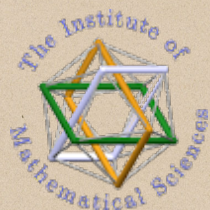


Course IMSc, Chennai, India

January-March 2018



The cellular ansatz:
bijective combinatorics and quadratic algebra

Xavier Viennot

CNRS, LaBRI, Bordeaux

www.viennot.org

mirror website

www.imsc.res.in/~viennot

Chapter 3
Tableaux for the PASEP quadratic algebra

Ch3c

(first part)

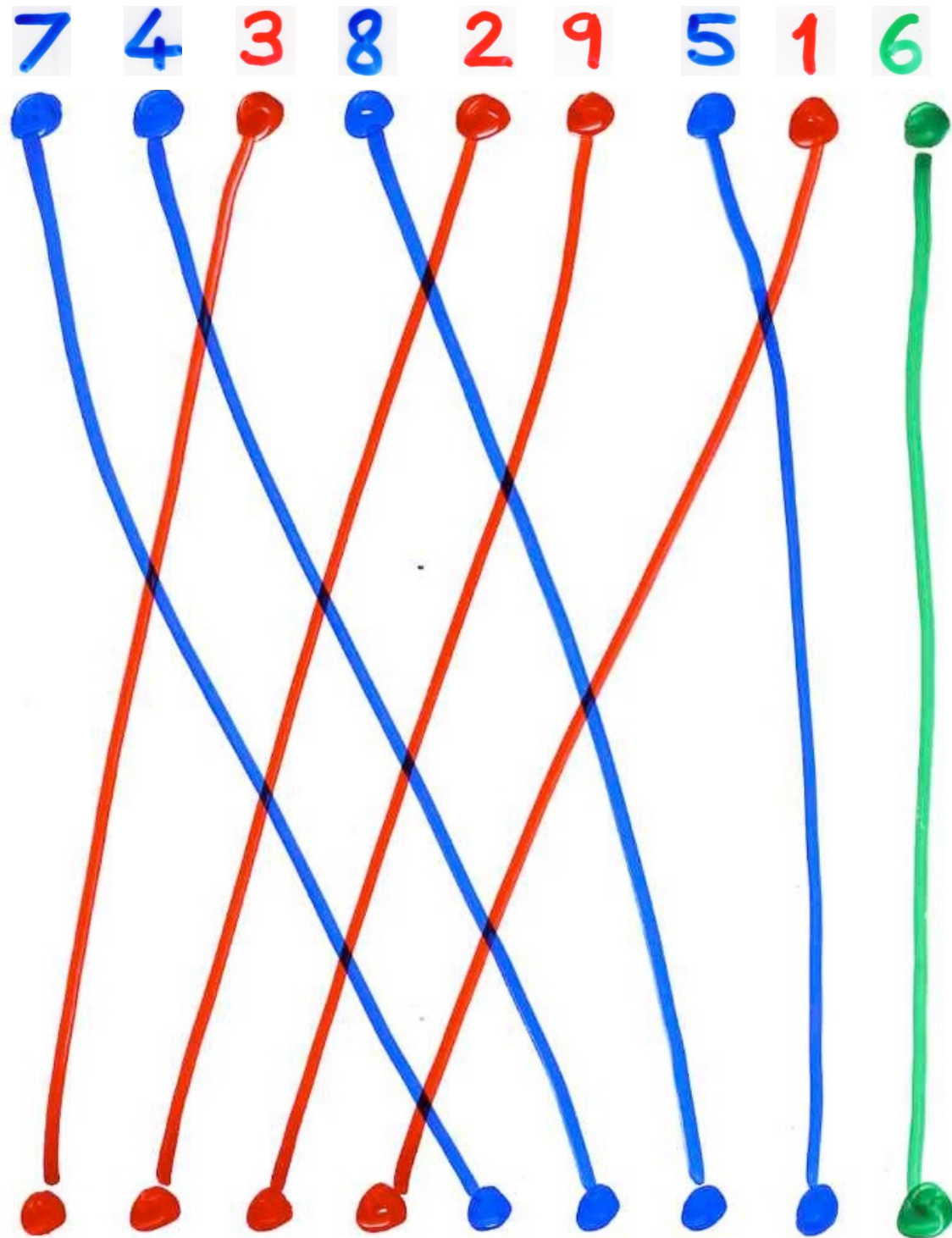
