Course IMSc, Chennai, India

January-March 2018

The cellular ansatz:
bijective combinatorics and quadratic algebra

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Chapter 5

Tableaux and orthogonal polynomials

Ch5a

IMSc, Chennai
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some trigonometry ...

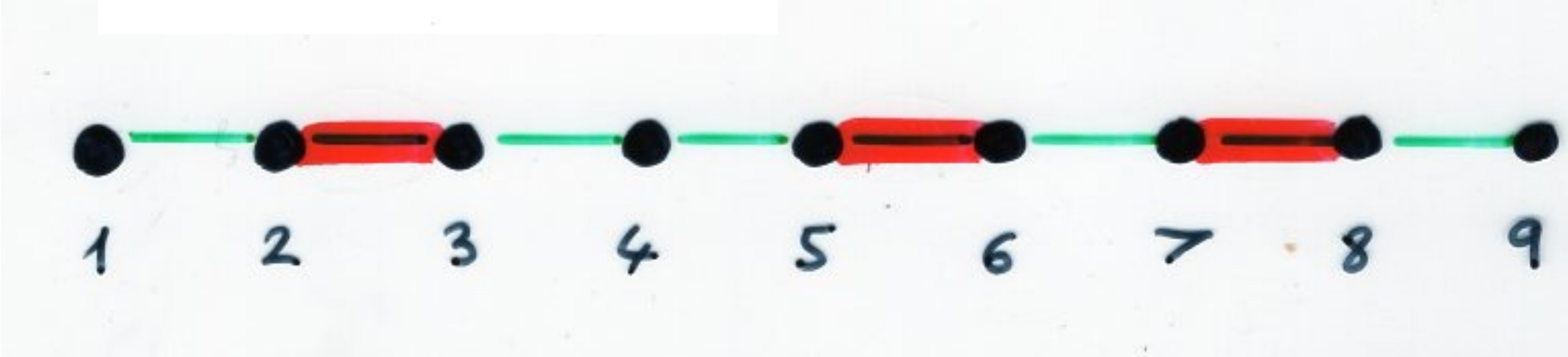
$$\sin\big((n+1)\theta\big) = \sin\theta \; U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2nd kind

sequence of orthogonal polynomials

$$\frac{2}{\pi}\int_{-1}^{+1} U_n(x)\, U_m(x)\, (1-x^2)^{1/2}\, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{else} \end{cases}$$

$$U_n(x) = F_n(2x)$$



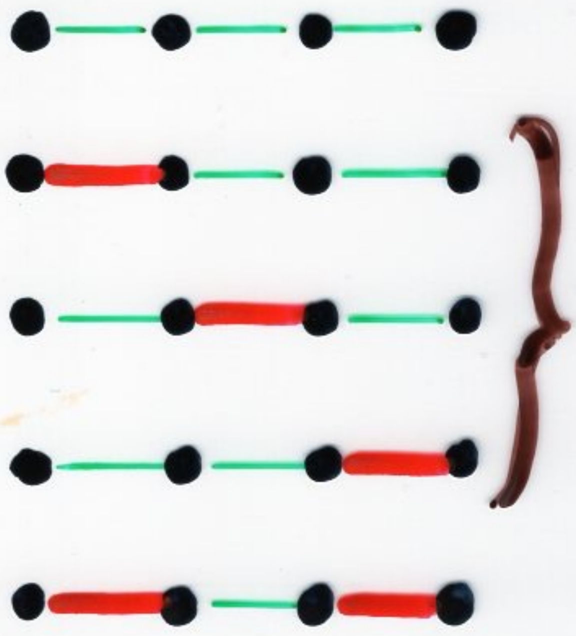
$= n$

Fibonacci polynomials

$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k$$

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1, -, n\}}} (-x)^{|M|}$$

$a_{n,k}$ = number of matchings of $\{1, 2, \dots, n\}$ with k dimers



x^4 $(2\cos\theta)^4$

$-3x^2$ $-3(2\cos\theta)^2$

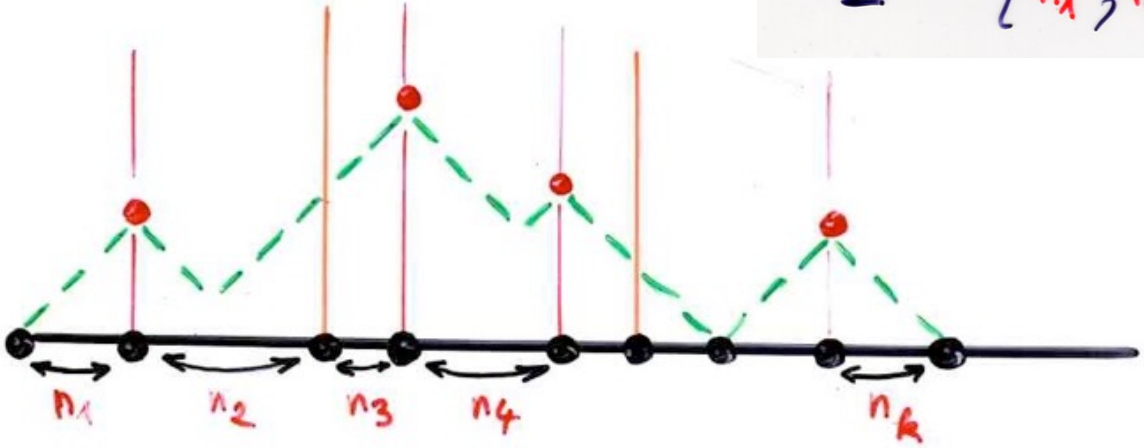
$+1$ 1

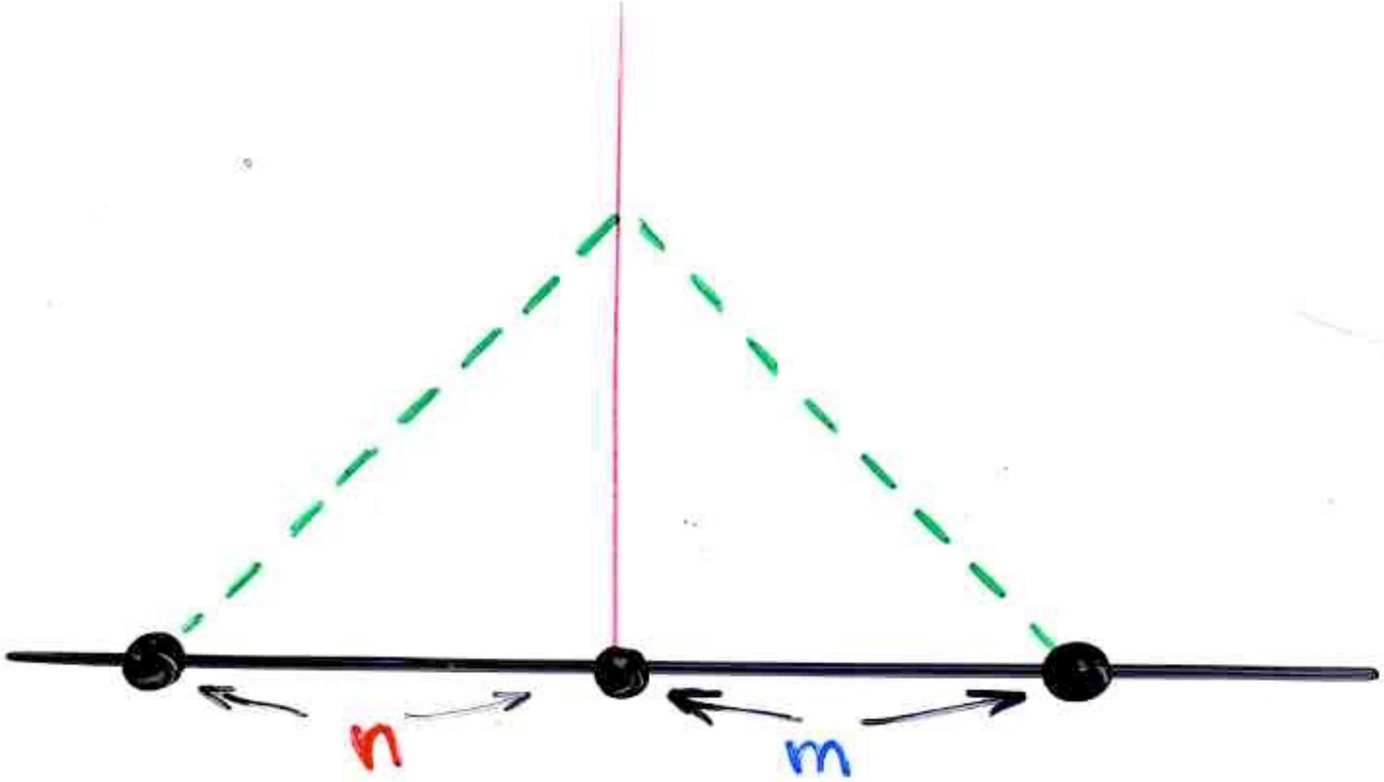
$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$$

$$\frac{2}{\pi}\int_{-1}^{+1} U_{n_1}(x)\,U_{n_2}(x)\dots U_{n_k}(x)\,(1-x^2)^{1/2}\,dx =$$

number of Dyck paths such that set of indices of peaks 

$$\subseteq \{n_1, n_1+n_2, \dots, n_1+n_2+\dots+n_k\}$$



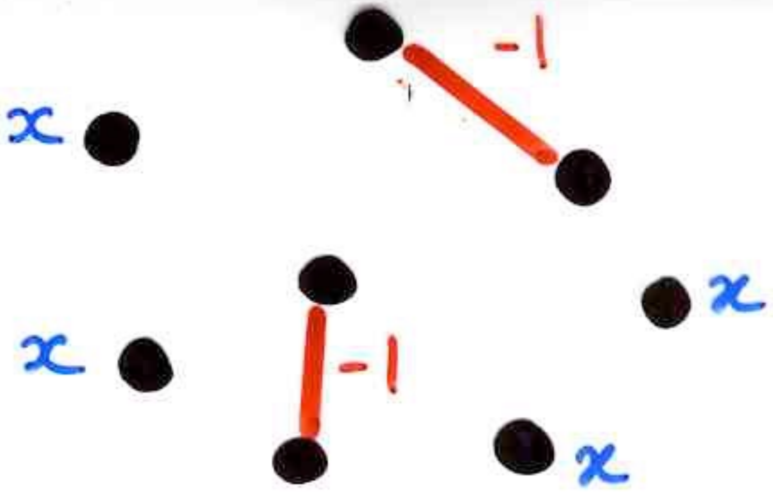


$$\frac{2}{\pi}\int_{-1}^{+1} U_n(x)\,U_m(x)\,(1-x^2)^{1/2}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{else} \end{cases}$$

combinatorial interpretations

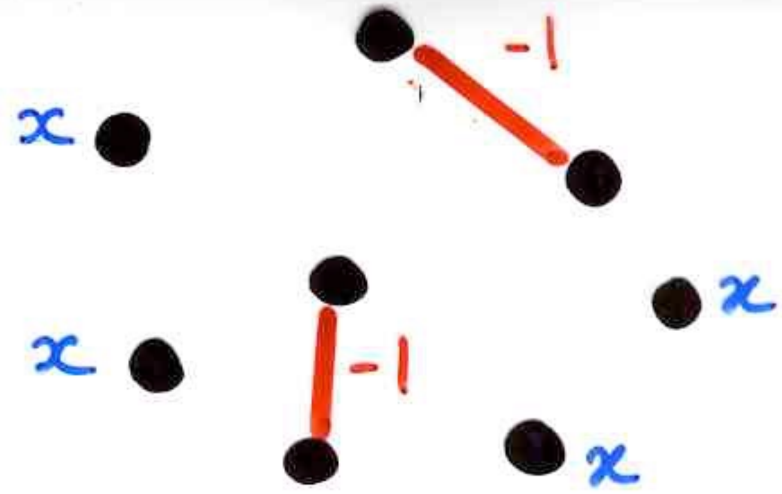
Hermite polynomials

$$H_n(x) = \sum_{\substack{\text{involutions } \sigma \\ \text{on } \{1,2,\dots,n\}}} (-1)^{\mathrm{cyc}(\sigma)} \, x^{\mathrm{fix}(\sigma)}$$



matching of the complete graph K_n

$$H_n(x) = \sum_{0\le 2k\le n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!}\, x^{n-2k}$$



$$\int_{-\infty}^{+\infty} H_n(x)\, H_m(x)\, e^{-x^2/2}\, dx = \sqrt{\pi}\ \ n!\ \ \delta_{n,m}$$

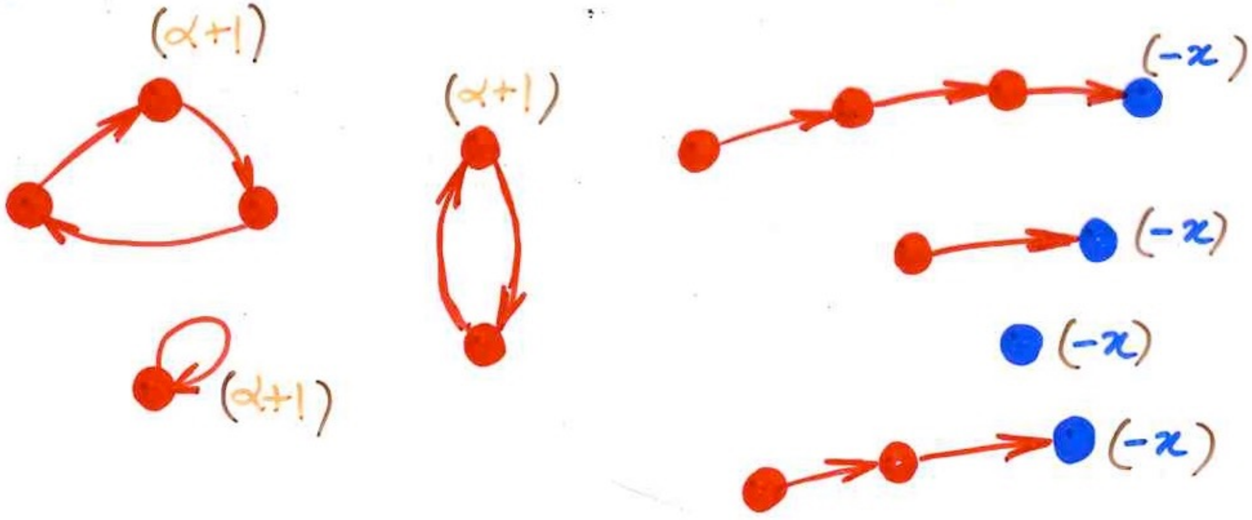
Hermite

Laguerre
polynomial

Laguerre $L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



Laguerre $\quad L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

$$L_n^{(\alpha)}(x) = (\alpha+1)_n \;{}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right]$$

$$= \sum_{i+j=n} \binom{n}{i}(\alpha+1+j)_i\,(-x)_j$$

$$\int_0^\infty L_m^{(\alpha)}(x)\,L_n^{(\alpha)}(x)\,x^\alpha e^{-x}\,dx = n!\,\Gamma(n+\alpha+1)\,\delta_{n,m}$$

Laguerre

classical
analysis

formal orthogonal polynomials:
définition, moments

$\mathbb{K}$ ring (domain) field $\mathbb{R}, \mathbb{C}$
or $\mathbb{Q}[\alpha, \beta, \cdots]$

$\mathbb{K}[x]$ $\{P_k(x)\}_{k \geqslant 0}$

$\deg(P_k) = k$

Orthogonal polynomials

Def. $\{P_n(x)\}_{n \geqslant 0}$ $P_n(x) \in \mathbb{K}[x]$
orthogonal iff $\exists$ $f : \mathbb{K}[x] \longrightarrow \mathbb{K}$
linear functional

$$\begin{cases} (i) & \deg(P_n(x)) = n \quad (\forall n \geqslant 0) \\ (ii) & f(P_k P_\ell) = 0 \quad \text{for } k \neq \ell \geqslant 0 \\ (iii) & f(P_k^2) \neq 0 \quad \text{for } k \geqslant 0 \end{cases}$$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$f(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

classical
analysis

remark. if $\mu_0 = \mu_1 = \mu_2 = 1$

no sequence of orthogonal polynomials

lemma $\{P_k(x)\}_{k\geq 0}$, $\{Q_k(x)\}_{k\geq 0}$

two sequences orthogonal for f

then $P_k(x) = Q_k(x)\, c_k$ $\qquad c_k \neq 0$

unicity

lemma if $\{P_k(x)\}_{k\geq 0}$ orthogonal for f and g

unicity

then $f = c\, g$ $\qquad c \neq 0$

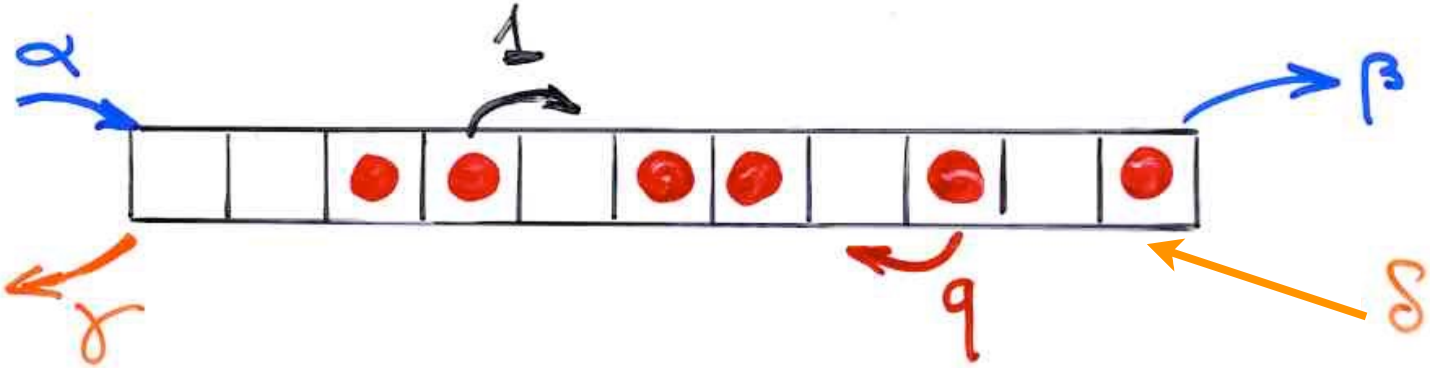
Orthogonal polynomials, the PASEP model in physics and ASM

partially asymmetric exclusion model

alternating sign matrices

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

$A_n(x)$

enumeration of ASM
according to the number of (-1)

1-, 2-, 3- enumeration
formula for $A_n(x)$

alternating sign matrices

Colomo, Pronko (2004)

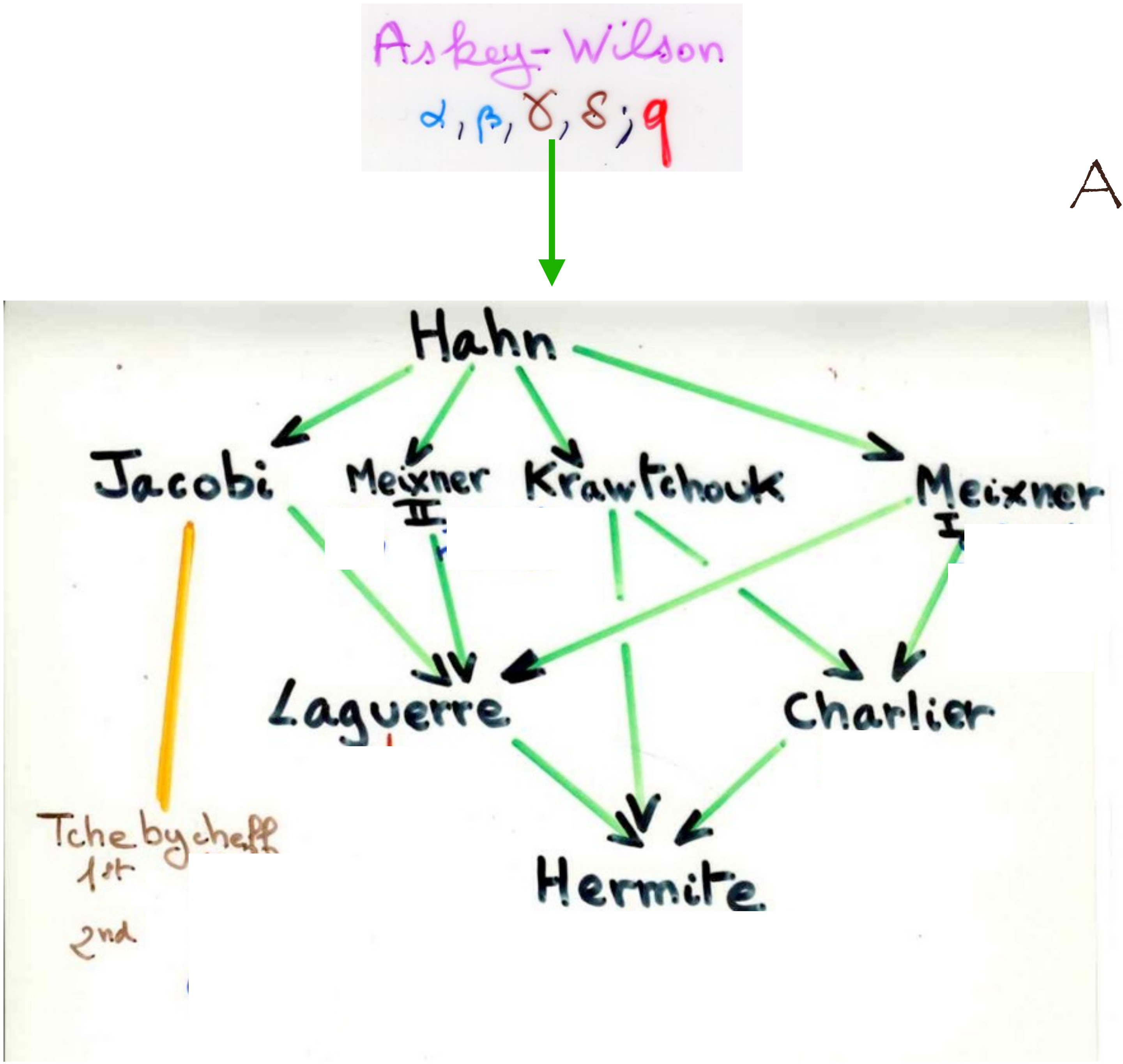
Hankel determinants

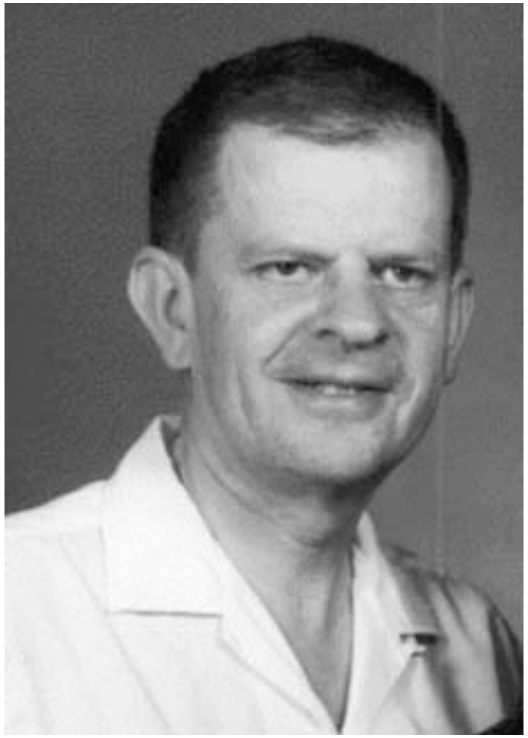
orthogonal polynomials

(continuous) Hahn Meixner-Pollaczek

(continuous) dual Hahn

Askey tableau





Yeh, Labelle, Strehl

Limit Formulas

example Jacobi → Laguerre

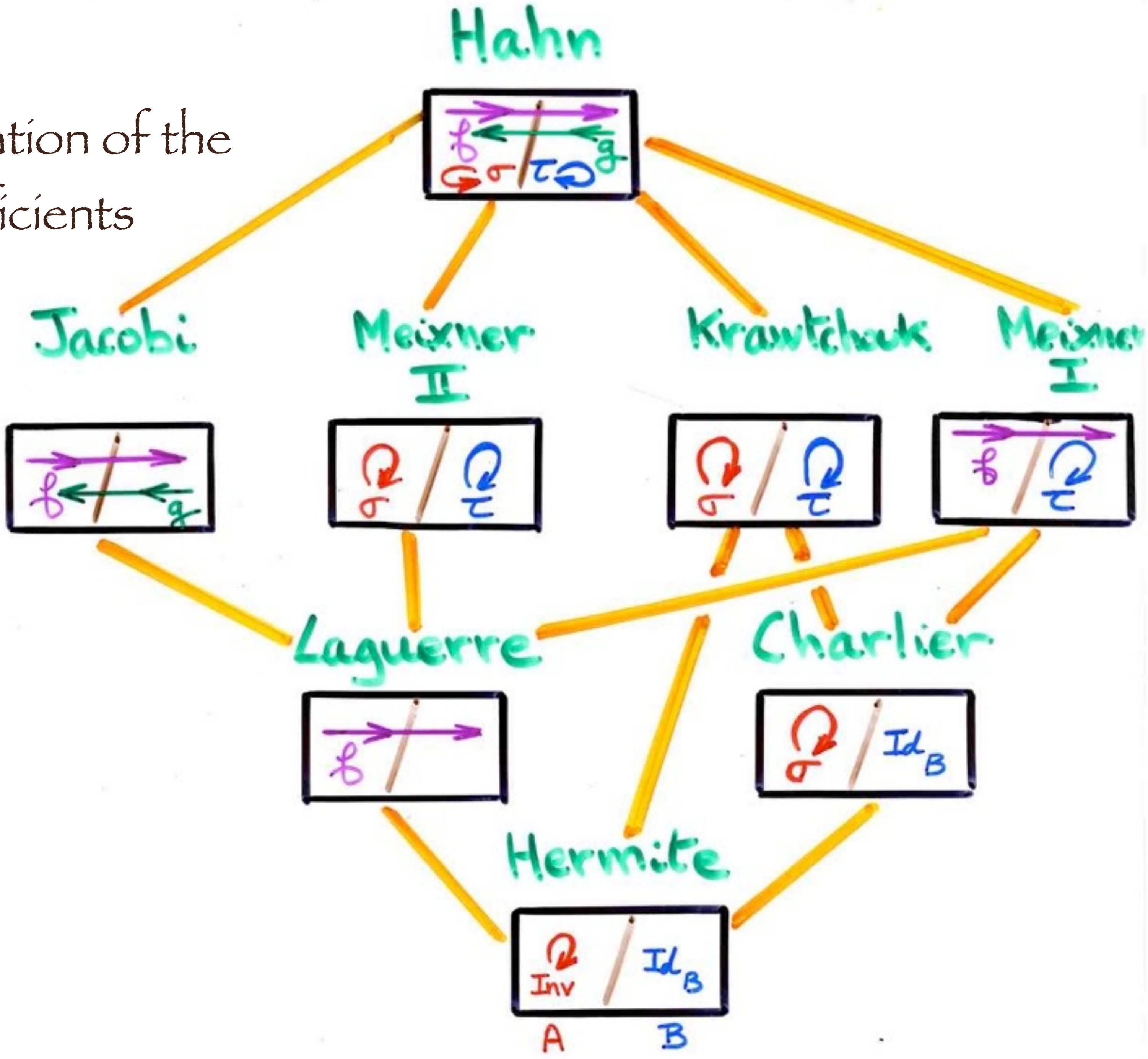
$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = L_n^{(\alpha)}(x)$$

