Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 5
Tableaux and orthogonal polynomials

Ch5b

IMSc, Chennai
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Reminding Ch5a

orthogonal polynomials

(analytic) continued fraction

Laguerre and Hermite polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \to \mathbb{K}$ linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

Thm. (Favard)

- $\{P_n(x)\}_{n\geq 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ coeff. in $\mathbb{K}$

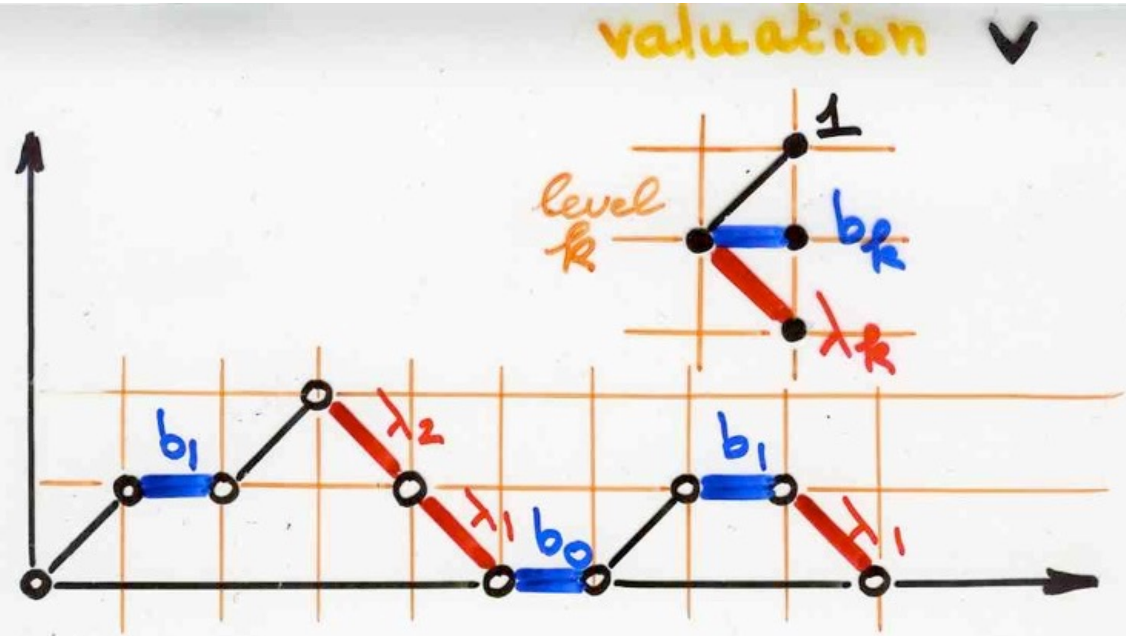
orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k \geq 1)$

3 terms linear recurrence relation

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$$J(t; b, \lambda)$$

Jacobi continued fraction

$$b = \{b_k\}_{k\geq 0} \qquad \lambda = \{\lambda_k\}_{k\geq 1}$$

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{\cdots\cdots \cfrac{}{1 - \cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$\mu_0 = 1$

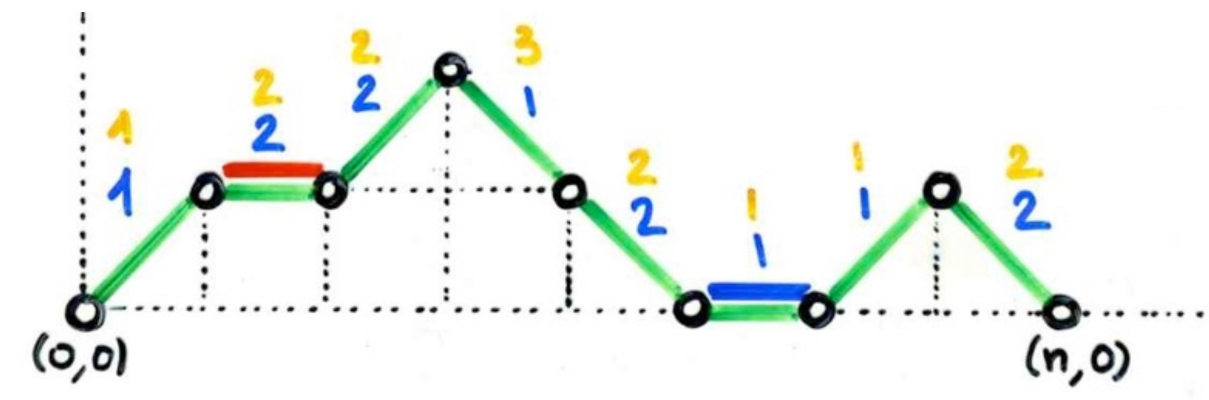
$S(t;\lambda)$

Stieltjès continued fraction

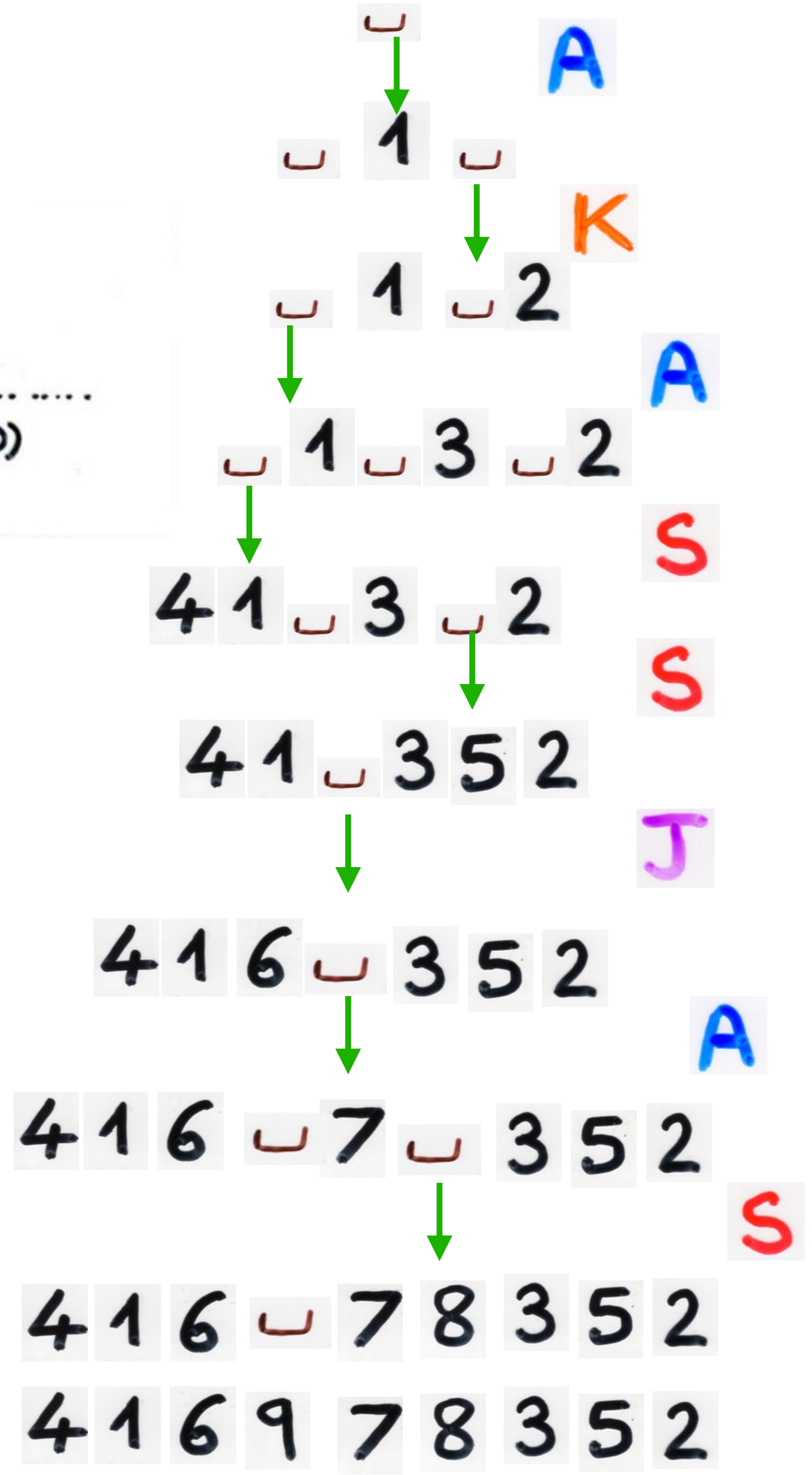
Laguerre polynomial

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$



(0,0)
(n,0)



A
K
A
S
S
J
A
S
4 1 6 9 7 8 3 5 2

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

moments
Hermite
polynomials

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

q PASEP

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

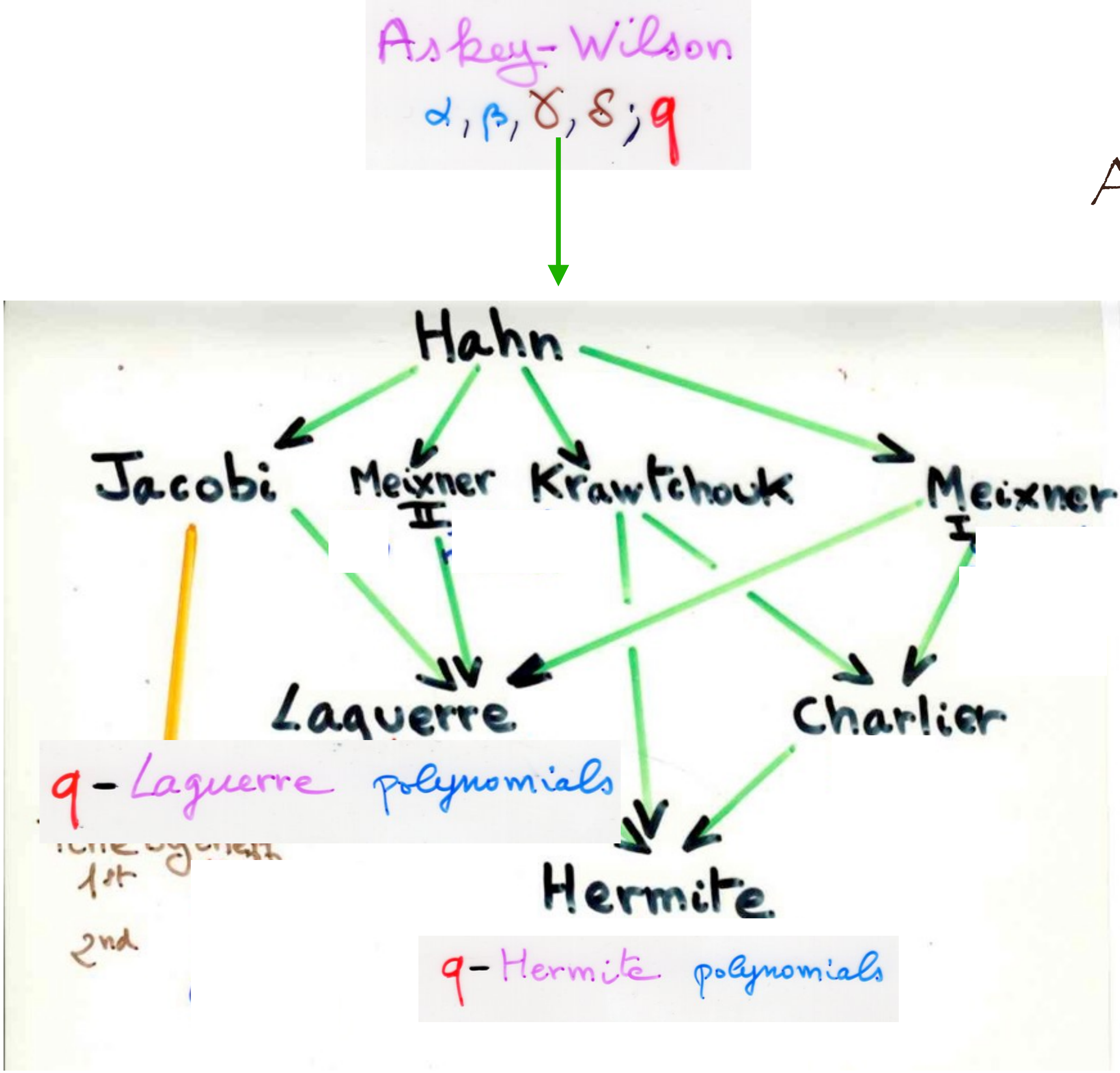
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

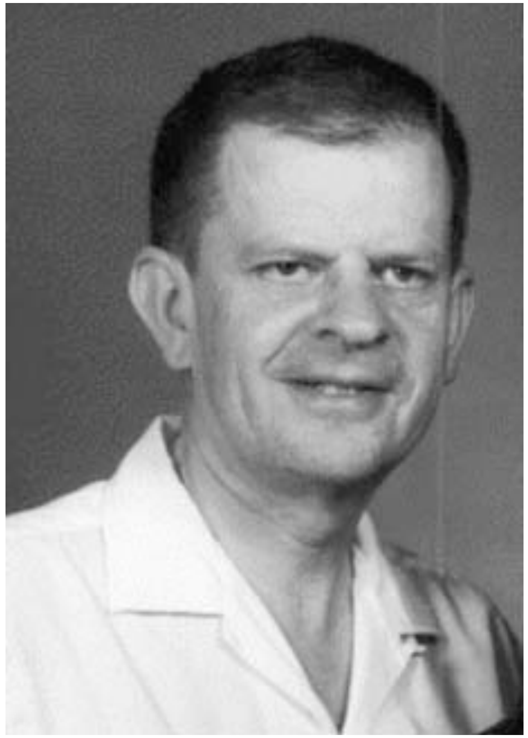
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

Askey tableau





Inversion table q-analogue

From BJC 1, Ch 4a

Definition

sub-exceedant functions

$$f : [1, n] \longrightarrow [0, n-1]$$

for every i, $1 \leqslant i \leqslant n$, $0 \leqslant f(i) < i$

$\mathcal{F}_n$ set of sub-exceedant functions

$$|\mathcal{F}_n| = n!$$

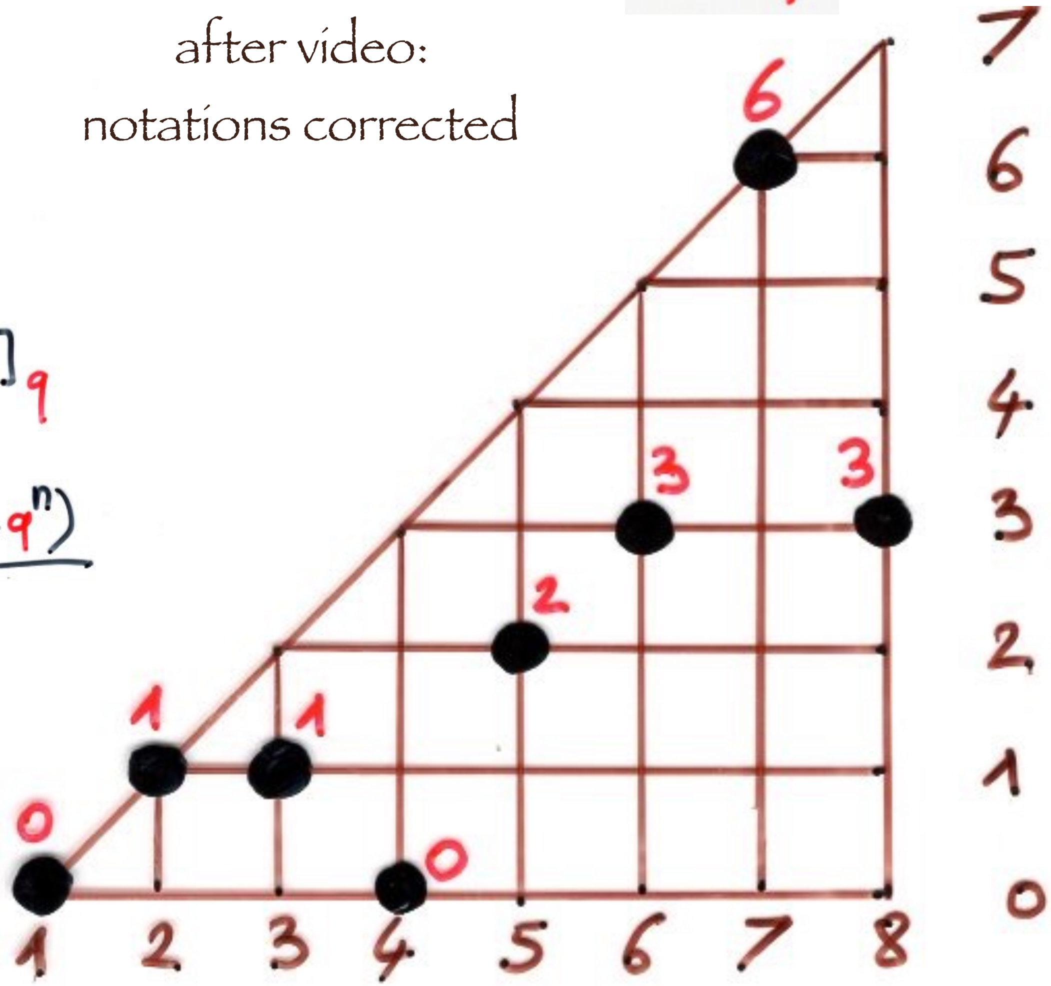
$$\sum_{f \in \mathcal{F}} q^{sum(f)} = 1(1+q)\cdots(1+q+q^2+\cdots+q^{n-1})$$

$$= [n]! \quad \text{or} \quad [n]_q!$$

after video:
notations corrected

$$[i]_q = 1+q+\cdots+q^{i-1} = \frac{1-q^i}{1-q}$$

$$[n]_q! = [1]_q \times [2]_q \times \cdots \times [n]_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$



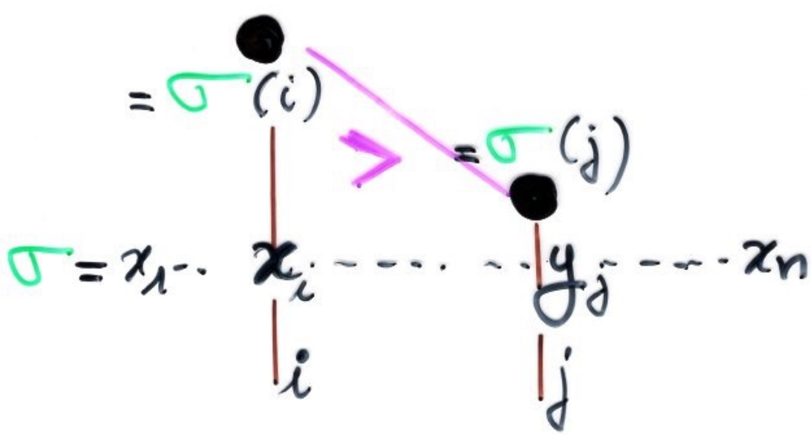
$\sigma \in \mathfrak{S}_n \longrightarrow f \in \mathfrak{F}_n$ **Inversion table**

$$\sigma = \frac{7\;2\;3\;6\;8\;5\;1\;4}{6\;1\;1\;3\;3\;2\;0\;0}$$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3

$1 \leq x \leq n$ $\quad x = \sigma(i)$

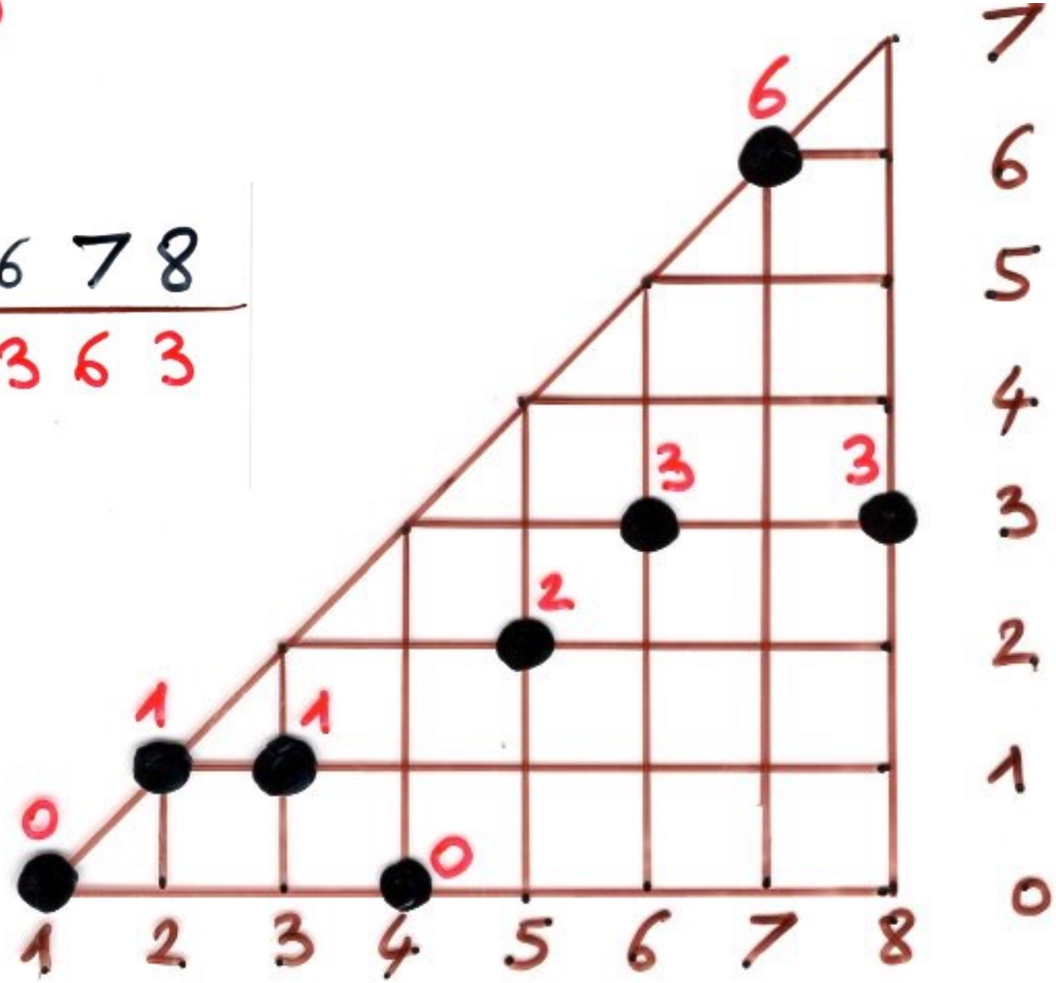
$f(x) =$ number of j, $i < j \leq n$ with $\sigma(j) < \sigma(i)$



Inversion table

$\sigma = \dfrac{7\ 2\ 3\ 6\ 8\ 5\ 1\ 4}{6\ 1\ 1\ 3\ 3\ 2\ 0\ 0}$

x	1	2	3	4	5	6	7	8
$f(x)$	0	1	1	0	2	3	6	3



reverse bijection

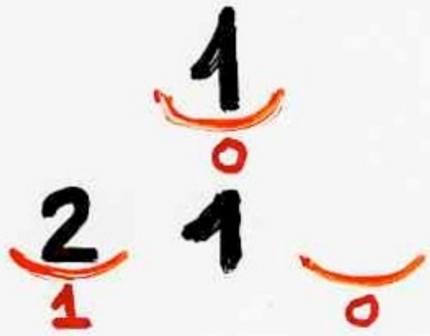
and

q-analogs of histories

1
0

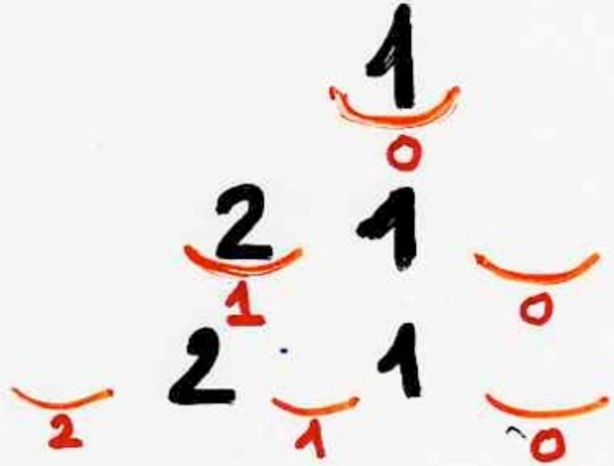
9°

9°



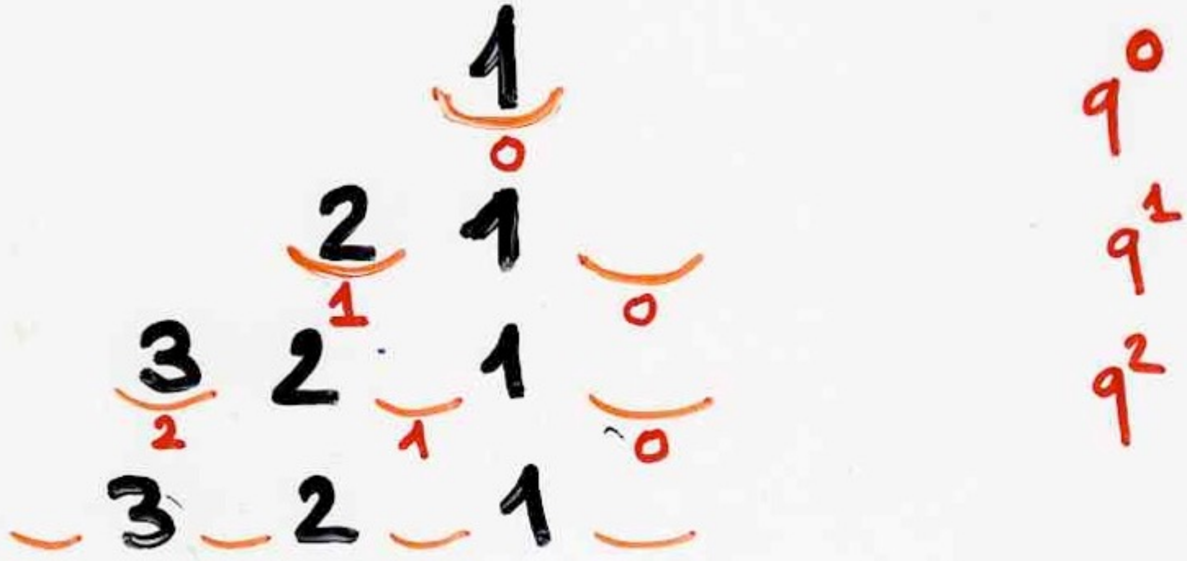
9^0

9^1

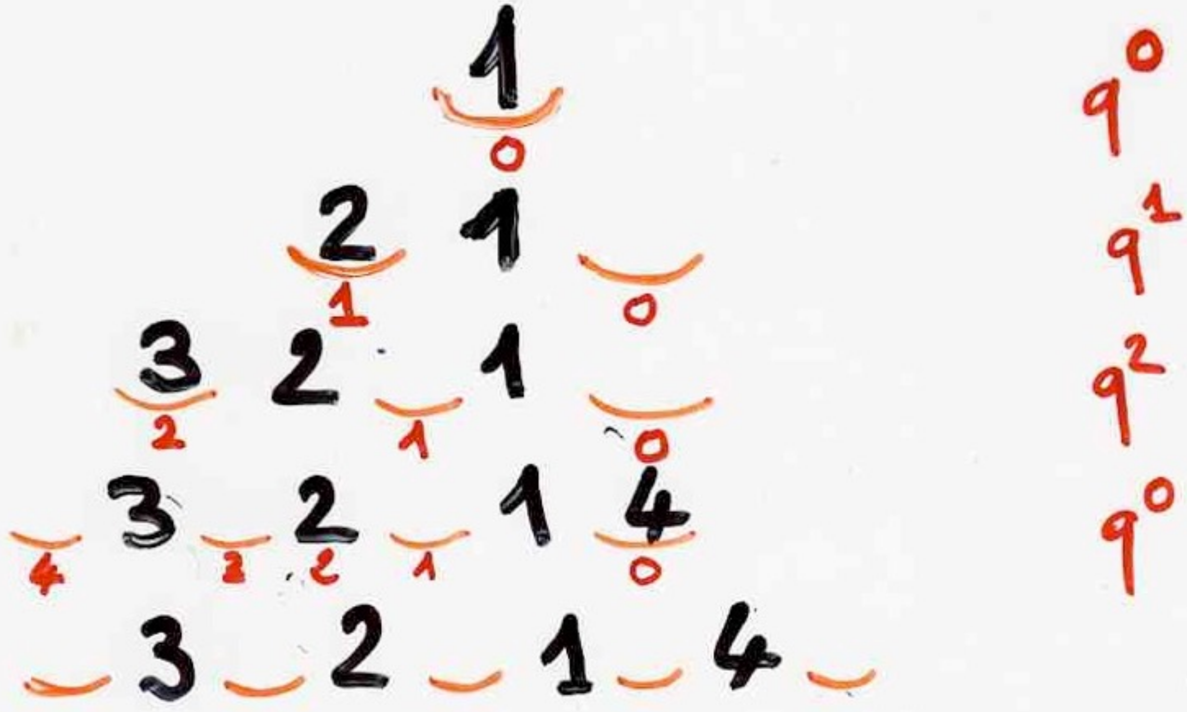


9^0

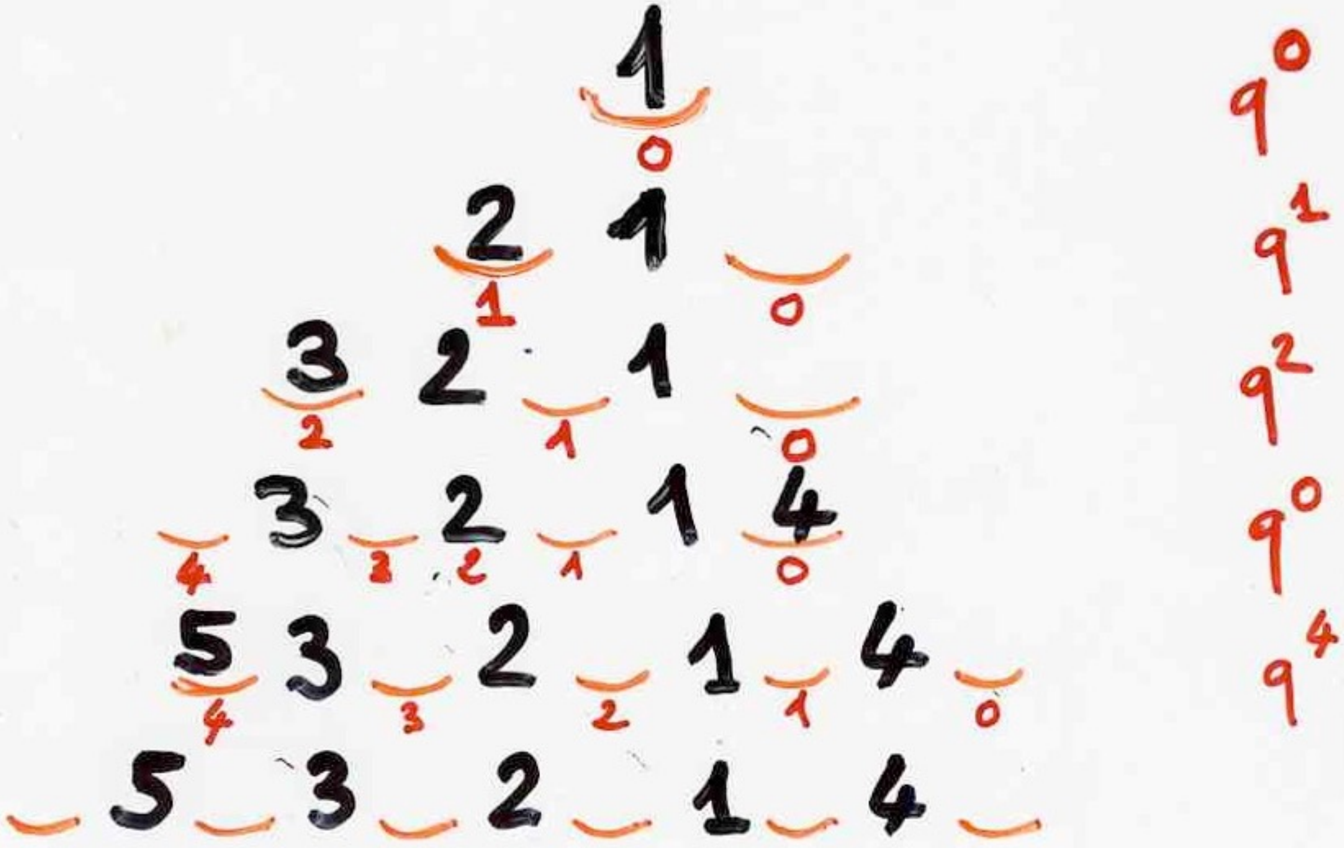
9^1

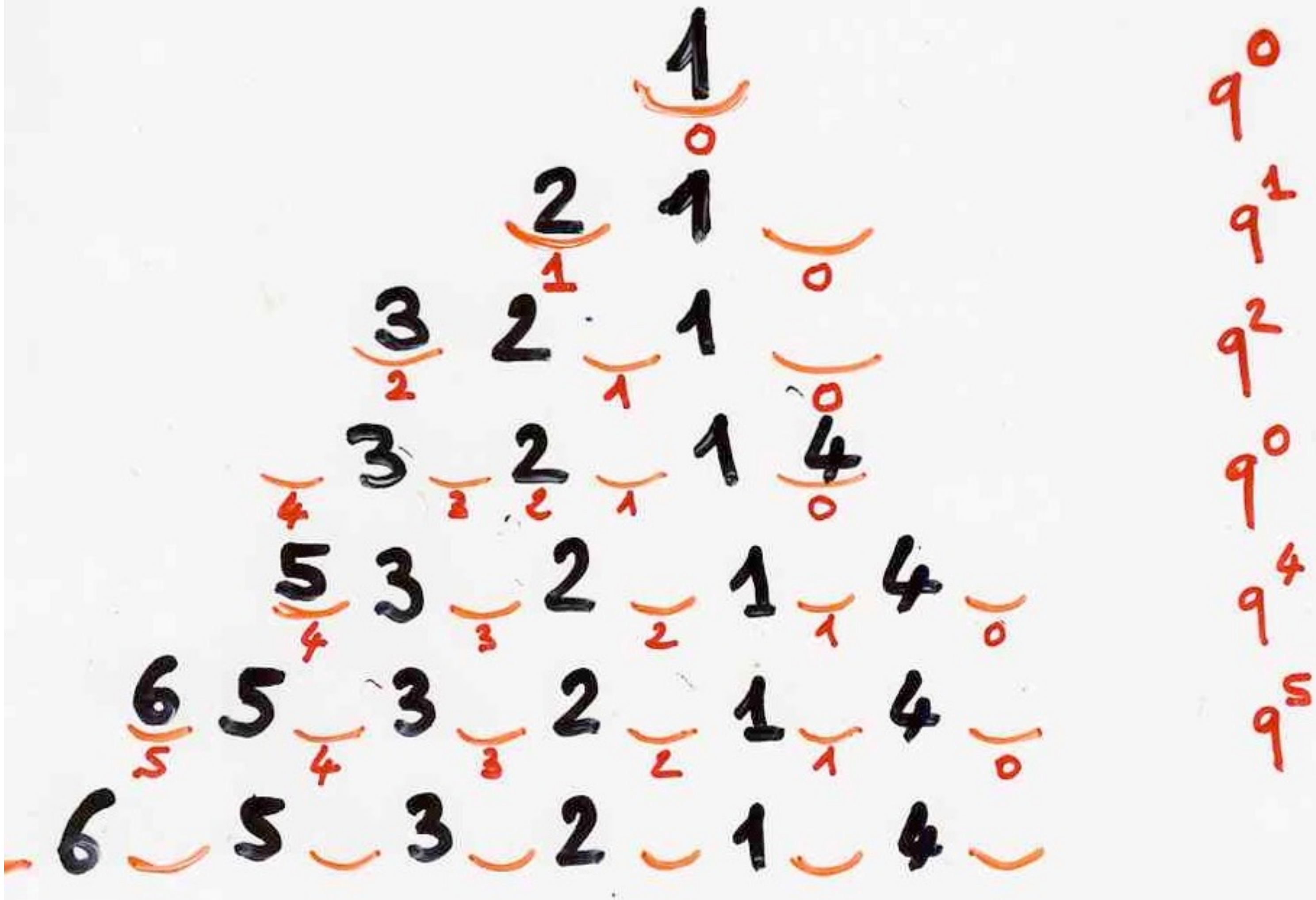


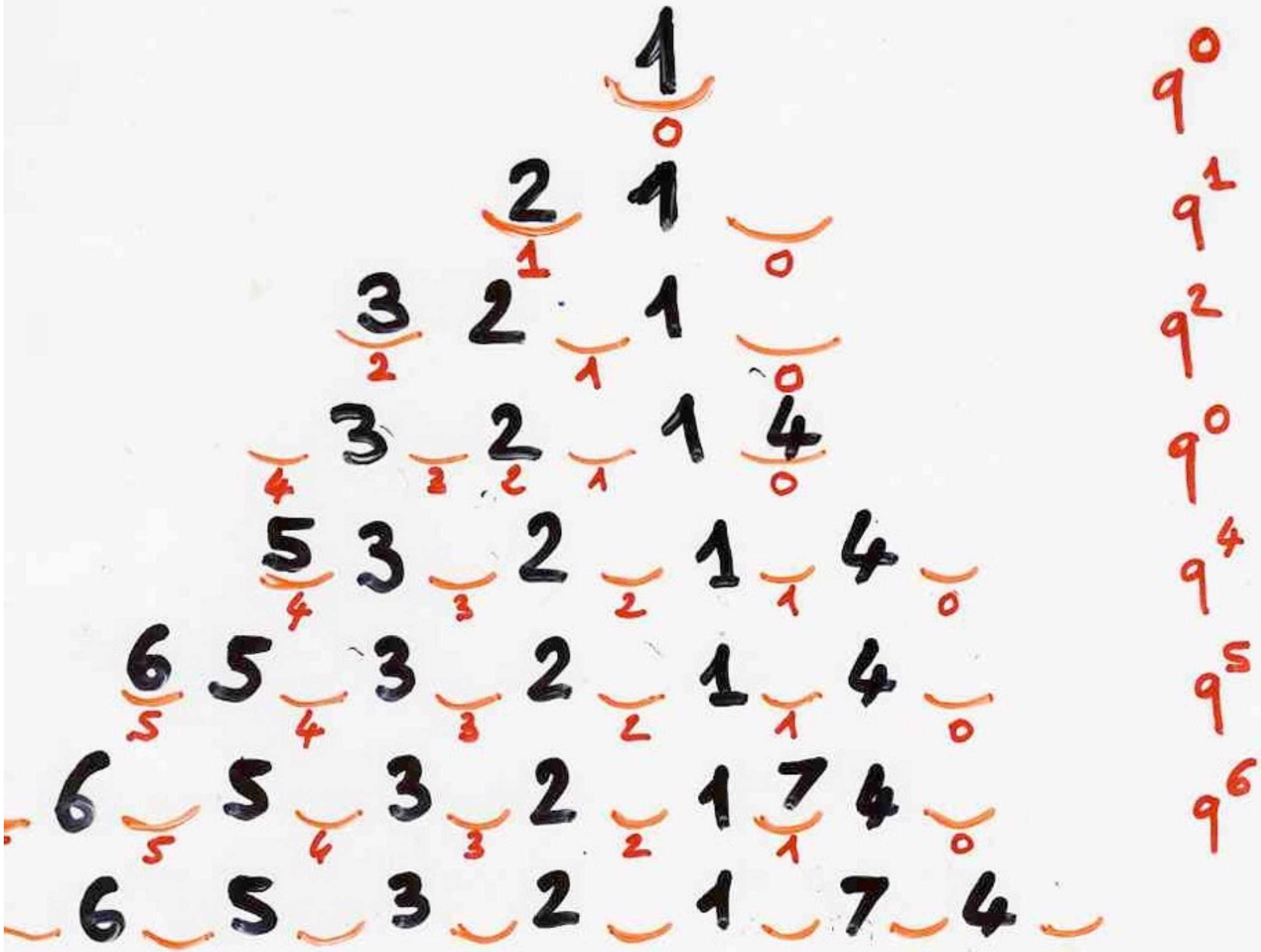
1
0
2 1
1 0
3 2 1
2 1 0
3 2 1
q^0
q^1
q^2



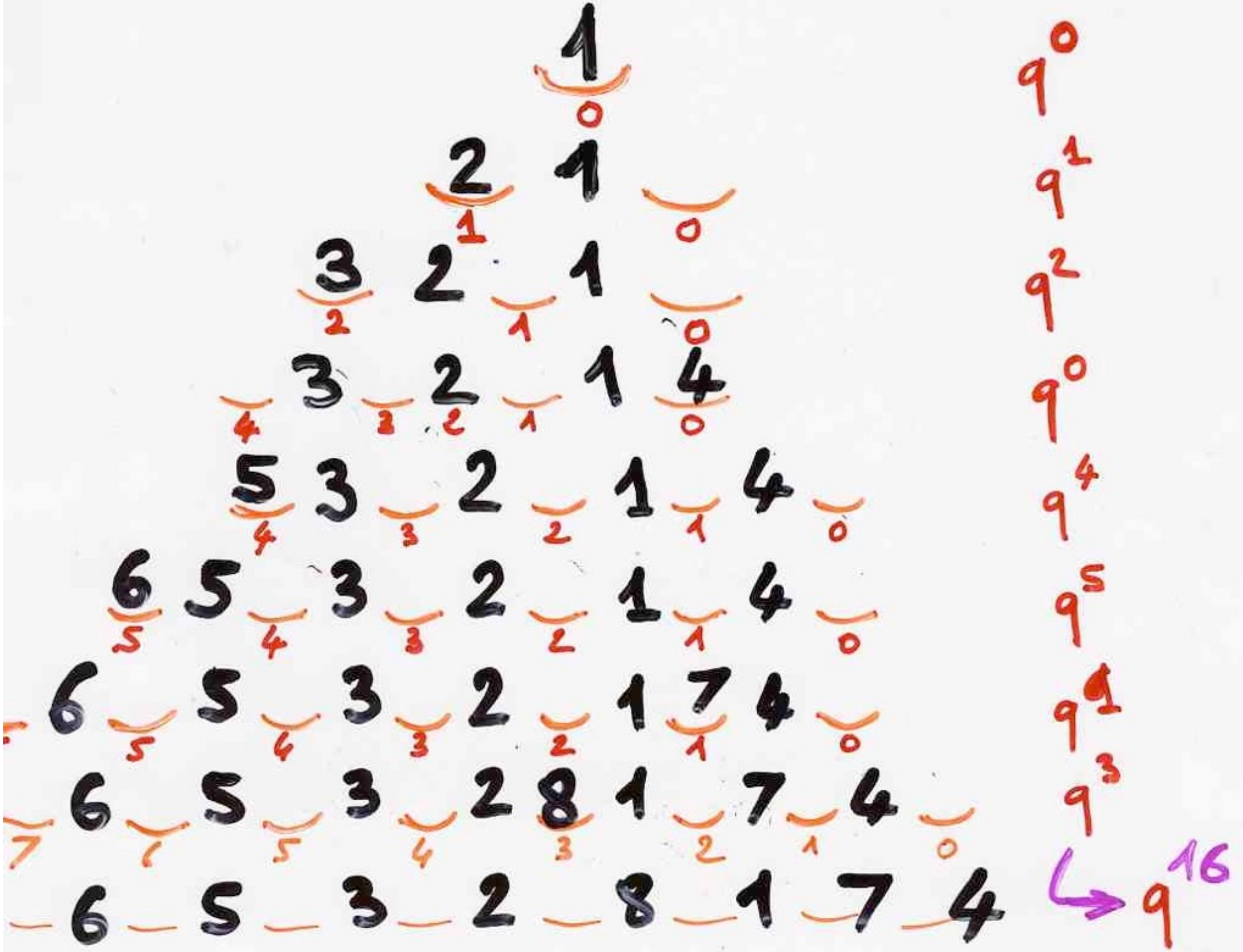
1
0
2 1
1 0
3 2 1
2 1 0
3 2 1 4
4 3 2 1 0
3 2 1 4
q^0
q^1
q^2
q^0


1
0
2 1
1 0
3 2 1
2 1 0
3 2 1 4
4 3 2 1 0
5 3 2 1 4
4 3 2 1 0
5 3 2 1 4
9^0
9^1
9^2
9^0
9^4

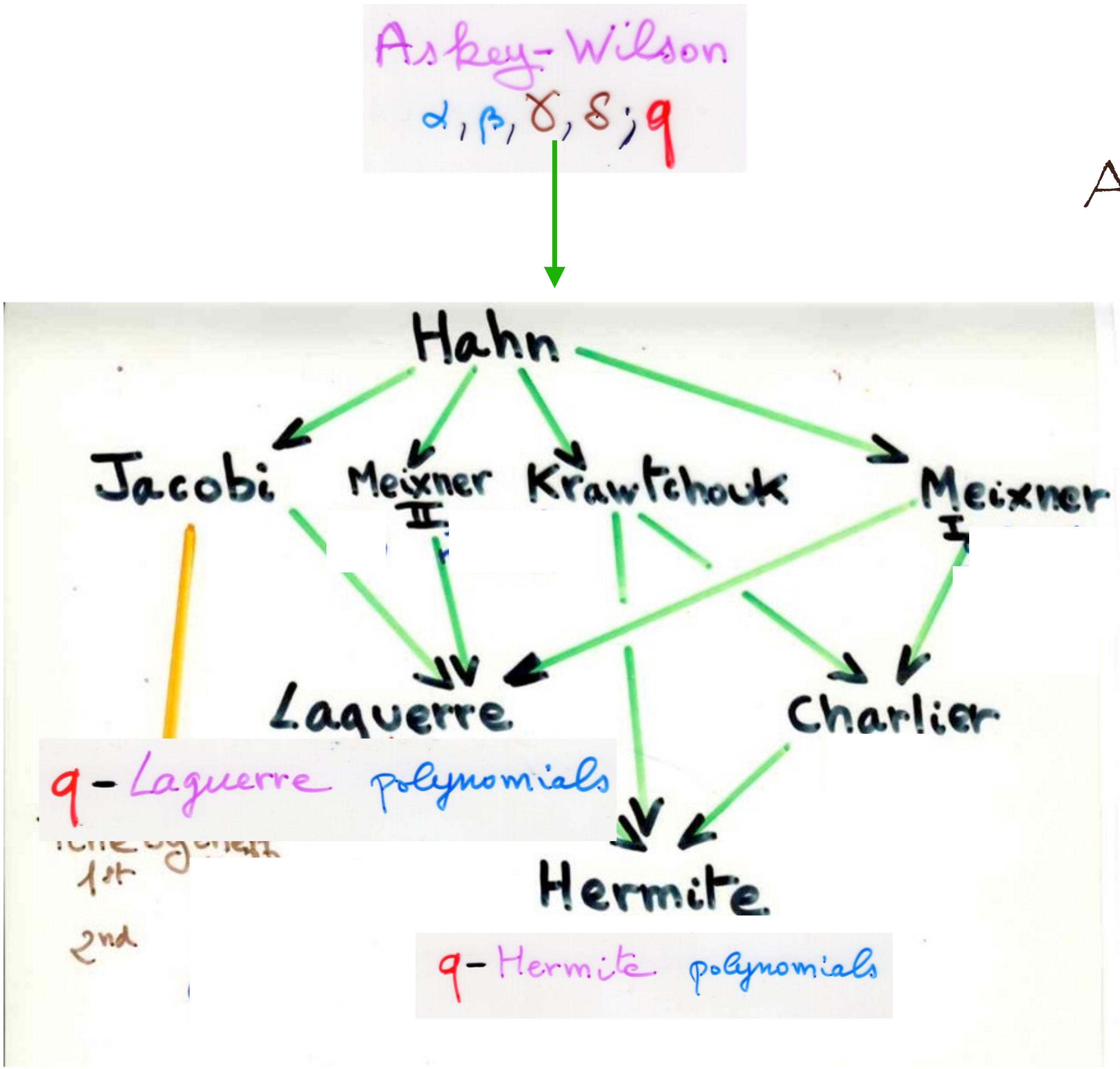

1
0
2 1
1 0
3 2 1
2 1 0
3 2 1 4
4 3 2 1 0
5 3 2 1 4
4 3 2 1 0
6 5 3 2 1 4
5 4 3 2 1 0
6 5 3 2 1 4
9^0
9^1
9^2
9^0
9^4
9^5

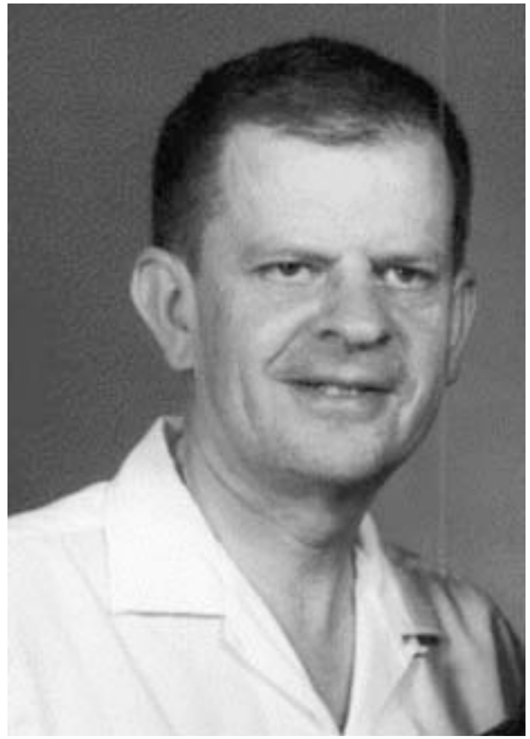


1 q^0
2 1 q^1
3 2 1 q^2
3 2 1 4 q^0
5 3 2 1 4 q^4
6 5 3 2 1 4 q^5
6 5 3 2 1 7 4 q^6
6 5 3 2 1 7 4


1 — q^0
2 1 — q^1
3 2 1 — q^2
3 2 1 4 — q^0
5 3 2 1 4 — q^4
6 5 3 2 1 4 — q^5
6 5 3 2 1 7 4 — q^1
6 5 3 2 8 1 7 4 — q^3
6 5 3 2 8 1 7 4 — q^{16}

Askey tableau





q - Laguerre II

if $\mu_n = [n!]_q$

then $\begin{cases} b_k = q^k\left([k]_q + [k+1]_q\right) \\ \lambda_k = q^{2k-1}\,[k]_q \times [k]_q \end{cases}$

→ subdivided Laguerre histories
A. de Médicis, X.V. (1994)

q PASEP

q- Laguerre I

then $\begin{cases} b_k = ([k]_q + [k+1]_q) \\ \lambda_k = [k]_q \times [k]_q \end{cases}$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008) y

q-Hermite

$$H_n^{I}(x;q) \qquad b_k = 0 \qquad \lambda_k = [k]_q = 1 + q + \dots + q^{k-1}$$

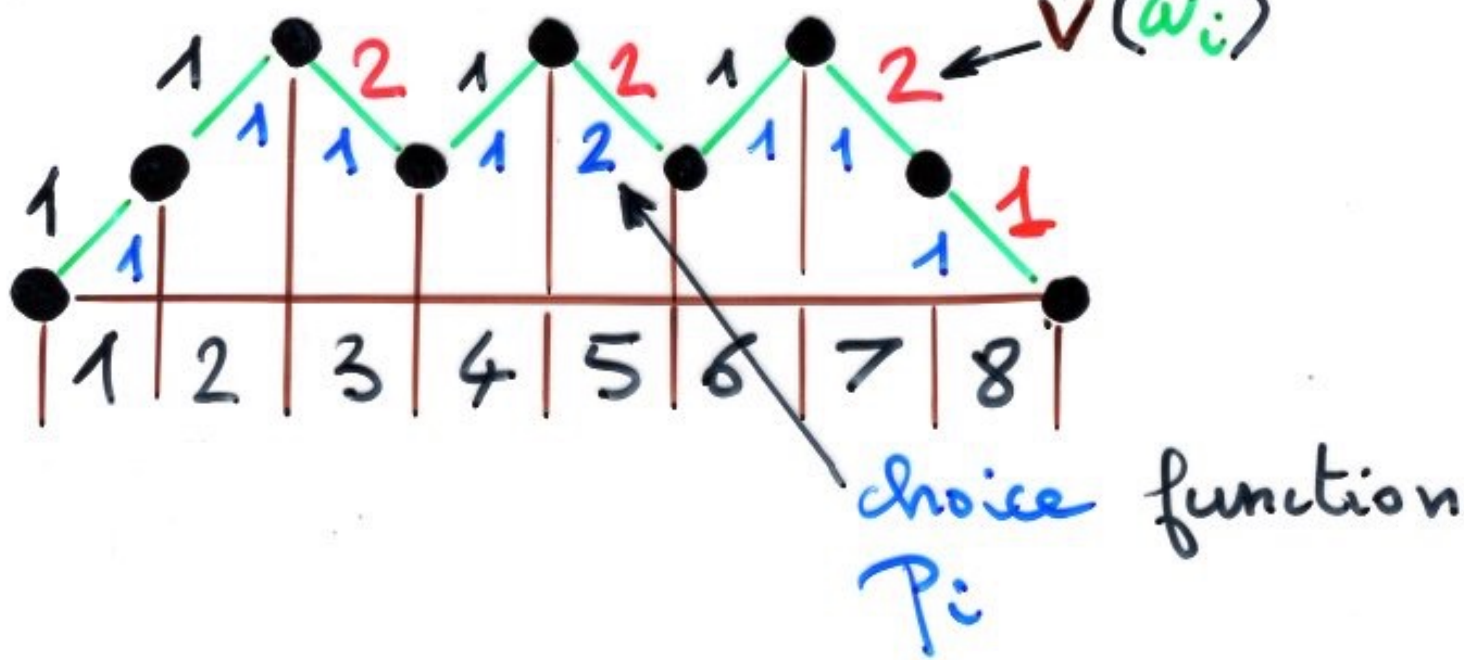
$$\begin{cases} \mu^{I}_{2n+1,q} = 0 \\ \mu^{I}_{2n,q} = \dfrac{1}{(1-q)^n} \displaystyle\sum_{j=0}^{n} (-1)^j t_{n,j} \, q^{j(j+1)/2} \end{cases}$$

$$t_{n,j} = \binom{2n}{n-j} - \binom{2n}{n+j+1}$$

Riordan (1975) Touchard (1952)
Penaud (1995)

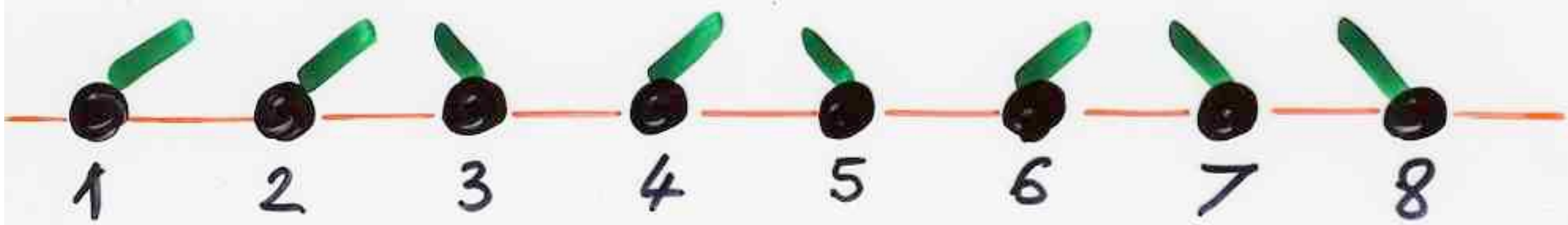
Reminding Ch5a

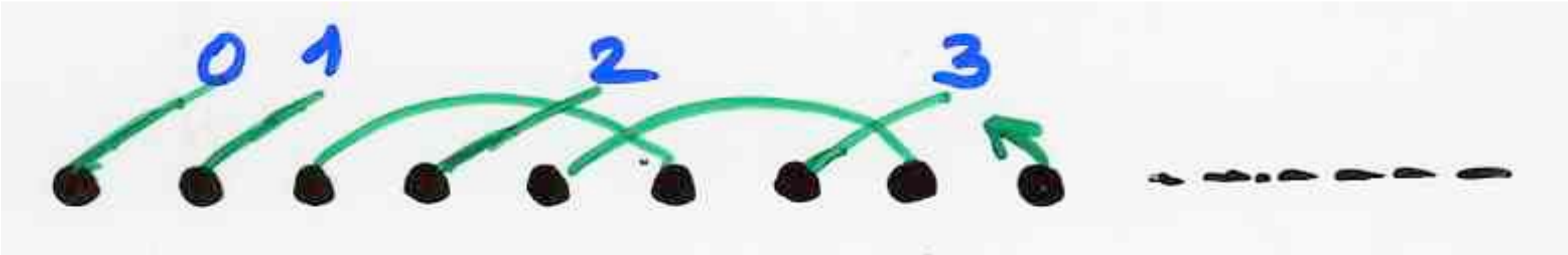
Hermite histories

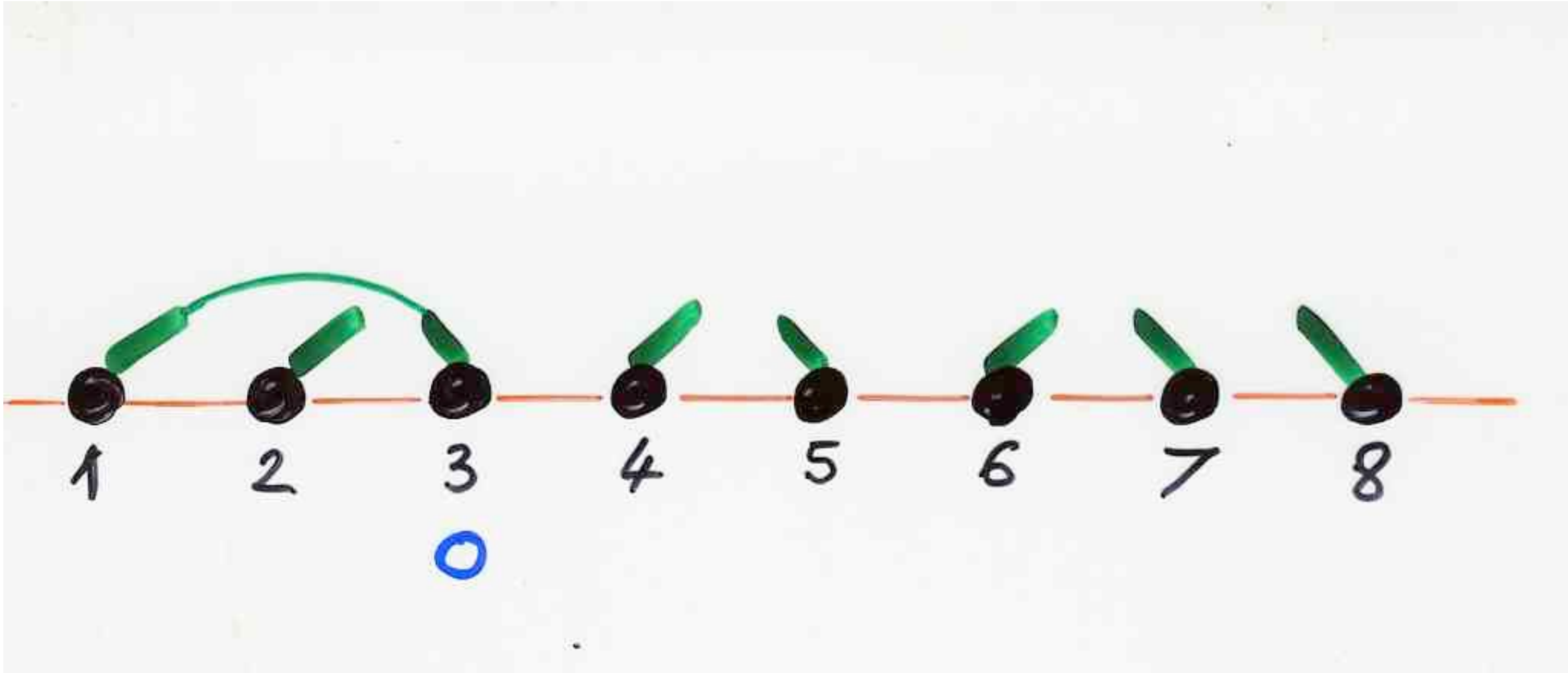


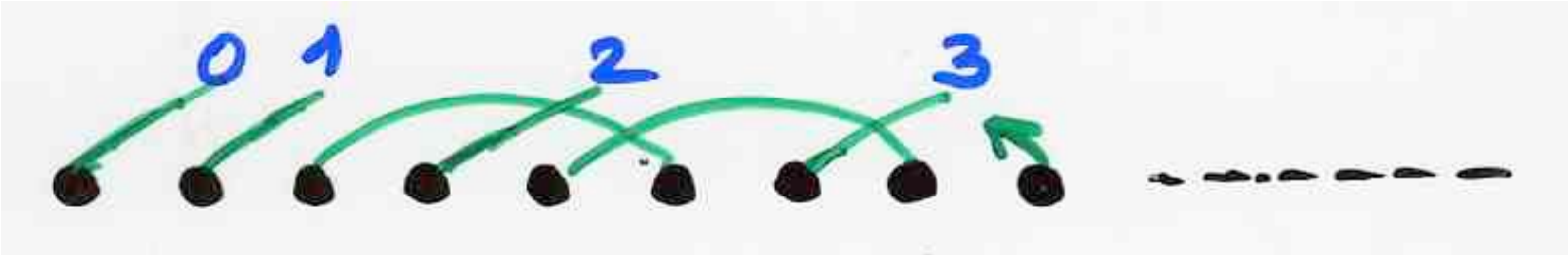
V(ωi)
choice function
Pi
1 2 3 4 5 6 7 8