IMSc, Chennai
February 22, 2018

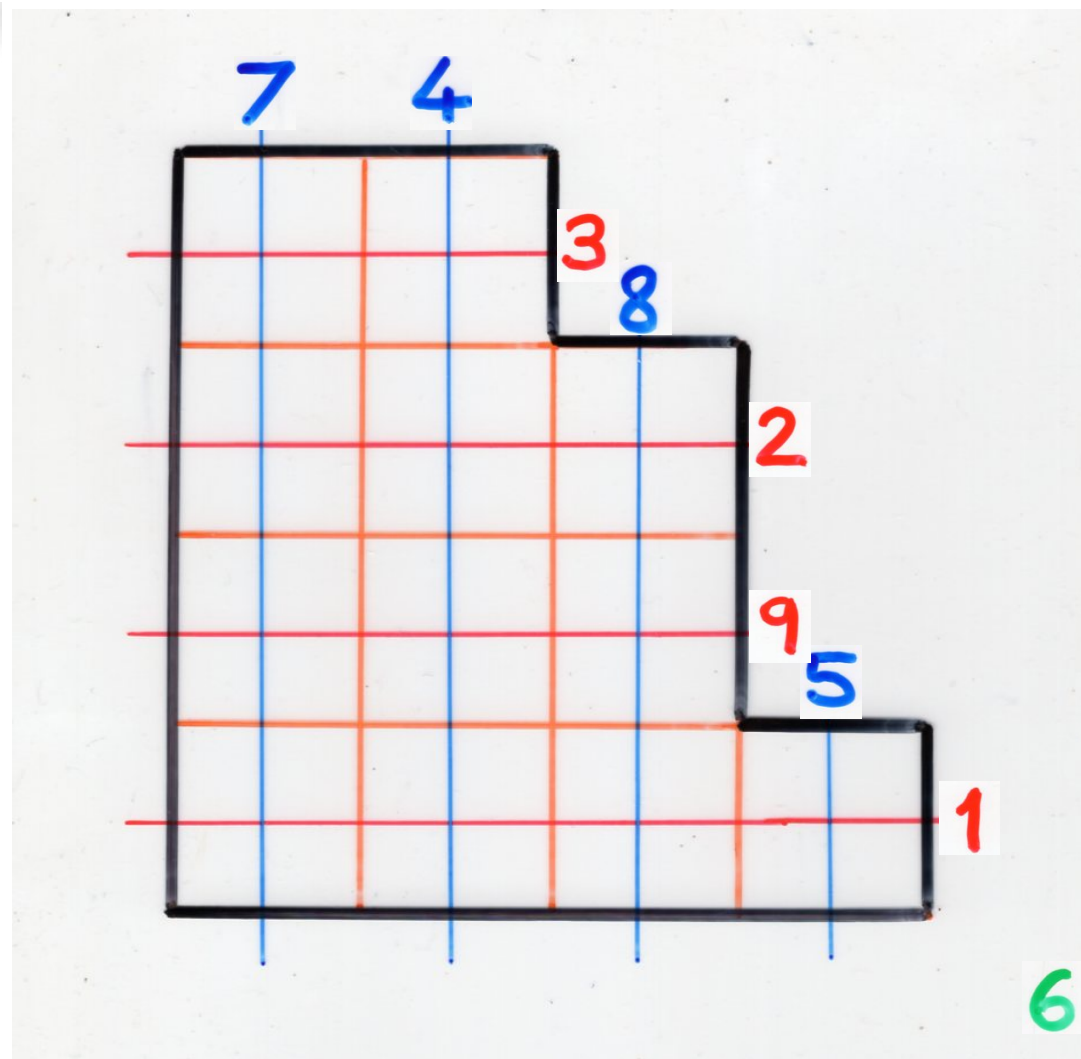
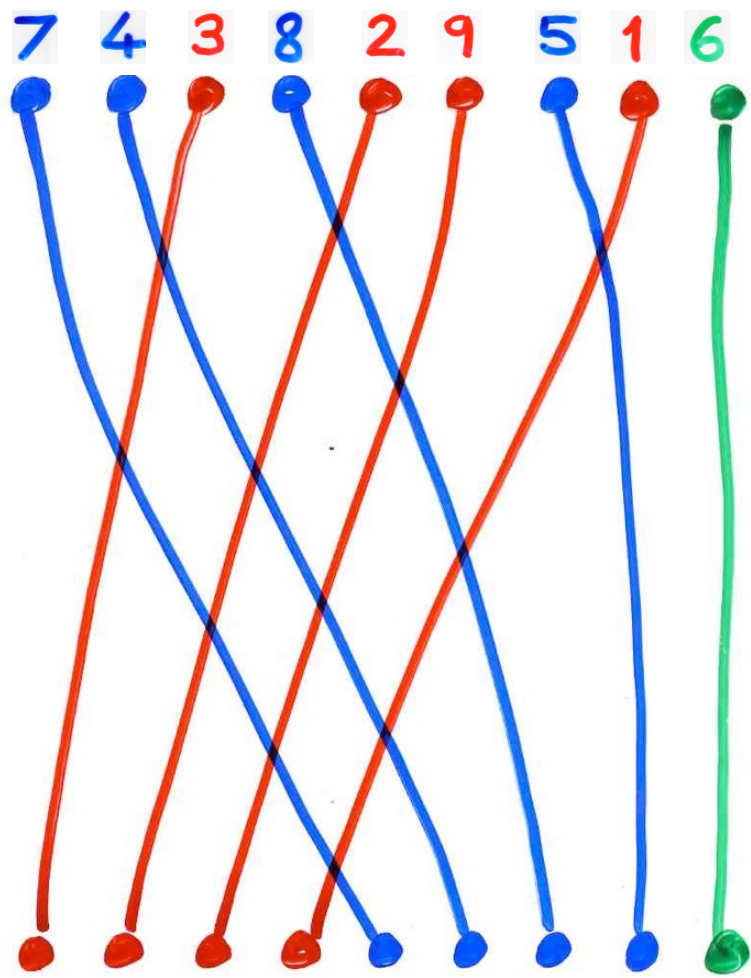
Xavier Viennot
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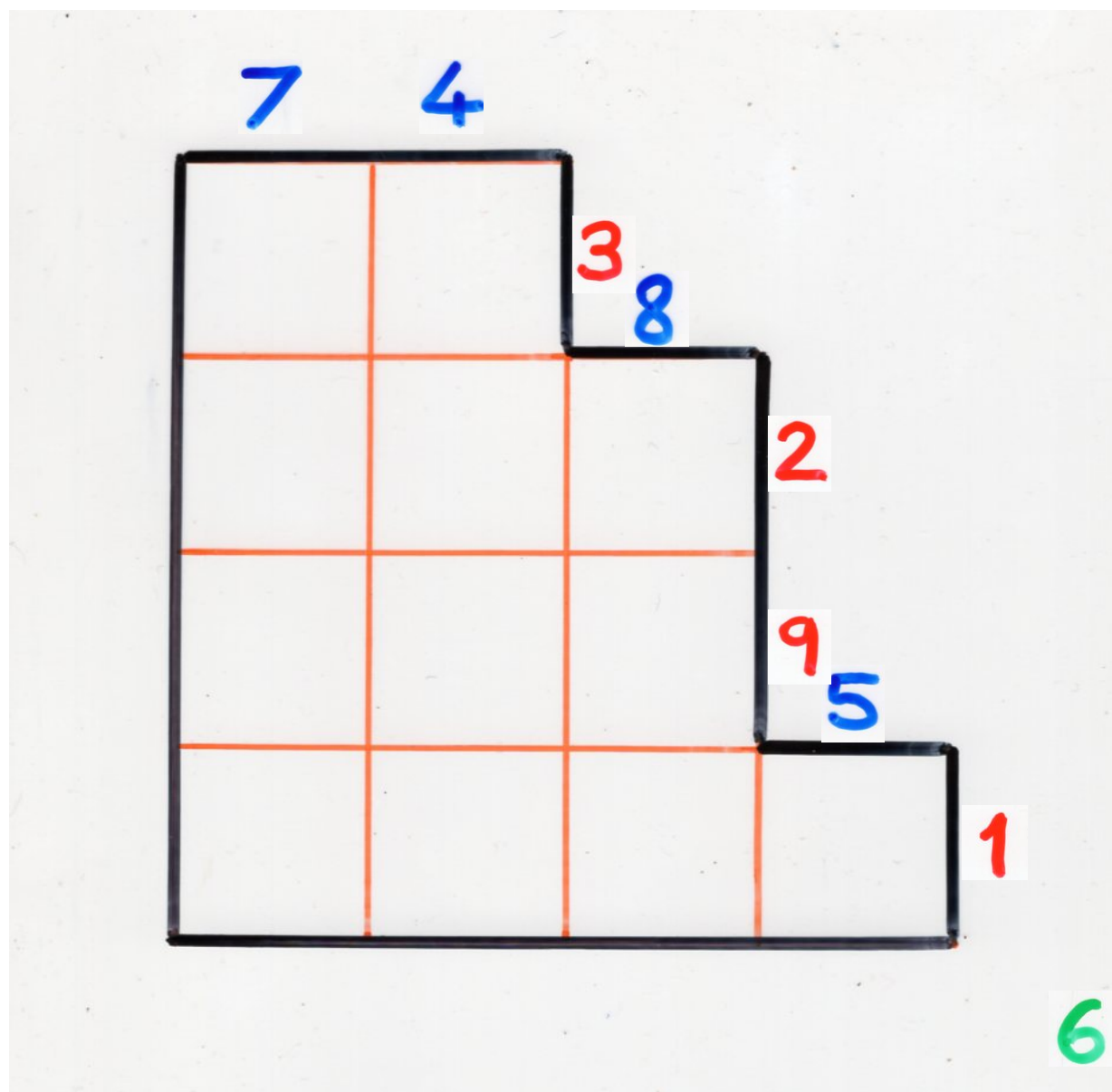
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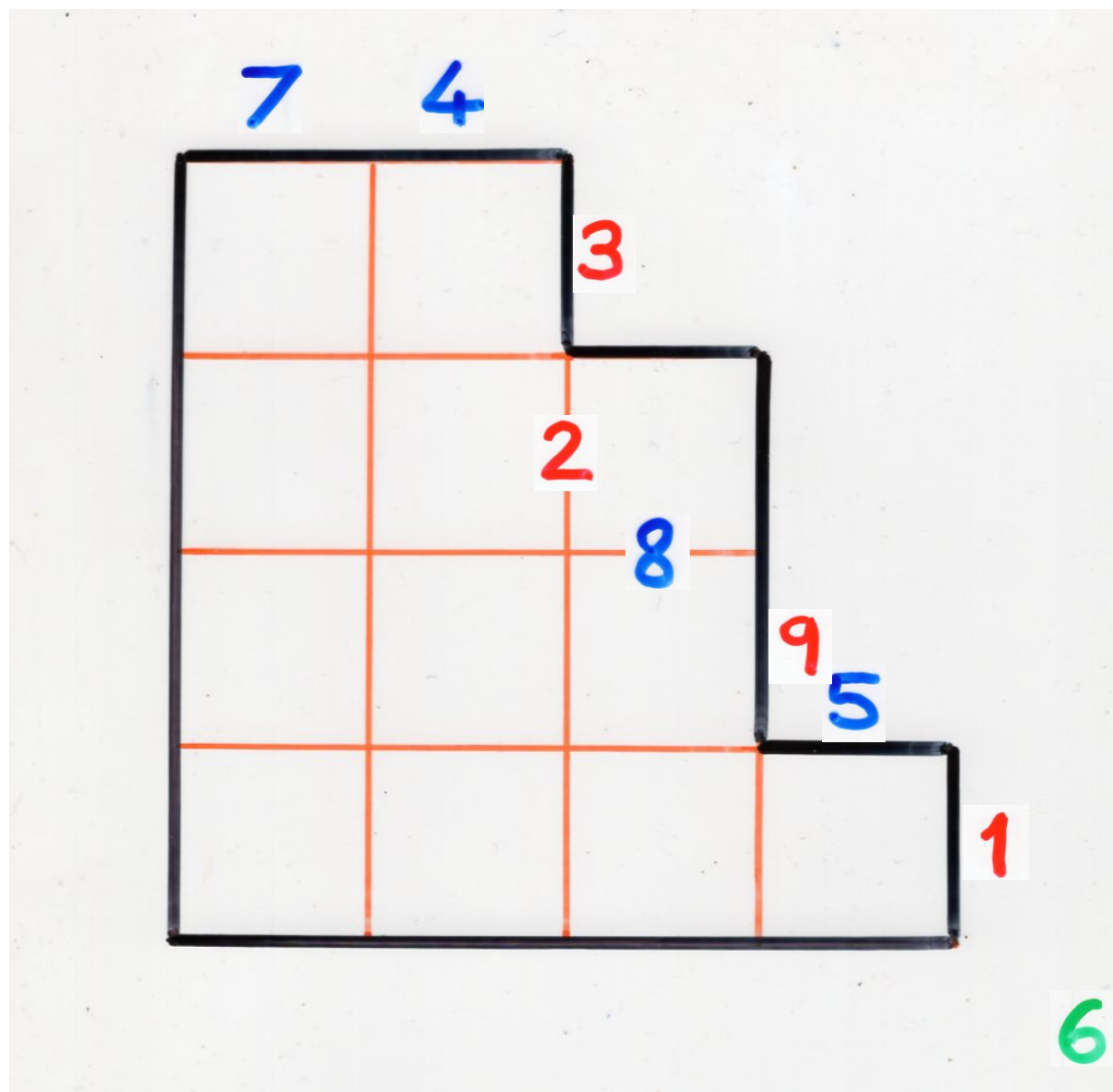
Reminding Ch3b

