Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 4
Trees and tableaux

Ch4c

IMSc, Chennai
April 19, 2018

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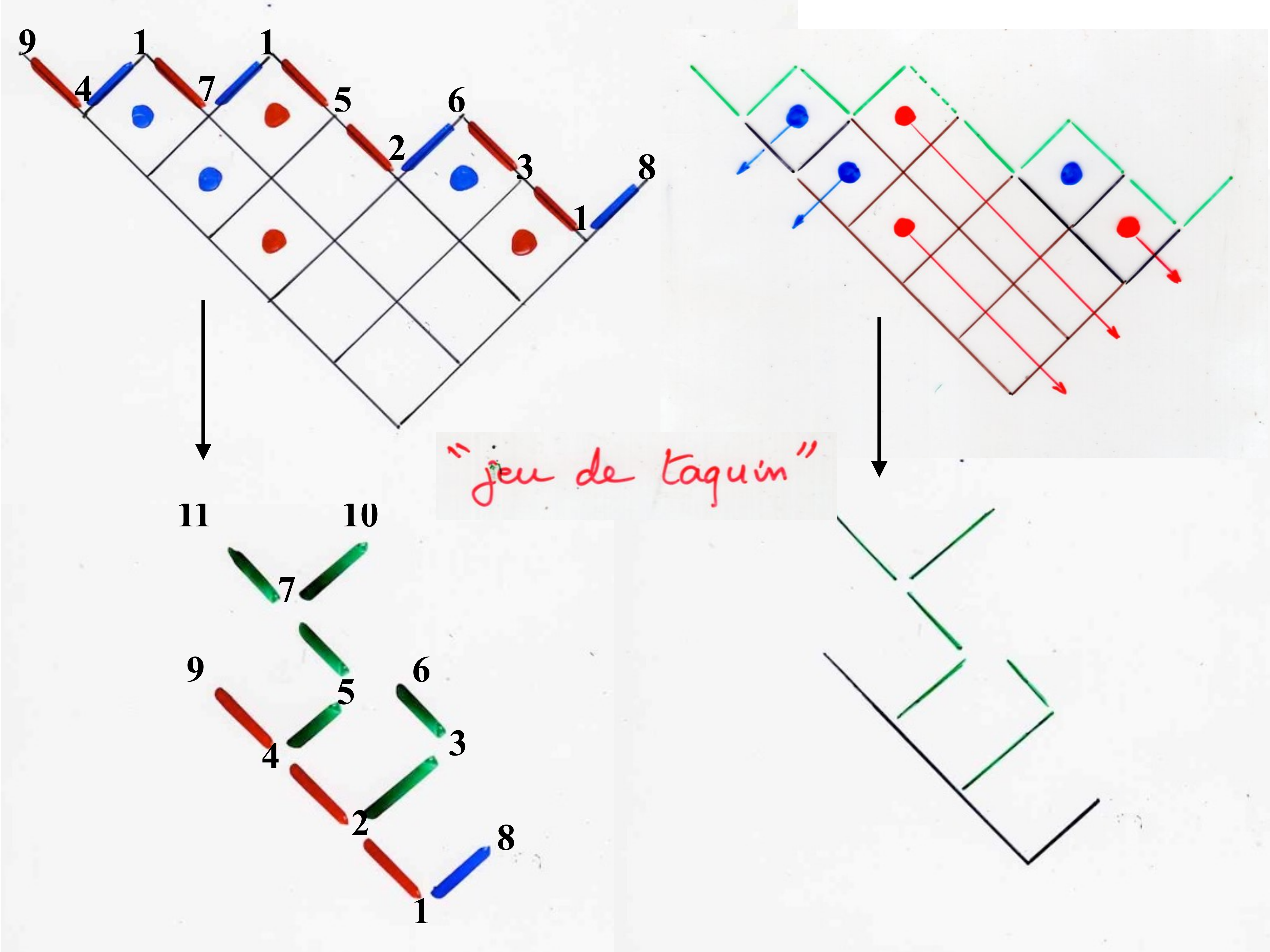
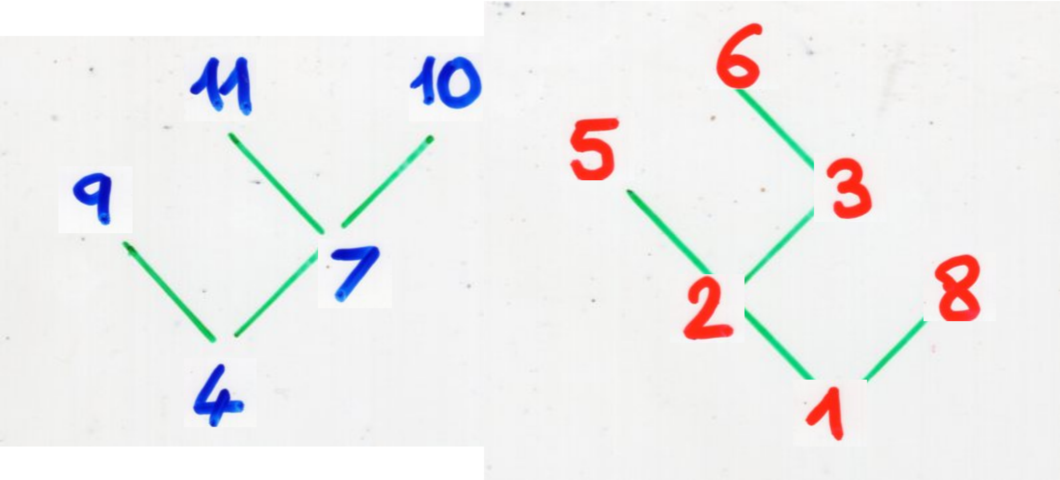
From Ch 4b

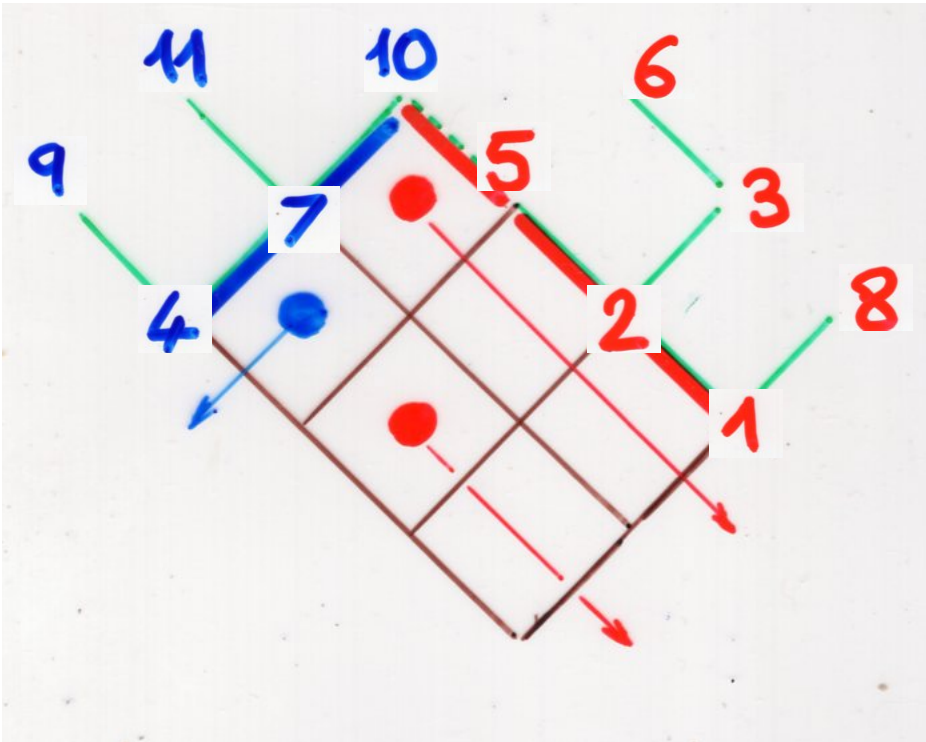
Jeu de taquin for an increasing wood

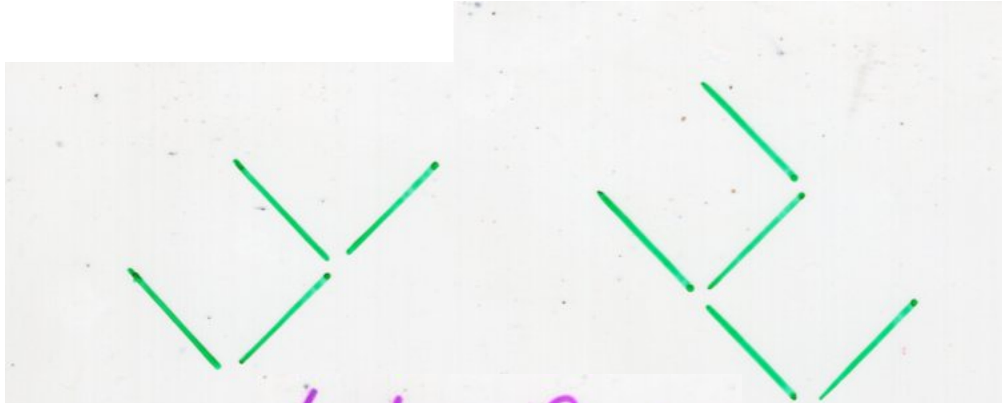


9 1 1
4 7 5 6
2 3 8
1
"jeu de taquin"
11 10
7
9 5 6
4 3
2 8
1

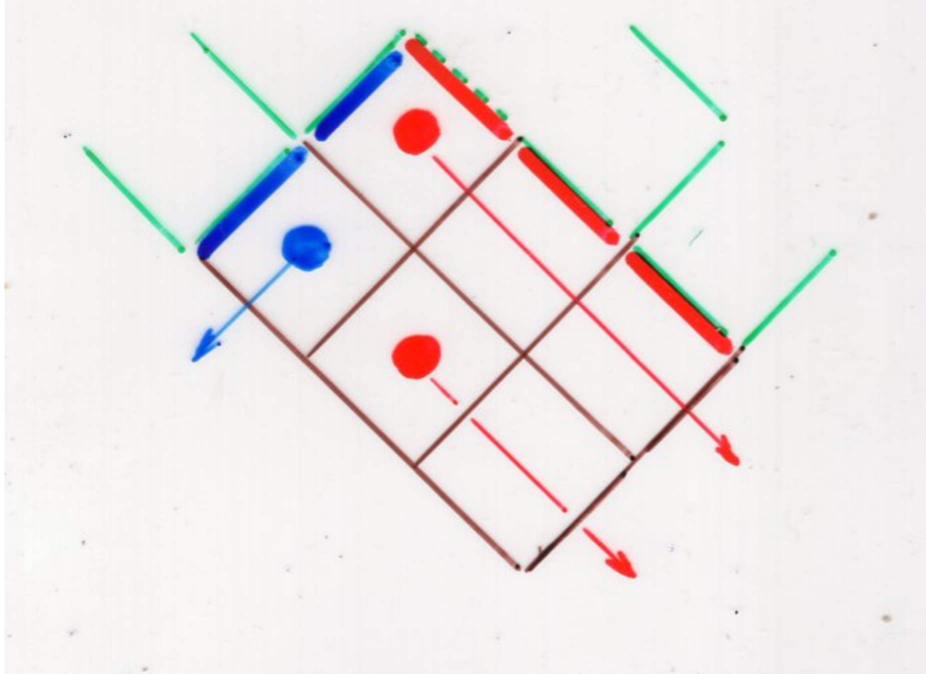


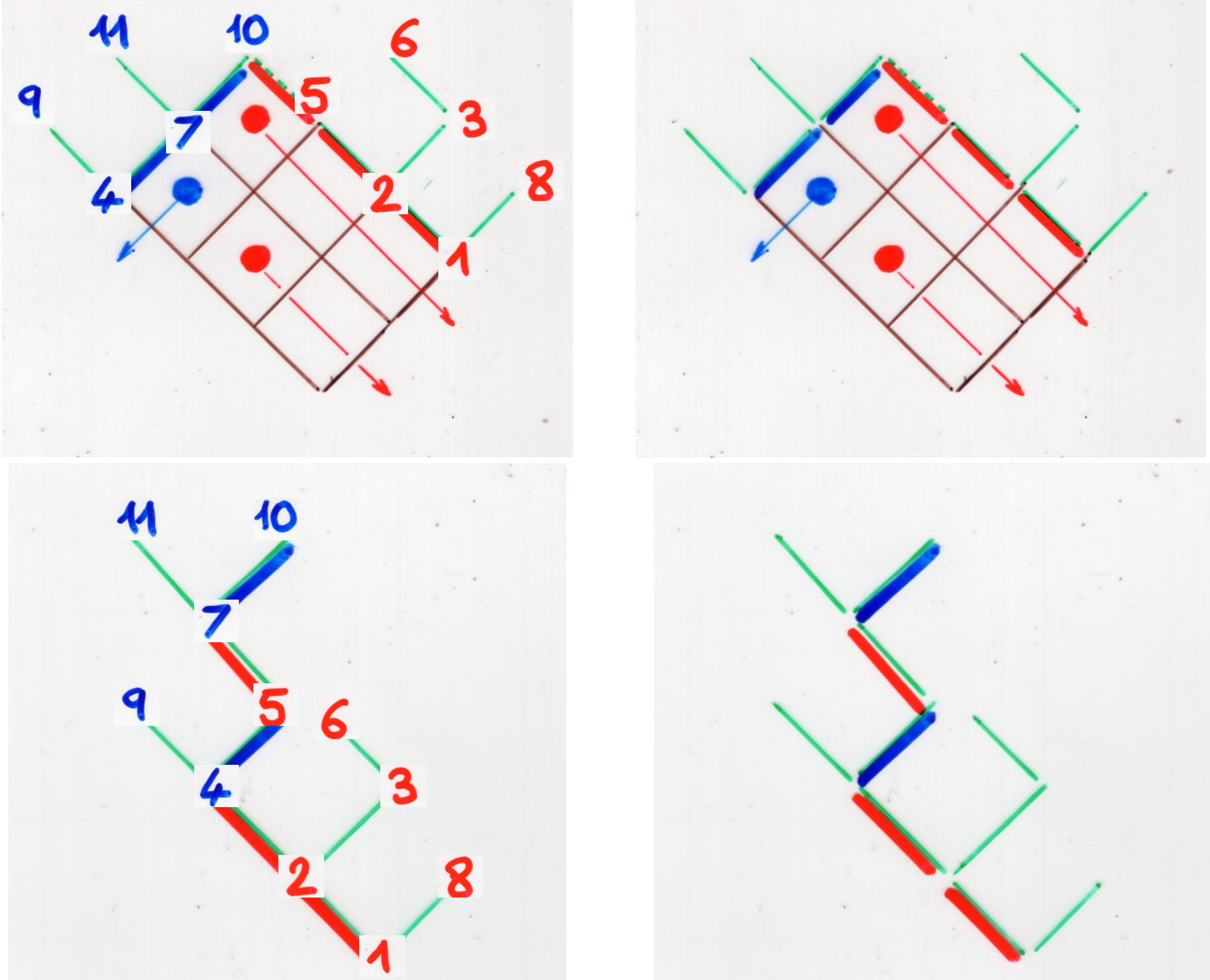
Malvenuto-Reutenauer product $*$ in $\mathbb{K}[S_\infty]$





Loday-Ronco product $*$




11 10 6
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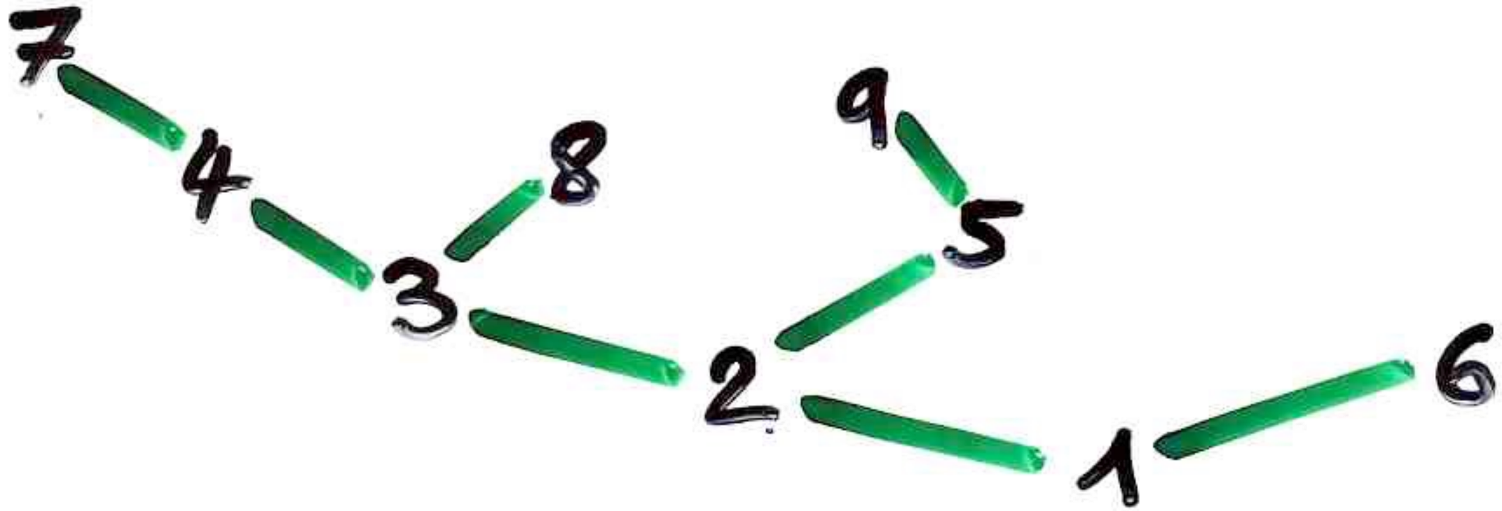
Bijection increasing binary tree $\longleftrightarrow$ permutation

T $\qquad\qquad$ σ



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

Bijection   increasing binary tree $\longleftrightarrow$ permutation

T σ

$T \xrightarrow{\pi} \sigma$ symmetric order of vertices (or "projection")

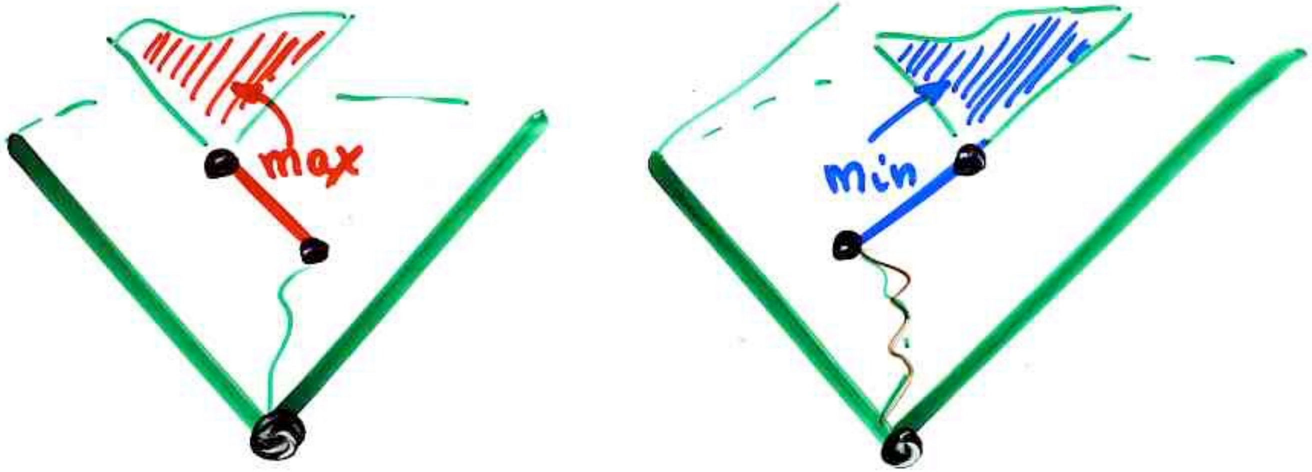
$\sigma \xrightarrow{\delta} T$ "déployé"

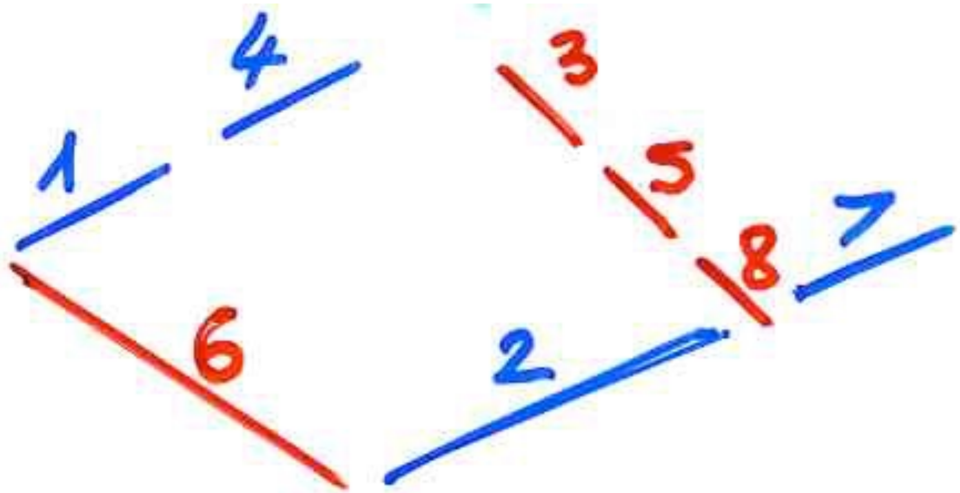
word $w = u\, m\, v$ m minimum letter (unique)

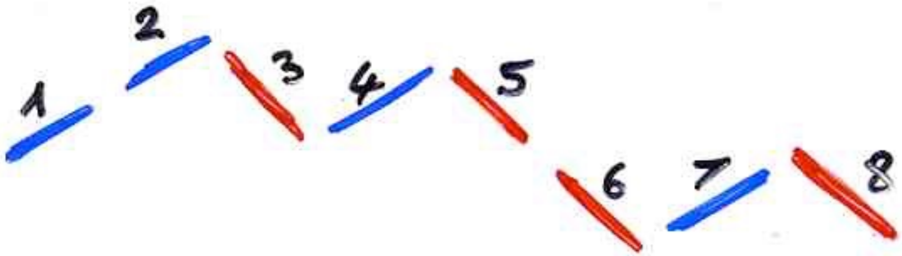
$$\delta(w) = \begin{tikzcd} \delta(u) \arrow[dr, dash] & & \delta(v) \arrow[dl, dash] \\ & m & \end{tikzcd}$$

alternative binary trees

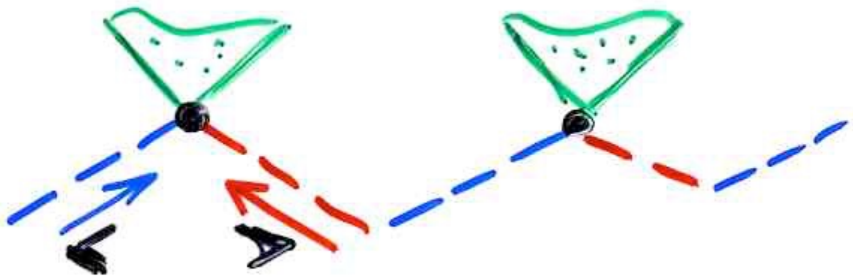
Definition edge-alternative binary trees



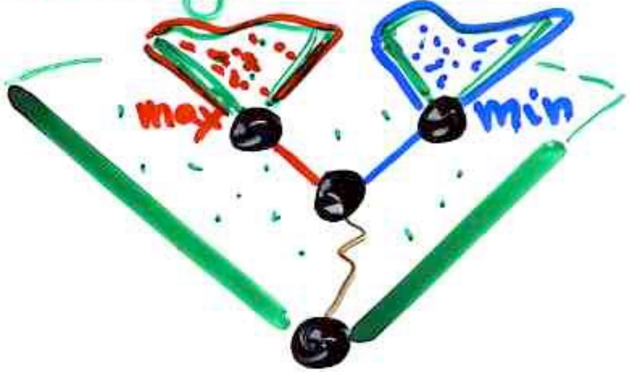




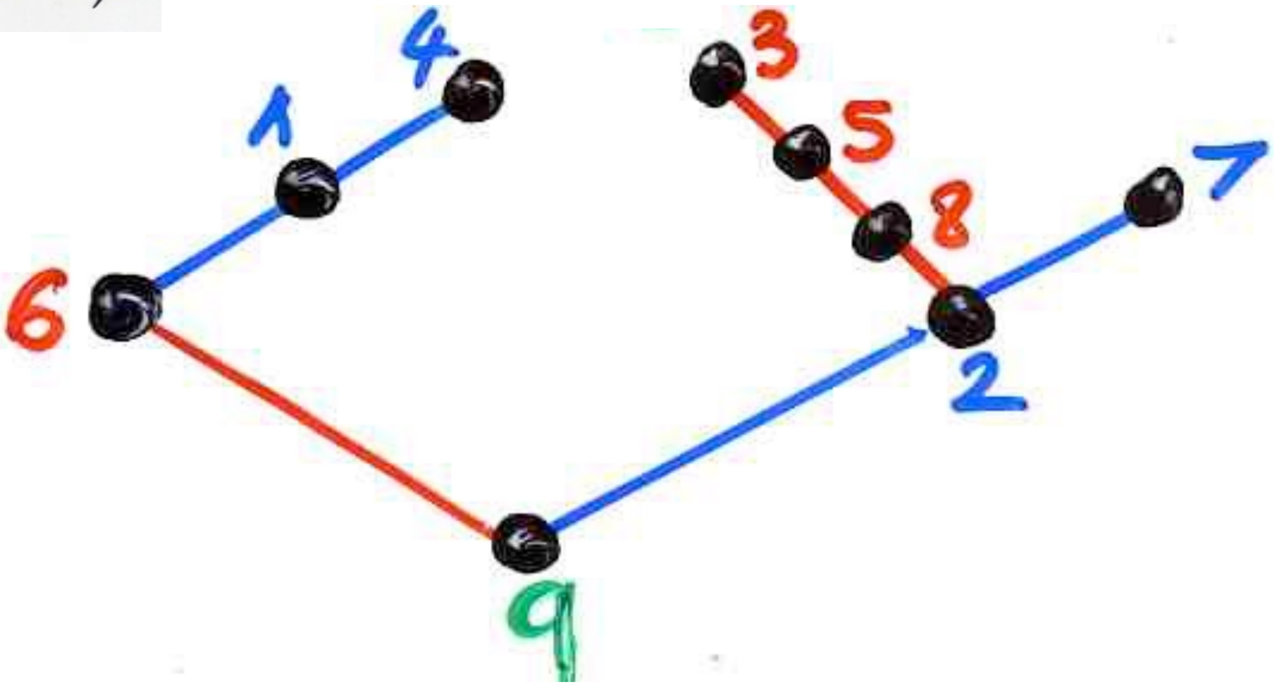
Definition edge-alternative woods

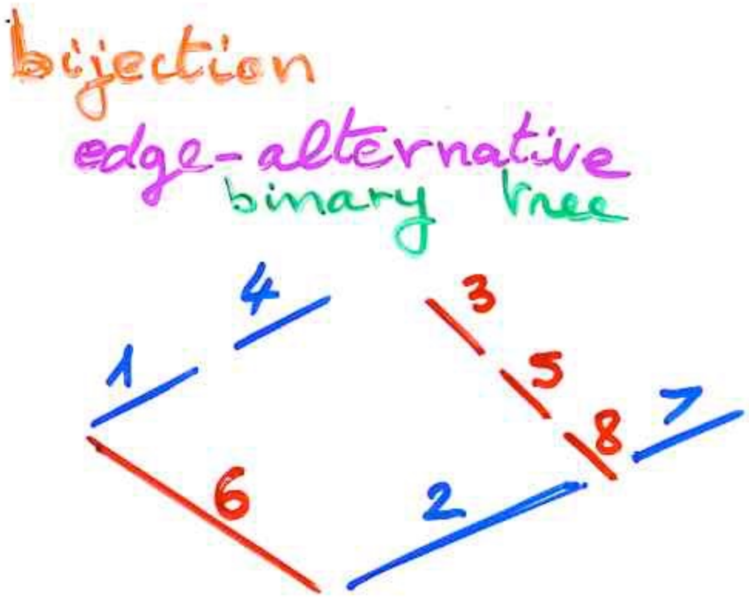


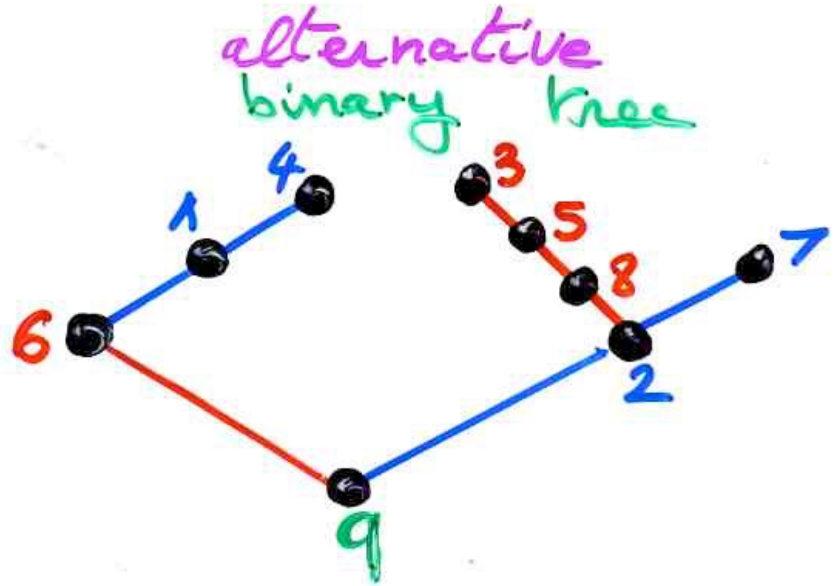
Definition alternative binary tree



Nadeau (2011)





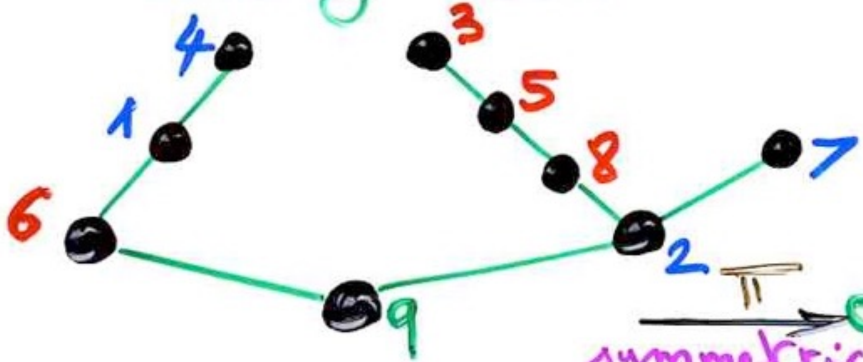


edge-alternative binary trees

alternative binary trees

bijection

alternative binary trees $\qquad\qquad$ permutations



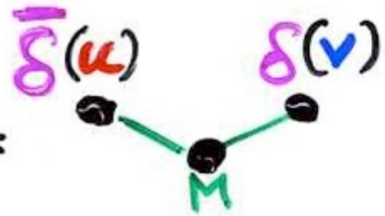
$\xrightarrow[\text{symmetric order}]{\pi} \sigma = (6\ 1\ 4\ 9\ 3\ 5\ 8\ 2\ 7)$

symmetric order

(projection)

$M = \max(\sigma)$

$\bar{\delta}(\sigma) =$

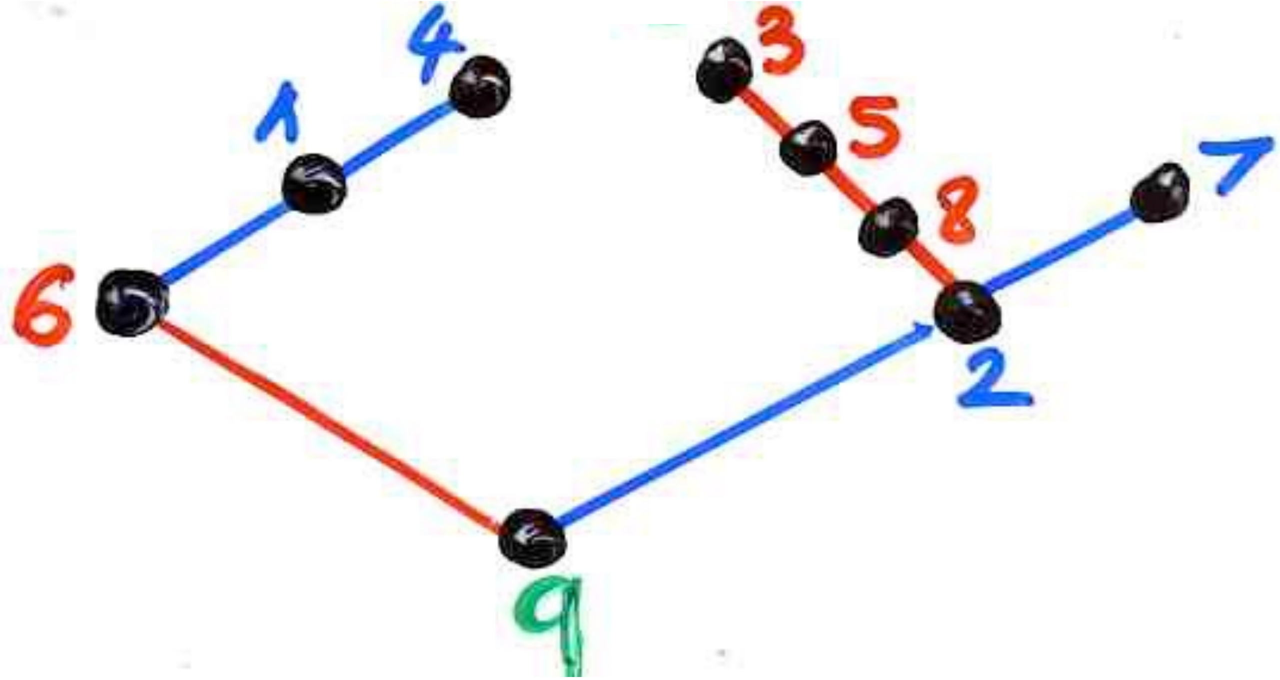


$\xleftarrow{\bar{\delta}}$ $\quad \sigma = u\,M\,v$

$\bar{\delta}$ alternative "déployé"

"hook length"
formula
same as for
increasing
binary tree

$$\frac{n!}{\prod_x h_x}$$



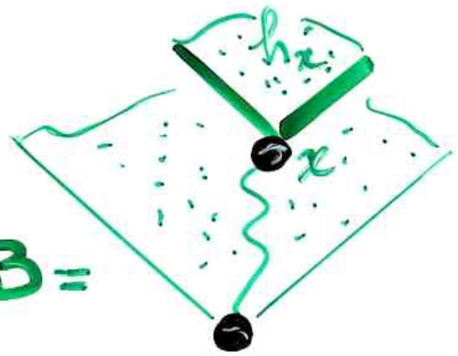
"hook-length formula"

$$\frac{n!}{\prod_x h_x}$$

n nb of vertices

product of size of sub-trees

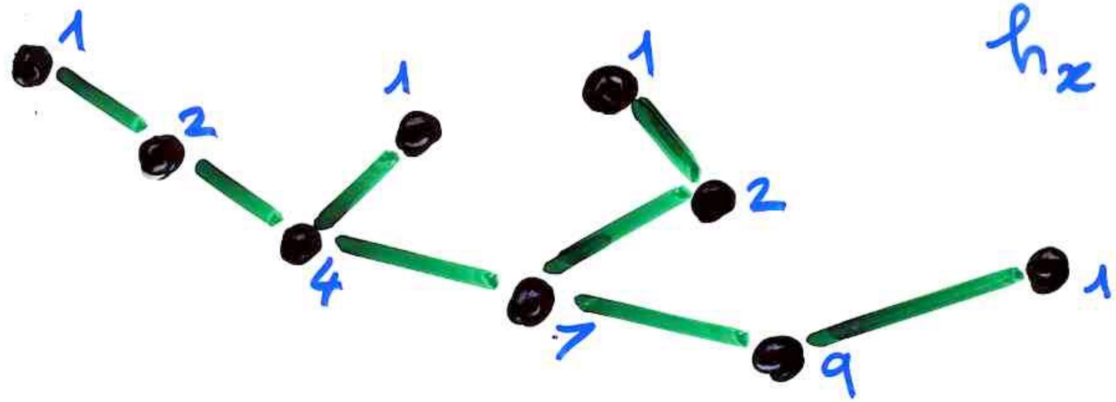
$\mathbf{B} =$



nb of increasing binary tree for a binary tree $\mathbf{B}$

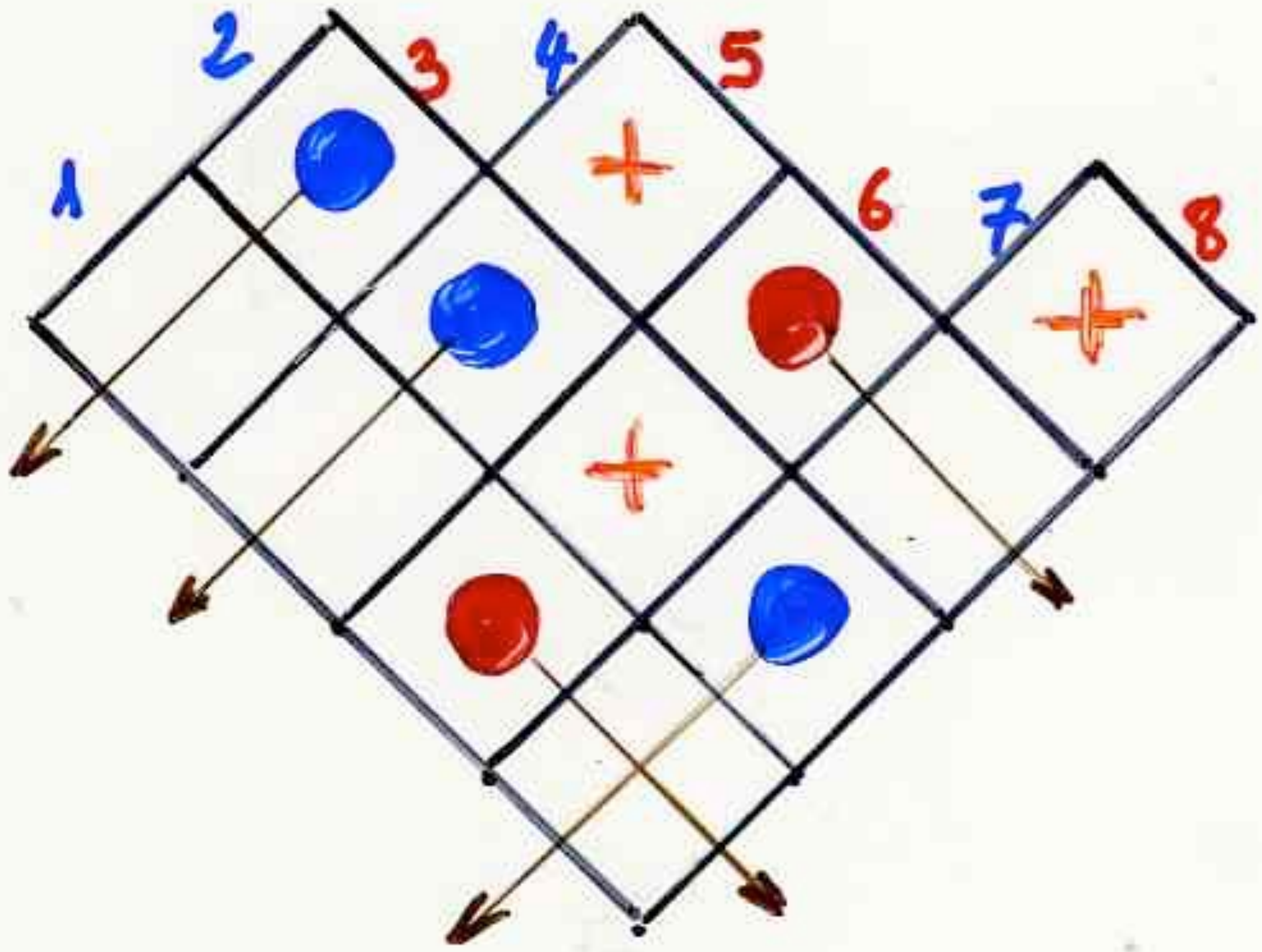
<u>example</u>

"hook-length" h_x



$$\frac{9!}{2^2 \cdot 4 \cdot 7 \cdot 9} = 360$$

jeu de taquin
for
edge alternative binary trees and woods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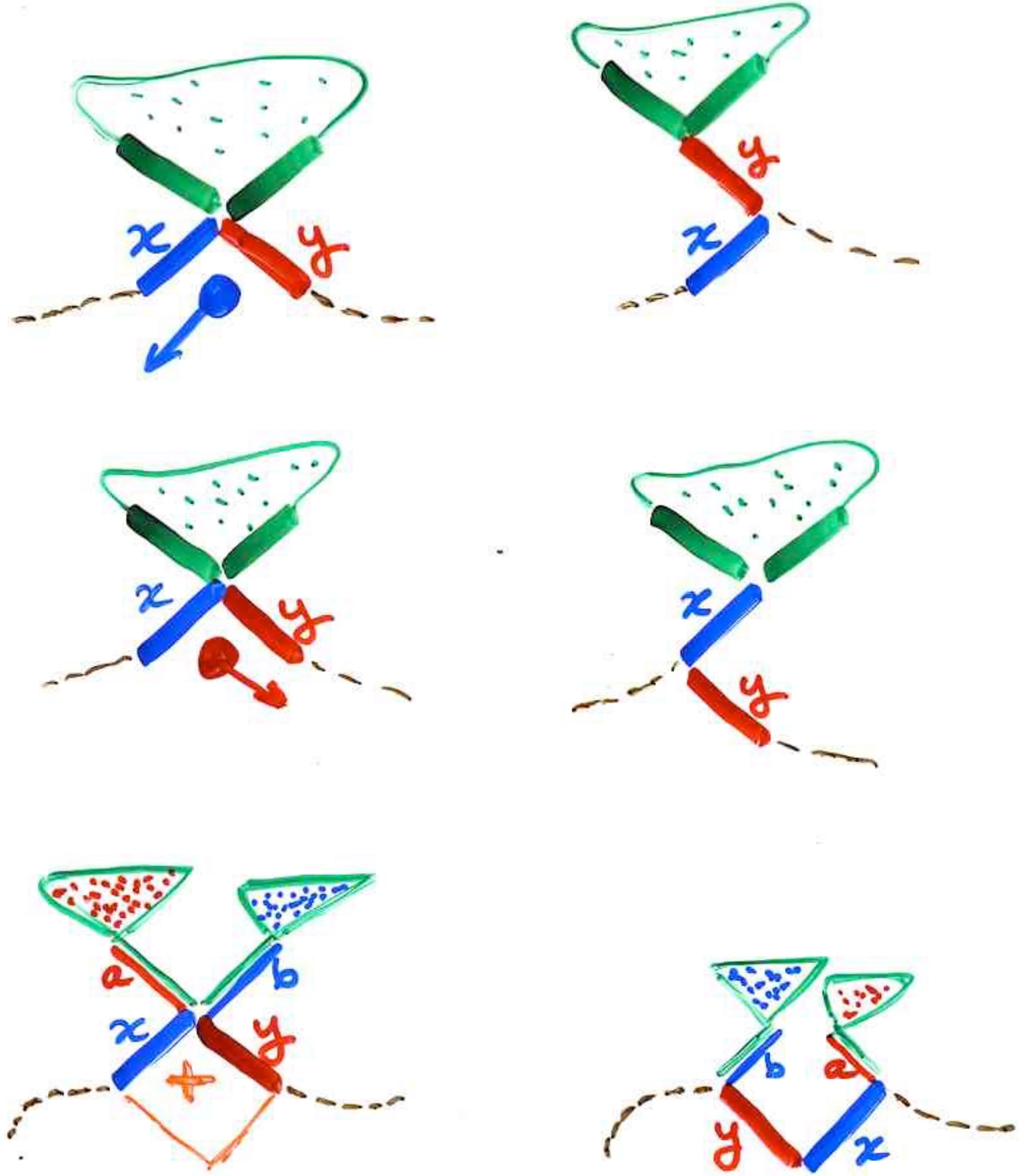

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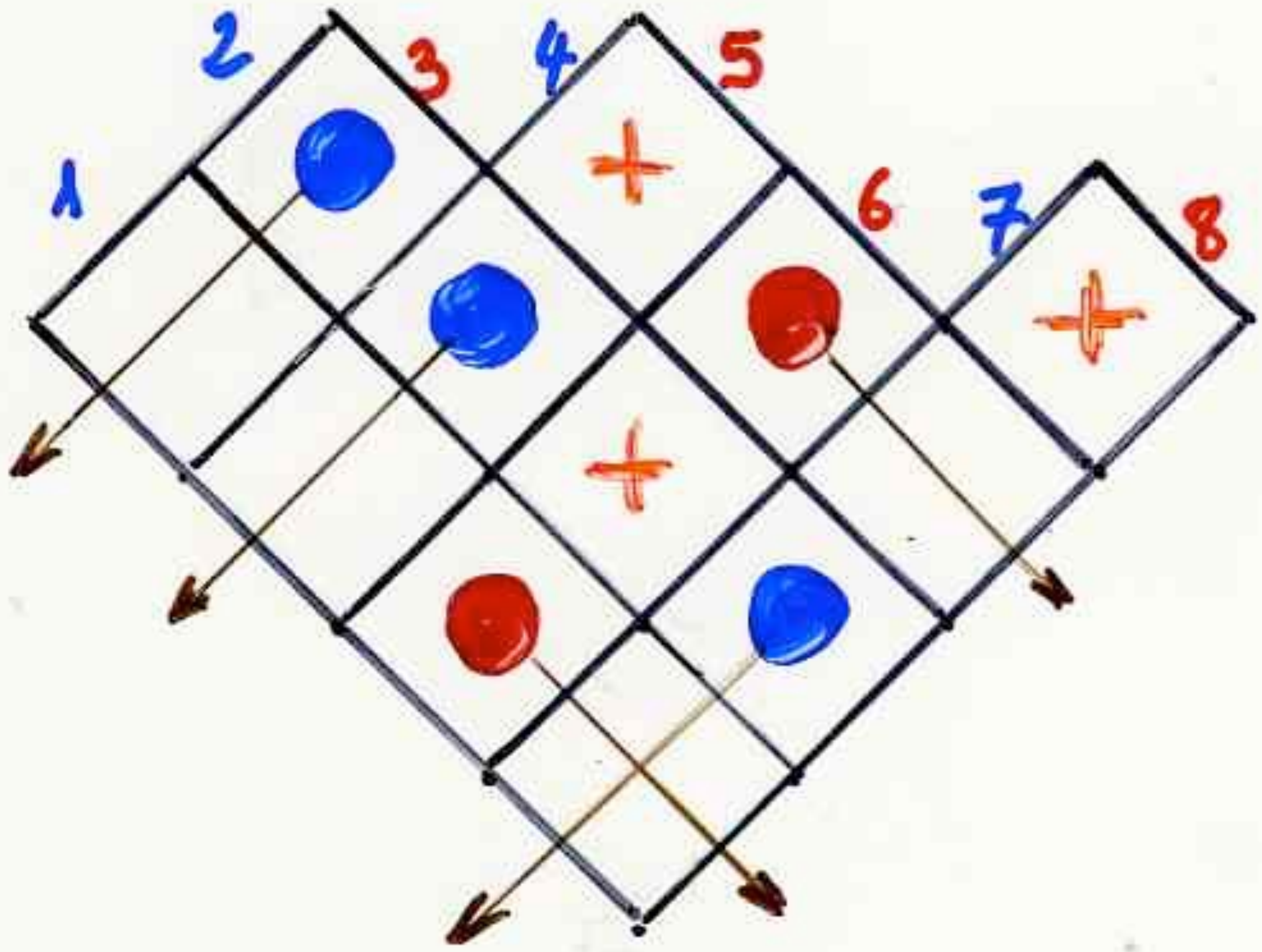
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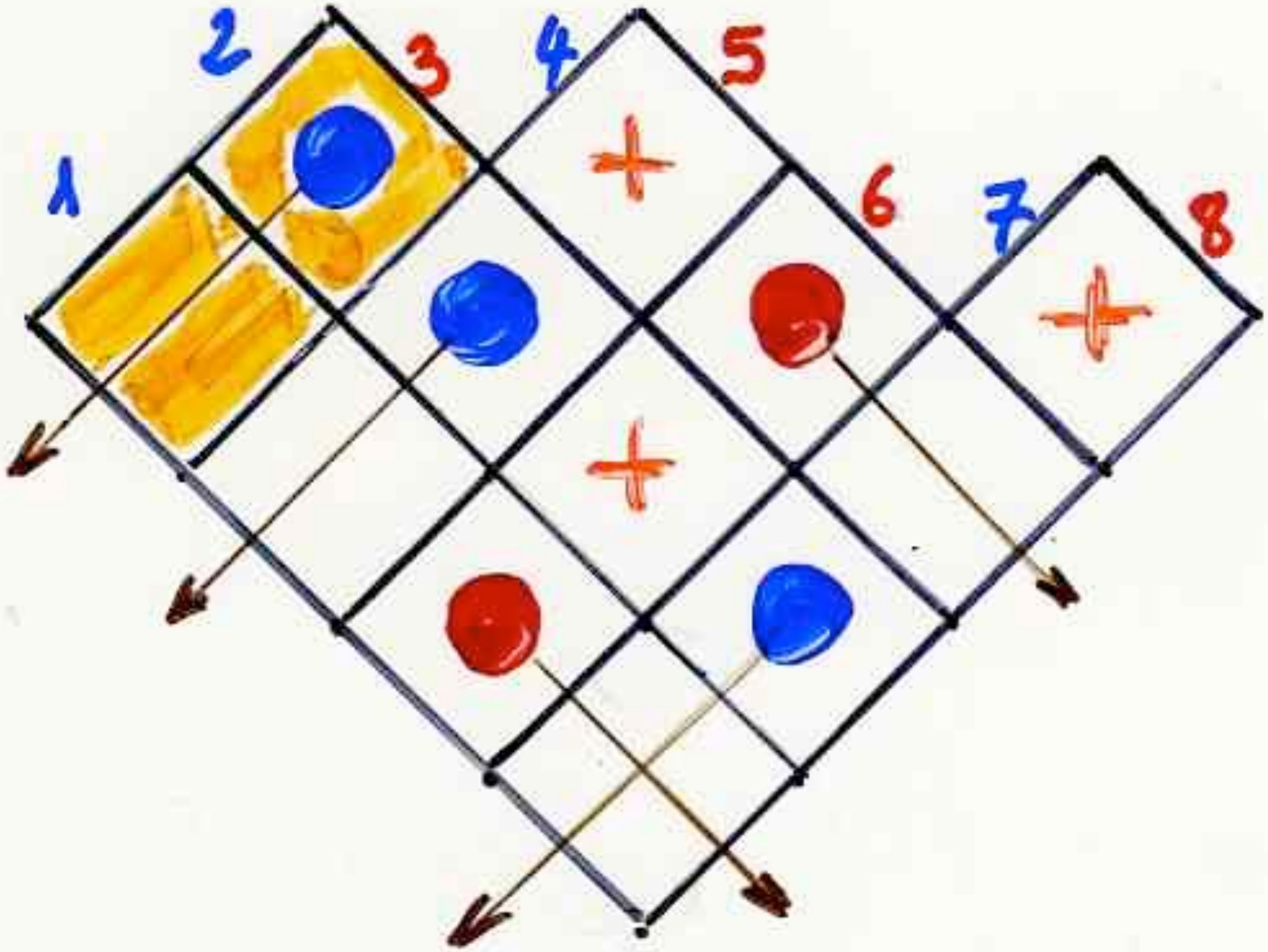
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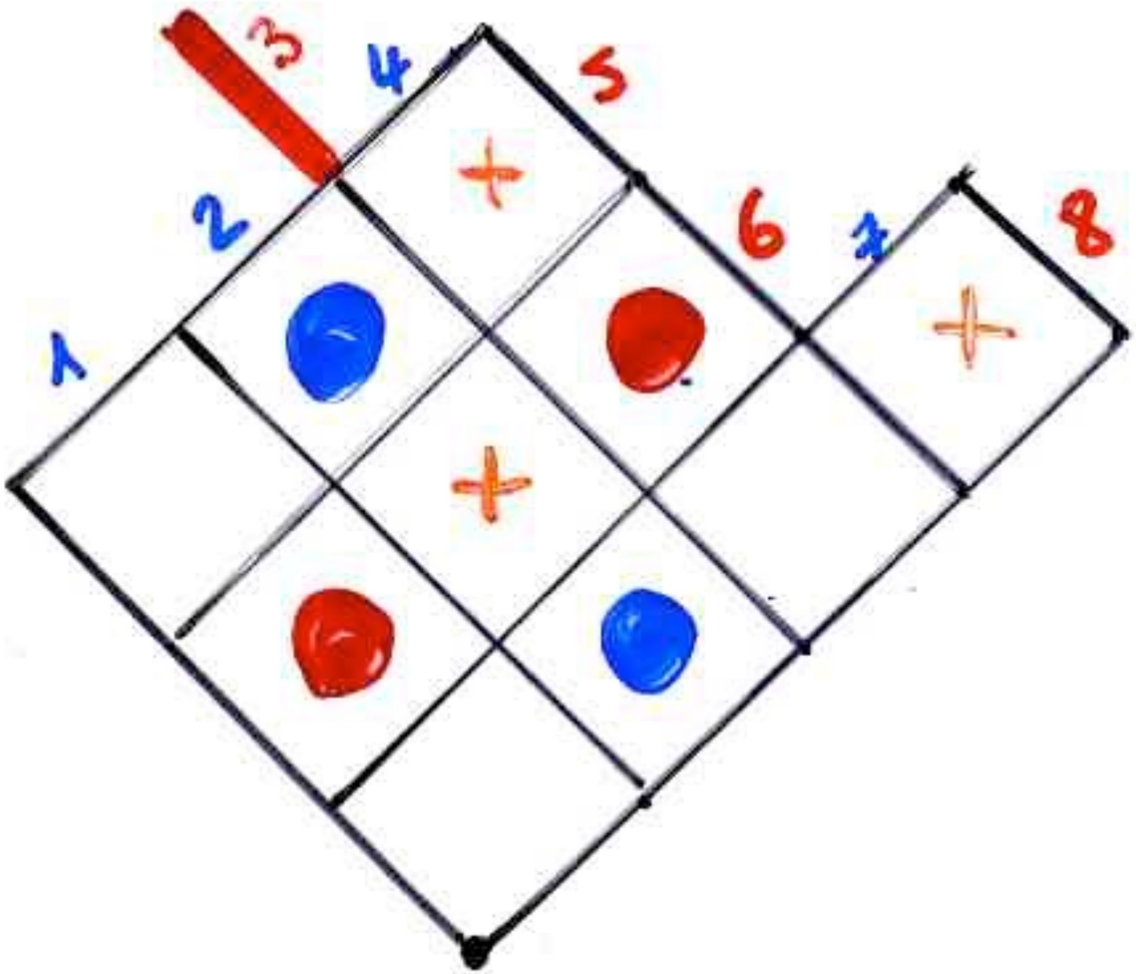
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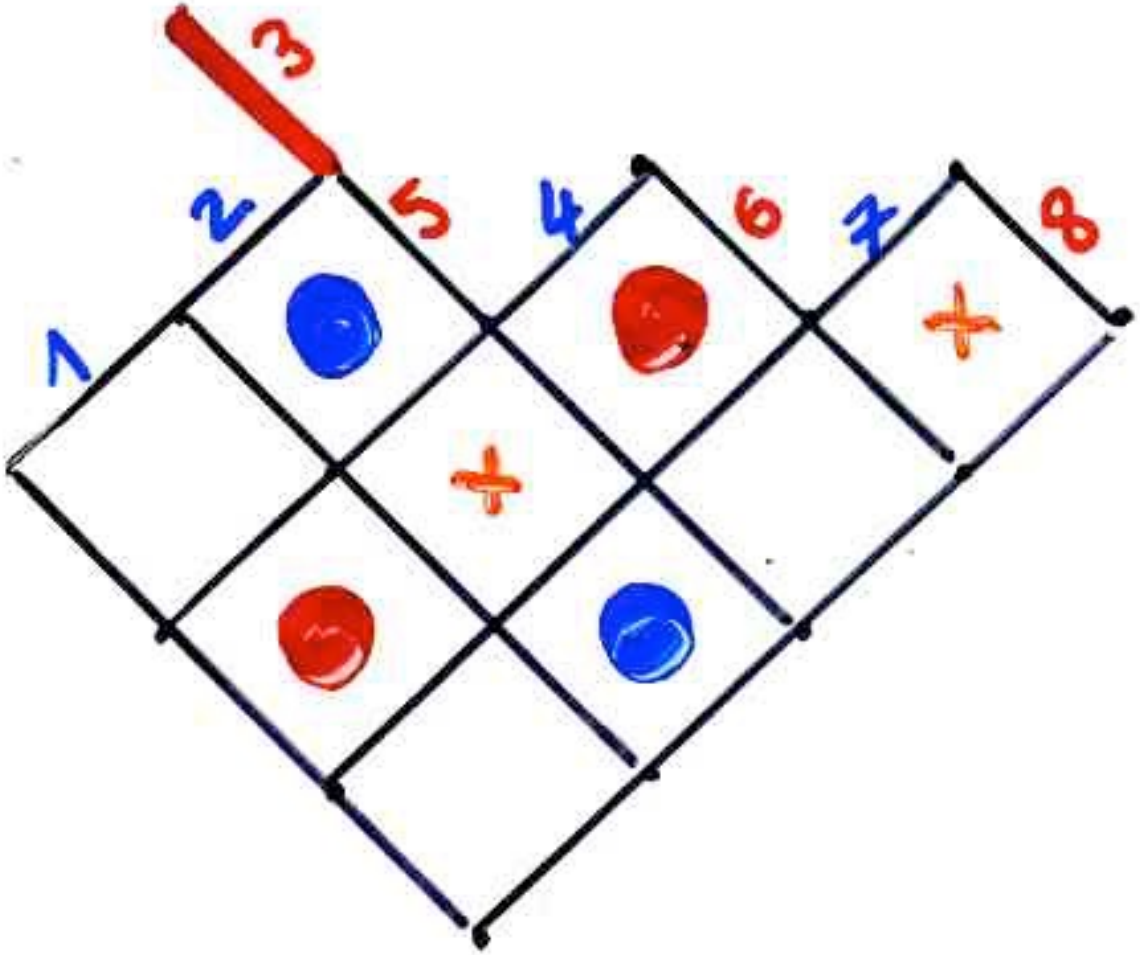
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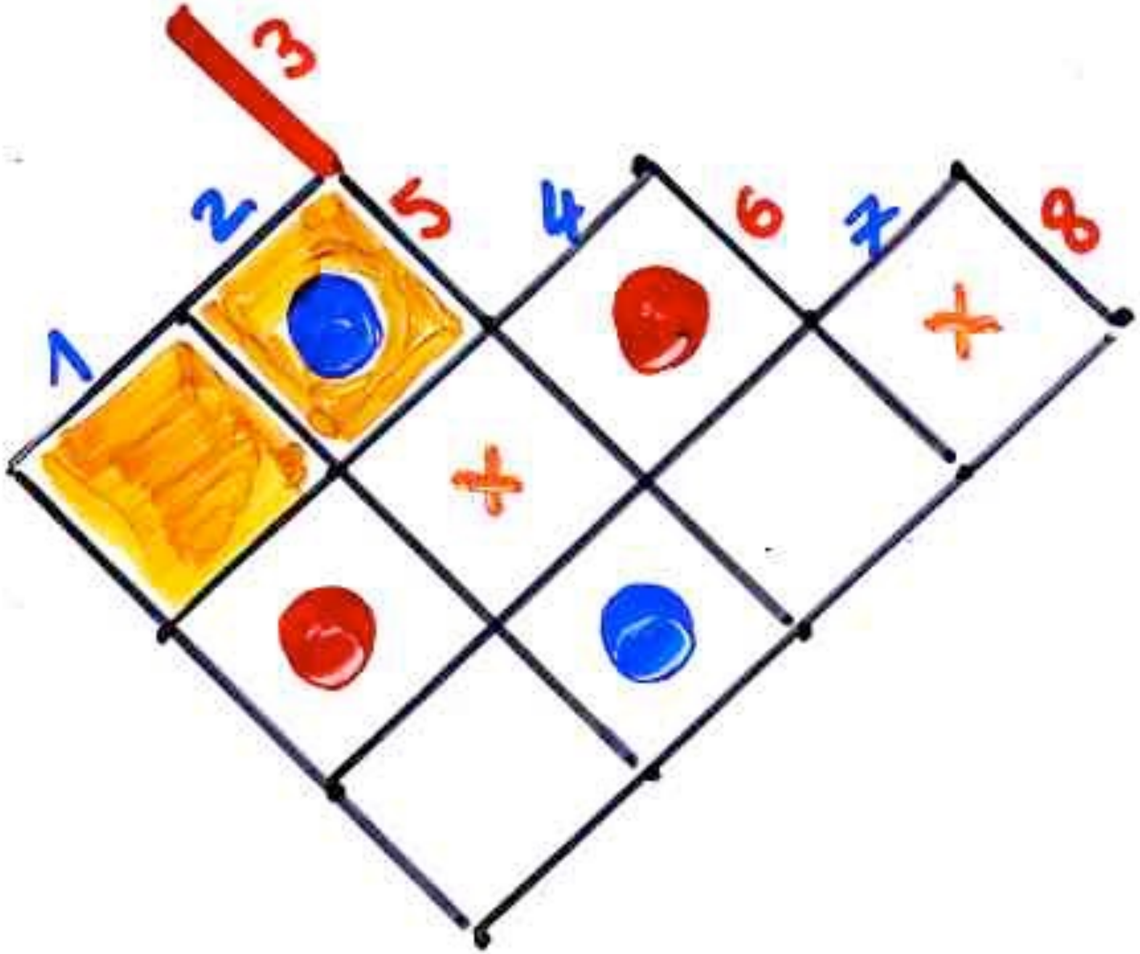

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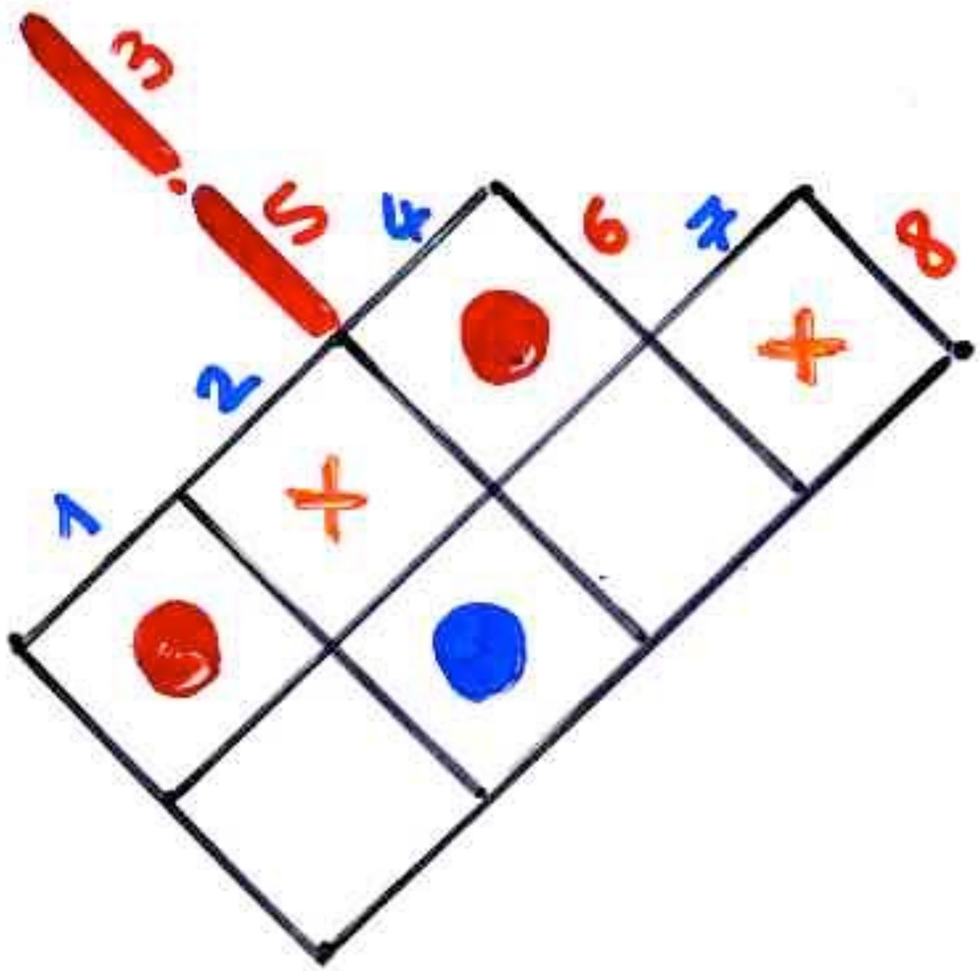

