Chapter 3

Tableaux for the PASEP quadratic algebra

Ch3c

(second part)

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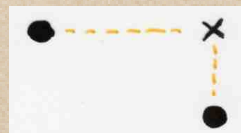
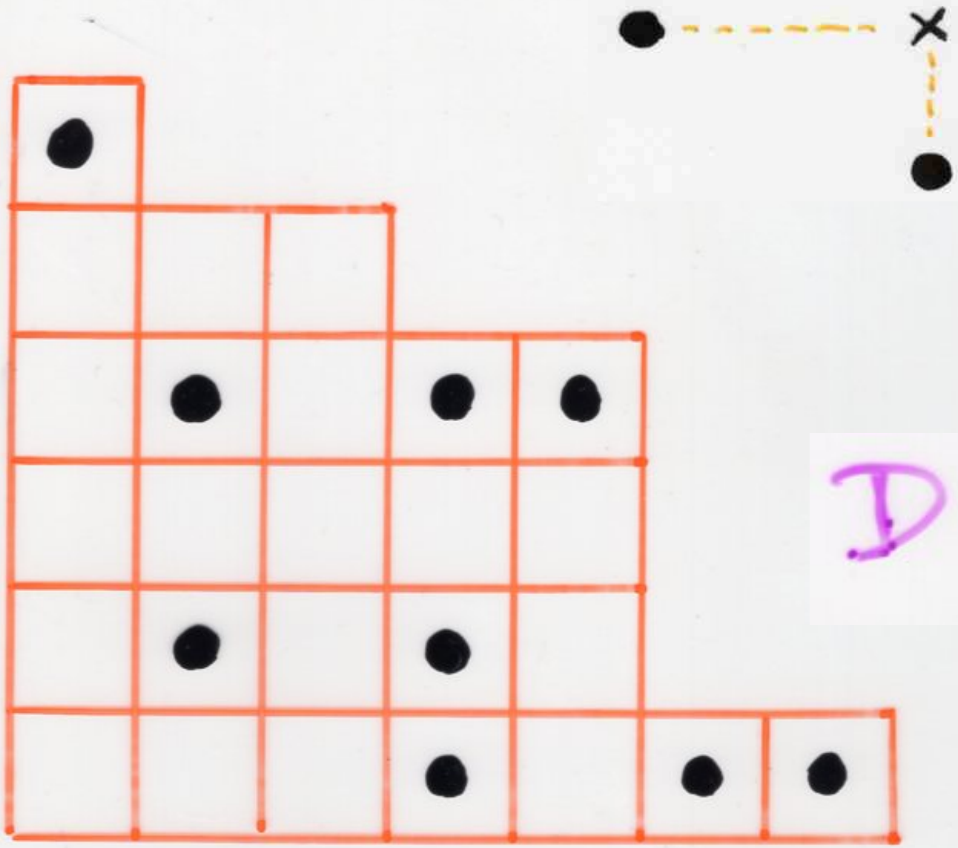
Complements

Postnikov bijection

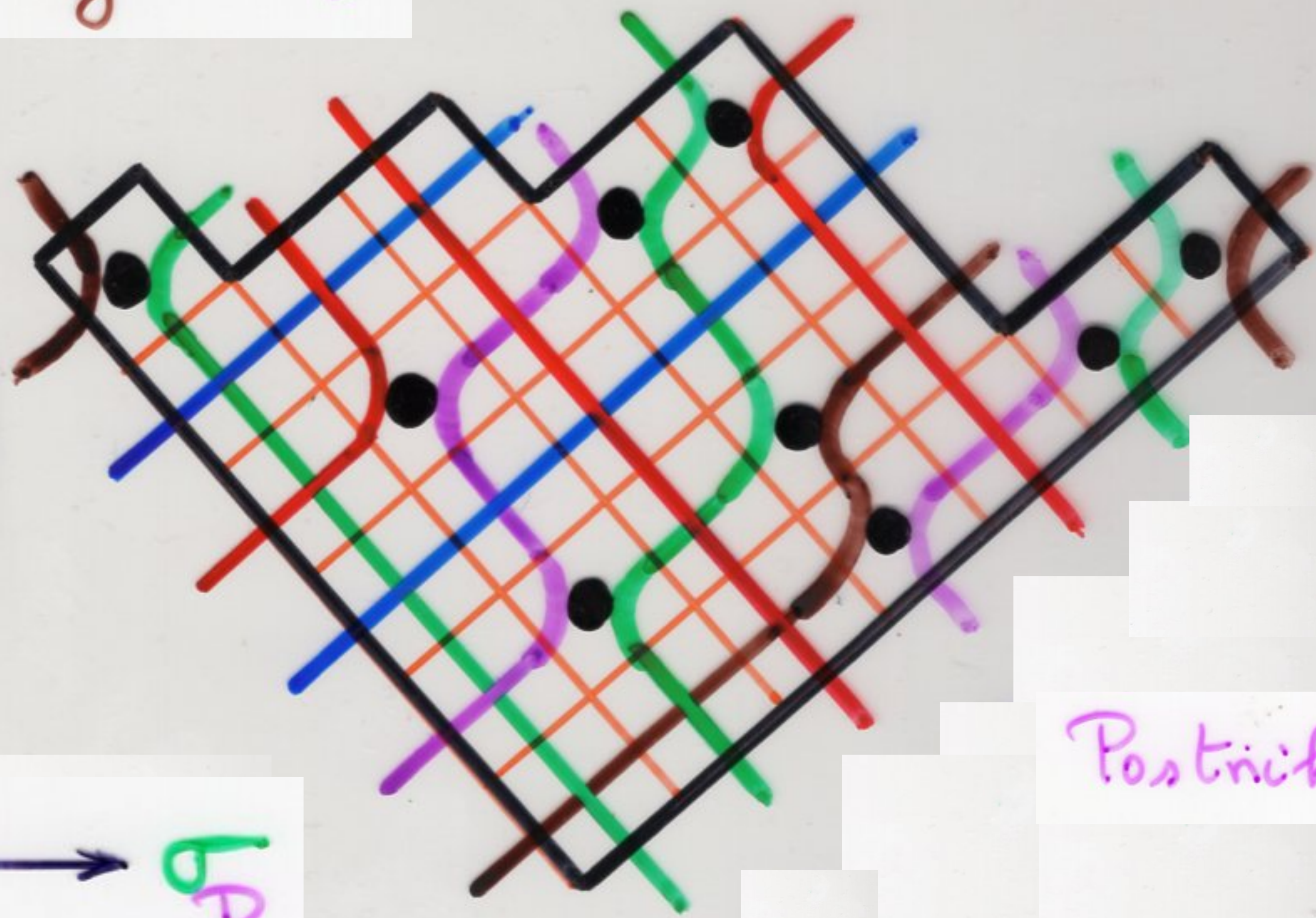
pipe dreams and  -diagrams

⅂- diagrams D



$D \longrightarrow \sigma_D$

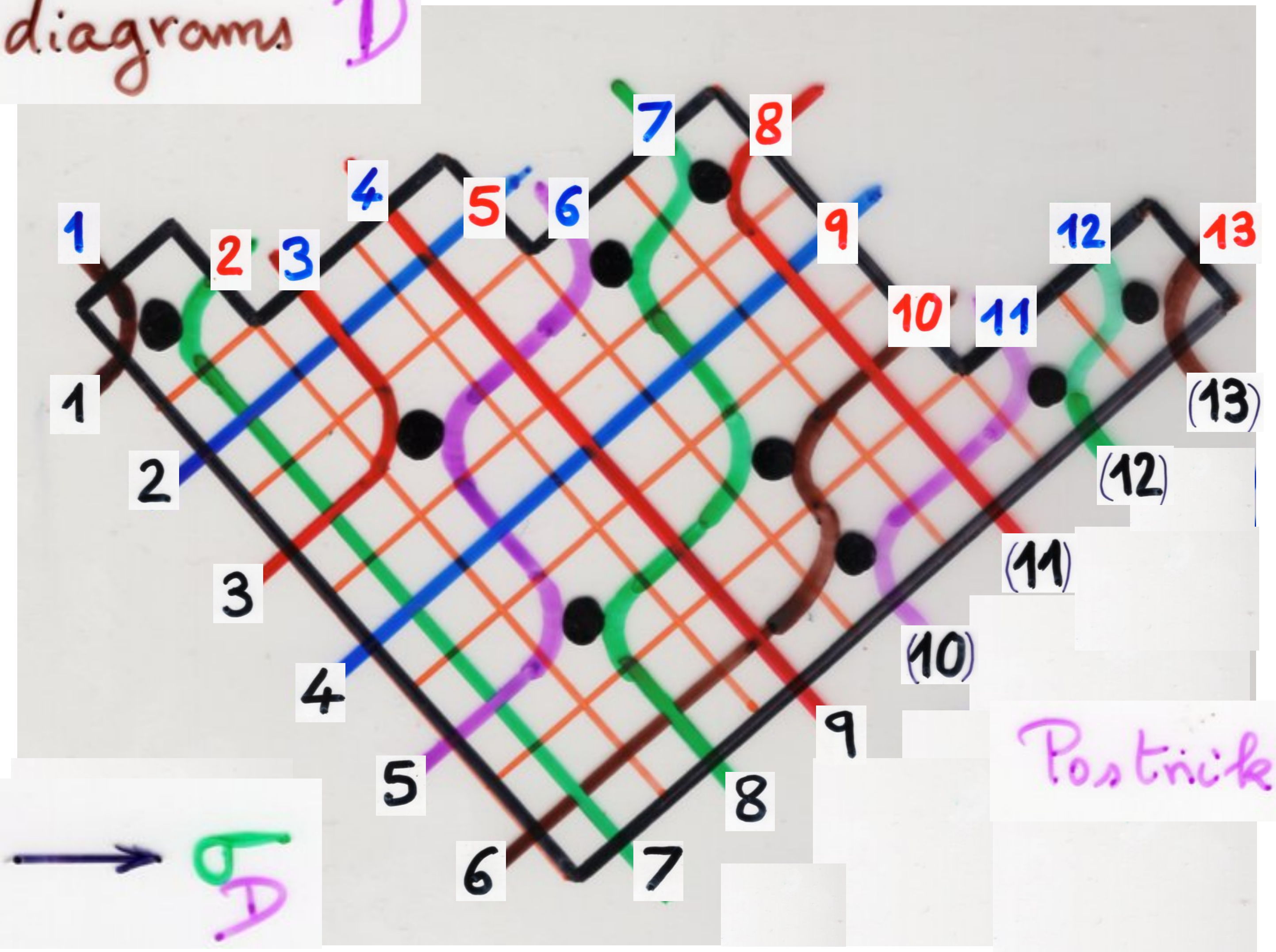
⅂- diagrams D



Postnikov (2006)

$D \longrightarrow \sigma_D$

⅂- diagrams D



Postnikov (2006)

$D \longrightarrow \sigma_D$

1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
1	5	3	9	6	10	2	7	4	11	8	12	13

Postnikov (2006)

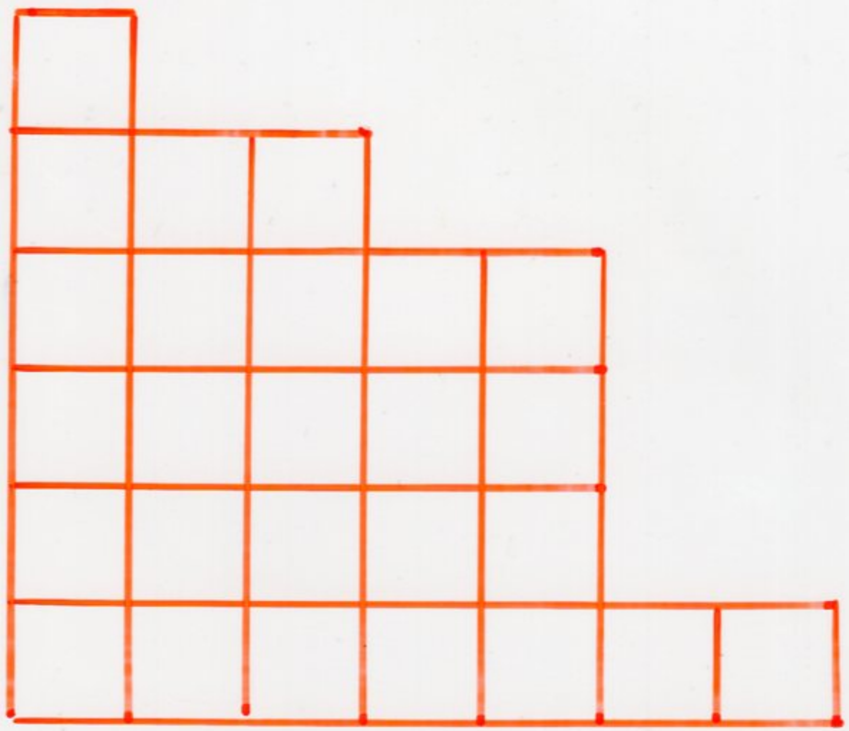
bijection $D \longrightarrow \sigma_D$

Ꞁ- diagrams D shape λ $\longrightarrow$ permutation $\sigma \in \mathfrak{S}_n$

$\sigma \leq \omega_\lambda$

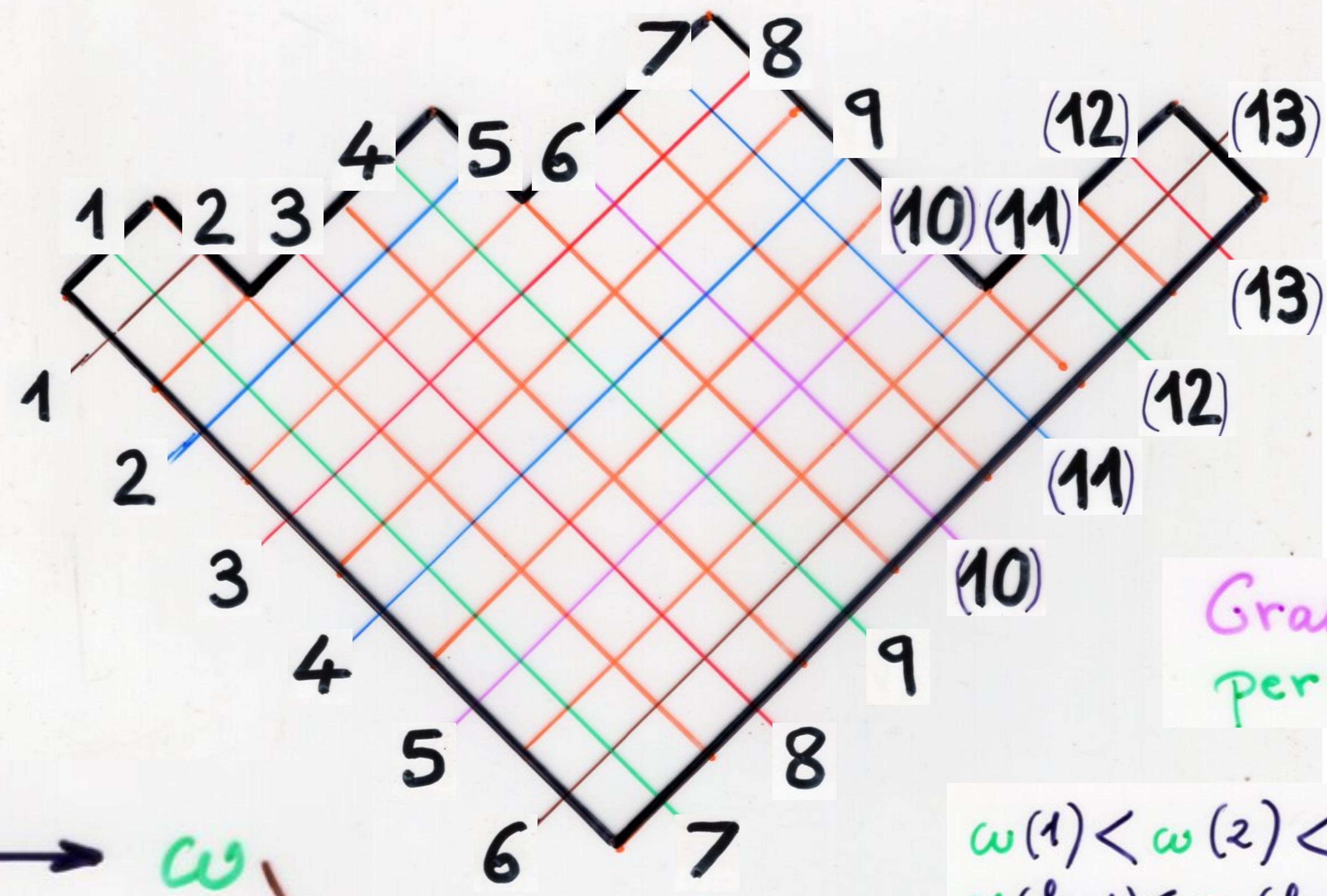
(Bruhat order)

(k, n) $\lambda \subseteq (n-k)^k$



$\lambda \longrightarrow \omega_\lambda$

Grassmanian permutation



Grassmanian permutation

$\lambda \longrightarrow \omega_\lambda$

$\omega(1) < \omega(2) < \cdots < \omega(k)$

$\omega(k+1) < \omega(k+2) < \cdots < \omega(n)$

1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
2	5	8	9	(10)	(13)	1	3	4	6	7	(11)	(12)

Postnikov (2006)

bijection $\quad D \longrightarrow \sigma_D$

⅂-diagrams D, shape λ $\longrightarrow$ permutation $\sigma \in \mathfrak{S}_n$

$\sigma \preccurlyeq \omega_\lambda$

(Bruhat order)

number of ● $= \ell(\omega_\lambda) - \ell(\sigma_D)$

Complements

reminding Ch6a, BJC2

Heaps of dimers

and the symmetric group

Symmetric group $\mathfrak{S}_n$

S_n

$n!$ permutations

$$\sigma_i = (i, i+1) \qquad i = 1, 2, \dots, n-1$$

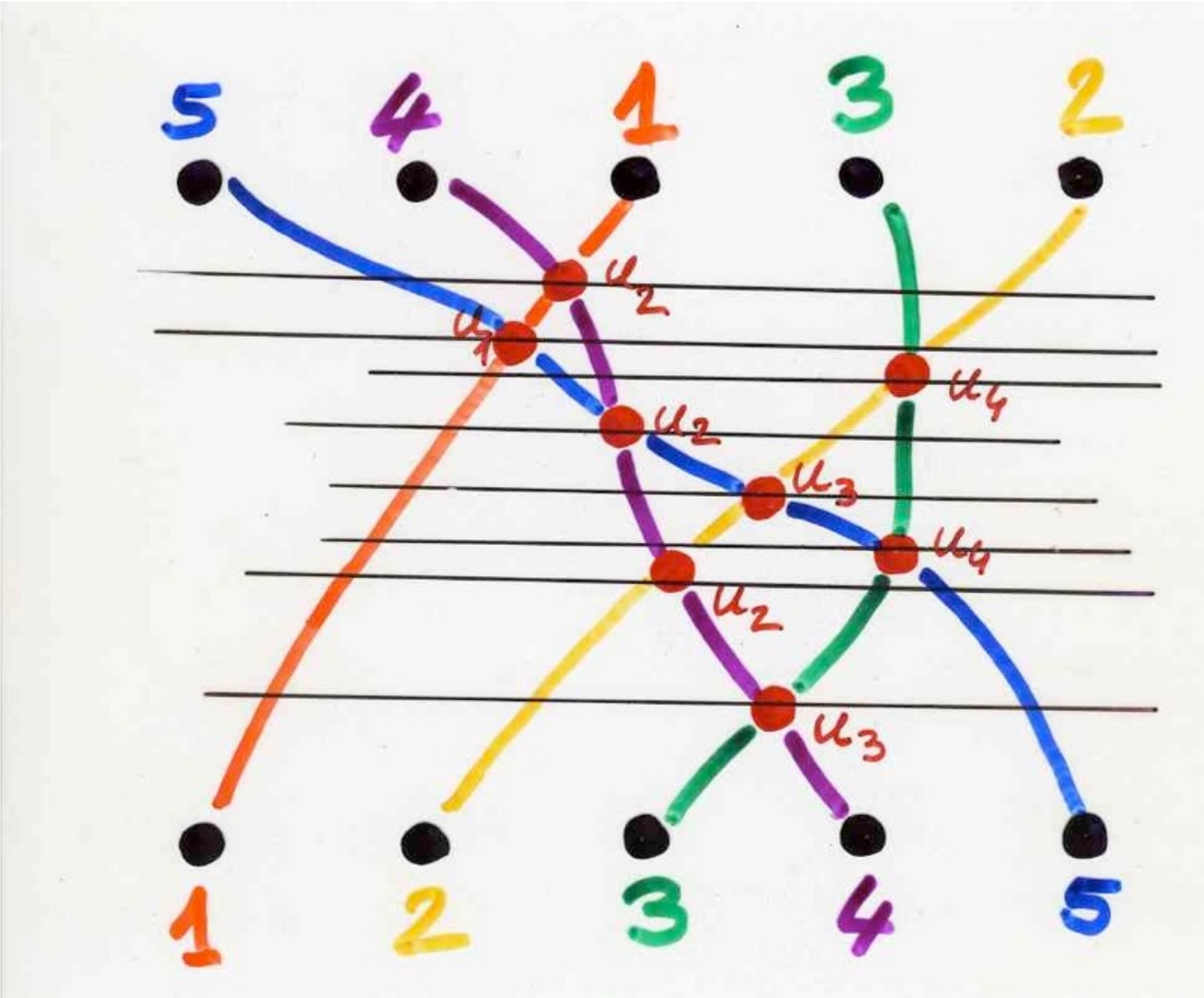
transposition of two consecutive elements

$$\begin{cases} \text{(i)} & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2 \\ \text{(ii)} & \sigma_i^2 = 1, \\ \text{(iii)} & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Moore-Coxeter

Yang-Baxter

Coxeter graph

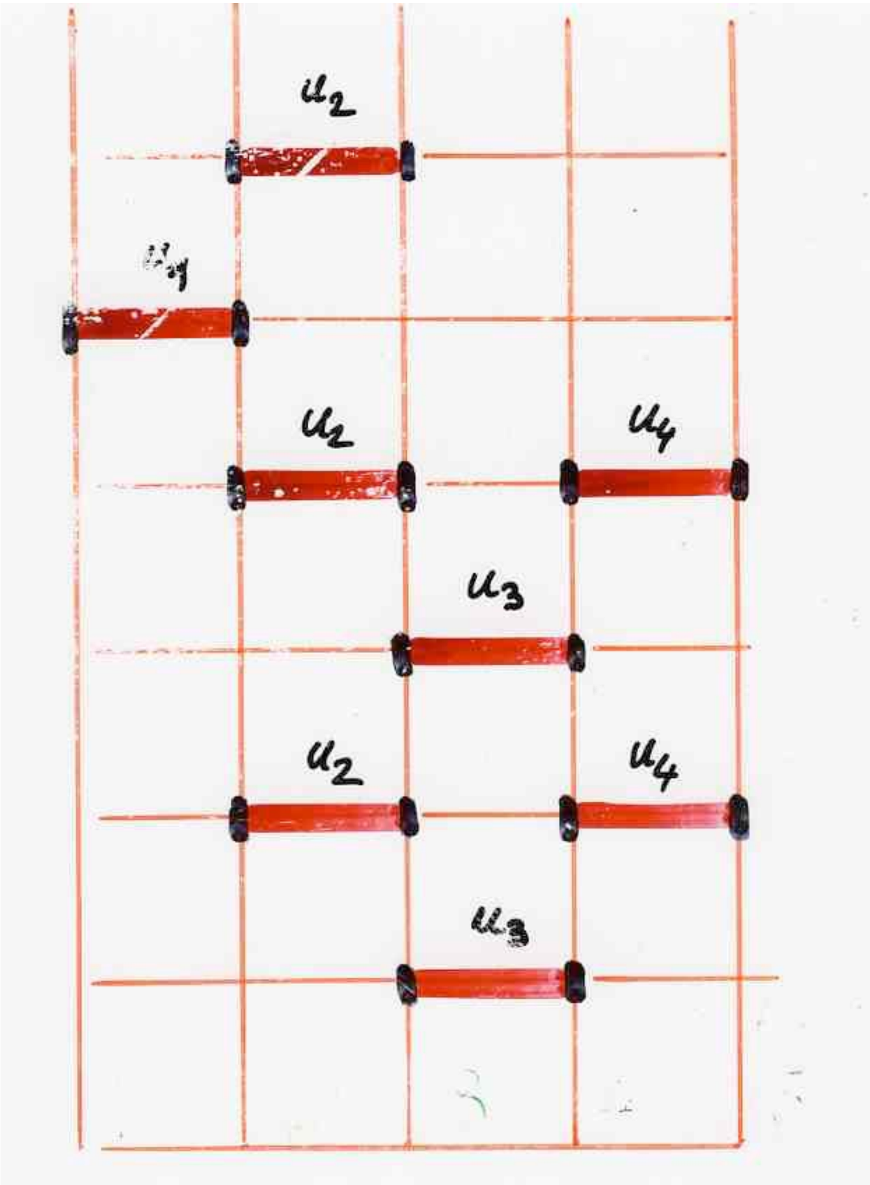


reduced decomposition of a permutation

$$\sigma = u_{i_1} \dots u_{i_k}$$

k minimum

(nb of inversion)