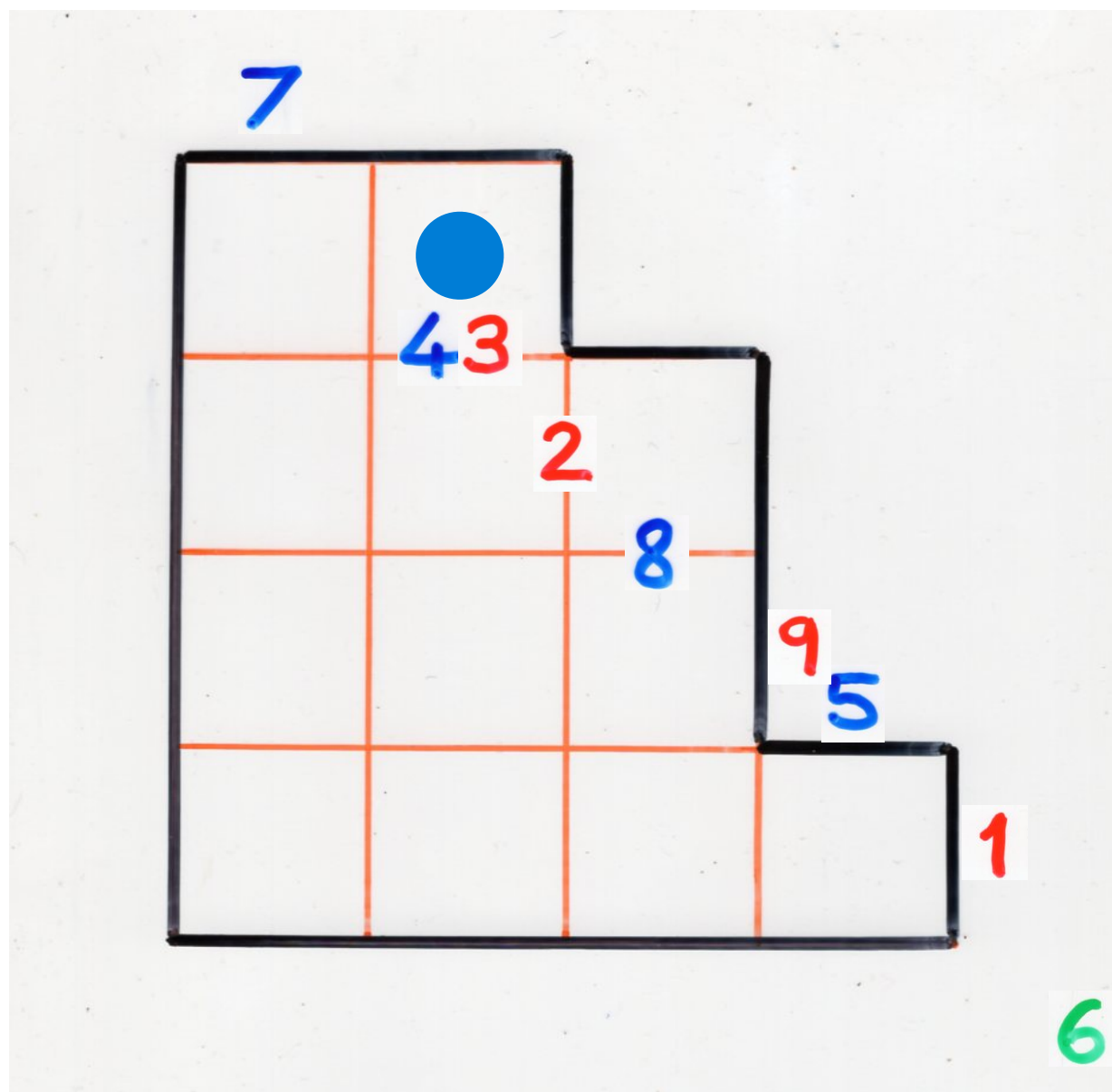
The “exchange-fusion” algorithm

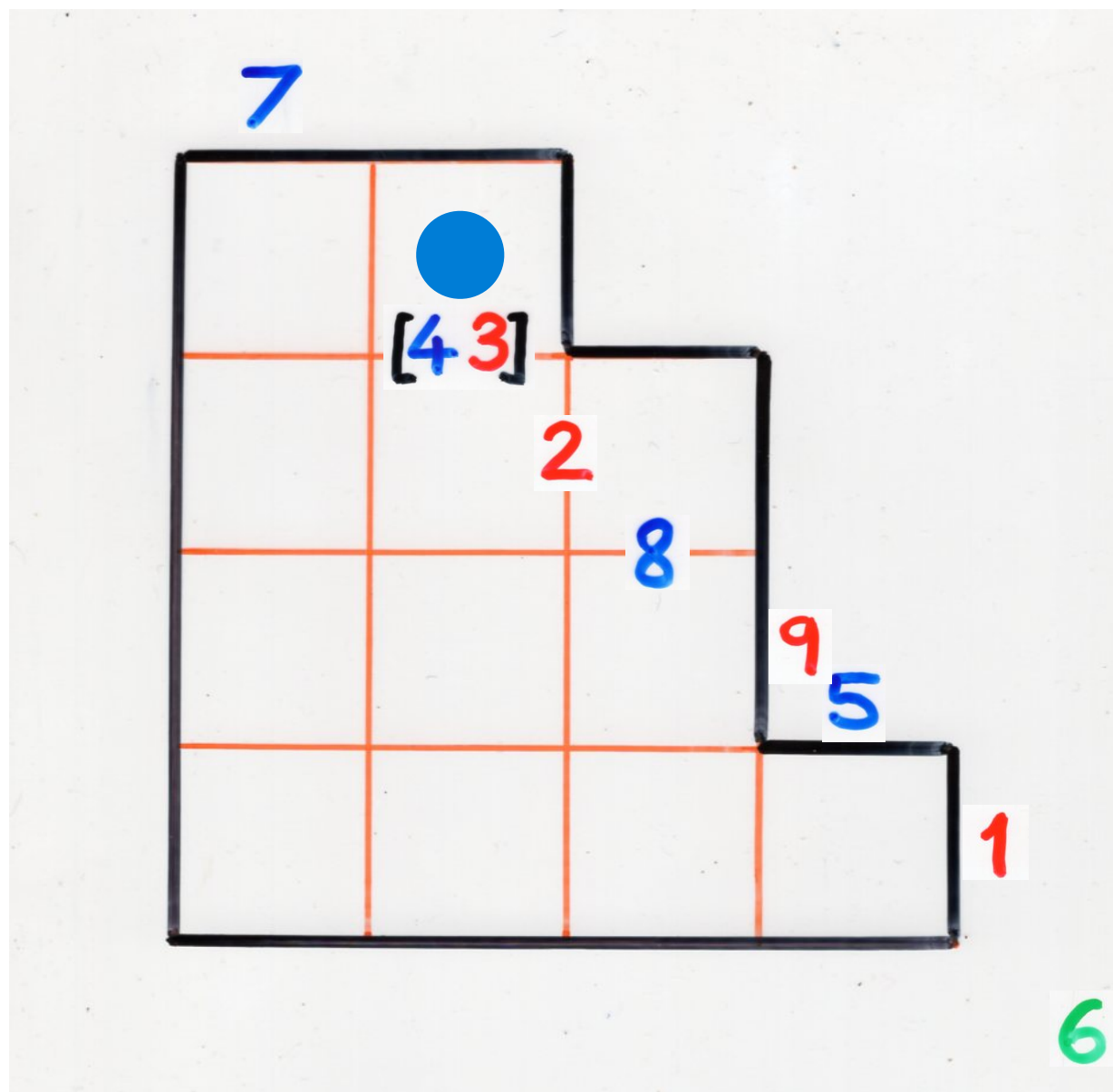