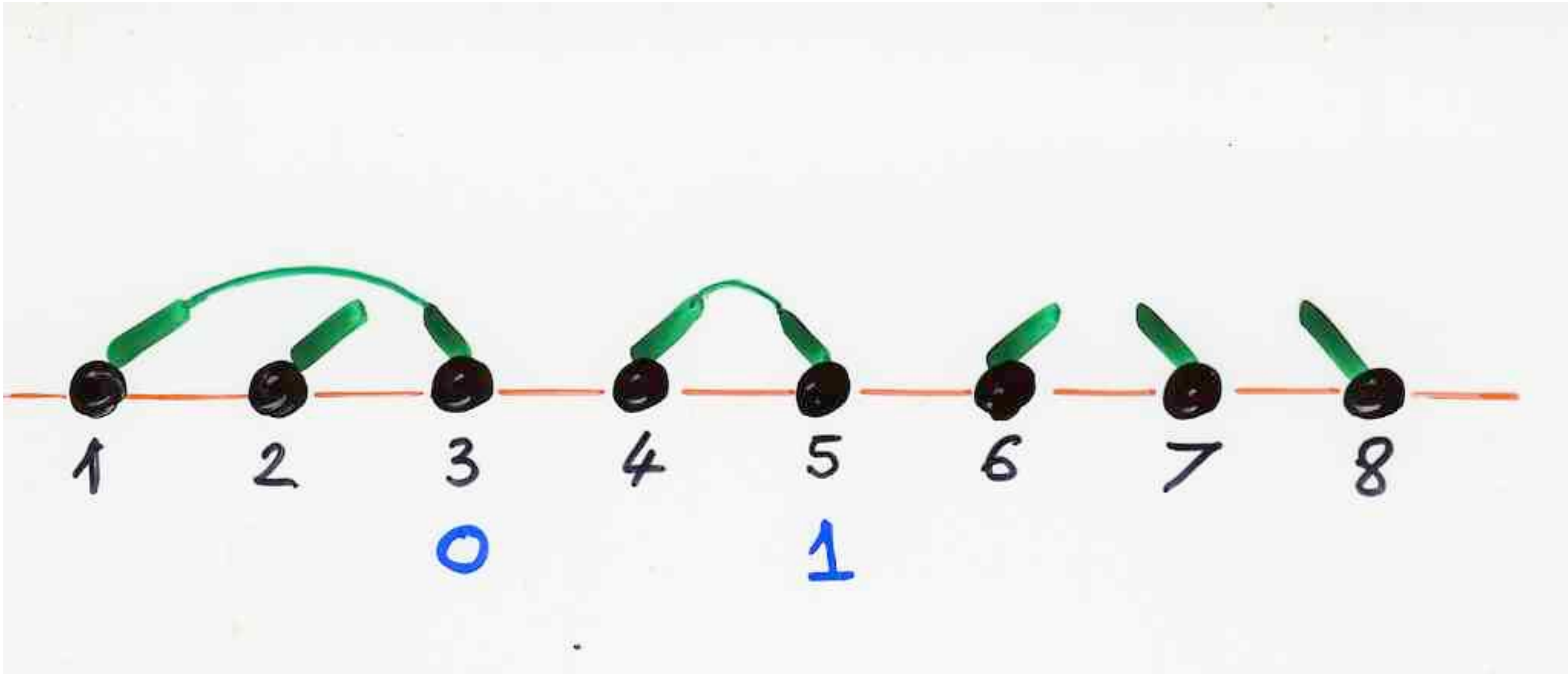
1 2 3 4 5 6 7 8


1
2
3
4
5
6
7
8


0
1
2
3


1
2
3
4
5
6
7
8
0


0
1
2
3


1
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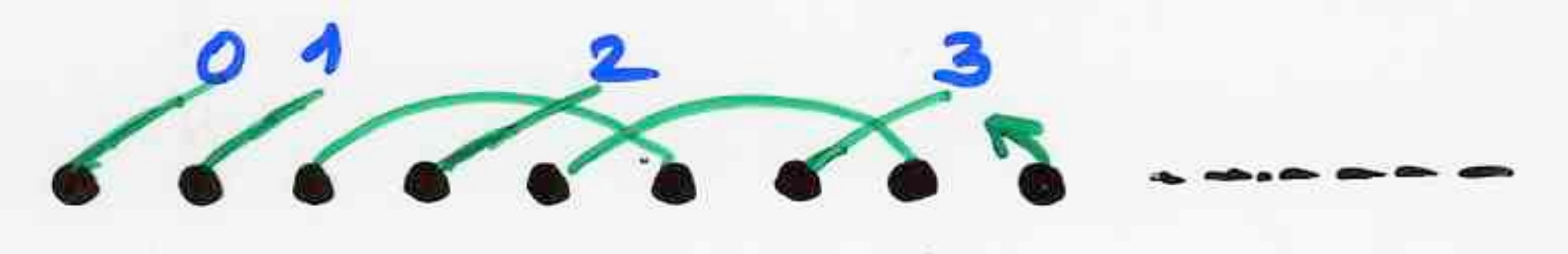
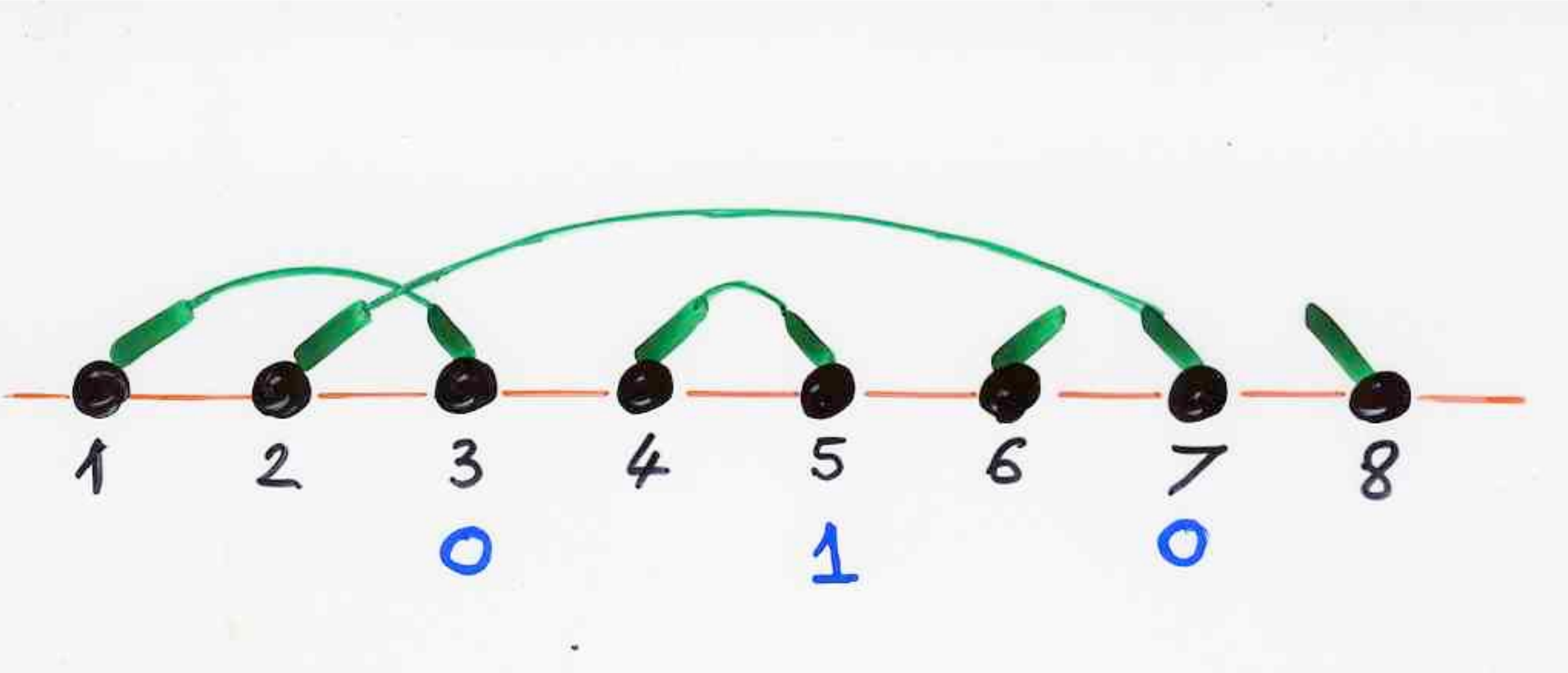
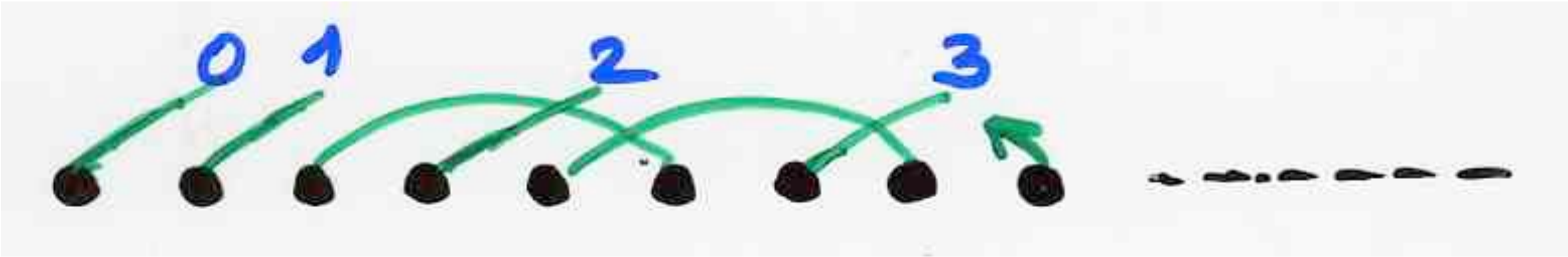
4
5
6
7
8
0
1


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0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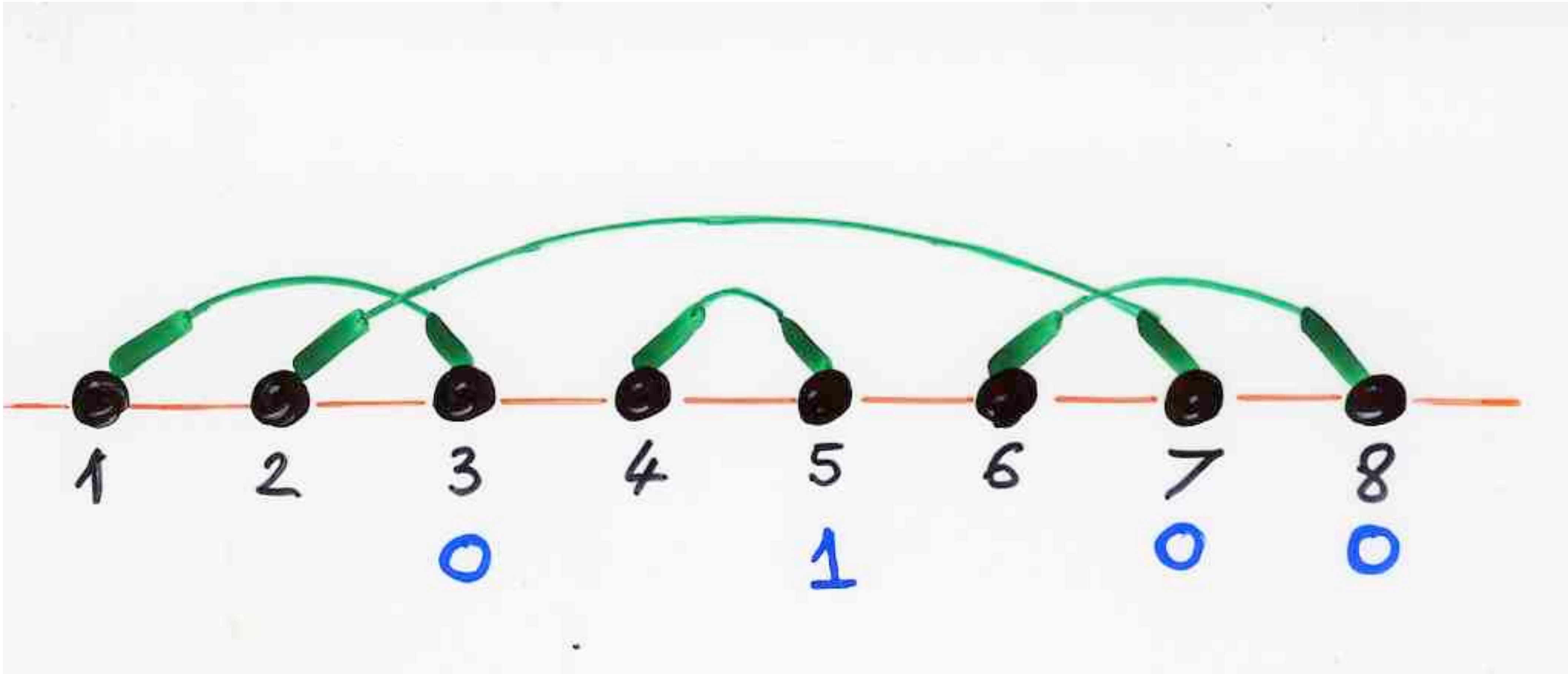

1
2
3
4
5
6
7
8
0
1
0
0

subdivided Laguerre histories

$\mu_n = (n+1)!$

$\mu_n = n!$

$b_k = (2k+2)$
$\lambda_k = k(k+1)$

$b_k = (2k+1)$
$\lambda_k = k^2$

Laguerre histories

restricted Laguerre histories

$\sigma(1) = (n+1)$

$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots\cdots}}}}$$

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

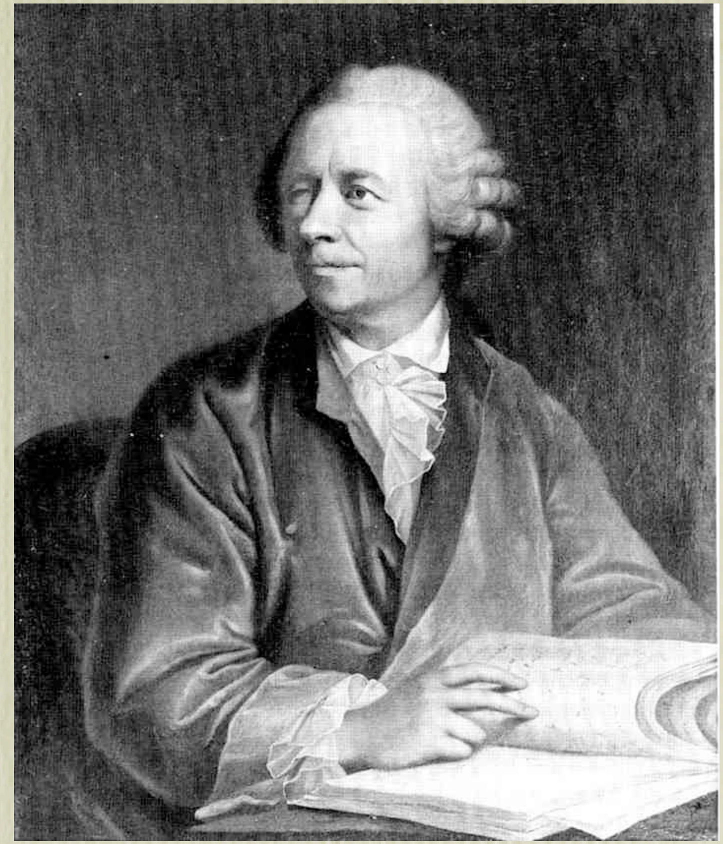
AVCTORE

Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.\ 4.6.\ 6.8.\ 8.10.\ 10}{3.3.\ 5.5.\ 7.7\ \ 9.\ 9.\ 11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-



atque feries infinita ita fe habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis eft huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

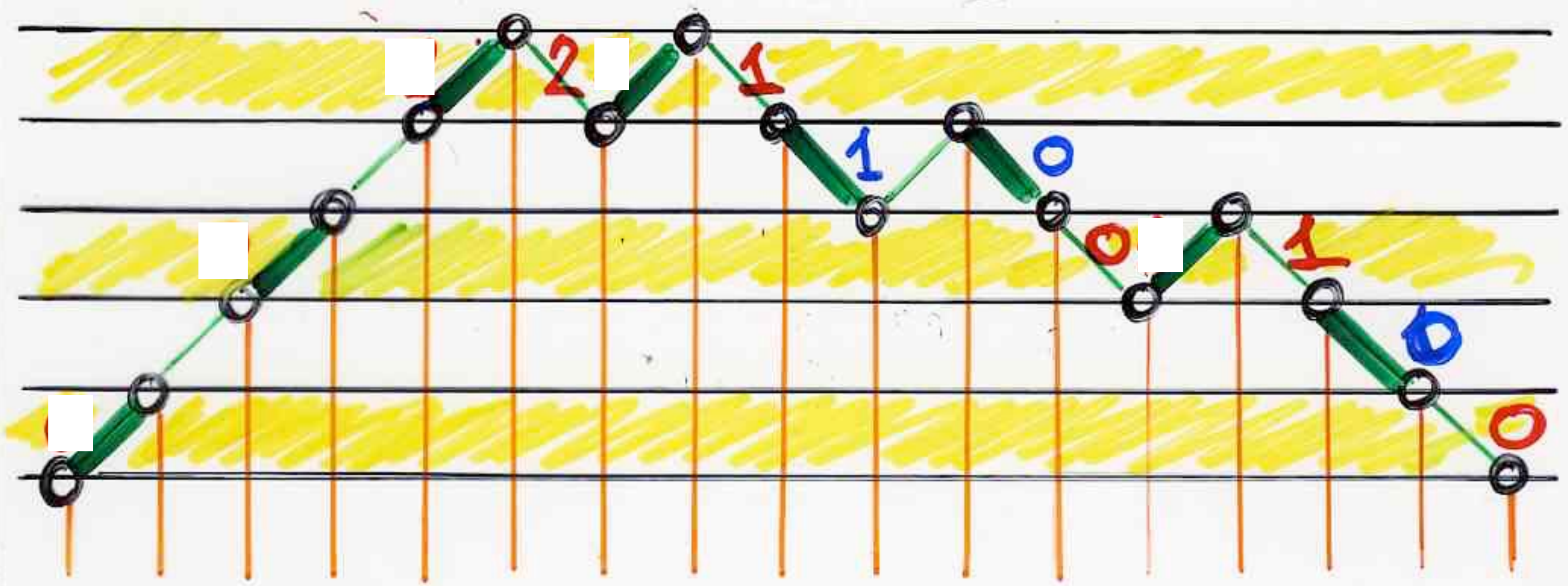
subdivided
Laguerre
histories

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n \geq 0} n!\, t^n =$$

$$\cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}}}$$

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

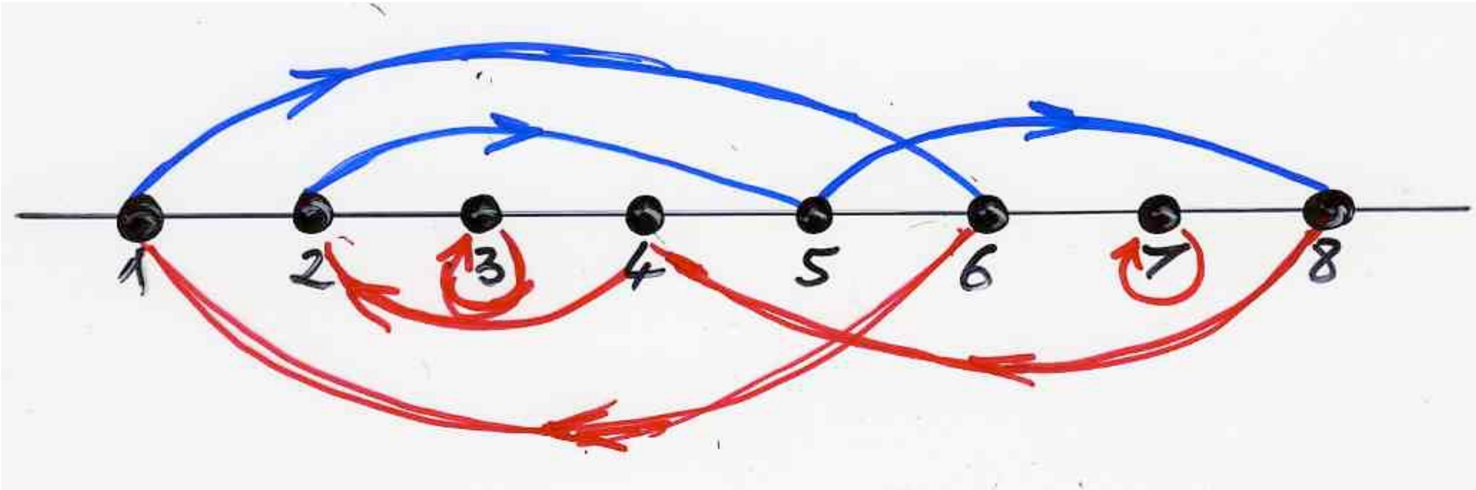


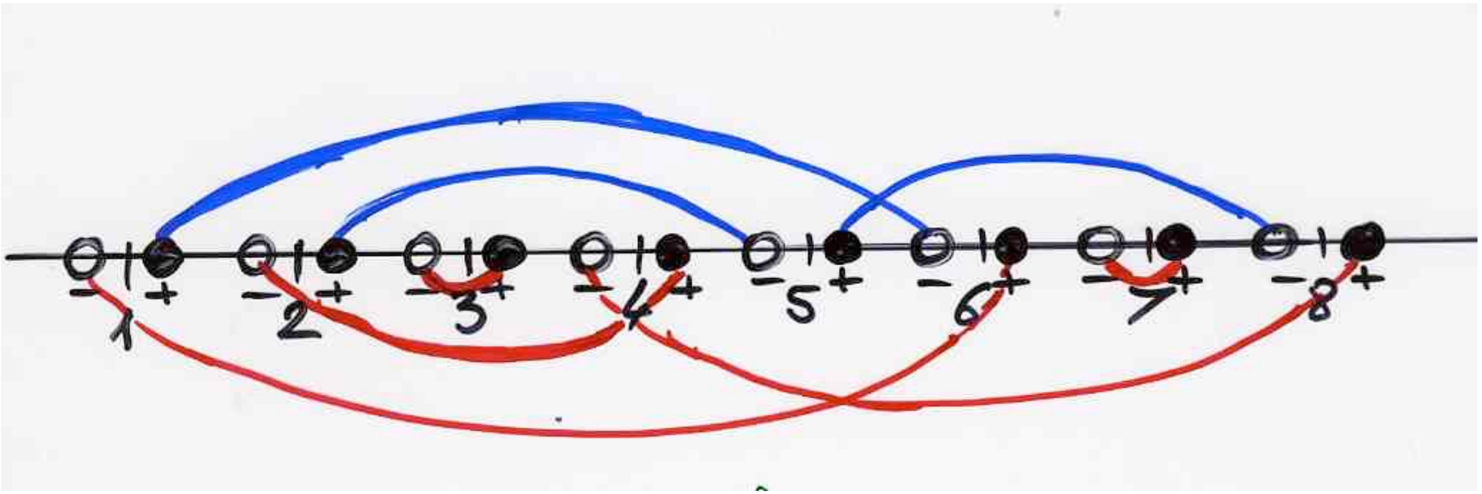
subdivided Laguerre history

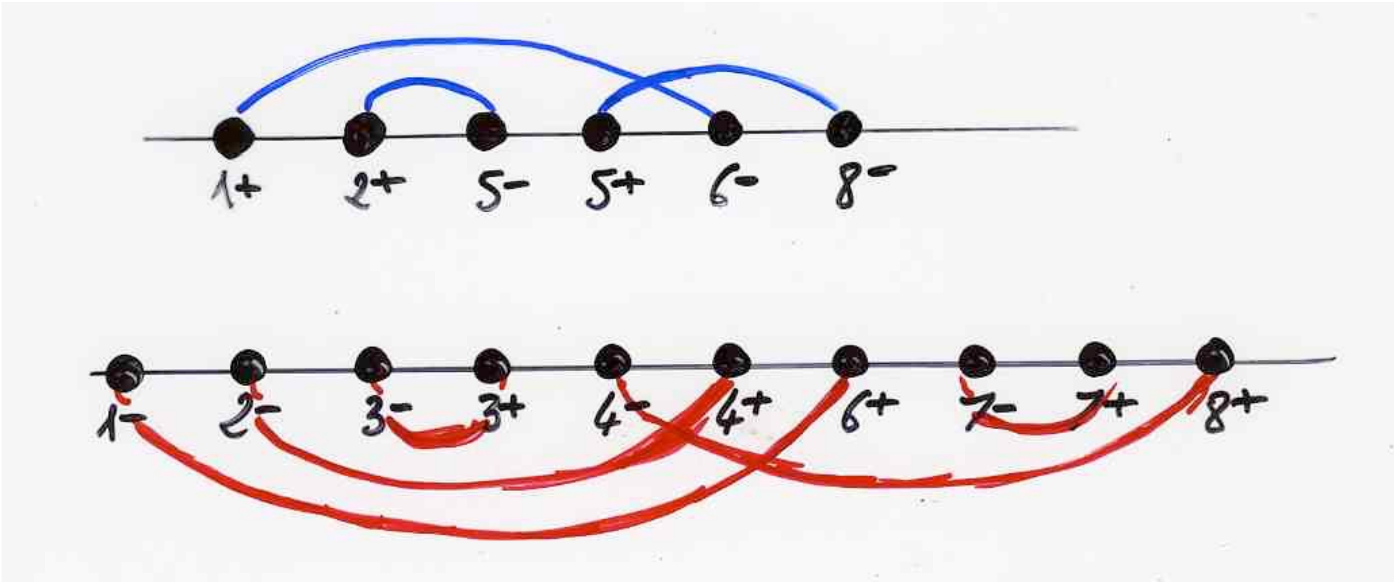
bijection permutations
subdivided Laguerre histories

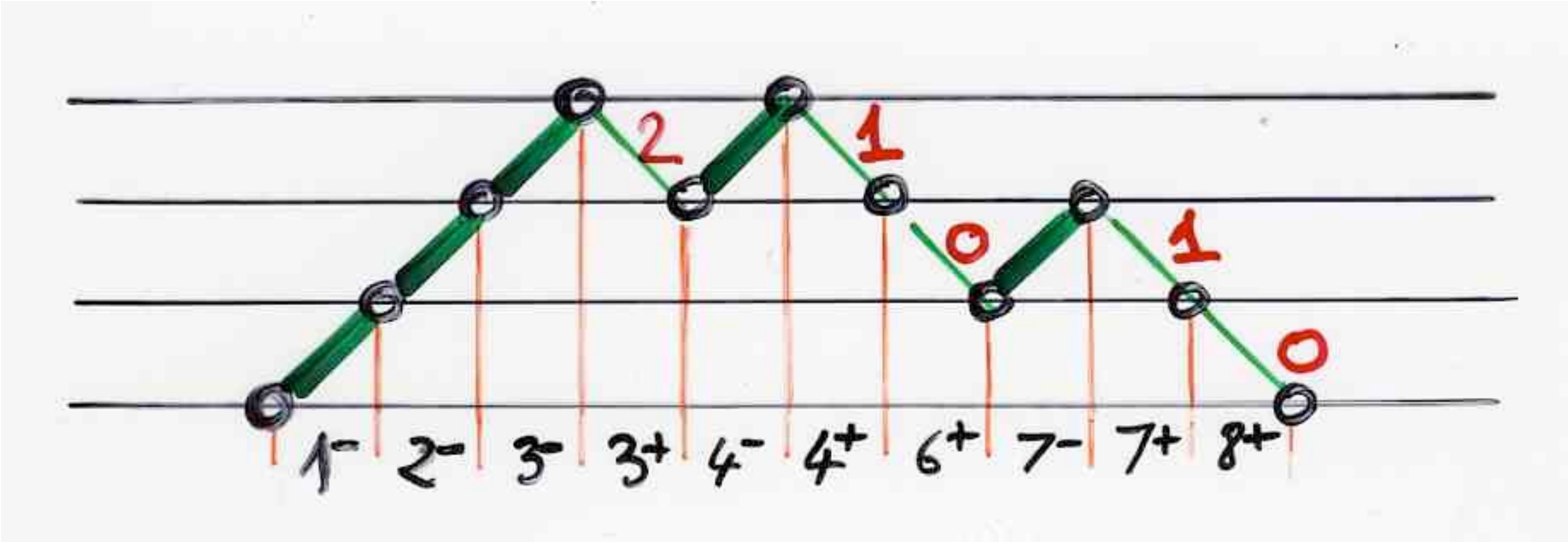
A. de Médicis, X.V.
(1994)

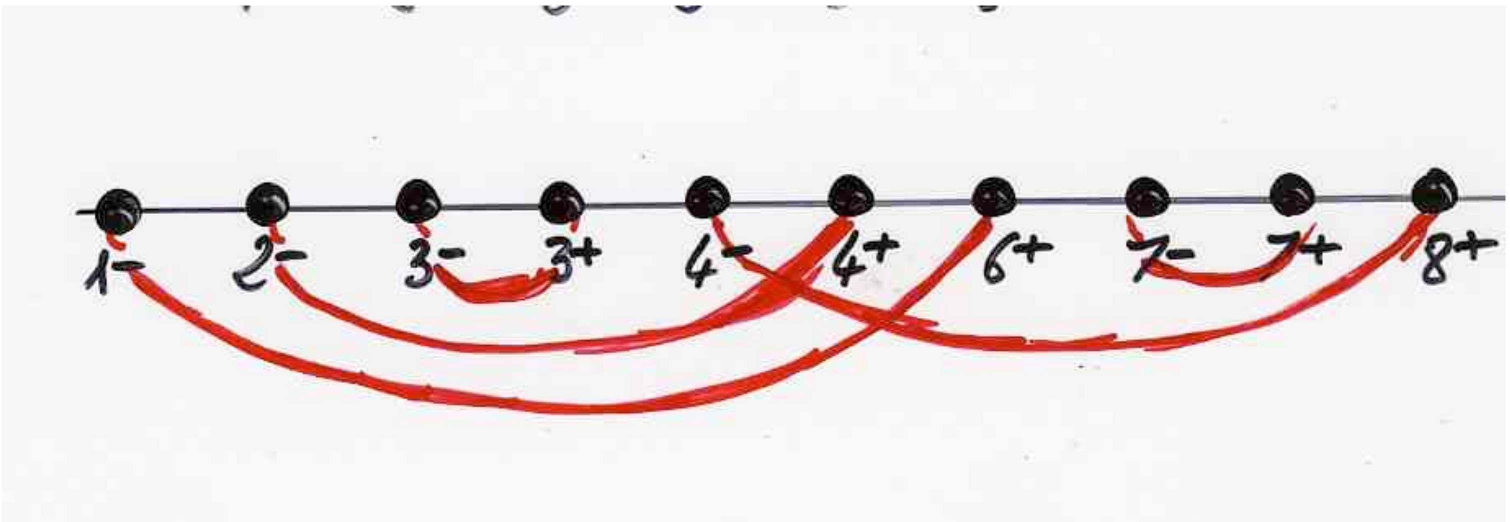
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$

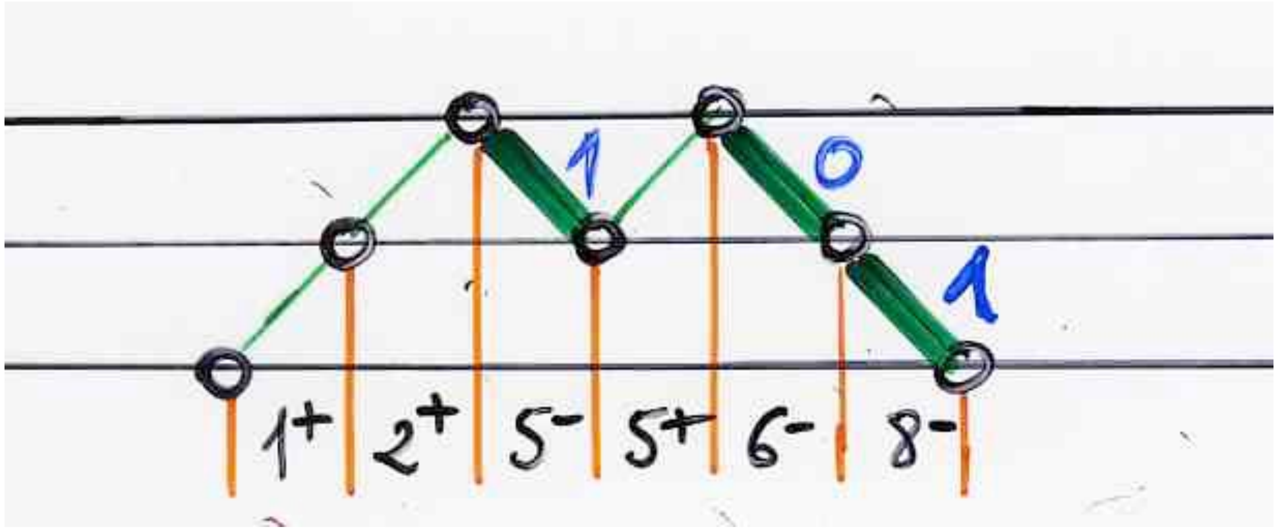


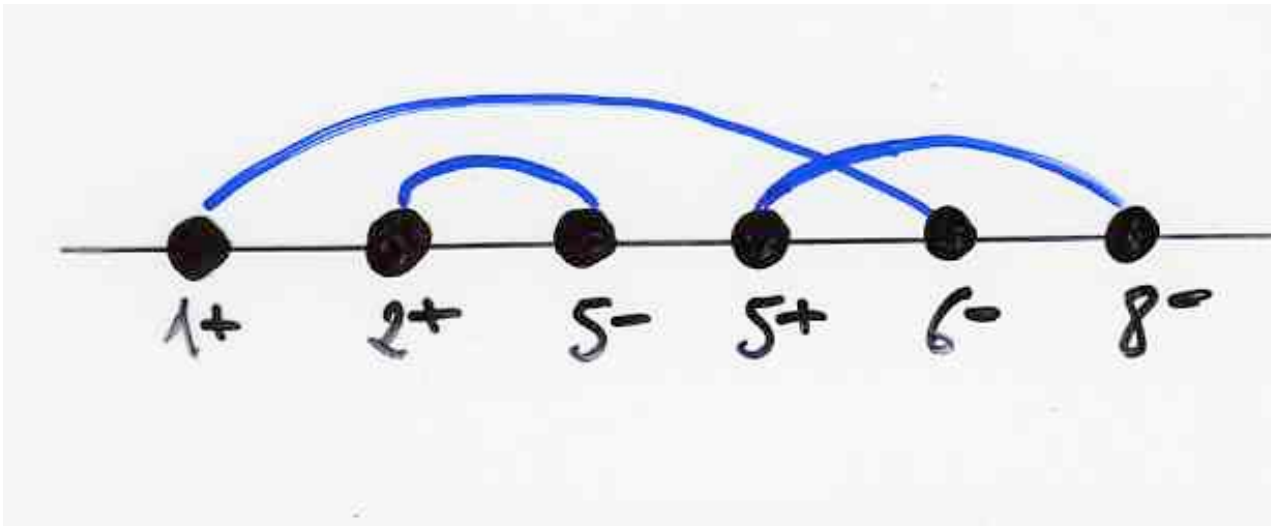


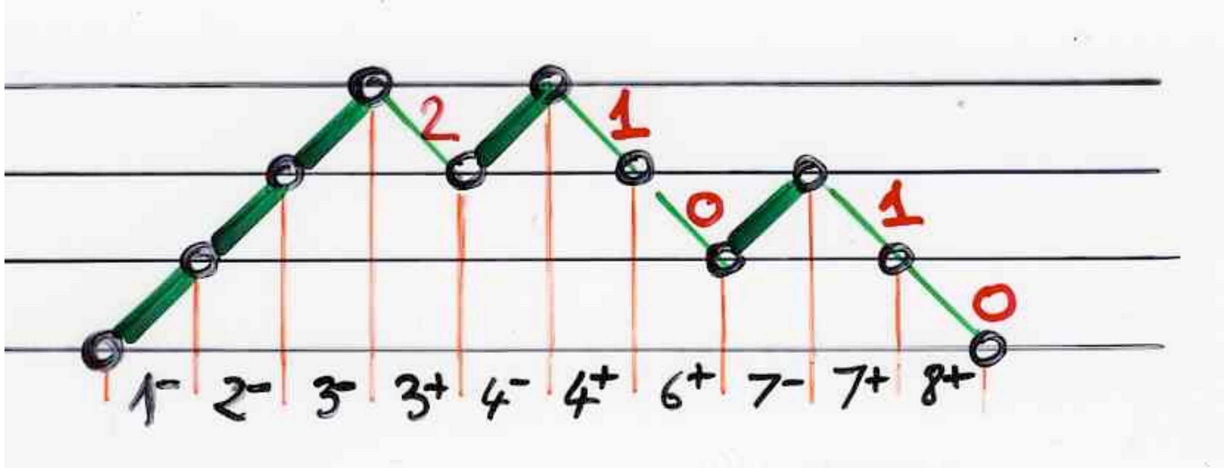

1+
2+
5-
5+
6-
8-
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+


2
1
0
1
0
1⁻ 2⁻ 3⁻ 3+ 4⁻ 4+ 6+ 7⁻ 7+ 8+


1- 2- 3- 3+ 4- 4+ 6+ 7- 7+ 8+

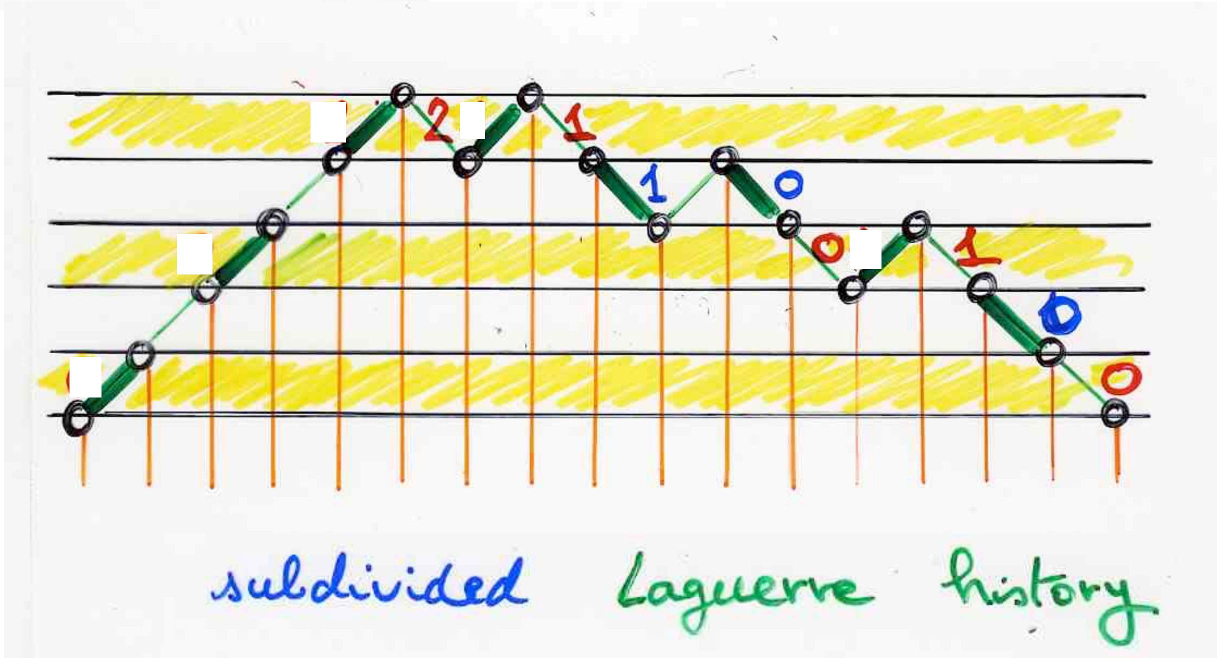

1
0
1
1+ 2+ 5- 5+ 6- 8-

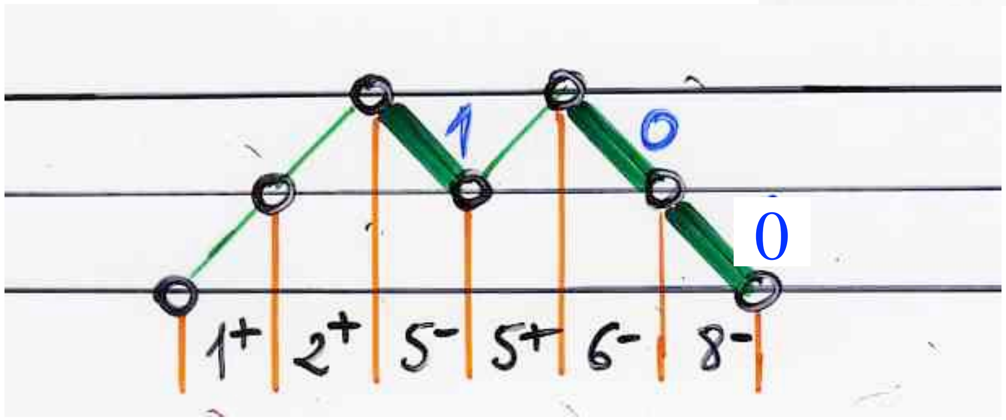

1+ 2+ 5- 5+ 6- 8-

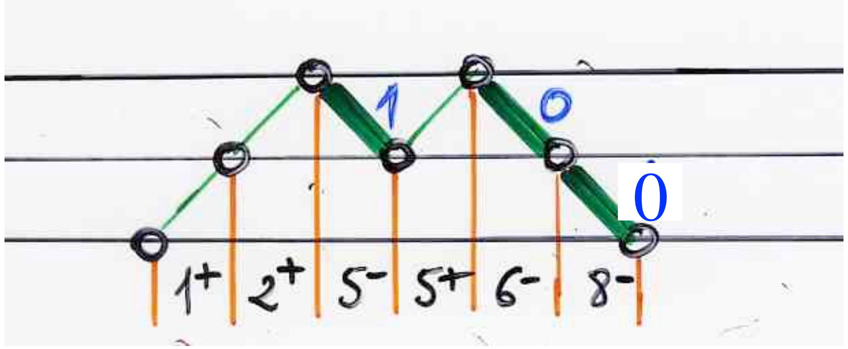


pair of two Hermite histories ("shuffle")

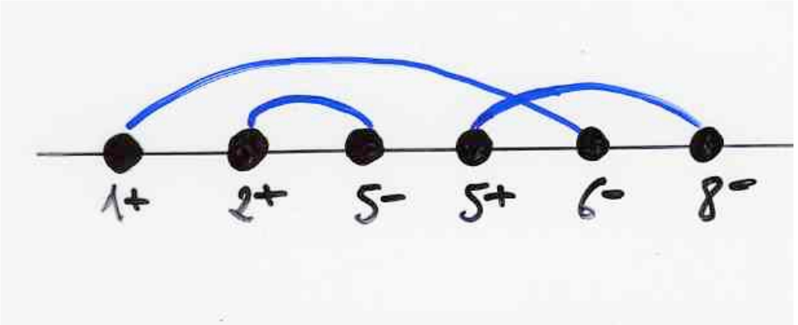
=

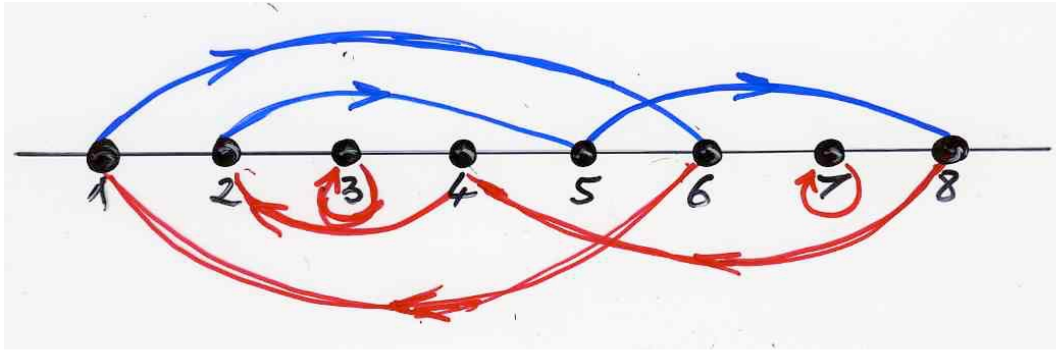


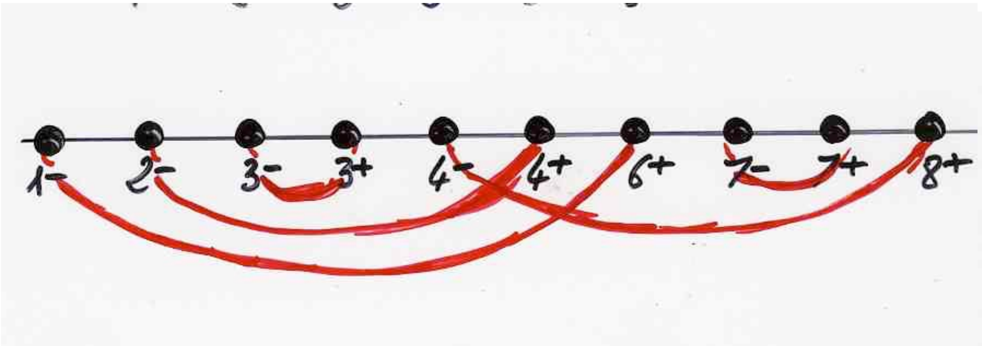


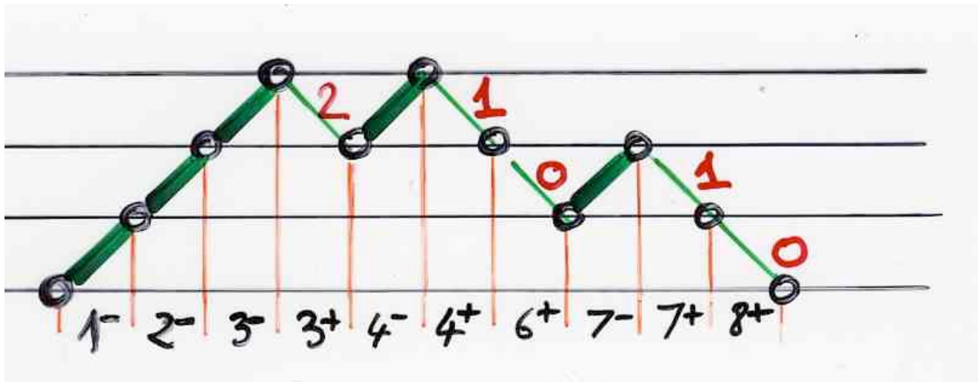


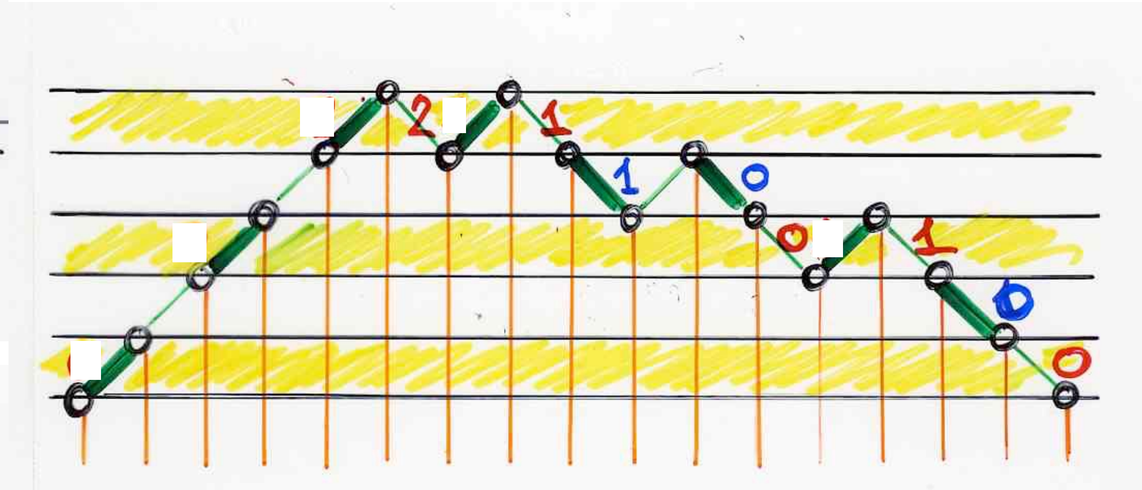
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 3 & 2 & 8 & 1 & 7 & 4 \end{pmatrix}$$



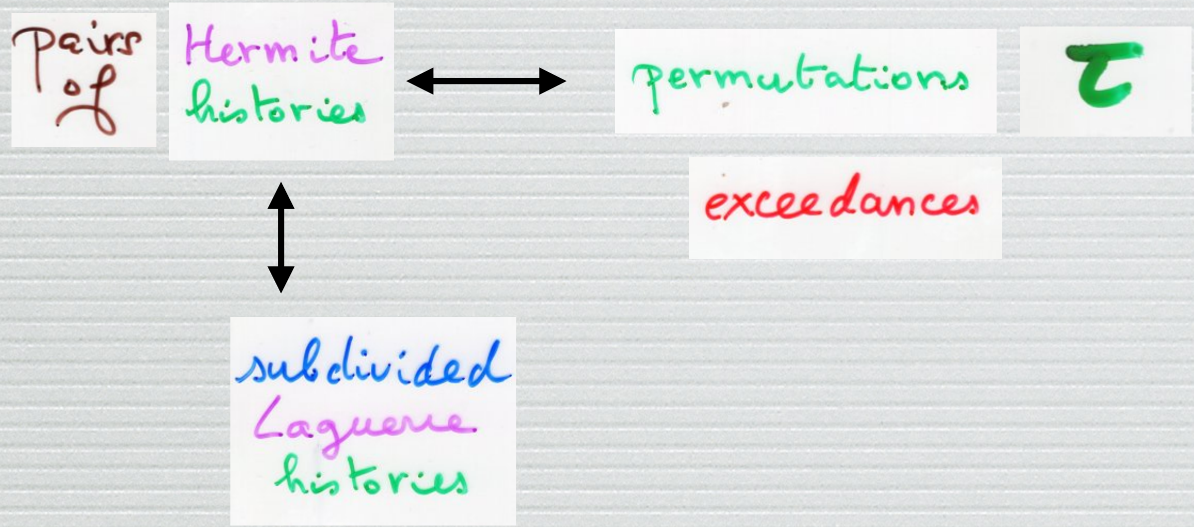








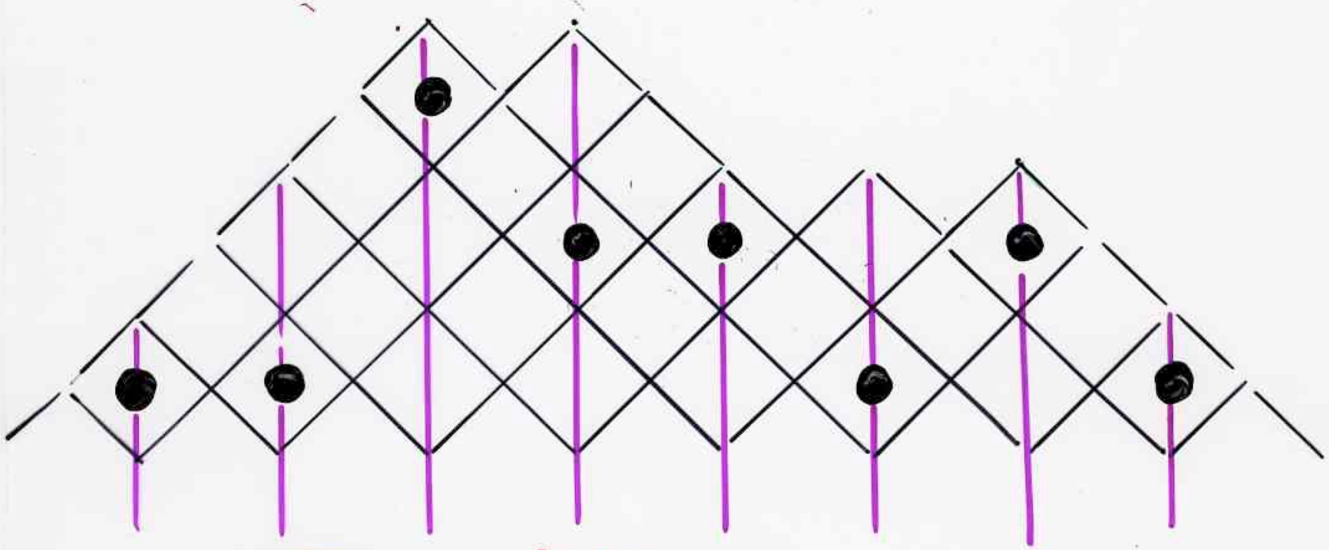
subdivided Laguerre history


Pairs of Hermite histories
permutations τ
exceedances
subdivided Laguerre histories

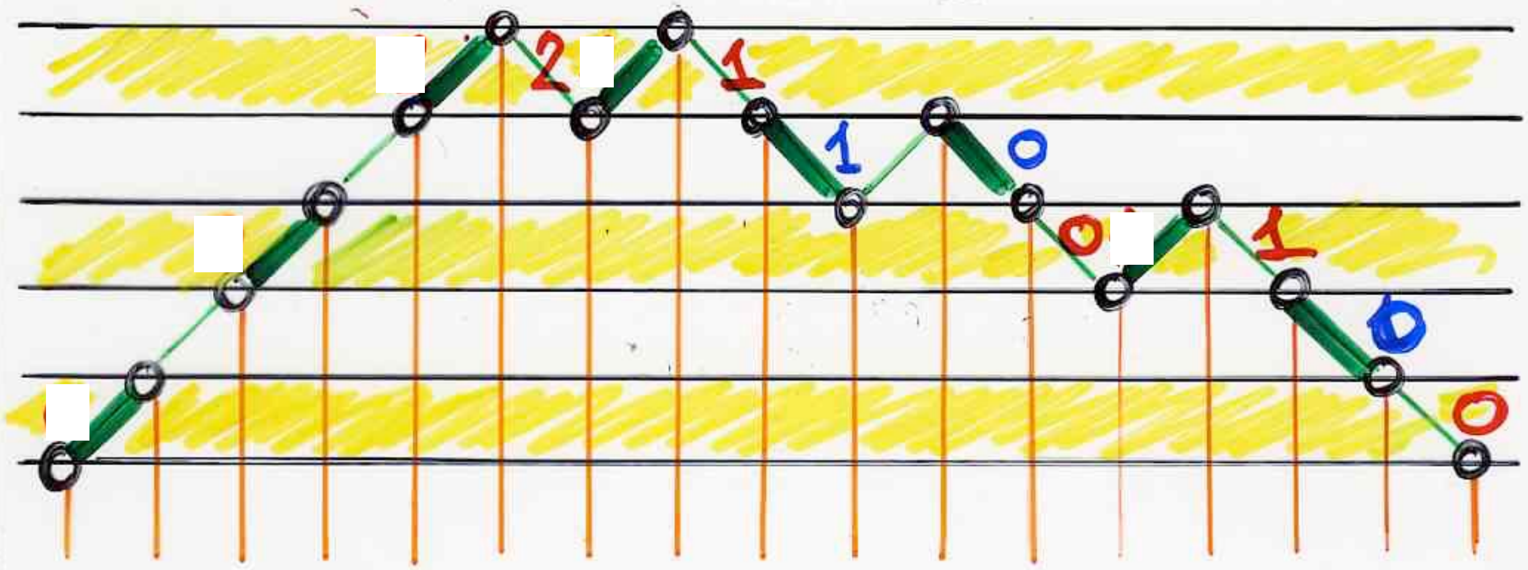
Dyck tableaux

as

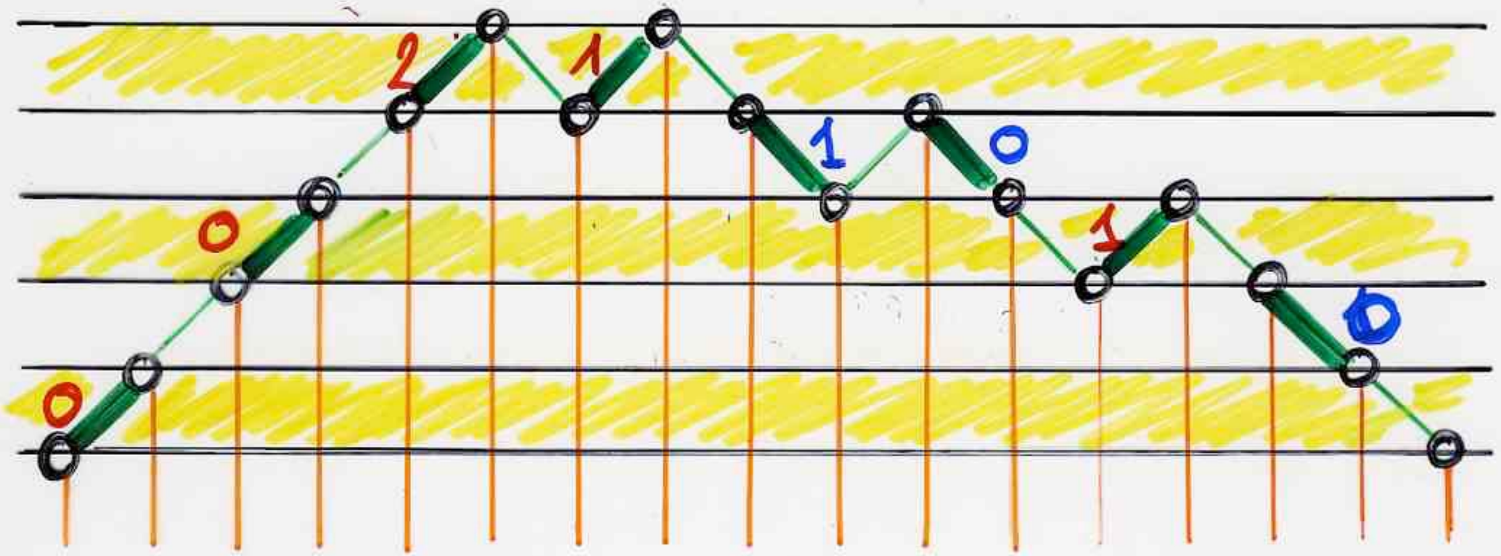
subdivided Laguerre histories



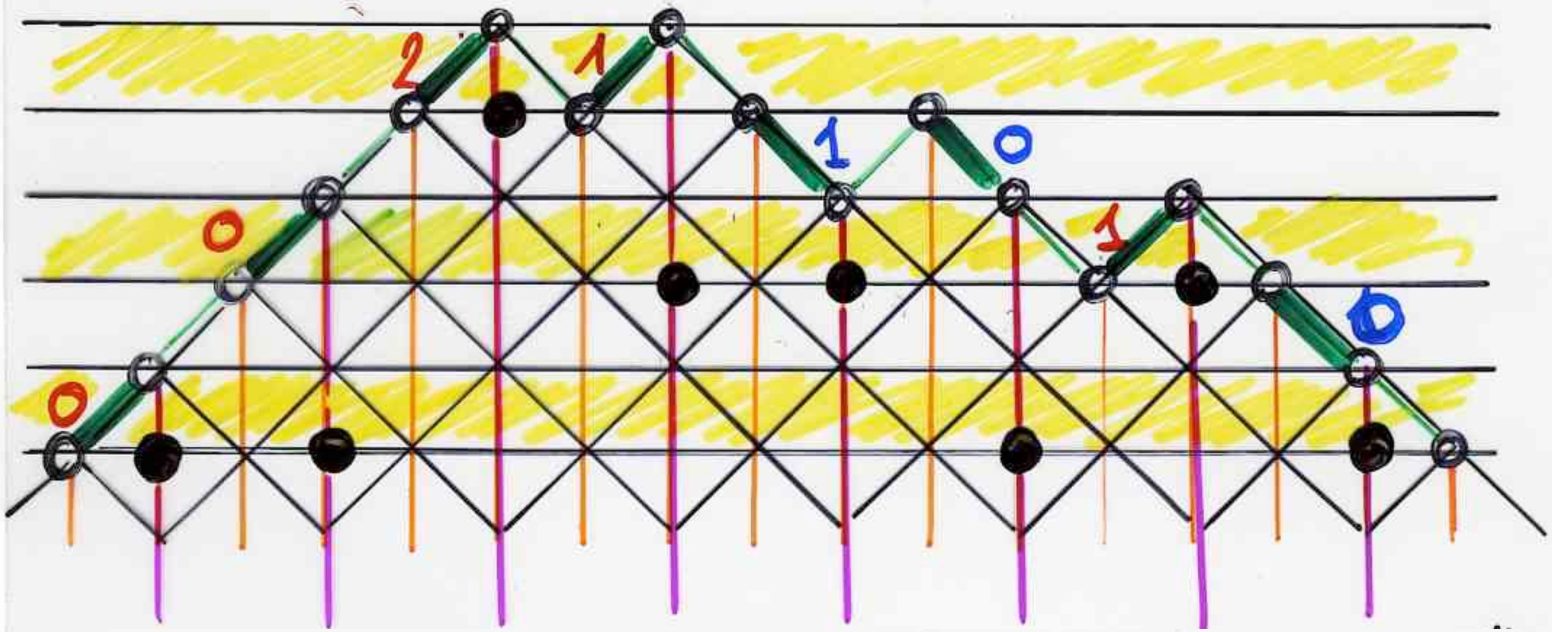
J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)