heaps of **dimers** $(i, i+1)$

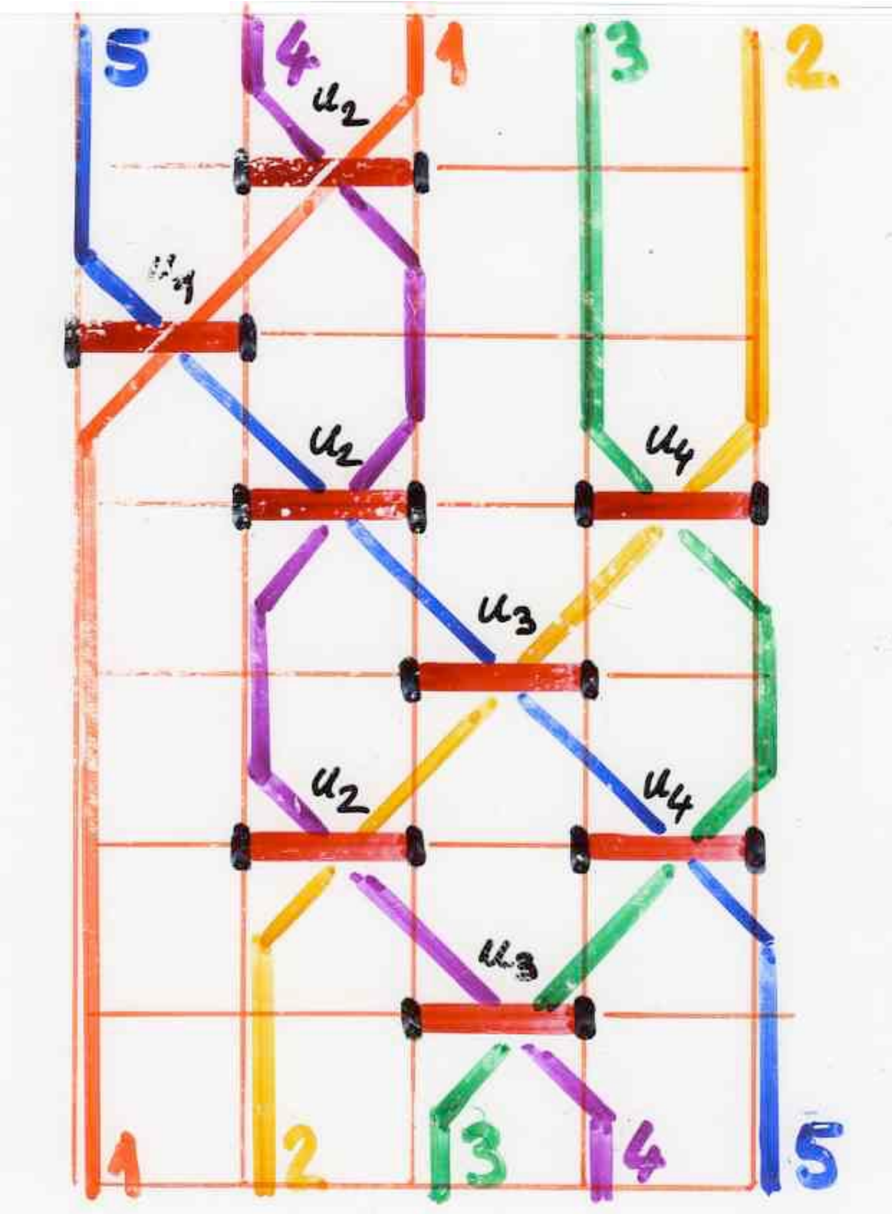
on $\{0, 1, \ldots, n-1\}$

generators $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$

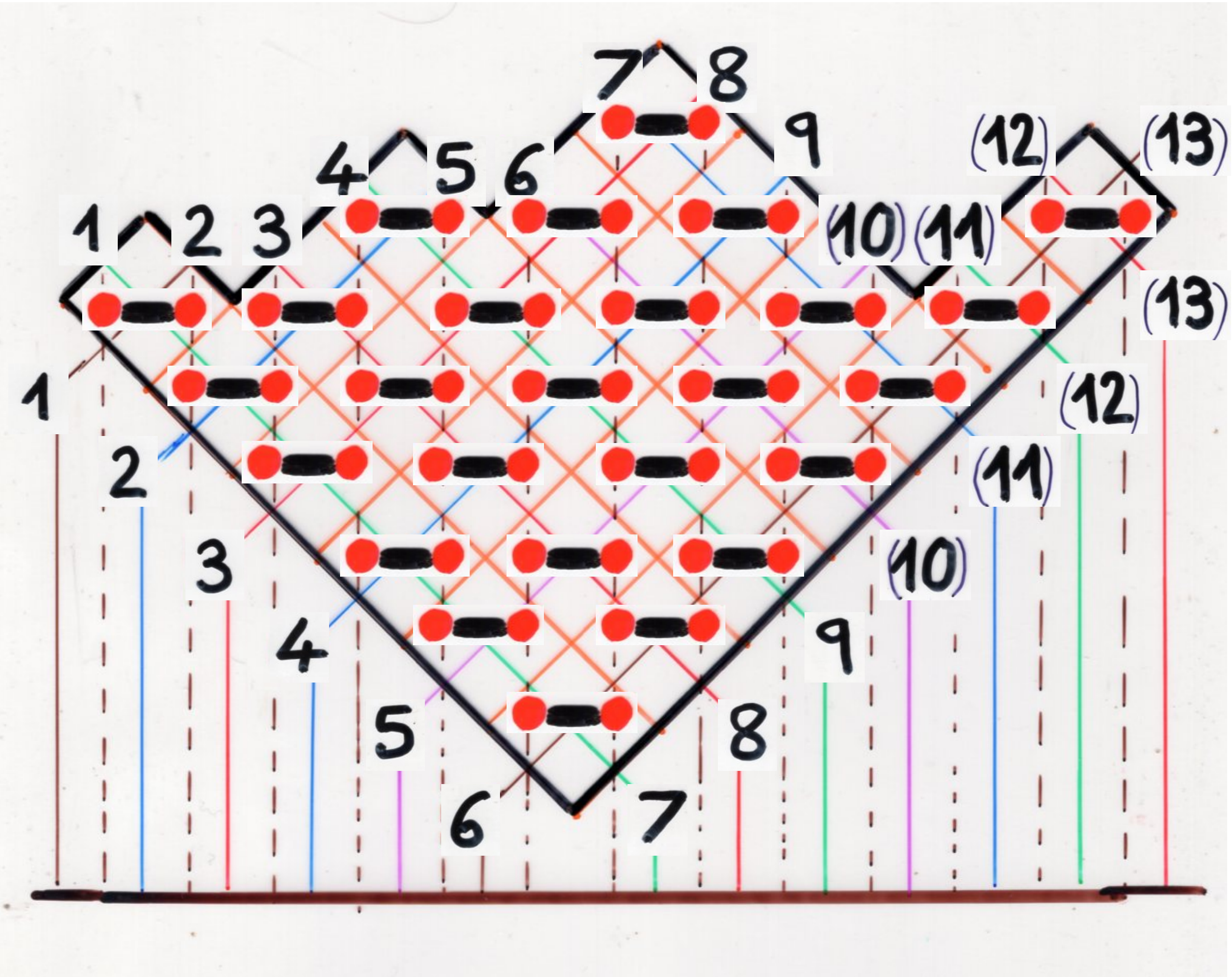
$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

iff $|i-j| \geq 2$

heap of dimers $[1, n]$ → **permutation** S_n



heap of dimers $[1,n]$ → permutation S_n



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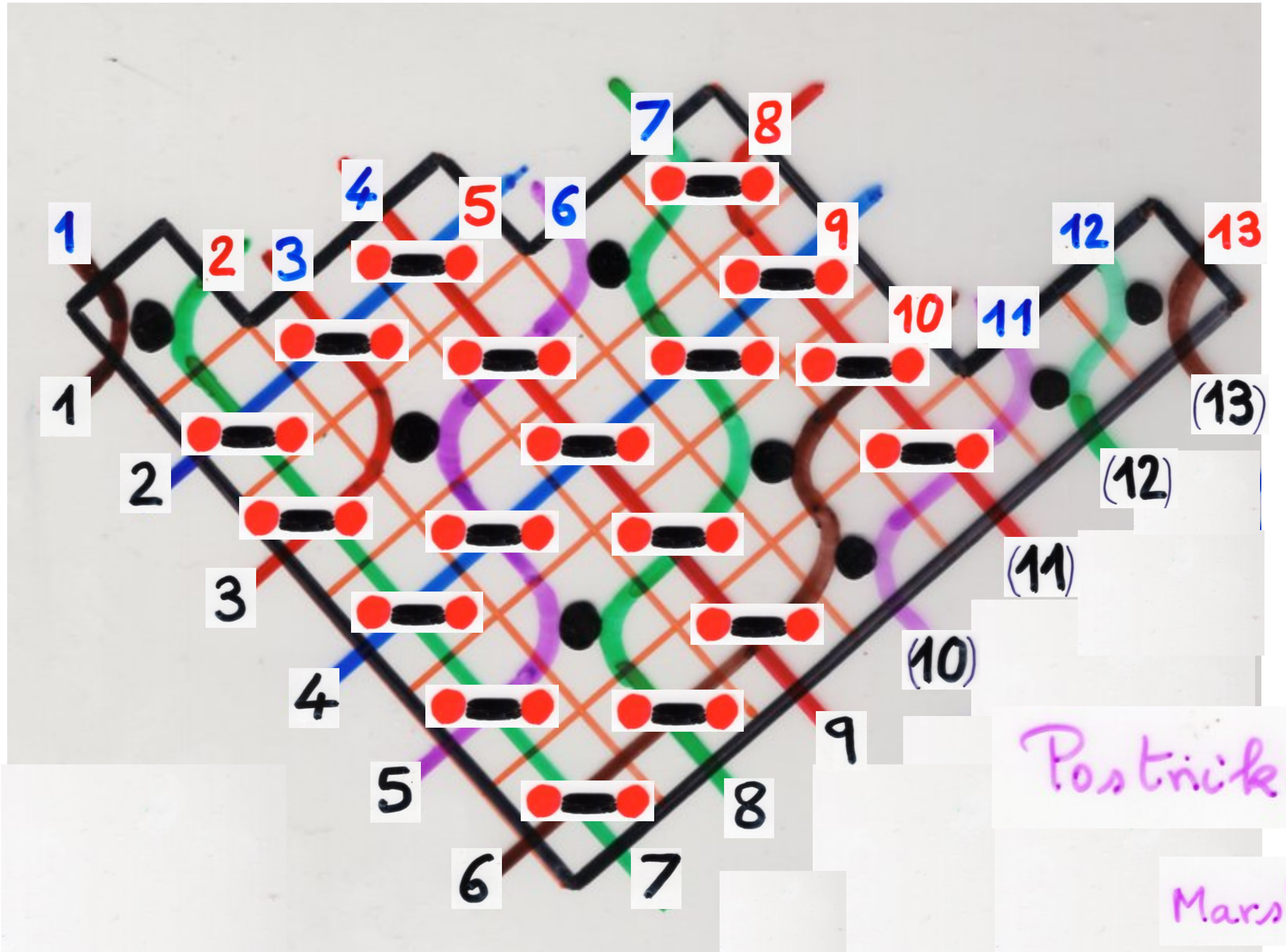


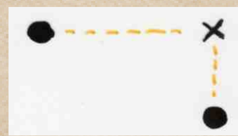
Postnikov (2006)

Marsh-Rietsch (2003)

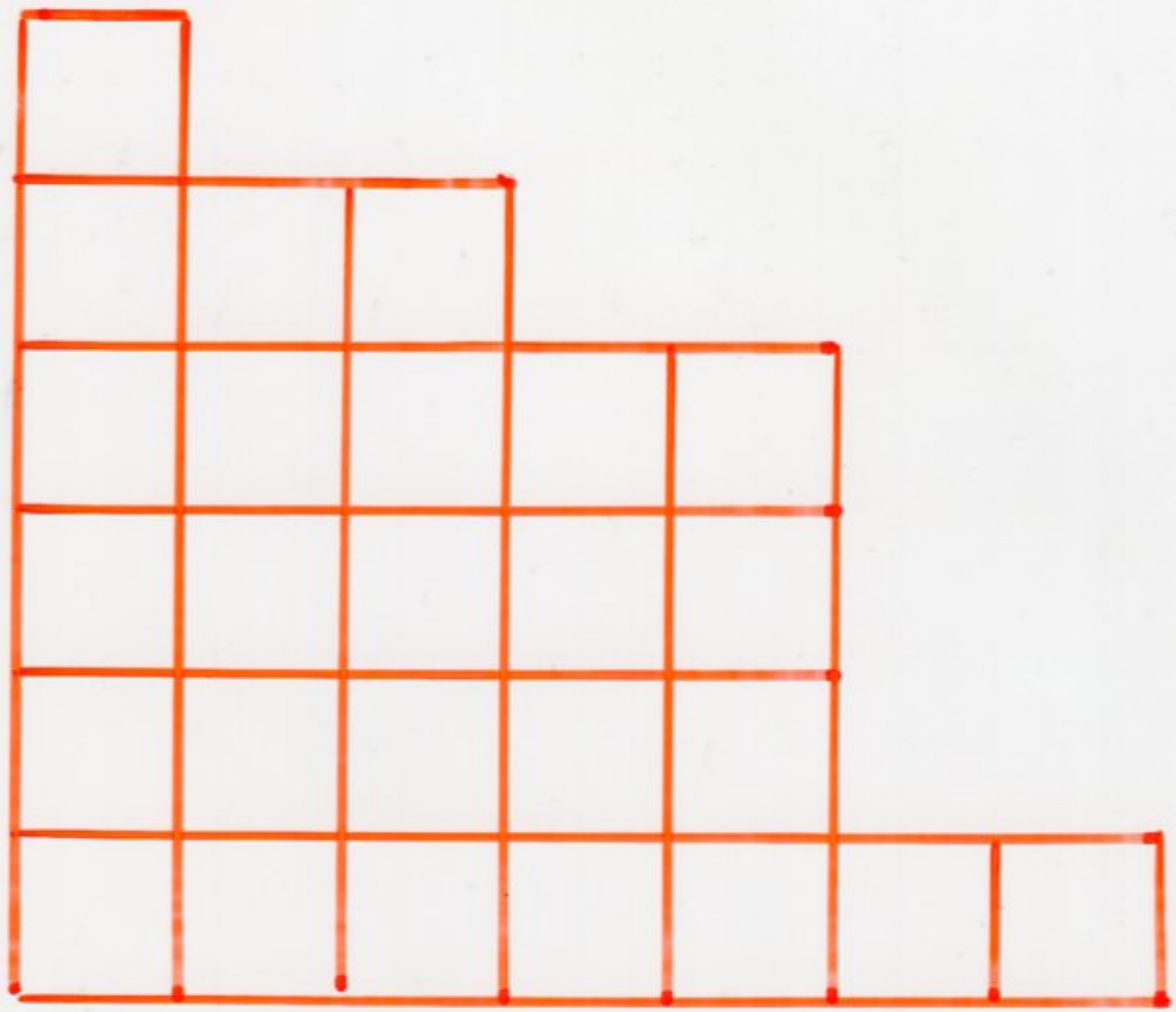
1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
1	5	3	9	6	10	2	7	4	11	8	12	13

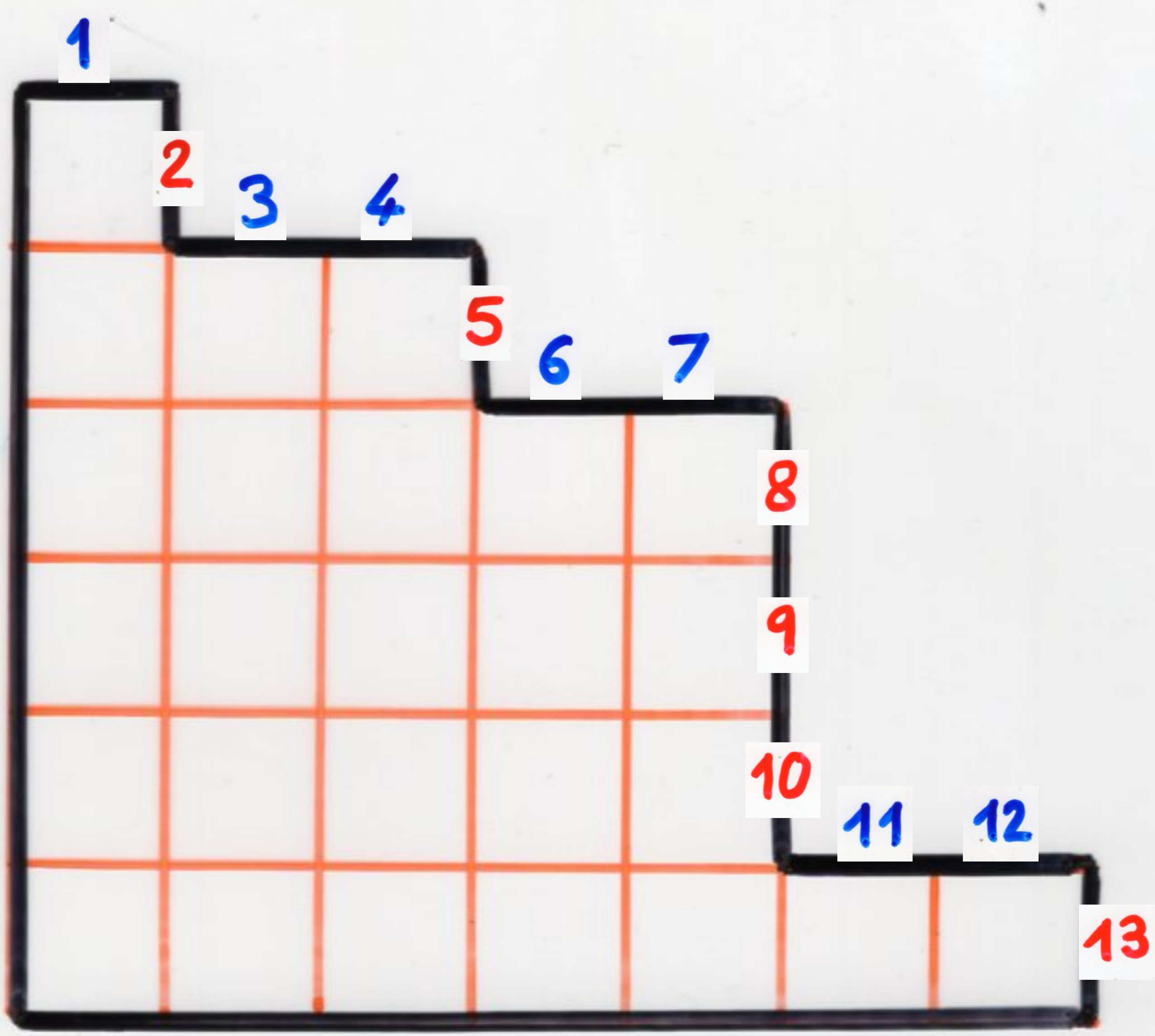
Complements

Postnikov bijections

decorated permutations ⟷  -diagrams

permutations ⟷ permutation tableaux

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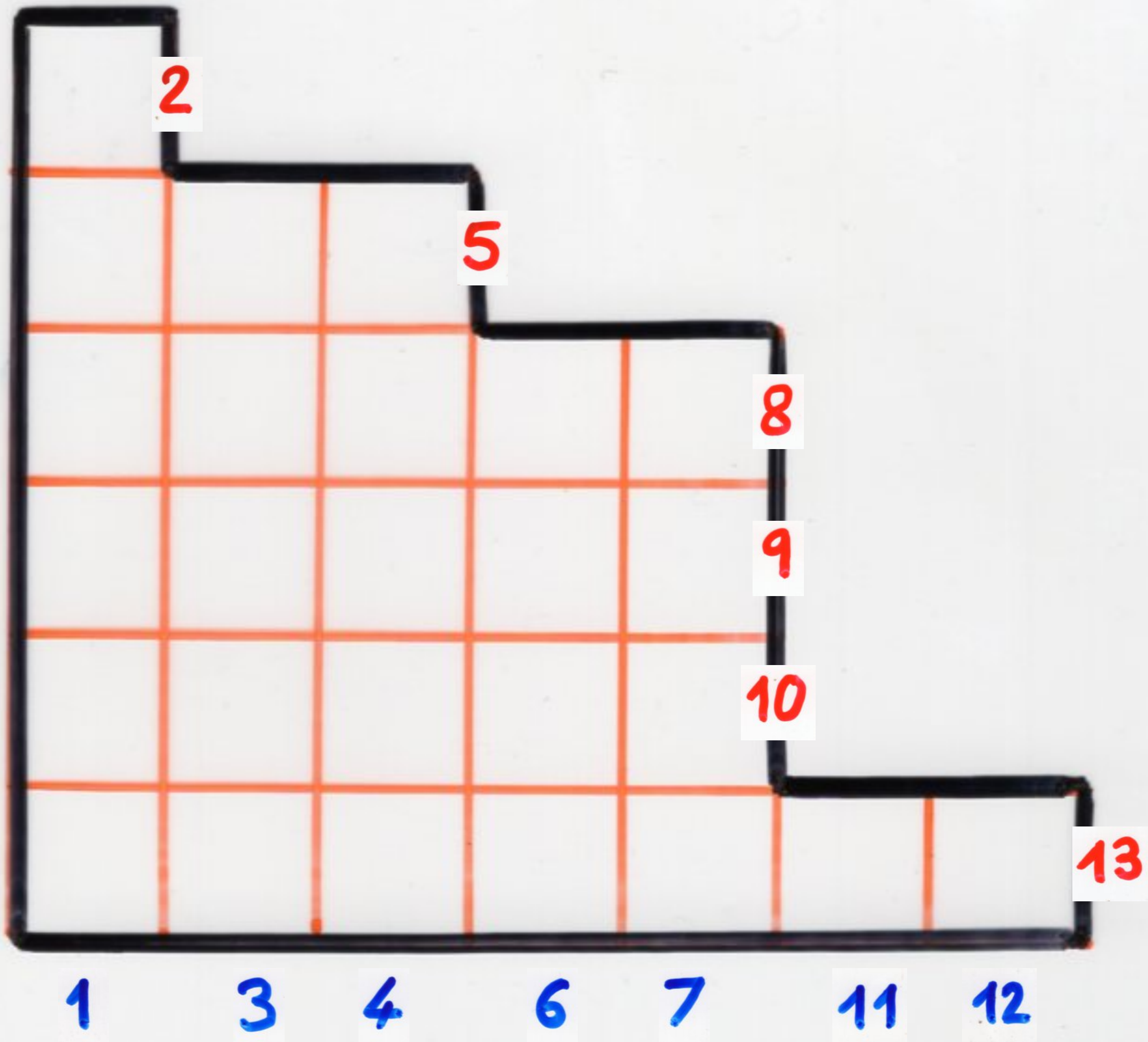

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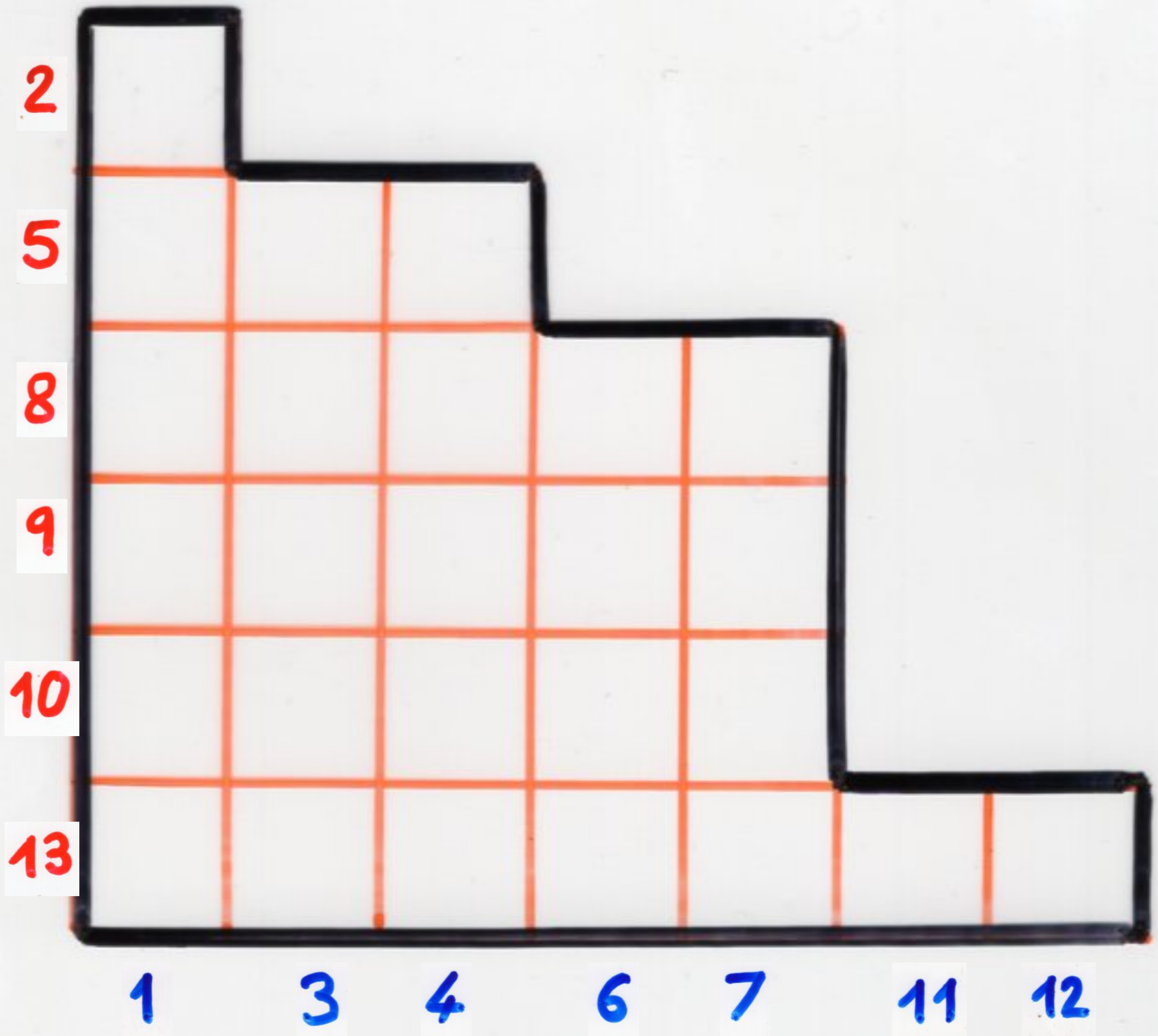
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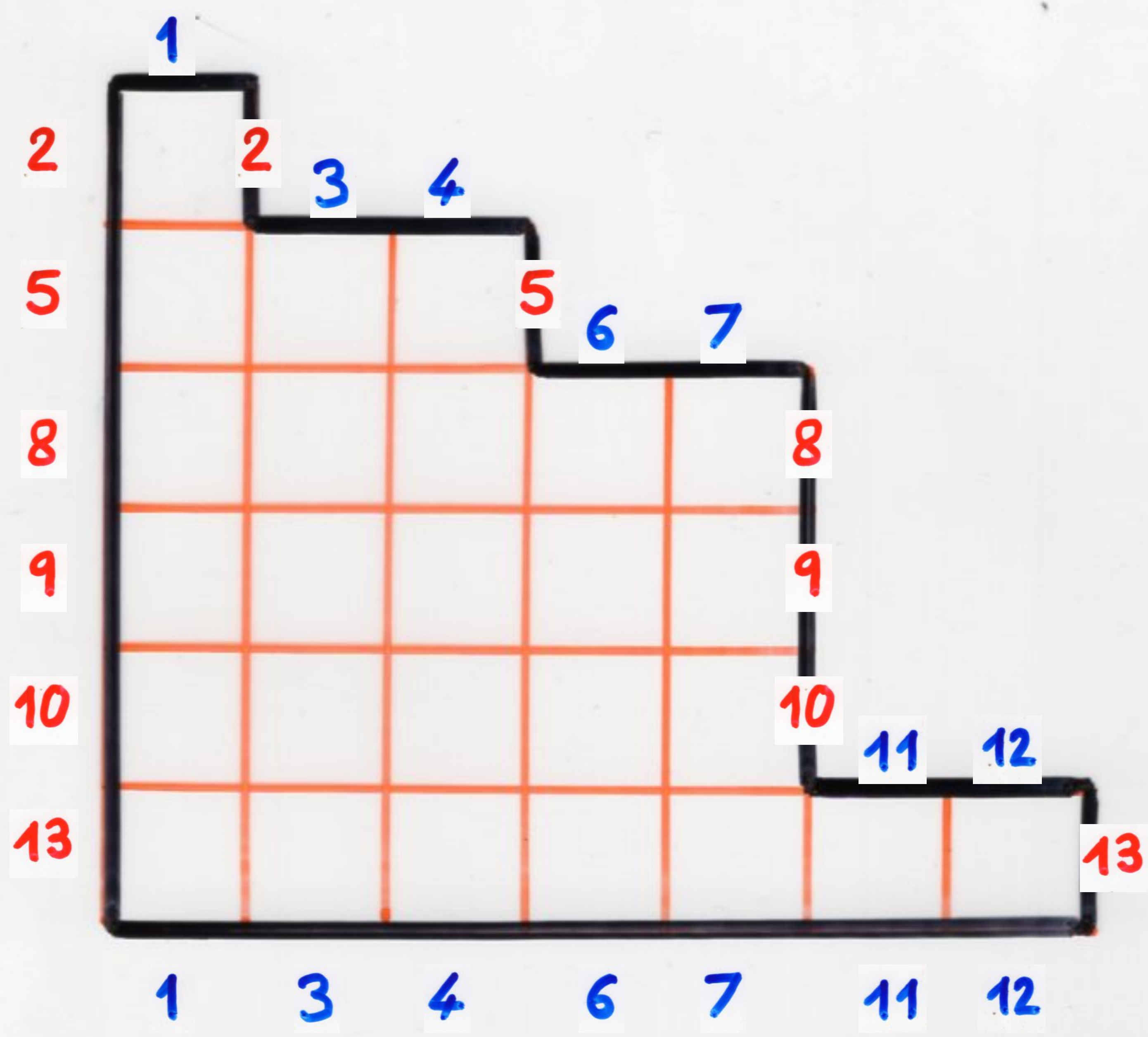
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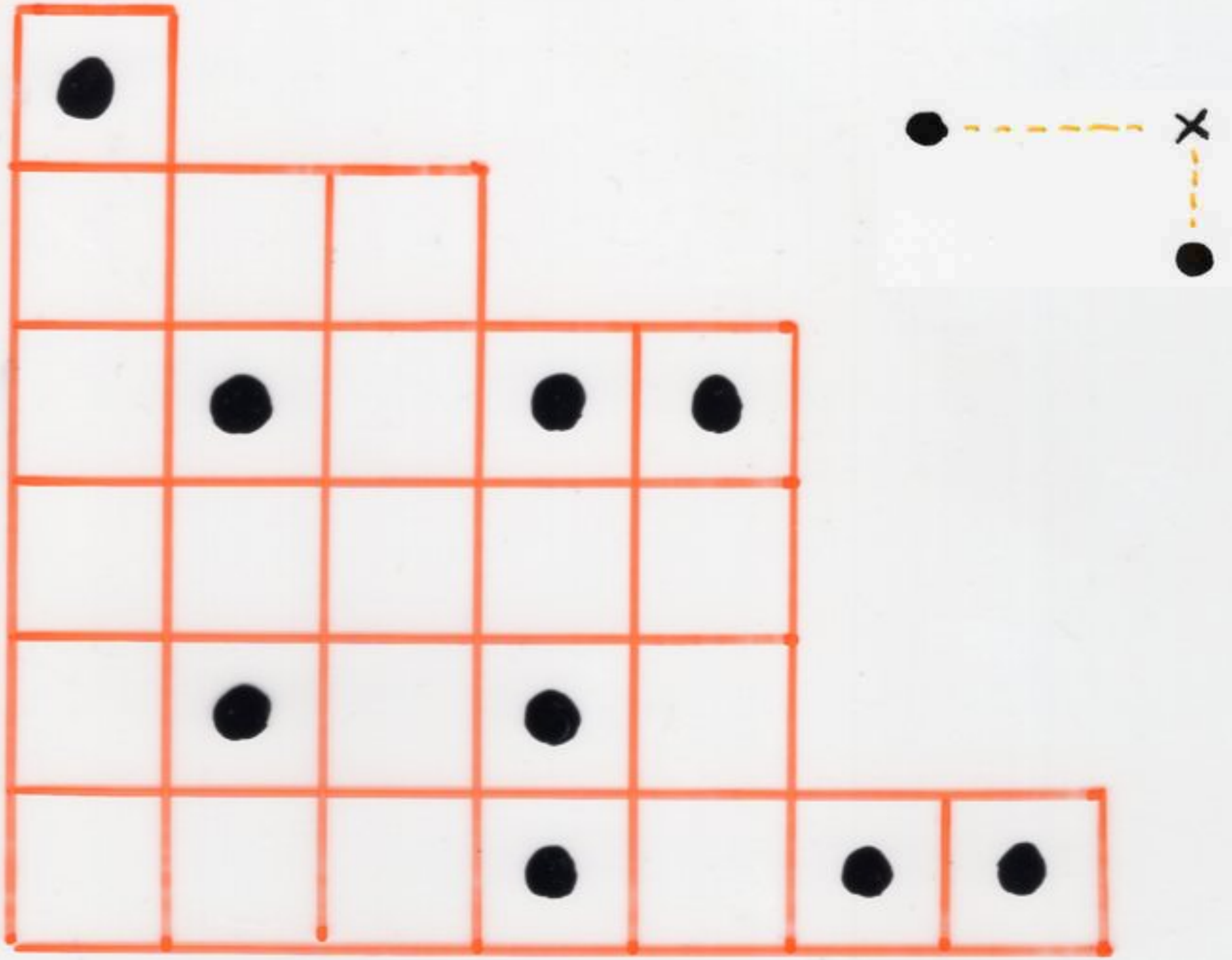
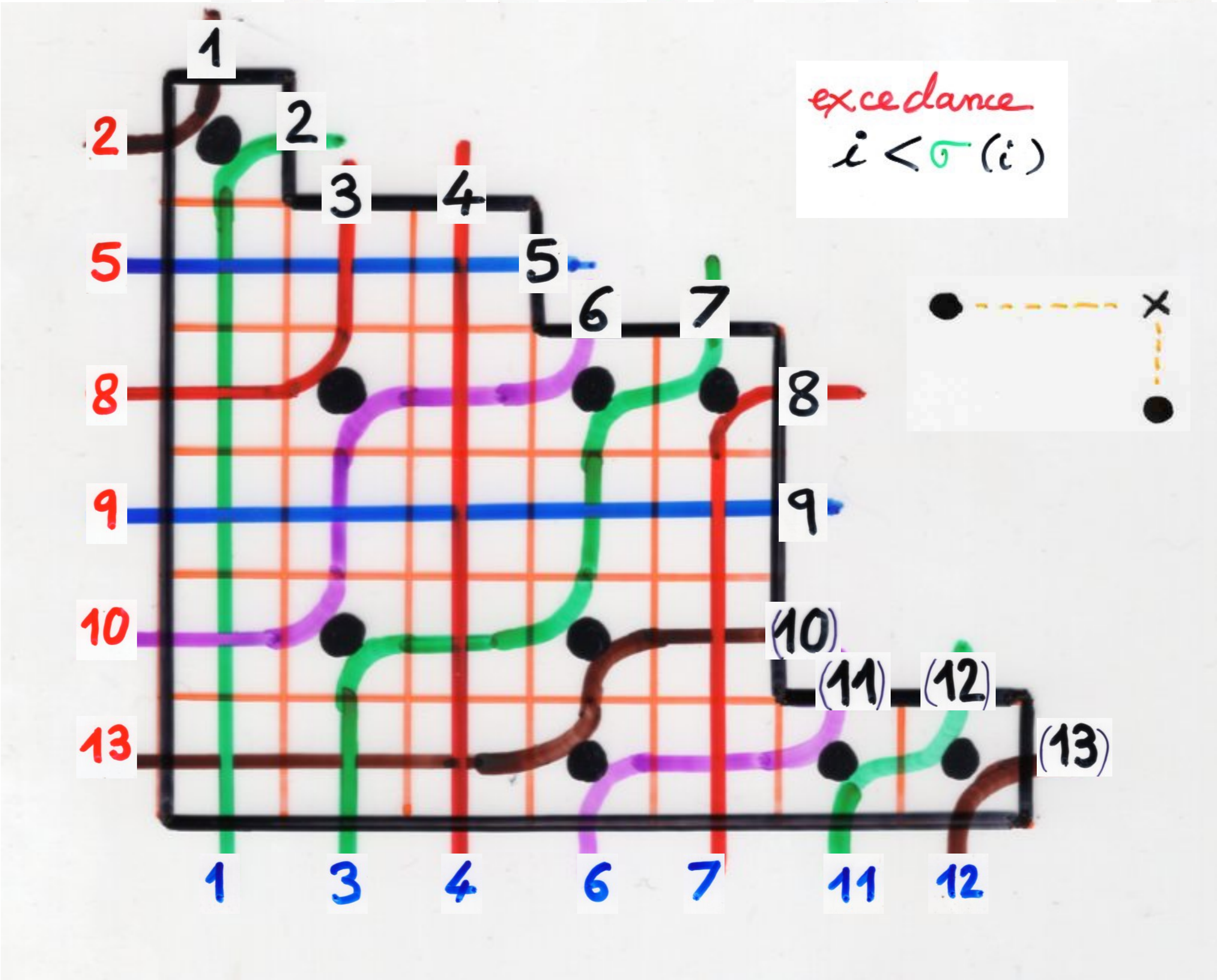