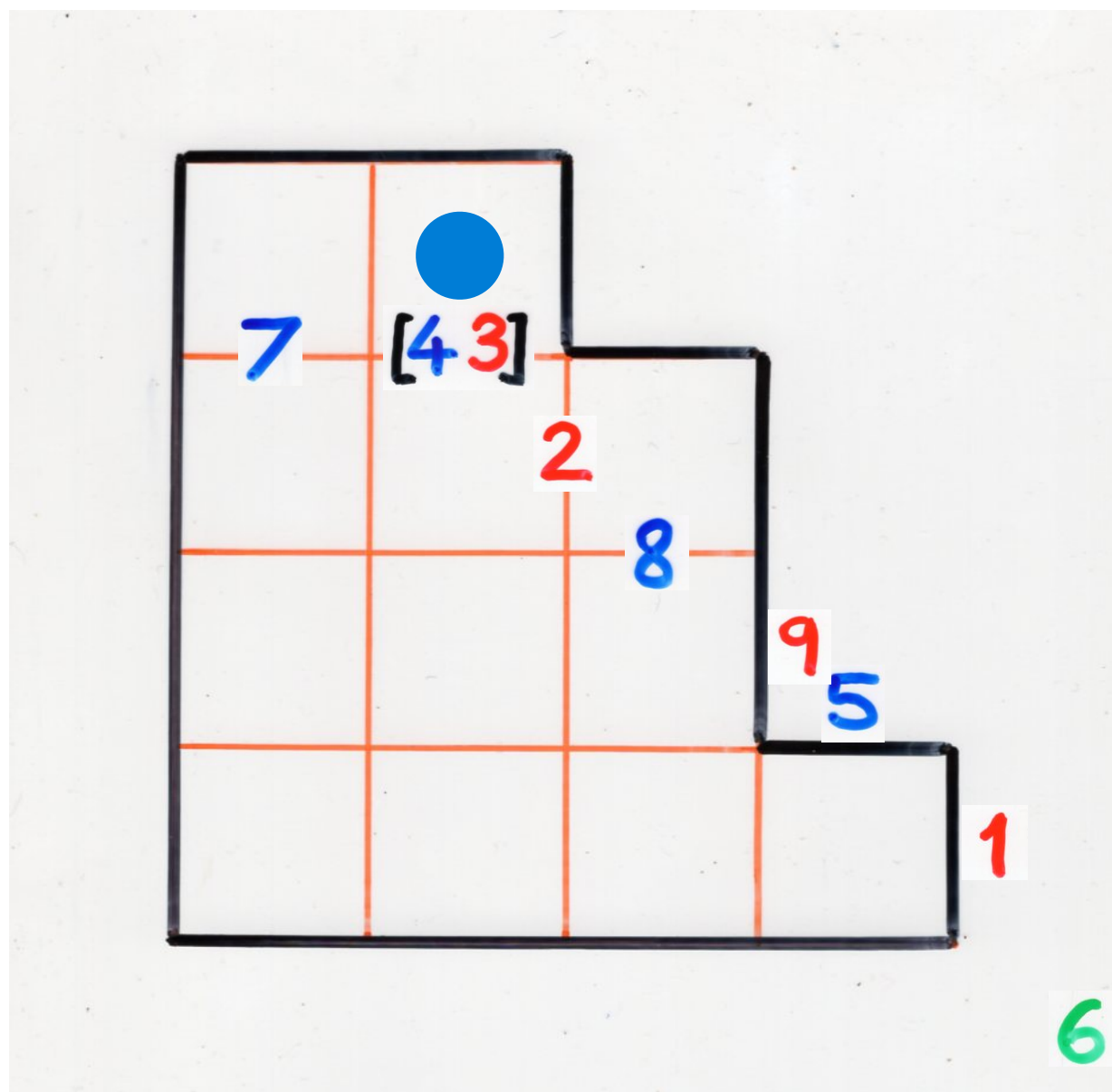


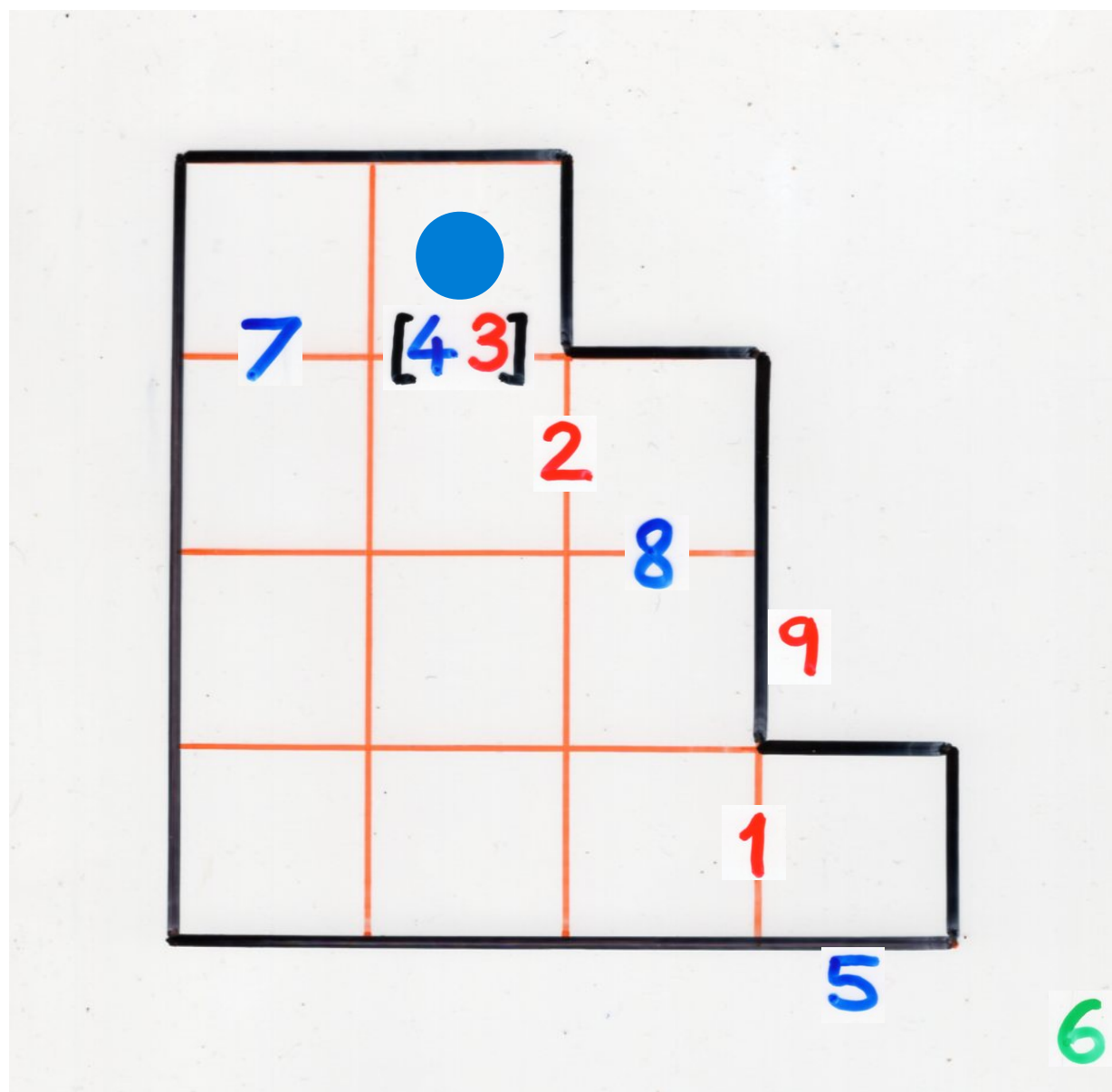