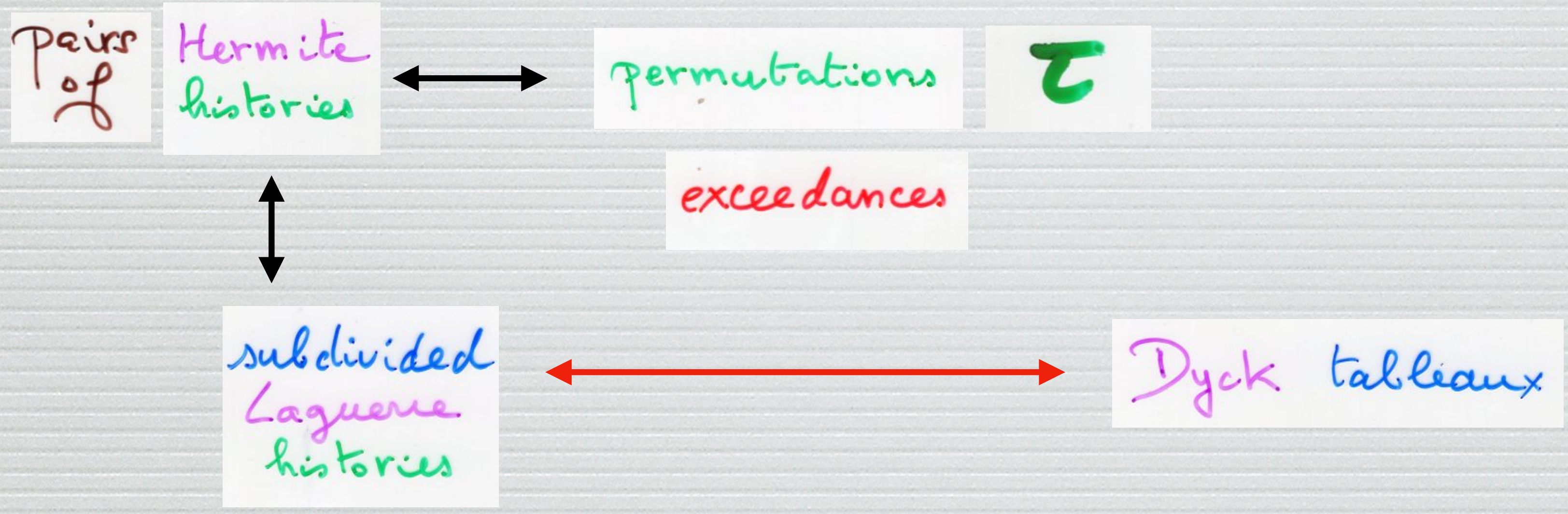
"subdivided Laguerre history"



subdivided Laguerre history



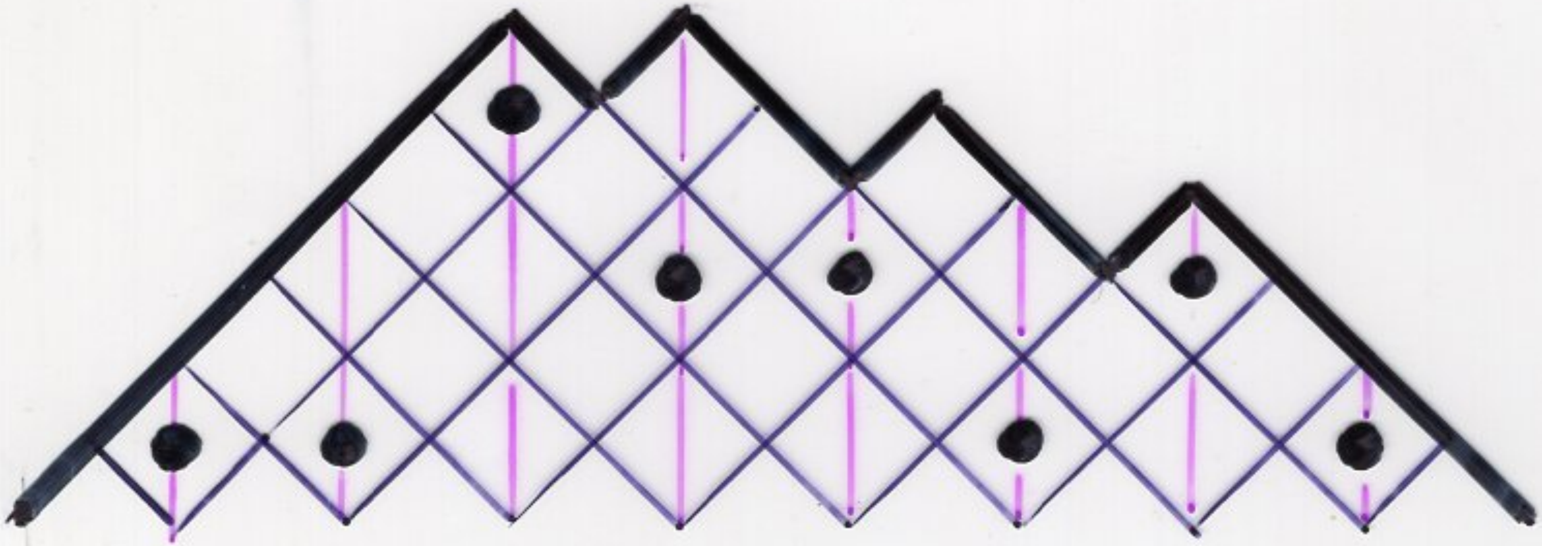
Dyck tableau as a subdivided Laguerre history

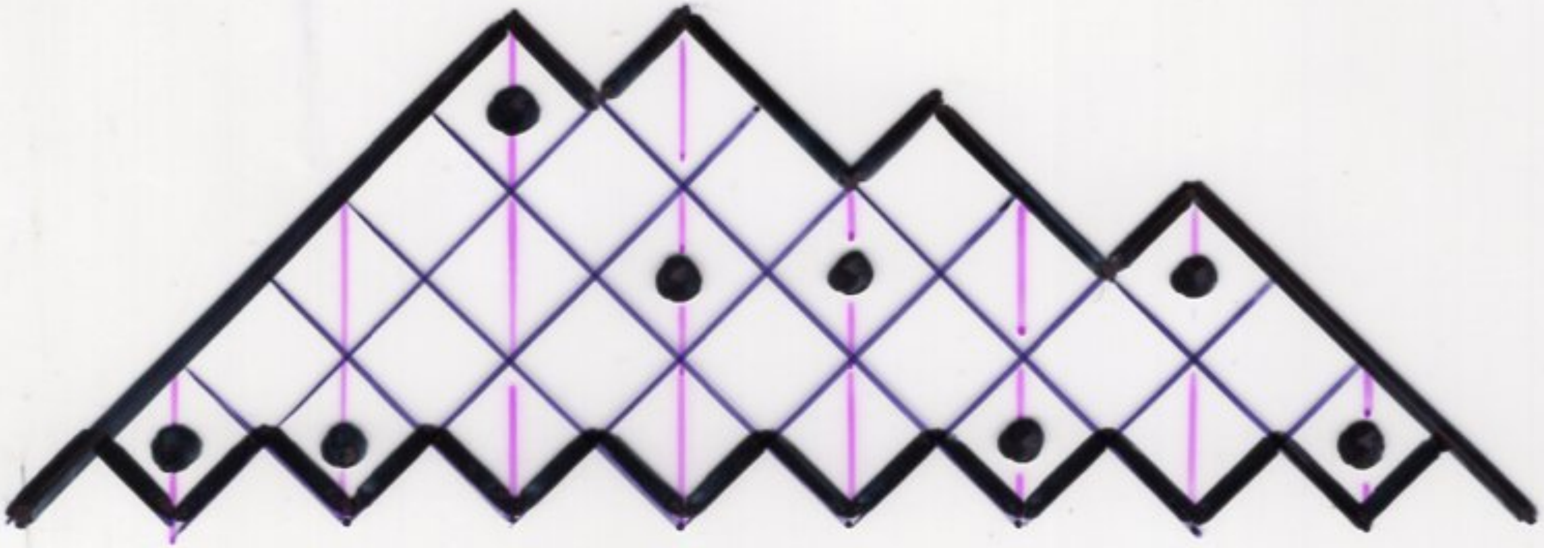


Pairs of
Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux

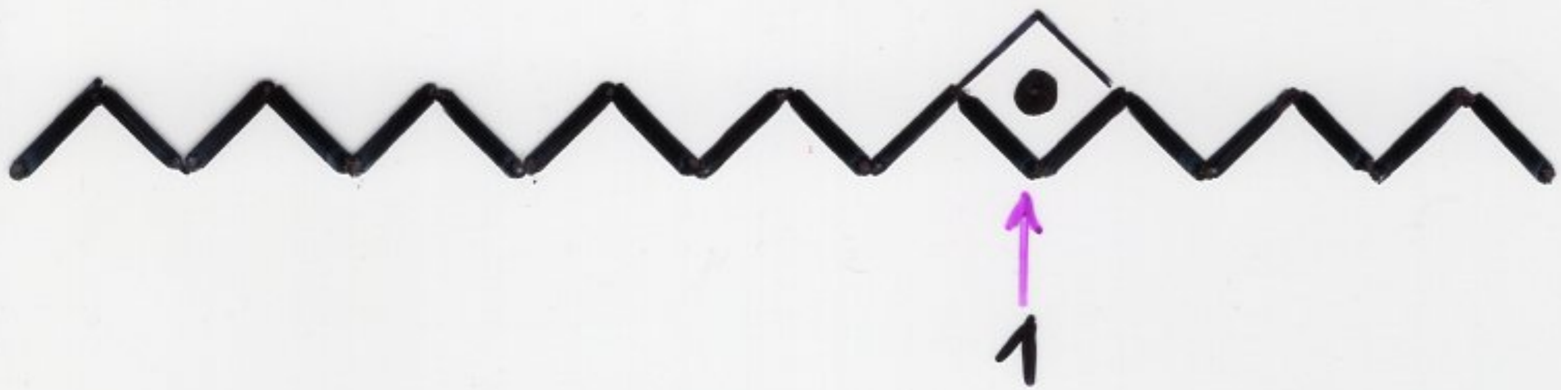
(direct) bijection

permutations ⟶ Dyck tableaux

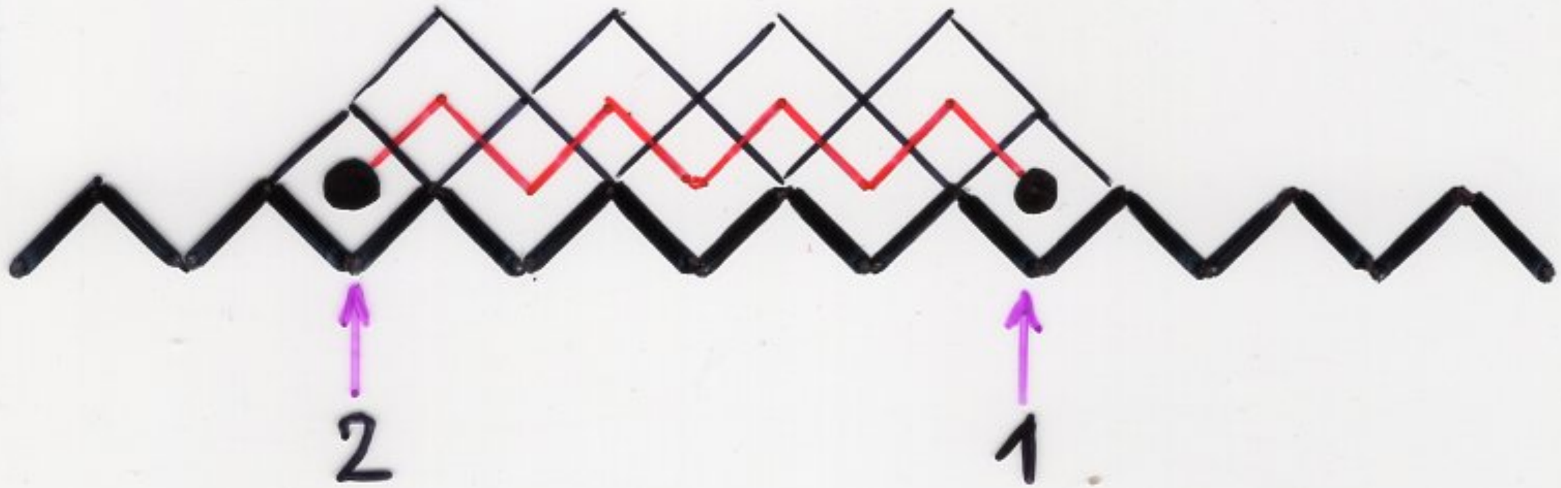




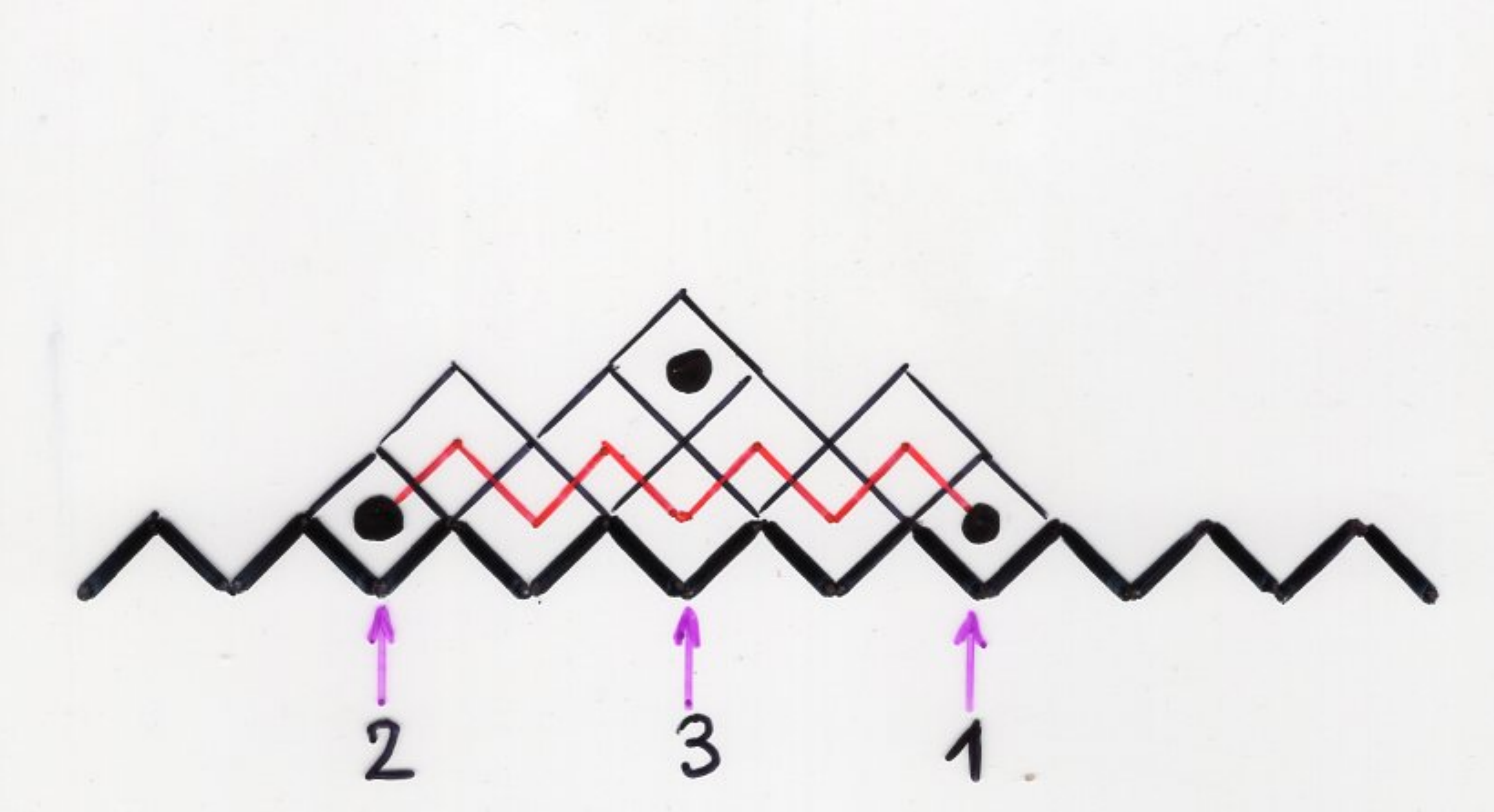
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



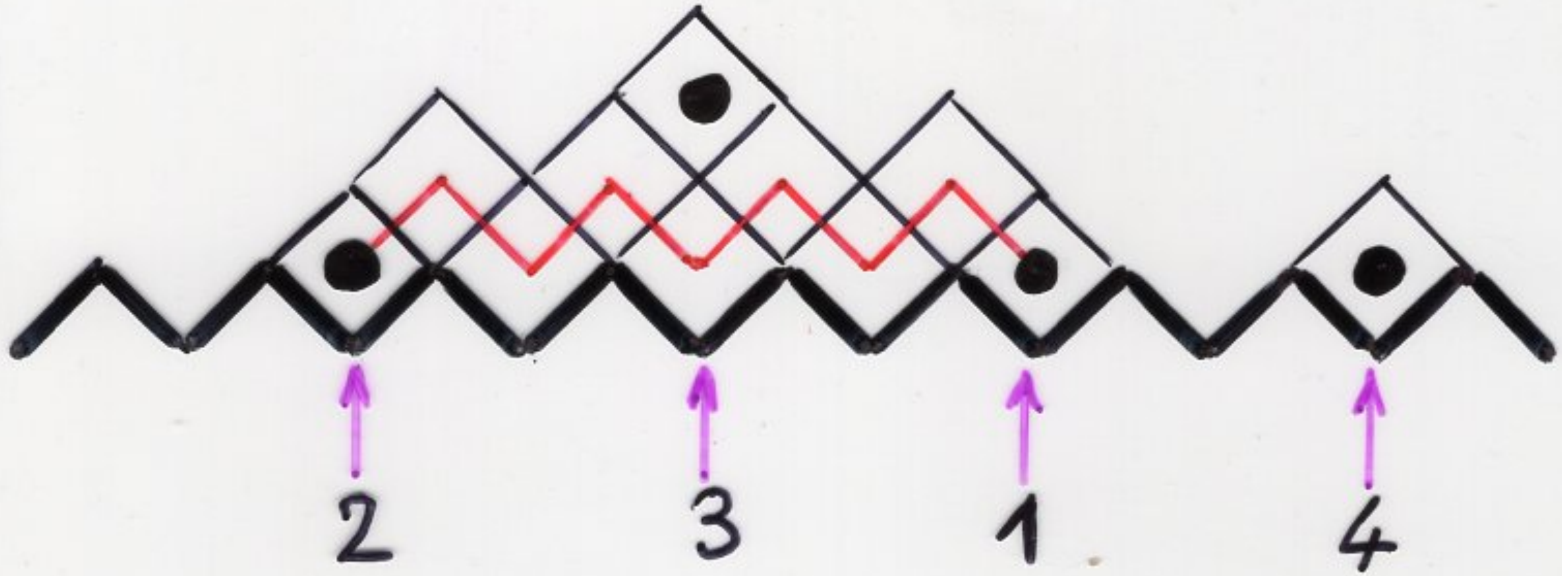
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



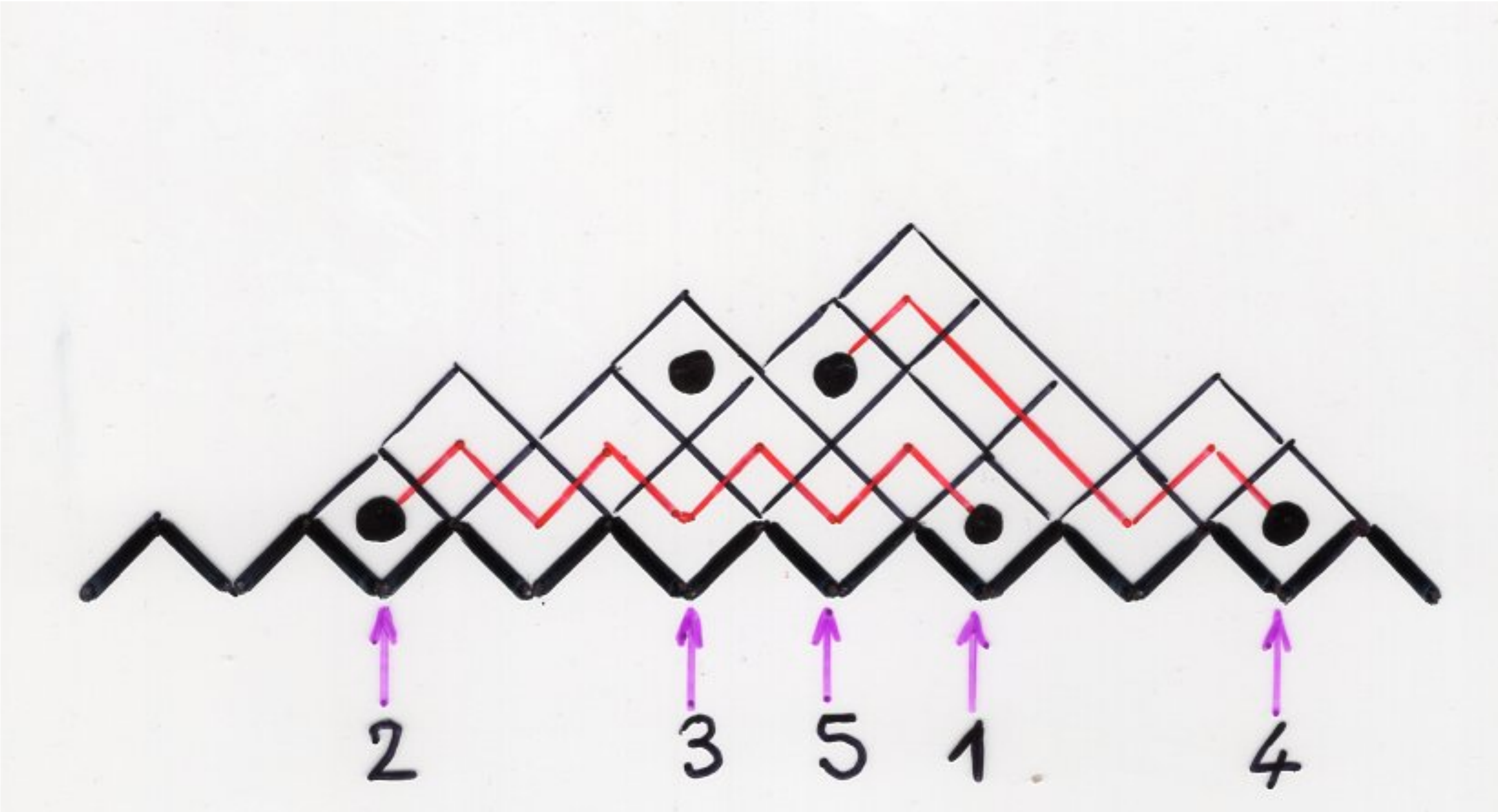
$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$



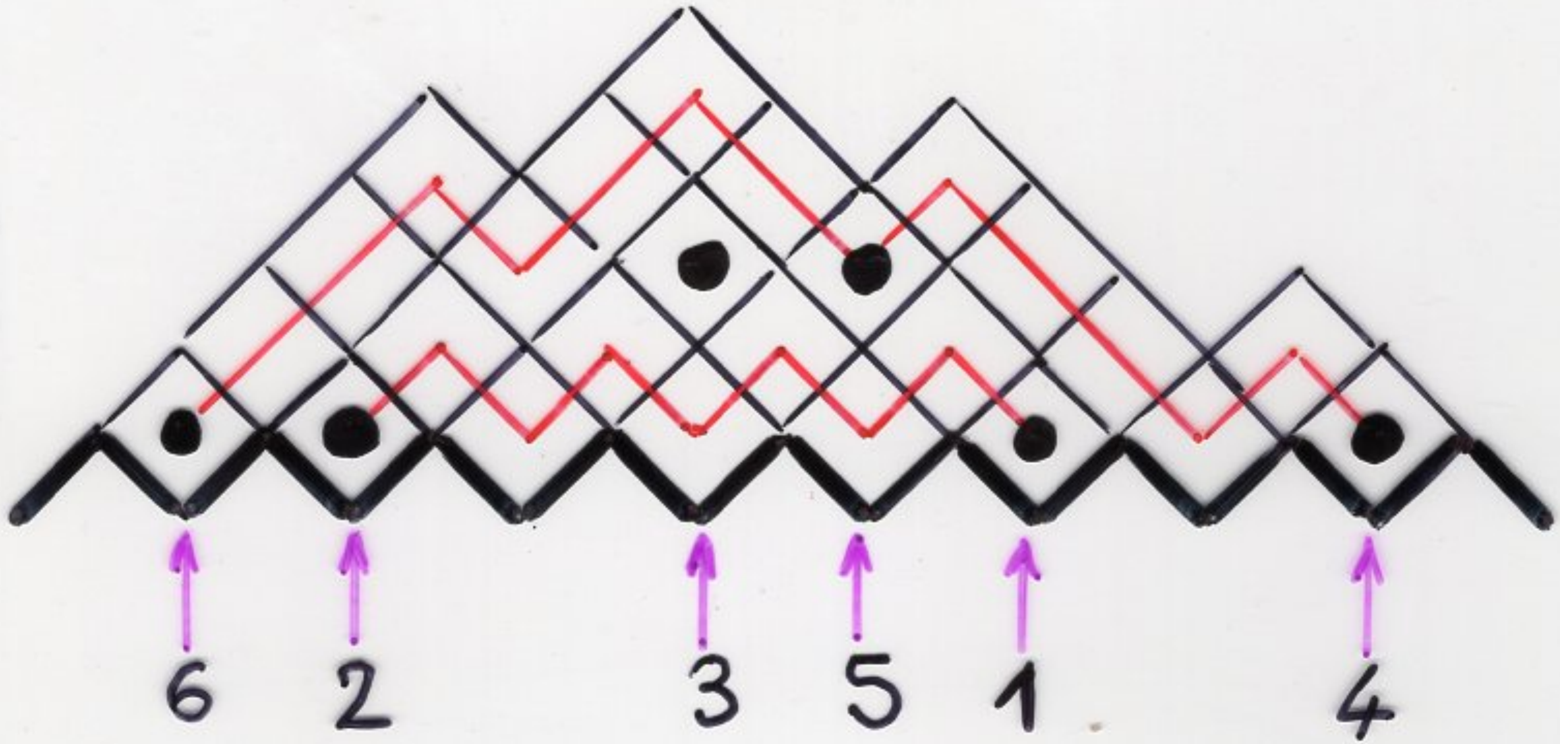
$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$



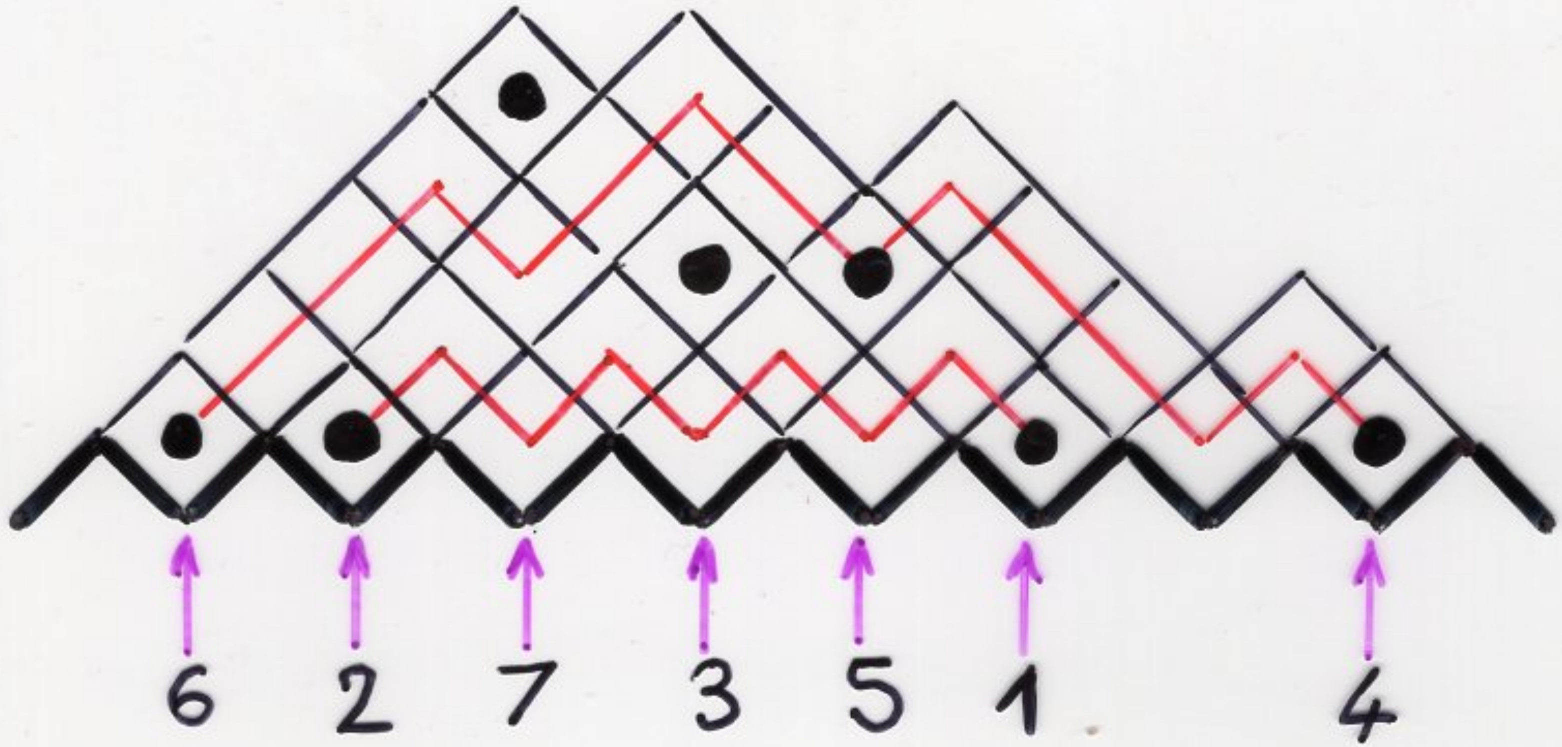
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



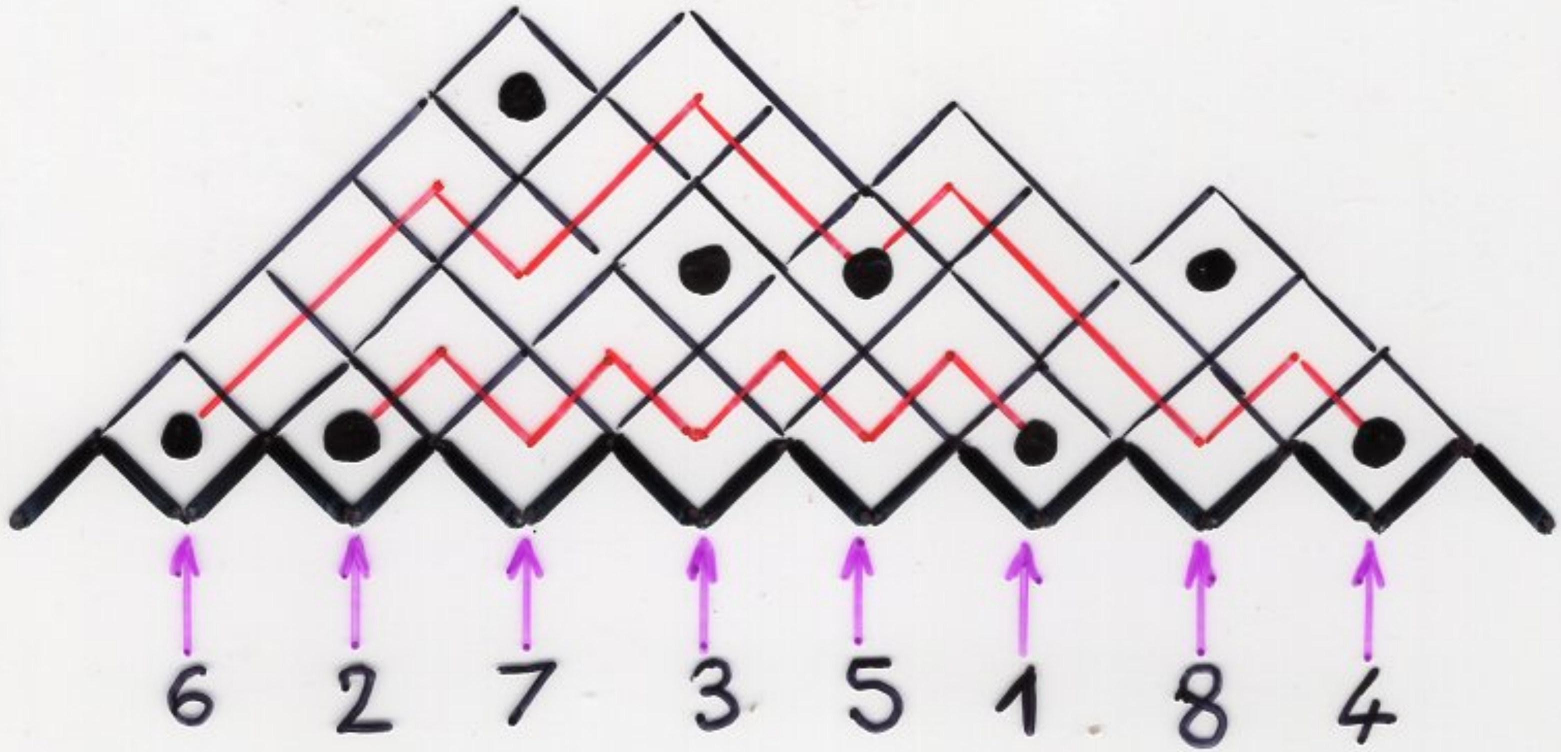
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

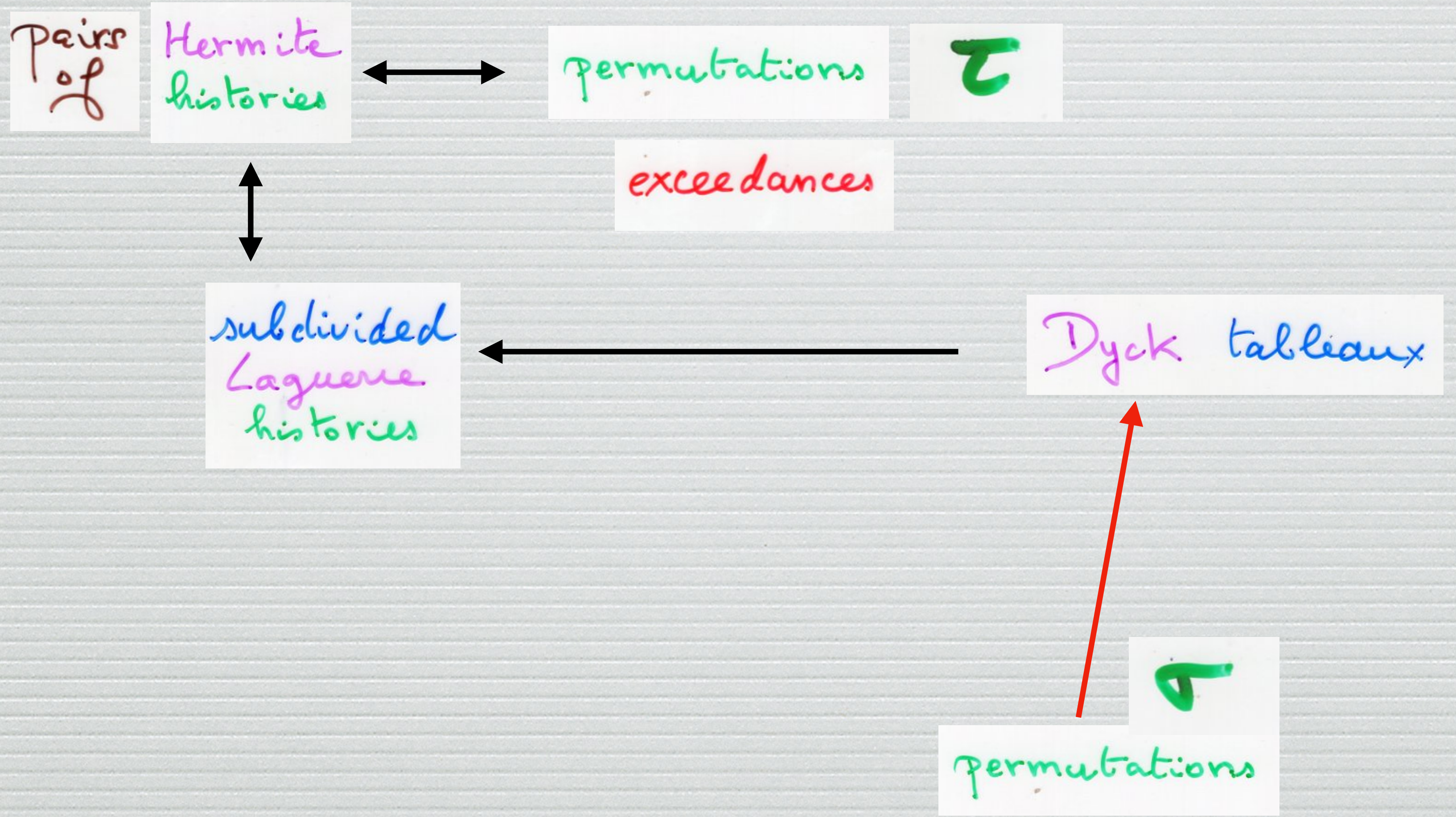


$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$



$$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$$





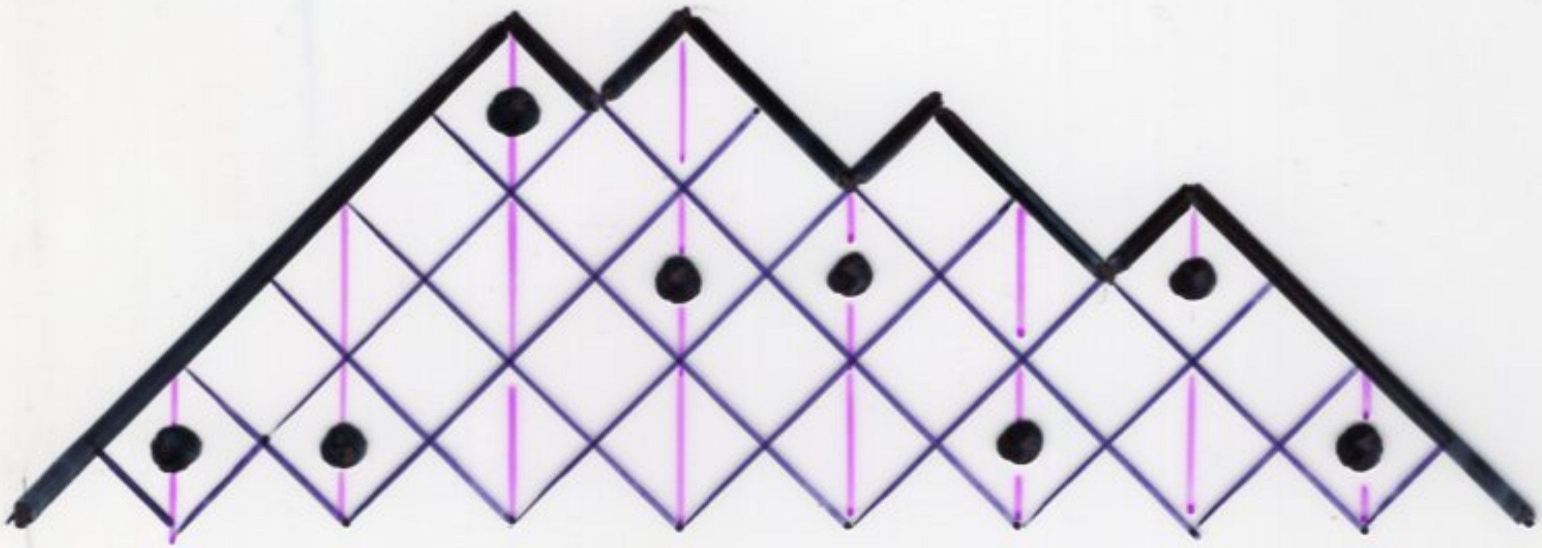
Pairs of
Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
σ
permutations