Combinatorial theory of orthogonal polynomials

Two aproaches:

- coefficients, generating functions of OP, proof of formulae
- Combinatorial interpretations of the moments

Interpretation of the coefficients



Combinatorial proof of formulae

Mehler identity

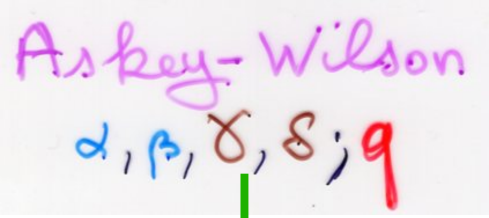
$$\sum_{n\geq 0} \mathbf{H}_n(x)\,\mathbf{H}_n(y)\,\frac{t^n}{n!}$$

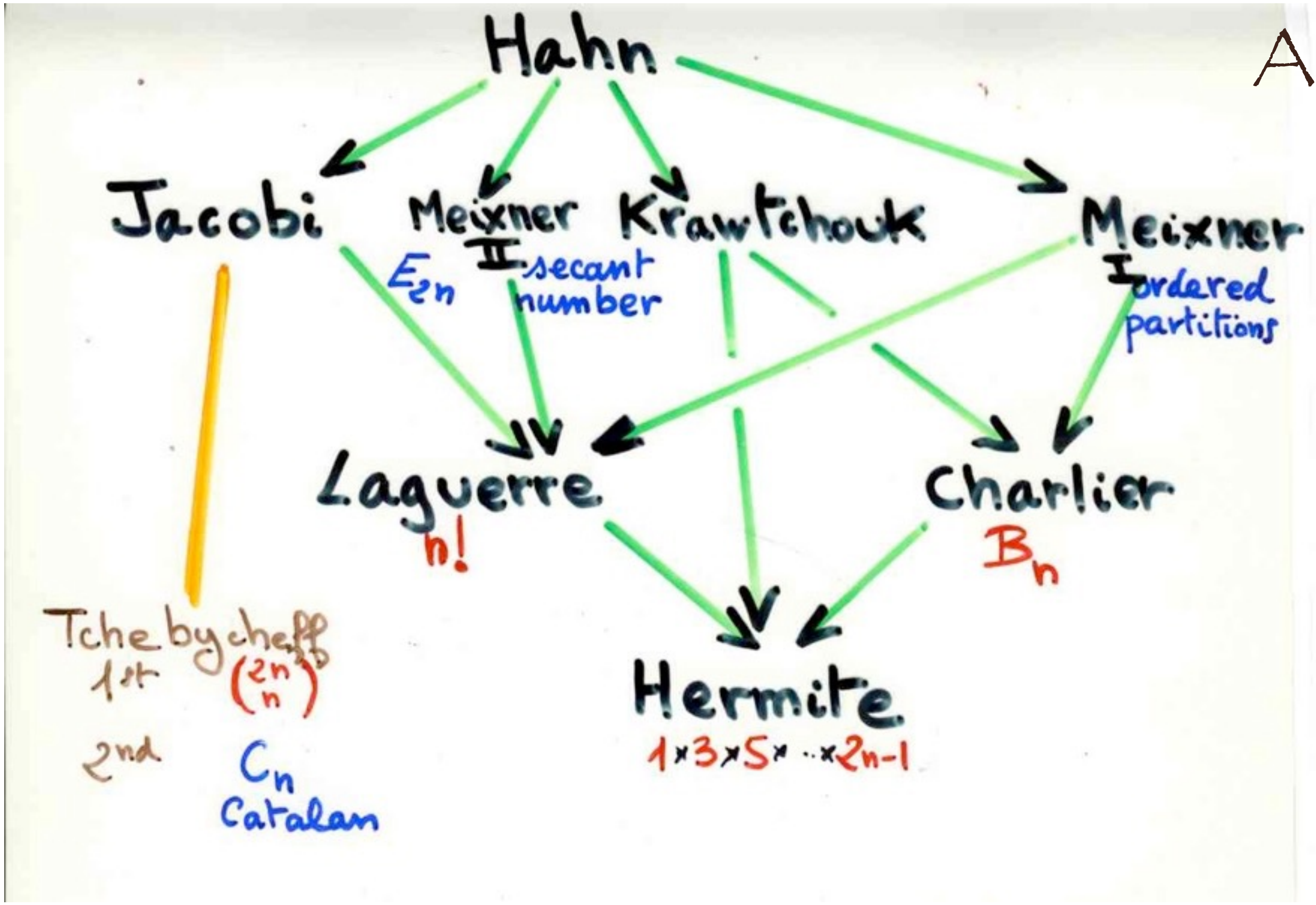
$$= (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

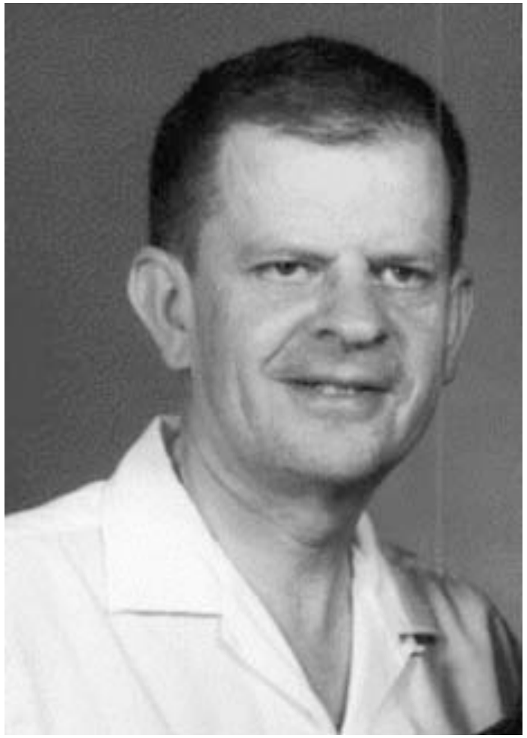
BJC1, Ch3b, p.26

Askey tableau

Interpretation of moments







Favard theorem

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k\geqslant 1)$

3 terms linear recurrence relation

μ_n ?

Combinatorial theory of (formal) orthogonal polynomials

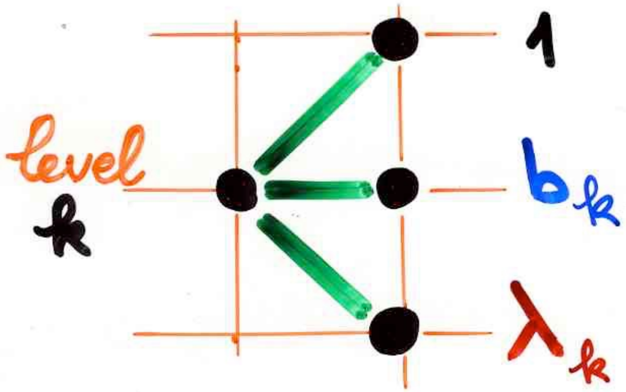
Combinatorial theory of (analytic) continued fractions

Flajolet, 1980. X.V. 1984

moments
and
Motzkin weighted paths

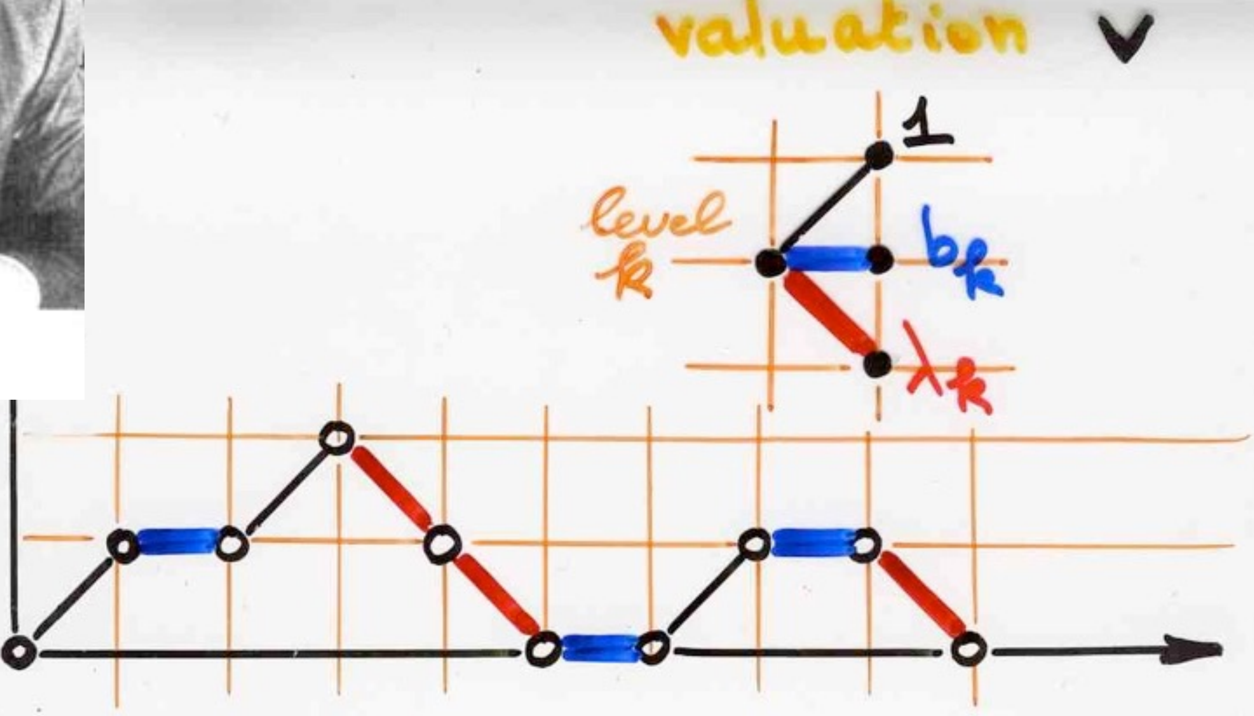
$\{b_k\}_{k \geqslant 0}$ $\{\lambda_k\}_{k \geqslant 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring



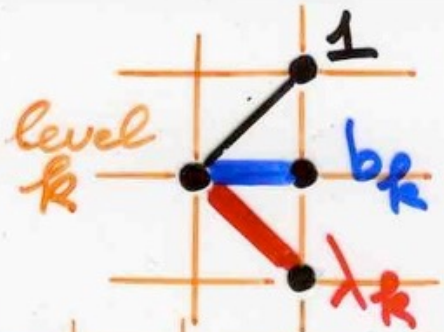
valuation

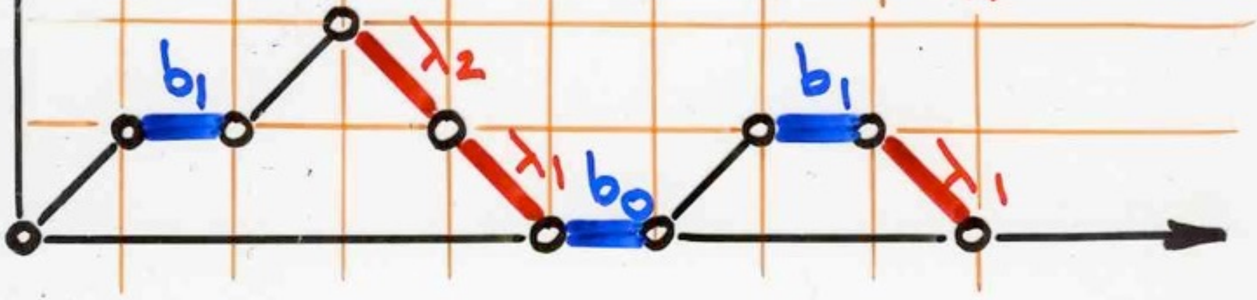
(weigth)



ω Motzkin path

valuation v





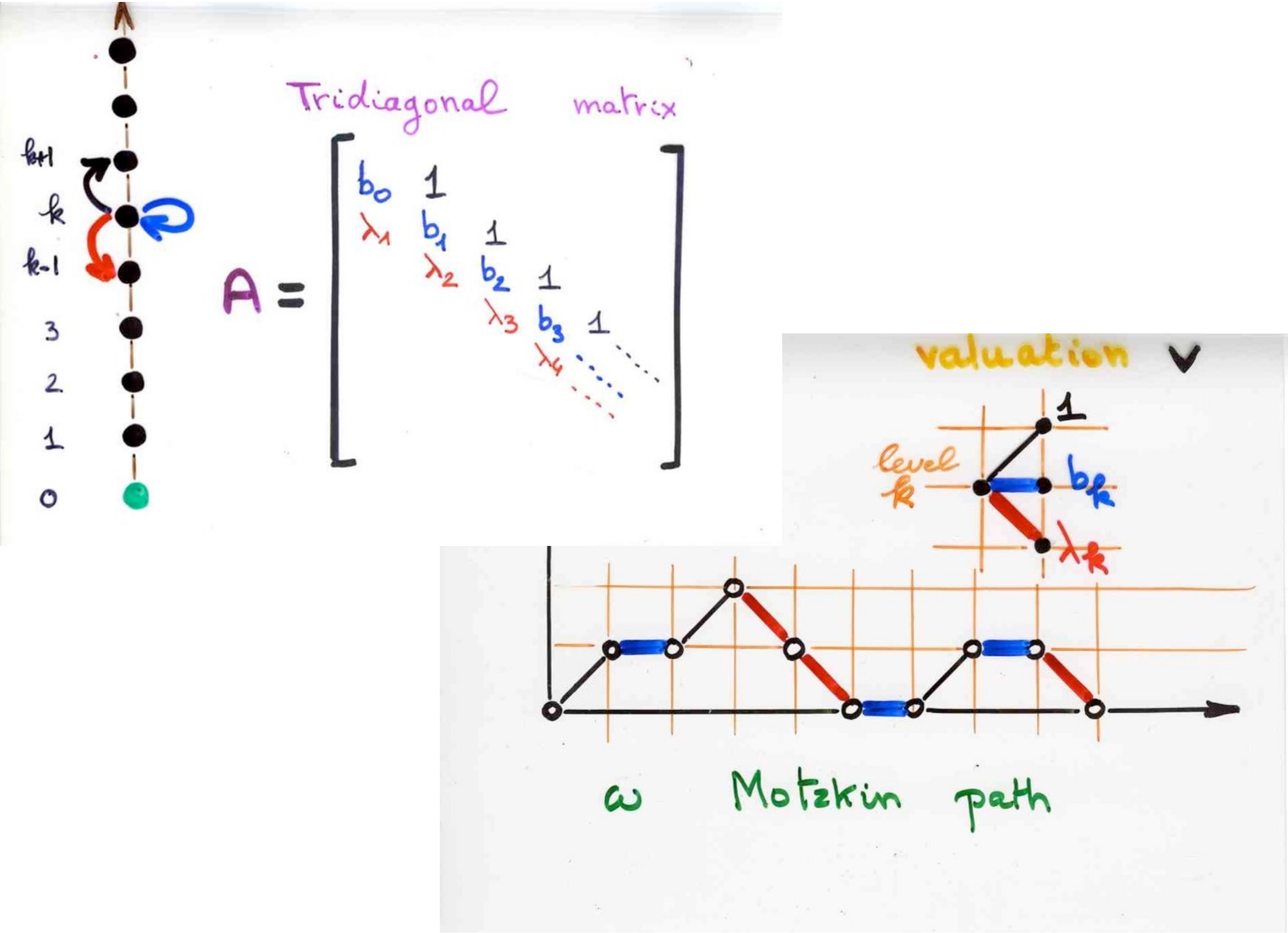
ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

$$f(x^n) = \mu_n \qquad (n \geq 0)$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$



Tridiagonal matrix

$$A = \begin{bmatrix} b_0 & 1 & & & & \\ \lambda_1 & b_1 & 1 & & & \\ & \lambda_2 & b_2 & 1 & & \\ & & \lambda_3 & b_3 & 1 & \\ & & & \lambda_4 & \ddots & \ddots \\ & & & & \ddots & \end{bmatrix}$$

equivalence
with
analytic continued fractions

$$\sum_{n\geqslant 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{1-b_1 t - \lambda_2 t^2 \cfrac{}{\cfrac{\cdots\cdots}{1-b_k t - \lambda_{k+1} t^2 \cfrac{}{\cdots\cdots}}}}}$$

$$J(t\,;b,\lambda)$$

Jacobi continued fraction

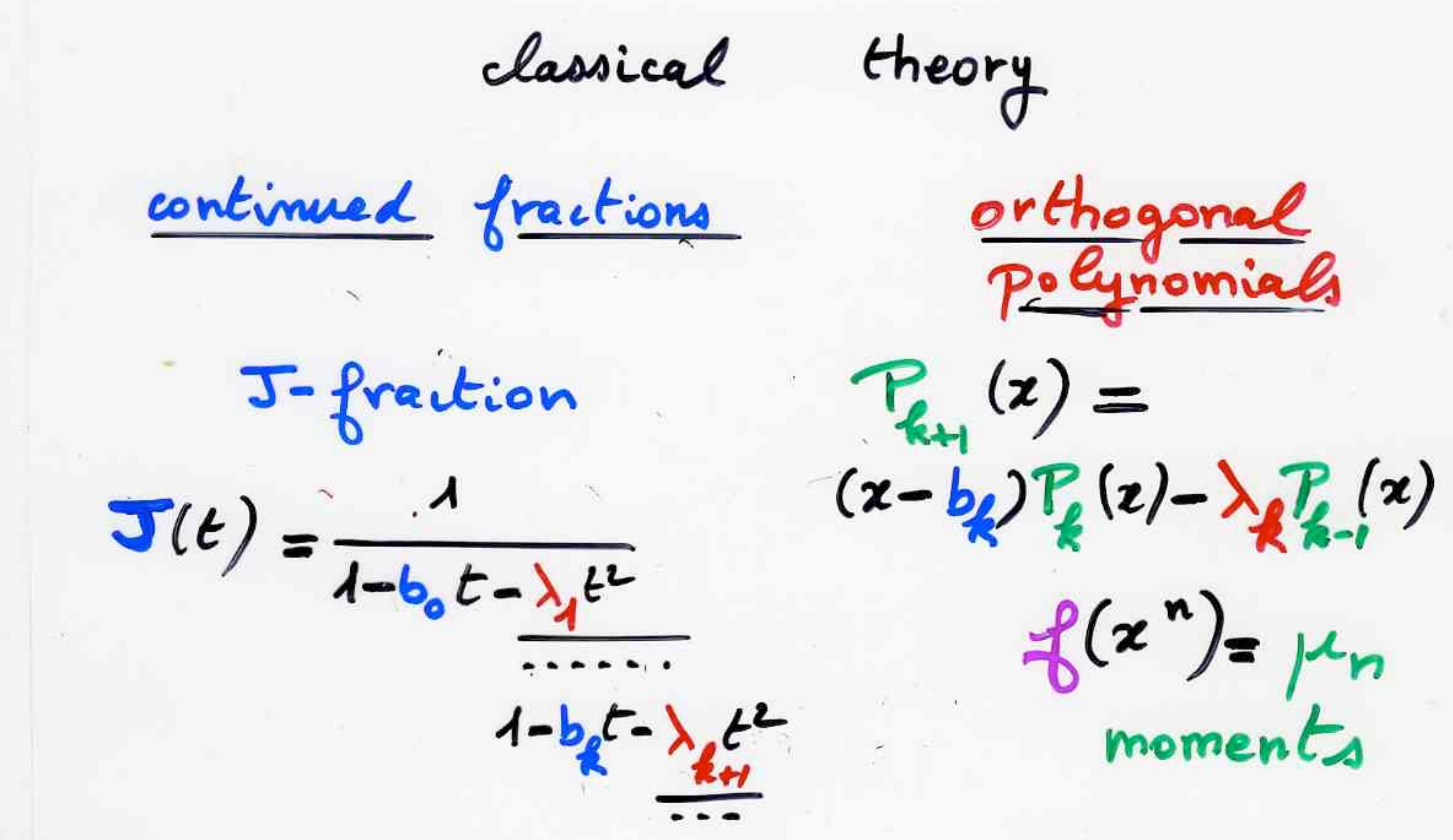
$$b=\{b_k\}_{k\geqslant 0} \qquad \lambda = \{\lambda_k\}_{k\geqslant 1}$$

continued fractions

$$\sum_{n\geq 0} \mu_n t^n = \underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cdot \cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}}_{S(t;\lambda)}$$