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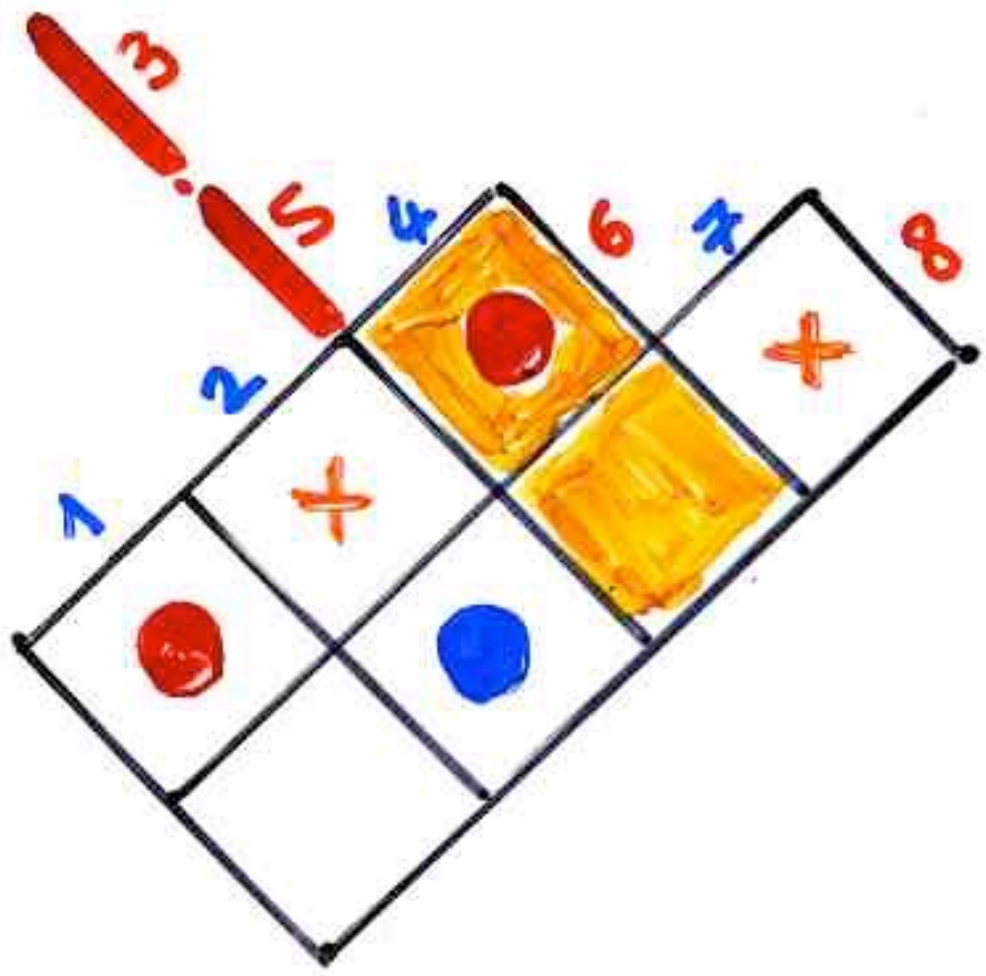
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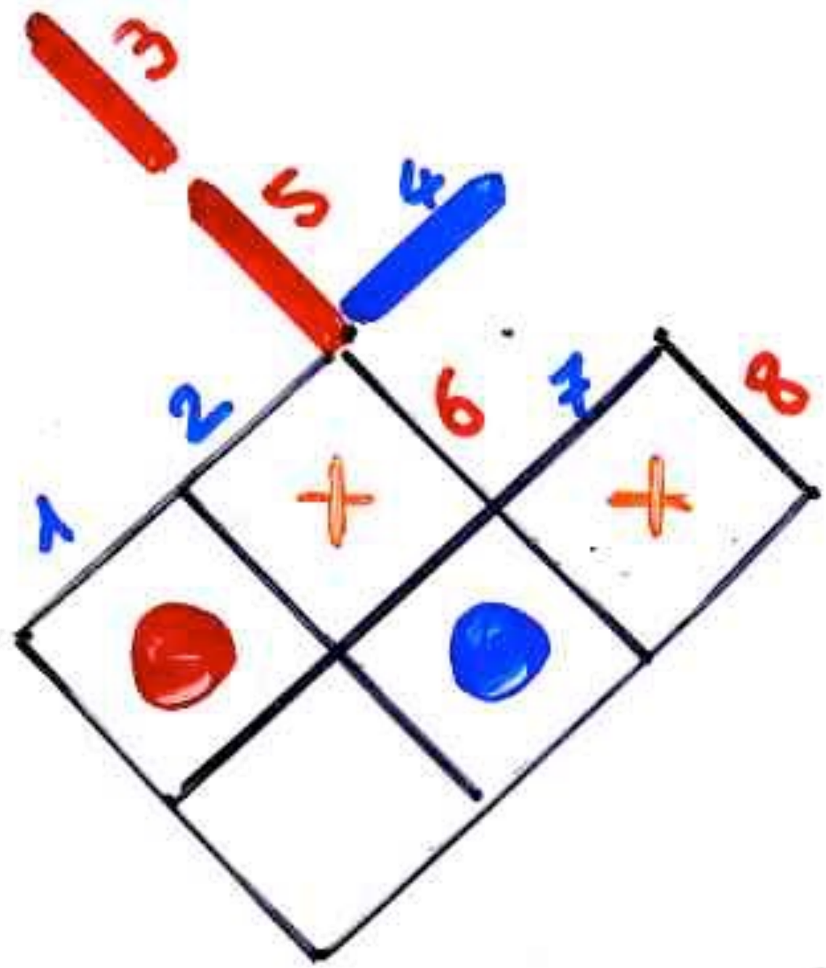
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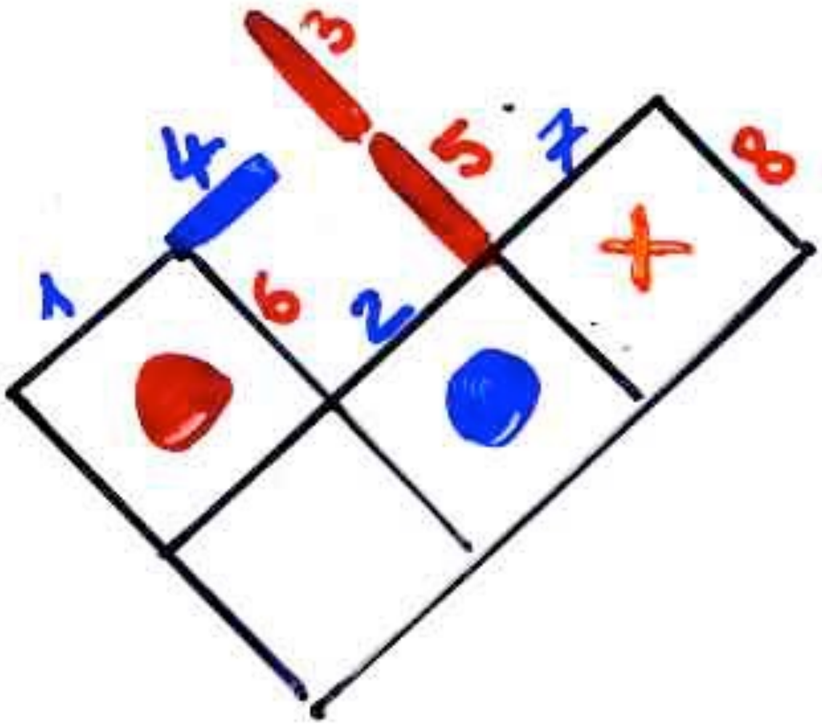

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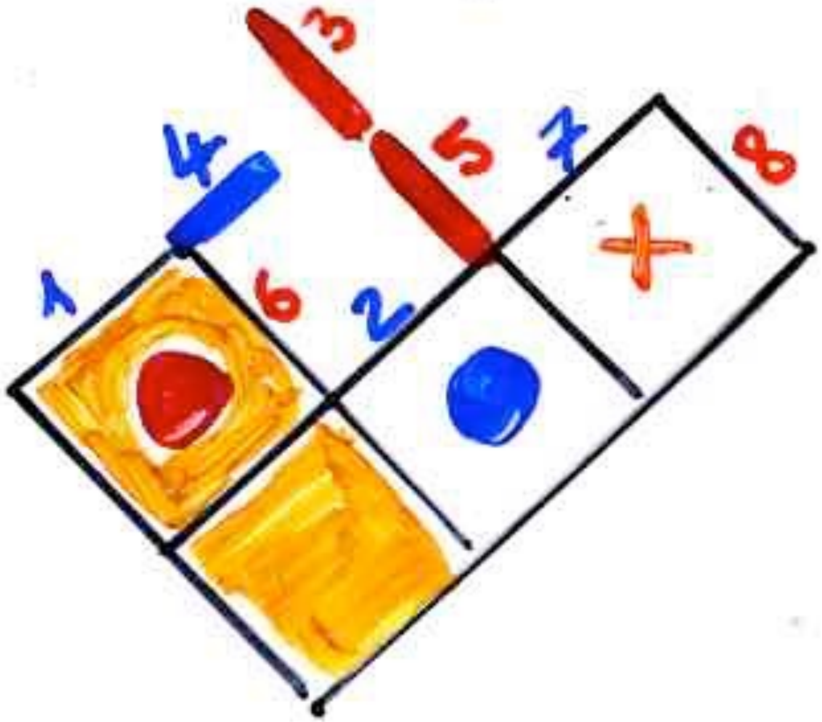
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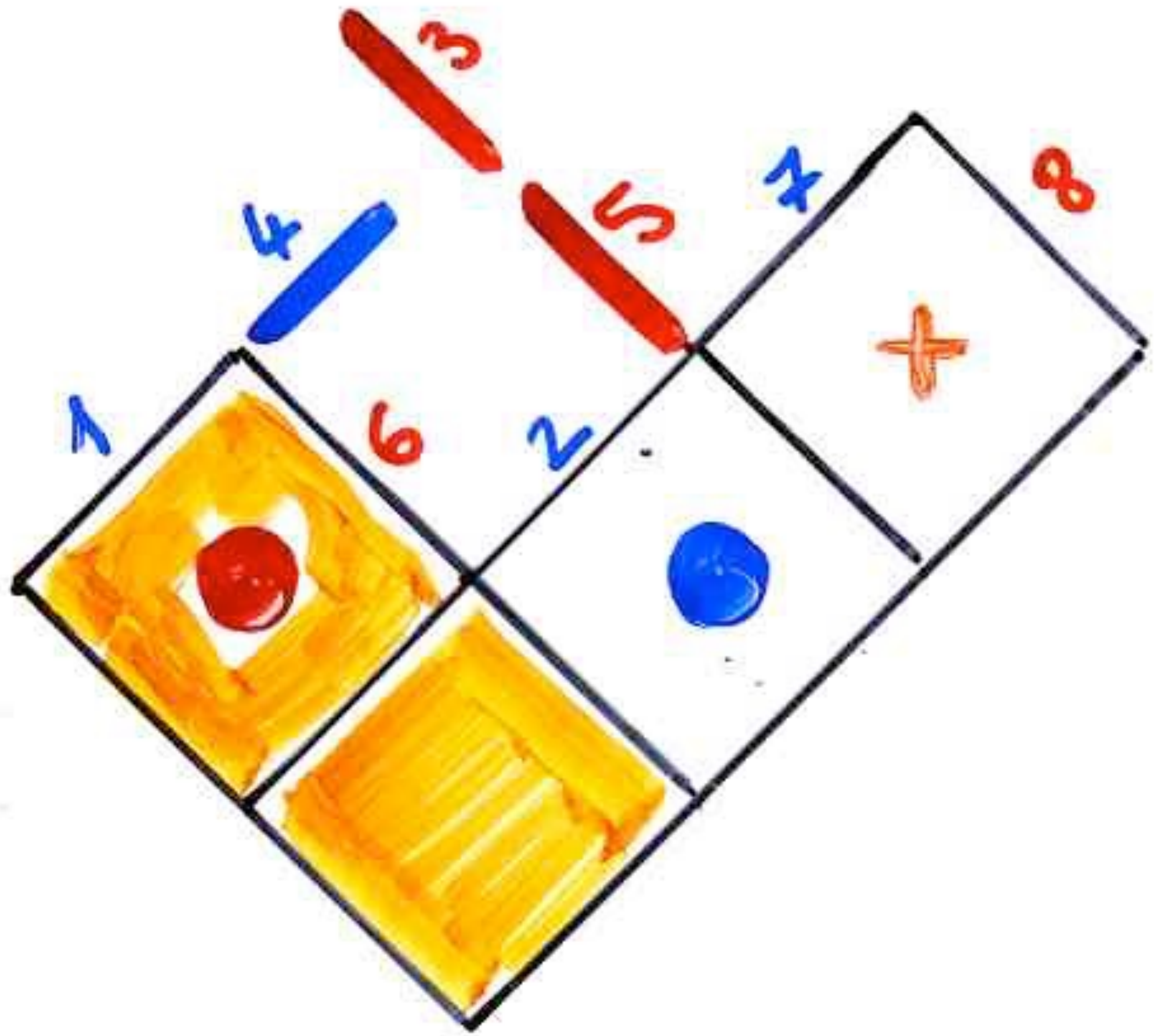
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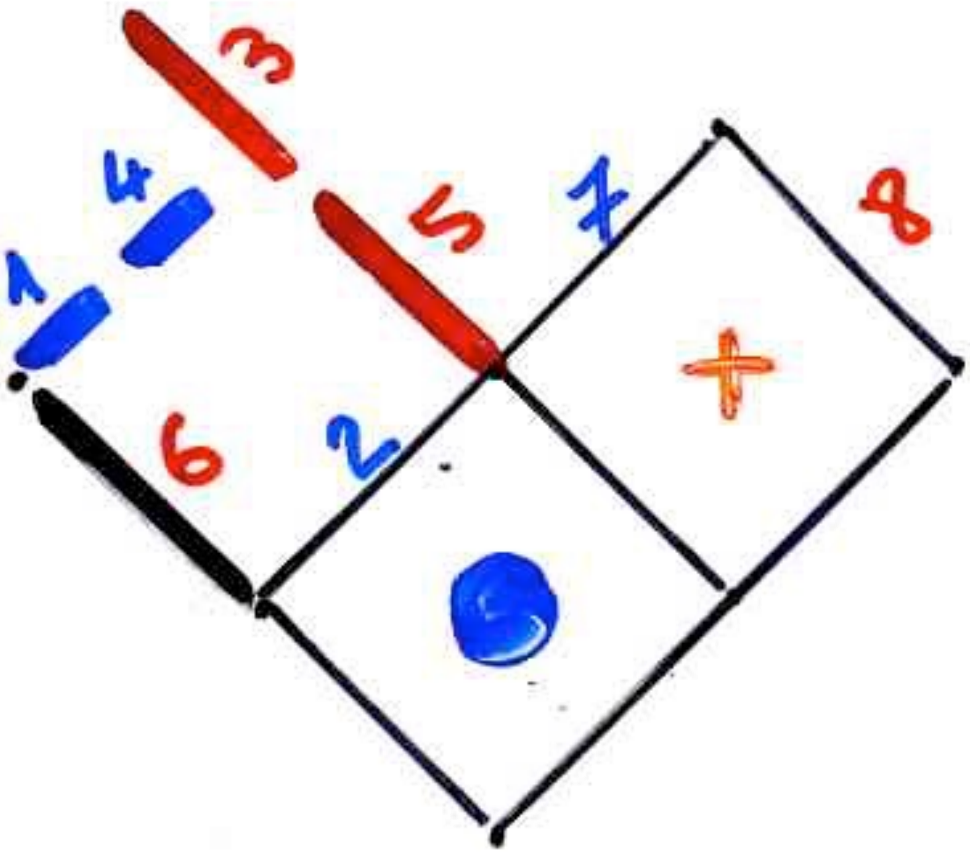

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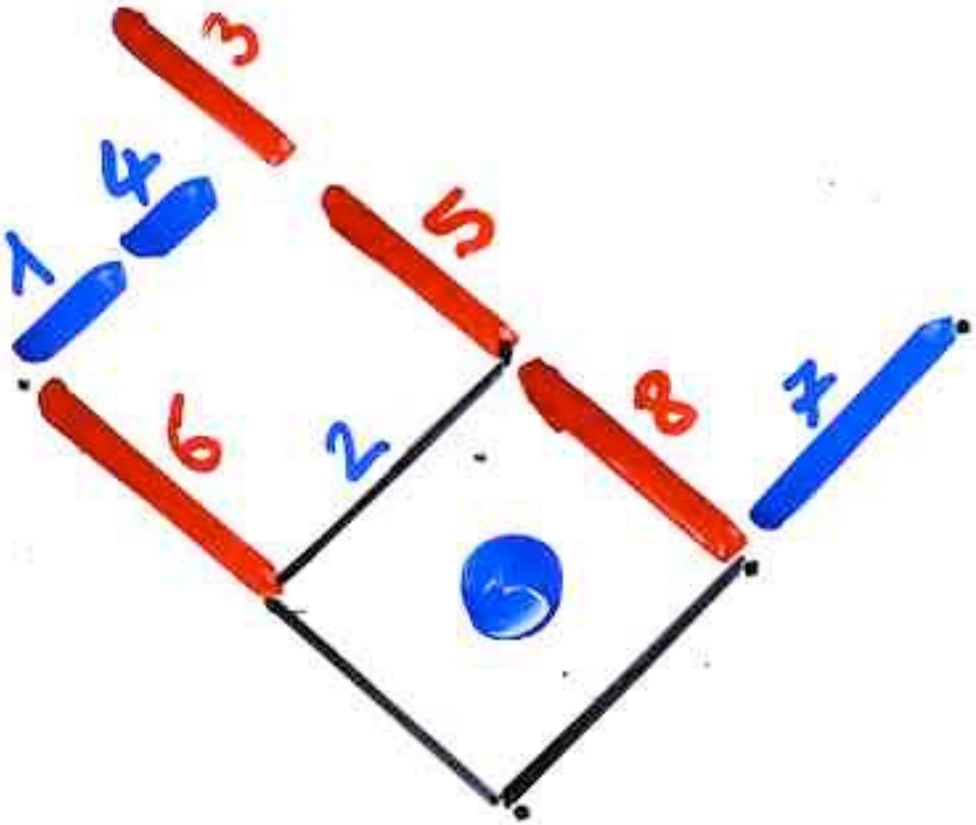

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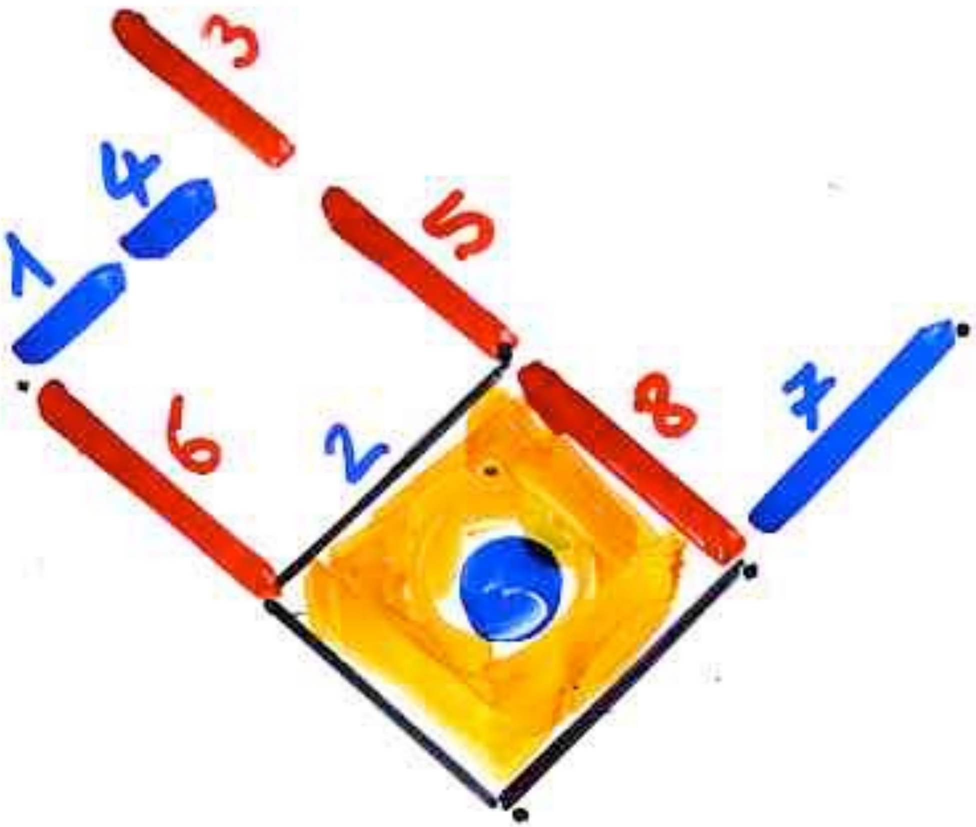

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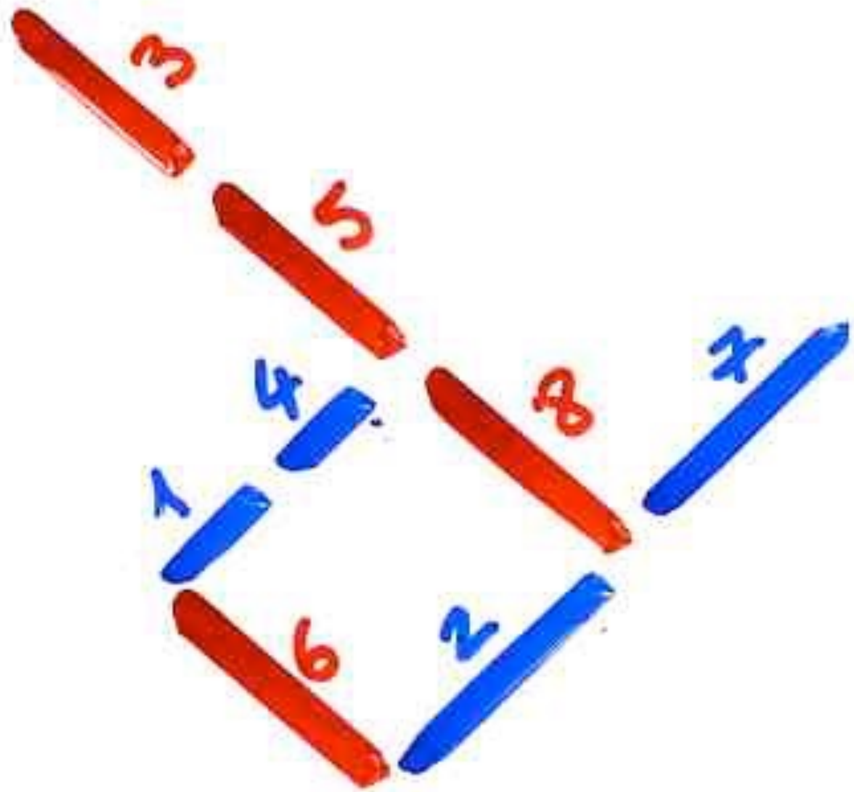
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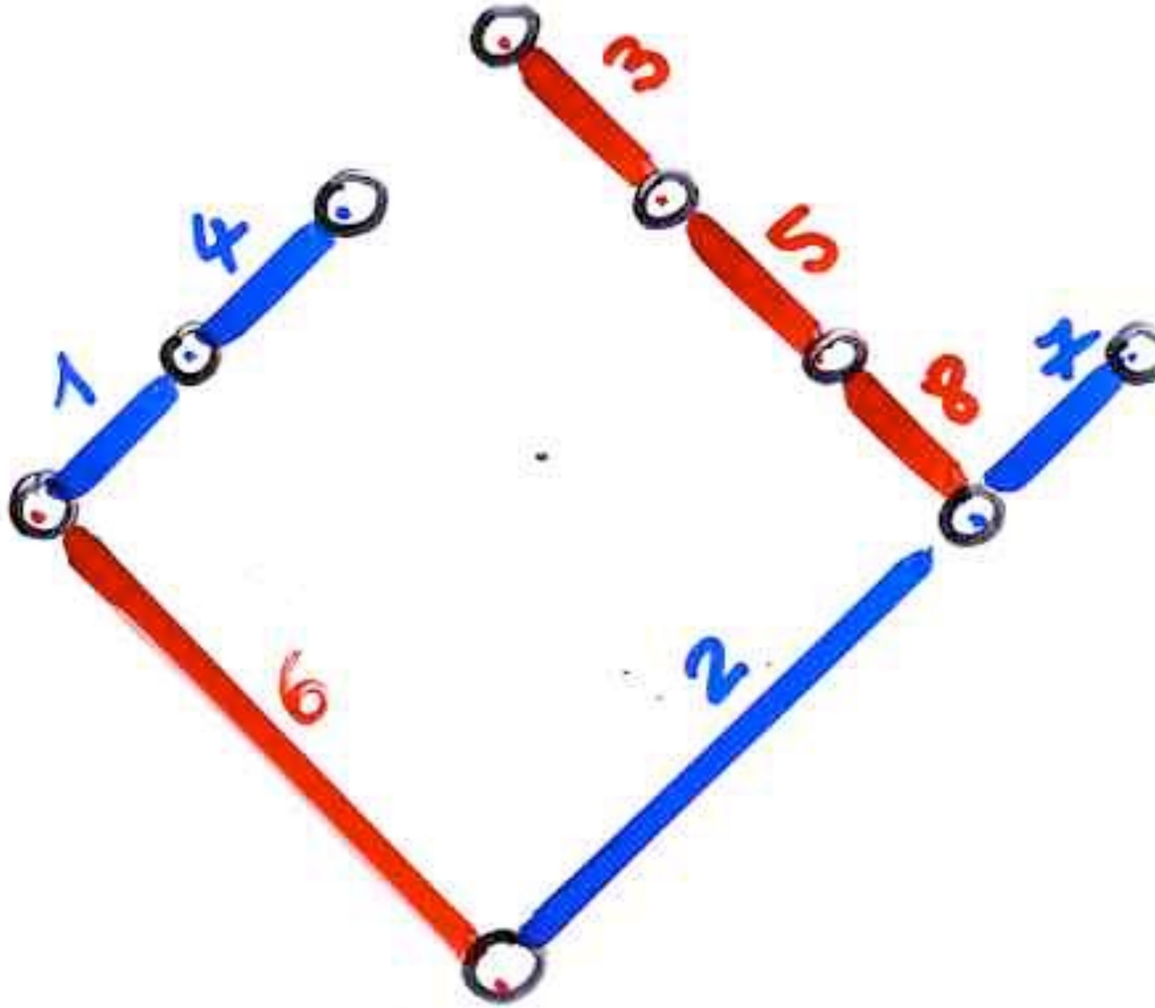

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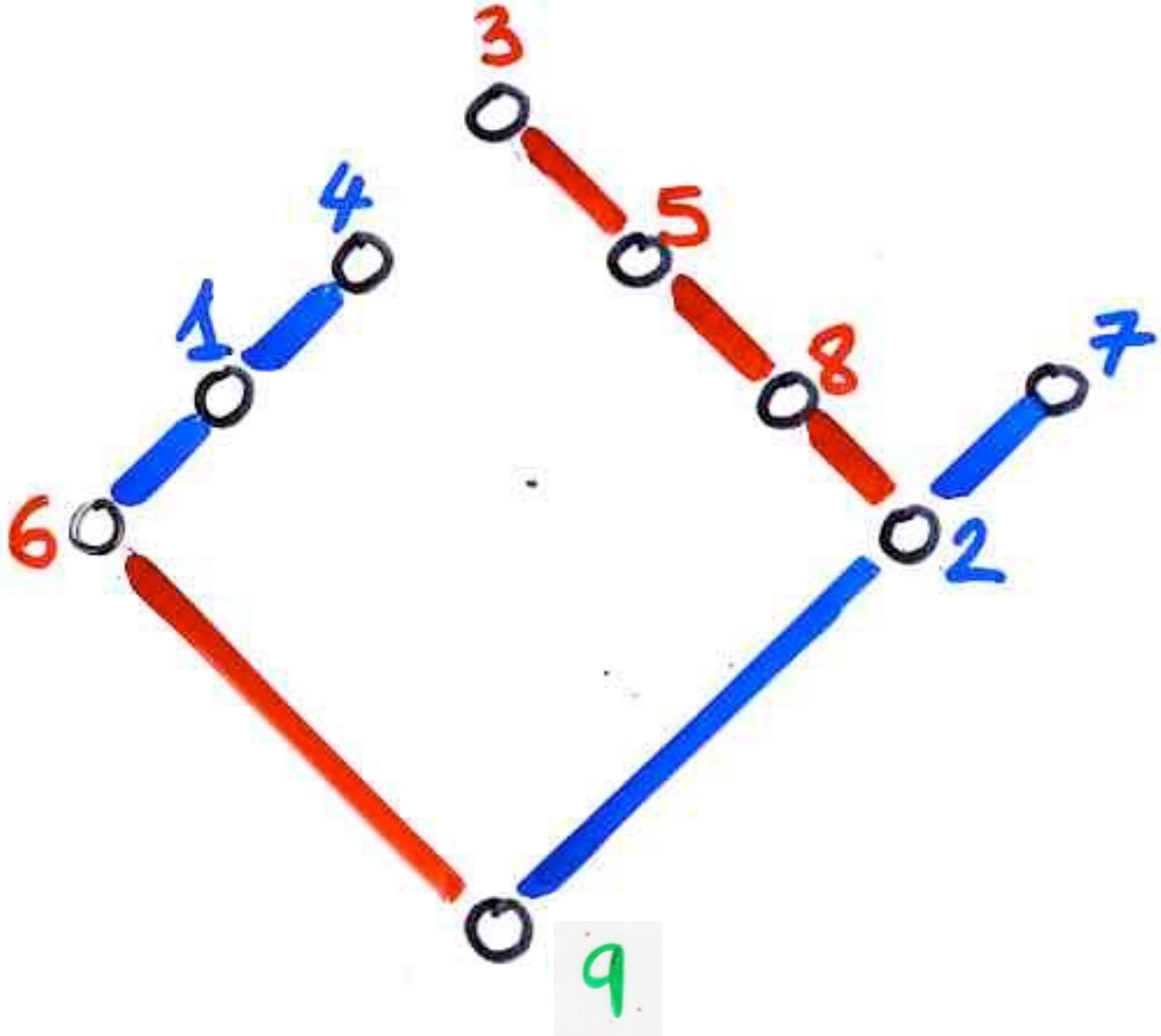
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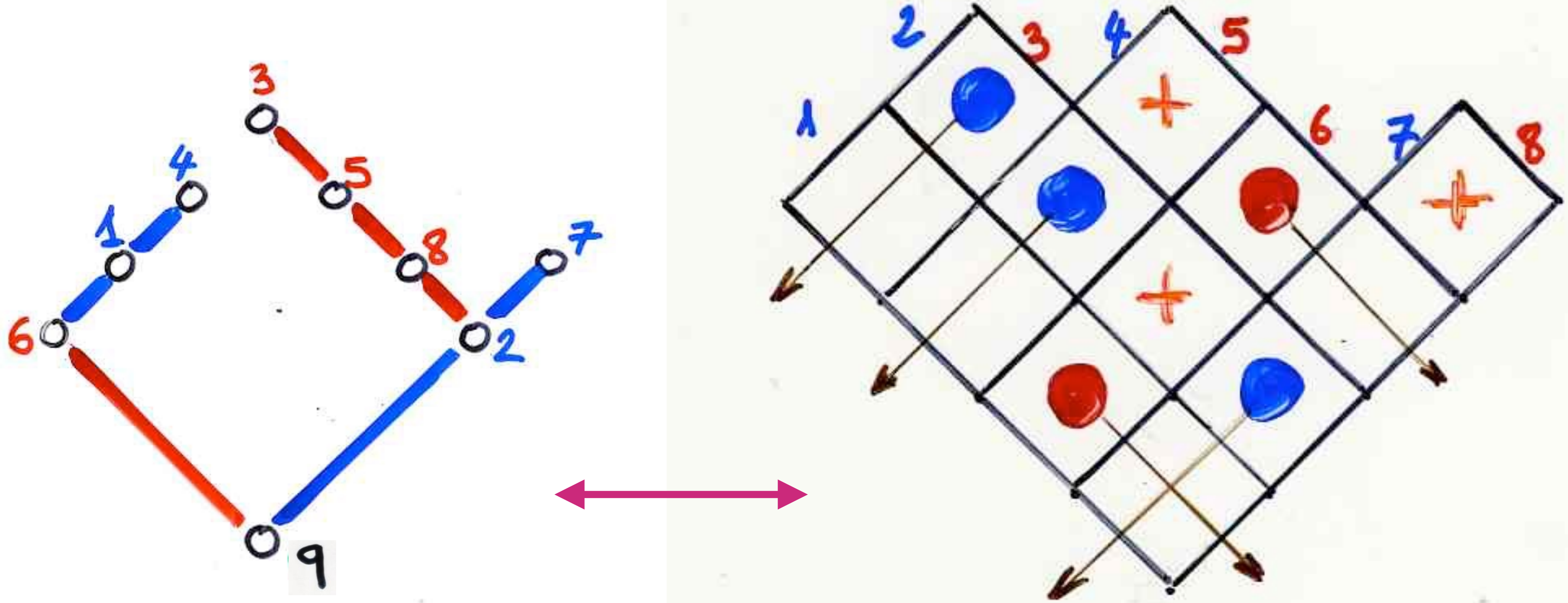
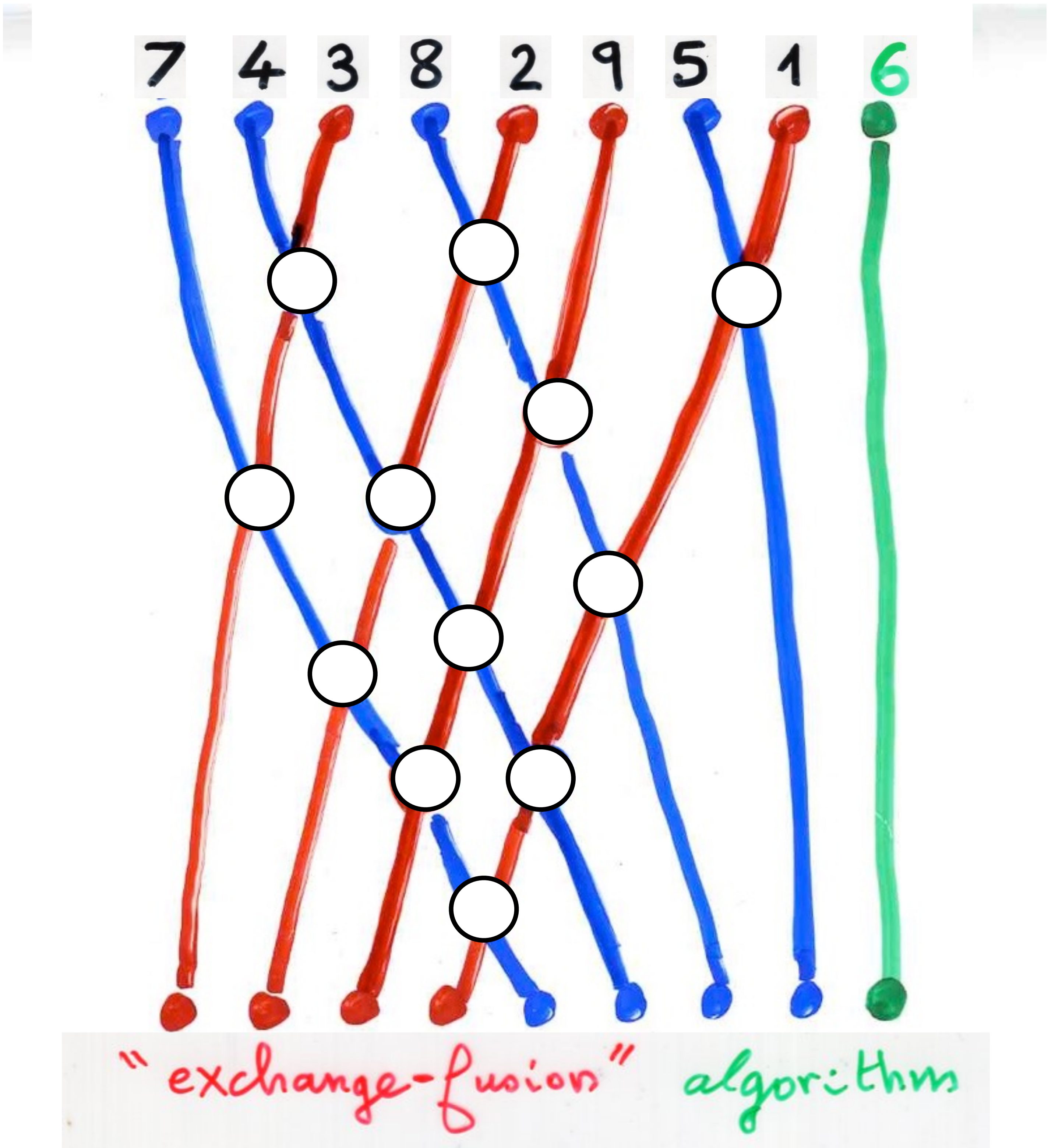
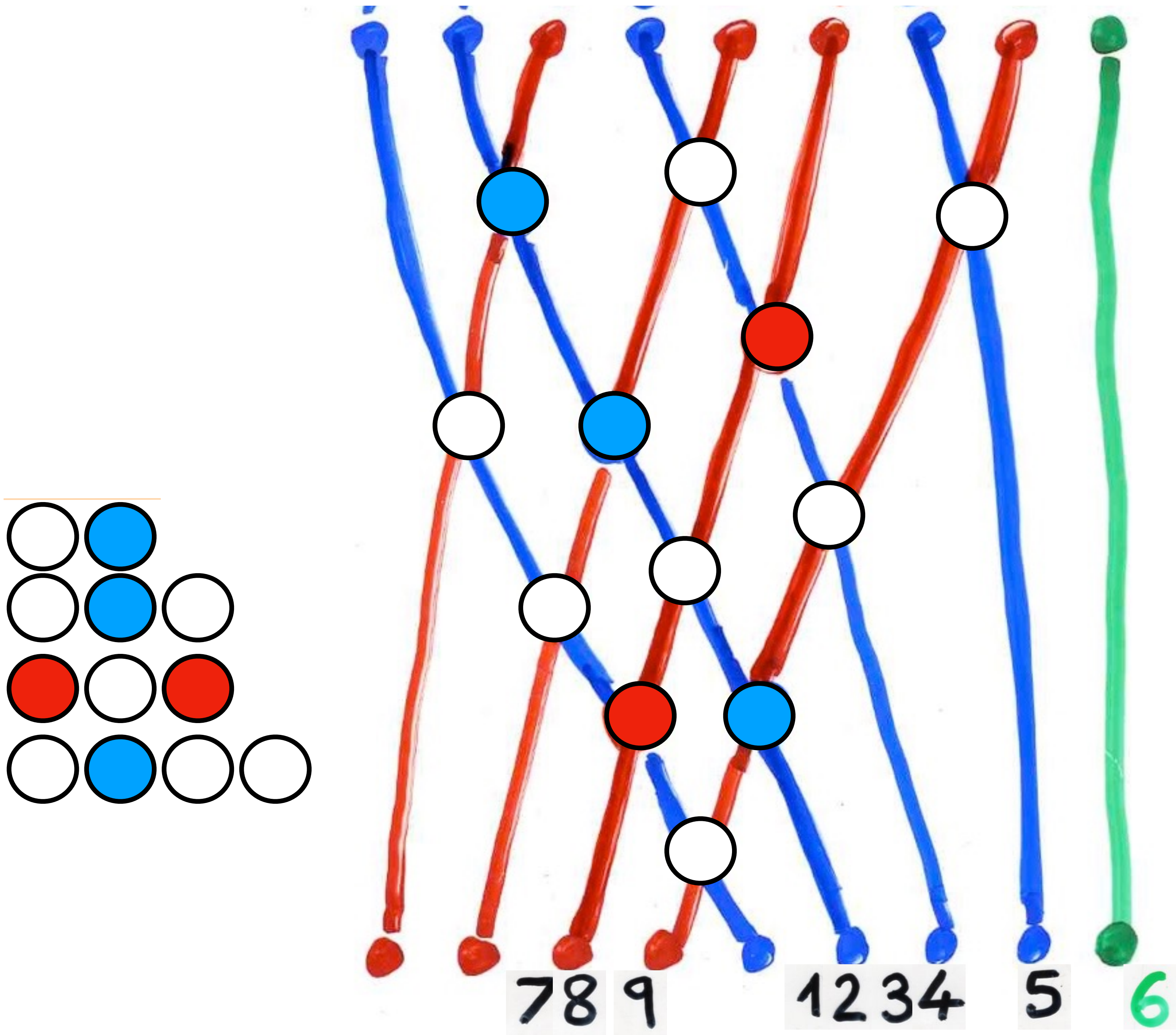


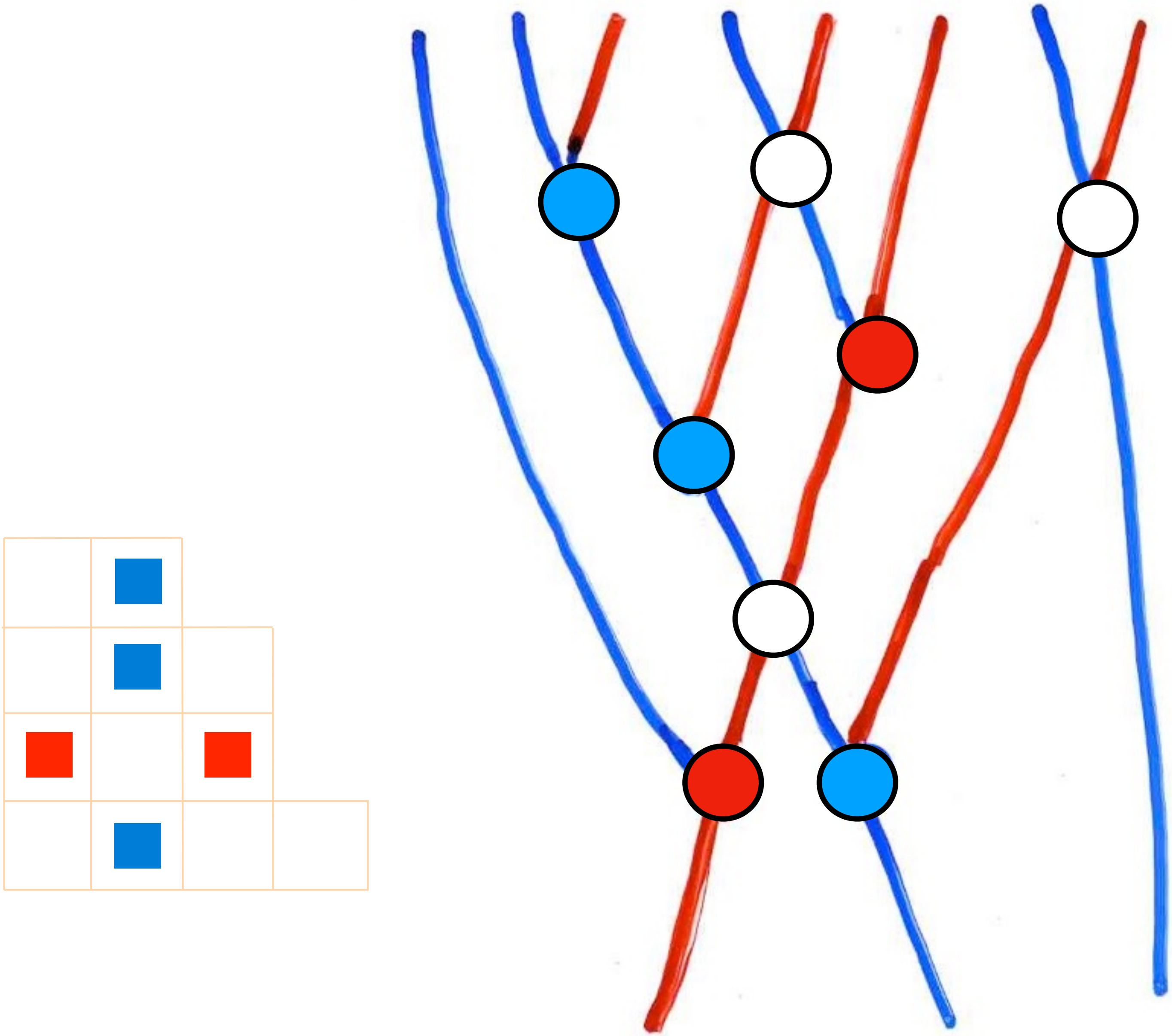
alternative binary trees

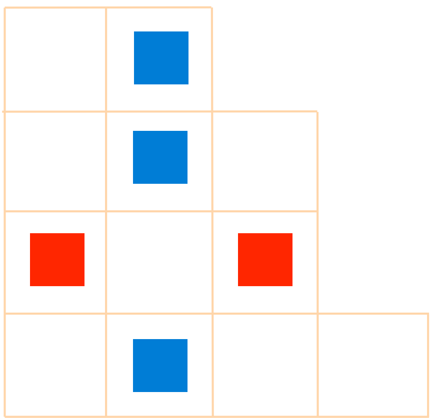
alternative tableaux

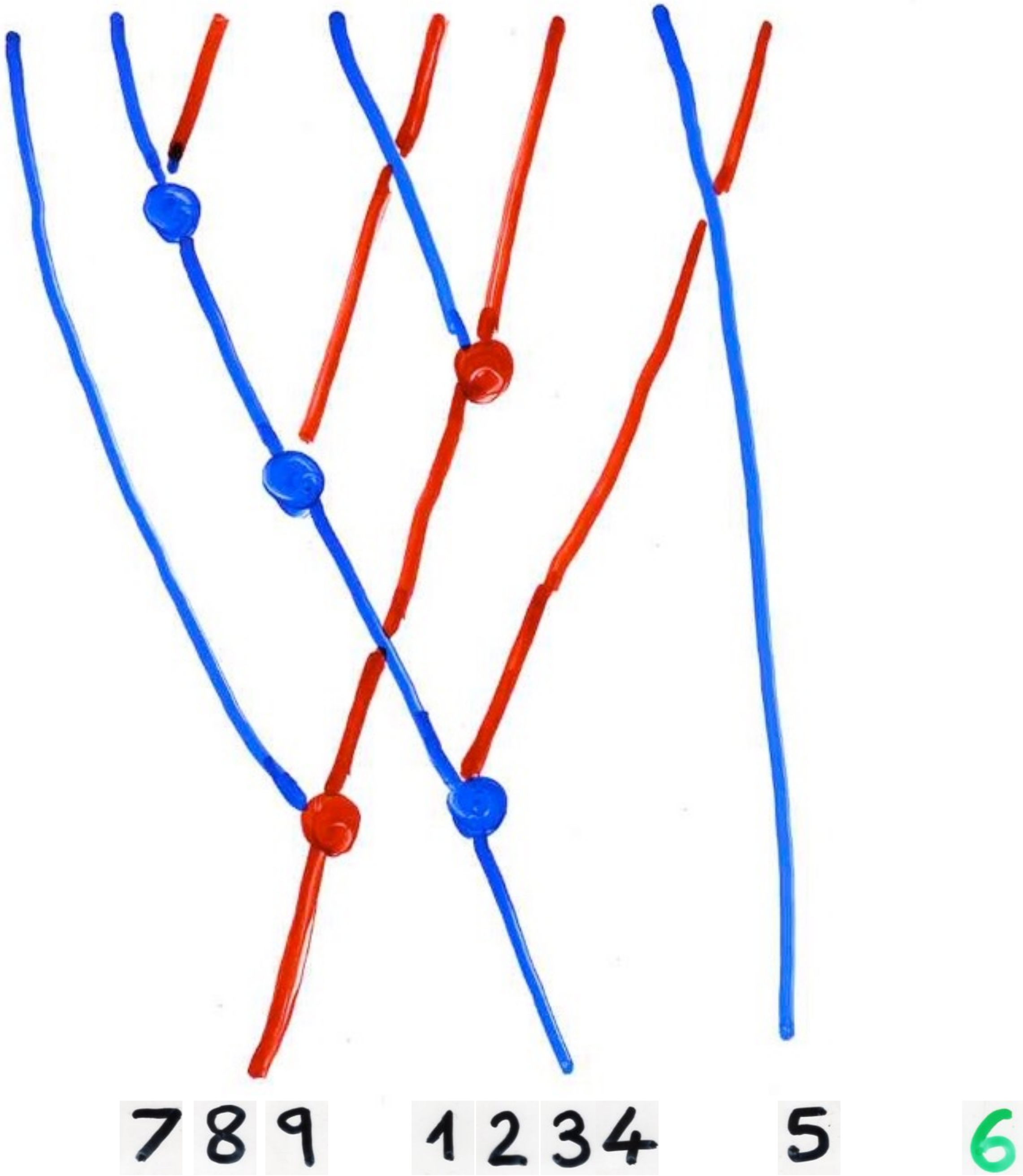
comparison with the exchange-fusion algorithm


7 4 3 8 2 9 5 1 6
"exchange-fusion" algorithm


78 9 1234 5 6



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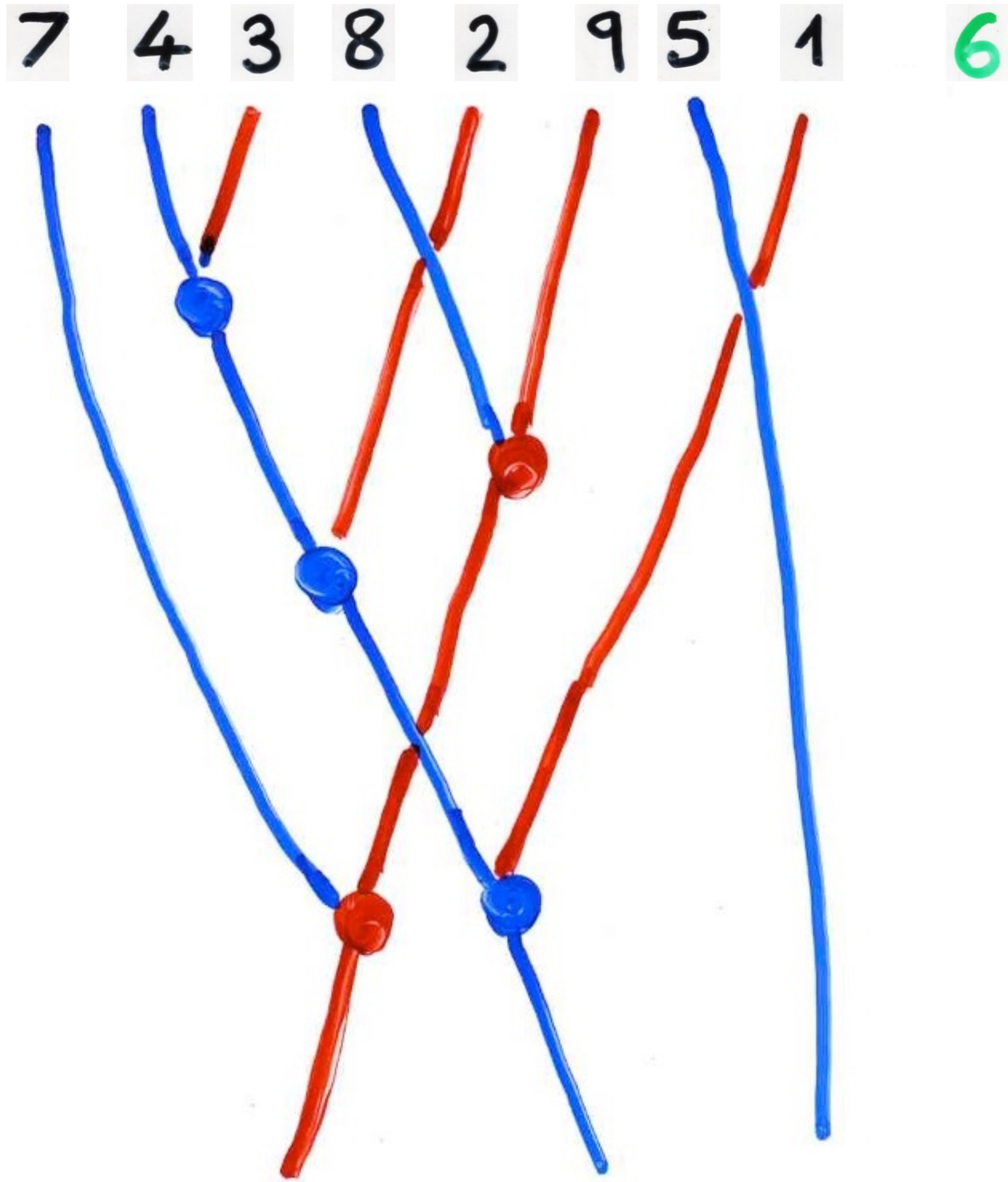
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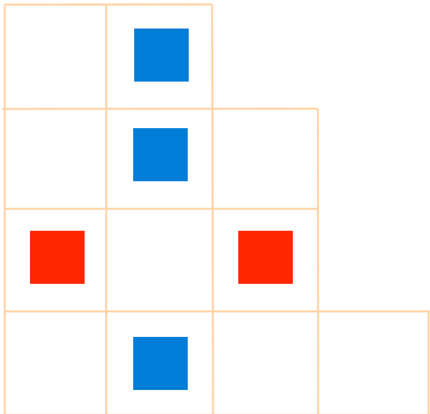
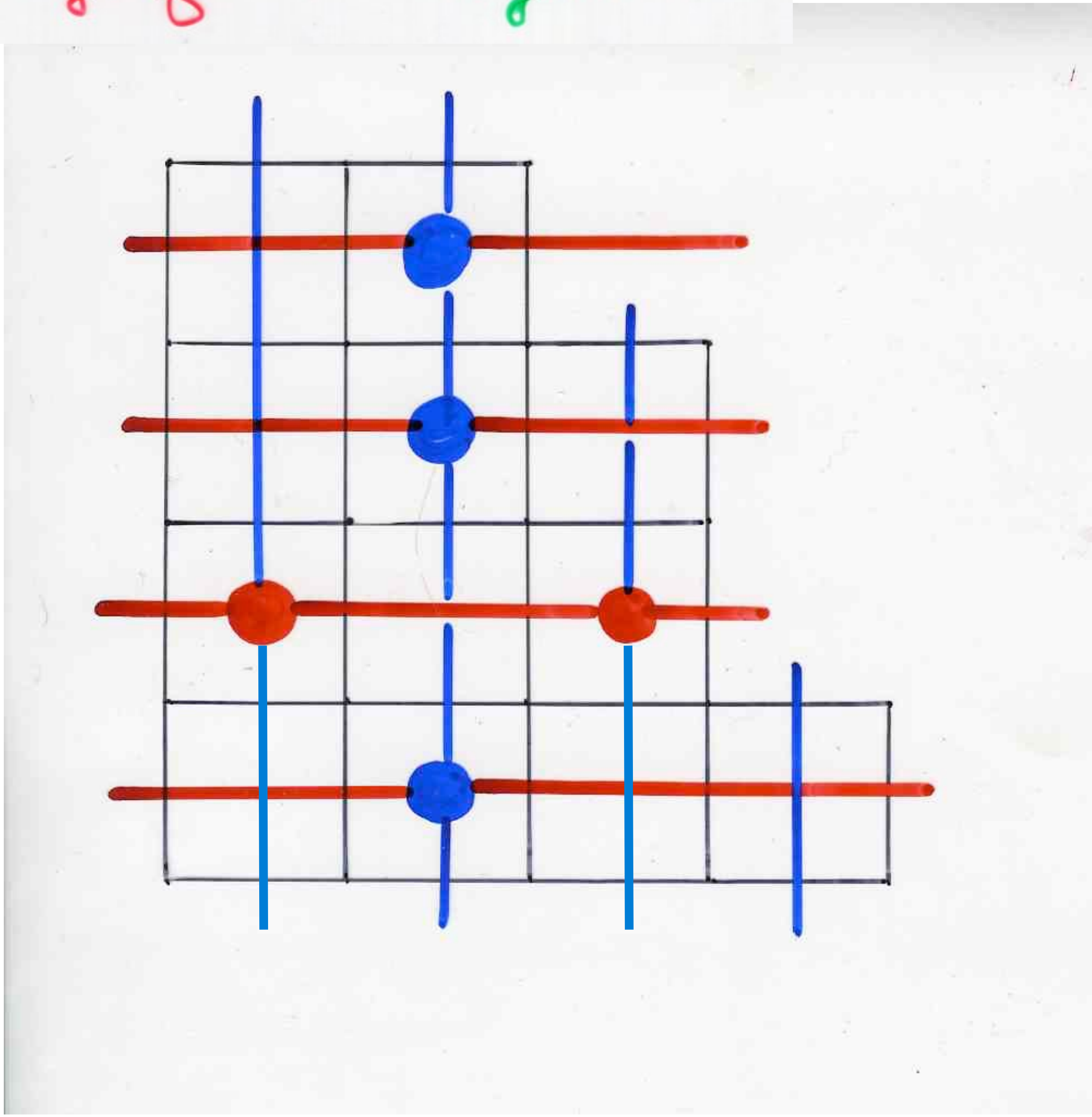
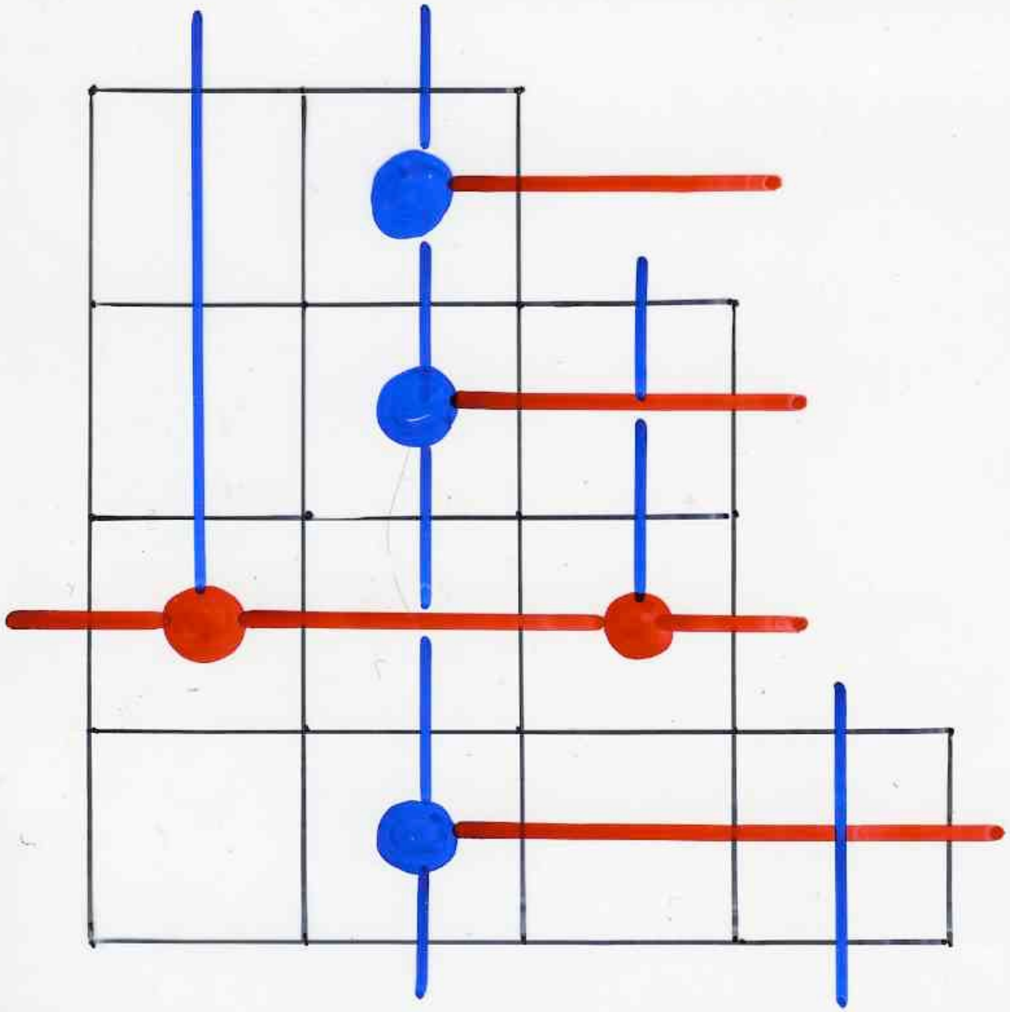


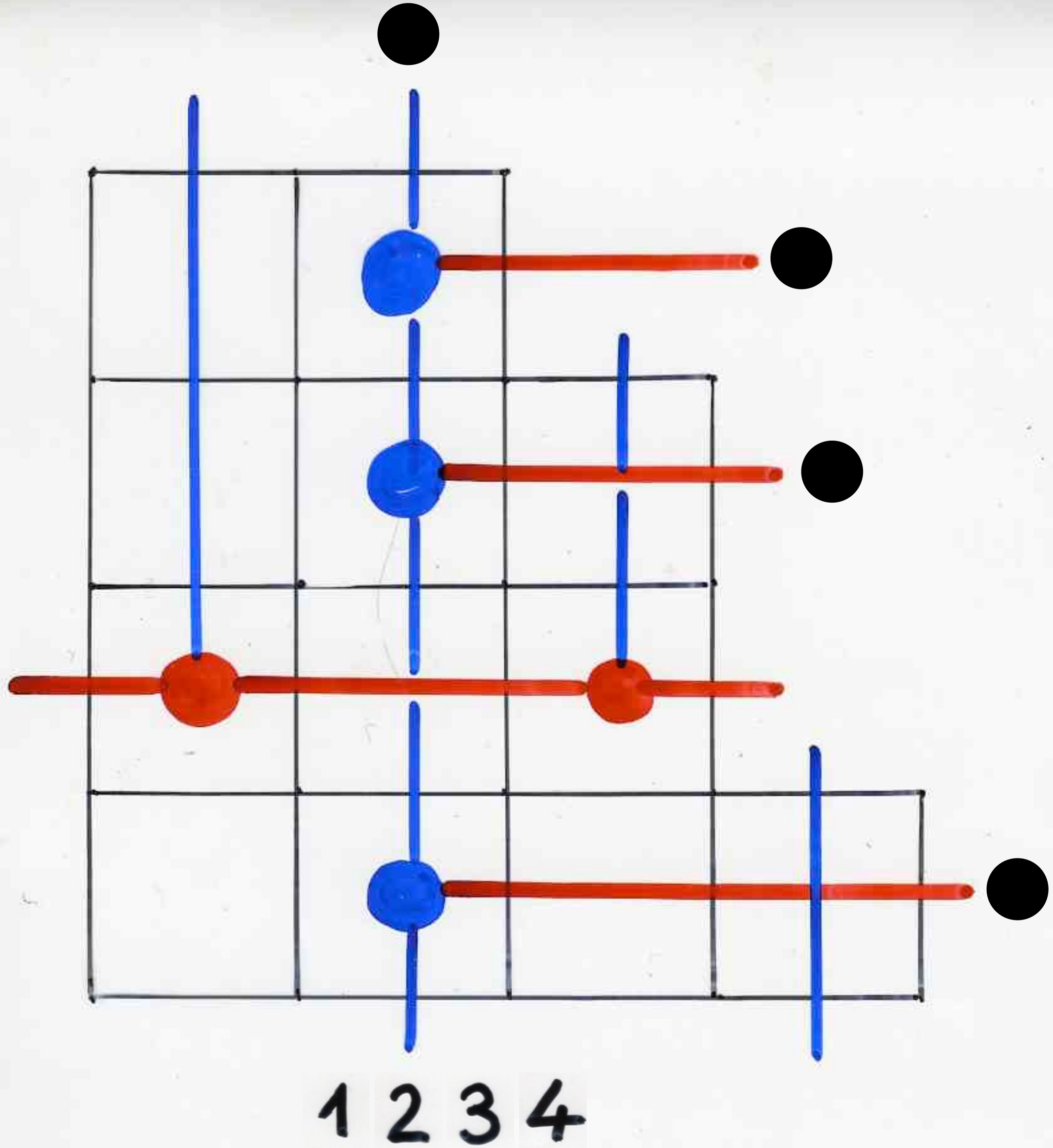
bijection alternative tableaux size n

permutations on (n+1)

"exchange-fusion" algorithms



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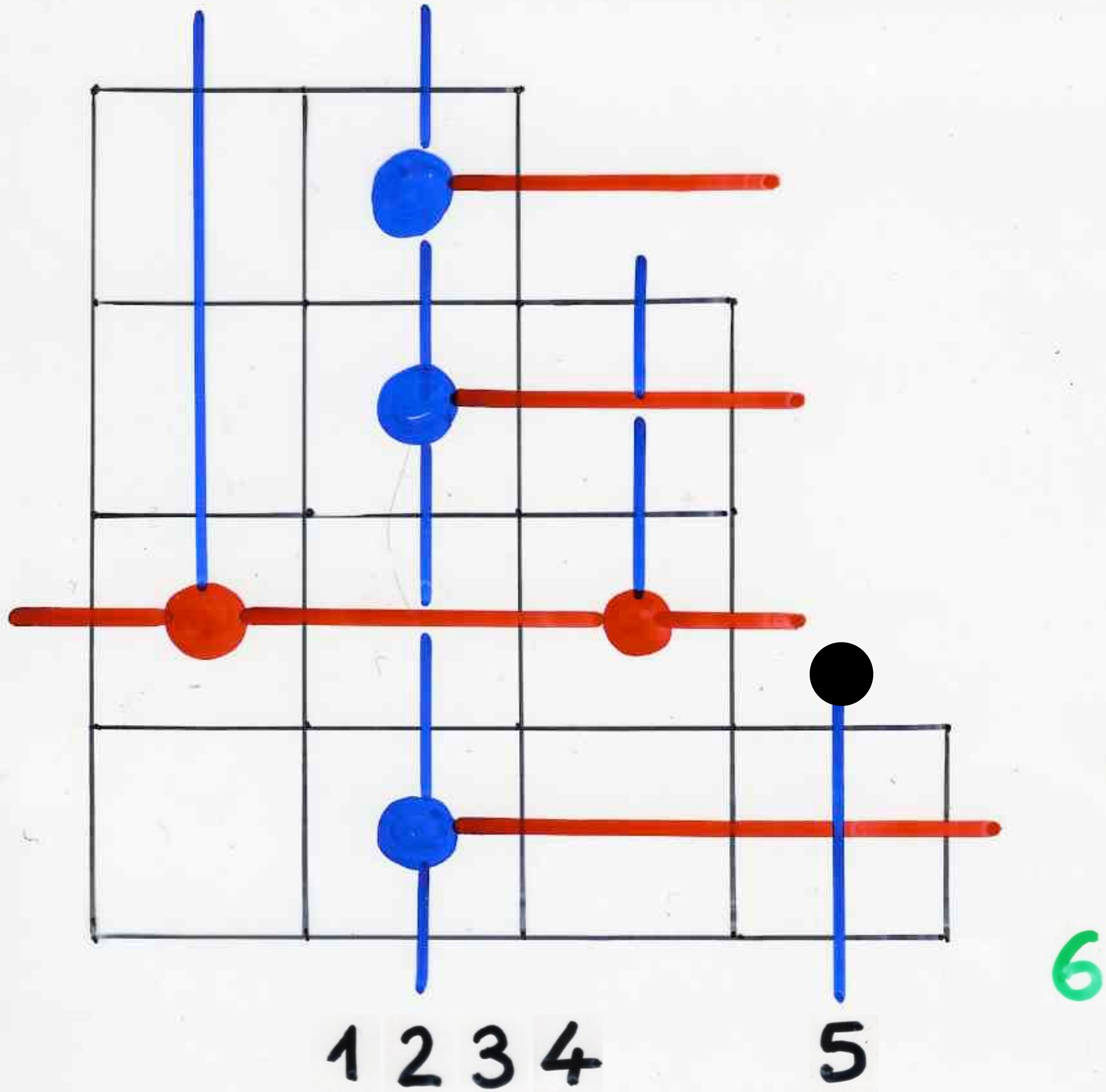
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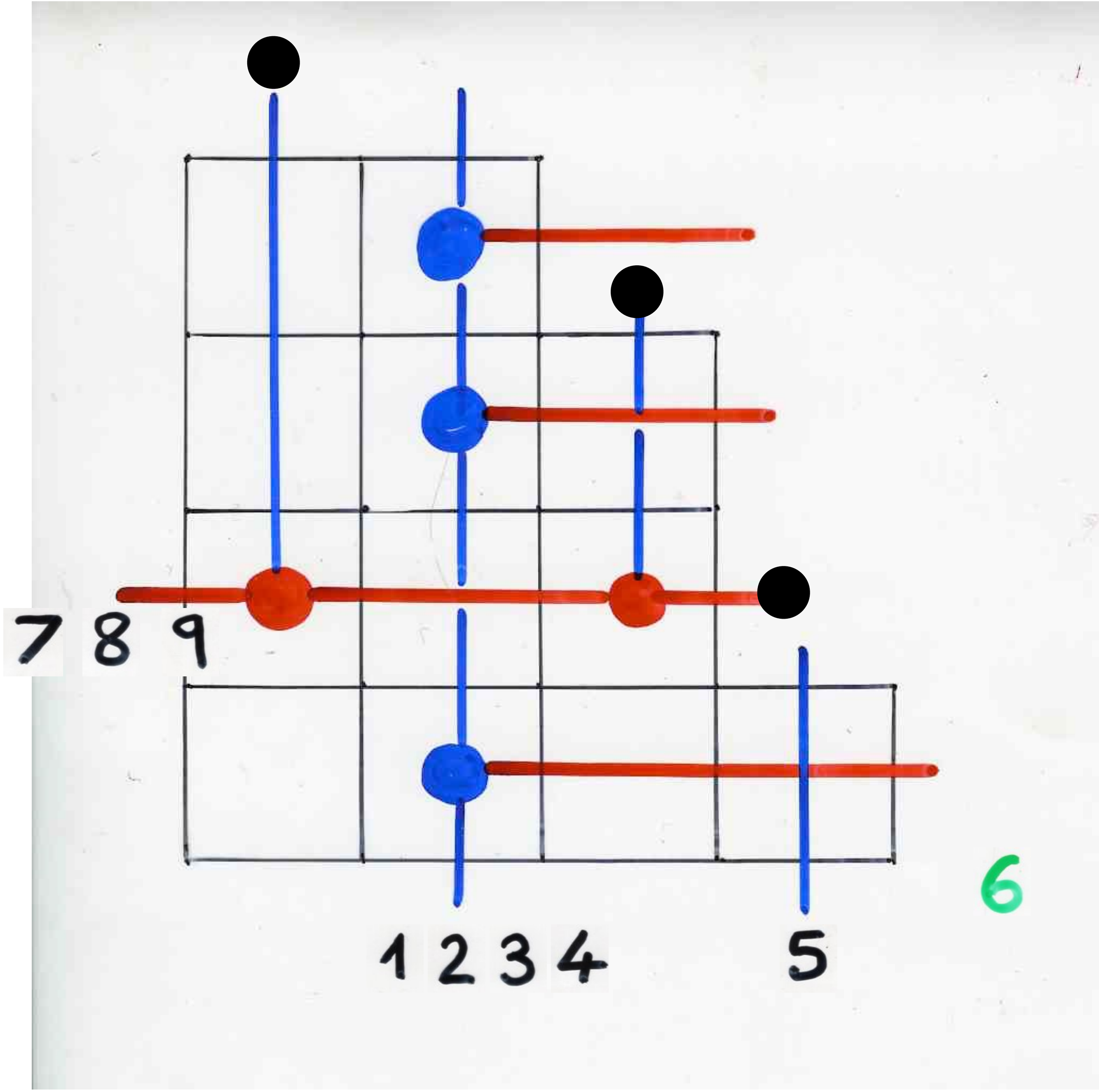