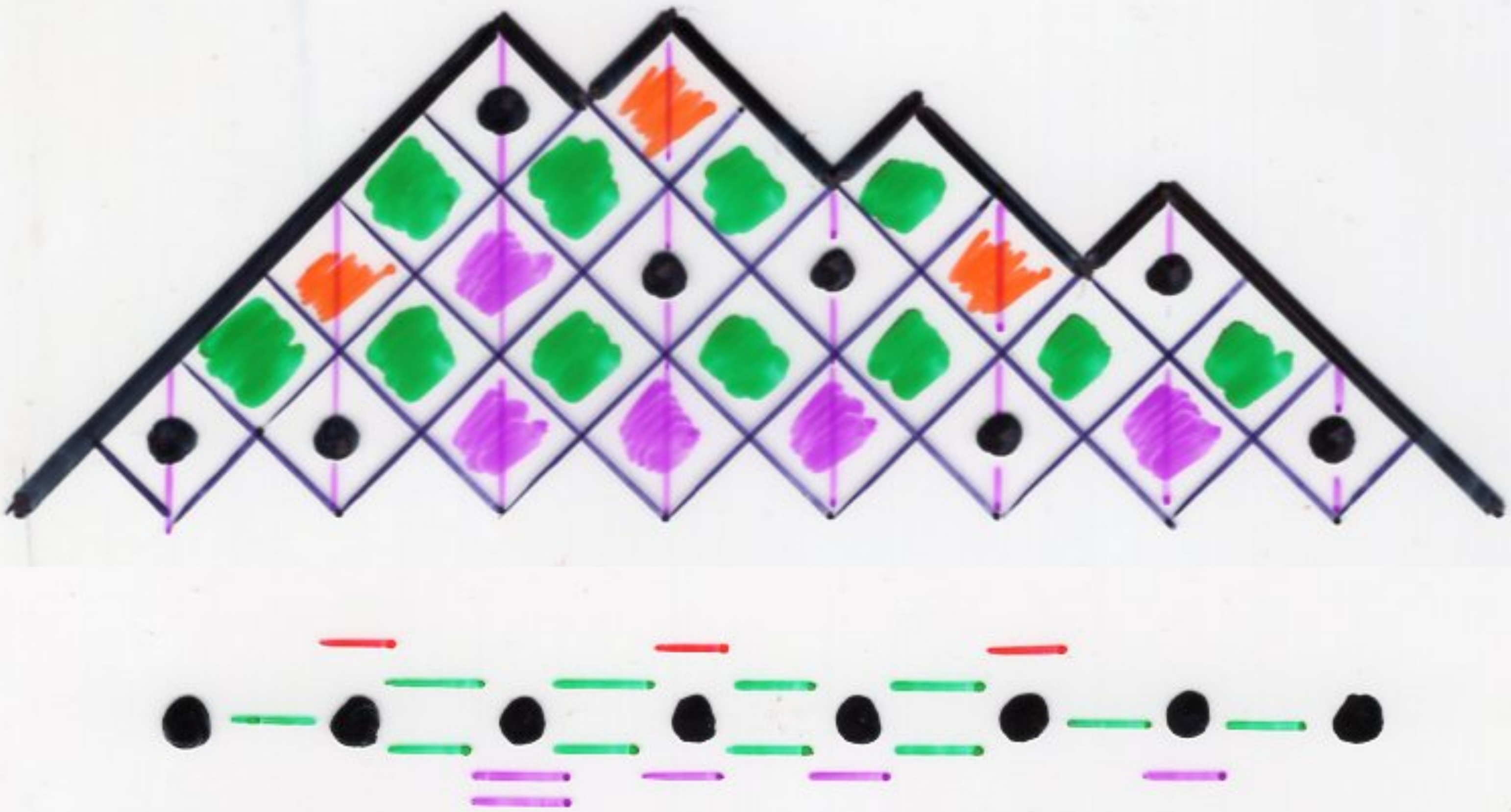
Reverse bijection

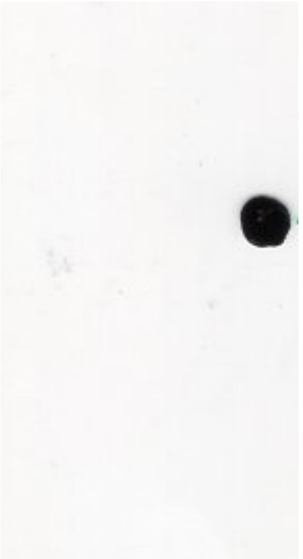
Dyck tableaux

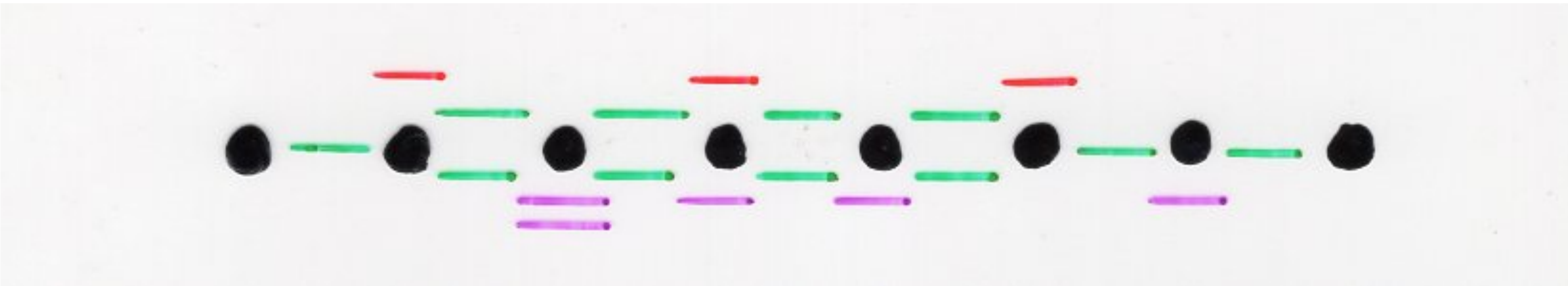
permutations

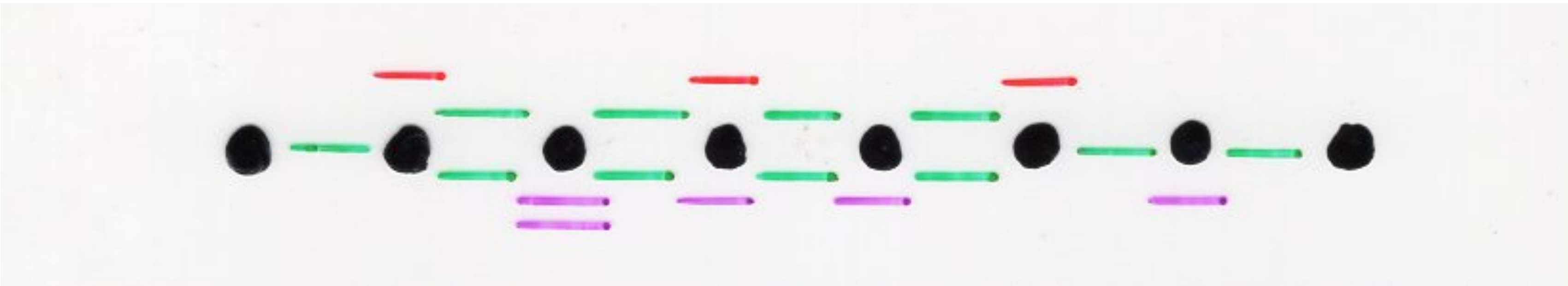
multilinear heaps of pointed segments

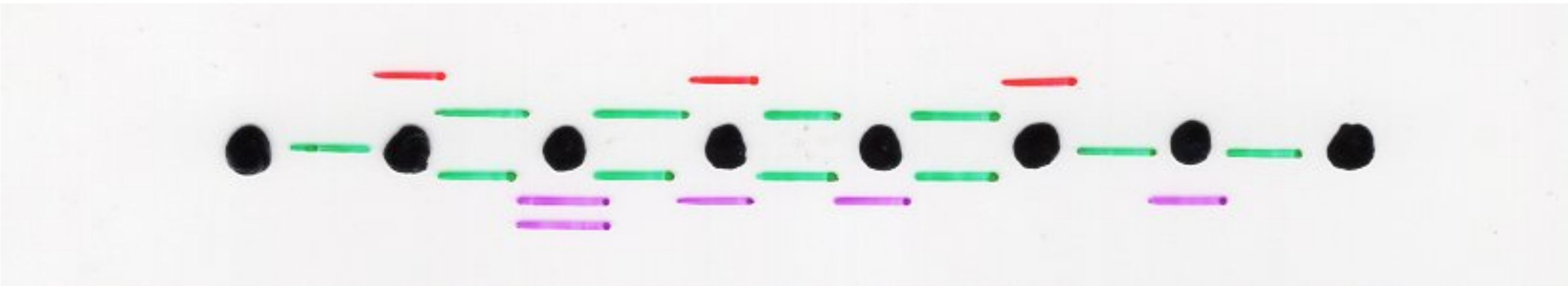


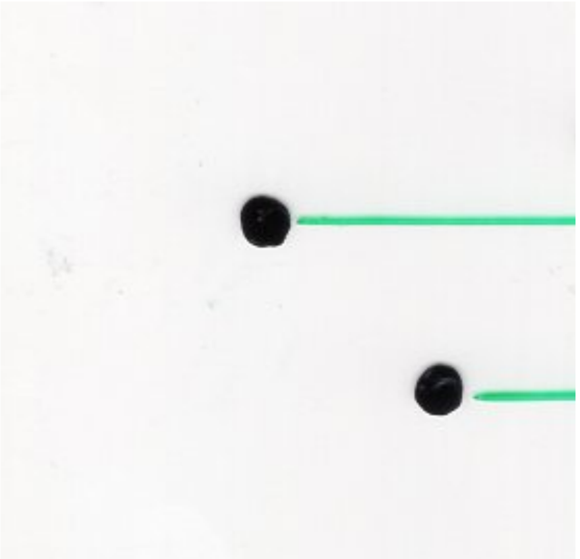


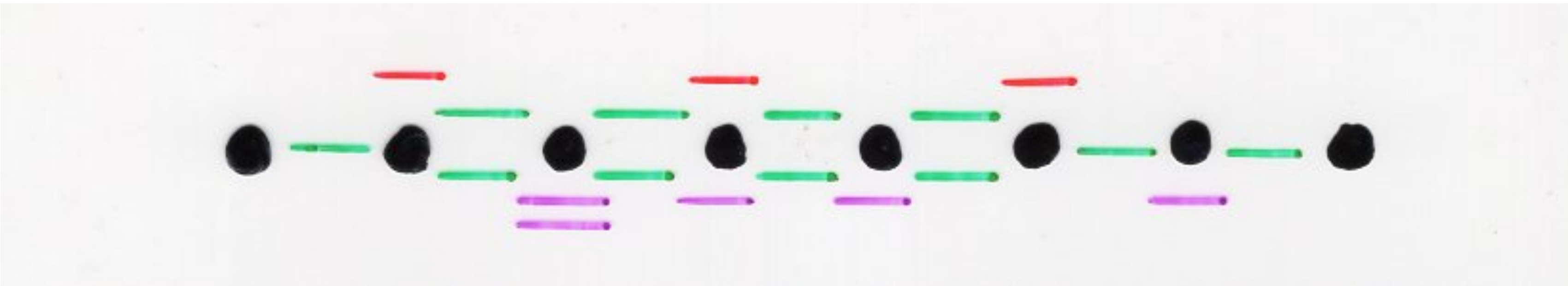


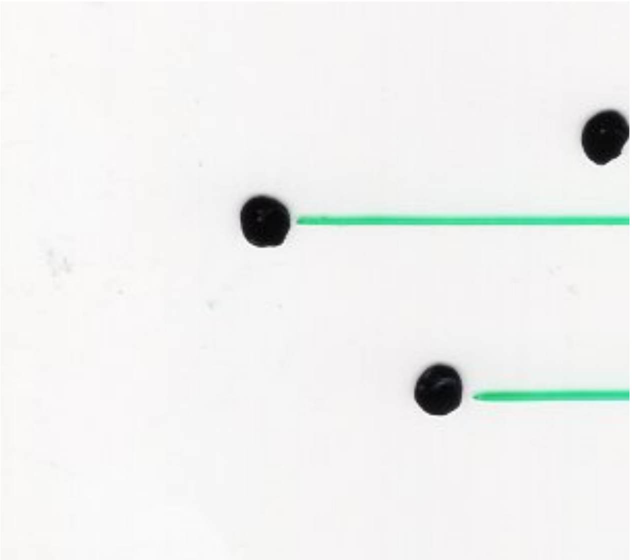


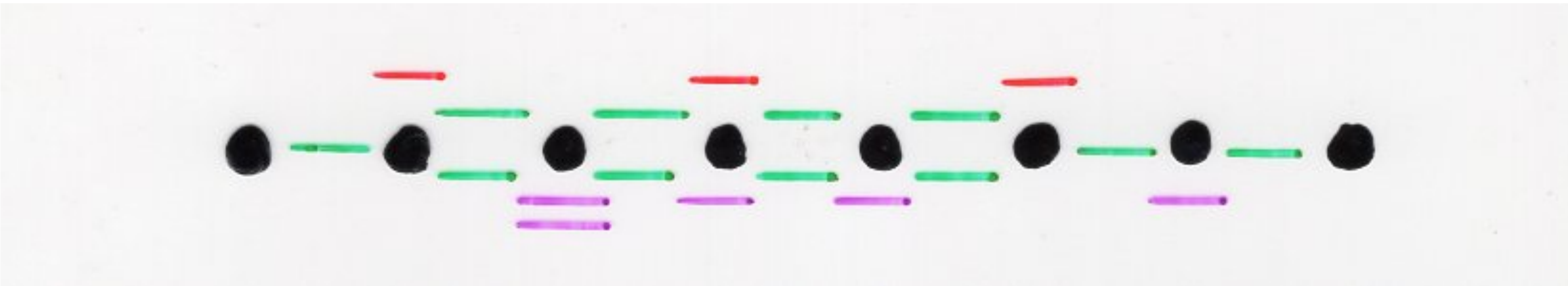


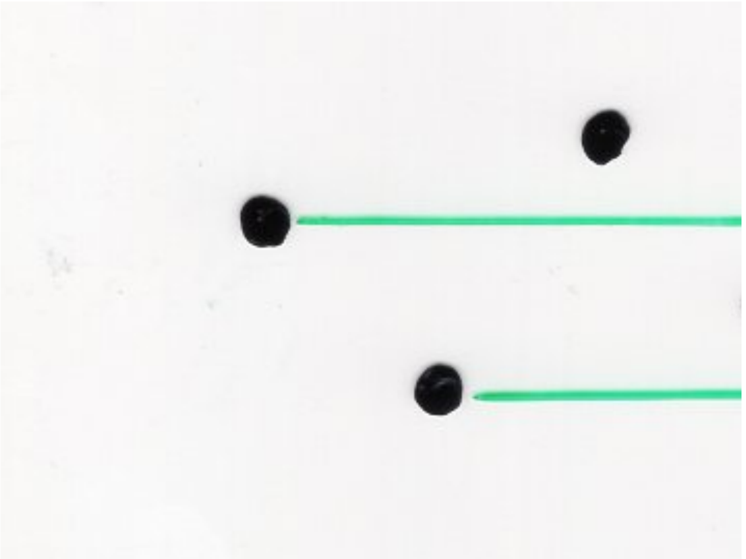


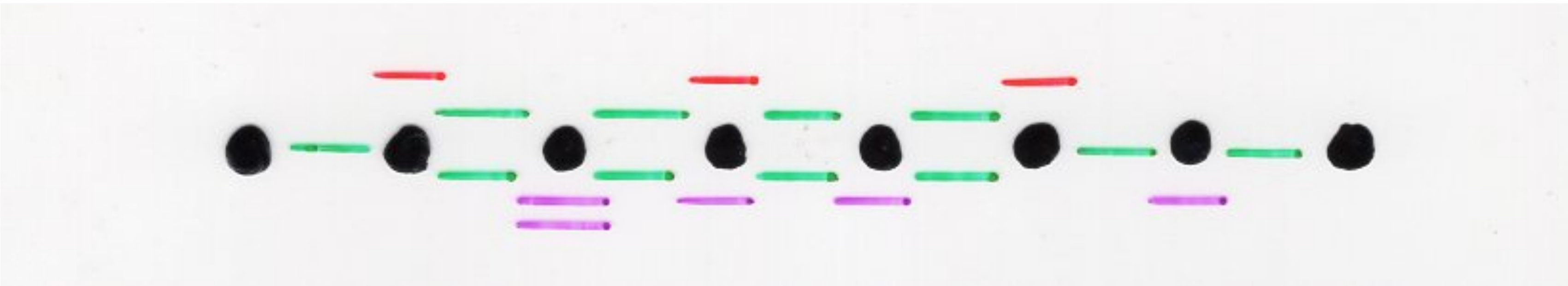


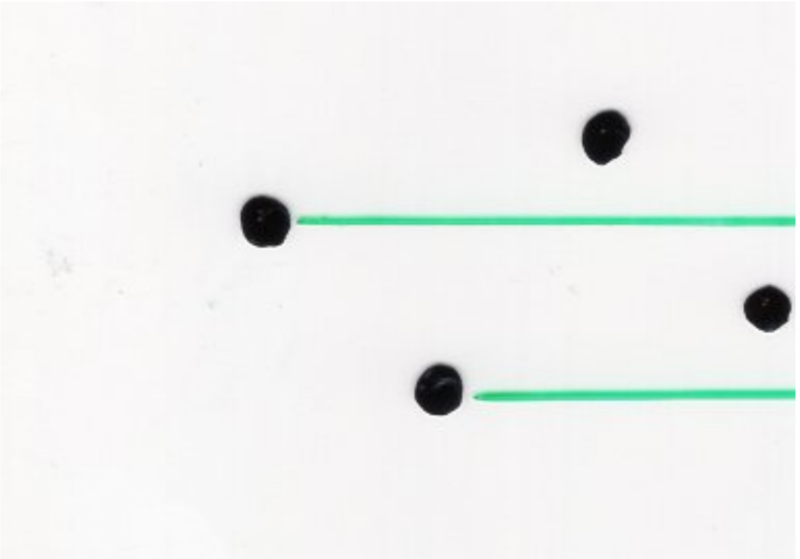


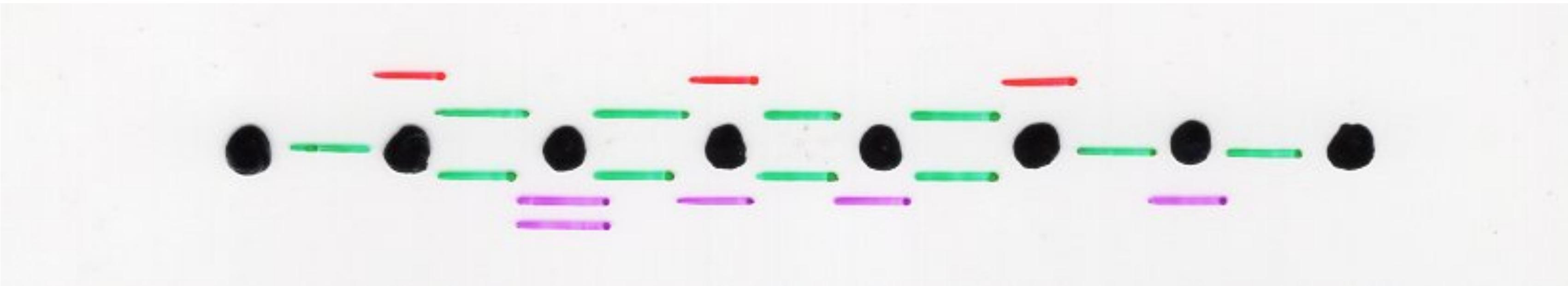


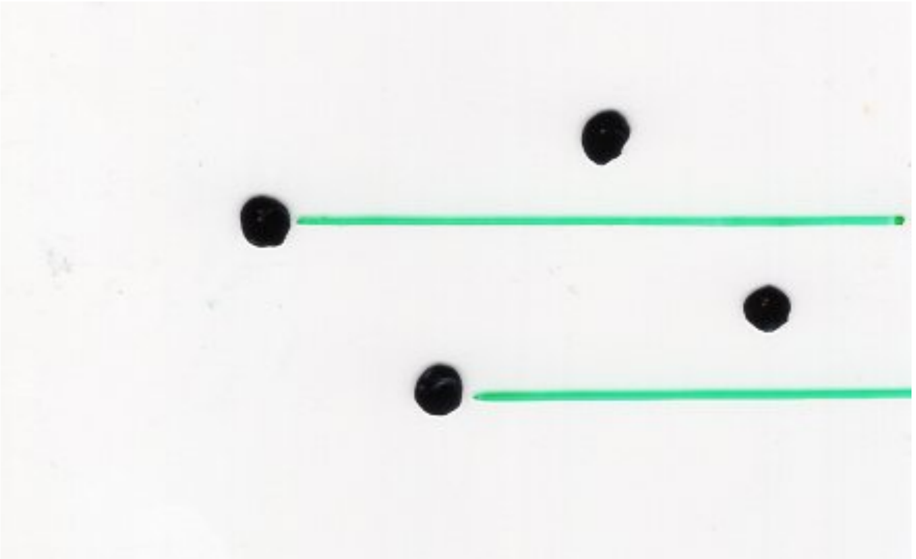


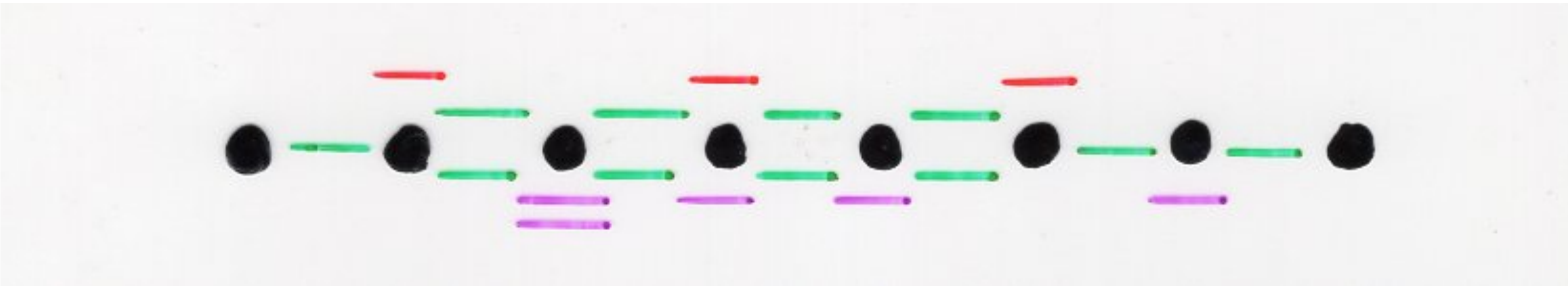


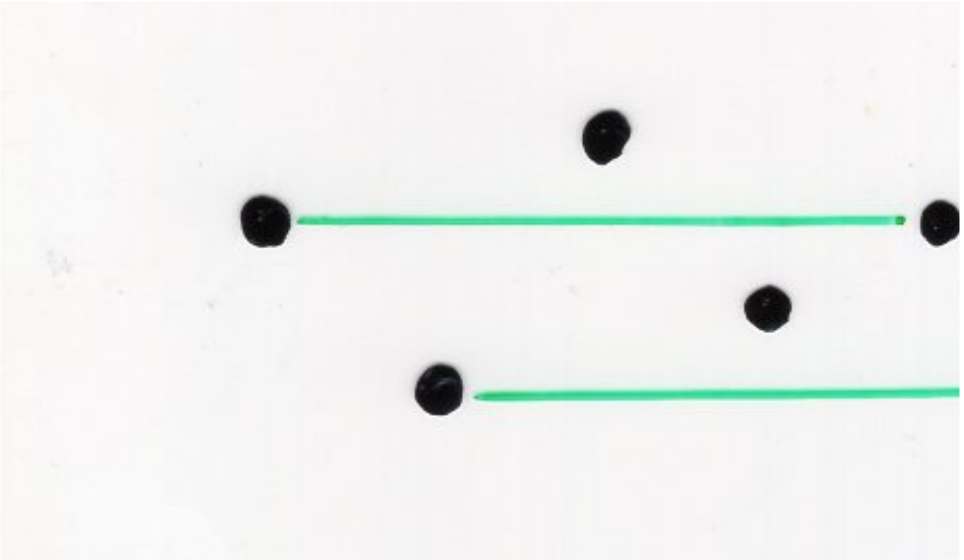


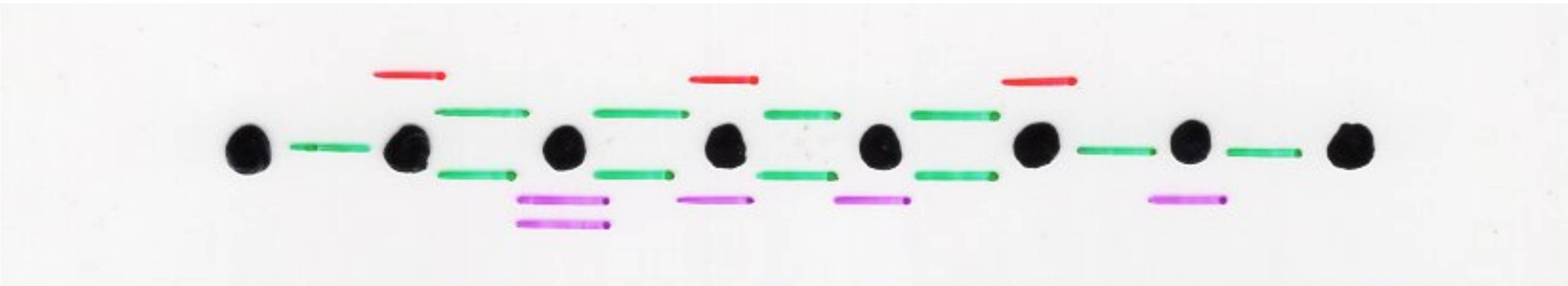


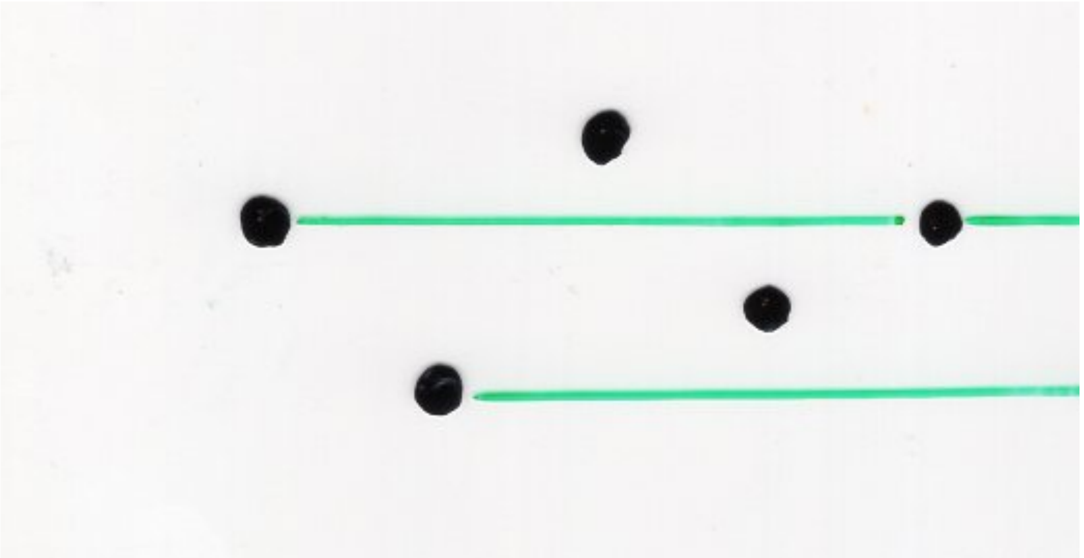


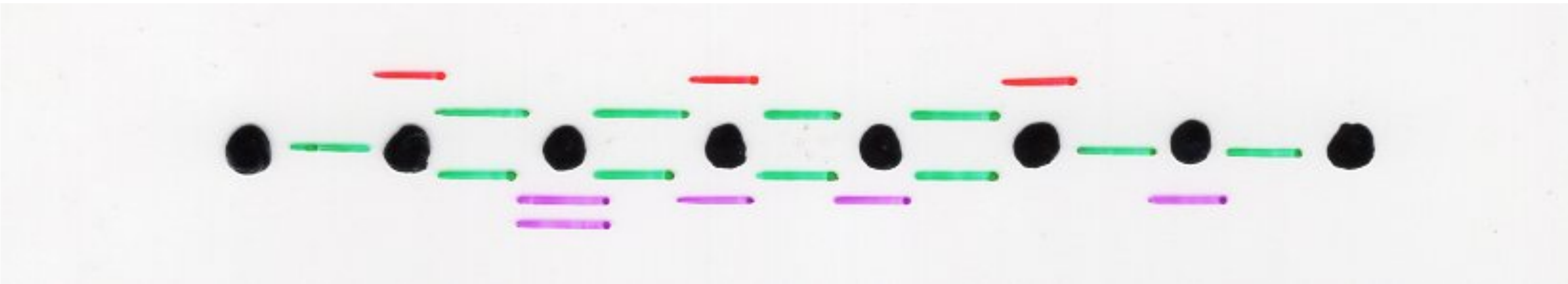


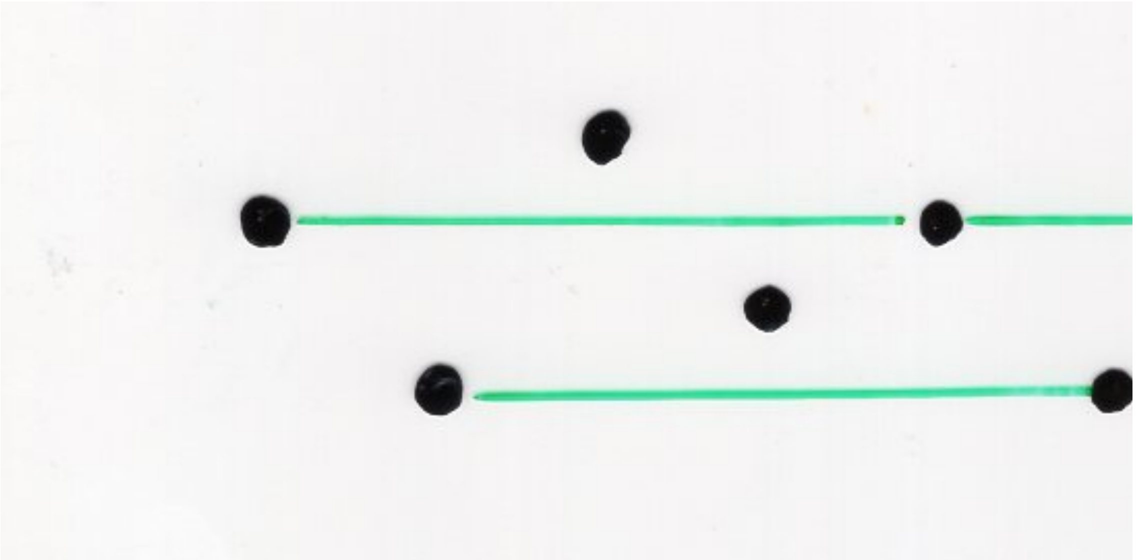


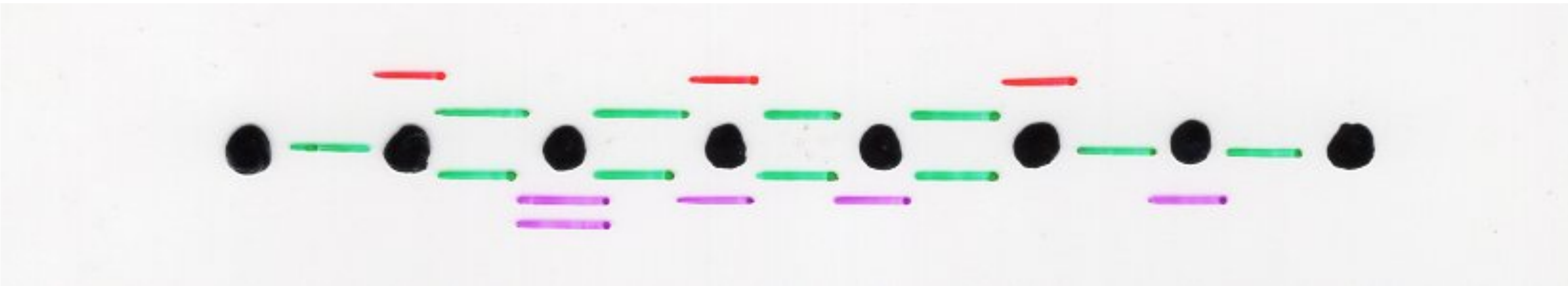


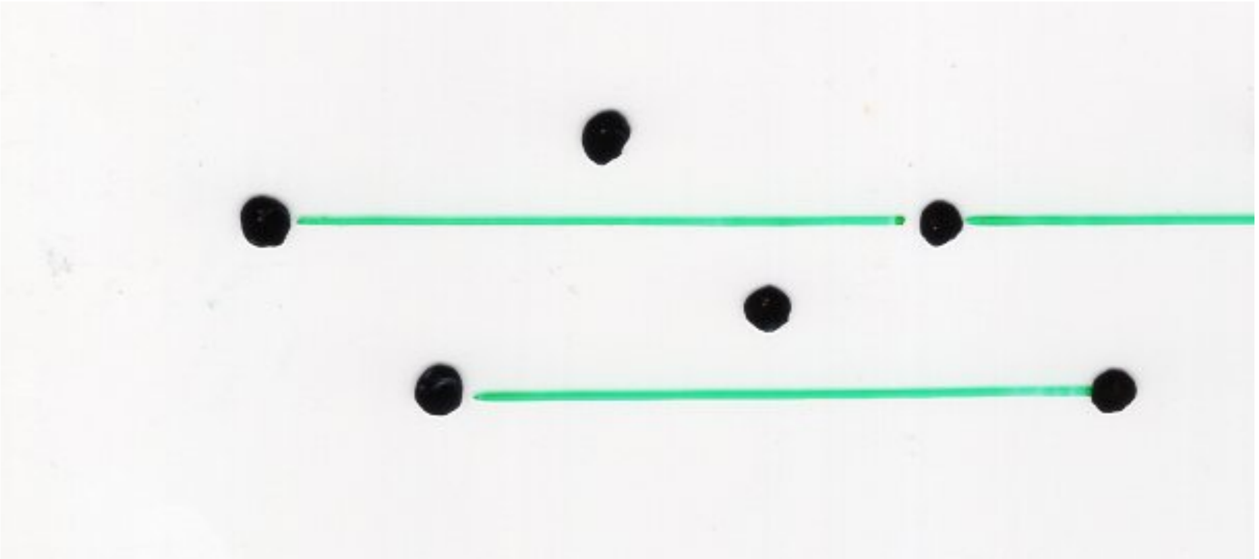


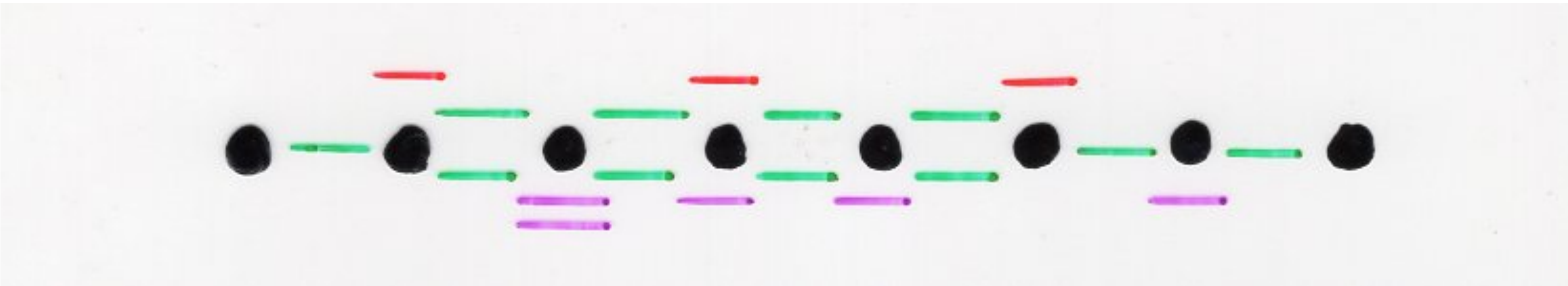


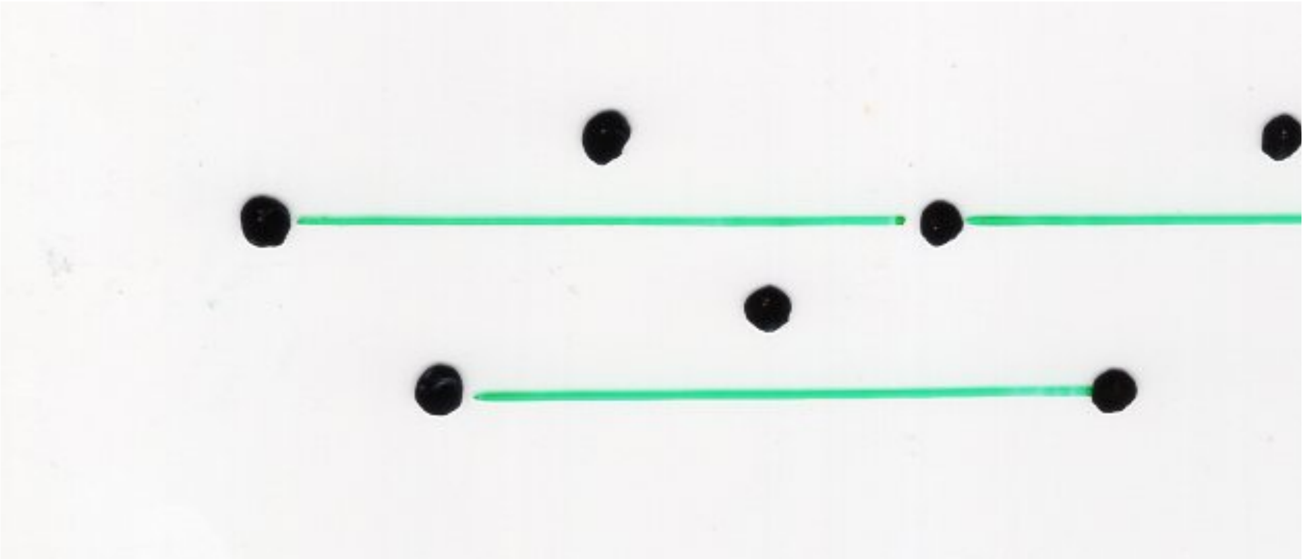


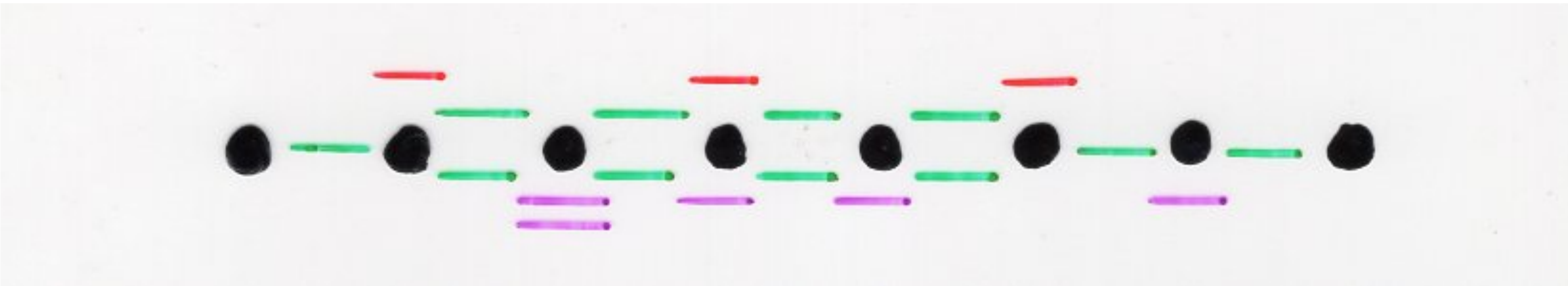


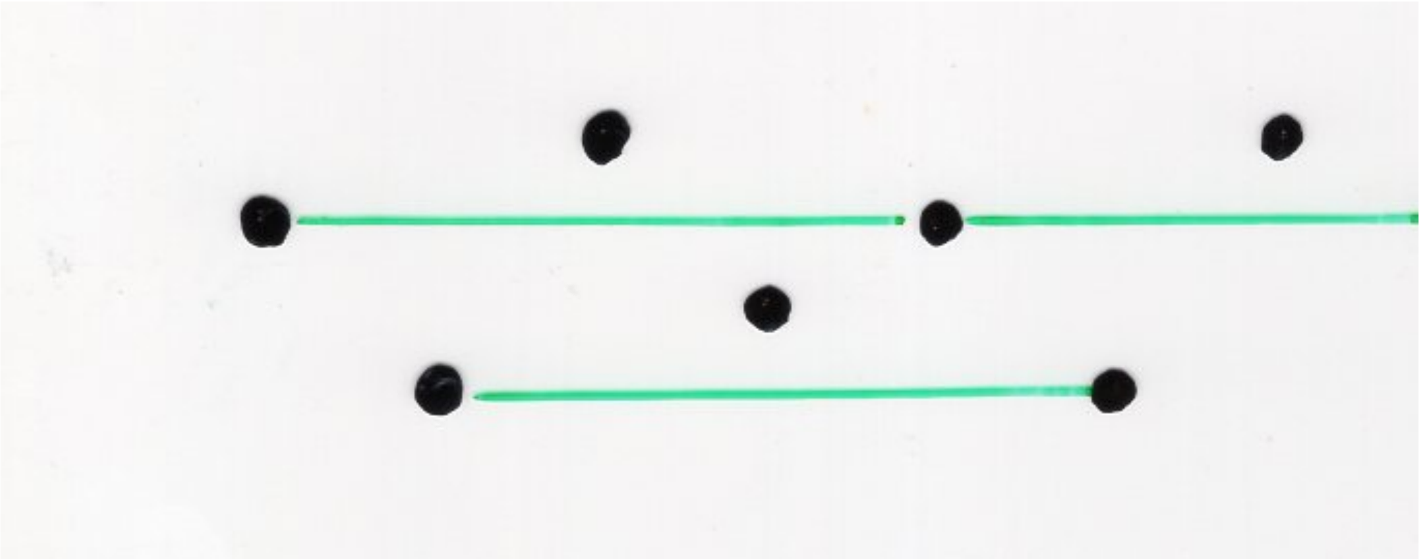


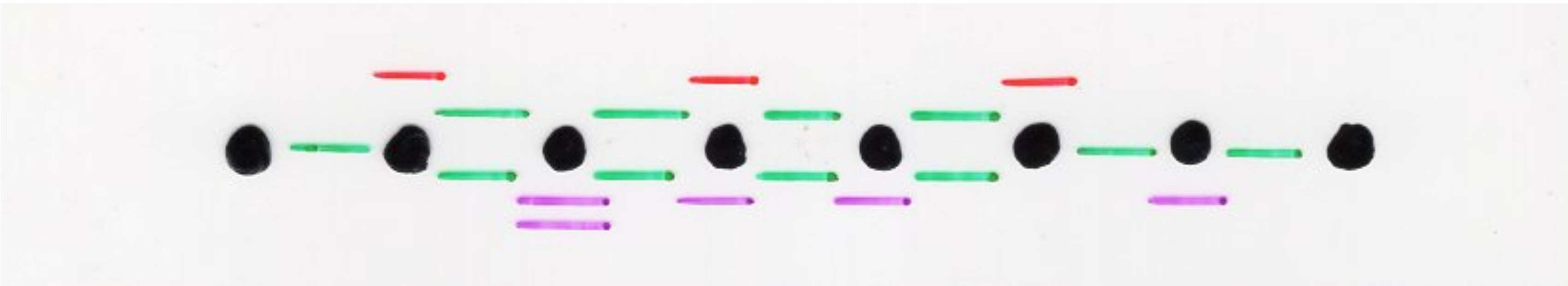


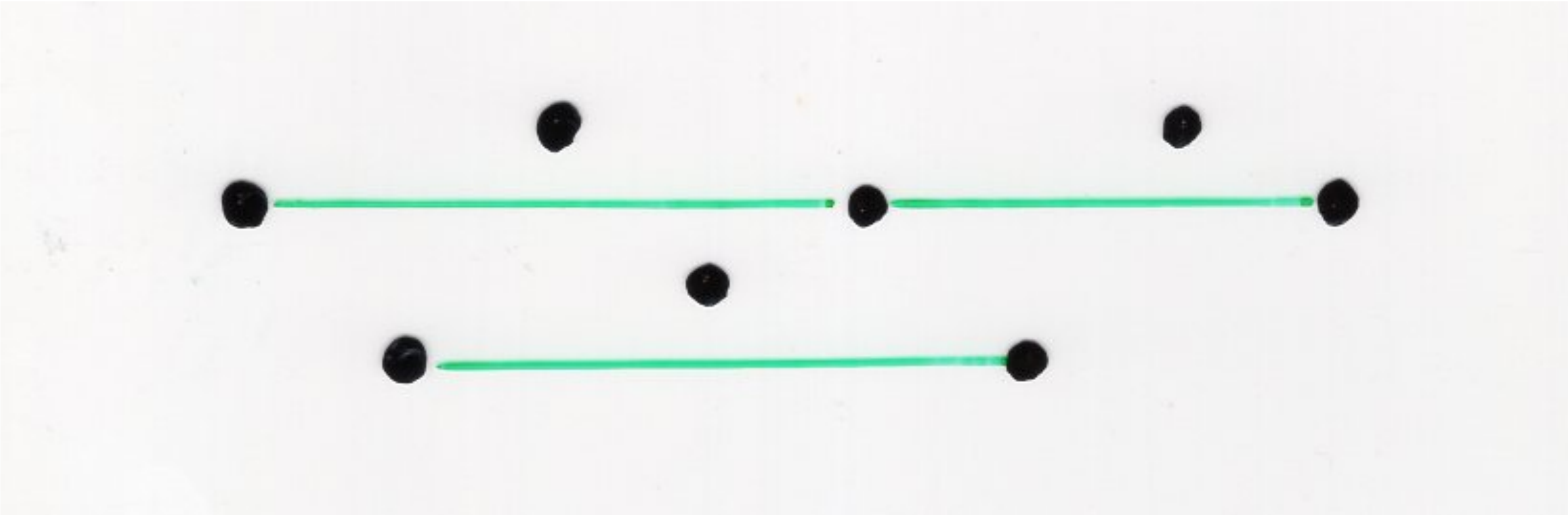


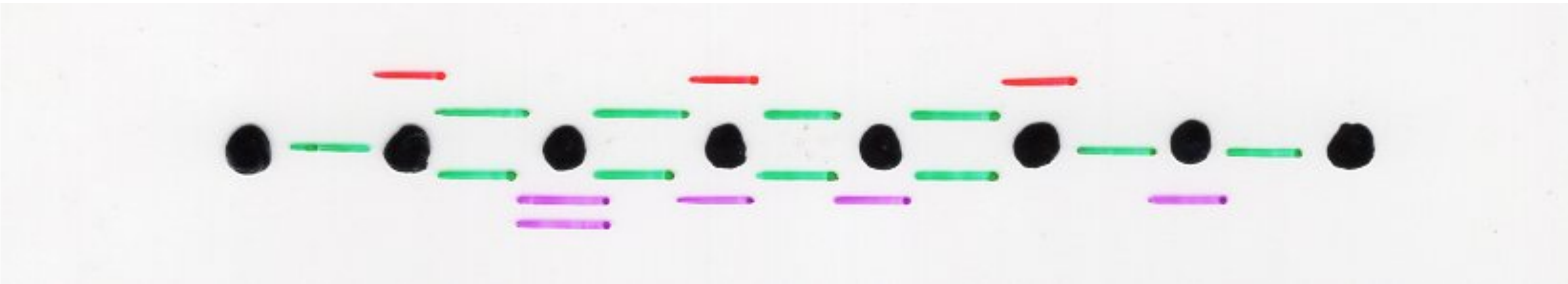


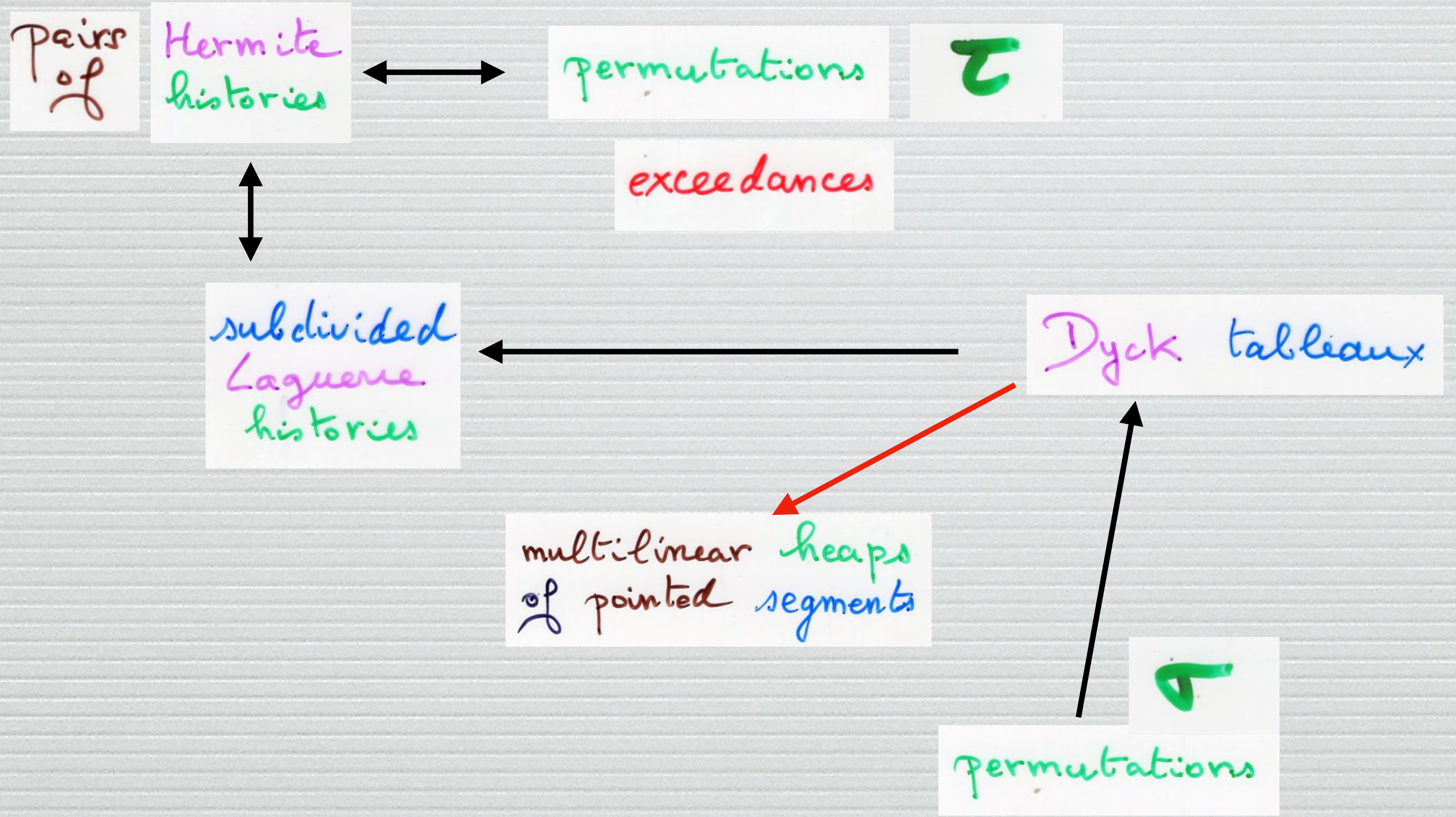






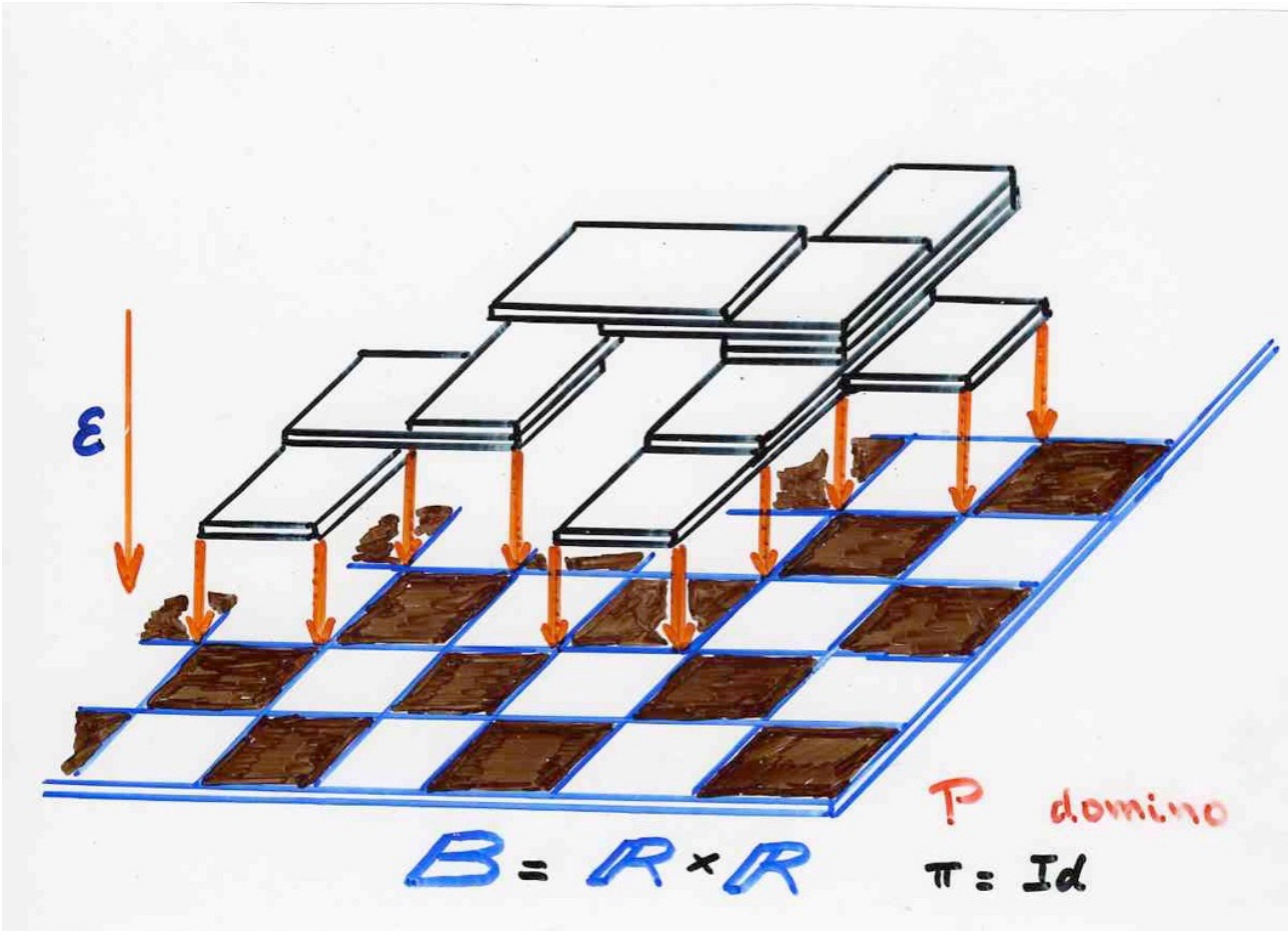


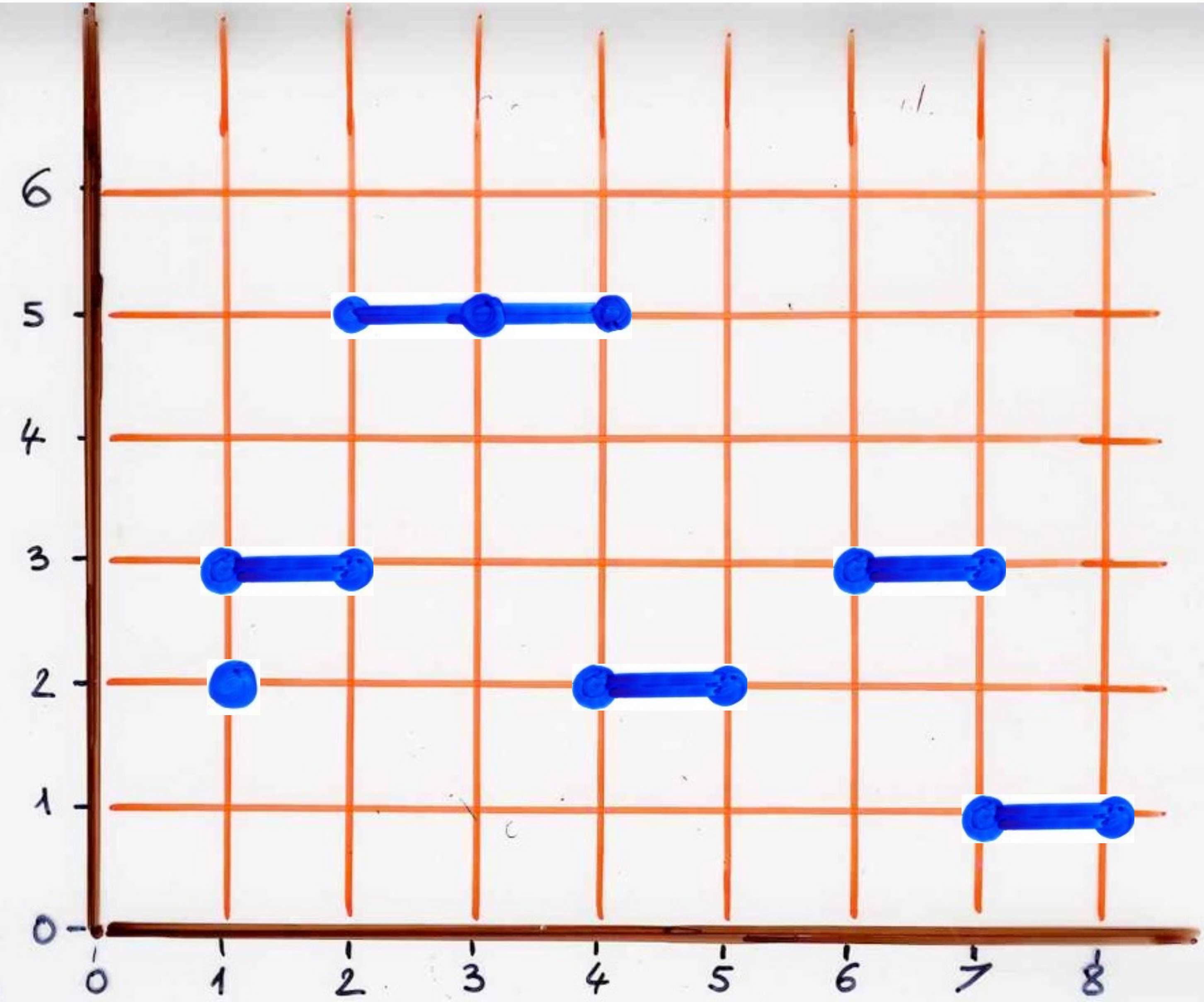



Pairs of
Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations

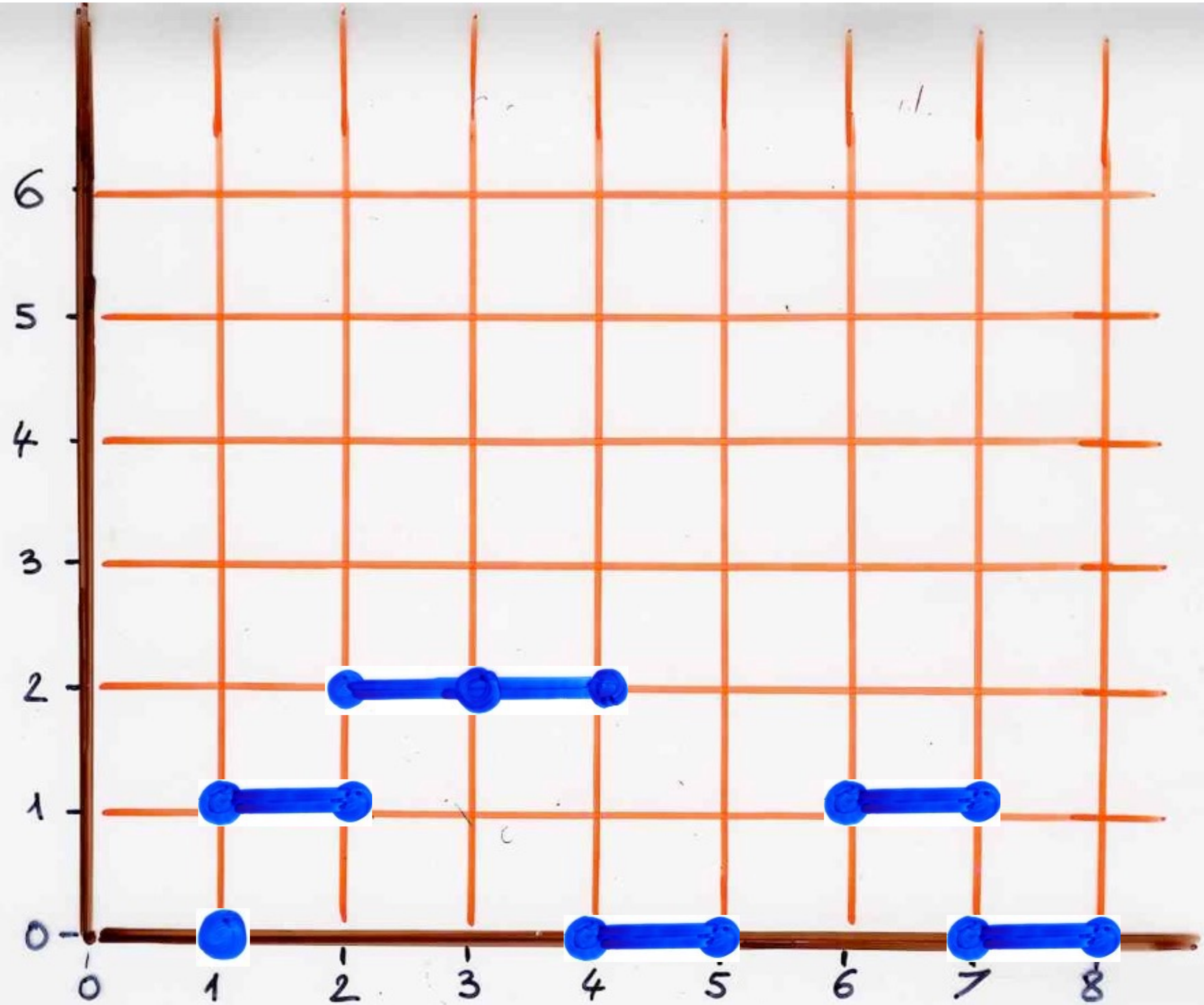
Introduction
Heaps

From BJC 2, Ch 1a





0
1
2
3
4
5
6
0
1
2
3
4
5
6
7
8

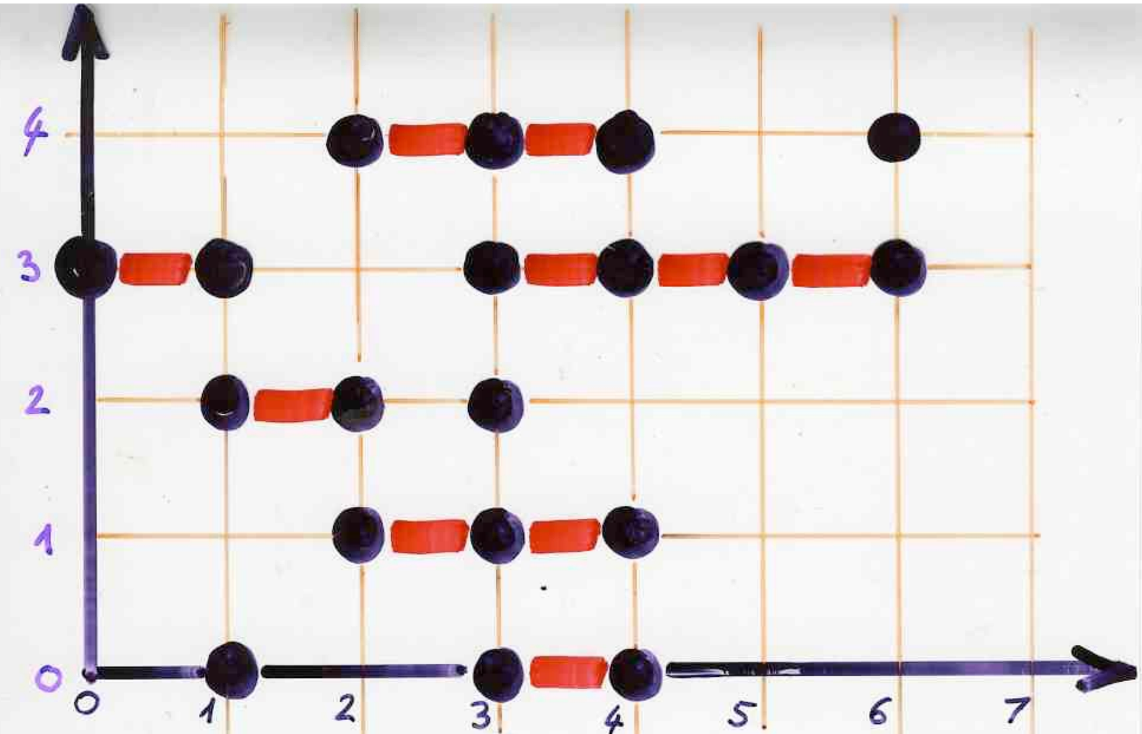


6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8

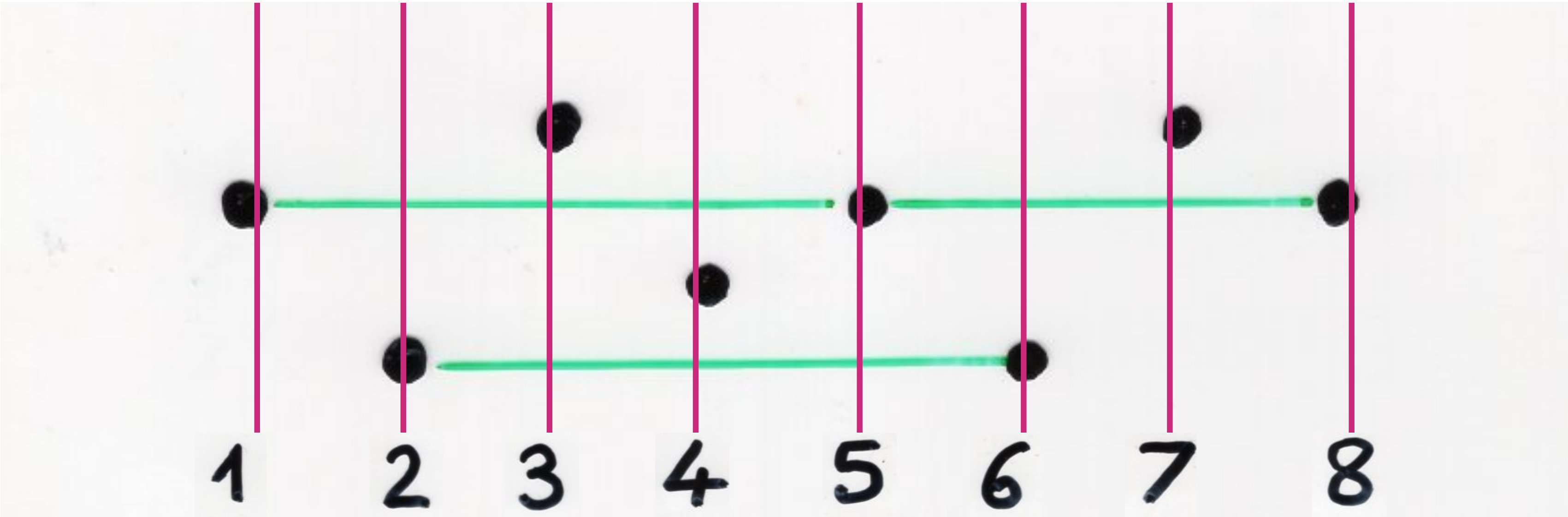
ex: heap of segments over $\mathbb{N}$

$$P = \{ [a,b] = \{a, a+1, \dots, b\}, \; 0 \leq a \leq b \}$$

$$\mathcal{C} \qquad [a,b] \, \mathcal{C} \, [c,d] \iff [a,b] \cap [c,d] \neq \emptyset$$



heap of segments



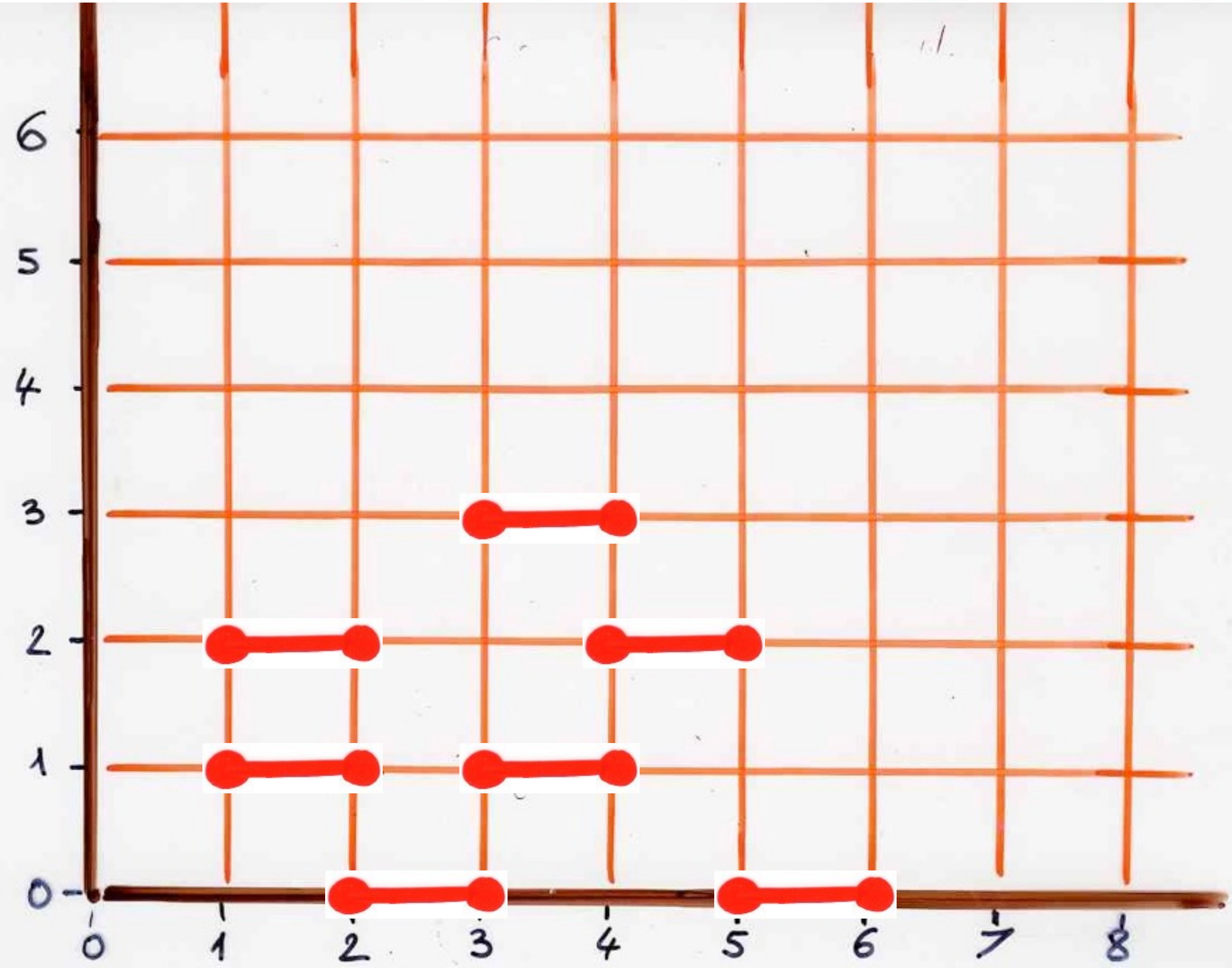
Definition heaps of pointed segments

Definition multilinear heap of pointed segments

Lexicographic normal form of a heap

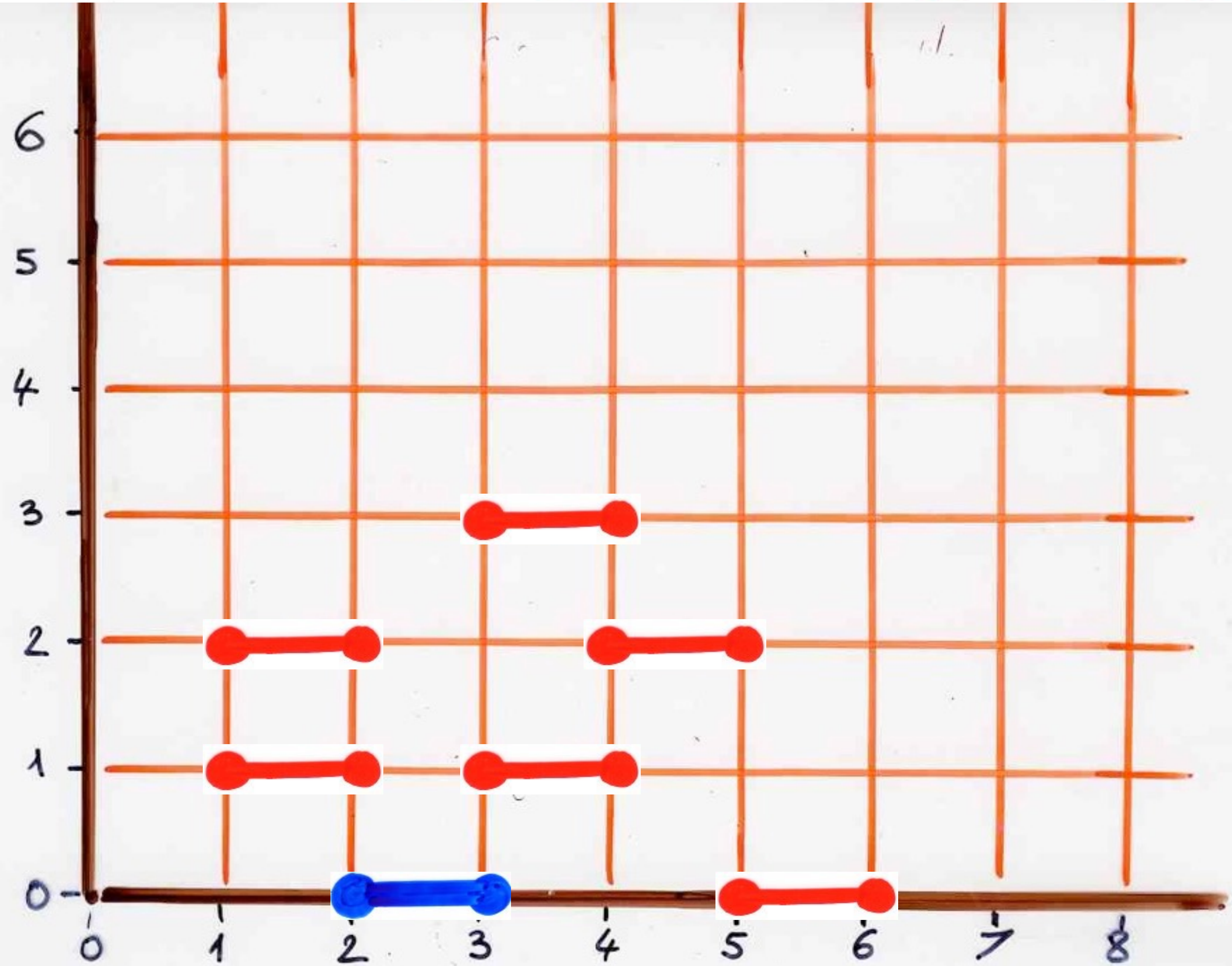
From BJC 2, Ch 1b, p44

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



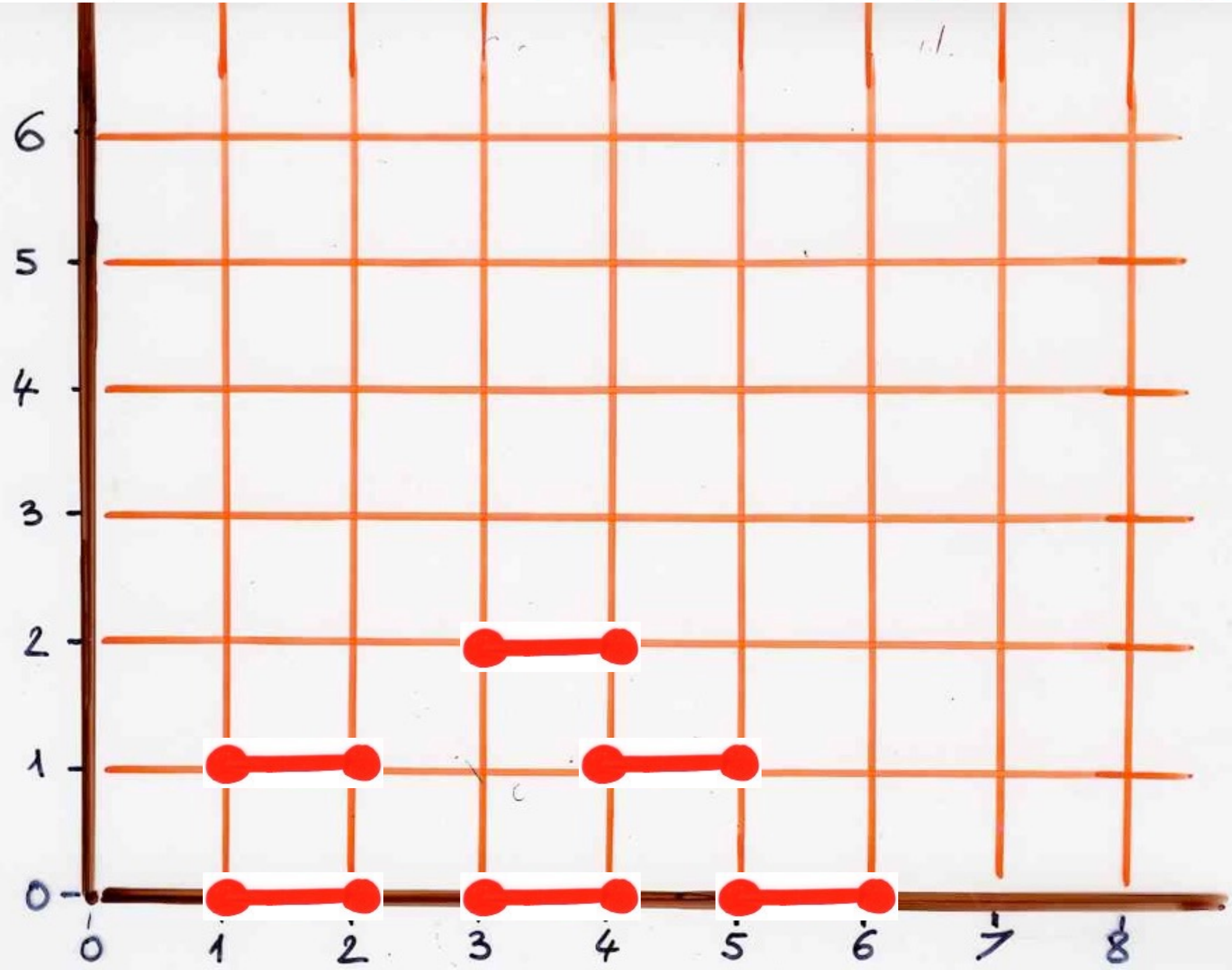
$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_5$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