$\mu_0 = 1$

Stieltjès continued fraction



classical theory

continued fractions

orthogonal polynomials

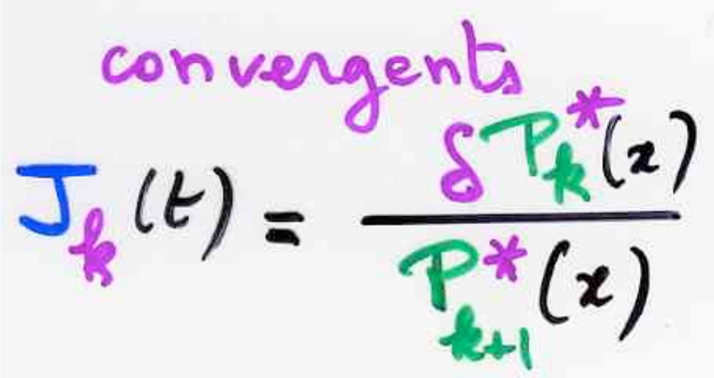
J-fraction

$$J(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cdots\cdots \cfrac{}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments



convergents

$$J_k(t) = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}$$

classical theory

continued fractions

orthogonal polynomials

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}$$

moments

generating function

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

convergents

$$J_k(t) = \frac{\delta P^*_k(x)}{P^*_{k+1}(x)}$$

The fundamental Flajolet Lemma

combinatorial interpretation of a
continued fraction with weighted paths

Discrete Mathematics 32 (1980) 125–161

Discrete Maths (1980)

COMBINATORIAL ASPECTS OF CONTINUED FRACTIONS

P. FLAJOLET

IRIA, 78150 Rocquencourt, France

Received 23 March 1979
Revised 11 February 1980

We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers.... We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

Introduction

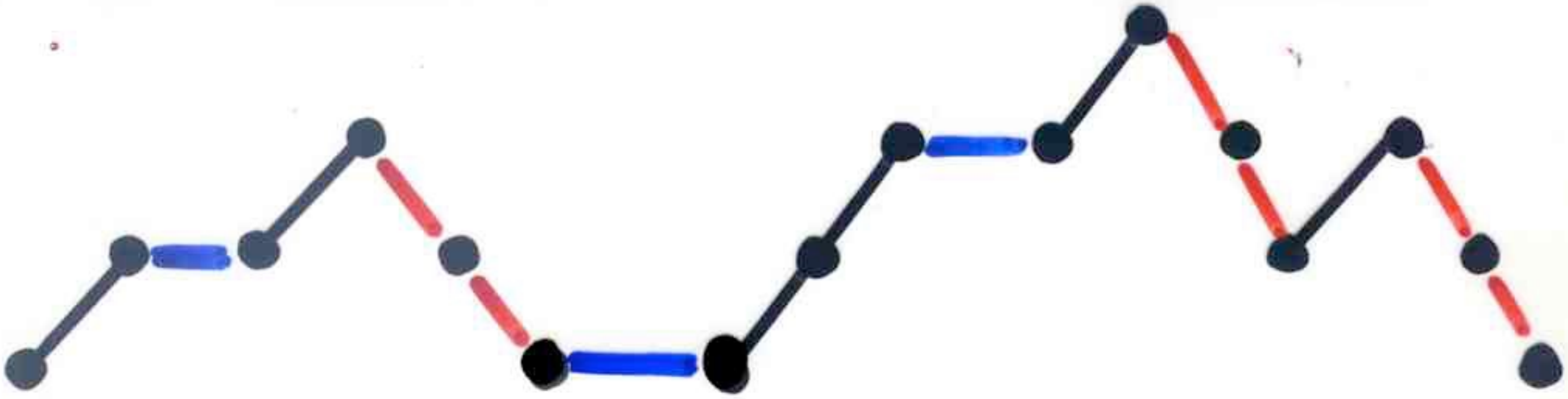
In this paper we present a geometrical interpretation of continued fractions together with some of its enumerative consequences. The basis is the equivalence

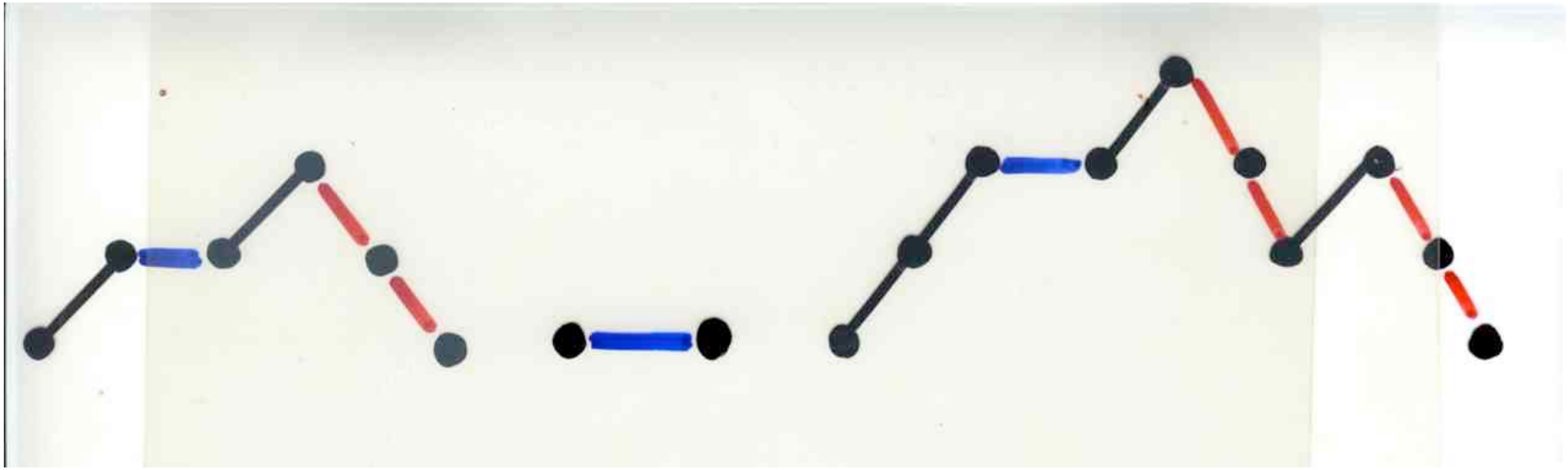
Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cfrac{\cdots}{\cdots}}}}}}$$

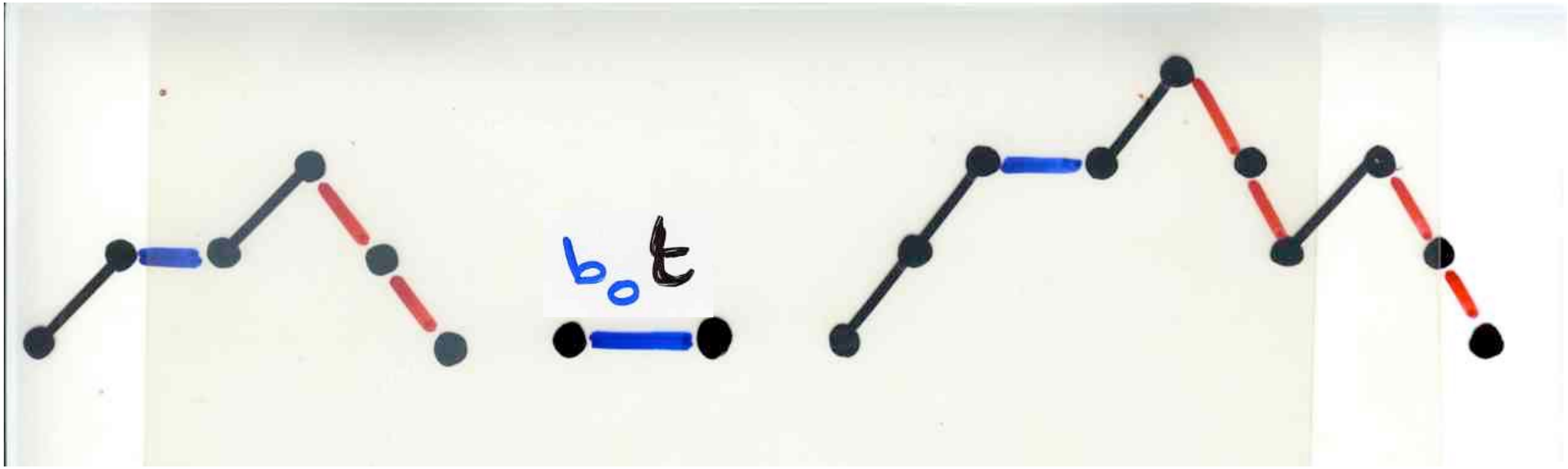
Philippe Flajolet
fundamental
Lemma

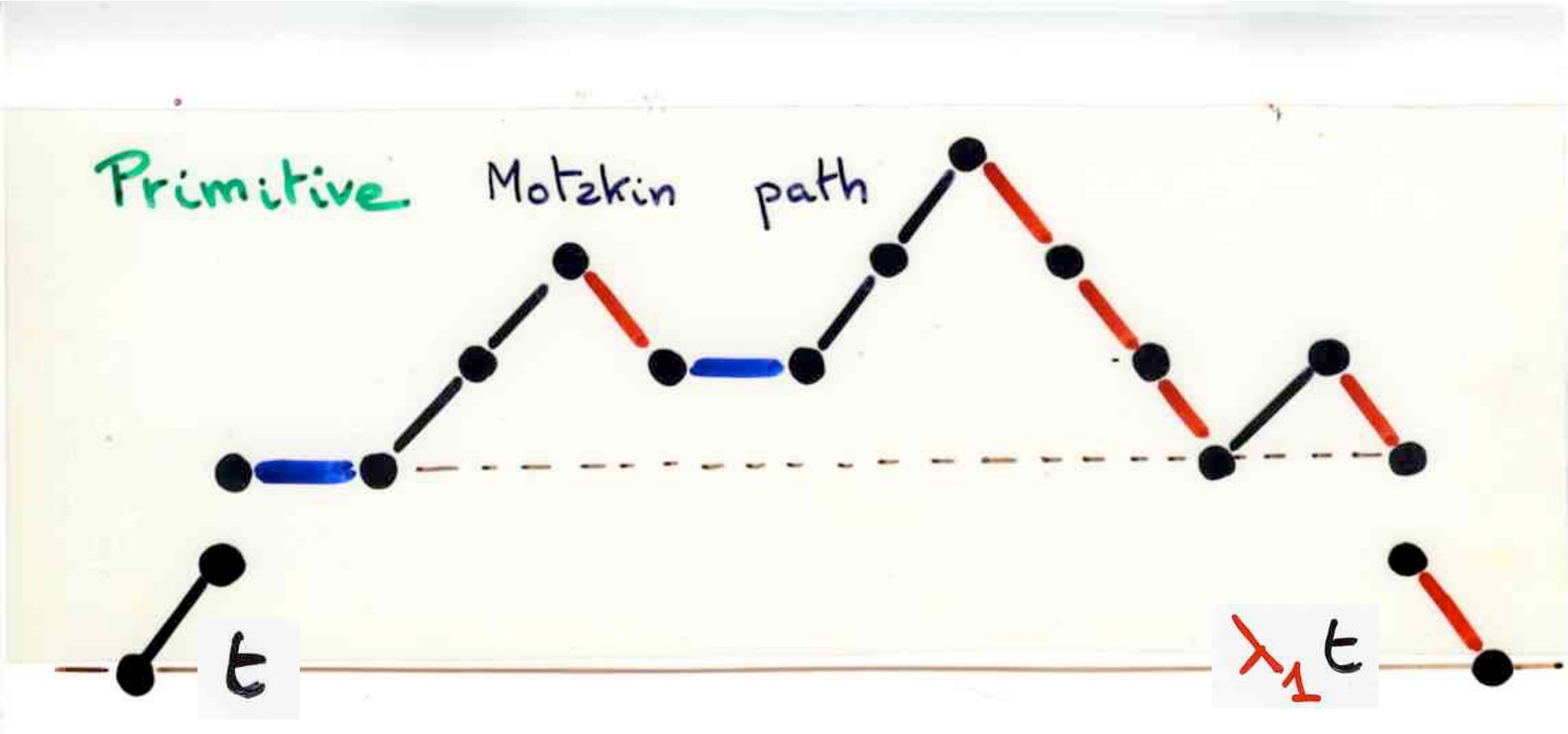
proof:





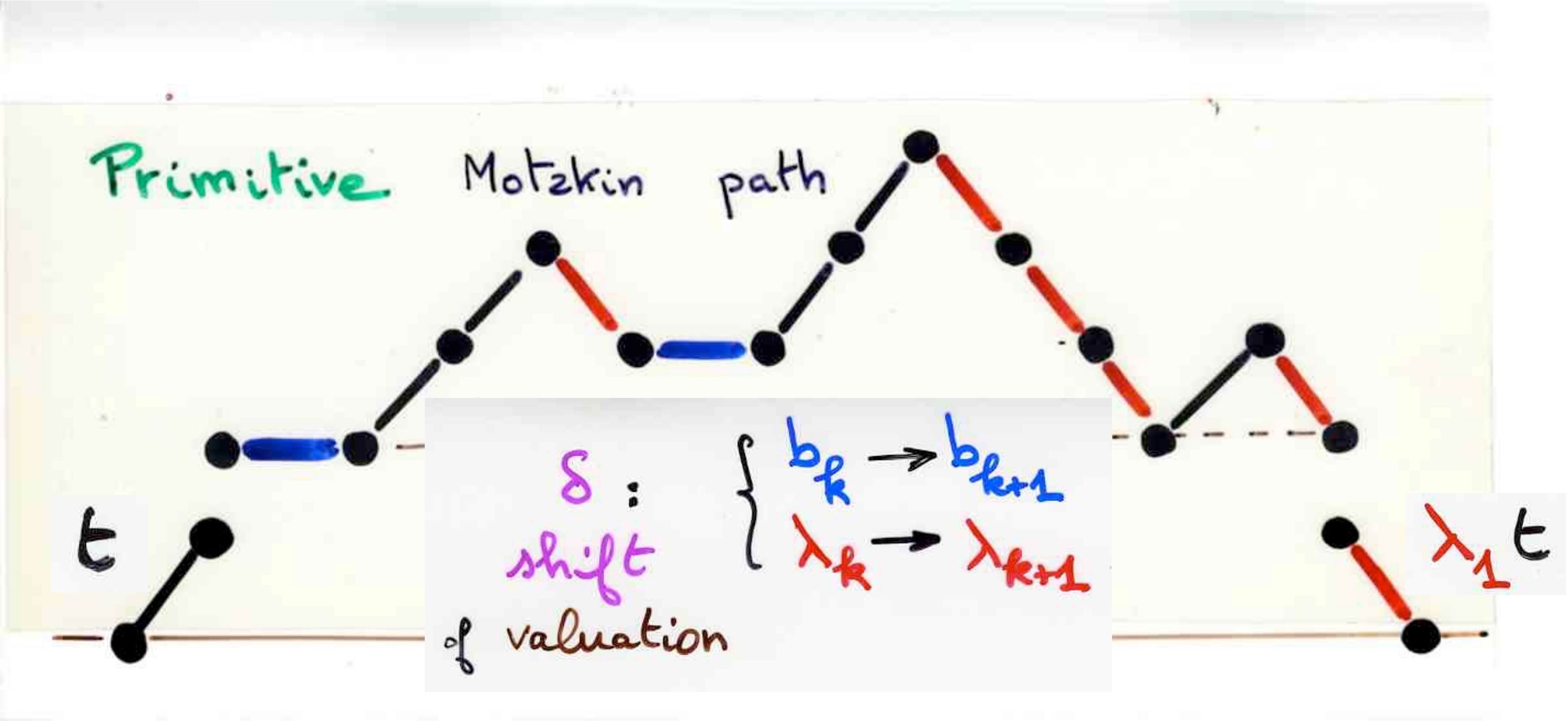
$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \frac{1}{1 - \displaystyle\sum_{\substack{\omega \\ \text{primitive} \\ \text{Motzkin} \\ \text{path}}} v(\omega)}$$


$b_0 t$


Primitive Motzkin path
t
$\lambda_1 t$

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \frac{1}{1 - b_0 t - \lambda_1 t^2 (\text{same})}$$

δ : shift of valuation $\left\{ \begin{array}{l} b_k \rightarrow b_{k+1} \\ \lambda_k \rightarrow \lambda_{k+1} \end{array} \right.$

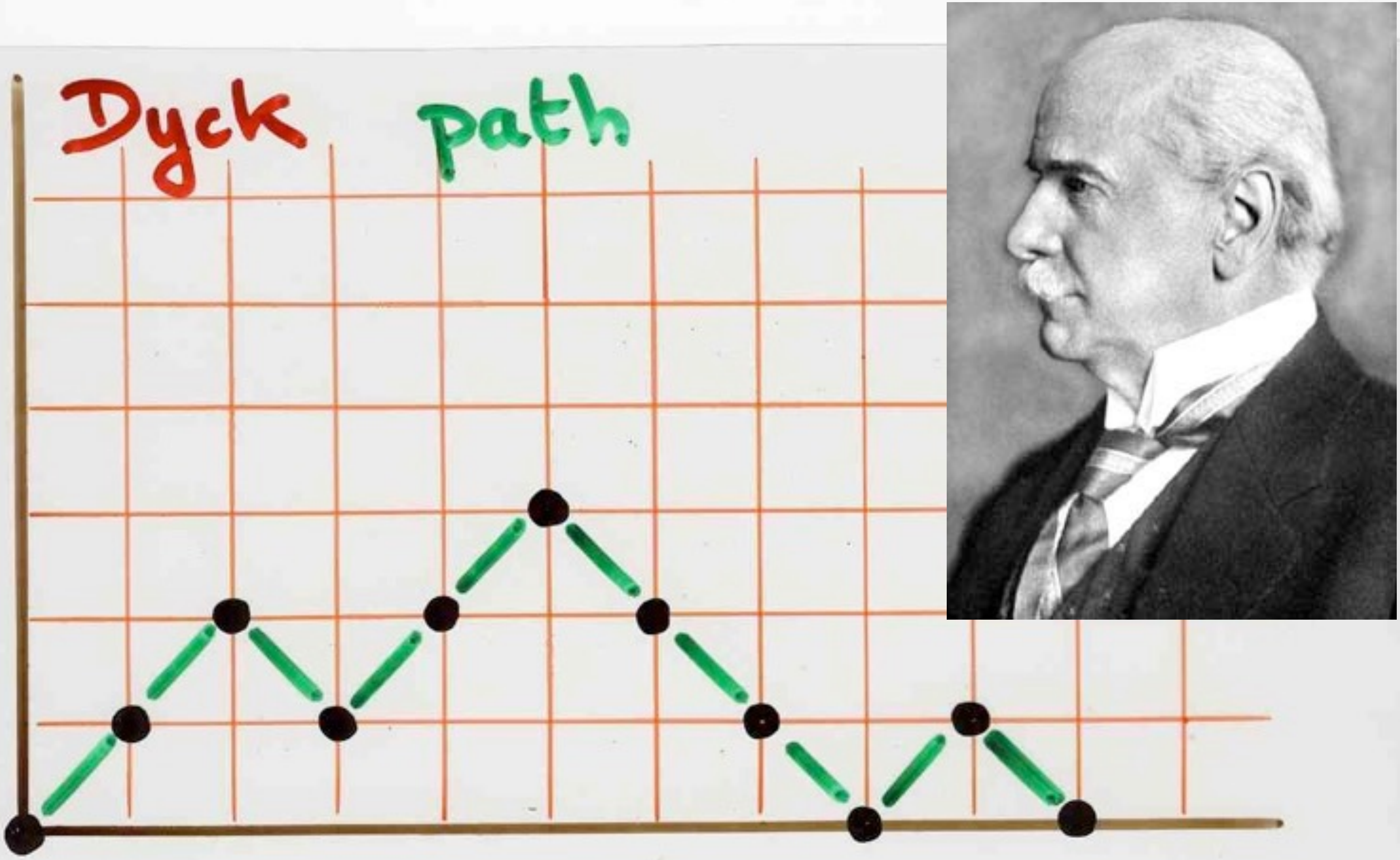

Primitive Motzkin path
t
$\lambda_1 t$
δ : shift of valuation
$b_k \to b_{k+1}$
$\lambda_k \to \lambda_{k+1}$

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\,t^{|\omega|} = \cfrac{1}{1-b_0t-\lambda_1t^2\cfrac{}{1-b_1t-\lambda_2t^2\left(\underset{\nearrow \atop \delta^2}{''}\right)}}$$

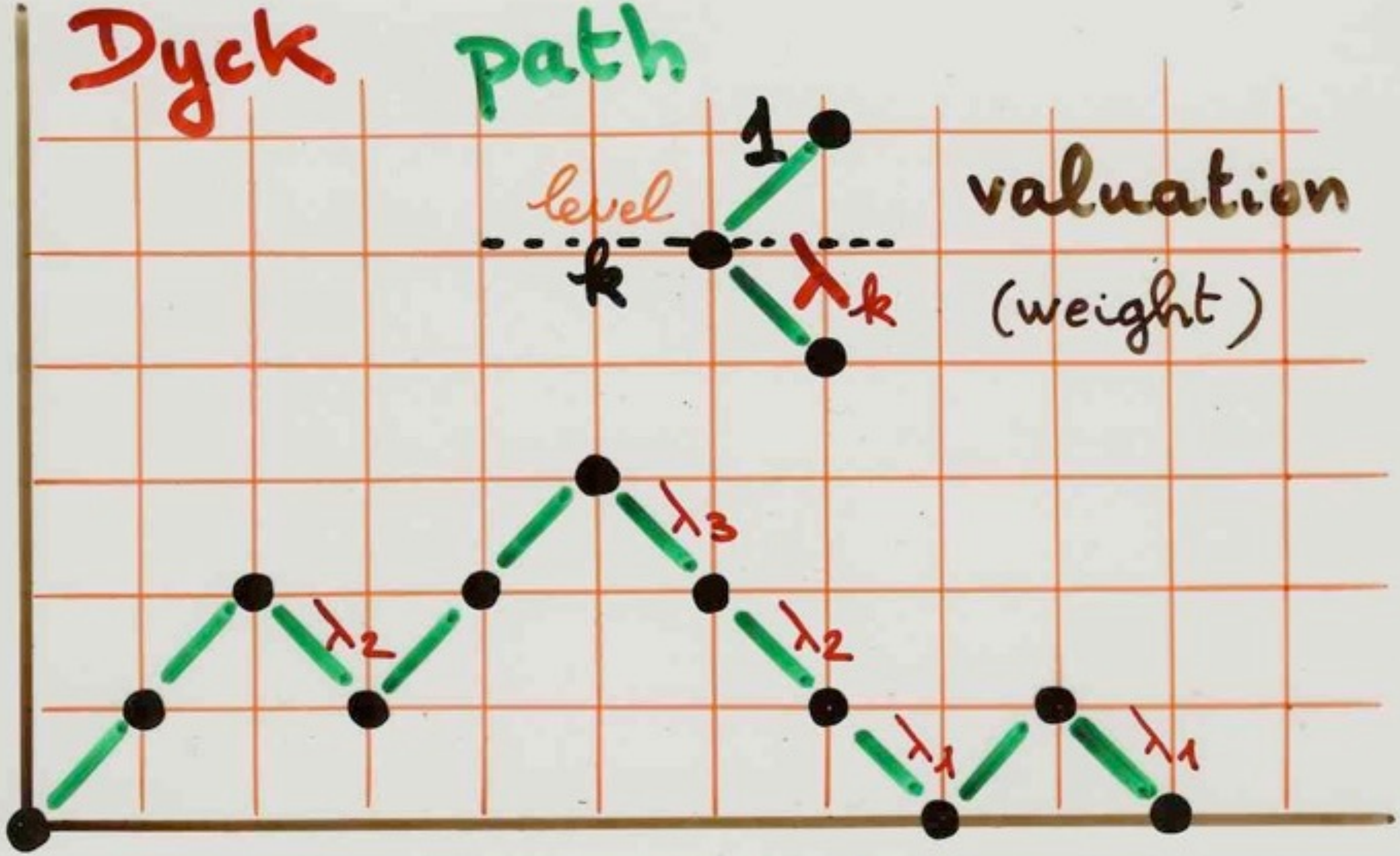
Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots\cdots}}}}}$$

Philippe Flajolet
fundamental
Lemma


Dyck path

Dyck path



weight $v(\omega) = \lambda_1^2 \lambda_2^2 \lambda_3$

continued fractions

$$\sum_{\substack{\omega \\ \text{Dyck} \\ \text{path}}} v(\omega)\, t^{|\omega|/2} = \underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cdot \cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdot}}}}}}_{S(t;\lambda)}$$

Stieljès continued fraction

classical theory

continued fractions

J-fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{\cfrac{\cdots\cdots}{1-b_kt-\cfrac{\lambda_{k+1}t^2}{\cdots}}}}$$

moments

generating function

orthogonal polynomials

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path

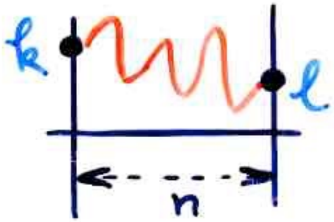
$|\omega| = n$

combinatorial
theory of
orthogonal polynomials

moments X.V. (1983)
Françon, X.V. (1978)

and
continued fractions
Flajolet (1980)