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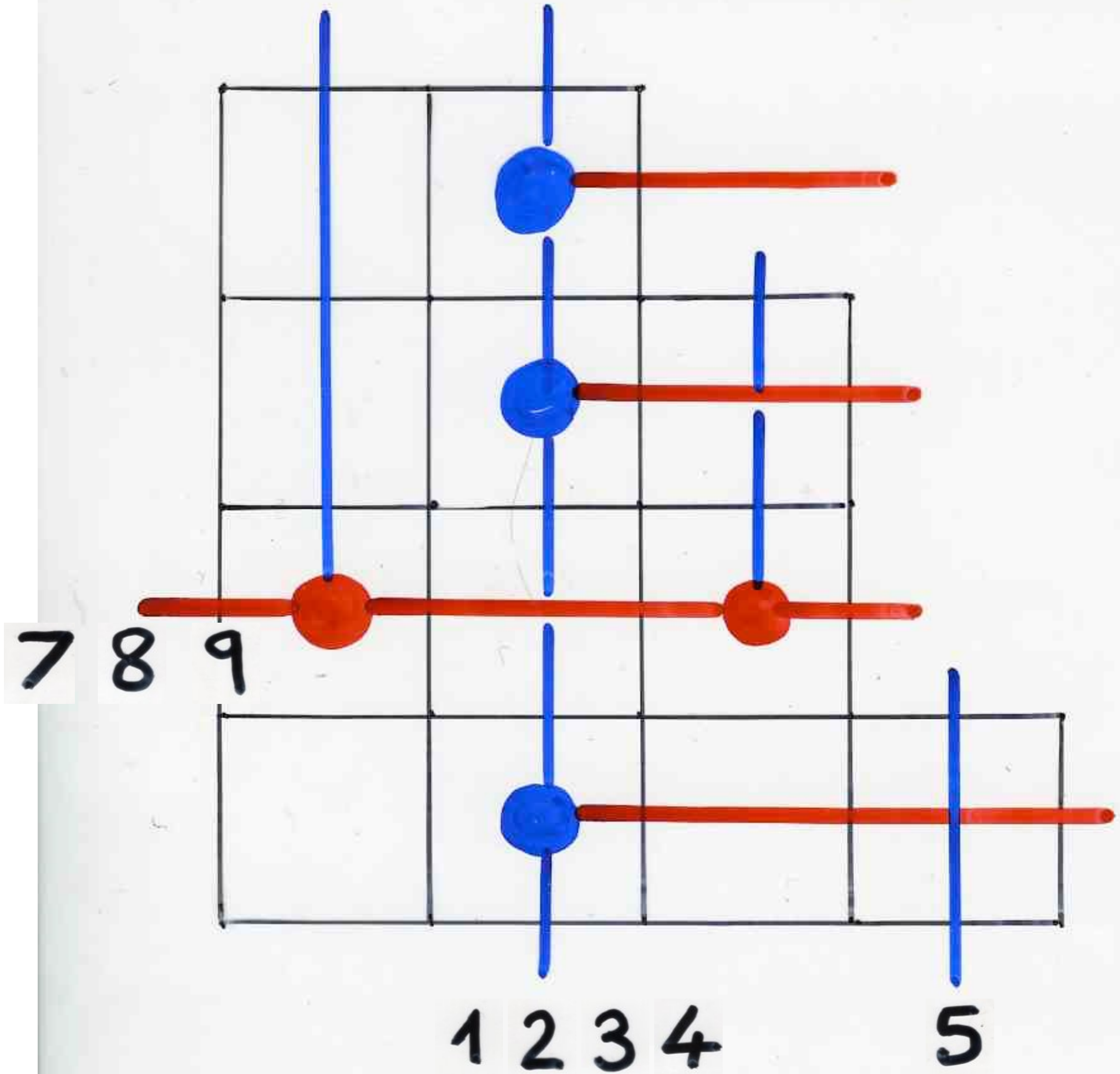

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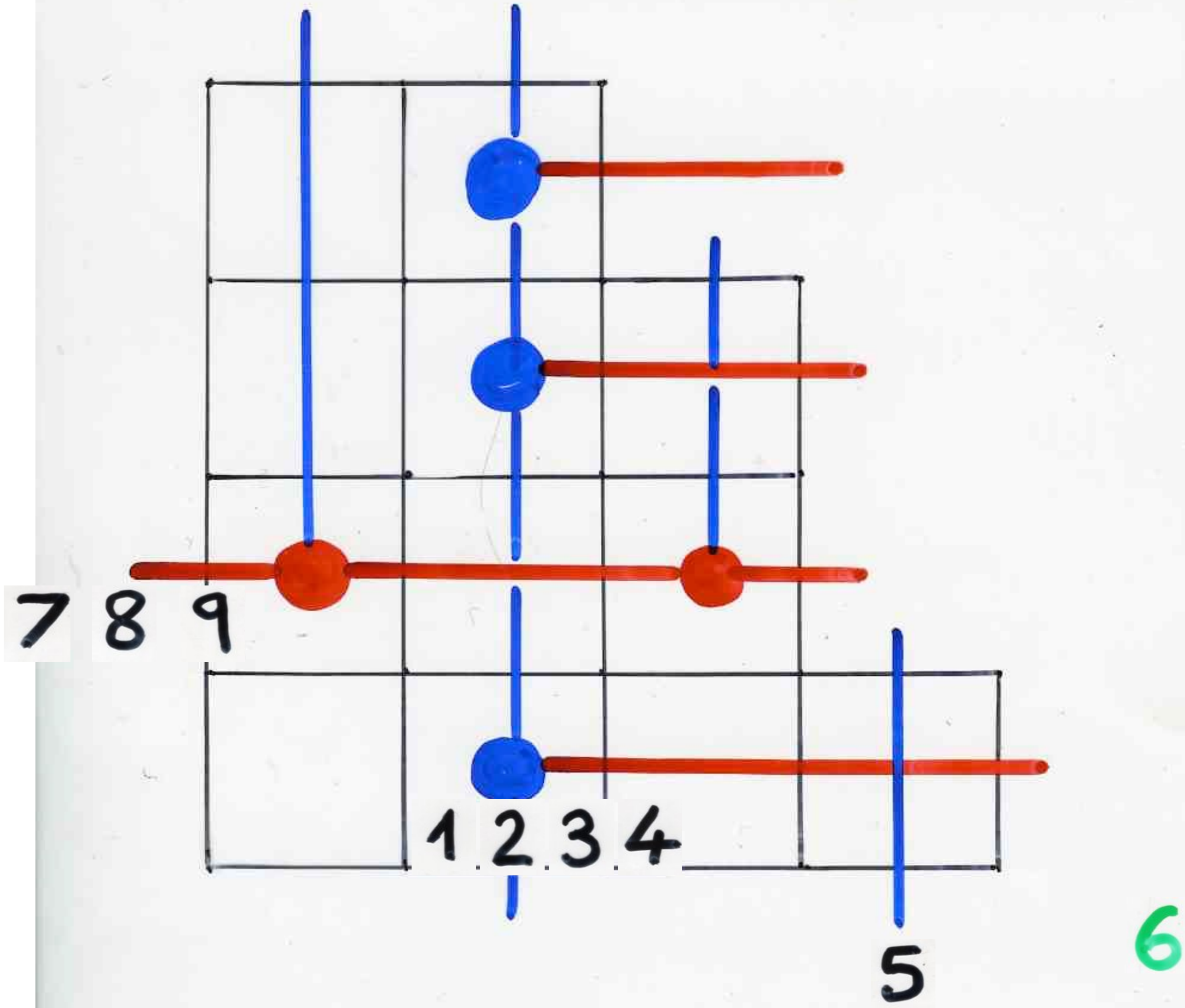
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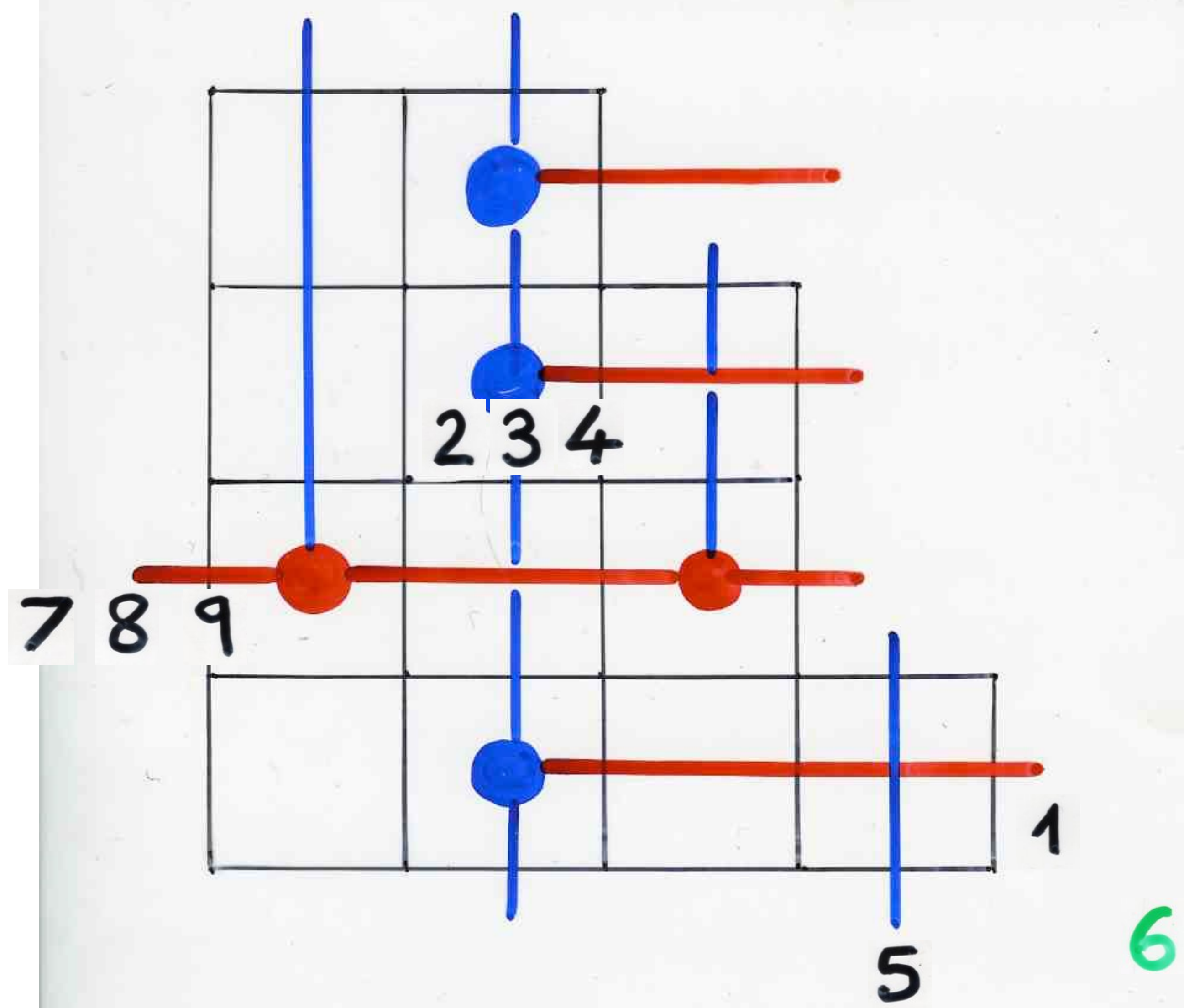

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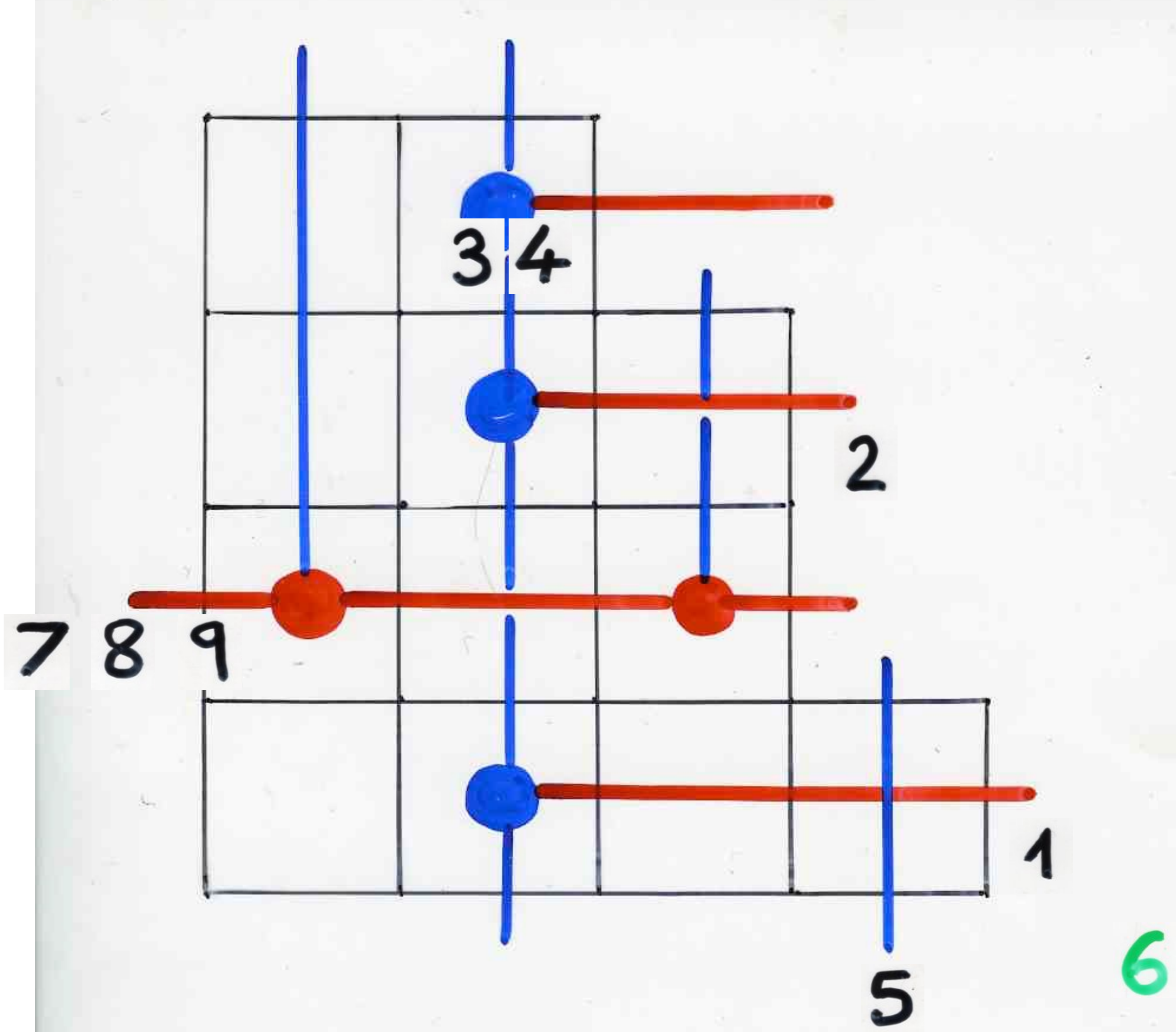
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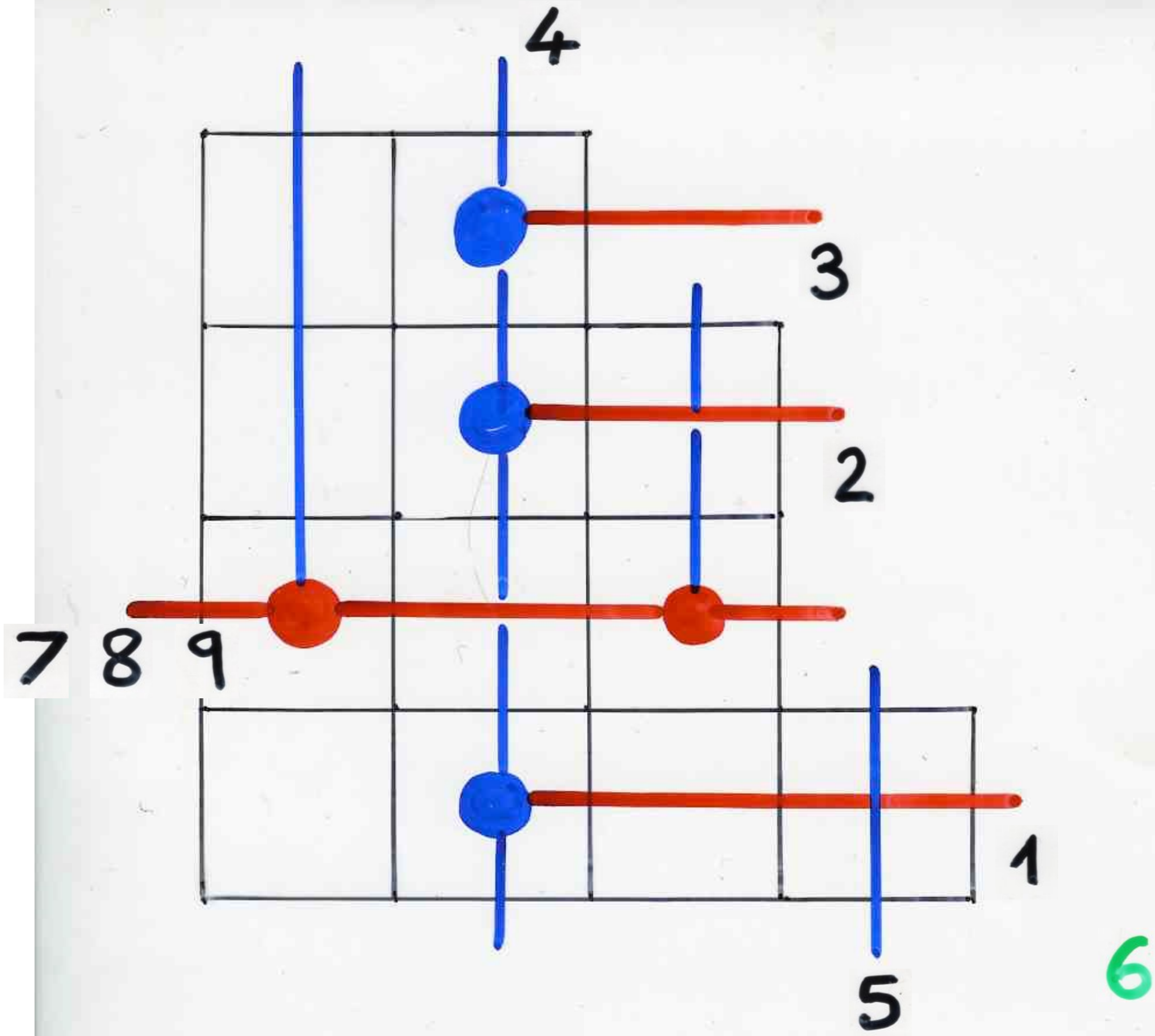
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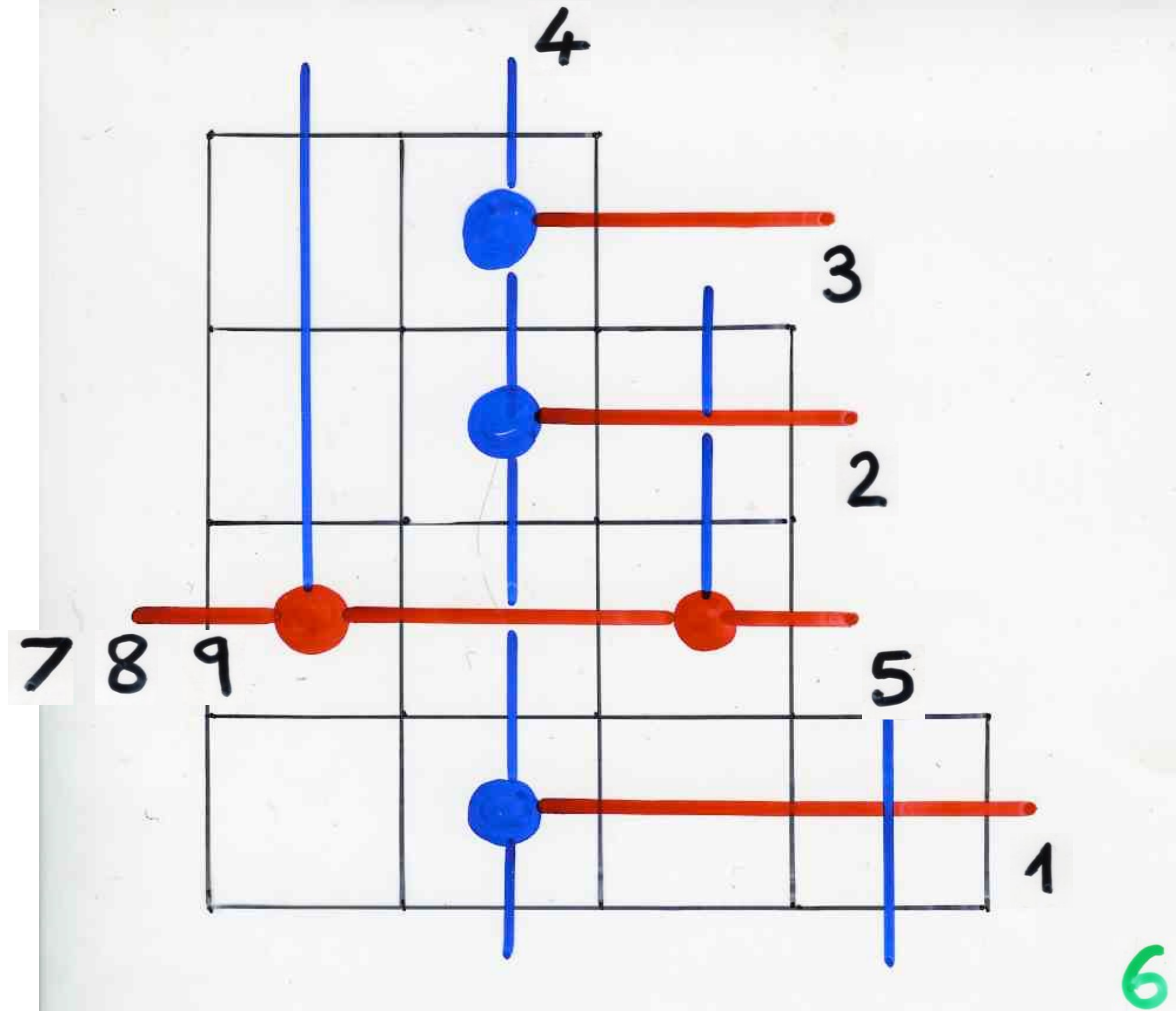
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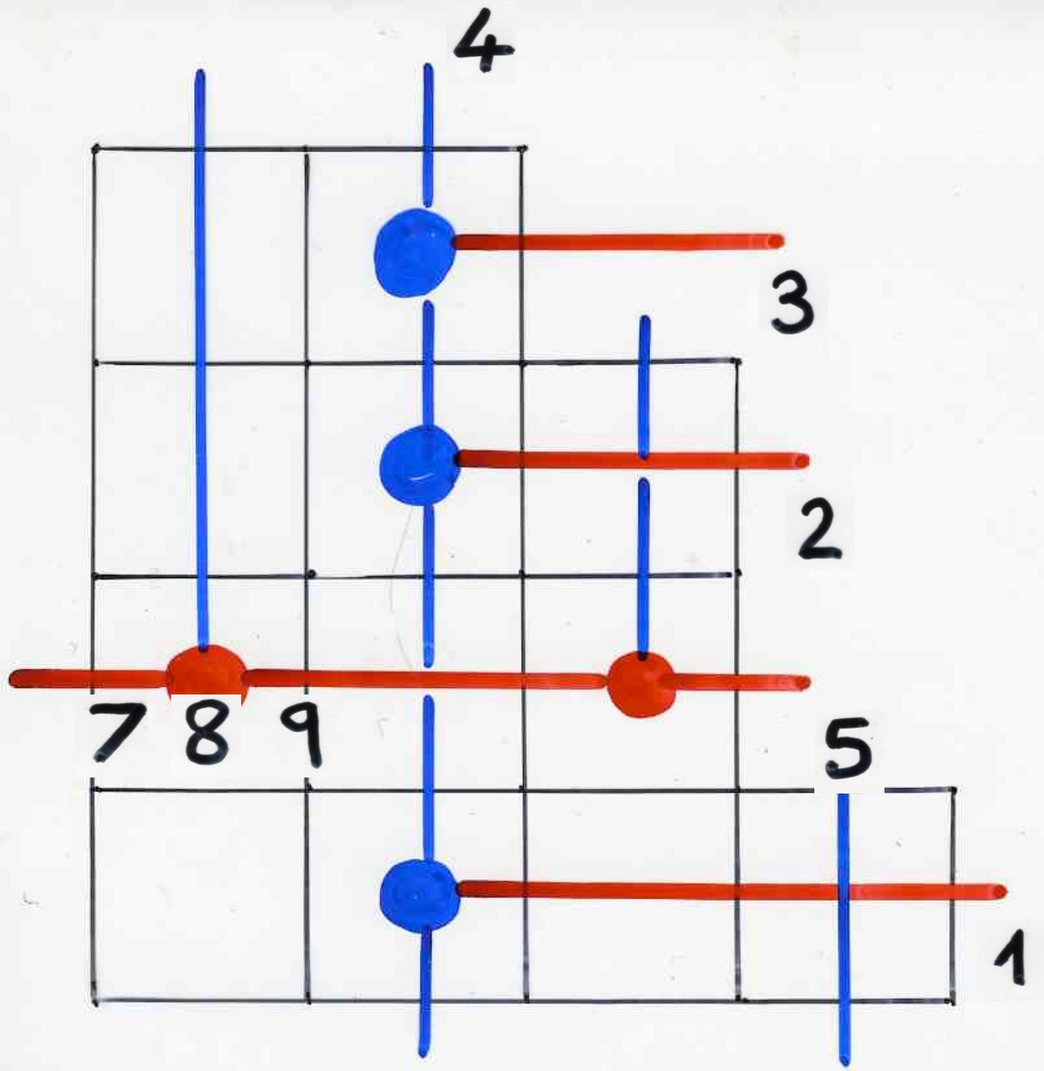

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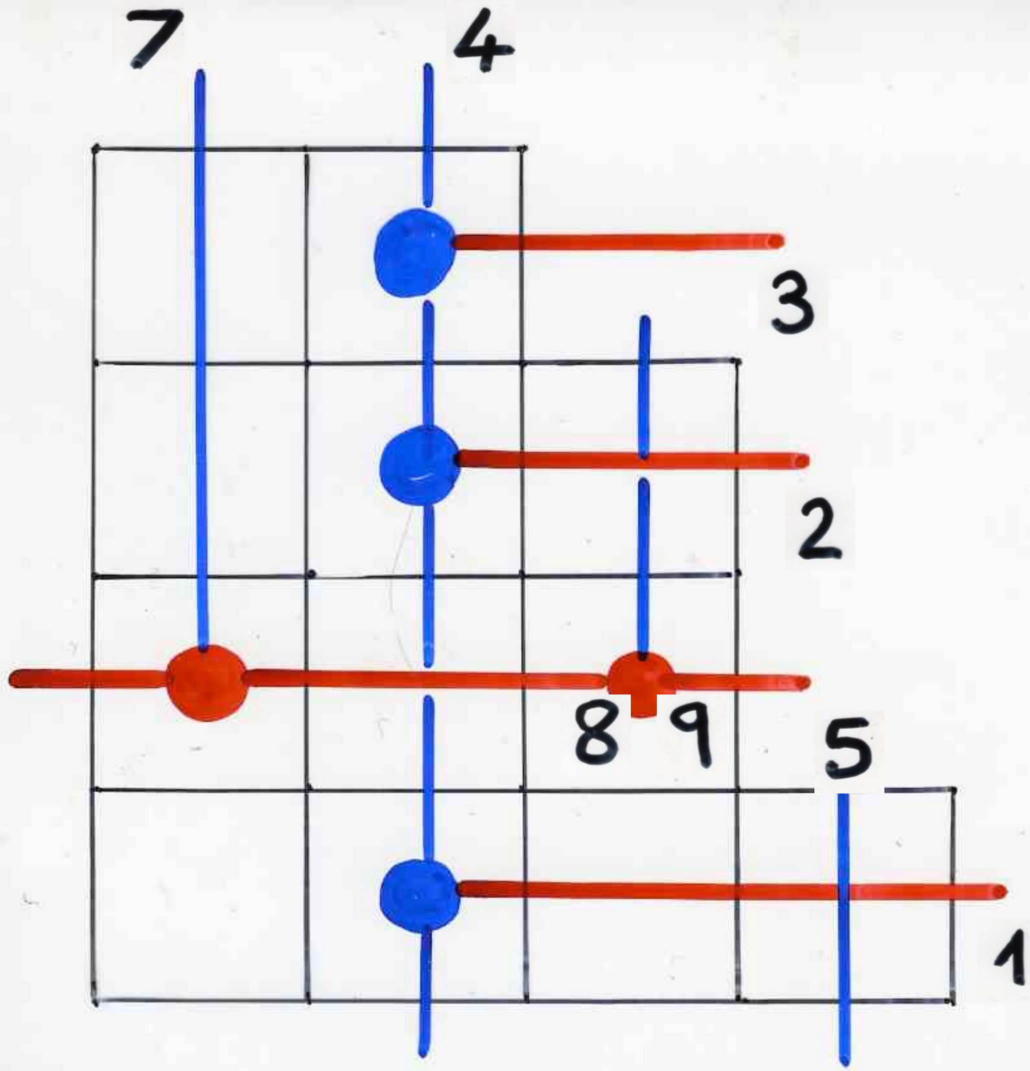

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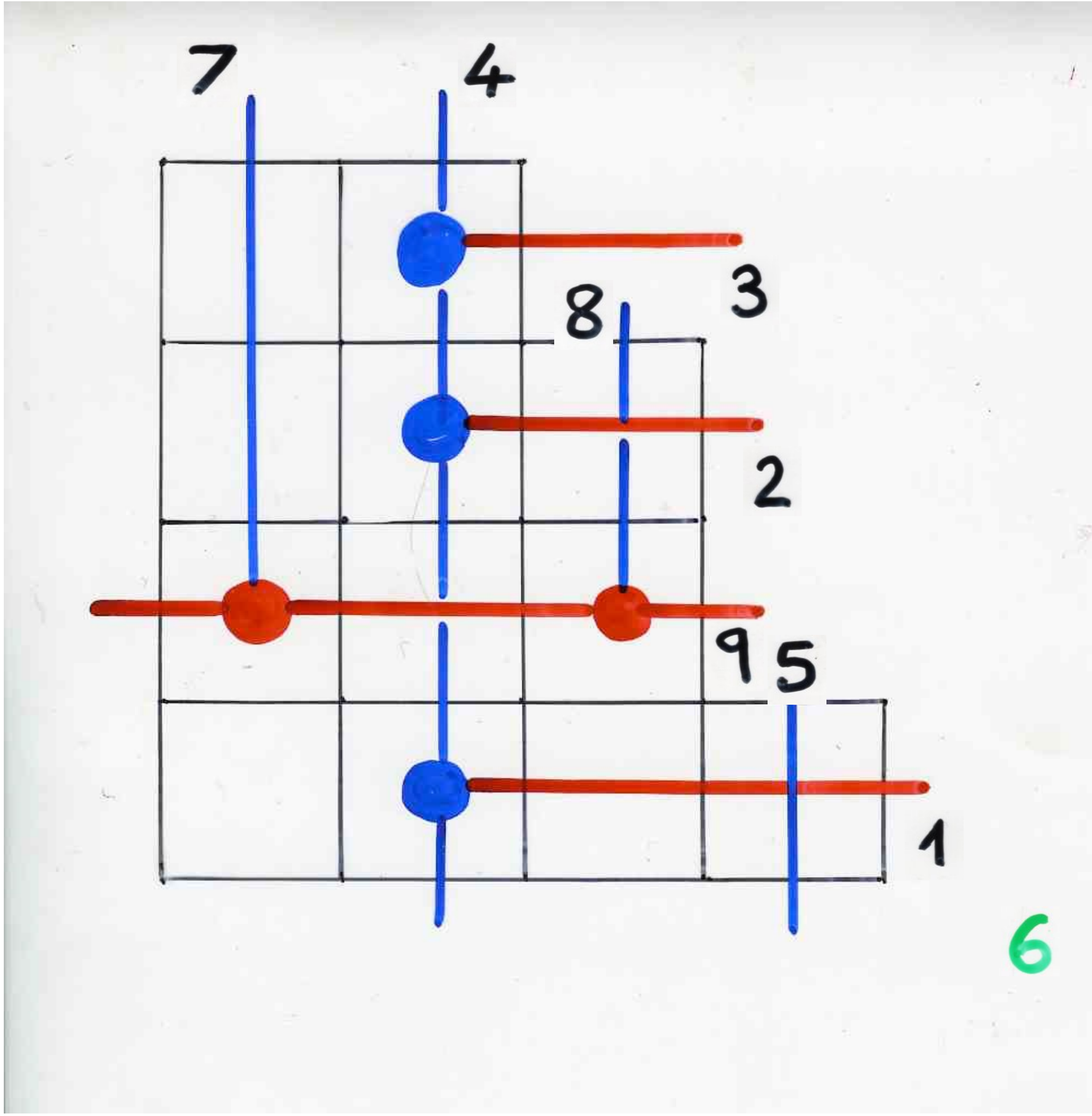

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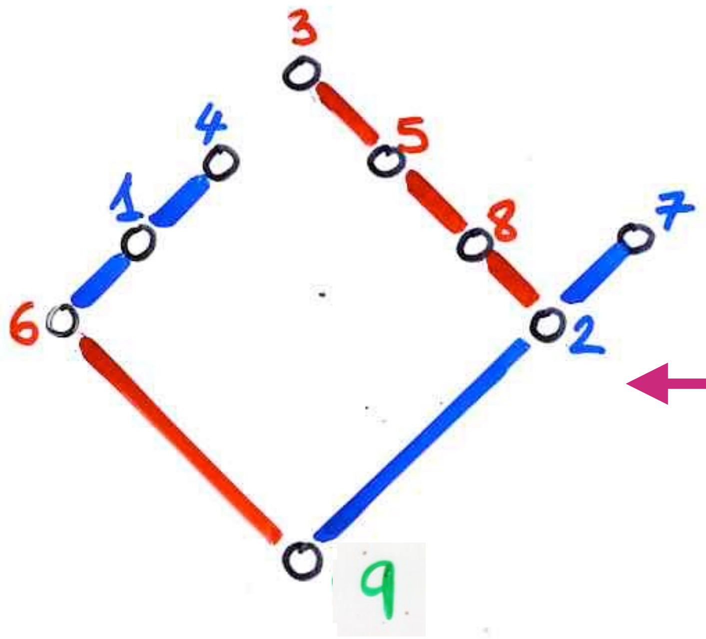
alternative binary trees



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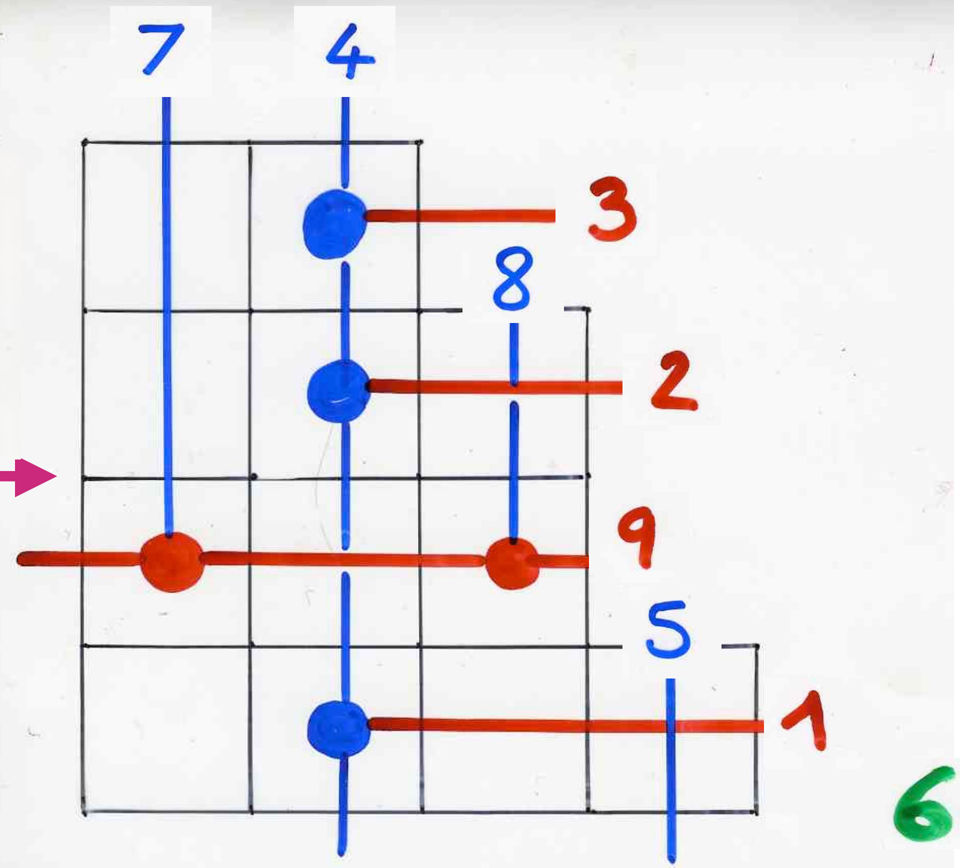
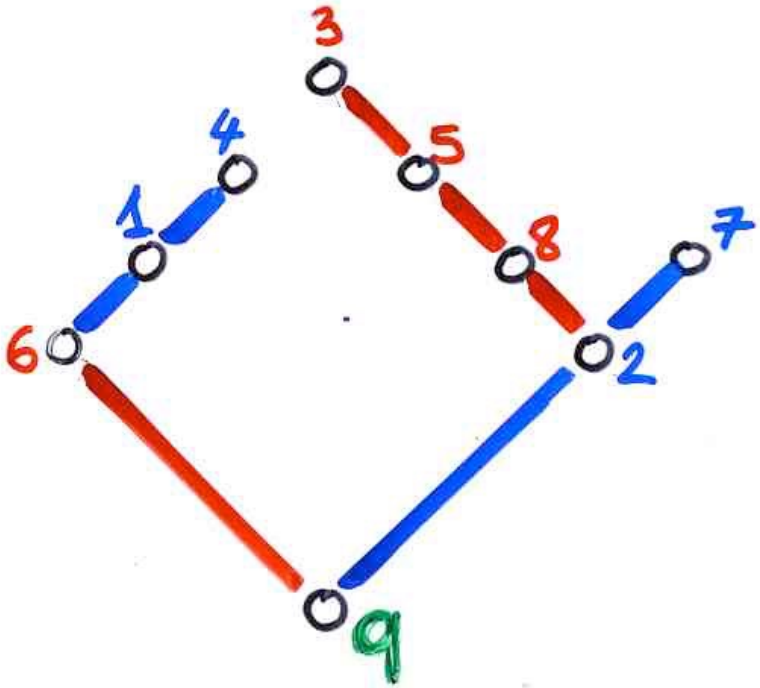


"jeu de taquin"

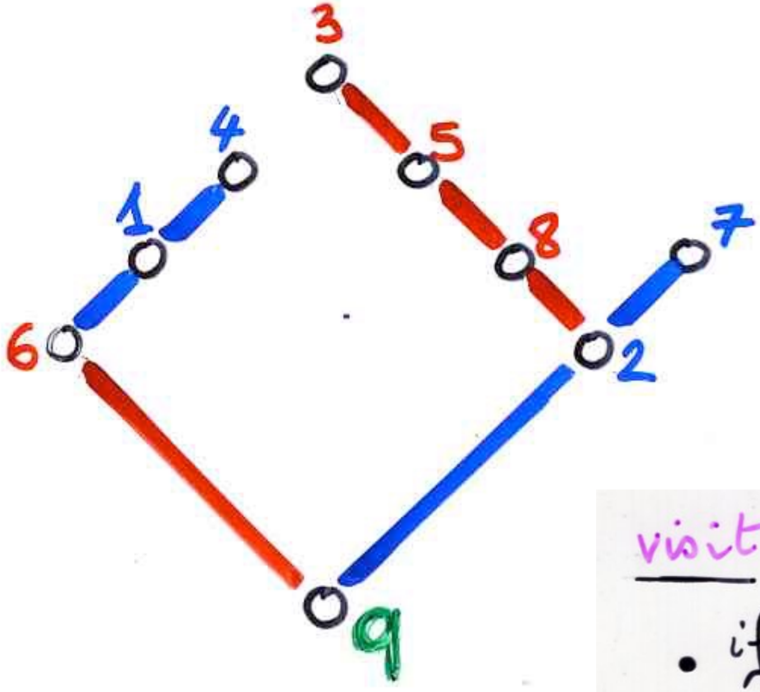
"exchange-fusion" algorithm

comparison with the
exchange-fusion algorithm

the twisted symmetric order



"twisted"
symmetric
order



$$B = \langle L, r, R \rangle$$

left subtree — root — right subtree

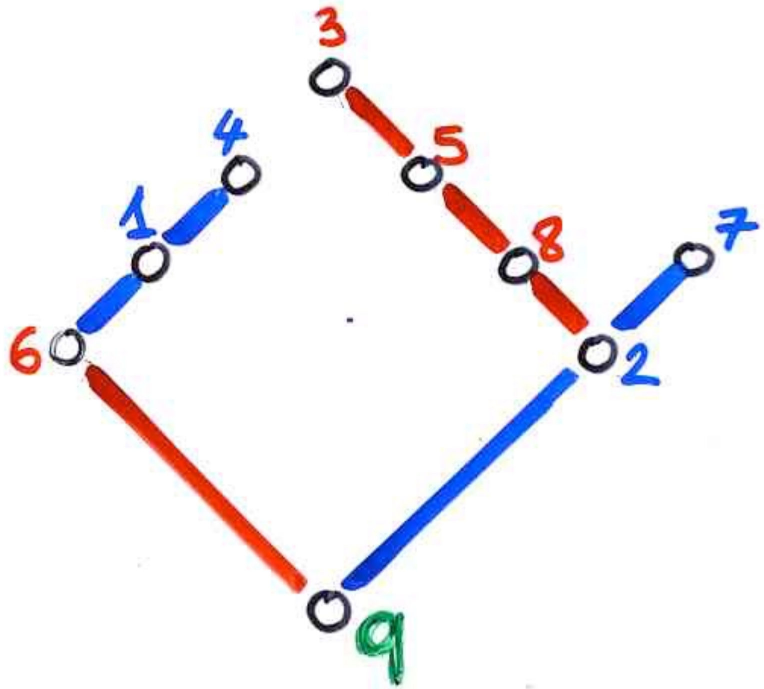
visit(B)

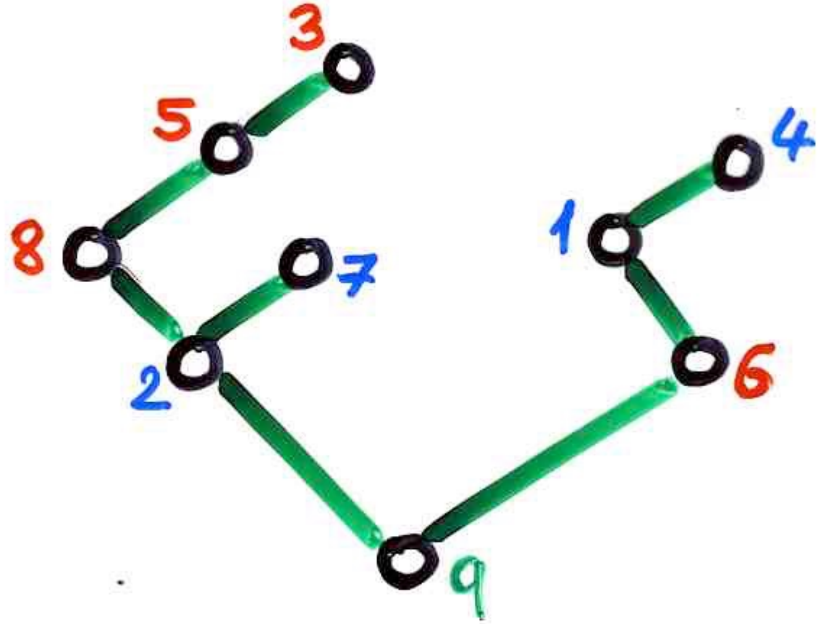
- if r root is a left son
  visit(R), visit(r), visit(L)
- if r root is a right son
  visit(L), visit(r), visit(R)

"twisted" symmetric order

- The root n of the principal binary tree is considered as a "left son"

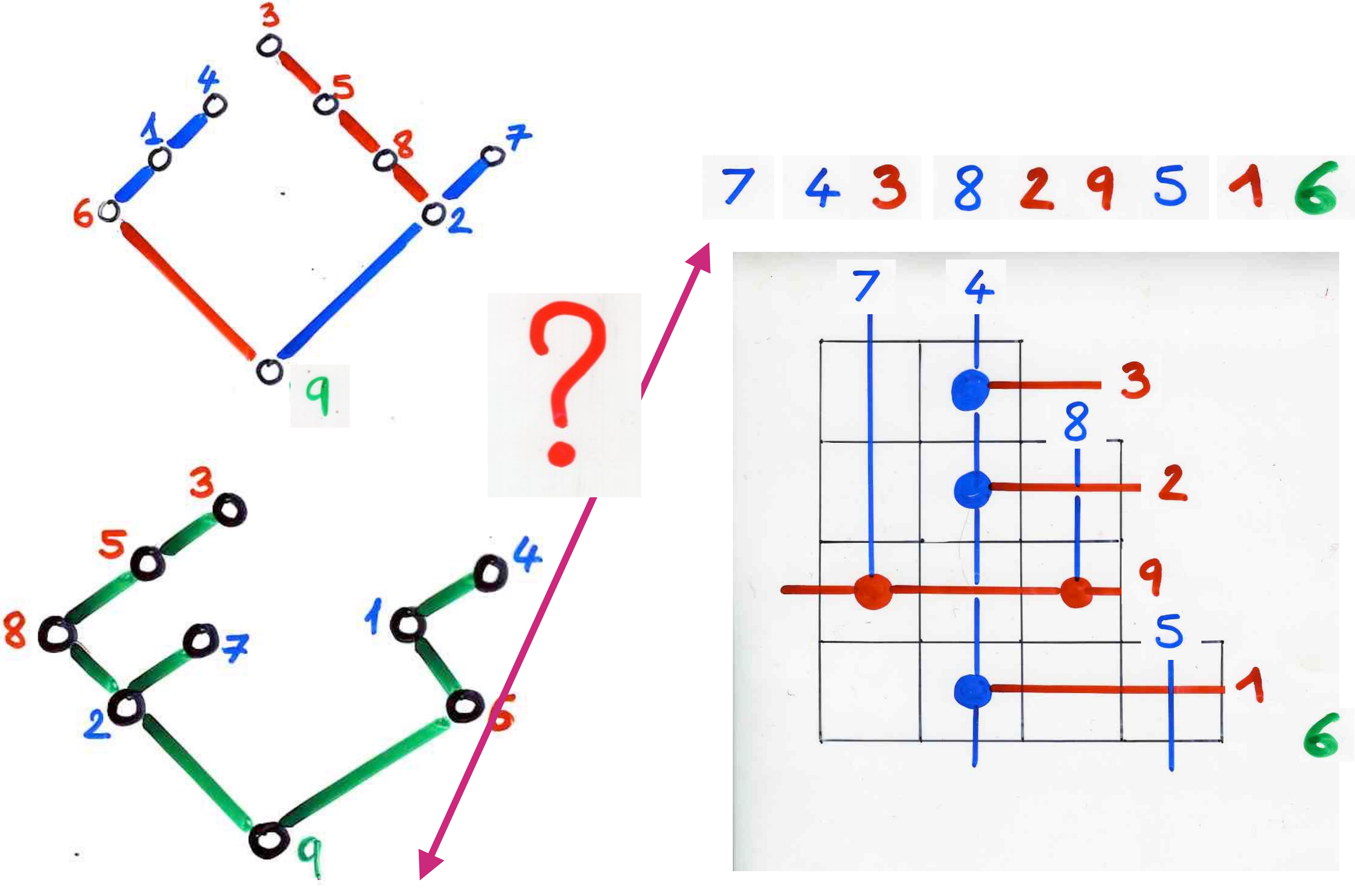
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$





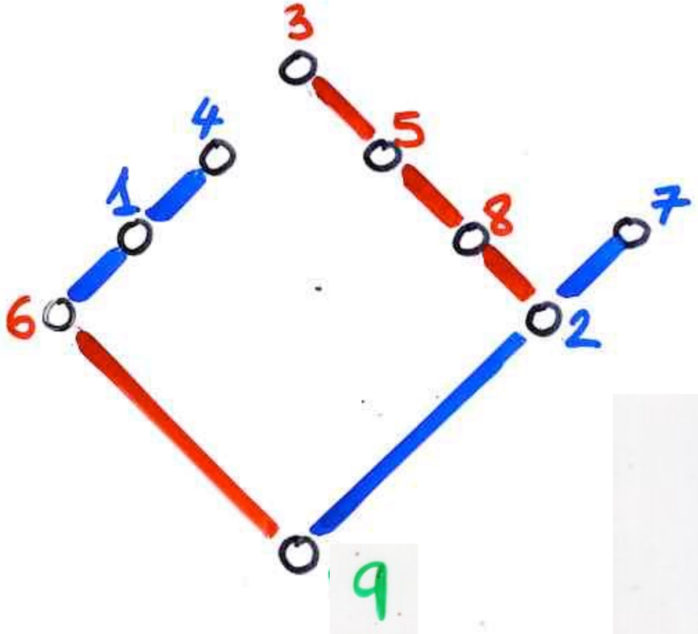
"twisted"
symmetric
order

$$\sigma = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{ccccccccc} 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{array} \right) \end{array}$$

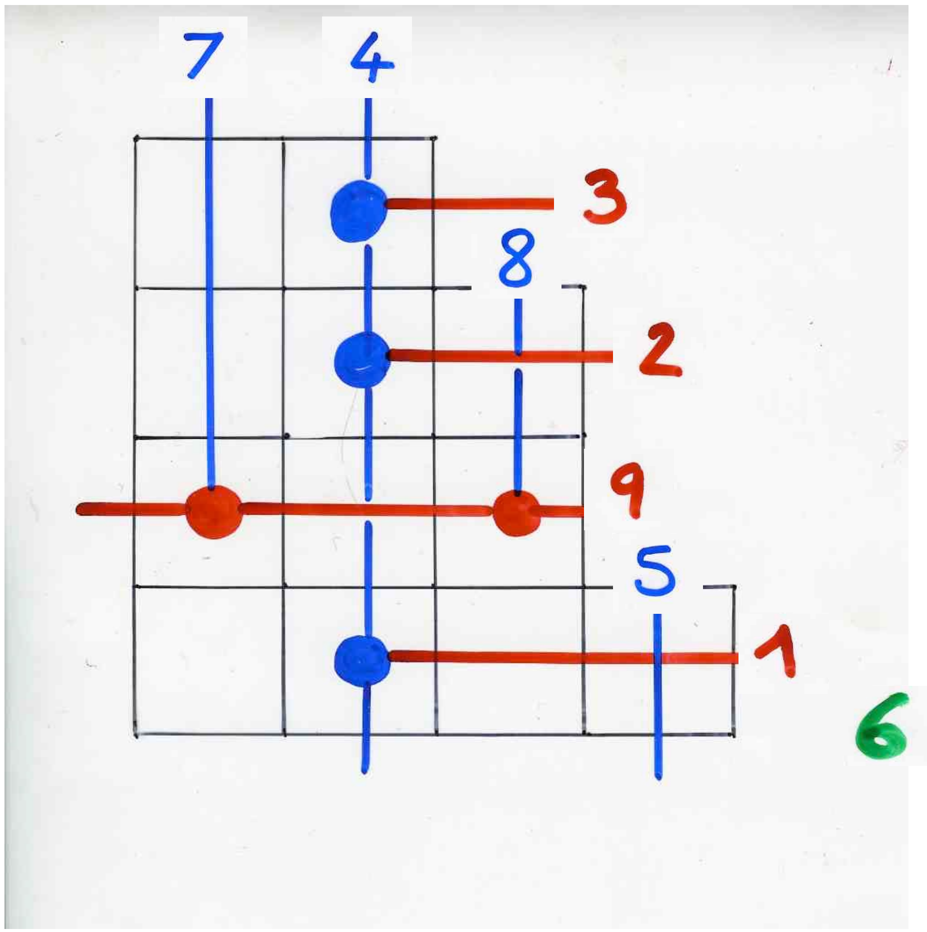


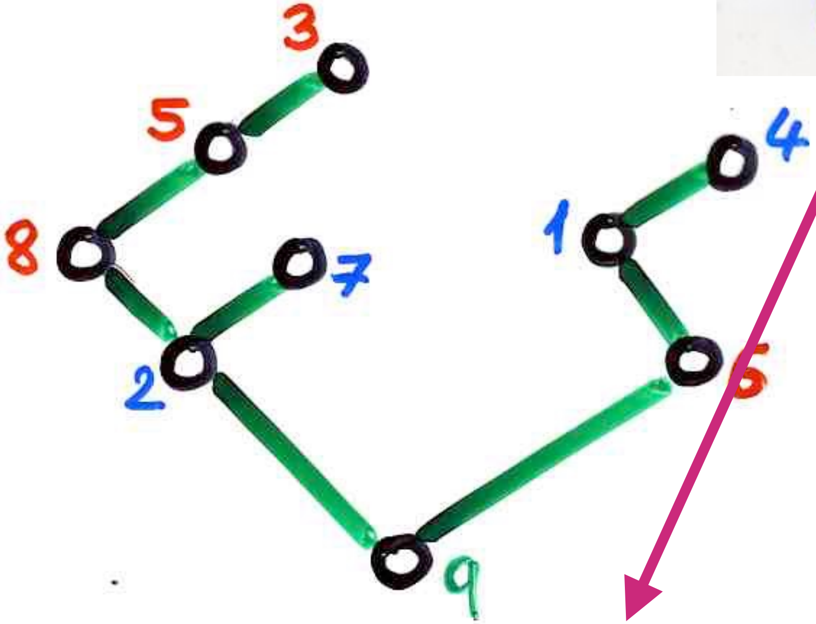
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$



7 4 3 8 2 9 5 1 6





$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

Def - Genocchi sequence of a permutation

$$\sigma = \sigma(1) \quad \sigma(n)$$

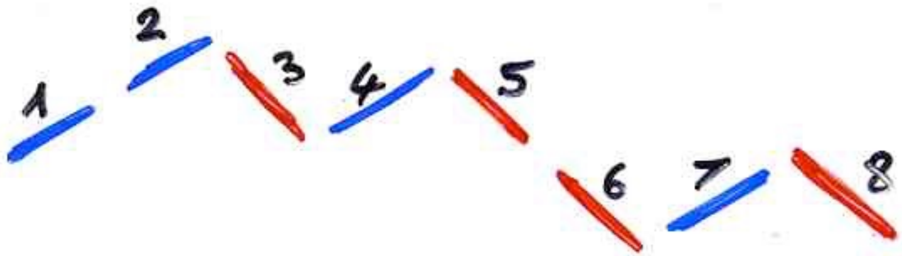
$$G(\sigma) = z_1 \ldots z_{n-1}$$

$$z_x = \begin{cases} a & \text{(ascent)} \\ d & \text{(descent)} \end{cases} \quad \underset{\text{"value"}}{x} = \sigma(\underset{\text{"index"}}{i}) \begin{matrix} < \sigma(i+1) \\ > \sigma(i+1) \end{matrix}$$

$$1 \leqslant x \leqslant n-1$$

convention: $\sigma(n+1) = 0$ ($\sigma(n)$ is a descent)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$



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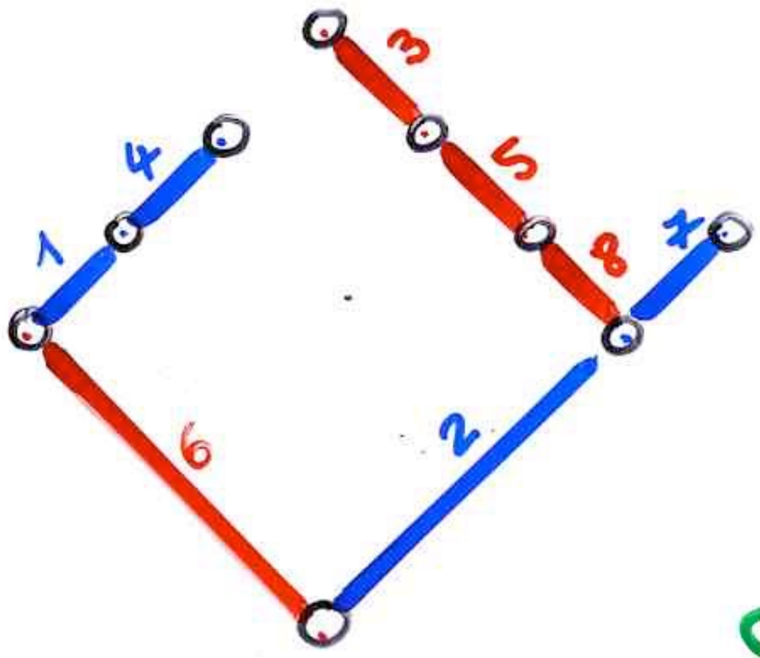
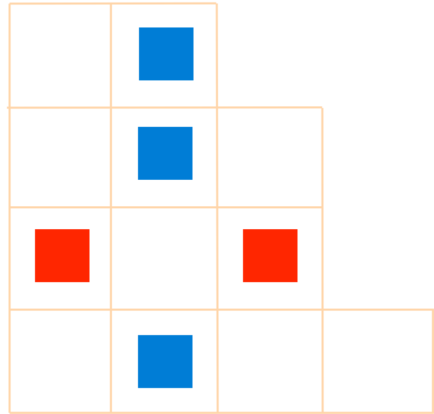
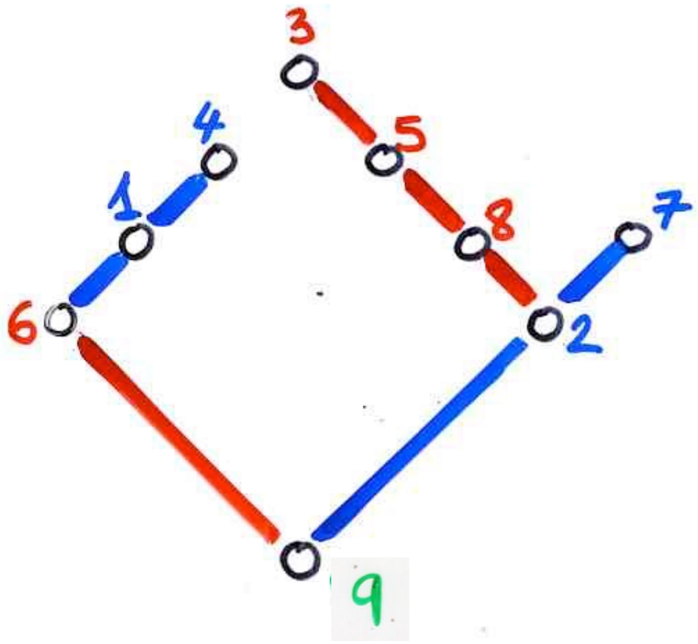




$$\sigma^{-1} = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{array}\right)$$

$$\sigma = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{array}\right)$$

comparison with the underlying binary trees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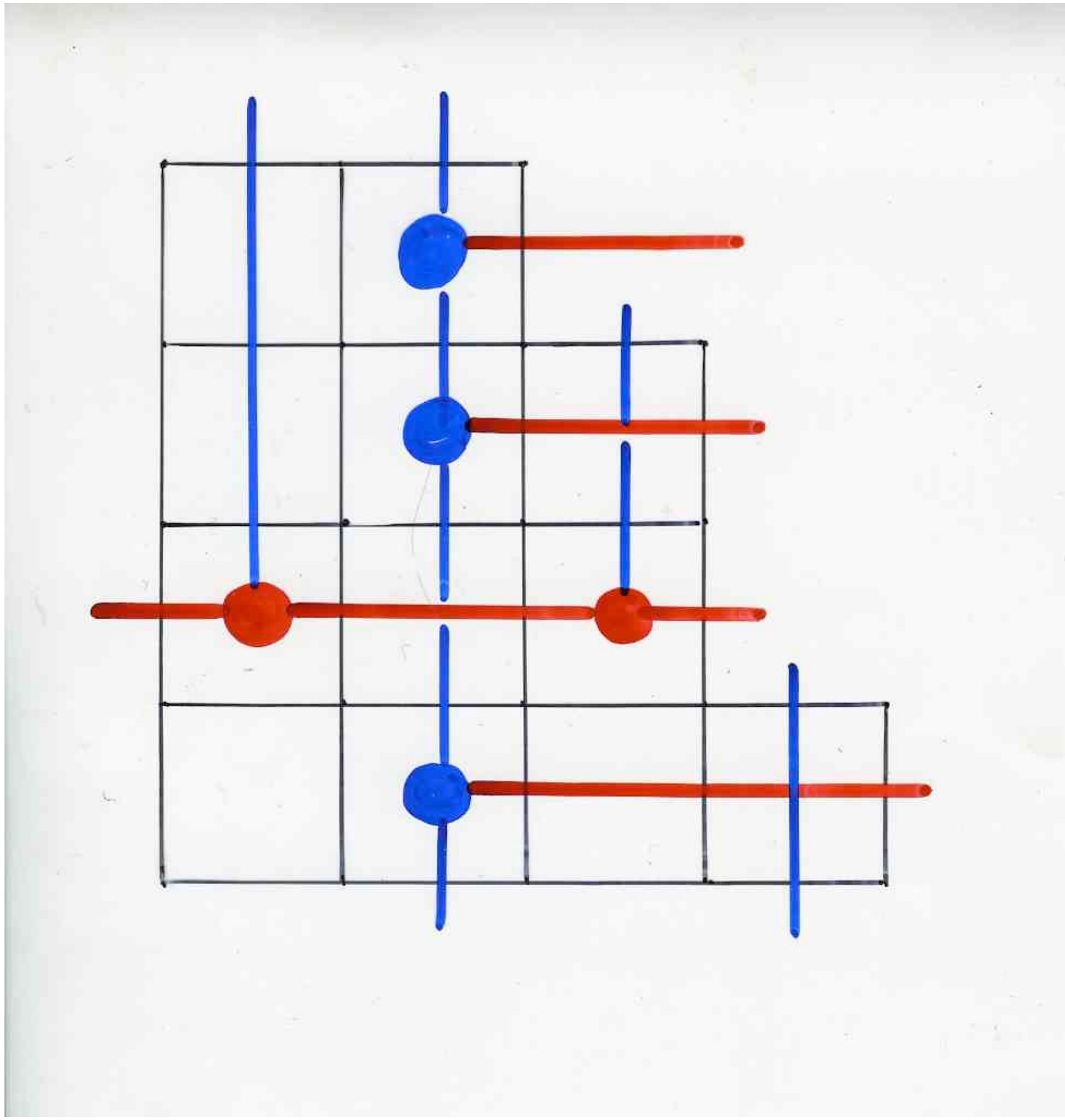
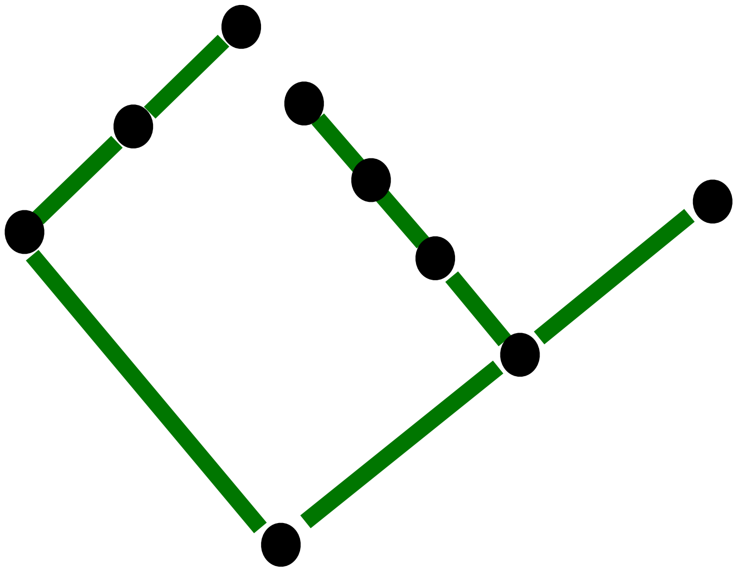
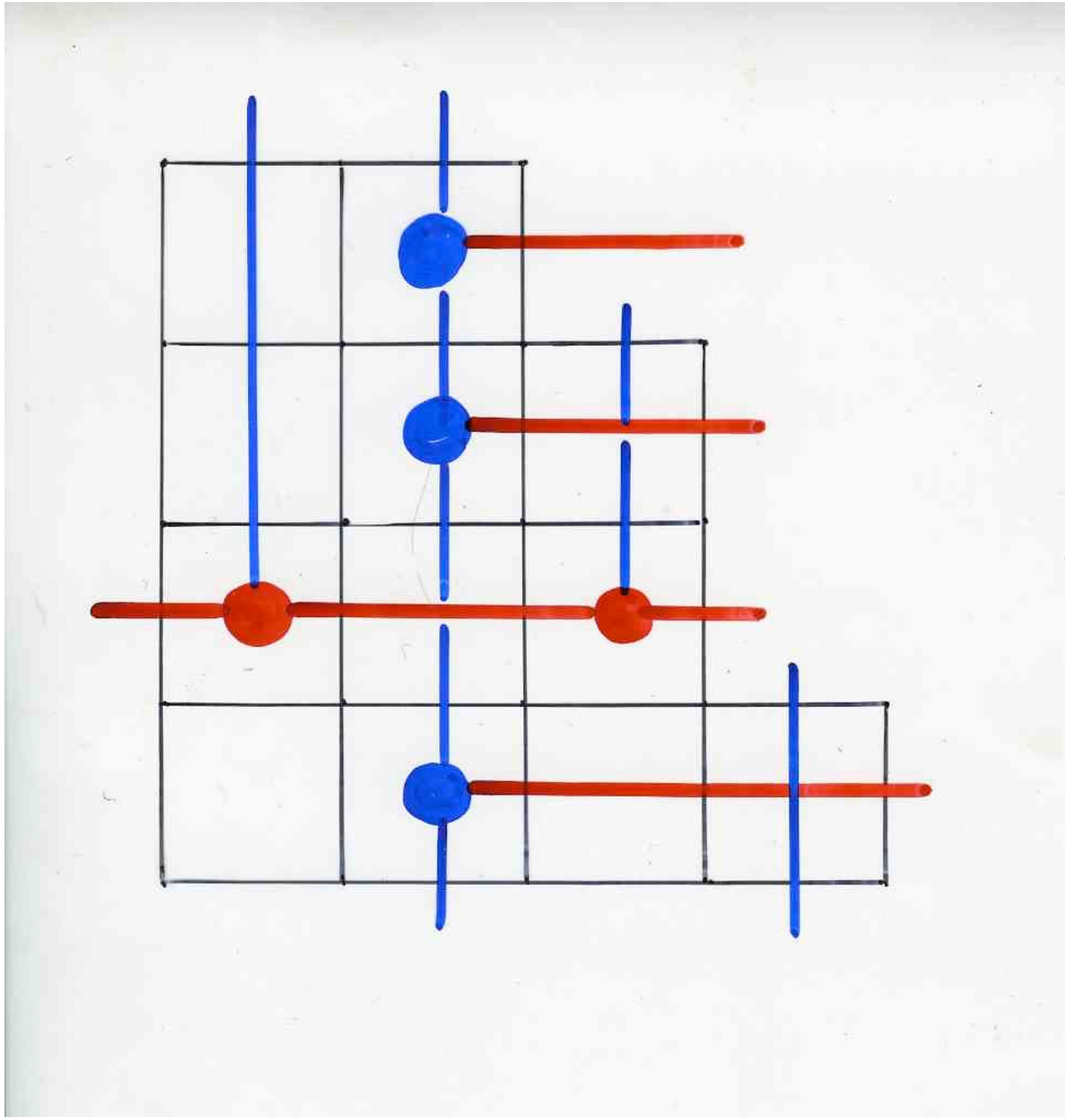
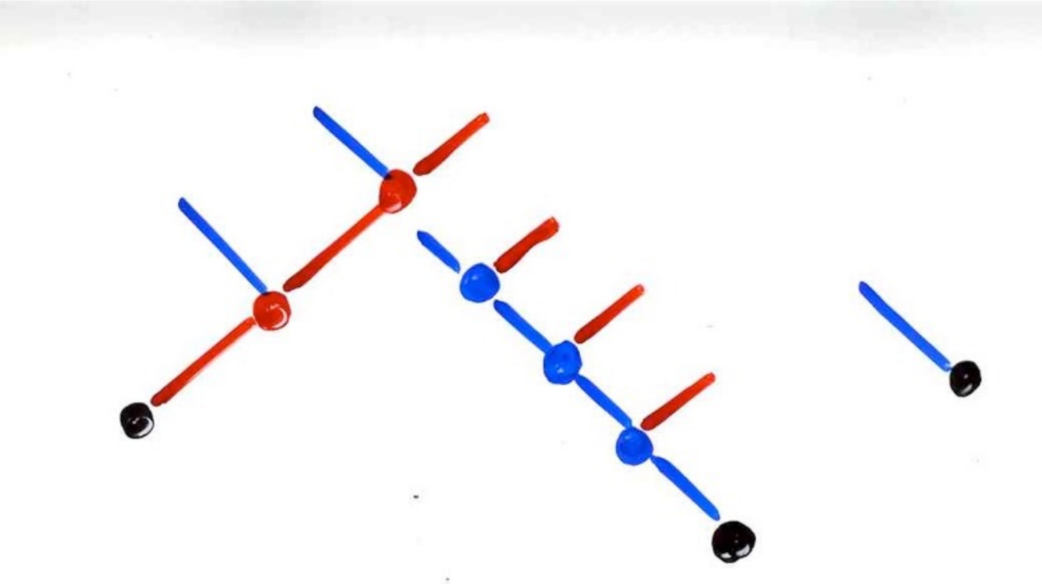
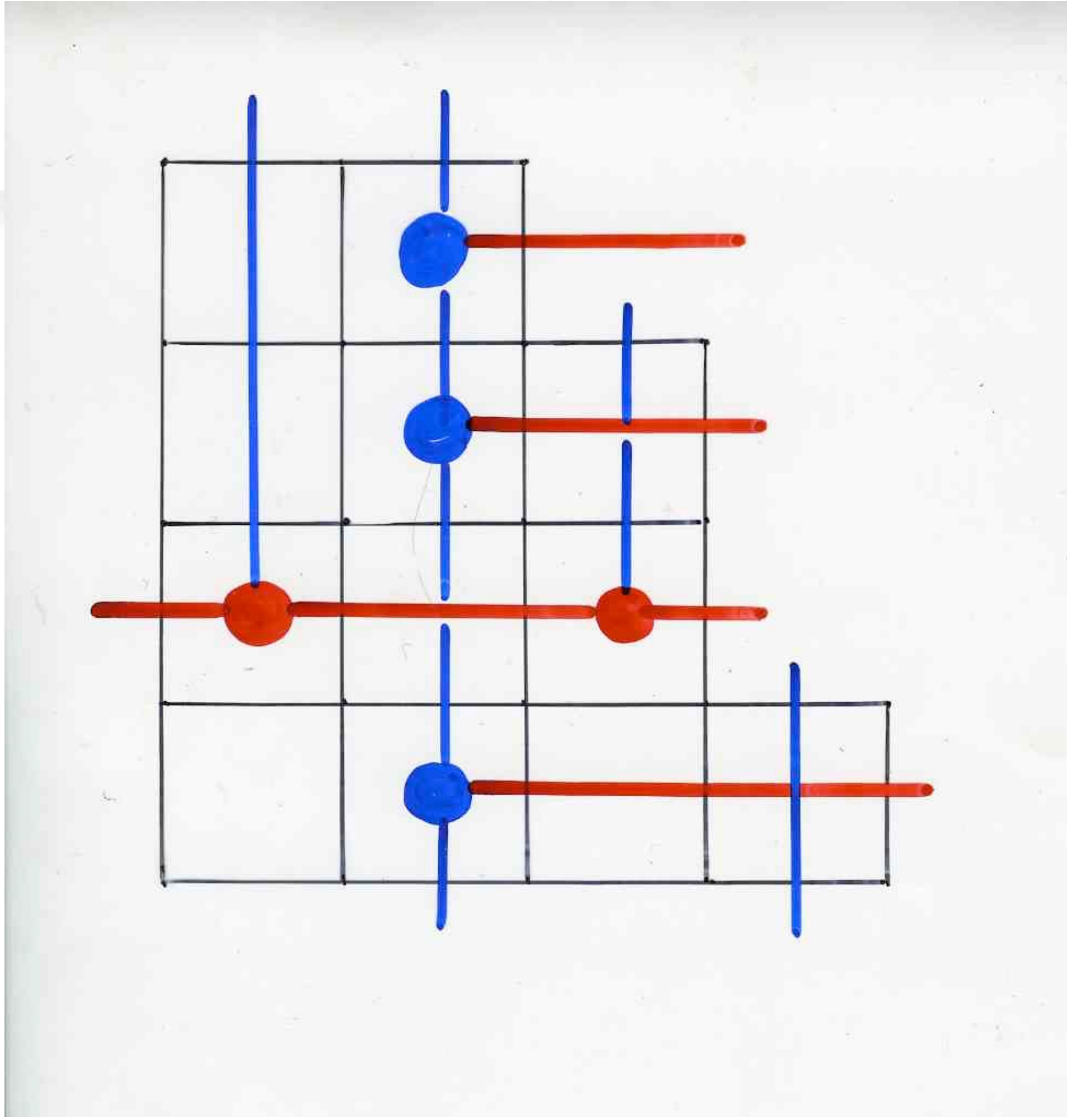
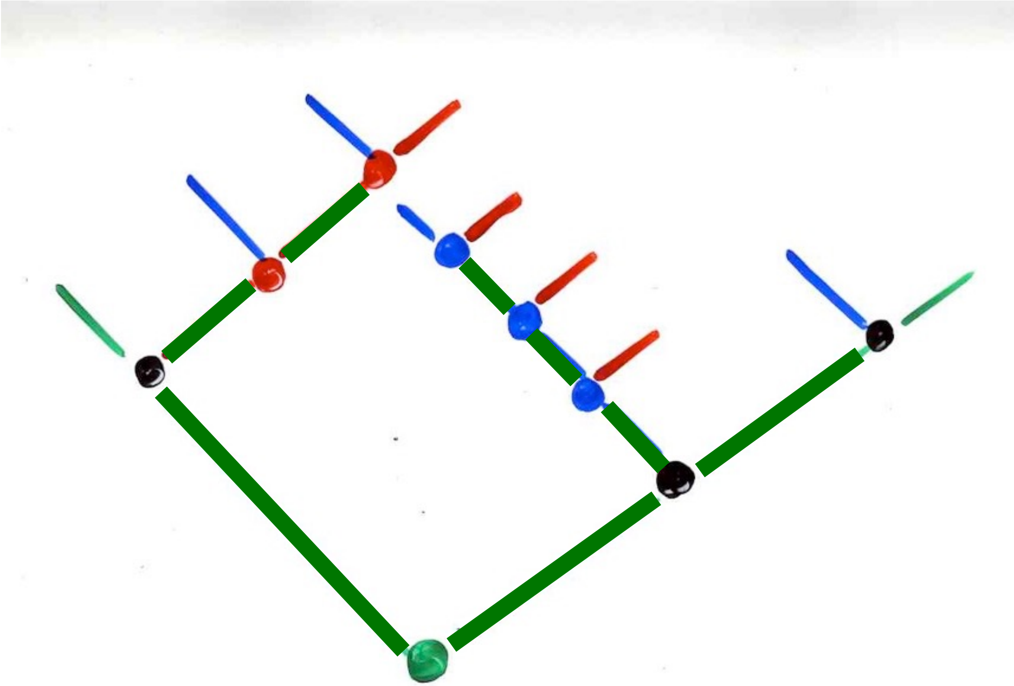