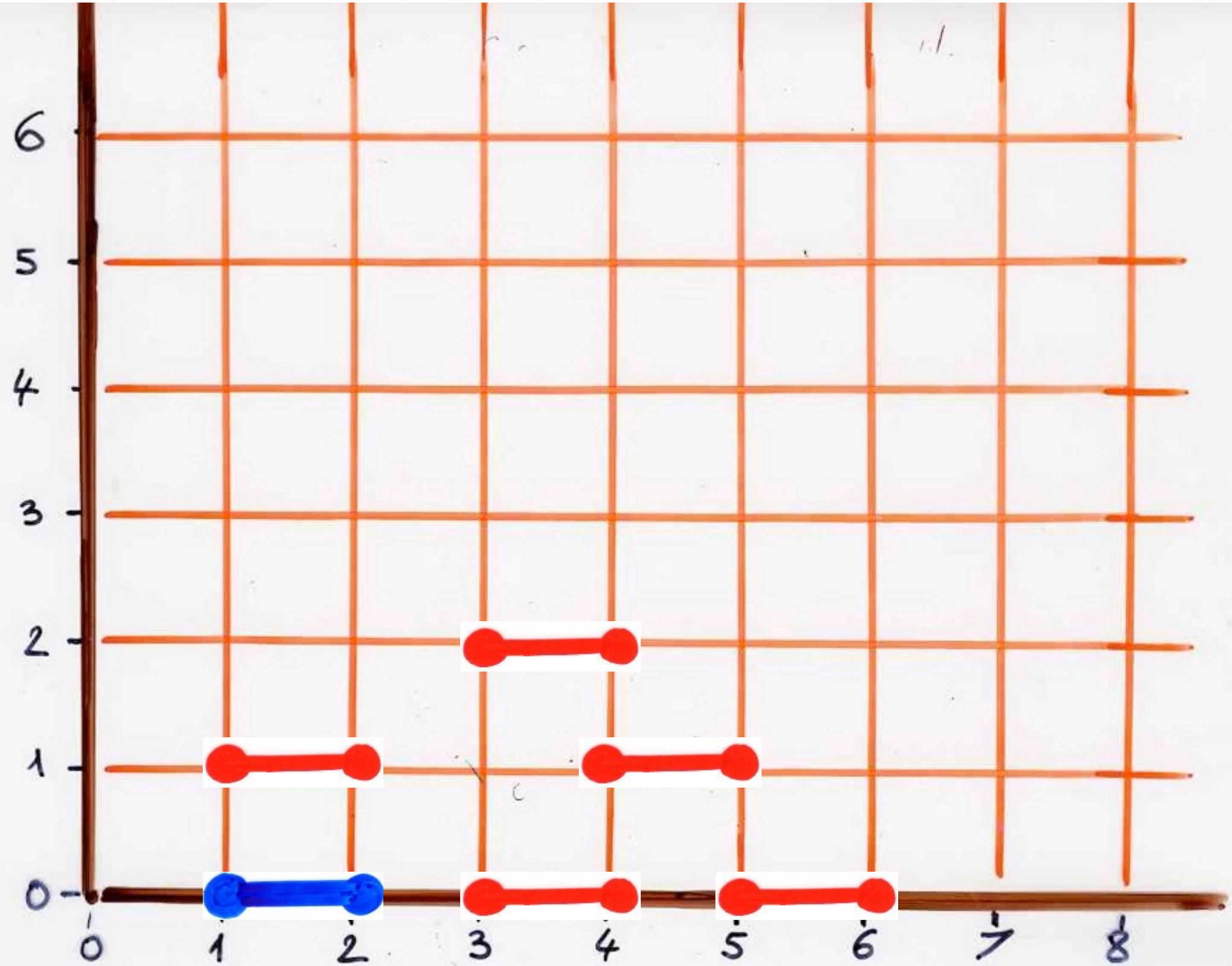
σ_2

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



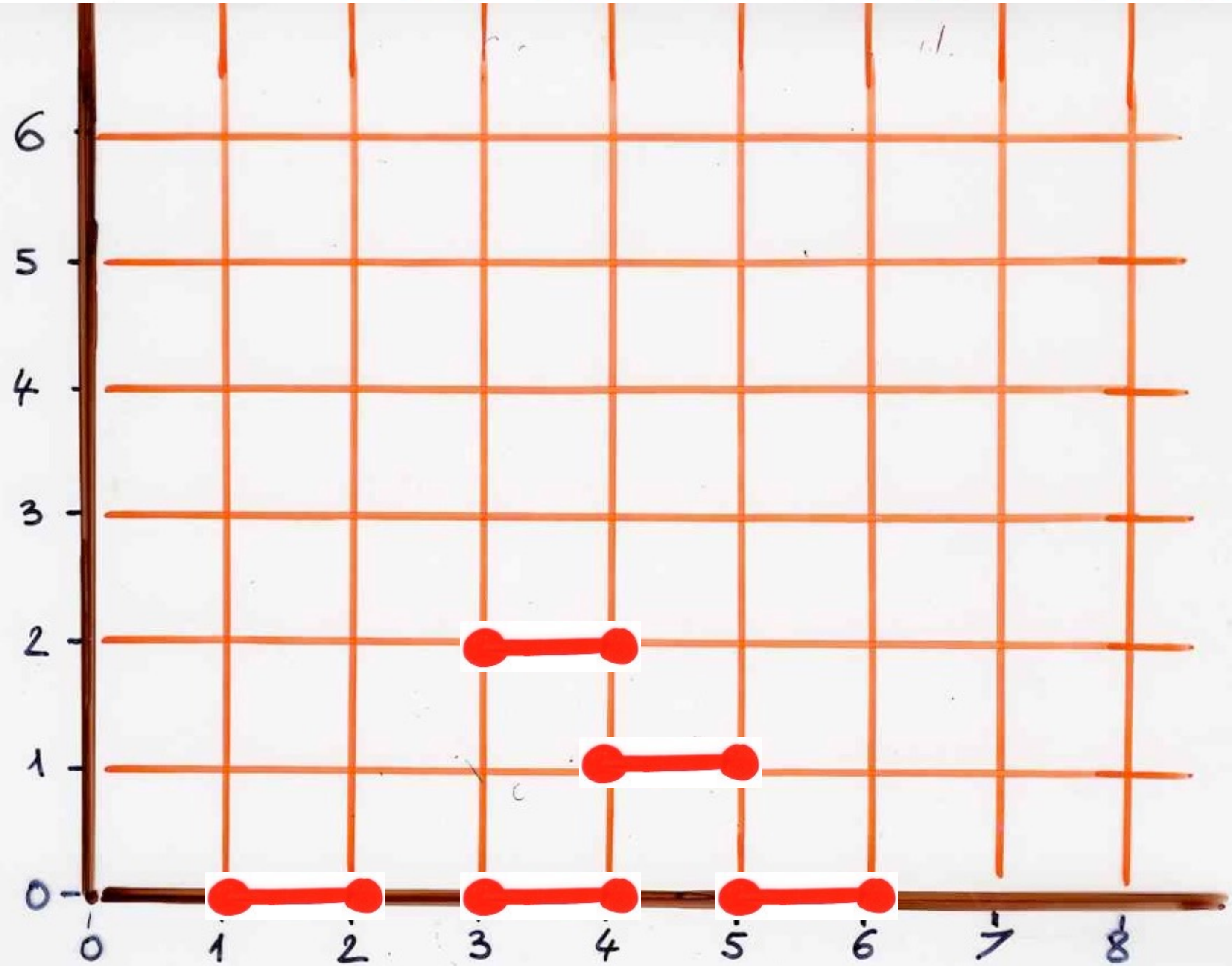
σ_2

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



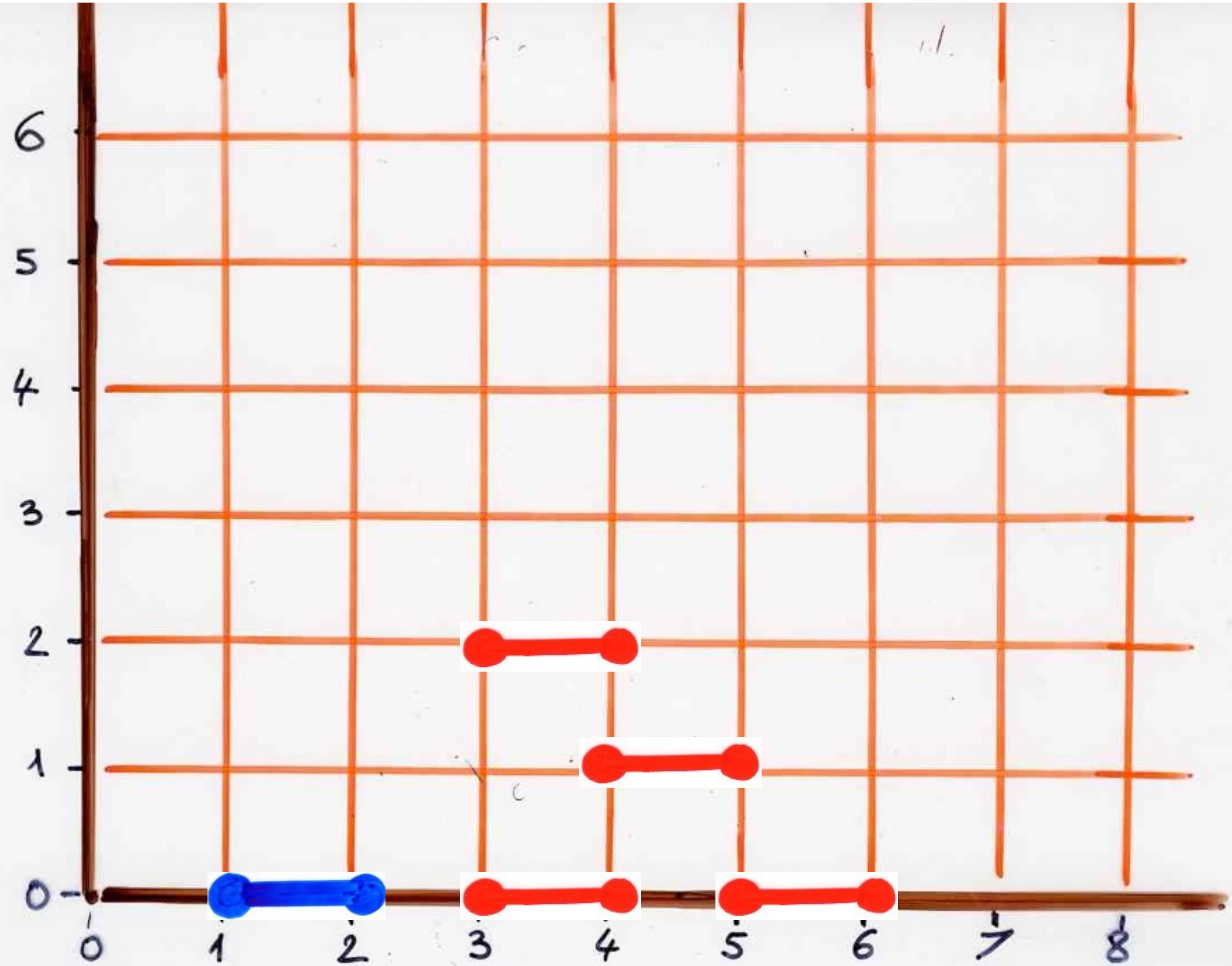
$\sigma_2 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



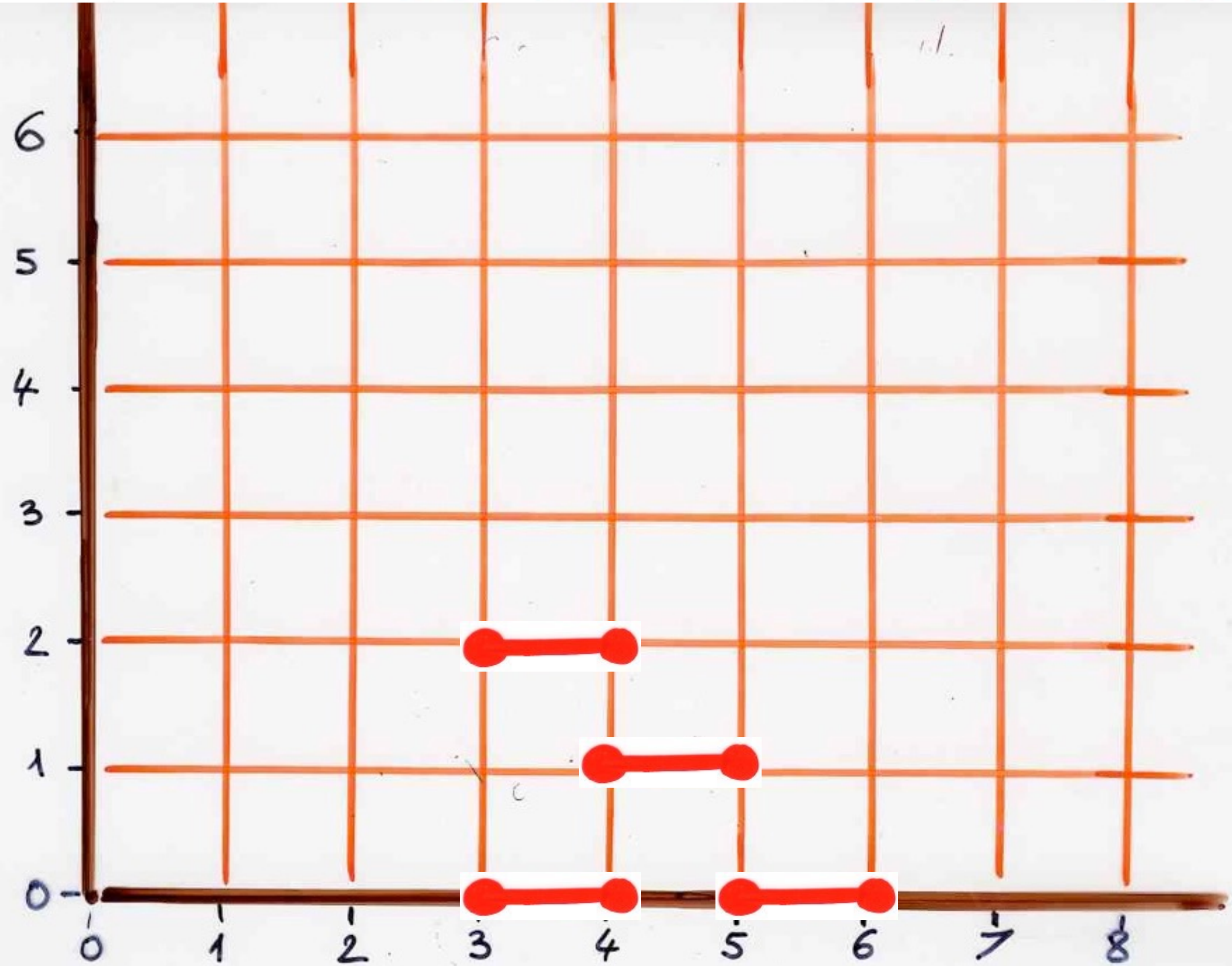
$\sigma_2 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



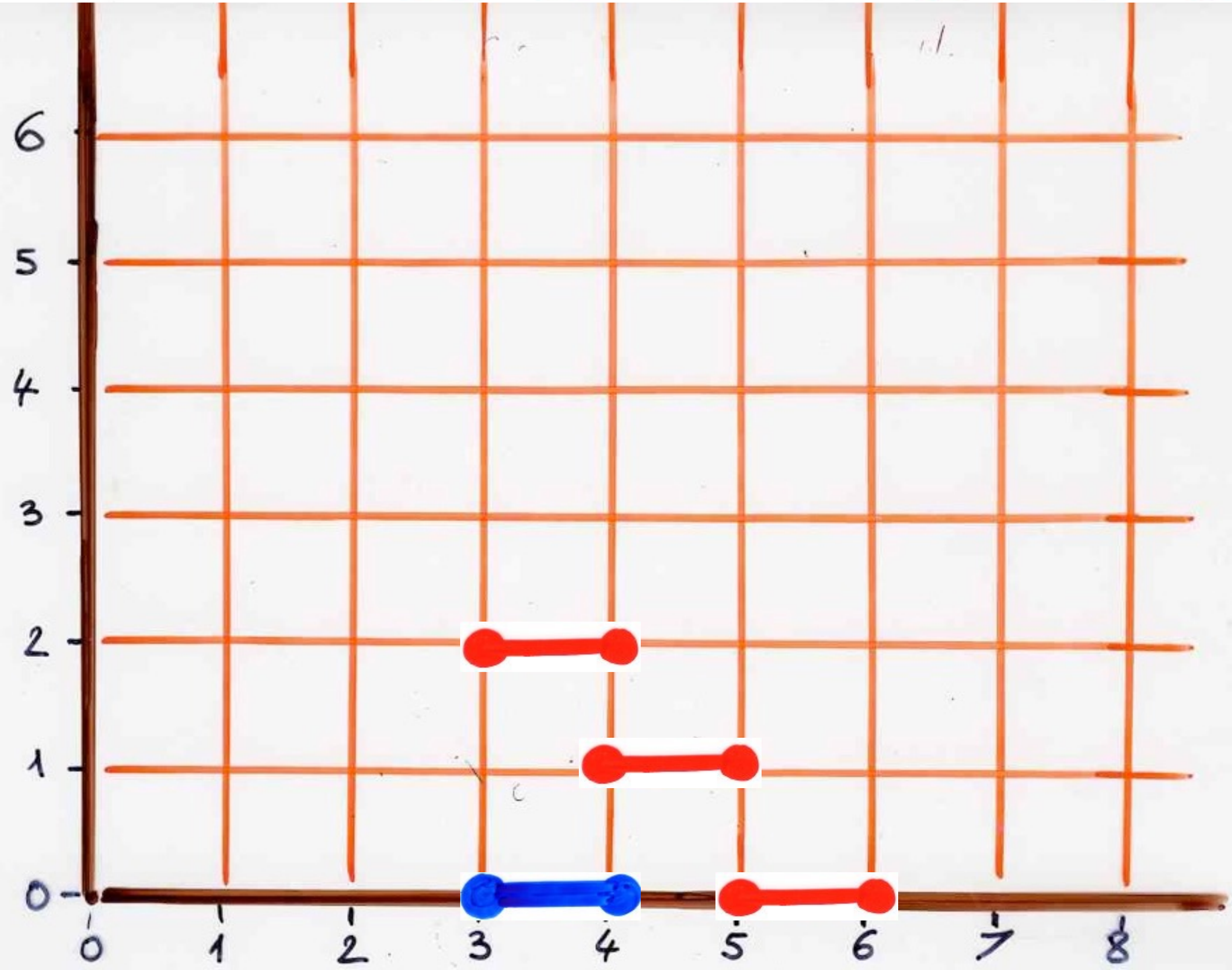
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



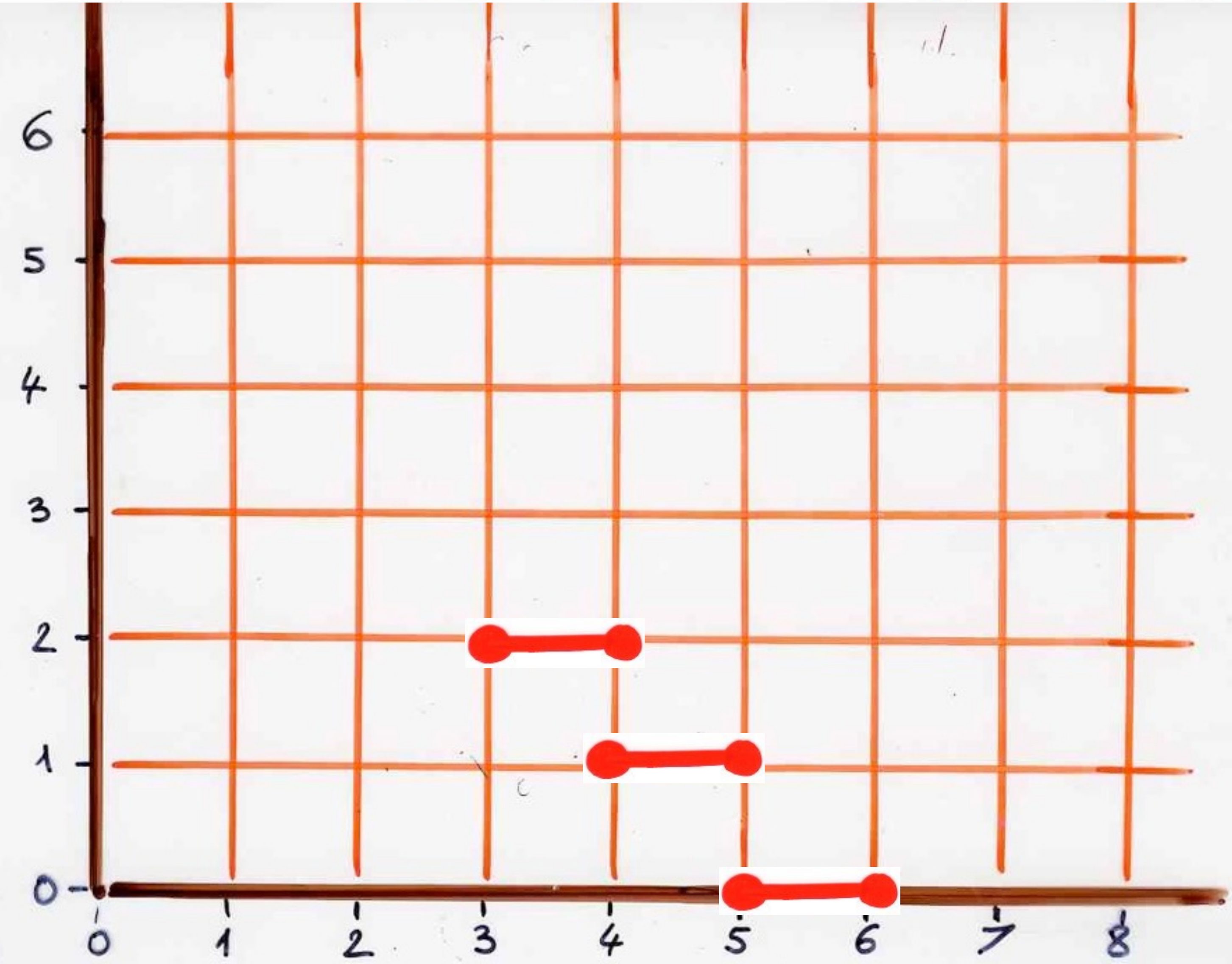
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



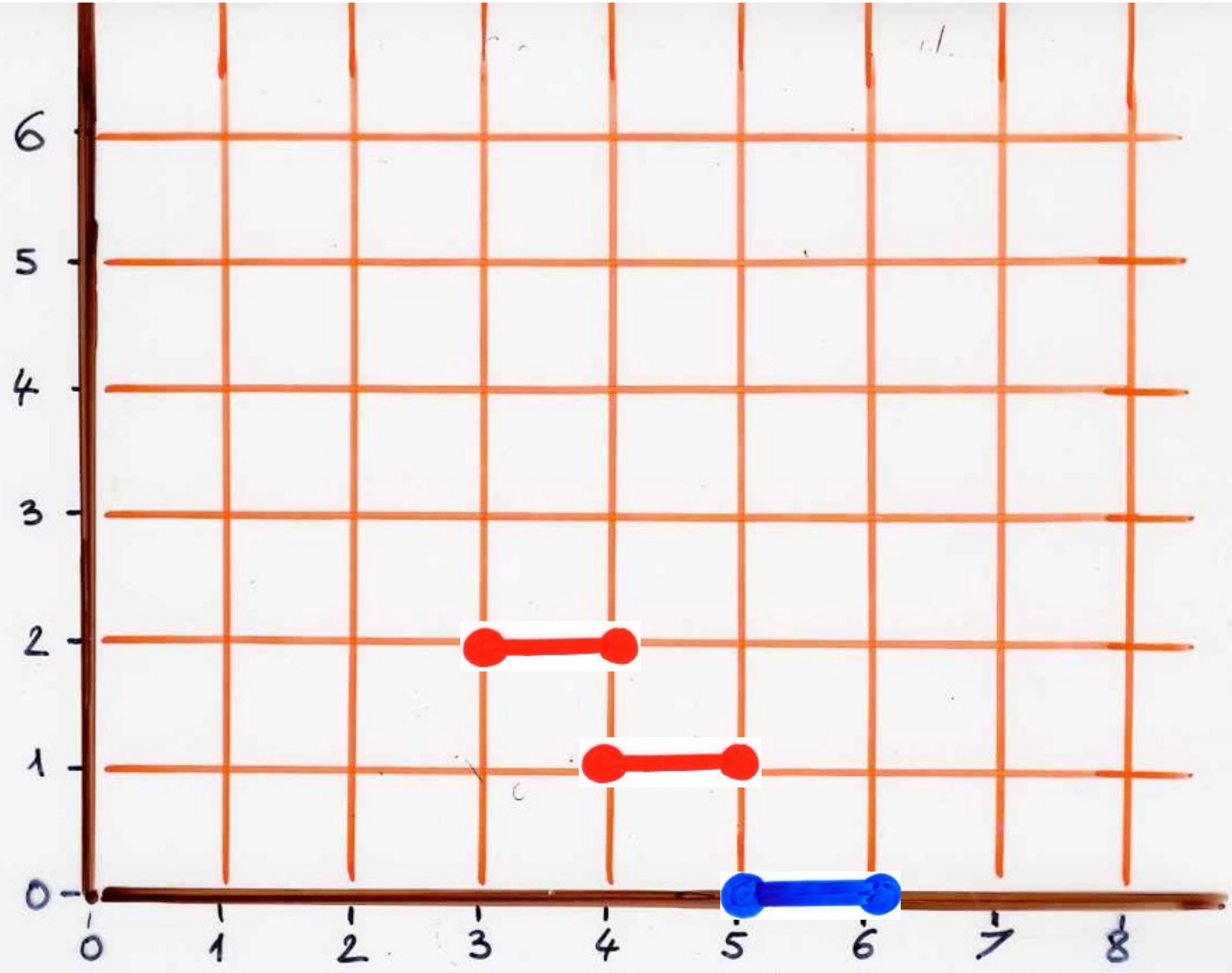
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



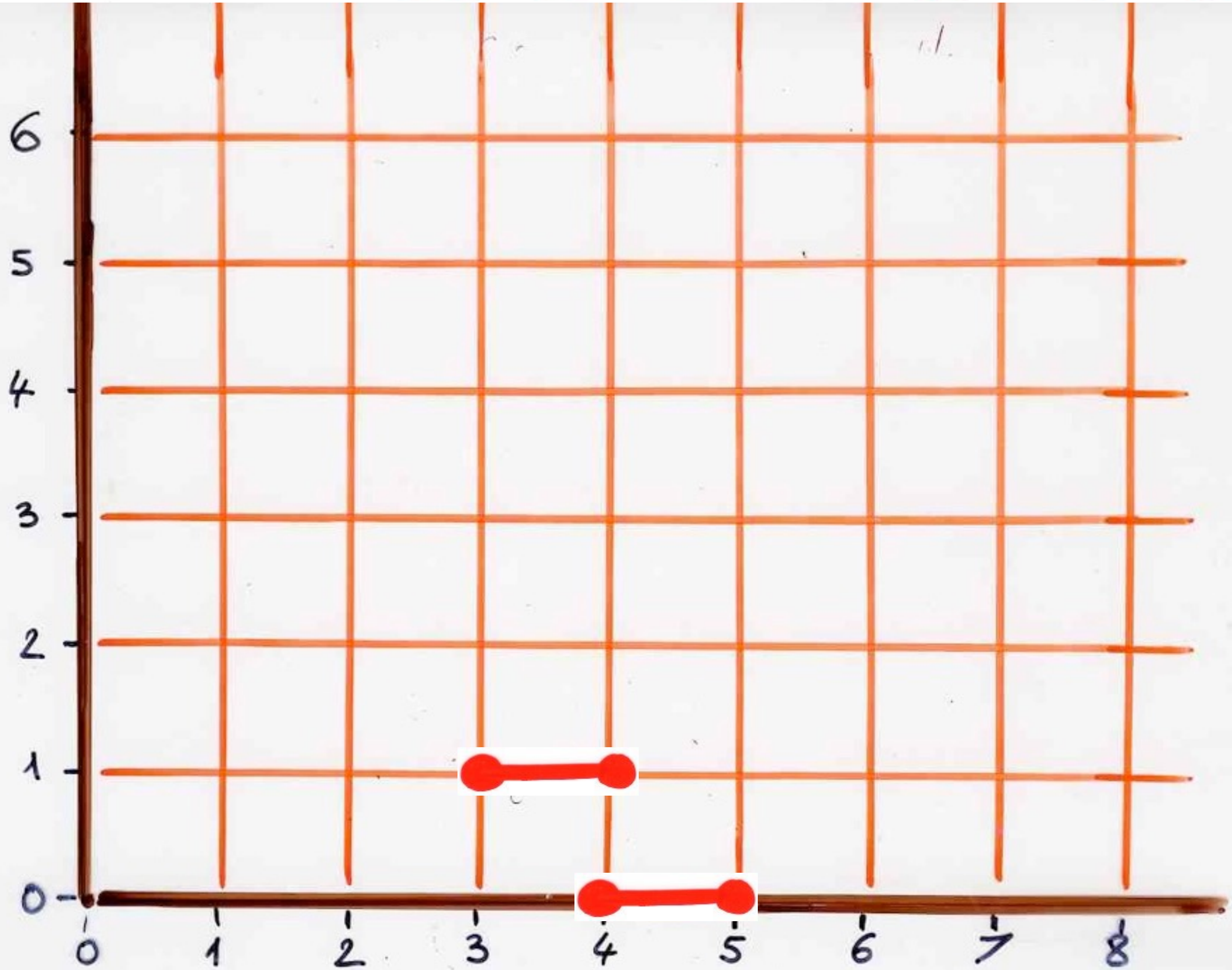
$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



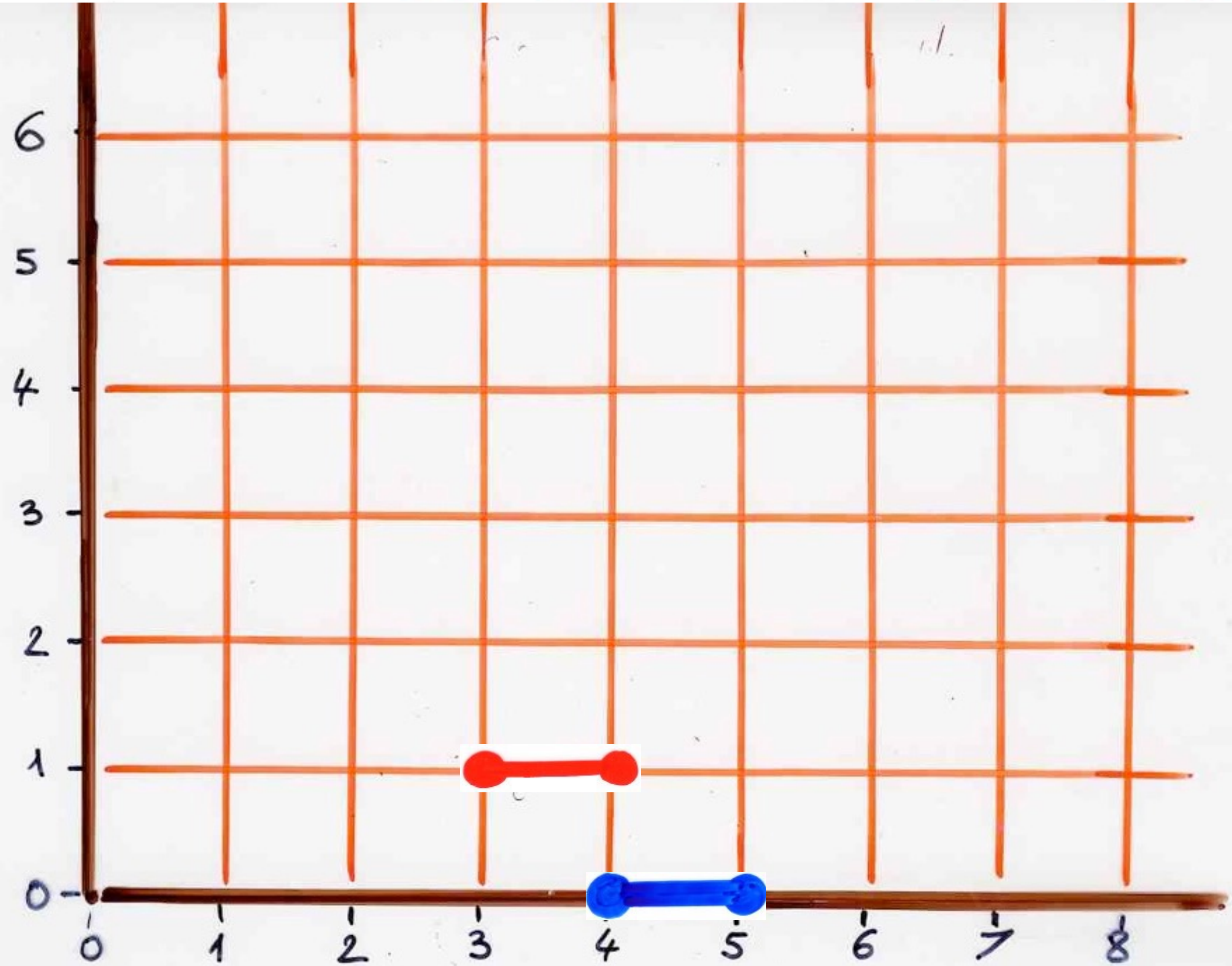
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



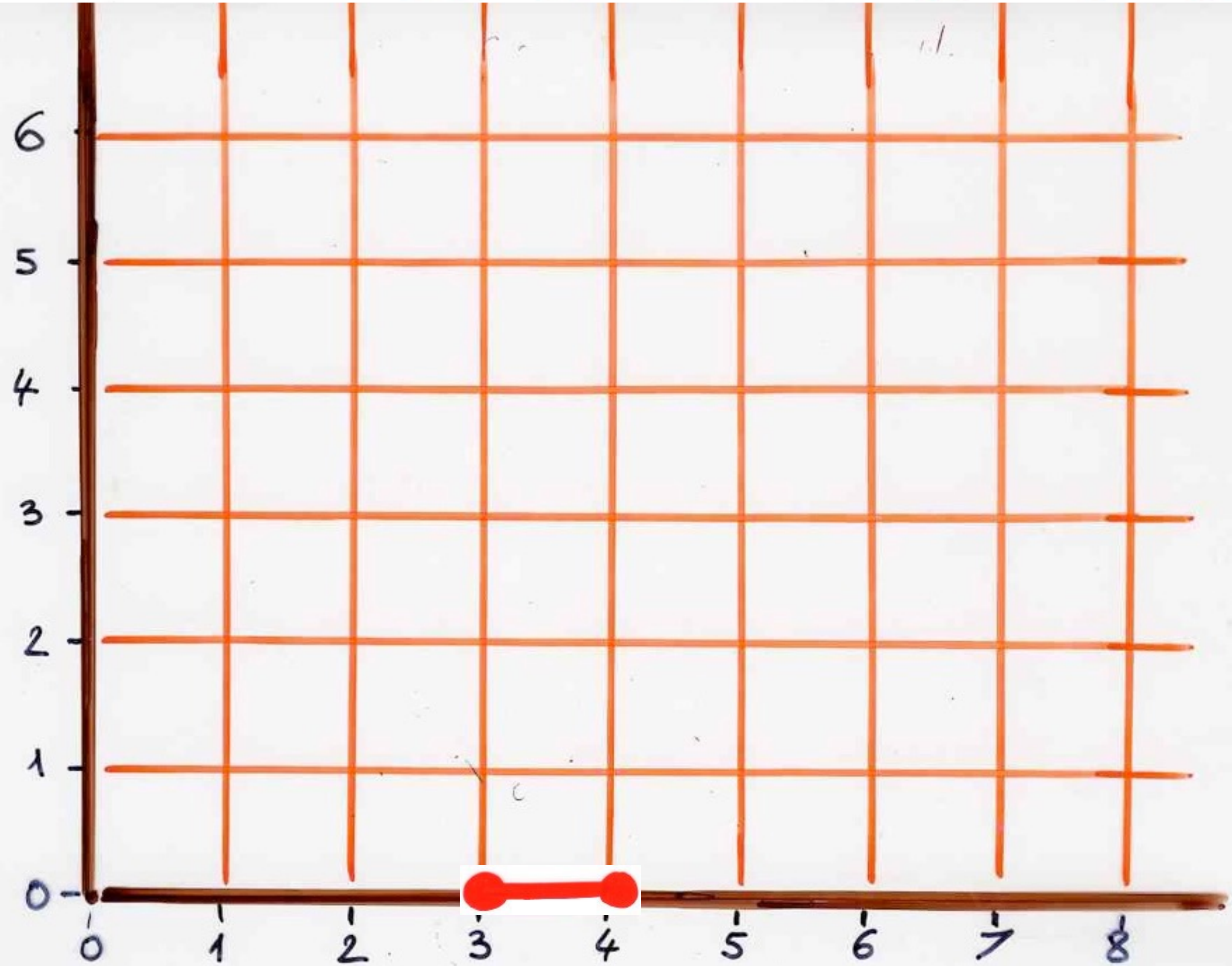
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



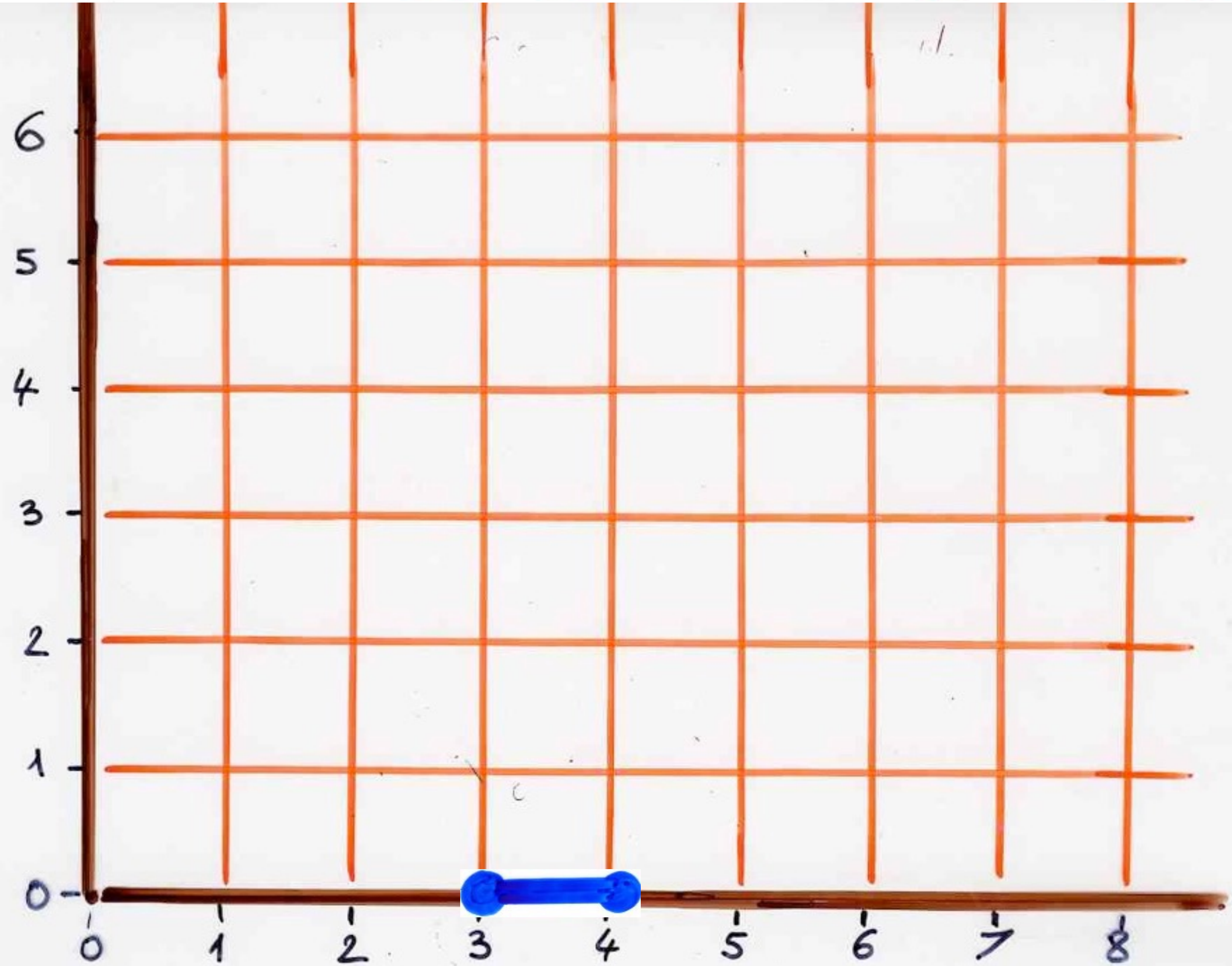
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



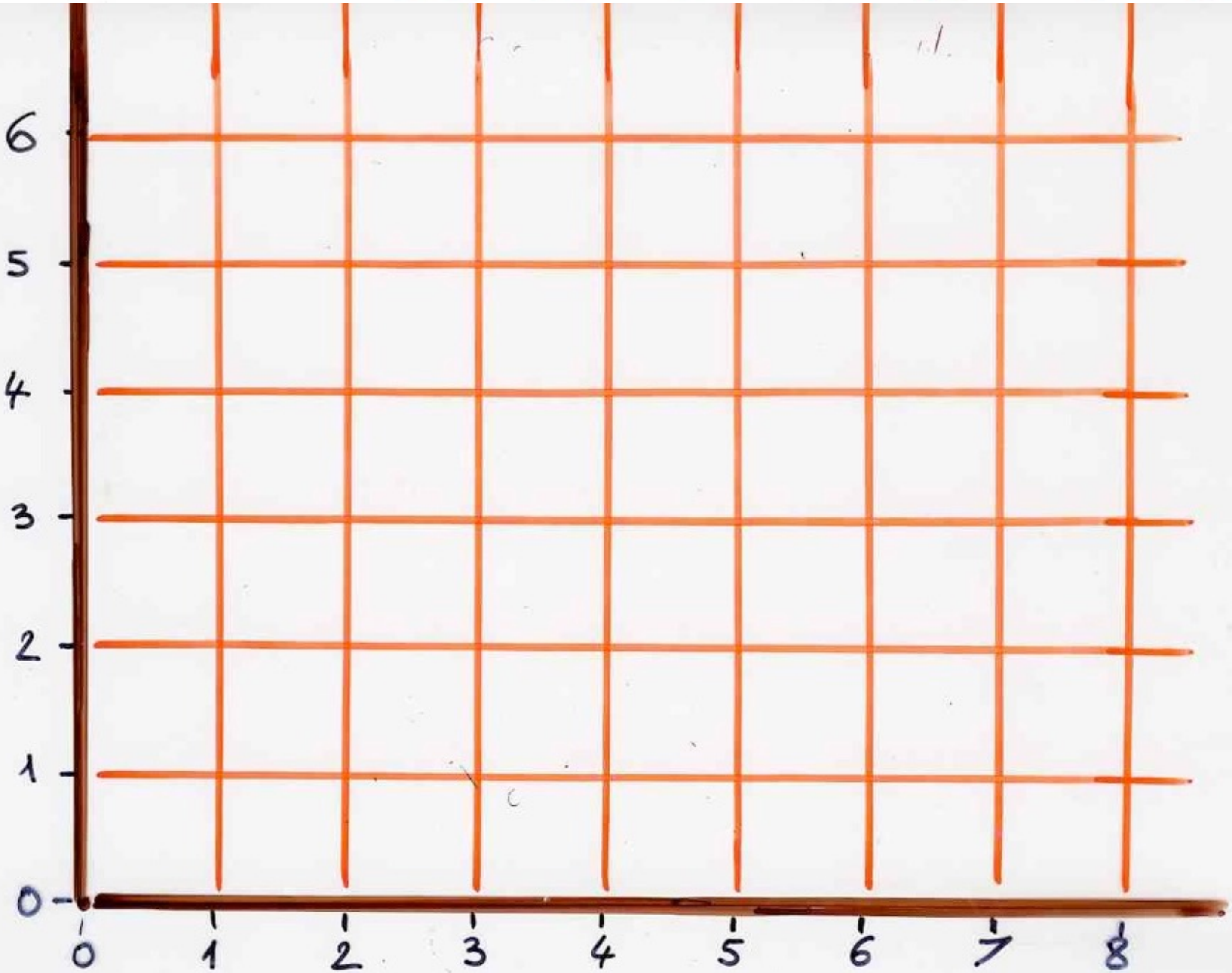
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4$

<u>example</u> $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



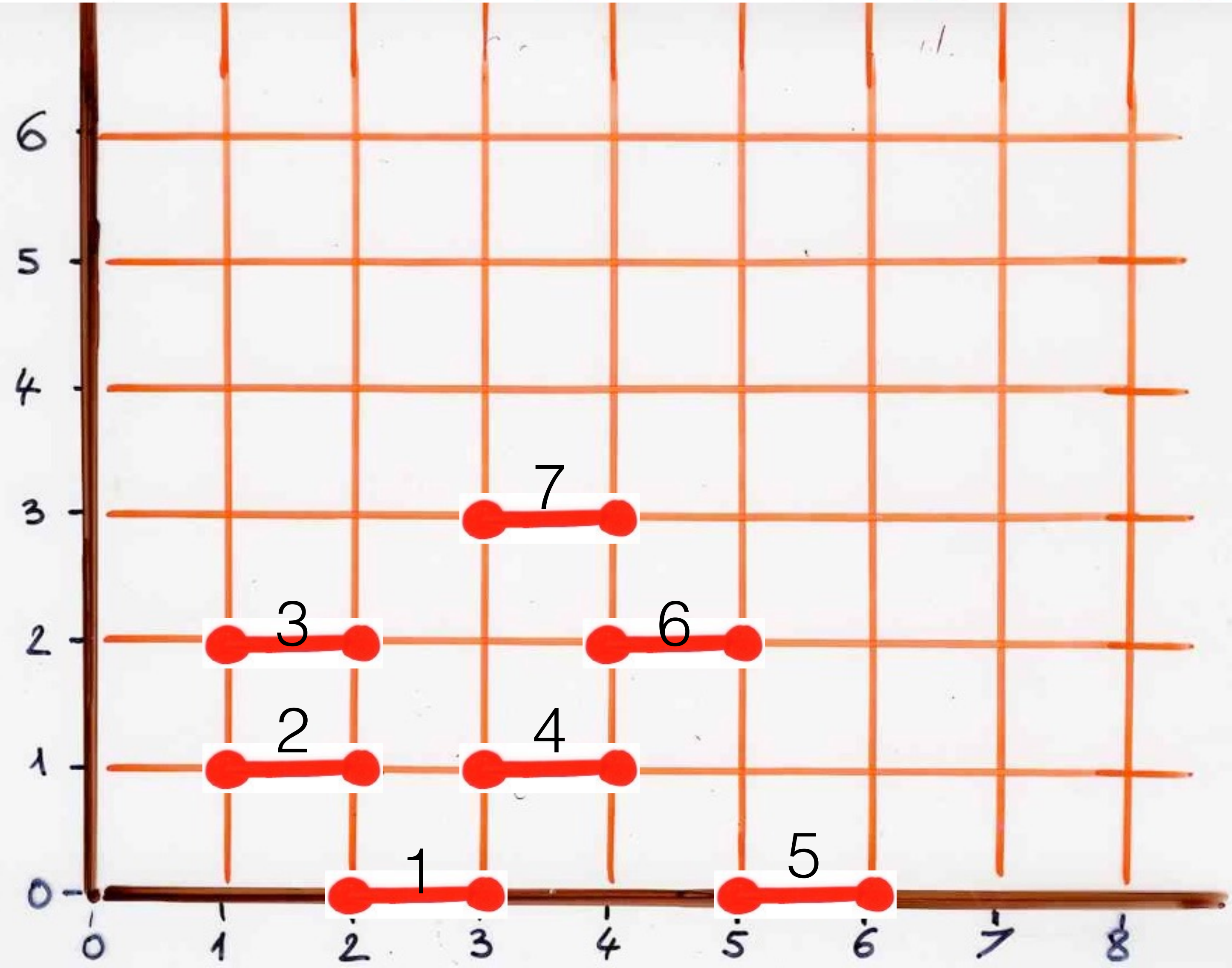
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4 \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



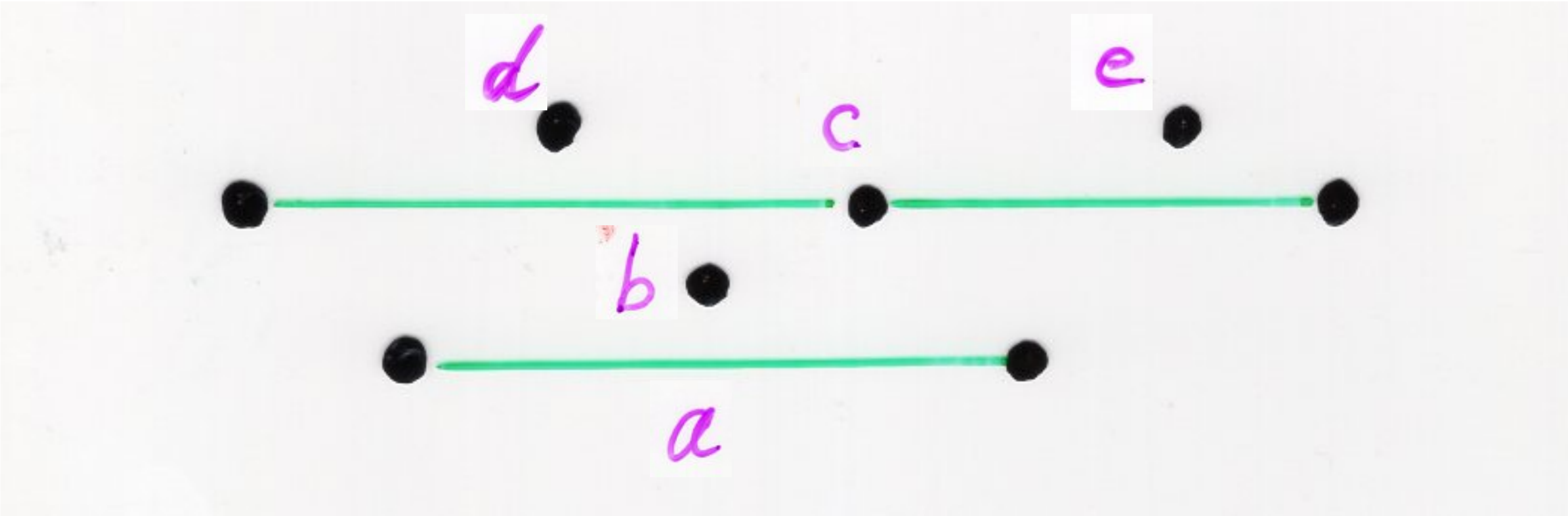
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4 \sigma_3$

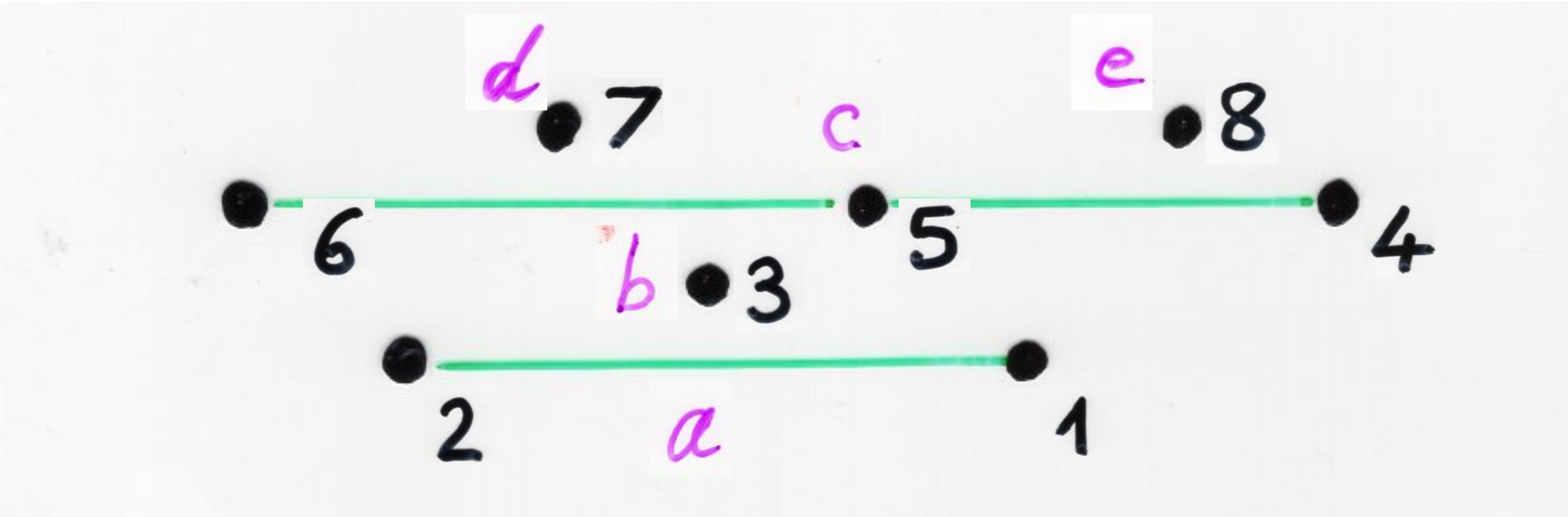
Reverse bijection

Dyck tableaux

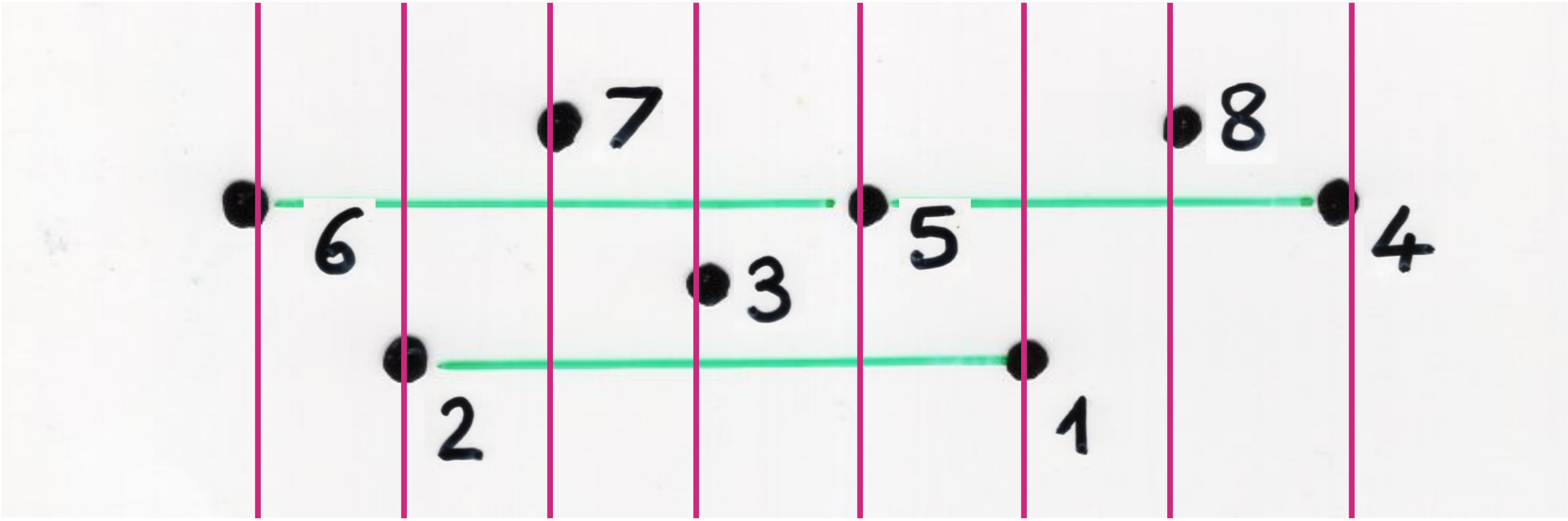
permutations

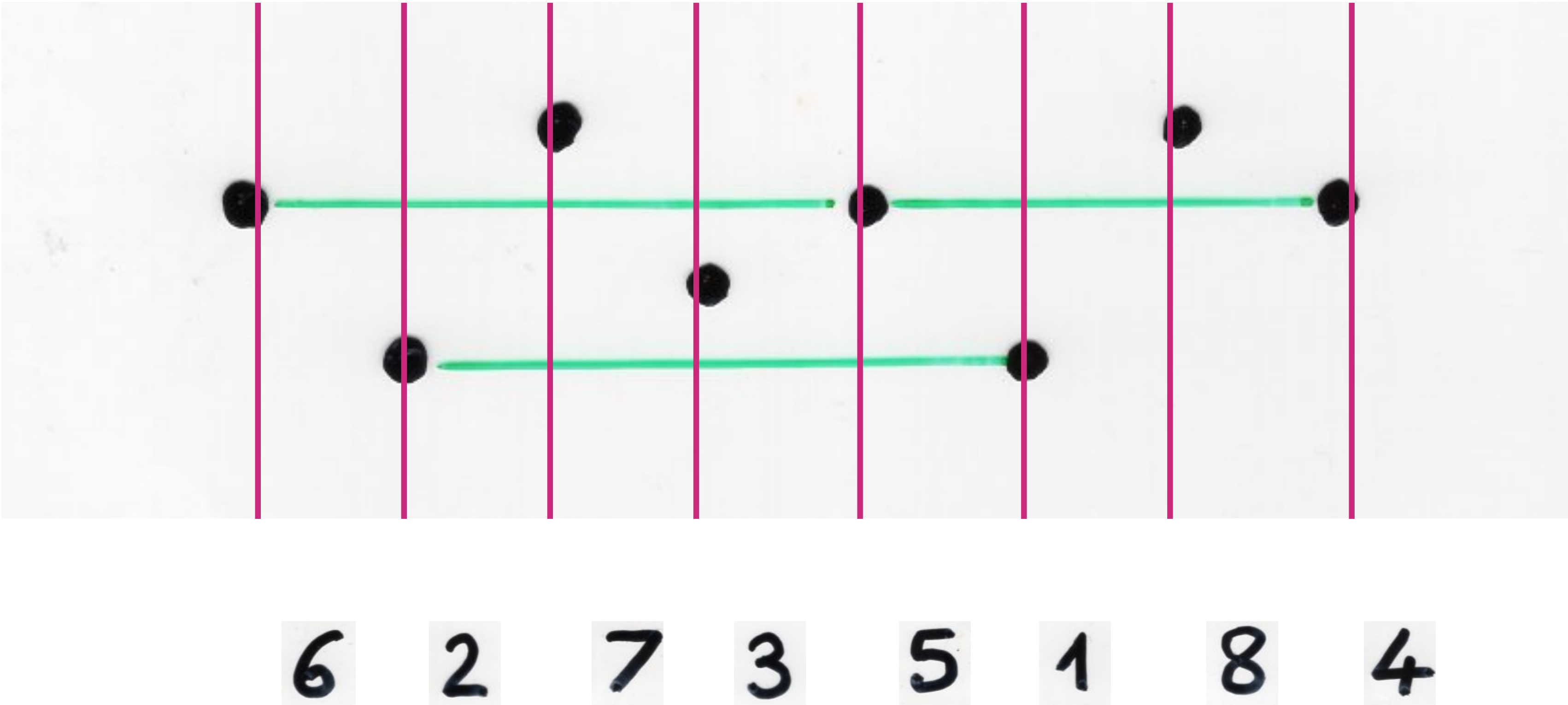
multinear heaps of pointed segments

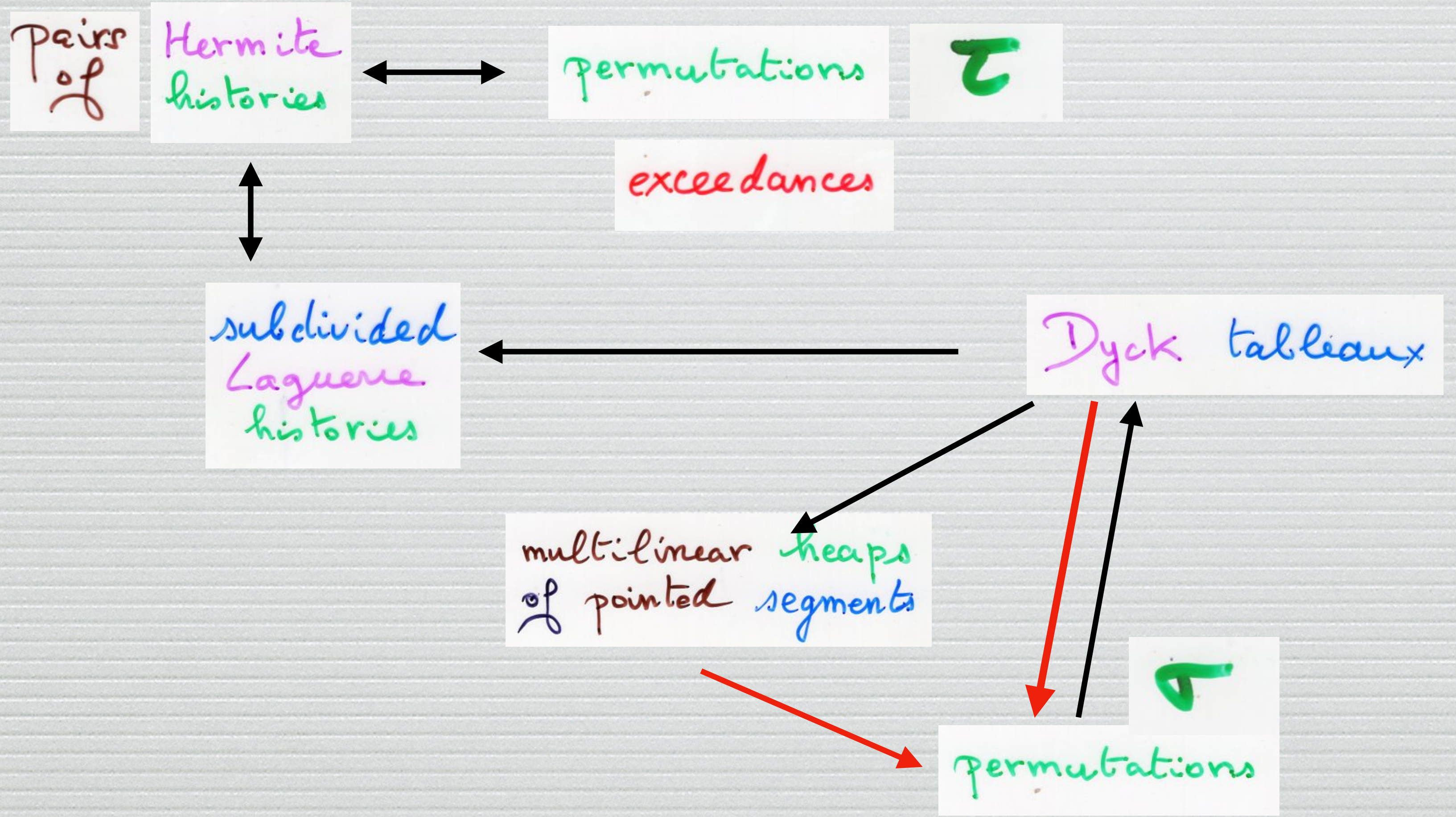

d
e
c
b
a



d
7
c
e
8
6
5
4
b
3
2
a
1

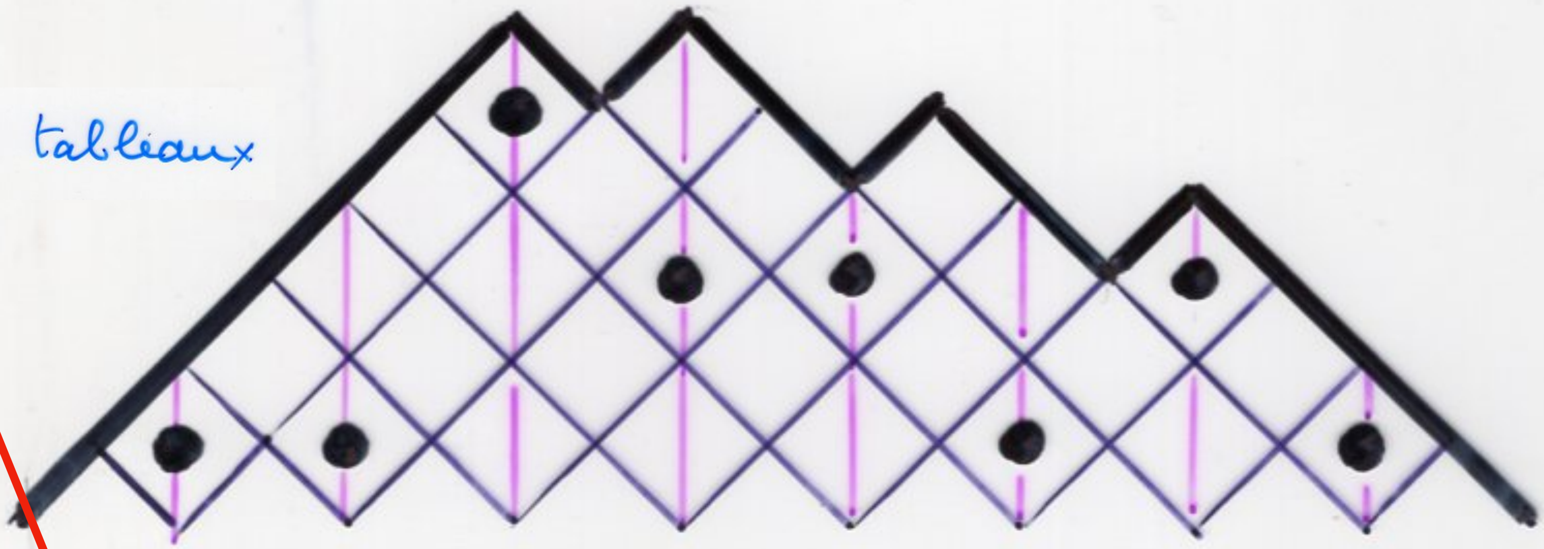

7
8
6
5
4
3
2
1


6 2 7 3 5 1 8 4



Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations

Dyck tableaux



permutations

σ = 6 2 7 3 5 1 8 4

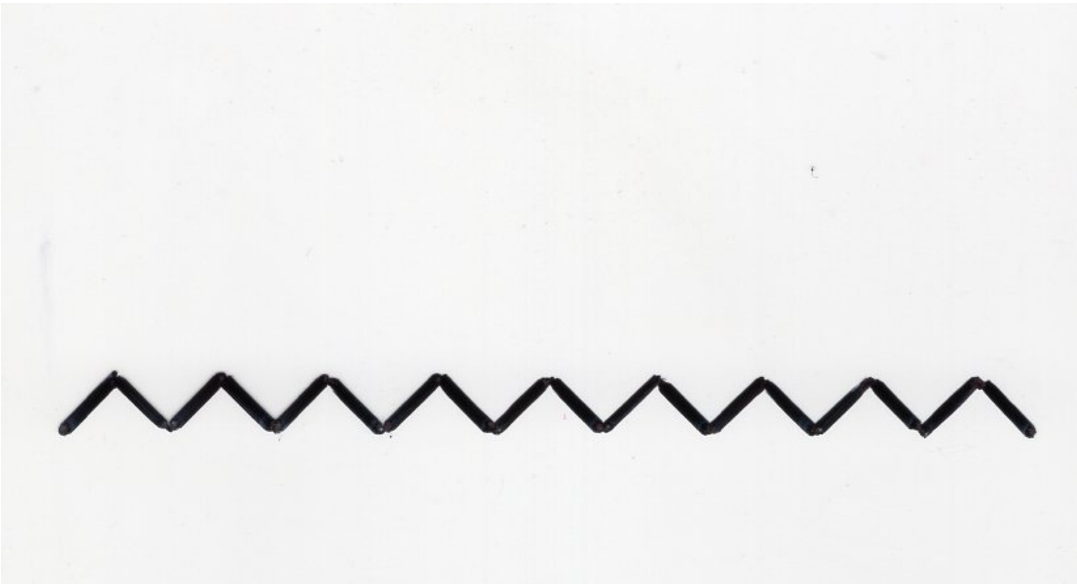
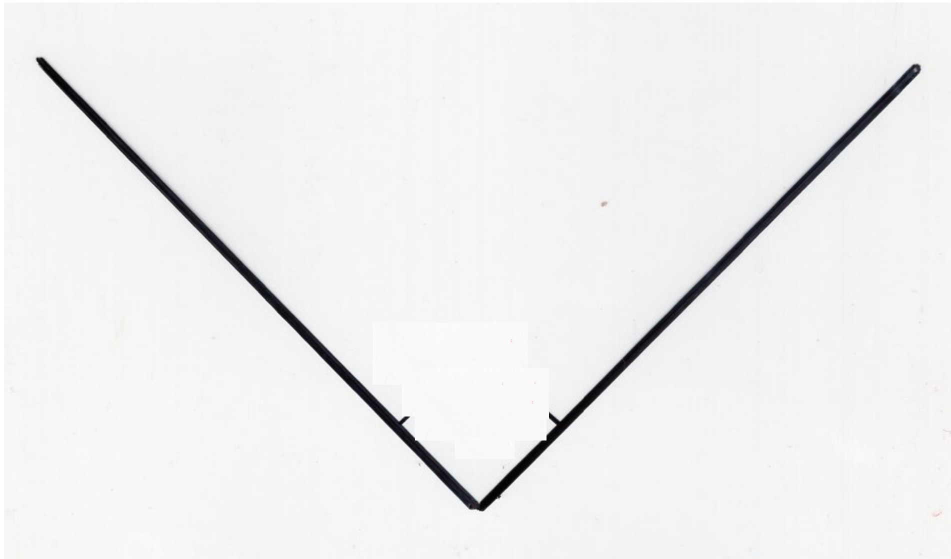
Bijections