1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
2	1	8	4	5	10	3	7	9	13	6	11	12



1	2	3	4	5	6	7	8	9	(10)	(11)	(12)	(13)
2	1	8	4	5	10	3	7	9	13	6	11	12

D → decorated permutations

Г-diagrams — exceedance set $I(\lambda)$

excedance

$i < \sigma(i)$

bijection

permutations ⟷ permutation tableaux

Postnikov (2006)

Steingrimsson, Williams (2005, 2007)

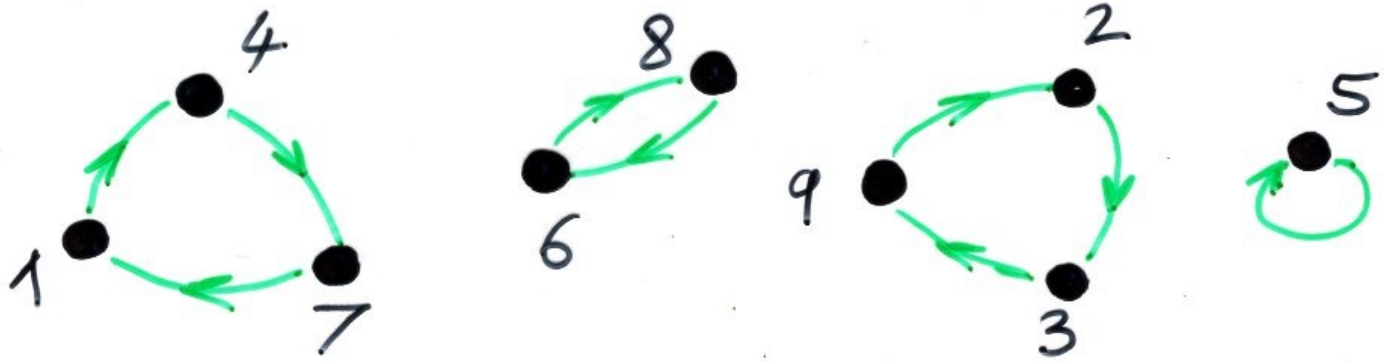
exceedances

and

Genocchi shape of a permutation

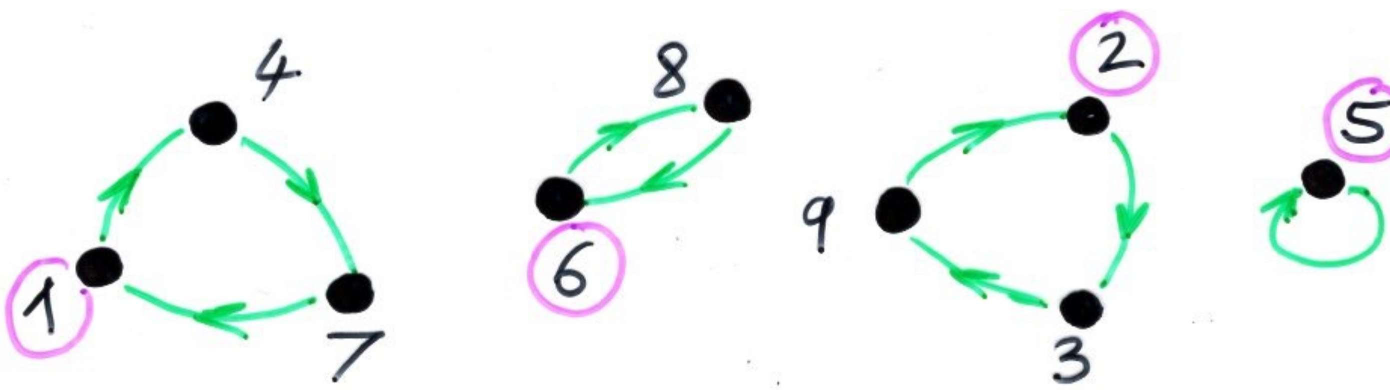
a classical bijection very classic !

σ cycles notation $\xrightarrow{f}$ word $\tau = f(\sigma)$



a classical bijection very classic !

σ cycles notation $\xrightarrow{f}$ word $\tau = f(\sigma)$



τ = / 6 8 / 5 / 2 3 9 / 1 4 7

excedance

$i < \sigma(i)$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

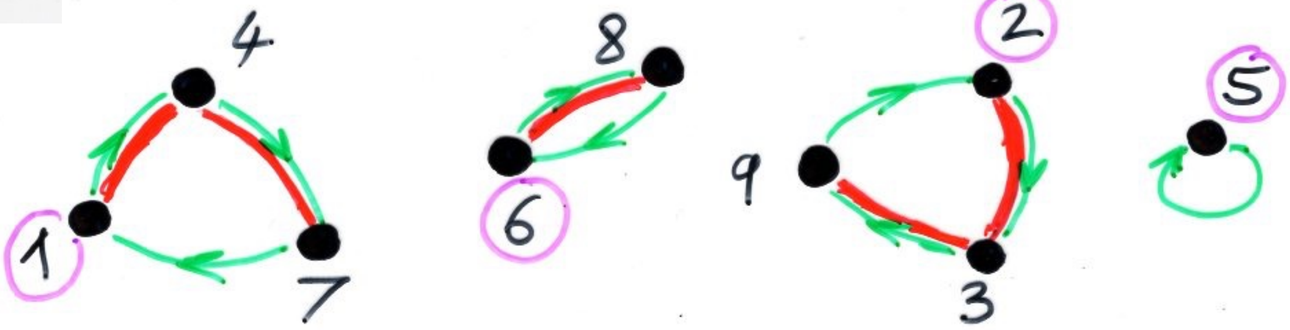
excedance

$$i < \sigma(i)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 9 & 7 & 5 & 8 & 1 & 6 & 2 \end{pmatrix}$$

"excedance" sequence of σ

1	2	3	4	5	6	7	8	(9)
a	a	a	a	d	a	d	d	



Genocchi sequence of τ

$$\tau = / 6 - 8 / 5 / 2 - 3 - 9 / 1 - 4 - 7$$

The direct bijection

subexceedant functions ⟷ Tree-like tableaux

Two bijections

- from a combinatorial representation of the PASEP algebra (X.V., 2008)

  equivalent to a bijection Corteel, Nadeau (2007)

  (with permutation tableaux)

  Steingrímsson, Williams (2005, 2007)

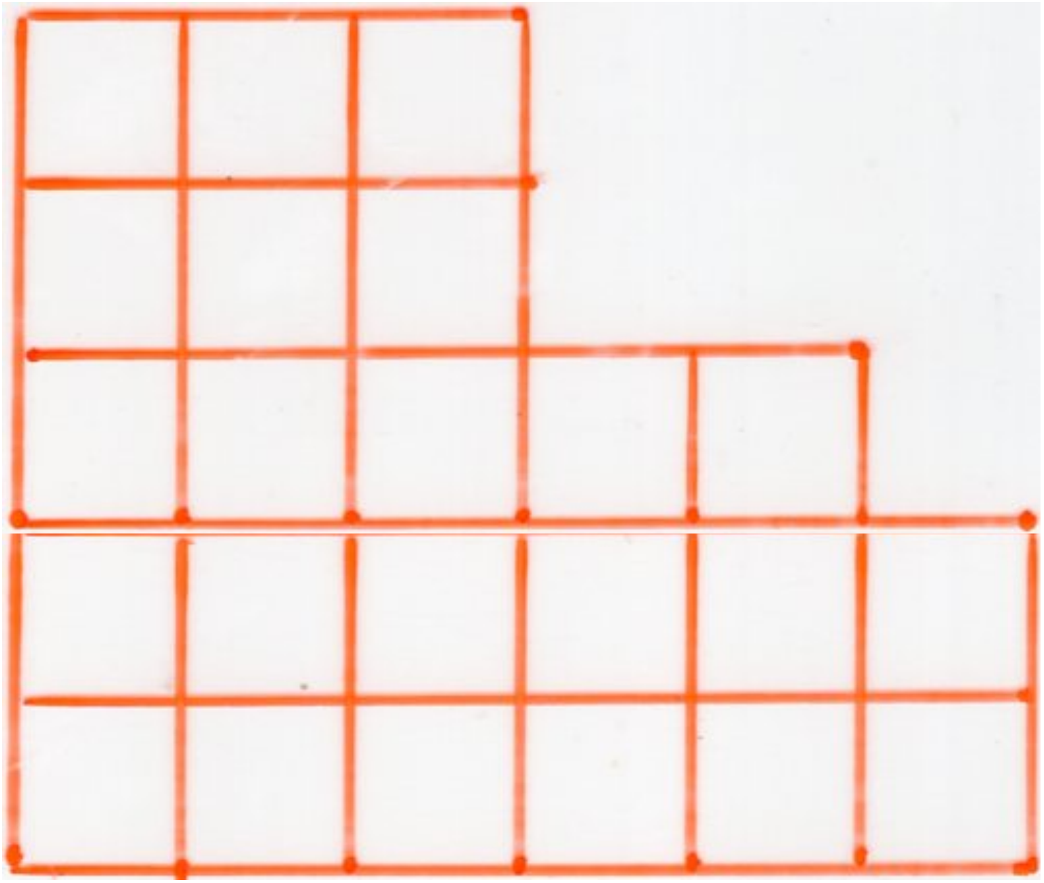
  Postnikov

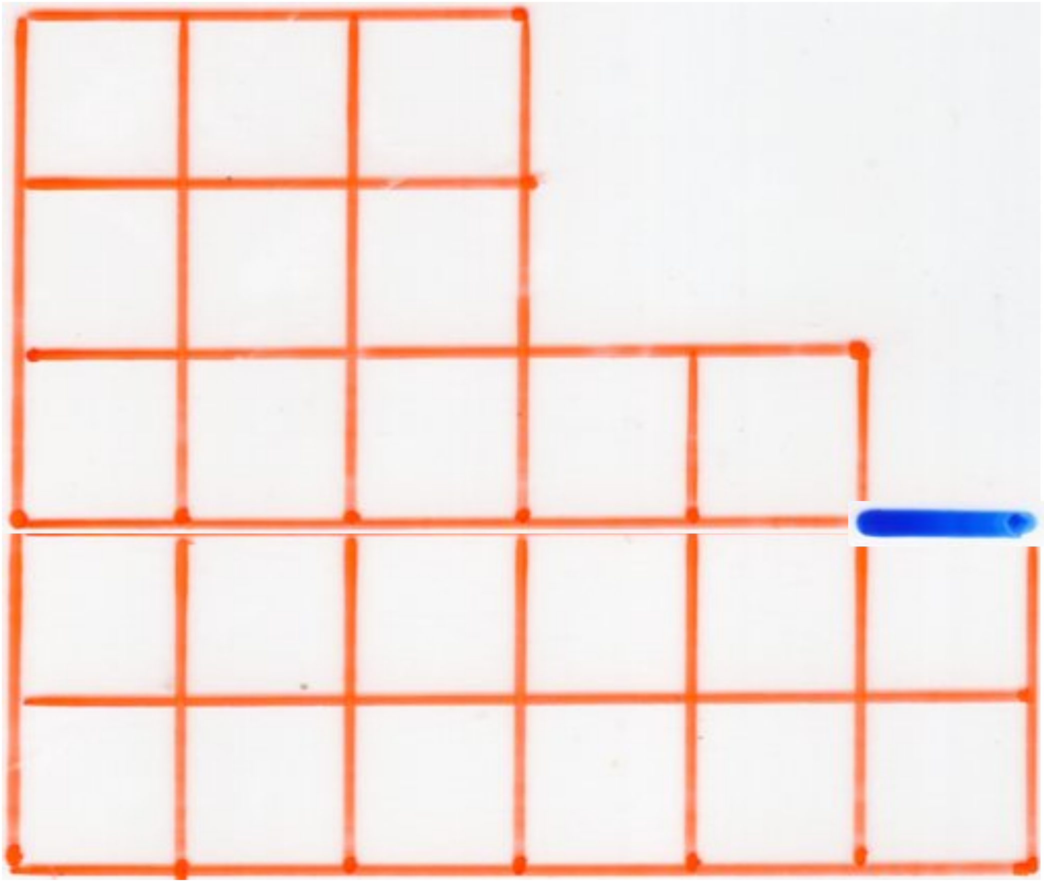
- direct bijection (with tree-like tableaux) Aval, Boussicault, Nadeau (2011)

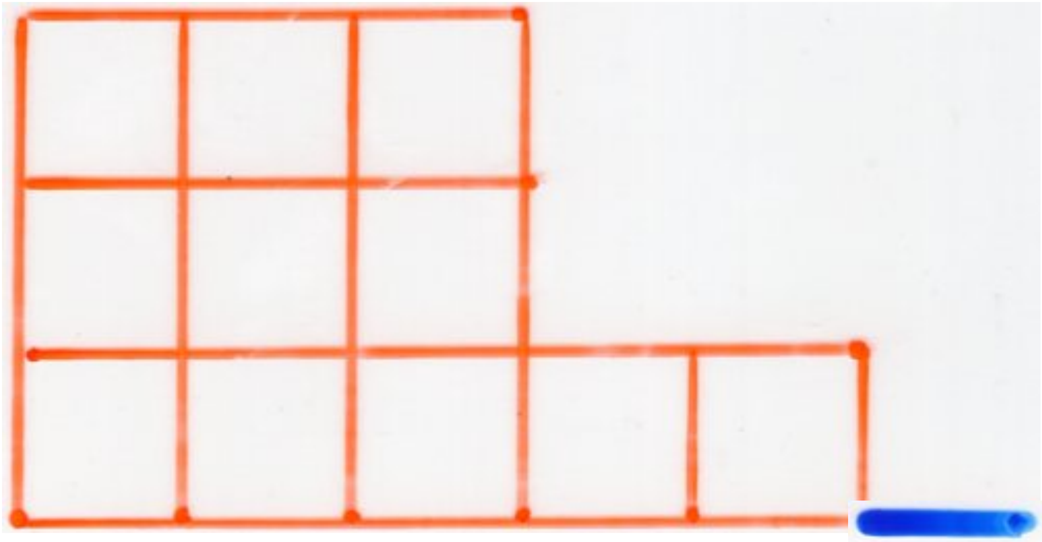
$$\text{tableaux size } (n+1) \longleftrightarrow \left(\text{tableaux size } n,\ 1 \leq i \leq n+1 \right)$$

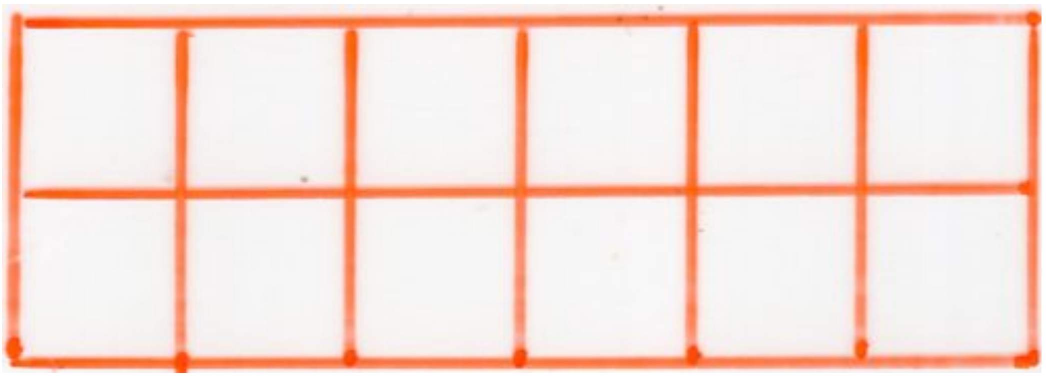
$$(n+1)!$$

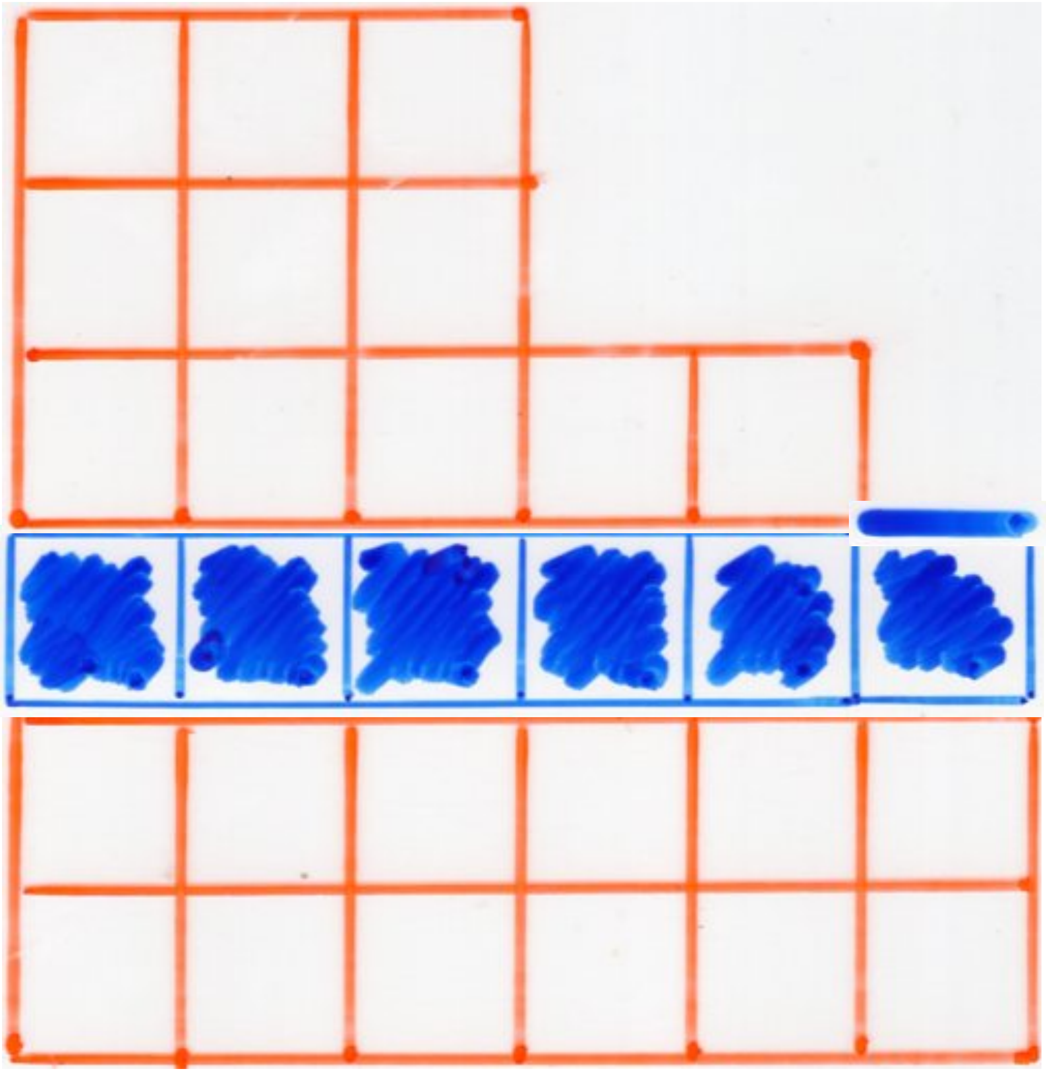
row/column insertion
in a Ferrers diagram