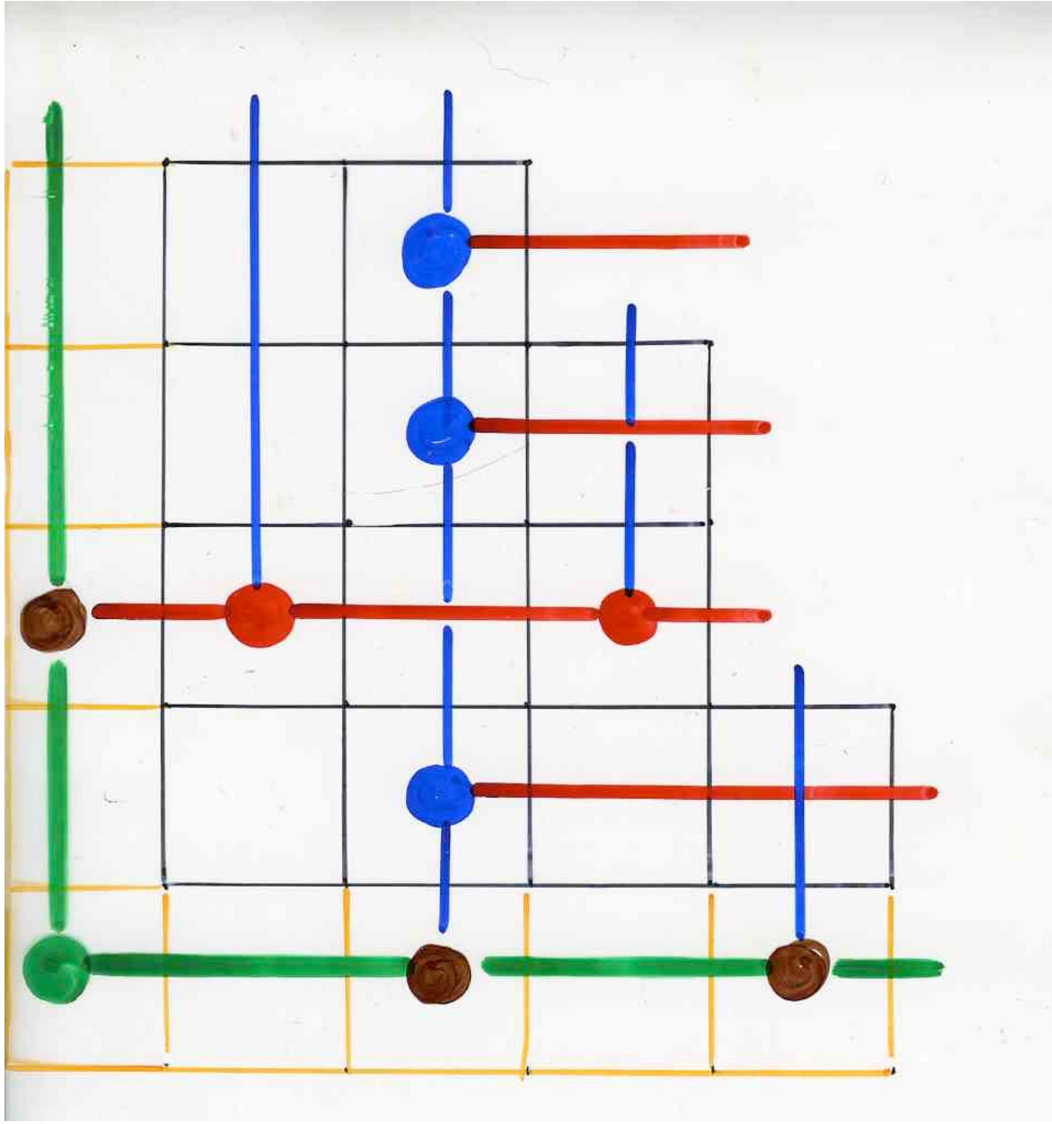






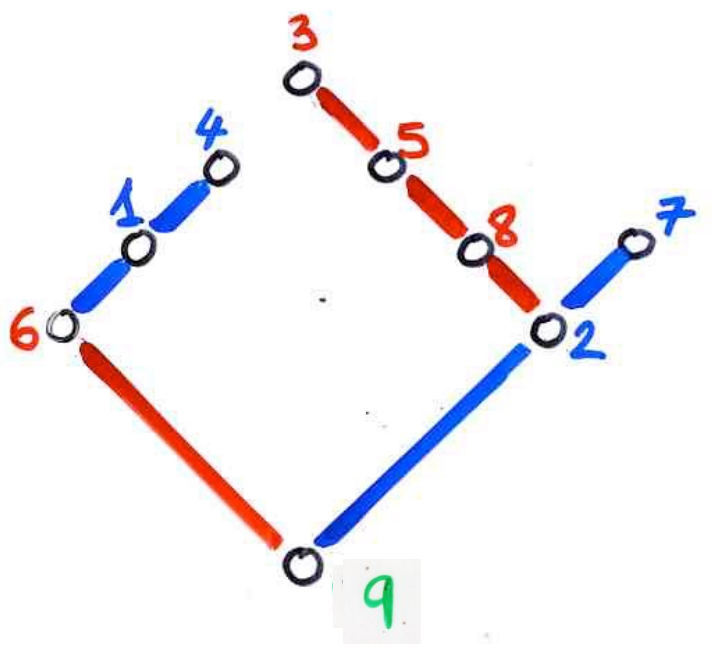






Second version of the bijection

alternative tableaux
alternative binary trees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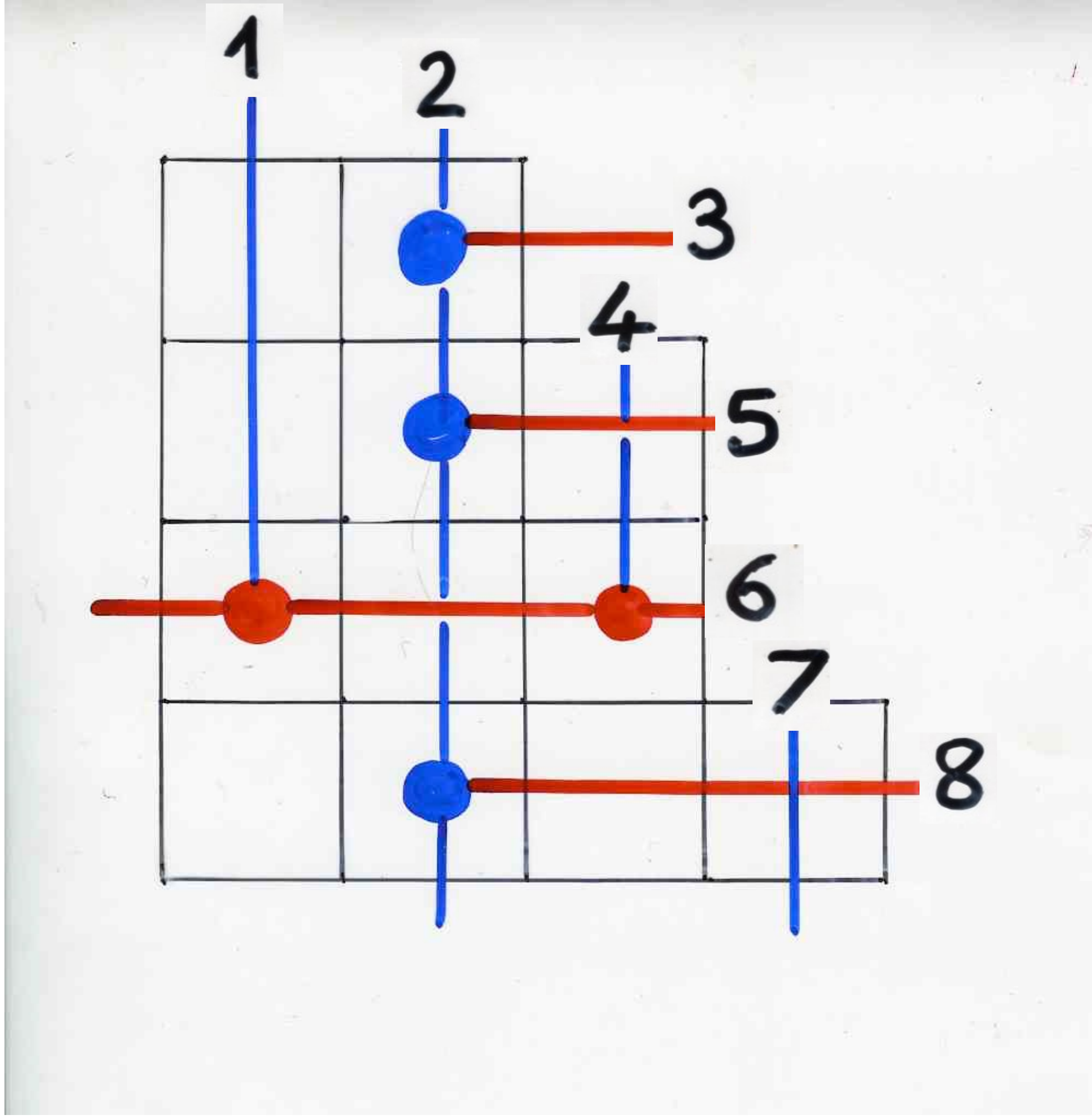
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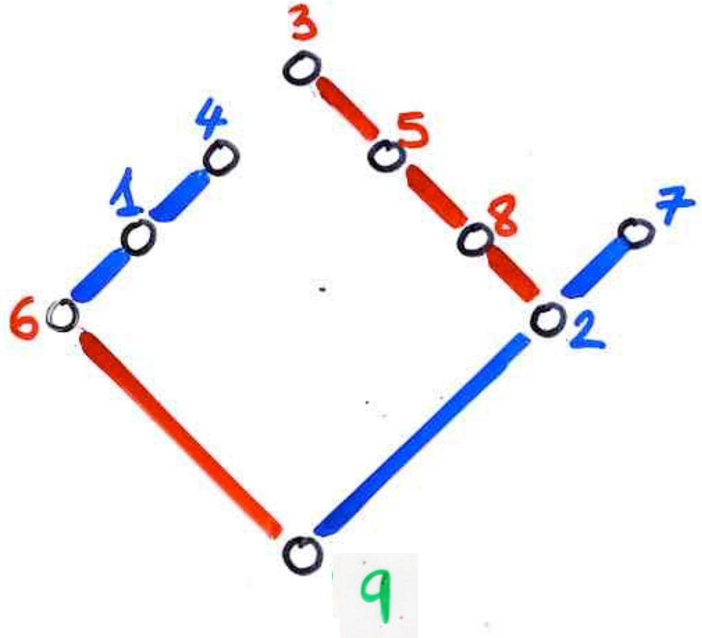

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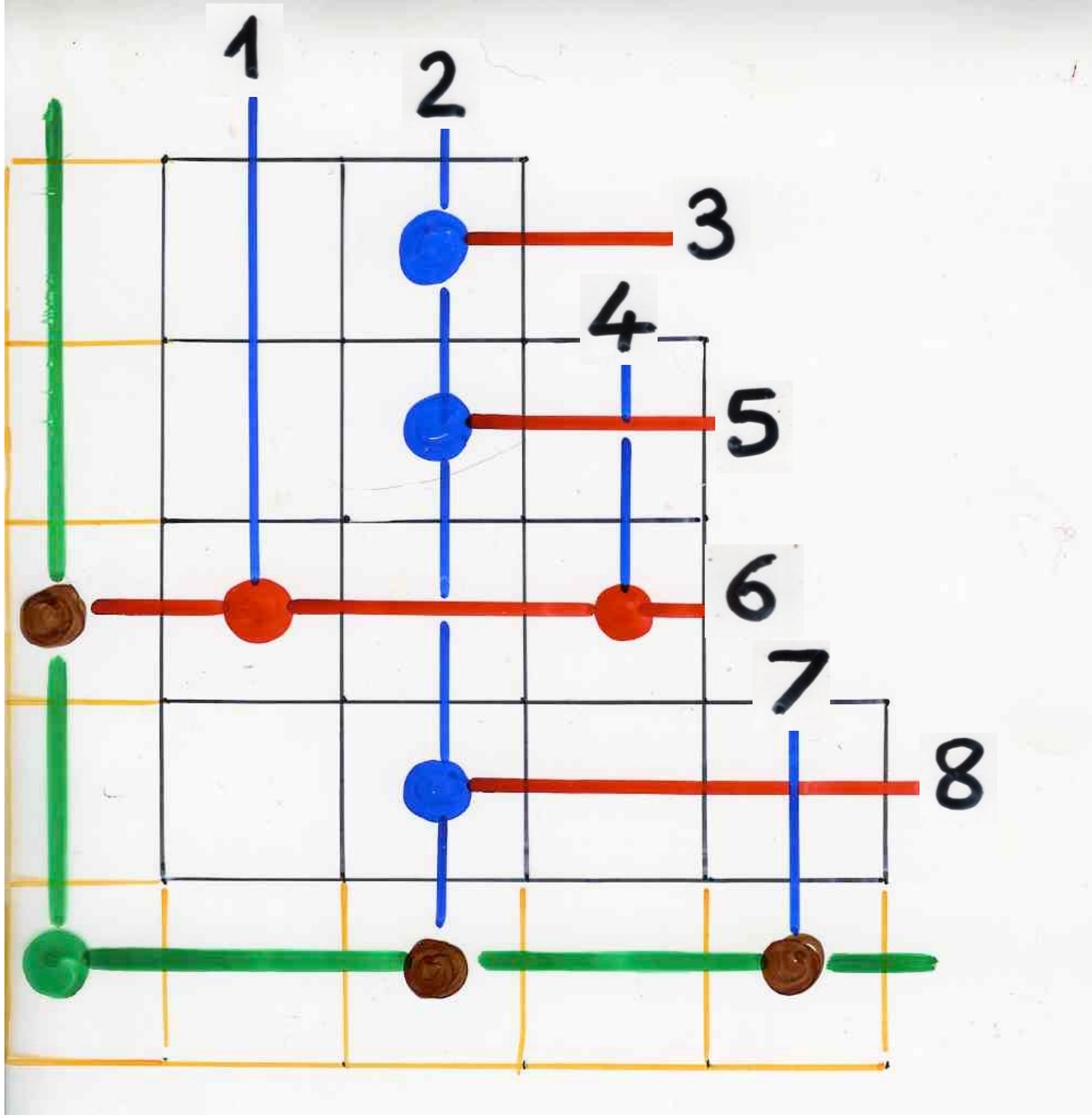

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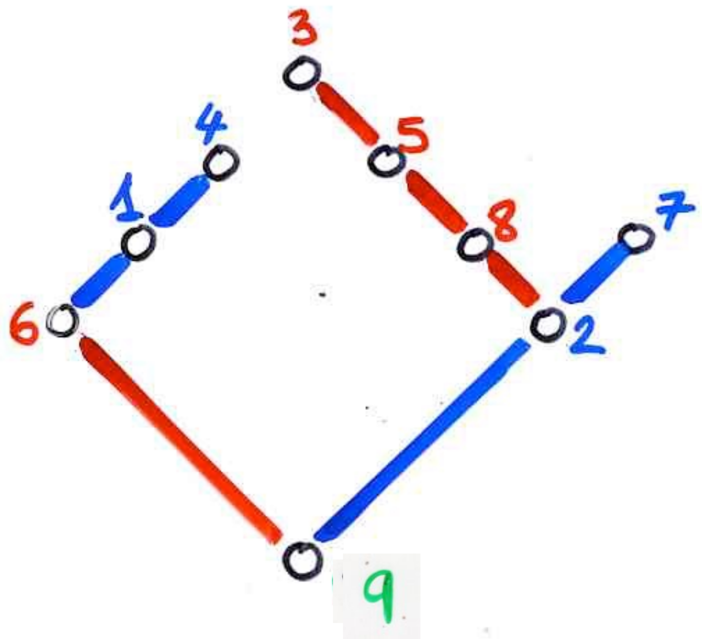
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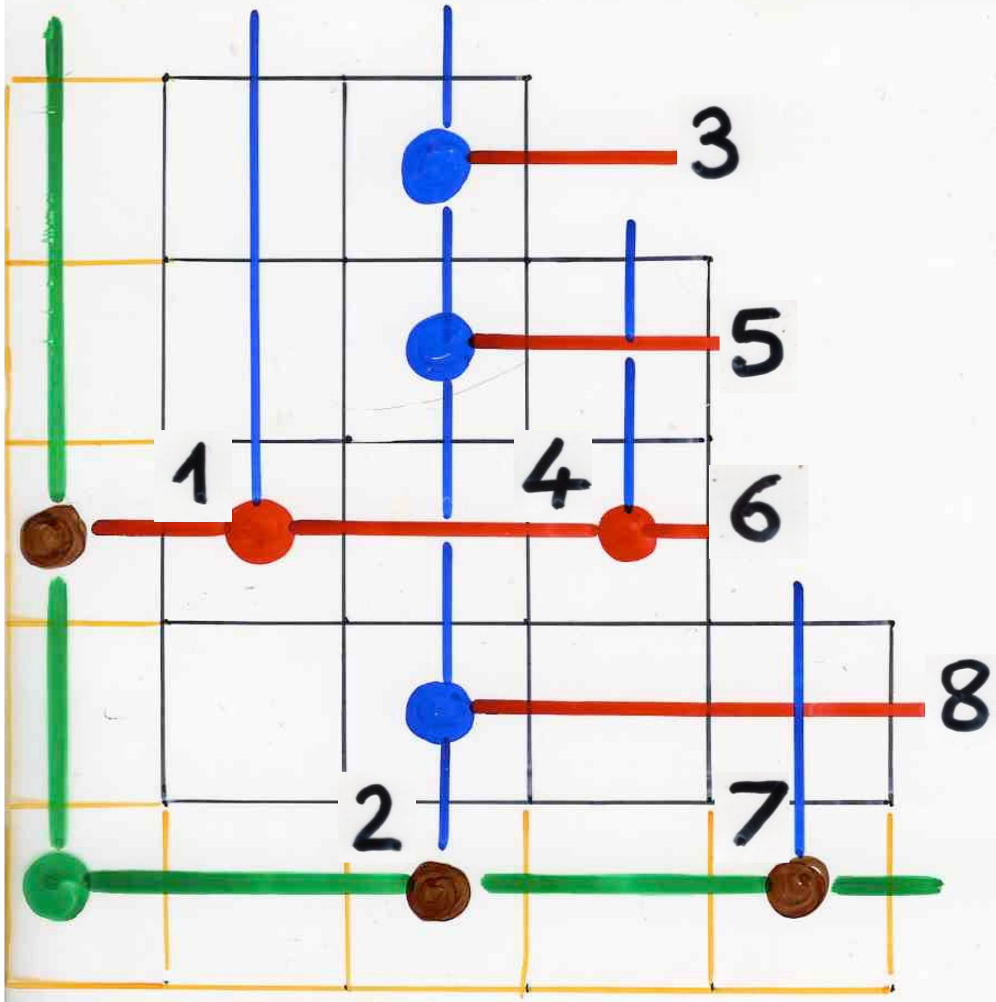
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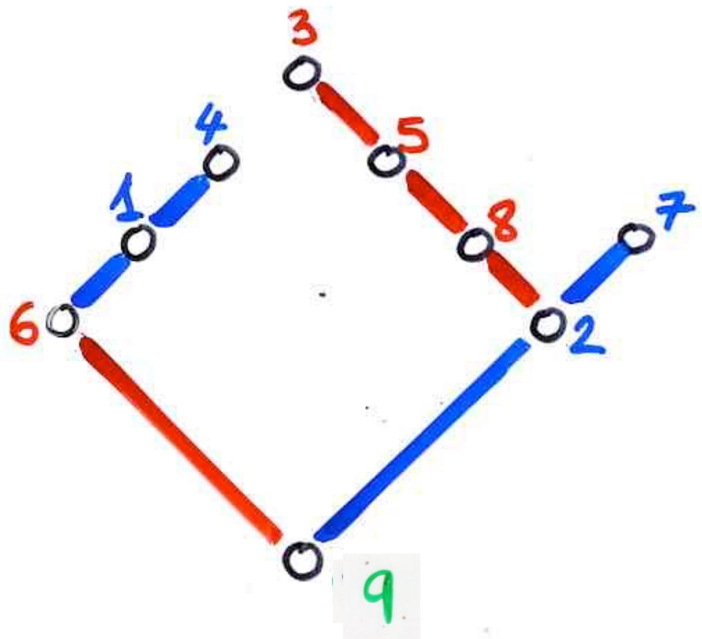
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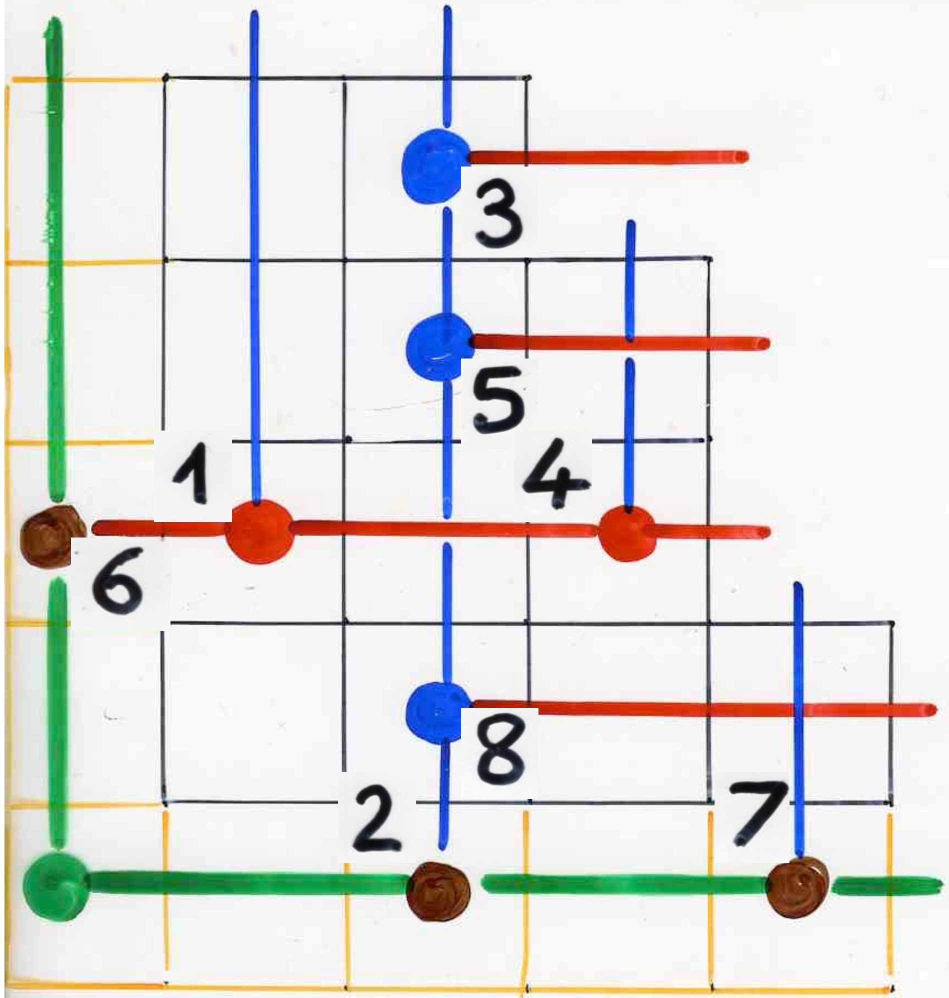
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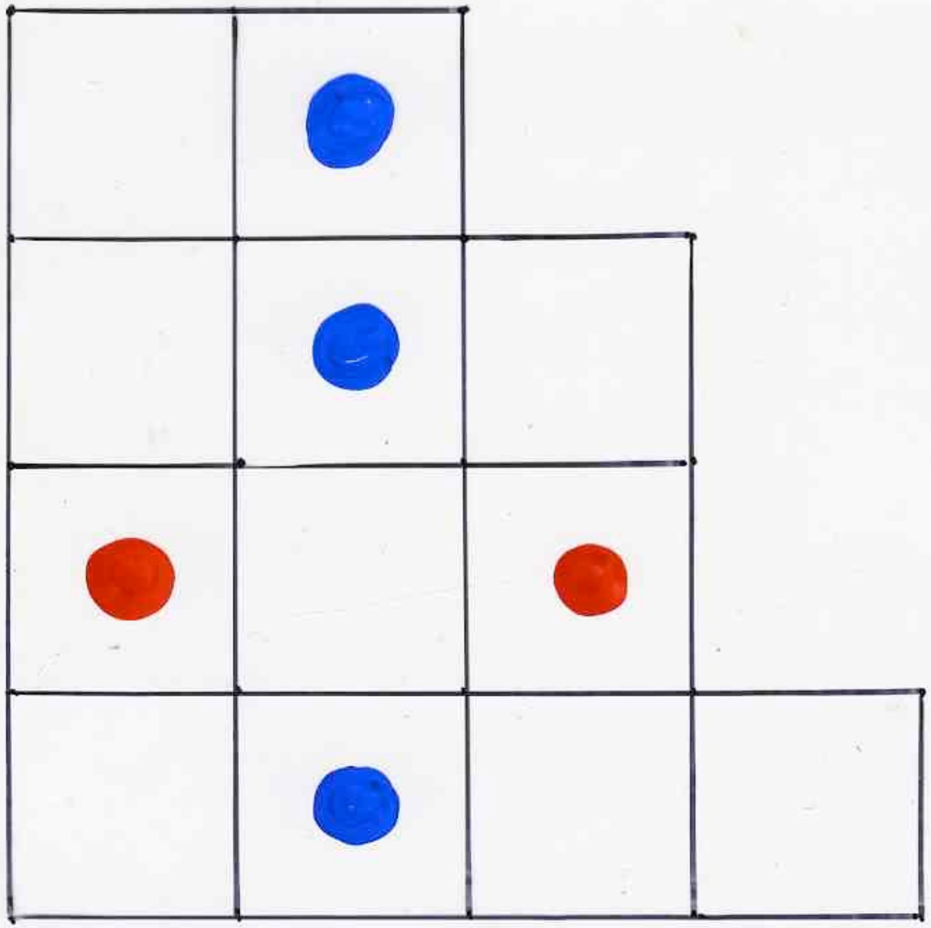
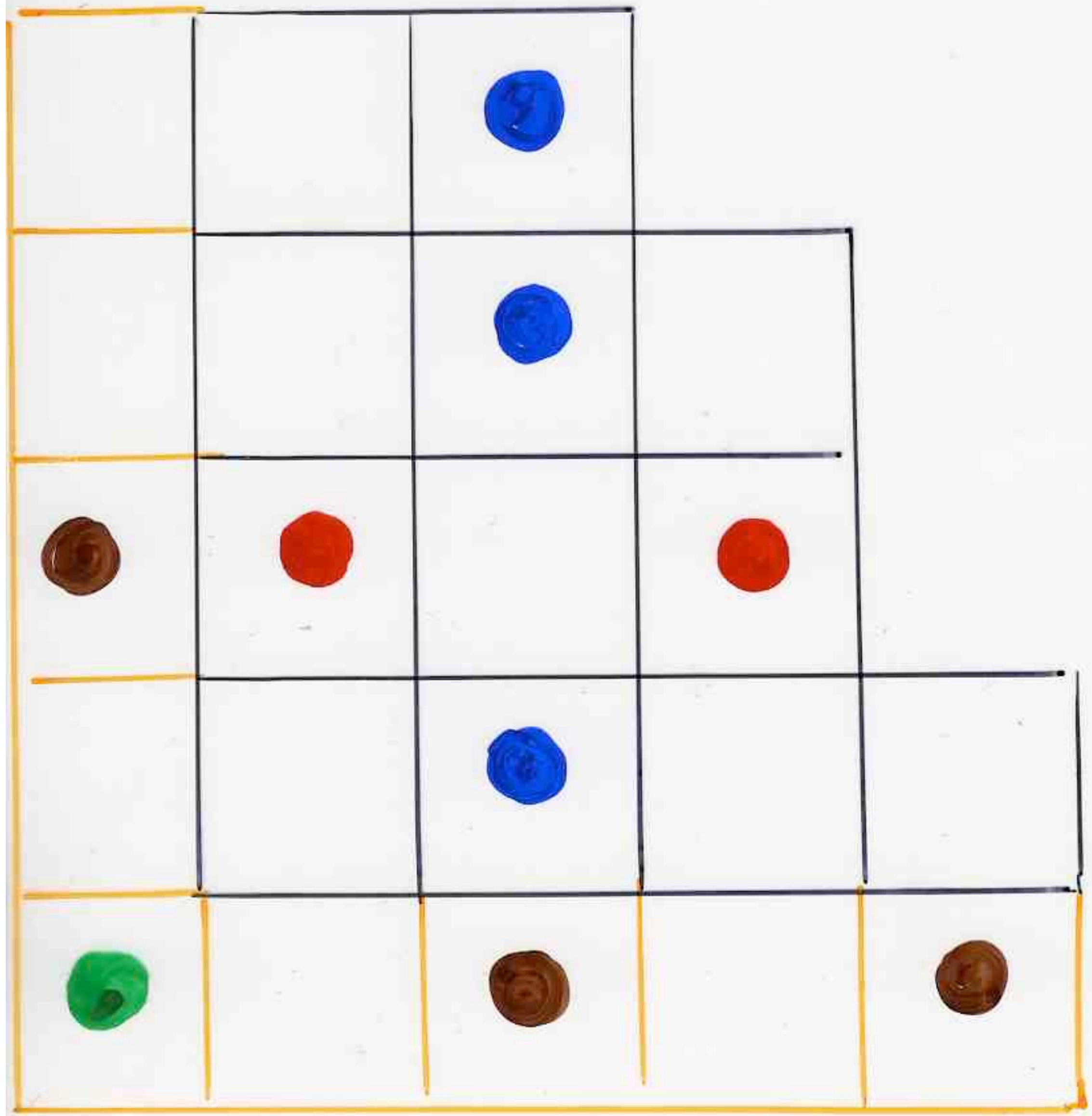
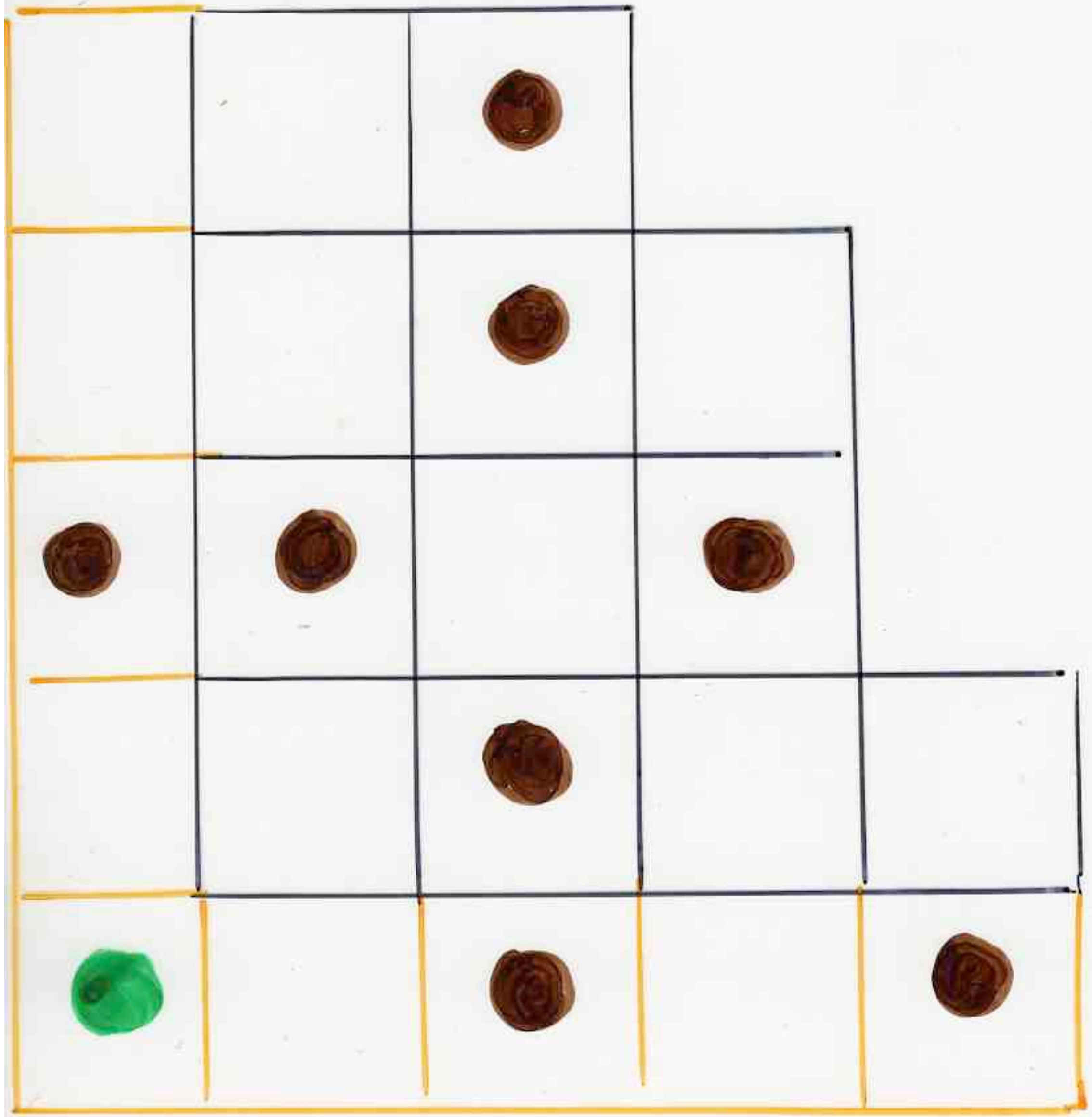
tree-like tableaux

alternative
tableaux







tree-like
tableaux

<u>Definition</u> Tree-like tableaux

Aval, Boussicault, Nadeau (2011)

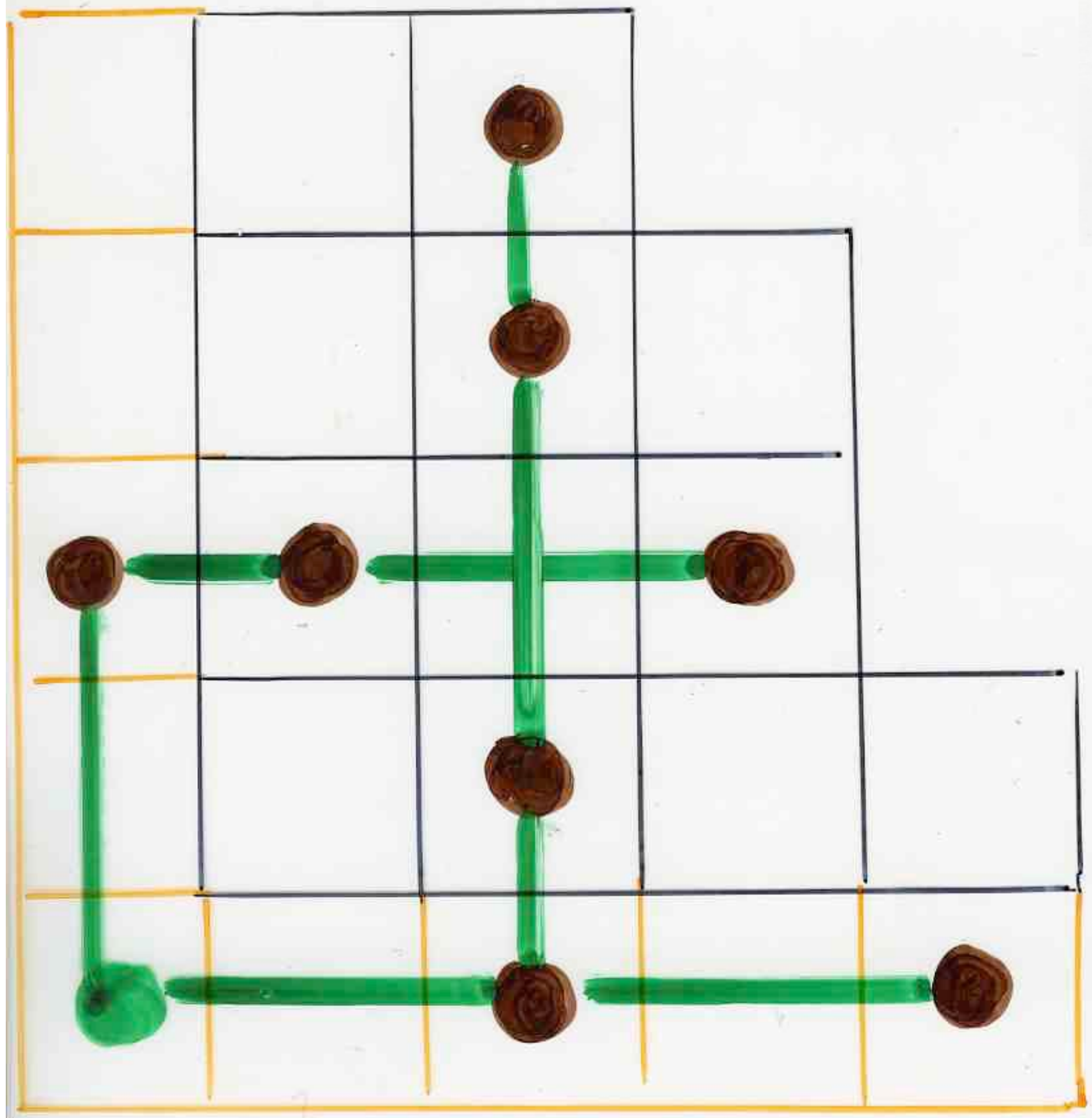
Ferrers diagram F with cell

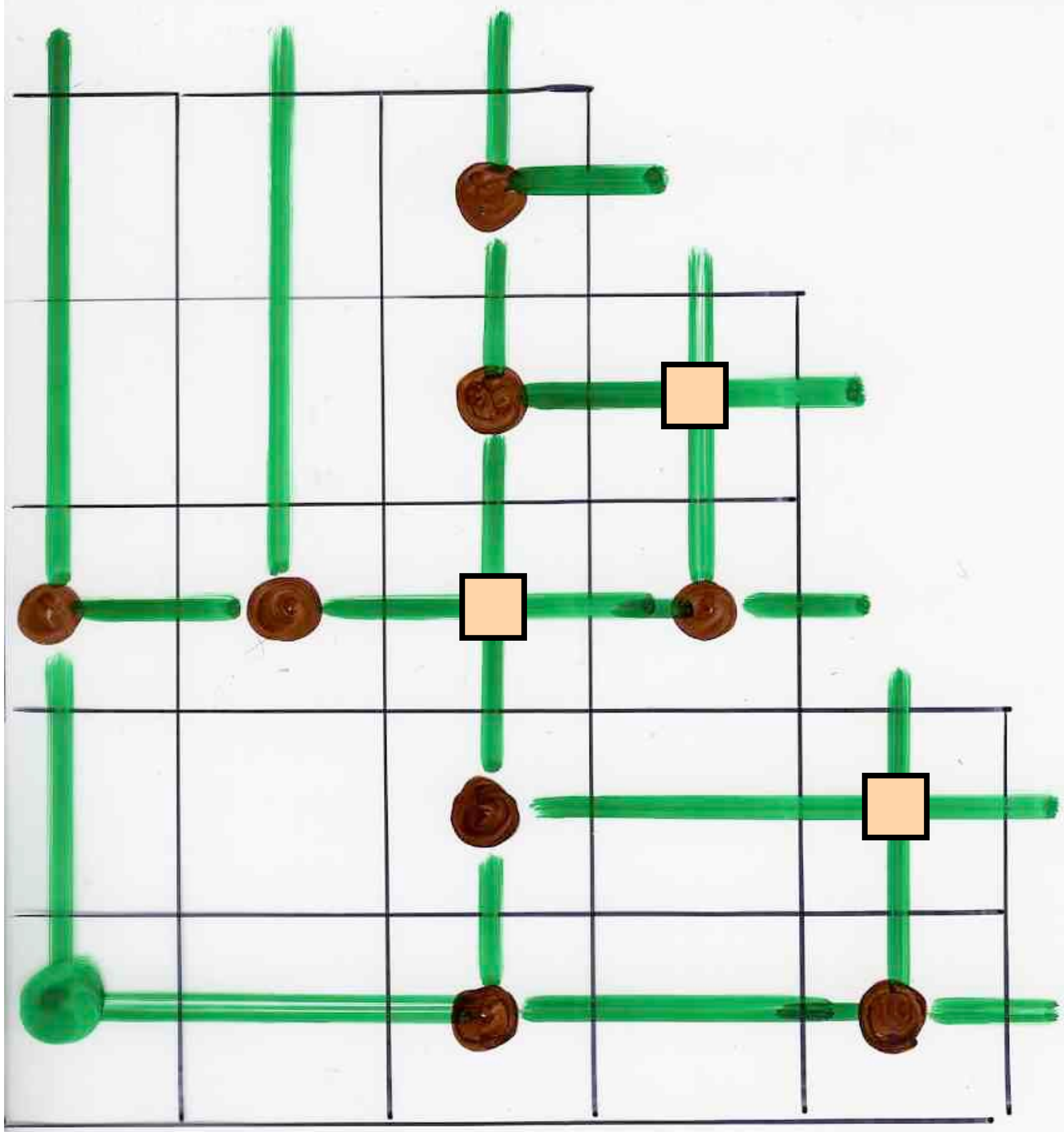
empty cell   pointed cell

(i) the botom left cell is pointed (called the root cell)

(ii) for every non-root pointed cell c, there exist a pointed cell below c in the same column, or a pointed cell to its left in the same row, but not both

(iii) every column and every row passes at least one pointed cell

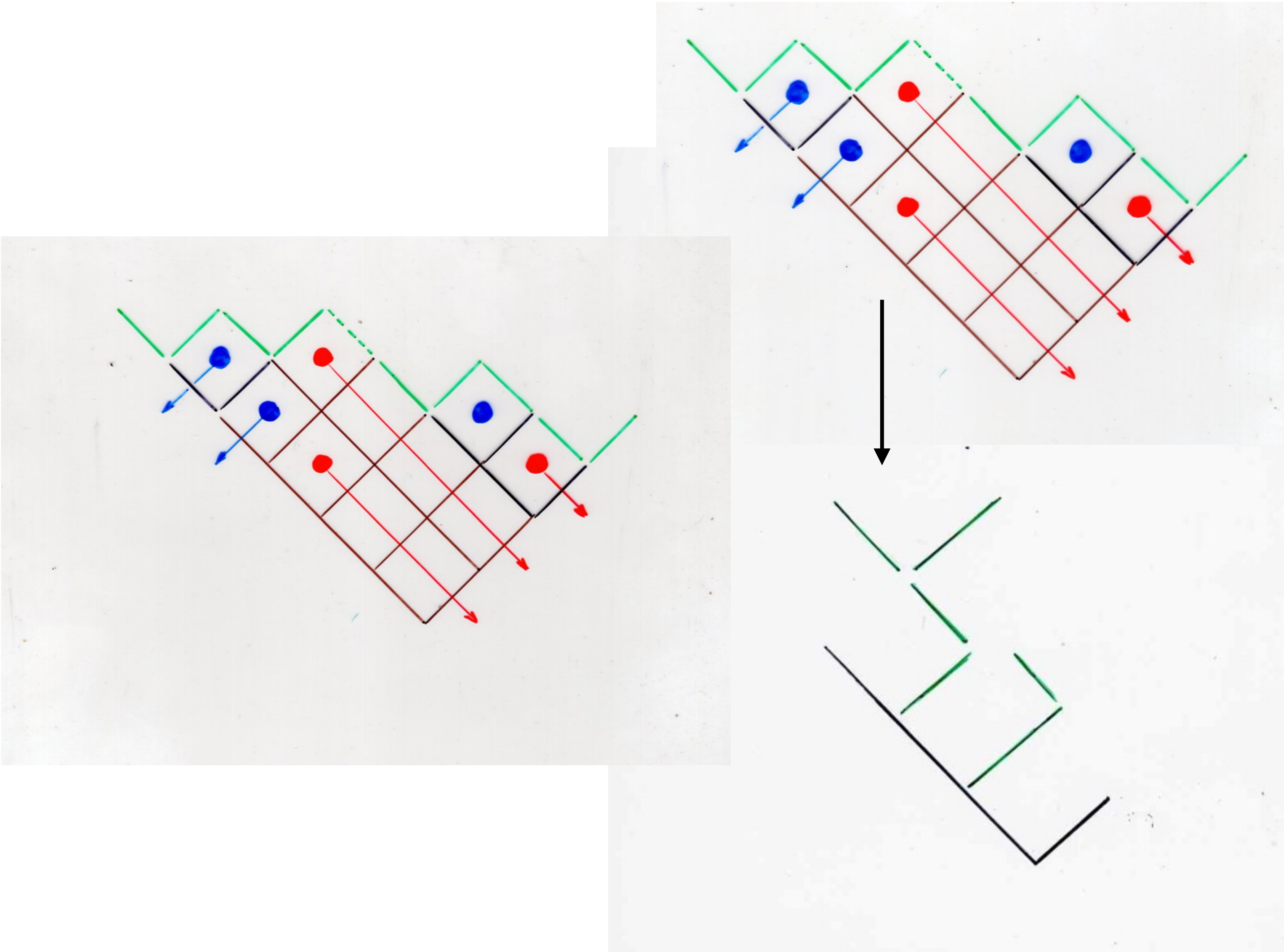


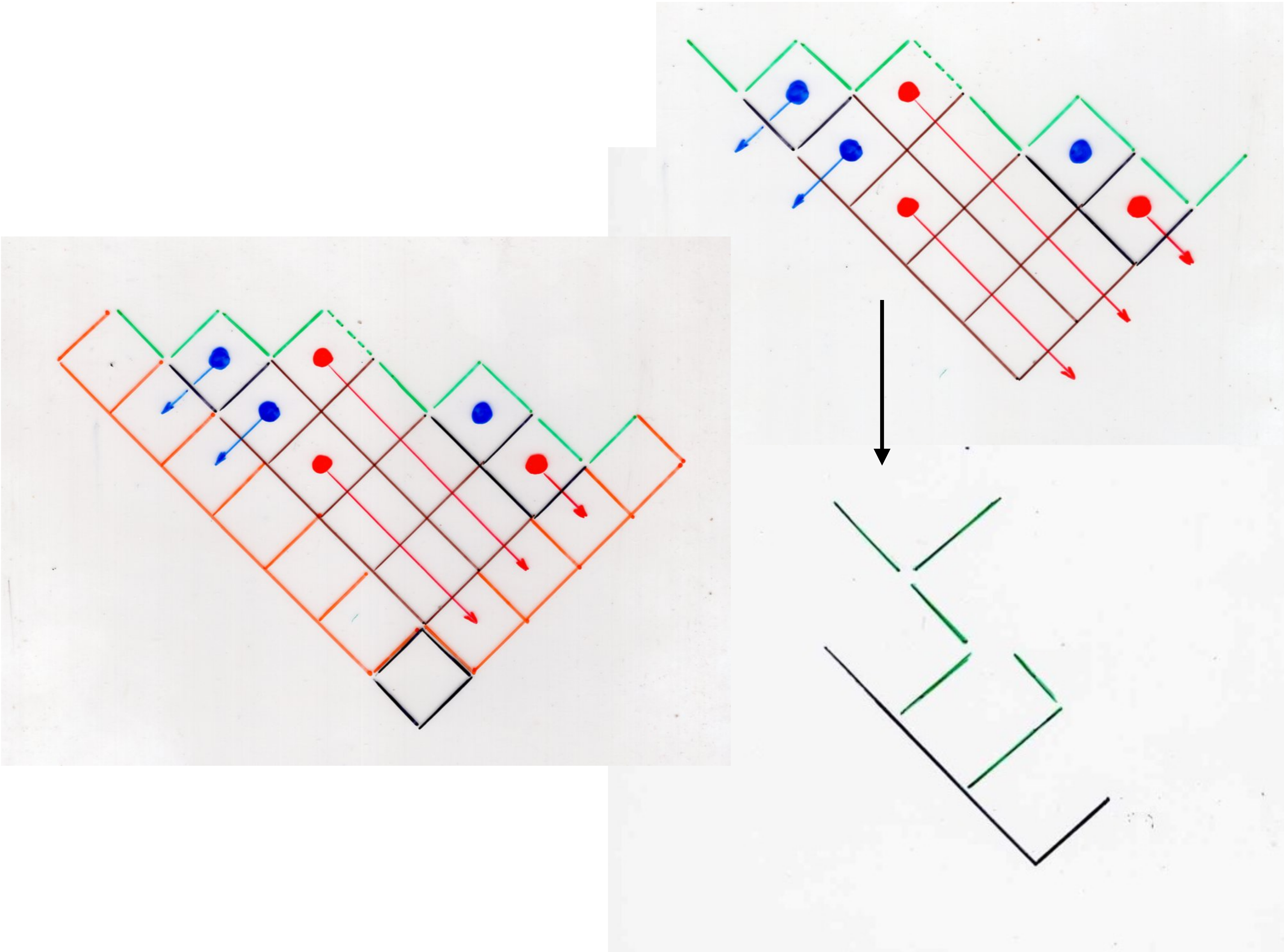



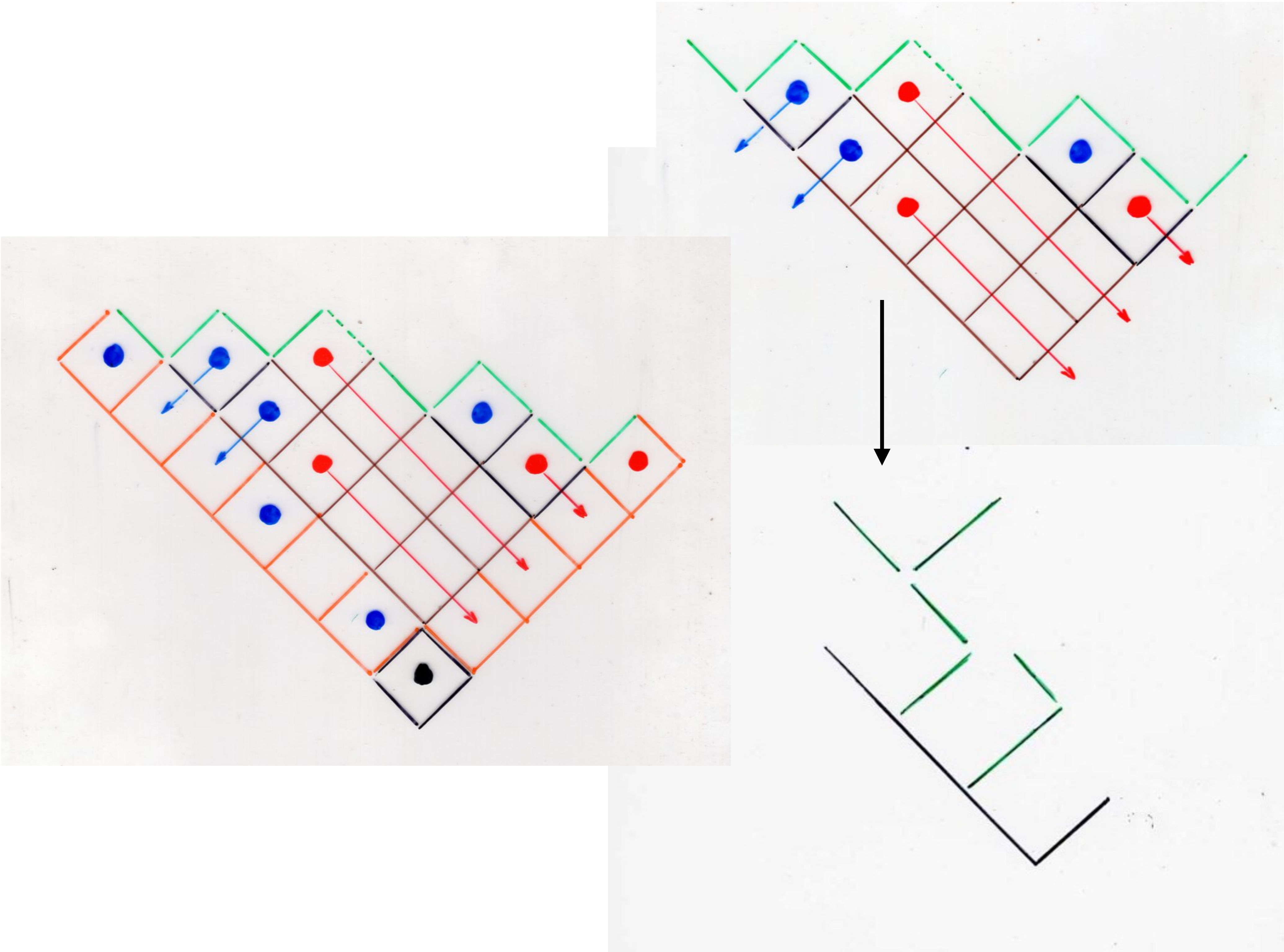
Catalan tree-like tableaux

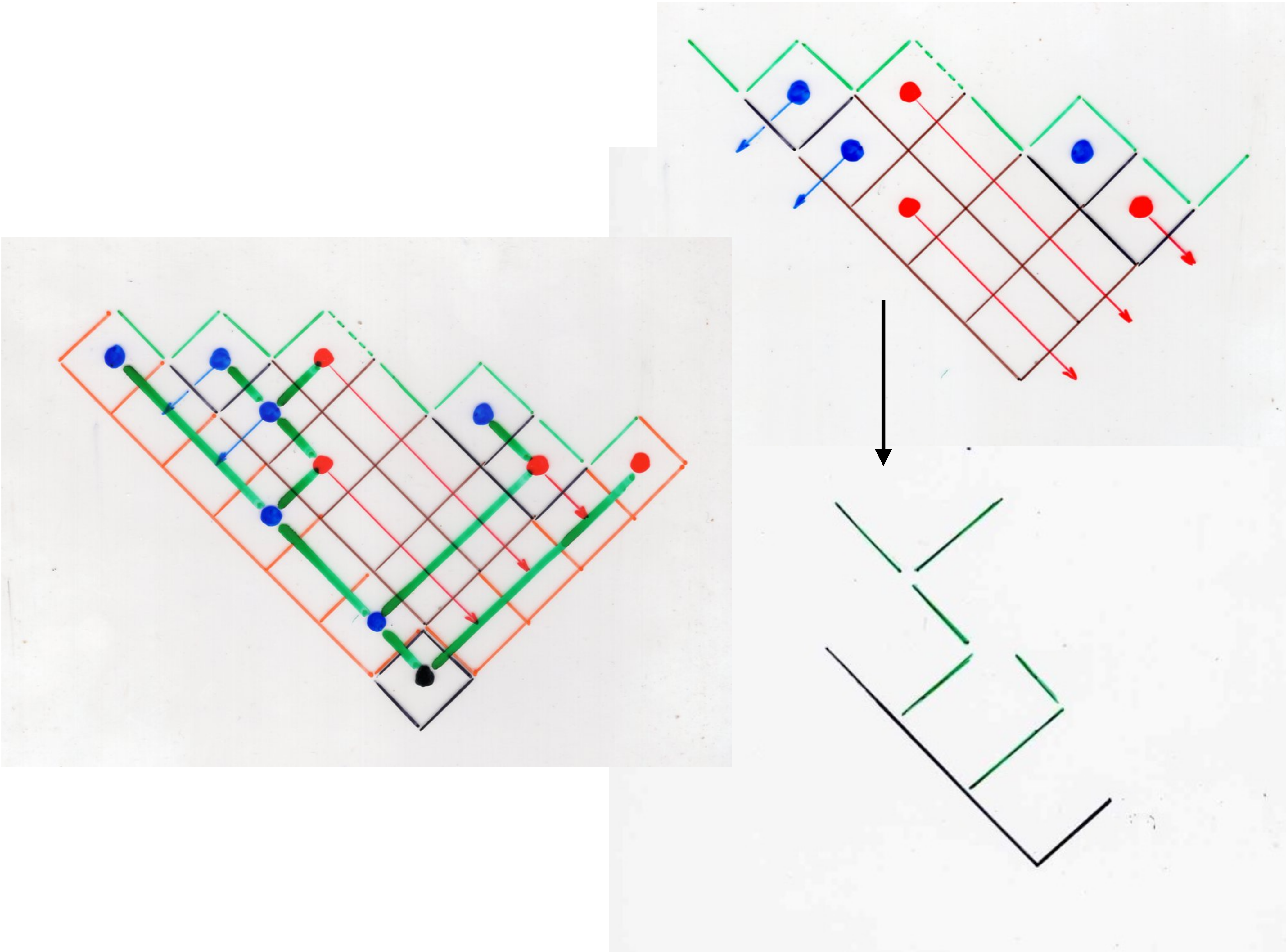
Two Bijections Catalan alternative tableaux