Permutations
(inversion tables)

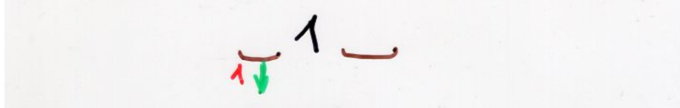
Dyck tableaux

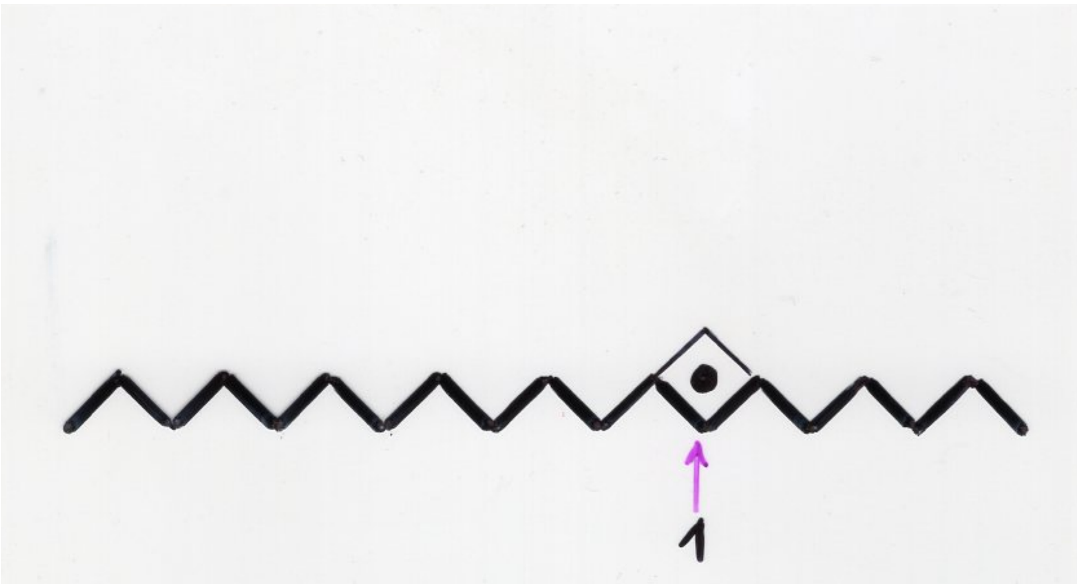
tree-like tableaux
(or alternative tableaux)

$\sqcup$

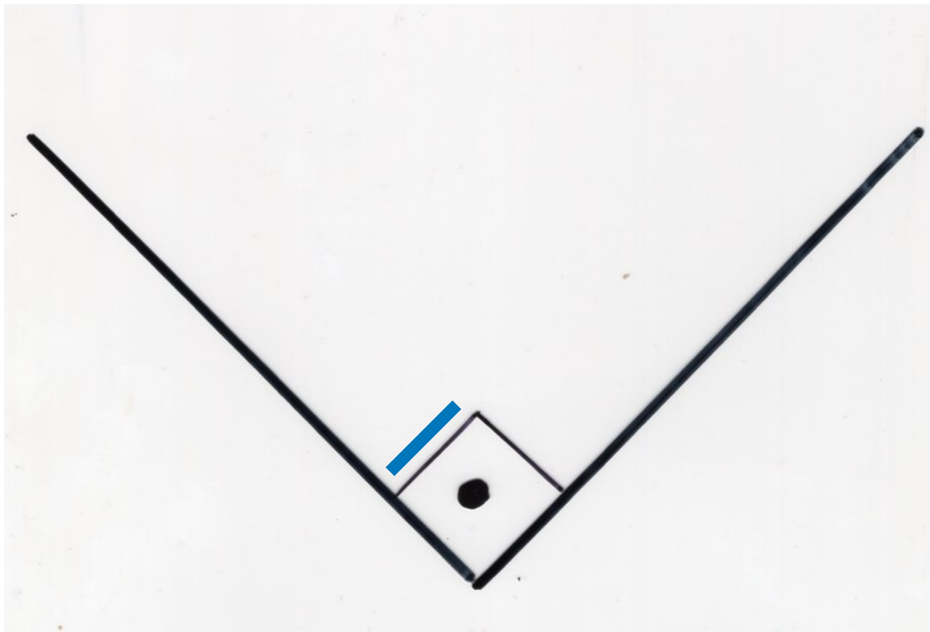
 

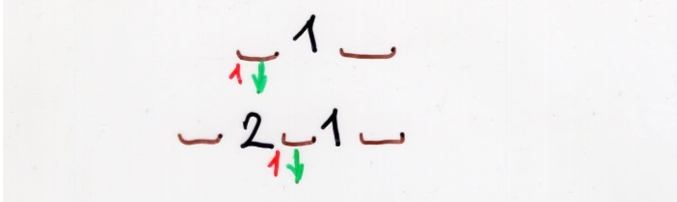
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

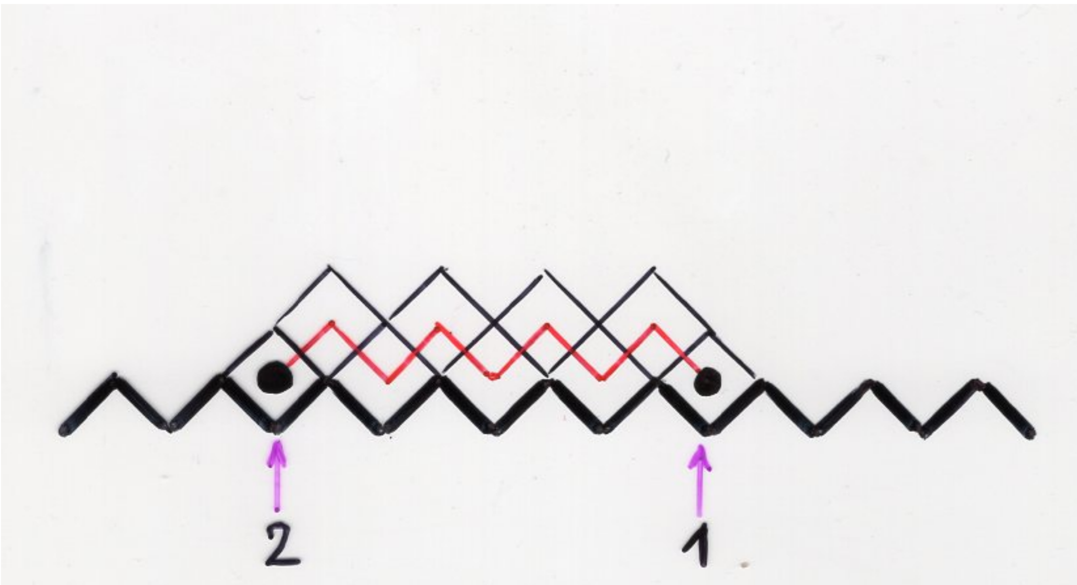




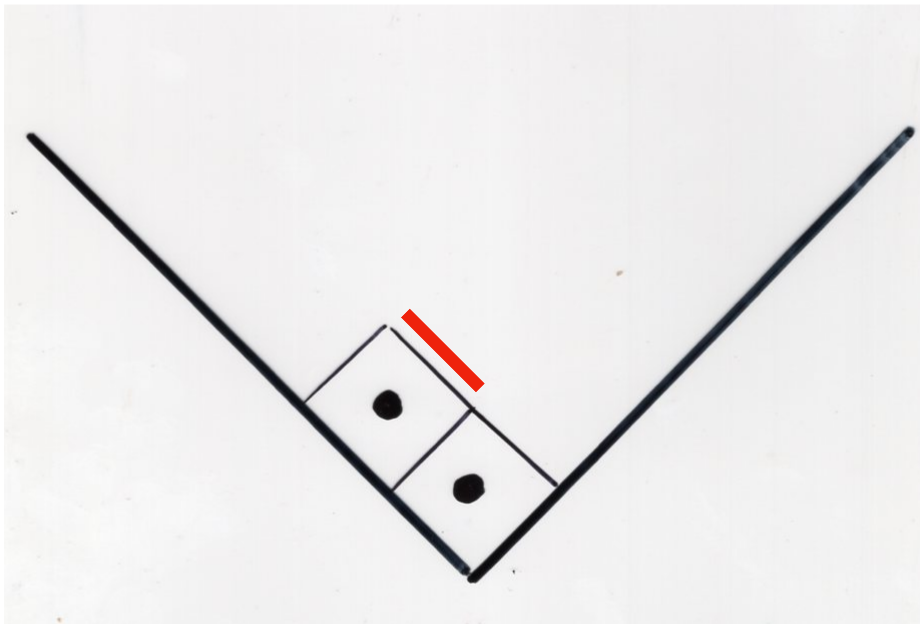
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

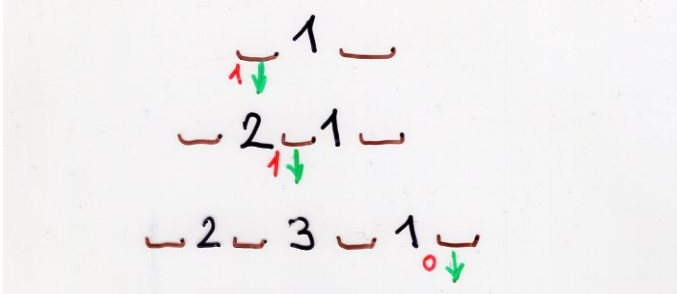


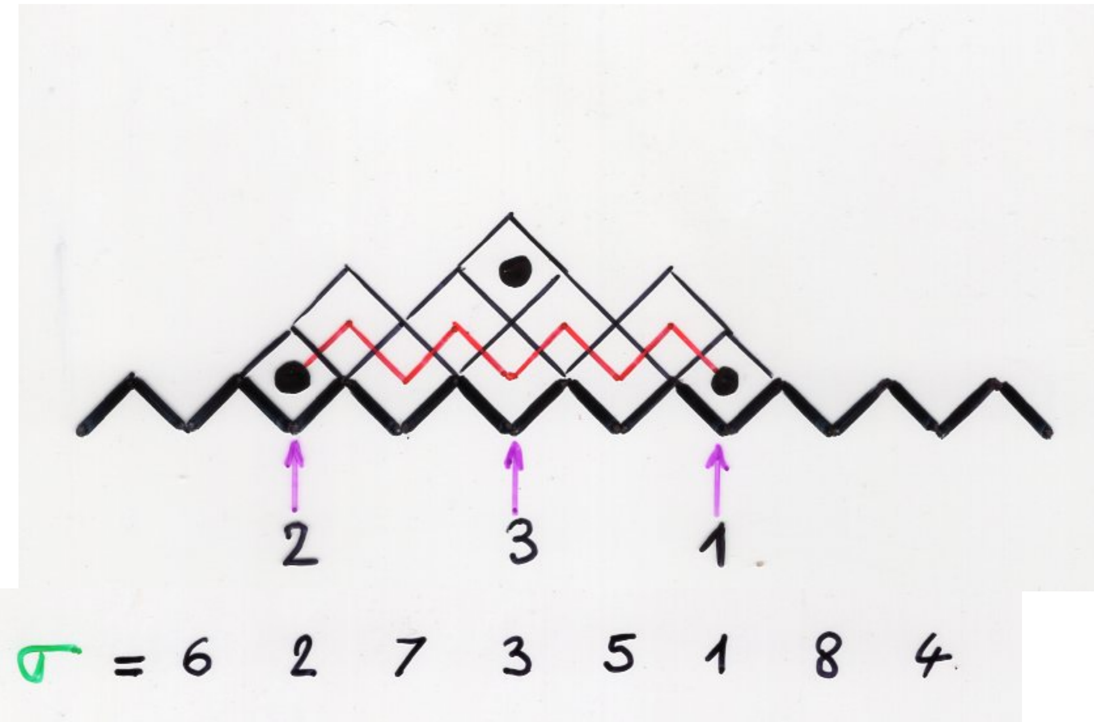


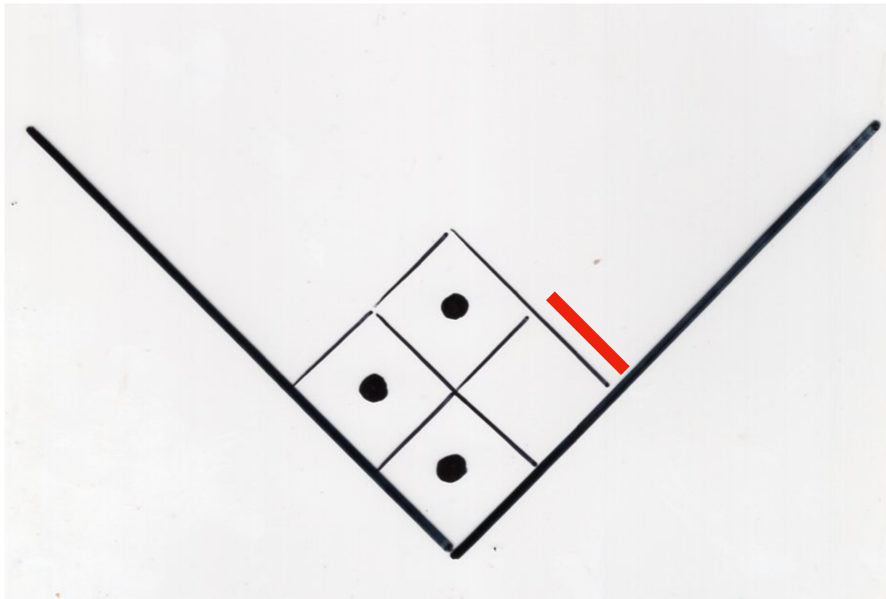


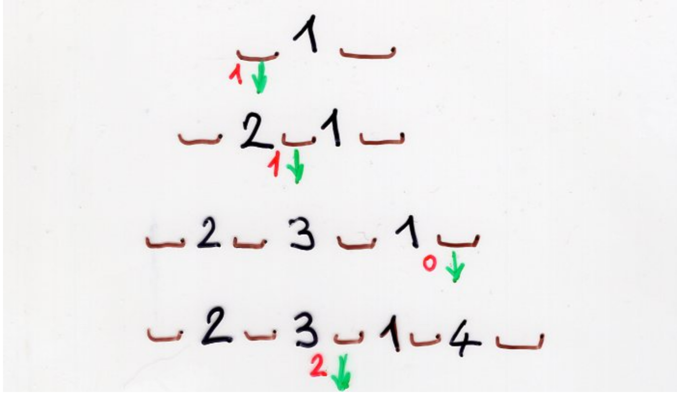
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

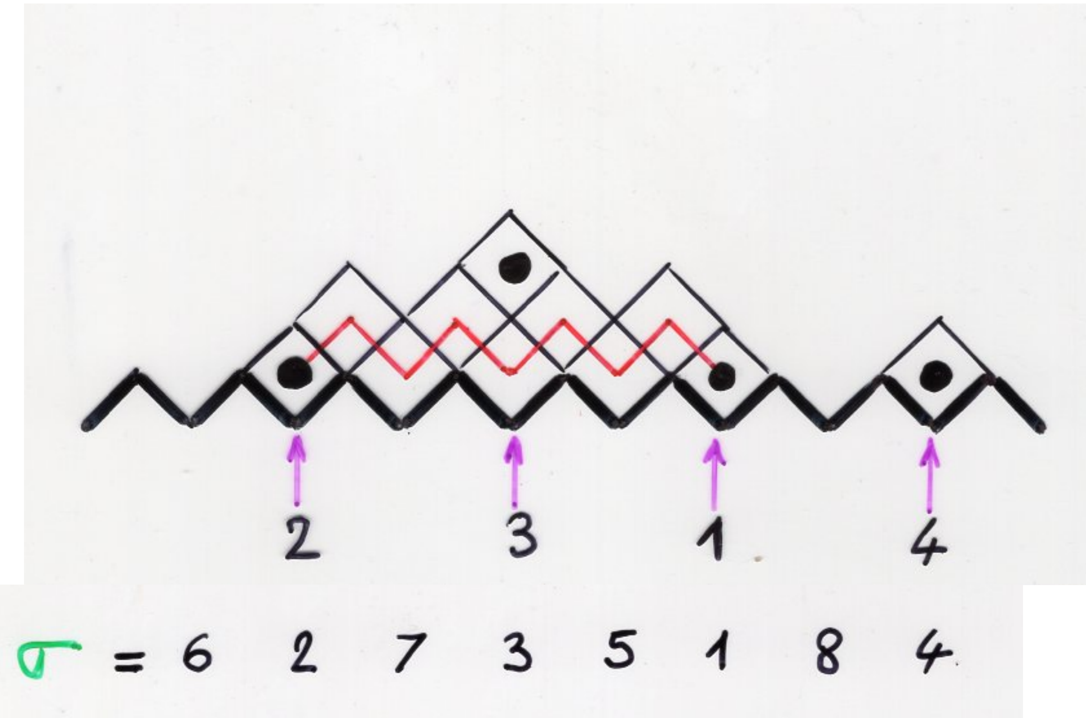


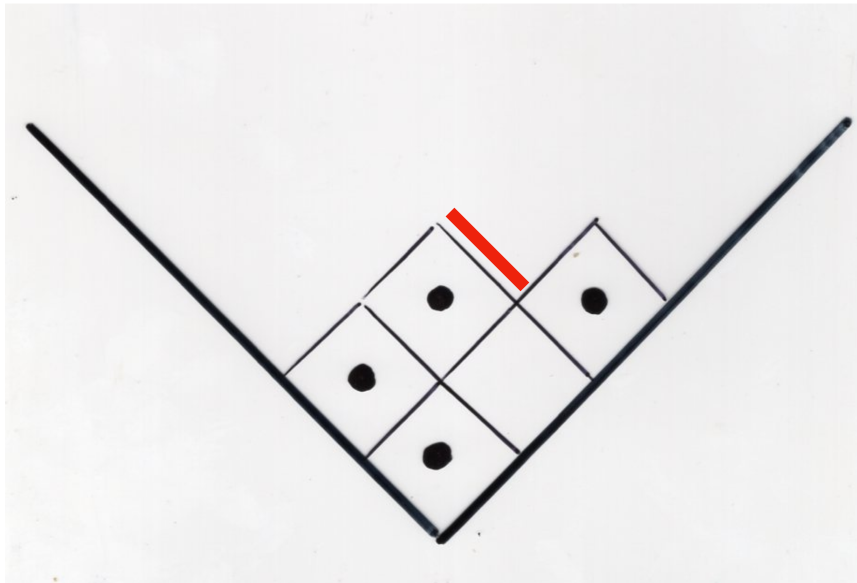

1
2 1
2 3 1

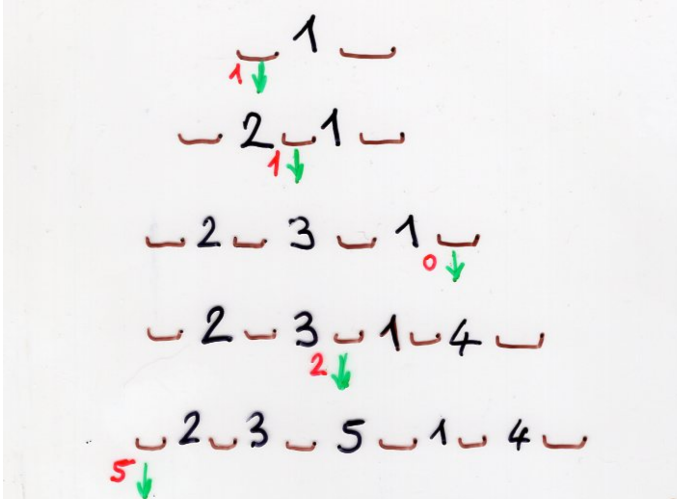

2 3 1
σ = 6 2 7 3 5 1 8 4

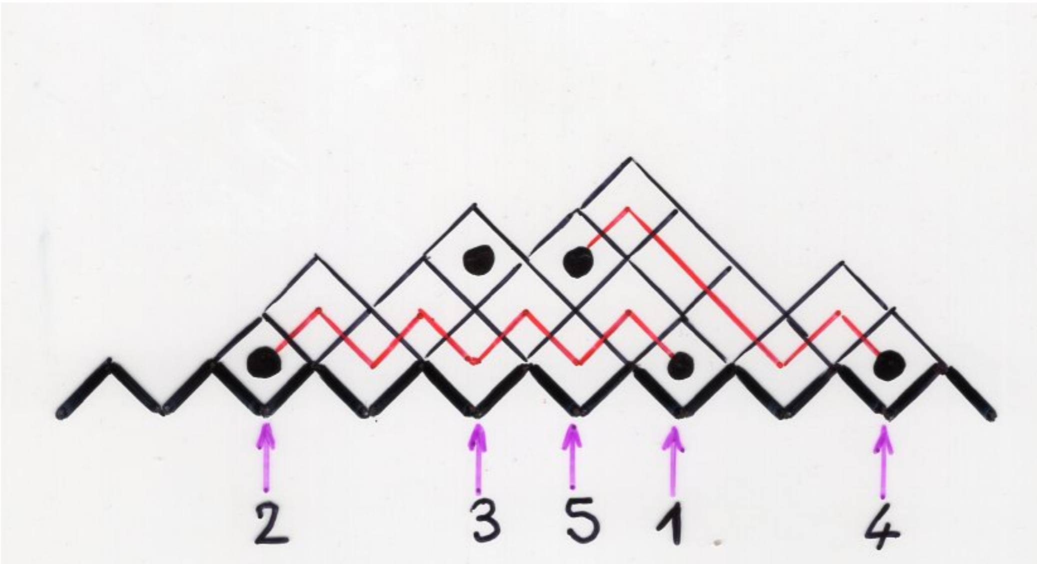



1
1
2 1
1
2 3 1
0
2 3 1 4
2

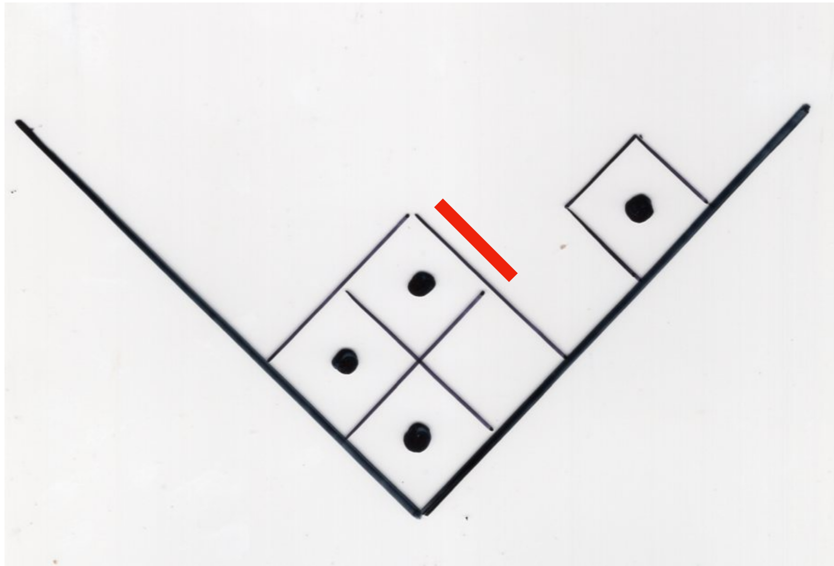

2 3 1 4
σ = 6 2 7 3 5 1 8 4

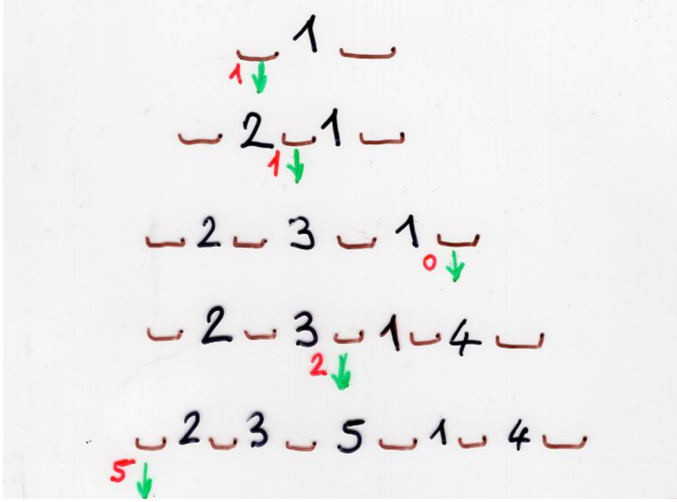


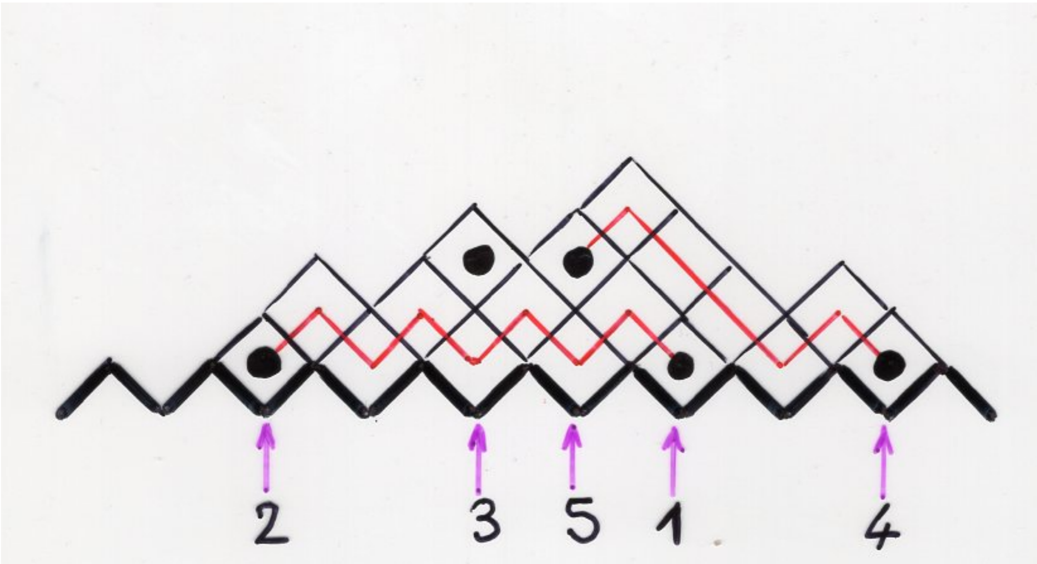


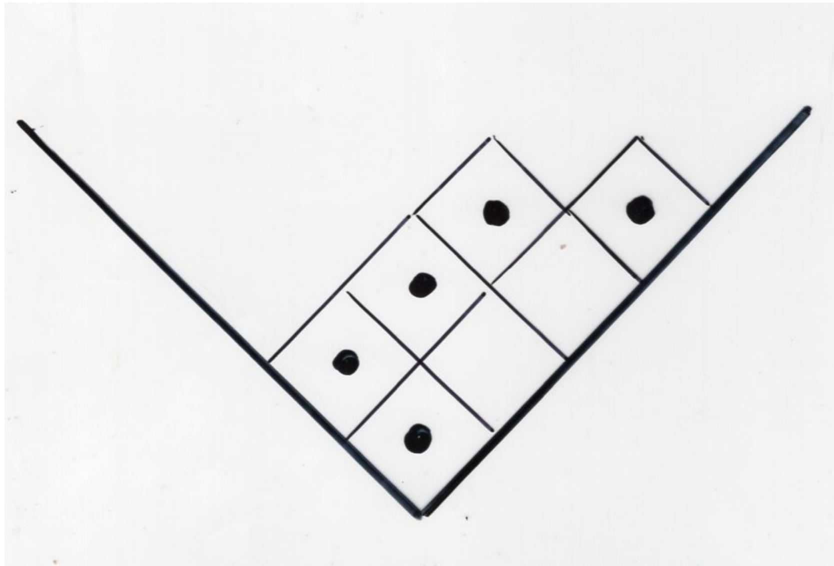


σ = 6 2 7 3 5 1 8 4

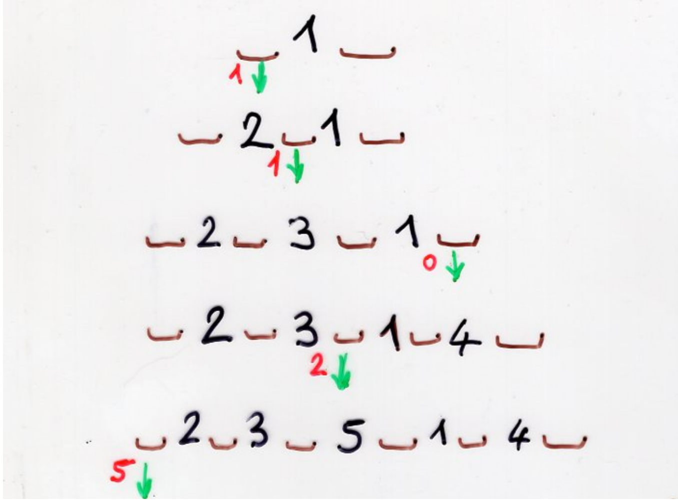


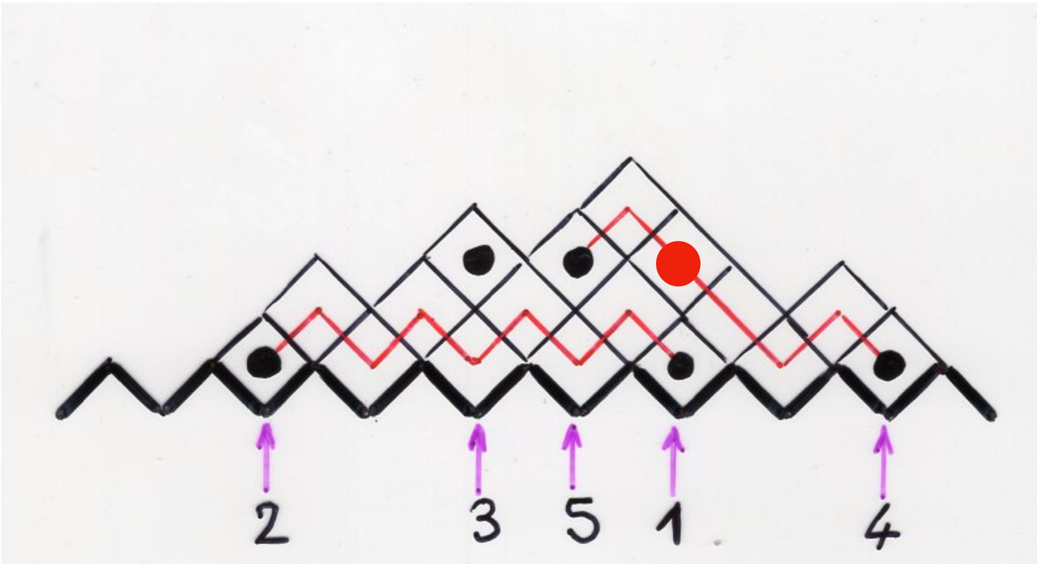




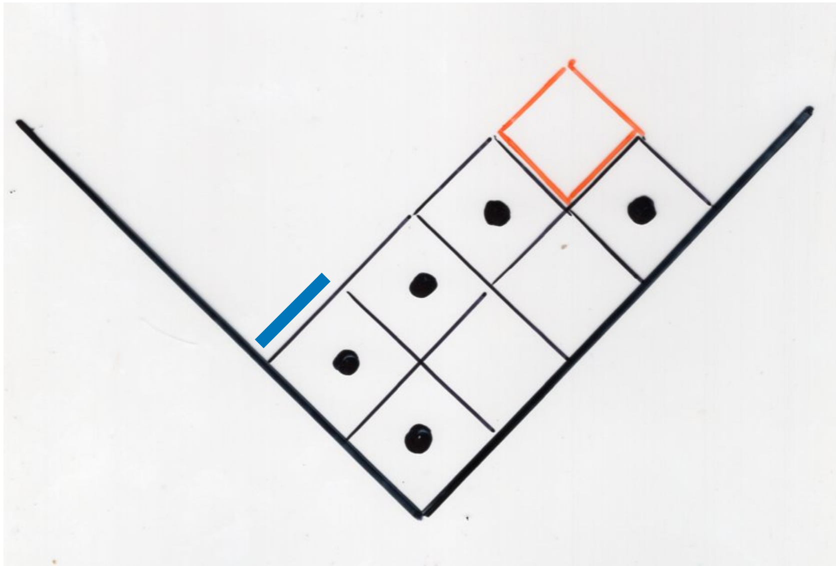


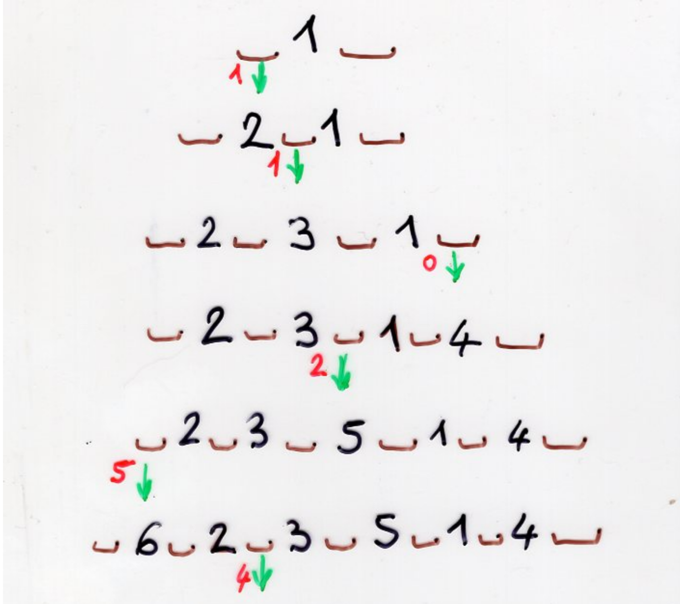
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

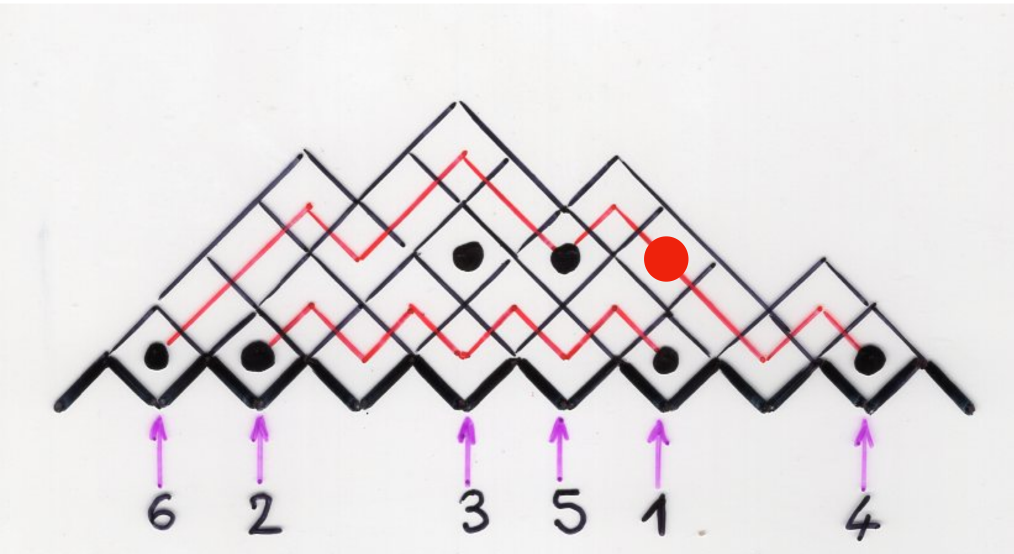




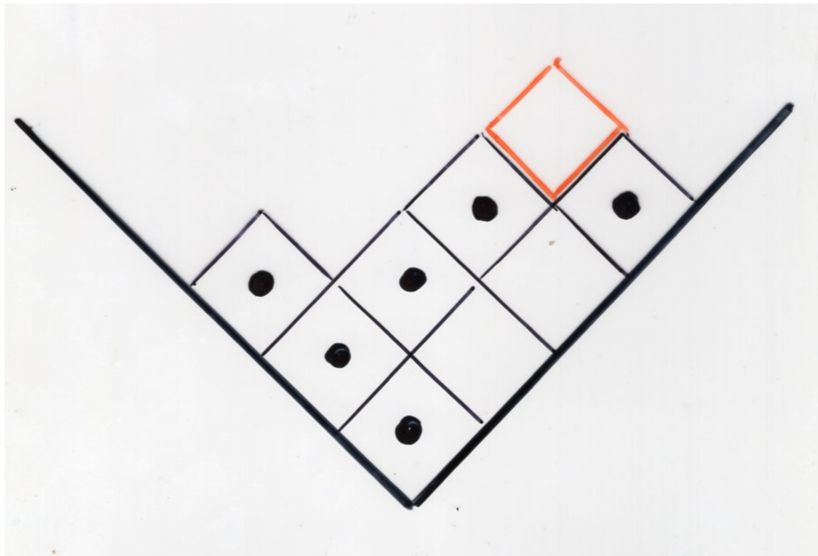
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

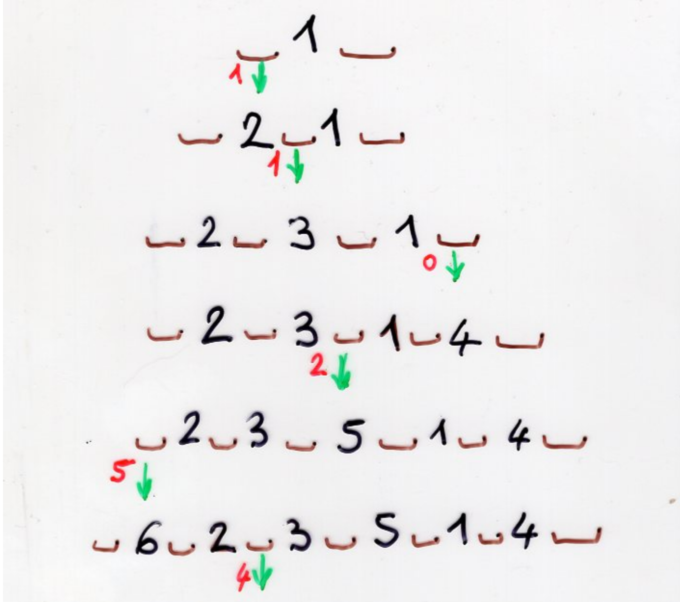


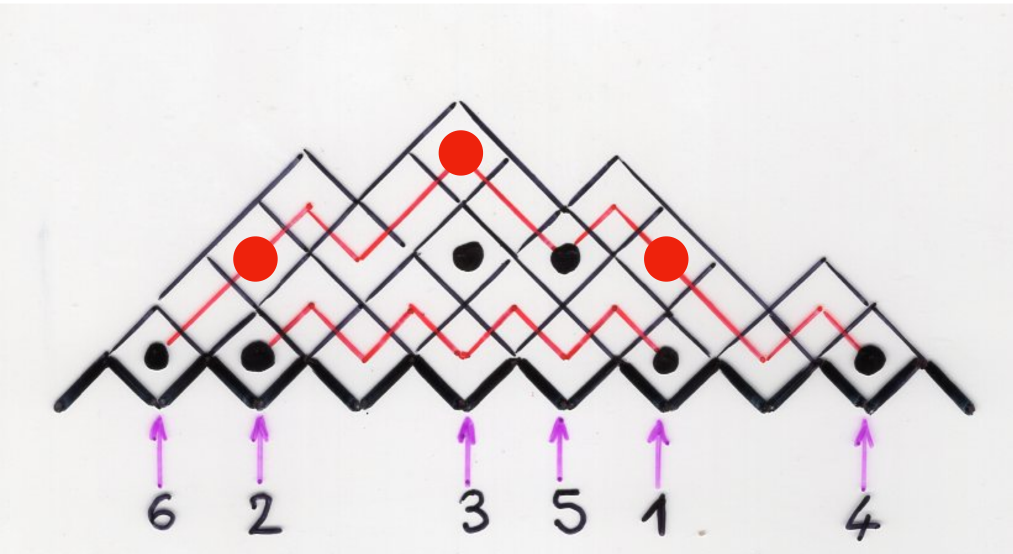




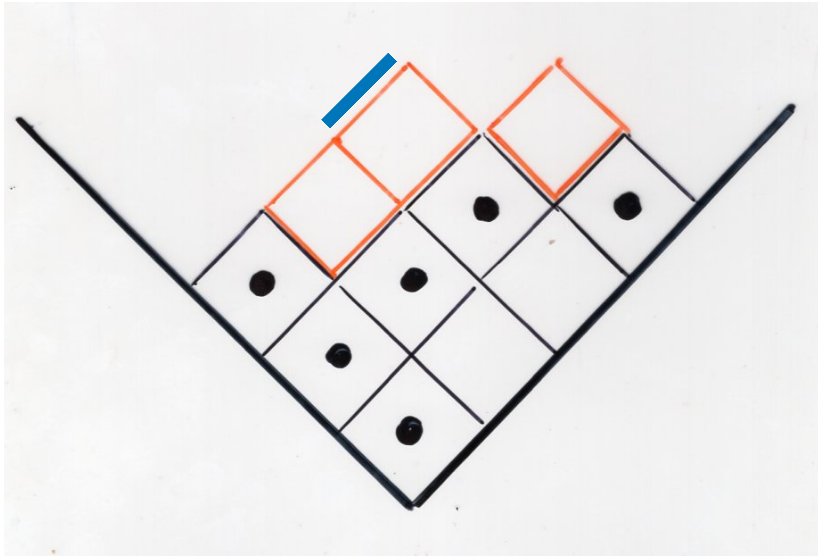
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

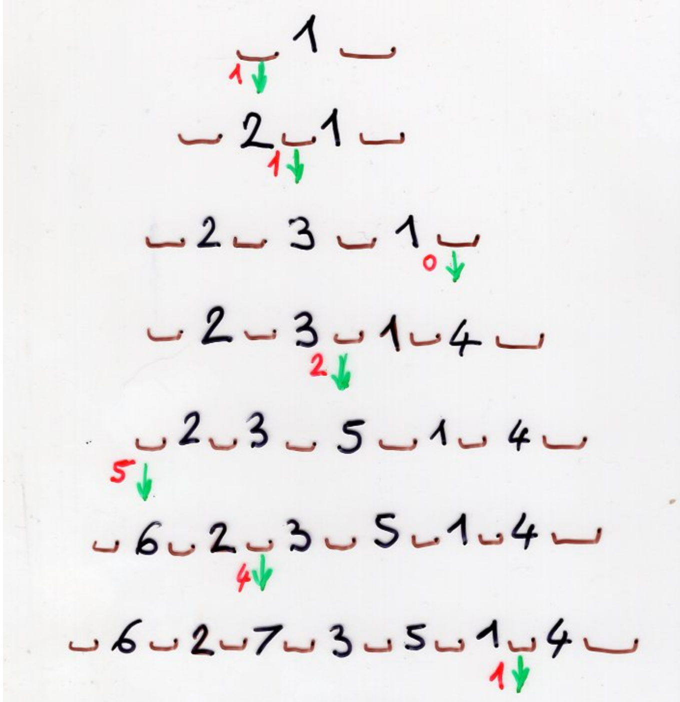


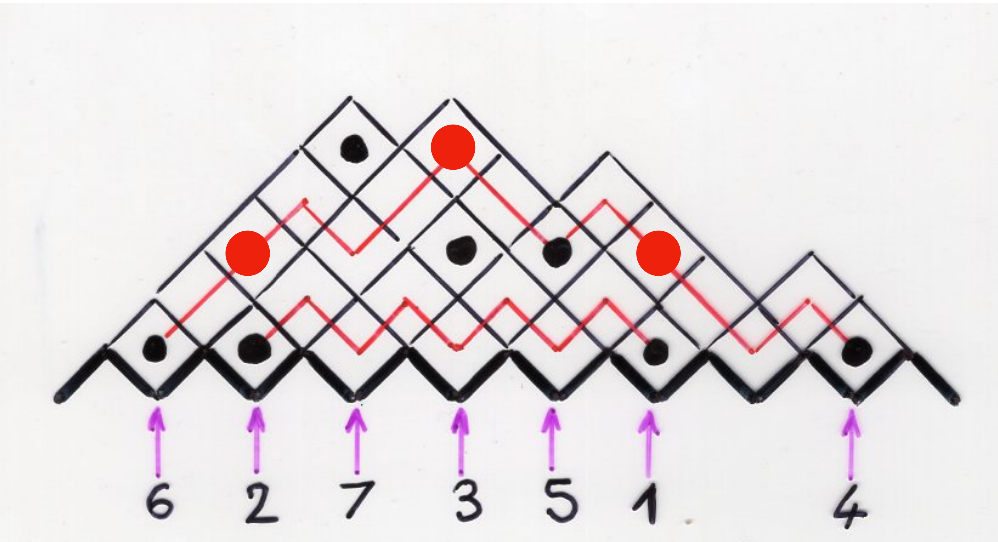


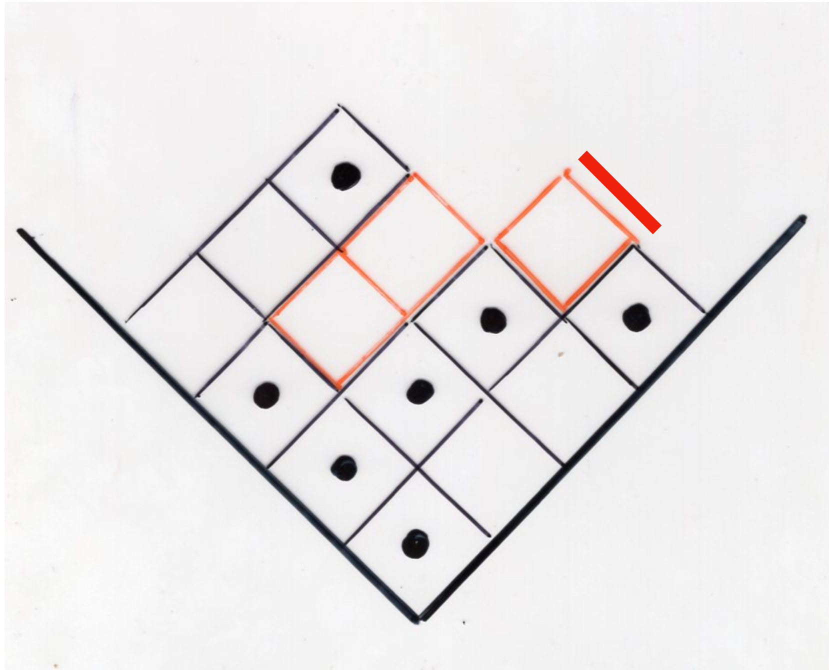


$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

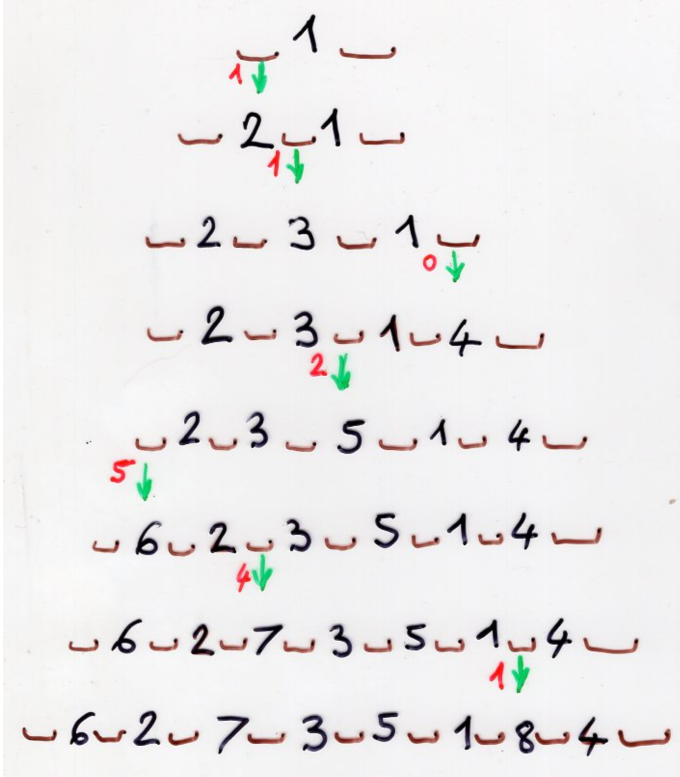


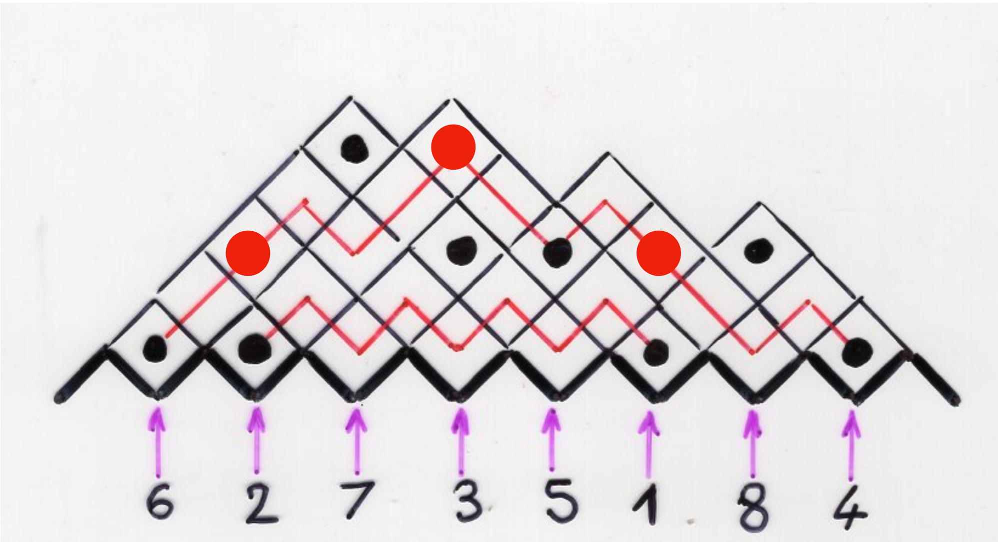


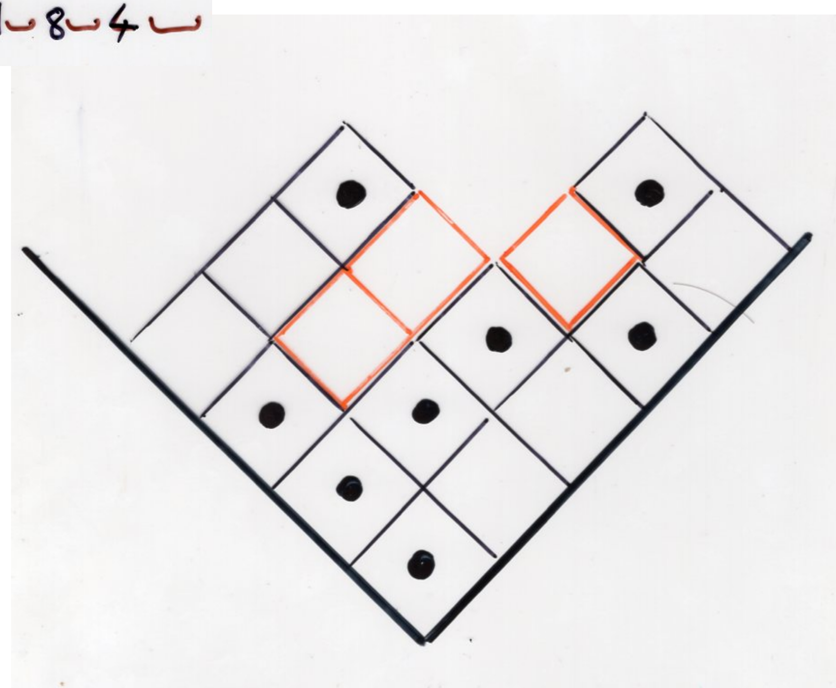




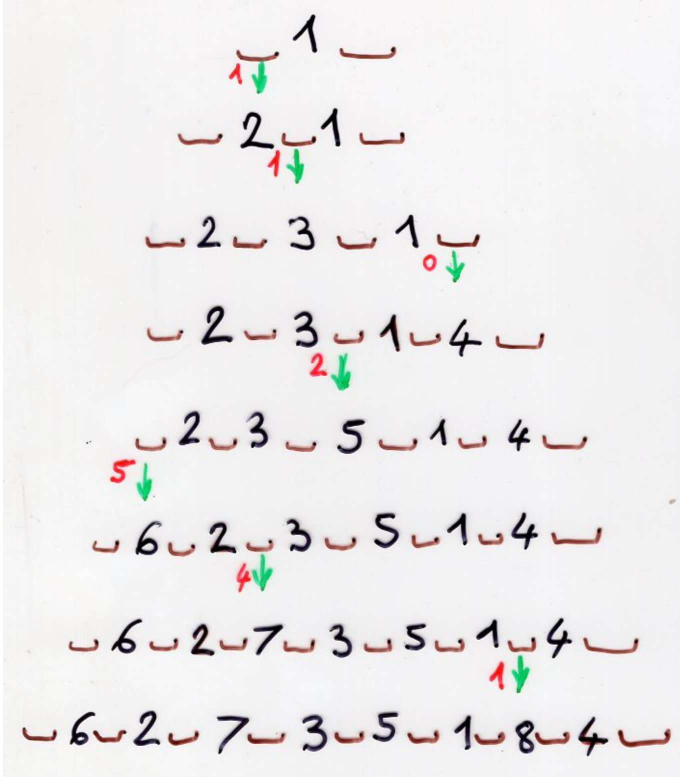
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$

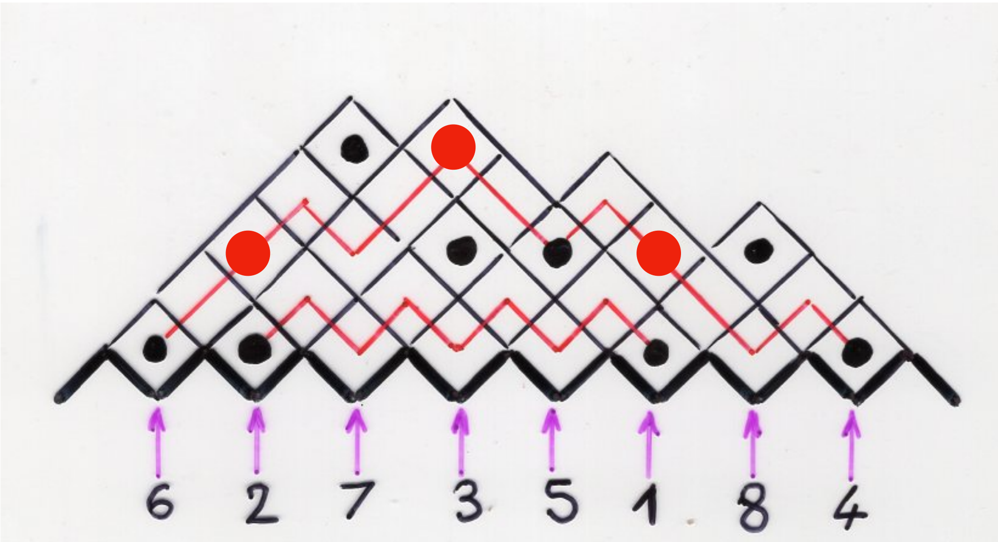


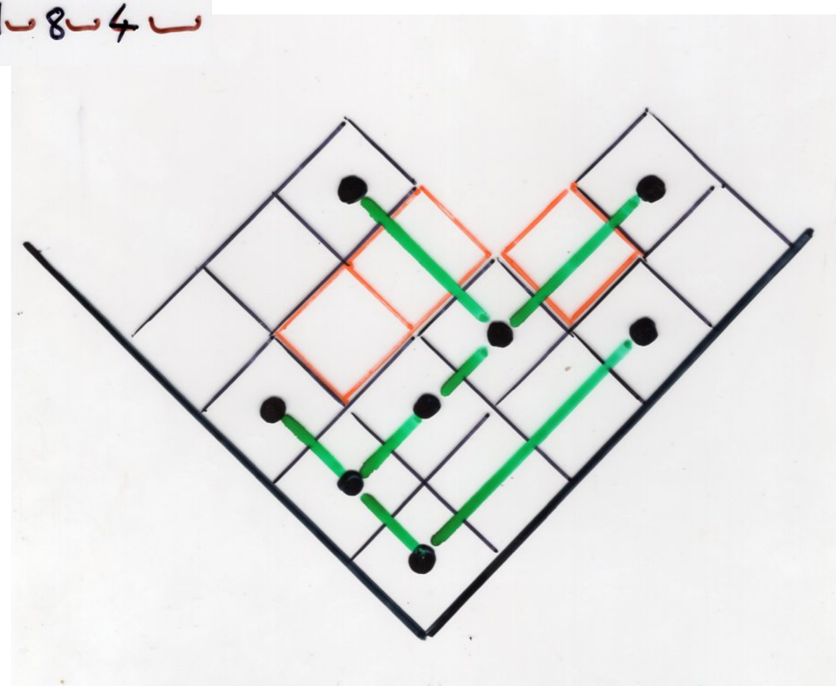




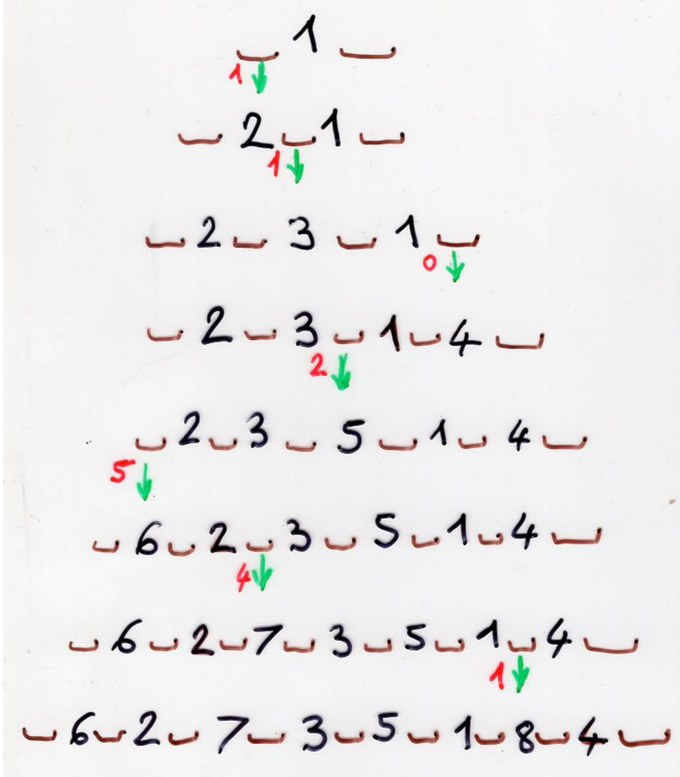
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$

