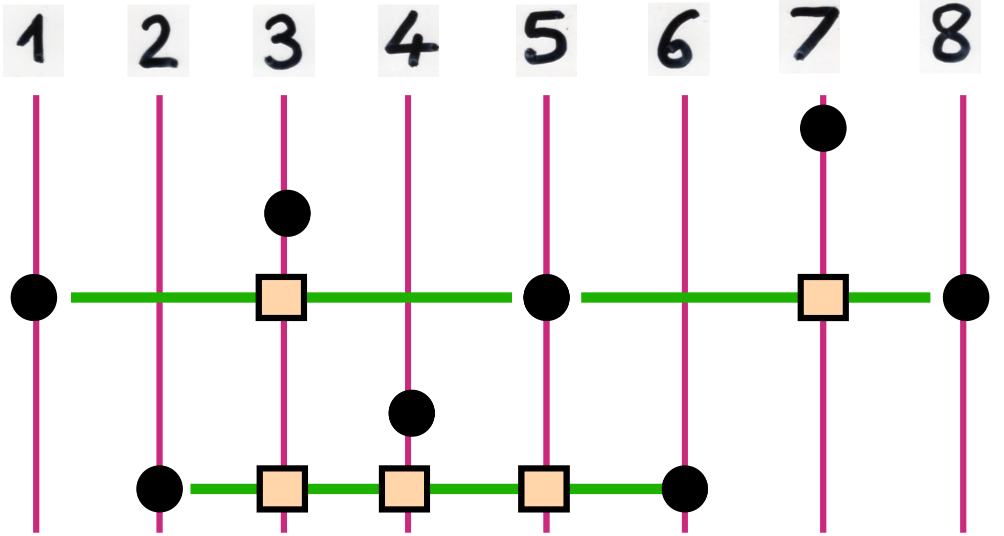

1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4

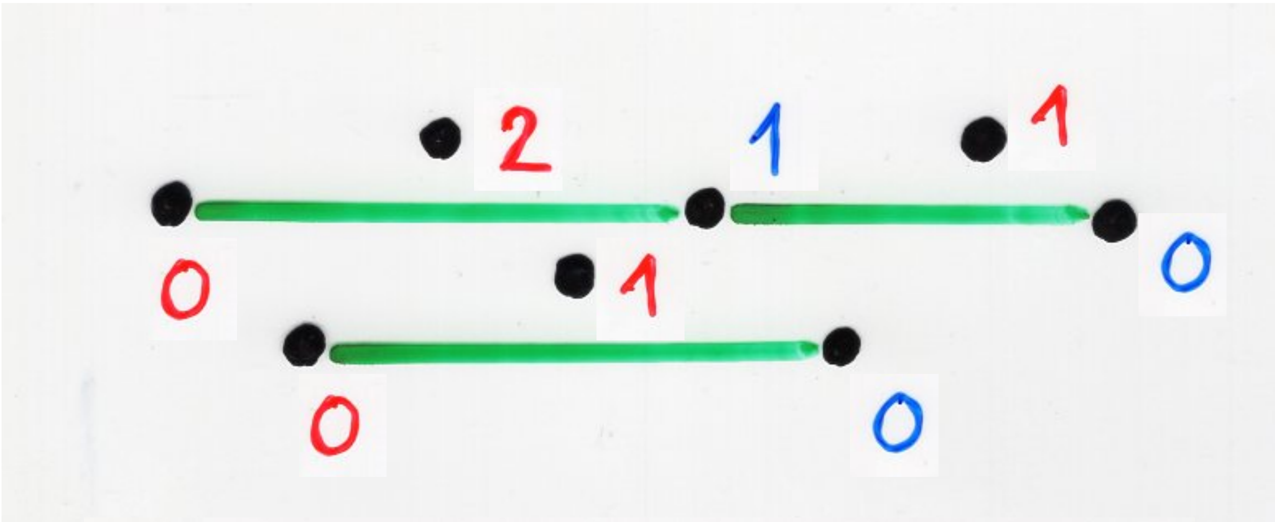

6 2 7 3 5 1 8 4

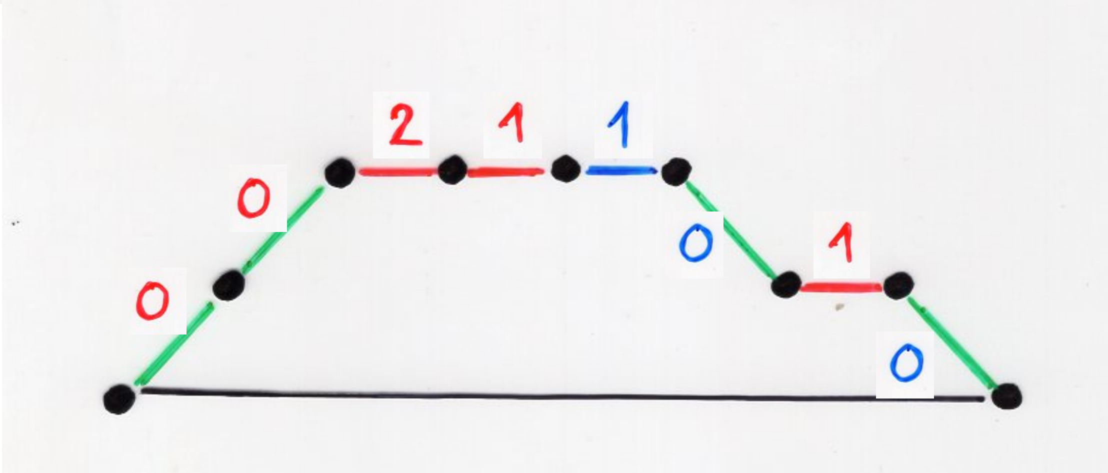

1
2
3
4
5
6
7
8


2
1
1
0
1
0
0
0


K
K
J^b
A
A
S^b
K
S^b


1
2
3
4
5
6
7
8


2
1
1
0
0
1
0
0

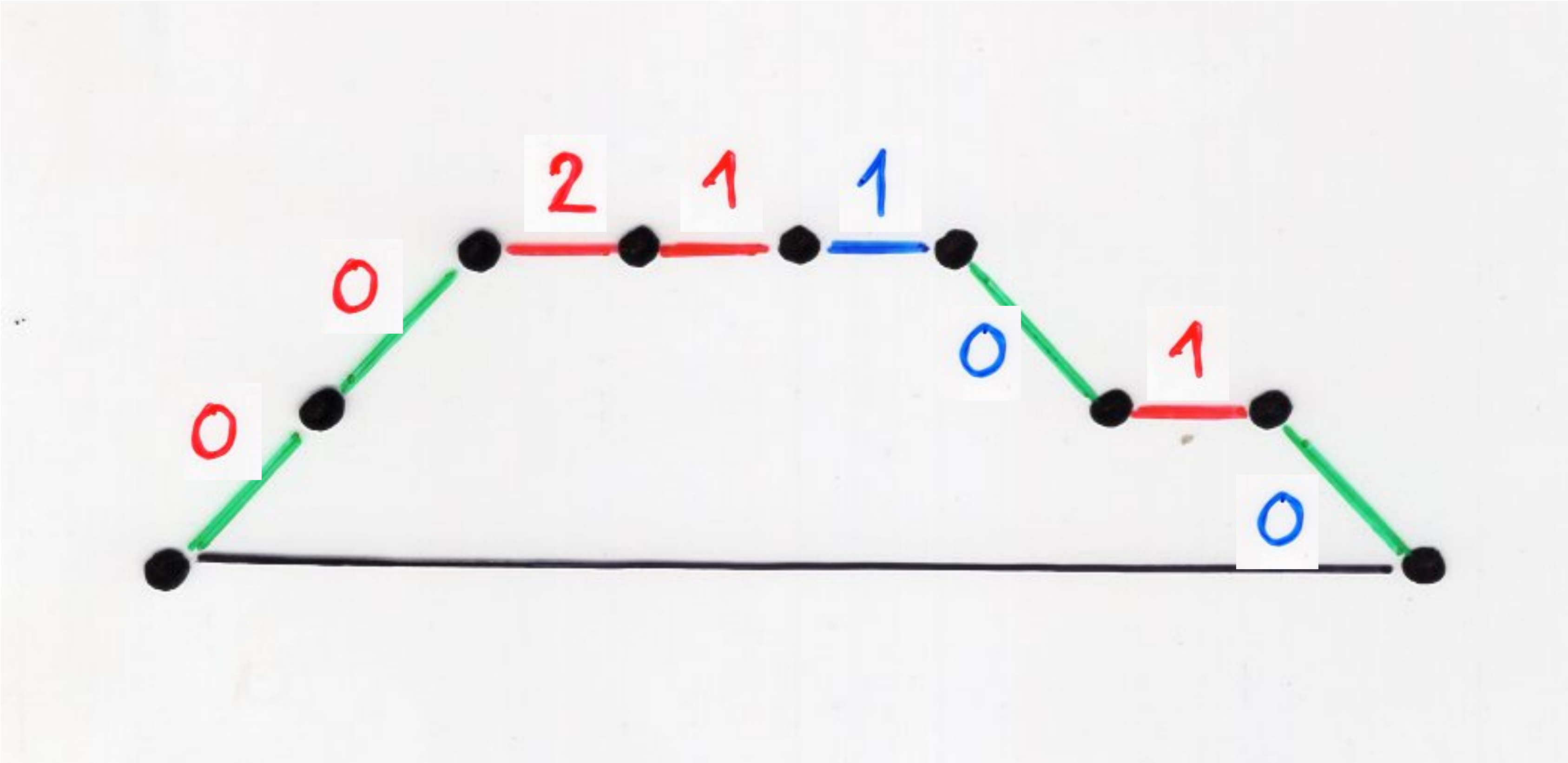

2
1
1
0
0
0
1
0

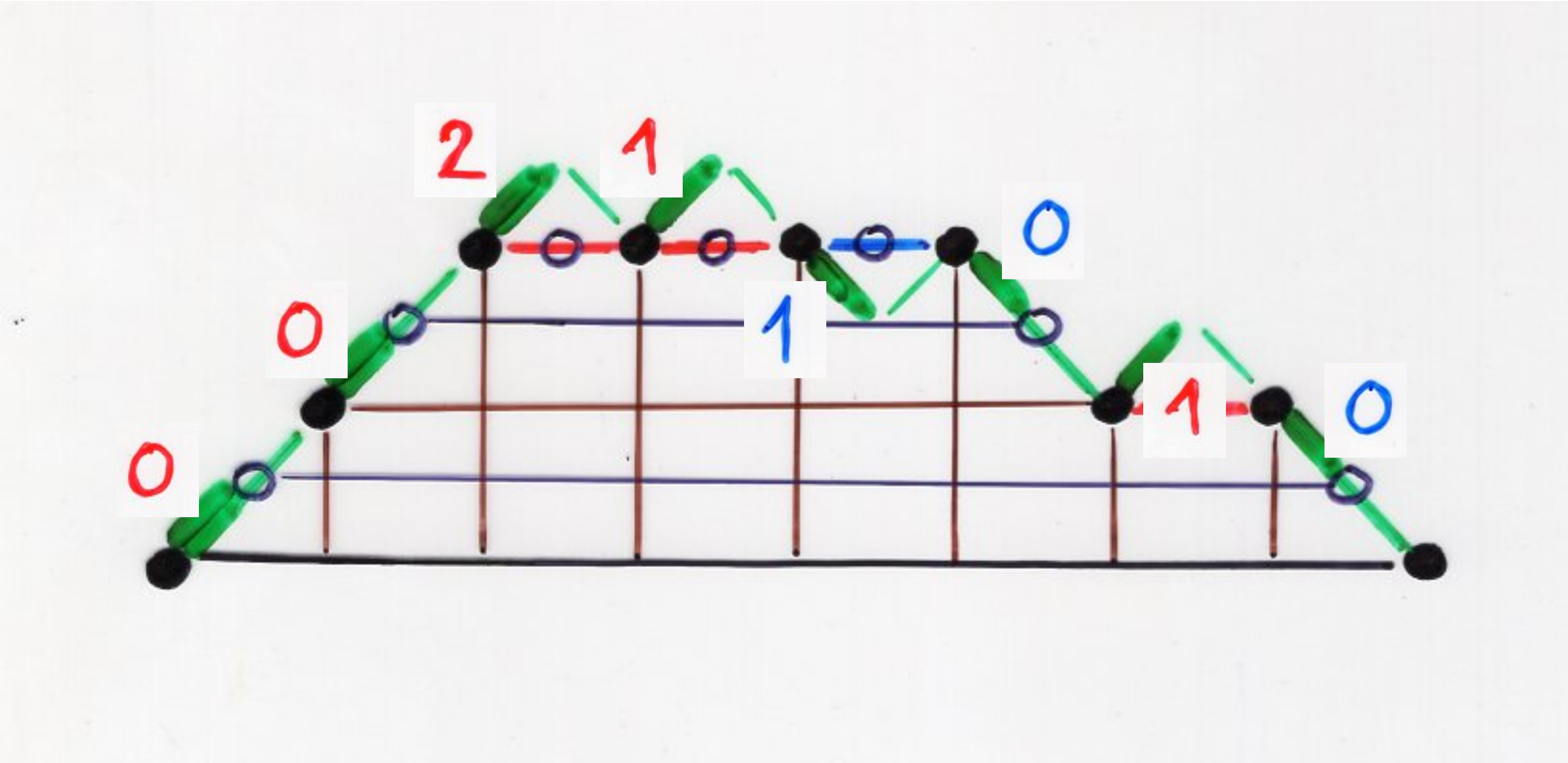
Same q-parameter

q-(restricted) Laguerre histories

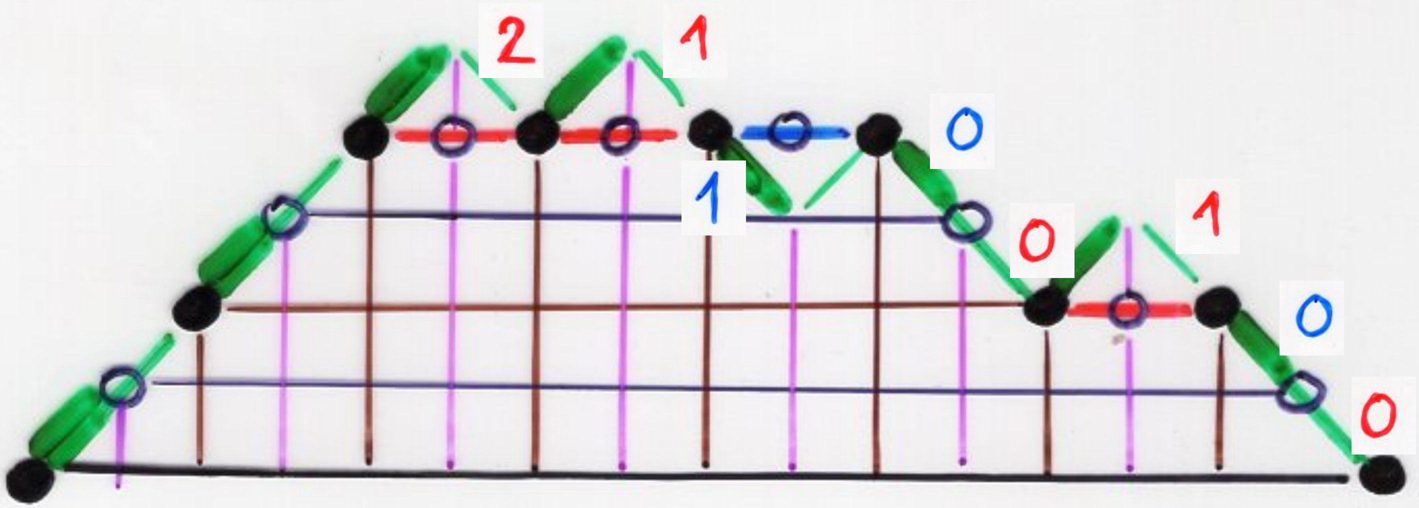
Number of crossing in

subdivided Laguerre histories

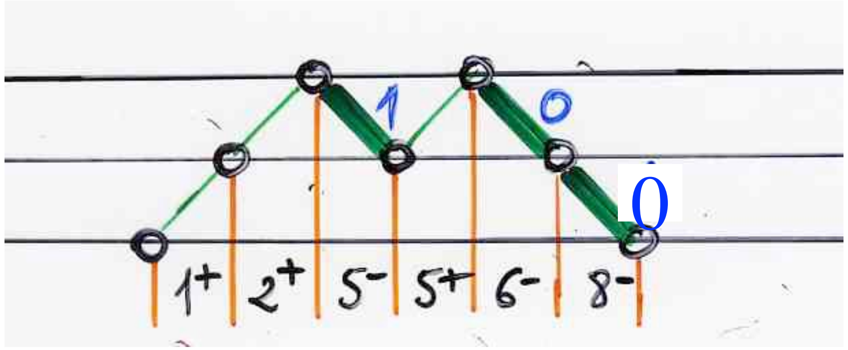

0
0
2
1
1
0
1
0



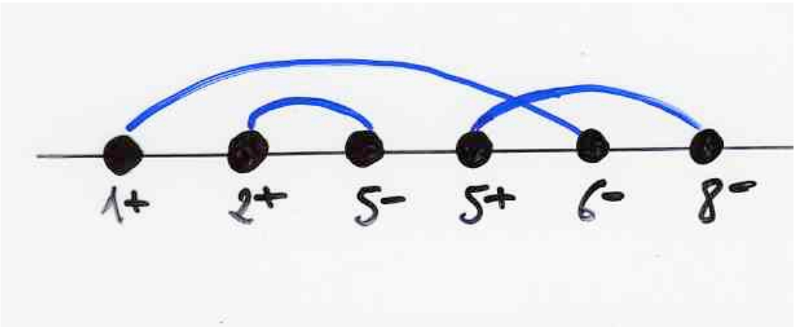
2
1
0
0
1
0
1
0

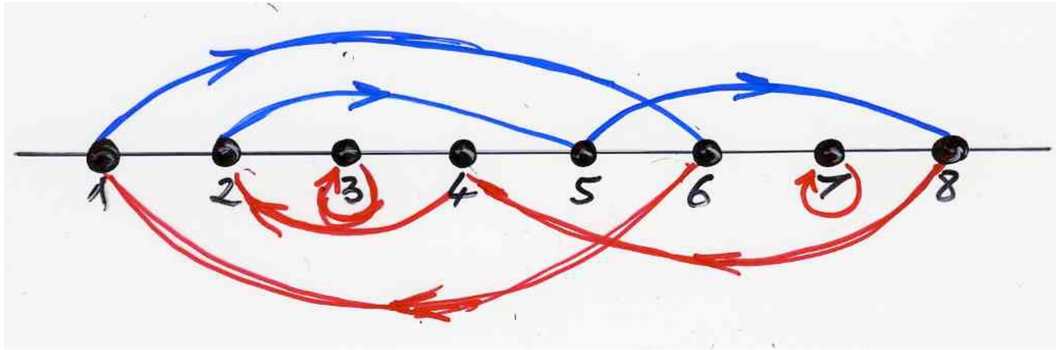


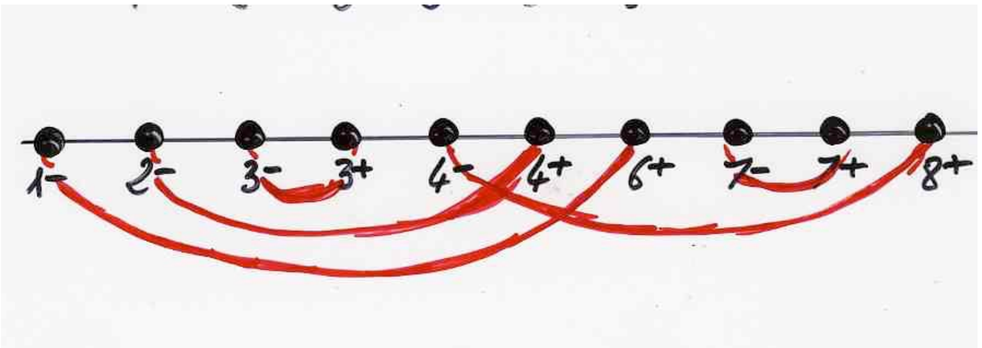
subdivided Laguerre history

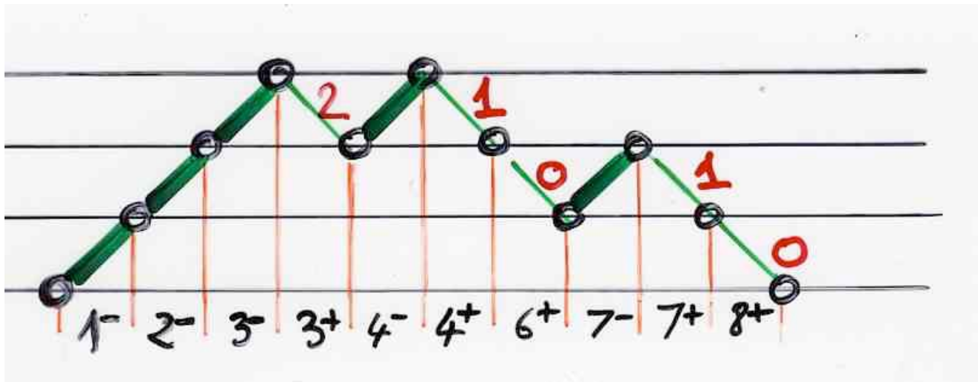


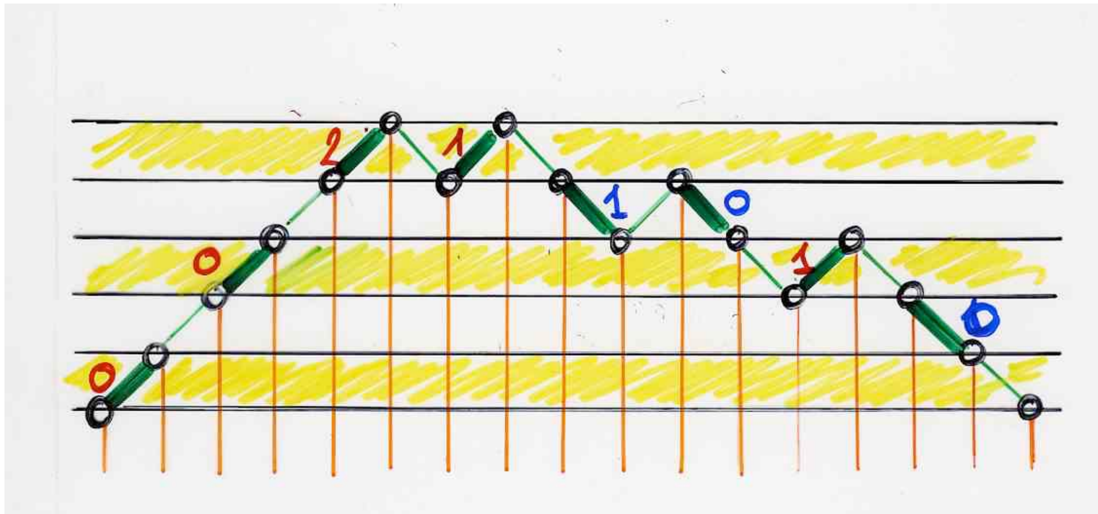
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$