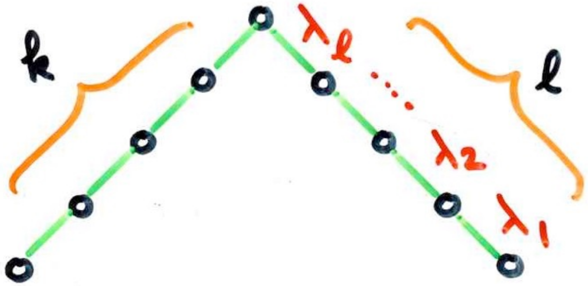
$$f(P_k P_l x^n) = \sum_{\omega} v(\omega) \quad (\lambda_1 \lambda_2 \cdots \lambda_l)$$



Orthogonality

Favard's theorem

$$f(P_k P_l) = 0 \qquad k \neq l$$
$$= \lambda_1 \cdots \lambda_l \qquad k = l$$

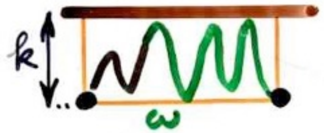


- Ramanujan's theorem (entry 17, ch.12)
- Favard's theorem (orthogonality)
- Convergents of continued fractions

same bijective proof

convergents order k

Prop $$J_k(t) = \sum_{\substack{\omega \\ H(\omega) \leqslant k}} v(\omega)$$



Prop $$J_k(t) = \frac{\delta P_k^*(t)}{P_{k+1}^*(t)}$$

Reciprocal

$$P_k^*(t) = t^k P_k\left(\frac{1}{t}\right)$$

$$\{\delta P_k\}_{k \geqslant 0}$$

orthogonal polynomials
defined by

$$\begin{cases} \lambda_k = \delta\lambda_k = \lambda_{k+1} \\ b_k = \delta b_k = b_{k+1} \end{cases}$$

Laguerre histories
and
moment of Laguerre polynomials

Laguerre
polynomial

$\mu_n = (n+1)!$

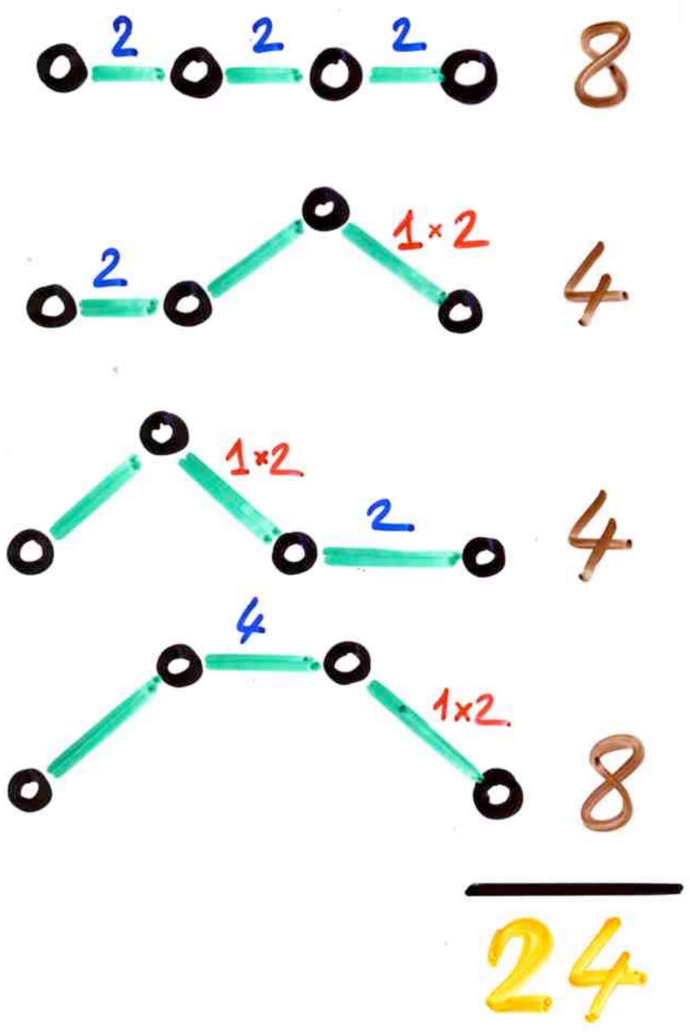
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

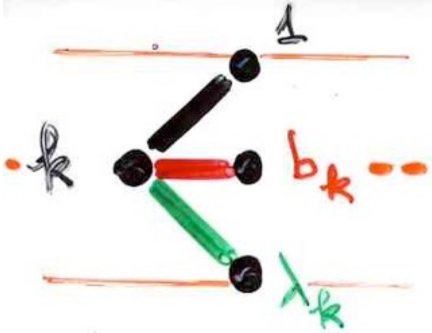
Laguerre $L_n^{(1)}(x)$

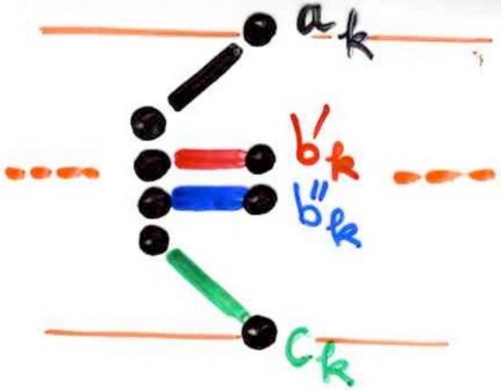
moment $\mu_n = (n+1)!$

$b_k = 2k+2$

$\lambda_k = k(k+1)$

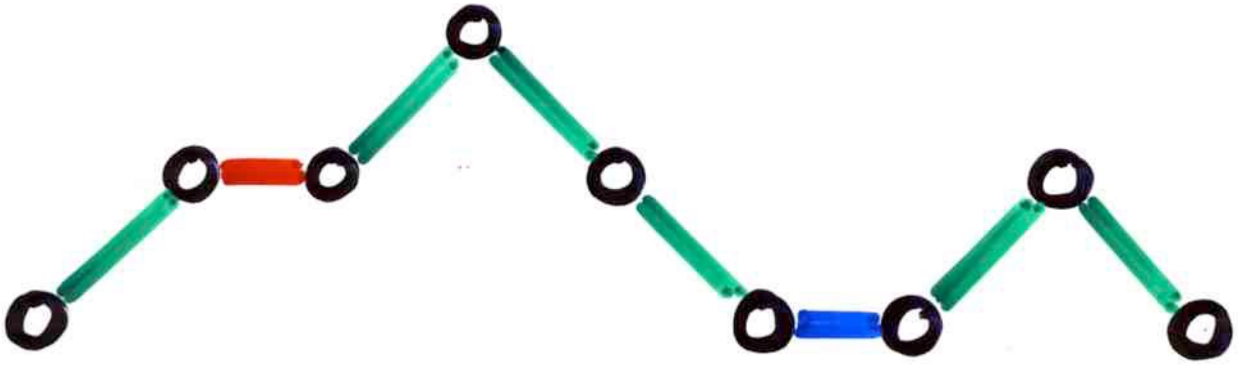


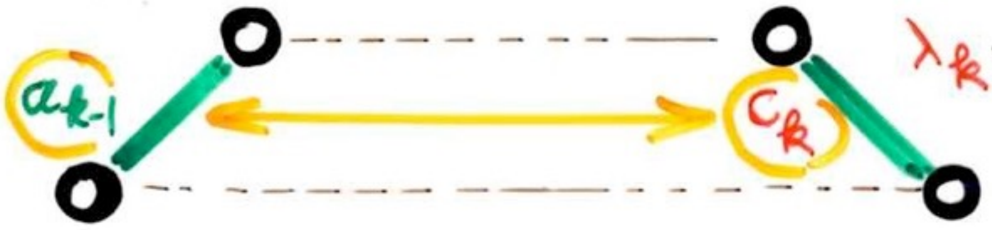




$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$





$\lambda_k = a_{k-1} c_k$

$$(n+1)! = \sum_{\substack{|\omega|=n \\ \text{Motzkin}}} v(\omega)$$

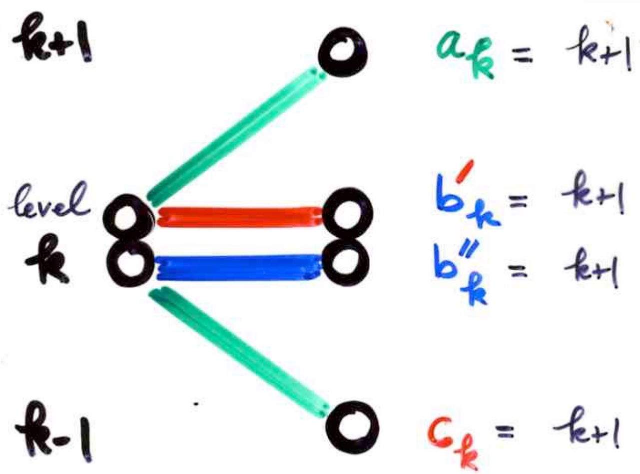
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$= \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin}}} v^{*}(\omega)$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

$$\lambda_k = a_{k-1}\, c_k$$

$$b_k = b'_k + b''_k$$



$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

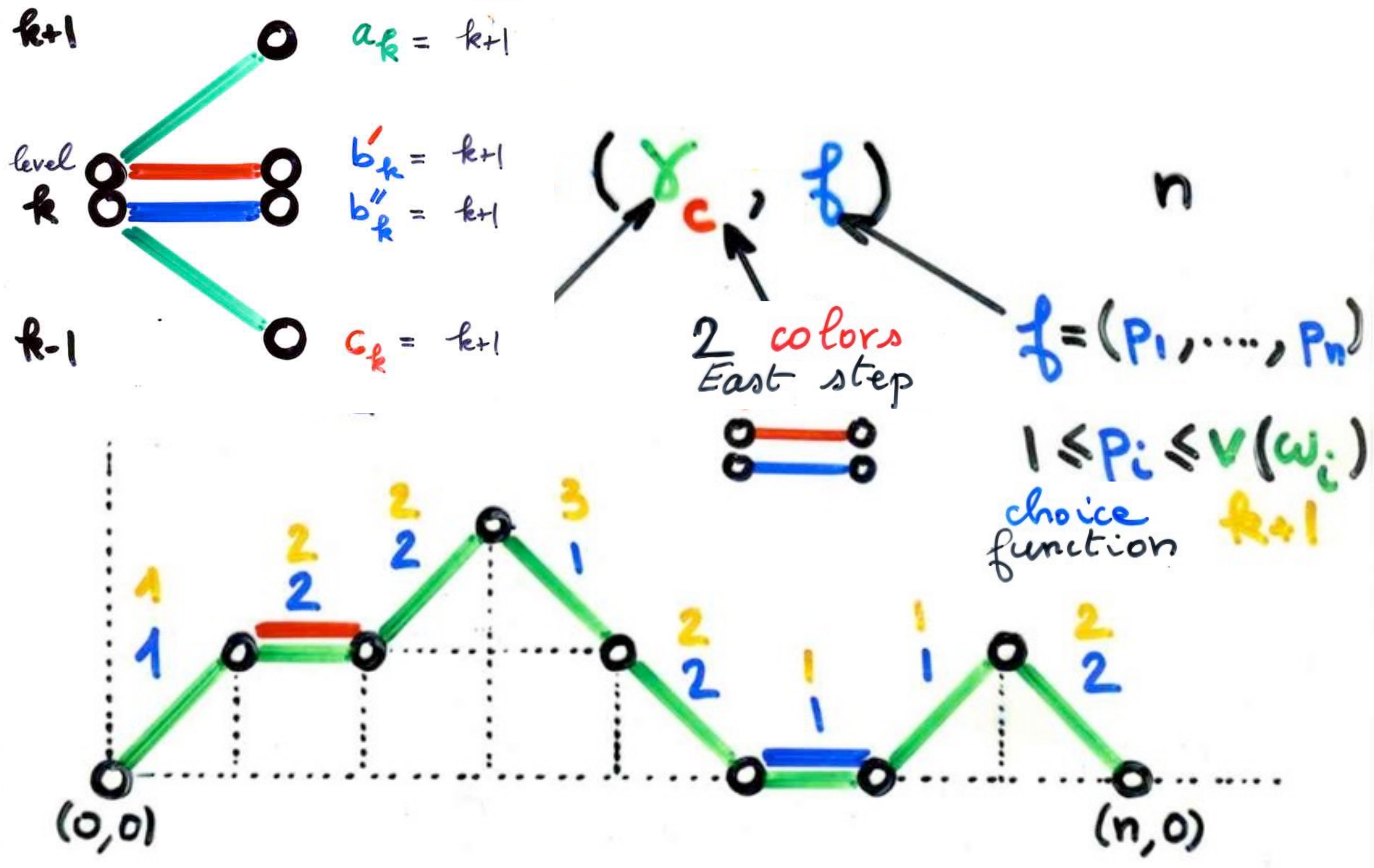
$$J(t) = \cfrac{1}{1 - 2t - \cfrac{1\cdot 2t^2}{1 - 4t - \cfrac{2\cdot 3t^2}{\ldots\ldots}}}$$

$$b_k = 2k+1 \qquad \lambda_k = k^2$$

$$\sum_{n\geqslant 0} n!\, t^n = \cfrac{1}{1 - 1t - \cfrac{1^2 t^2}{1 - 3t - \cfrac{2^2 t^2}{1 - 5t - \cfrac{3^2 t^2}{\ldots\ldots}}}}$$

reminding Laguerre histories

Ch3b (1st part), slides 23-56



$a_k = k+1$

$b'_k = k+1$

$b''_k = k+1$

$c_k = k+1$

(γ_c, f)