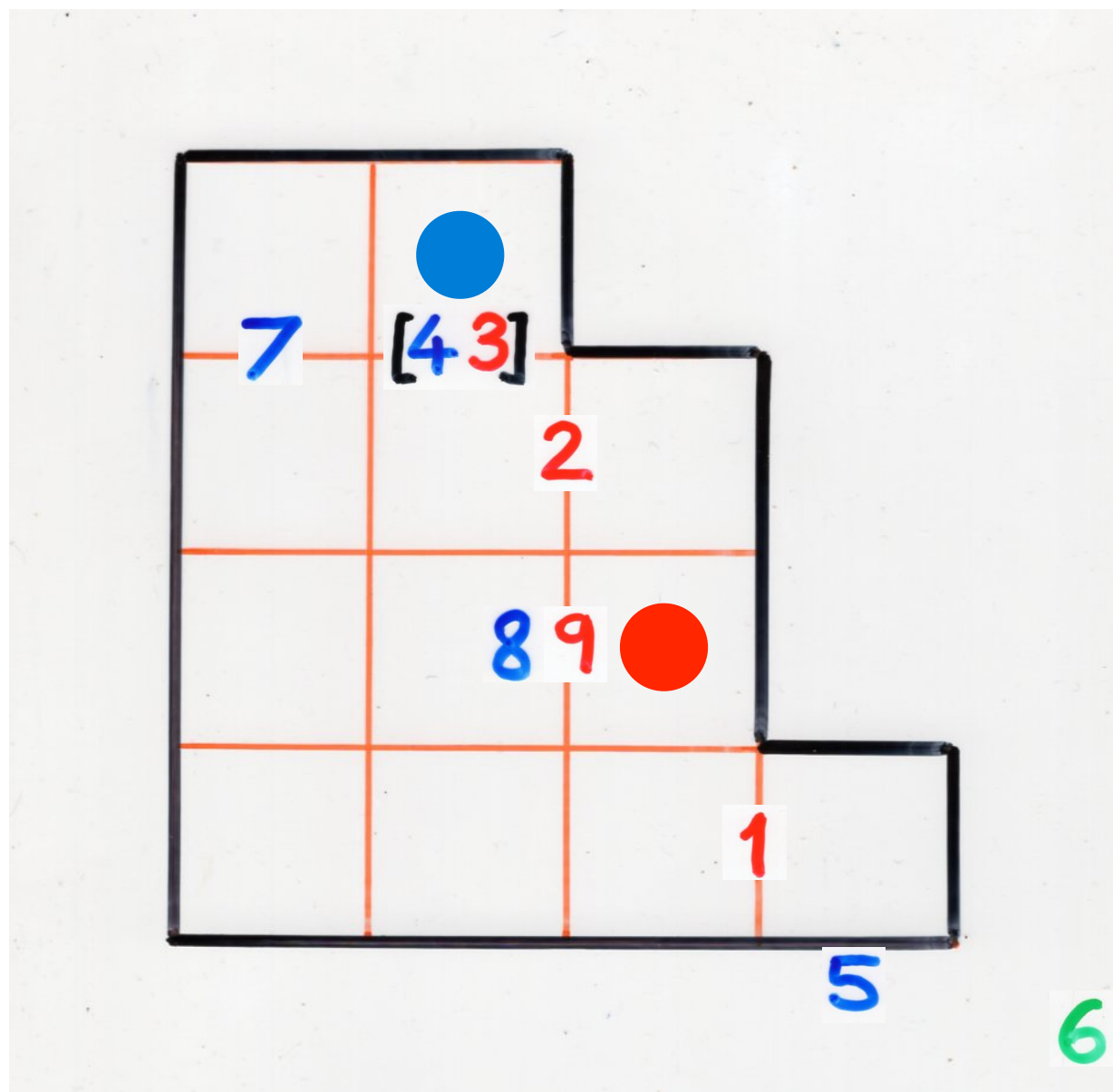


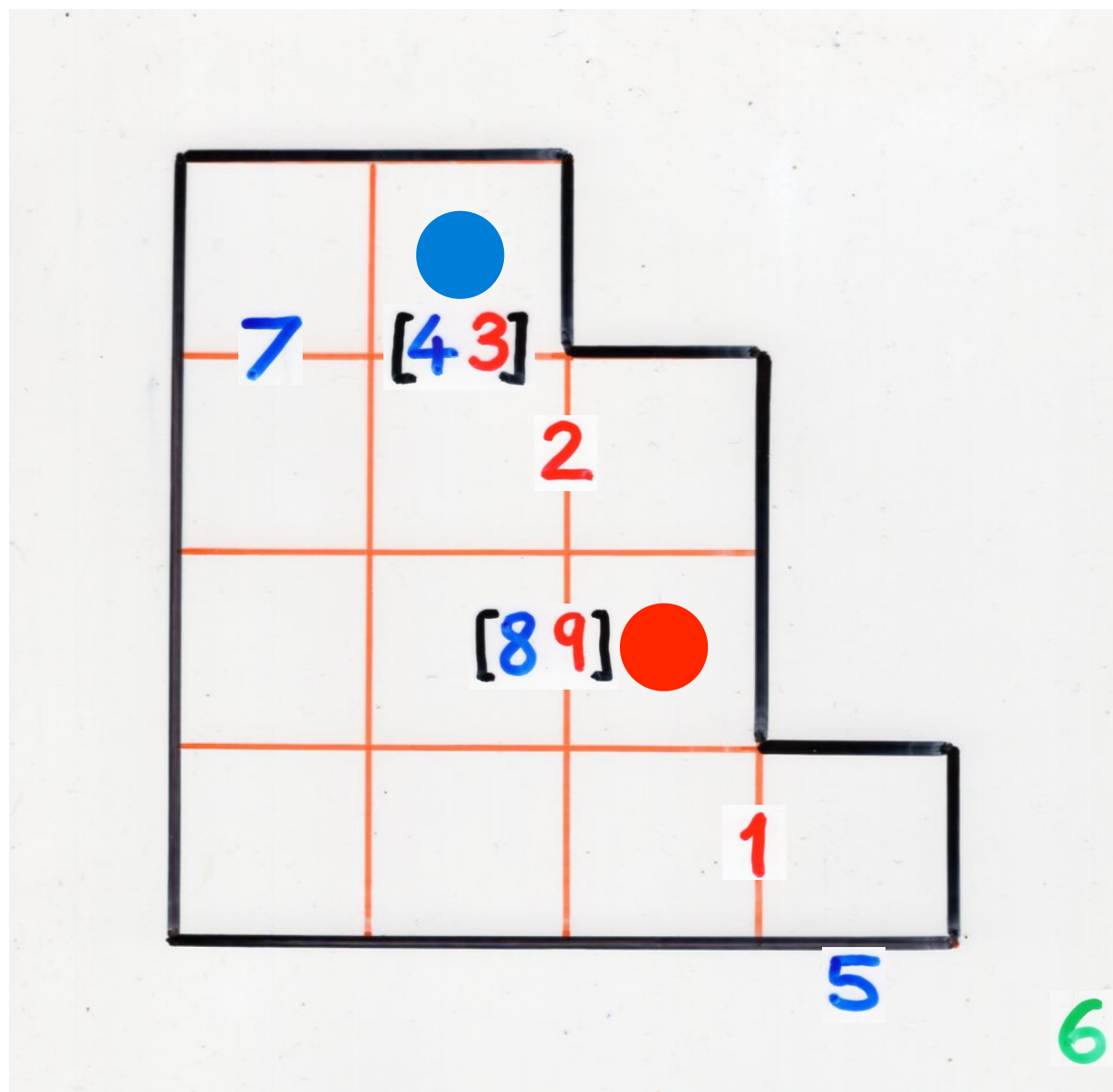


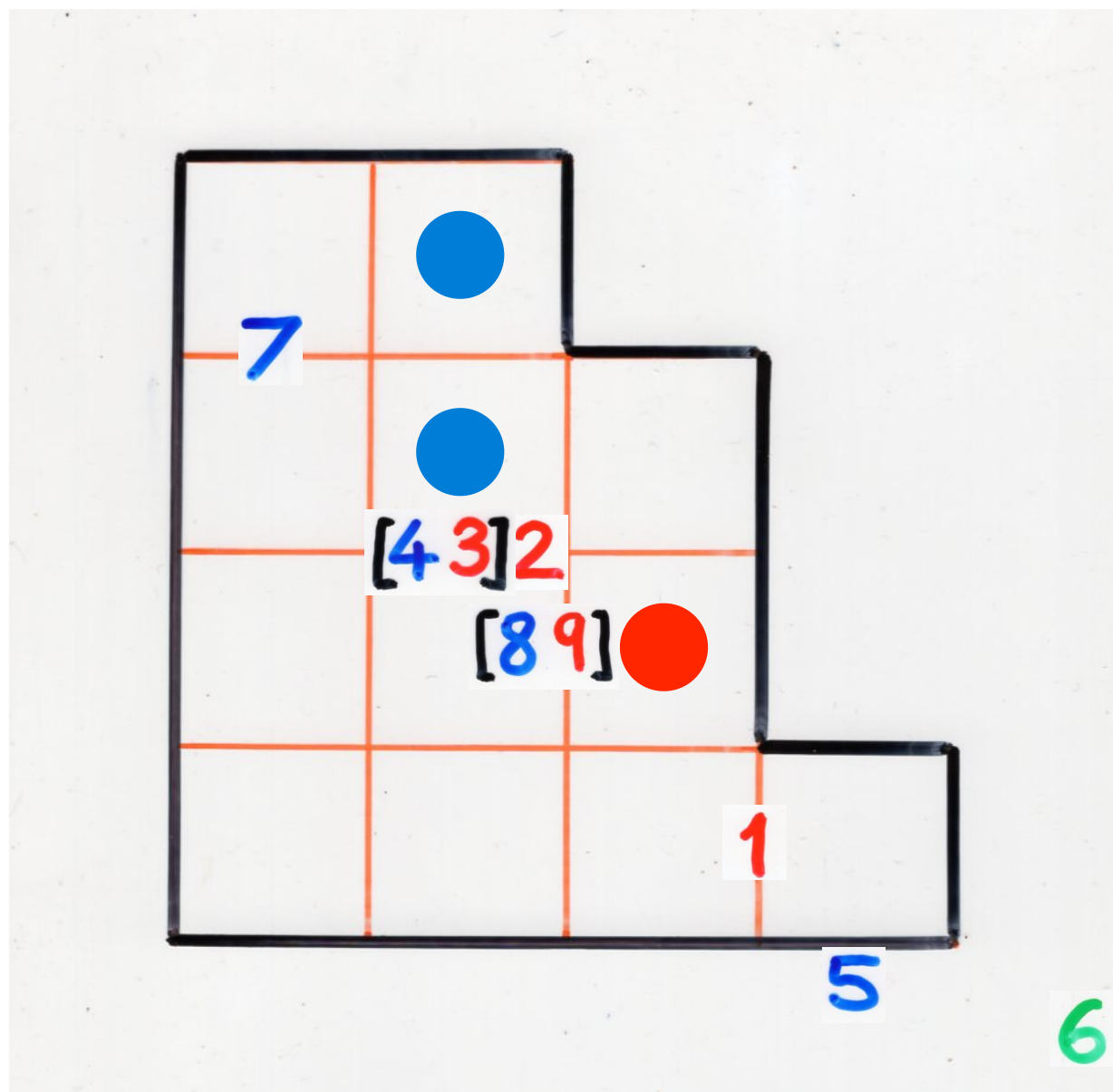