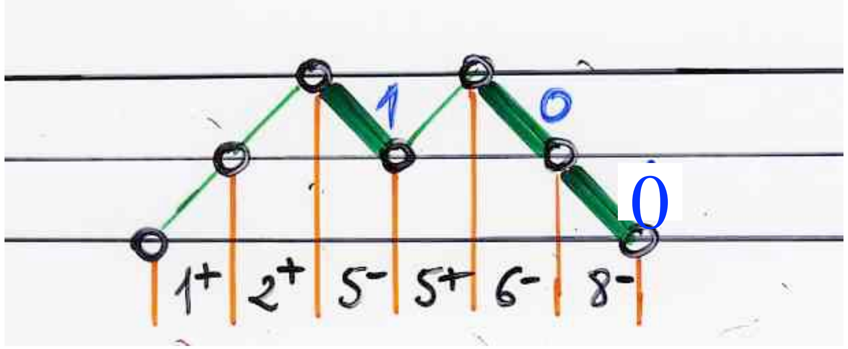




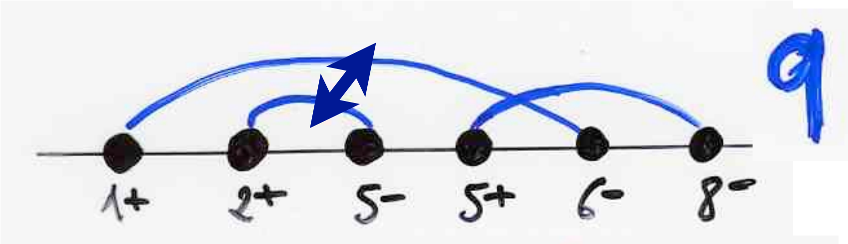


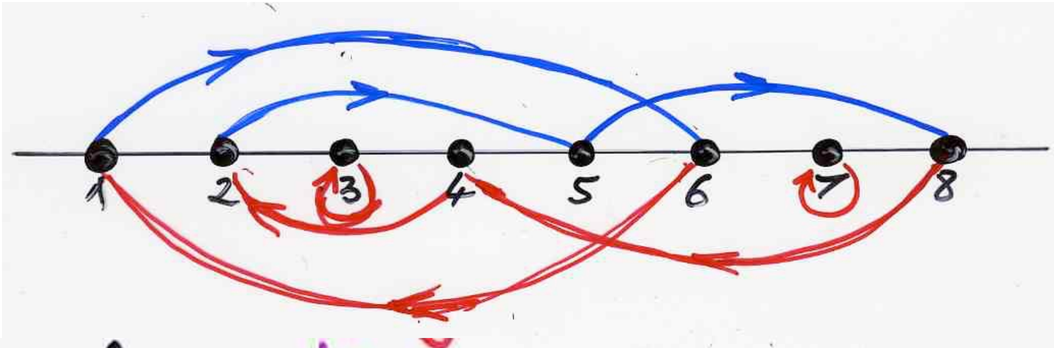


subdivided Laguerre history

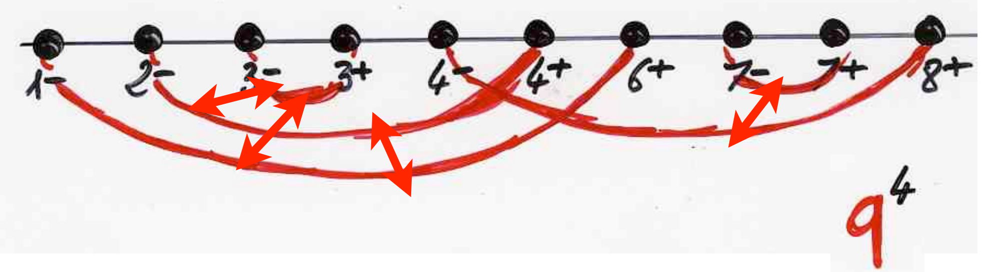


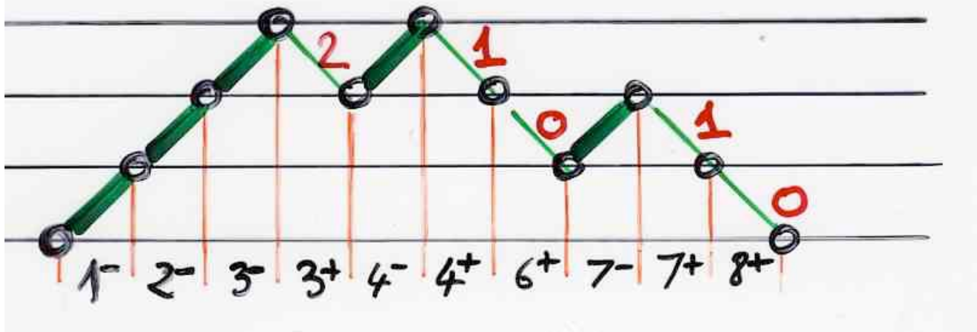
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

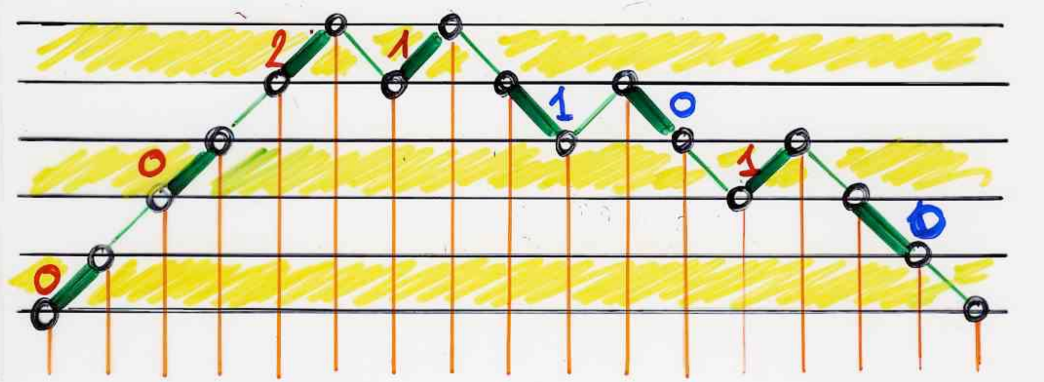




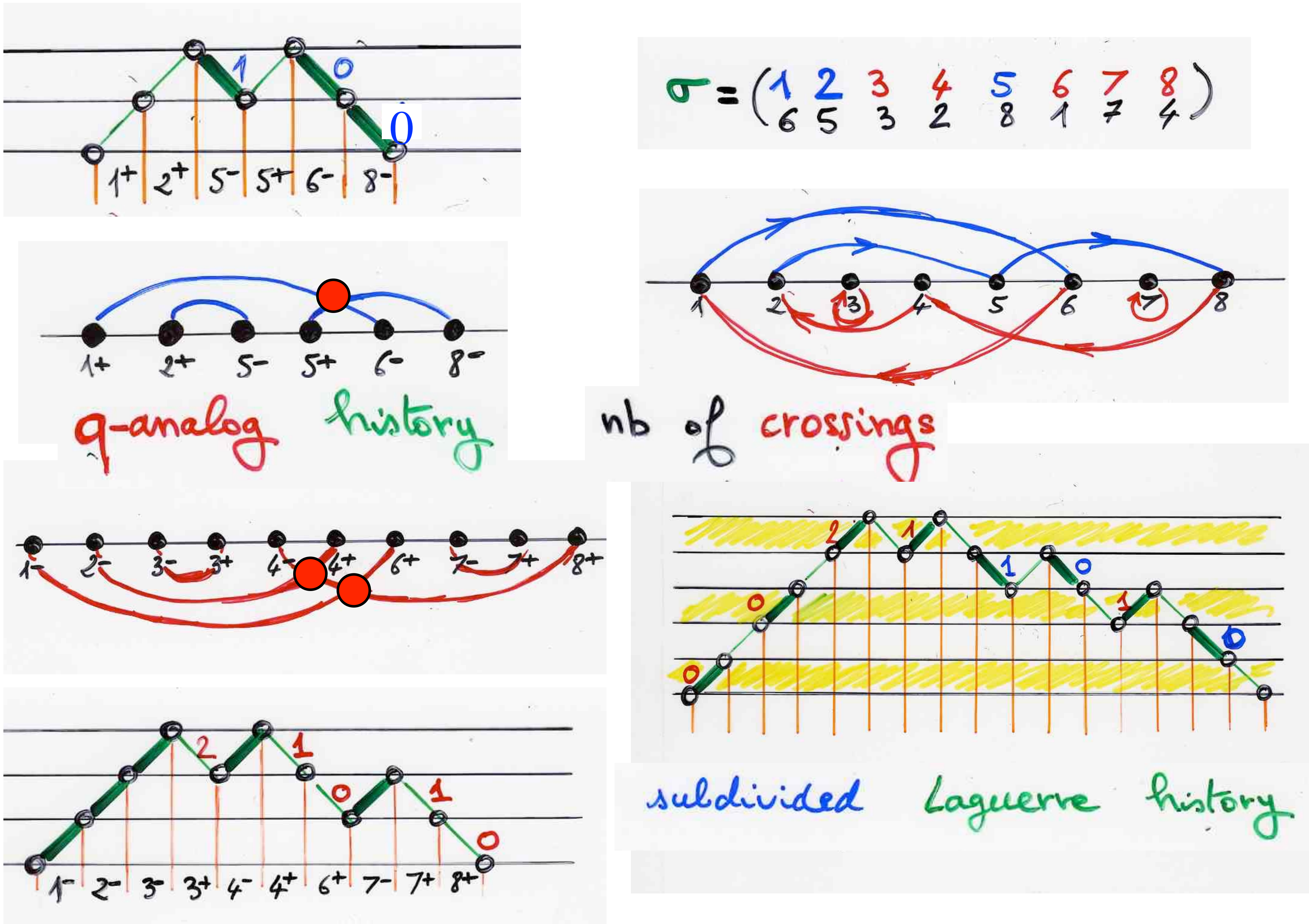
q-analog history nb of nestings





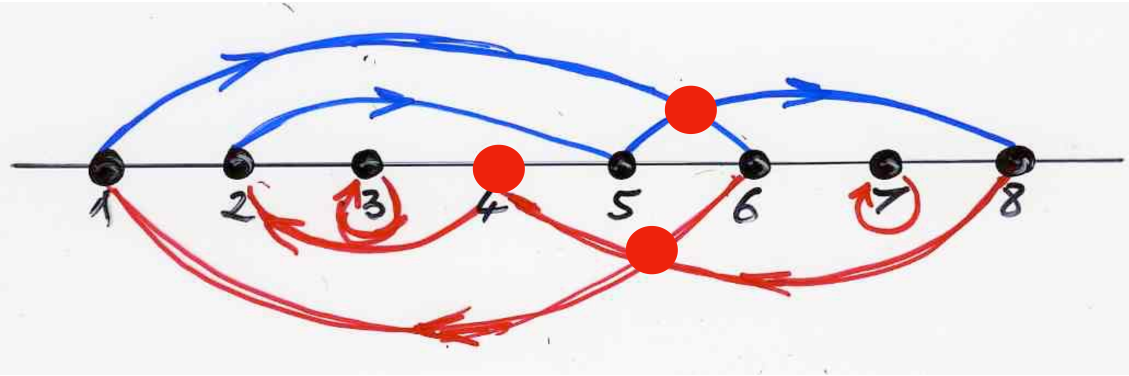


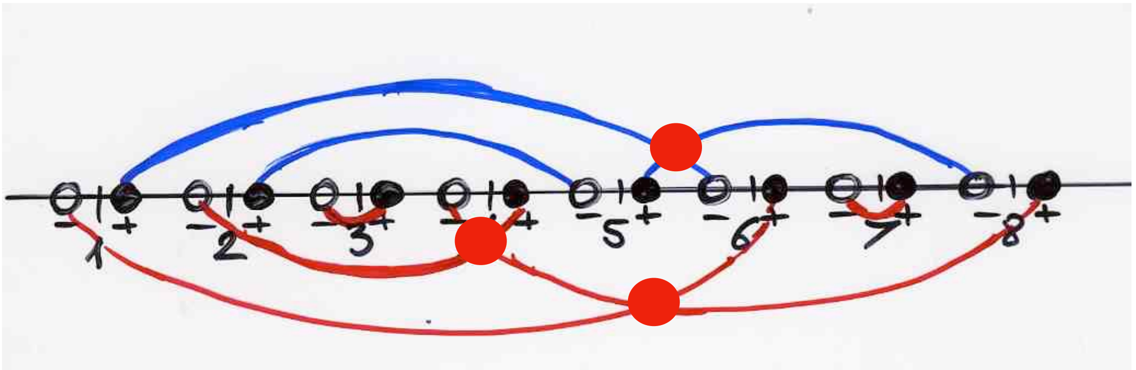
subdivided Laguerre history

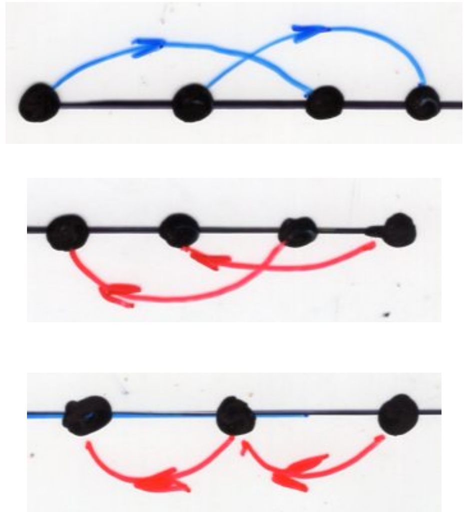


1+ 2+ 5- 5+ 6- 8-
σ = (1 2 3 4 5 6 7 8 / 6 5 3 2 8 1 7 4)
q-analog history
nb of crossings
1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+
subdivided Laguerre history

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$







number of crossings of a permutation

Corteel (2007)

Corteel, Williams (2007)

q with permutation tableaux

Steingrimsson, Williams (2007)

bijection

permutation tableaux → permutations

q number of crossing

What about

the triple of parameters (alpha, beta; q) ?

$\alpha = 0 \qquad L_n^{(0)}(x) \qquad \sigma(1) = n+1$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

3-terms linear recurrence relation

$\mu_n = n!$

$\beta = \alpha + 1 \qquad \alpha = 0$

$a_k = k + \beta \qquad \begin{cases} b'_k = k \\ b''_k = k + \beta \end{cases} \qquad c_k = k$

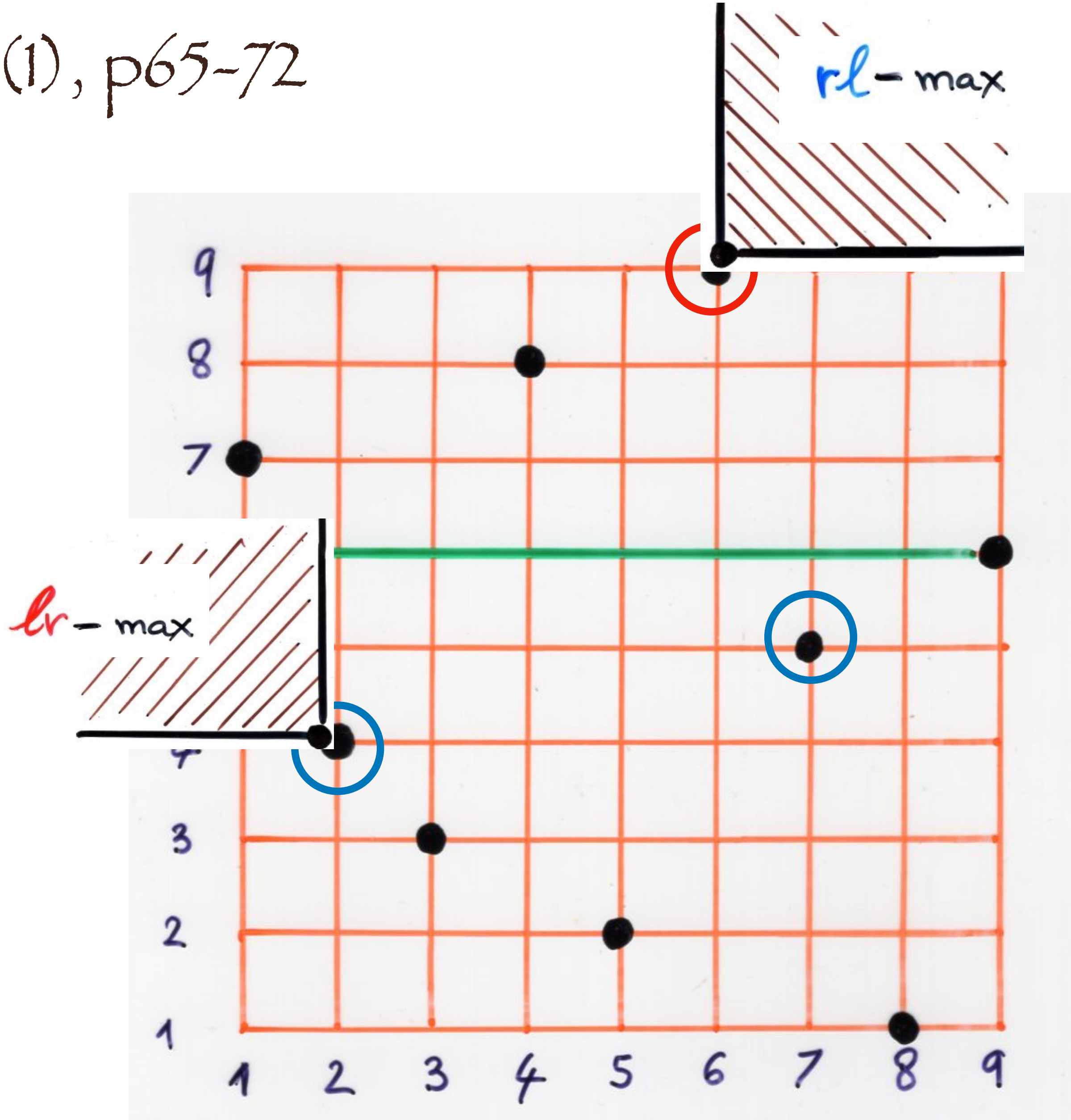
$(k \geq 0) \qquad\qquad (k \geq 1)$

For restricted Laguerre histories put a weight β for each choice $p_i = 1$ with $\omega_i = \{$ or

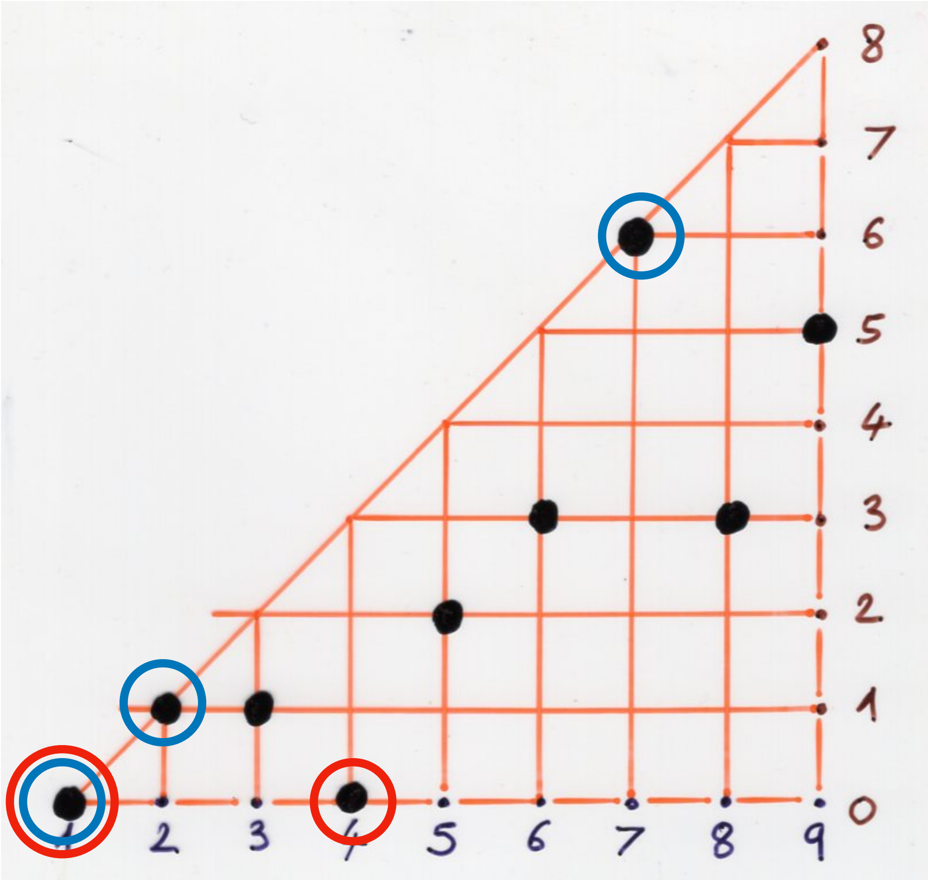
this is equivalent to say that the element i is a lr-min element of the permutation σ

Corollary $\qquad \mu_n = \beta(\beta+1)\cdots(\beta+n-1)$

Ch3c(1), p65-72



7 4 3 8 2 9 5 1 6



Ch3c(1), p65-72

generating polynomial:

$xy(x+y)(x+1+y)\ldots(x+n-1+y)$

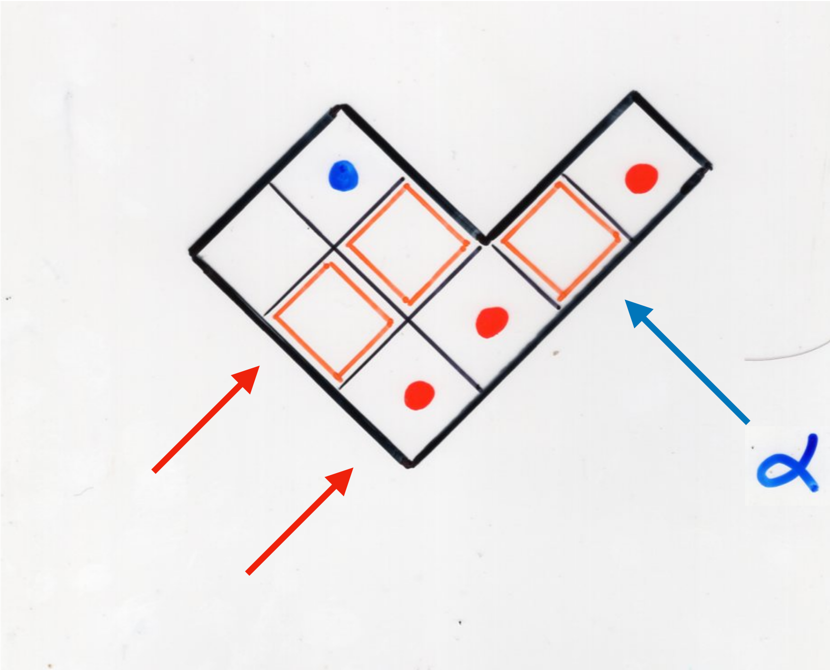
σ permutation = 7 2 3 9 6 8 5 1 4 word

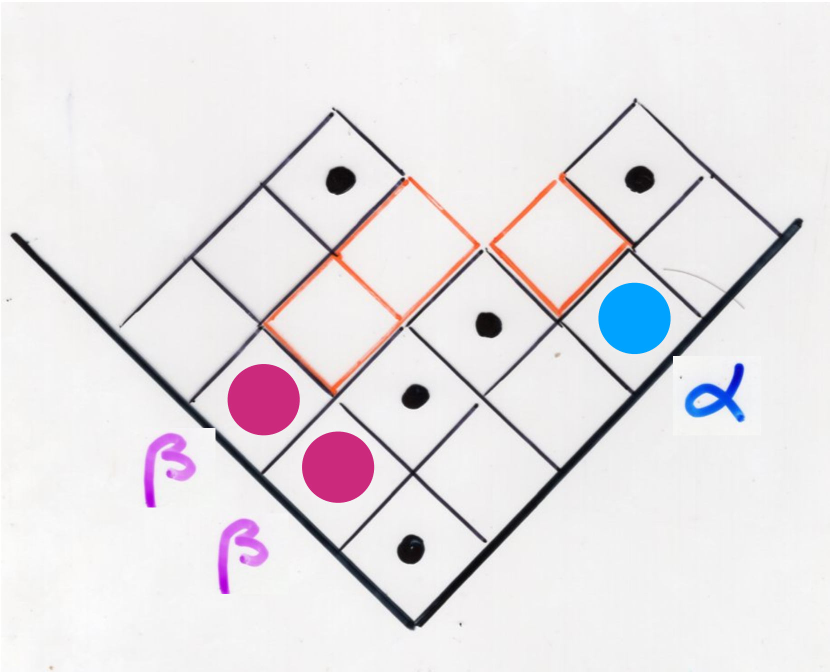
left-to-right
right-to-left

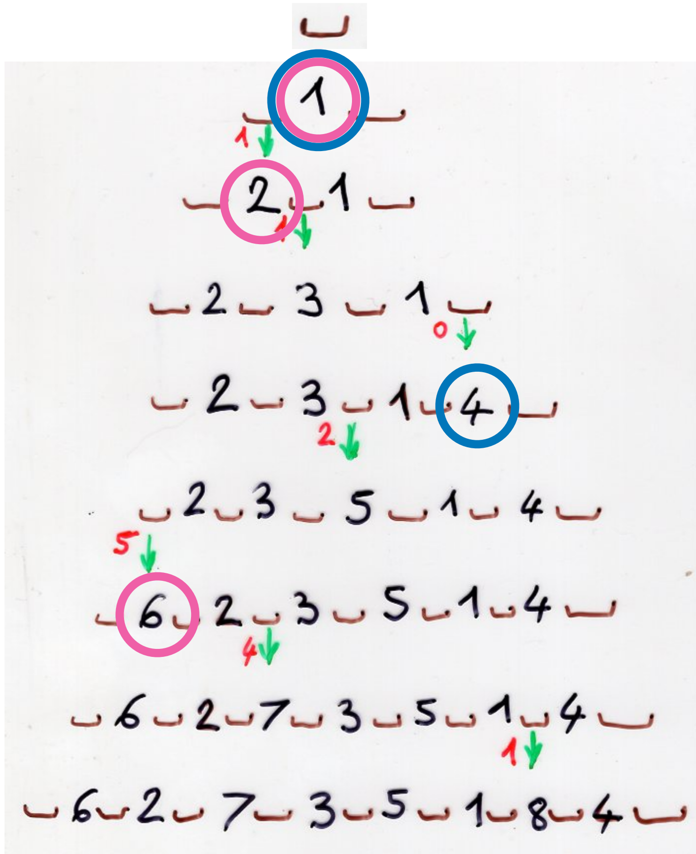
minimum

maximal

elements


α


α
β
β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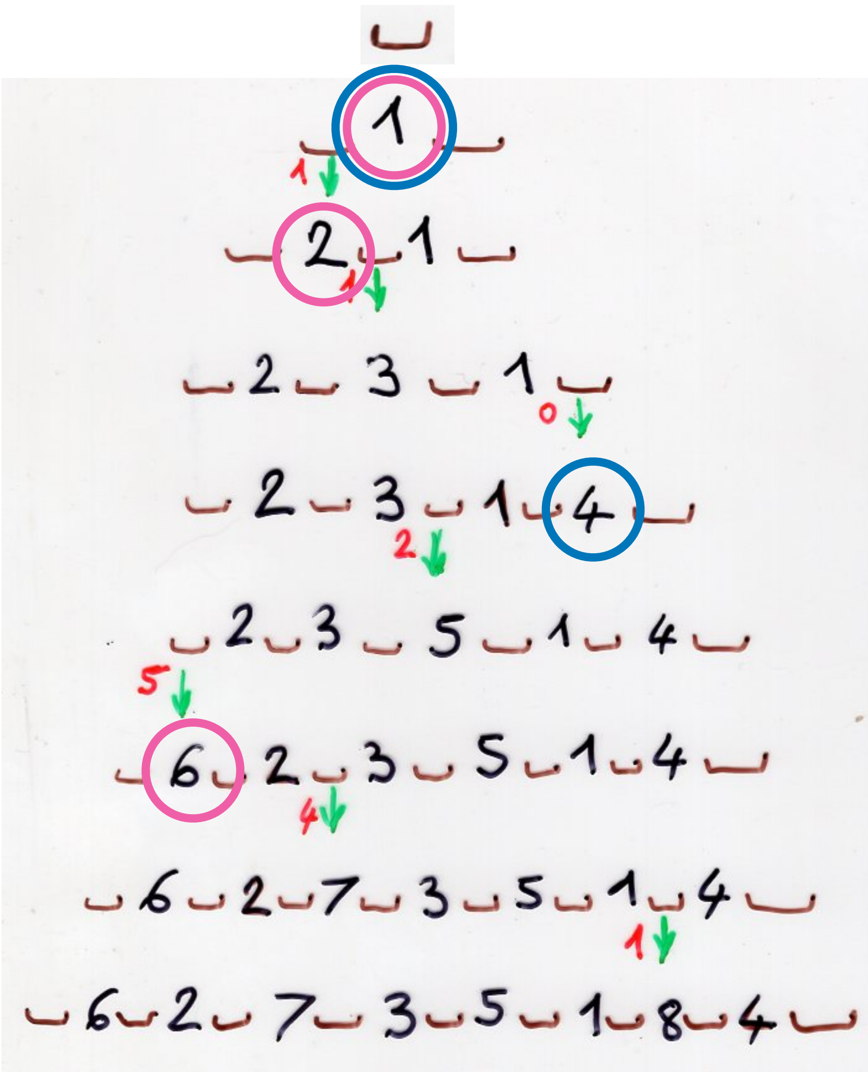
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4