n

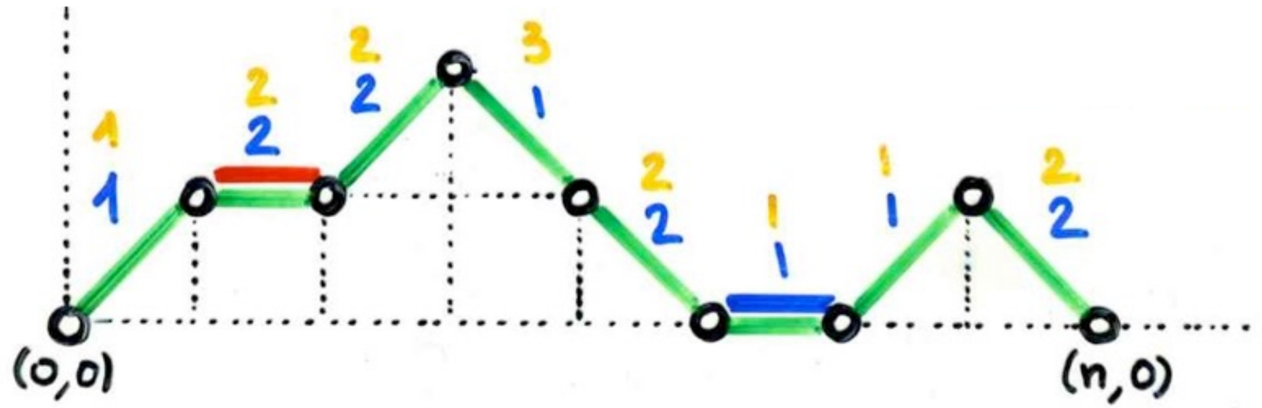
2 colors
East step

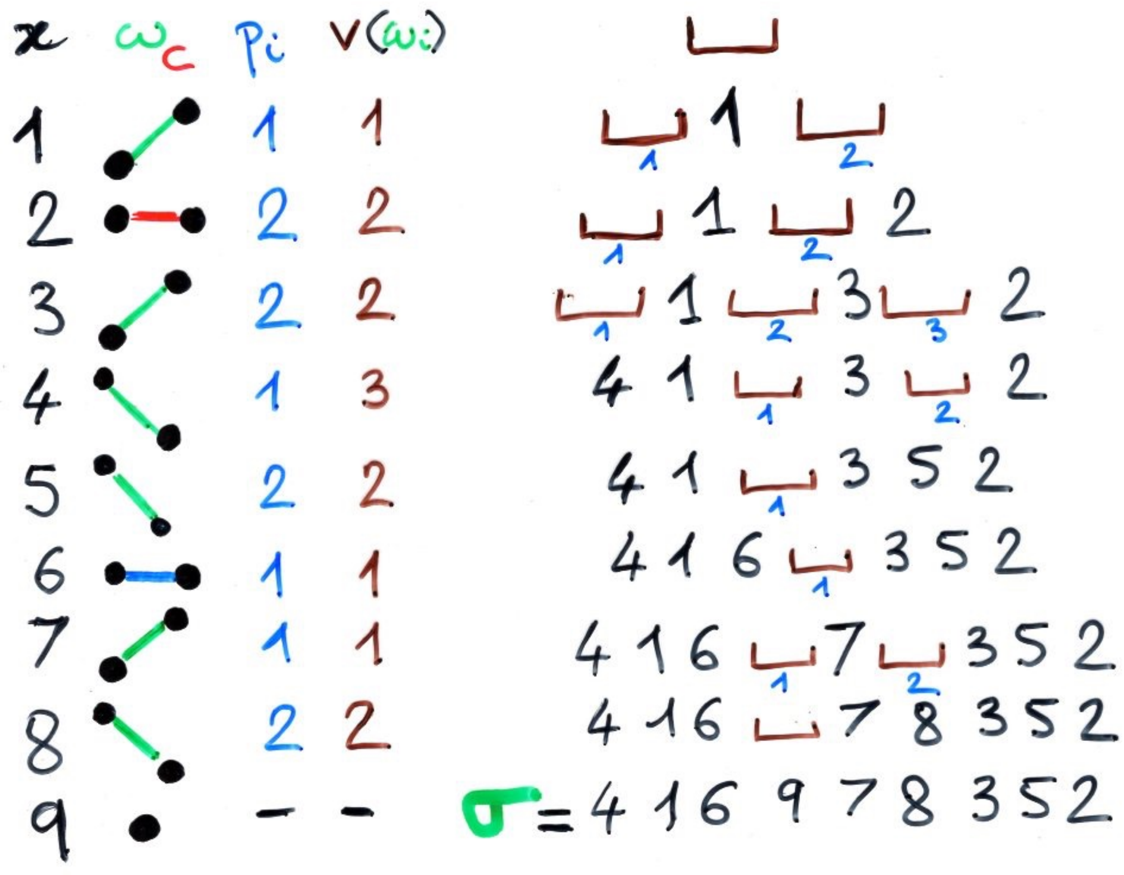
$f = (p_1, \dots, p_n)$

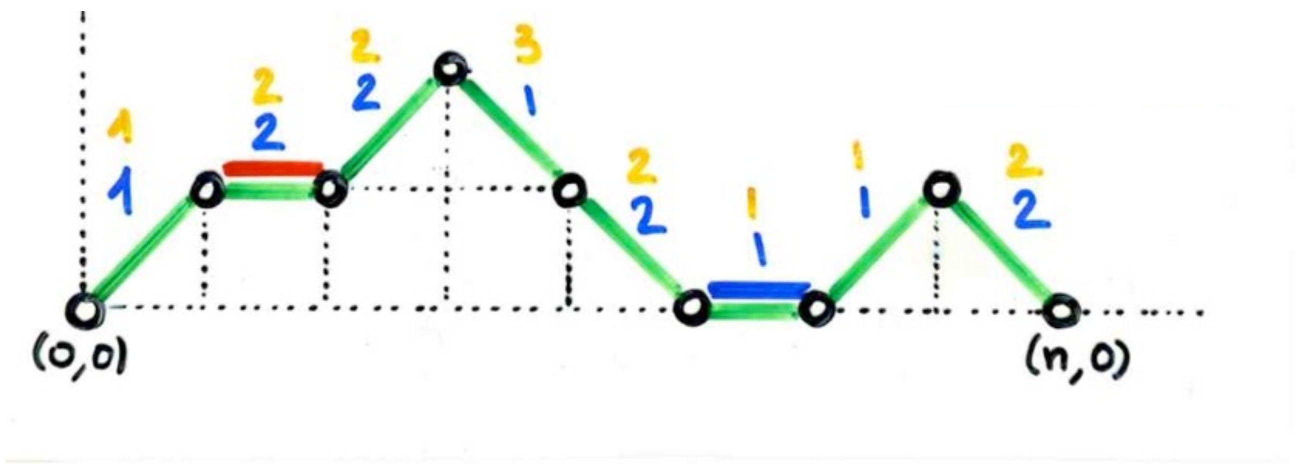
$1 \leq p_i \leq v(\omega_i)$

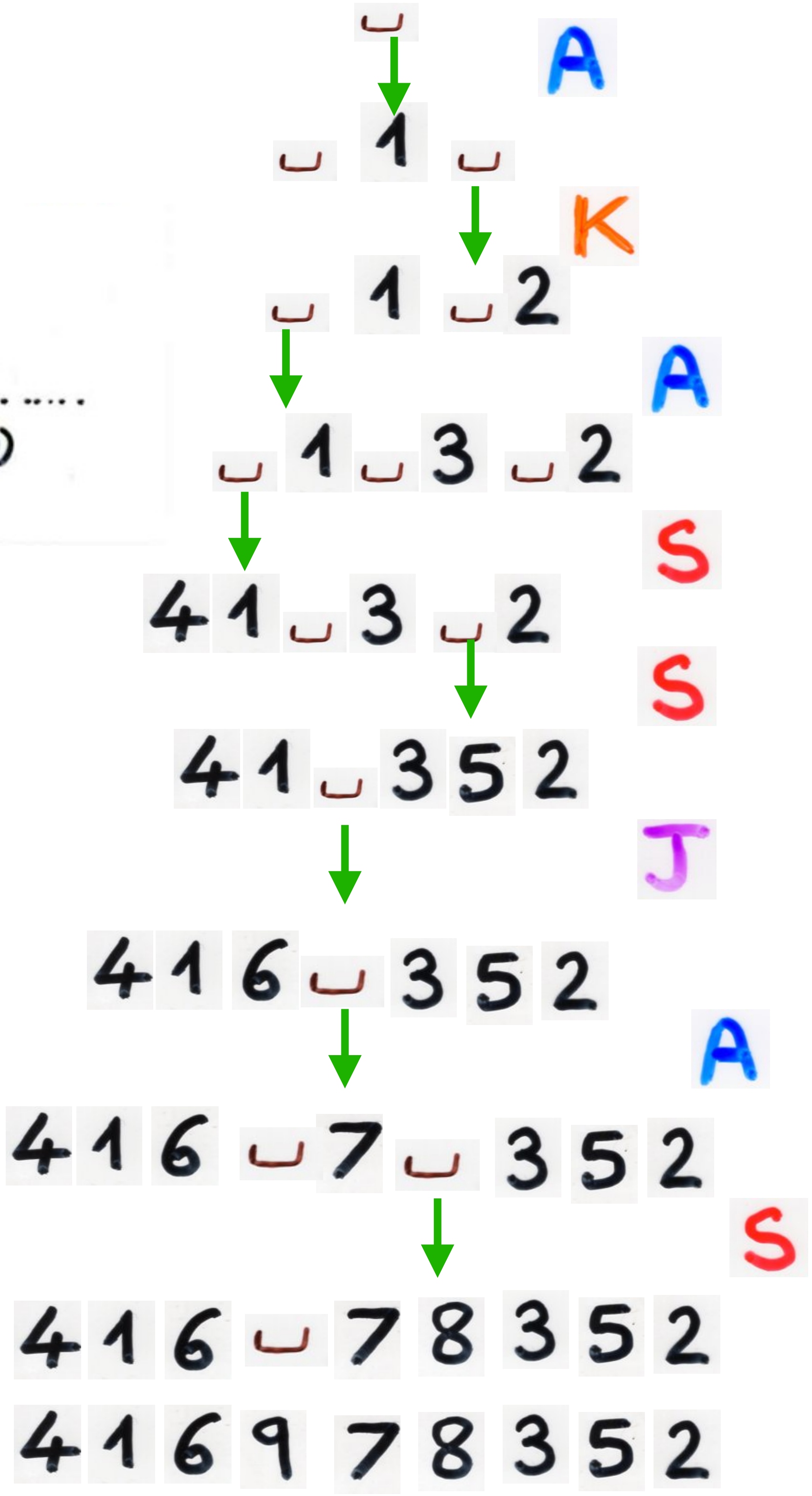
choice
function

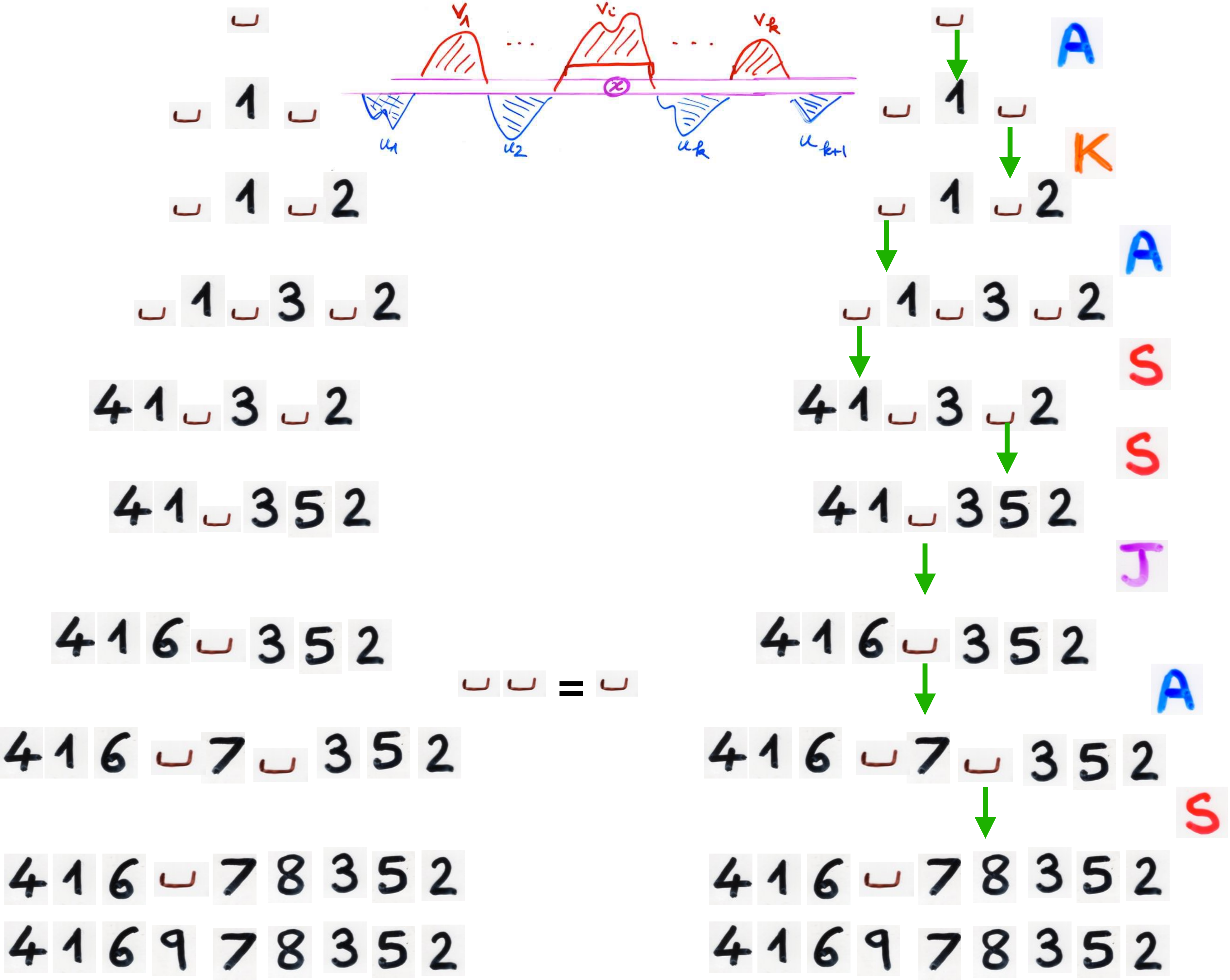
$k+1$


1
1
2
2
2
2
3
1
2
2
1
1
1
1
2
2
(0,0)
(n,0)


x ω_c p_i v(ω_i)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
6 1 1
7 1 1
8 2 2
9 – –
⊔
⊔ 1 ⊔
1 2
⊔ 1 ⊔ 2
1 2
⊔ 1 ⊔ 3 ⊔ 2
1 2 3
4 1 ⊔ 3 ⊔ 2
1 2
4 1 ⊔ 3 5 2
1
4 1 6 ⊔ 3 5 2
1
4 1 6 ⊔ 7 ⊔ 3 5 2
1 2
4 1 6 ⊔ 7 8 3 5 2
σ = 4 1 6 9 7 8 3 5 2


1
1
2
2
2
2
3
1
2
2
1
1
1
1
2
2
(0,0)
(n,0)


␣
A
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2


v_1
…
v_i
…
v_k
x
u_1
u_2
u_k
u_{k+1}
␣
␣ 1 ␣
␣ 1 ␣ 2
␣ 1 ␣ 3 ␣ 2
4 1 ␣ 3 ␣ 2
4 1 ␣ 3 5 2
4 1 6 ␣ 3 5 2
␣ ␣ = ␣
4 1 6 ␣ 7 ␣ 3 5 2
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2
A
K
A
S
S
J
A
S

weigthed Laguerre histories

Laguerre $L_n^{(\alpha)}$

$$b_k = 2k+\alpha+1 \quad ; \quad \lambda_k = k(k+\alpha)$$

$$(n+1)! = \sum_{\substack{|\omega|=n \\ \text{Motzkin}}} v(\omega) \qquad \begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$= \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin}}} v^{*}(\omega) \qquad \begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

$$\lambda_k = a_{k-1}\, c_k$$

$$b_k = b'_k + b''_k$$

Laguerre $L_n^{(\alpha)}$

$$b_k = 2k+\alpha+1 \quad ; \quad \lambda_k = k(k+\alpha)$$

$a_k = k+1$ $(k \geq 0)$

$\begin{cases} b'_k = k+\alpha \\ b''_k = k+1 \end{cases}$ $(k \geq 0)$;

$c_k = k+\alpha$ $(k \geq 1)$

$$\lambda_k = a_{k-1}\, c_k$$

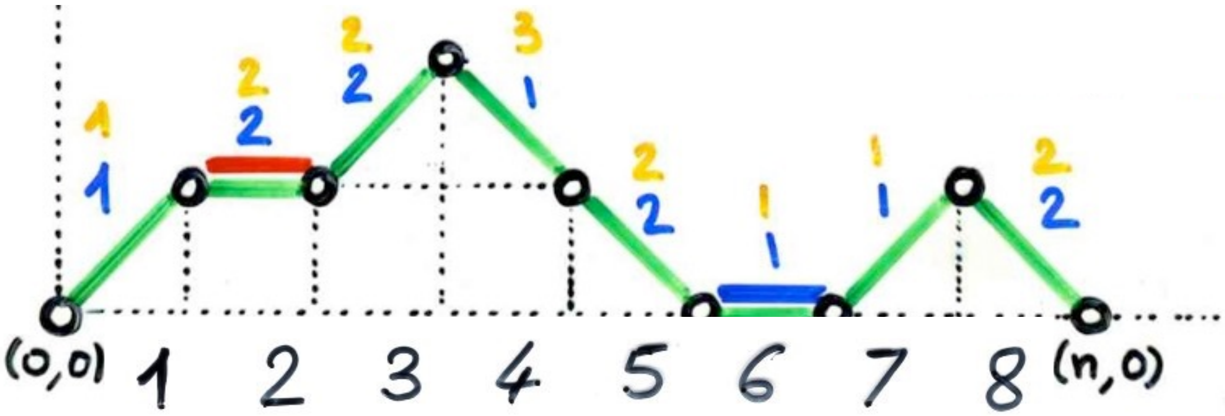
$$b_k = b'_k + b''_k$$

Laguerre polynomial $L_n^{(\alpha)}(x)$

$$h = (\omega_c ; (p_1, \dots, p_n))$$

$$\omega_c = \omega_1 \cdots \omega_n$$

weighted Laguerre histories



put a weight α for each choice $p_i = 1$
with $\omega_i = \begin{cases} \text{blue East step} \\ \text{or South-East step} \end{cases}$  

this is equivalent to say that
the element i is a lr-max element
of the permutation σ (except $i=n+1$)

Laguerre $L_n^{(\alpha)}$

$b_k = 2k+\alpha+1$; $\lambda_k = k(k+\alpha)$

 $a_k = k+1$  $\begin{cases} b'_k = k+\alpha \\ b''_k = k+1 \end{cases}$; $c_k = k+\alpha$ 

$(k\geq 0)$ $(k\geq 0)$ $(k\geq 1)$

put a weight α for each choice $p_i = 1$

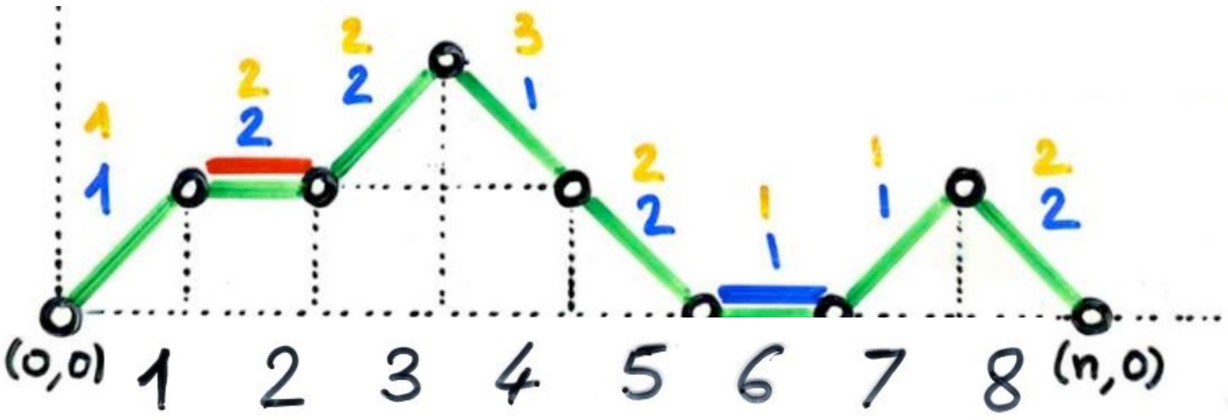
with $\omega_i = \begin{cases} \text{blue East step } ●\!—\!● \\ \text{or South-East step } ╲ \end{cases}$

this is equivalent to say that the element i is a lr-max element of the permutation σ (except $i = n+1$)

example $$\sigma = \underset{\uparrow}{\textcircled{4}}\ 1\ \underset{\nearrow}{\textcircled{6}}\ \textcircled{9}\ 7\ 8\ 3\ 5\ 2$$

lr-max

$$\begin{cases} \omega_4 = ╲, & p_4 = 1 \\ \omega_6 = ●\!—\!●, & p_6 = 1 \end{cases}$$



Corollary The moments of the Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n\geq 0}$ are :

$$\mu_n = (\alpha+1)(\alpha+2)\ldots(\alpha+n)$$

restricted
Laguerre
histories

$$\mu_n = n!$$

restricted
Laguerre
histories

$\mu_n = n!$

$b_k = 2k+1$ $\lambda_k = k^2$

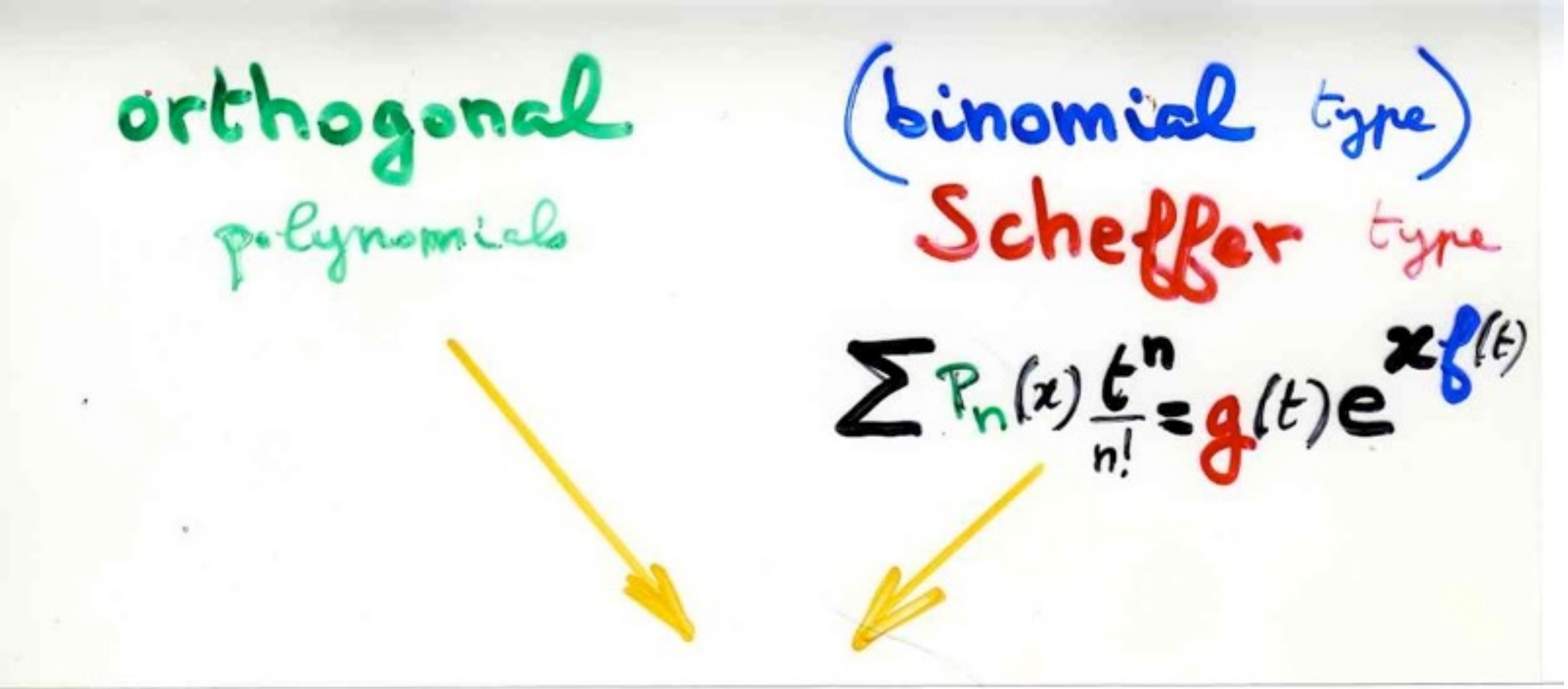
$\sigma(1) = (n+1)$

	operator
$a_k = k+1$	A
$b'_k = k+1$	K
$b''_k = k$	J
$c_k = k$	S

restricted Laguerre histories

$$\sum_{n\geqslant 0} n!\, t^n = \cfrac{1}{1-1t-\cfrac{1^2t^2}{1-3t-\cfrac{2^2t^2}{1-5t-\cfrac{3^2t^2}{\cdots\cdots,}}}}$$

Sheffer orthogonal polynomials



orthogonal
polynomials

(binomial type)
Scheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

orthogonal
polynomials

(**binomial** type)
Scheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

Hermite	H_n
Laguerre	$L_n^{(\alpha)}$
Charlier	$C_n^{(a)}$
Meixner I	$M_n^{I\,(\alpha,)}$
Meixner II	$M_n^{II\,(\delta,\eta)}$

Charlier histories

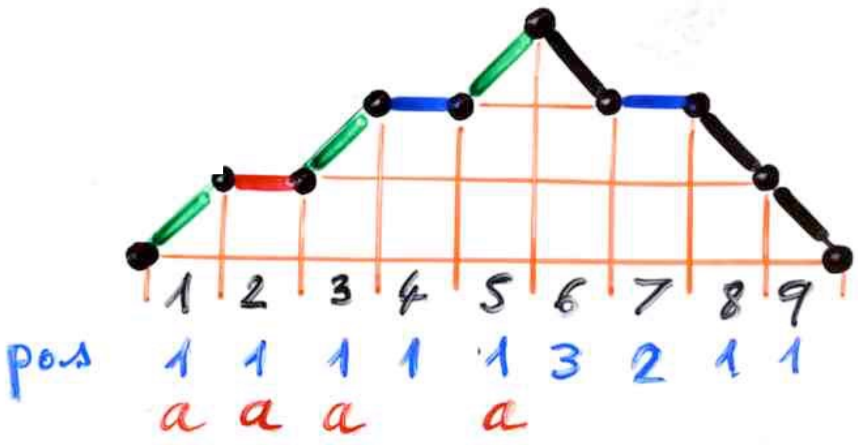
Charlier polynomials

$$\begin{cases} \lambda_k = ak & (k \geqslant 1) \\ b_k = a + k & (k \geqslant 0) \end{cases}$$

$$\mu_n = \sum_{1 \leqslant k \leqslant n} S(n,k)\, a^k$$

moments

Stirling
numbers
(2nd kind)



Charlier histories

$$\begin{cases} \lambda_k = ak & (k \geq 1) \\ b_k = a + k & (k \geq 0) \end{cases}$$

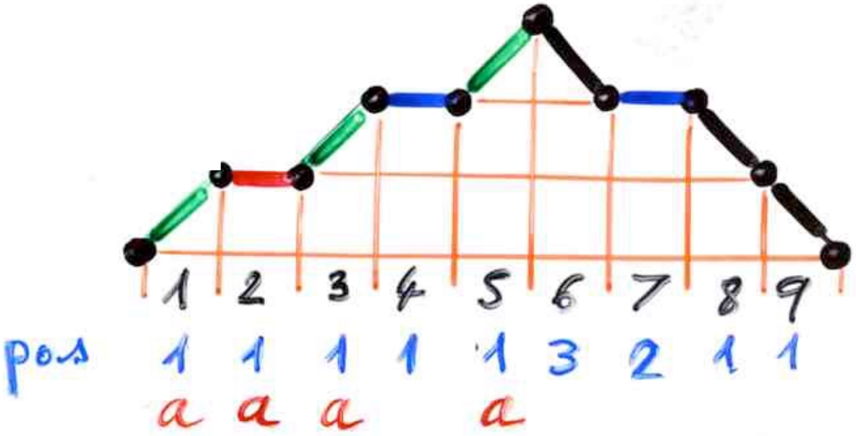
$$a_k = a$$

$$\begin{cases} b'_k = k \\ b''_k = a \end{cases}$$

$(k \geq 0)$

$$c_k = k$$

$(k \geq 1)$



Charlier histories

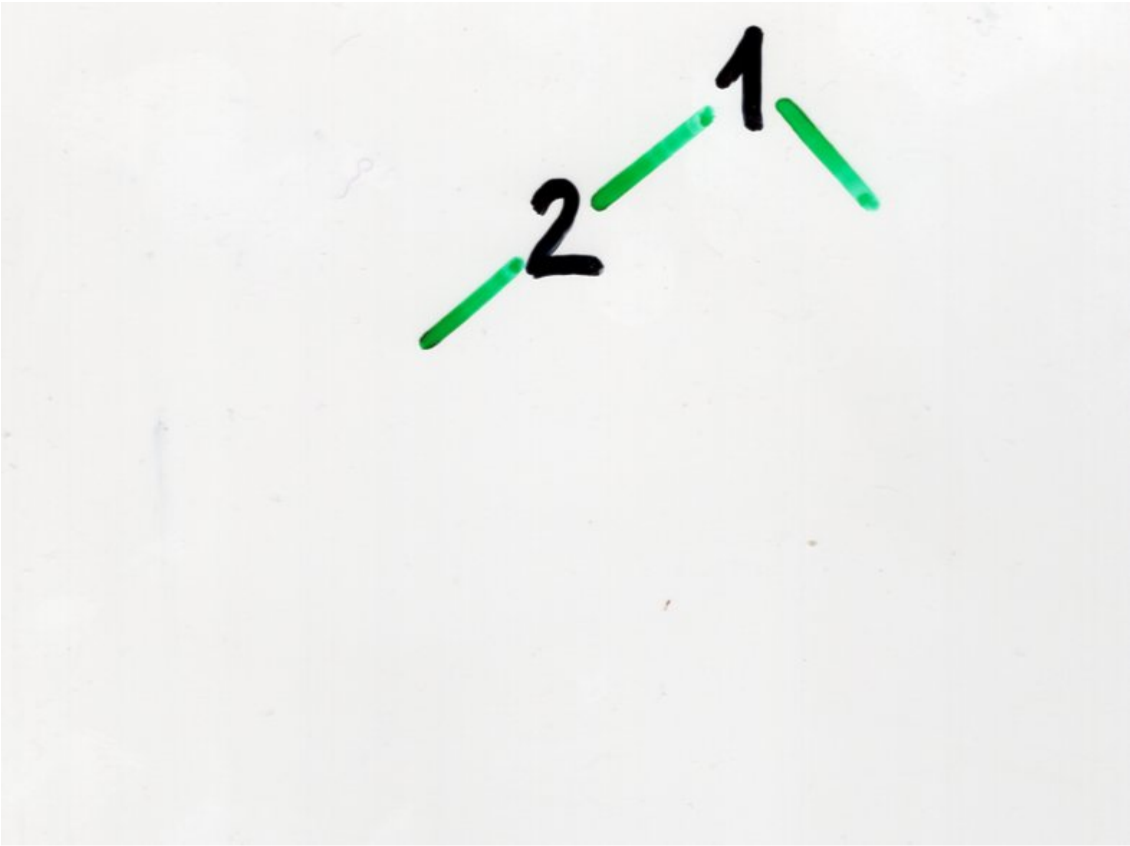
$$a_k = a$$

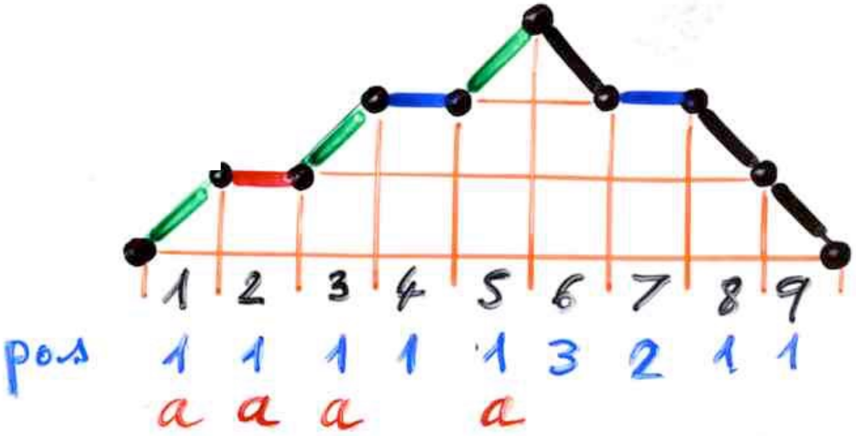
$$\begin{cases} b'_k = k \\ b''_k = a \end{cases}$$

$(k \geqslant 0)$

$$c_k = k$$

$(k \geqslant 1)$





Charlier histories

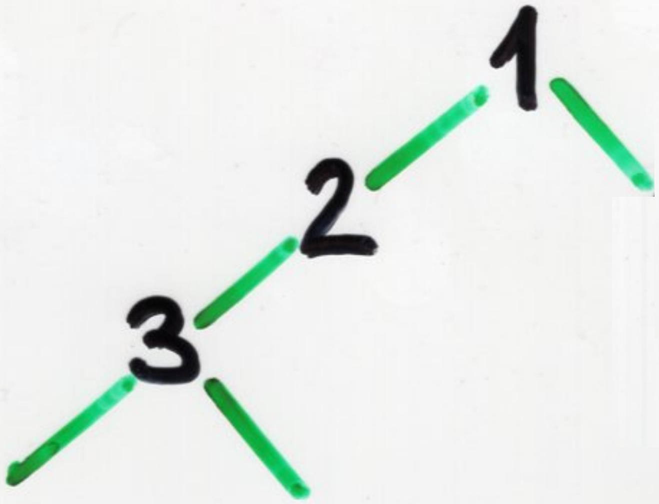
$$a_k = a$$

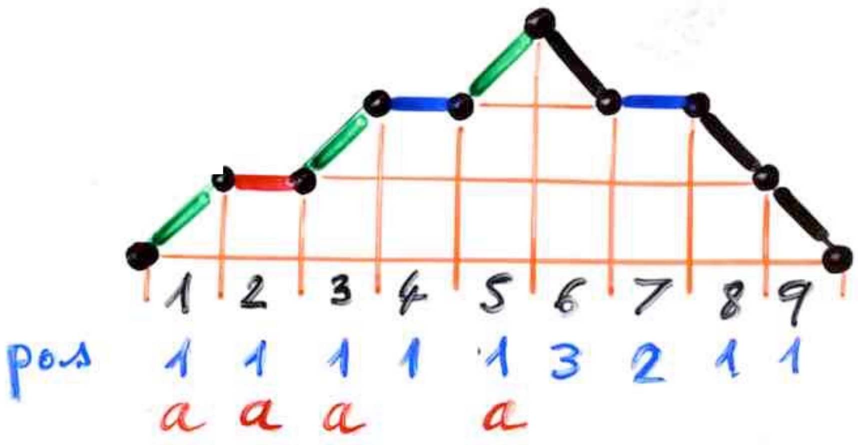
$$\begin{cases} b'_k = k \\ b''_k = a \end{cases}$$