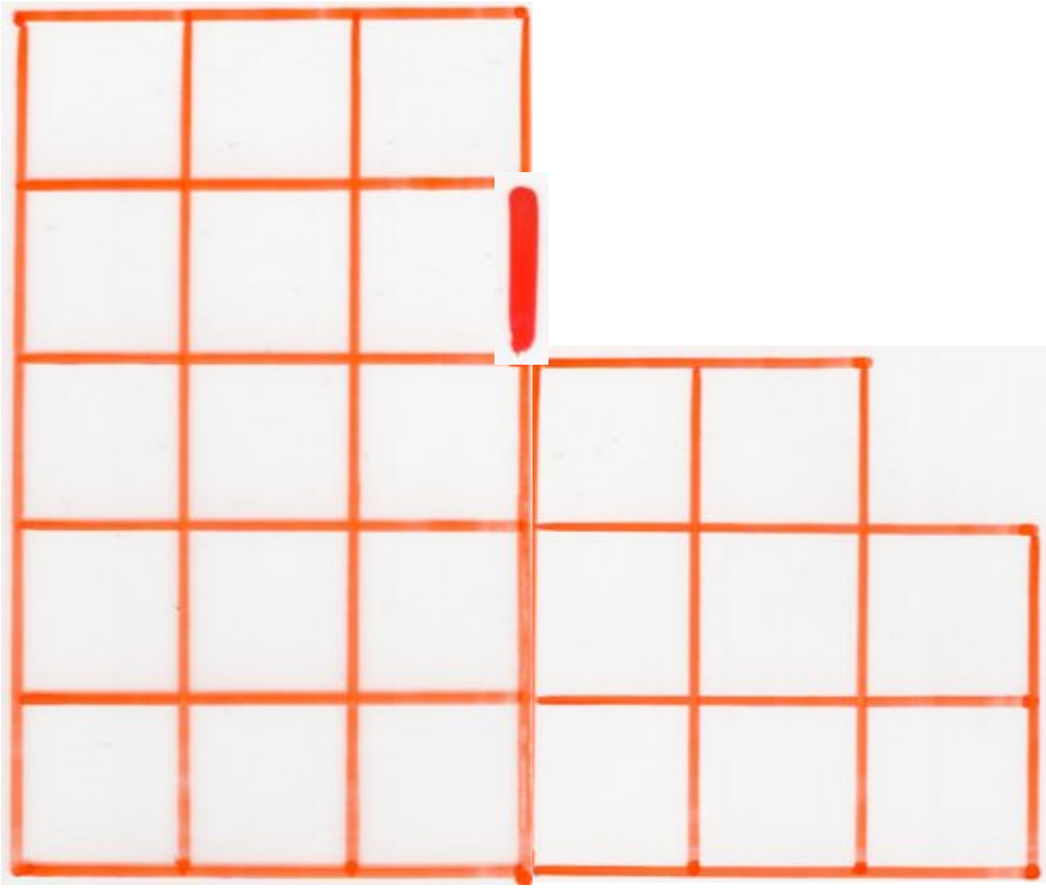


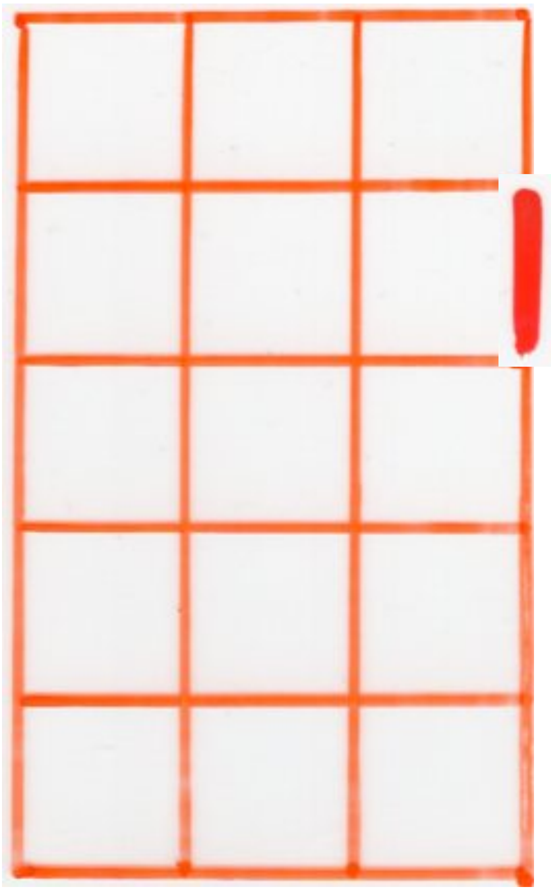


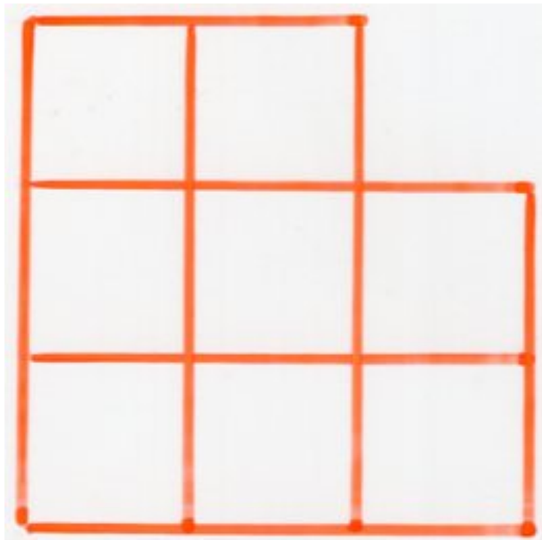


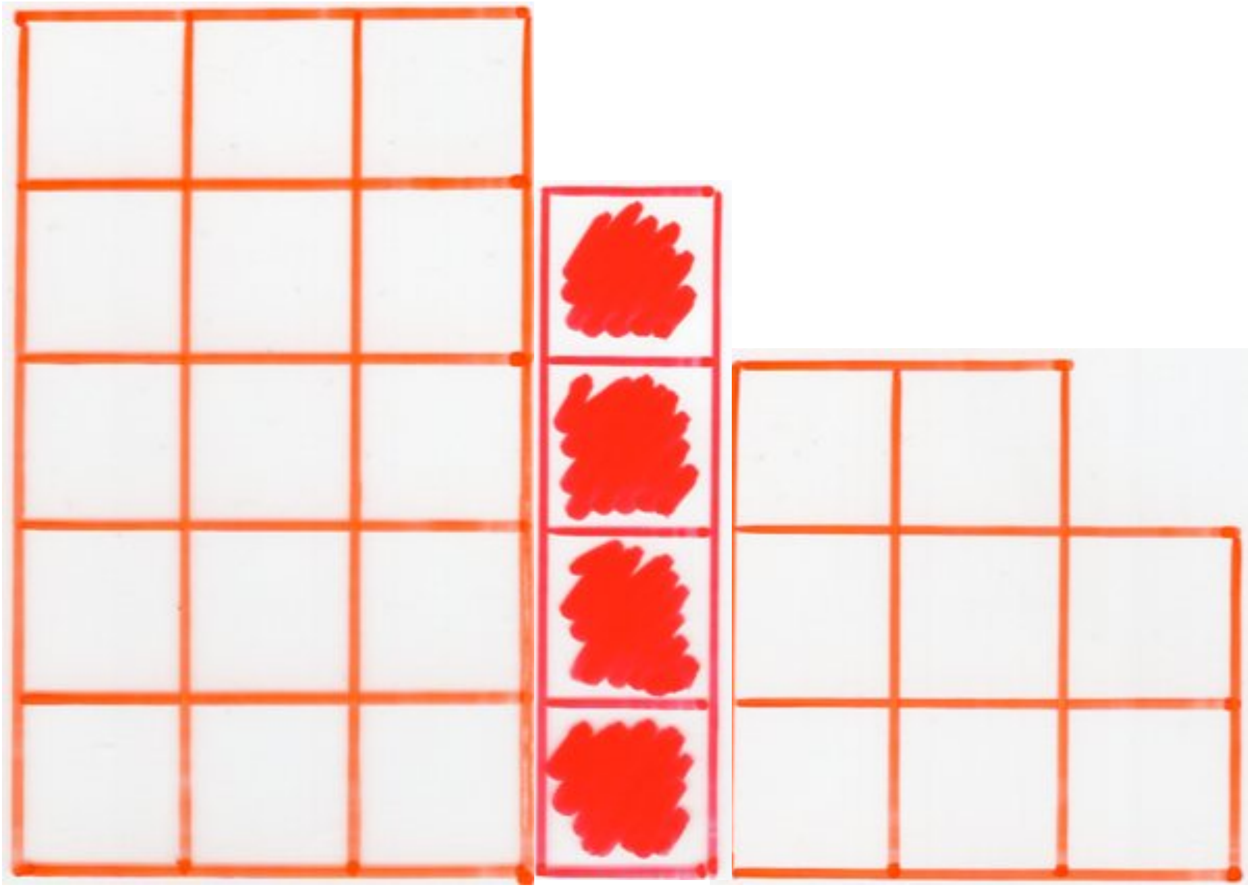




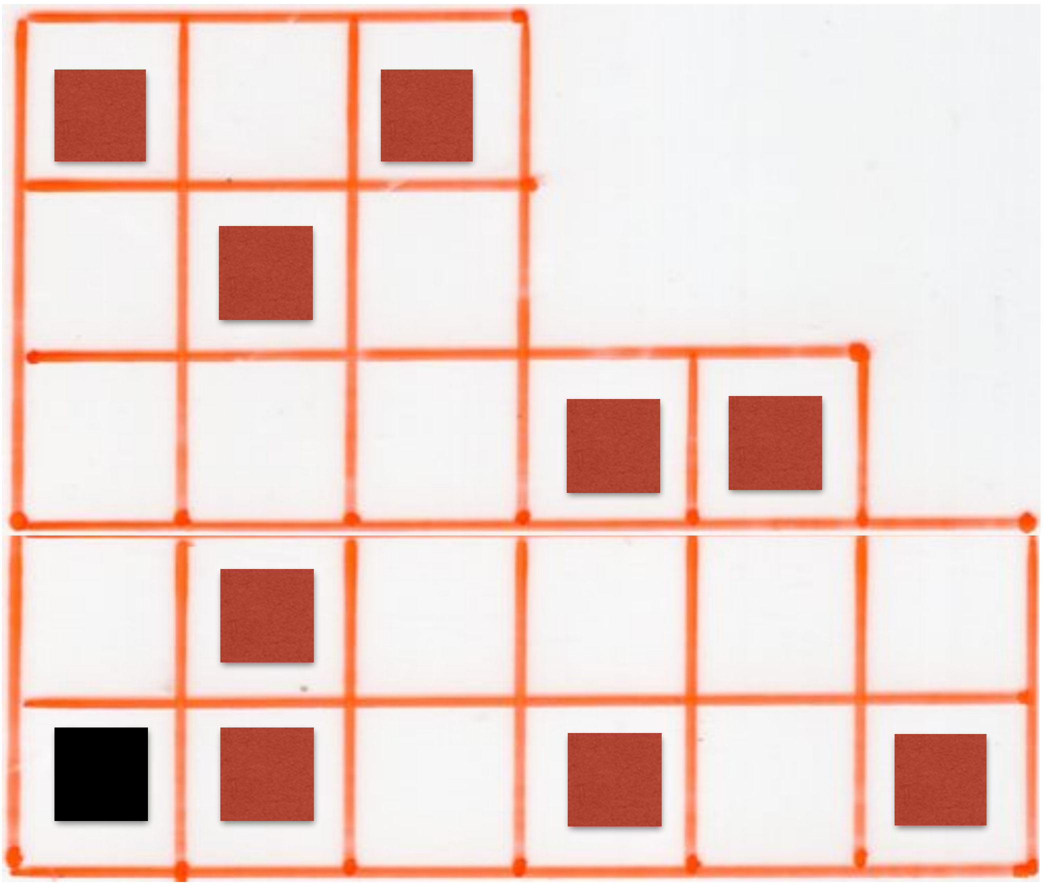


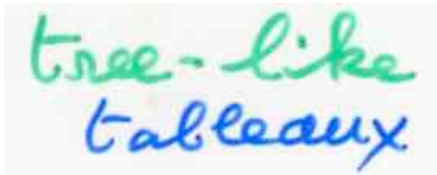


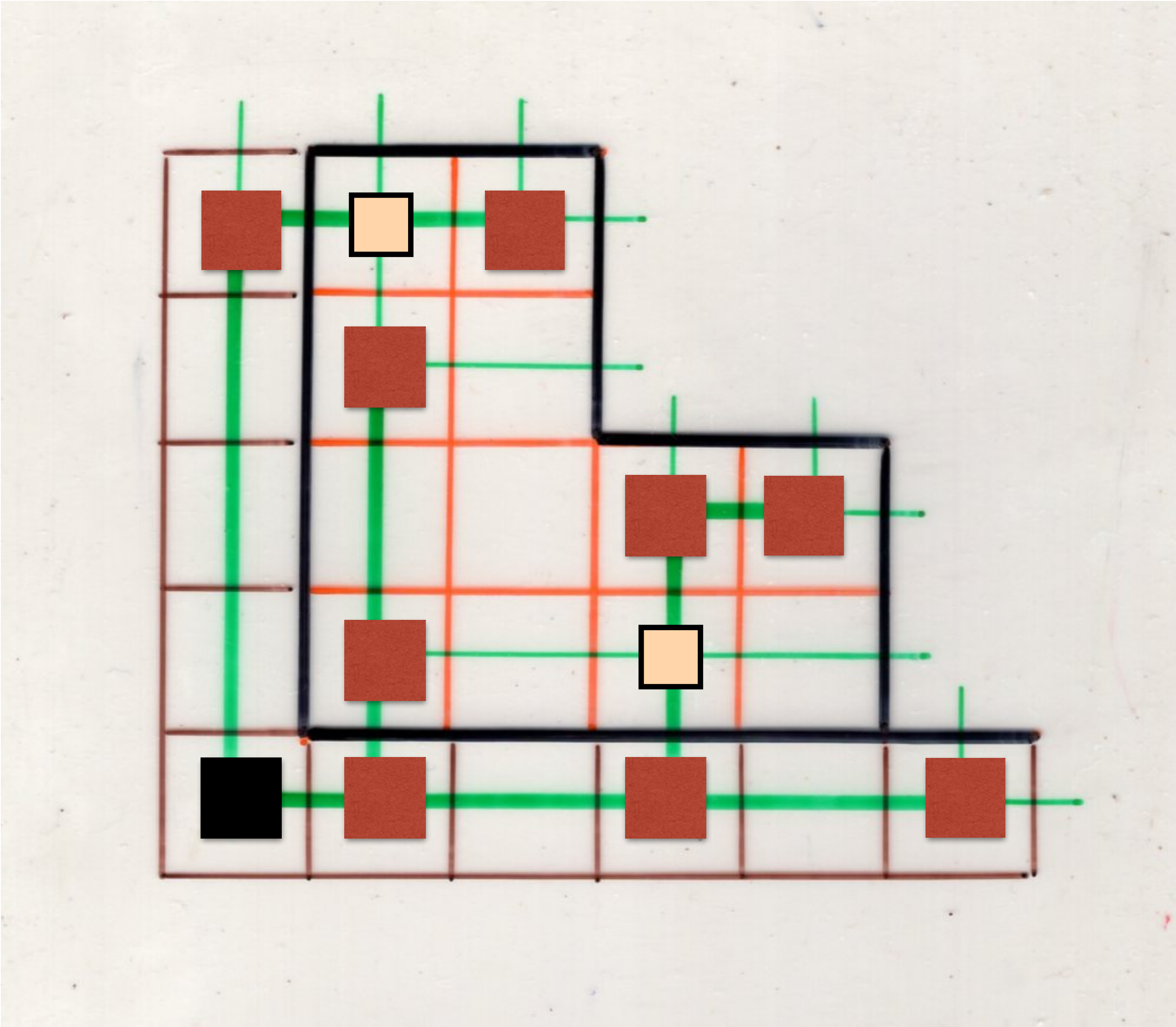


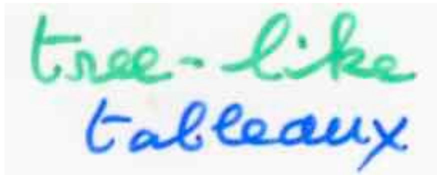


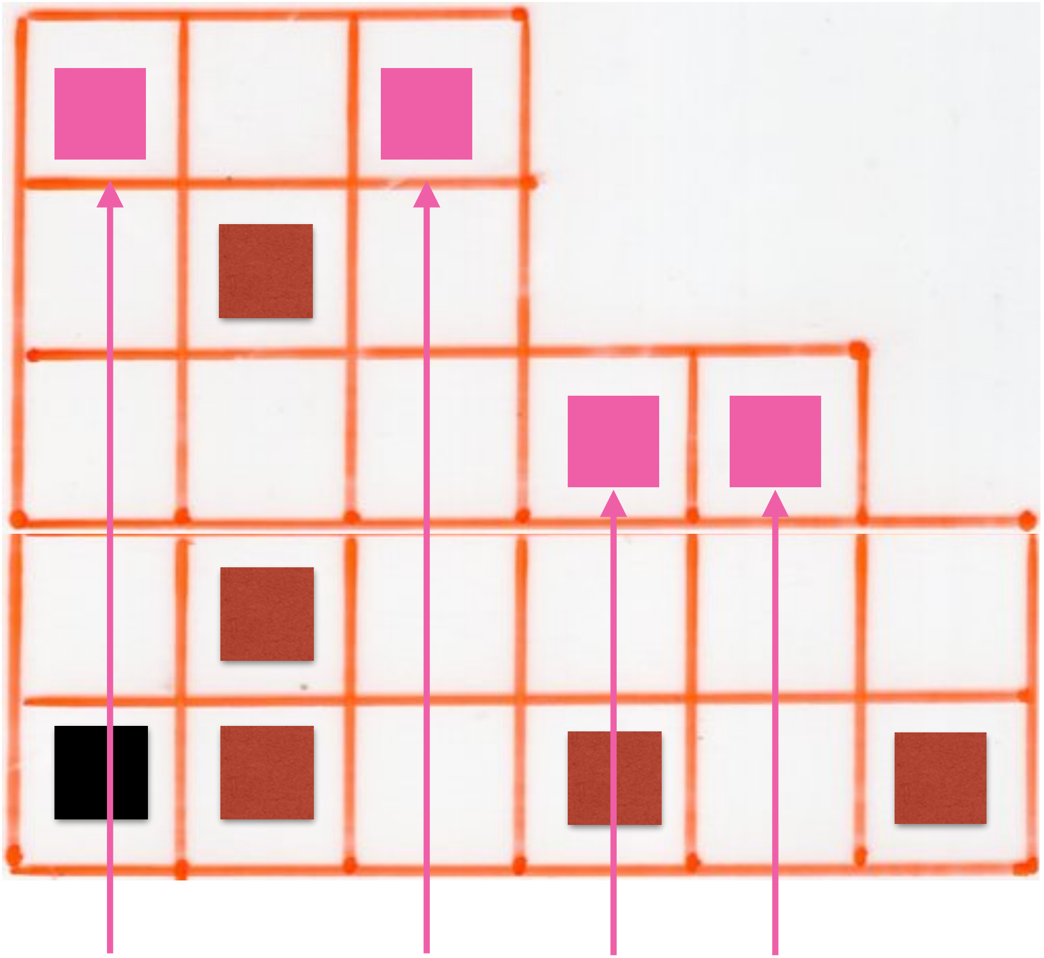
special point
of a tree-like tableau



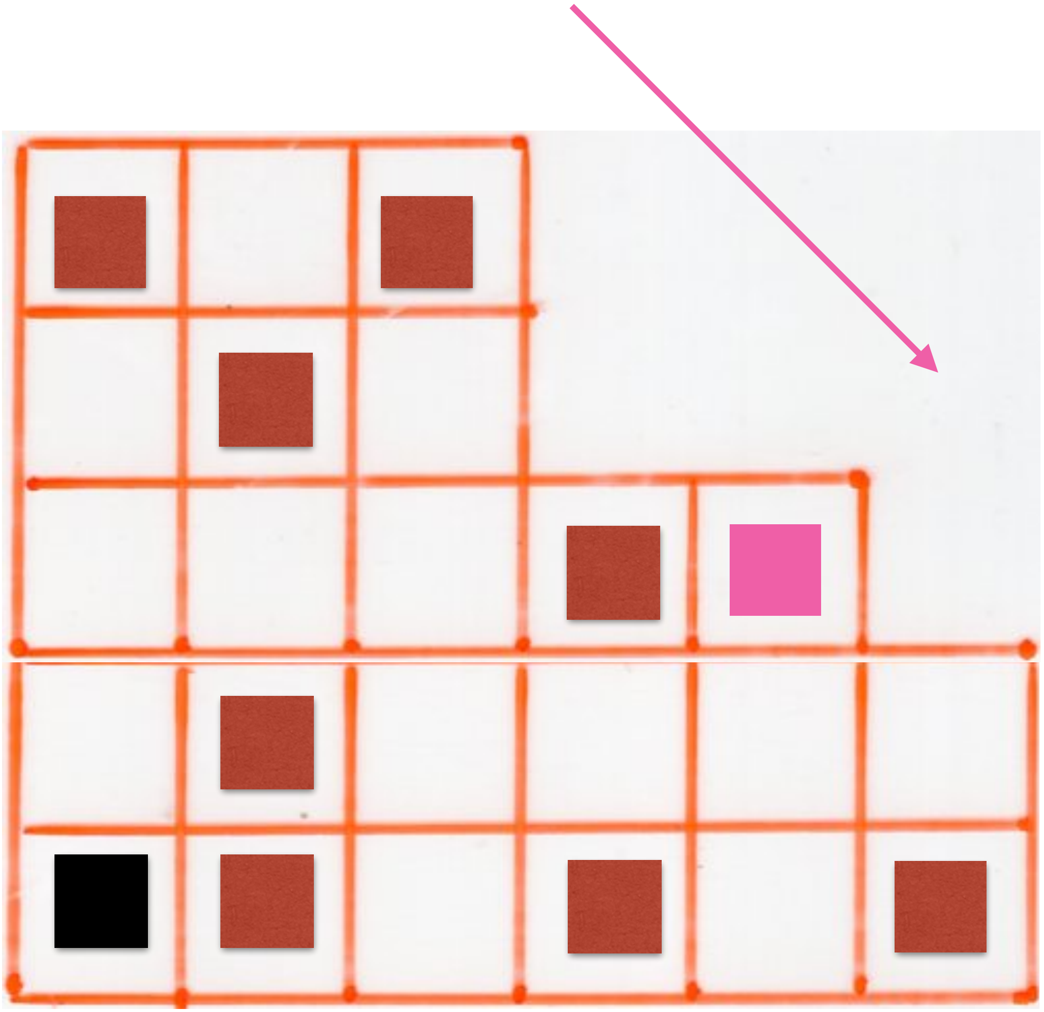

tree-like
tableaux




tree-like
tableaux



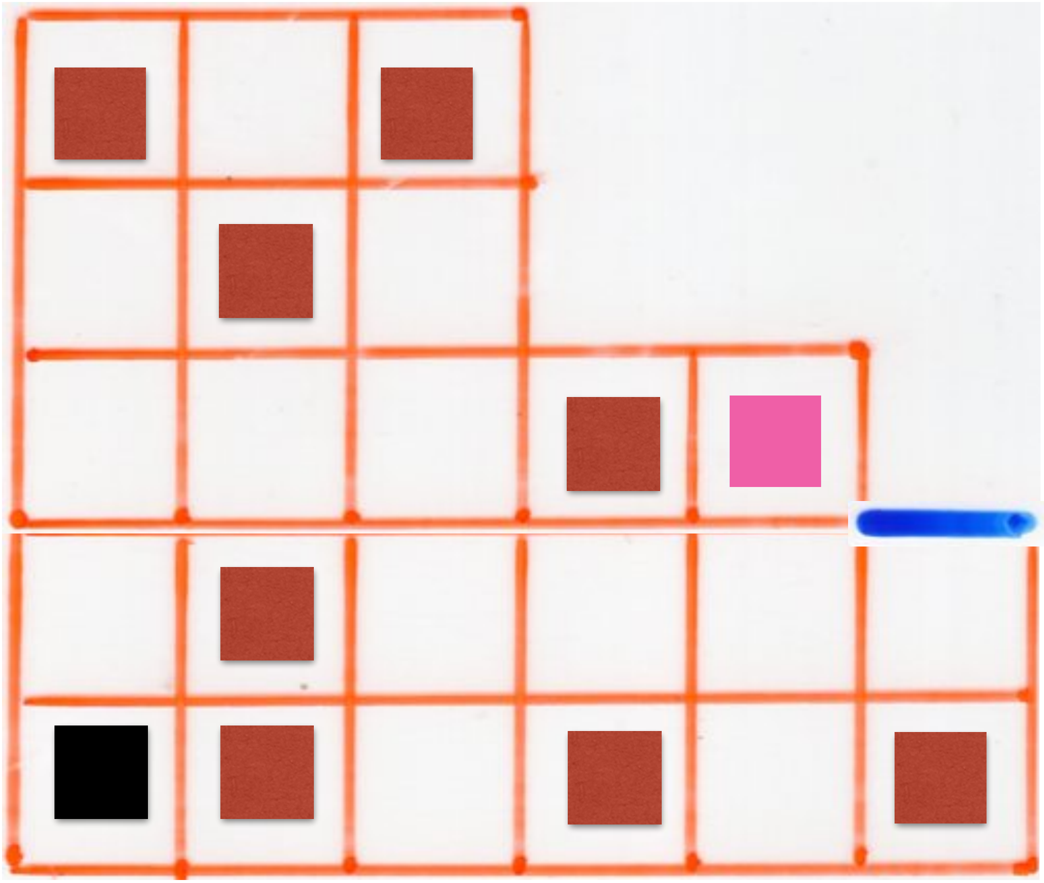
« end point »
in each column



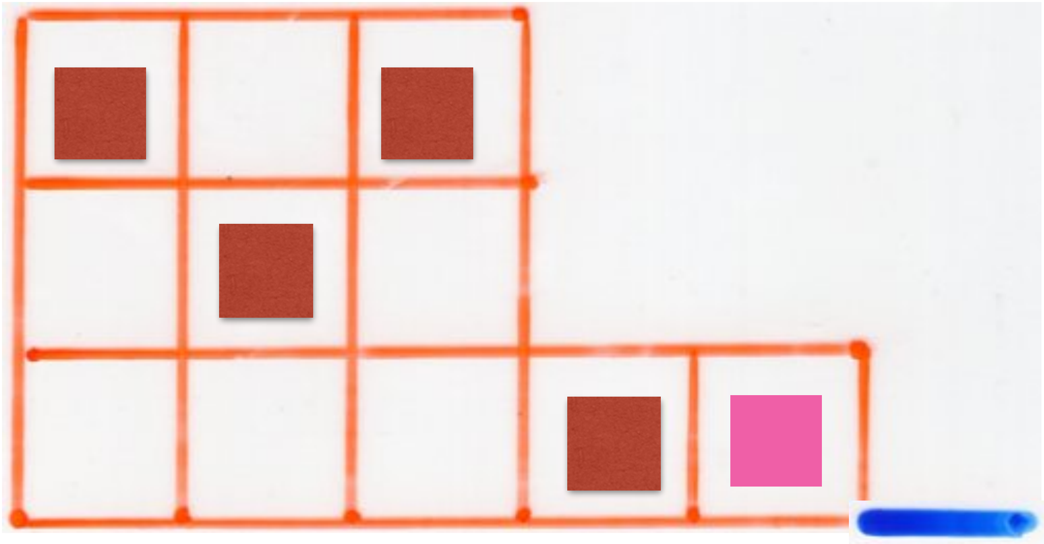
special point
of a tree-like
tableau

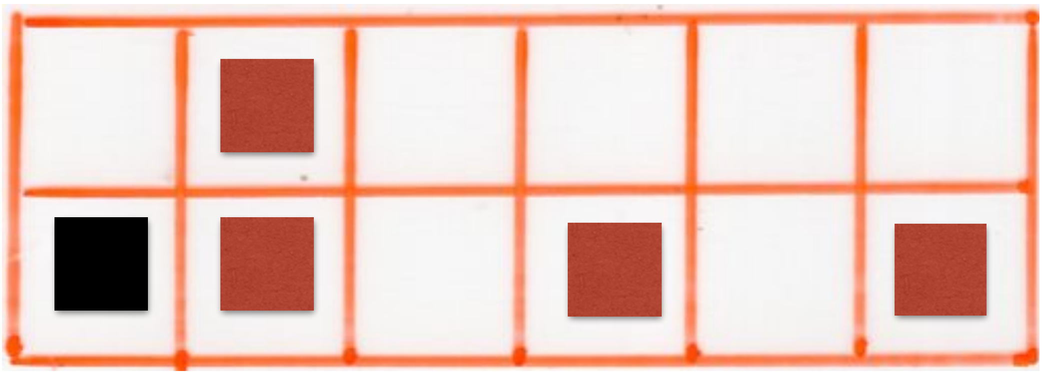
Insertion algorithm
in a tree-like tableau