1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
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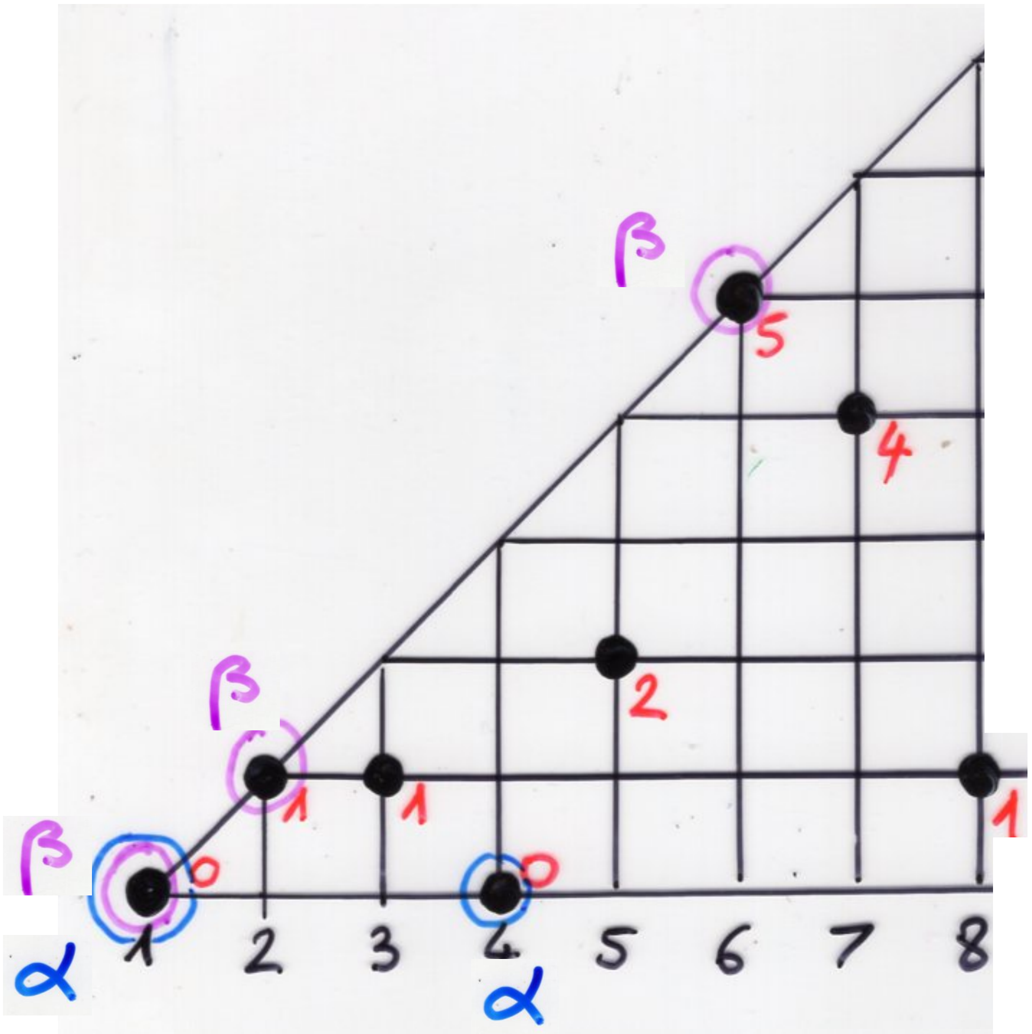
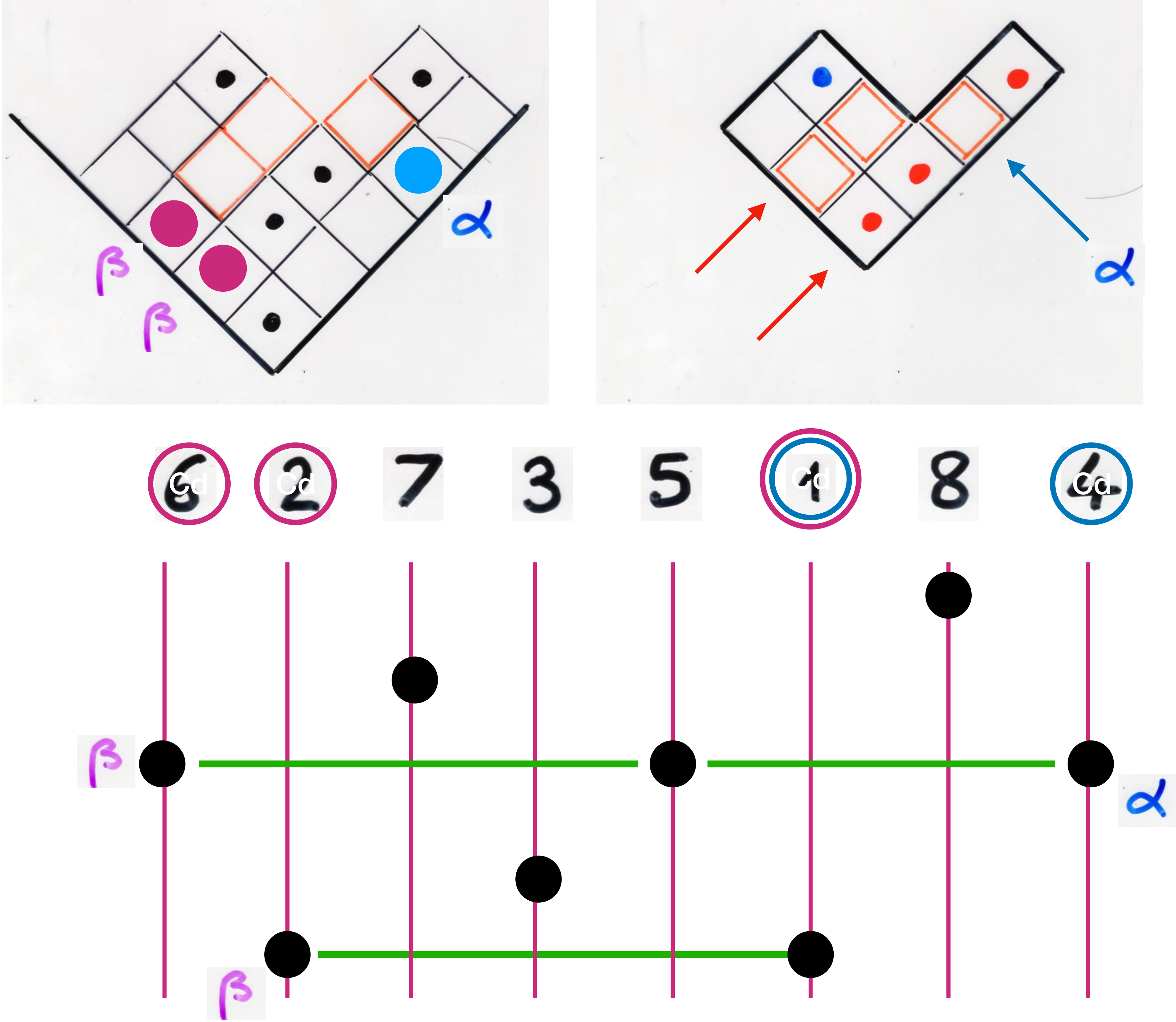
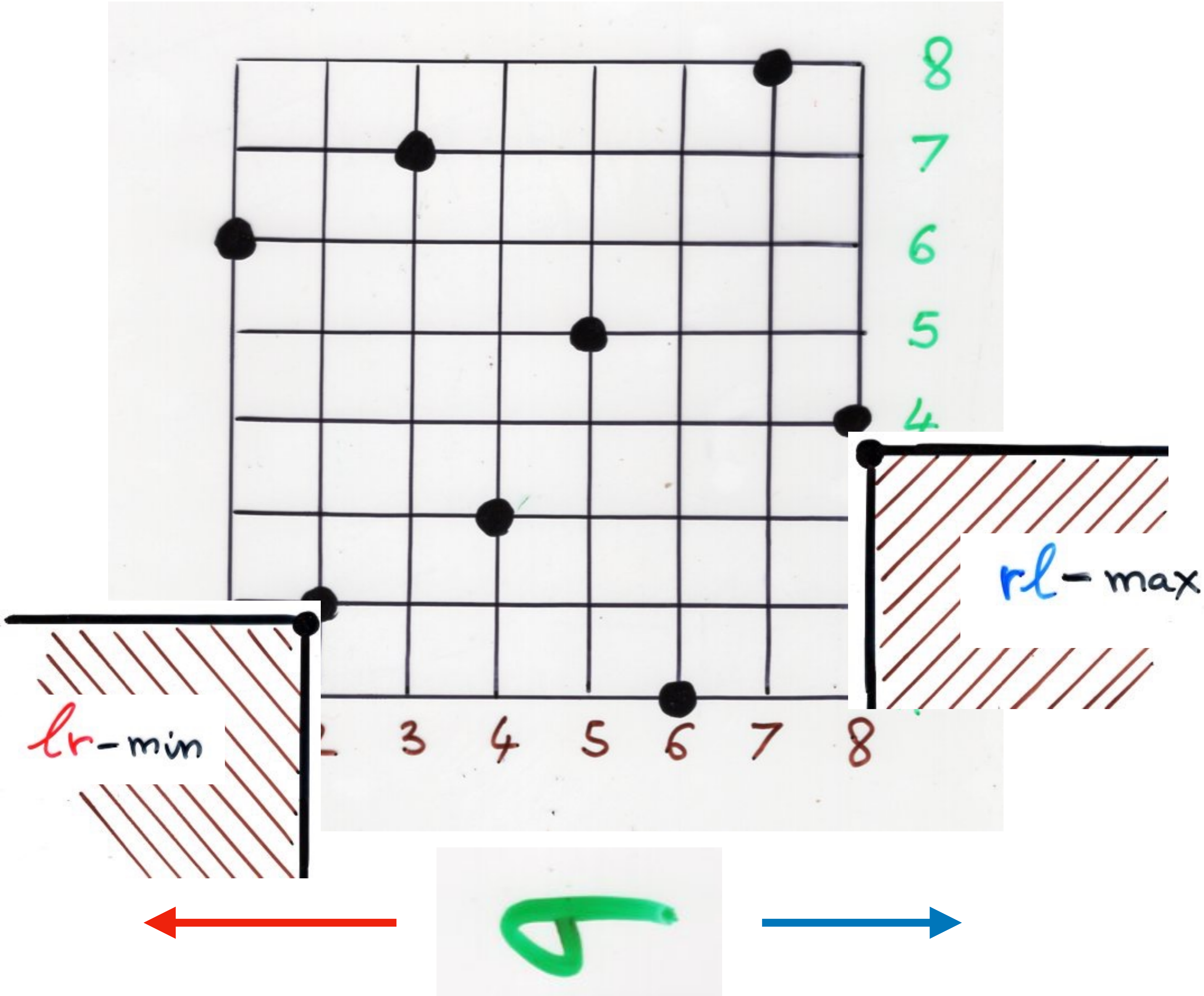
6 2 7 3 5 1 8 4

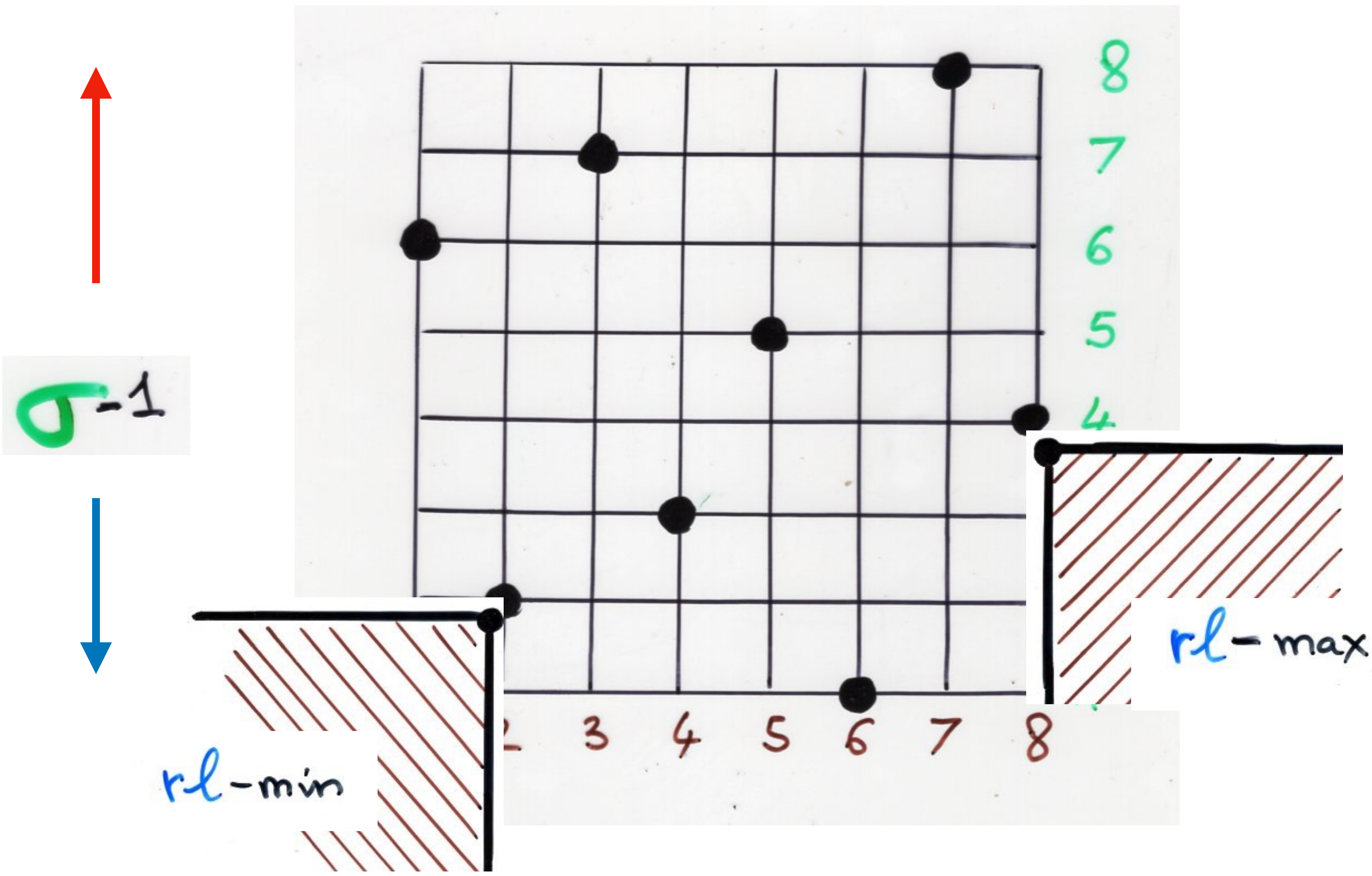

β
β
β
α
α
1 2 3 4 5 6 7 8



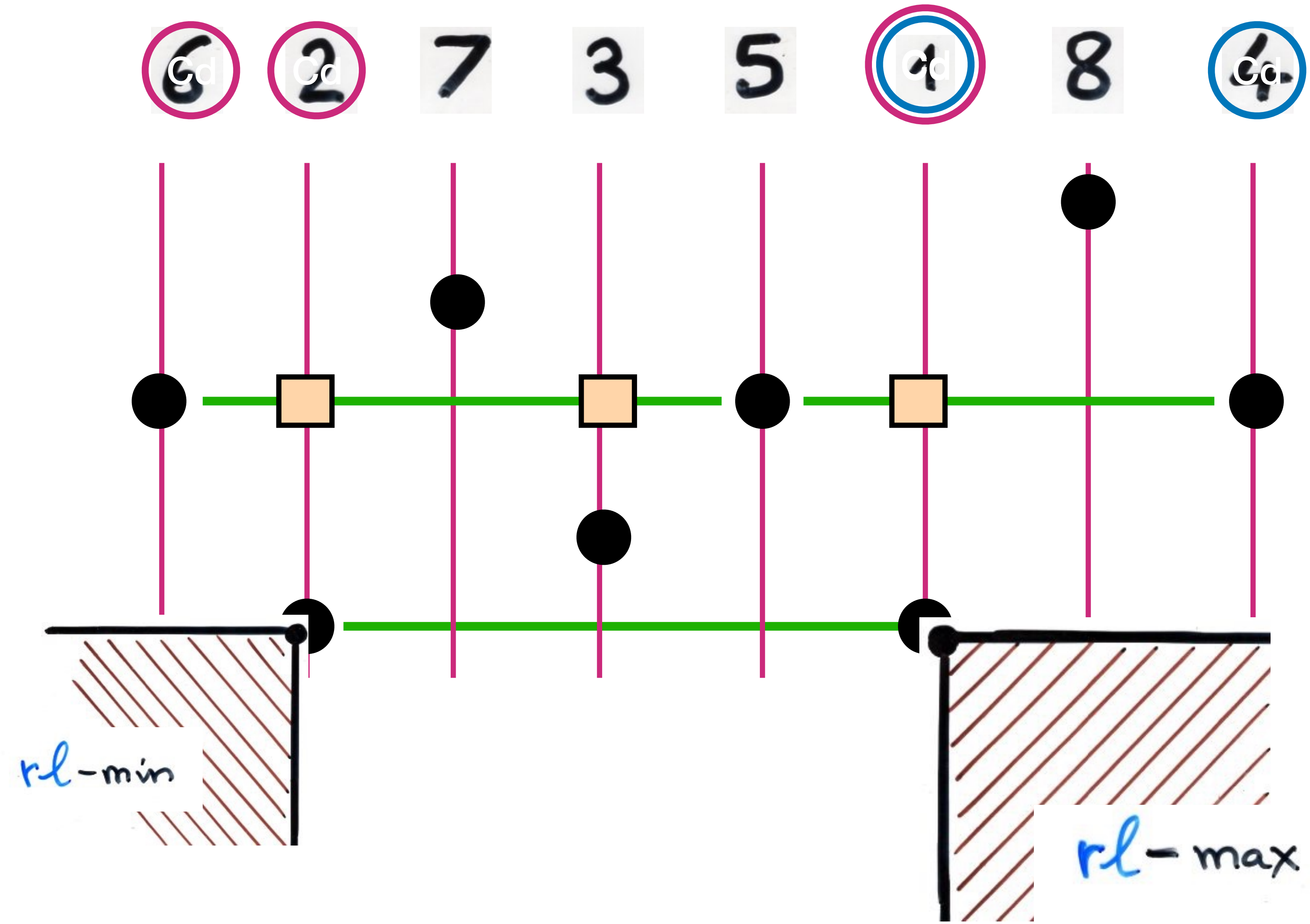
β
β
α
α
6
2
7
3
5
1
8
4
β
α
β



8
7
6
5
4
2 3 4 5 6 7 8
lr-min
rl-max



σ-1
8
7
6
5
4
2
3
4
5
6
7
8
rl-max
rl-min



6
2
7
3
5
1
8
4
rl-min
rl-max

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Al-Salam-Chihara polynomials

$$2x\,Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})\,Q_{n-1}(x)$$

What about

the approach by physicists ?

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

$$\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$$

$$\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$$

$$D = (D_{i,j})_{i,j \in \mathbb{N}}$$

$$E = (E_{i,j})_{i,j \in \mathbb{N}}$$

$$(1-q) D_{i,i} = 1 + \tilde{\beta} q^i$$

$$(1-q) D_{i,i+1} = 1 - \tilde{\alpha}\tilde{\beta} q^i$$

$$(1-q) E_{i,i} = 1 + \tilde{\alpha} q^i$$

$$(1-q) E_{i+1,i} = 1 - q^{i+1}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(\ \ , q)\, B_n(\tilde{\alpha}, \tilde{\beta}, y, q)$$

$$R_{N,n}(\ \ q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha}, \tilde{\beta}, \ \ q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (\ \ \tilde{\beta})^{n-k}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(y,q) B_n(\tilde{\alpha},\tilde{\beta},y,q)$$

$$R_{N,n}(1,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha},\tilde{\beta},y,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (y\,\tilde{\beta})^{n-k}$$

Josuat-Vergès (2011)

$$R_{N,n}(y,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-y)^i q^{\binom{i+1}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \sum_{j=0}^{N-n-2i} y^j \left(\binom{N}{j}\binom{N}{n+2i+j} - \binom{N}{j-1}\binom{N}{n+2i+j+1} \right)$$

Complements

q-Hermite and q-Laguerre
second kind

recalling: q-Hermite I

q-Hermite

$H_n^I(x;q)$ $\quad b_k = 0 \quad \lambda_k = [k]_q$

$= 1+q+\dots+q^{k-1}$

h histoire d'Hermite $\rightarrow$ τ involution sans point fixe

$\mathrm{som}(h) = \mathrm{pbr}(\tau)$

paires imbriquées (nesting)

$\mathrm{csom}(h) = \mathrm{cr}(\tau)$

croisements (crossing)

recalling: q-Hermite I

q-Hermite

$H_n^{I}(x;q)$

$b_k = 0 \qquad \lambda_k = [k]_q$

$= 1 + q + \dots + q^{k-1}$

$$\begin{cases} \mu^{I}_{2n+1,q} = 0 \\ \mu^{I}_{2n,q} = \dfrac{1}{(1-q)^n} \displaystyle\sum_{j=0}^{n} (-1)^j \, t_{n,j} \, q^{j(j+1)/2} \end{cases}$$

$$t_{n,j} = \binom{2n}{n-j} - \binom{2n}{n+j+1}$$

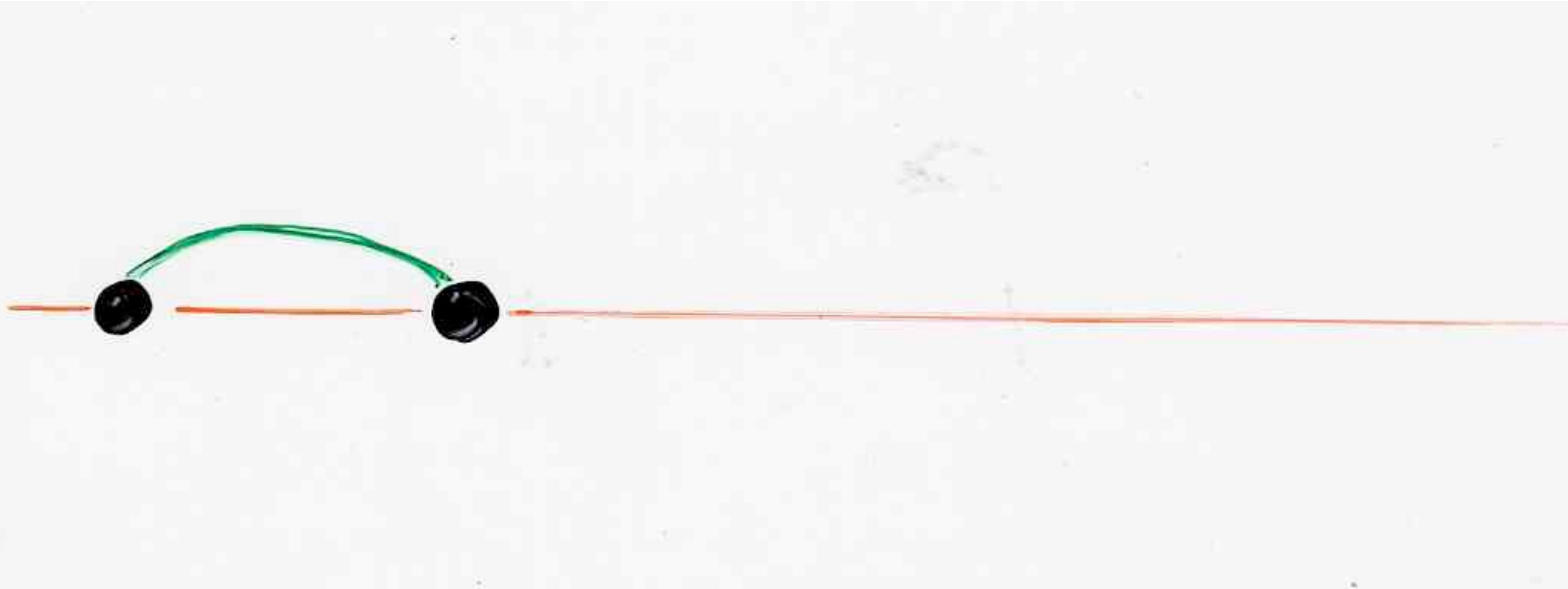
Riordan (1975) Touchard (1952)