Binary trees

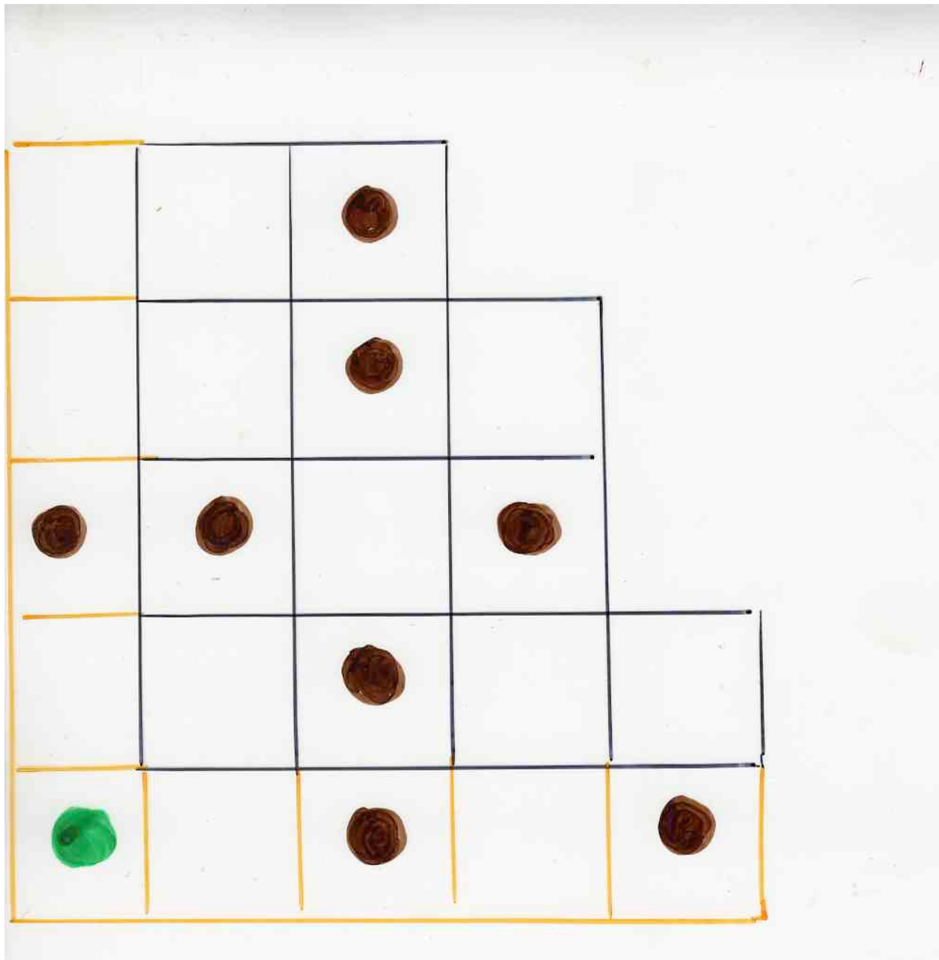


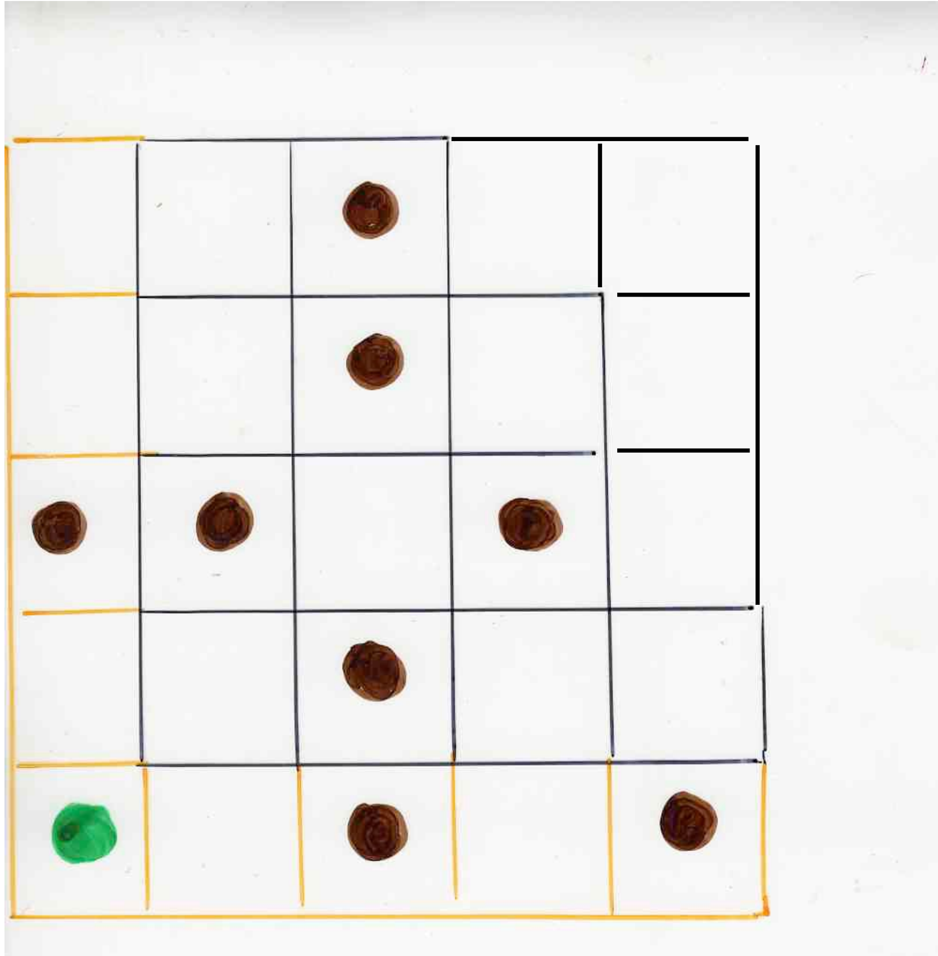




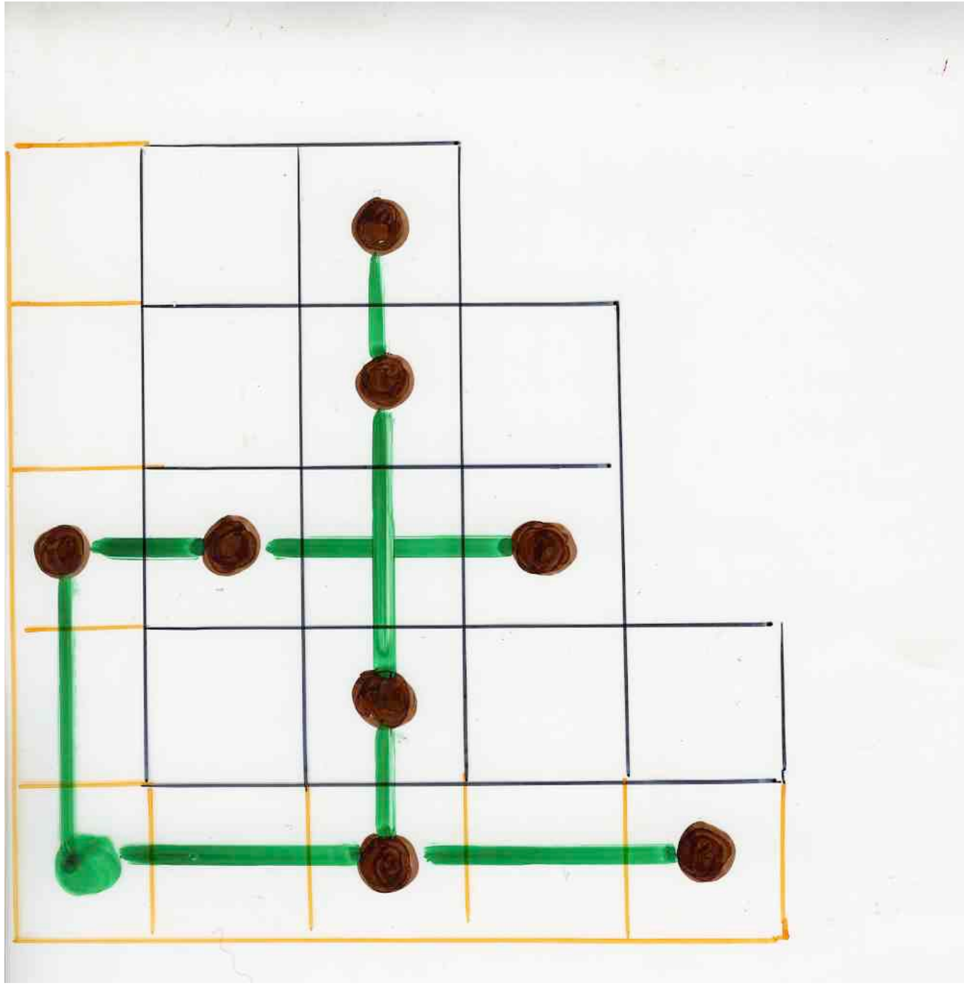


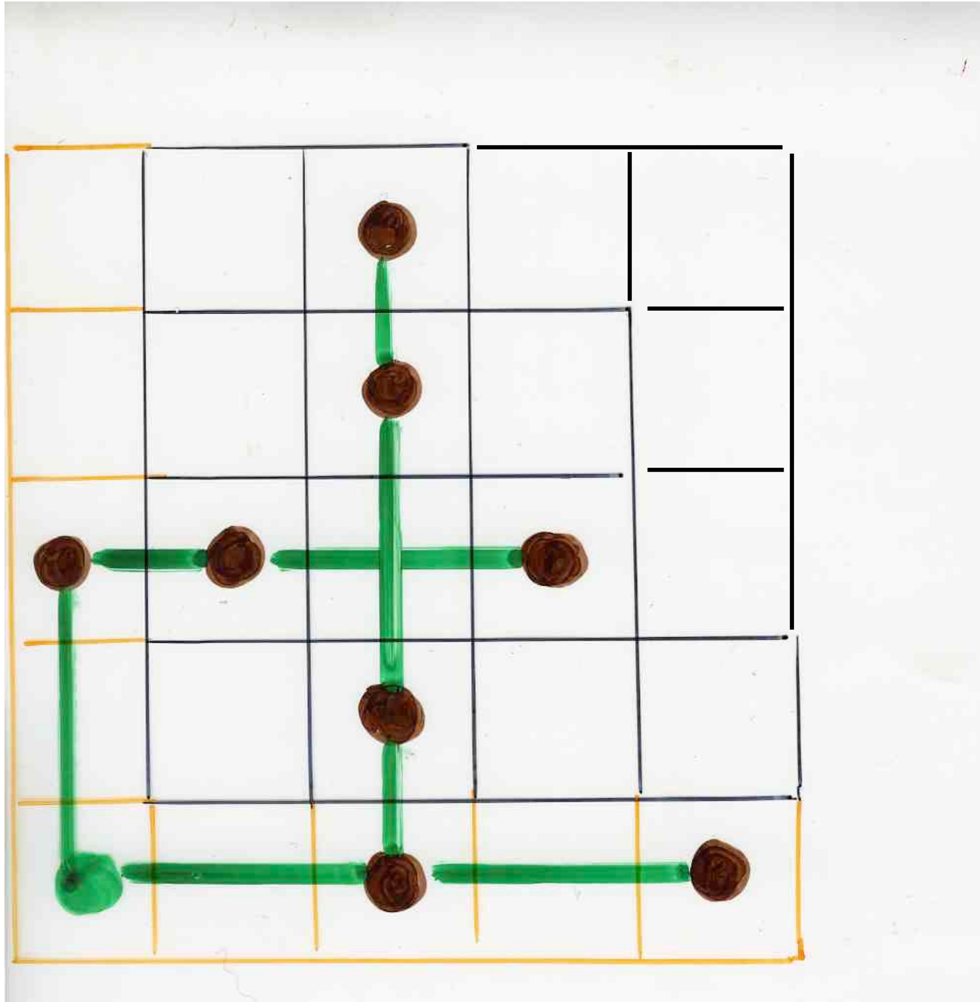
Non-ambiguous trees

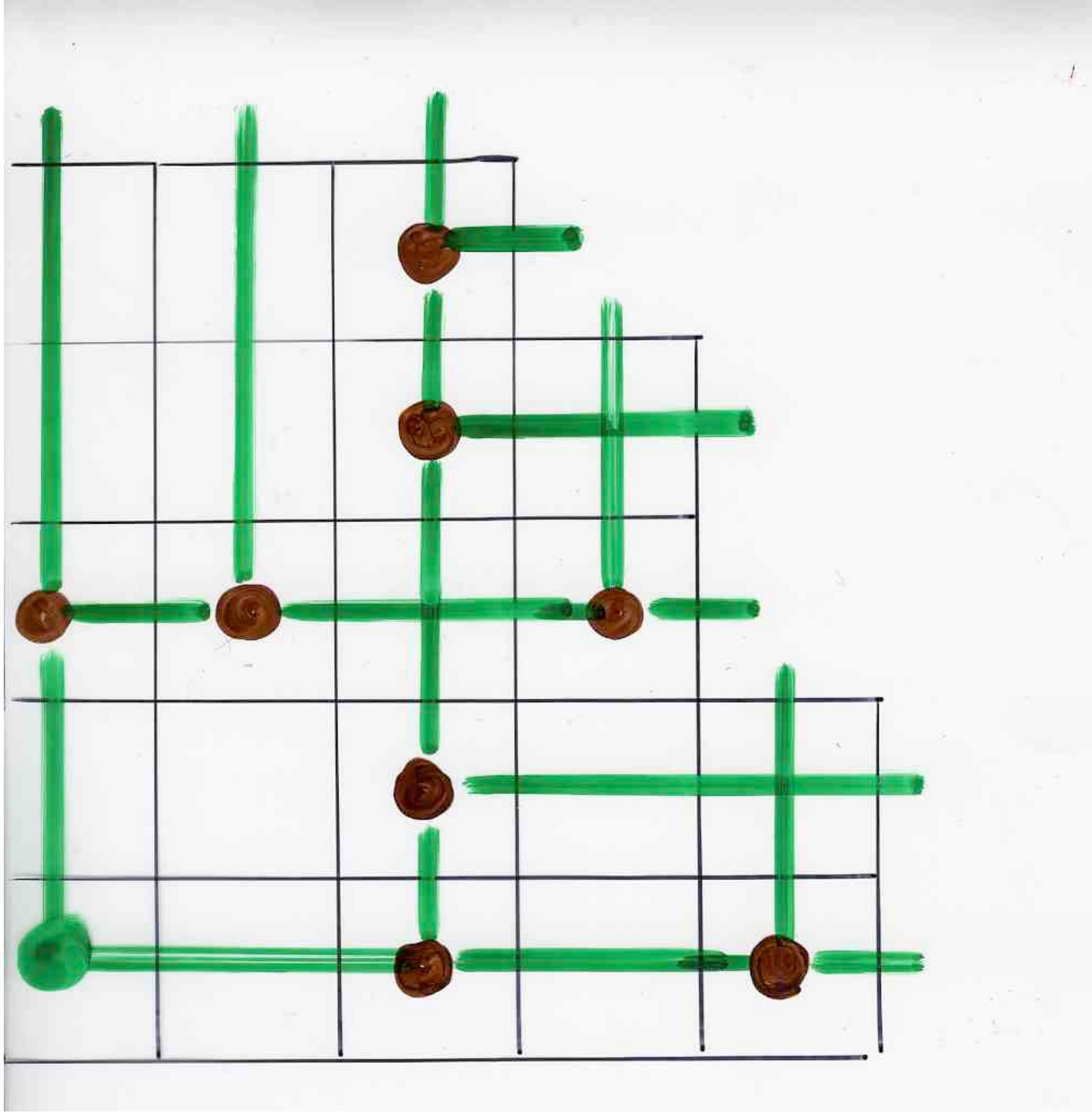


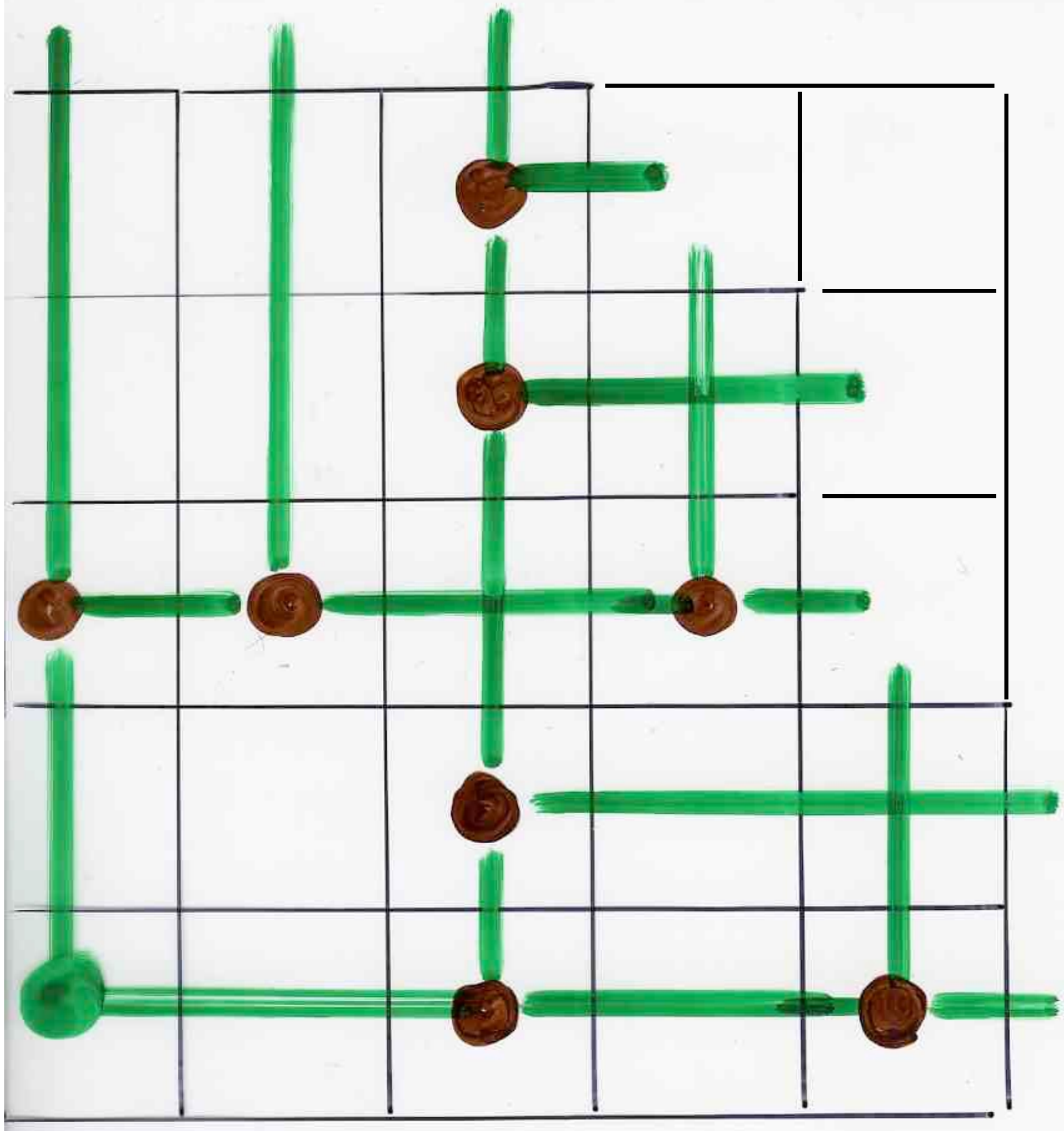


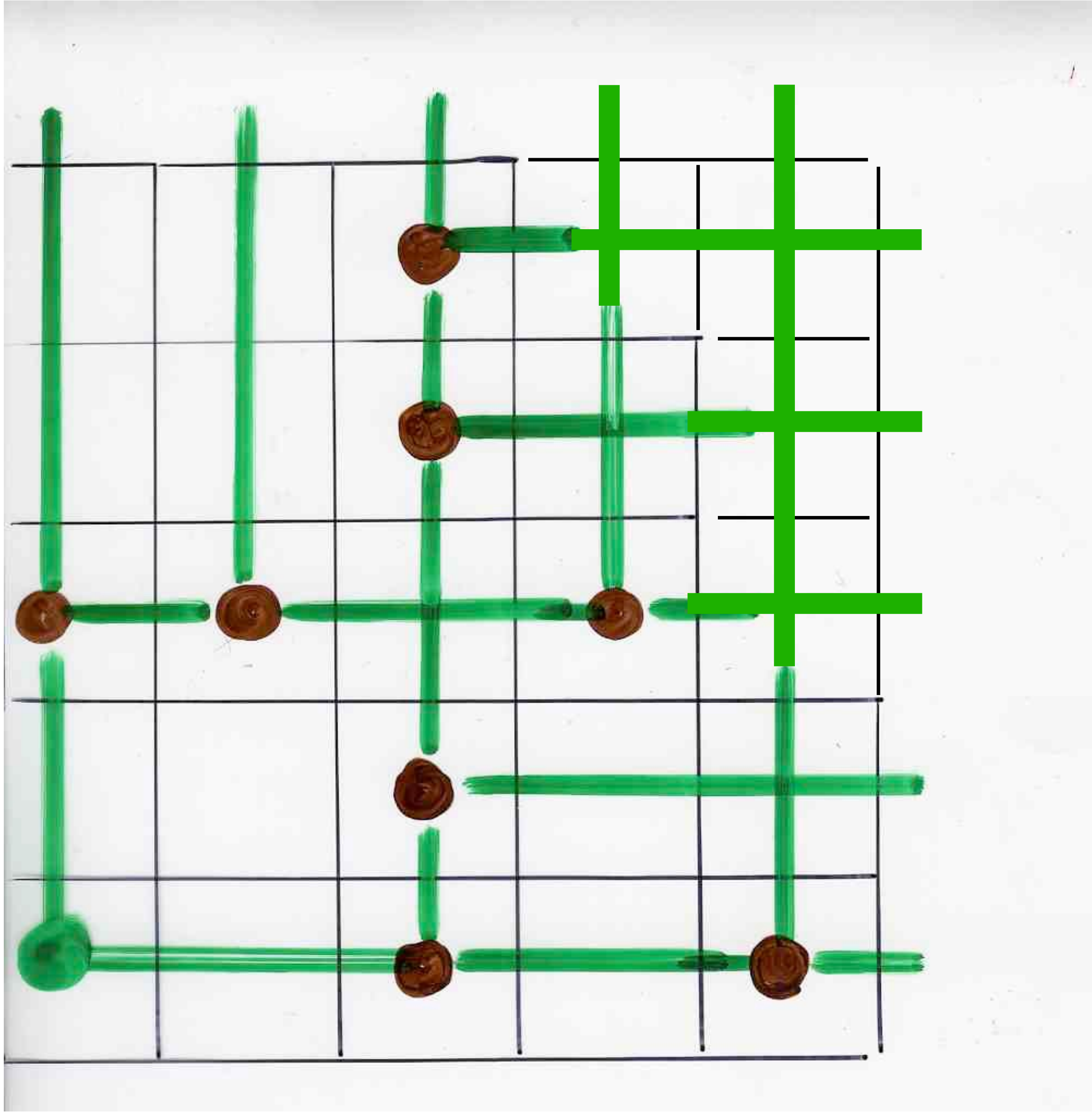
Aval, Boussicault, Bouvel, Silimbani
(2013) (2014)







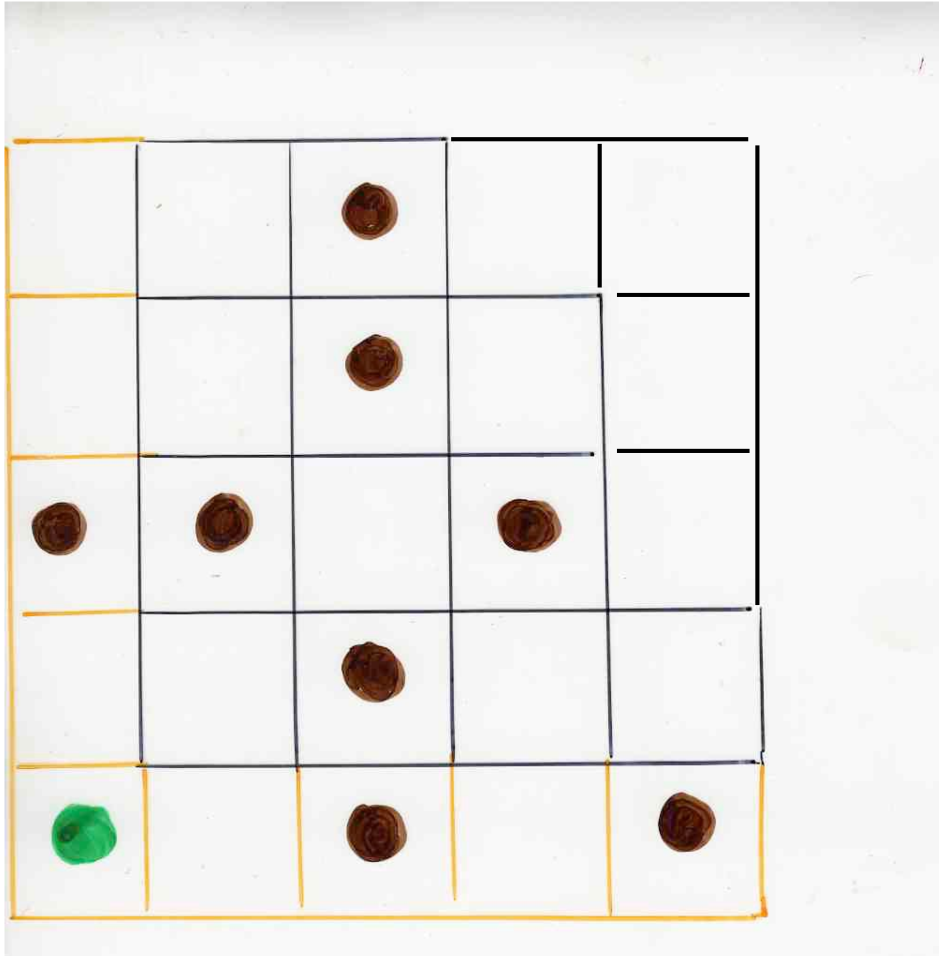




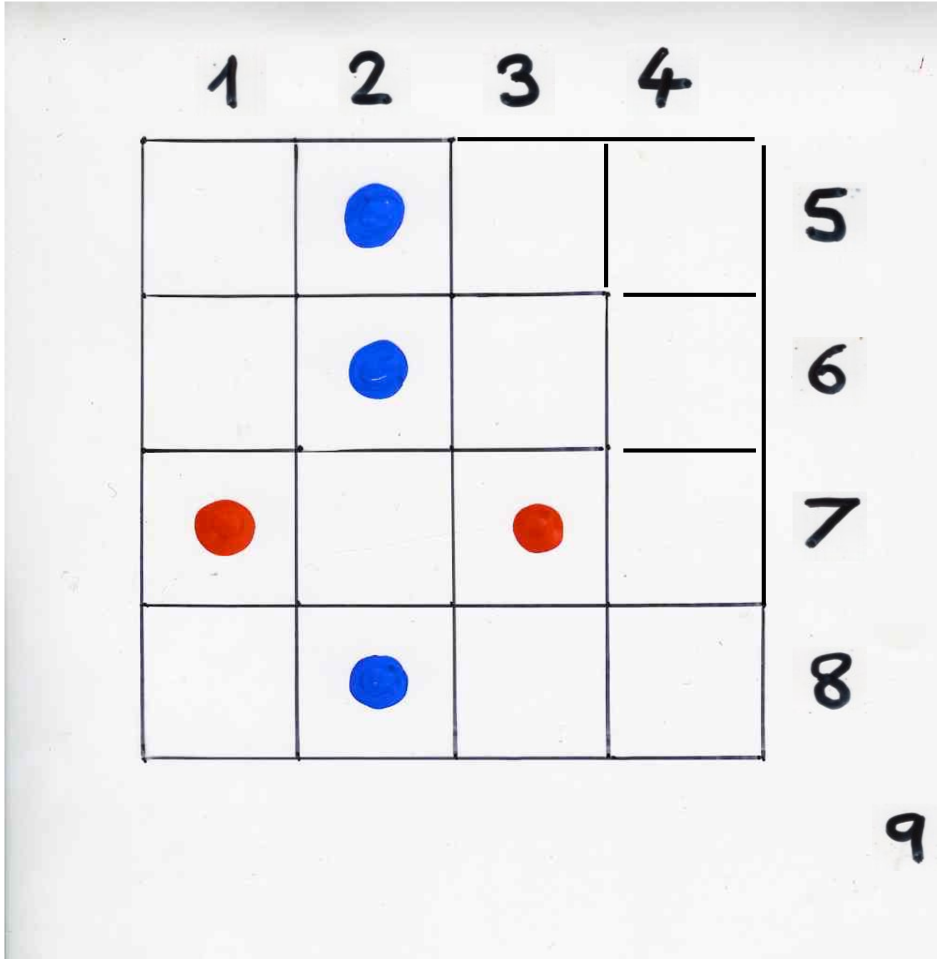
non-ambiguous trees

in bijection with permutations
with Genocchi sequence

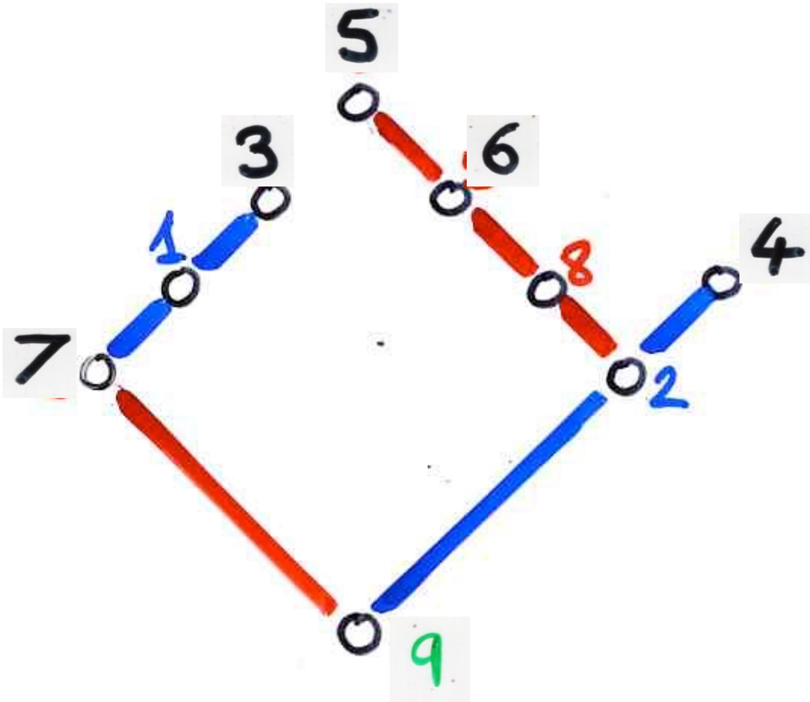
$\underbrace{a\,a \cdots a}_{k}\,\underbrace{d \cdots d}_{l}$

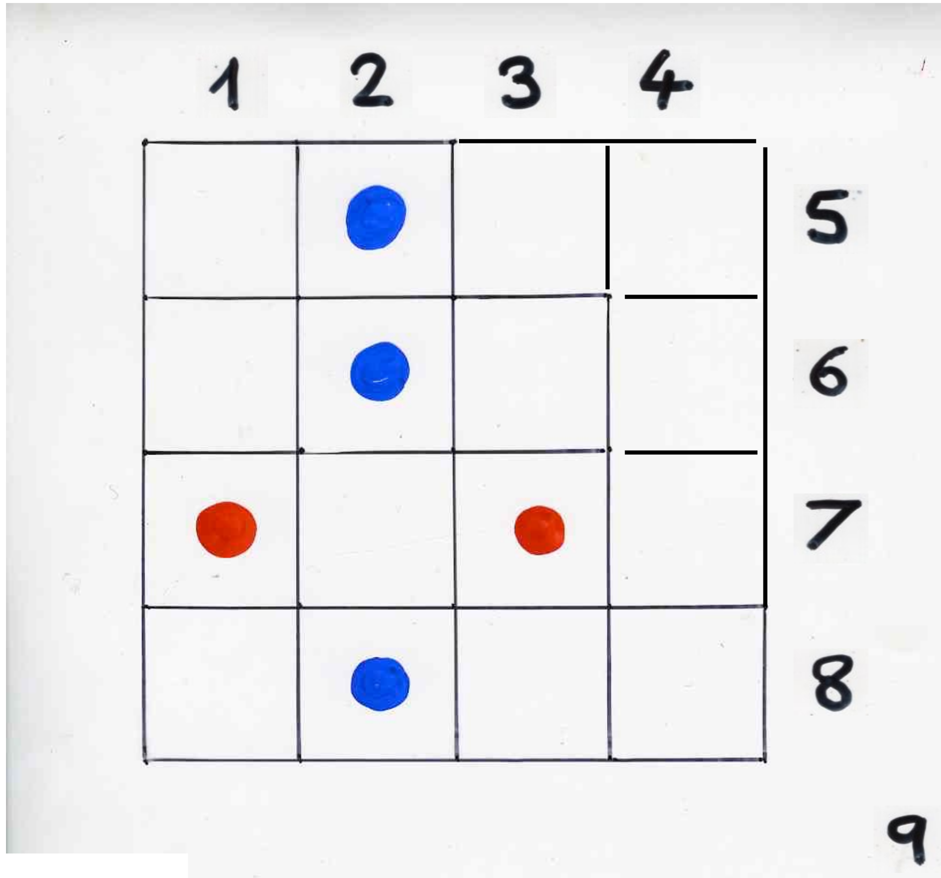


tree-like
tableaux



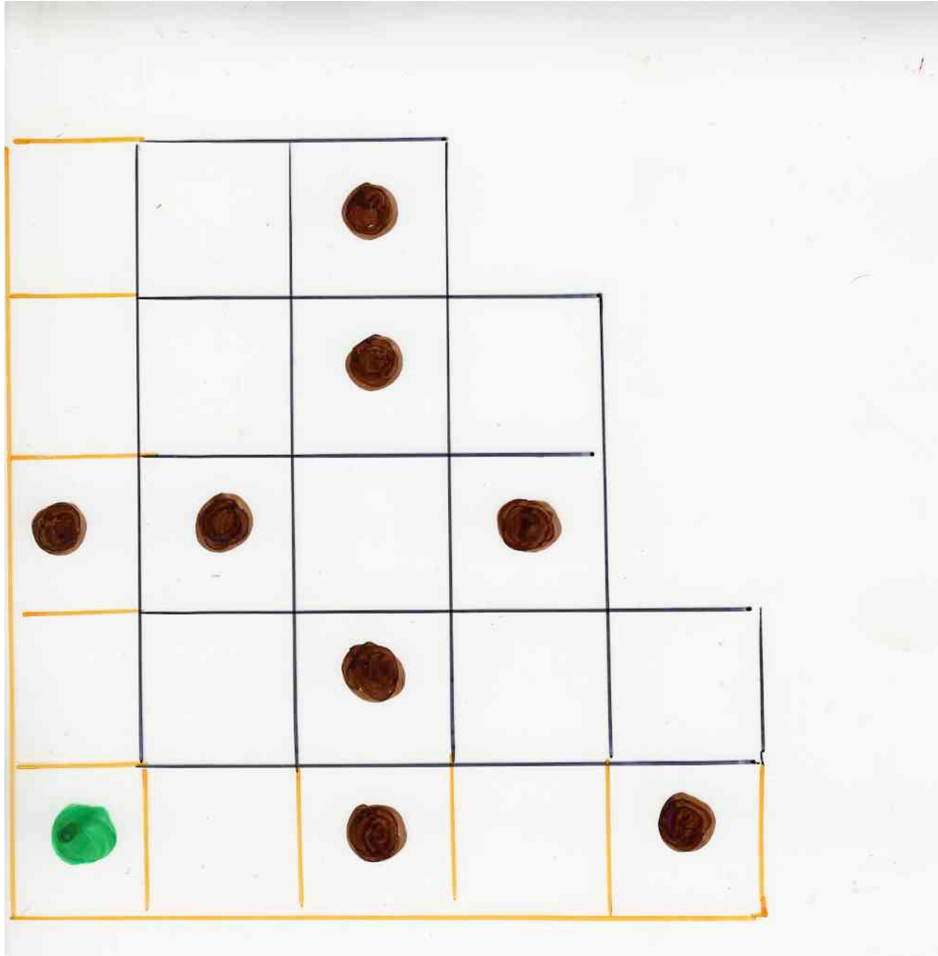
alternative
tableaux

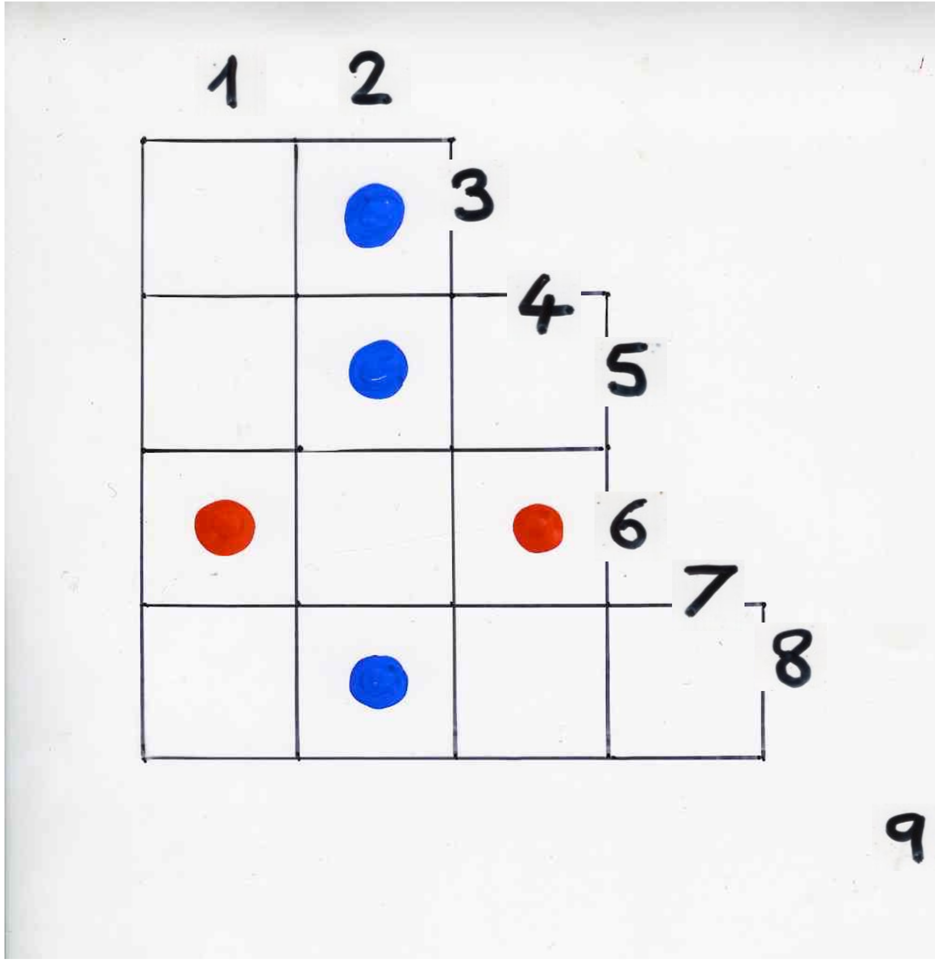




$$\sigma = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 5 & 2 & 4 & 9 & 1 & 3 & 7 \end{array}\right)$$

alternative
tableaux

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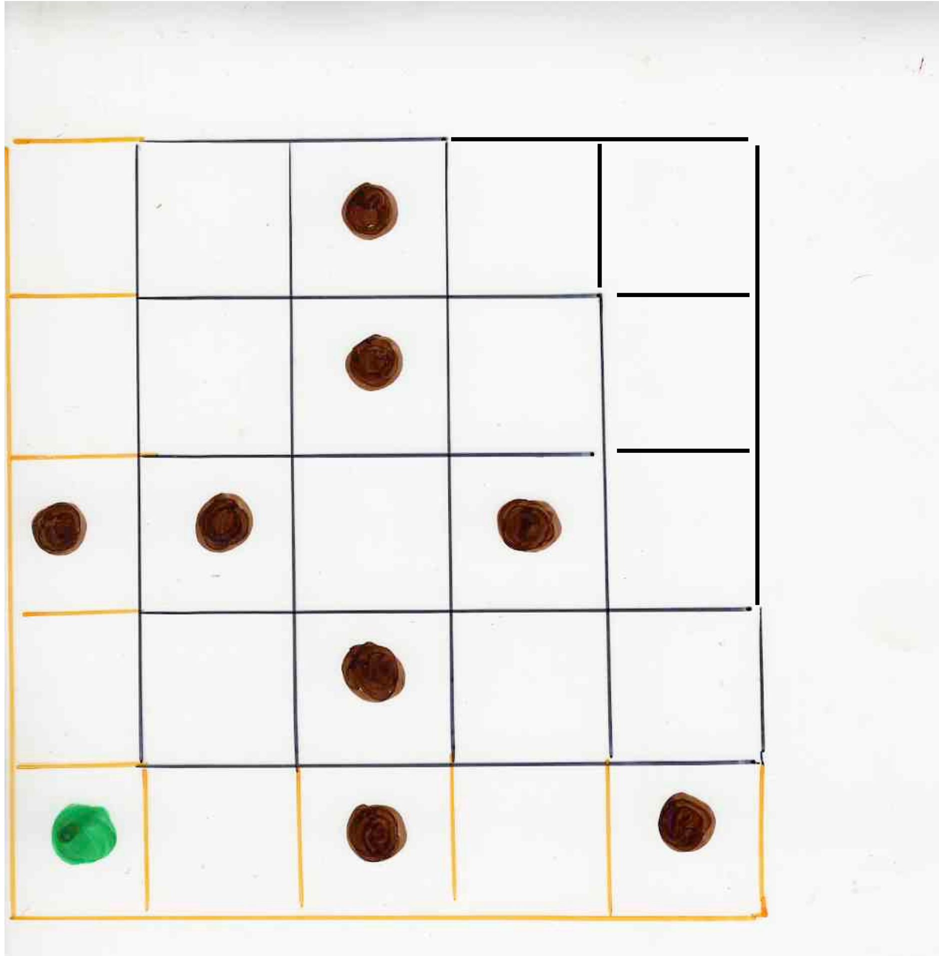
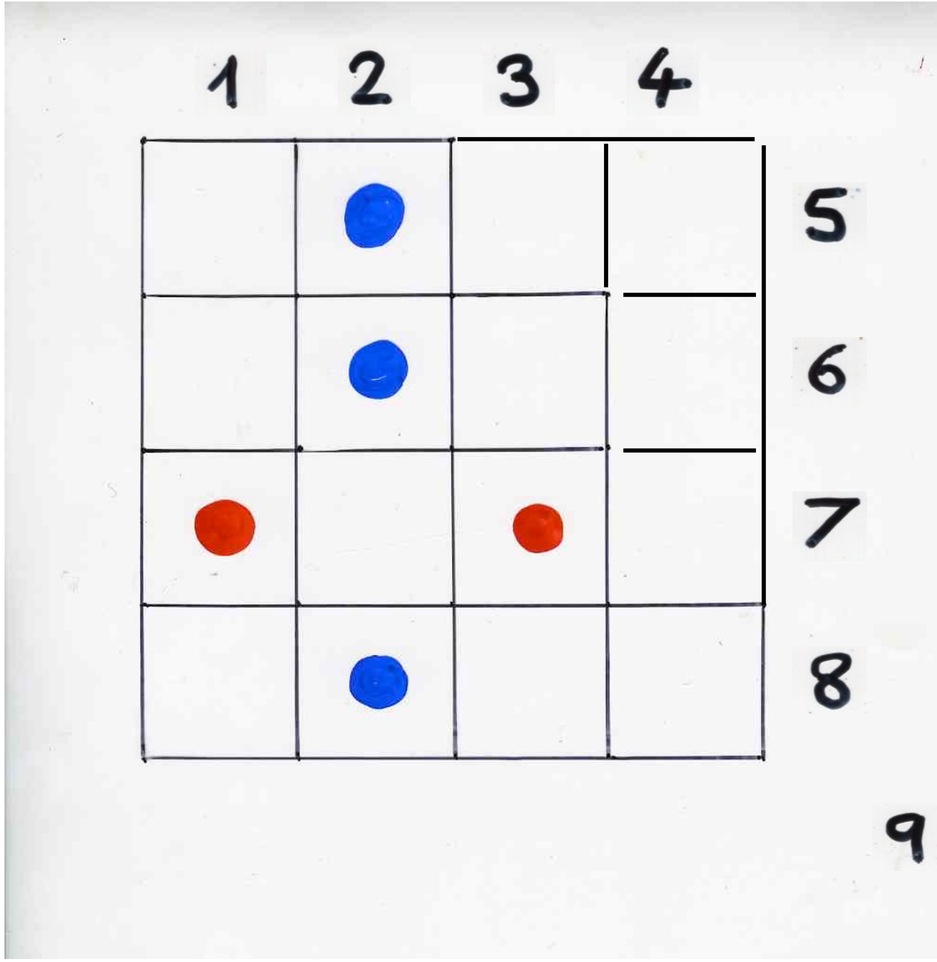
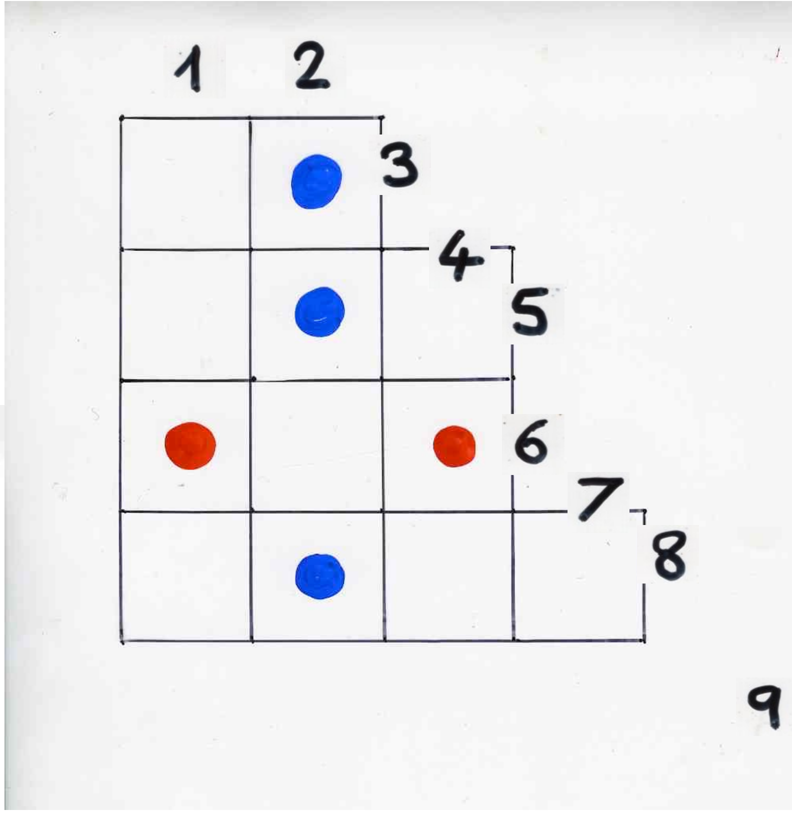
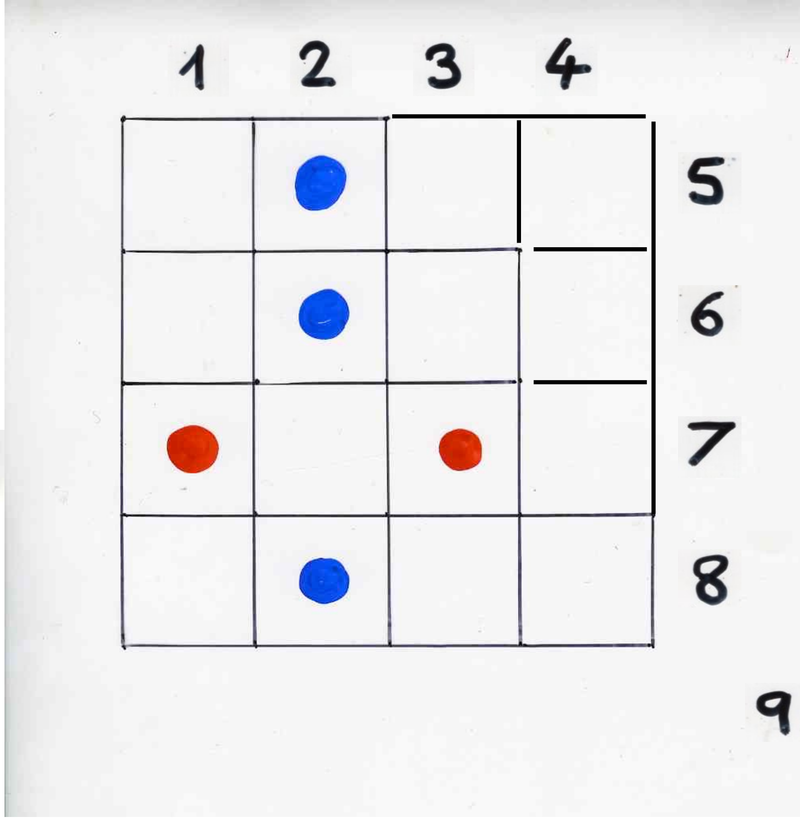


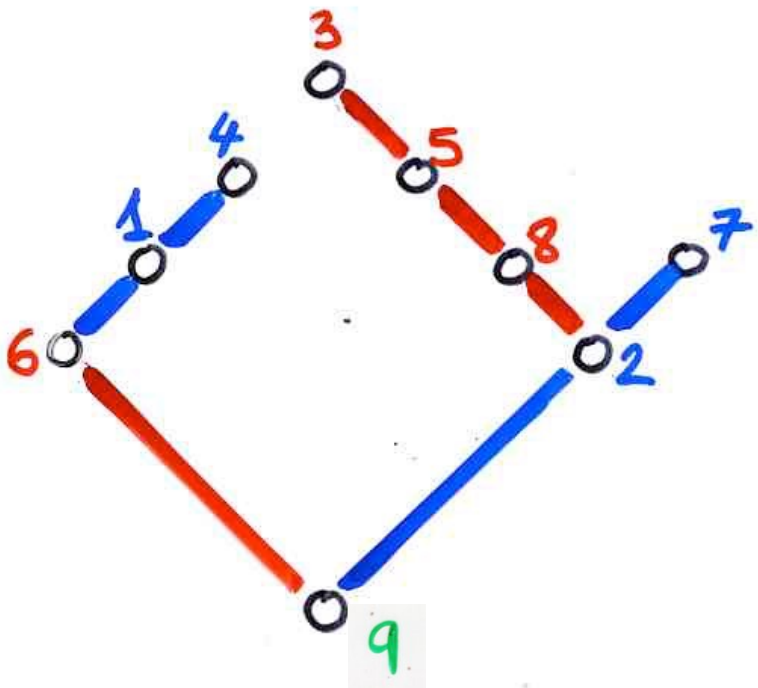
tree-like
tableaux



alternative
tableaux







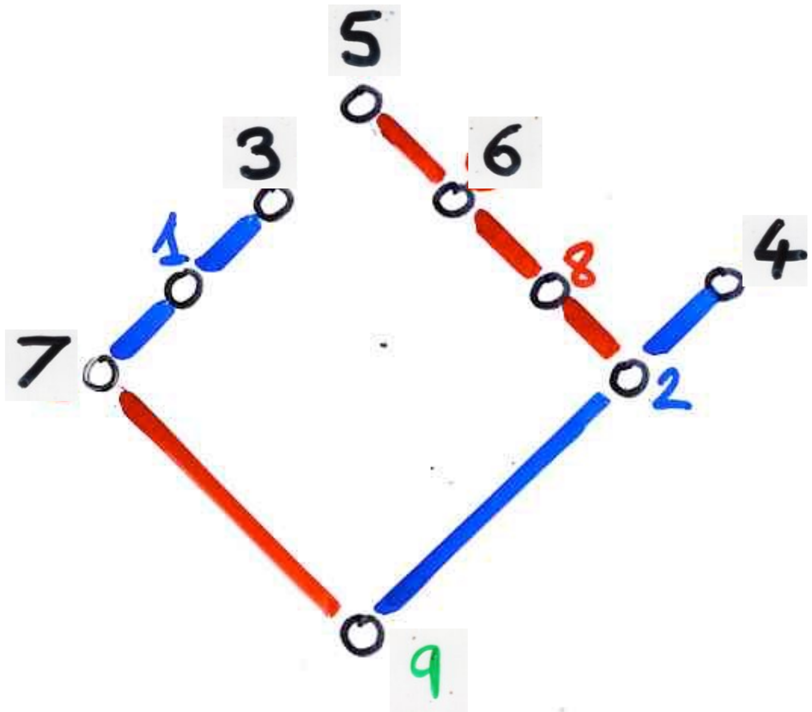
$4 \longrightarrow 3$

$7 \longrightarrow 4$

$3 \longrightarrow 5$

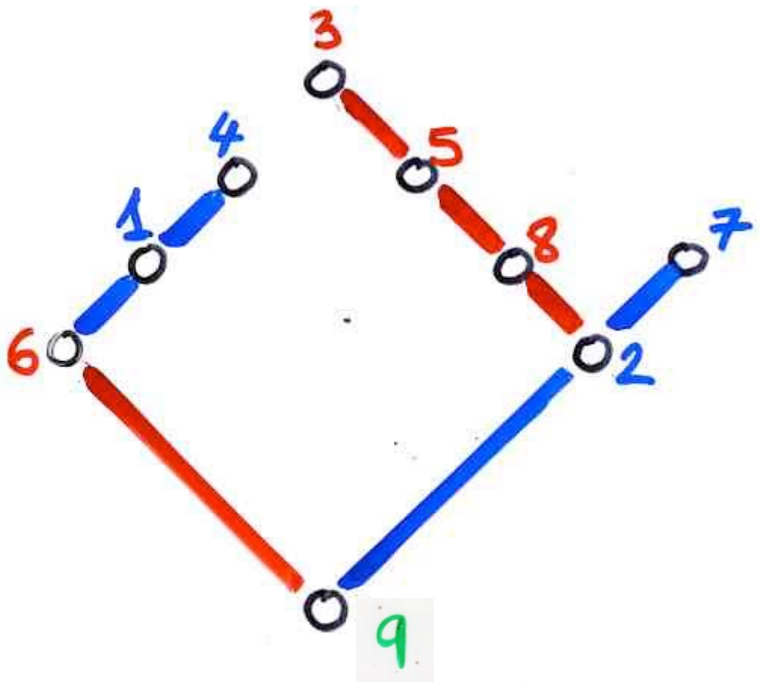
$5 \longrightarrow 6$

$6 \longrightarrow 7$



$$\sigma = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{array}\right)$$

$$\sigma = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 5 & 2 & 4 & 9 & 1 & 3 & 7 \end{array}\right)$$



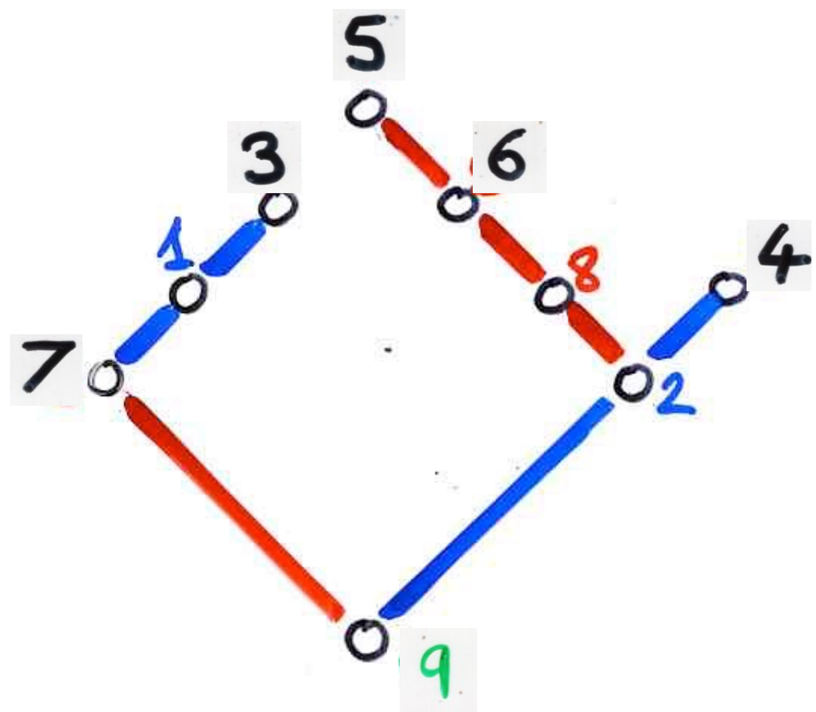
$4 \longrightarrow 3$

$7 \longrightarrow 4$

$3 \longrightarrow 5$

$5 \longrightarrow 6$

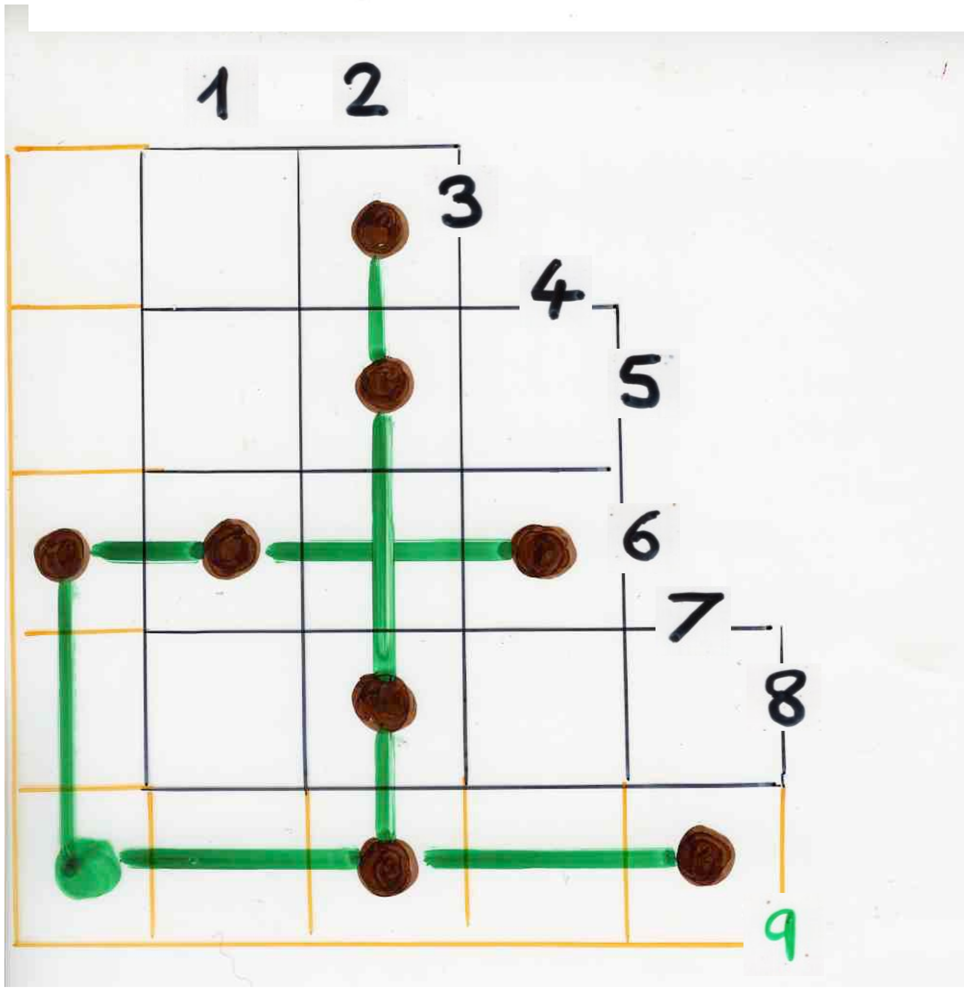
$6 \longrightarrow 7$

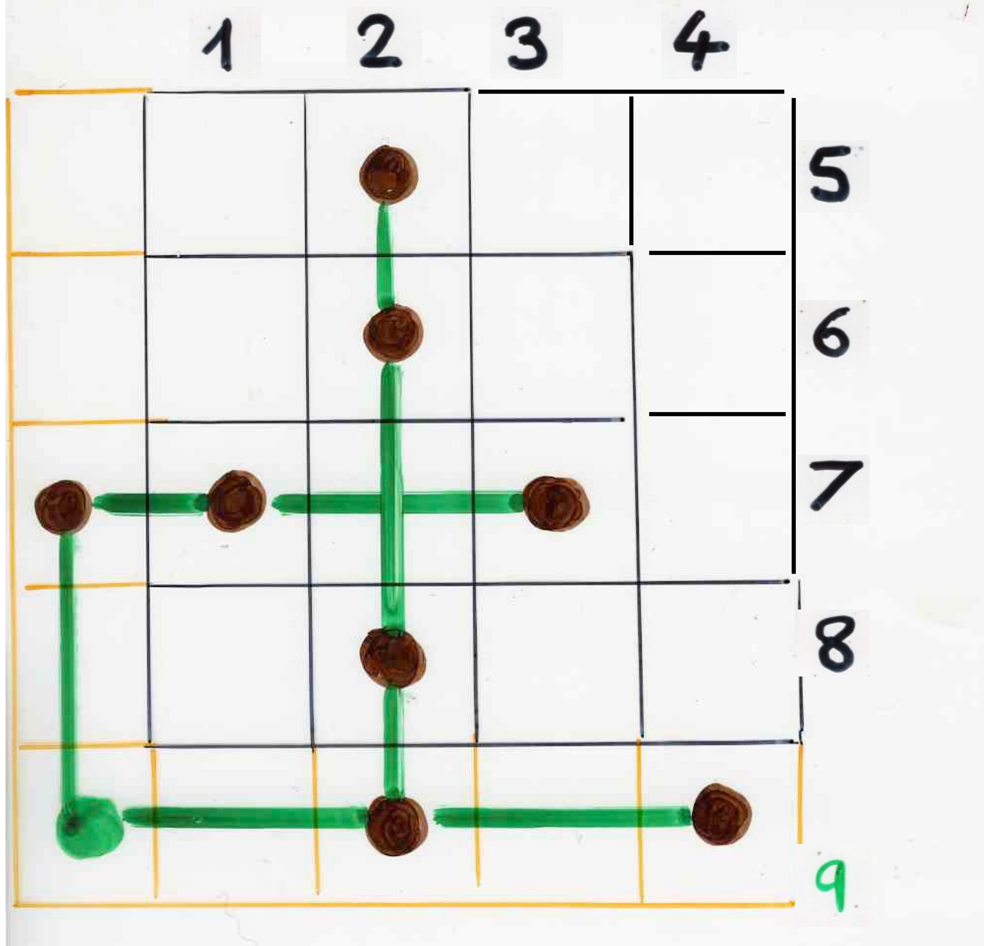


$$\sigma = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{array}$$

interval

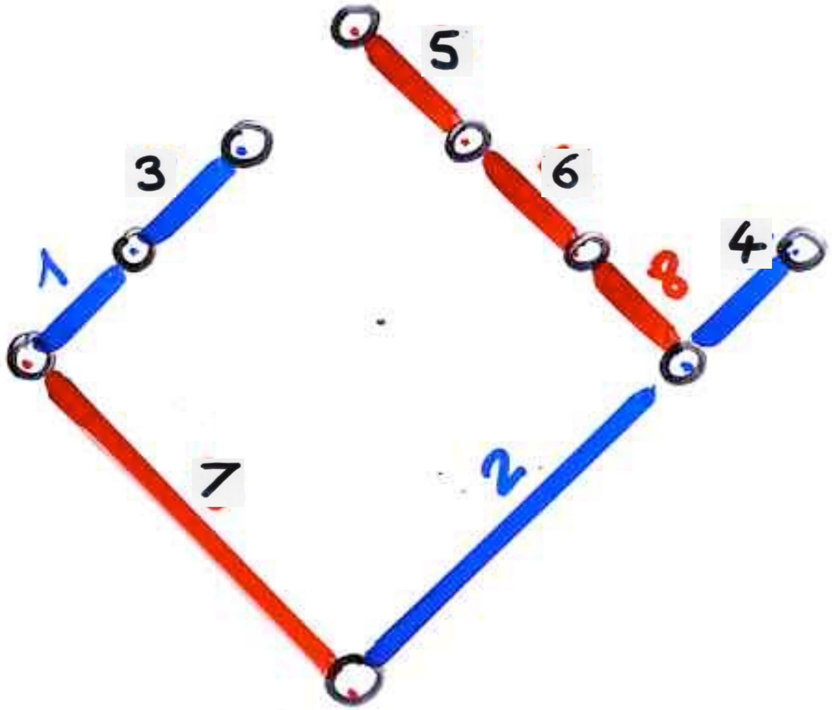
(weak) Bruhat order





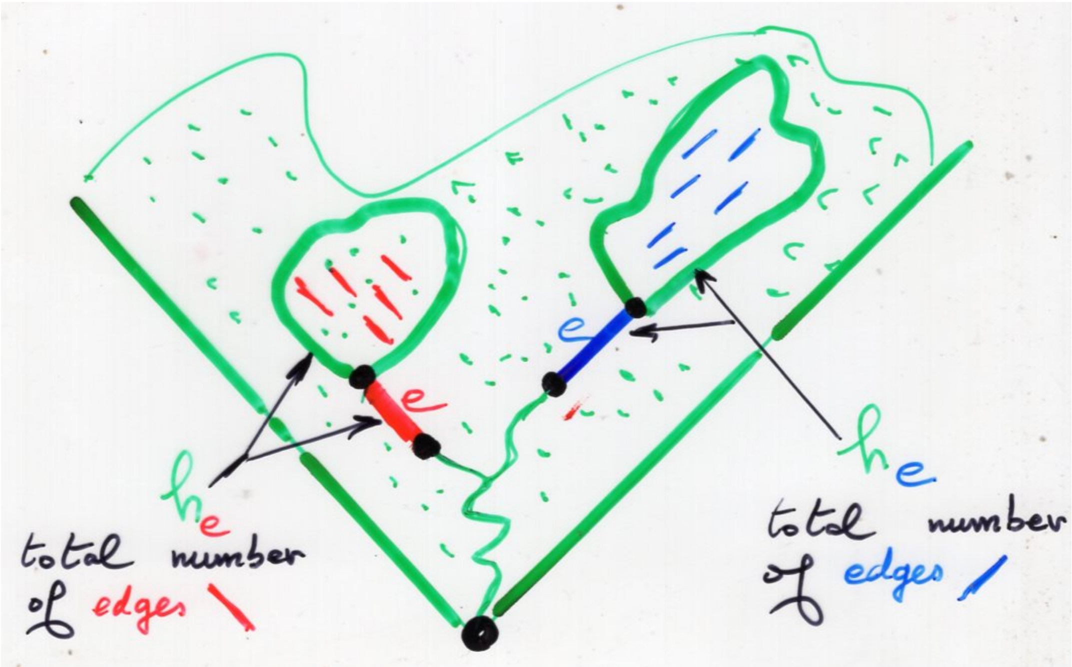
$$\sigma = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 6 & 5 & 2 & 4 & 9 & 1 & 3 & 7 \end{array}$$

hooks formula



The number of non-ambiguous trees having B as underlying binary tree is

$$\frac{k!\,\ell!}{\prod\limits_{\substack{e \\ \text{edge } /}} (h_e) \prod\limits_{\substack{e \\ \text{edge } \backslash}} (h_e)}$$



non-ambiguous trees

in bijection with permutations
with Genocchi sequence

$$\underbrace{a\,a\cdots a}_{k}\ \underbrace{d\cdots d}_{l}$$

in bijection with permutations
on $(k+l+1)$ such that all exceedances
are exactly $1, 2, \ldots, k$.