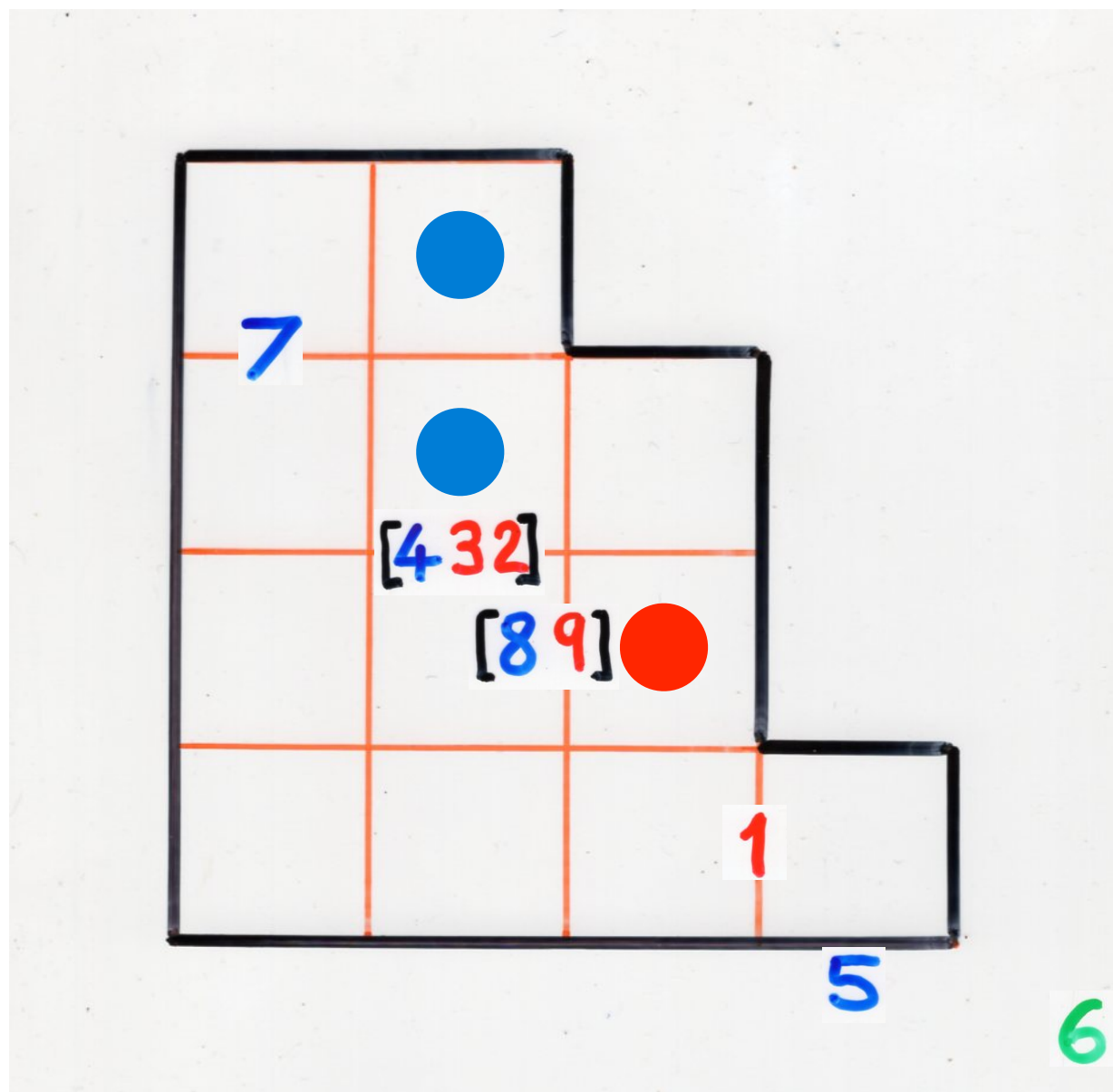


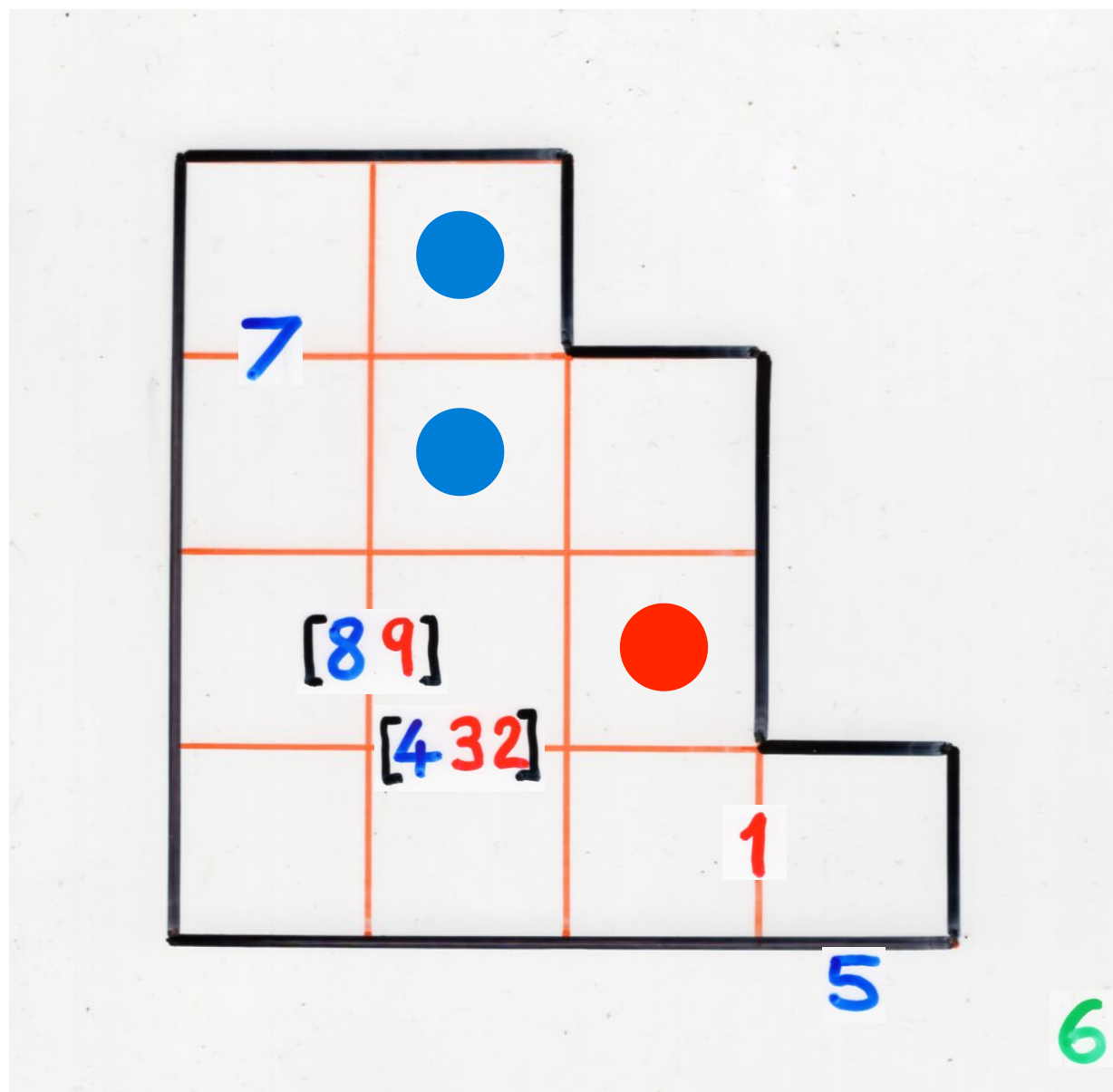