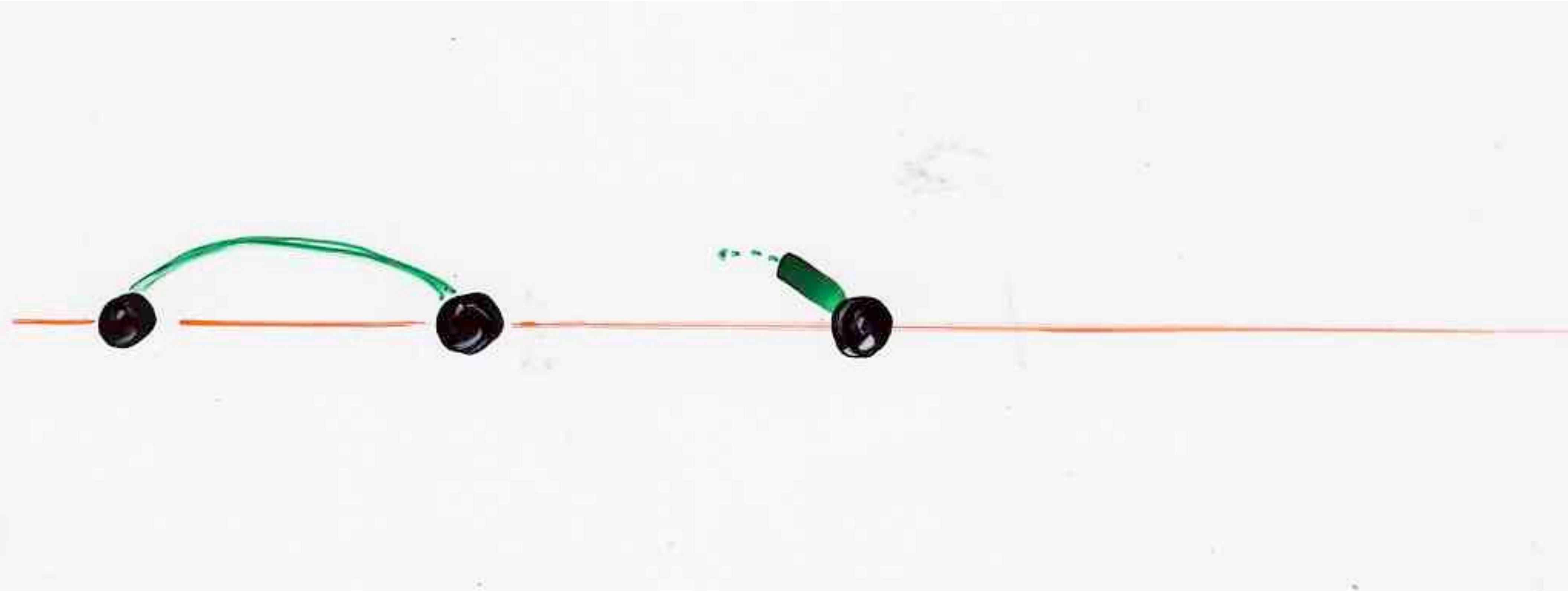
Penaud (1995)

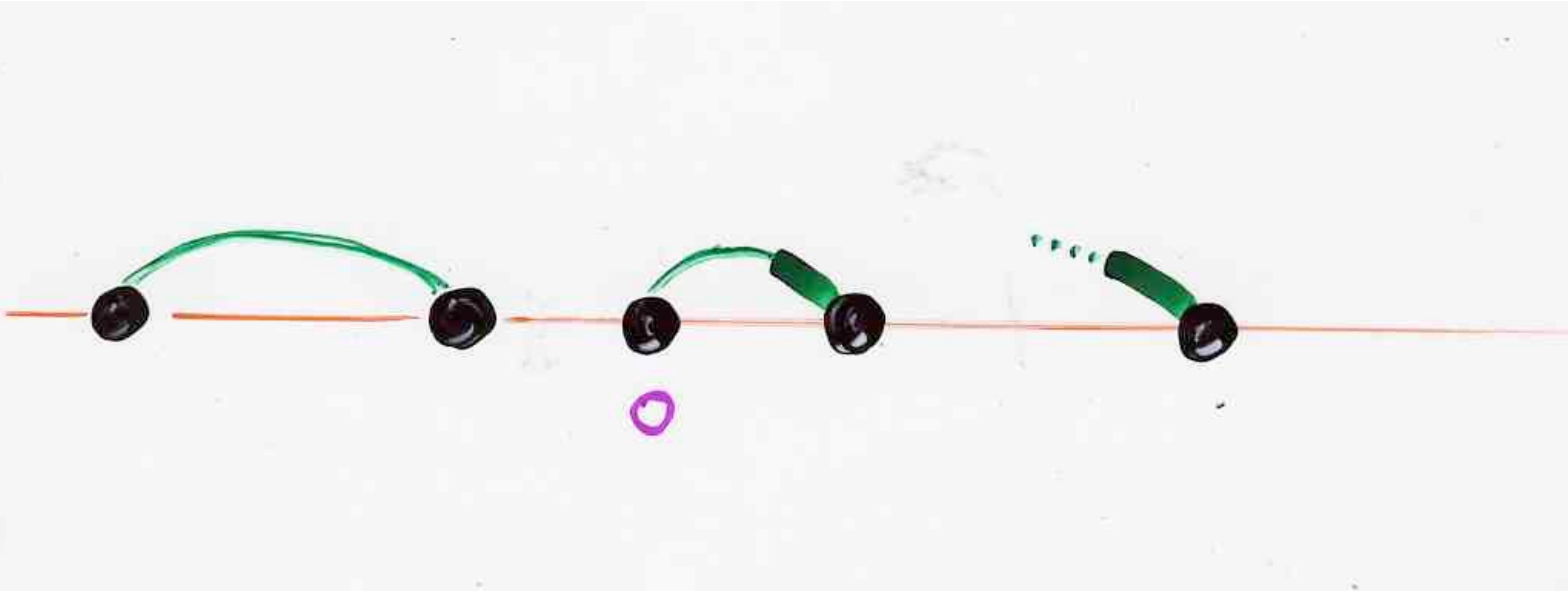
q-Hermite II

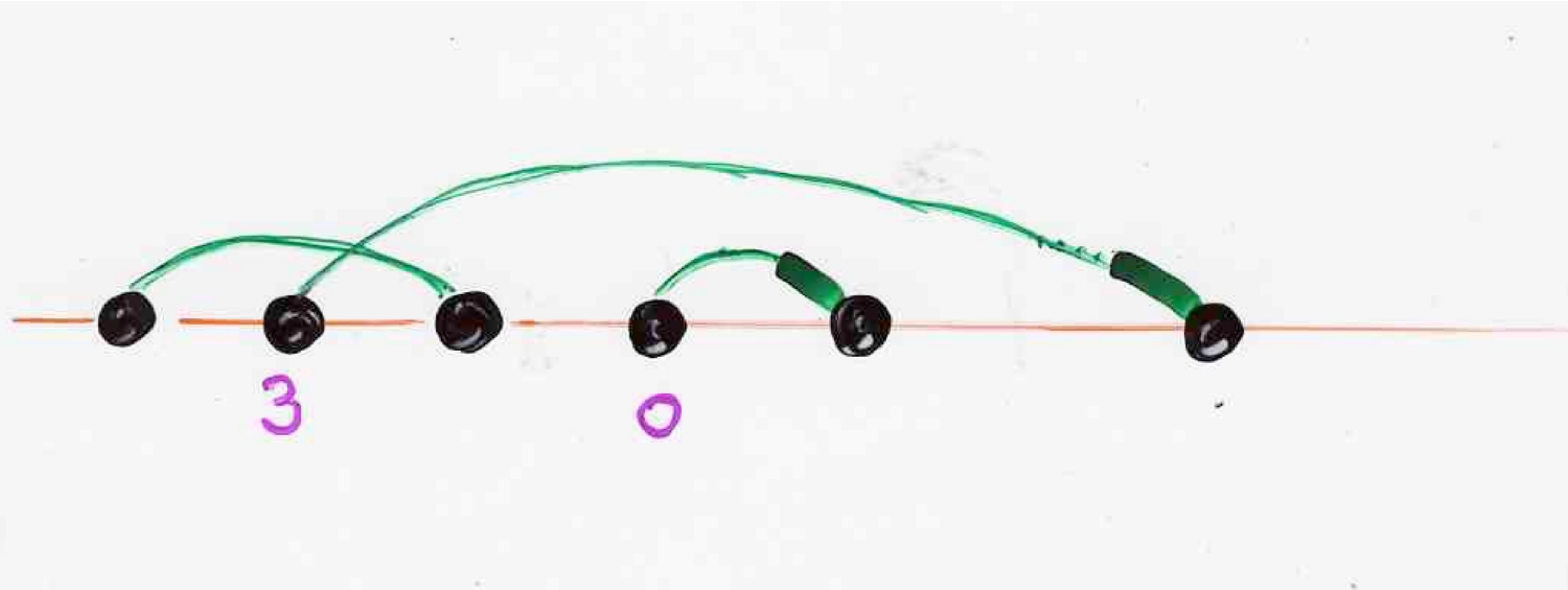
$H_n^{II}(x;q) \qquad b_k = 0 \qquad \lambda_k = q^{k-1}[k]_q$

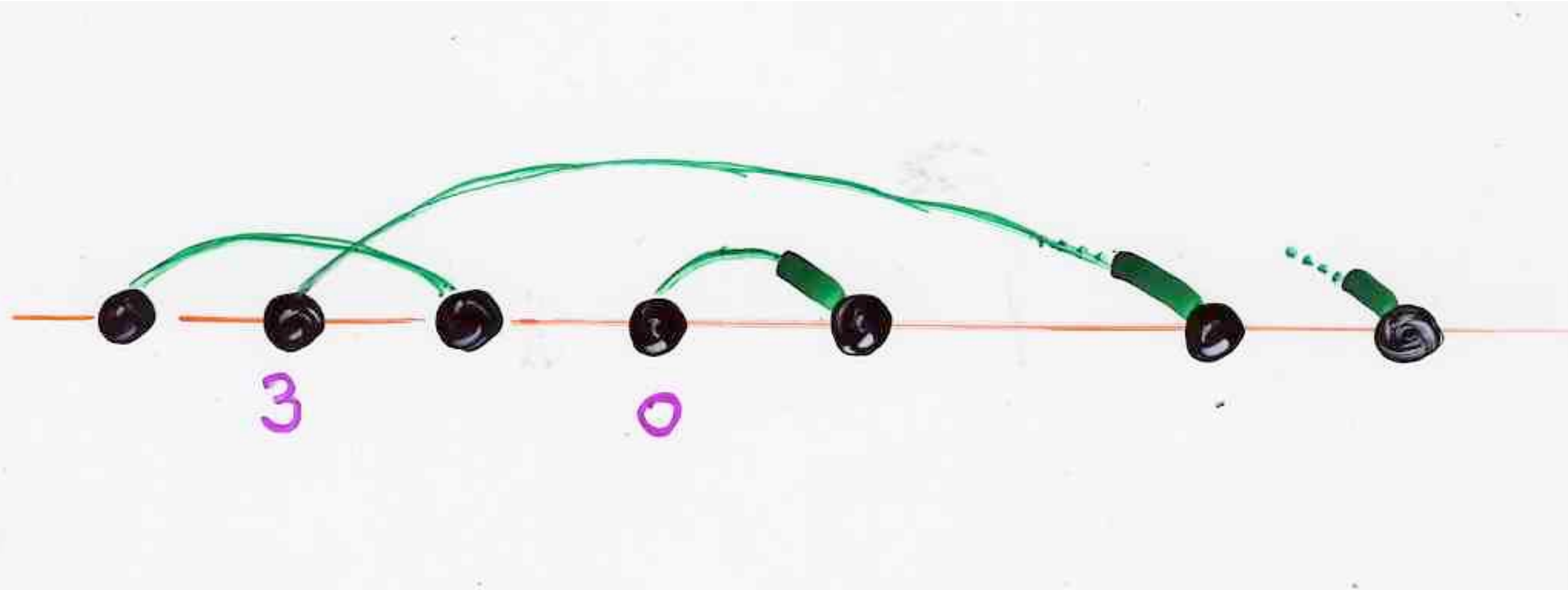
$$\begin{cases} \mu^{II}_{2n+1,q} = 0 \\ \mu^{II}_{2n,q} = [1]_q \cdot [3]_q \cdots [2n-1]_q \end{cases}$$

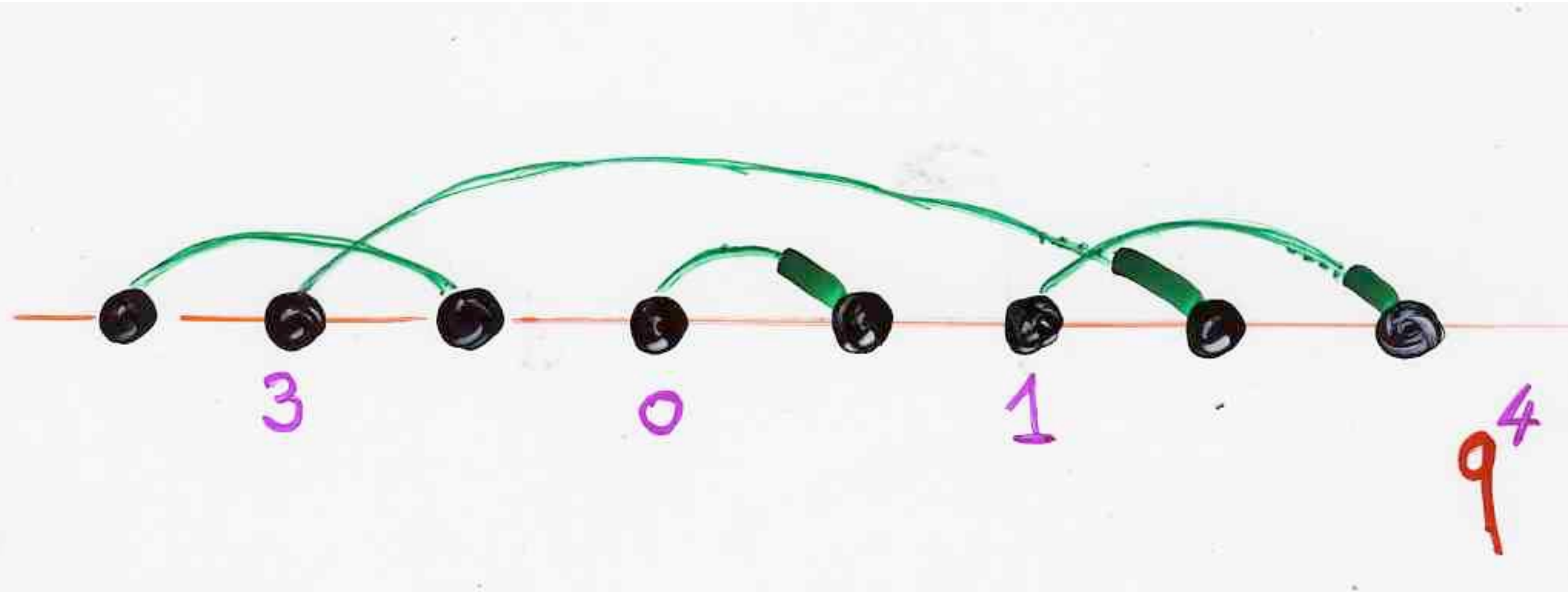







3
0


3
0



$$\mathrm{Inv}(\tau) = \mathrm{cr}(\tau) + 2\,\mathrm{nest}(\tau)$$

$$H_n^{II}(x;q) \qquad b_k = 0 \qquad \lambda_k = q^{k-1}[k]_q$$

$$\mathrm{Inv}(\tau) = \mathrm{cr}(\tau) + 2\,\mathrm{nest}(\tau)$$

$$\begin{cases} \mu^{II}_{2n+1,q} = 0 \\ \mu^{II}_{2n,q} = [1]_q \cdot [3]_q \cdots [2n-1]_q \end{cases}$$

q - Laguerre II

if $\mu_n = [n!]_q$

then
$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

→ subdivided Laguerre histories

A. de Médicis, X.V. (1994)

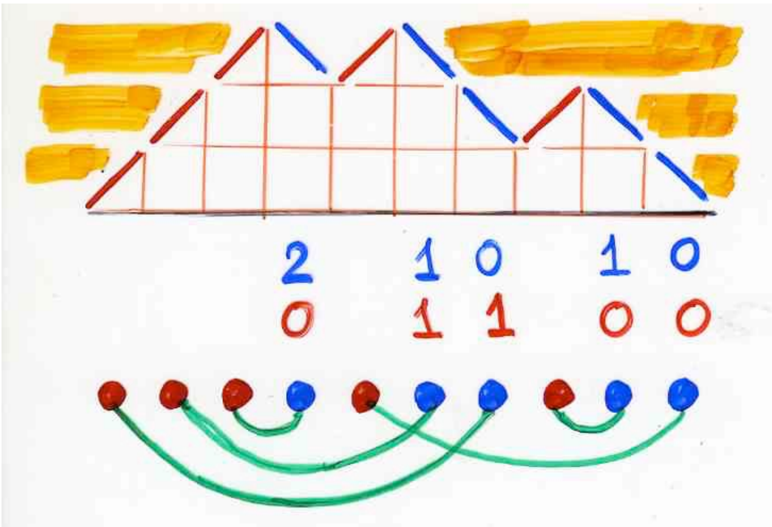
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

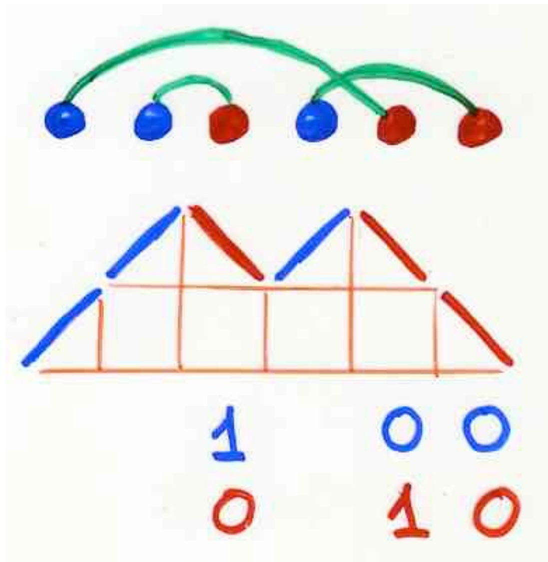
Inv 5 4 2 1 3 1 → 16

$$\mathrm{Inv}(\sigma) = \mathrm{exc}(\sigma) + \mathrm{Inv}(\tau_{exc}) + \mathrm{Inv}(\tau_{nex})$$

3 1+2 2+8 → 16

$$\mathrm{Inv}(\tau) = \mathrm{cr}(\tau) + 2\,\mathrm{nest}(\tau)$$





q-Laguerre II

$$\mu^{(\beta)}_{n,q} = [n;\beta]_q !$$

$$= [1;\beta]_q [2;\beta]_q \cdots [n;\beta]_q$$

q-Laguerre II

$$L^{II\,\beta}_n(x;q) \qquad \begin{cases} b^{(\beta)}_{k,q} = q^k\left([k]_q + [k+1;\beta]_q\right) \\ \lambda^{(\beta)}_{k,q} = q^{2k-1}[k]_q \cdot [k;\beta]_q \end{cases}$$

$$\mu^{(\beta)}_{n,q} = [n;\beta]_q!$$

$$= [1;\beta]_q\,[2;\beta]_q \cdots [n;\beta]_q$$

PASEP
with 5 parameters

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009)

staircase tableaux

Z_n partition function

Askey-Wilson polynomials

$$P_n(x) = P_n(x; a, b, c, d \mid q)$$

$$P_n(x) = a^{-n}(ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k}$$

$$(a_1, a_2, \ldots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k)$$

${}_4\phi_3$ basic hypergeometric function

$$A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) = 2x P_n(x)$$

$$A_n = \frac{1 - q^{n-1} abcd}{(1 - q^{2n-1} abcd)(1 - q^{2n} abcd)}$$

$$B_n = \frac{q^{n-1}}{(1 - q^{2n-2} abcd)(1 - q^{2n} abcd)}$$

$$\left[(1 + q^{2n-1} abcd)(qs + abcd\, s') - q^{n-1}(1+q) abcd\, (s + qs') \right]$$

$$C_n = \frac{(1-q^n)(1-q^{n-1}ab)(1-q^{n-1}ac)(1-q^{n-1}ad)(1-q^{n-1}bc)(1-q^{n-1}bd)(1-q^{n-1}cd)}{(1 - q^{2n-2} abcd)(1 - q^{2n-1} abcd)}$$

$$s = a+b+c+d \qquad s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}$$

$$D = \frac{1}{1-q}(1+d)$$

$$E = \frac{1}{1-q}(1+e)$$

$$de - qed = (1-q)\,\mathrm{Id}$$

$$DE = qED + E + D$$

$$d = \begin{bmatrix} d_0^{\natural} & d_0^{\sharp} & 0 & & \\ d_0^{\flat} & d_1^{\natural} & d_1^{\sharp} & & 0 \\ 0 & d_1^{\flat} & d_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$e = \begin{bmatrix} e_0^{\natural} & e_0^{\sharp} & 0 & & \\ e_0^{\flat} & e_1^{\natural} & e_1^{\sharp} & & 0 \\ 0 & e_1^{\flat} & e_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$d^{4}_{k} = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[bd(a+c) + (b+d)q - abcd(b+d)q^{k-1} - \Big(bd(a+c) + (abcd)(b+d) \Big) q^{k} - bd(a+c)q^{k+1} + ab^{2}cd^{2}(a+c)q^{2k-1} + abcd(b+d)q^{2k} \Big]$$

$$e^{4}_{k} = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[ac(b+d) + (a+c)q - abcd(a+c)q^{k-1} - \Big(ac(b+d) + abcd(a+c) \Big) q^{k} - ac(b+d)q^{k+1} + a^{2}bc^{2}d(b+d)q^{2k-1} + abcd(a+c)q^{2k} \Big]$$

$$d_k^{\sharp} = \frac{1}{1 - q^k ac} \mathcal{A}_k$$

$$d_k^{\flat} = - \frac{q^k bd}{1 - q^k bd} \mathcal{A}_k$$

$$e_k^{\sharp} = - \frac{q^k ac}{1 - q^k ac} \mathcal{A}_k$$

$$e_k^{\flat} = \frac{1}{1 - q^k bd} \mathcal{A}_k$$

$$\mathcal{A}_k = \left[\frac{(1-q^{k-1}abcd)(1-q^{k+1})(1-q^{k}ab)(1-q^{k}ad)}{(1-q^{2k-1}abcd)(1-q^{2k}abcd)^2} \, \frac{(1-q^{k}ac)(1-q^{k}bc)(1-q^{k}bd)(1-q^{k}cd)}{(1-q^{2k+1}abcd)} \right]^{1/2}$$