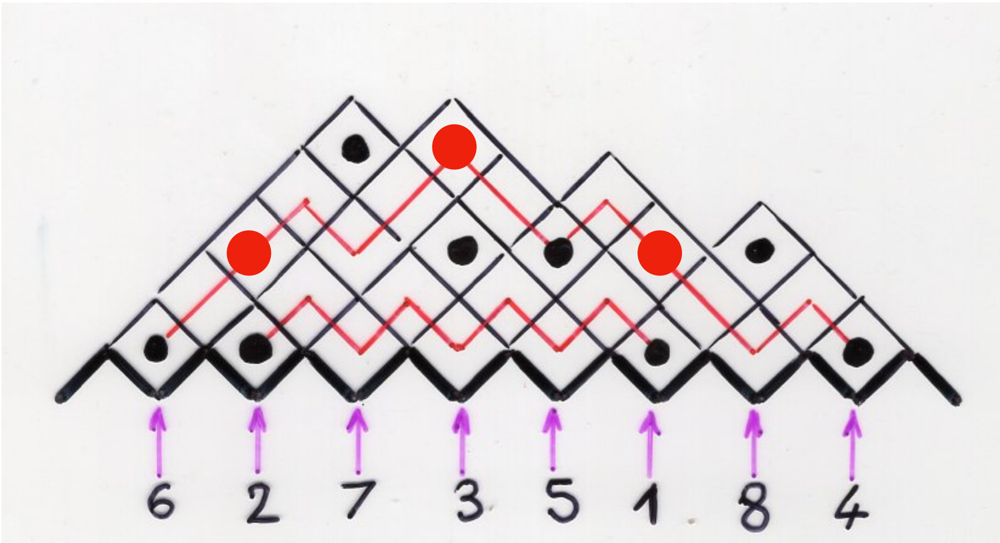


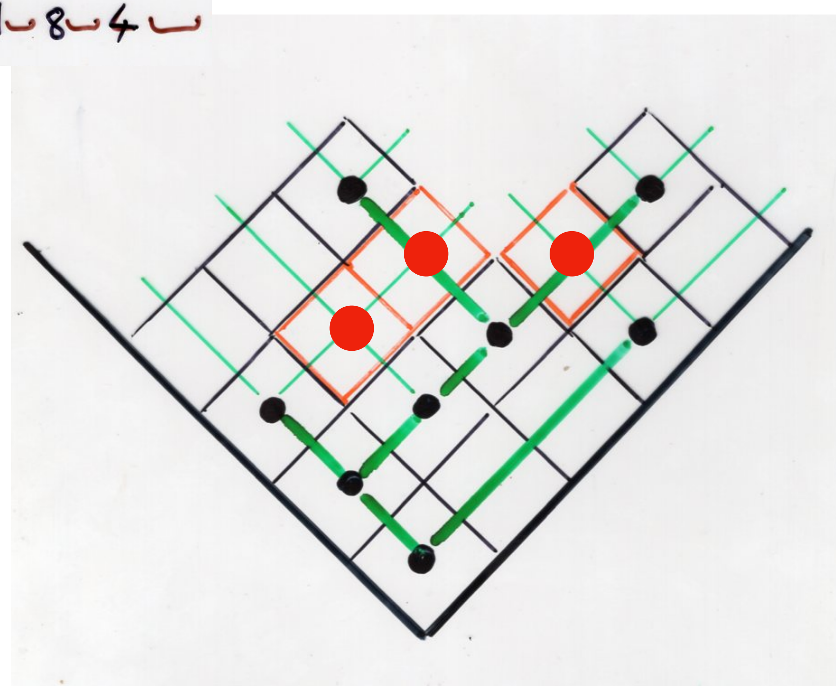




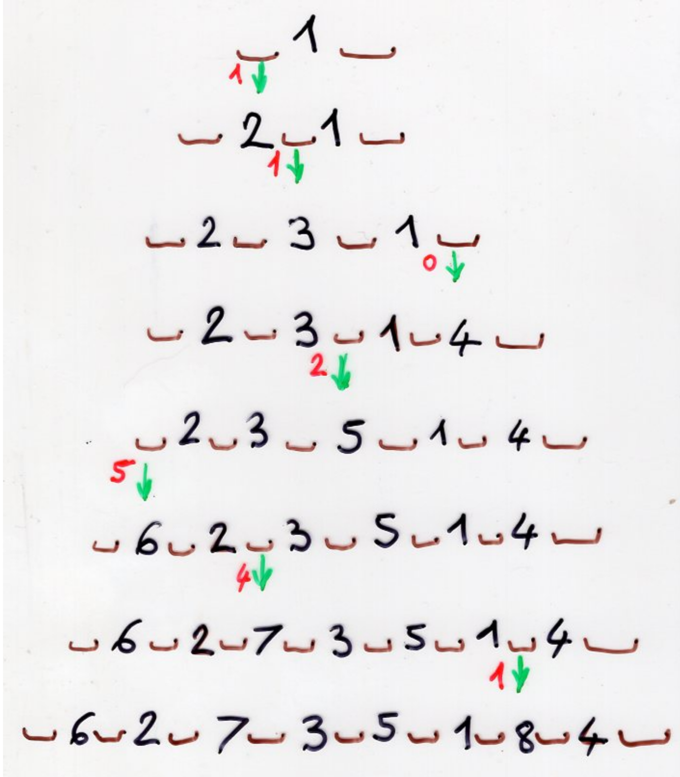
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

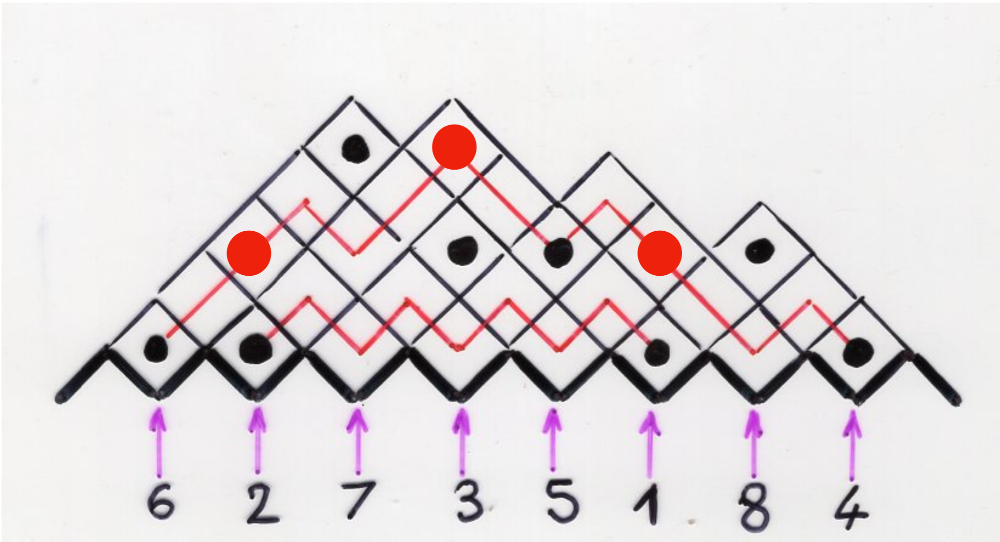


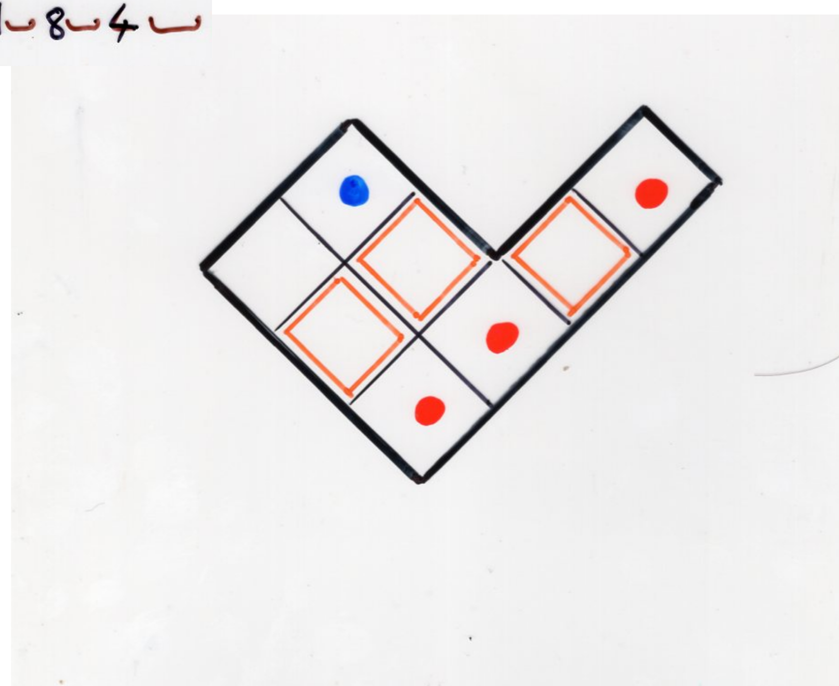




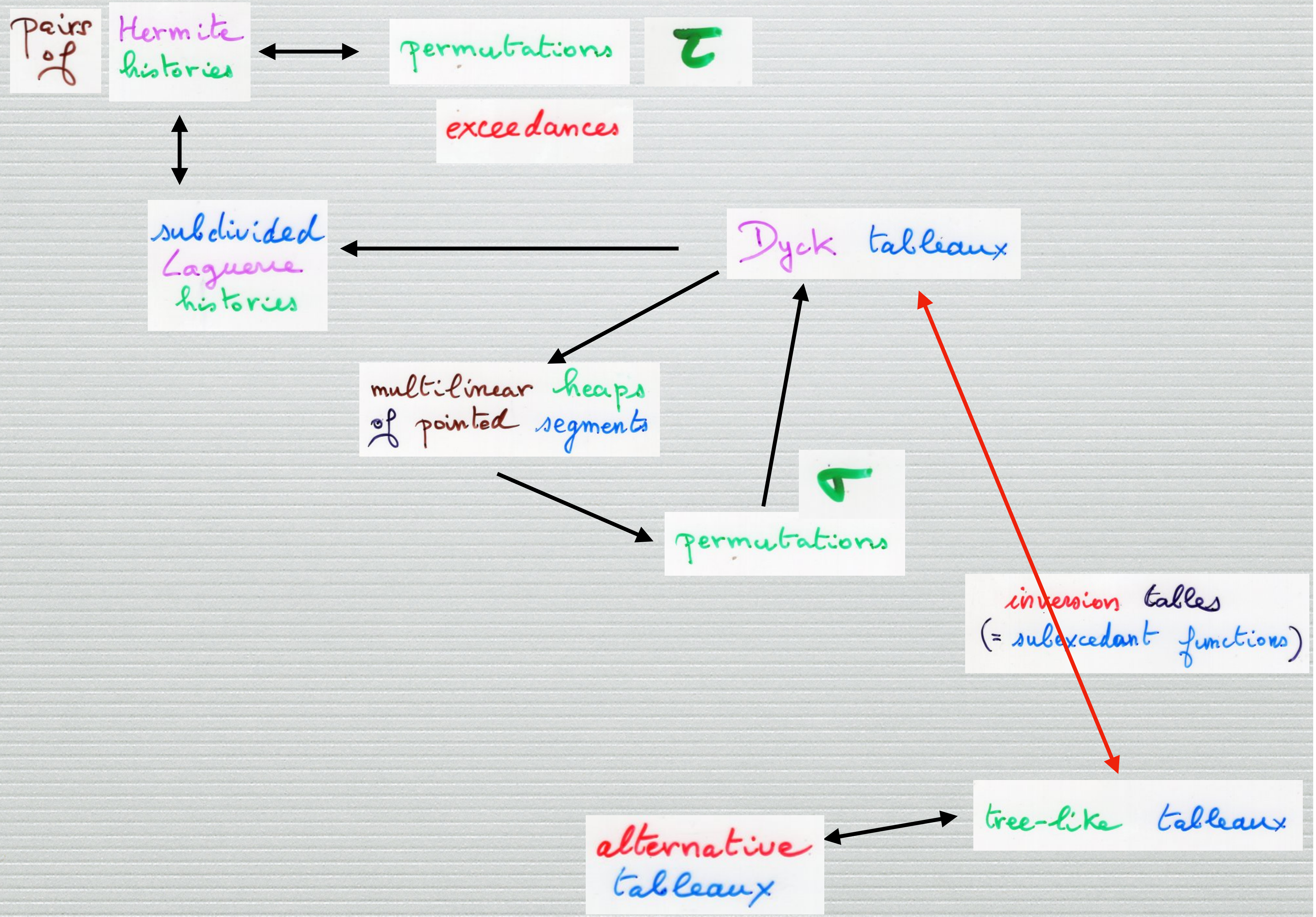
σ = 6 2 7 3 5 1 8 4







$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
inversion tables
(= subexcedant functions)
tree-like tableaux
alternative tableaux

Complements

Dyck tableaux

(cover-inclusive)
Dyck tilings

Shigechi, Zinn-Justin
(2012)

R. Kenyon, D. Wilson
(2011)

FPL

Kazhdan-Lusztig
polynomials

J. S. Kim (2012)

J.S. Kim, K. Mészáros
G. Panova, D. Wilson (2013)

Hermite
histories

Bijection (restricted) Laguerre histories

(of the inverse permutation)

σ^{-1}

and multilinear heaps

D, E "large"

From Ch3a, p69-71
And Ch3b(1), p11-12

$$\langle k | A = (k+1) \langle (k+1)|$$
$$\langle k | K = (k+1) \langle k|$$
$$\langle k | J = (k+1) \langle k|$$
$$\langle k | S = (k+1) \langle (k-1)|$$

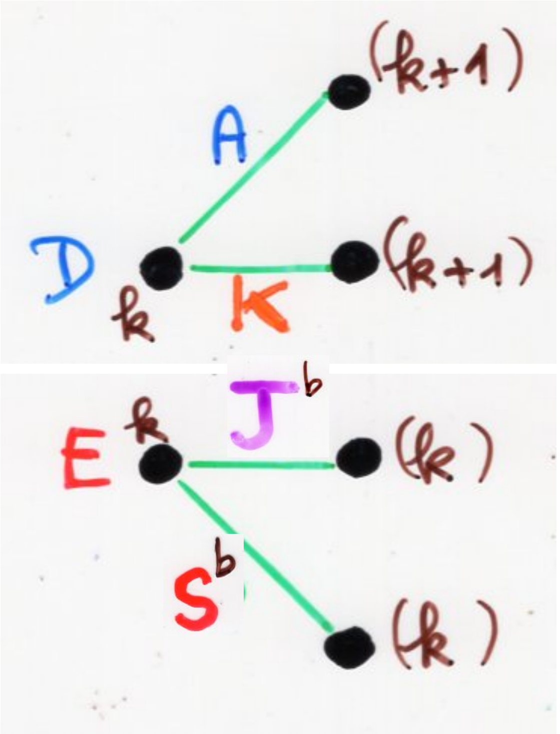
$$D = A + K$$
$$E = S + J$$

$$DE = ED + E + D$$

$$\mu_n = (n+1)!$$

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

Laguerre histories



D, E "restricted"

$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

$$\mu_n = n!$$

$$b_k = (2k+1)$$
$$\lambda_k = k^2$$

restricted
Laguerre
histories

$$\sigma(1) = (n+1)$$

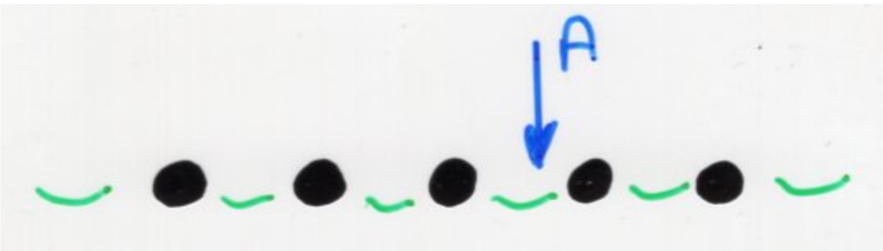
dictionnary data structure

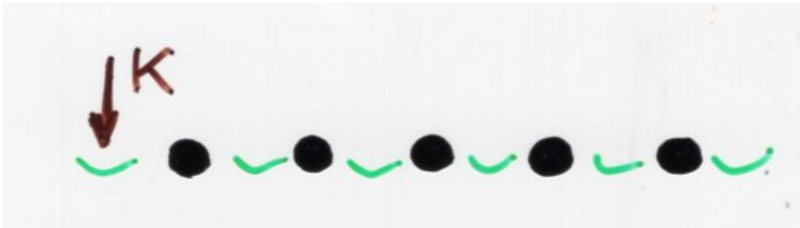
add or delete any element

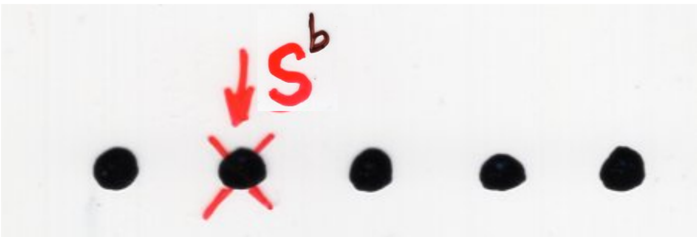
ask questions

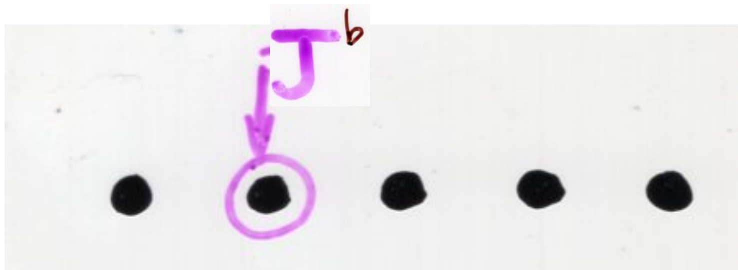
J^b positive

K negative









$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

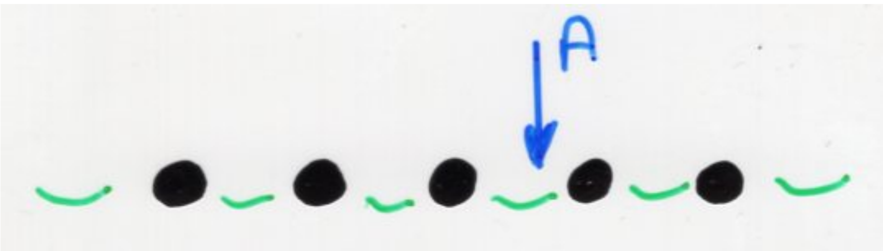
$$DE = ED + E + D$$

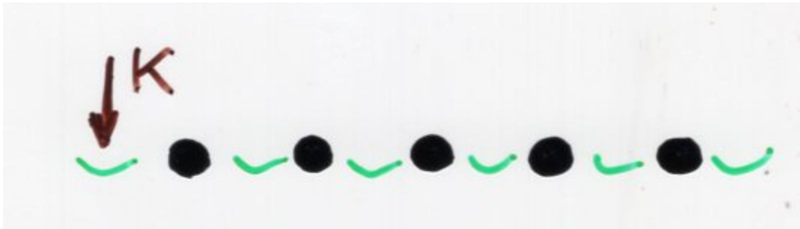
$$A \, | k \rangle = (k+1) \, | (k+1) \rangle$$

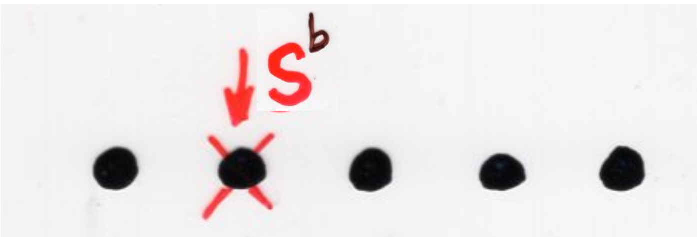
$$J^b \, | k \rangle = k \, | k \rangle$$

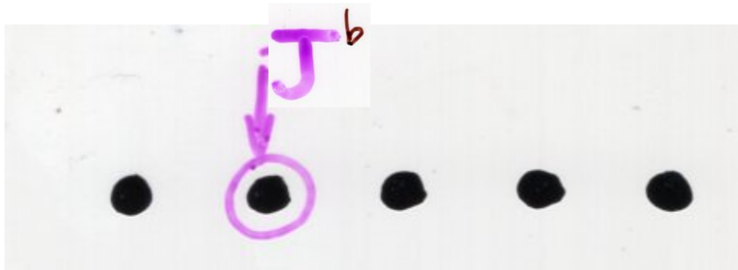
$$K \, | k \rangle = (k+1) \, | k \rangle$$

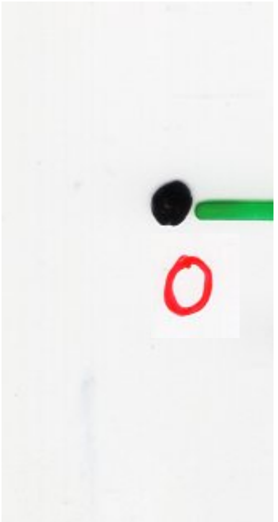
$$S^b \, | k \rangle = k \, | (k-1) \rangle$$



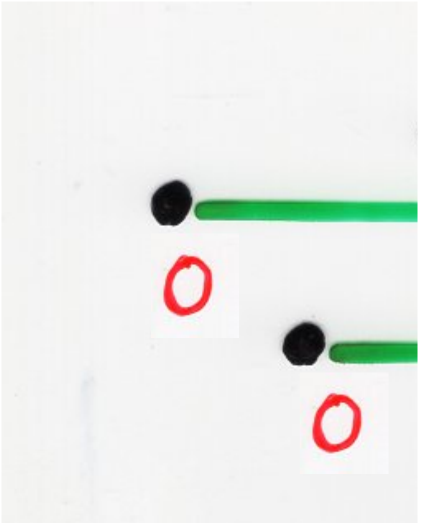


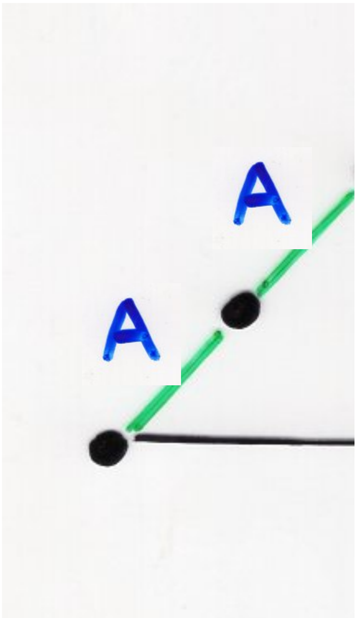


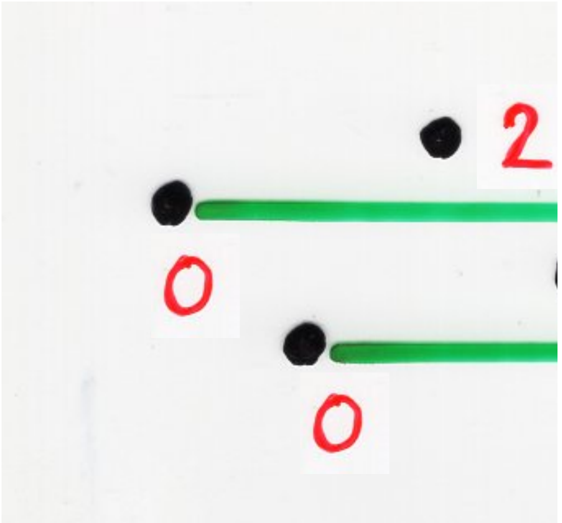



O

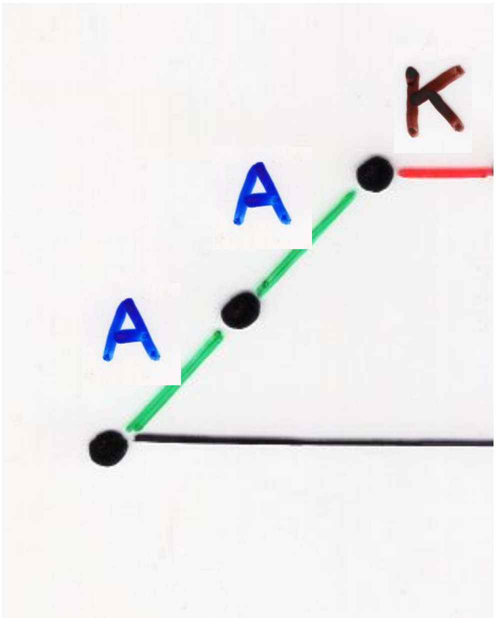

A


O
O

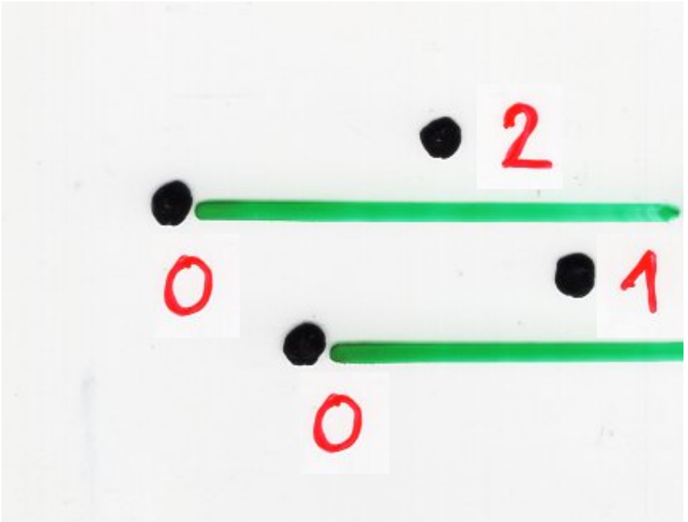

A
A



2
O
O



K
A
A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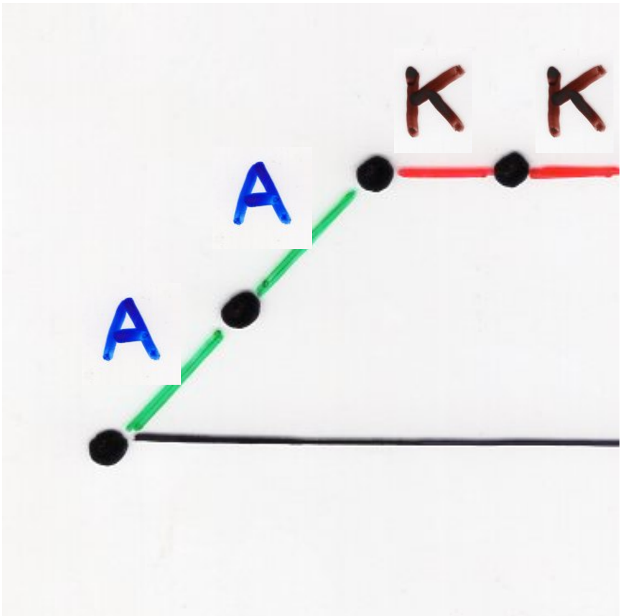
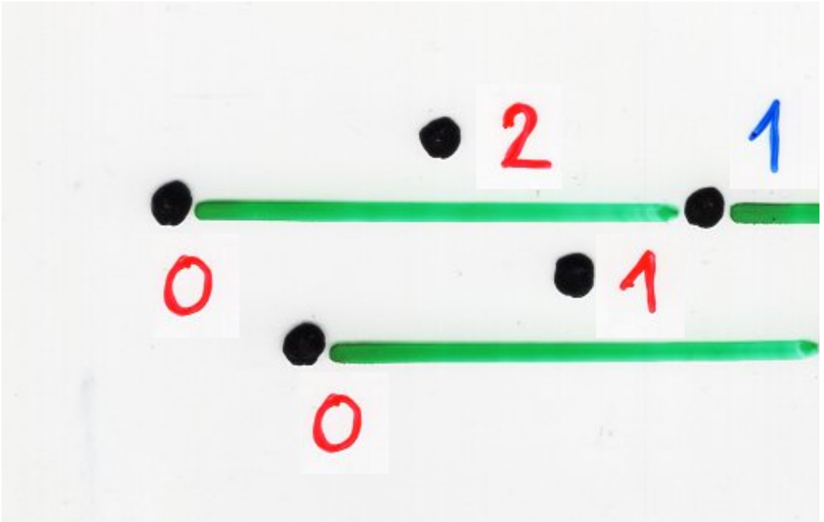
1
0


K
K
A
A



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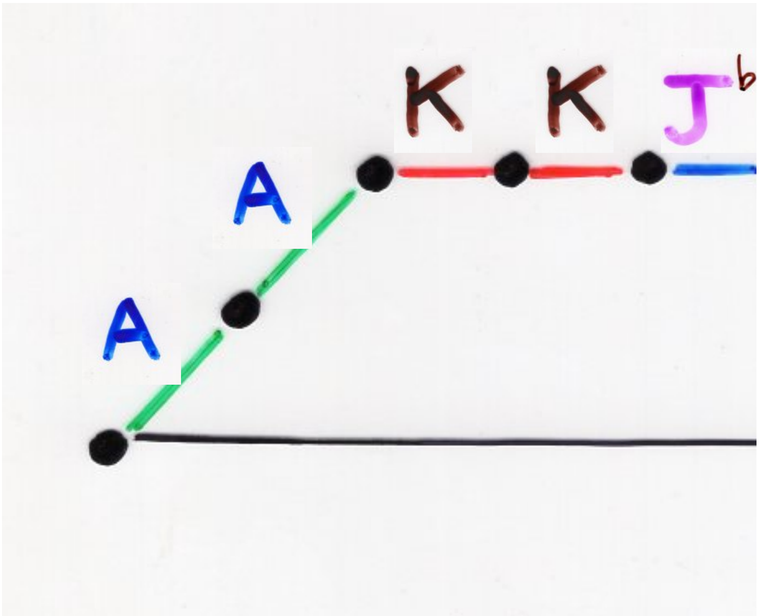
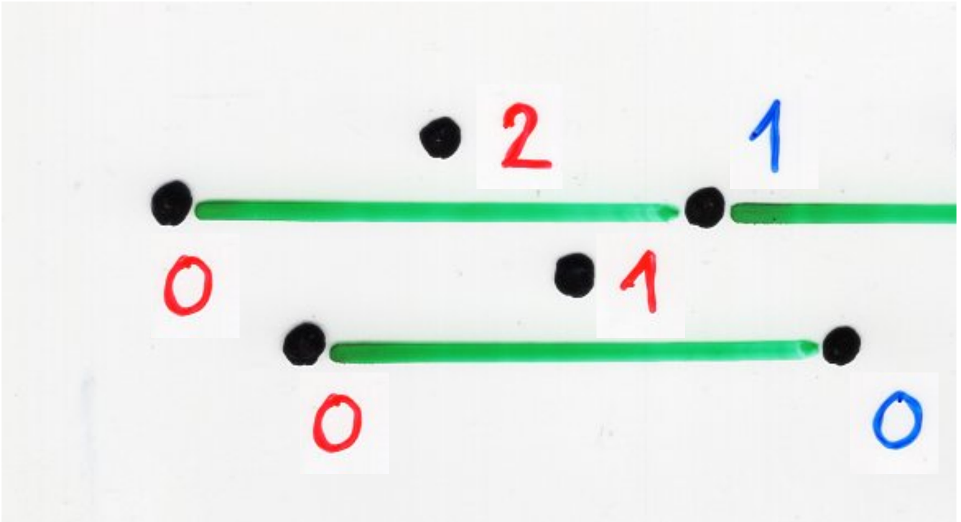


K
K
J
A
A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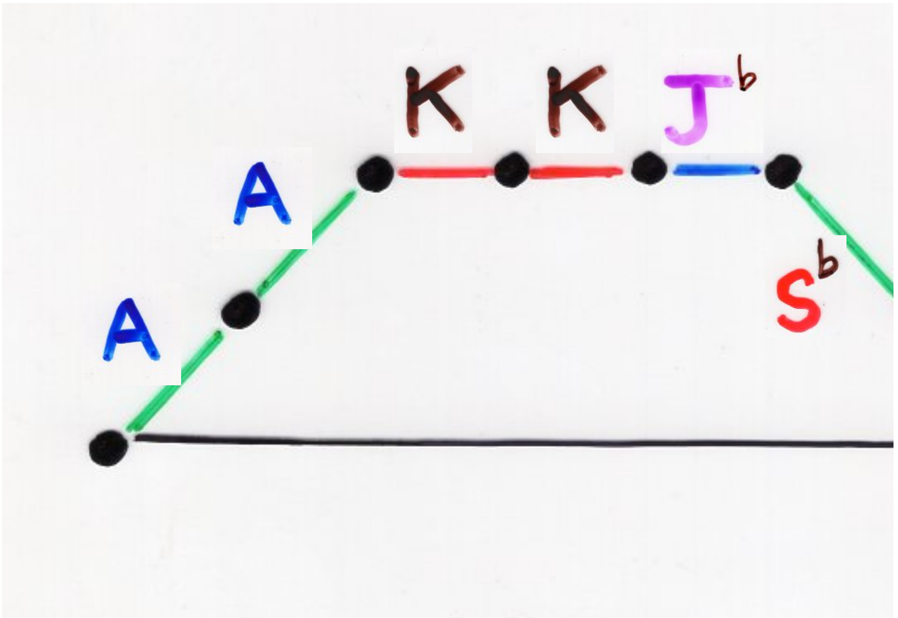
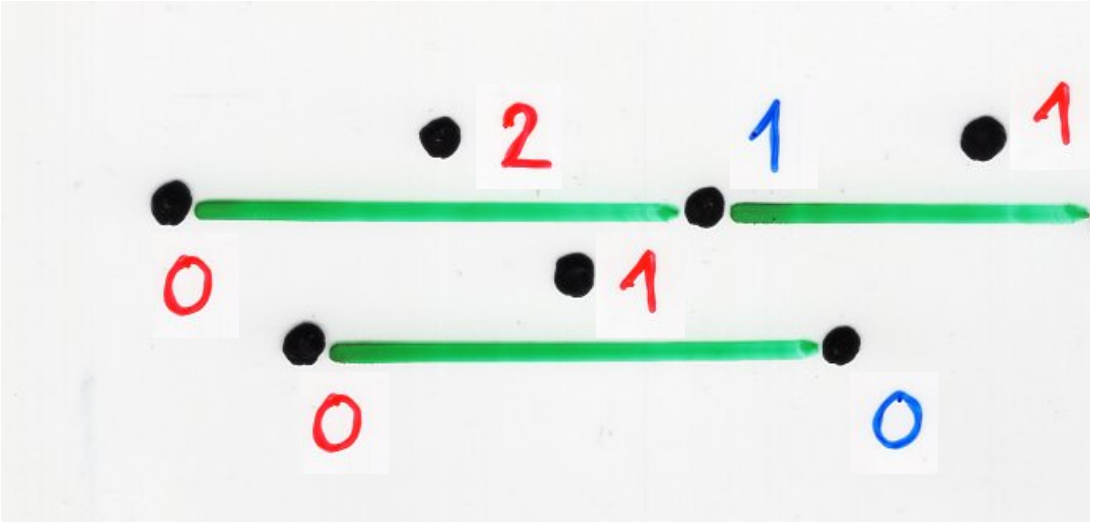
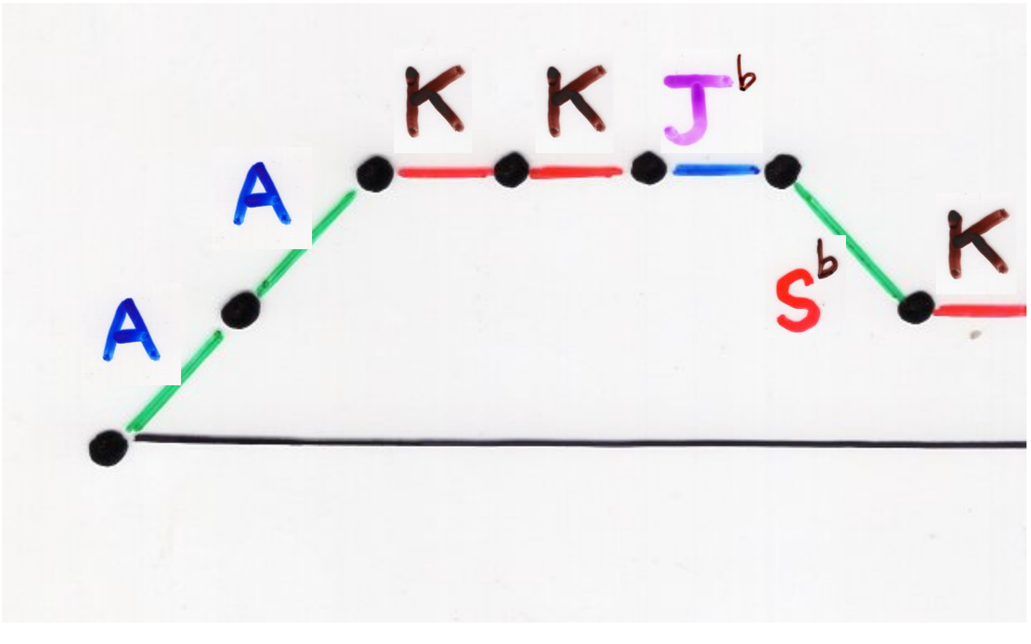
1
0
0

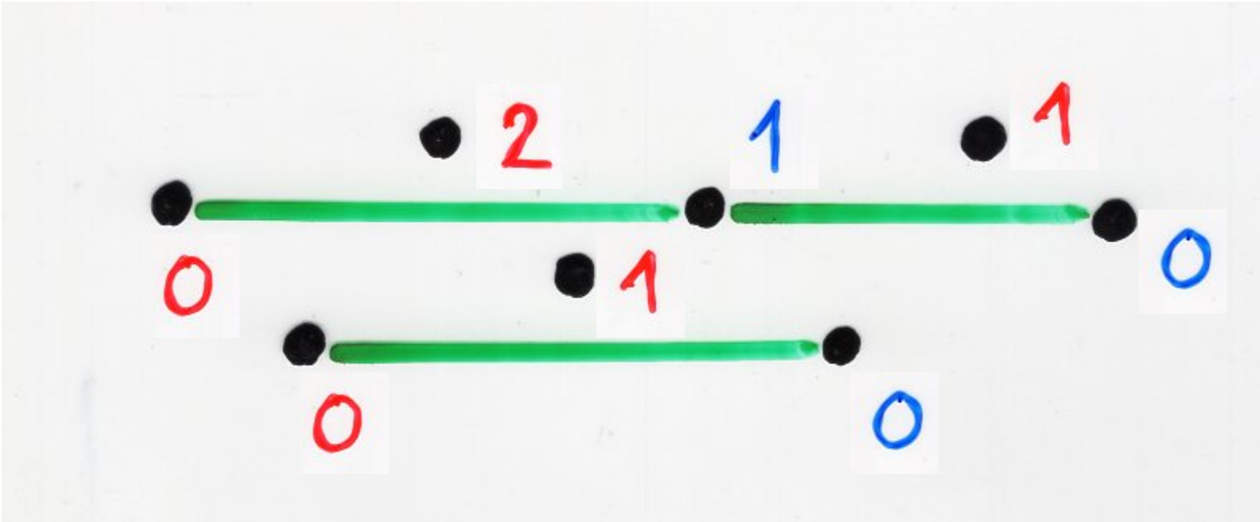

K
K
J^b
A
A
S^b



2
1
1
0
1
0
0



K
K
J^b
A
K
S^b
A



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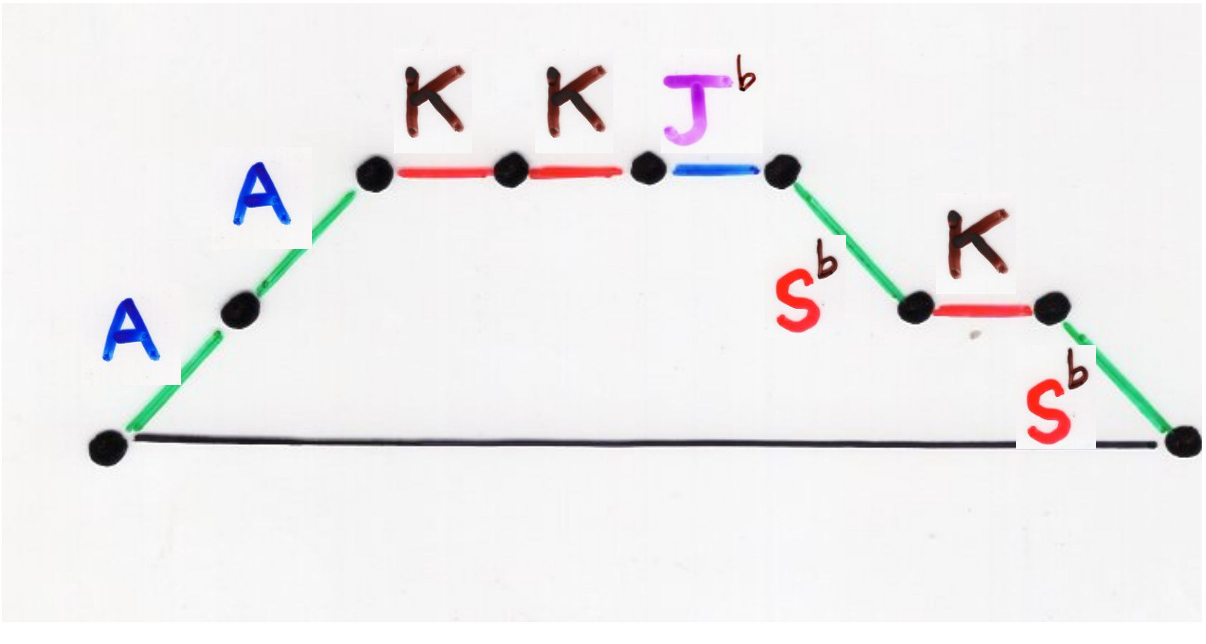
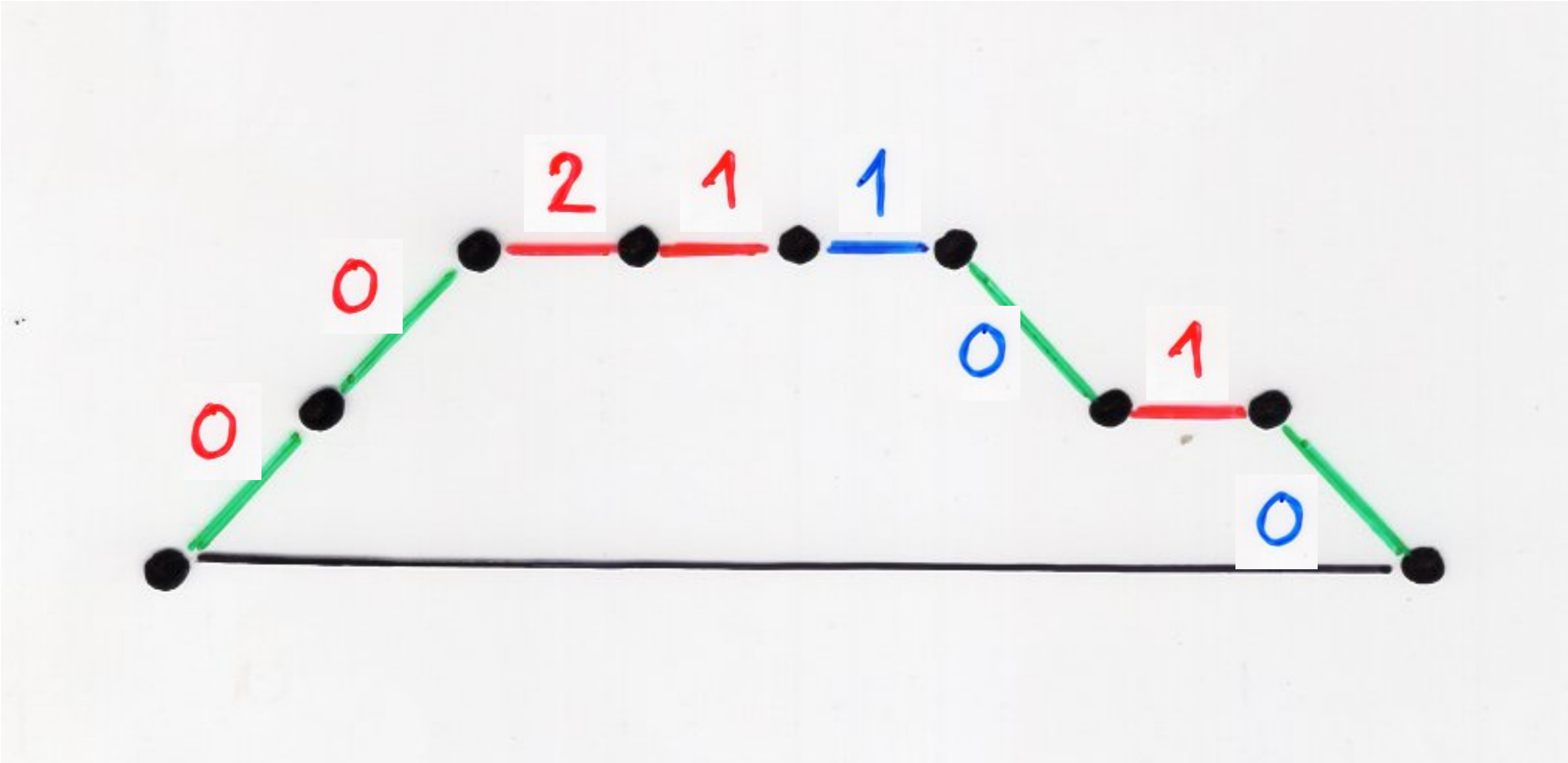
1
1
0
1
0
0
0



K
K
J^b
A
A
S^b
K
S^b



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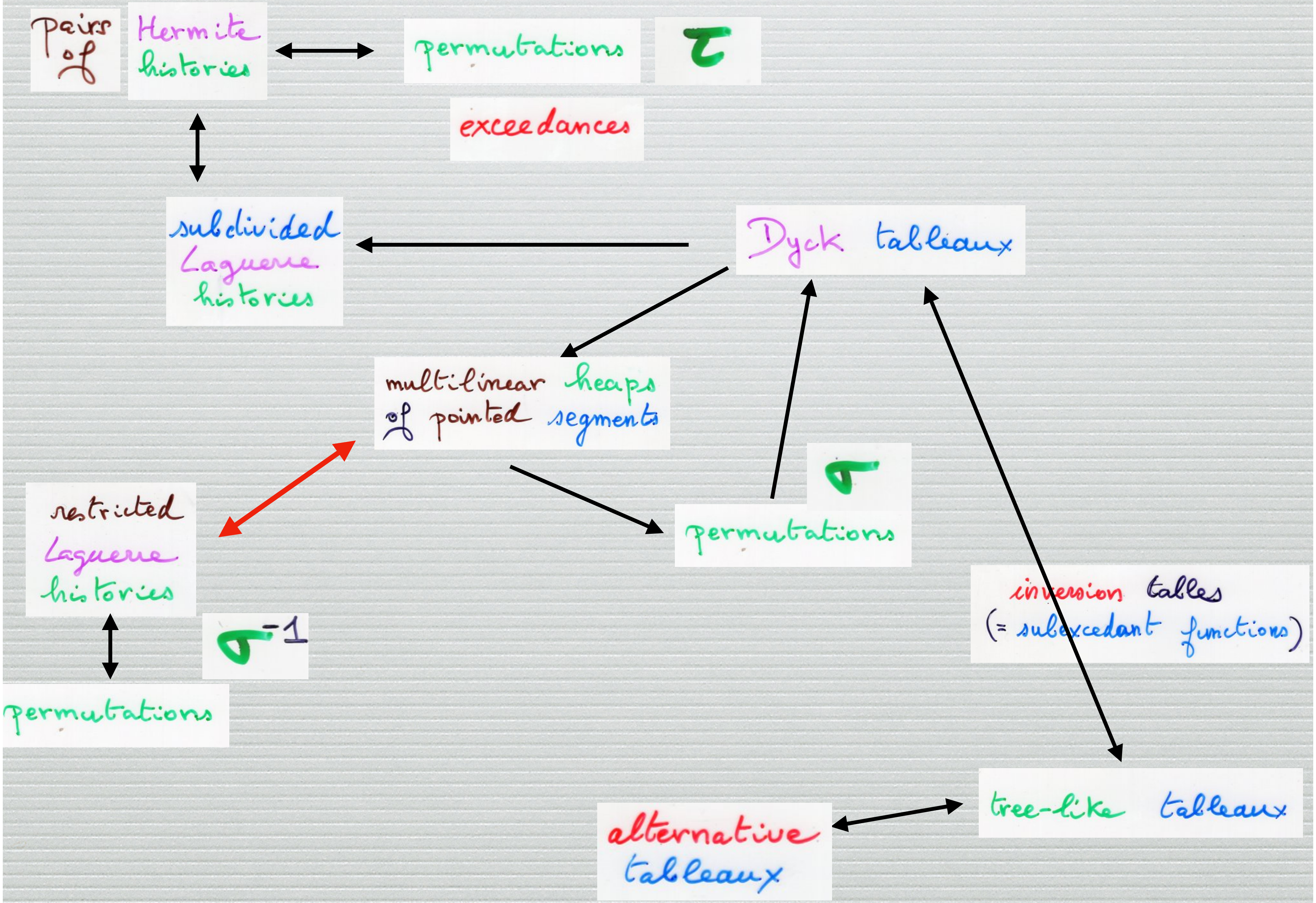
0
2
1
1
0
1
0


Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
multilinear heaps of pointed segments
σ
permutations
restricted Laguerre histories
σ⁻¹
permutations
inversion tables
(= subexcedant functions)
alternative tableaux
tree-like tableaux

contractions

in

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$J(t;b,\lambda)$

Jacobi continued fraction

$b = \{b_k\}_{k\geq 0} \quad \lambda = \{\lambda_k\}_{k\geq 1}$

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cfrac{\cdots\cdots\cdot}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$\mu_0 = 1$

$S(t;\lambda)$

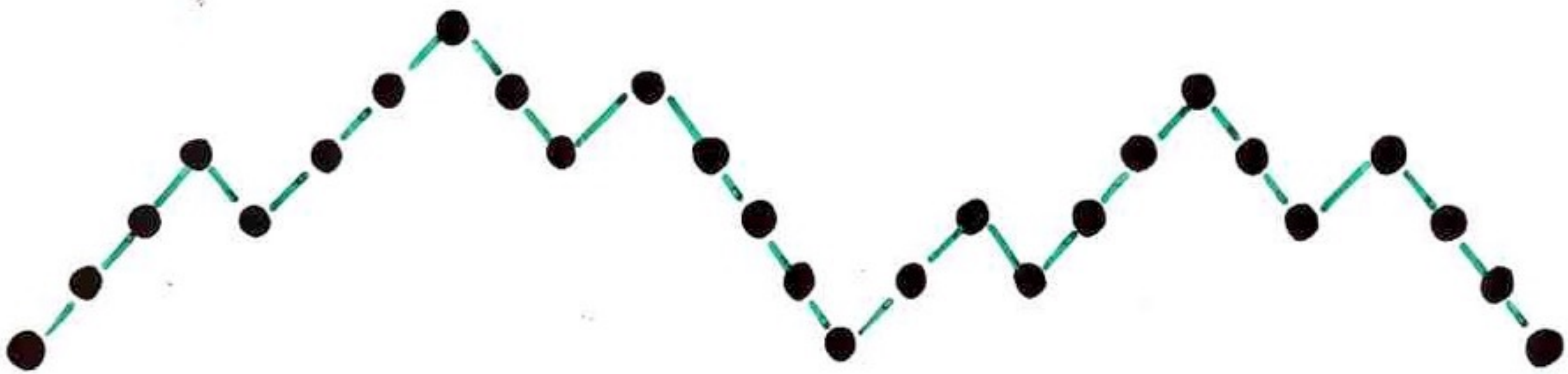
Stieltjès continued fraction

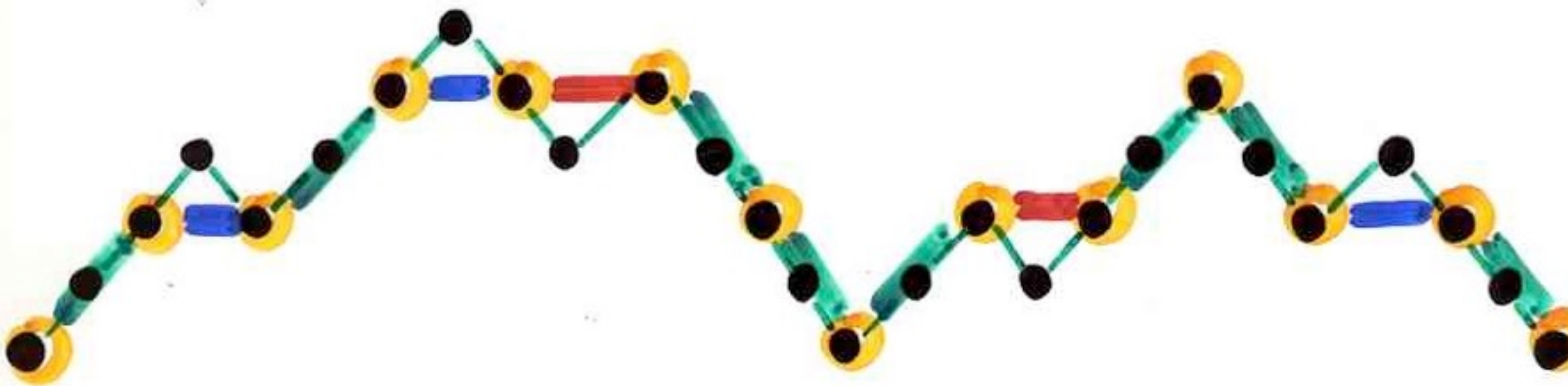
$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}}}$$

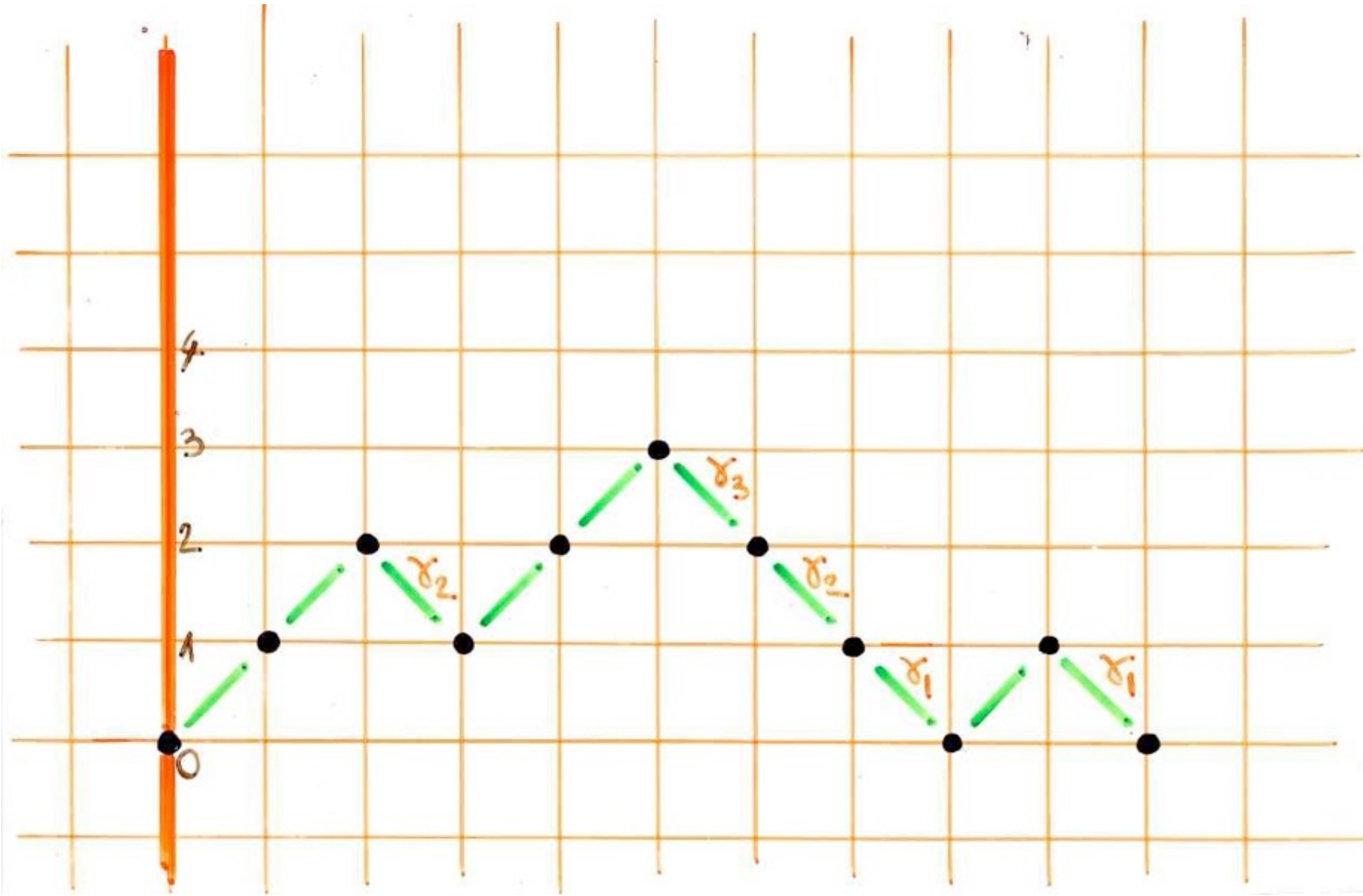
$$\sum_{n\geq 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots}}}}$$

Combinatorial proof

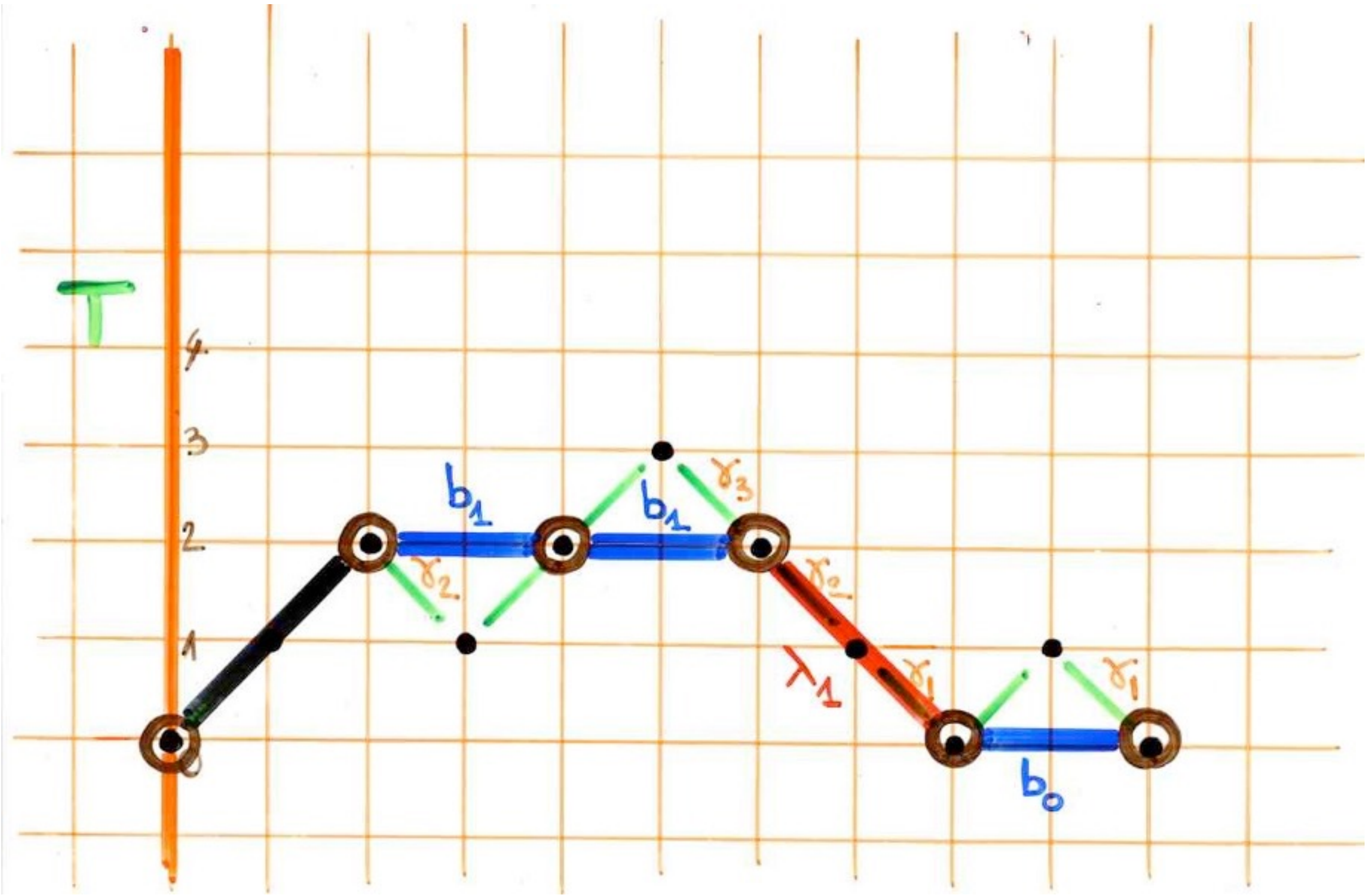
contractions
- Dyck paths
- Motzkin paths