$(k \geqslant 0)$

$$c_k = k$$

$(k \geqslant 1)$





Charlier histories

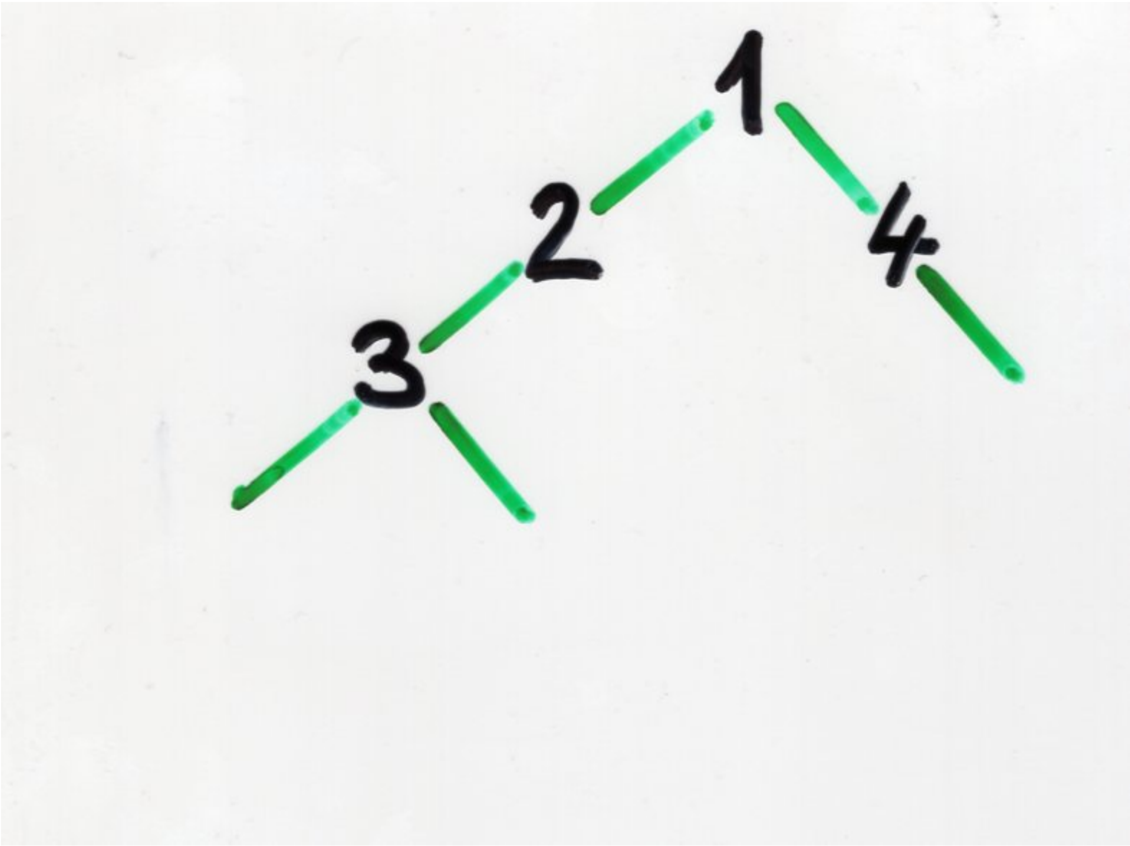
$$\left\{\begin{array}{l} b'_k = k \\ b''_k = a \end{array}\right.$$

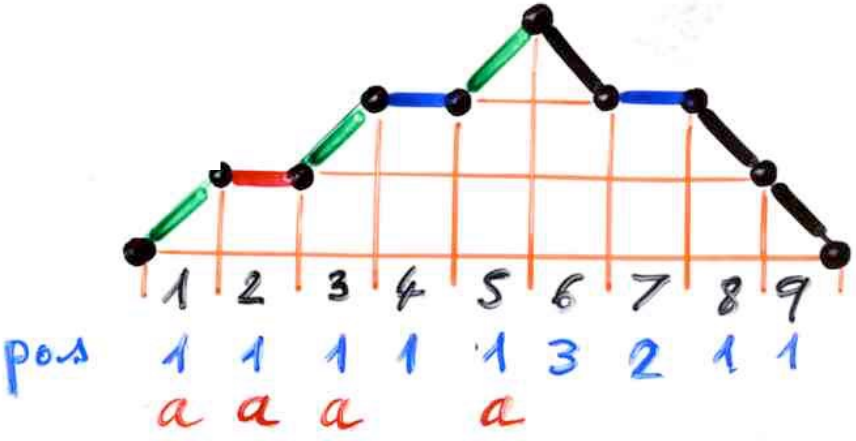
$a_k = a$

$(k \geq 0)$

$c_k = k$

$(k \geq 1)$





Charlier histories

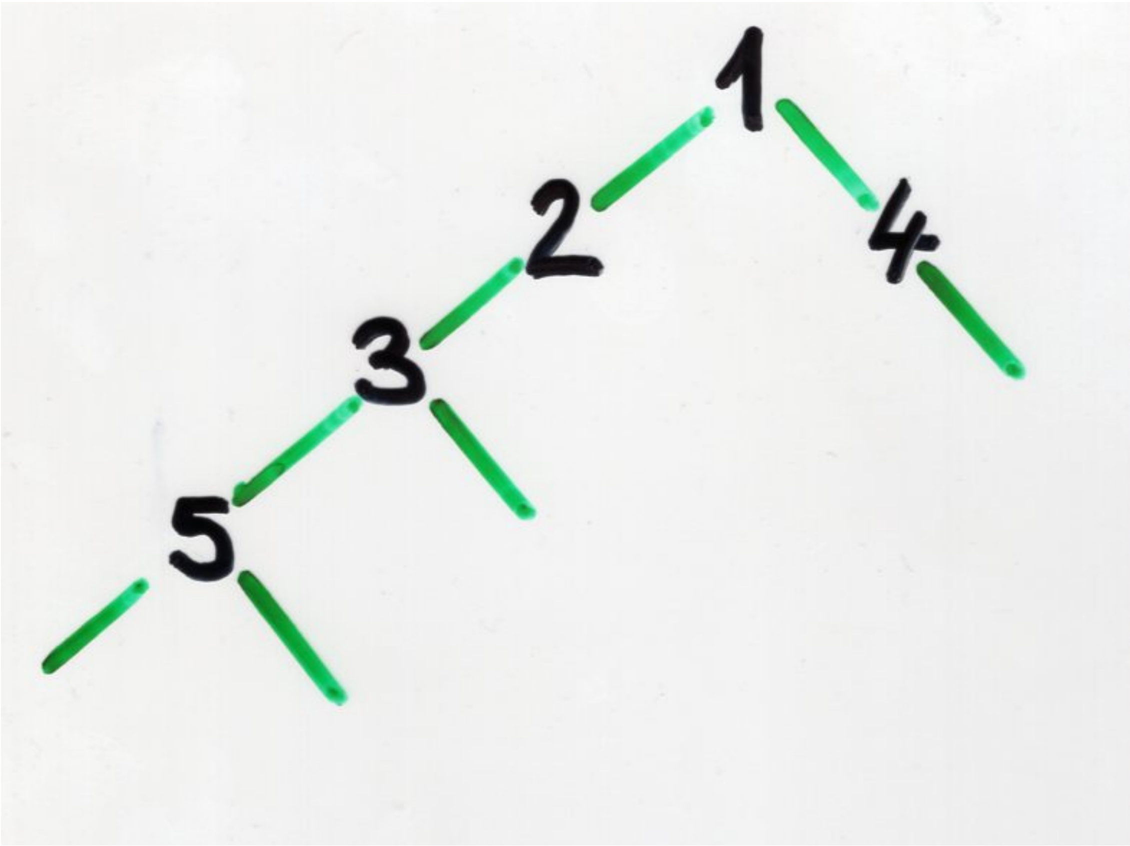
$a_k = a$

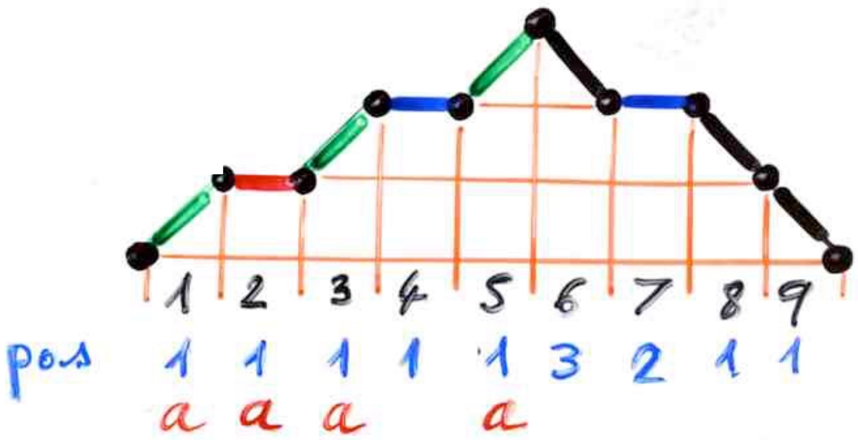
$$\left\{\begin{array}{l} b'_k = k \\ b''_k = a \end{array}\right.$$

$(k \geq 0)$

$c_k = k$

$(k \geq 1)$





Charlier histories

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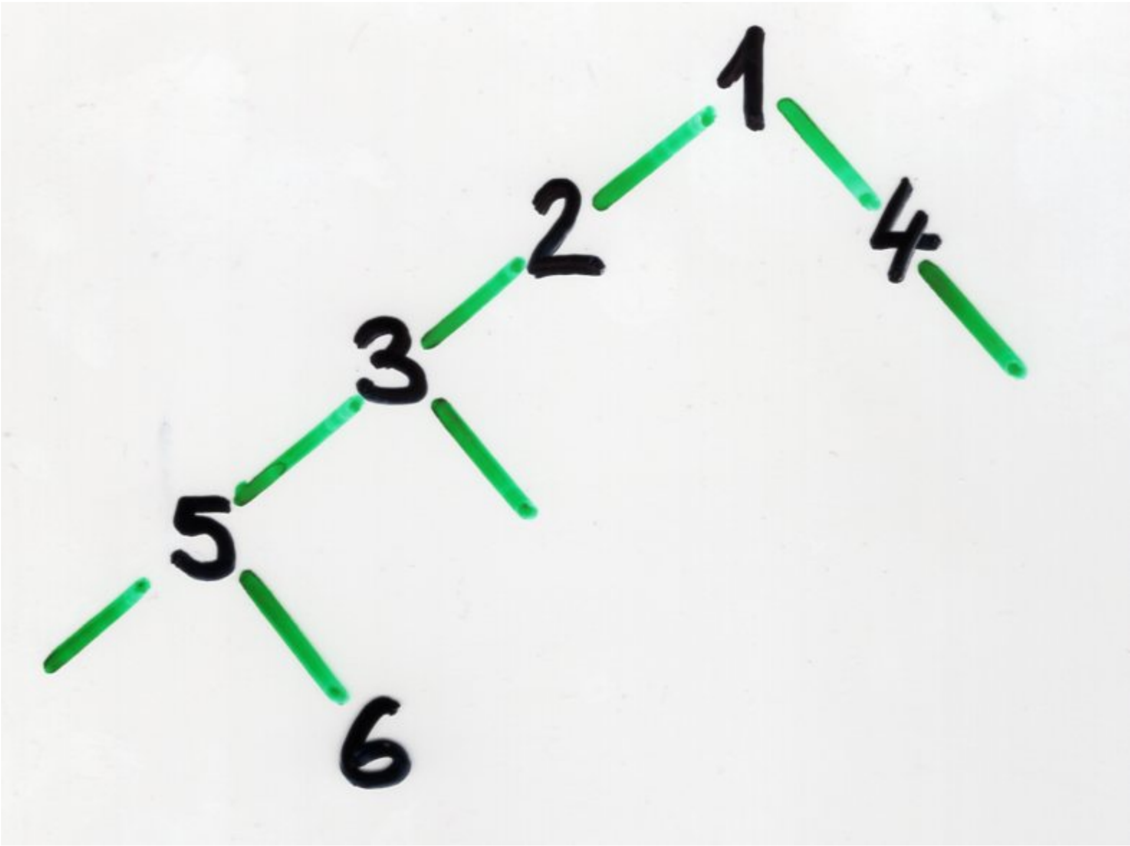
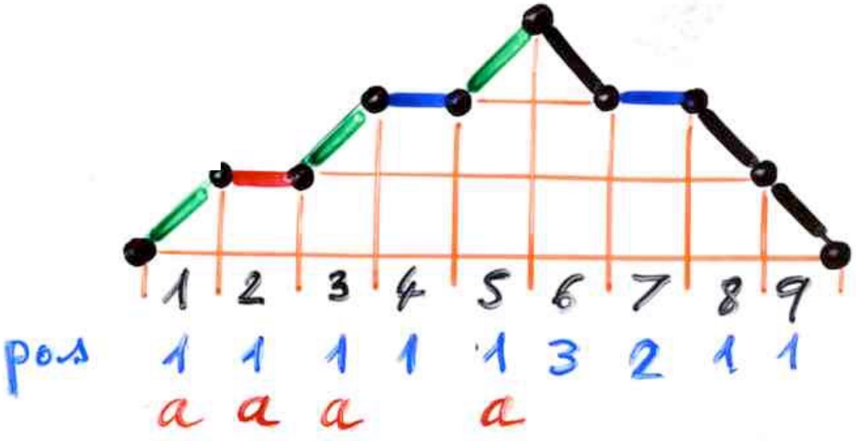
$(k \geq 1)$





Charlier histories

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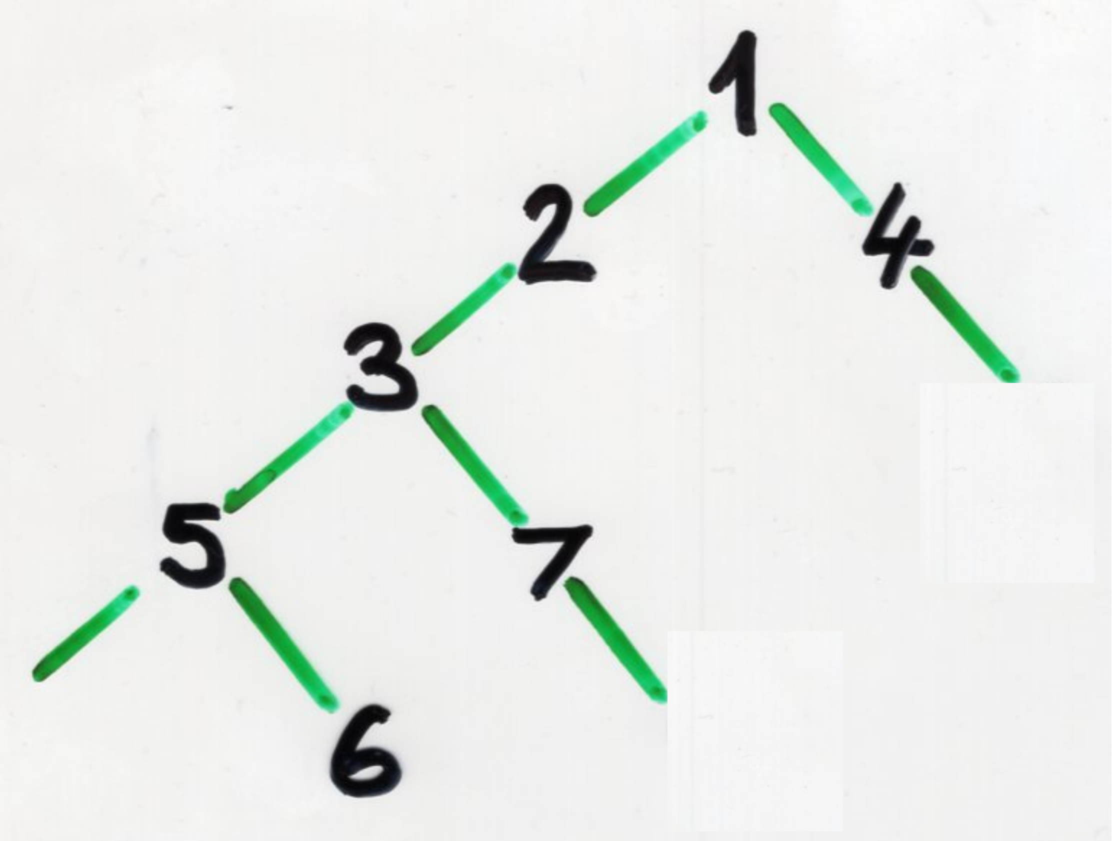
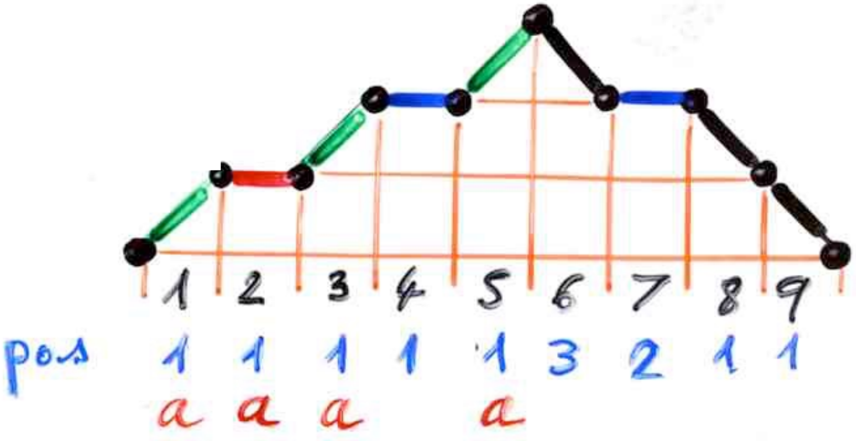
$(k \geq 1)$





Charlier histories

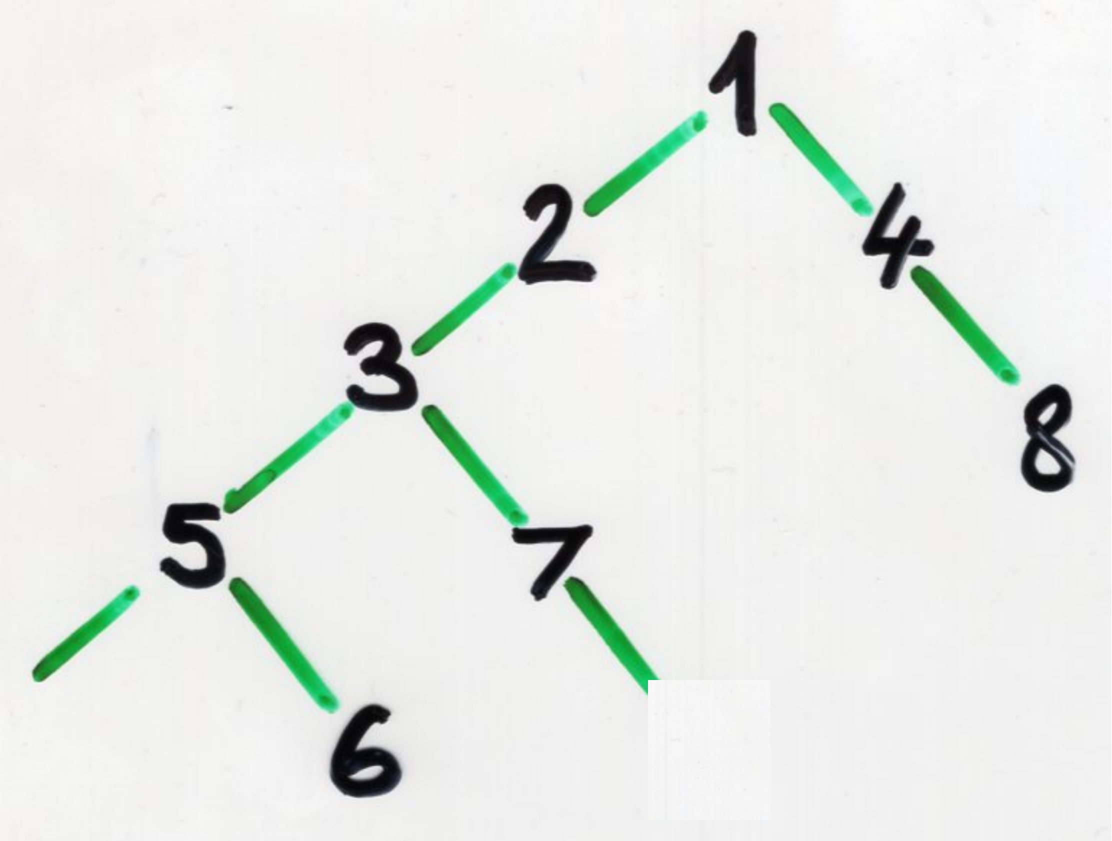
$a_k = a$

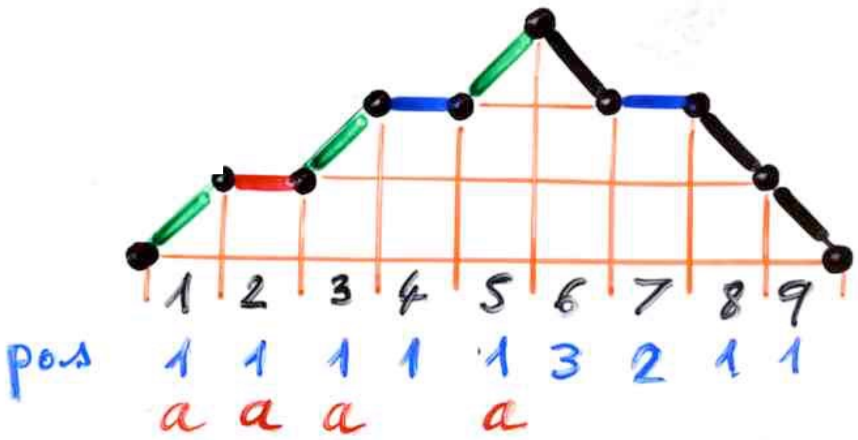
$$\begin{cases} b'_k = k \\ b''_k = a \end{cases}$$