$$\langle W| = h_0^{1/2}(1, 0, 0, \dots)$$

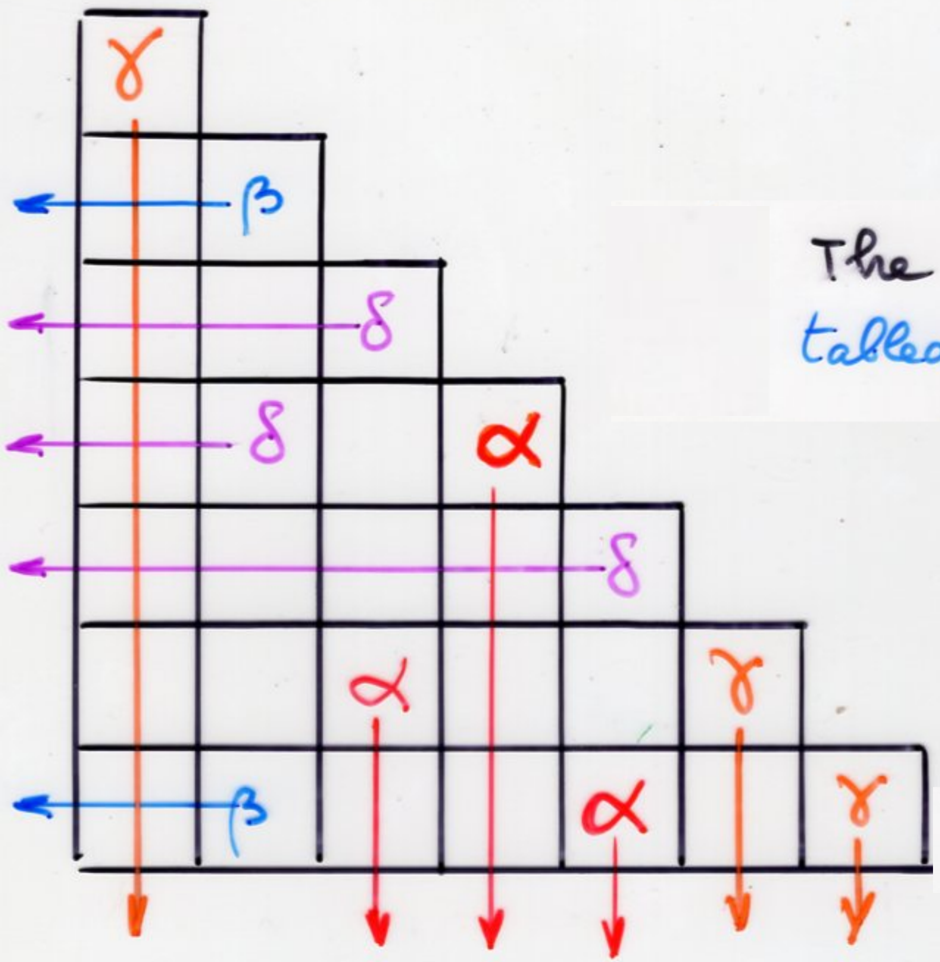
$$|V\rangle = h_0^{1/2}(1, 0, 0, \dots)^T$$

$$h_0 = \left[(1-q)(1-q^2)\dots\right]^{1/2}$$

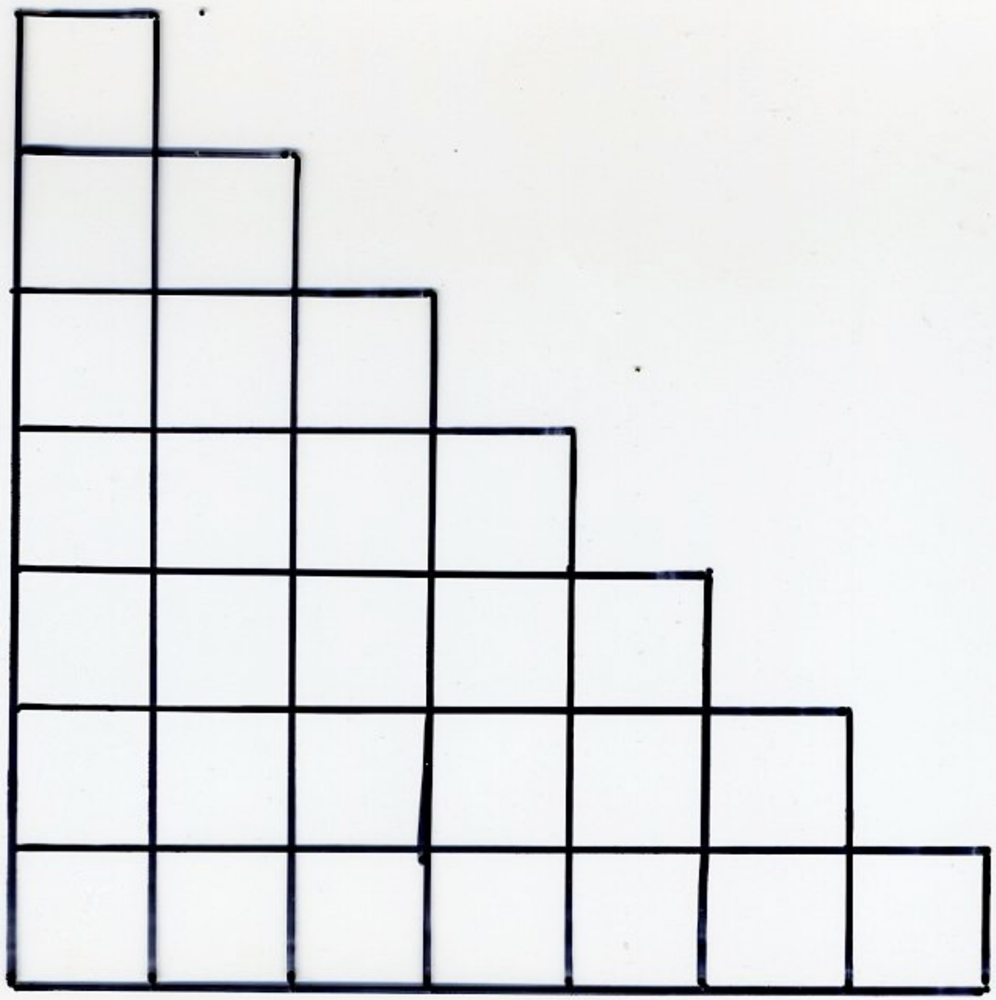
$$(q;q)_\infty$$

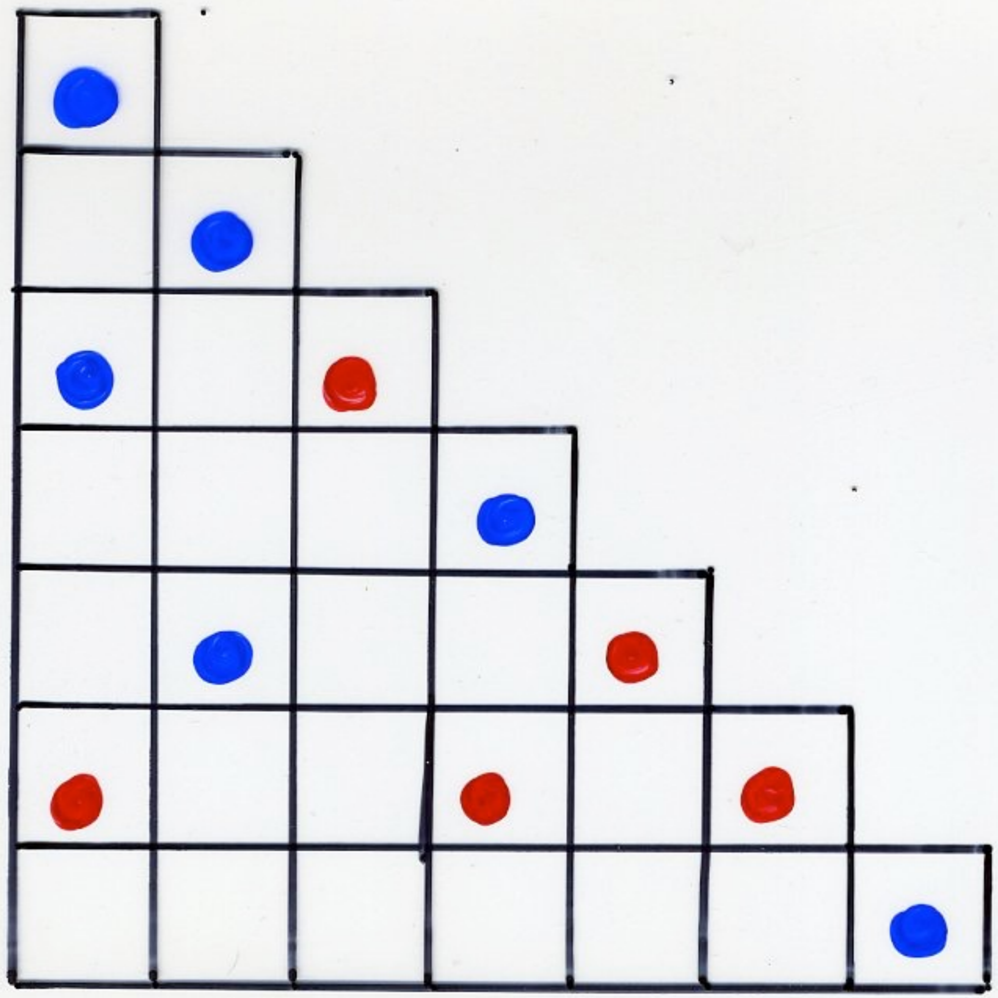
S. Corteel, L. Williams (2009)

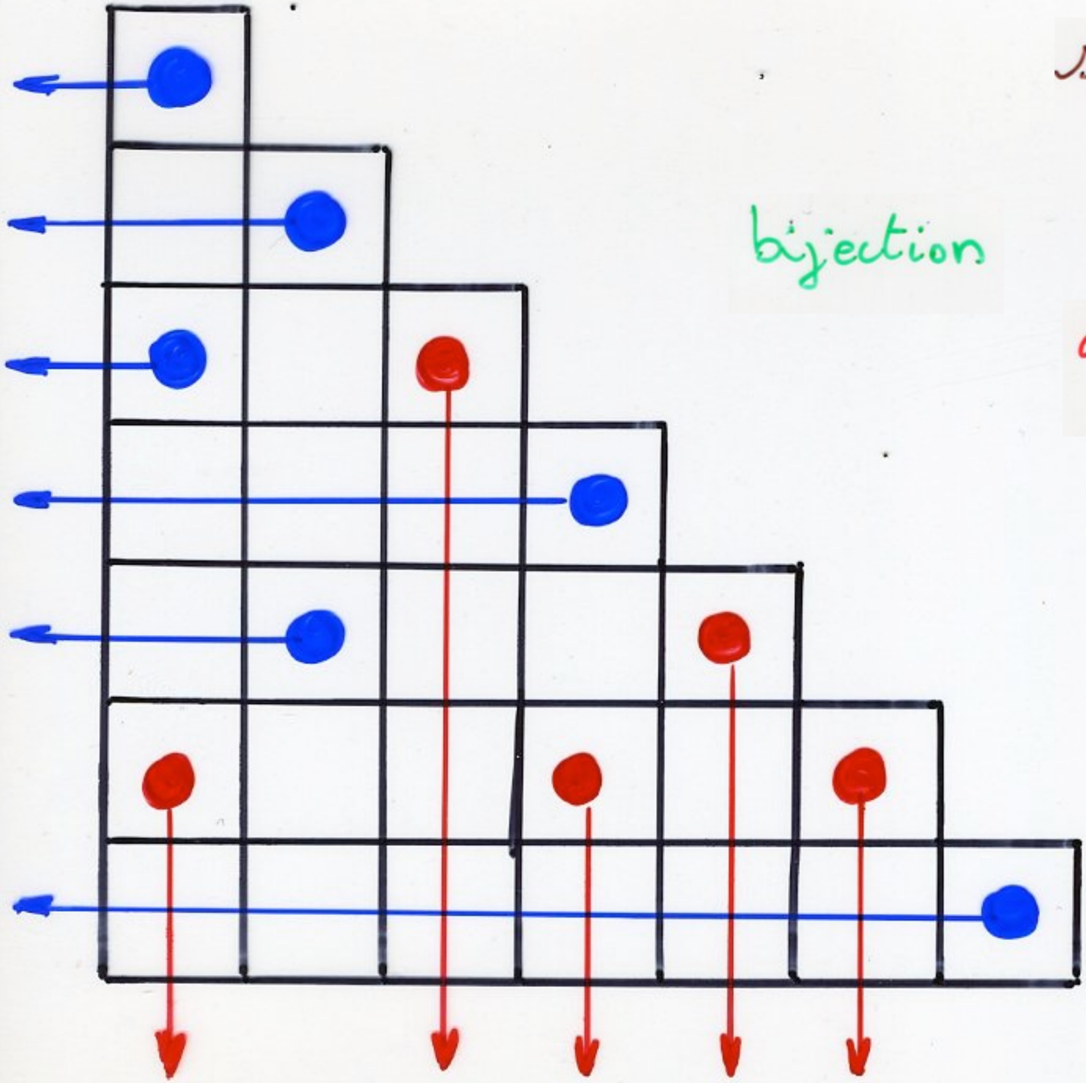
staircase tableaux



The number of staircase tableaux of size n is $4^n n!$



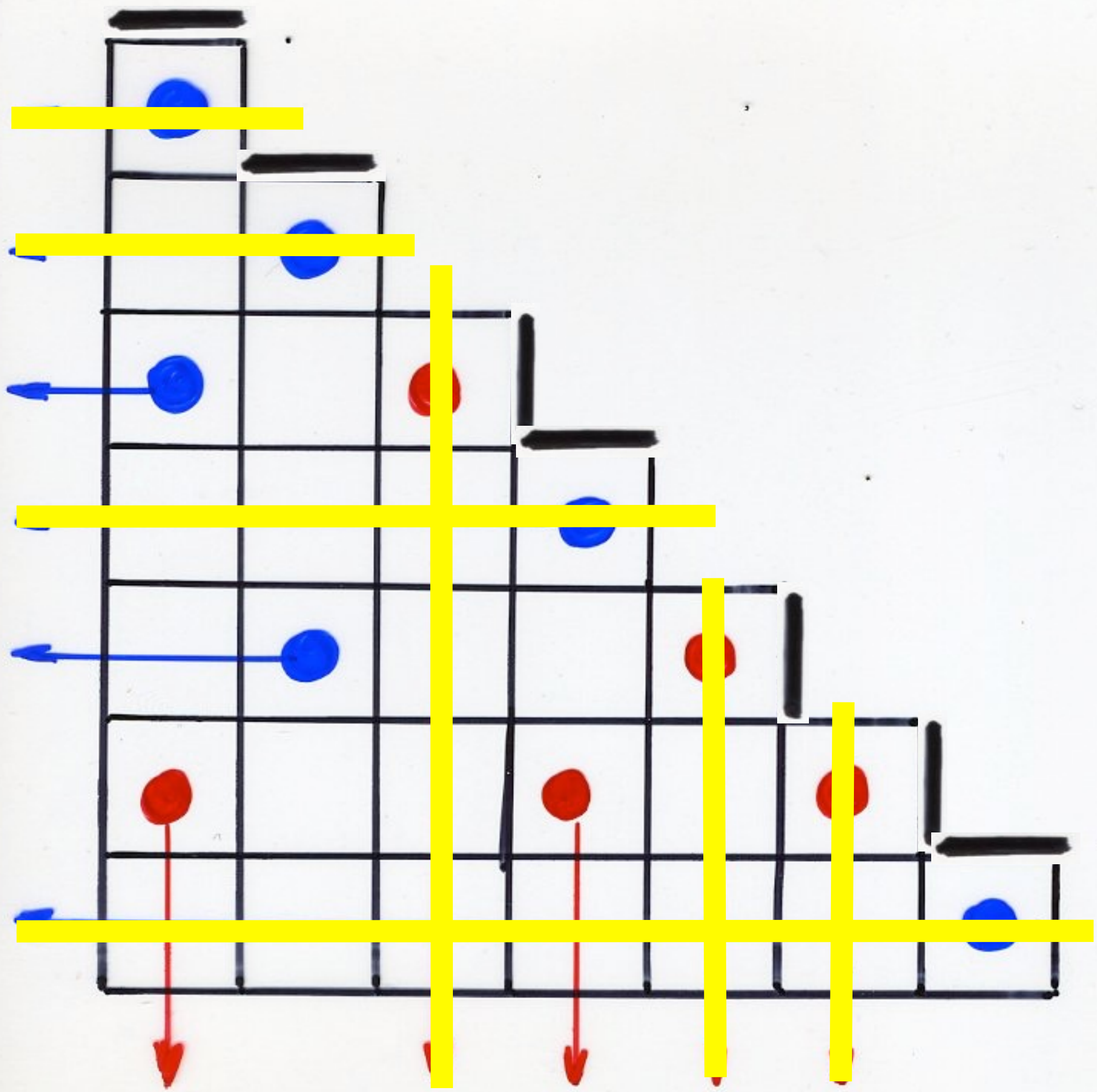


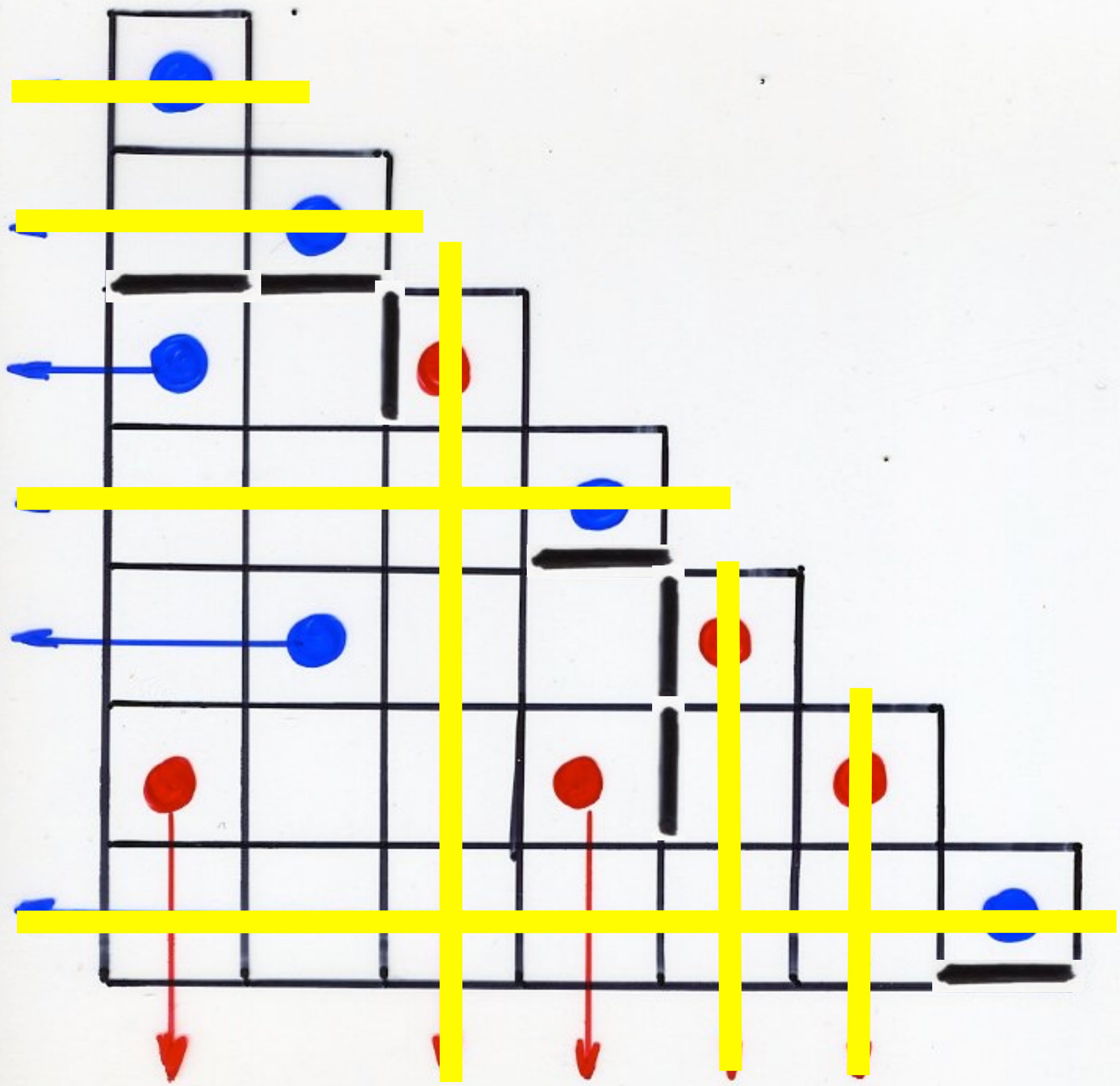


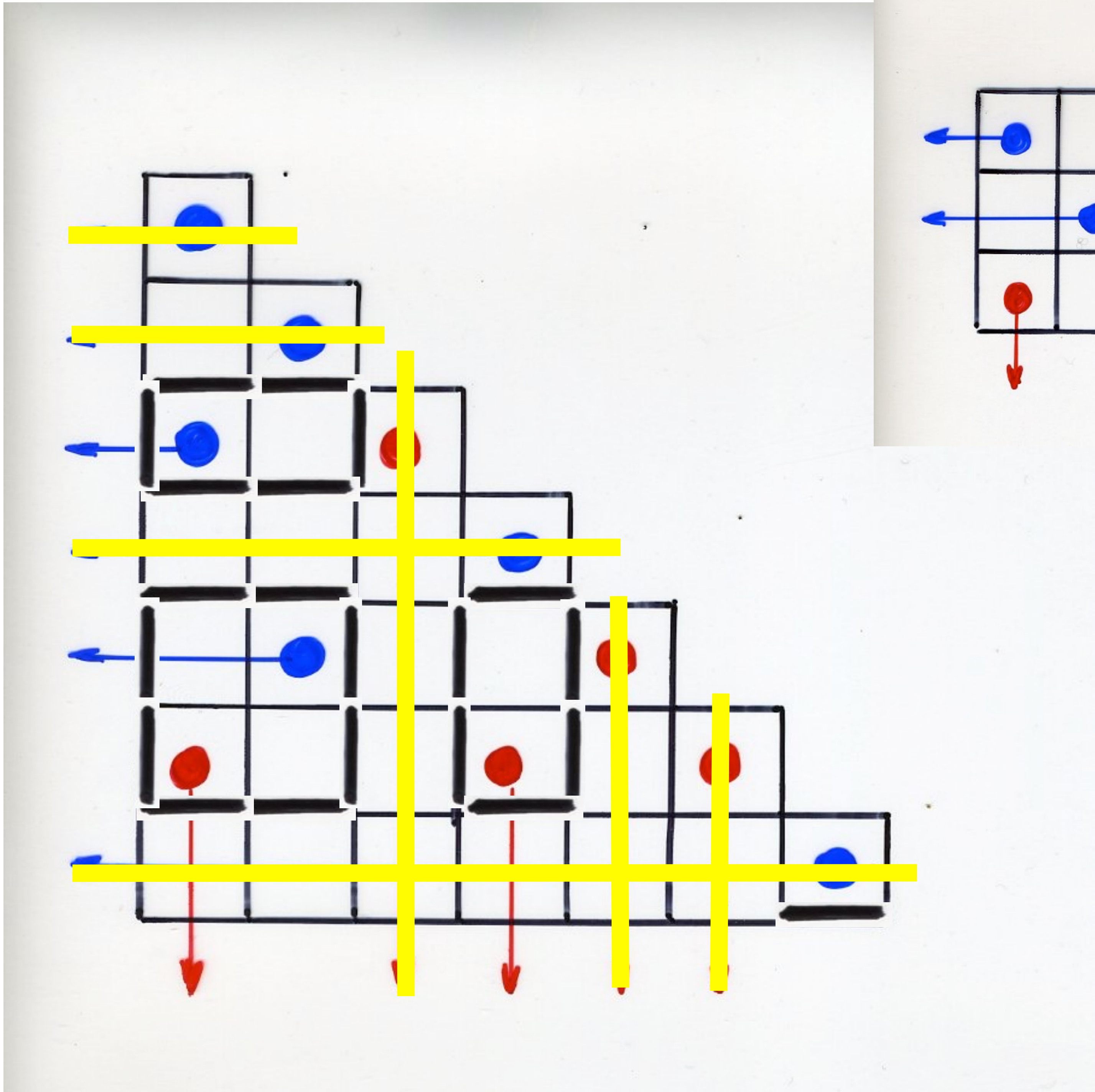
bijection

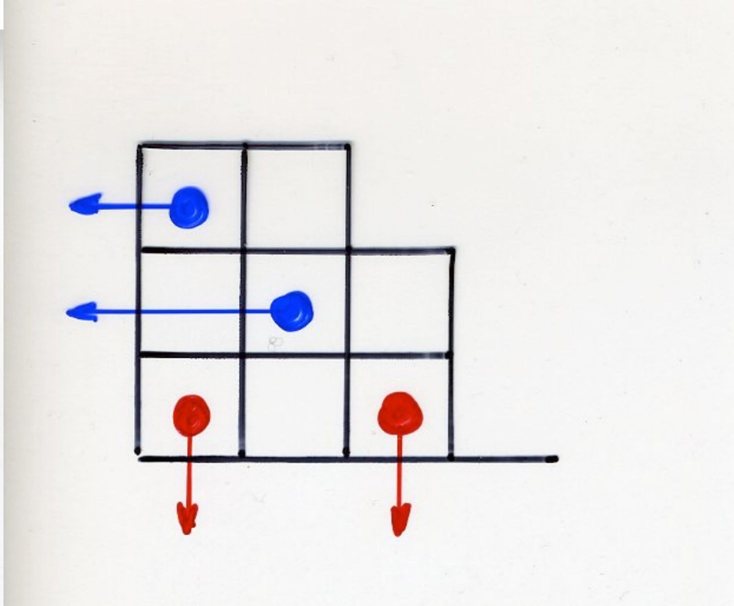
staircase alternative tableaux
"size 2n"