Ehrenborg, Steingrímsson (2000)

$$\sum_{k=1}^{n} |s(n,k)|\, A(k,\ell) = n^{\ell-1}\; n!$$

$$A(k,\ell) = \sum_{i=1}^{k} (-1)^{k-i}\, S(k,i)\, i!\, i^{\ell-1}$$

$s(n,k)$ Stirling 1st permutations, cycles

$S(k,i)$ Stirling 2nd partitions, block

numbers

Ehrenborg, Steingrimsson (2000)

complete non-ambiguous trees

Bessel functions

and heaps of pieces

b_n

number of complete non-ambiguous trees
of size n (number of internal vertices)
satisfies:

$$\sum_{n\geq 1} b_n \frac{t^n}{(n!)^2} = -\log\left(\sum_{n\geq 0} (-1)^n \frac{t^n}{(n!)^2}\right)$$

A002190 1, 1, 4, 33, 456, 9460

Carlitz (1963)
a sequence of integers related to the
Bessel functions

bijection

complete non-ambiguous trees size $(2n-1)$

pyramids Q_n having n elements

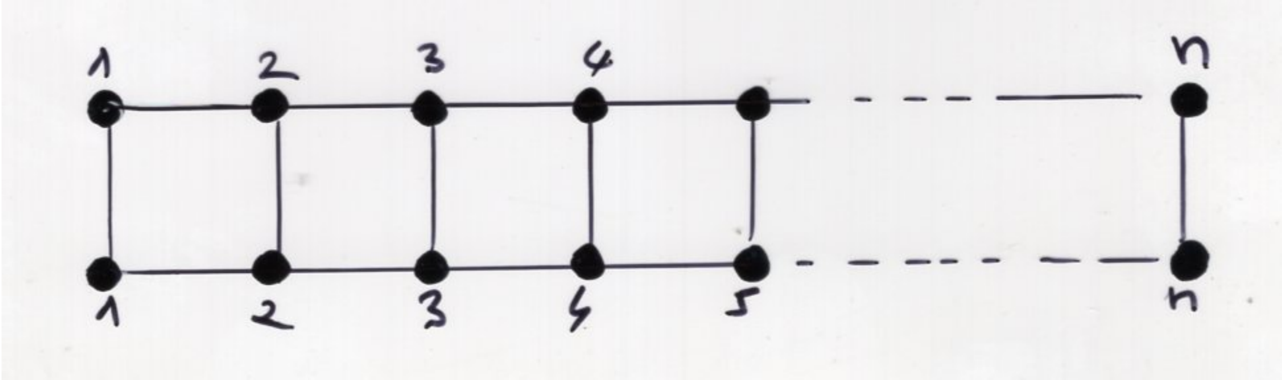
of the form $(i, \sigma(i))$ where $\sigma \in \mathfrak{S}_n$ permutations

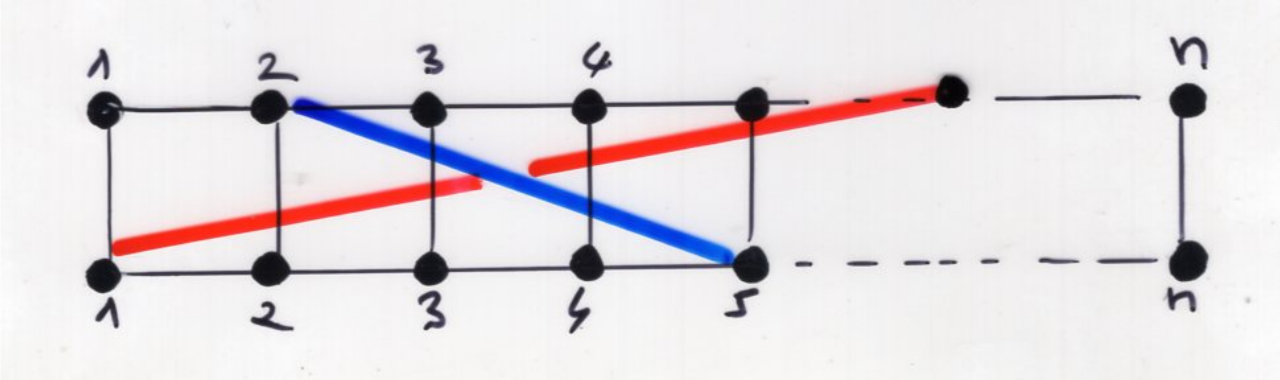
and the unique maximal element

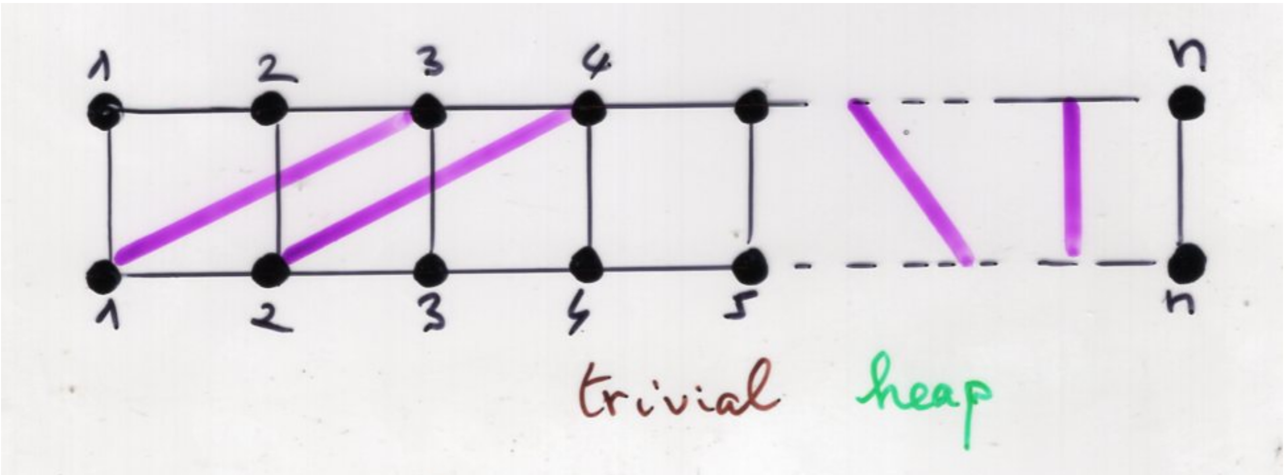
is $(1, \sigma(1))$

Emma Jin (2016)

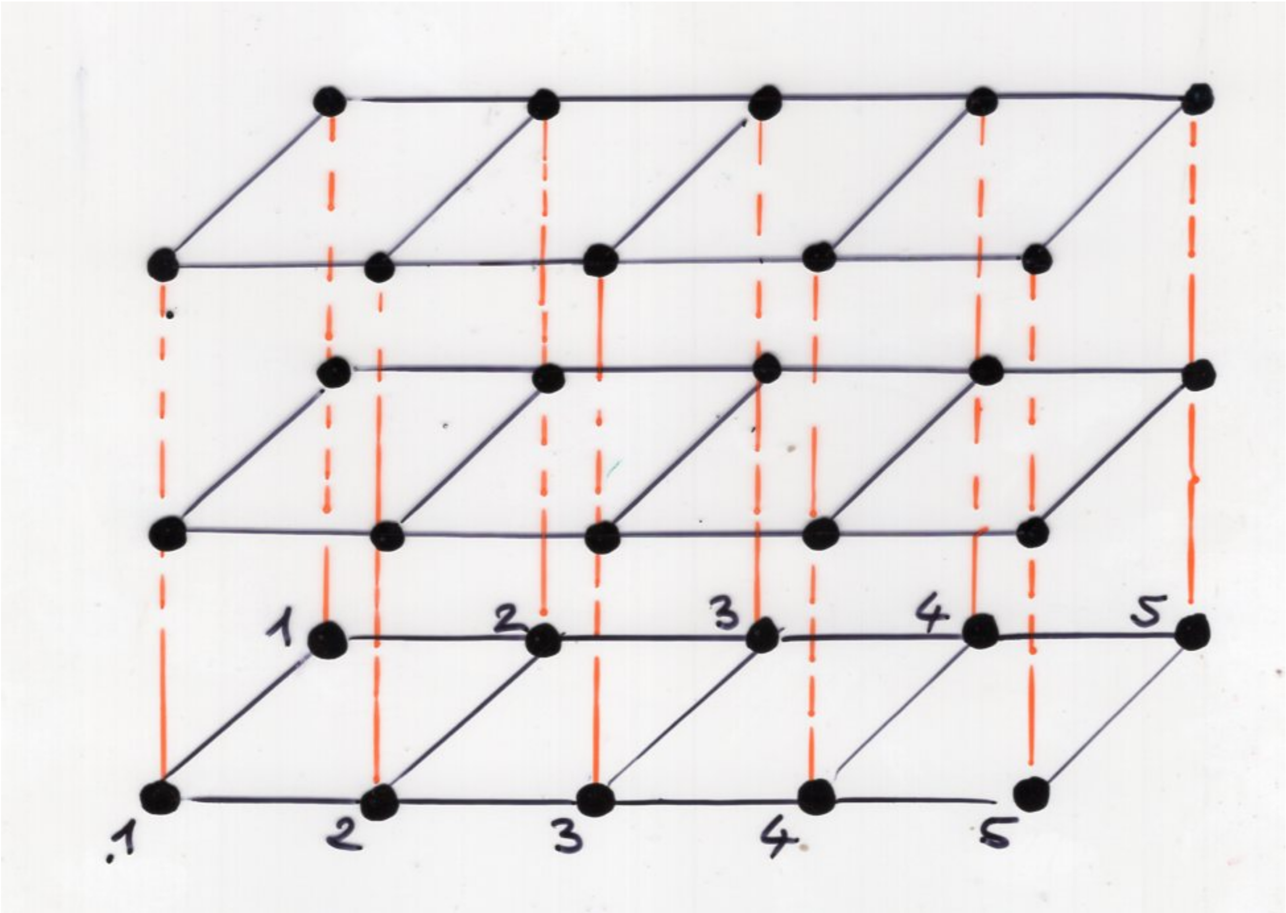
heaps of "sticks"

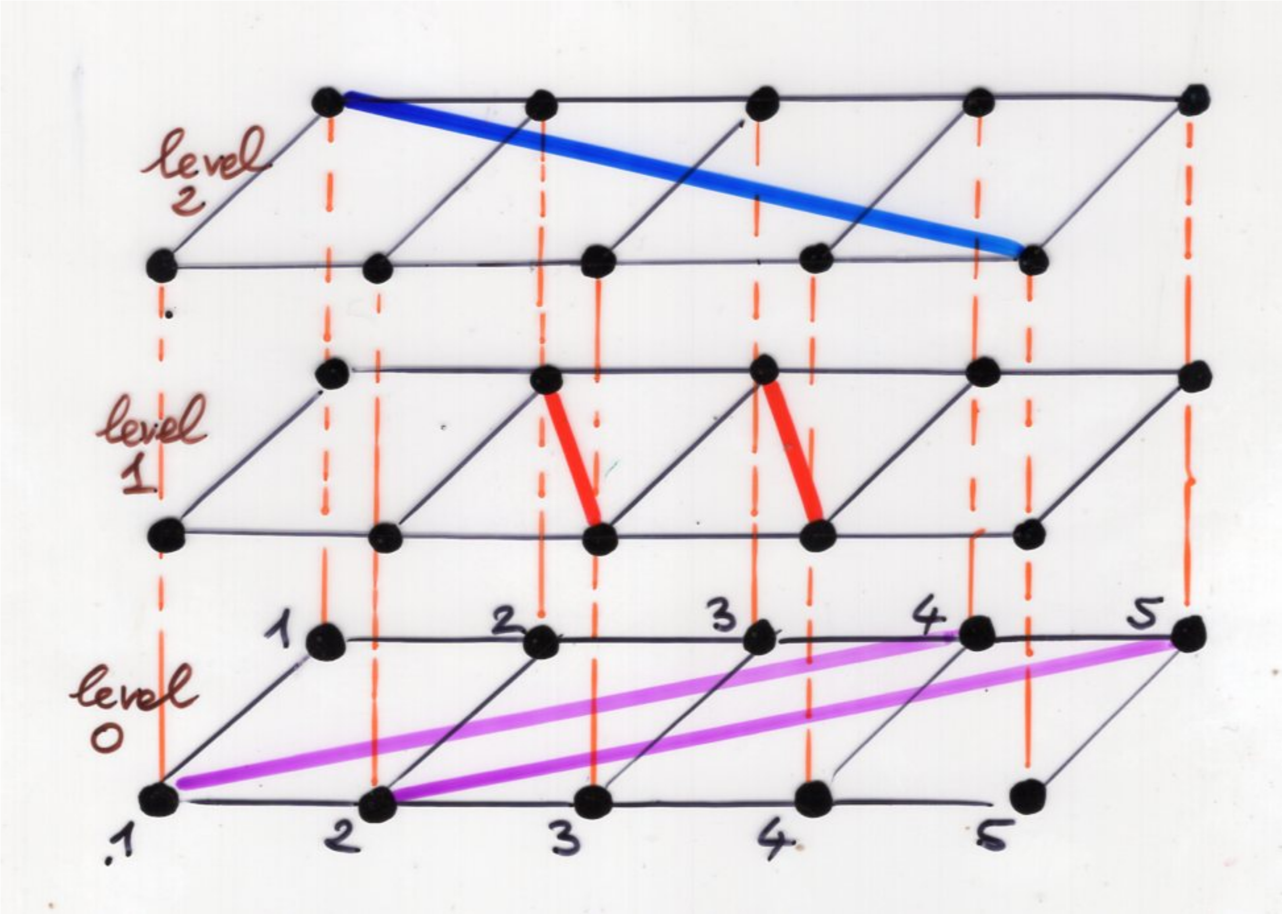

1
2
3
4
n
1
2
3
4
5
n



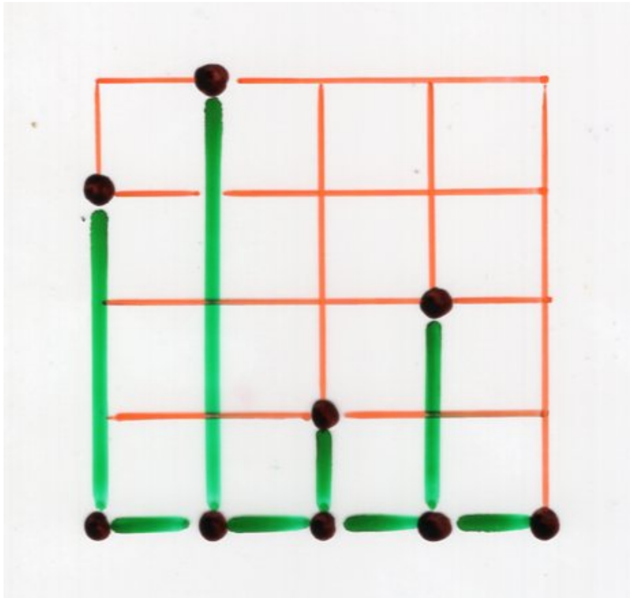


heaps of "sticks"

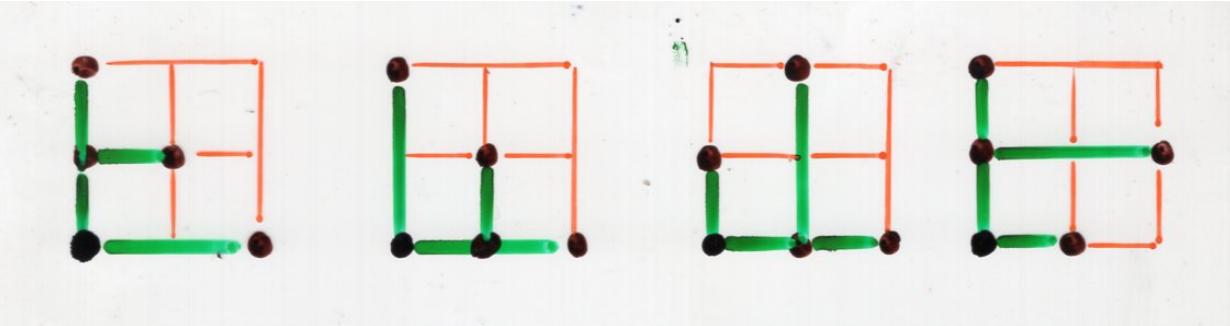

1
2
3
4
5
1
2
3
4
5

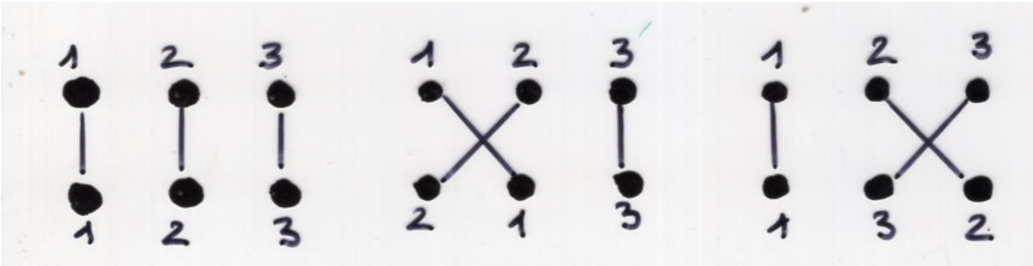


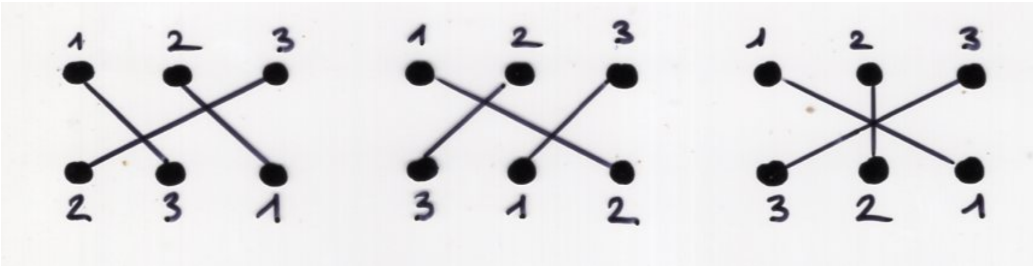
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$$

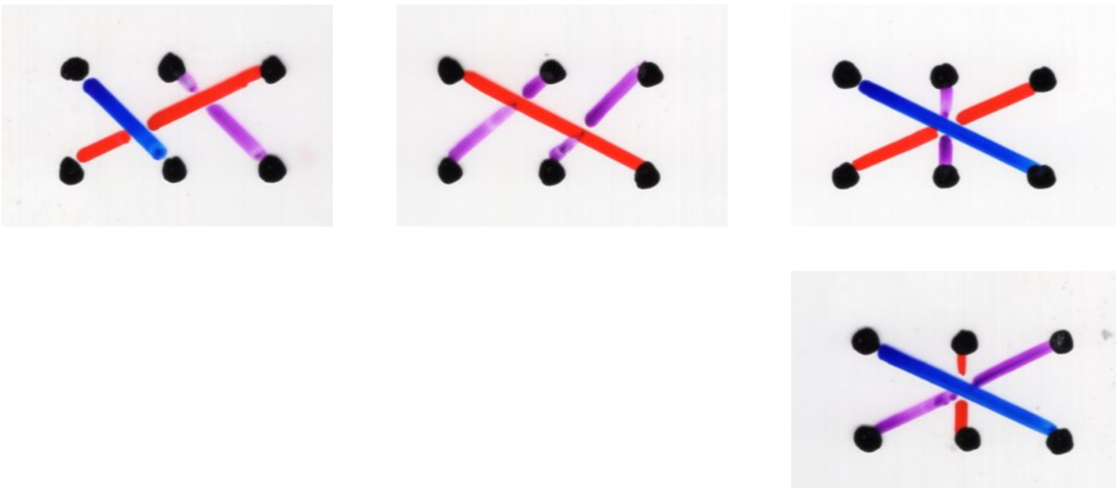


heaps of "sticks"




1 2 3
1 2 3
1 2 3
2 1 3
1 2 3
1 3 2


1 2 3
2 3 1
1 2 3
3 1 2
1 2 3
3 2 1




level
2
1
0

Aisbett (2015)

poset of vector partitions of [n] into s components $\Pi_{n,s}$

CL-shellable

$\Rightarrow$ homotopy type of wedge of spheres dimension $(n-2)$

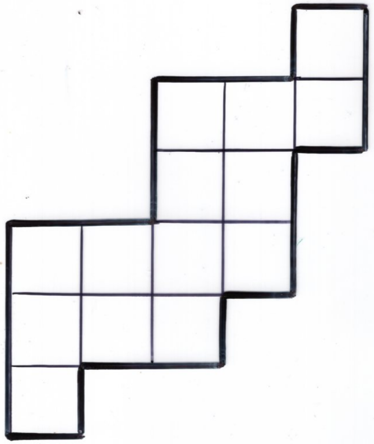
$s=1$, number of spheres $= b_n$

Carlitz, Scoville, Vaughan (1974)

number of pairs of permutations of $\{1, .., n\}$ with no common rise

Beáta Bényi, G. Nagy (2017)

Fedou, Rawlings (1995)



q-Polya enumeration

parallelogram polyominoes

(area, number of rows / columns)

= staircase polygons

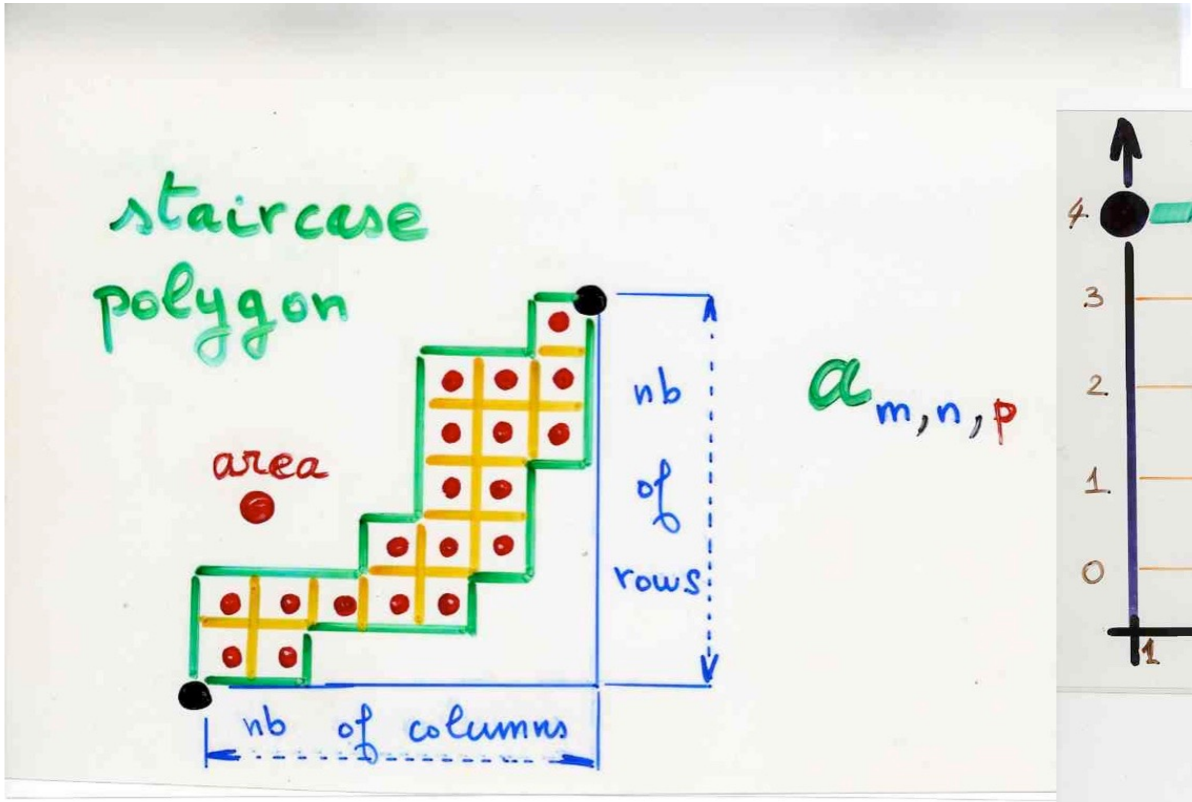
Delest, Fedou (1990)

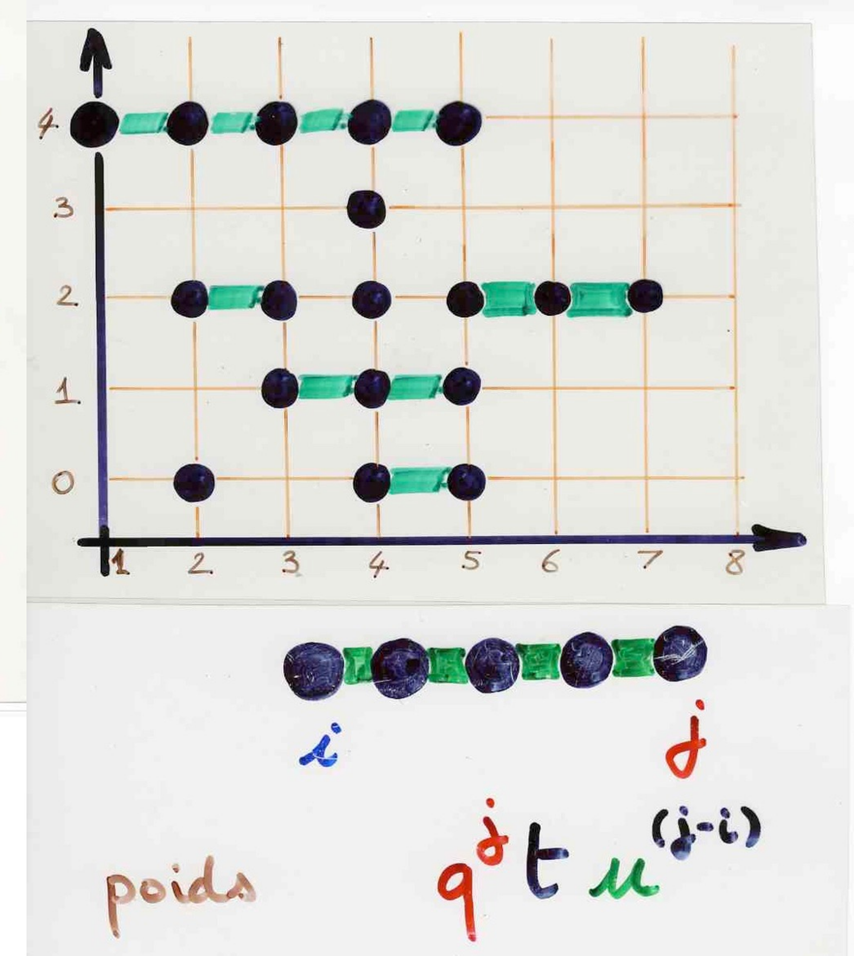
$$\frac{J_1(t;q)}{J_0(t;q)}$$

q-Bessel

heaps of "segments"

Bousquet-Mélou, X.V. (1992)





heaps of "segments" Bousquet-Mélou, X.V. (1992)

heaps of "sticks" Fedou (1992)

Bijection

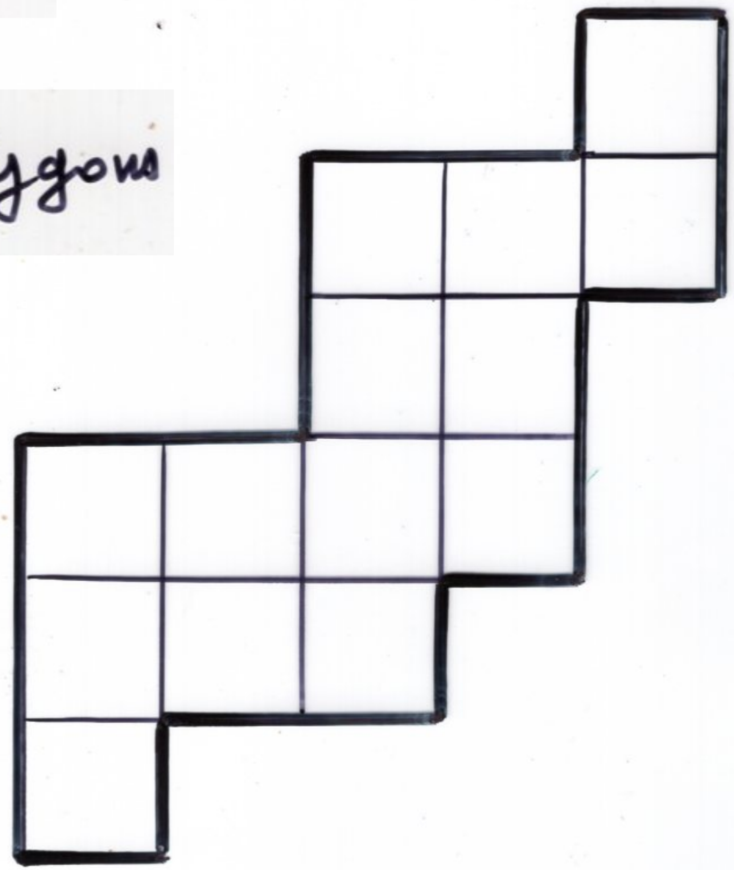
parallelogram polyominoes
(=staircase polygons)

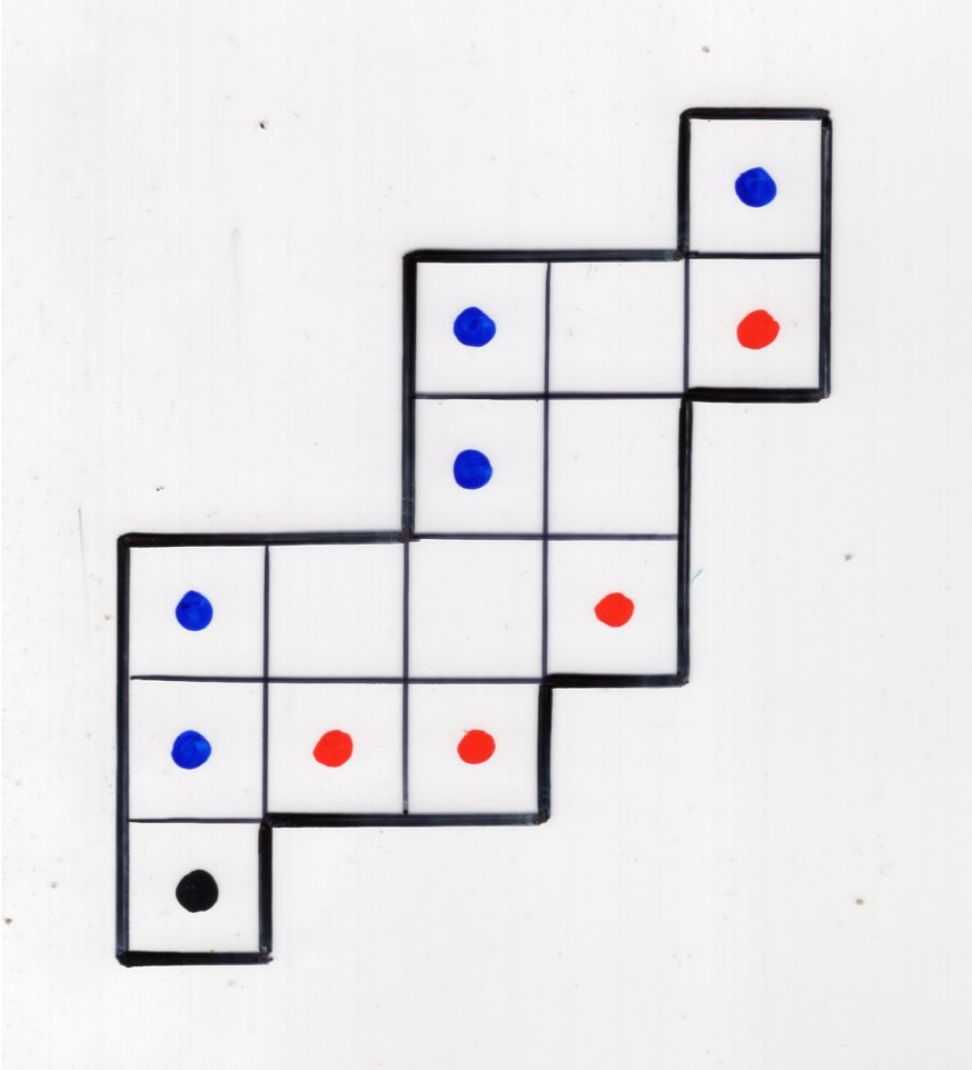
Binary trees

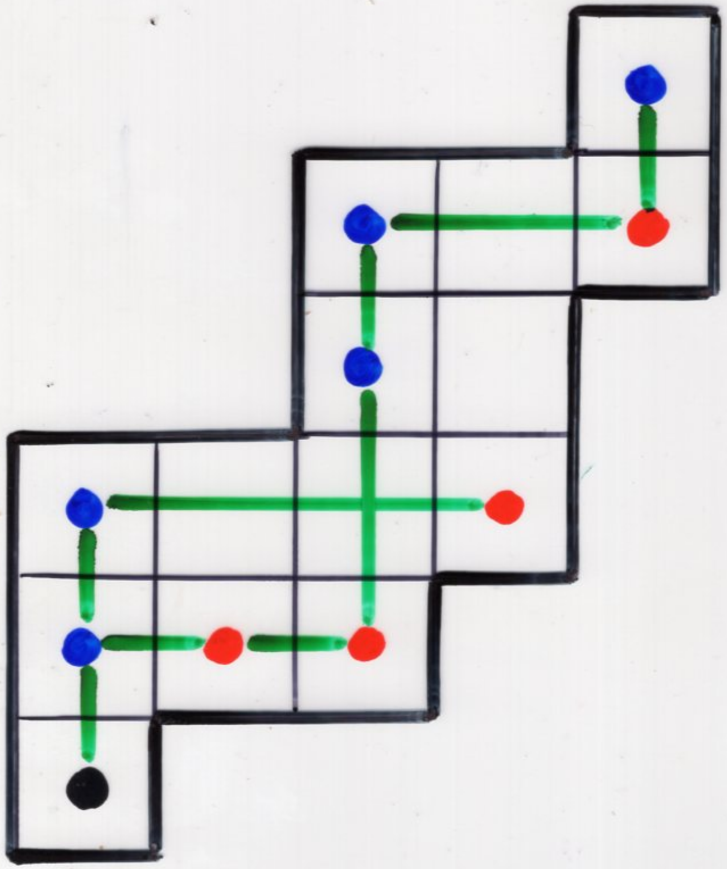
via non-ambiguous trees

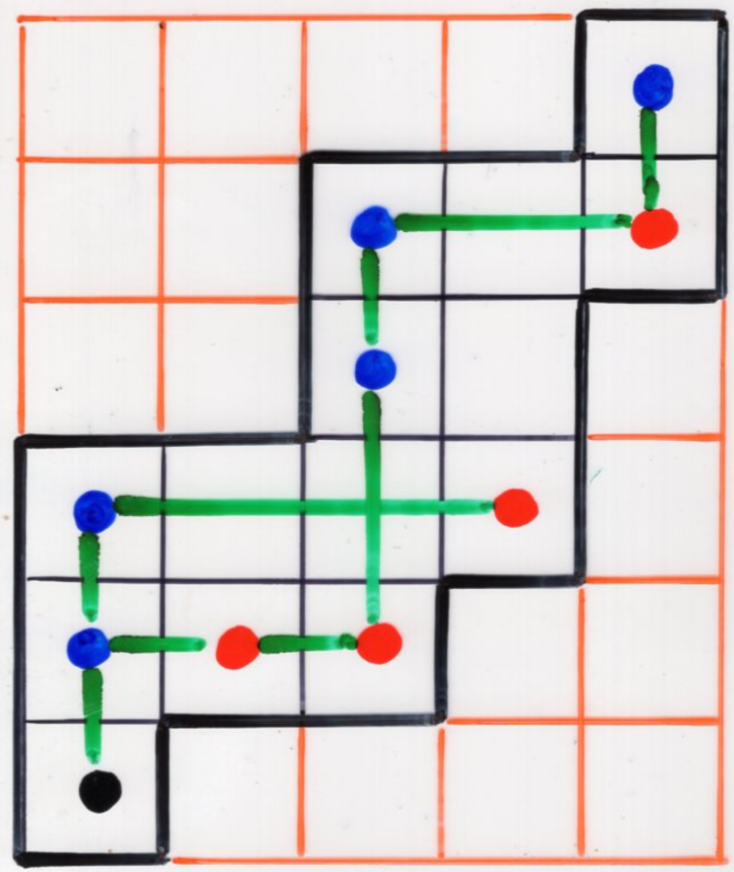
parallelogram
polyominoes

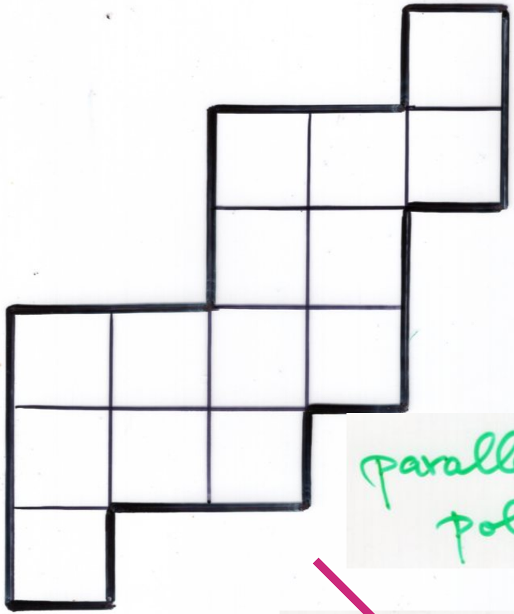
= staircase polygons



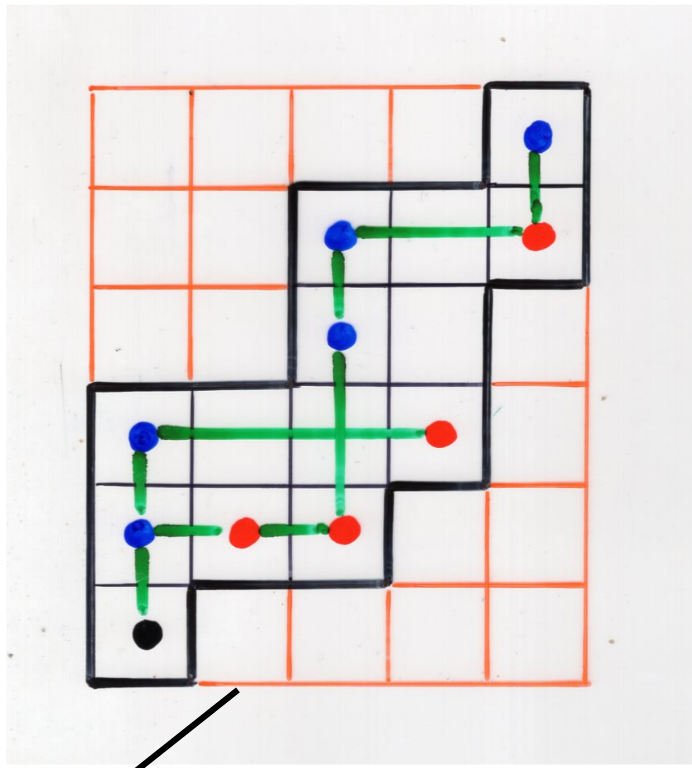






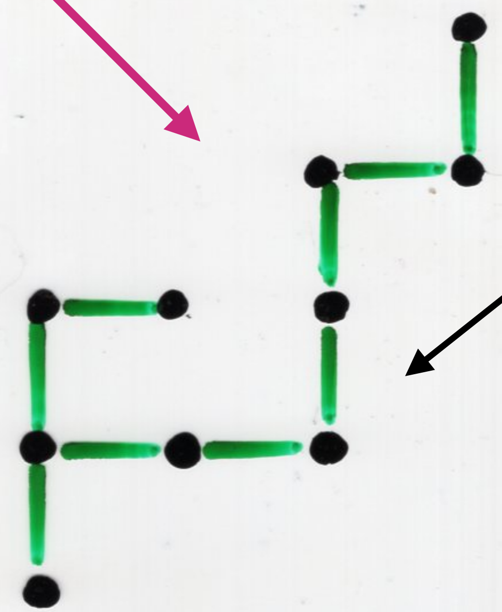


parallelogram polyominoes



bijection

binary trees



Aval, Boussicault, Bouvel, Silimbani
(2013) (2014)

Aval, Boussicault, Delcroix-Oger, Hivert, Laborde-Zubieta (2015)

Reverse Q-tableaux

example with the PASEP algebra:

tree-like tableaux

PASEP algebra

$$Q \begin{cases} DE = q\,ED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

$$Q \begin{cases} DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\ XE = \square\, EX \\ DY = \square\, YD \\ XY = \square\, YX \end{cases}$$

alternative tableaux

PASEP algebra

$$
Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}
$$

PASEP algebra

$$
Q\left\{
\begin{aligned}
DE &= \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE &= \square\, EX \\
DY &= \square\, YD \\
XY &= \square\, YX
\end{aligned}
\right.
$$

$$
Q \left\{ \begin{array}{l}
DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{array} \right.
$$

PASEP algebra

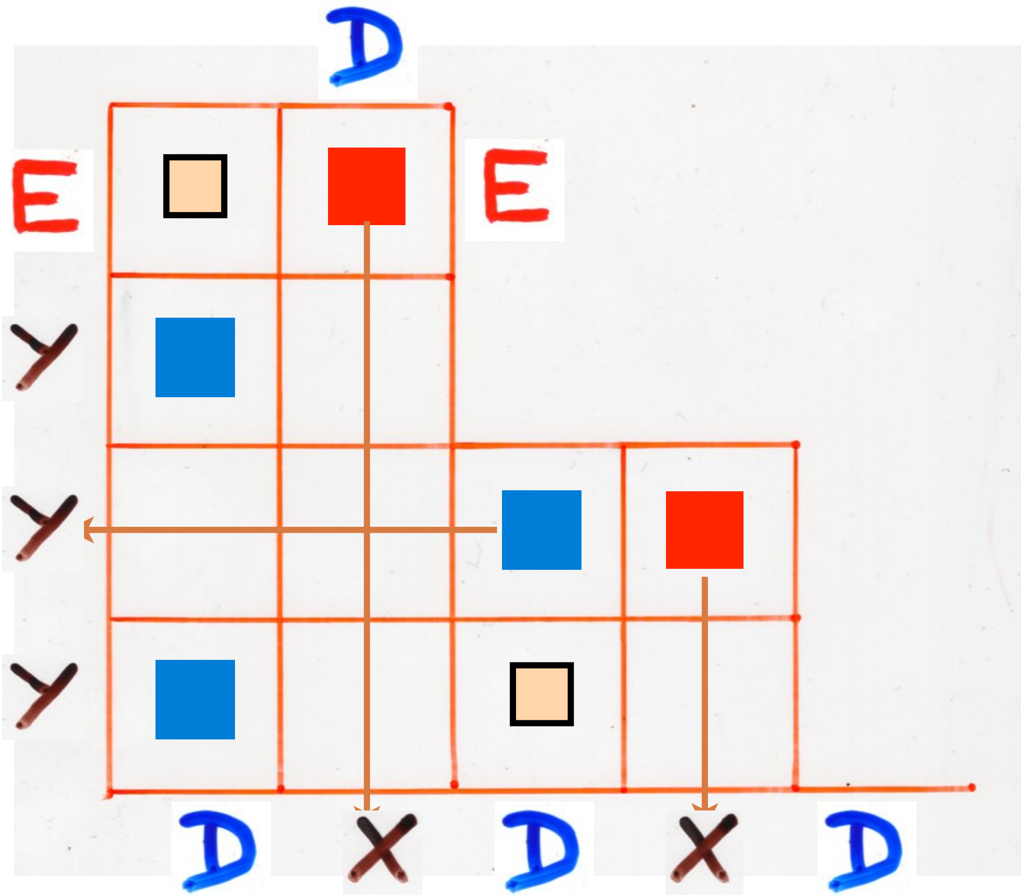
alternative tableaux

reverse PASEP algebra

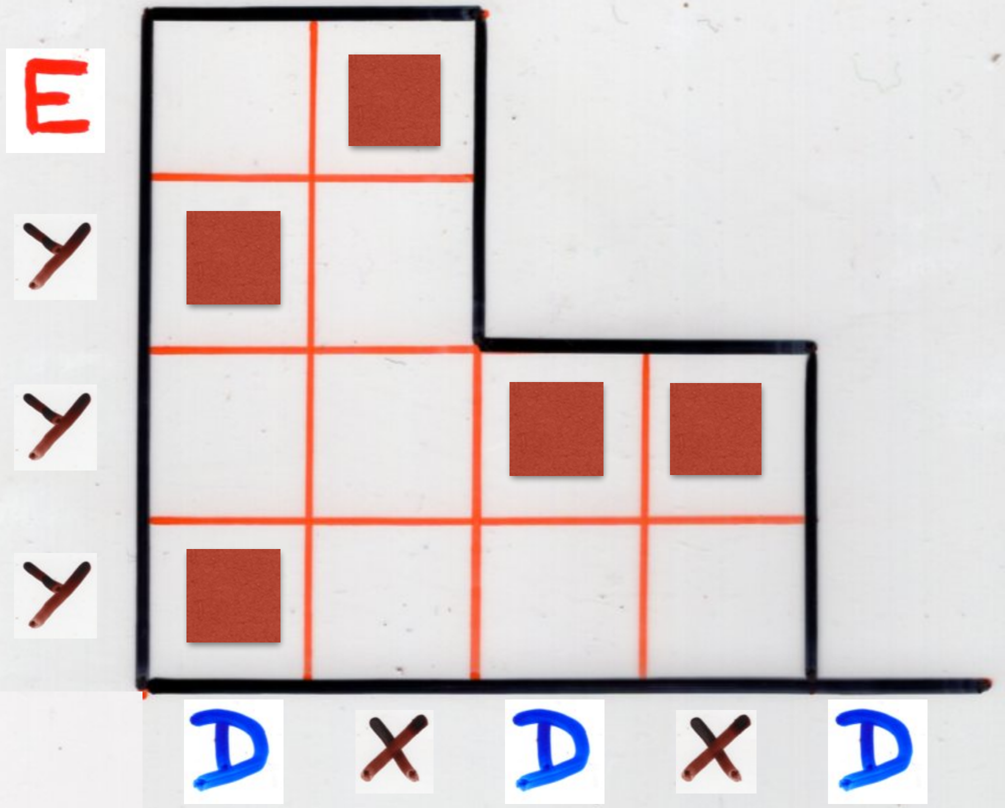
$$
Q^{+} \left\{ \begin{array}{l}
DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{array} \right.
$$

tree-like tableaux

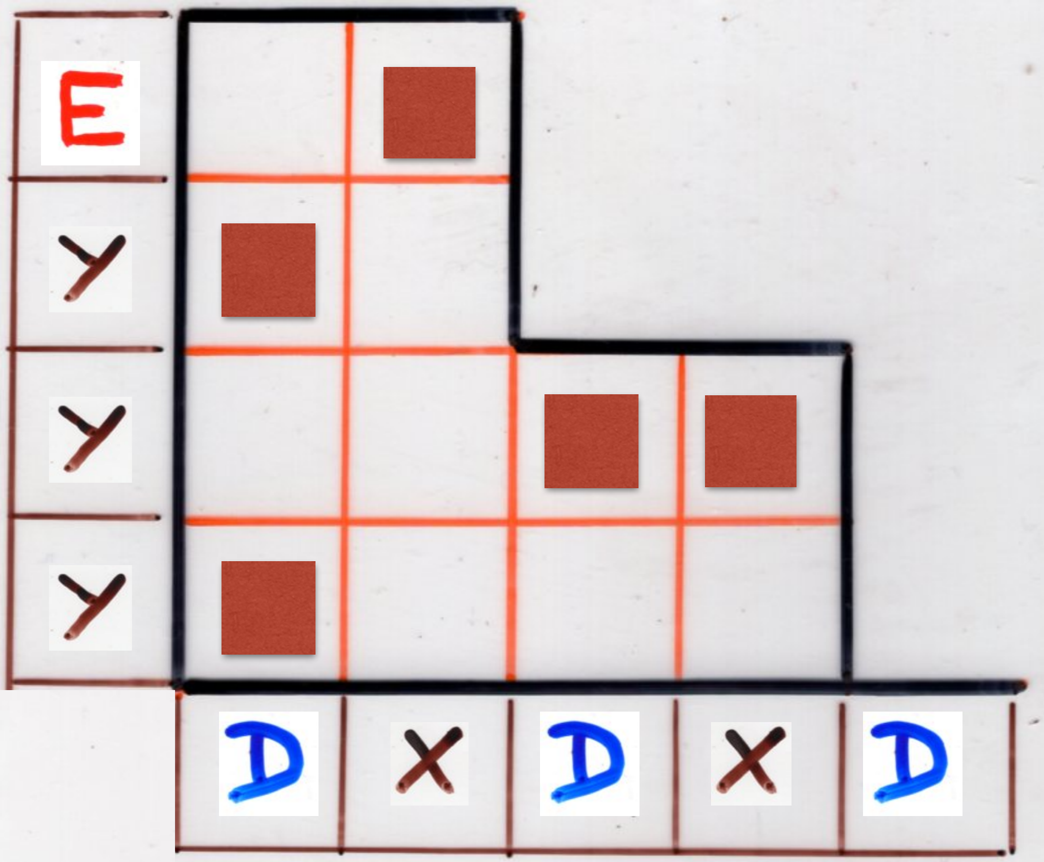
alternative
tableaux



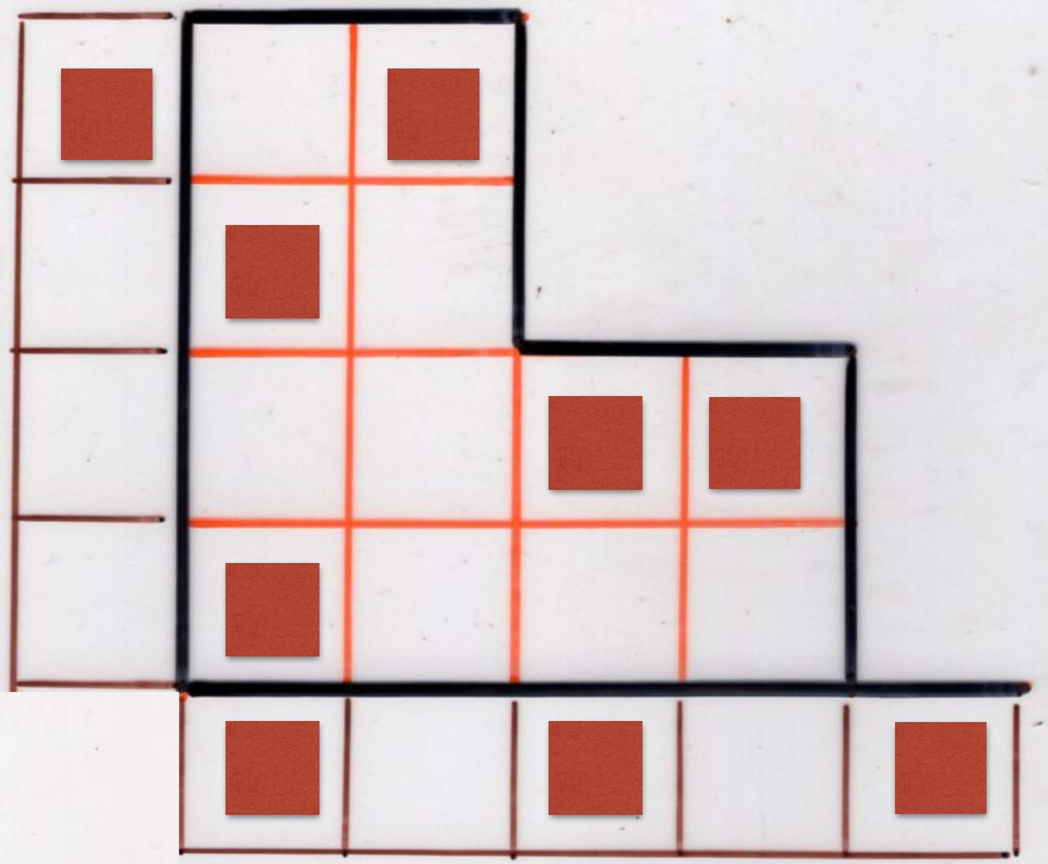
reverse Q-tableaux

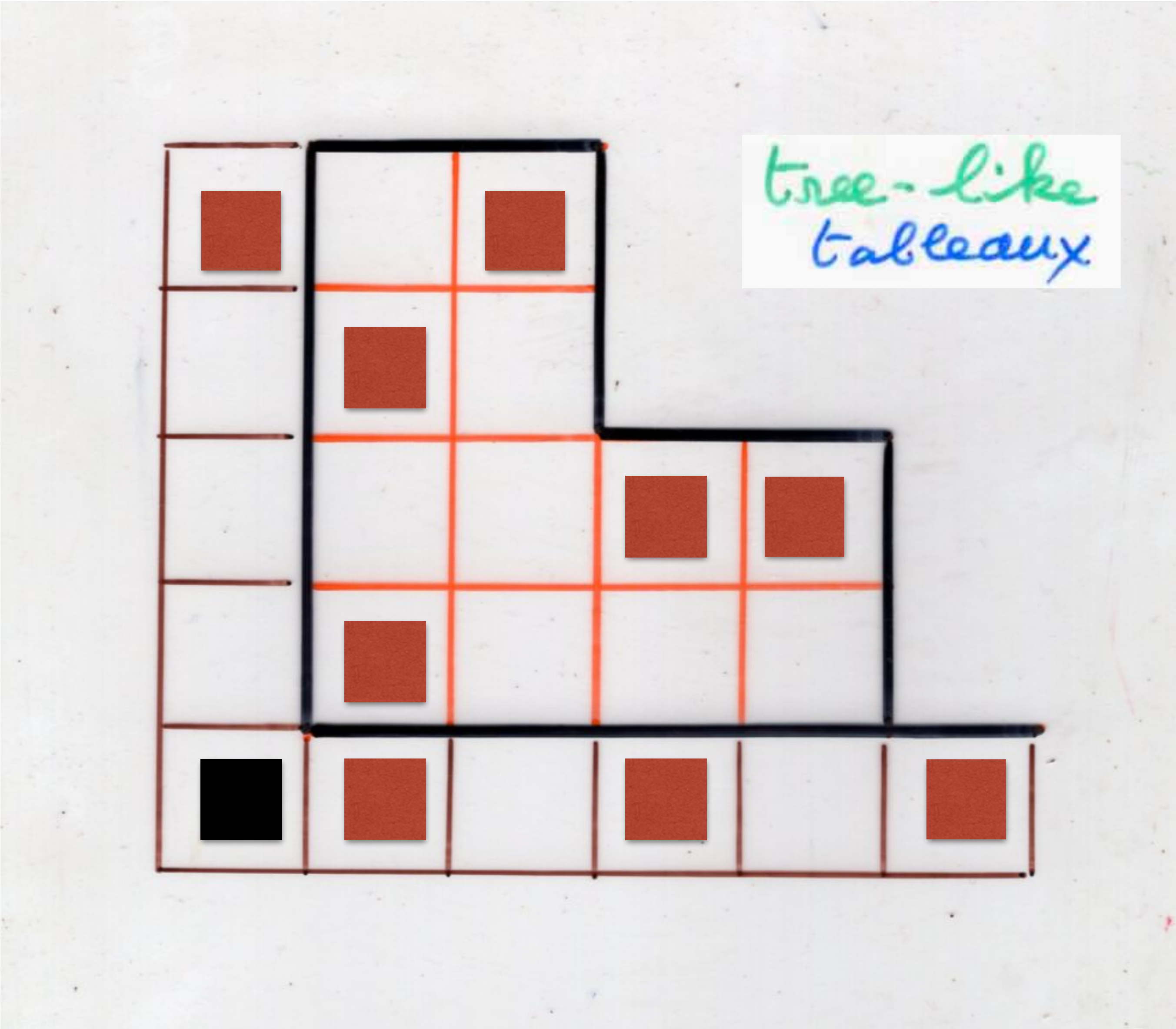


reverse Q-tableaux



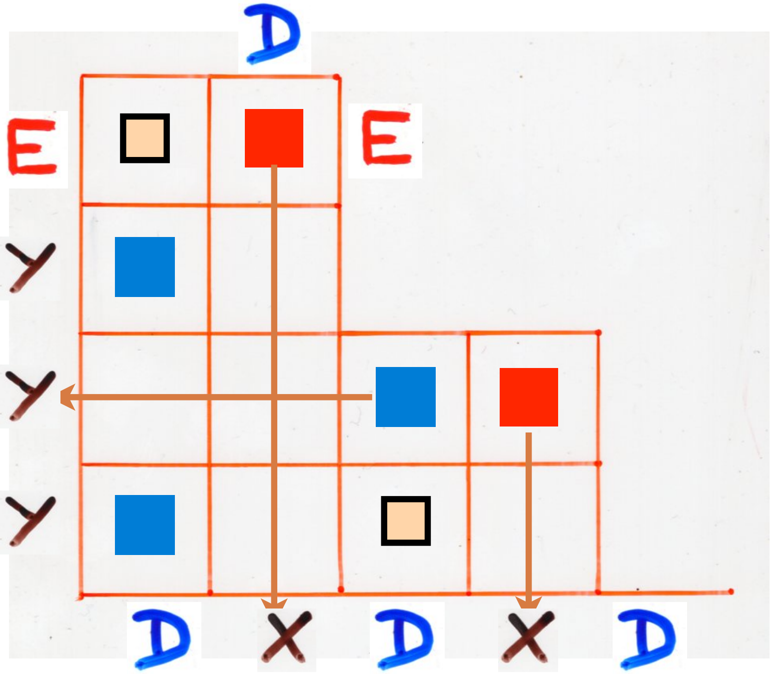
reverse Q- tableaux

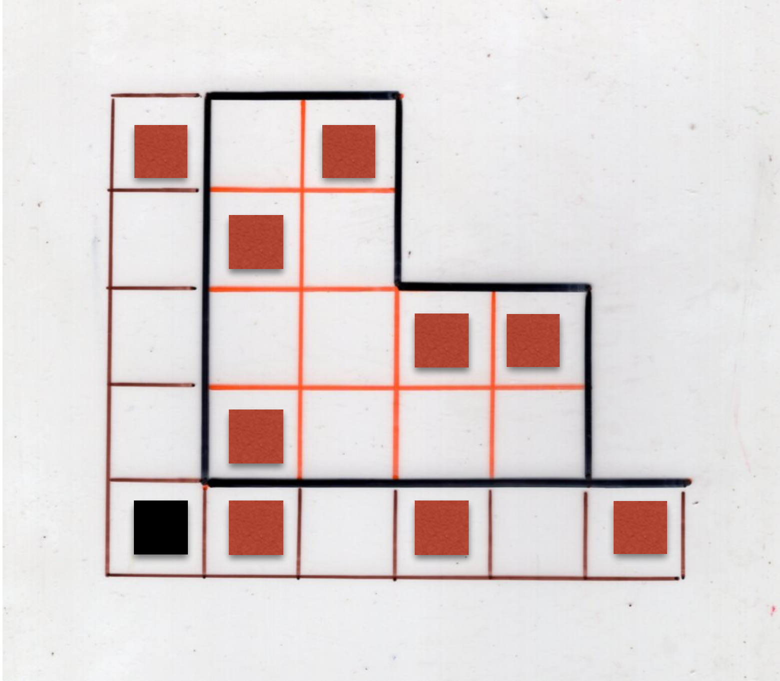



tree-like
tableaux

alternative tableaux

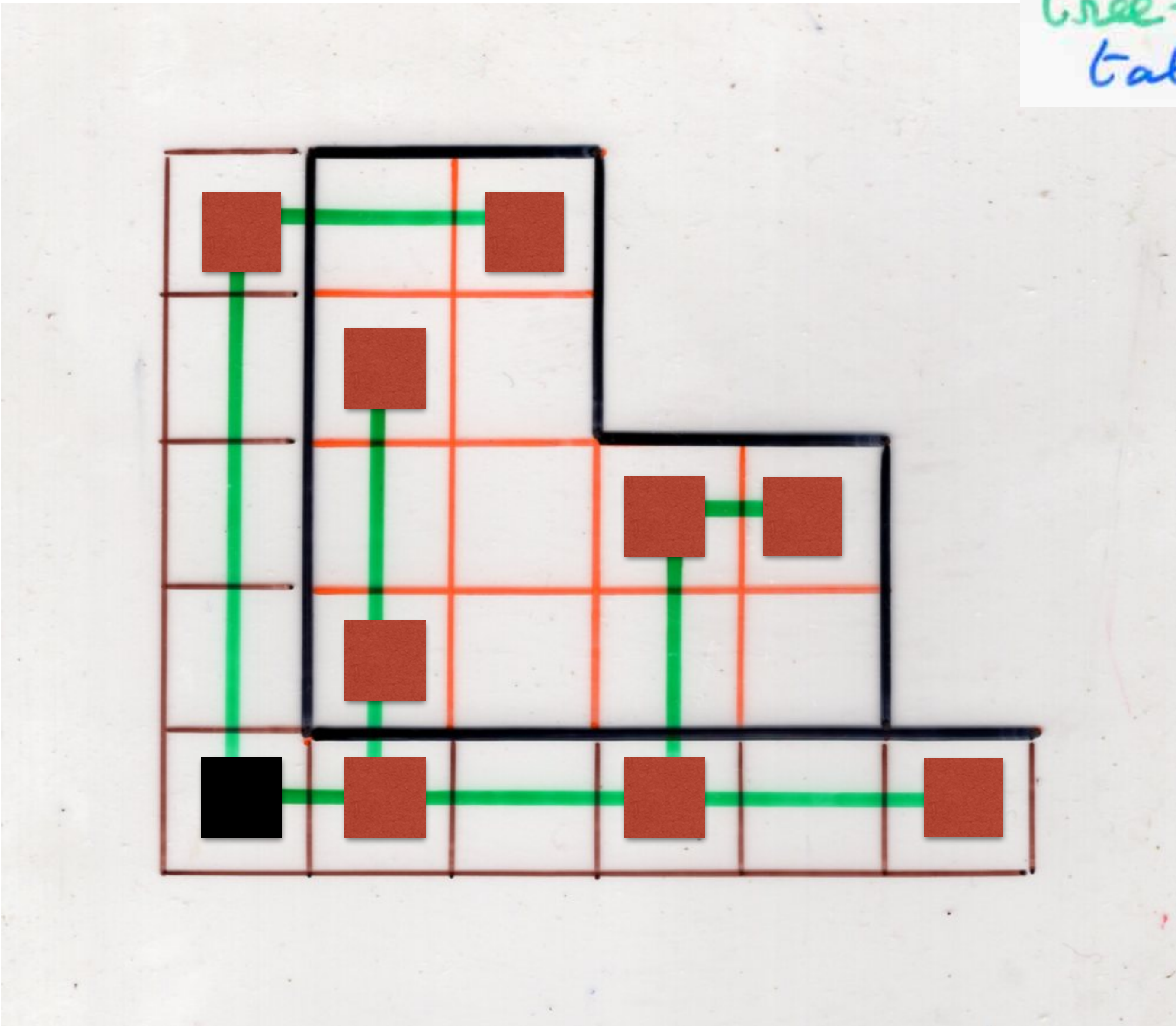
tree-like tableaux



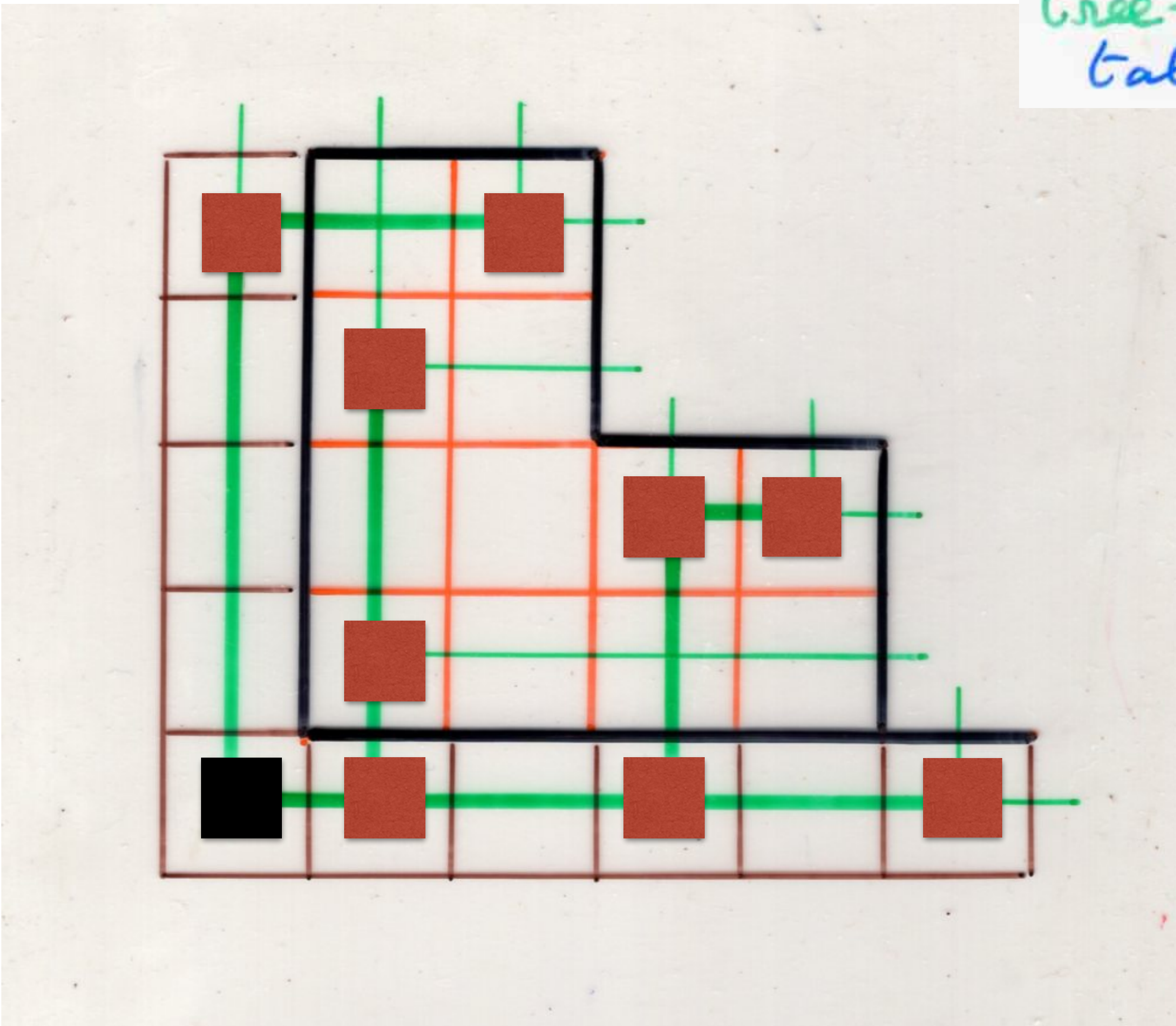


tree-like tableaux
are "the" reverse Q-tableaux
of alternative tableaux

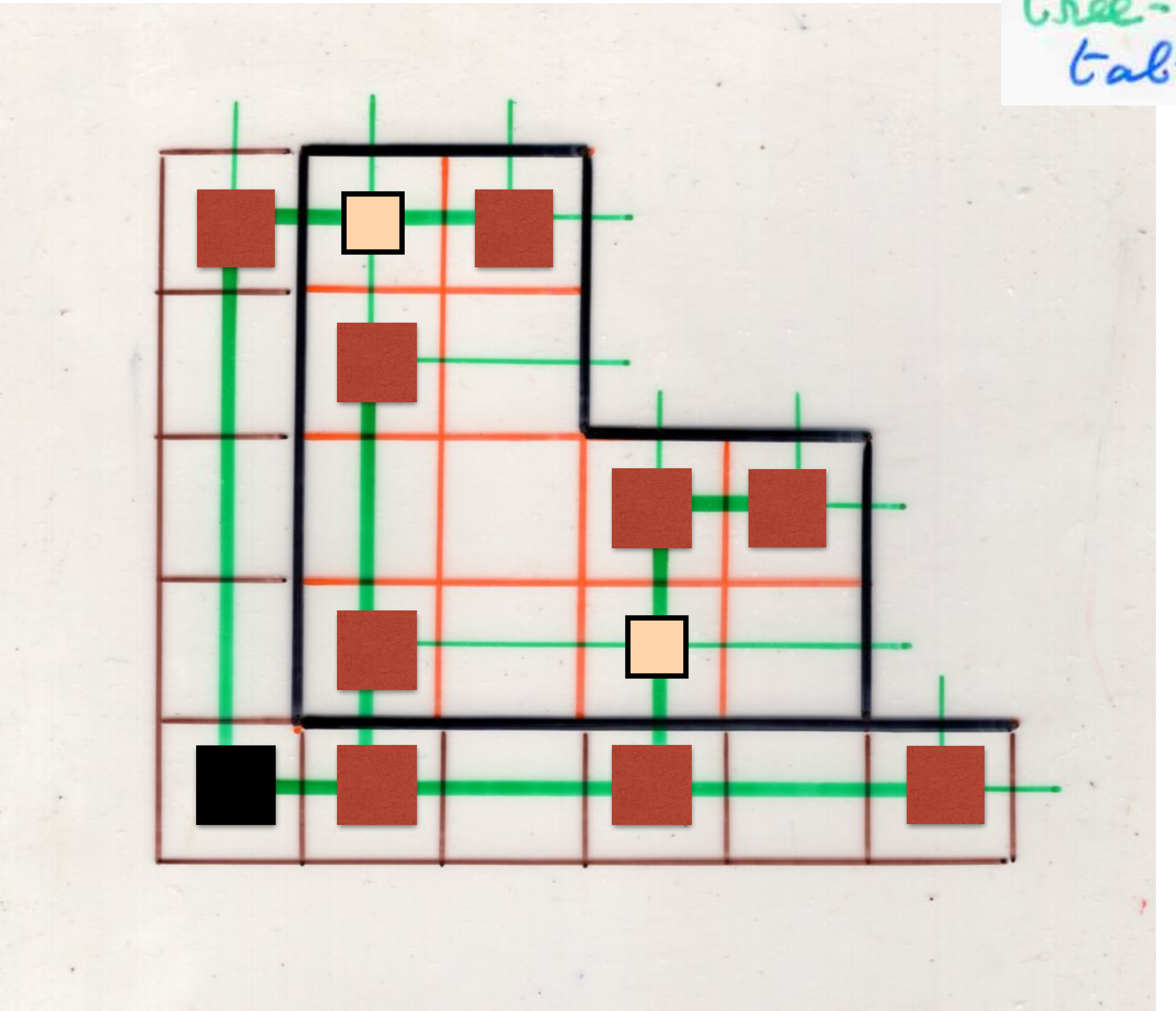
tree-like tableaux



tree-like tableaux



tree-like
tableaux



In conclusion

The philosophy of the cellular ansatz

The philosophy of the cellular ansatz

combinatorial representation
of the quadratic algebra

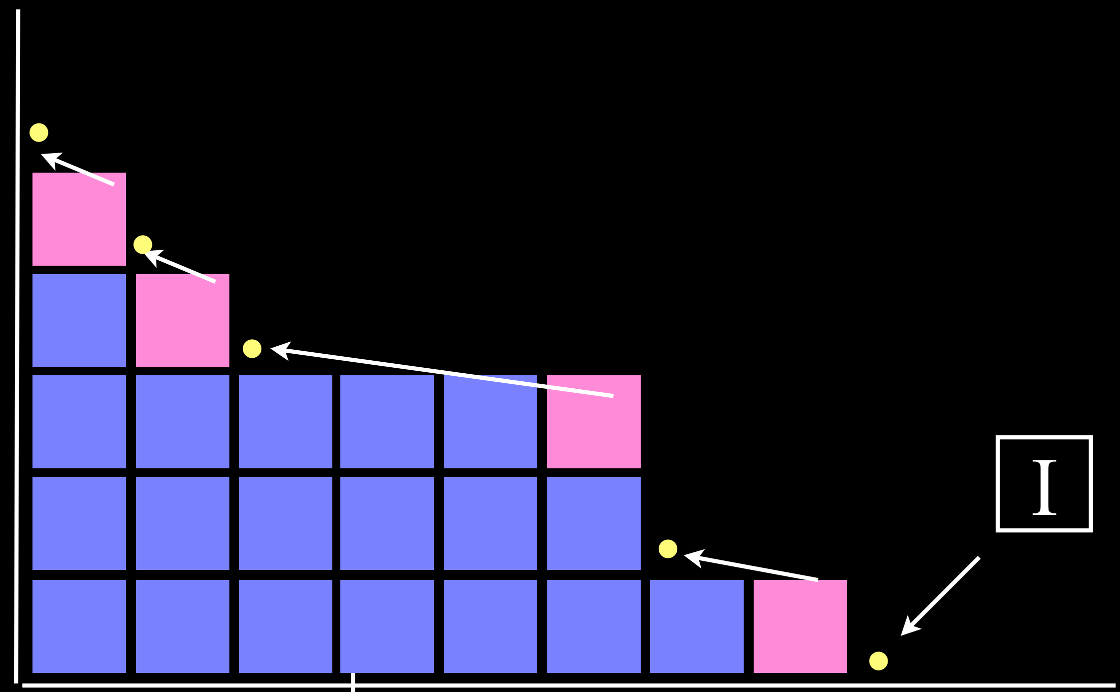
$UD = qDU + Id$ $\longrightarrow$

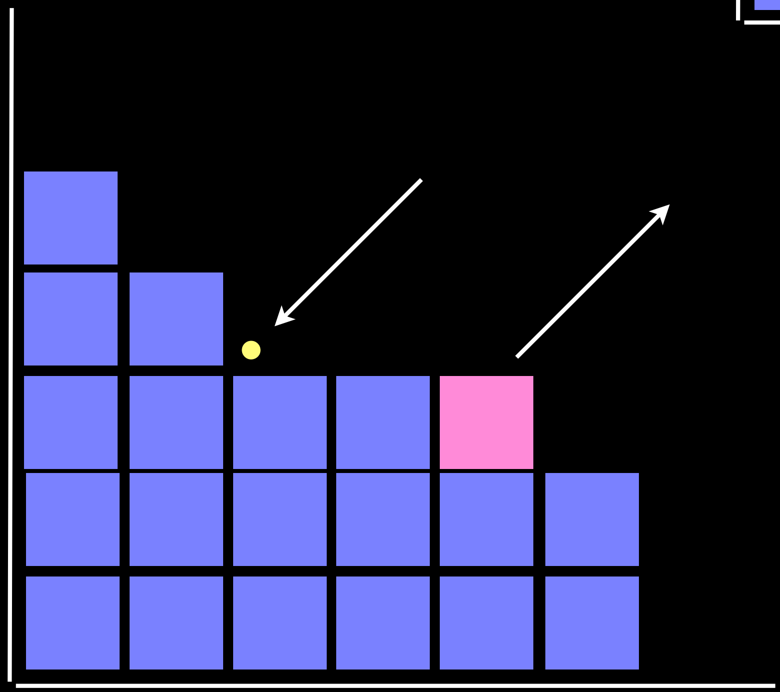
- commutations diagrams
  RSK
  Fomin growth diagrams