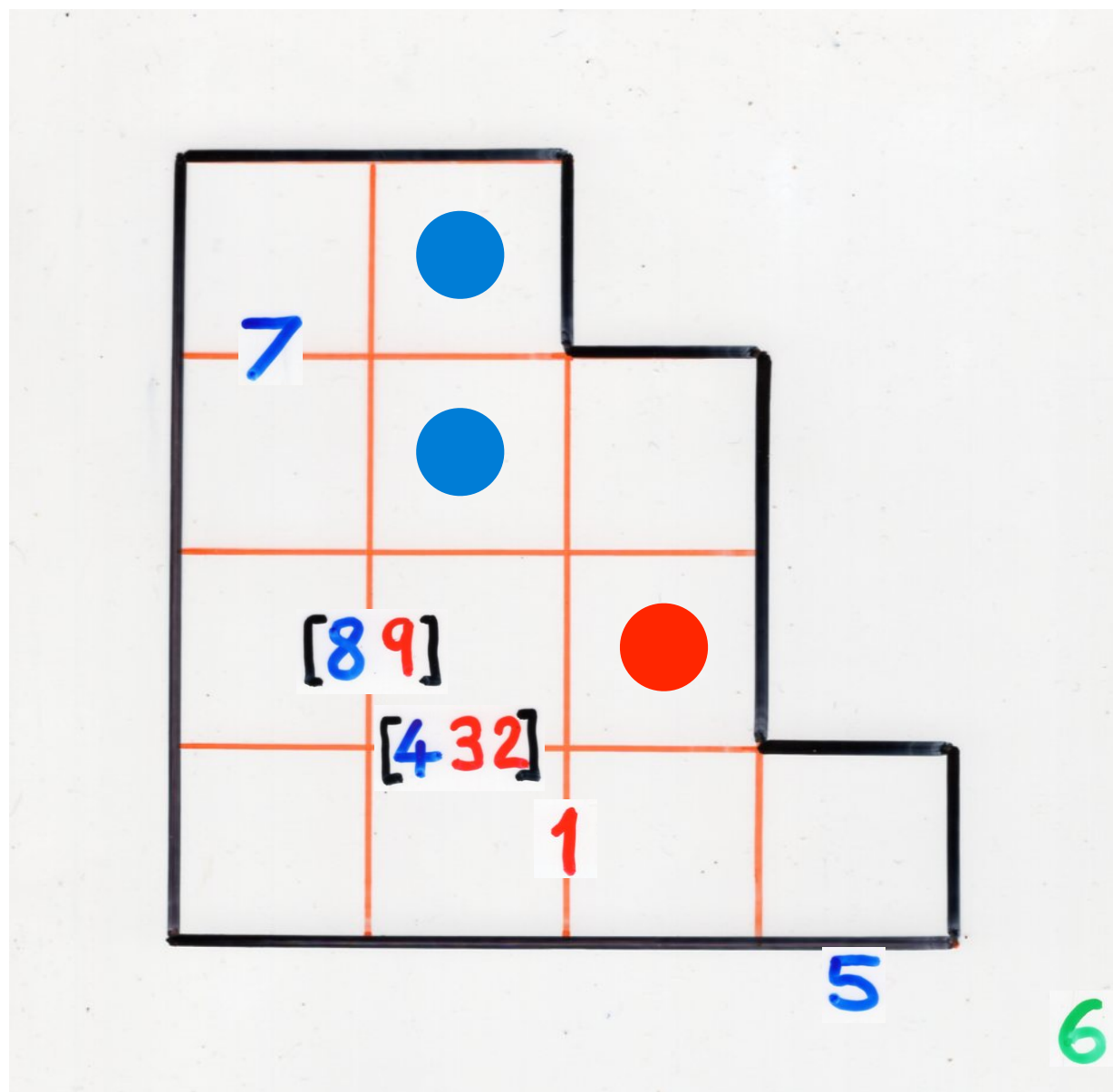


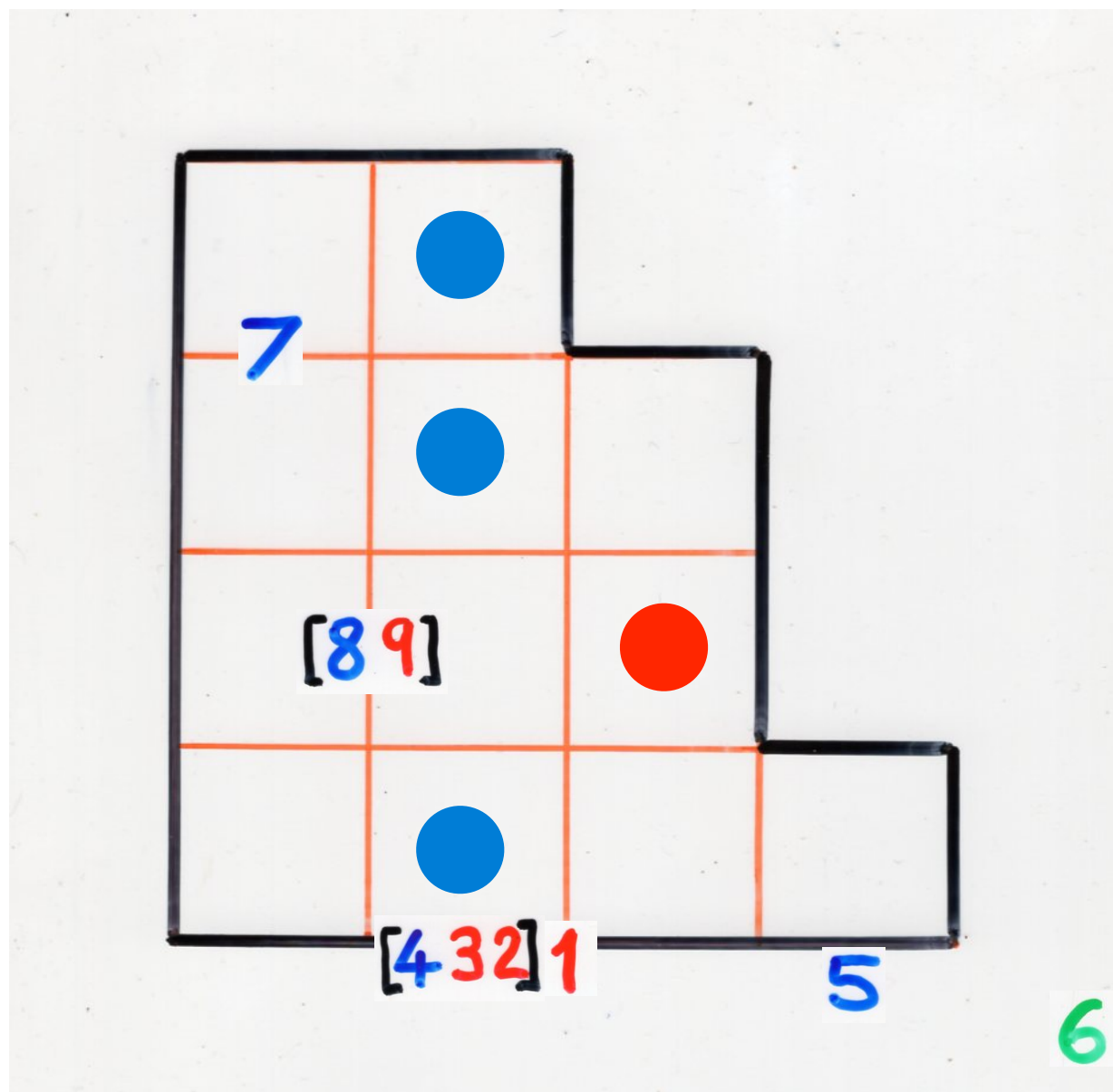


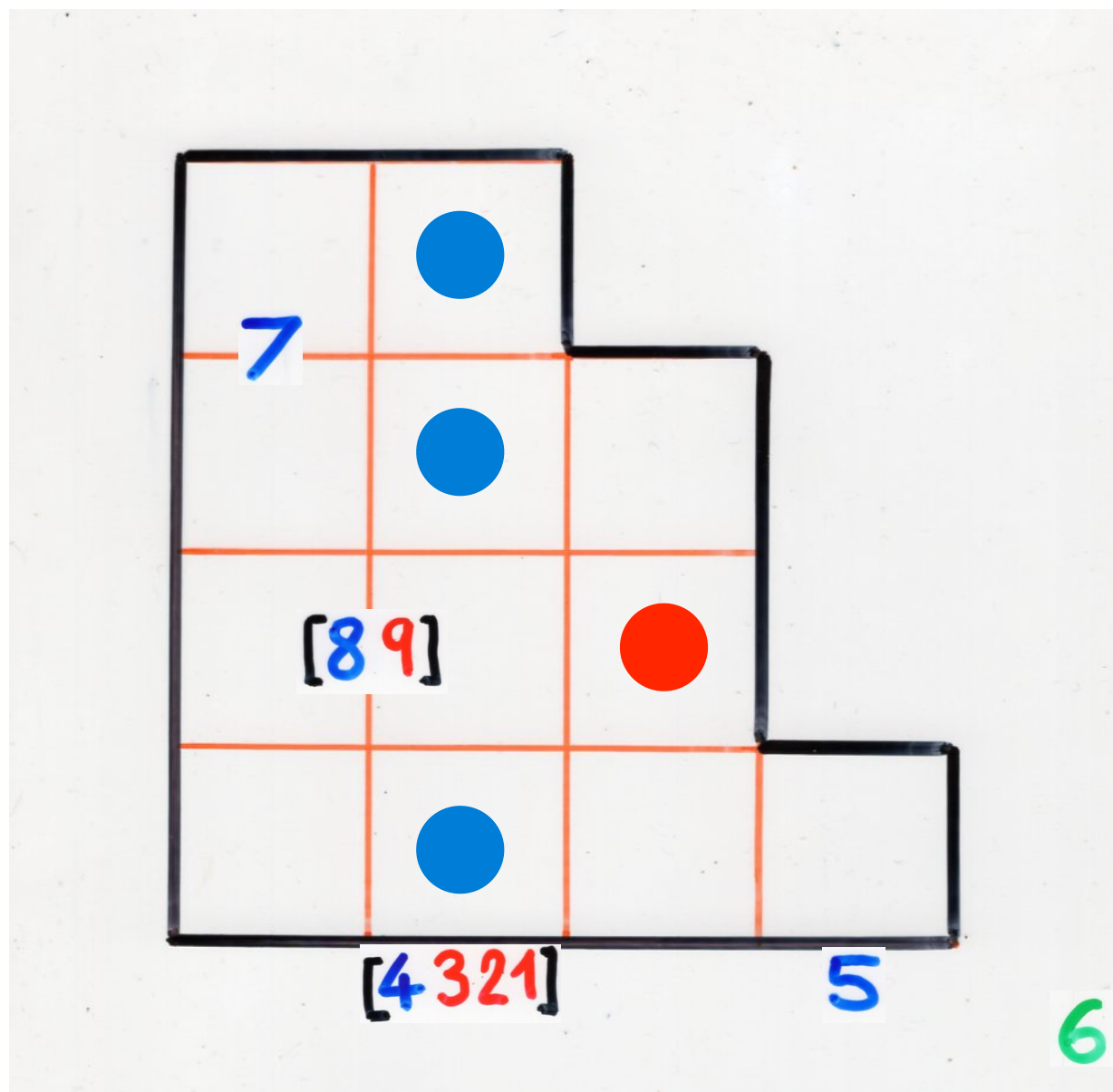