$(k \geqslant 0)$

$c_k = k$

$(k \geqslant 1)$





Charlier histories

$$\underset{\text{moments}}{\mu_n} = \sum_{1\leq k\leq n} S(n,k)\, a^k$$

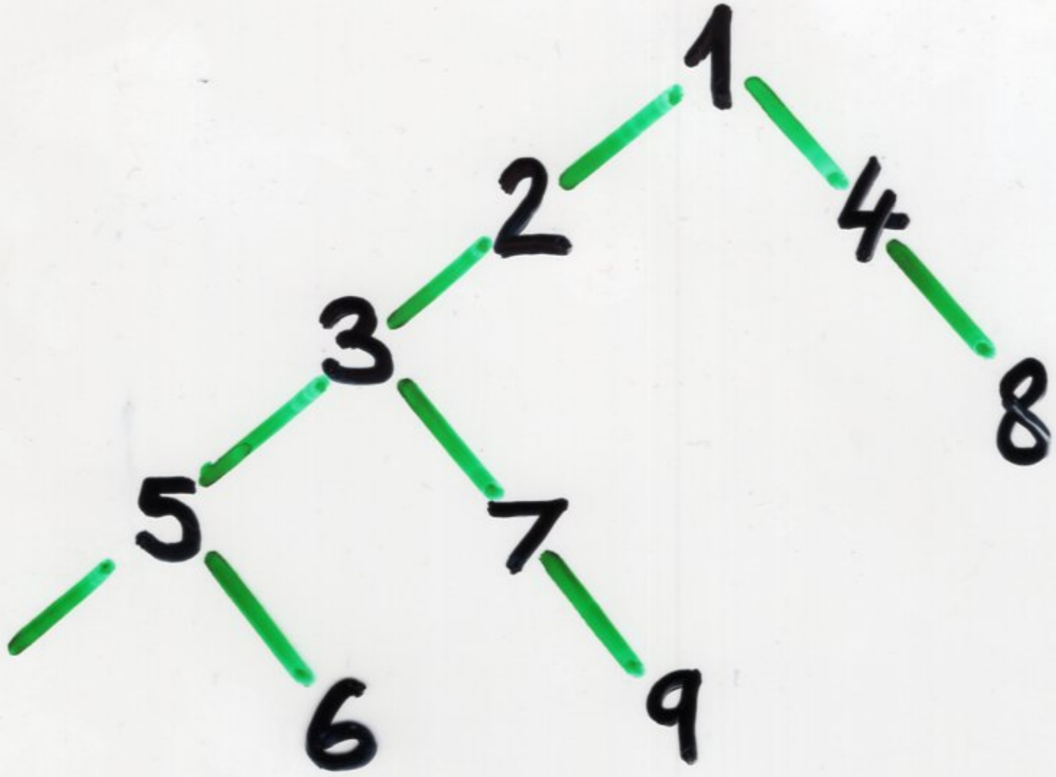
$a_k = a$

$$\begin{cases} b'_k = k \\ b''_k = a \end{cases}$$

$(k\geq 0)$

$c_k = k$

$(k\geq 1)$

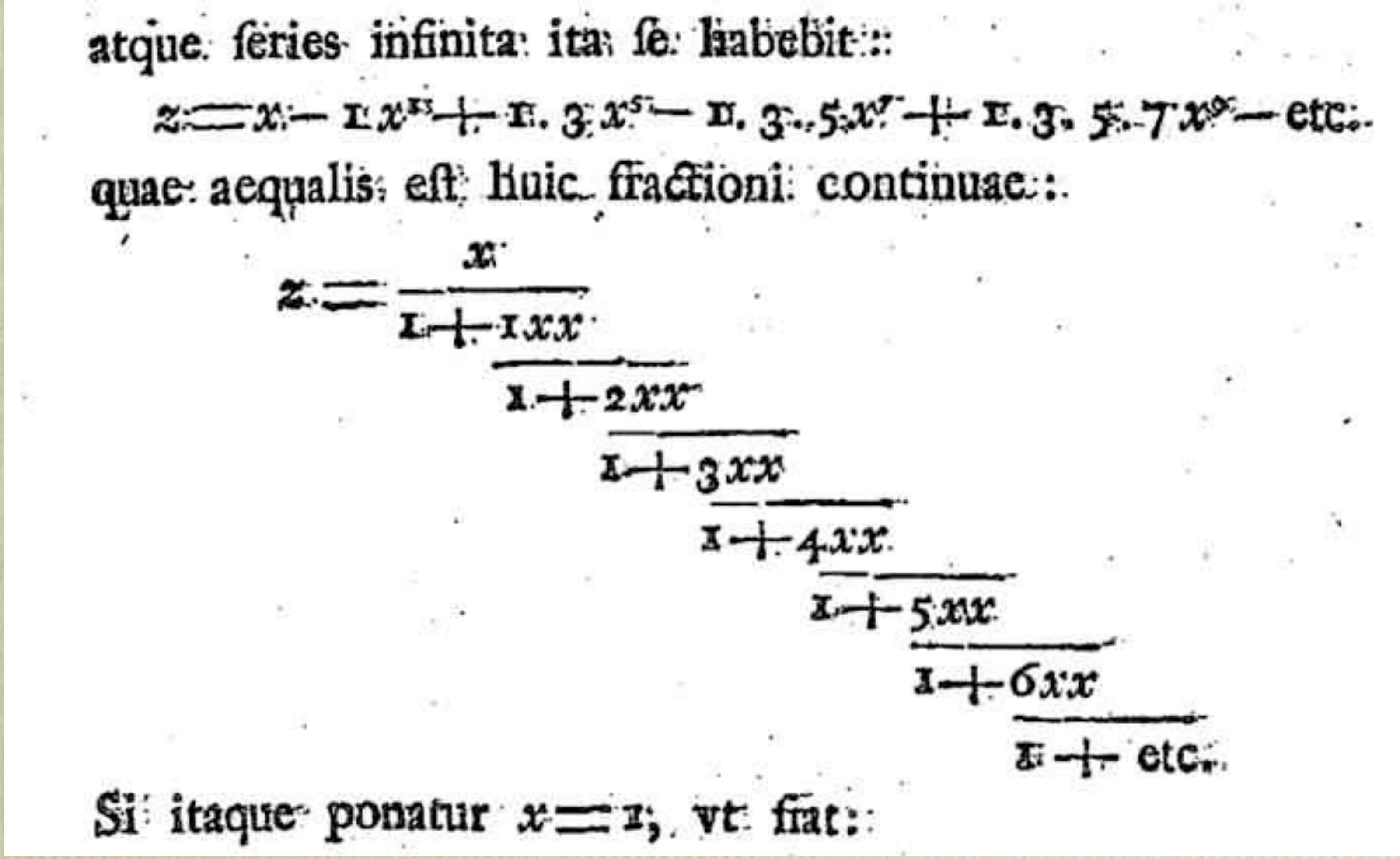


Hermite histories

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

Hermite polynomials

$$\cfrac{1}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{3t}{\cdots\cdots}}}}$$



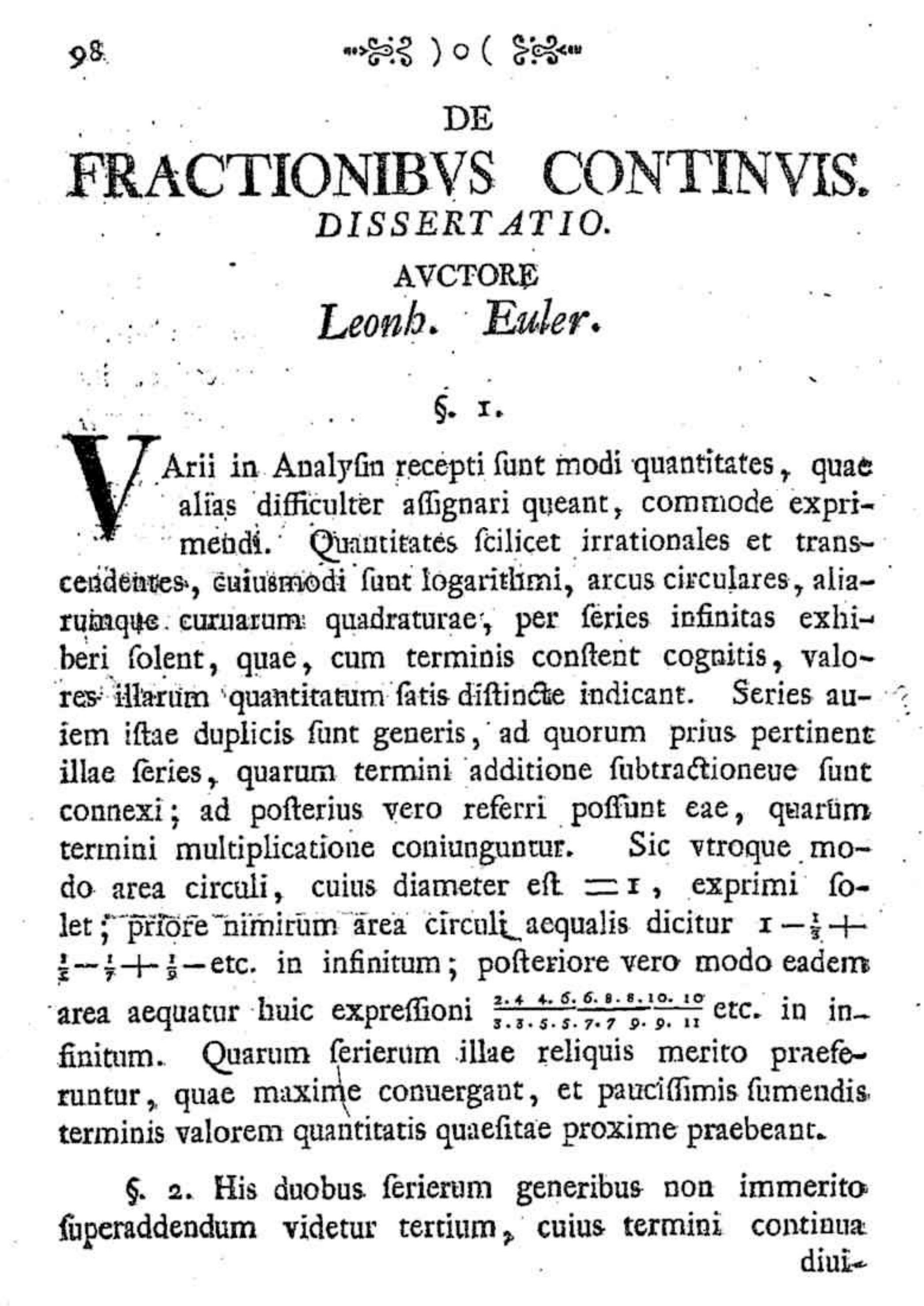
atque ſeries infinita ita ſe habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis eſt huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:



98)o(

DE FRACTIONIBVS CONTINVIS.

DISSERTATIO.

AVCTORE

Leonh. Euler.

§. 1.

Varii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4\ 4.6.\ 6.8.\ 8.10.\ 10}{3.3.\ 5.5.\ 7.7\ \ 9.\ 9.\ 11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

moments
Hermite
polynomials

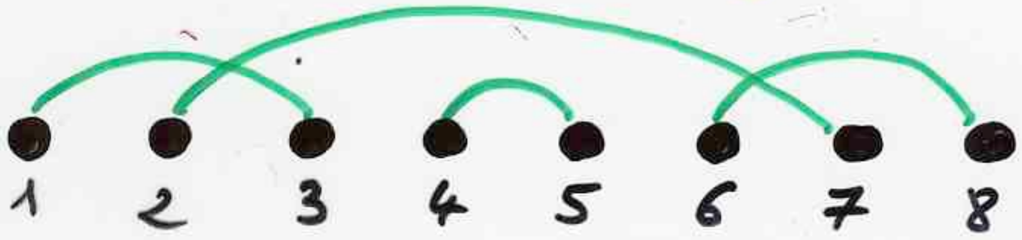
$$\text{Hermite} \begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching



Hermite history

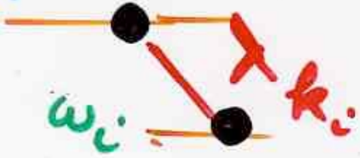
$$h = (\underset{\text{Dyck path}}{\omega} \; ; \; \underset{\text{choice function}}{f})$$

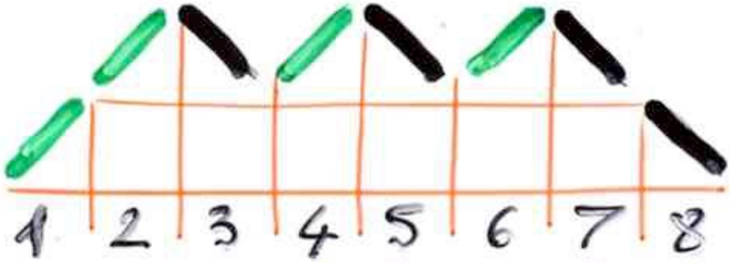
$\omega = \omega_1 \ldots \omega_{2n}$

$f = (p_1, \ldots, p_{2n})$

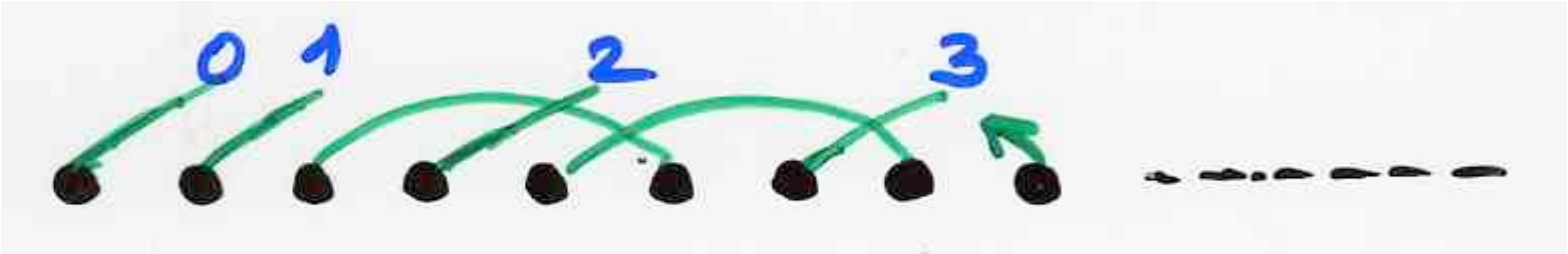
$p_i = 1$

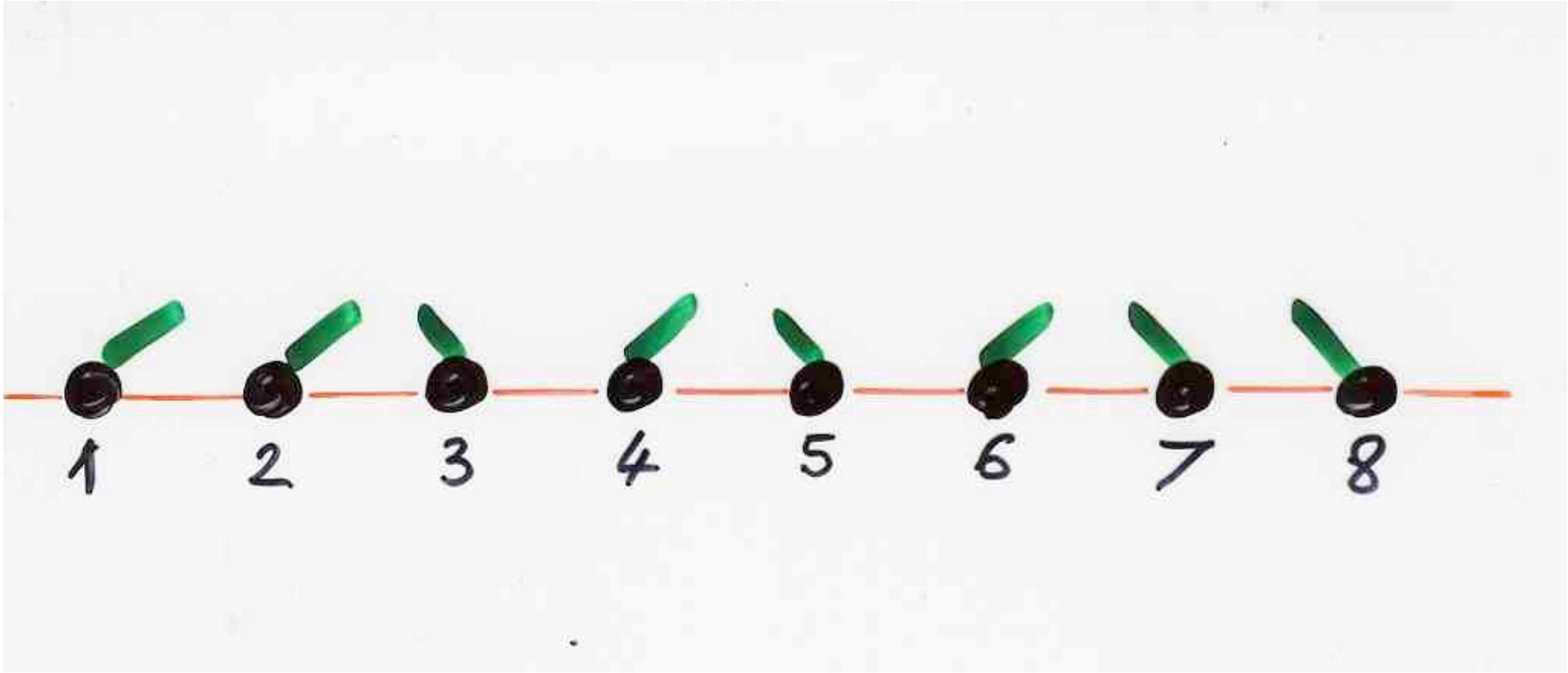
$1 \le p_i \le v(\omega_i) = \lambda_{k_i}$