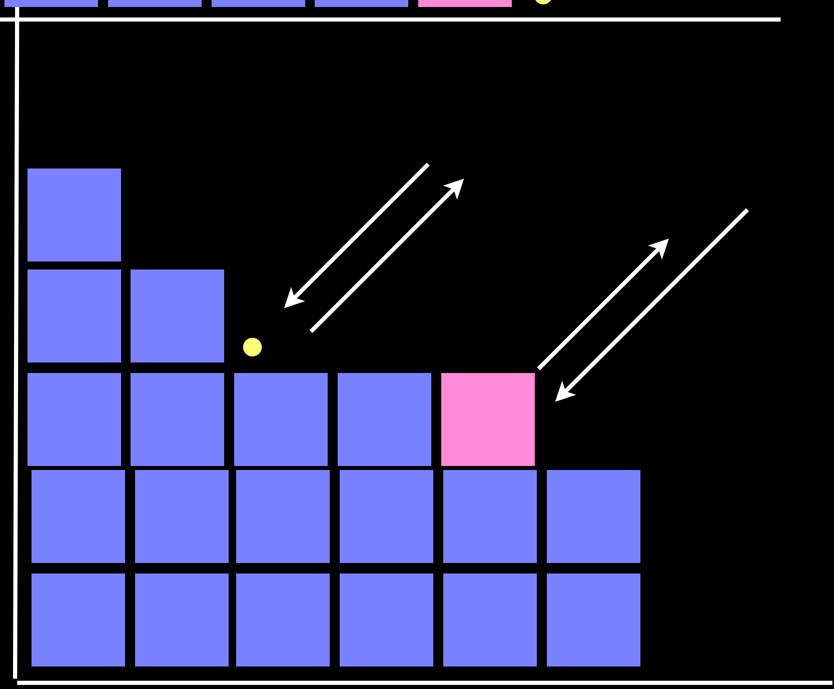
- local rules on edges
  RSK

- duplication of equations
  RSK

$$UD = DU + I_v I_h$$

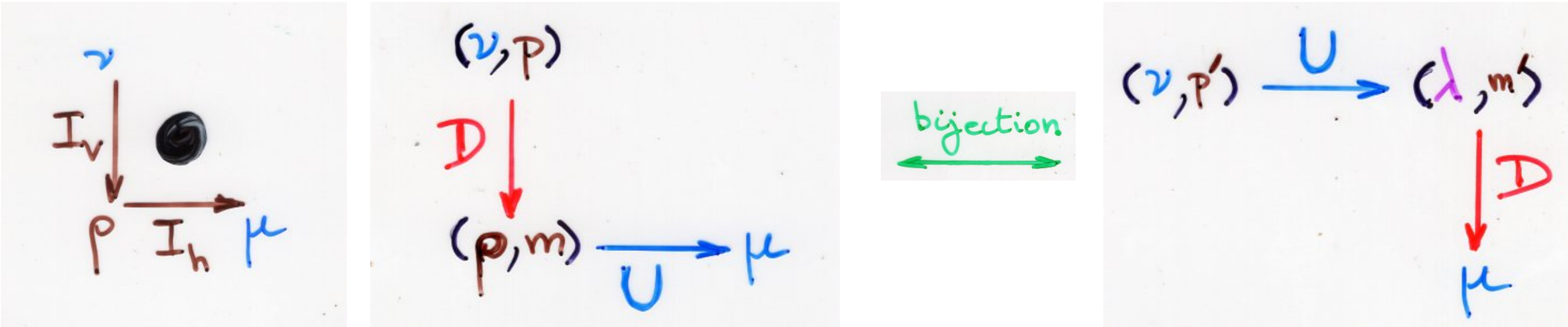


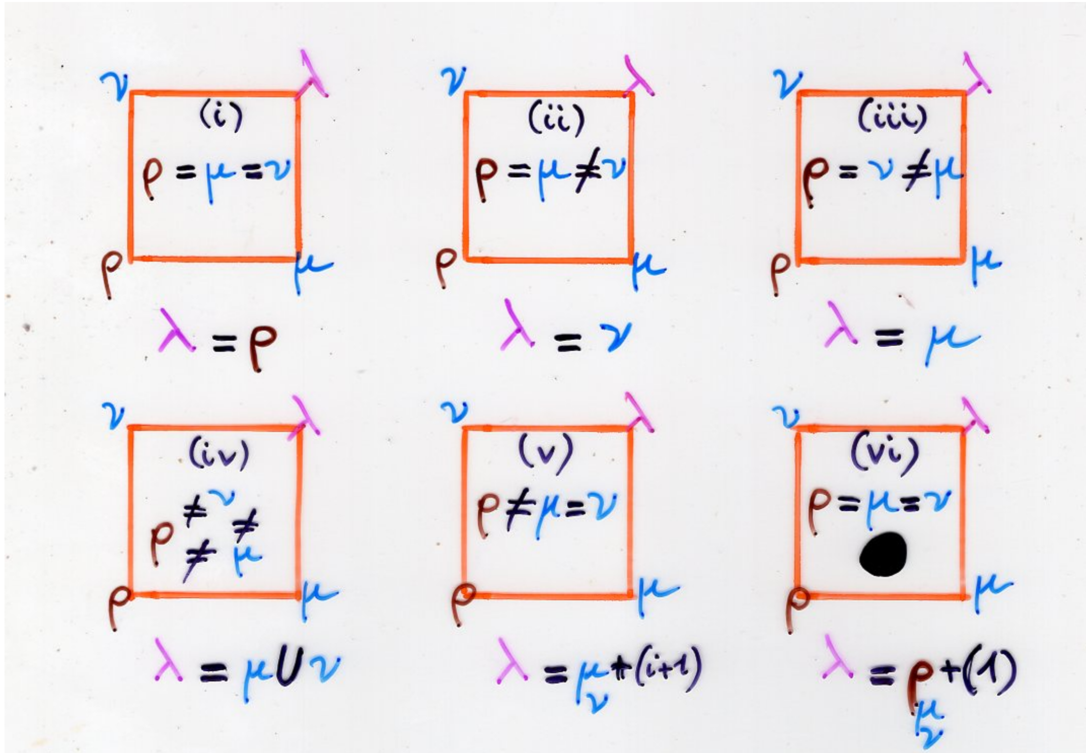




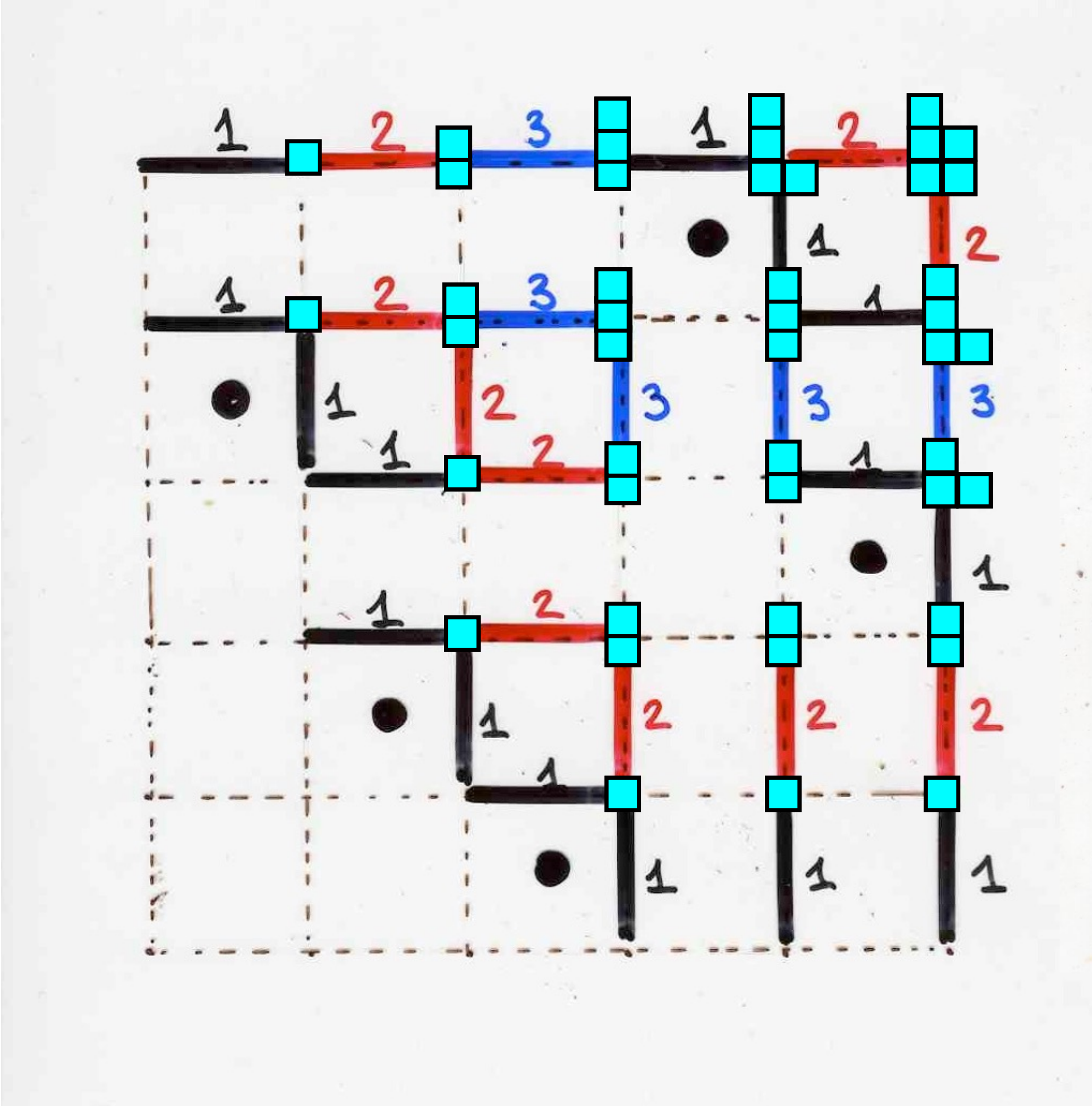
$$UD = DU + I_v I_h$$

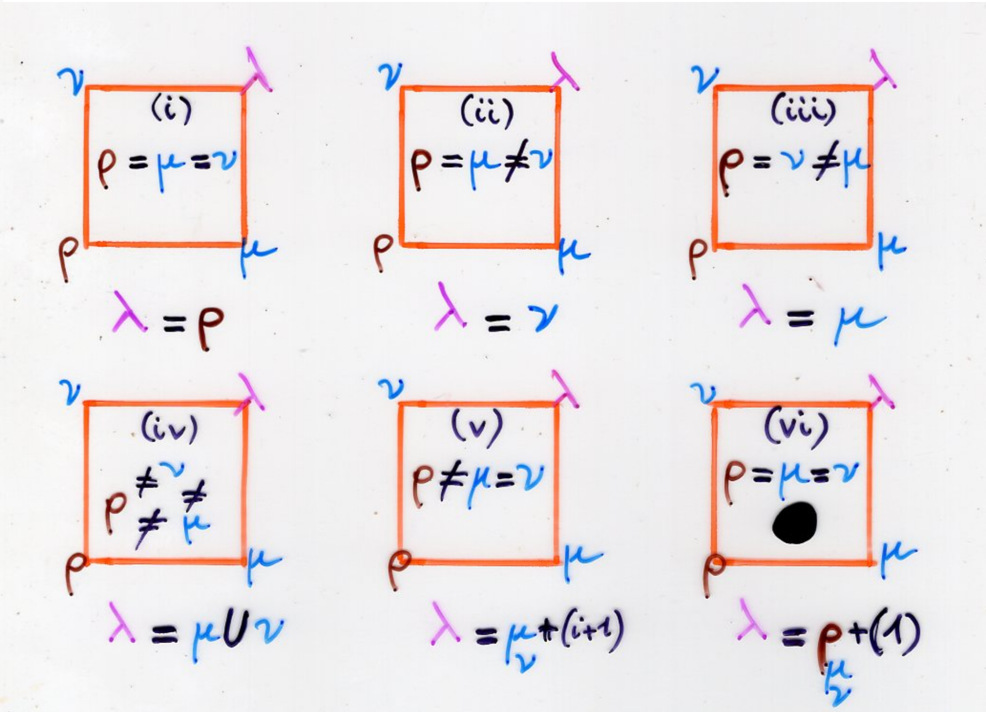
"commutation diagrams"

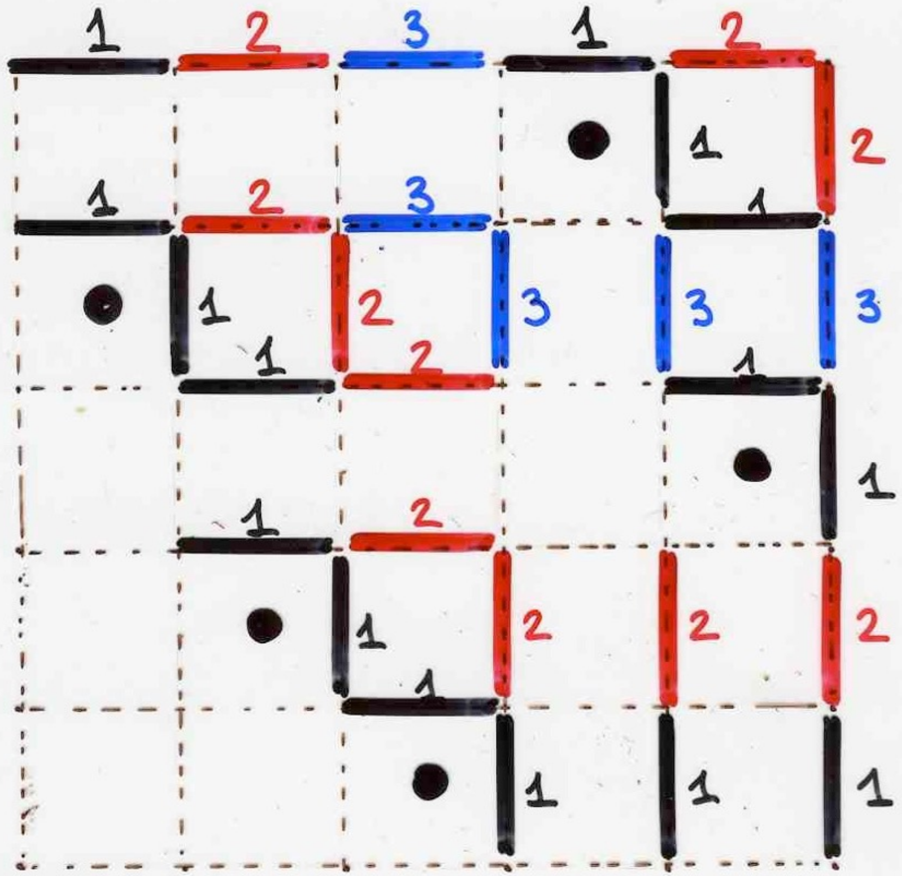




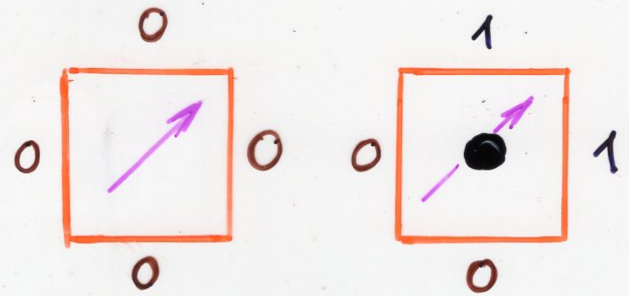
Fomin growth diagrams

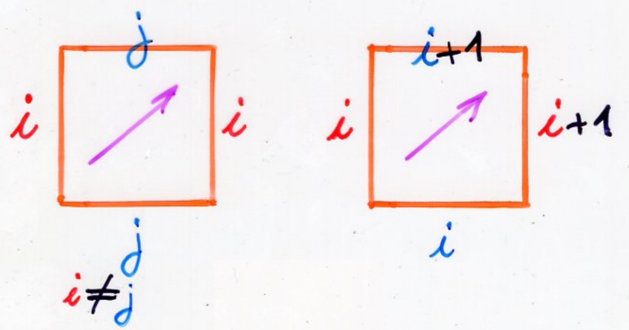






"local rules" on the edges



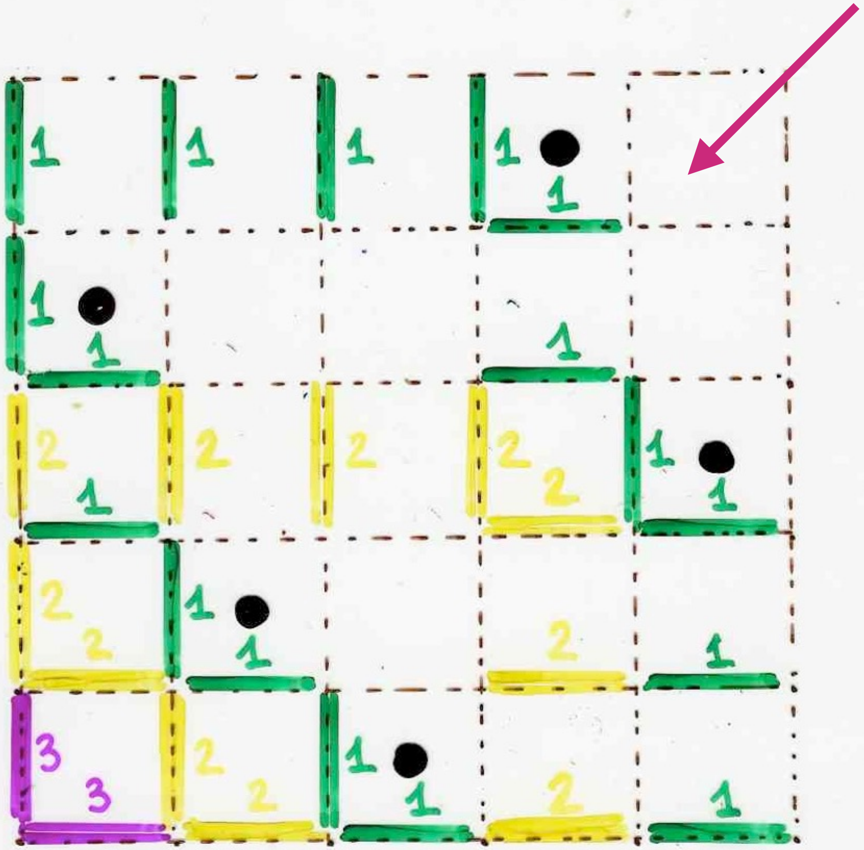


$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

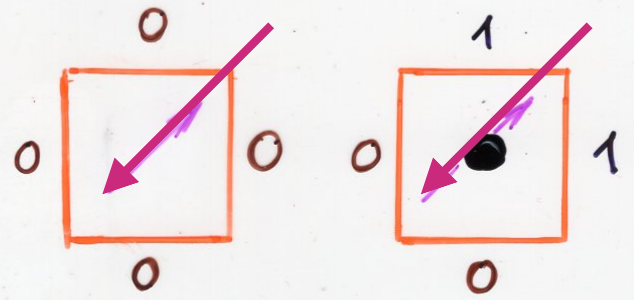
duplication of the commutation relations
defining the quadratic algebra Q

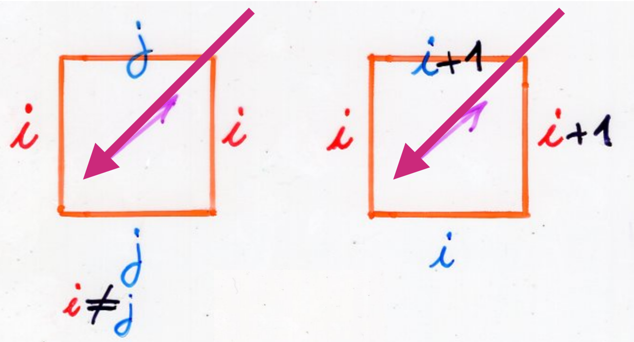
$$UD = DU + Y_1 X_1$$
$$X_1 Y_1 = Y_2 X_2$$
$$X_2 Y_2 = Y_3 X_3$$
$$\cdots\cdots\cdots$$
$$X_i Y_i = Y_{i+1} X_{i+1}$$
$$\cdots\cdots\cdots$$

$$U Y_i = Y_i U$$
$$X_j U = U X_j$$
$$\cdots\cdots\cdots$$



"local rules" on the edges





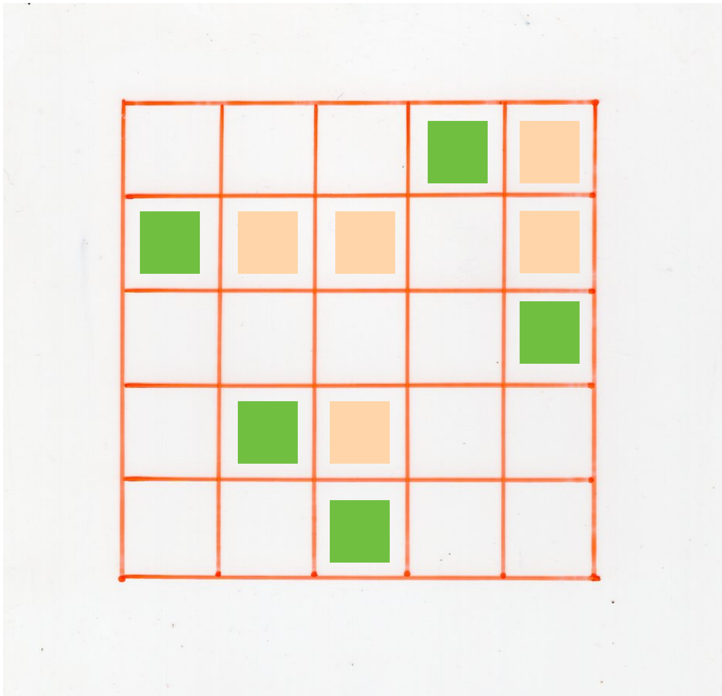
Weil-Heisenberg algebra

$$Q \begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2 \boxed{YX} \end{cases}$$

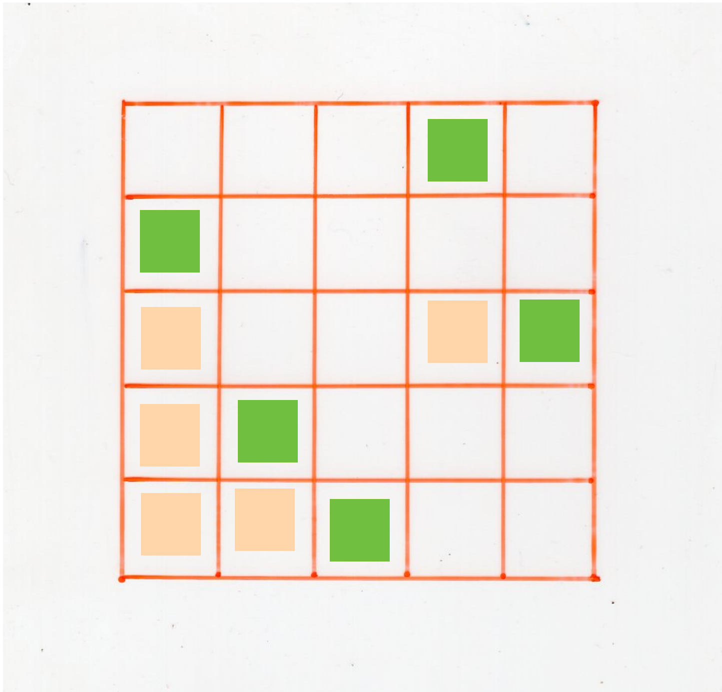
for the Weyl-Heisenberg algebra

$$Q^{+} \simeq Q$$

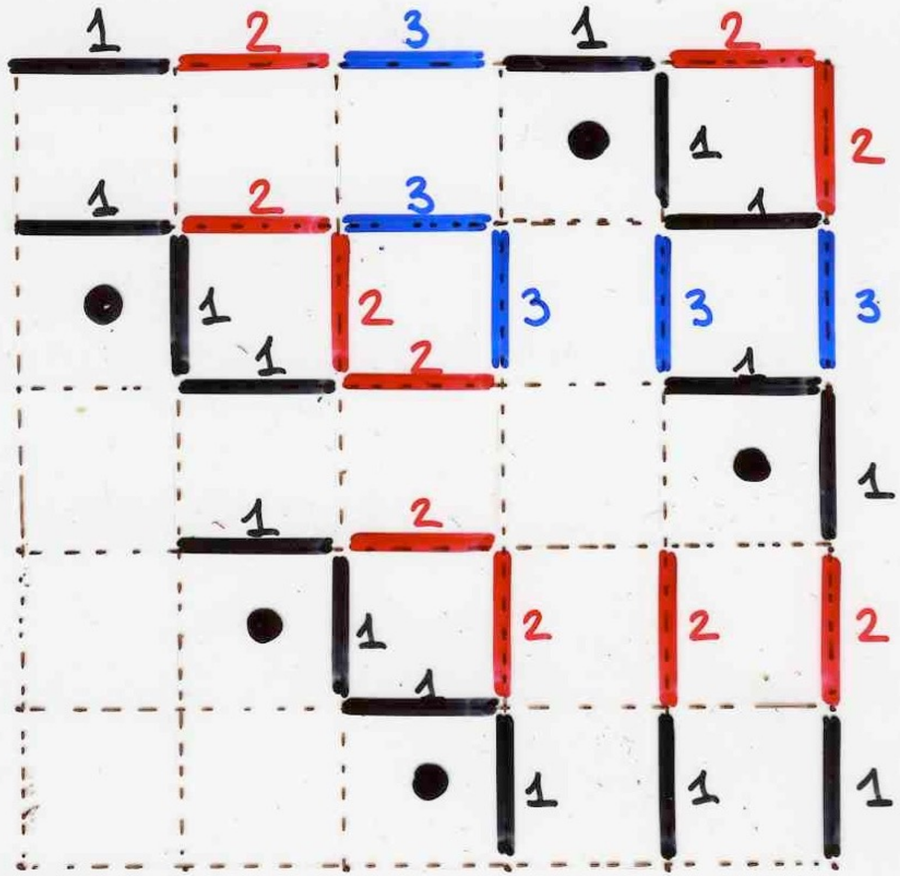
$$Q \begin{cases} UD = q_1 DU + t YX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$



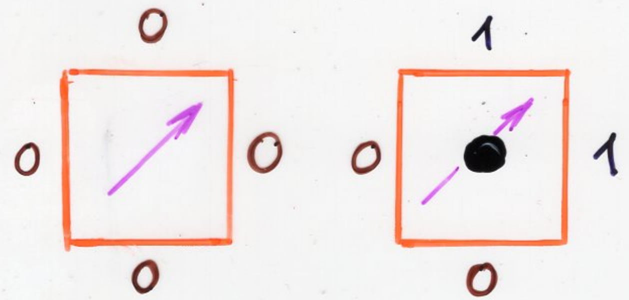
$$Q^{+} \begin{cases} YX = q_2 XY + t XY \\ YU = UY \\ DX = XD \\ DU = q_1 UD \end{cases}$$

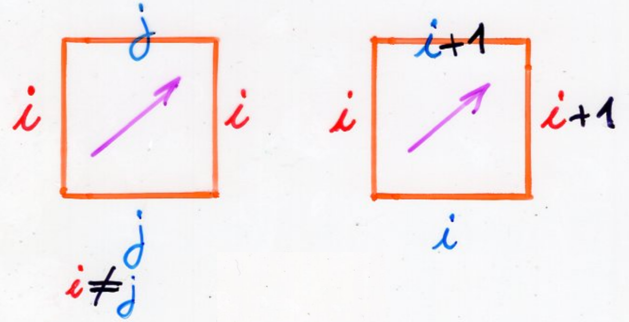


duplication of equations
in the reverse quadratic algebra Q^+



"local rules"
on the edges





The philosophy of the cellular ansatz

combinatorial representation
of the quadratic algebra

$UD = qDU + Id$

- commutations diagrams

RSK

Fomin growth diagrams

- local rules on edges

RSK

- duplication of equations

RSK

The philosophy of the cellular ansatz

combinatorial representation
of the quadratic algebra

$DE = qED + E + D$

alternative
tableaux

→

- commutations diagrams

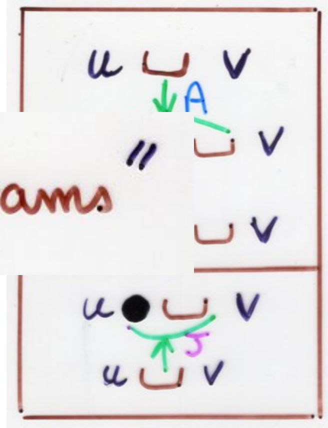
$E X F$

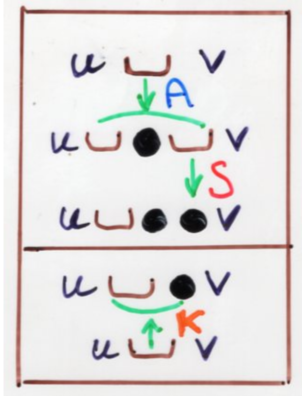
"Laguerre histories"

- duplication of equations

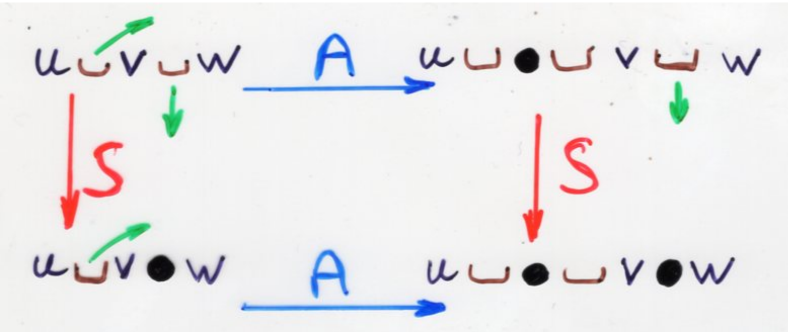
Adela bijection

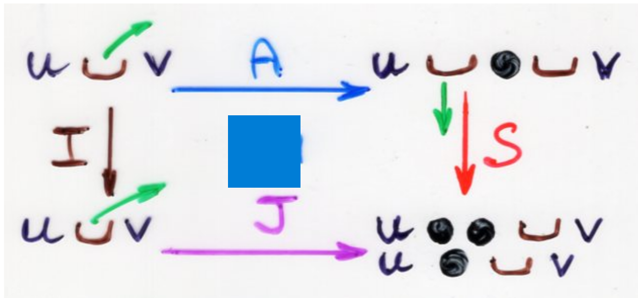
"commutation diagrams"

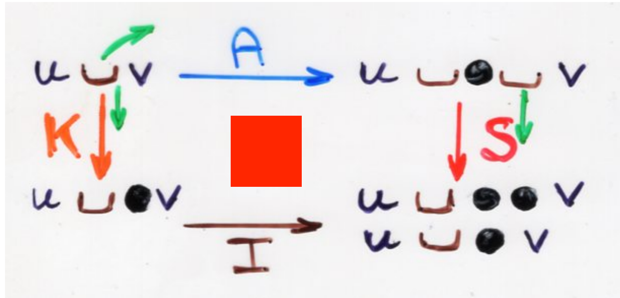




$$AS = SA + J + K$$



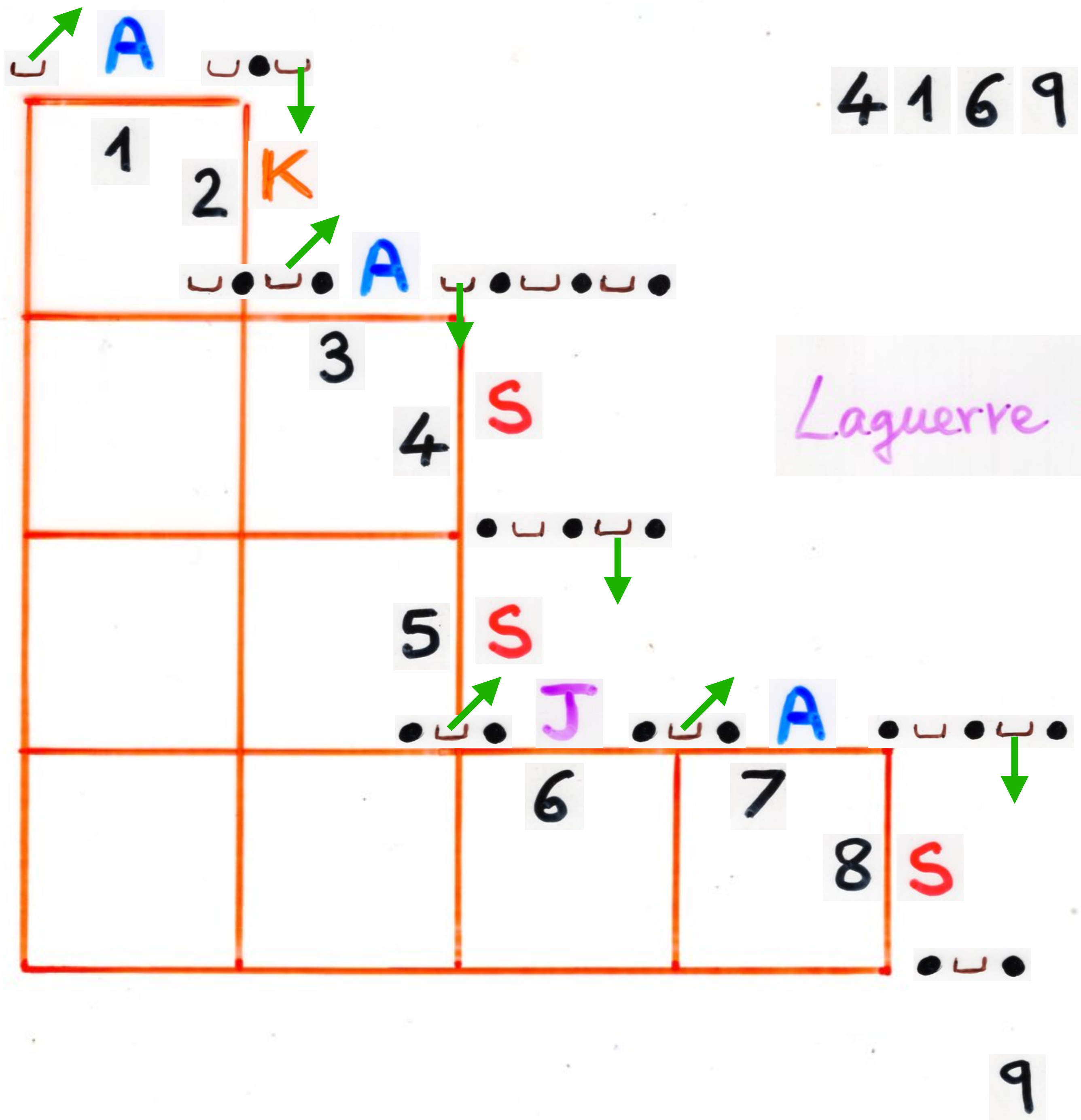


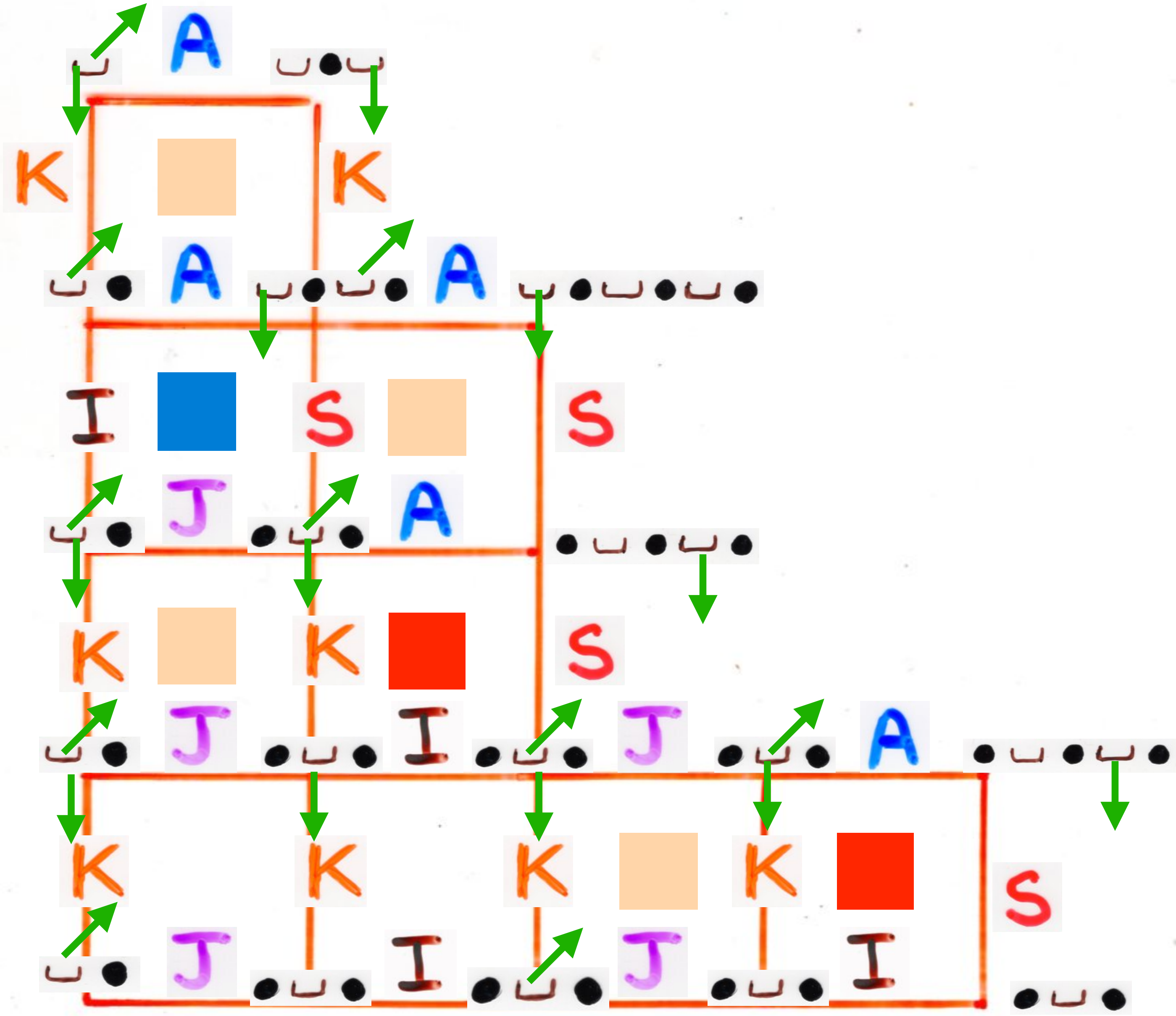


$$AS = SA + I_v J + K I_h$$

Laguerre histories

4 1 6 9 7 8 3 5 2




A
K K
A A
I S S
J A
K K S
J I J A
K K K K S
J I J I

Prop.

T alternative tableau

$\rightarrow \sigma$ "exchange-fusion" inverse algorithm

$\rightarrow \tau$ "local" algorithm from $DE = ED + E + D$

$$\sigma = \tau^{-1}$$

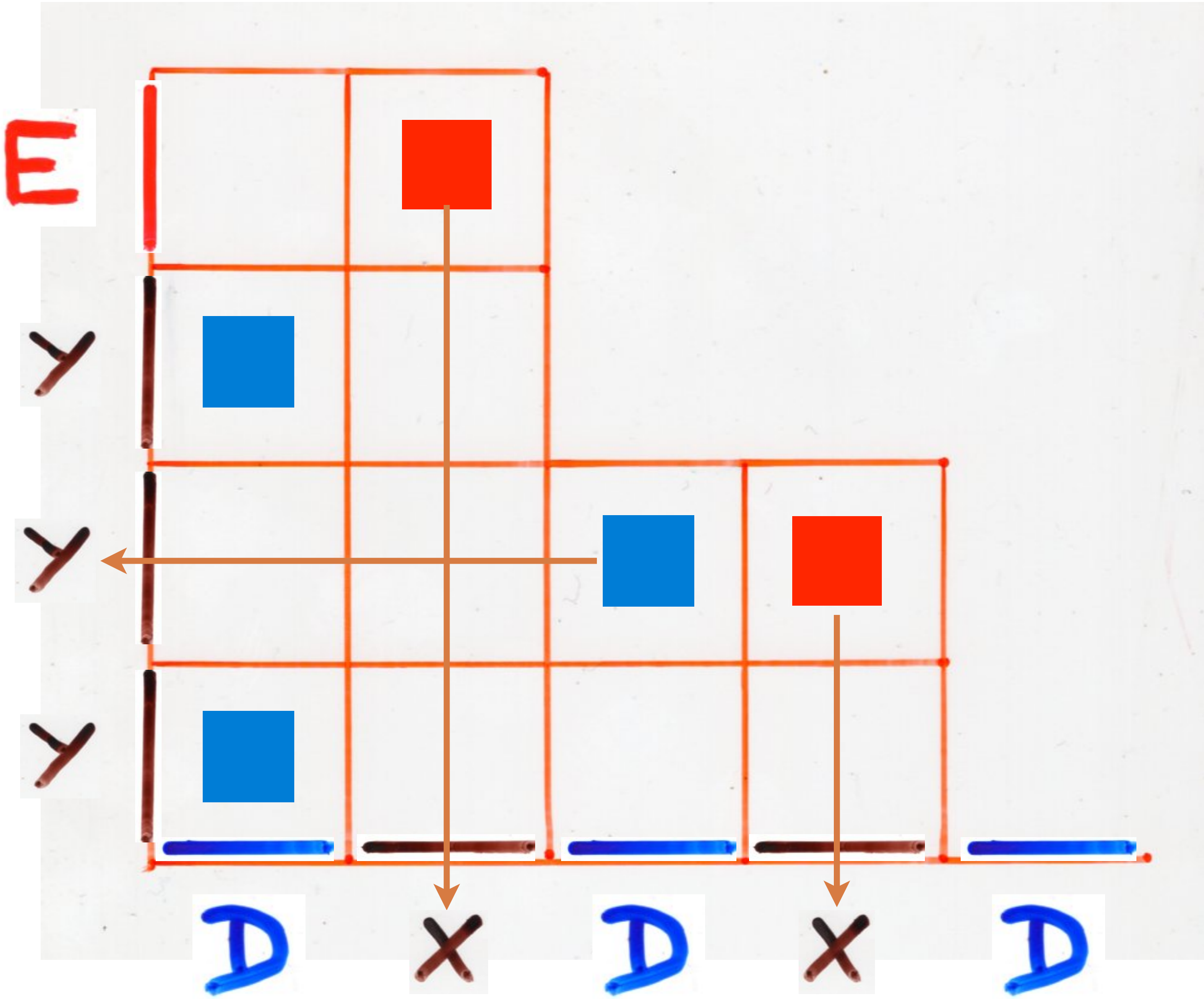
PASEP algebra

$$Q\begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

$$DE = ED + EX_1 + Y_1 D$$

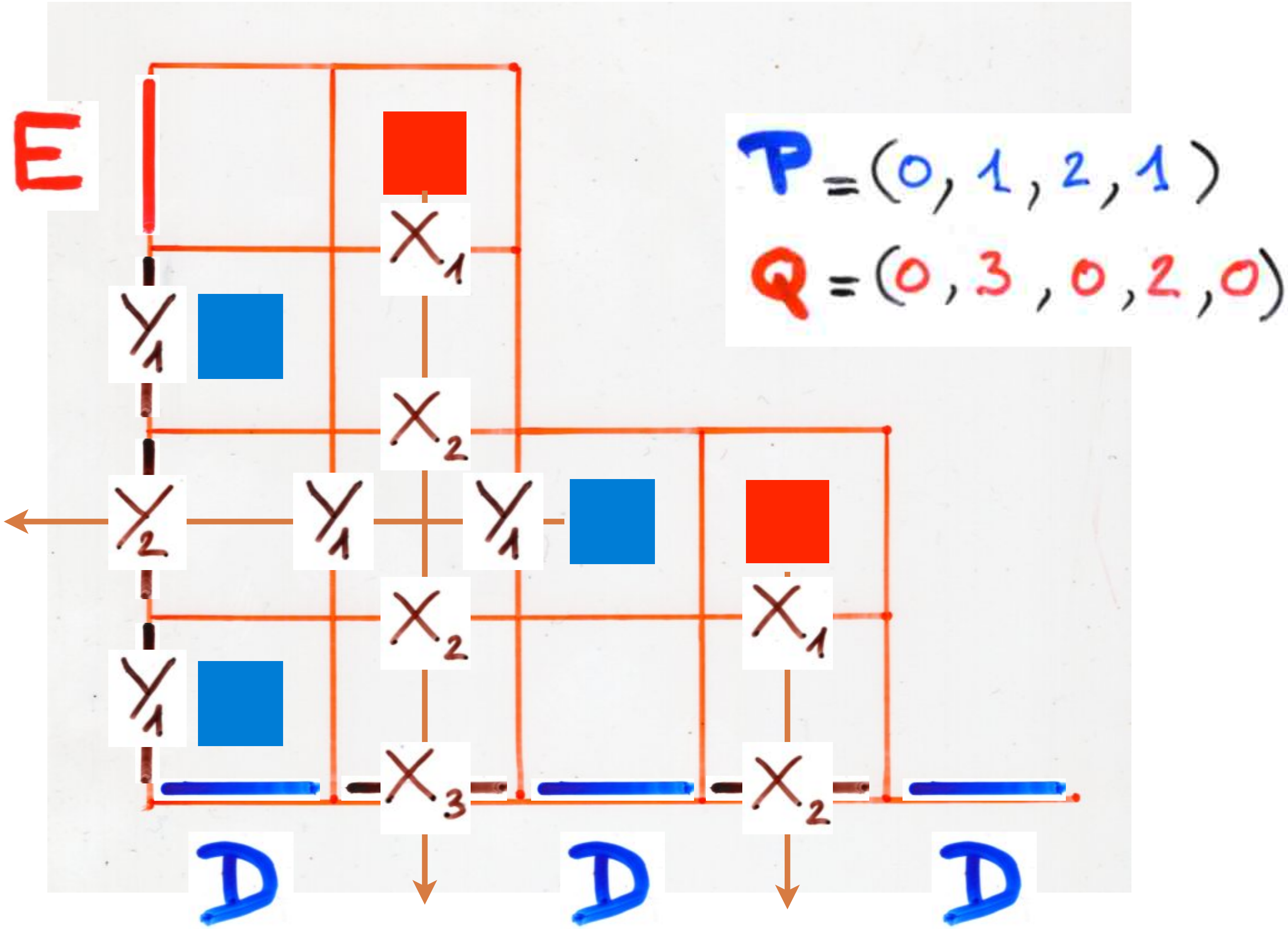
$$\begin{cases} X_1 E = E X_2 \\ \text{-----------} \\ X_i E = E X_{i+1} \\ \text{---} \end{cases} \qquad \begin{cases} D Y_1 = Y_2 D \\ \text{-----------} \\ D Y_i = Y_{i+1} D \\ \text{---} \end{cases}$$

$$X_i Y_j = Y_j X_i$$


E
Y
Y
Y
D
X
D
X
D

Adela bijection

$\text{Adela}(T) = (P, Q)$



The philosophy of the cellular ansatz

combinatorial representation
of the quadratic algebra

$DE = qED + E + D$

alternative tableaux

$\longrightarrow$

- commutations diagrams

EXF

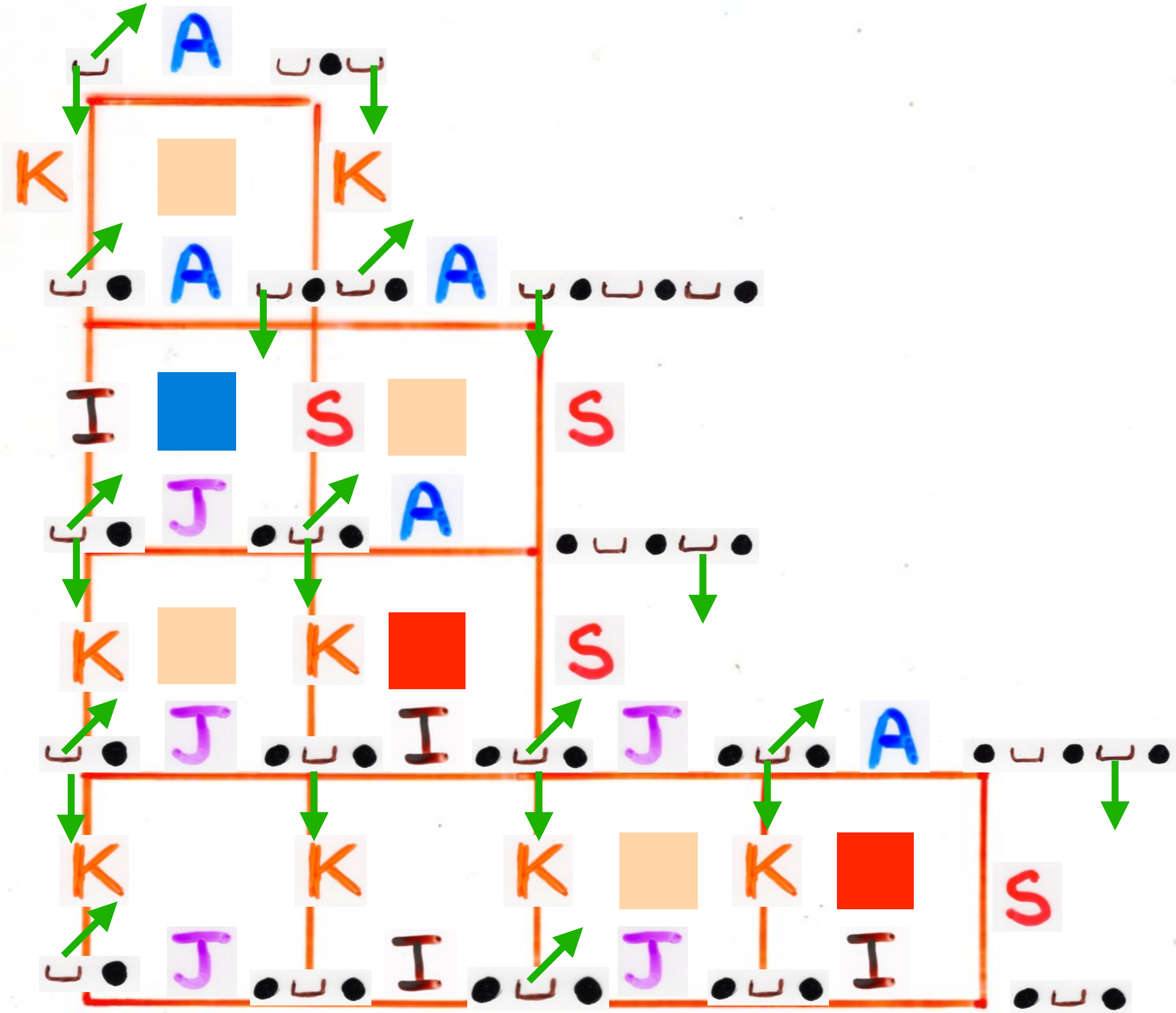
"Laguerre histories"

- duplication of equations

Adela bijection

- local rules on edges ?

with duplication of equations
in the reverse quadratic algebra



A
K
K
A
A
I
S
S
J
A
K
K
S
J
I
J
A
K
K
K
K
S
J
I
J
I

PASEP algebra

$$
Q \begin{cases}
DE = \square\, ED + \square\, EX + \square\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{cases}
$$

reverse PASEP algebra

$$
Q^{\dagger} \begin{cases}
ED = \square\, DE \\
EX = \square\, XE + \square\, DE \\
YD = \square\, DY + \square\, DE \\
YX = \square\, XY
\end{cases}
$$

$$AS = SA + I_v J + K I_h$$
$$AK = KA + I_v A$$
$$JS = SJ + S I_h$$
$$JK = KJ$$

$$A I_v = I_v A$$
$$J I_v = I_v J$$
$$I_h S = S I_h$$
$$I_h K = K I_h$$

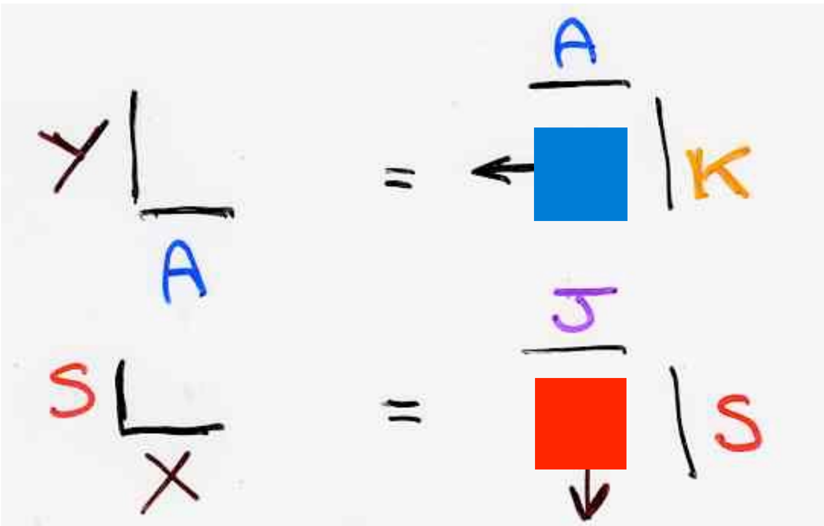
$$D = A + J$$
$$E = S + K$$

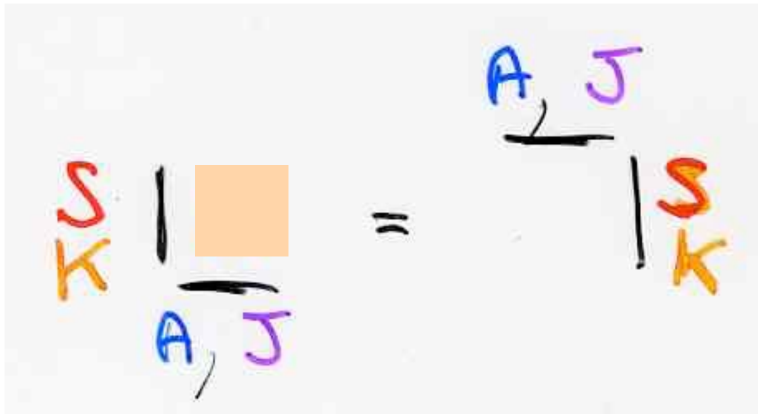
$$| \; I_v = Y$$

$$I_h = X$$

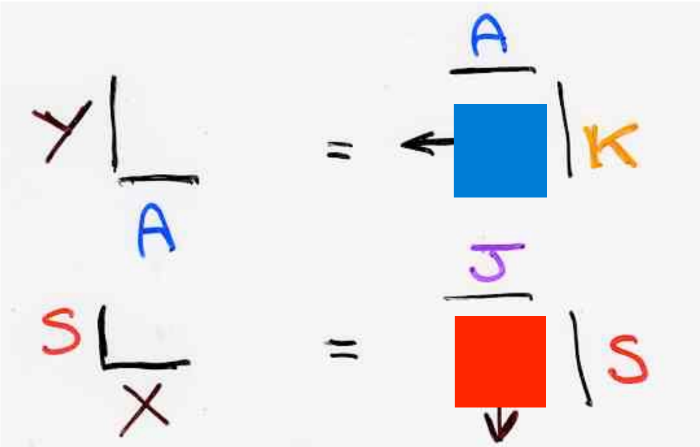
duplication of equations in the reverse quadratic algebra Q^+

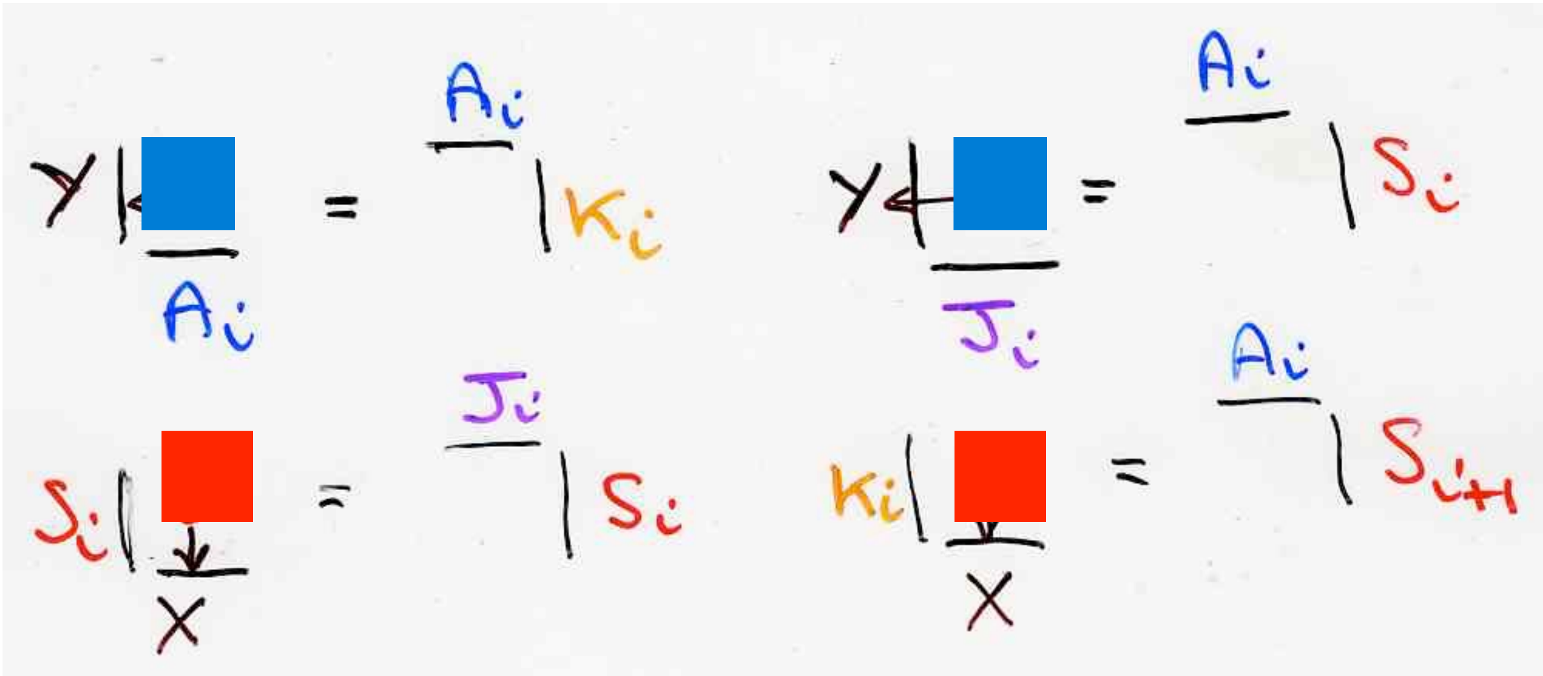
$$D \to \begin{cases} A \\ J \end{cases} \qquad\qquad E \to \begin{cases} S \\ K \end{cases}$$

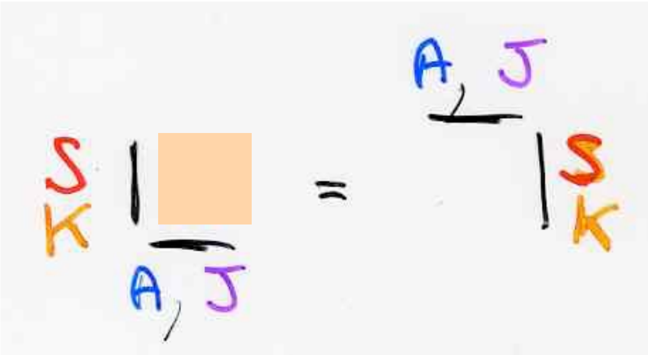


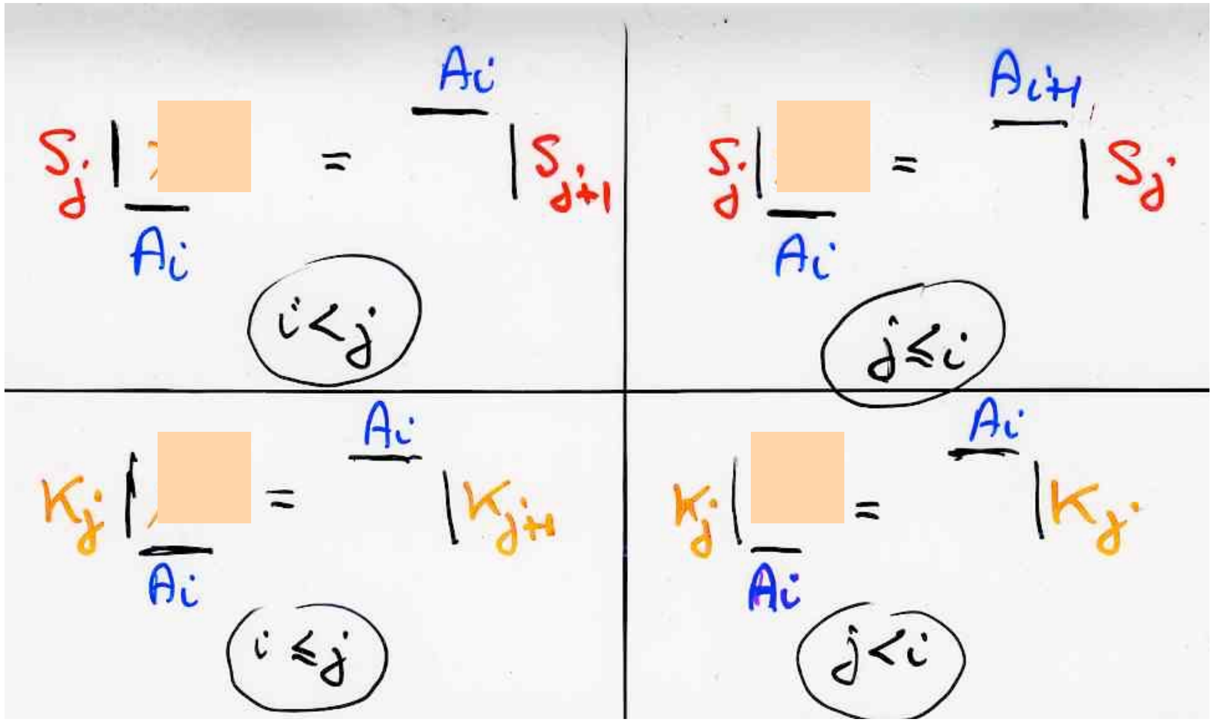


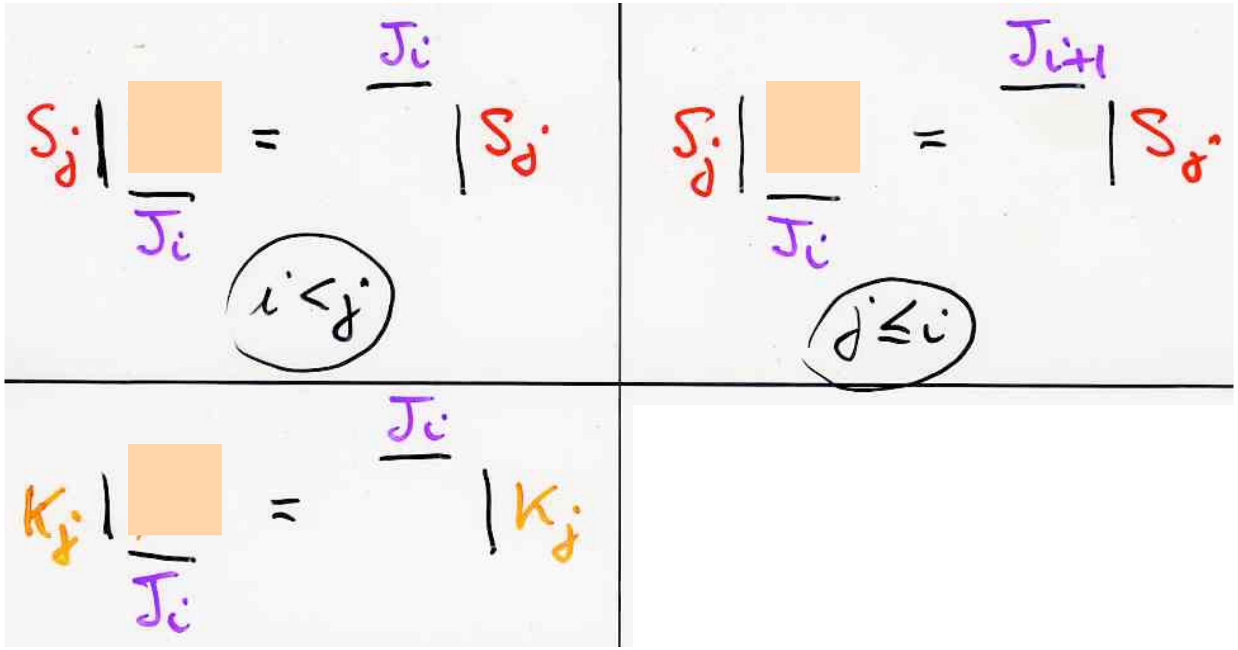
duplication of equations in the reverse quadratic algebra Q^+

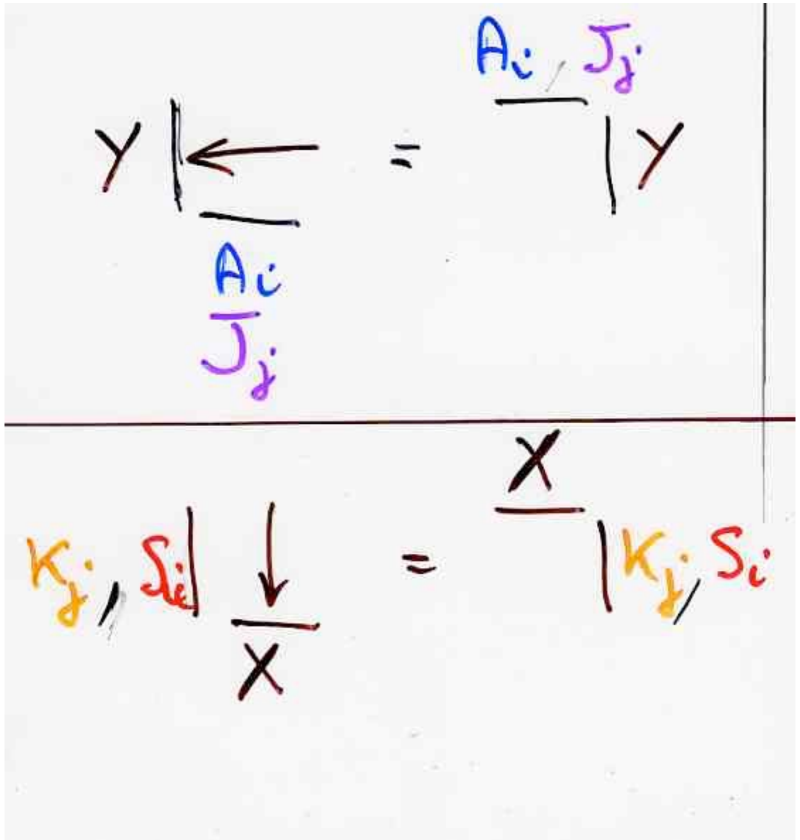


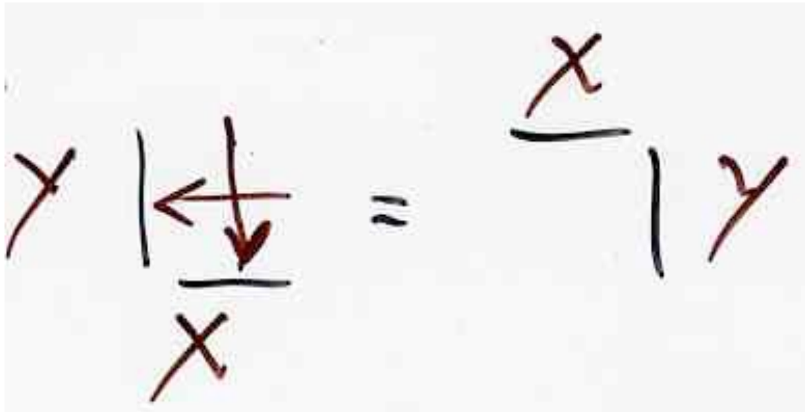



S
K
A, J
=
A, J
S
K


S_j
A_i
=
A_i
S_{j+1}
i<j
S_j
A_i
=
A_{i+1}
S_j
j≤i
K_j
A_i
=
A_i
K_{j+1}
i≤j
K_j
A_i
=
A_i
K_j
j<i