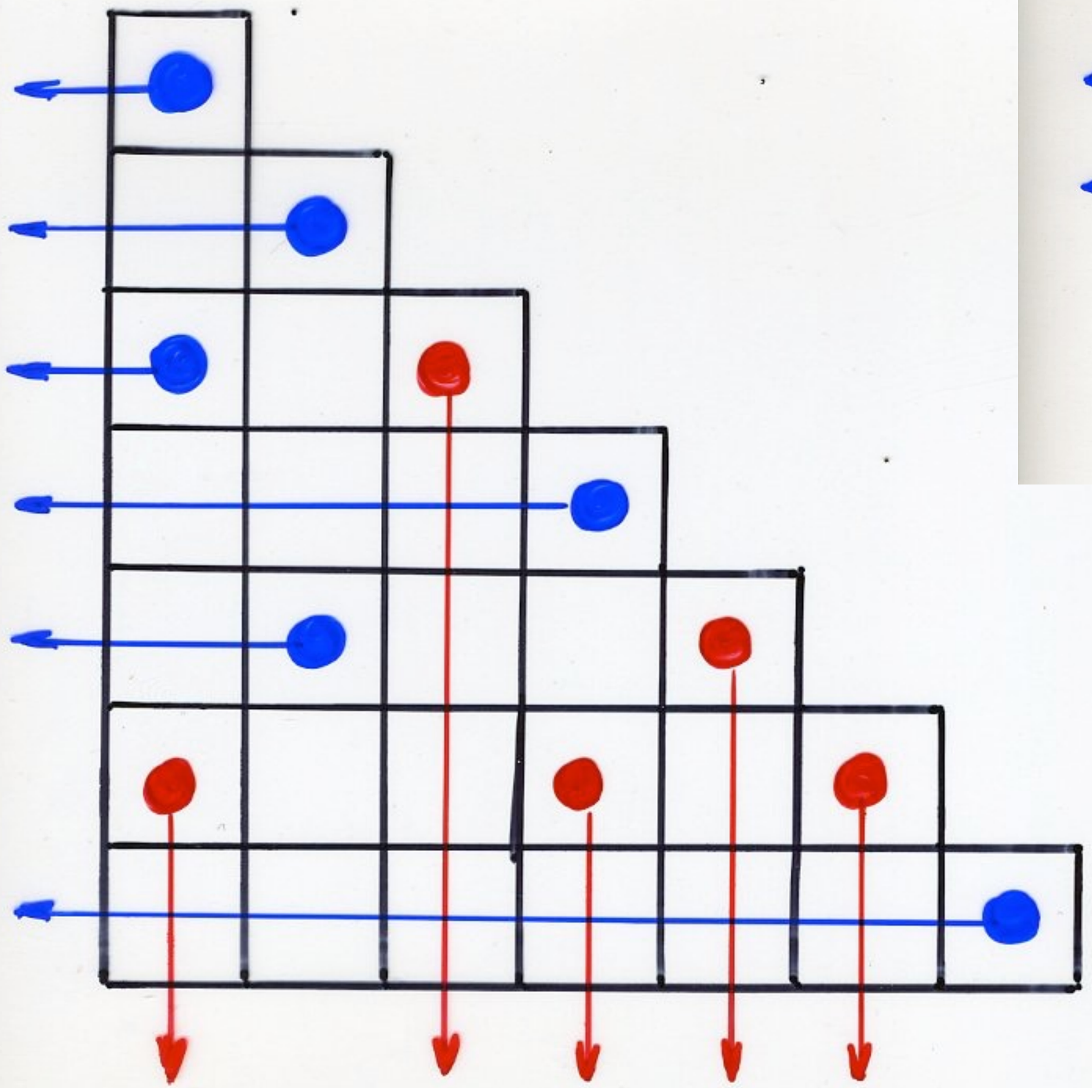
alternative tableaux
size n

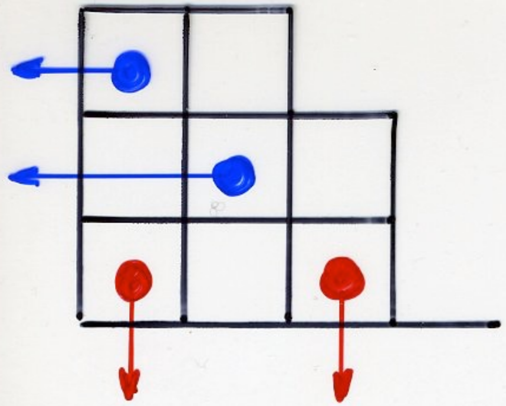


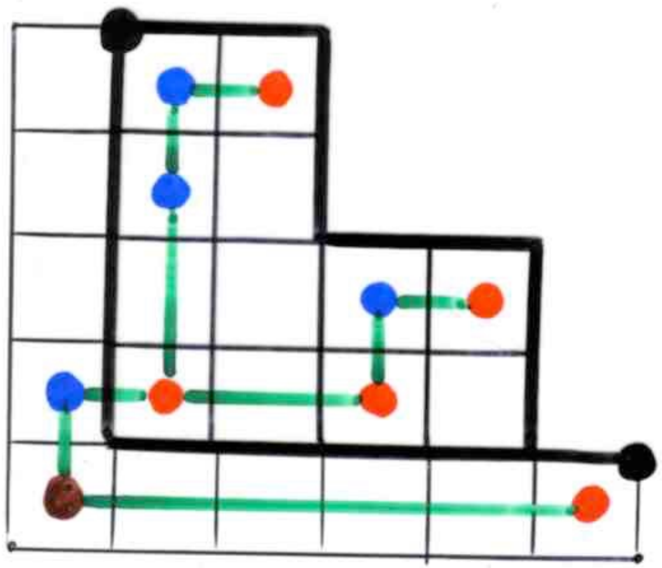


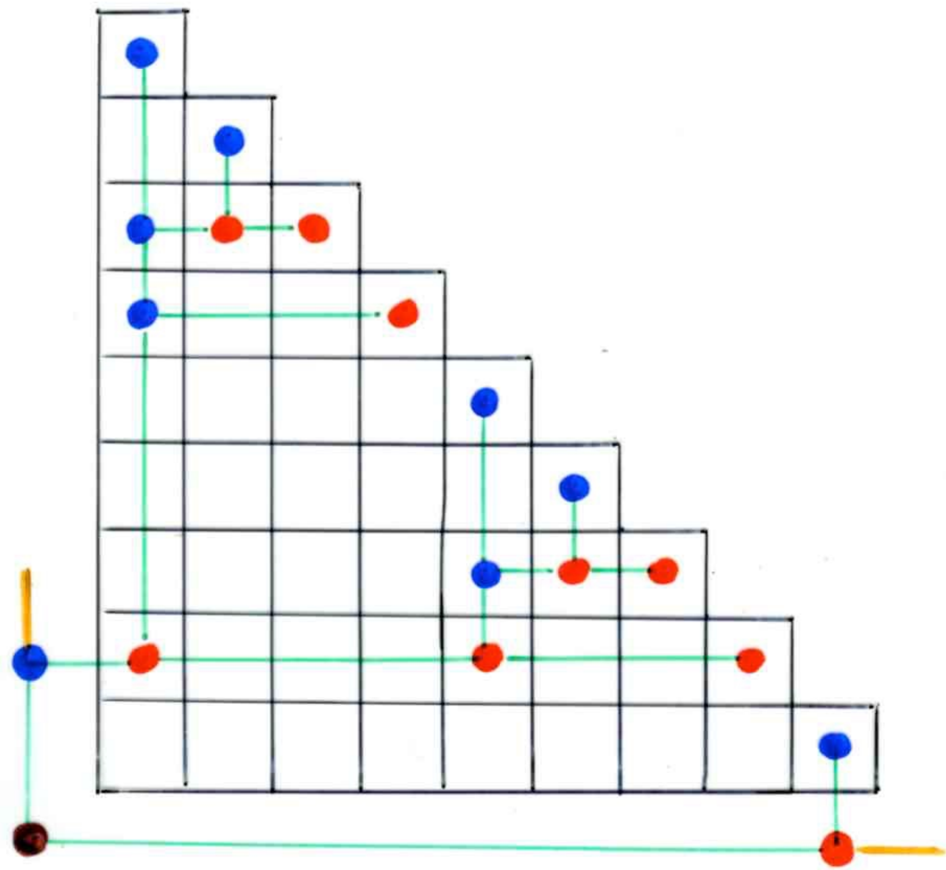


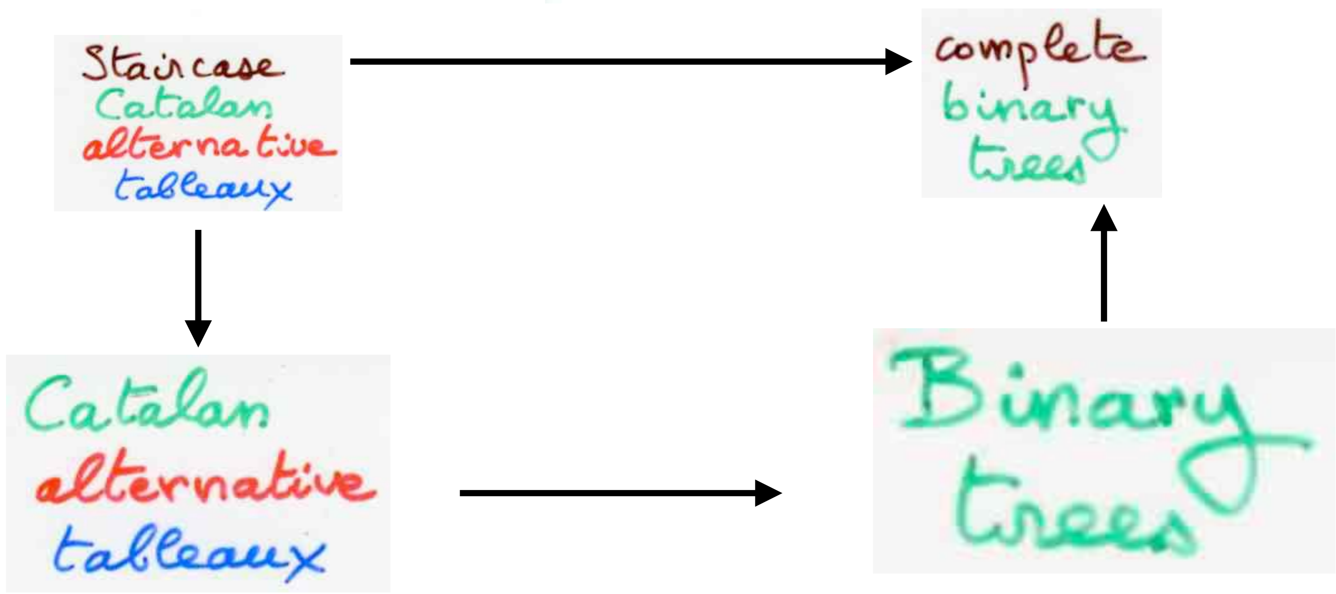




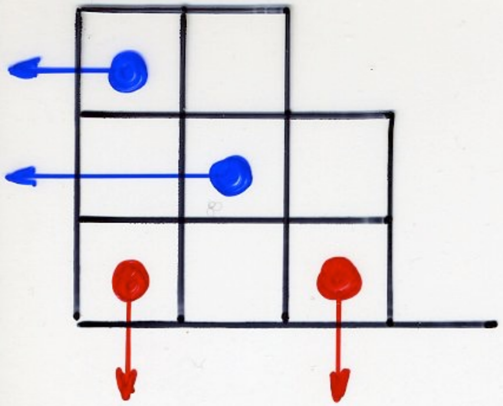







Staircase
Catalan
alternative
tableaux
complete
binary
trees
Catalan
alternative
tableaux
Binary
trees

$(n+1)!$

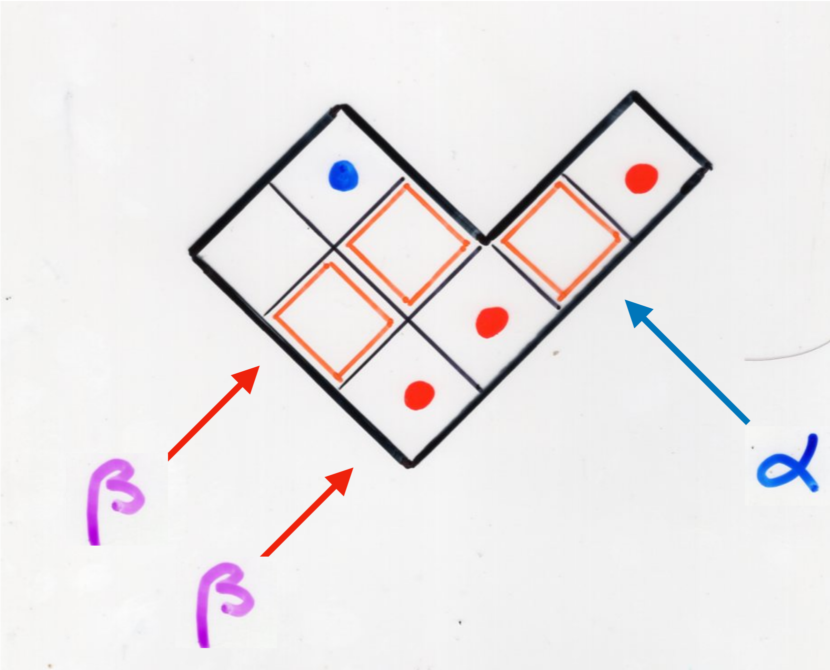


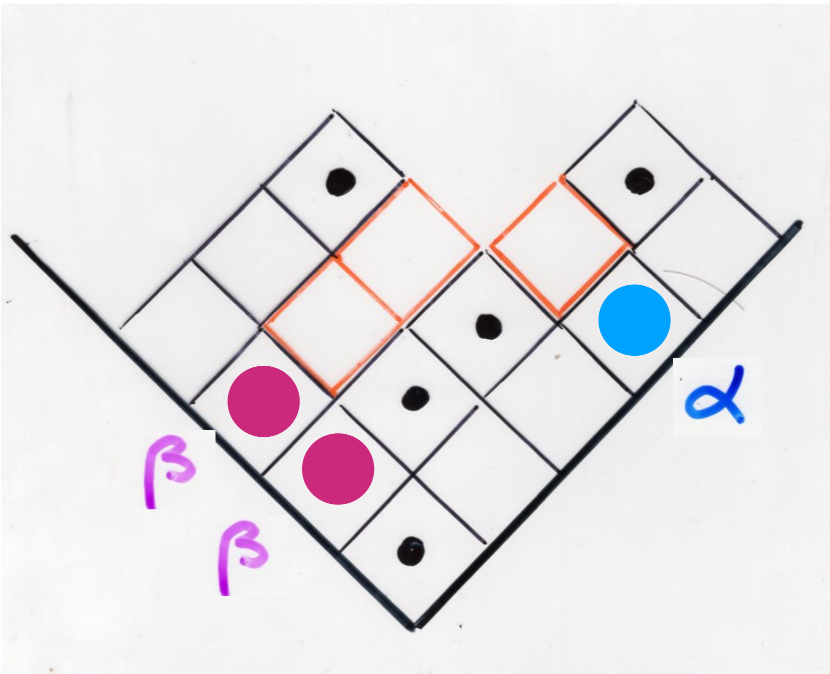
Proposition The number of alternative tableaux (size n) with 2 colors for the blue and red cells is:

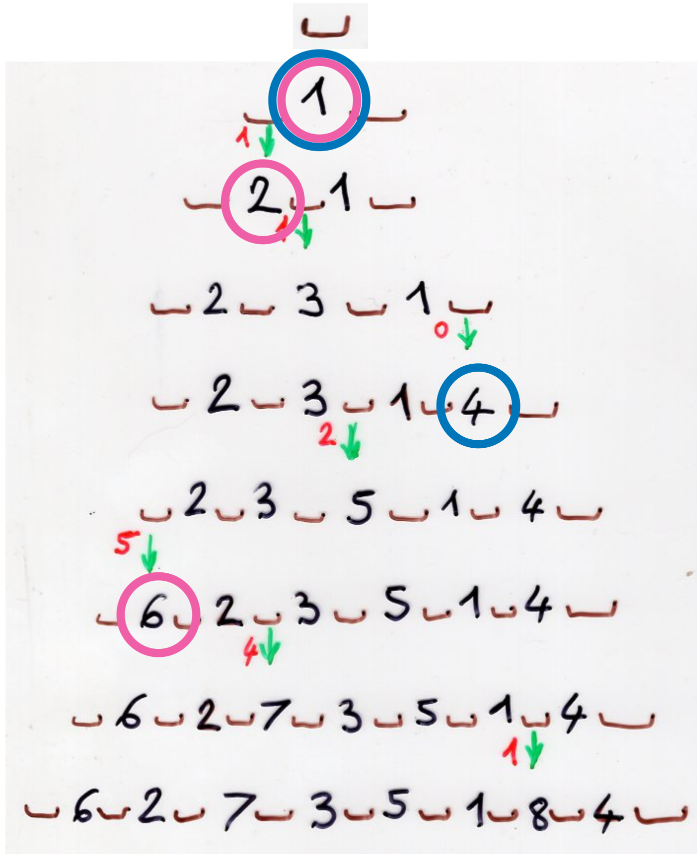
$$2^n n!$$

bijection

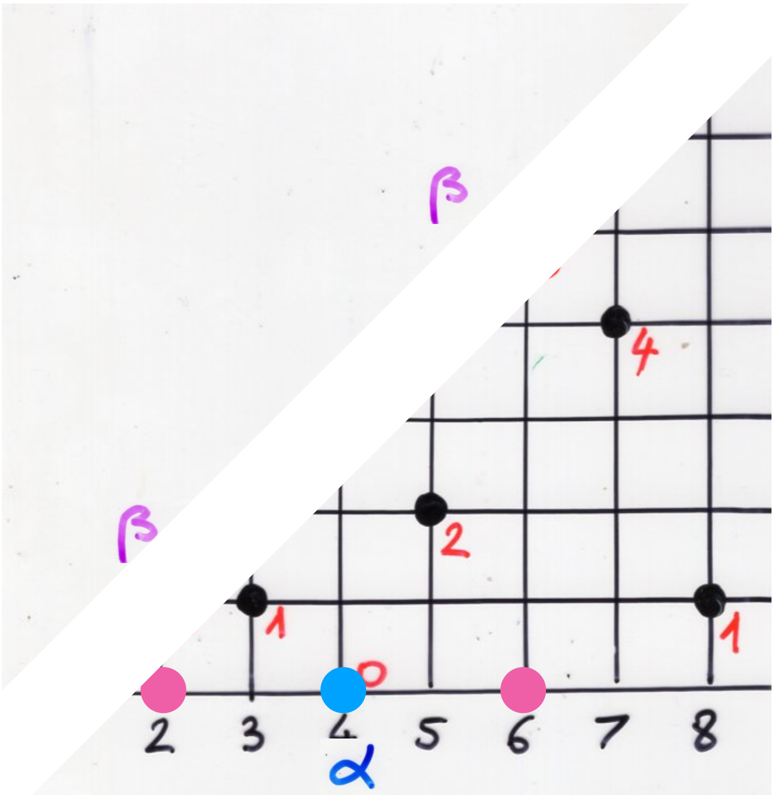
group B_n


β
β
α

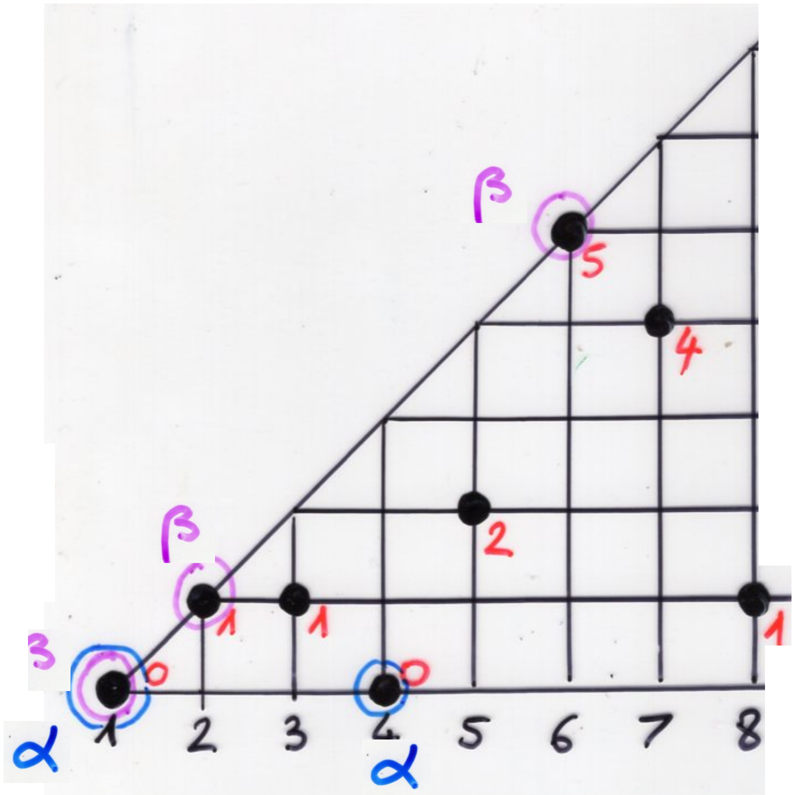

β
β
α

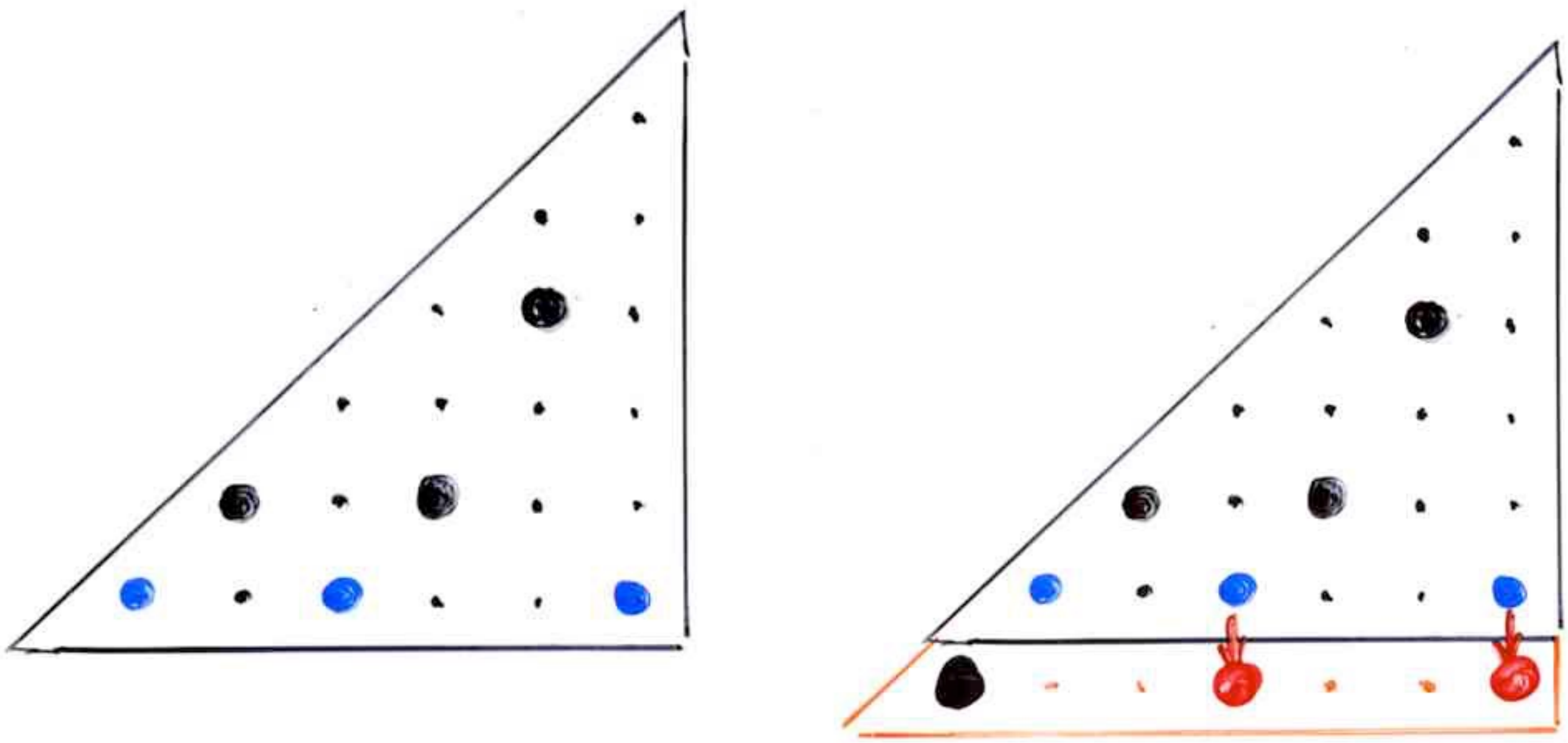

1
2 1
2 3 1
2 3 1 4
2 3 5 1 4
6 2 3 5 1 4
6 2 7 3 5 1 4
6 2 7 3 5 1 8 4

$2^n n!$

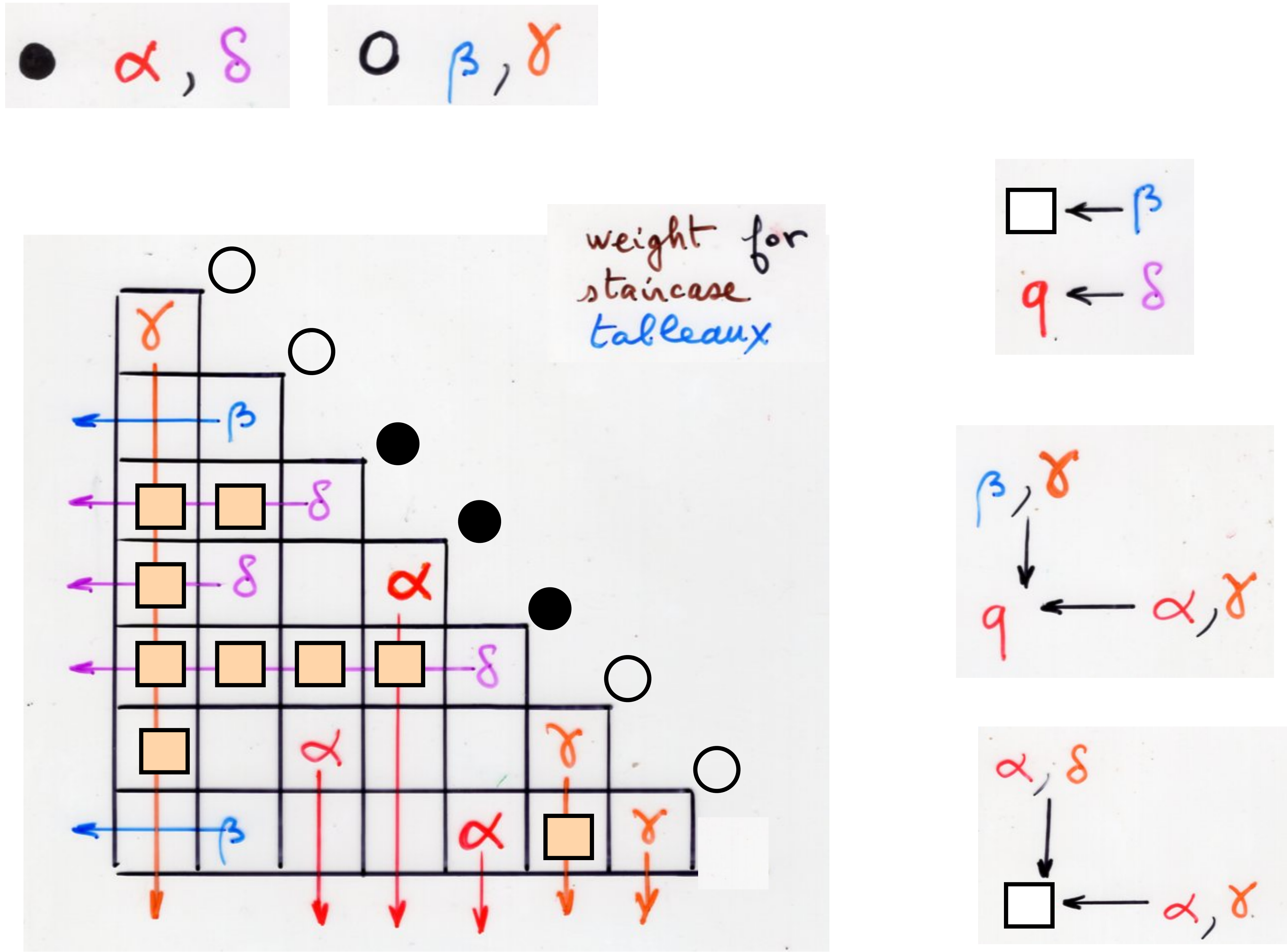


$(n+1)!$





$$(n+1)! = \sum_{k=1}^{n} s_{n,k}\, 2^{k}$$


α, δ
β, γ
weight for
staircase
tableaux
β
q
δ
β, γ
q
α, γ
α, δ
α, γ

$$\tau = (\tau_1, \dots, \tau_n)$$

$$Z_\tau = \sum_{T} v(T)$$

τ: profile of T

T: staircase tableaux size n

S. Corteel, L. Williams (2009)

$$Z_n(\alpha, \beta, \gamma, \delta; q) = \sum_{T} v(T)$$

Z_n: partition function

T: staircase tableaux size n

→ expression for the moments of the Askey-Wilson polynomials

S. Corteel, L. Williams
R. Stanley, D. Stanton
(2010)

The Askey-Wilson integral

The Askey-Wilson integral

$$w(\cos\theta, a, b, c, d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i \geq 0} (1 - aq^i)$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a, b, c, d \mid q)\, d\theta =$$

$$\frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

The Askey-Wilson integral

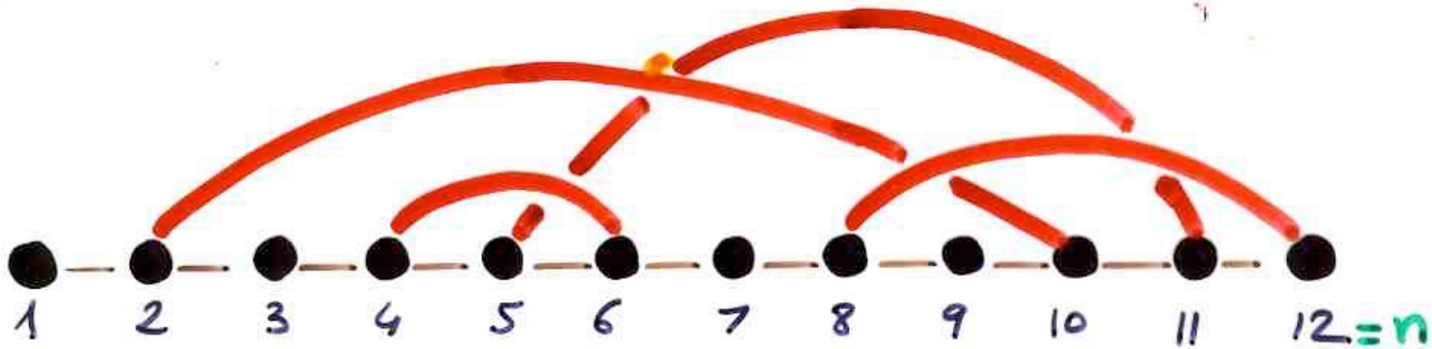
integral of the product
of 4 q-Hermite polynomials
(type II)

Ismail, Stanton, V. (1986)

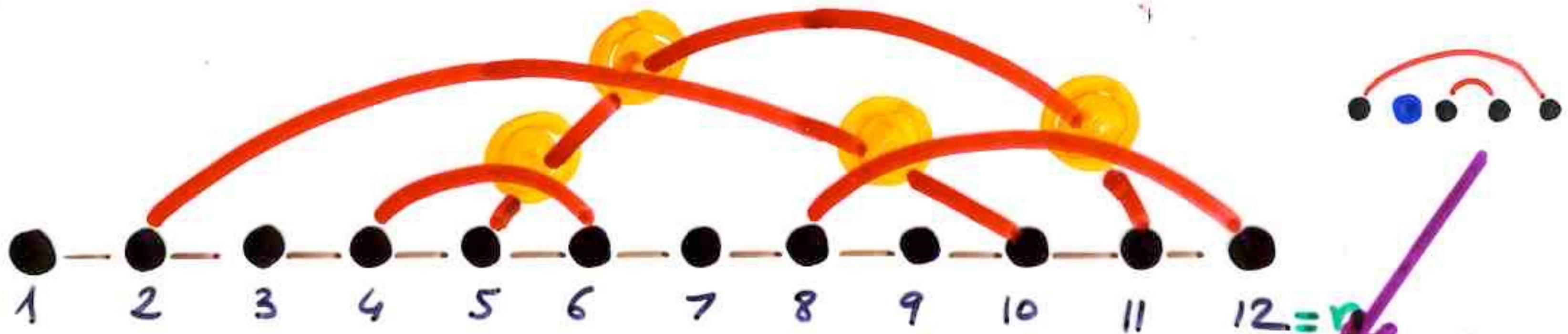
$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_k(\cos\theta\,|\,q)\,H_l(\cos\theta\,|\,q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty = (q)_k\,\delta_{kl}$$

(continuous)

q-Hermite



$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} \, q^{cr(\gamma)+e(\gamma)} \, x^{fix(\gamma)}$$



(continuous)

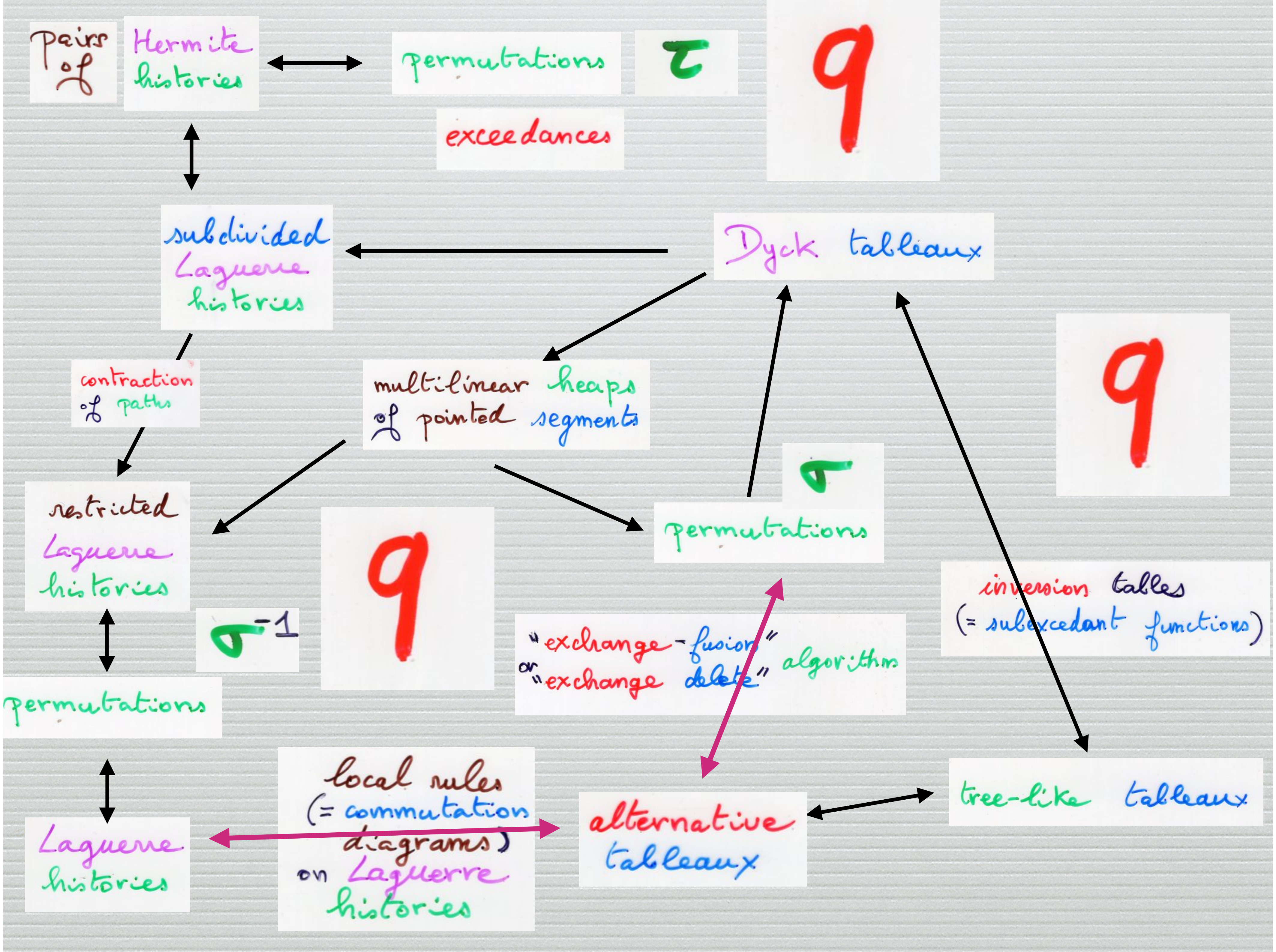
q-Hermite

$$H_n(x|q) = \sum_{\gamma} (-1)^{|\gamma|} \, q^{cr(\gamma) + e(\gamma)} \, x^{fix(\gamma)}$$

γ matching

crossings

The « essence » of bijections …



Pairs of Hermite histories
permutations
τ
q
exceedances
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
q
restricted Laguerre histories
σ
permutations
q
σ⁻¹
"exchange-fusion" or "exchange delete" algorithm
inversion tables (= subexcedant functions)
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories

"The art of bijective combinatorics"

Pairs of Hermite histories

permutations τ

exceedances

subdivided Laguerre histories

Dyck tableaux

contraction of paths

multilinear heaps of pointed segments

σ

permutations

restricted Laguerre histories

inversion tables (= subexcedant functions)

σ^{-1}

"exchange - fusion" or "exchange delete" algorithm

permutations

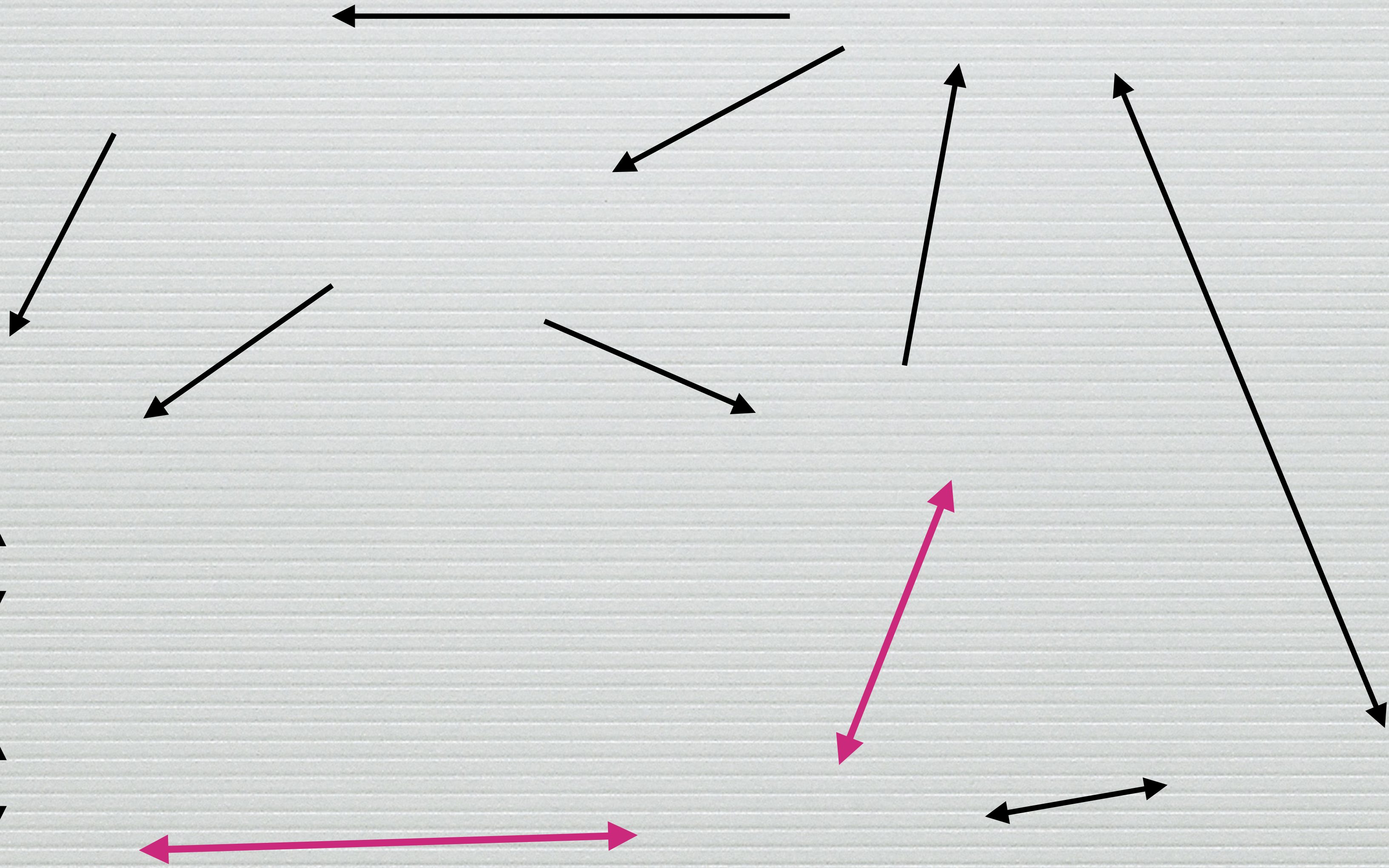
local rules (= commutation diagrams) on Laguerre histories

tree-like tableaux

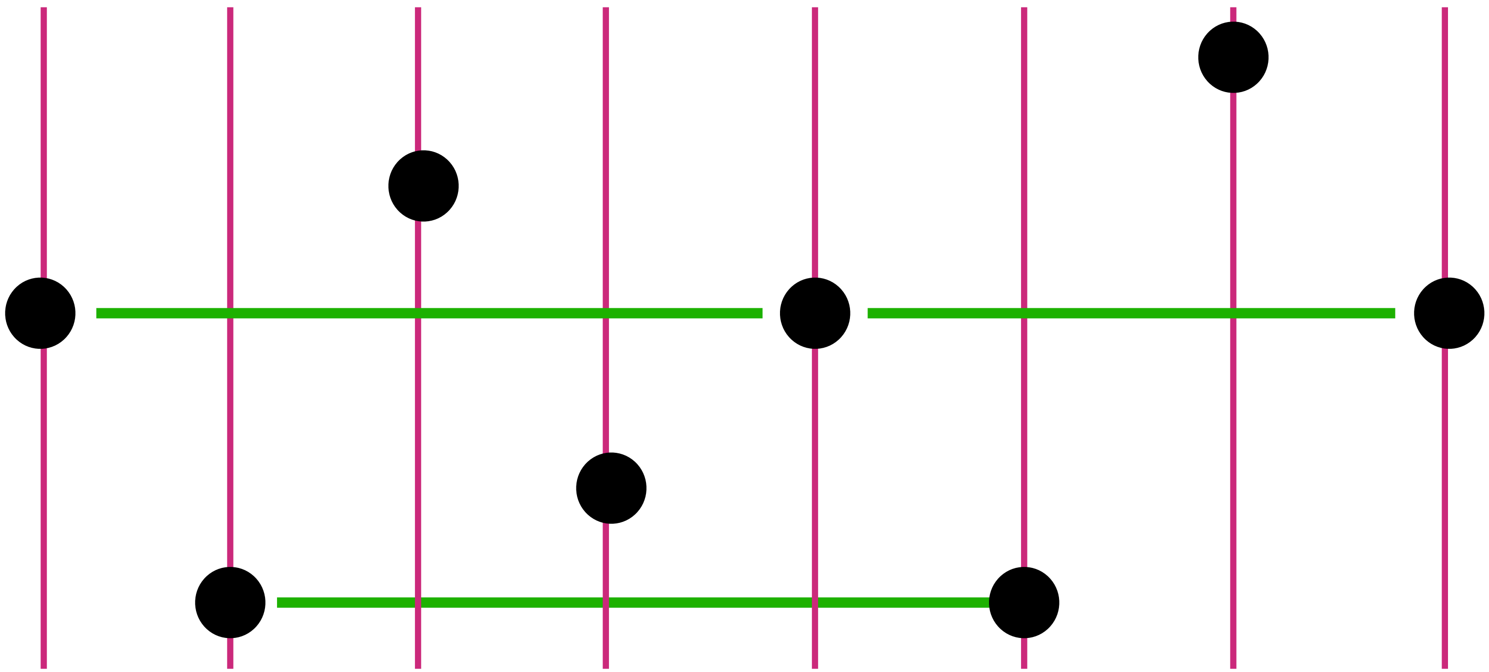
alternative tableaux

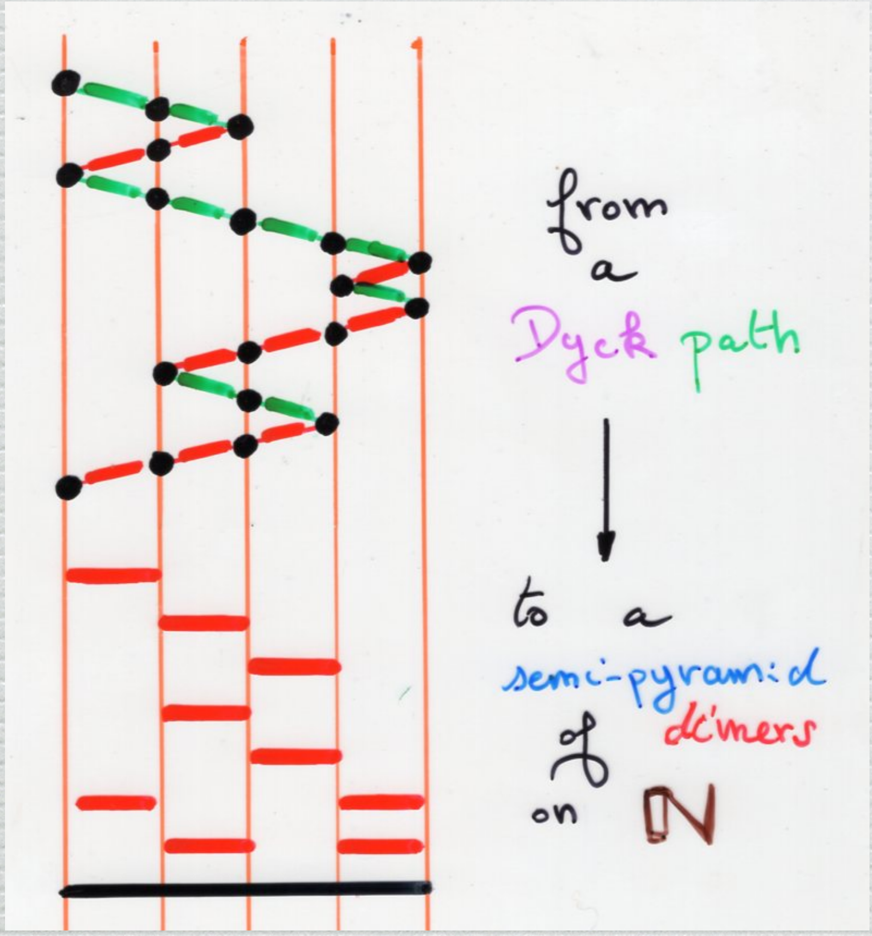
Laguerre histories

The « essence » of bijections ...