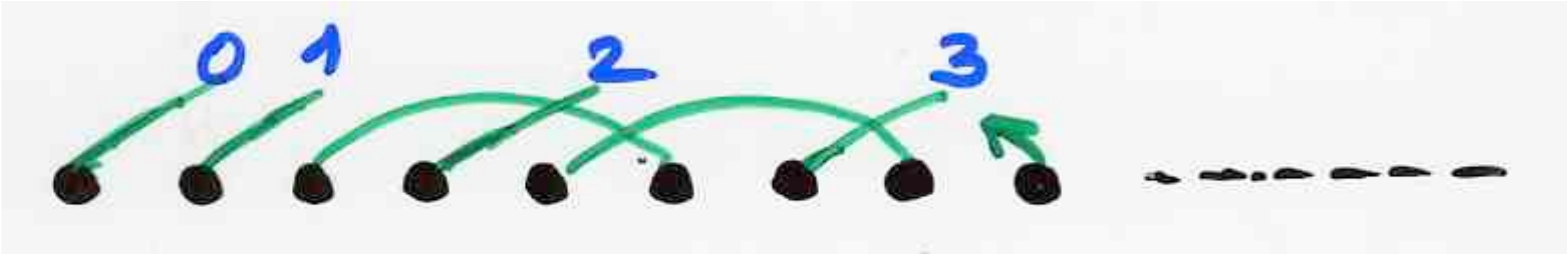


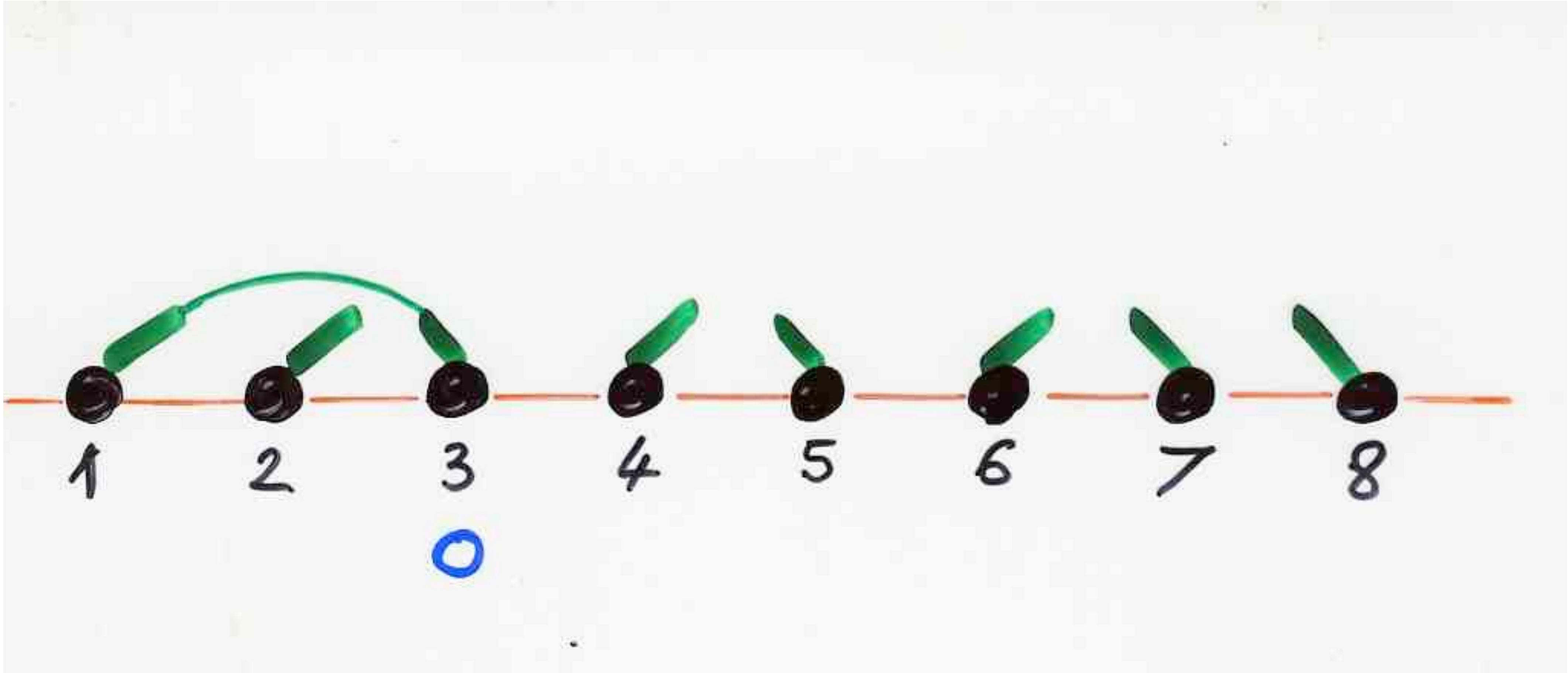


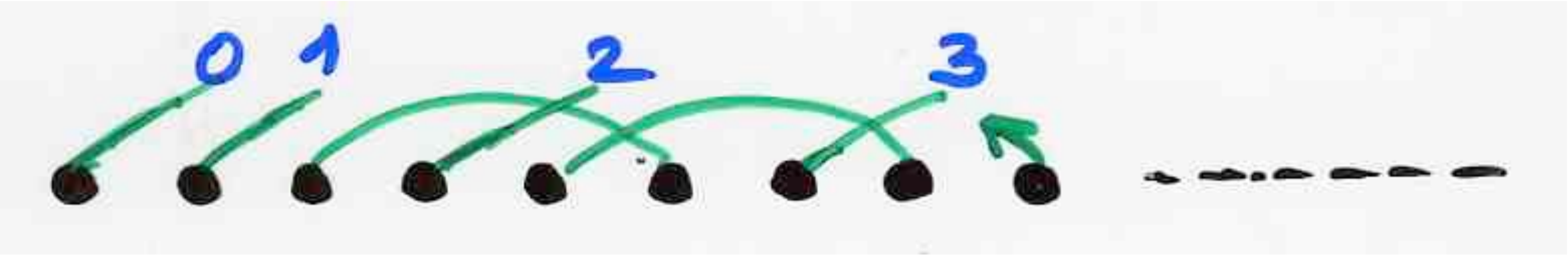
pos 1 1 1 1 2 1 1 1

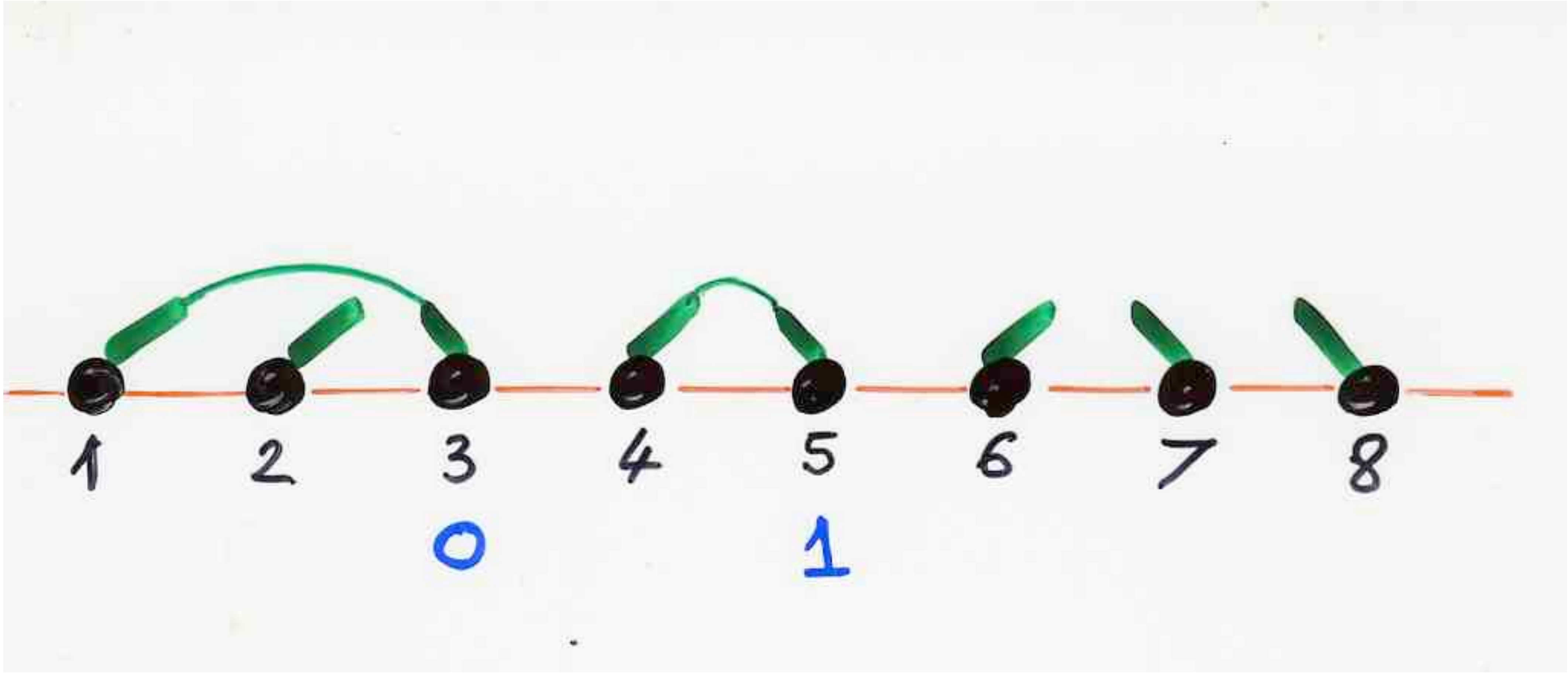

0
1
2
3

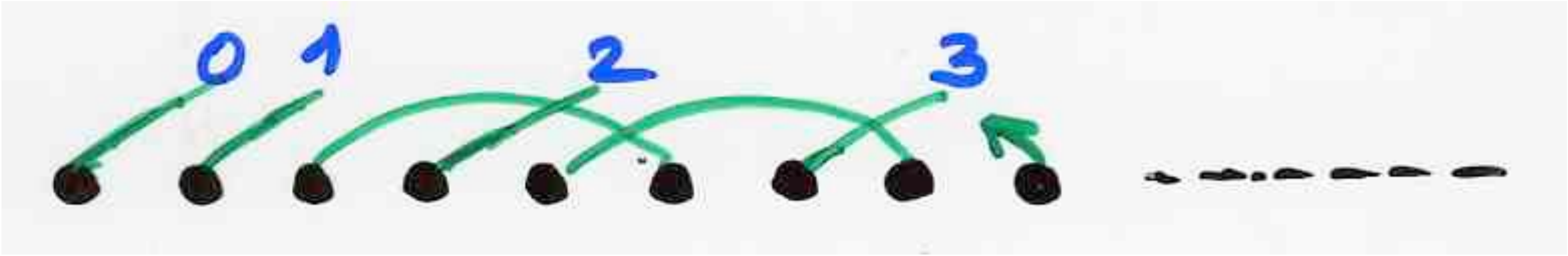

1
2
3
4
5
6
7
8

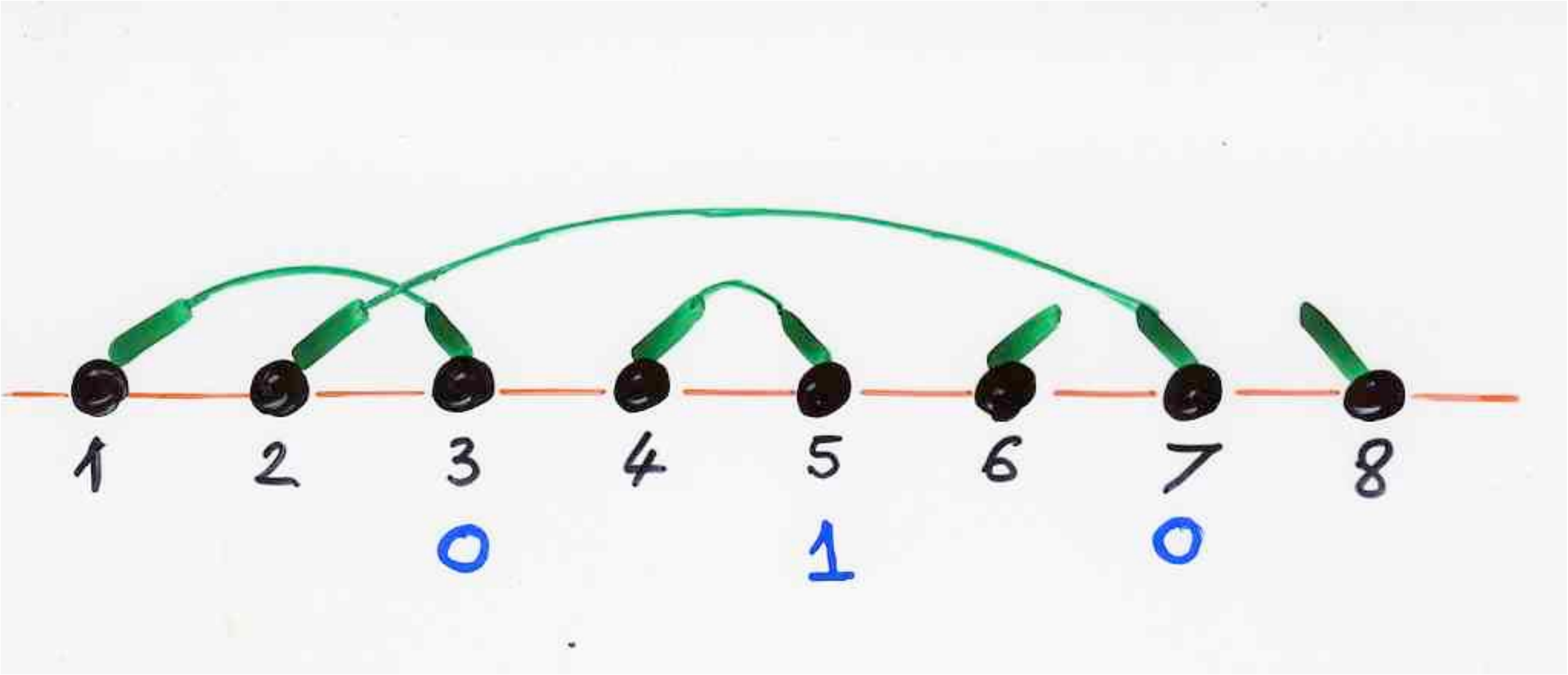

0
1
2
3

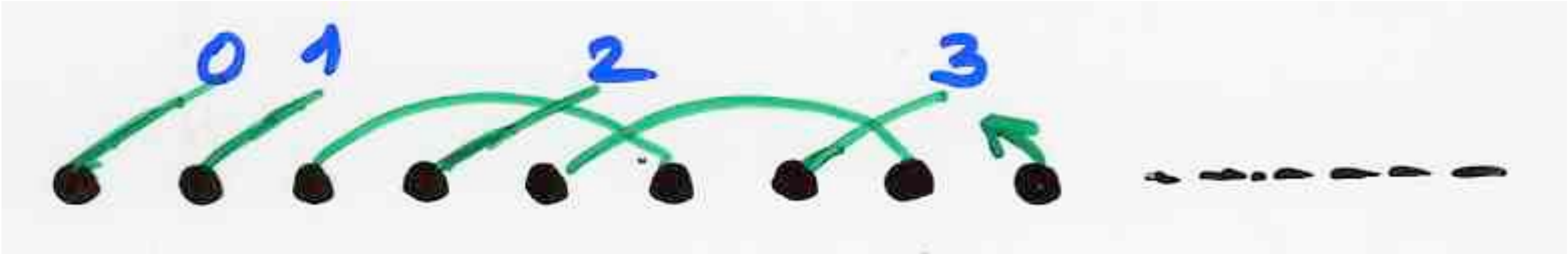

1
2
3
4
5
6
7
8
0

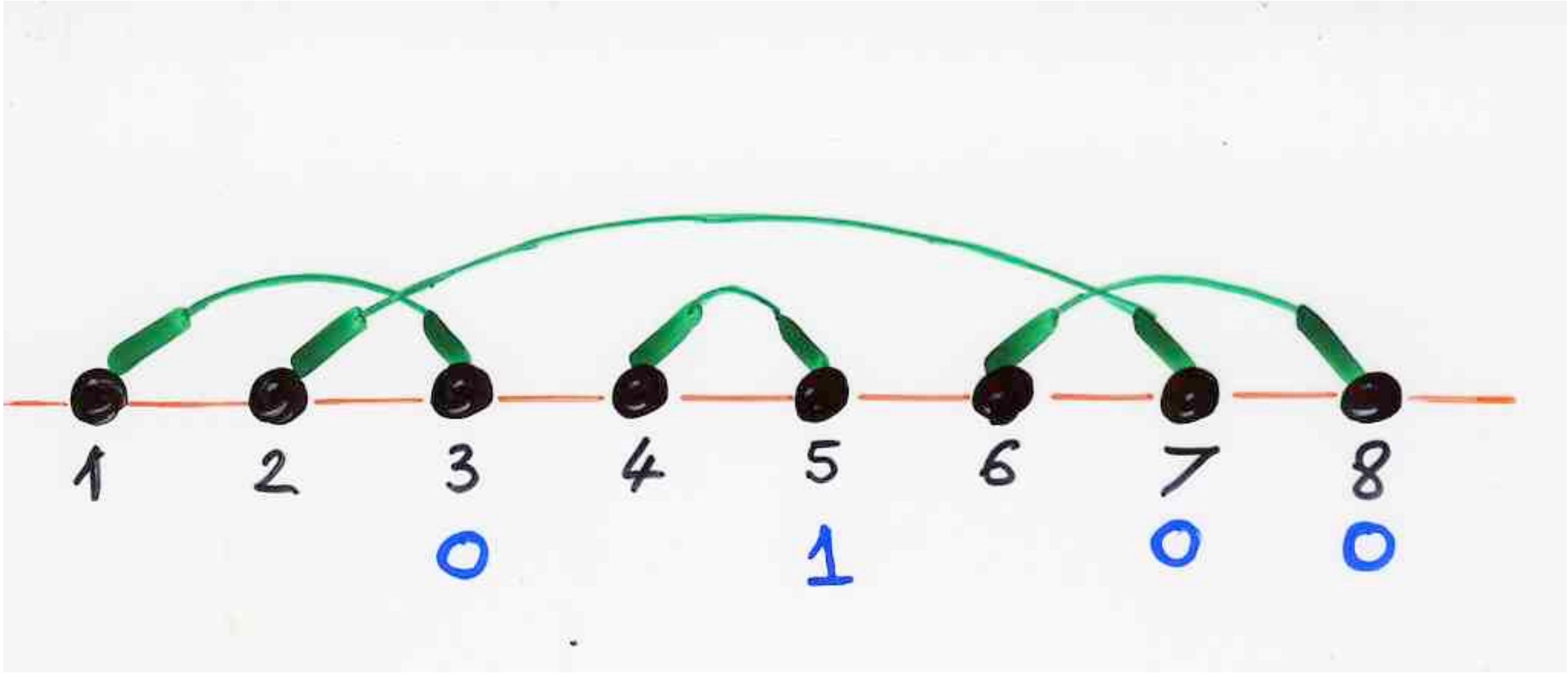

0
1
2
3


1
2
3
4
5
6
7
8
0
1


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0

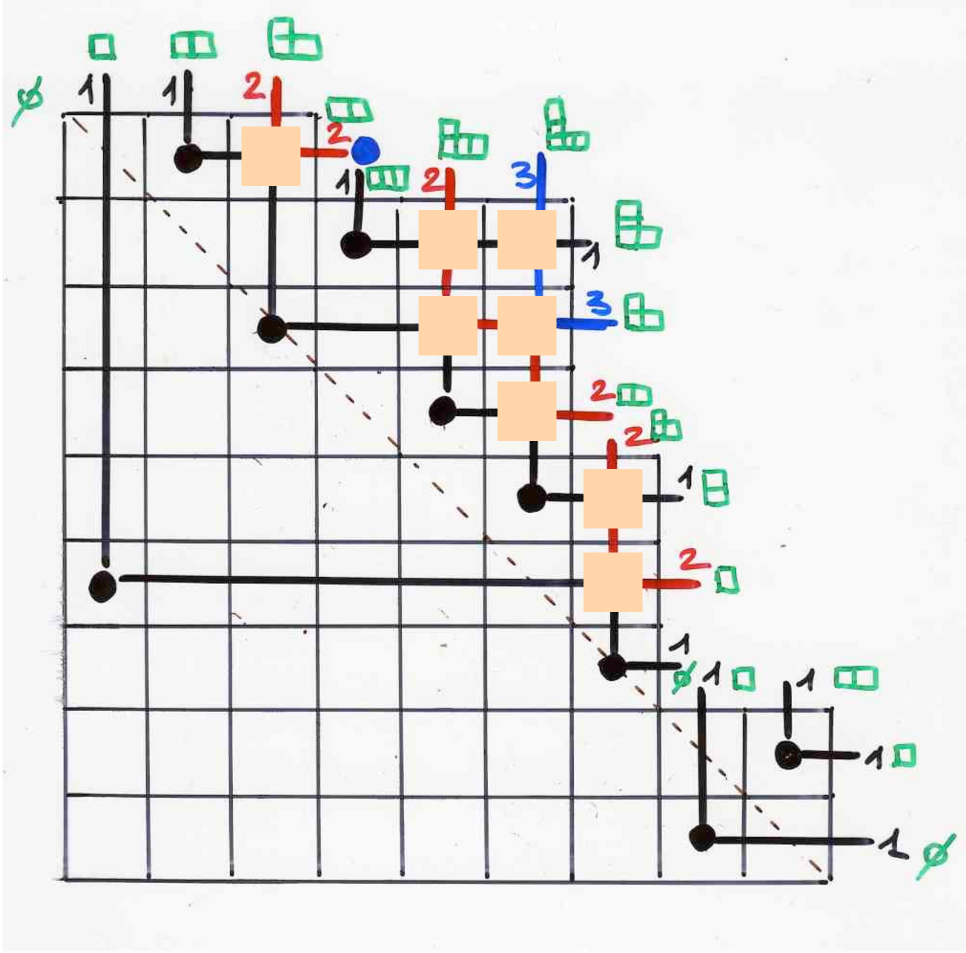

0
1
2
3


1
2
3
4
5
6
7
8
0
1
0
0

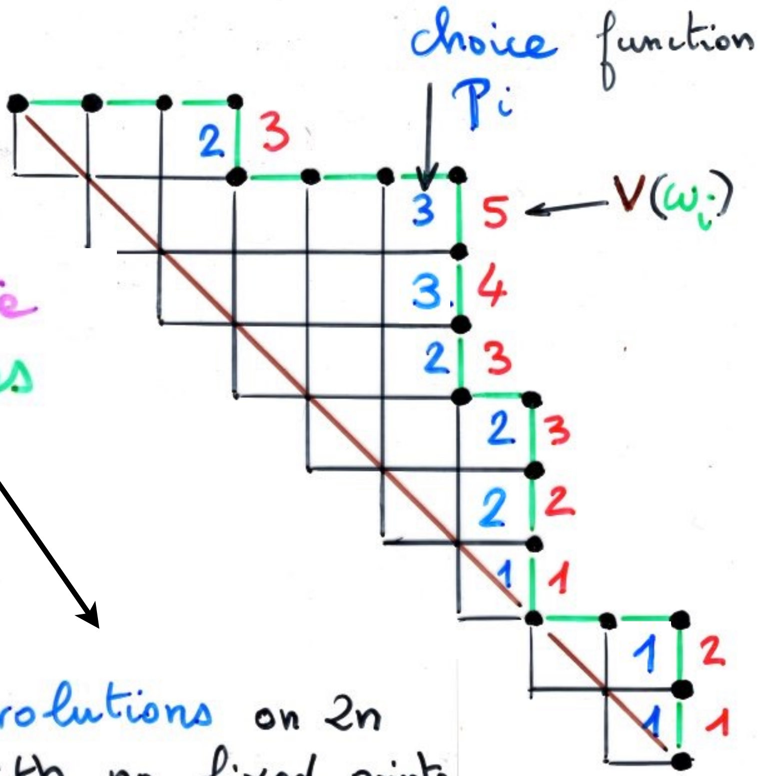
see Chle, p.90-117

sequences of oscillating tableaux
starting and ending
at ∅

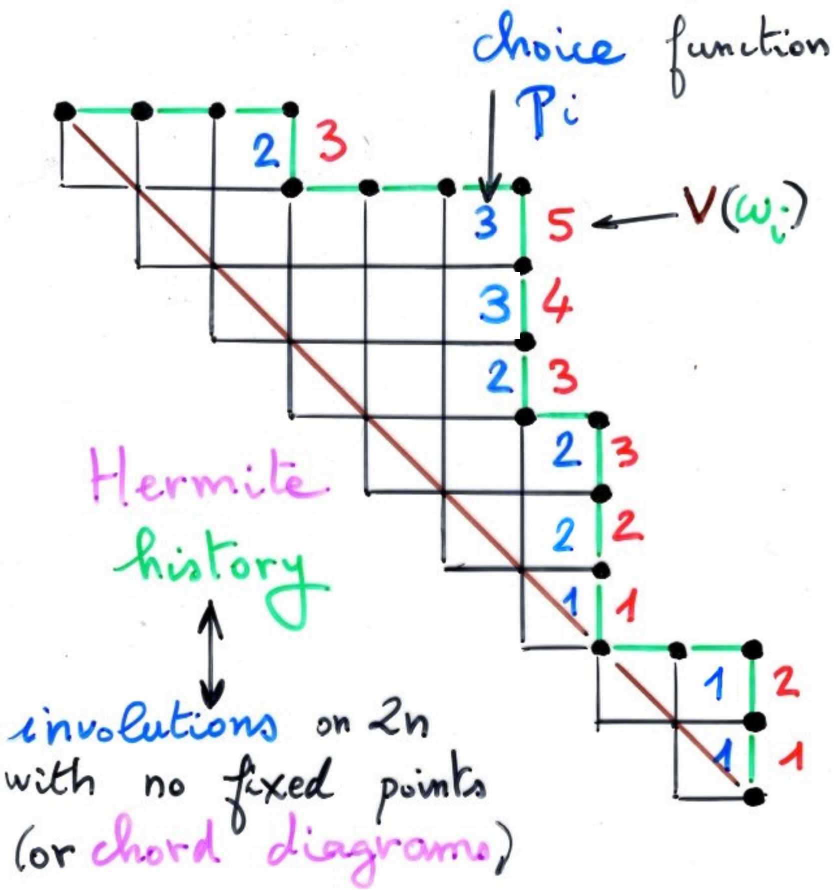
Rook placements
with
no empty row
no empty column

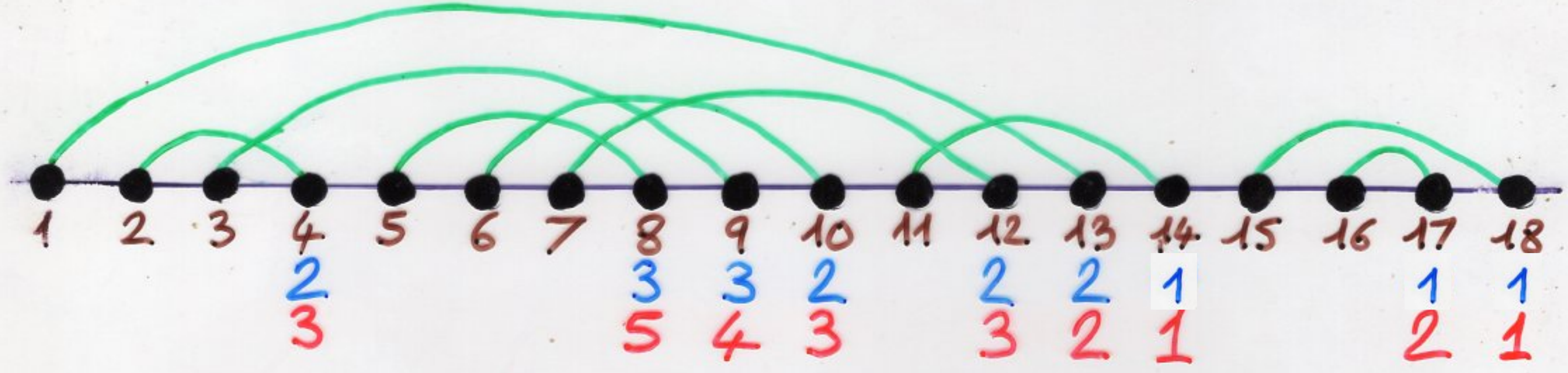


Hermite histories



involutions on 2n
with no fixed points
(or chord diagrams)


choice function
p_i
$V(\omega_i)$
2 3
3 5
3 4
2 3
2 3
2 2
1 1
1 2
1 1
Hermite history
involutions on 2n with no fixed points (or chord diagrams)


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
2 3 3 2 2 2 1 1 1
3 5 4 3 3 2 1 2 1

Sheffer orthogonal polynomials

orthogonal
polynomials

(binomial type)
Sheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

Hermite	H_n
Laguerre	$L_n^{(\alpha)}$
Charlier	$C_n^{(a)}$
Meixner I	$M_n^{I\,(\alpha,\beta)}$
Meixner II	$M_n^{II\,(\delta,\eta)}$

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1} c_k$	Moments
Tchebycheff unitaires $U_n(x)$	0	1/4	$\frac{1}{4^n} C_n$ Catalan
$T_n(x)$	0	1/4 $\lambda_0 = 1/2$	$\frac{1}{4^n}\binom{2n}{n}$
Laguerre $L^1_n(x)$ $L^\alpha_n(x)$			$(n+1)!$ $(\alpha+1)\dots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$			$\mu_{2n} = 1.3\dots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C^a_n(x)$			$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x;\beta,c)$ Kreweras $\beta = 1 \quad c = 1/2$			$\dfrac{\sum_{\sigma\in\mathfrak{S}_n} \beta^{a(\sigma)} c^{1+d(\sigma)}}{(1-c)^n}$ $= (1-c)^\beta \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$
Meixner II $M_n(x;\delta,\eta)$ $\delta = 0 \quad \eta = 1$			$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{a(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma}$ E_{2n} Sécant

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1}\, c_k$	Moments
Tchebycheff unitaires $U_n(x)$	0	$1/4$	$\frac{1}{4^n} C_n$ Catalan
$T_n(x)$	0	$1/4$ $\lambda_0 = 1/2$	$\frac{1}{4^n} \binom{2n}{n}$
Laguerre $L_n^1(x)$	$2k+2$	$k(k+1)$	$(n+1)!$
$L_n^{\alpha}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)\ldots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1.3\ldots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{a}(x)$	$k+a$	ak	$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x;\beta,c)$	$\frac{(1+c)k+\beta c}{1-c}$	$\frac{c\,k(k-1+\beta)}{(1-c)^2}$	$\sum_{\sigma\in\mathfrak{S}_n} \frac{\beta^{A(\sigma)}\, c^{1+d(\sigma)}}{(1-c)^n}$
Kreweras $\beta=1$ $c=1/2$	$3k+1$	$2k^2$	$=(1-c)^{\beta} \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$
Meixner II $M_n(x;\delta,\eta)$	$(2k+\eta)\,\delta$	$(\delta^2+1)\,k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{A(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{F(\sigma}$
$\delta=0$ $\eta=1$	0	k^2	E_{2n} Sécant

more on X.G.V. (old) website www.xavierviennot.org

see the page: «livres»

X.G.V.: Une théorie combinatoire des polynômes orthogonaux
Notes de cours, 217p., LACIM, UQAM, Montréal, 1984
(french)

also on the page «petite école»,
a series of lectures with slides given at LaBRI, Bordeaux, in 2006/2007 (mixture of french and english)

See the Bijective Course at IMSc, part IV (January-March 2019)

www.viennot.org or www.imsc.res.in/~viennot
(new website) (mirror image)