S_j
J_i
=
J_i
S_j
i<j
S_j
J_i
=
J_{i+1}
S_j
j≤i
K_j
J_i
=
J_i
K_j


Y
=
A_i, J_j
Y
A_i
J_j
K_j, S_i
=
X
K_j, S_i
X


Y
=
X
Y
X

The philosophy of the cellular ansatz

combinatorial representation
of the quadratic algebra

$DE = qED + E + D$

alternative tableaux

- commutations diagrams

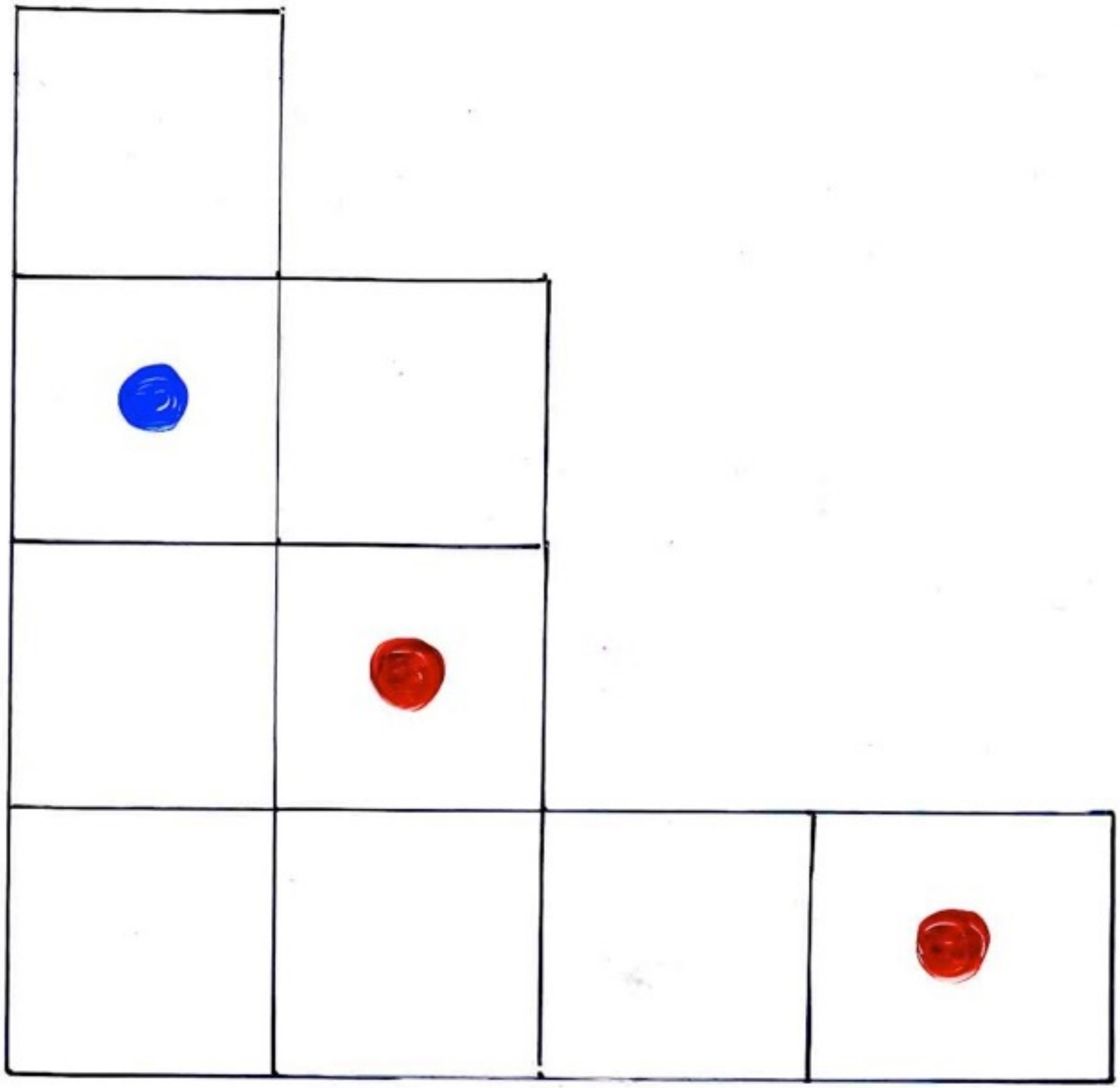
EXF

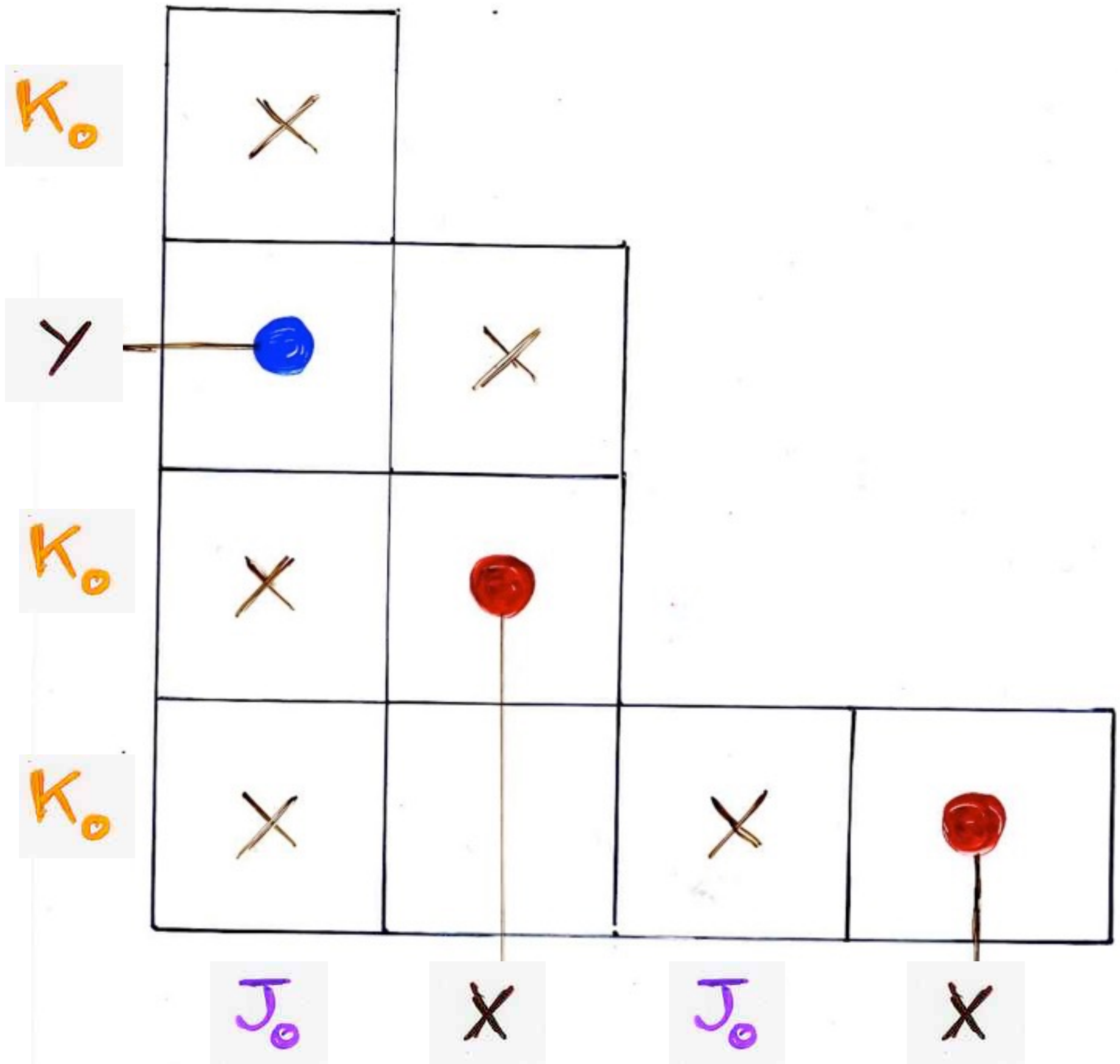
"Laguerre histories"

Equivalence

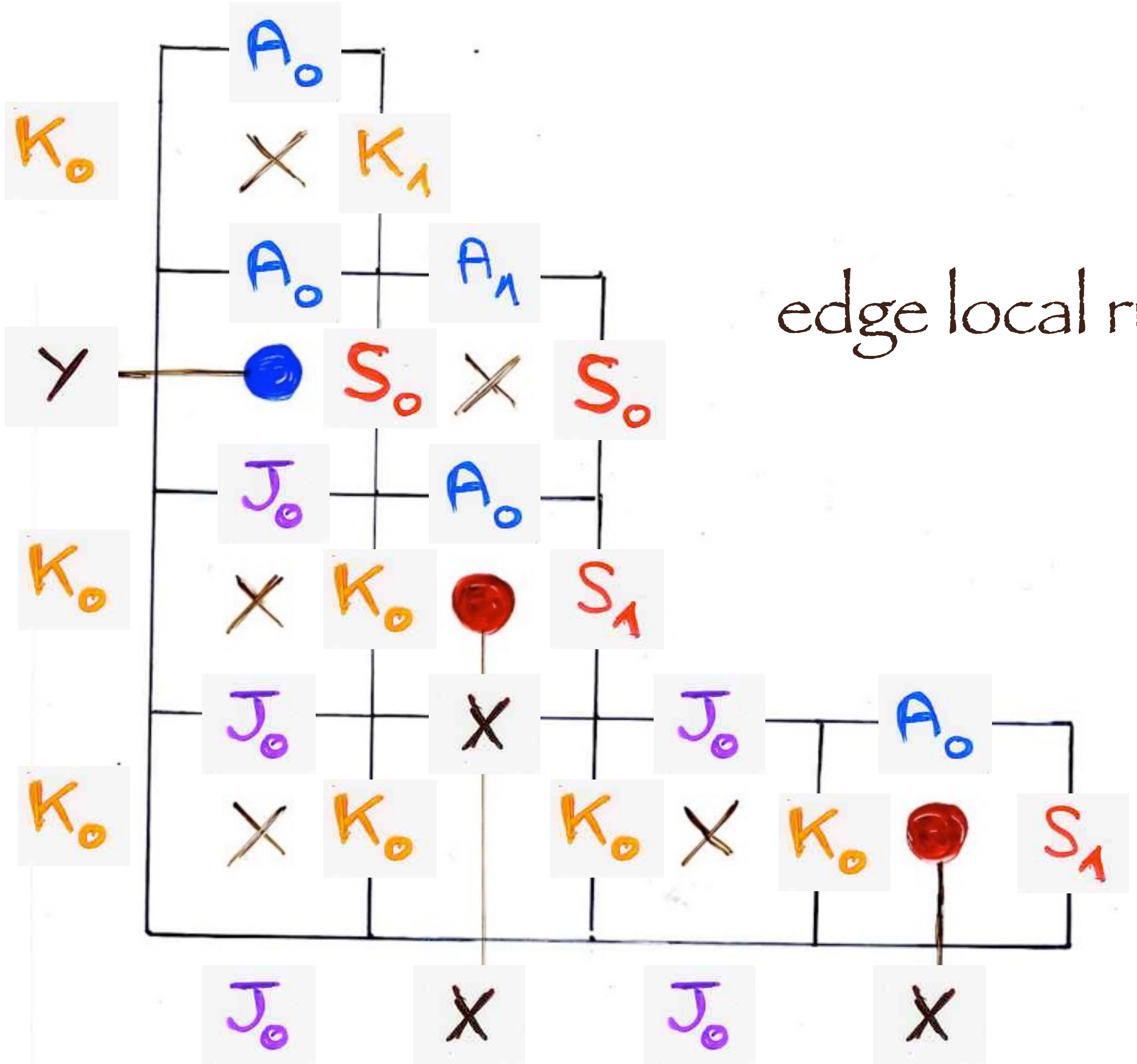
- local rules on edges

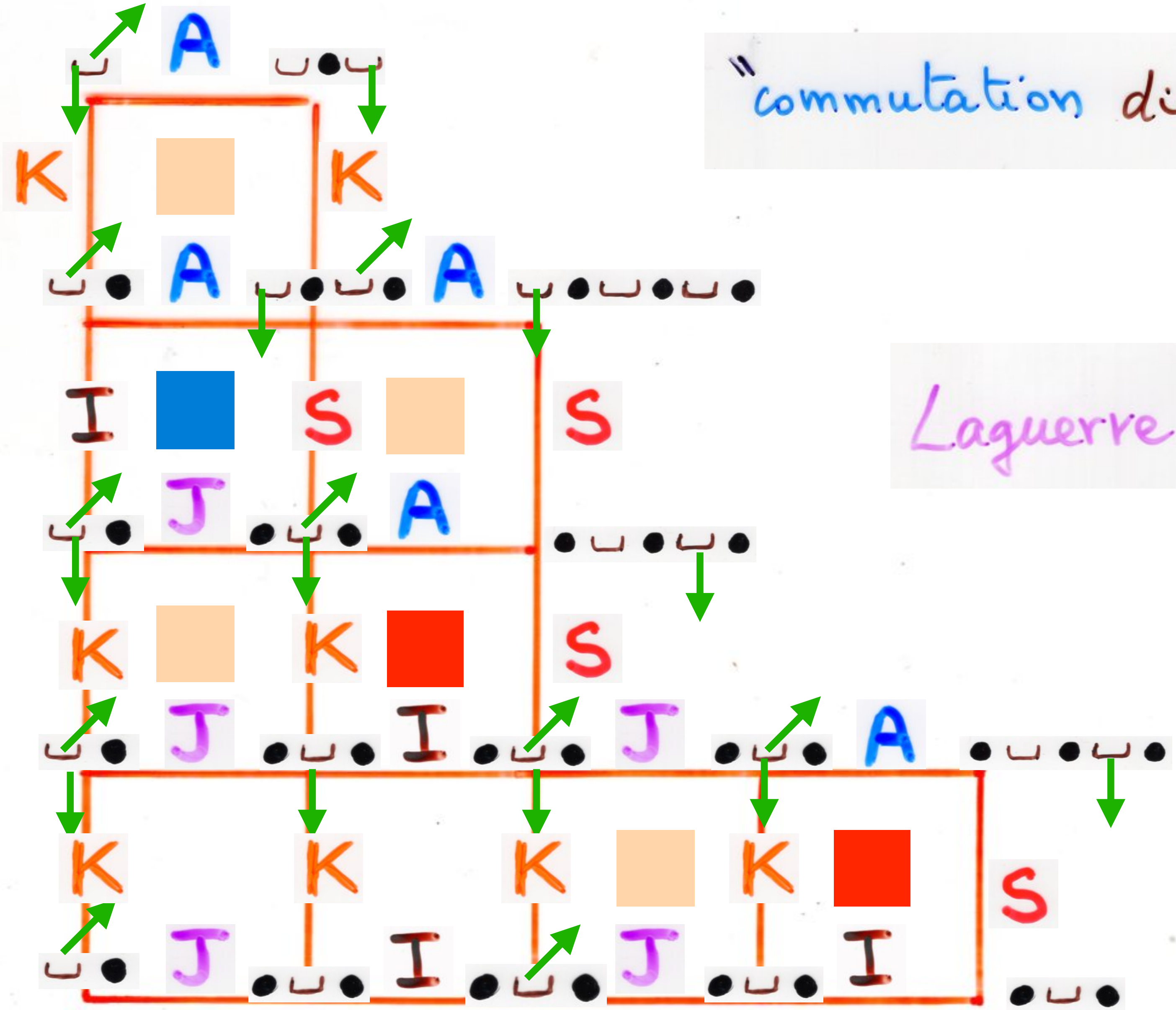
with duplication of equations
in the reverse quadratic algebra




Ko
Y
Ko
Ko
Jo
X
Jo
X

edge local rules





"commutation diagrams"

Laguerre histories

A new bijection

demultiplication of equations
in the
reverse PASEP quadratic algebra

(2)

PASEP algebra

$$
Q \begin{cases}
DE = \overset{\square}{q}\, ED + \blacksquare\, EX + \blacksquare\, YD \\
XE = \square\, EX \\
DY = \square\, YD \\
XY = \square\, YX
\end{cases}
$$

reverse PASEP algebra

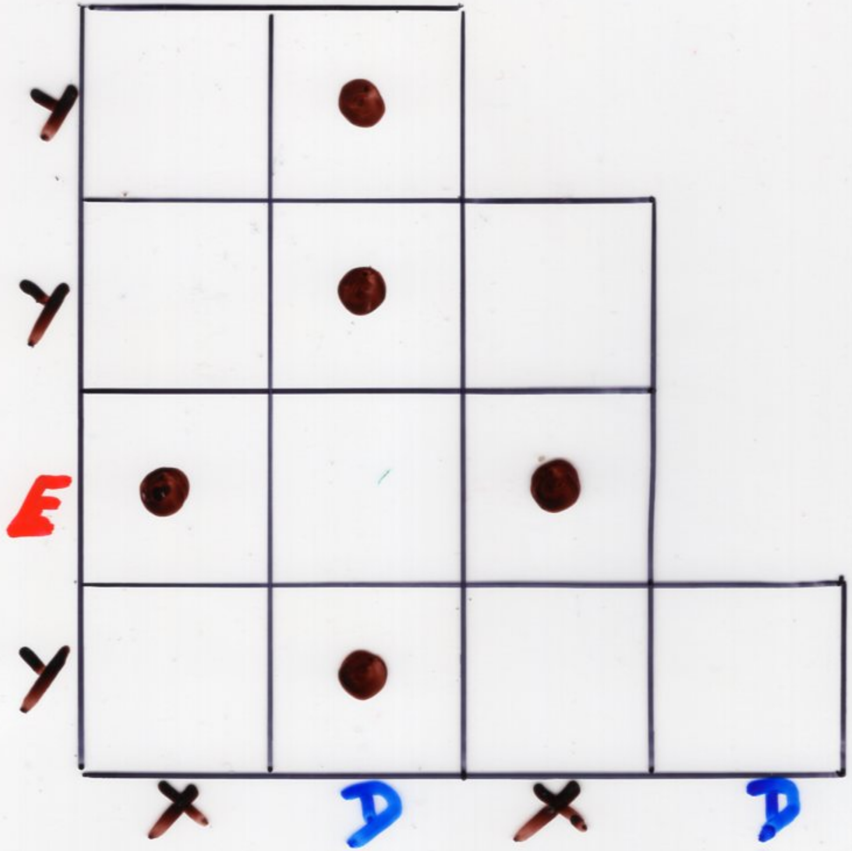
$$
Q^{+} \begin{cases}
ED = \overset{\square}{q}\, DE \\
EX = \square\, XE + \blacksquare\, DE \\
YD = \square\, DY \quad \blacksquare\, DE \\
YX = \square\, XY
\end{cases}
$$

reverse PASEP algebra

$$
Q^{+}\begin{cases}
ED = q\,DE \\
EX = XE + DE \\
YD = DY \quad DE \\
YX = XY
\end{cases}
$$

$$
\begin{cases}
E_i D_j = q\,\boxed{D_{j+1} E_{i+1}} \\
E_i X = X E_i + \boxed{D_1 E_{i+1}} \\
Y D_j = D_j Y + \boxed{D_{j+1} E_1} \\
Y X = X Y
\end{cases}
$$

$i, j \geq 1$

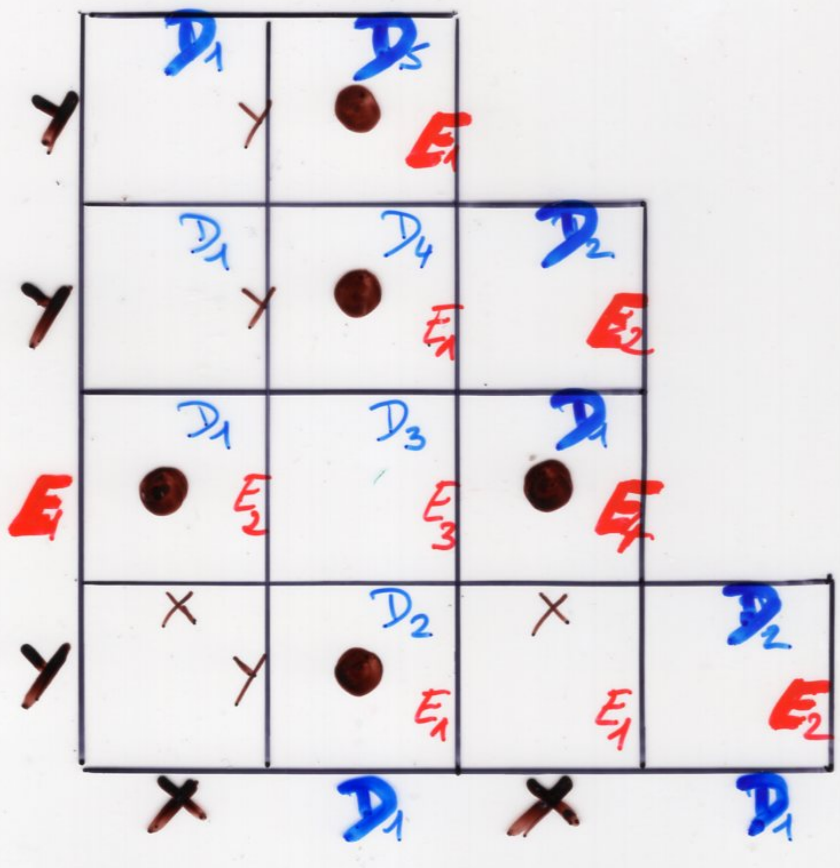

Y
Y
E
Y
X
D
X
D

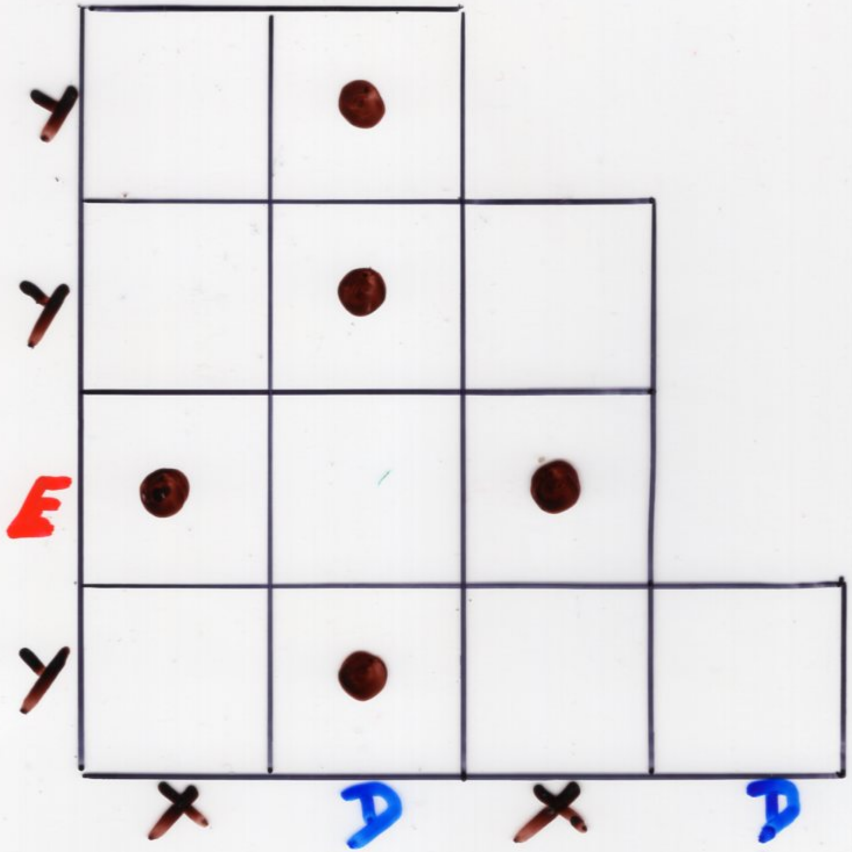
The Tamil bijection

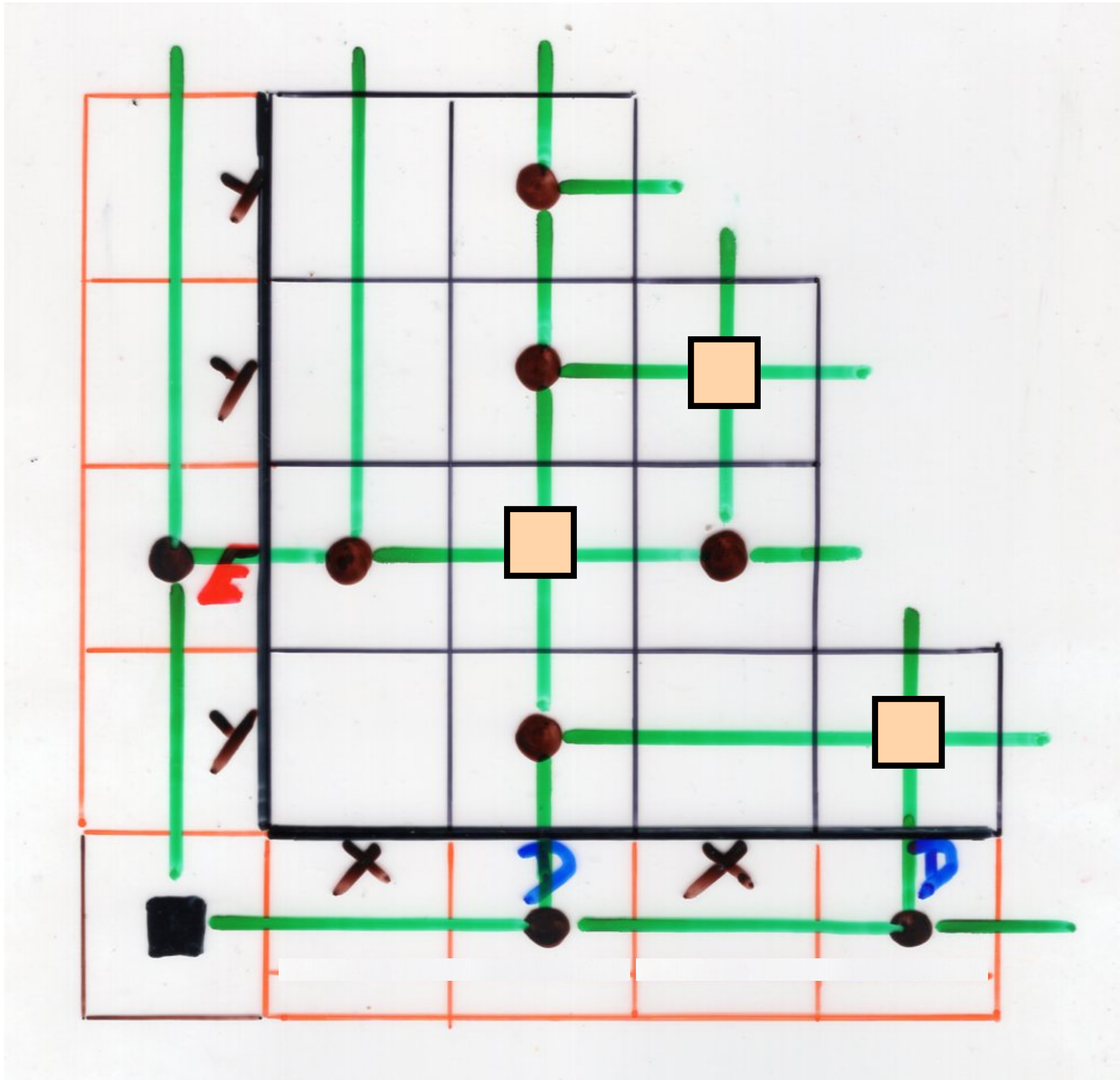
alternative or tree-like tableaux (size n)

some words $w \in \{D_i, E_j \; ; \; i,j \geq 1\}^*$, $|w| = n$

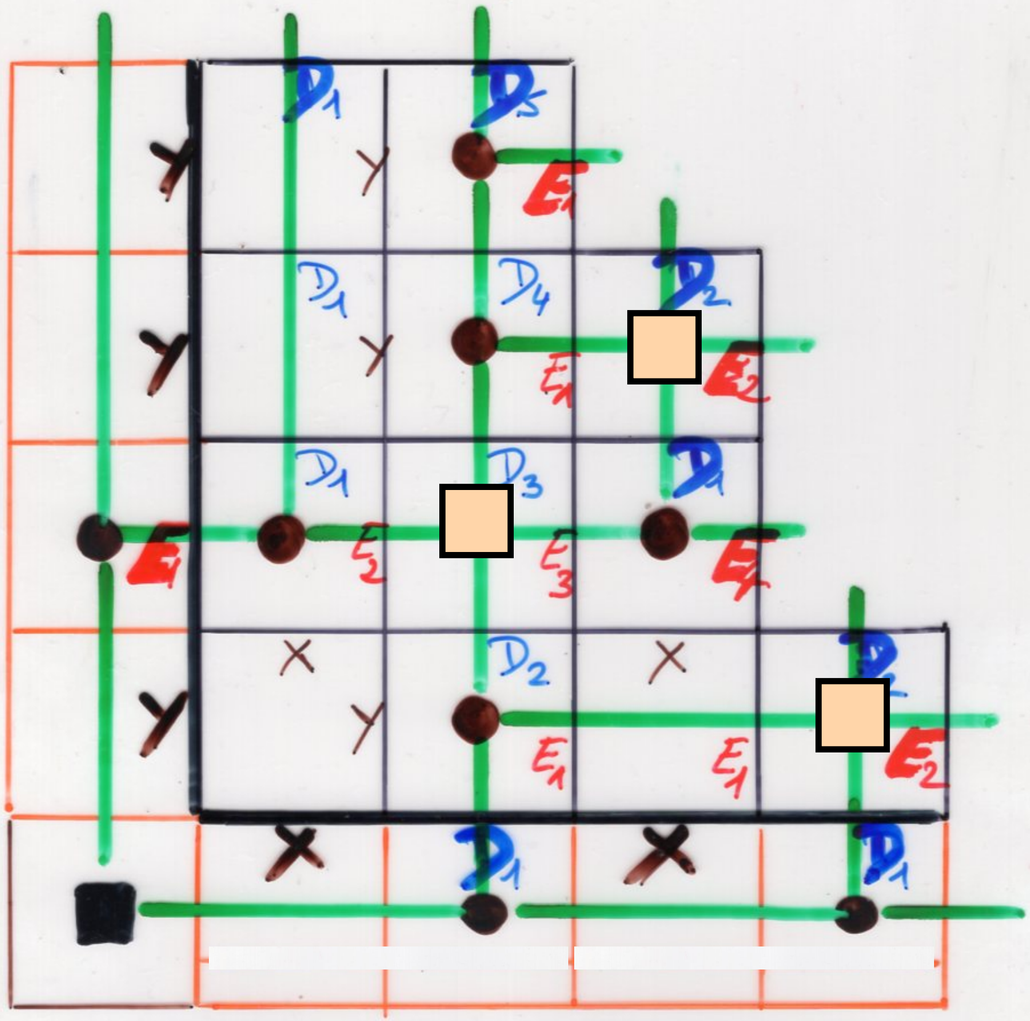
$w = D_1 D_5 E_1 D_2 E_2 E_4 D_2 E_2$



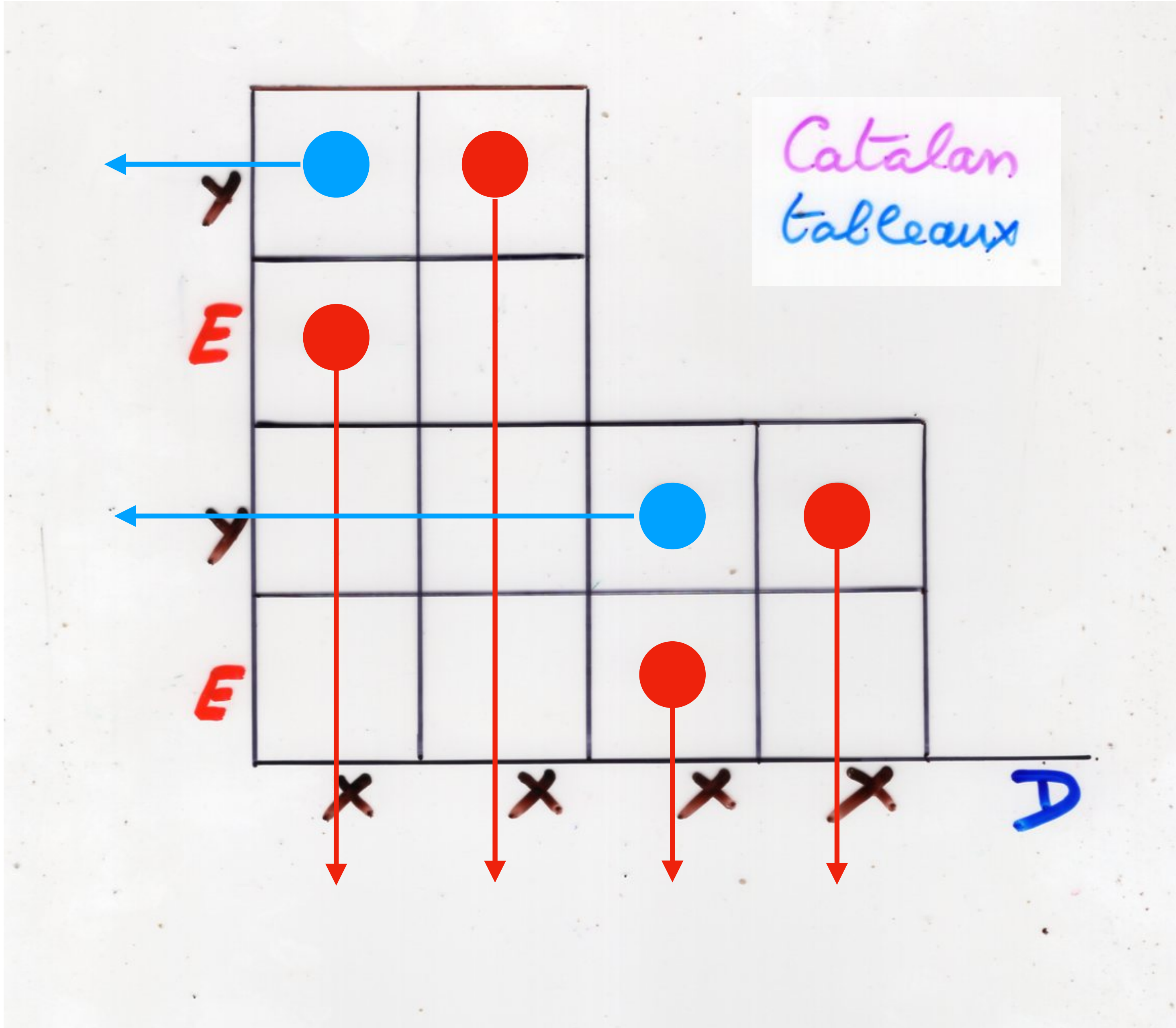

Y
Y
E
Y
X
D
X
D

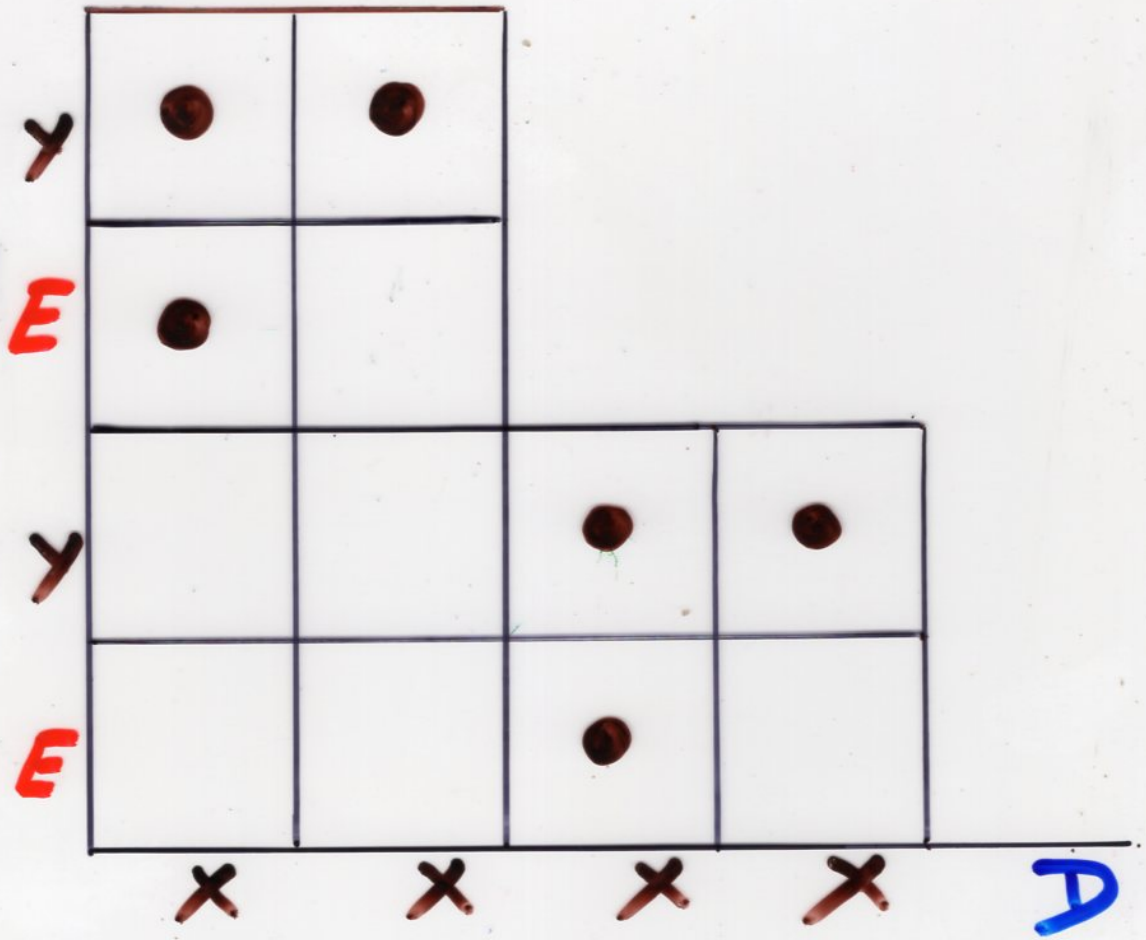


The Tamil bijection

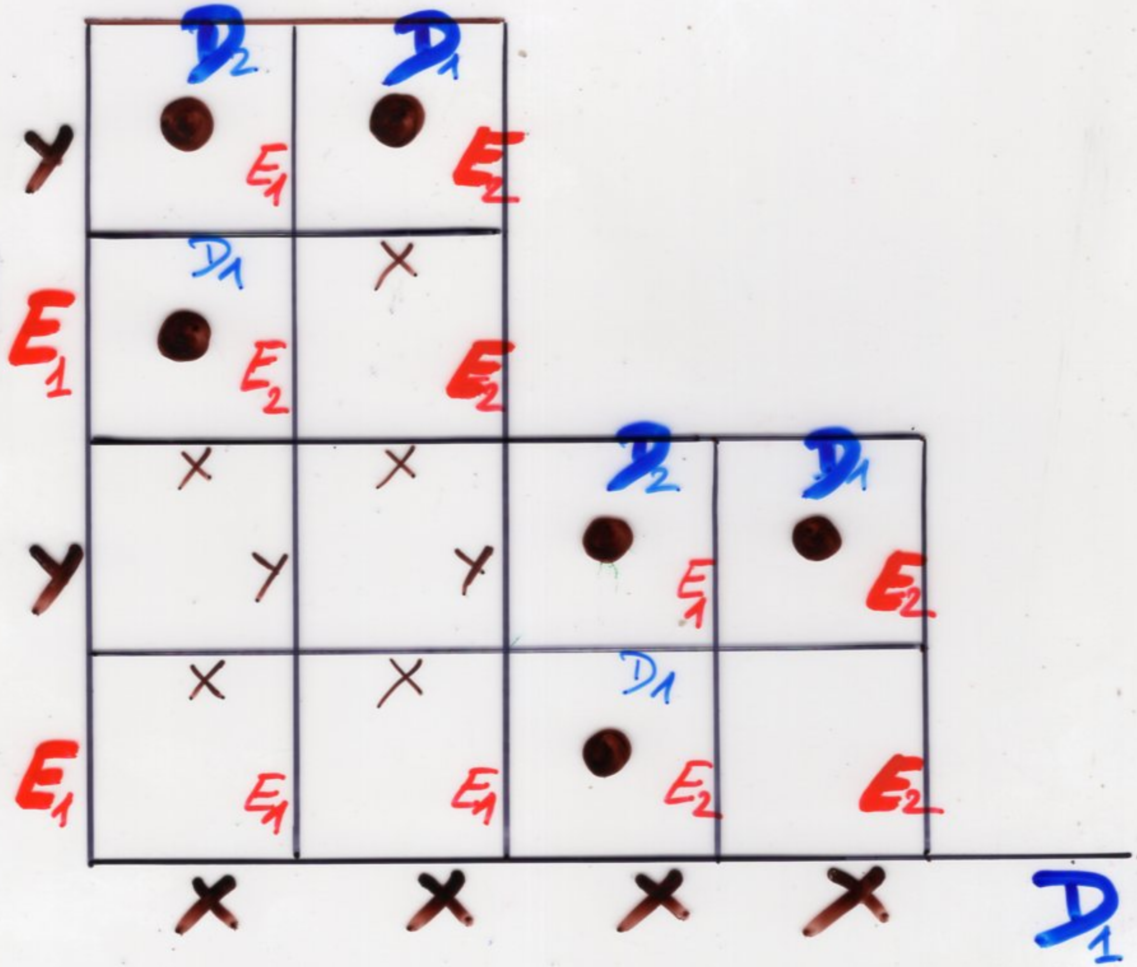


$w = D_1 \, D_5 \, E_1 \, D_2 \, E_2 \, E_4 \, D_2 \, E_2$


Catalan
tableaux
y
E
y
E
x
x
x
x
D

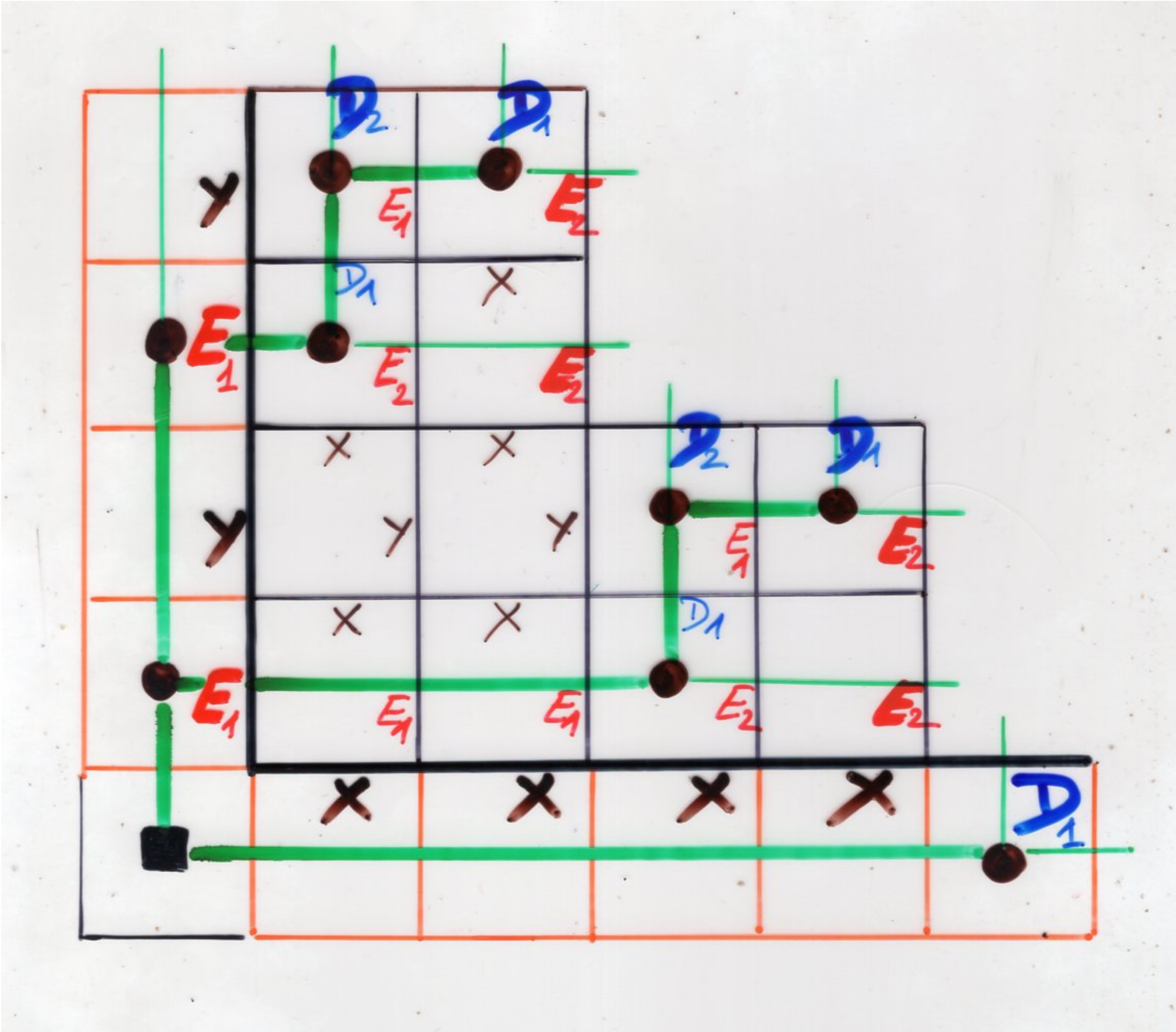


Y
E
Y
E
X X X X
D

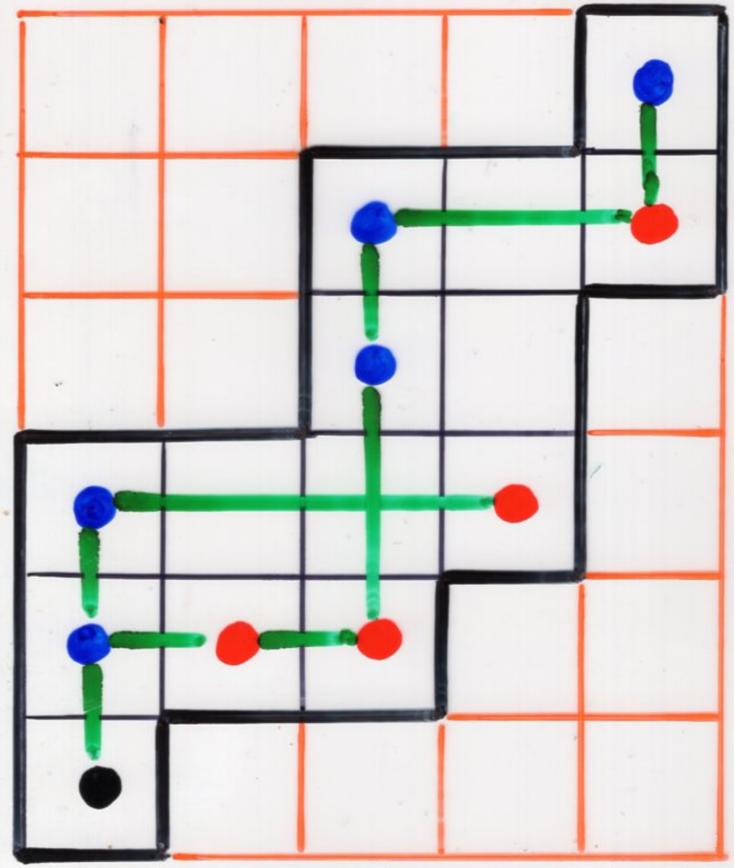


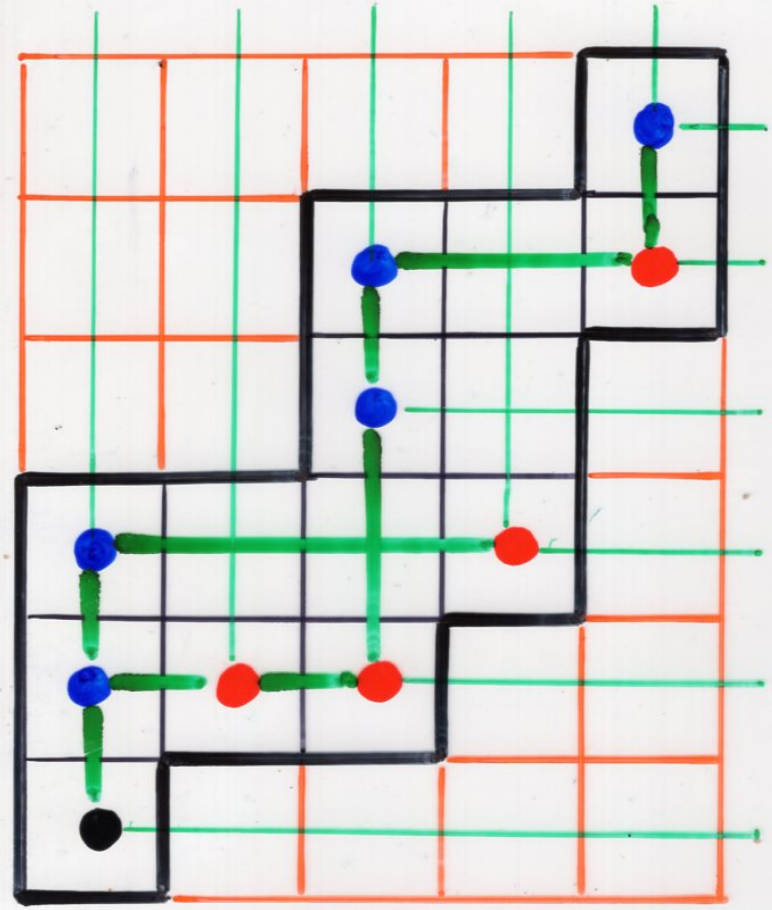
D_2
D_1
Y
E_1
E_2
D_1
X
E_1
E_2
E_2
X
X
D_2
D_1
Y
Y
Y
E_1
E_2
X
X
D_1
E_1
E_1
E_1
E_2
E_2
X
X
X
X
D_1

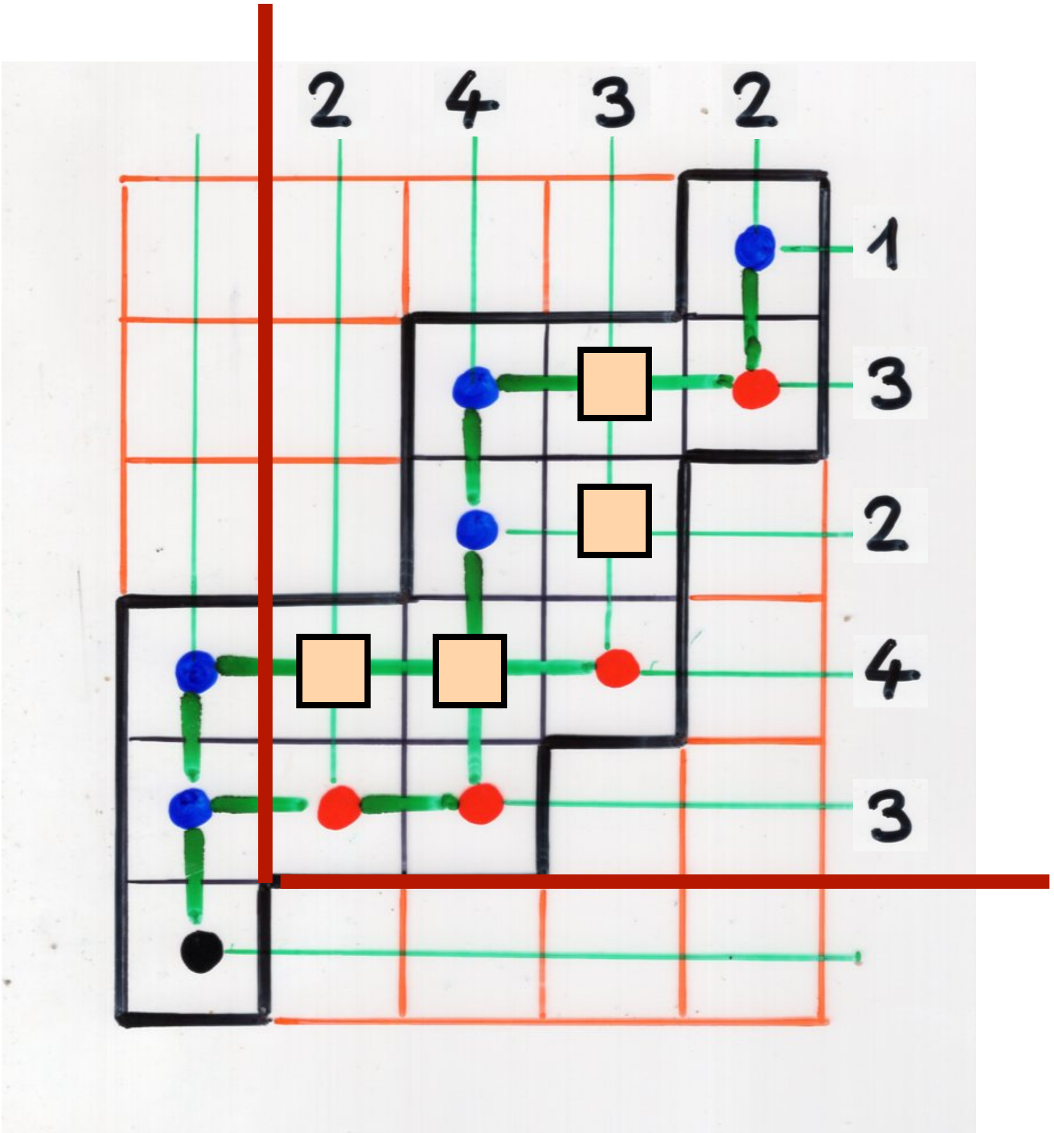
$$w = D_2\, D_1\, E_2\, E_2\, D_2\, D_1\, E_2\, E_2\, D_1$$

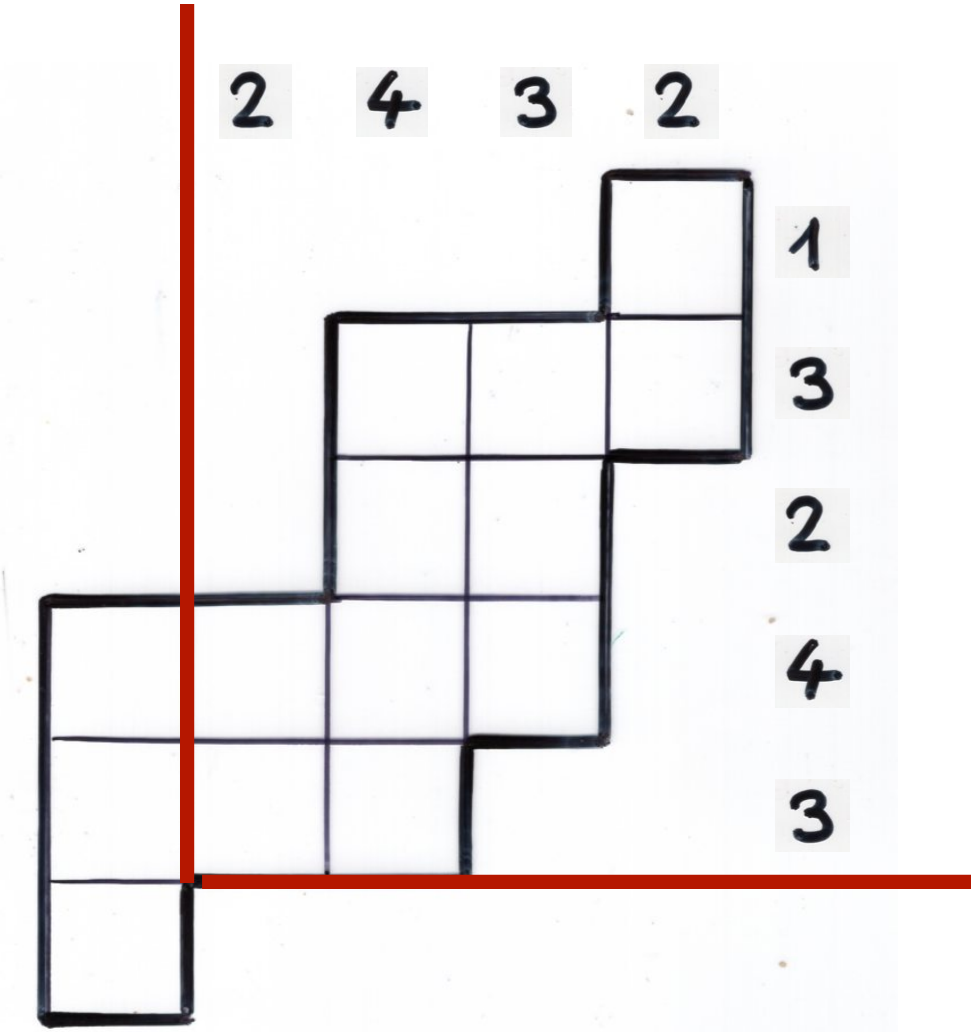


example



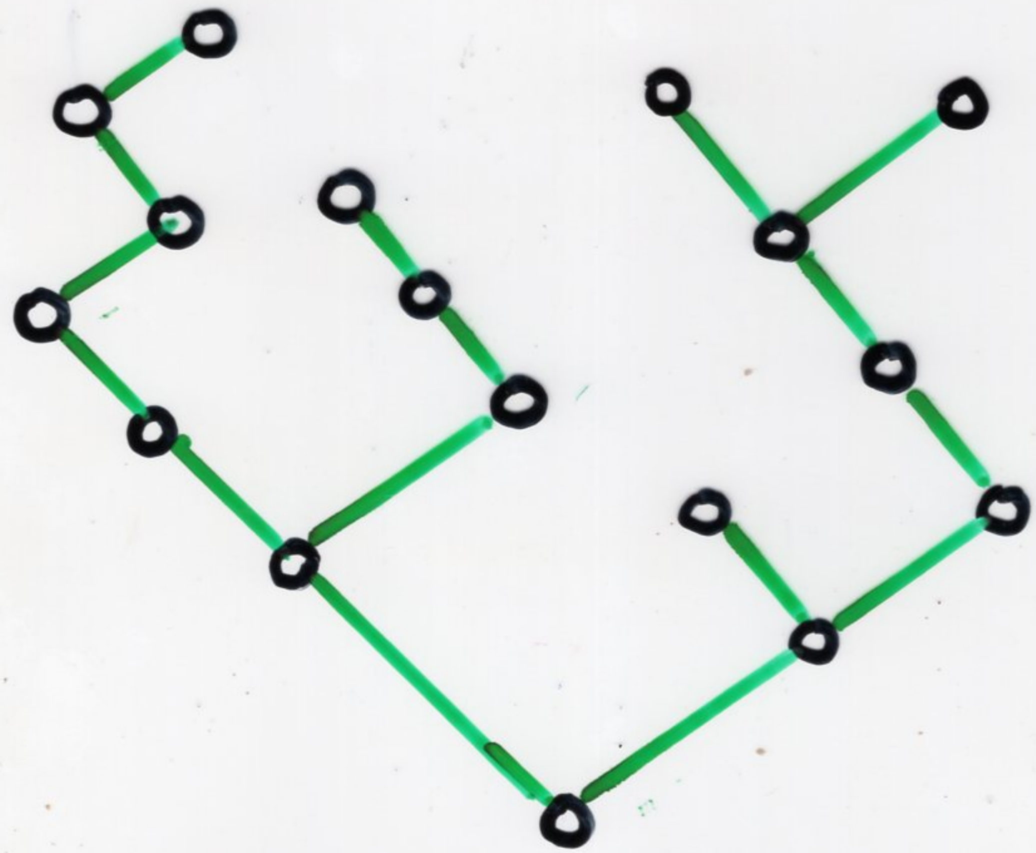



2 4 3 2
1
3
2
4
3


2 4 3 2
1
3
2
4
3

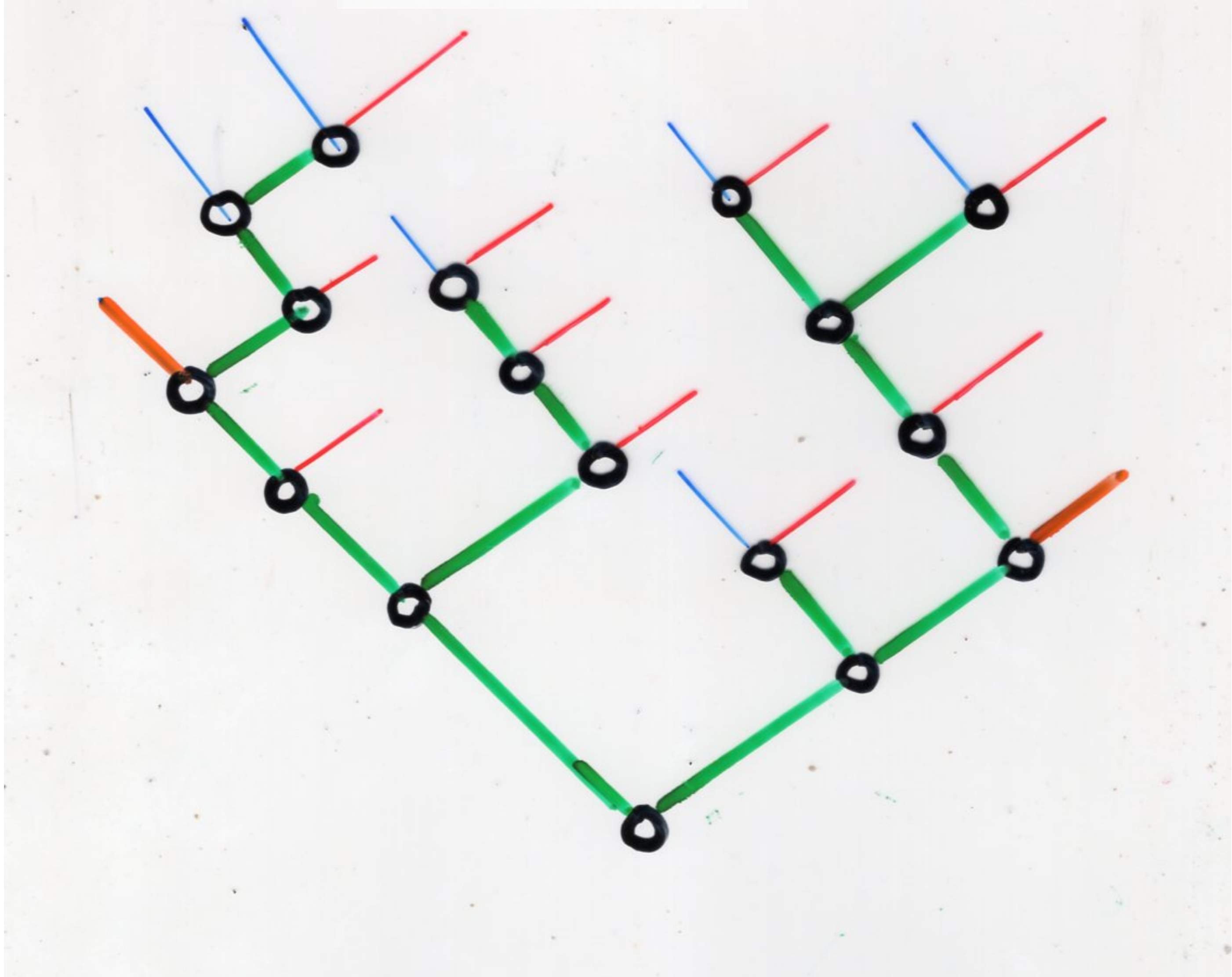
The Tamil bijection

binary trees

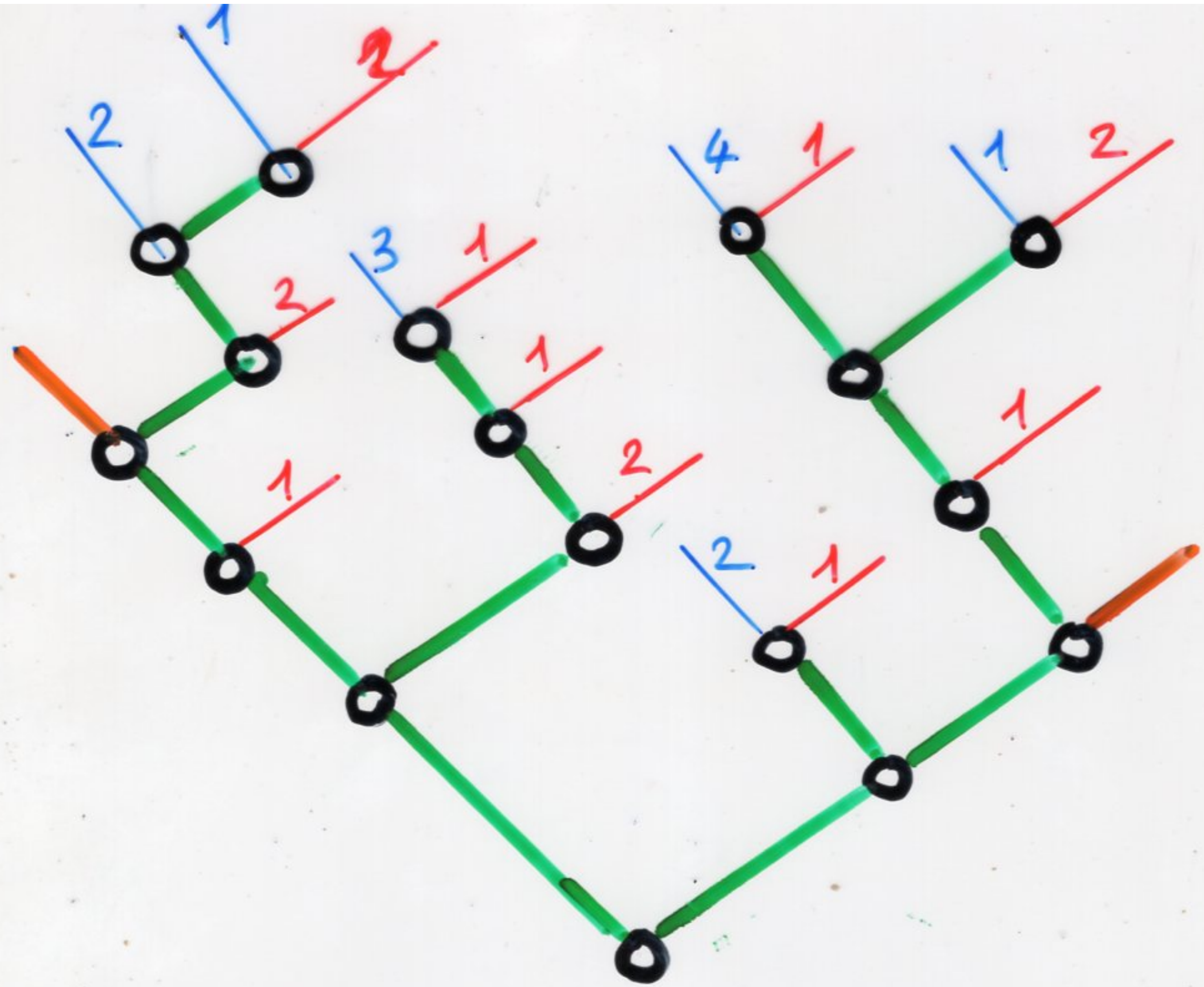


The Tamil bijection

binary trees



$w = 2\ 1\ 2\ 2\ 1\ 3\ 1\ 1\ 2\ 2\ 1\ 4\ 1\ 1\ 2\ 1$



The ultimate end of the bijective course III

Thank you very much !

for all of you, students, professors, friends,

For the videos:

Gayathri and Kirubananth

special thanks to Amri Prasad

ॐ सरस्वत्यै नमः।