$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$





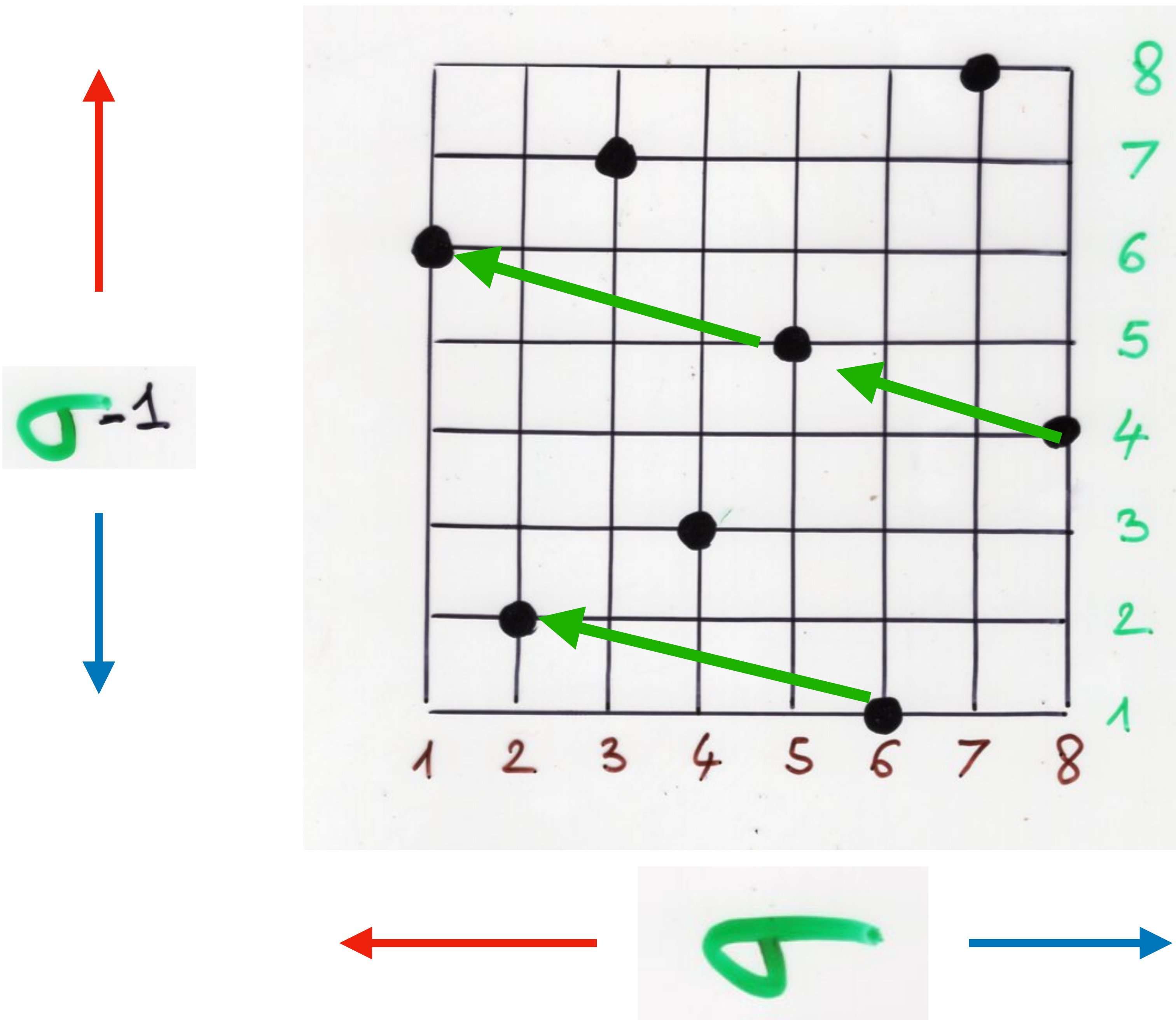
Video with violon:

violonist: Gérard Duchamp

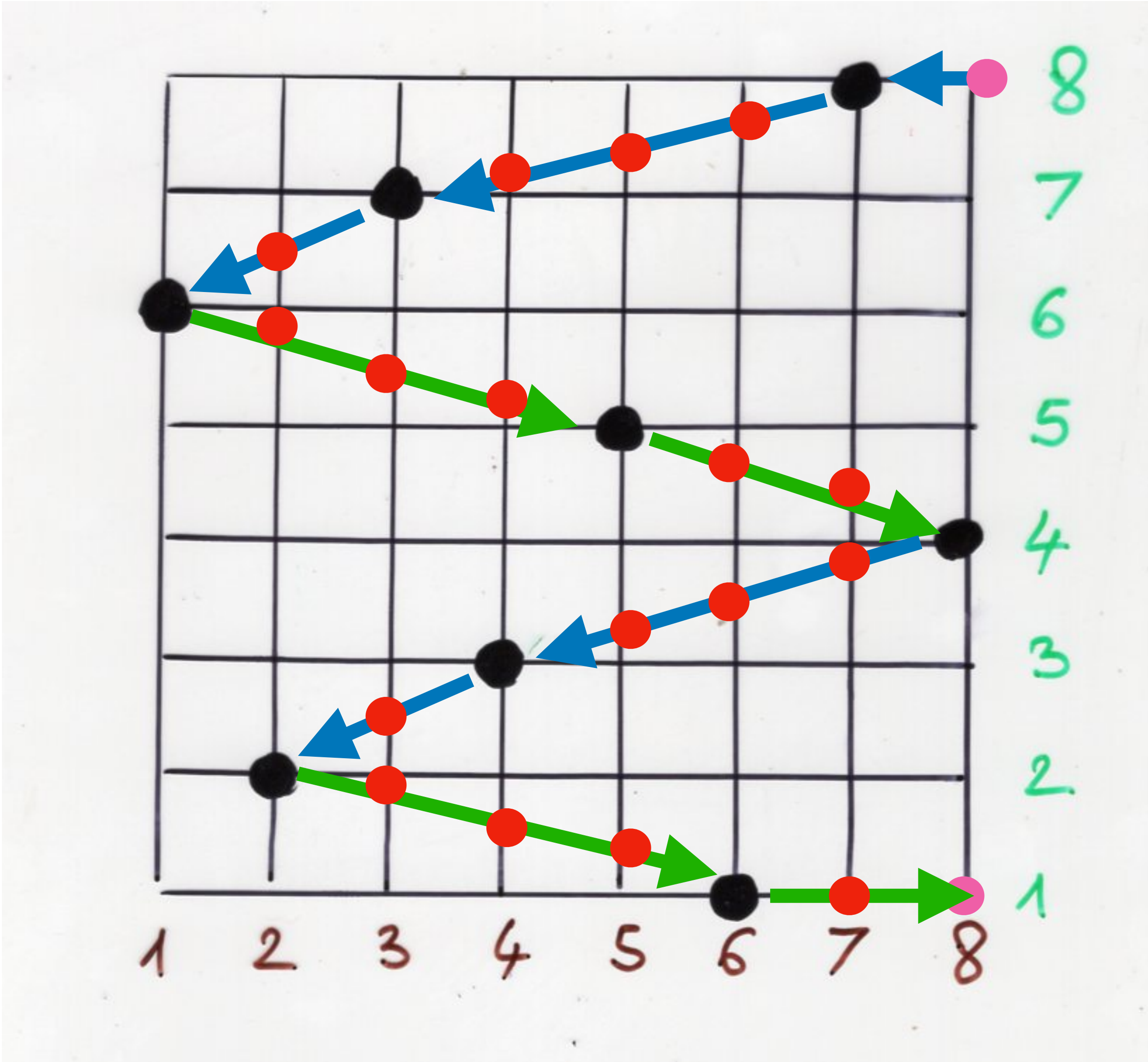
(association Cont'Science)

Bijection paths — heaps, see BJC II, Ch3b p 26-40,

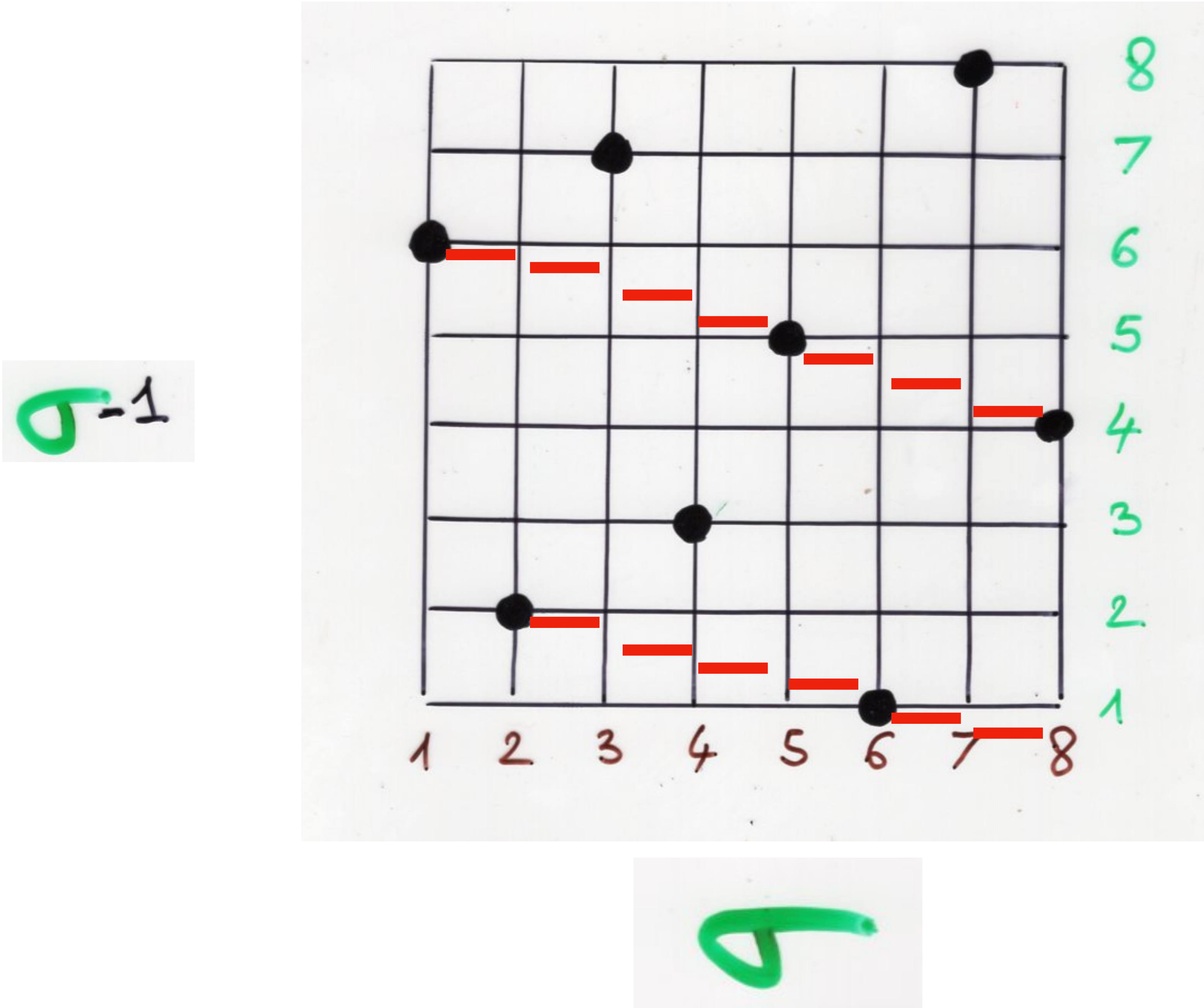
and p 42,60 in the case of Dyck paths.


8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ-1
σ

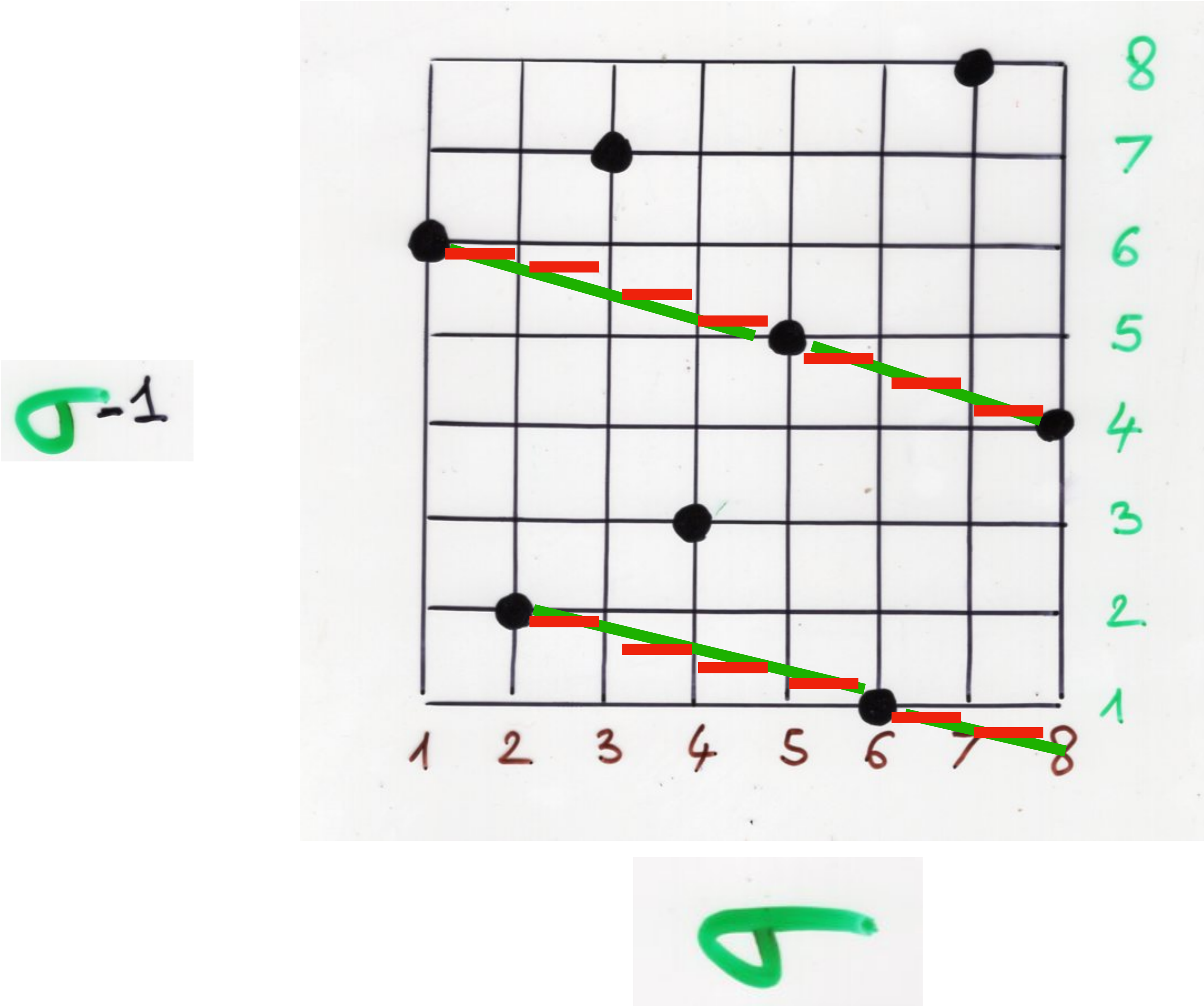
σ^{-1}



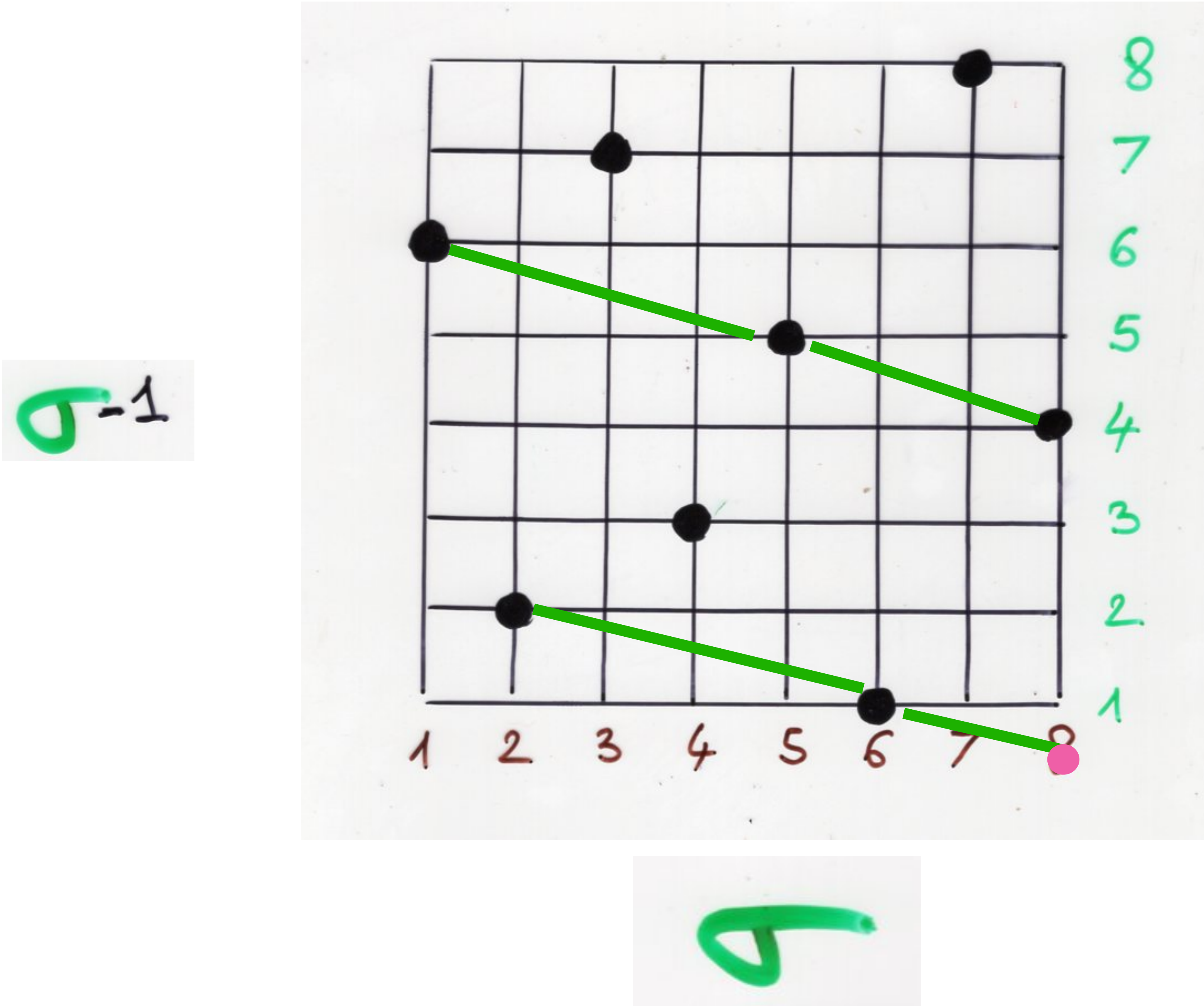
σ



1 2 3 4 5 6 7 8
8 7 6 5 4 3 2 1
σ-1
σ



8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8
σ-1
σ



8
7
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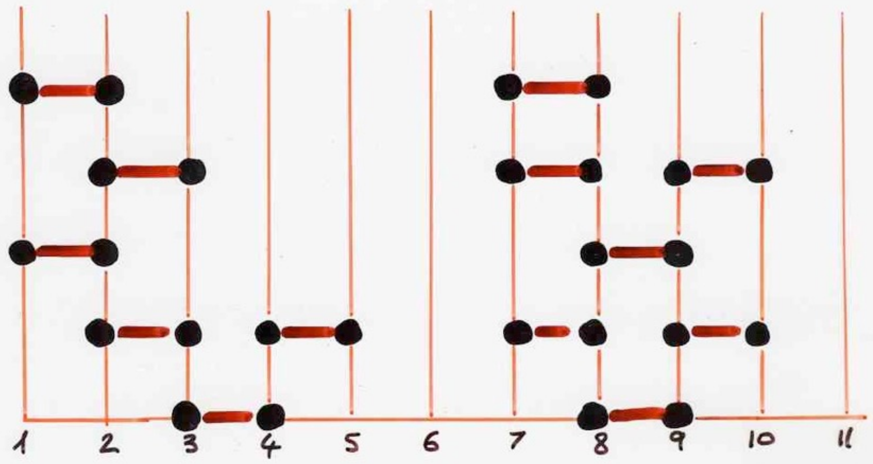
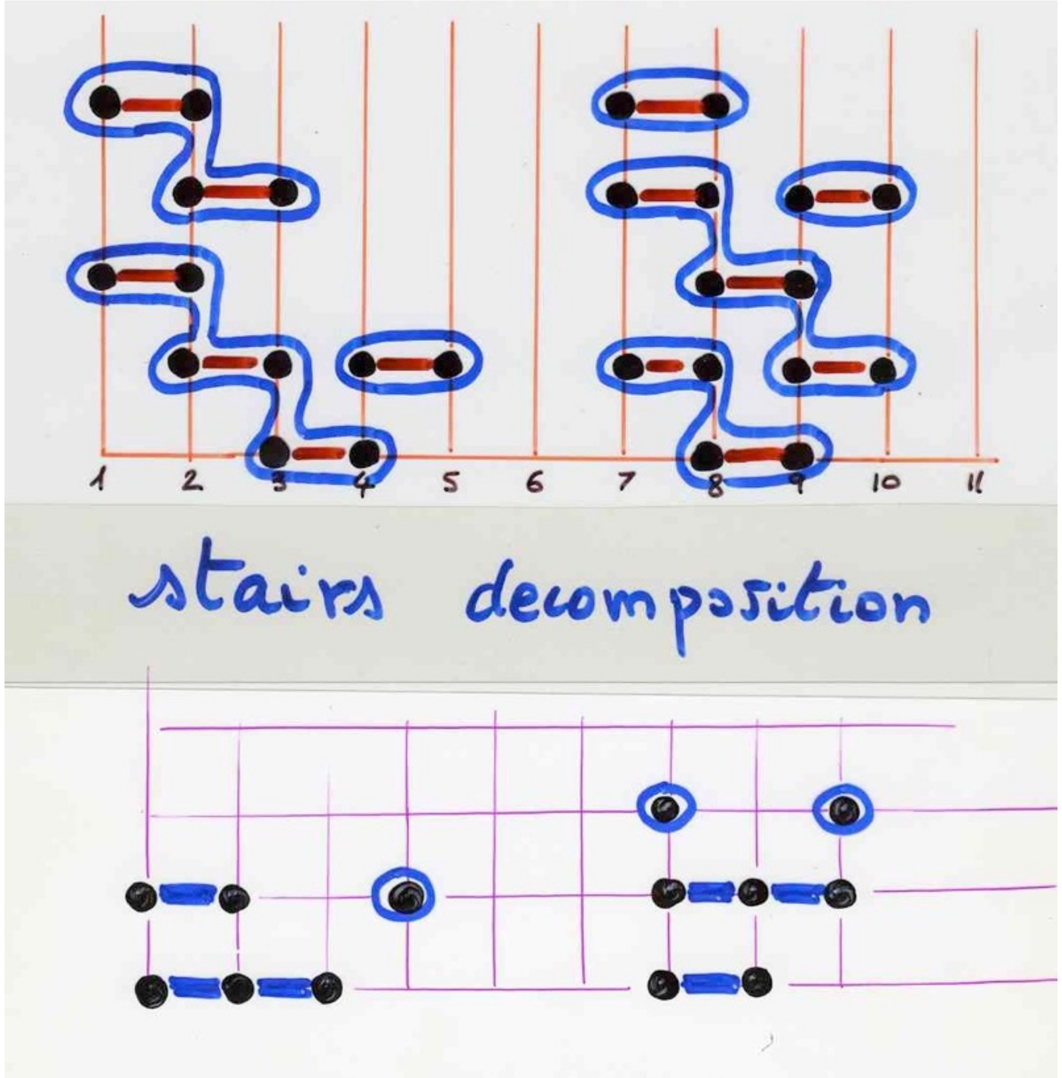
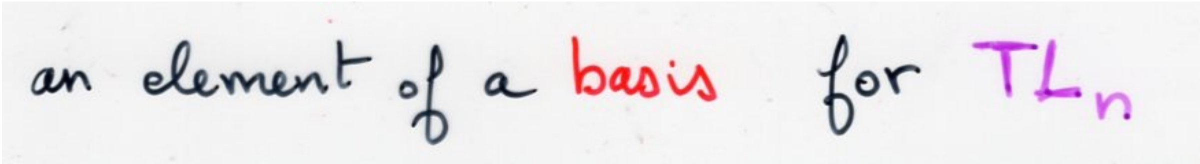
2
1
1 2 3 4 5 6 7 8
σ-1
σ

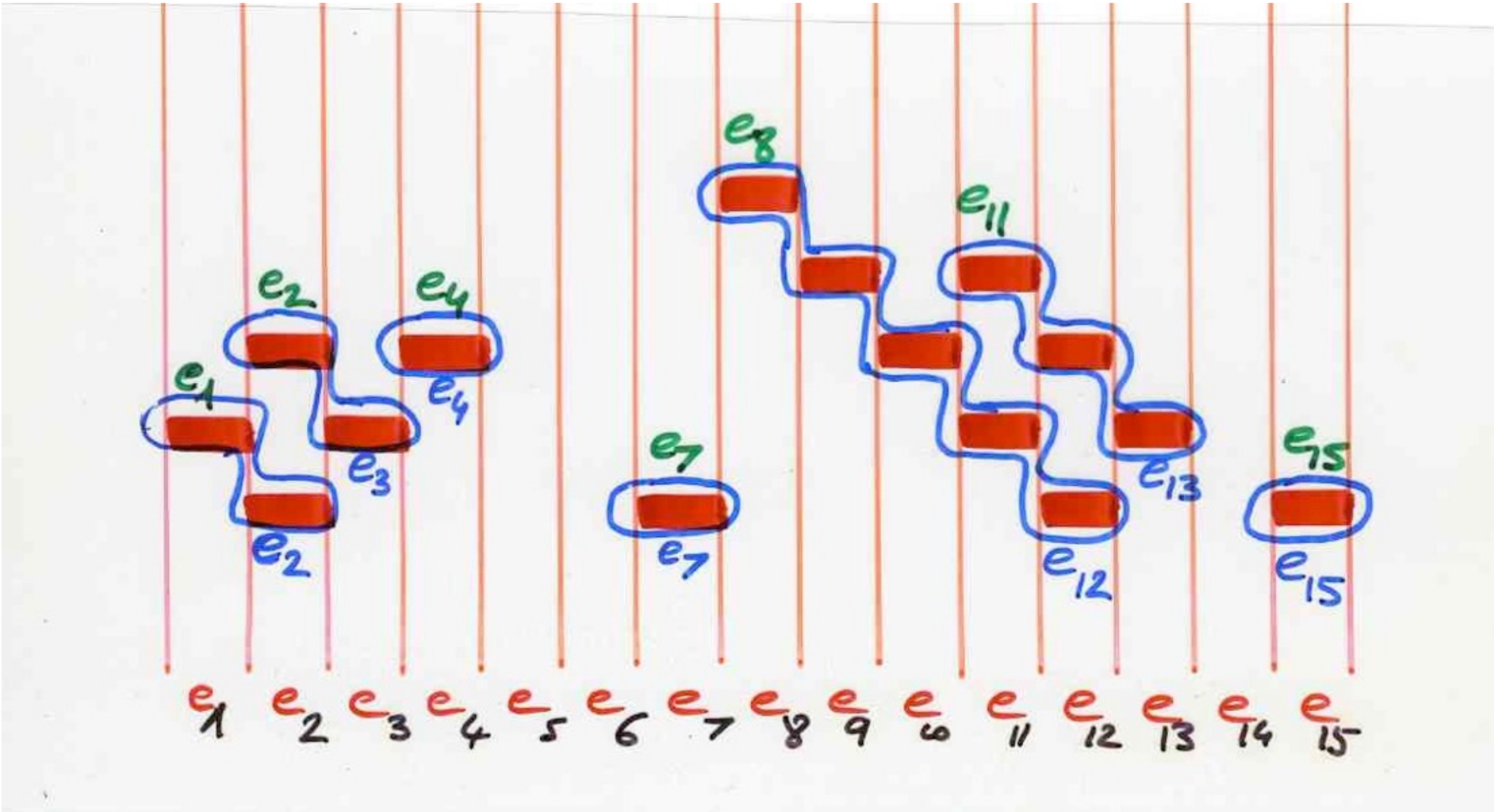


The stairs decomposition
of a heaps of dimers on N
with the bijection
heaps of dimers — heaps of segments
see BJC II, Ch6b, p50-57



leading to give a basis of
the Temperley-Lieb algebra
or fully commutative elements
see BJC II, Ch6b





basis of the Temperley-Lieb algebra
see BJC II, Ch6b

The (happy) end