Example 1

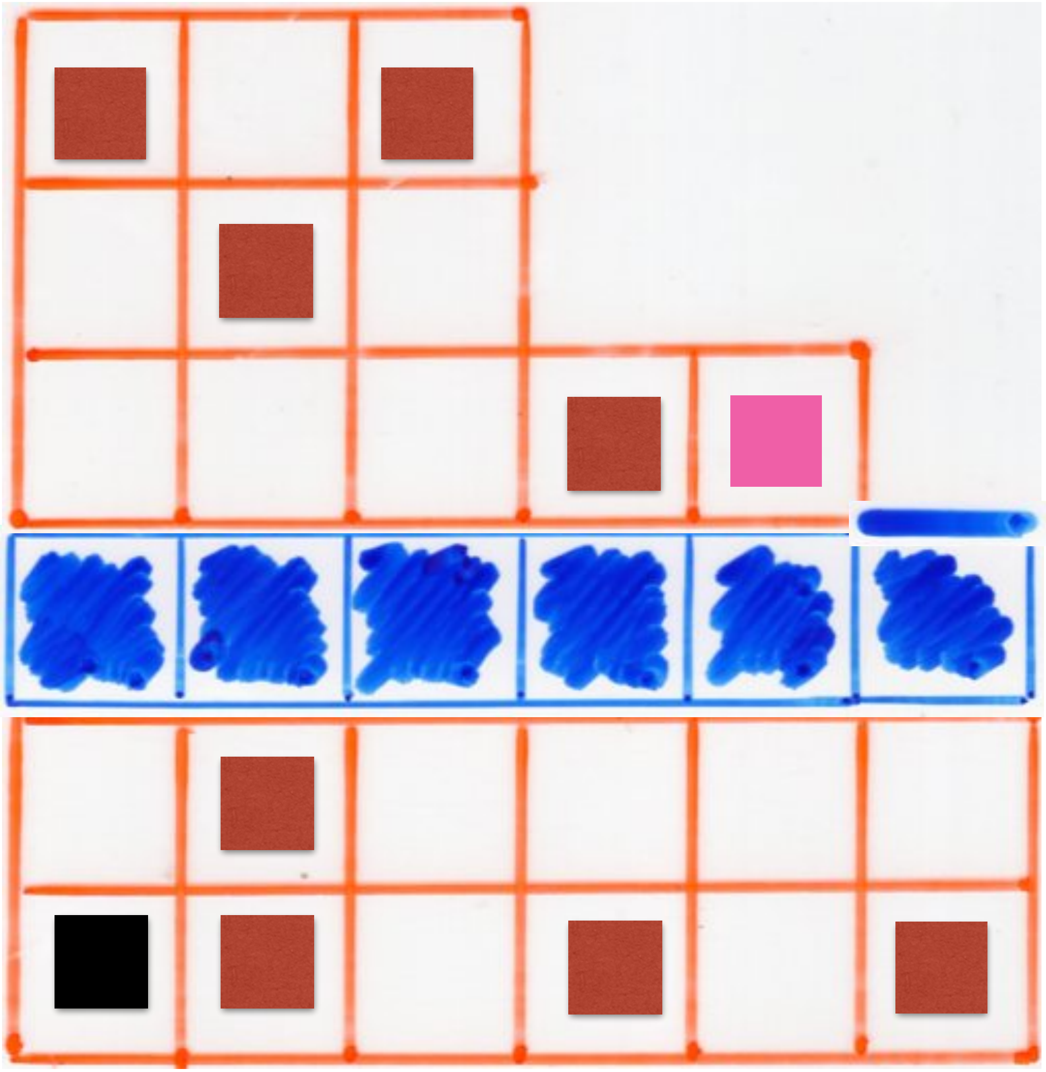


Example 1





Example 1



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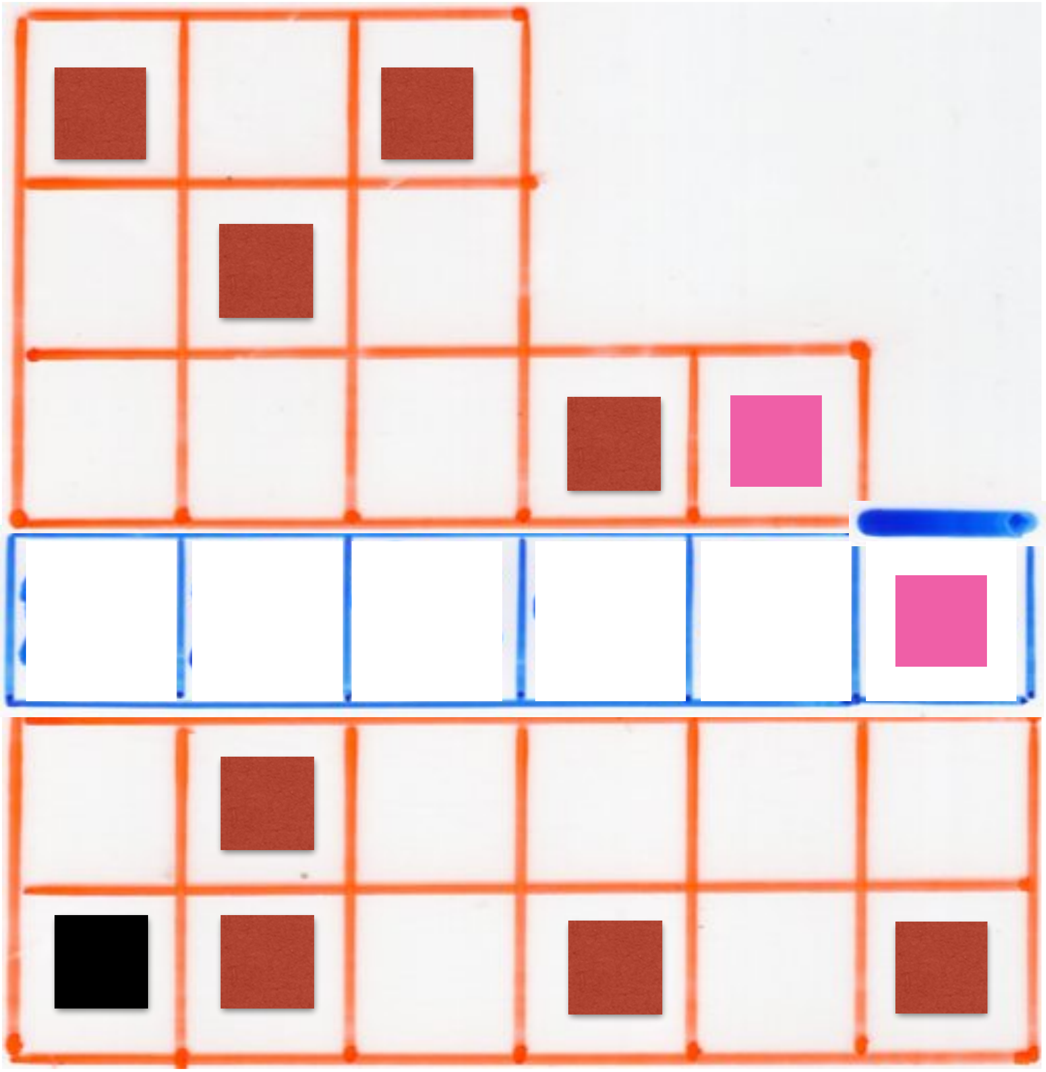
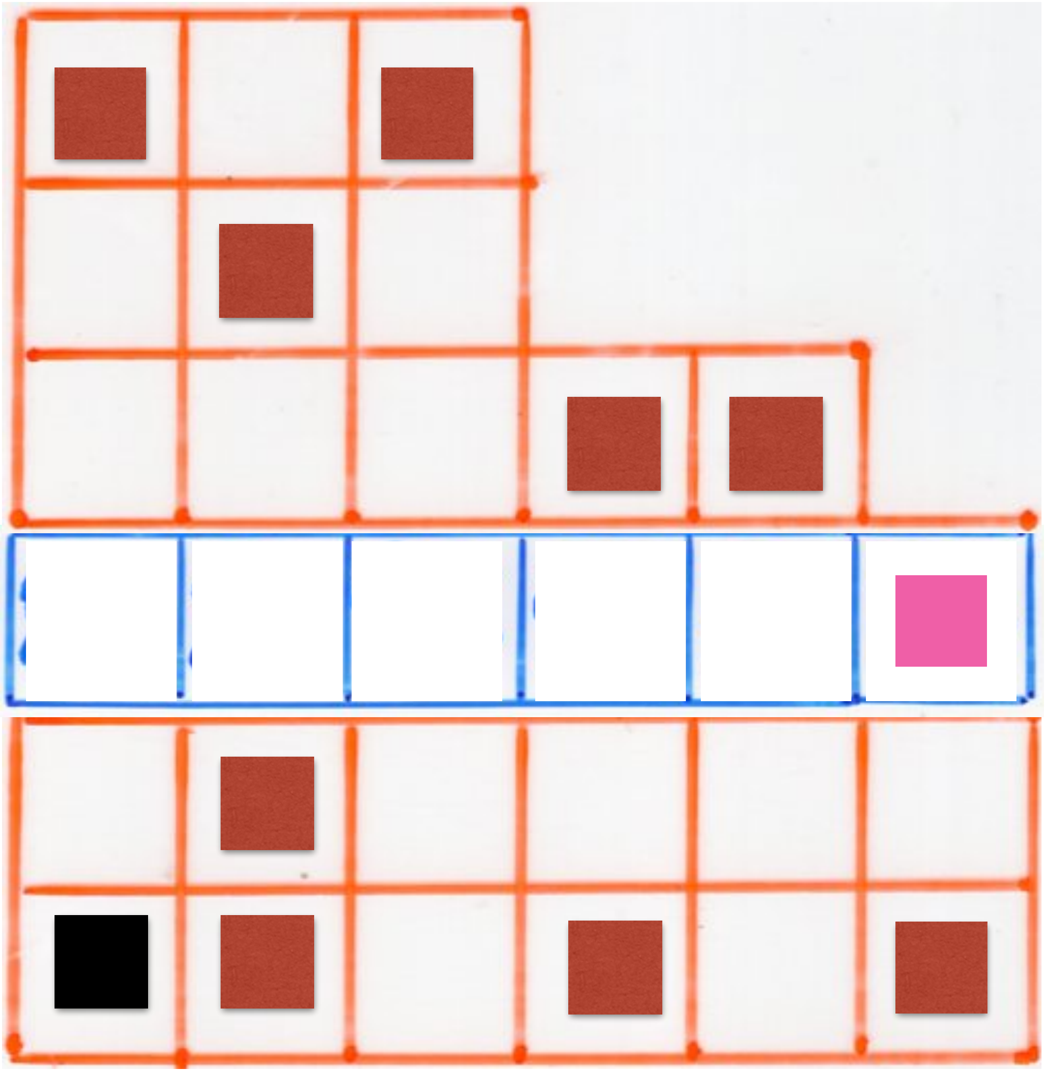
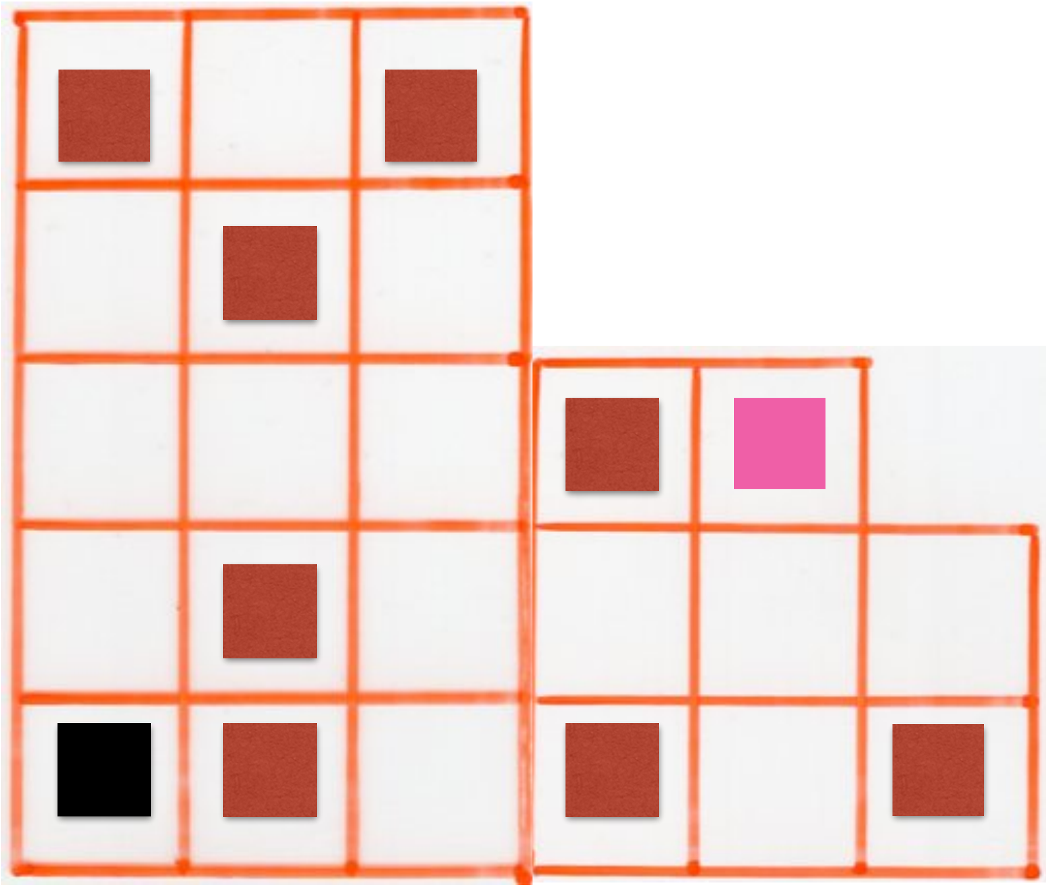


Case (i)

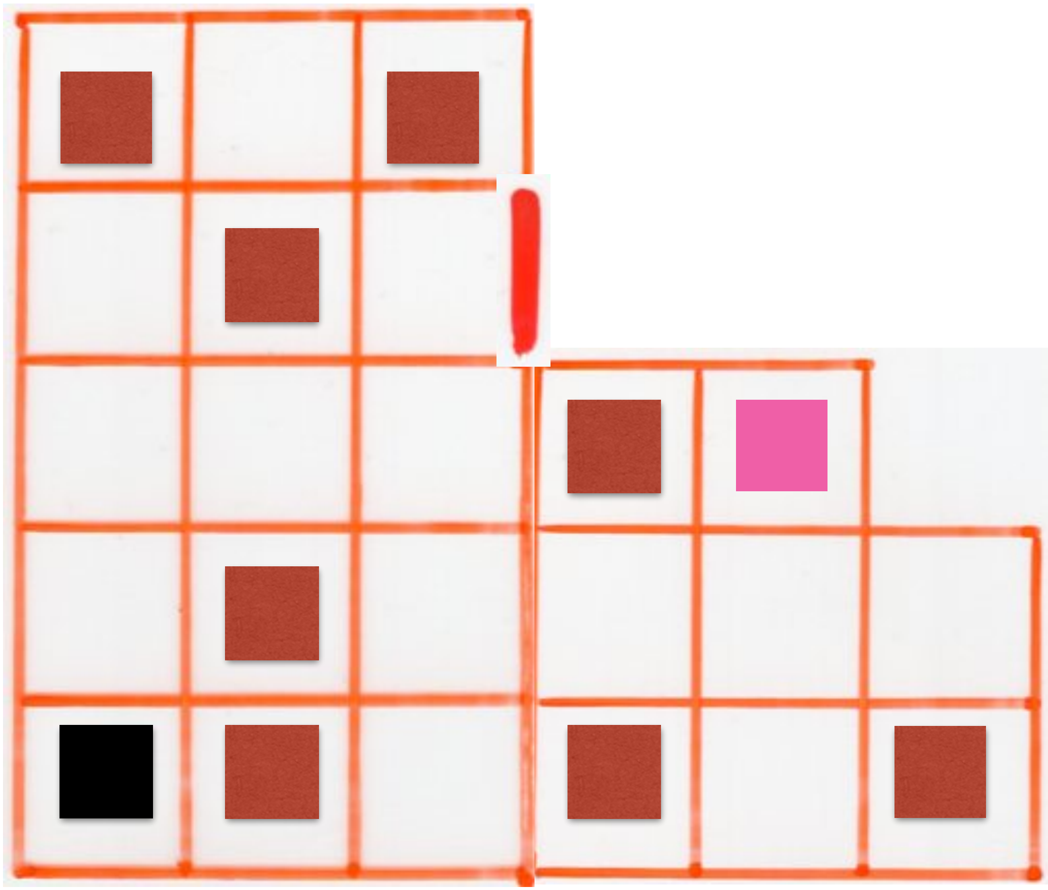
Example 1



Example 2



Example 2



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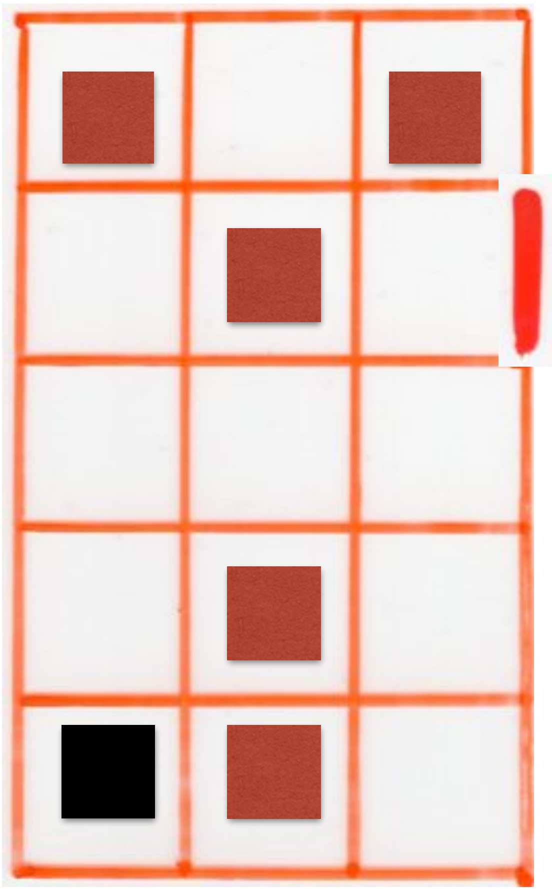

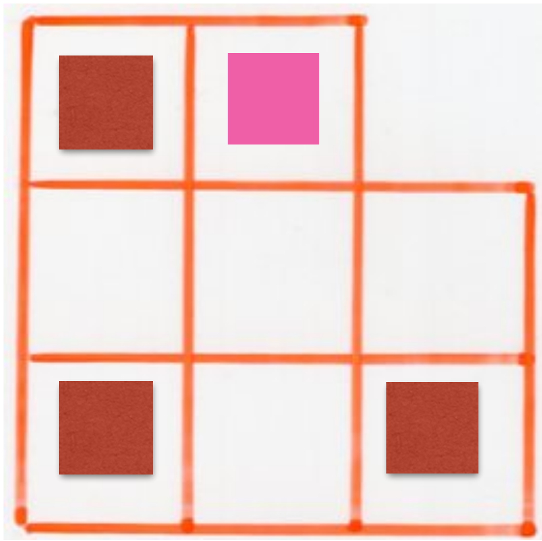

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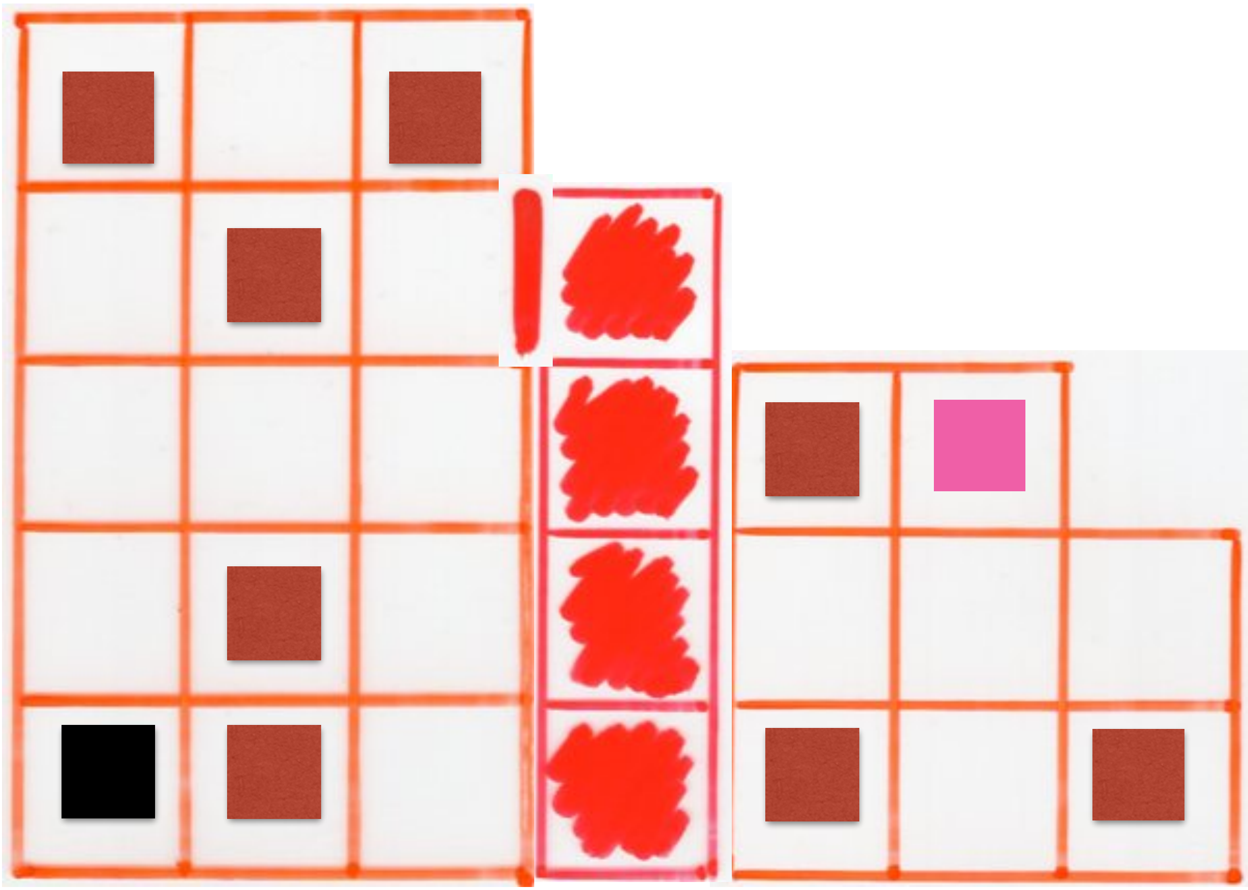
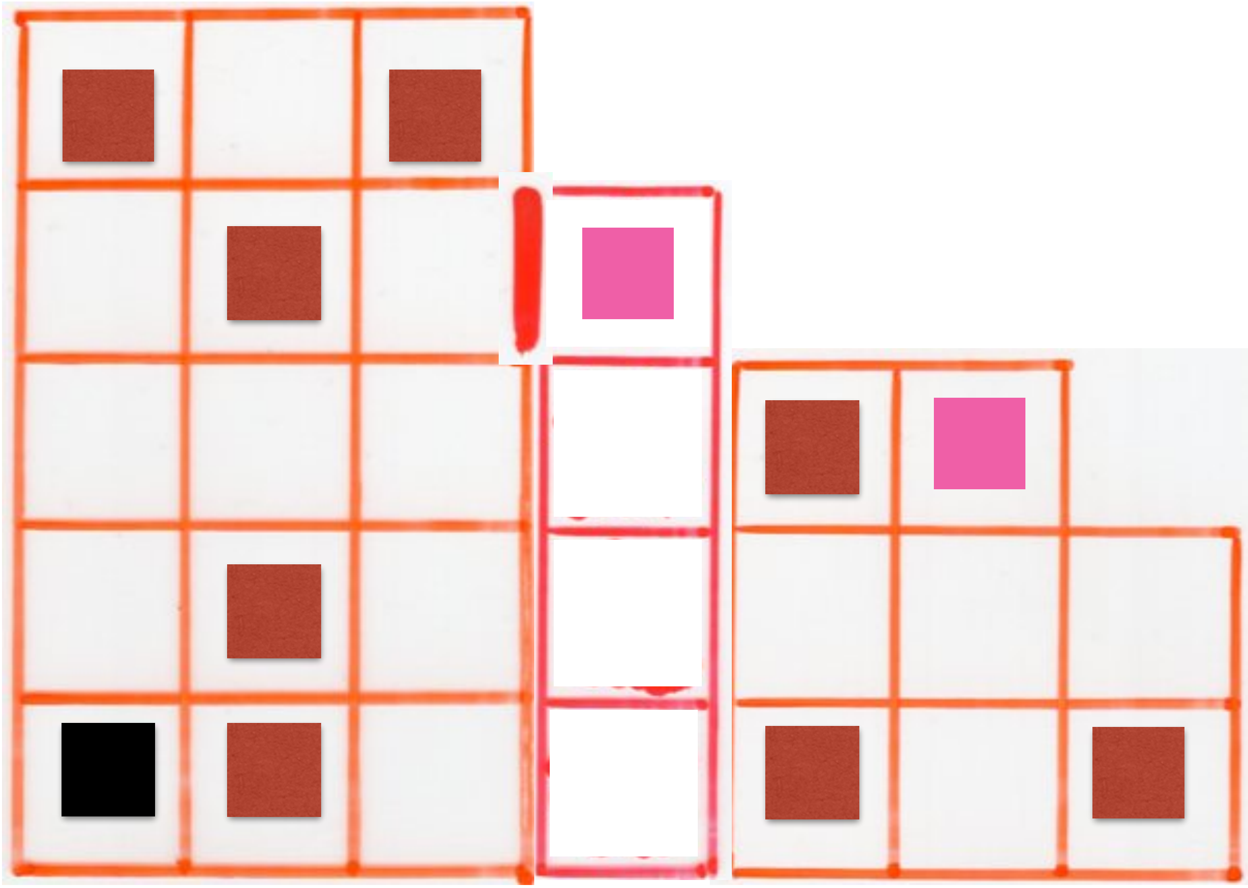
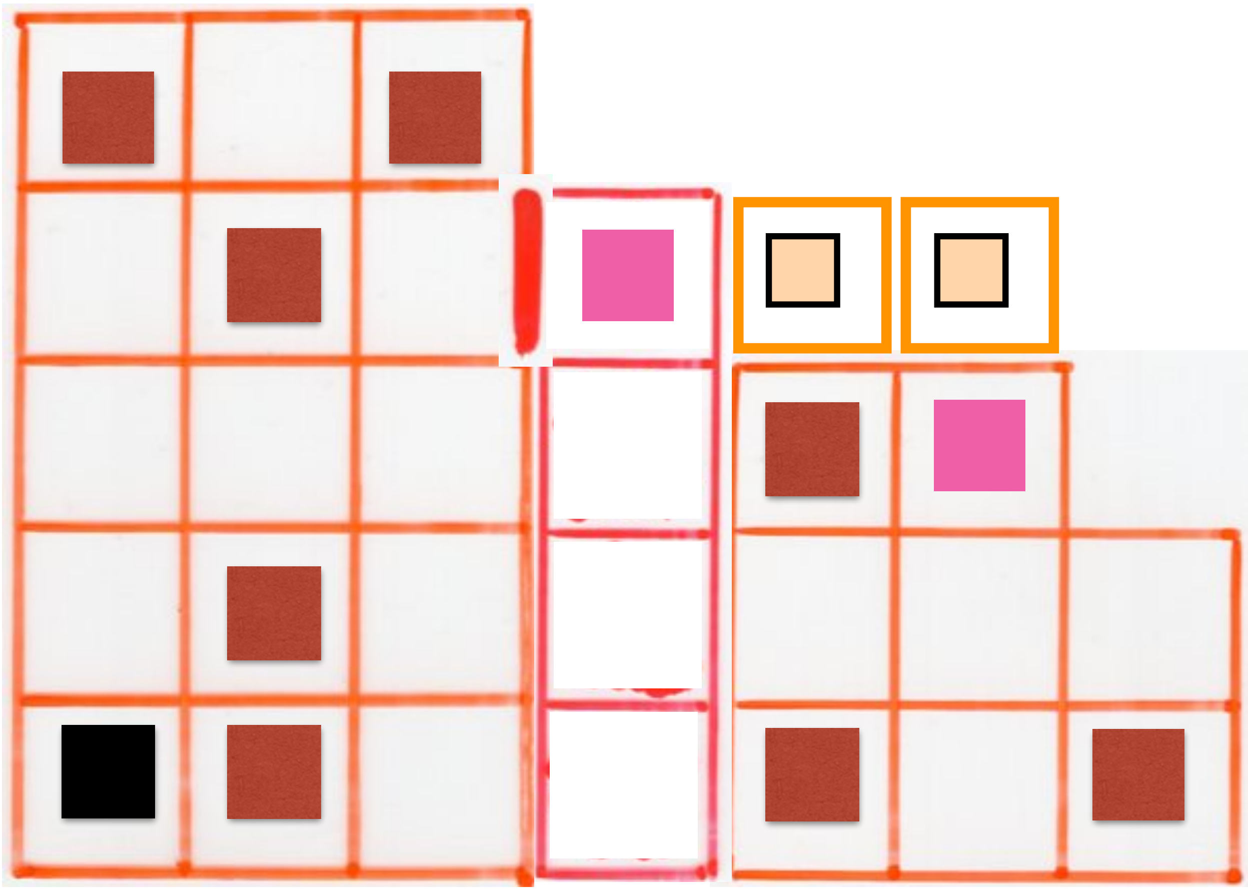