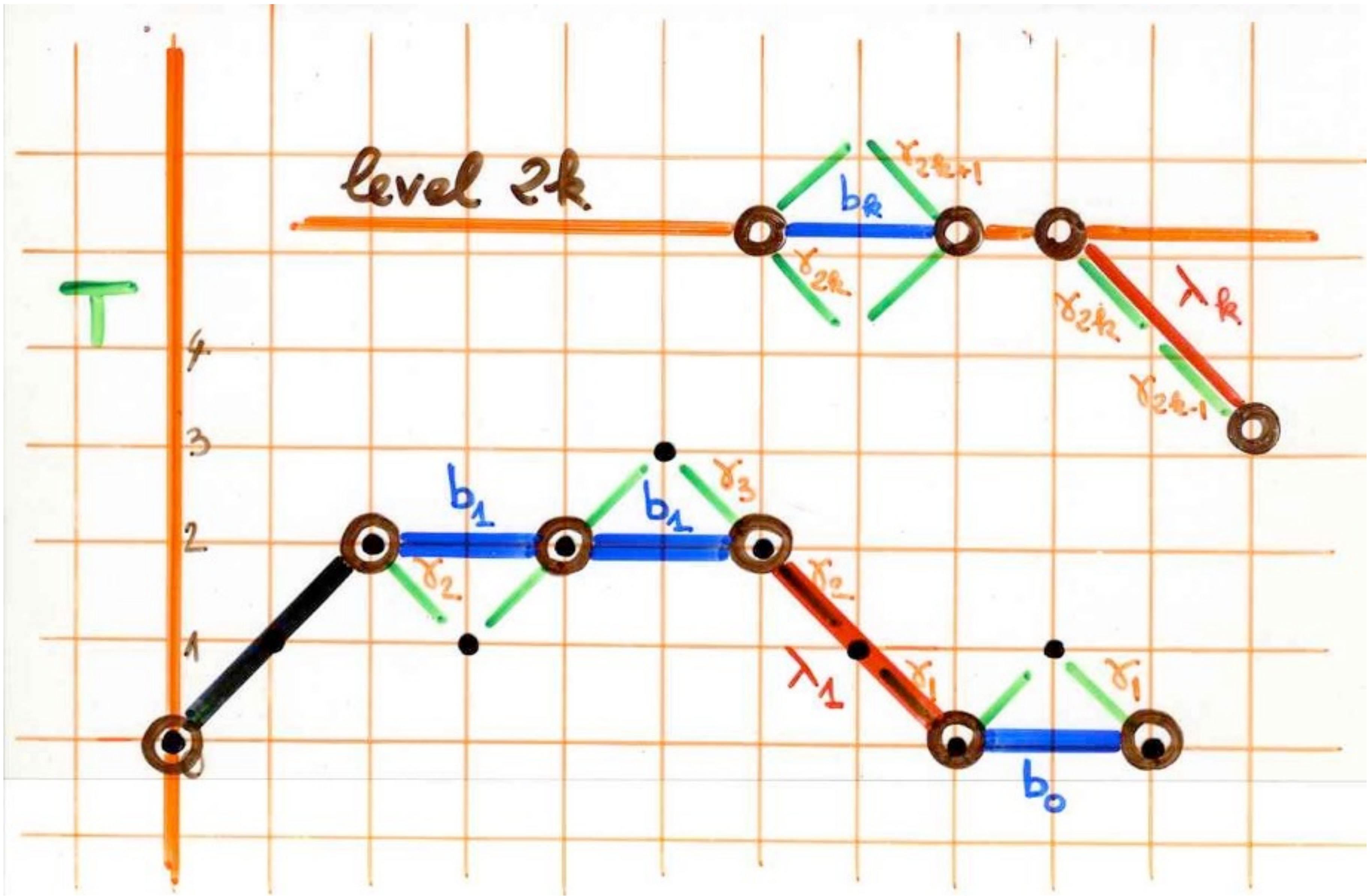






4
3
2
1
0
γ_3
γ_2
γ_2
γ_1
γ_1


T
4
3
2
1
0
b1
b1
γ2
γ3
γ2
γ1
γ1
T1
b0


level 2k
T
4
3
2
1
0
b_k
γ_{2k+1}
γ_{2k}
γ_{2k}
λ_k
γ_{2k-1}
b_1
b_1
γ_3
γ_2
γ_2
λ_1
γ_1
γ_1
b_0

$$b_k = \gamma_{2k} + \gamma_{2k+1} \qquad \lambda_k = \gamma_{2k}\gamma_{2k-1}$$

$$S(t;\gamma) = J(t;b,\lambda)$$

$$S(t;\gamma) = J(t;b,\lambda)$$

$$\gamma_k = \left\lceil \frac{k}{2} \right\rceil \longrightarrow \begin{cases} \lambda_k = k^2 \\ b_k = (2k+1) \end{cases}$$

$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}}}$$

subdivided
Laguerre
histories

$$\downarrow$$

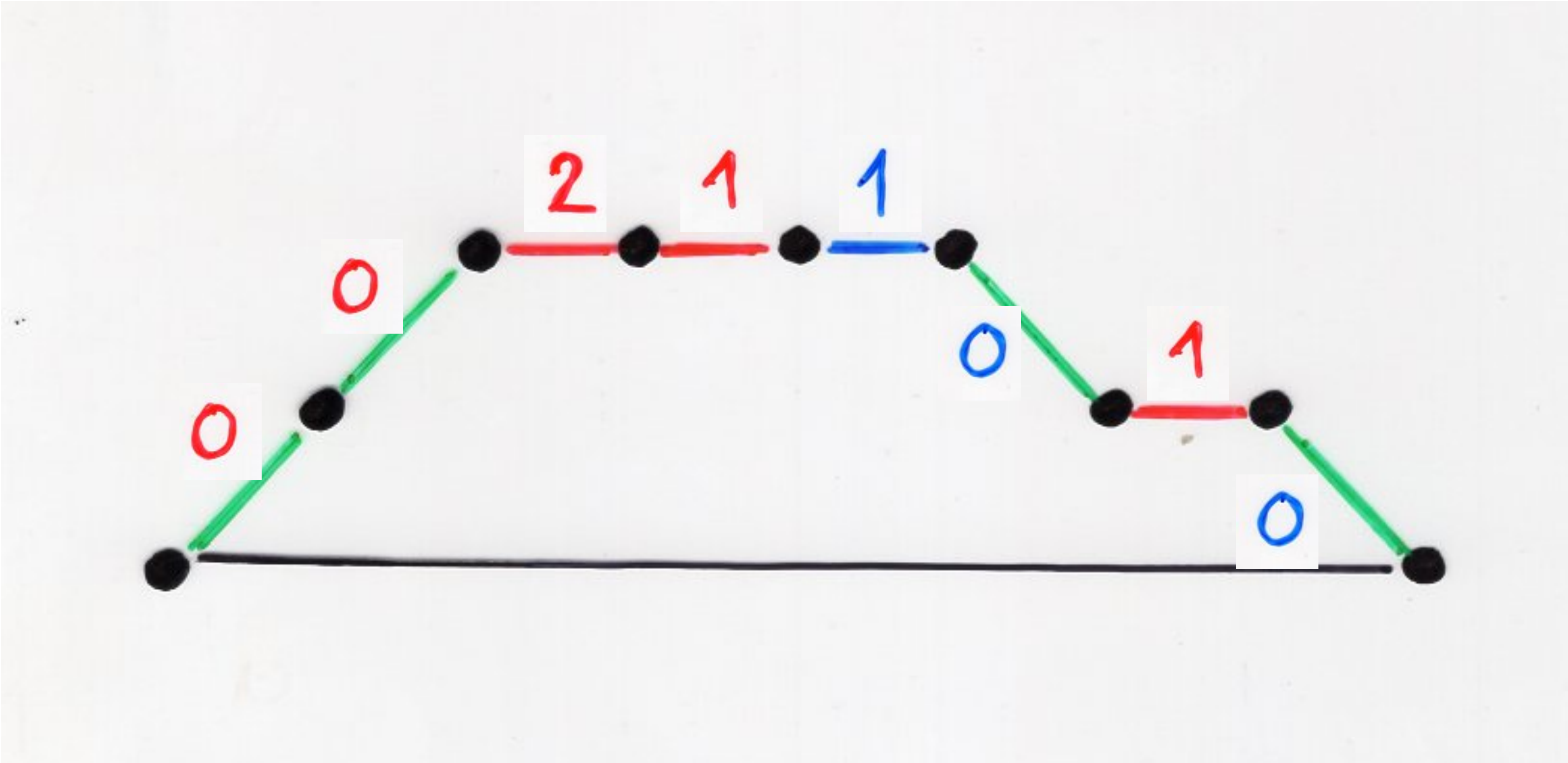
$$\sum_{n \geq 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\cdots}}}}$$

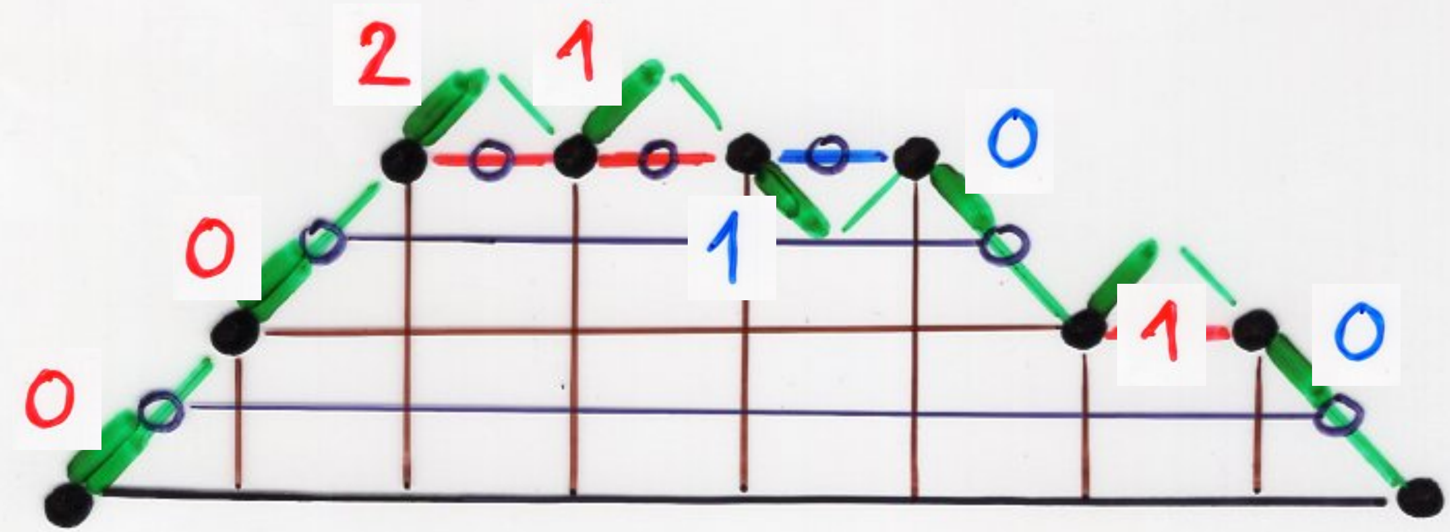
restricted
Laguerre
histories

From (restricted) Laguerre histories

to

subdivided Laguerre histories


0
0
2
1
1
0
1
0



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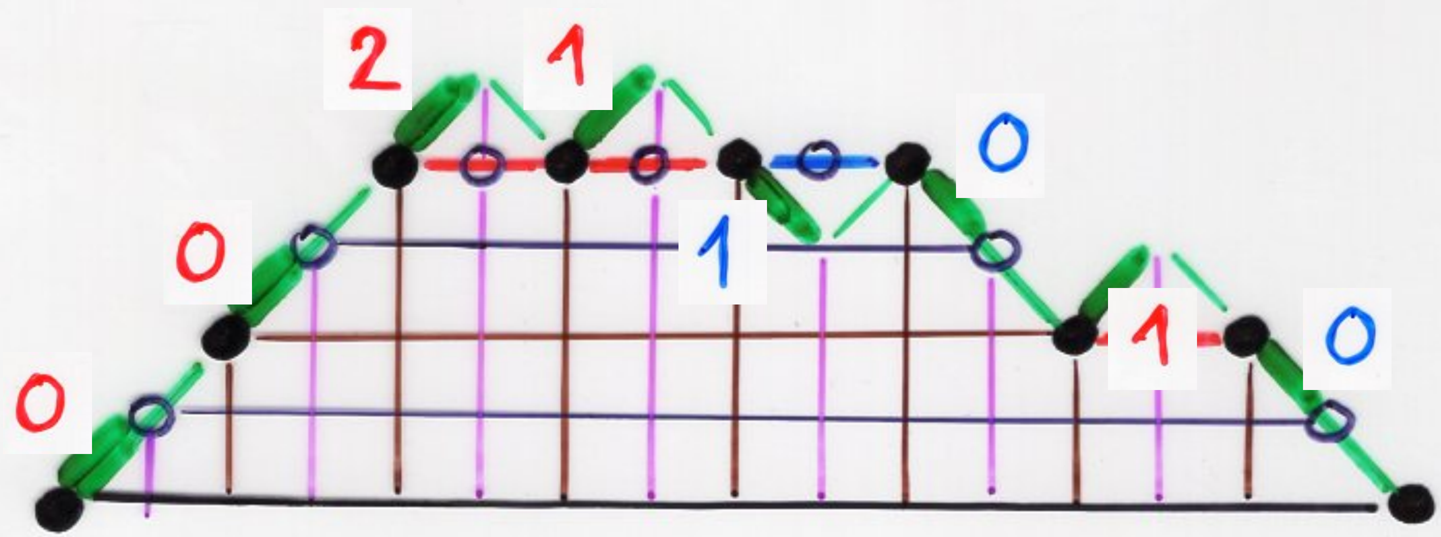
1
0
1
0
0
1
0
0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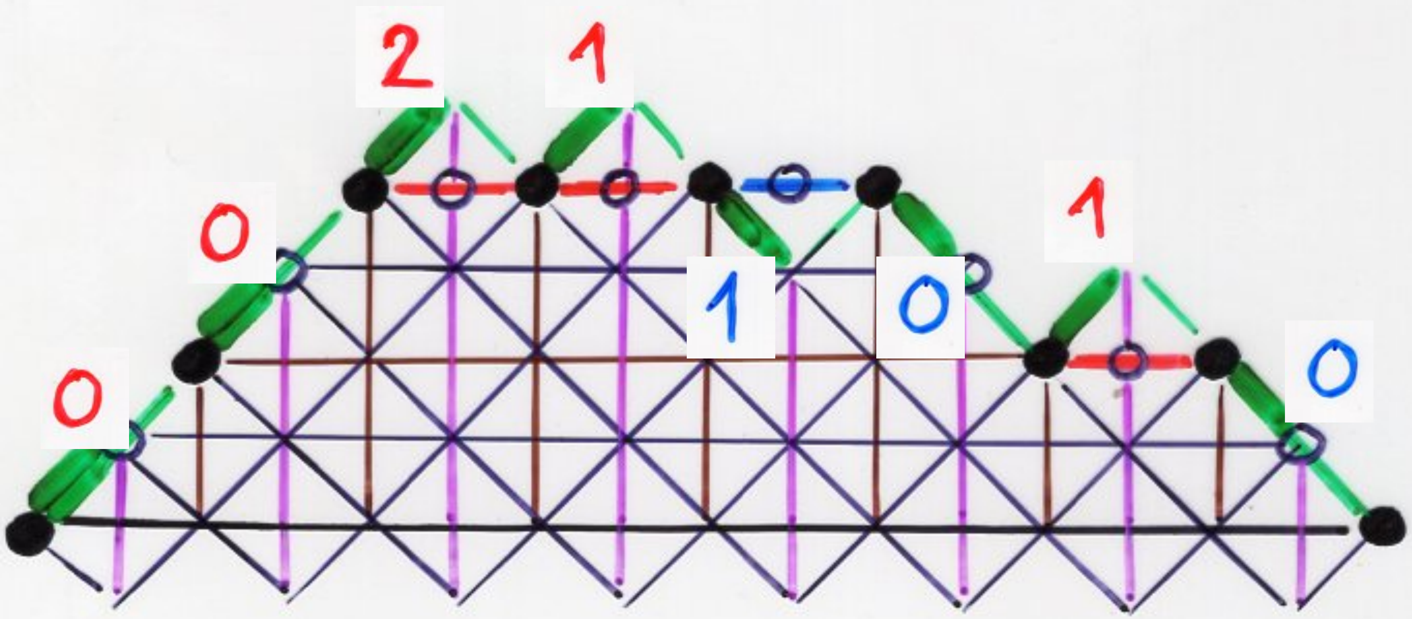
2
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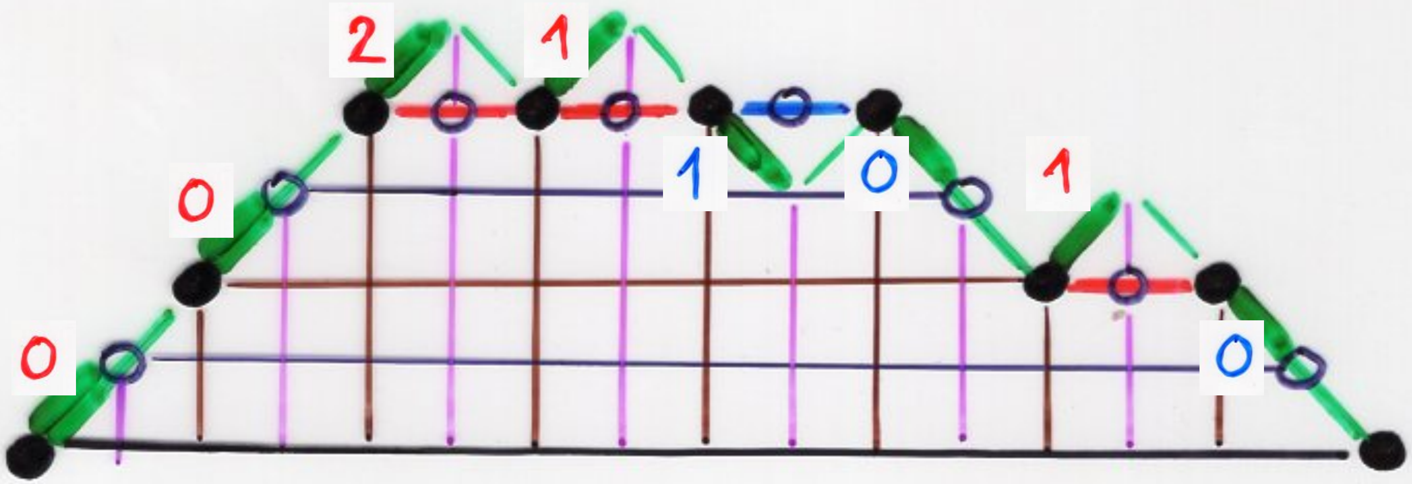

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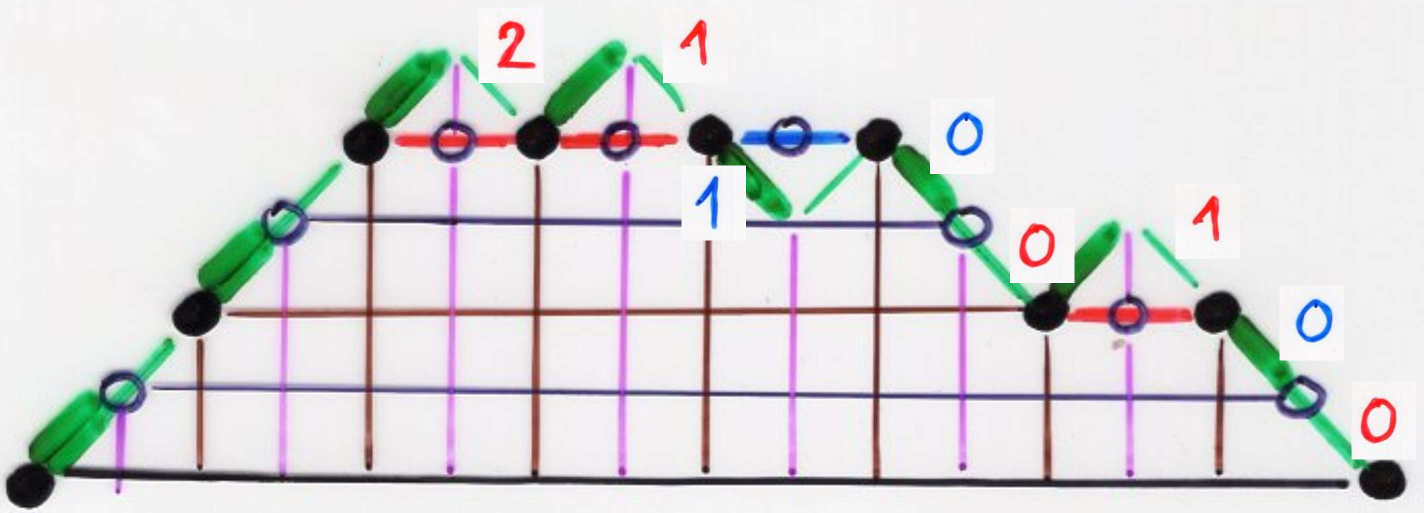
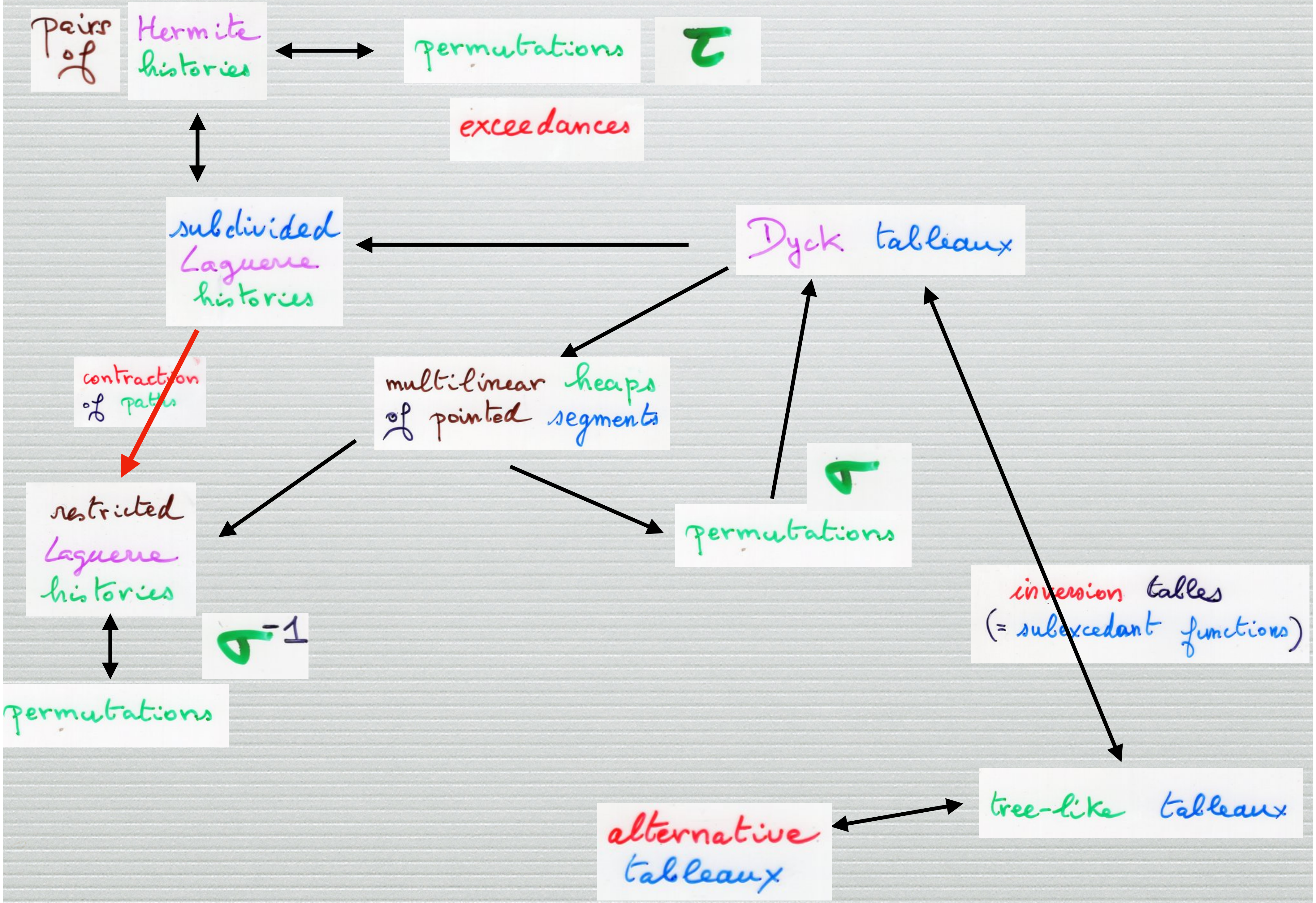
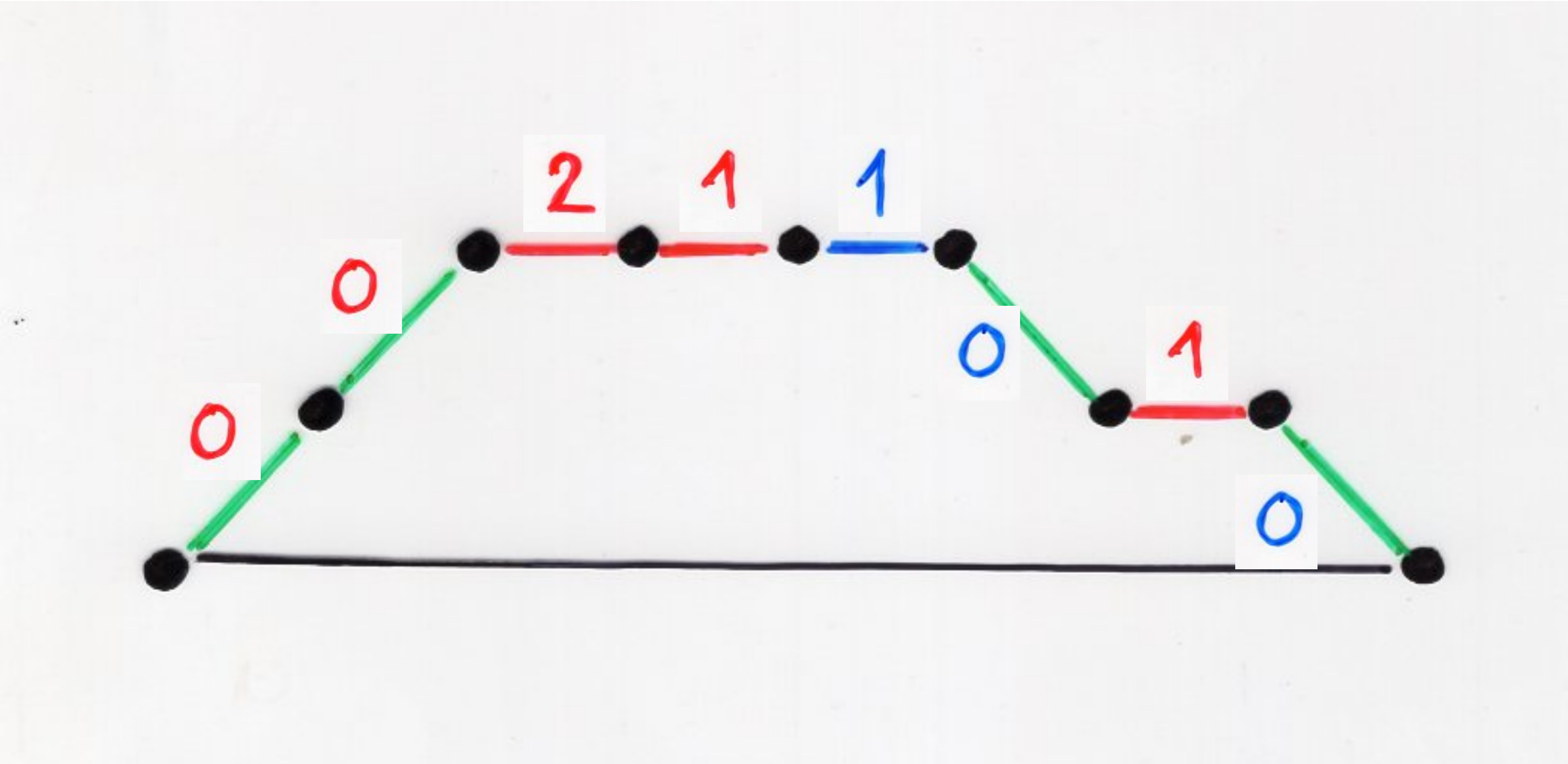


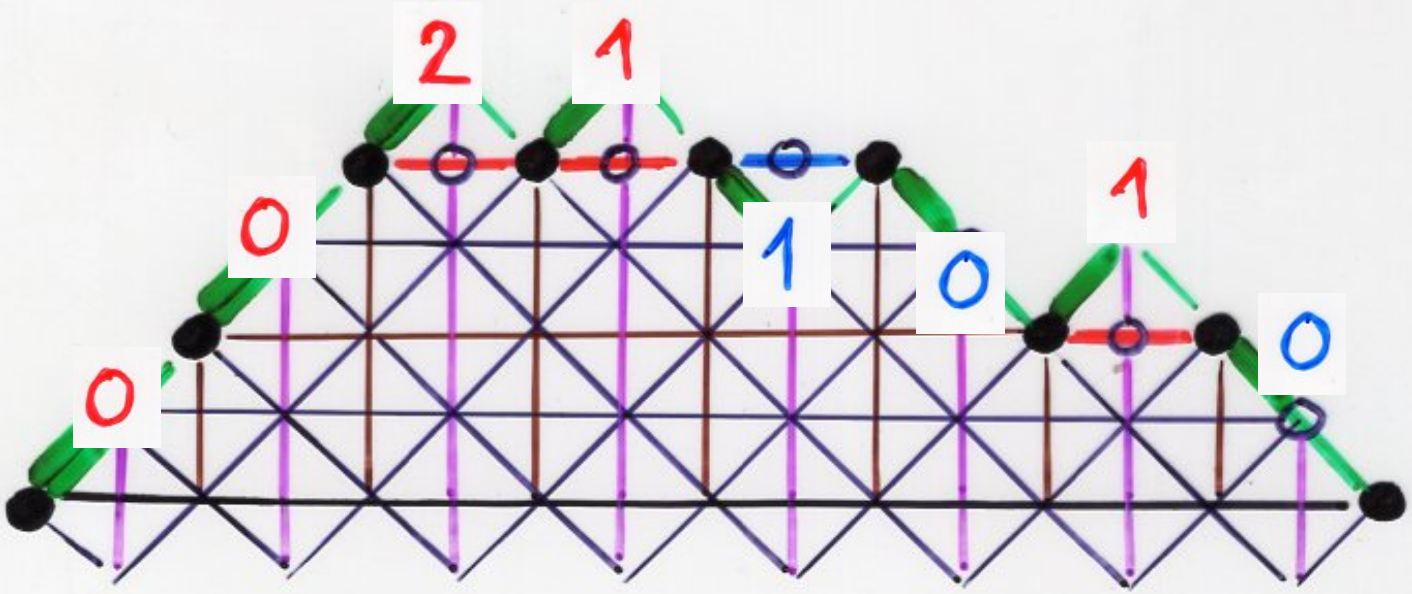
subdivided Laguerre history



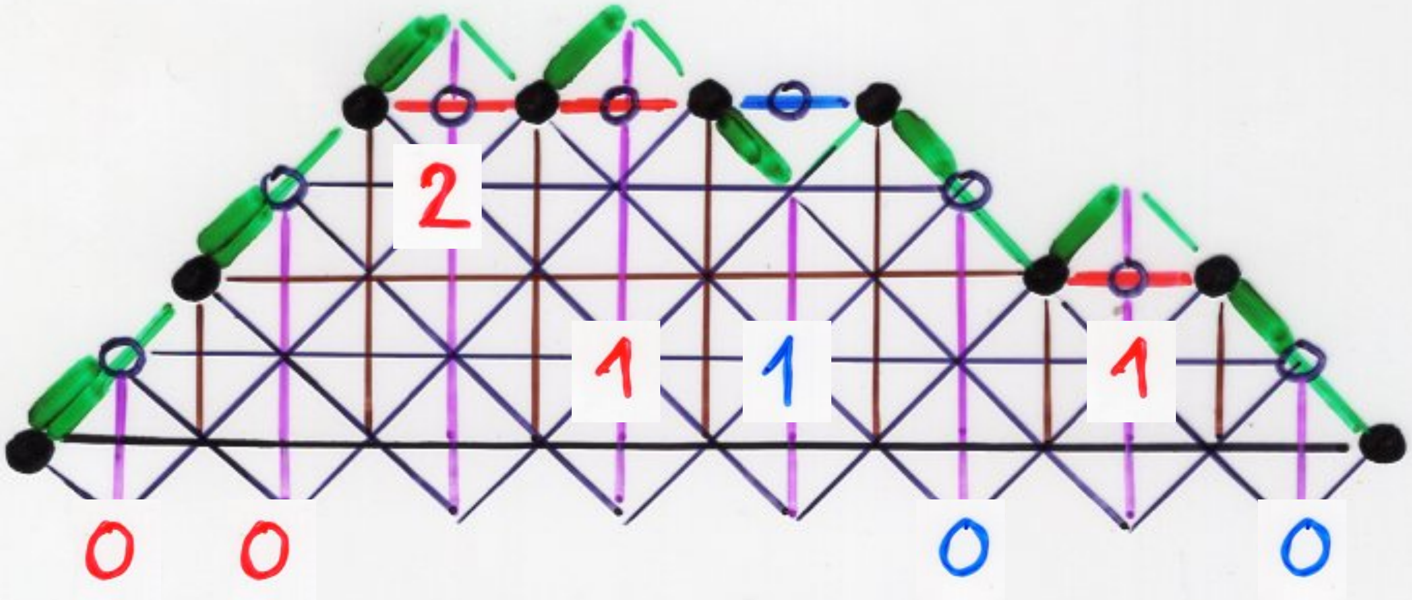
Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
σ
permutations
restricted Laguerre histories
σ^{-1}
permutations
inversion tables
(= subexcedant functions)
tree-like tableaux
alternative tableaux



0
0
2
1
1
0
1
0



2
1
0
1
0
1
0
0


2
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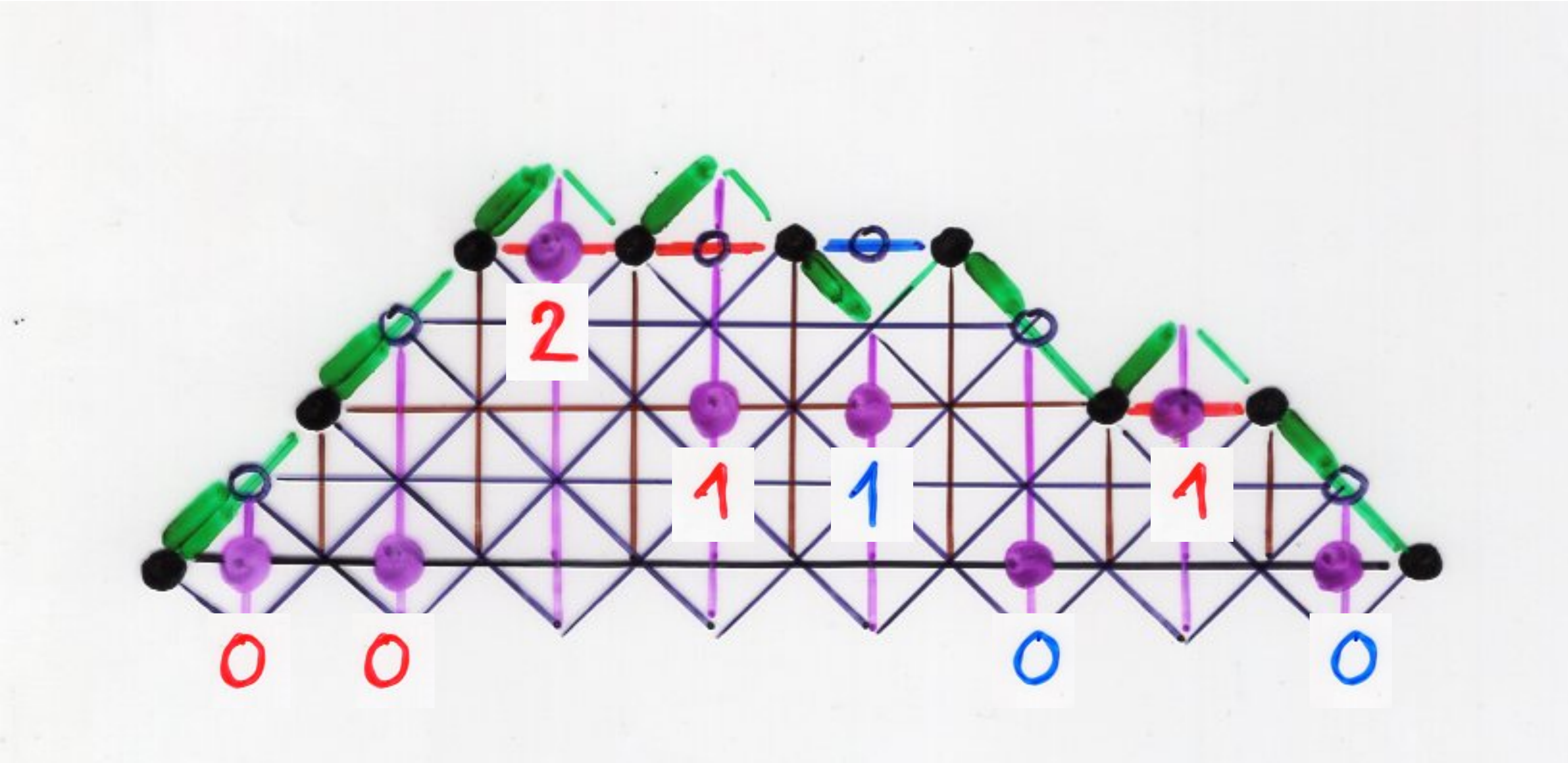
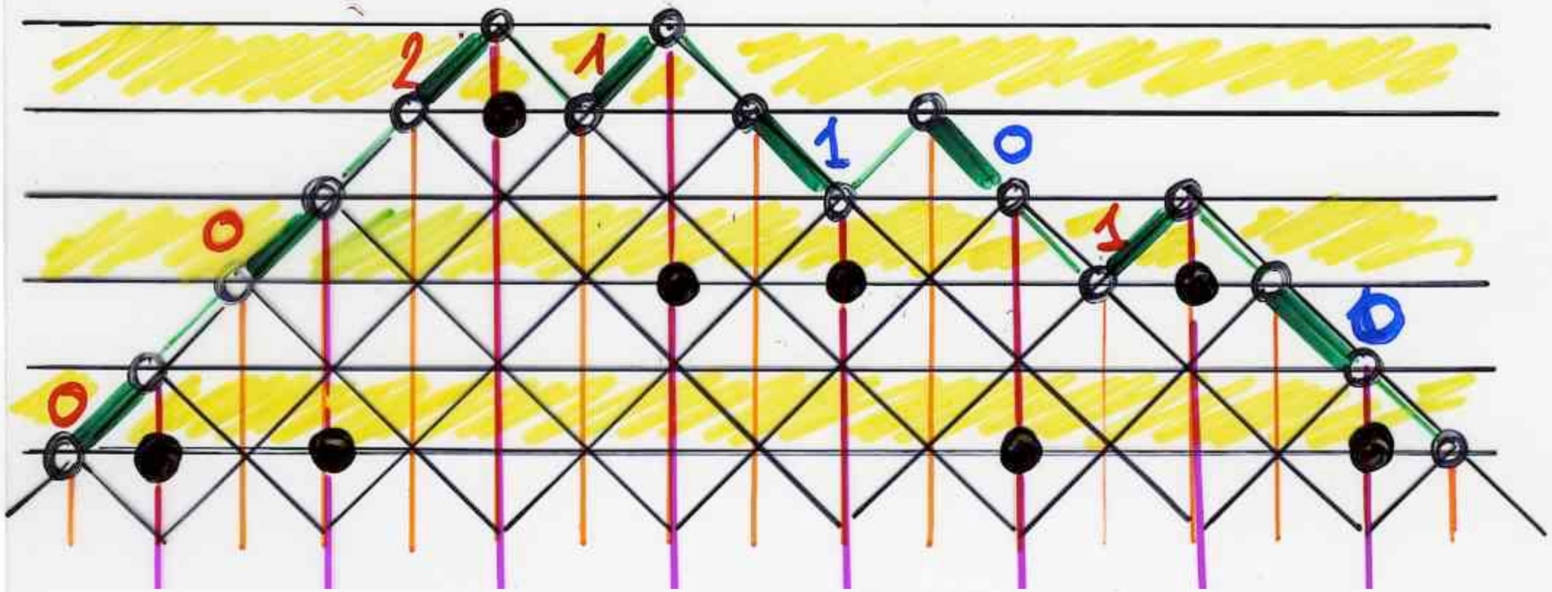
1
1
0
0
0
0



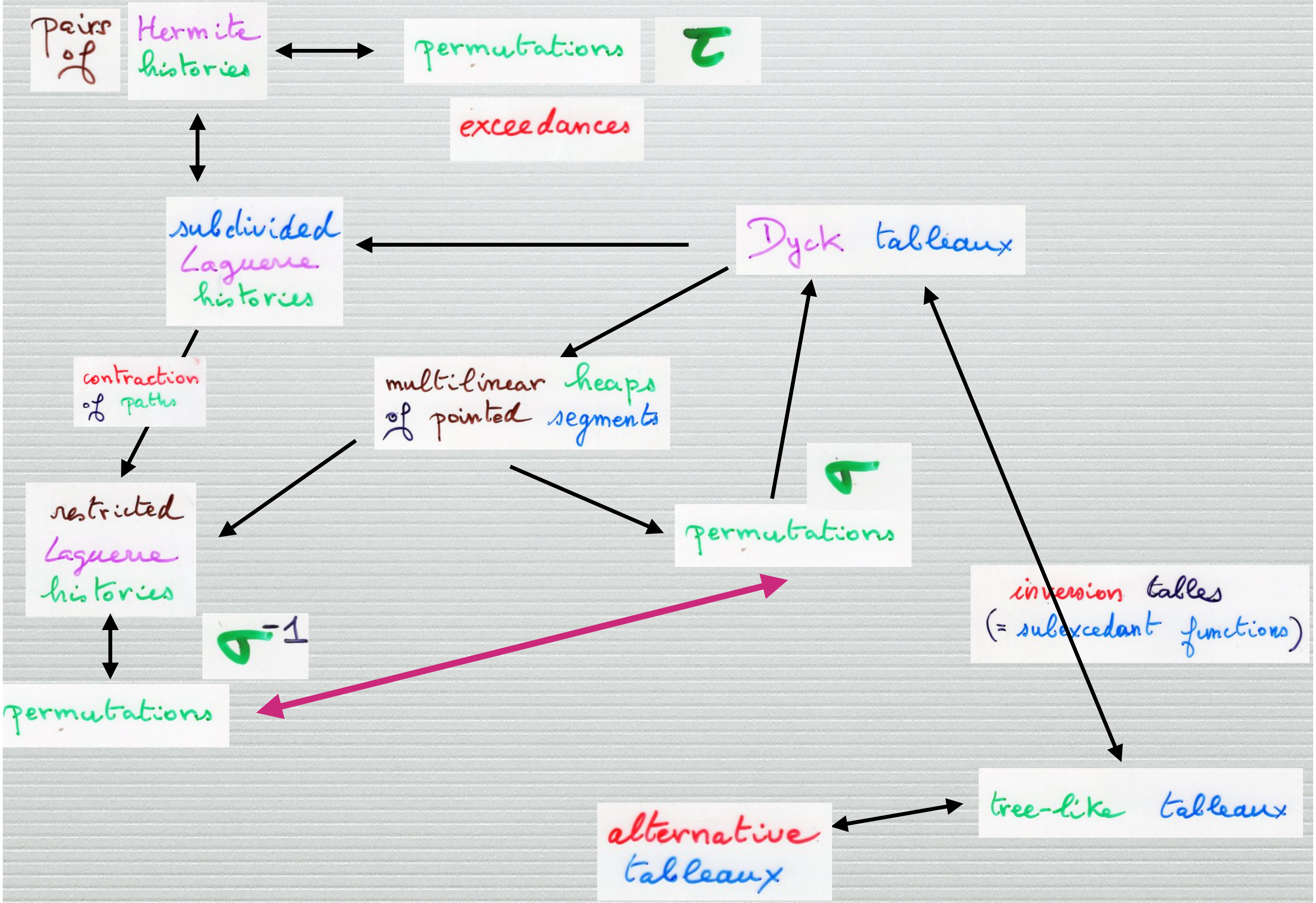
2
1
1
1
0
0
0
0



Dyck tableau as a subdivided Laguerre history

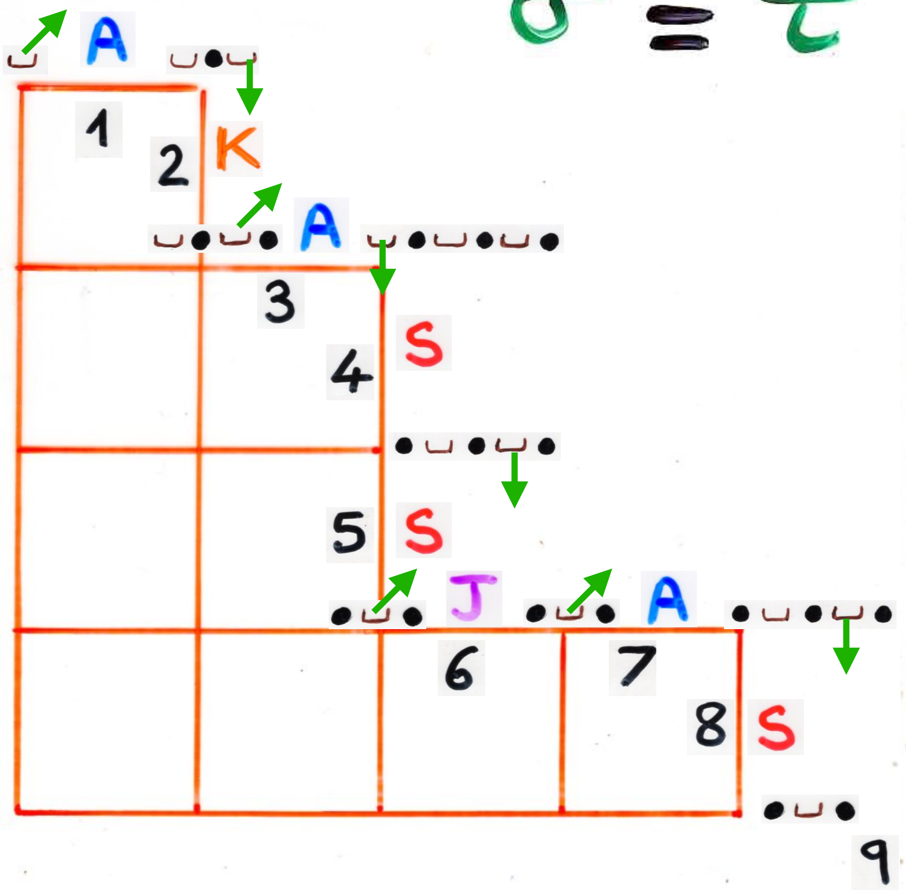
Commutations diagrams ?

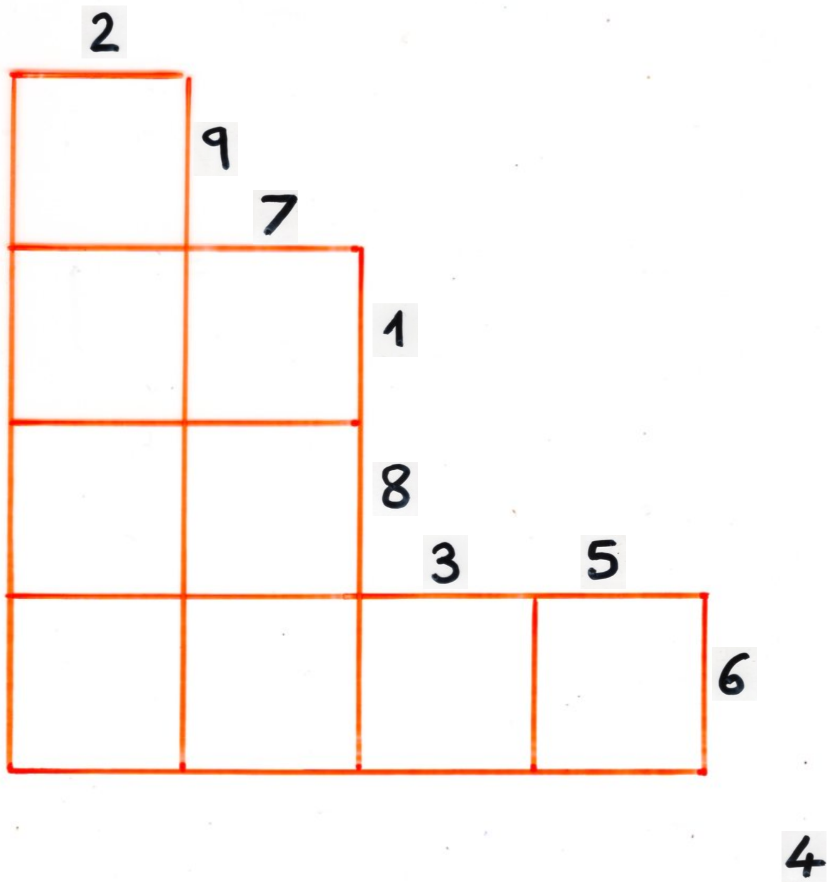
Local rules ?


Pairs of Hermite histories
permutations τ
exceedances
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
restricted Laguerre histories
σ
permutations
inversion tables
(= subexcedant functions)
σ^-1
permutations
tree-like tableaux
alternative tableaux

4 1 6 9 7 8 3 5 2 2 9 7 1 8 3 5 6 4

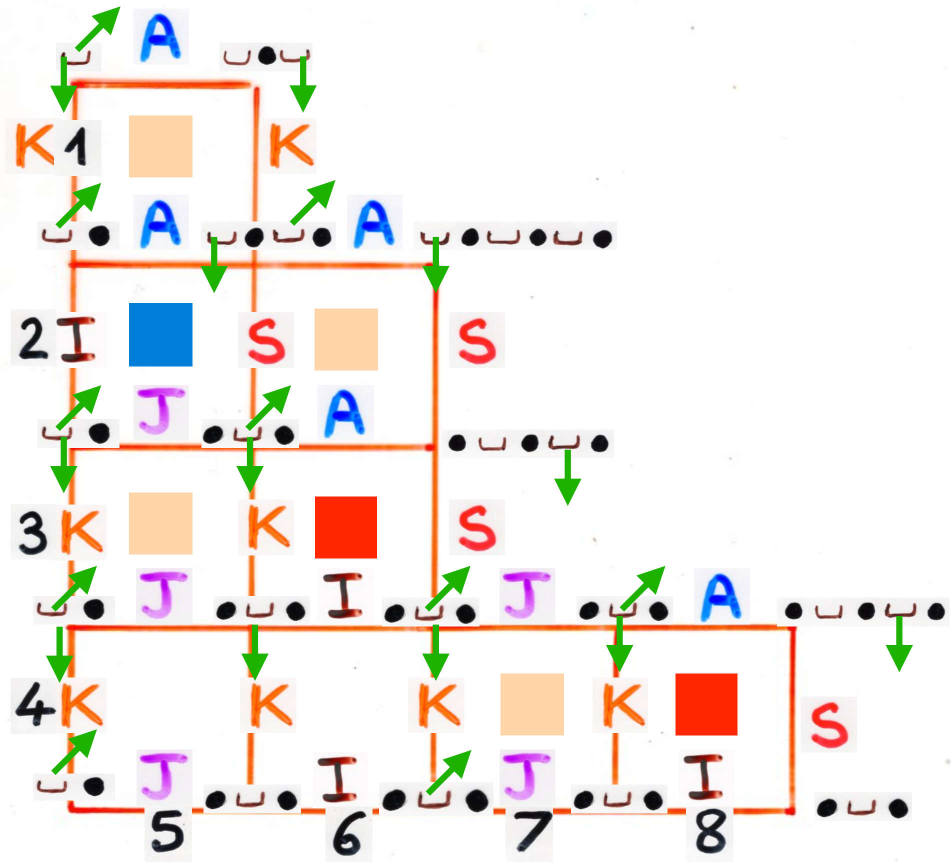
$$\sigma = \tau^{-1}$$

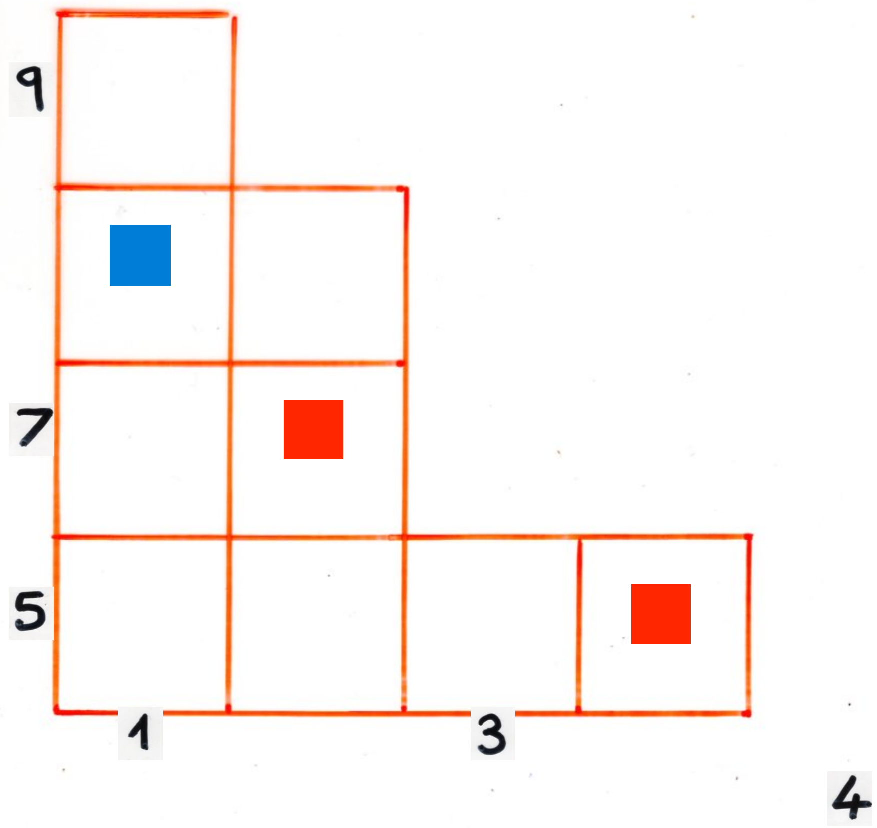


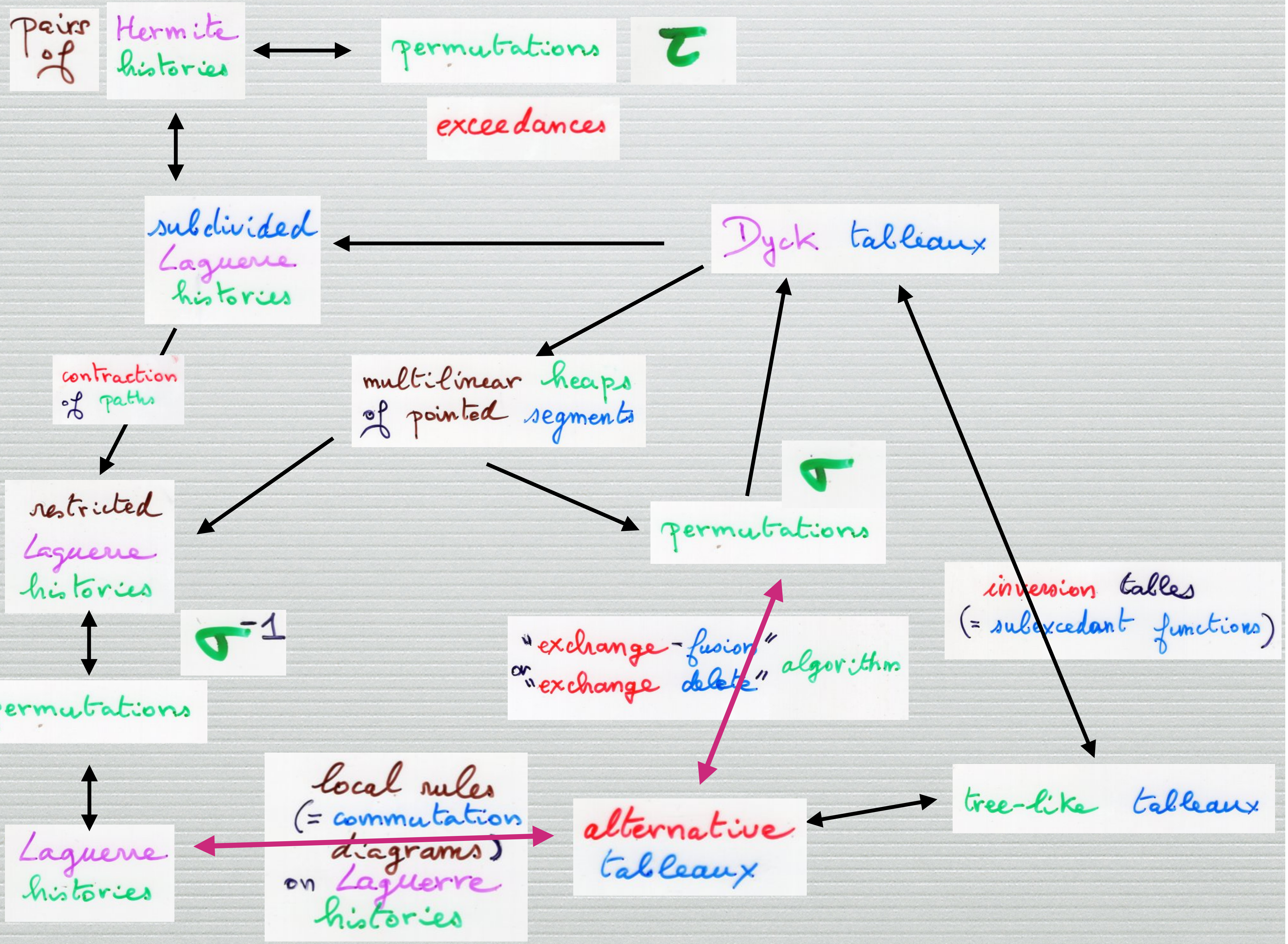


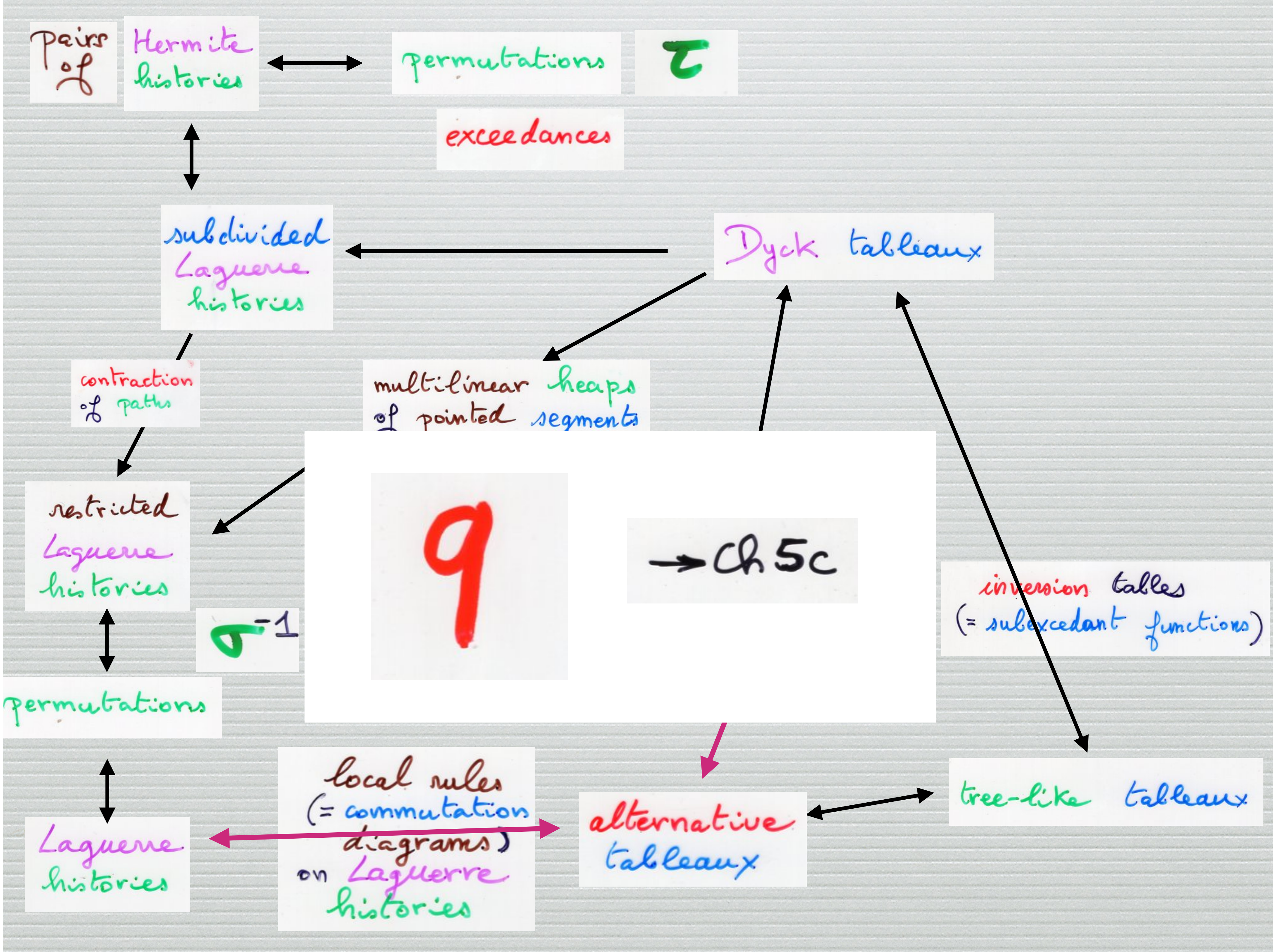
4 795 3 1 — 9 715 3 4

5 794 3 1 — 9 751 3 4






Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
σ
permutations
restricted Laguerre histories
σ⁻¹
inversion tables
(= subexcedant functions)
"exchange-fusion" algorithm
or "exchange delete"
permutations
local rules
(= commutation diagrams)
on Laguerre histories
Laguerre histories
alternative tableaux
tree-like tableaux


Pairs of Hermite histories
permutations
τ
exceedances
subdivided Laguerre histories
Dyck tableaux
contraction of paths
multilinear heaps of pointed segments
restricted Laguerre histories
σ-1
9
→ Ch5c
inversion tables (= subexcedant functions)
permutations
local rules (= commutation diagrams) on Laguerre histories
alternative tableaux
tree-like tableaux
Laguerre histories