Example 2

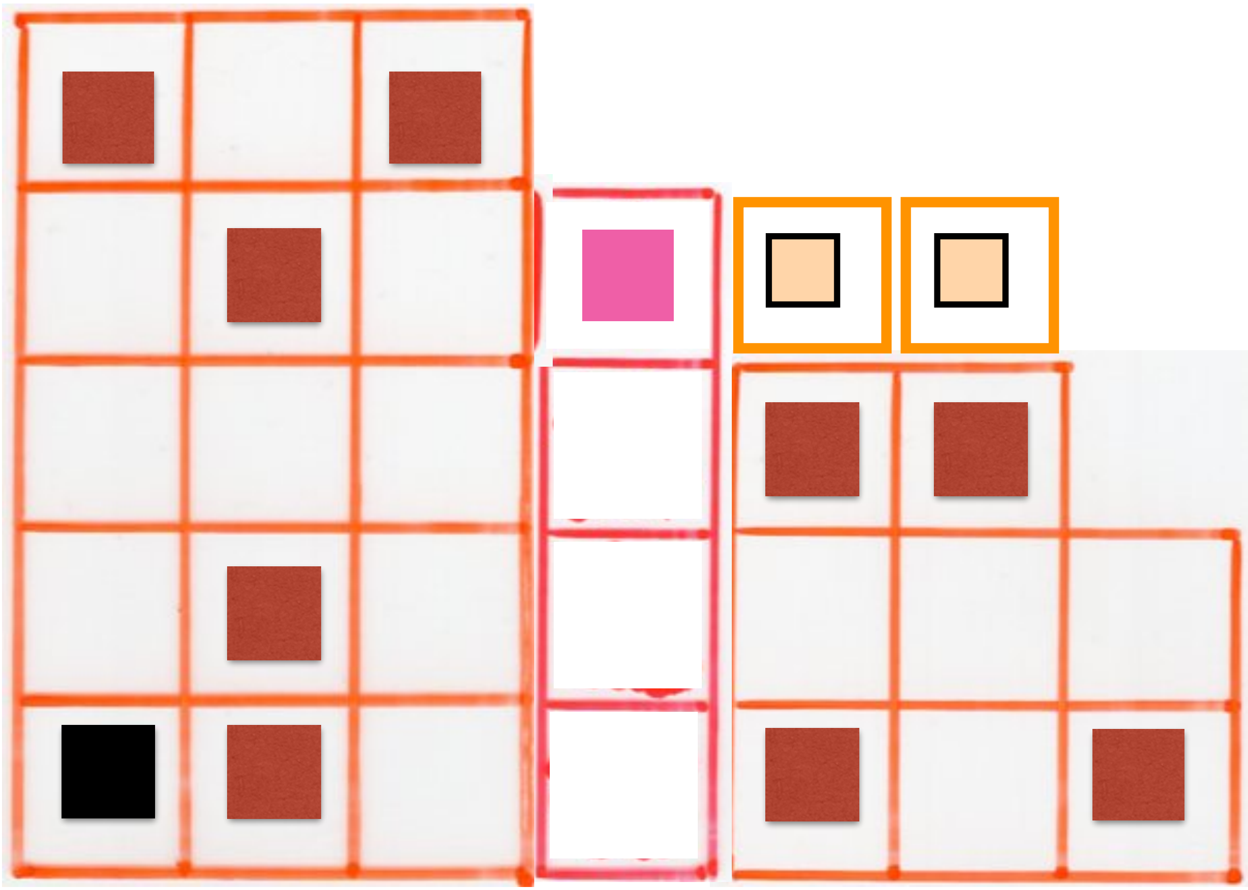


Case (ii)

Example 2



Example 2



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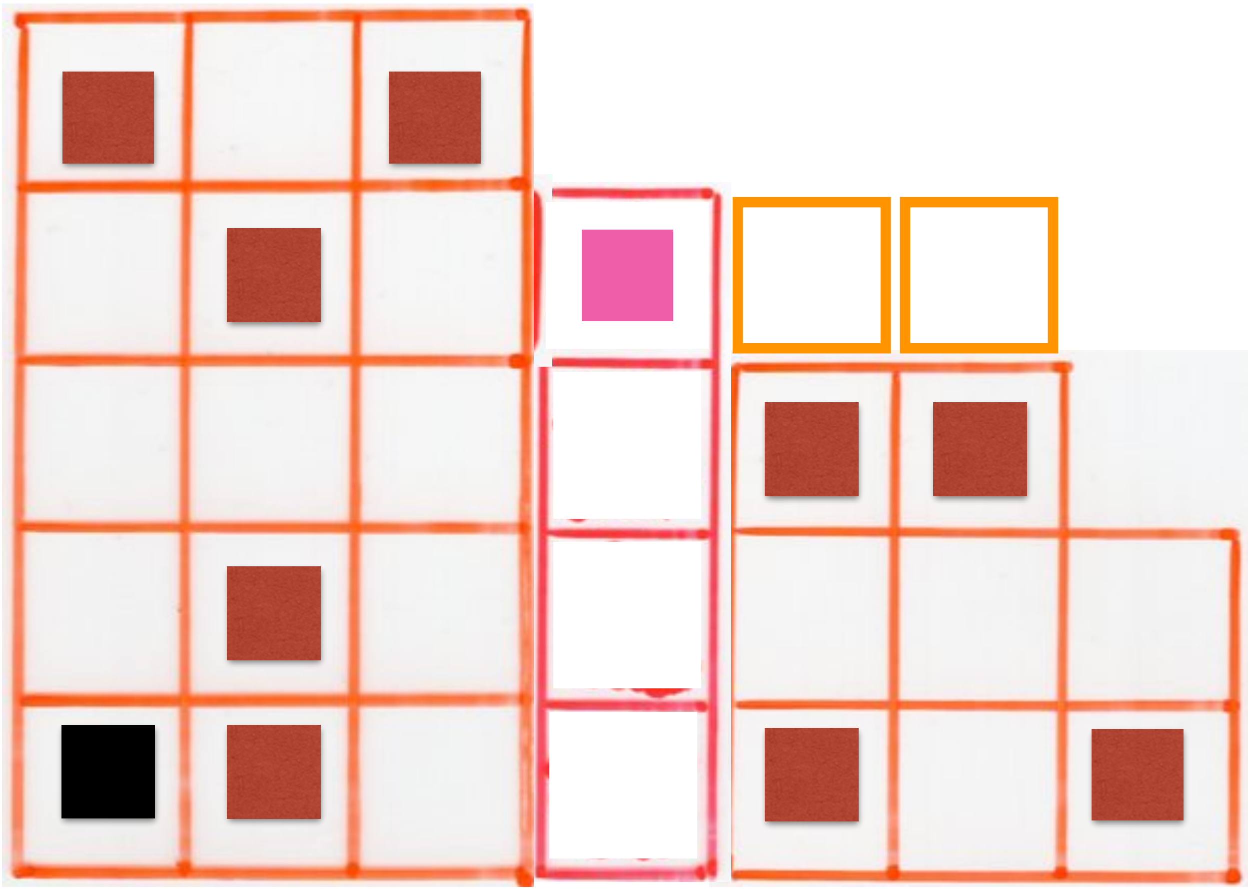
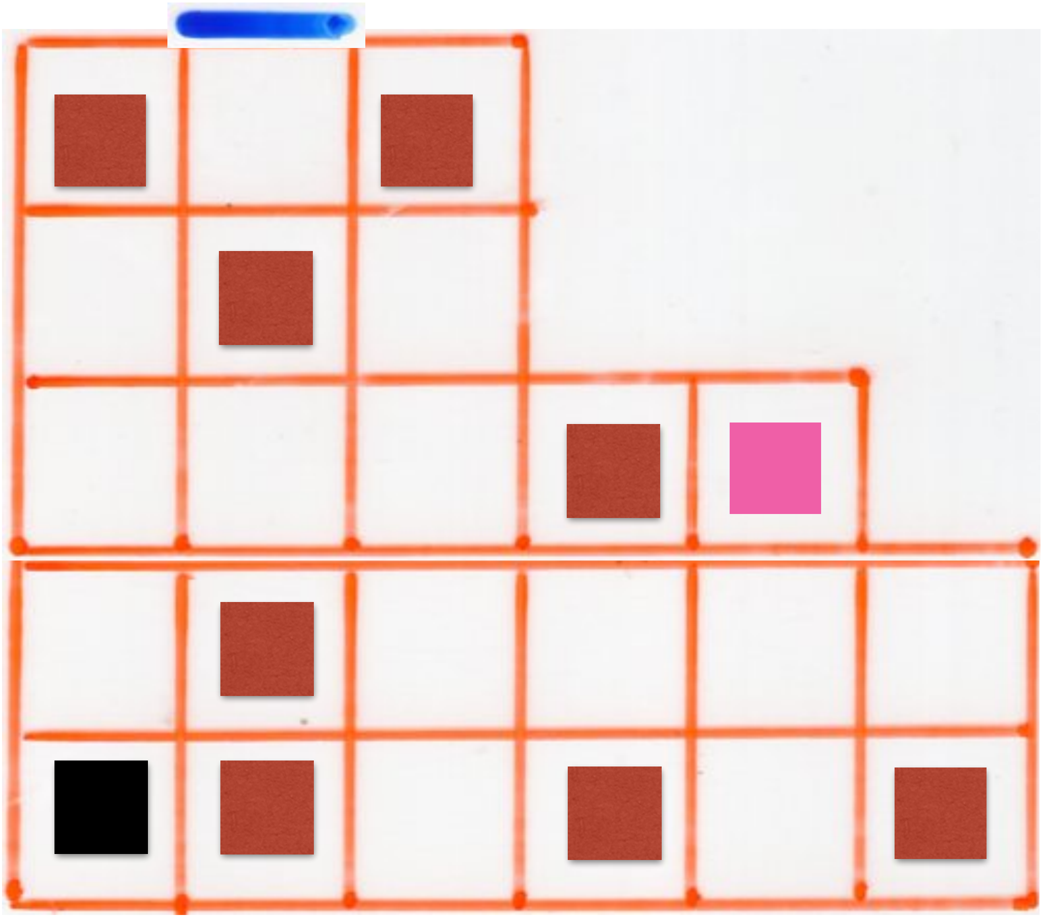
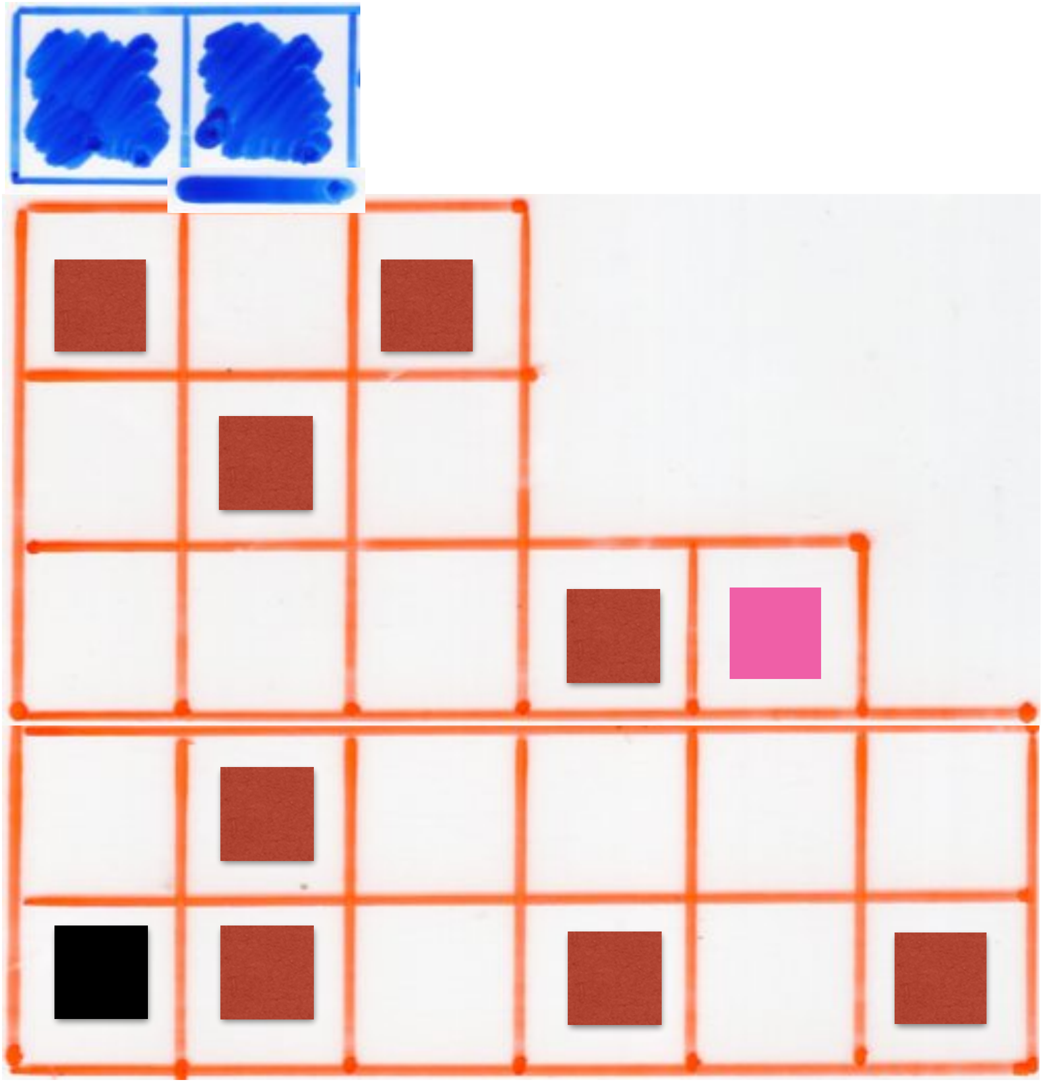
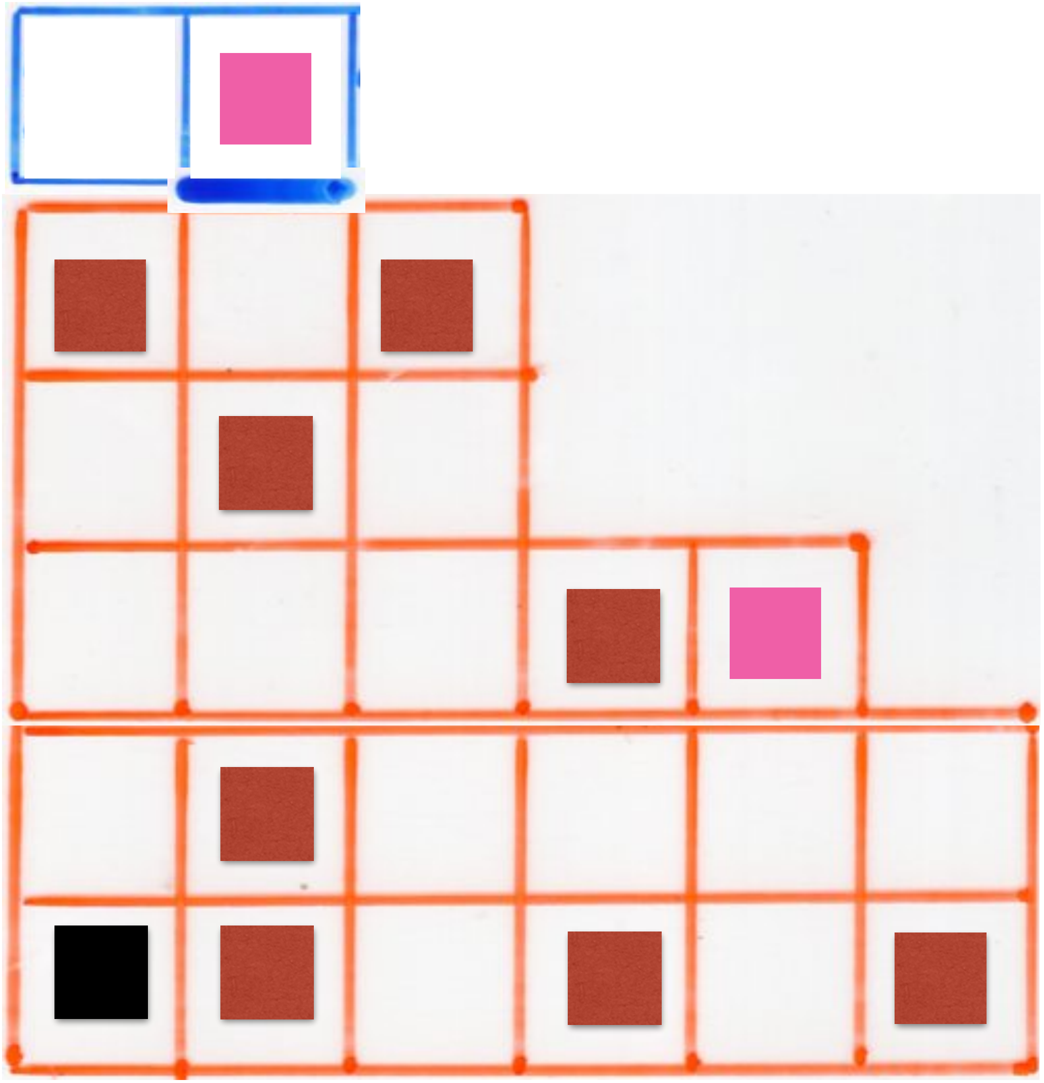


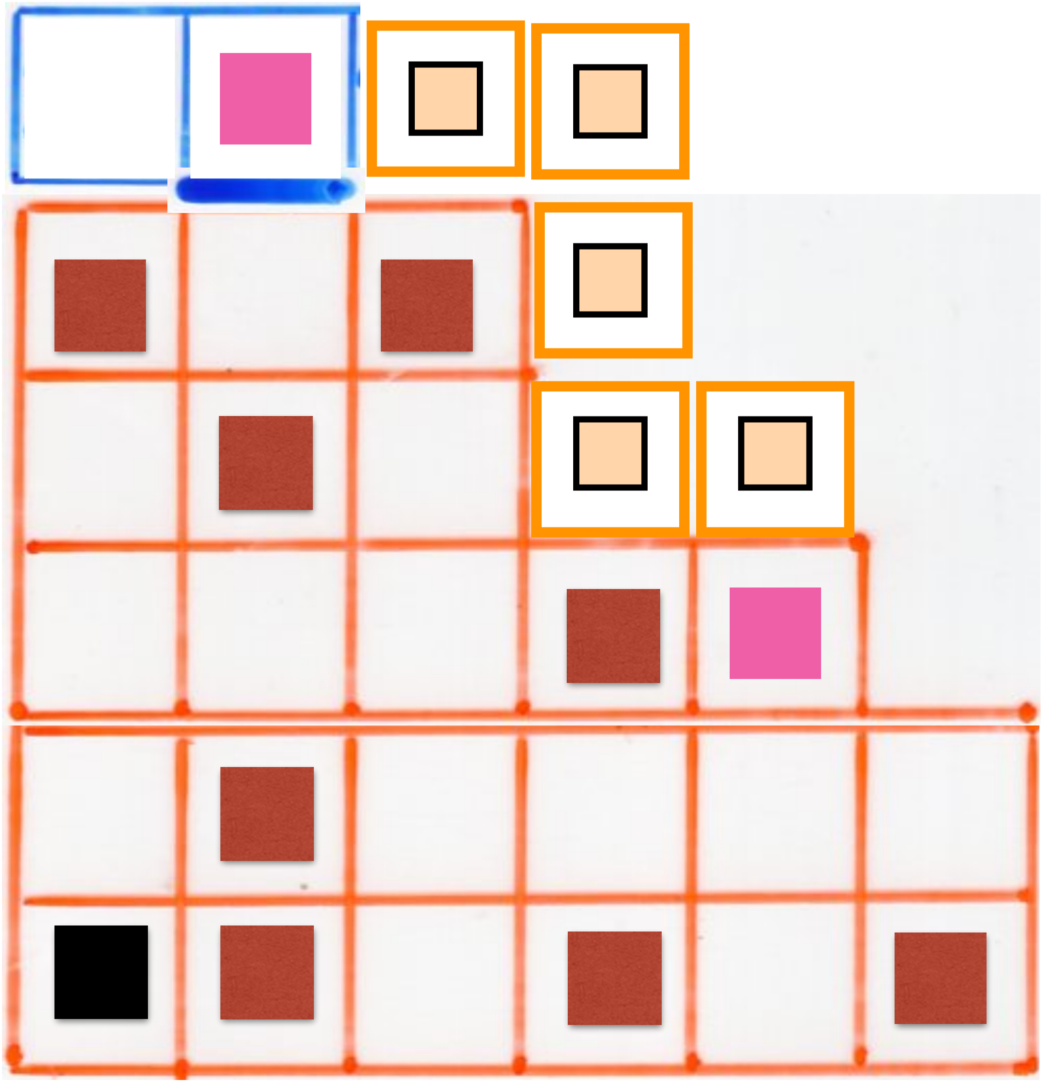
Example 3

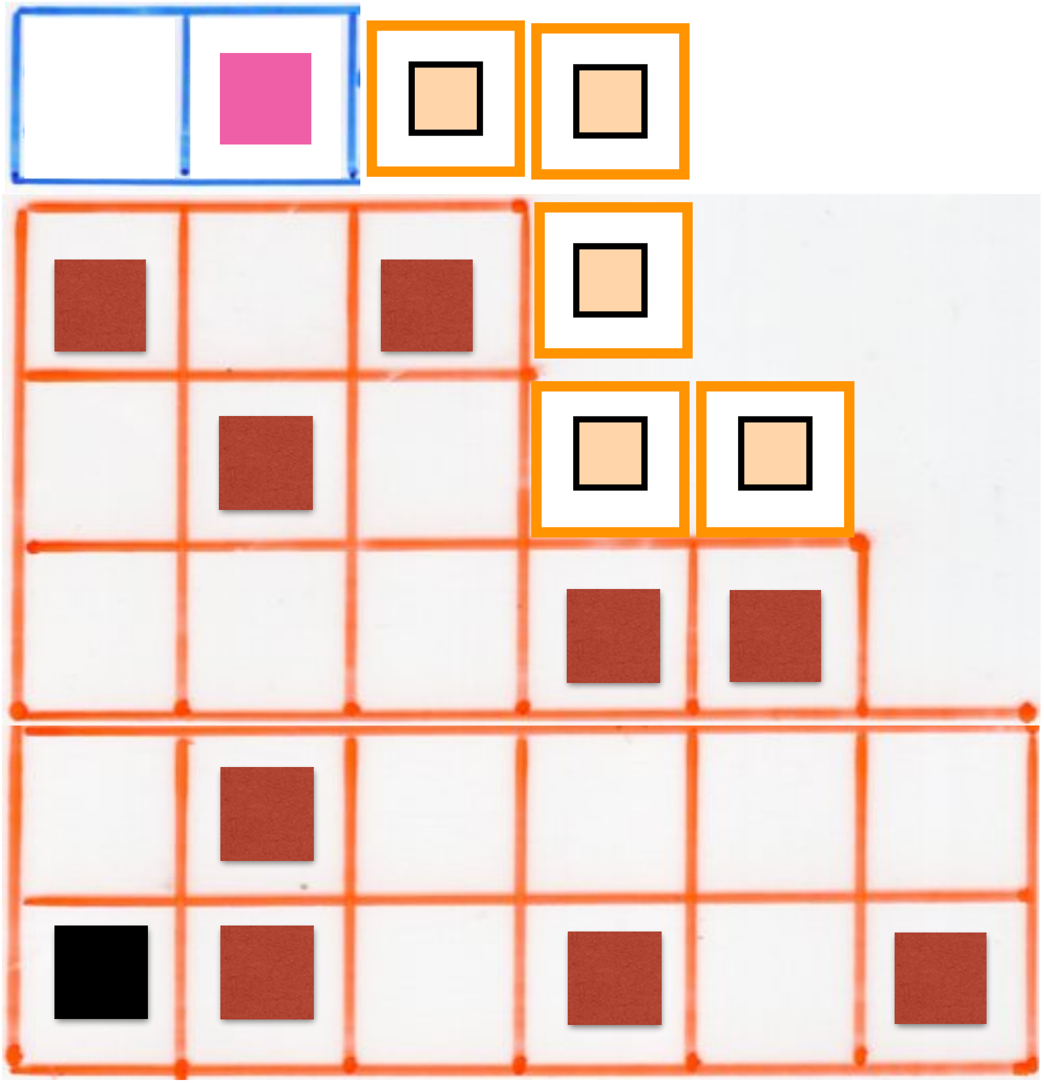


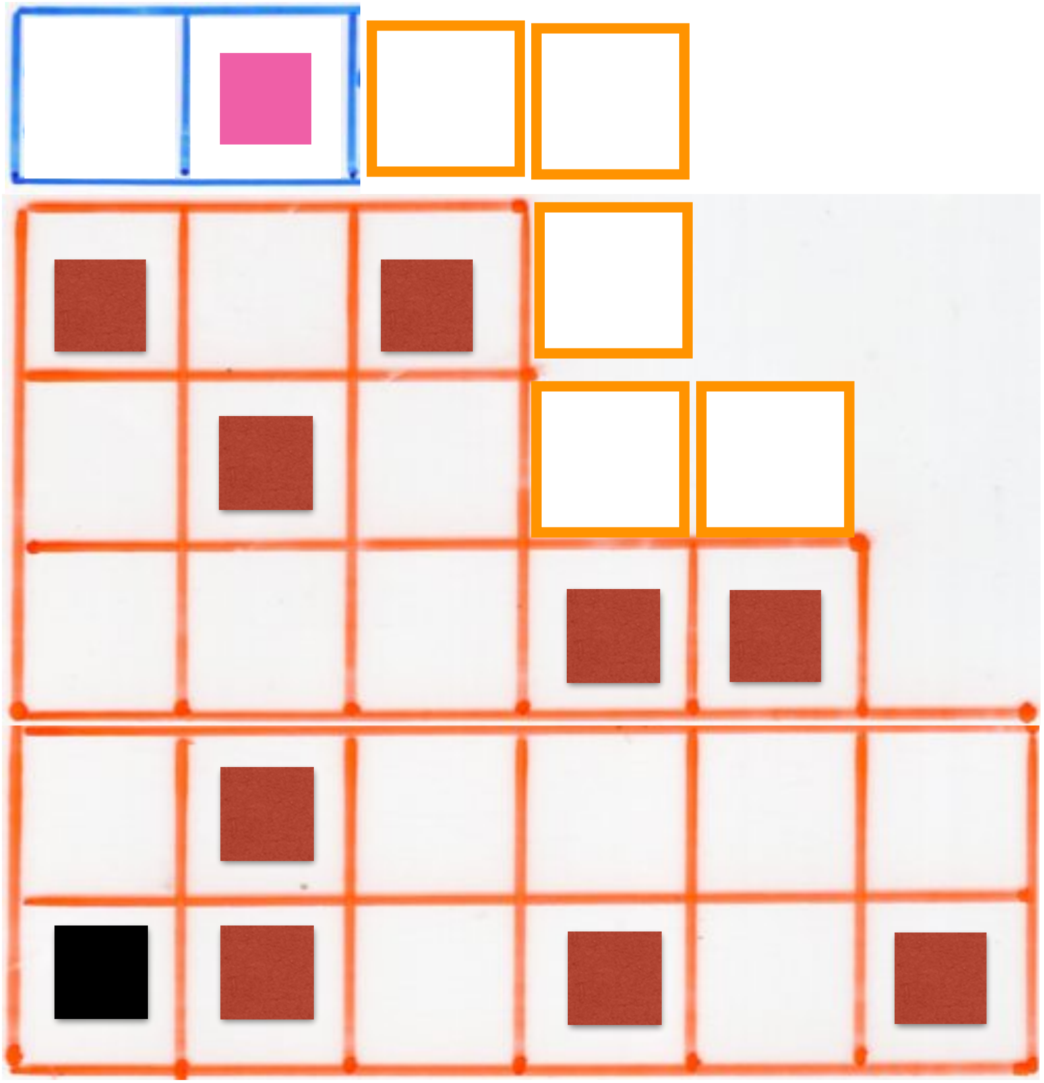




Case (ii)

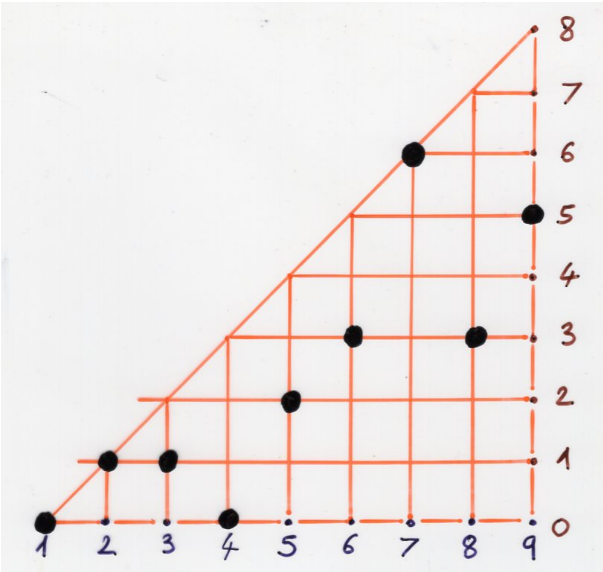






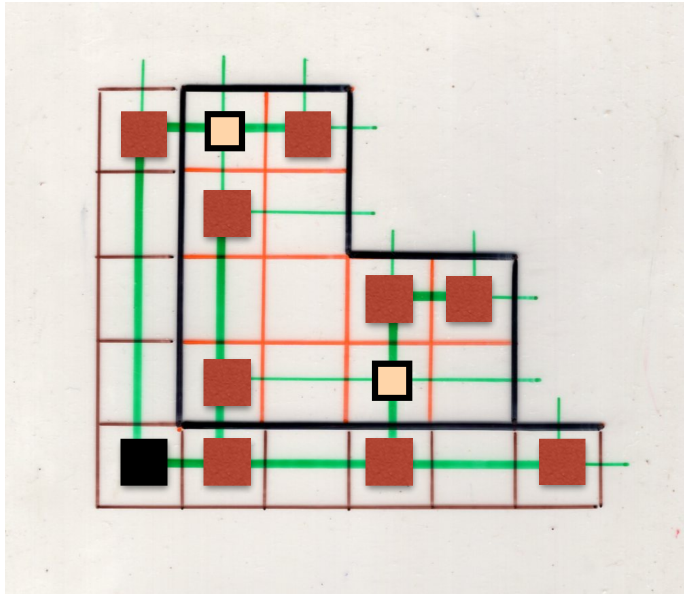
bijection

tree-like tableaux size $(n+1)$ ⟷ (tree-like tableaux size n , $1 \leq i \leq (n+1)$)



bijection

⟷



sub-excedant functions on [n]

$0 \leq f(i) \leq (i-1)$

tree-like tableaux size n

parameters

We have seen (first bijection):

Corollary for T alternative tableau (size n)

The double distribution $(i(T), j(T))$ of parameters

$i(T)$: number of open rows

$j(T)$: number of open column

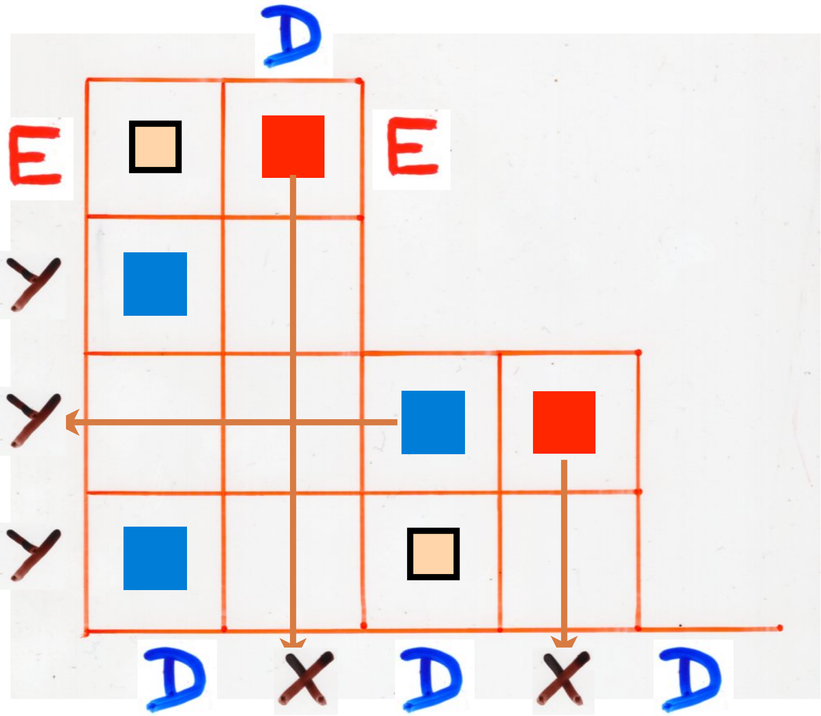
is the same as the double distribution of permutations of $\mathfrak{S}_{n+1}$ according to the parameters

(left-to right / right-to-left) minimum elements of σ

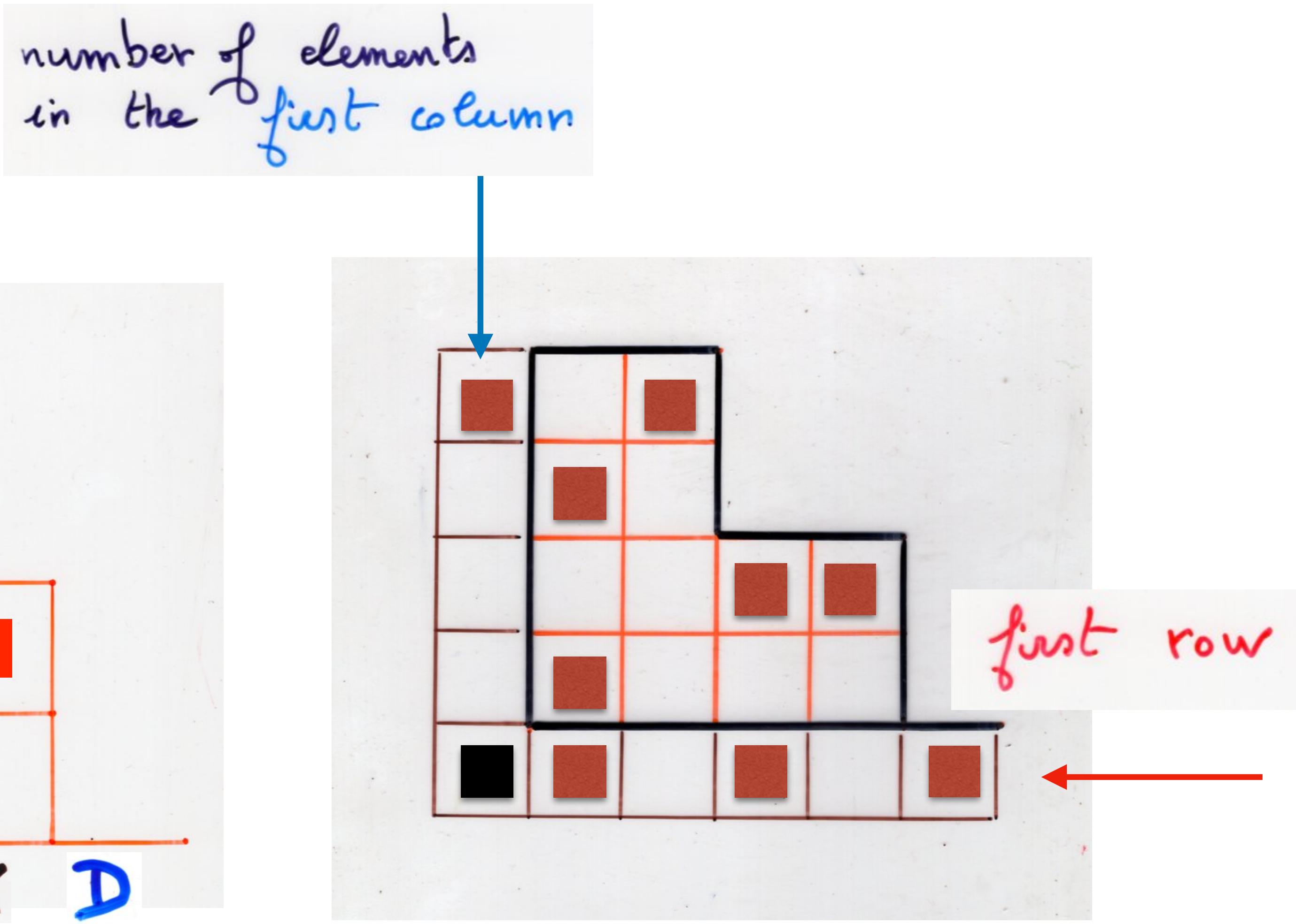
generating polynomial:

$$xy(x+y)(x+1+y)\cdots(x+n-1+y)$$

$i(T)$ = nb of rows without ●
$j(T)$ = nb of columns without ●



alternative tableaux



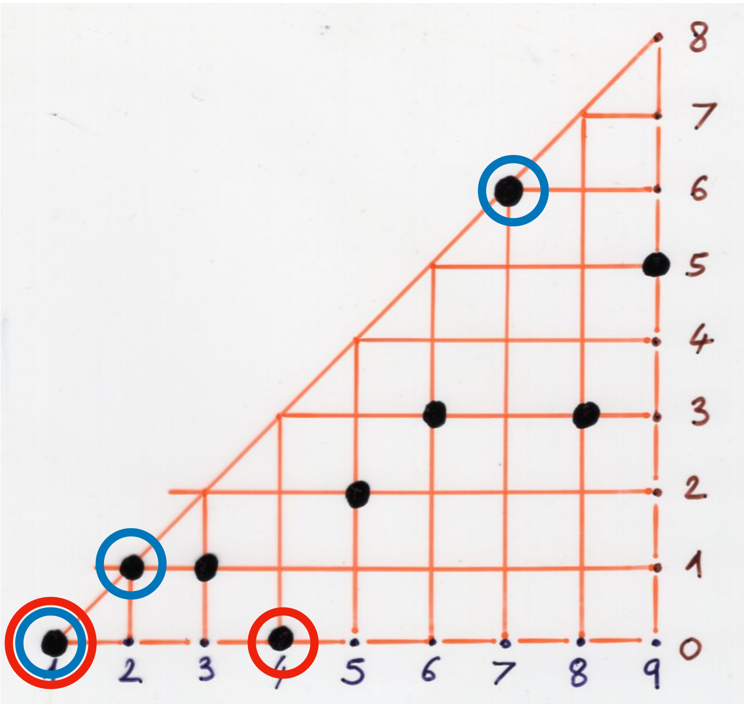
tree-like tableaux

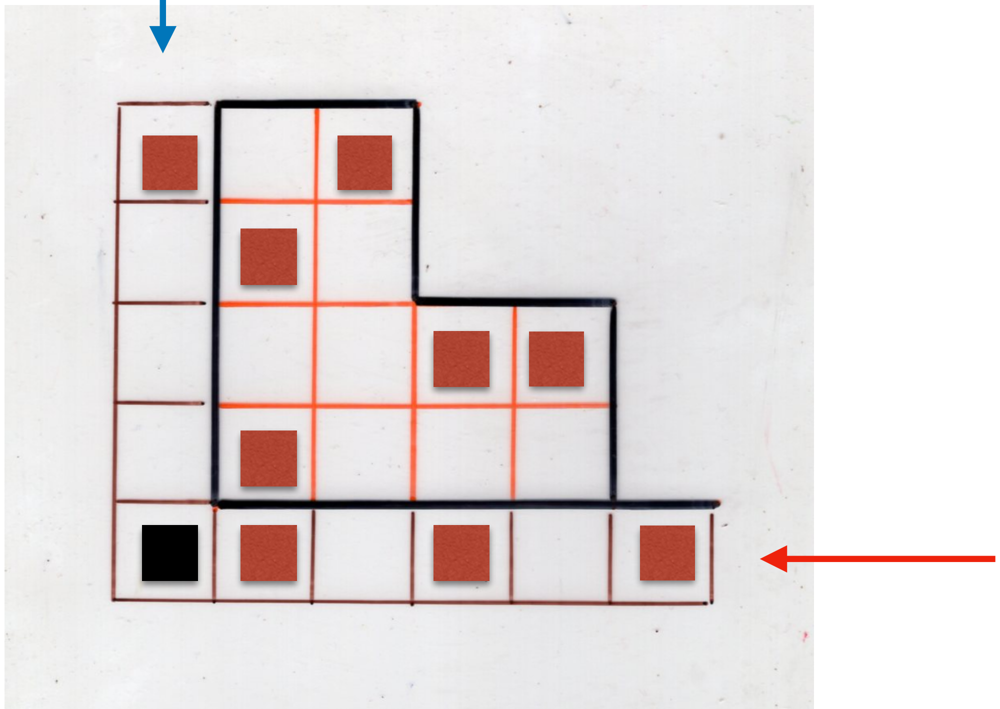
number of elements in the first column

number of i's $f(i) = (i-1)$

first row

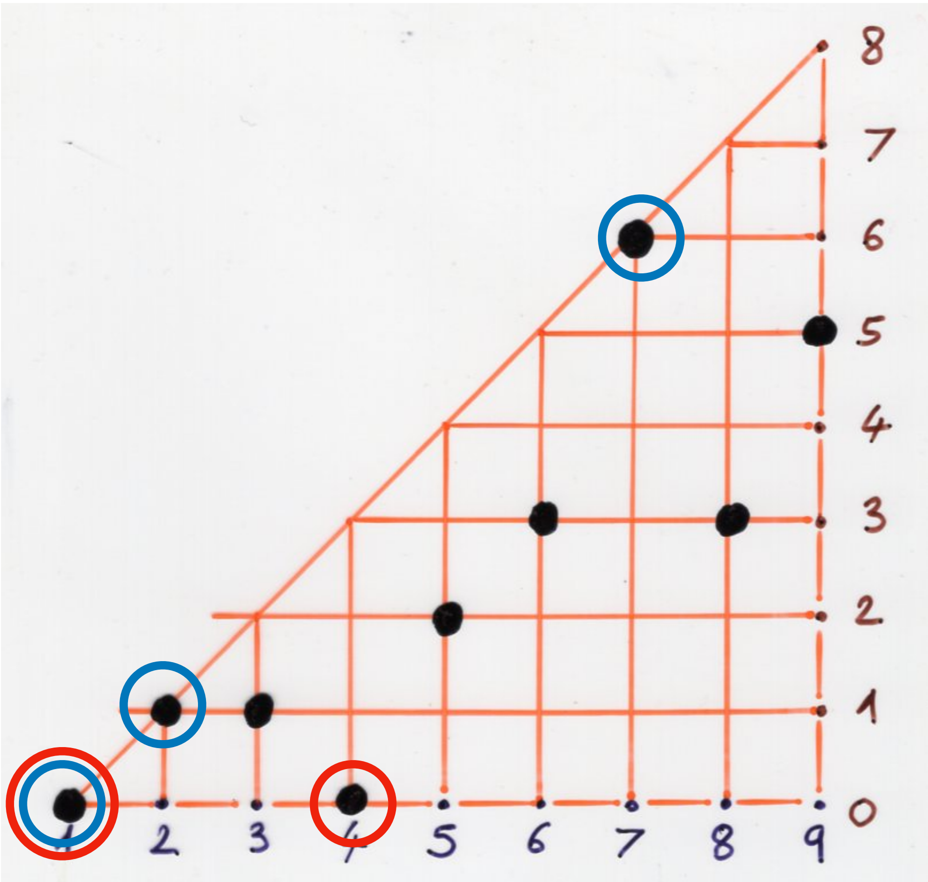
$f(i) = 0$





$f \longrightarrow T$

tree-like tableaux



generating polynomial:

$$xy\,(x+y)(x+1+y)\cdots(x+n-1+y)$$

σ permutation = 7 2 3 9 6 8 5 1 4 word

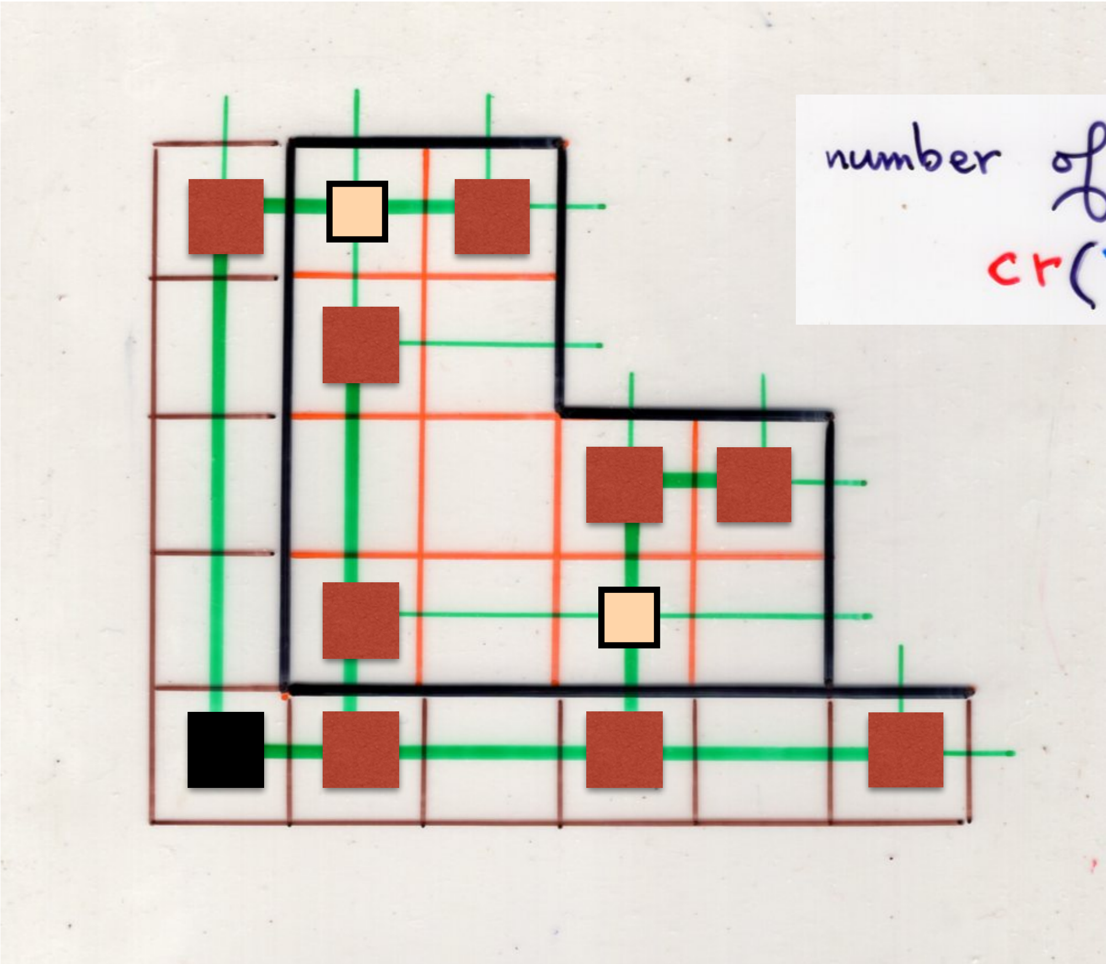
left-to-right
right-to-left

minimum

maximal

elements

the parameter q:

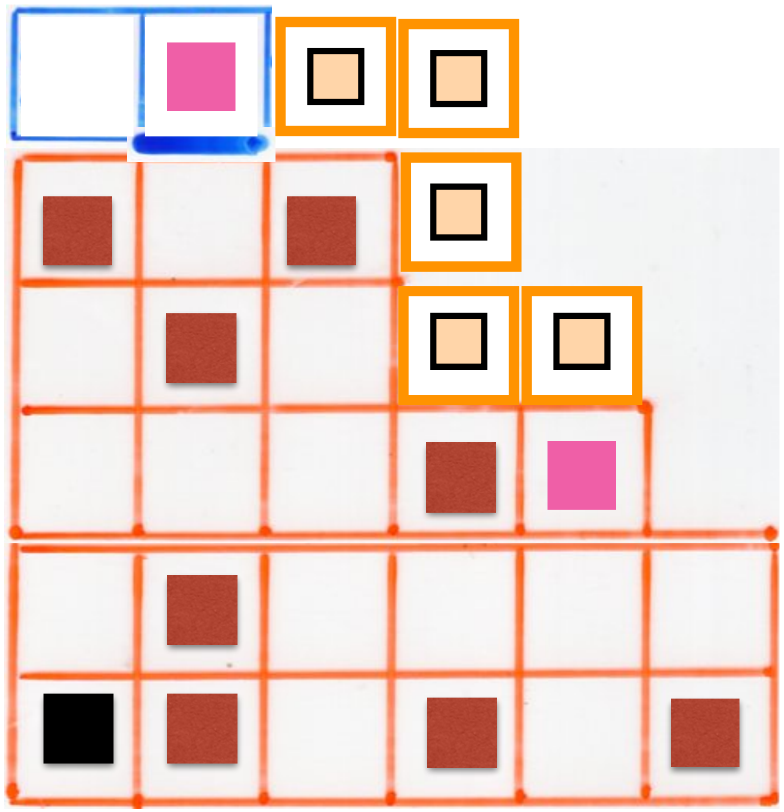


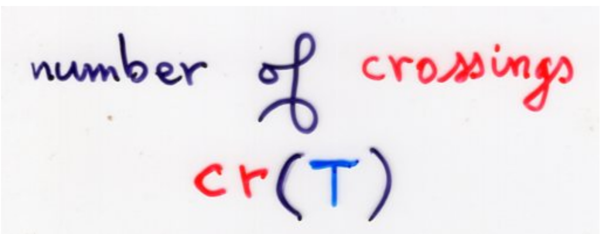
number of crossings

cr(T)

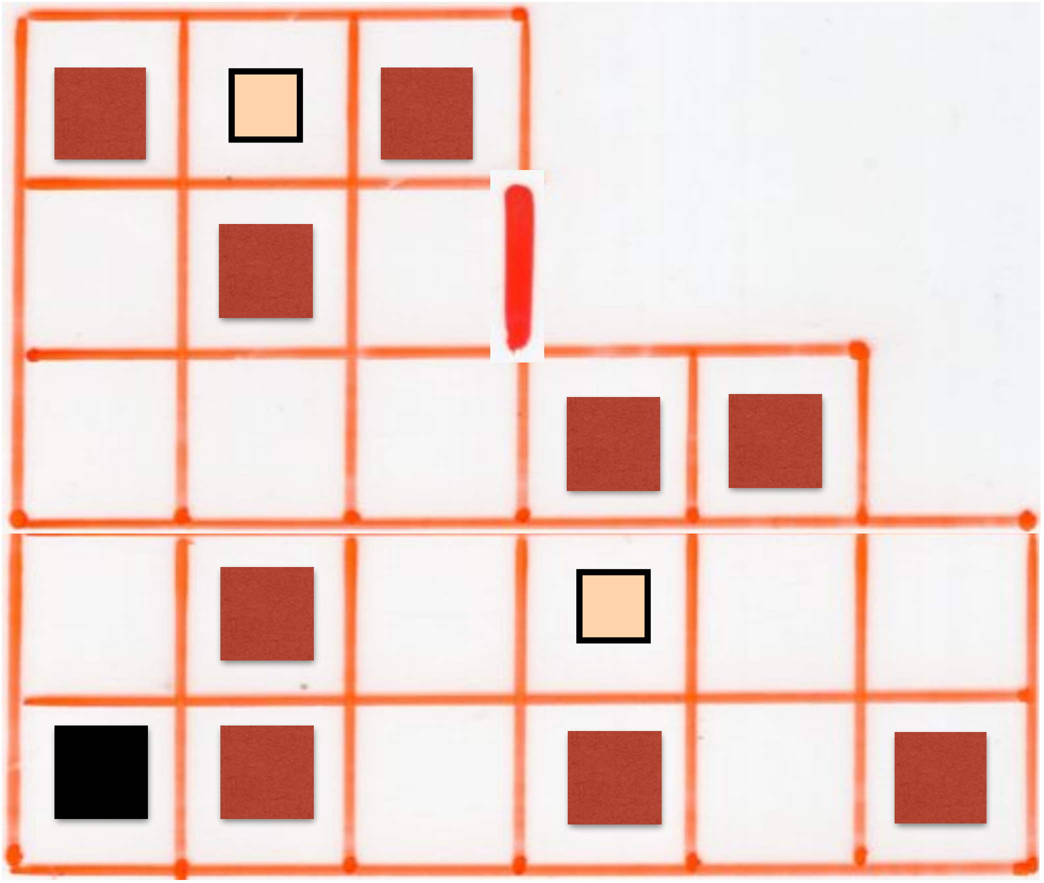
tree-like tableaux

the parameter q:



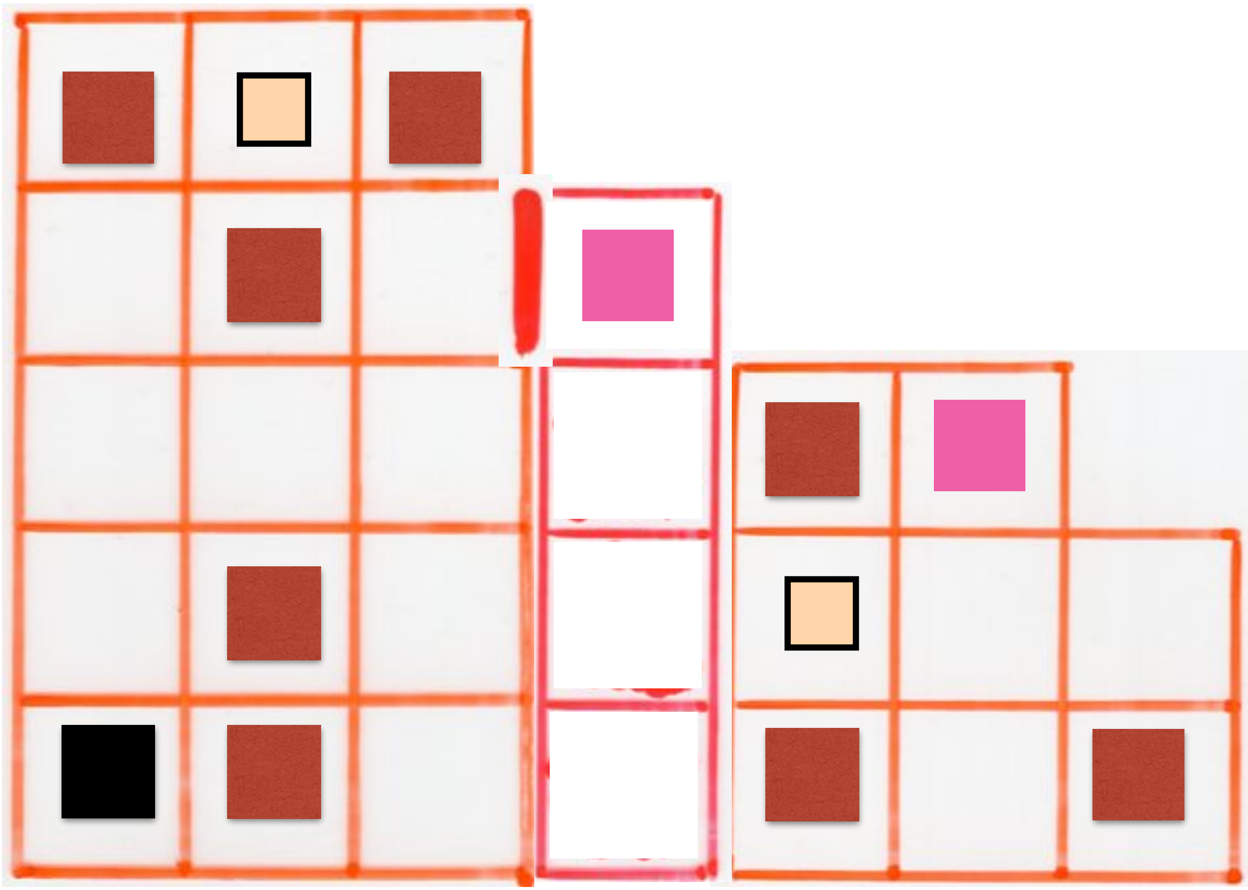


= sum of the length of all rim-hooks added in the algorithm

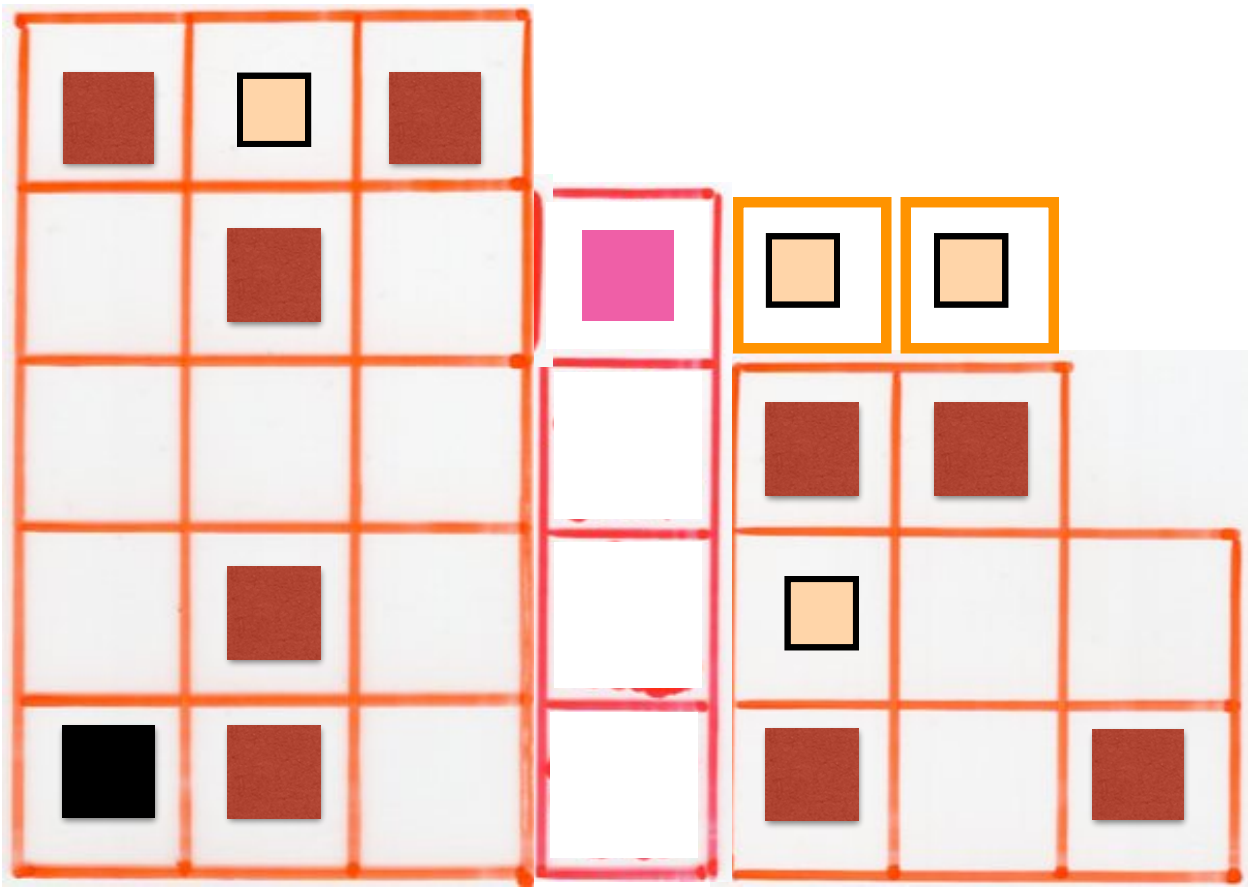


tree-like
tableaux

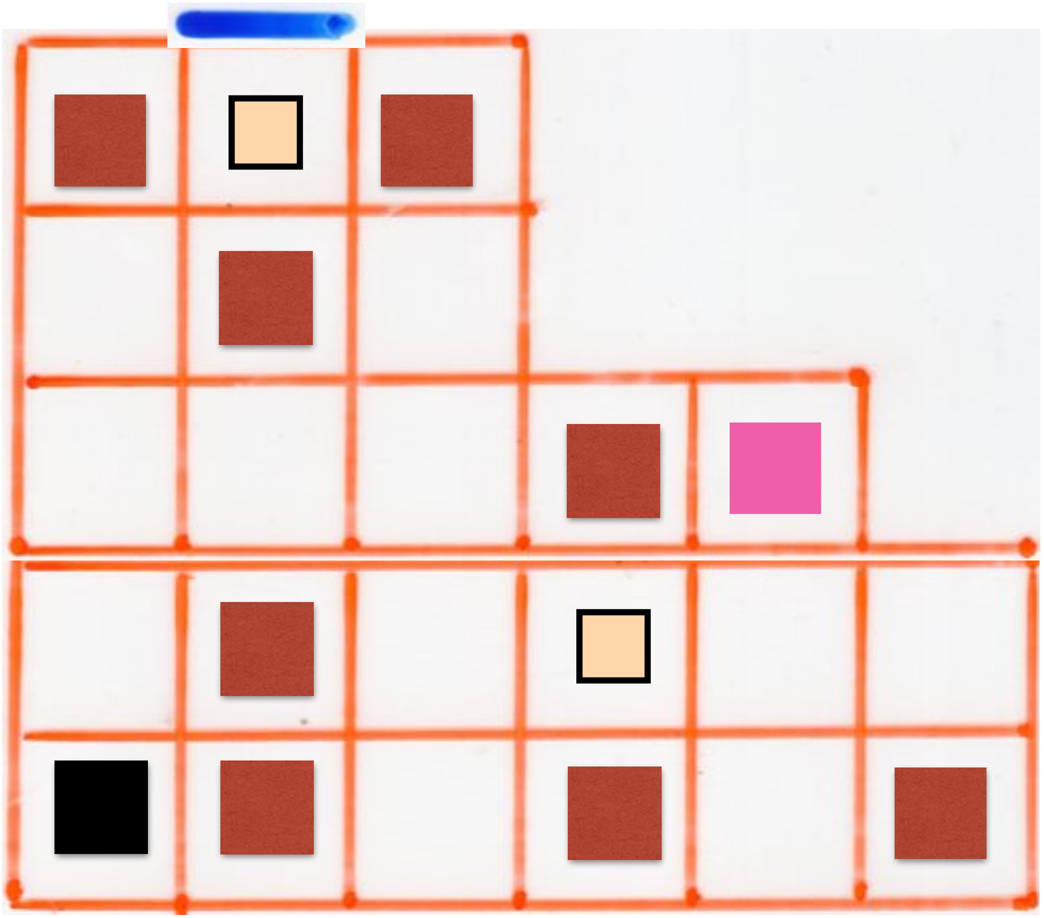
Example 2

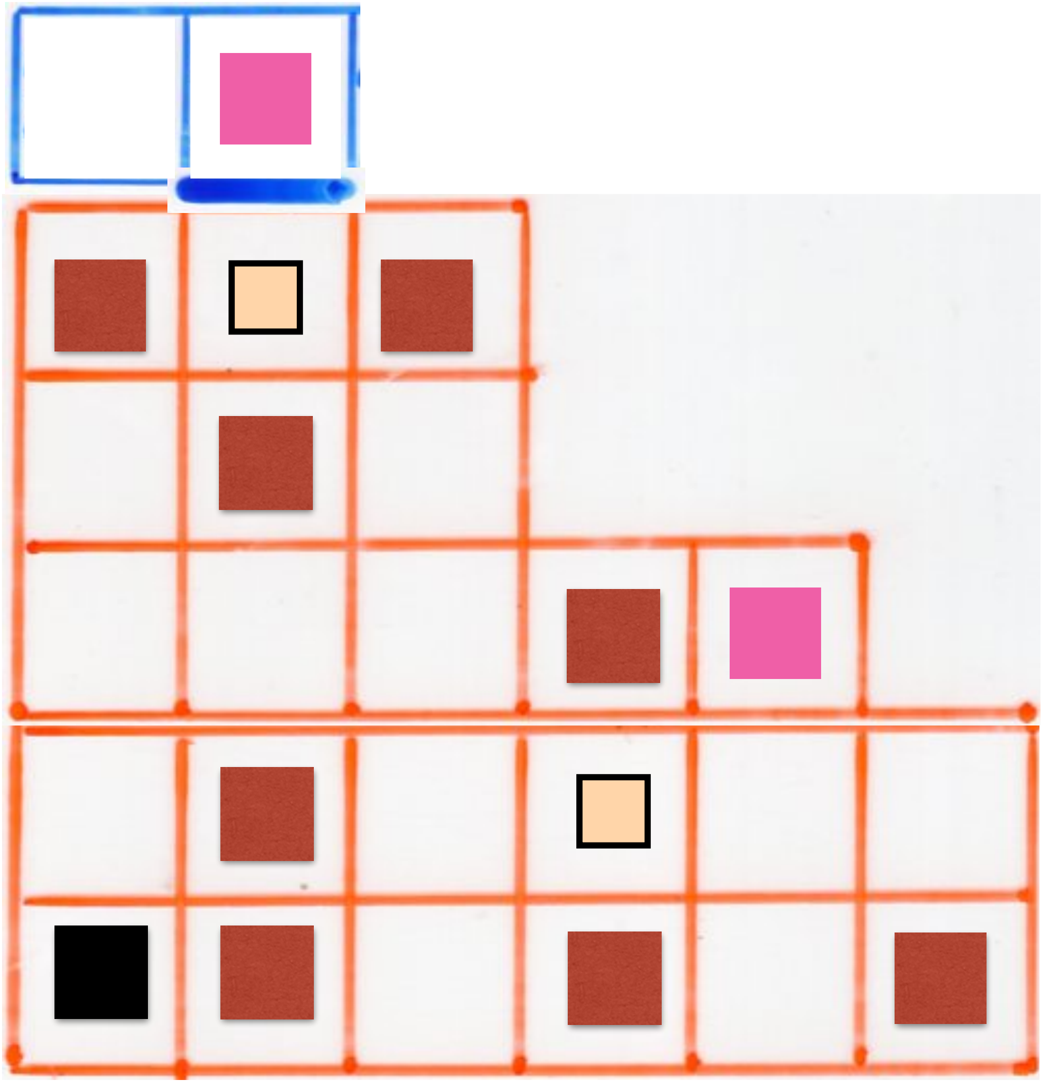


Example 2

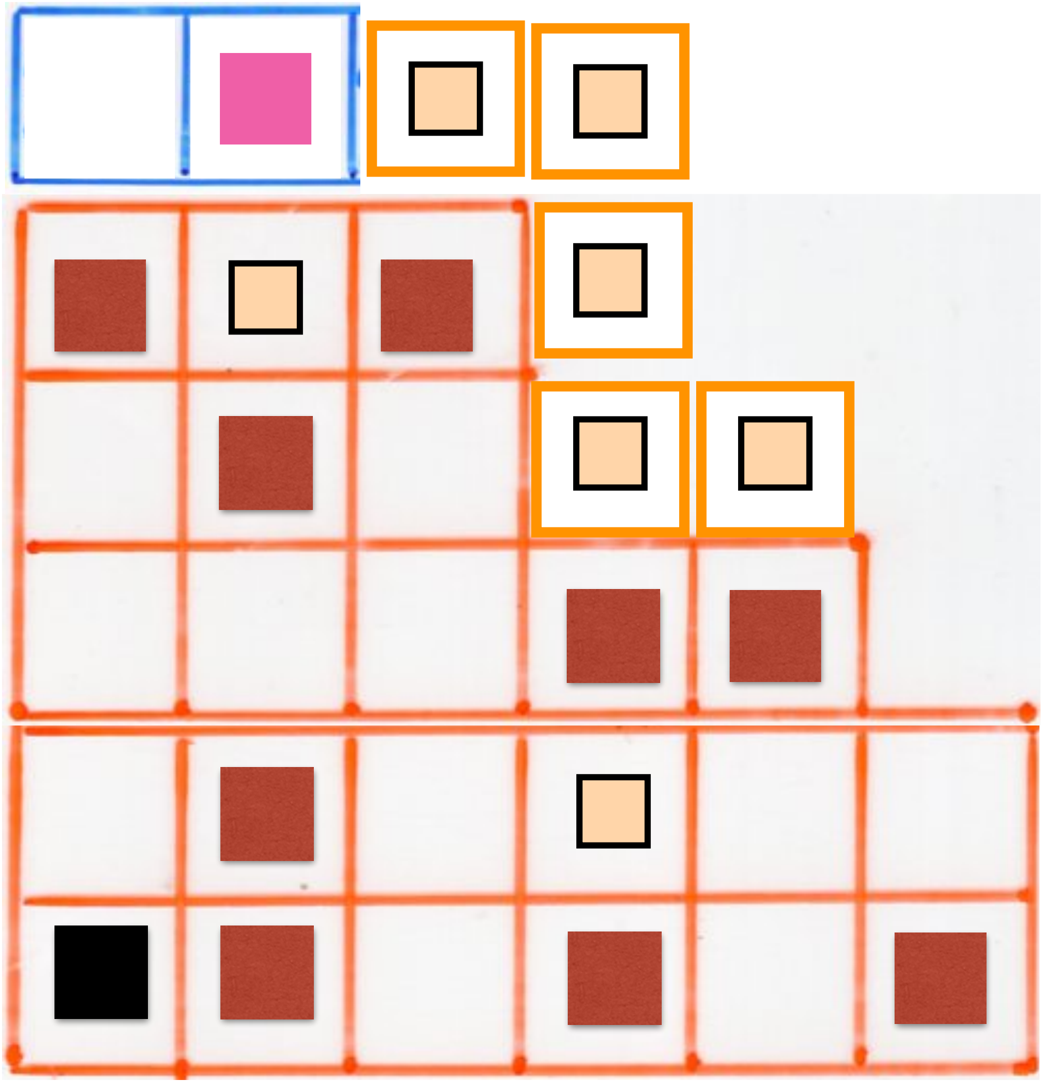


Example 3



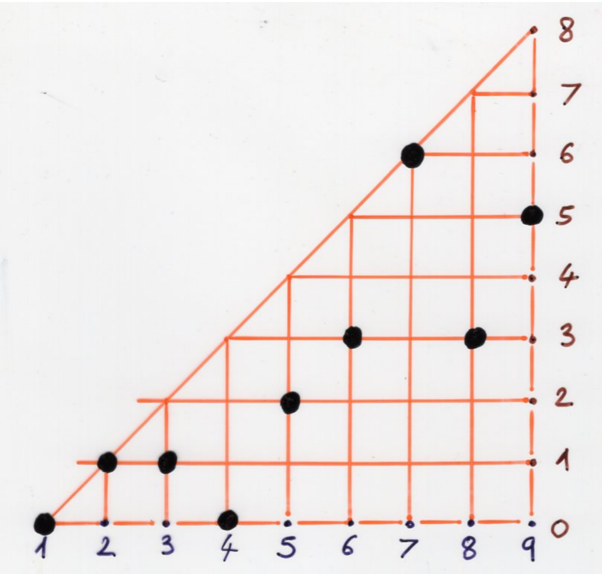


Case (ii)



bijection

$$f \longrightarrow T$$



number of crossings $cr(T)$

= sum of the length of all rim-hooks added in the algorithm

$$= \sum_{1 \leq i \leq (n-1)} \max\left[\big(f(i+1) - f(i)\big), 0\right]$$

the parameter q:

this is another story ! .....

.... related to q-analogue of Laguerre polynomials weighted histories

q-Laguerre

continuous q-Laguerre
discrete polynomials

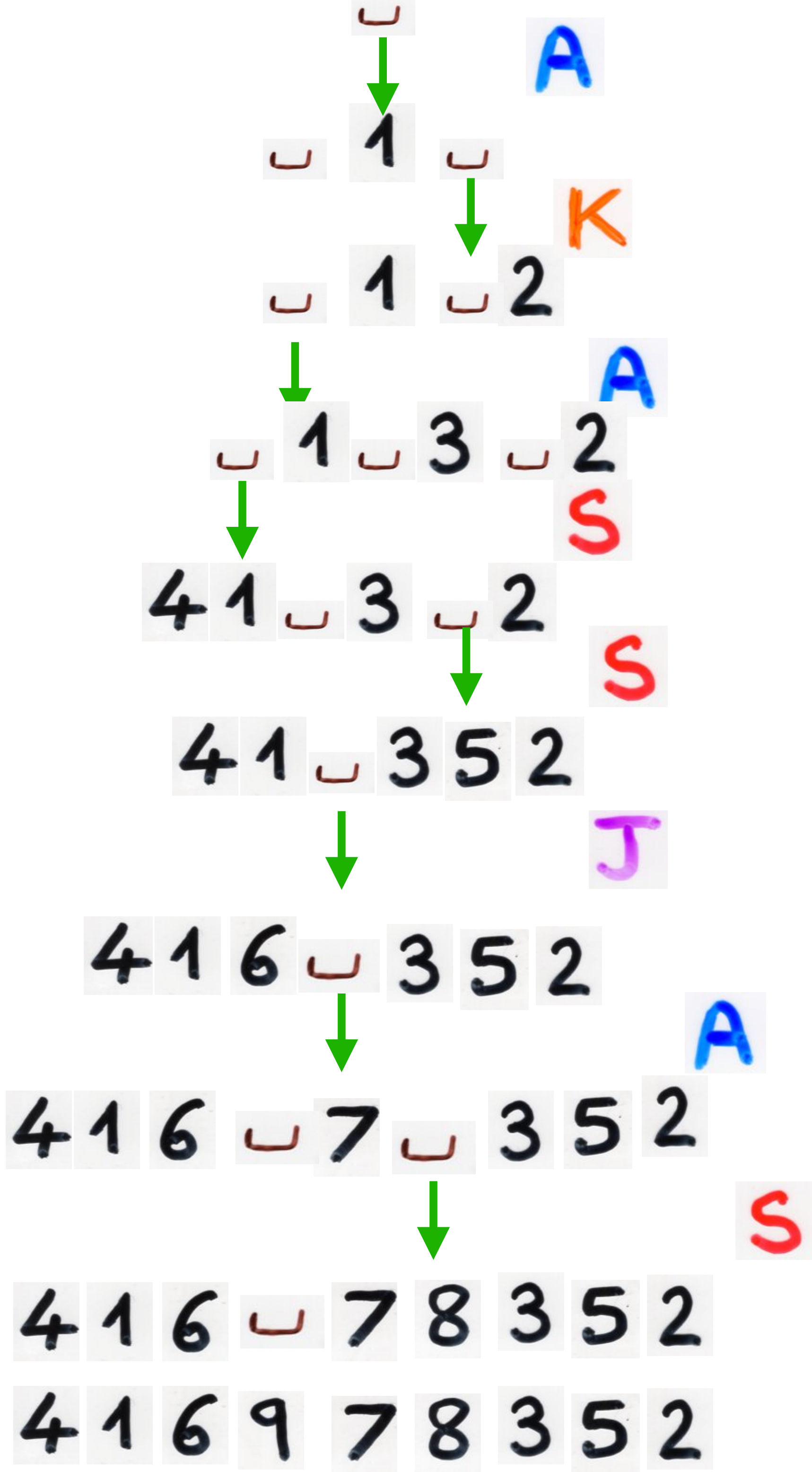
$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$

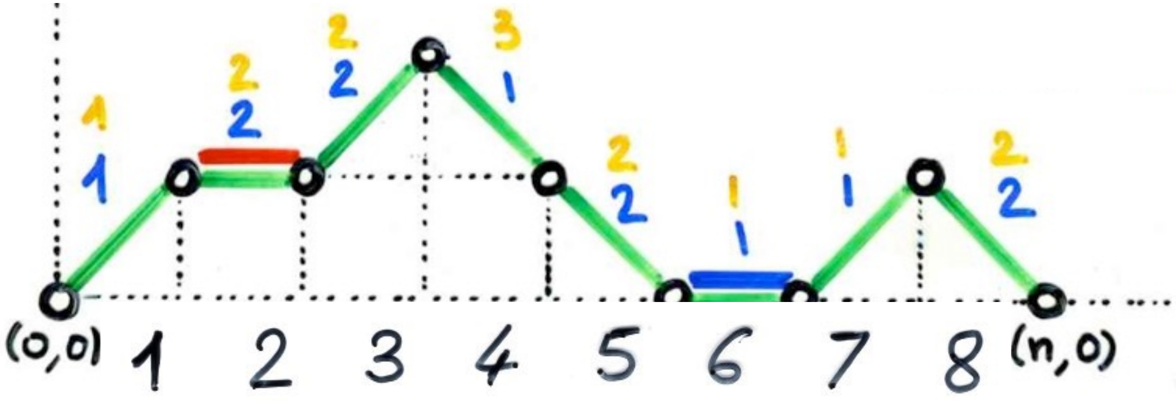
weighted
q-Laguerre
histories

$$q^{\left[\sum_{i=1}^{n} (p_i - 1)\right]}$$

choice function


␣
A
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

"q-analogue" of Laguerre histories



choice function

i =	1	2	3	4	5	6	7	8
p_i =	1	2	2	1	2	1	1	2
$p_i - 1$ =	0	1	1	0	1	0	0	1

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

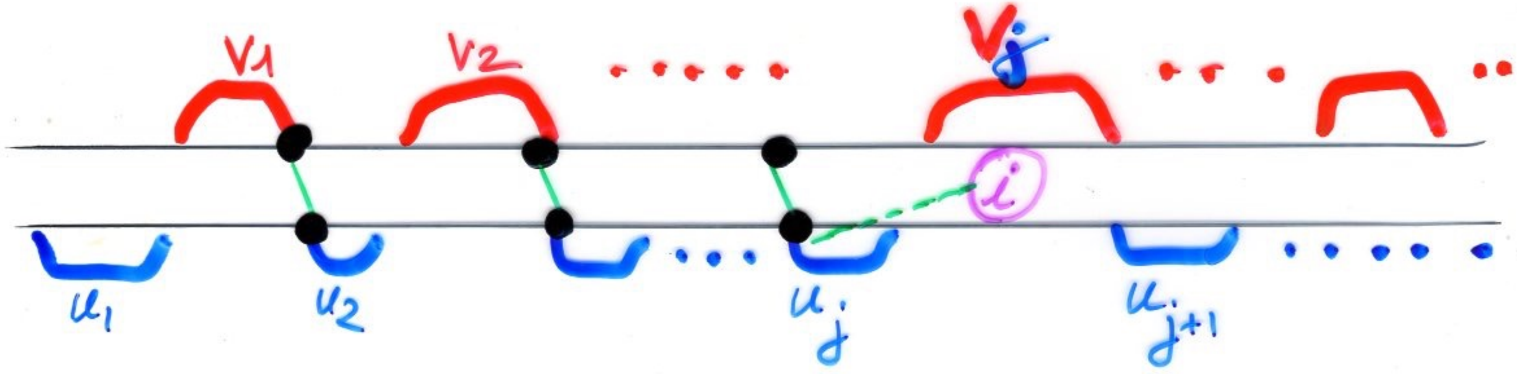
weighted q-Laguerre histories

q^4

weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern (31-2)



The parameter

« number of crossings » in alternating tableaux or in tree-like tableaux

has the same distribution as the parameter

« q-Laguerre » for permutations

$\sigma \longrightarrow T$

permutation $\mathfrak{S}_{n+1}$ — alternative tableau size n

O. Bernardi (2008)

The number of crossings of T is the number of pairs (x, y)

$x = \sigma(i)$, $y = \sigma(j)$, $1 \leq i < j \leq n+1$

such that there exist two integers $k, \ell \geq 0$

such that the set of values

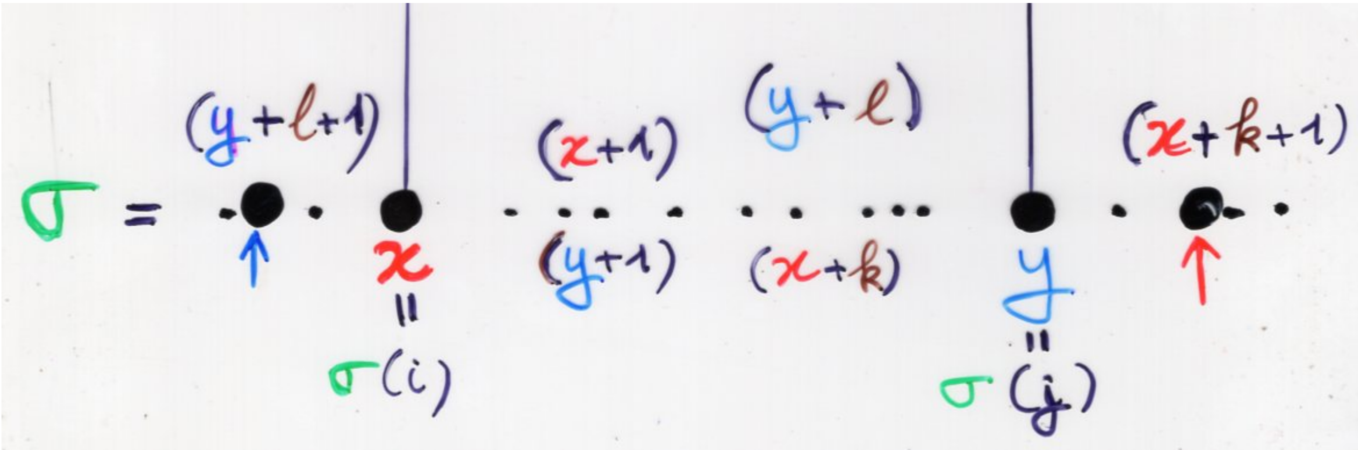
$x+1, x+2, .., x+k$, $y+1, y+2, .., y+\ell$

are located between x and y (in σ)

and $x+k+1$ is located (in σ) at the right of y

and $y+\ell+1$ is located (in σ) at the left of x

(convention: $(n+2)$ at the left of all values)


σ =
(y+ℓ+1)
(x+1)
(y+ℓ)
(x+k+1)
x
(y+1)
(x+k)
y
σ(i)
σ(j)

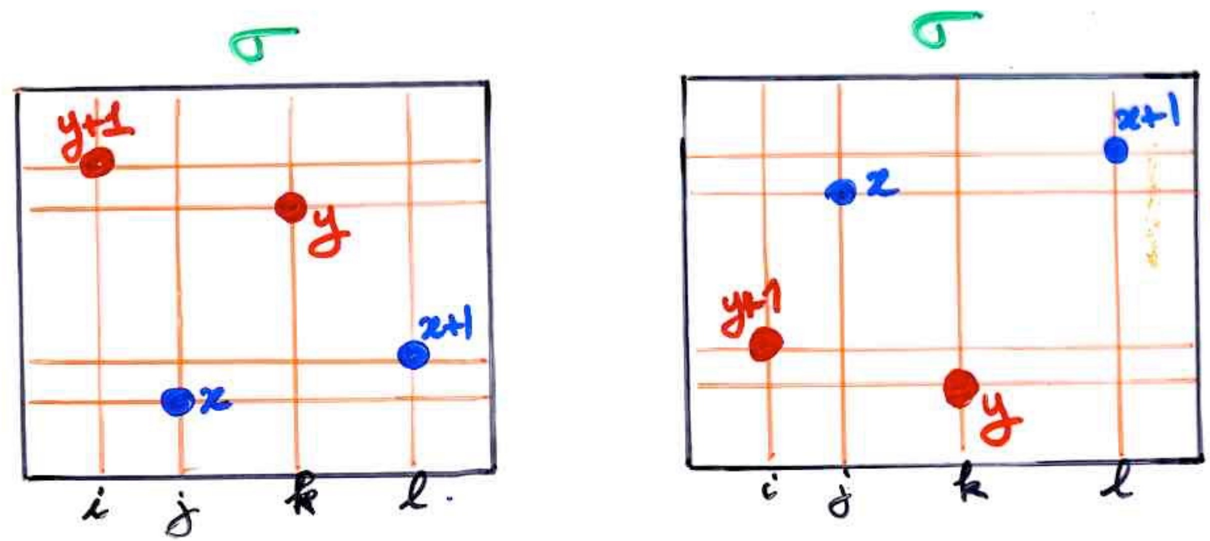
Proposition Bernardi (2008)

The number of permutations of $\mathfrak{S}_n$ with no subsequences of the type

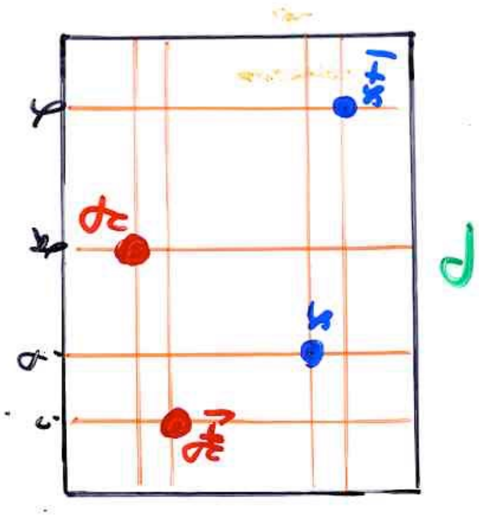
$$\cdots (y+1) \cdots x \cdots y \cdots (x+1) \cdots$$

is the Catalan number C_n

example $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

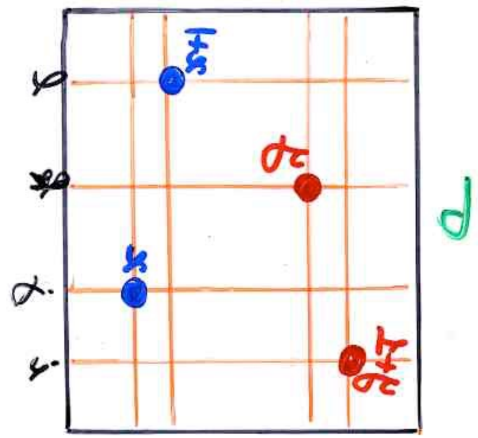


inverse permutation σ^{-1}



permutation with "forbidden patterns" :

31-24 and 24-31



Bernardi permutations

From work of Corteel, Nadeau, Steingrímsson, Williams

we know that parameter

"number of crossing" in alternating tableaux :

same distribution as

"q-analog of Laguerre histories"