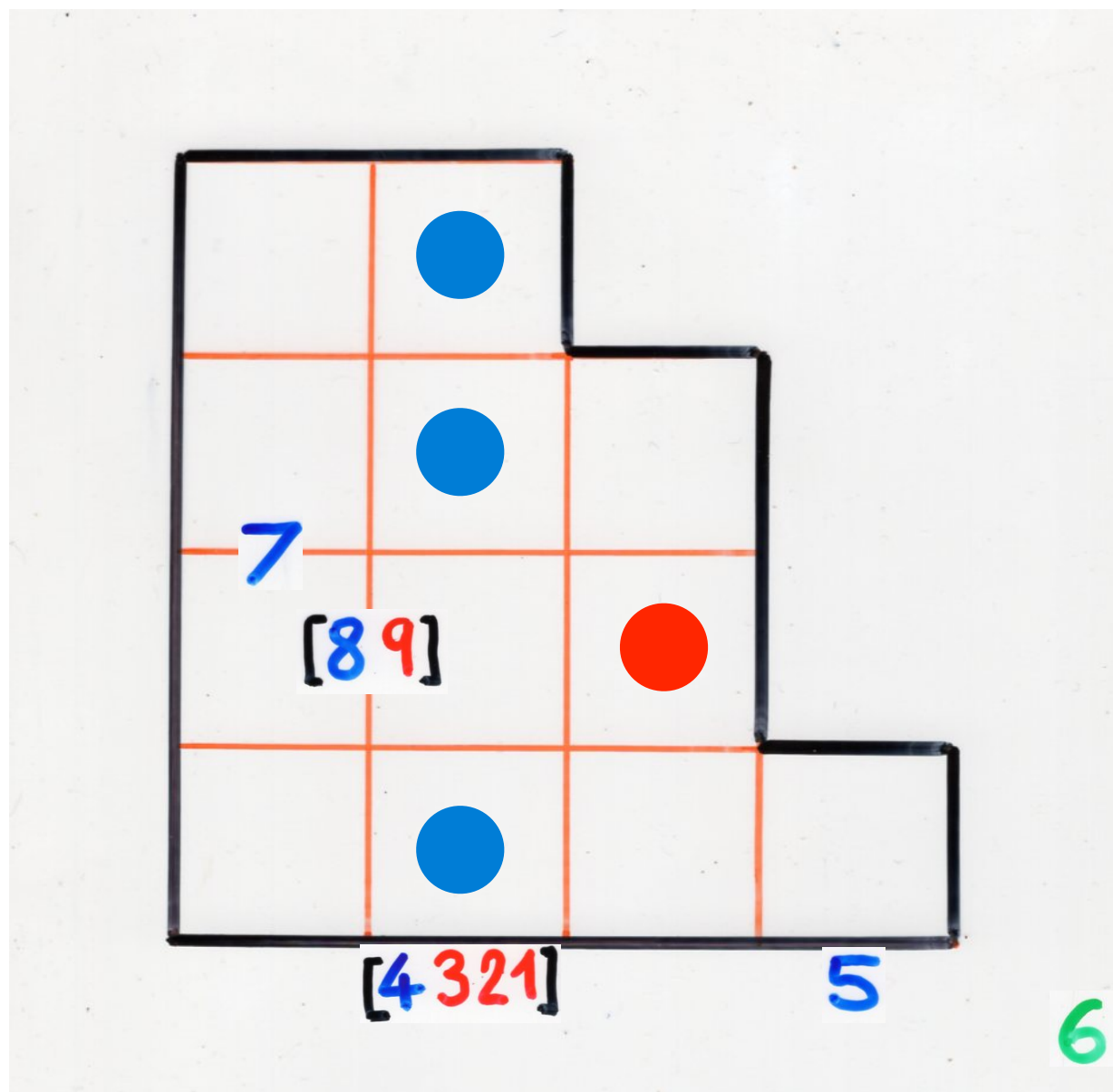


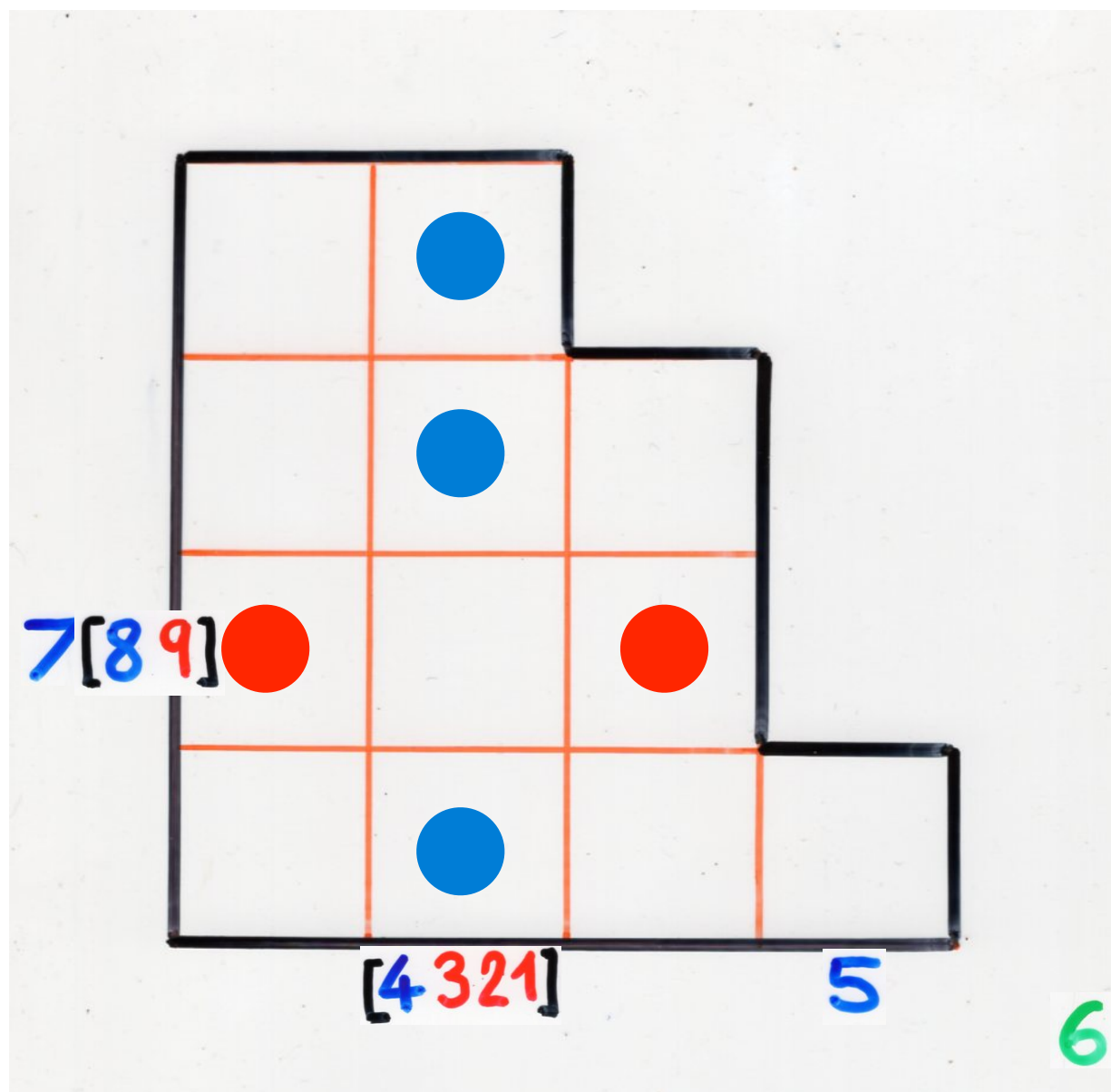


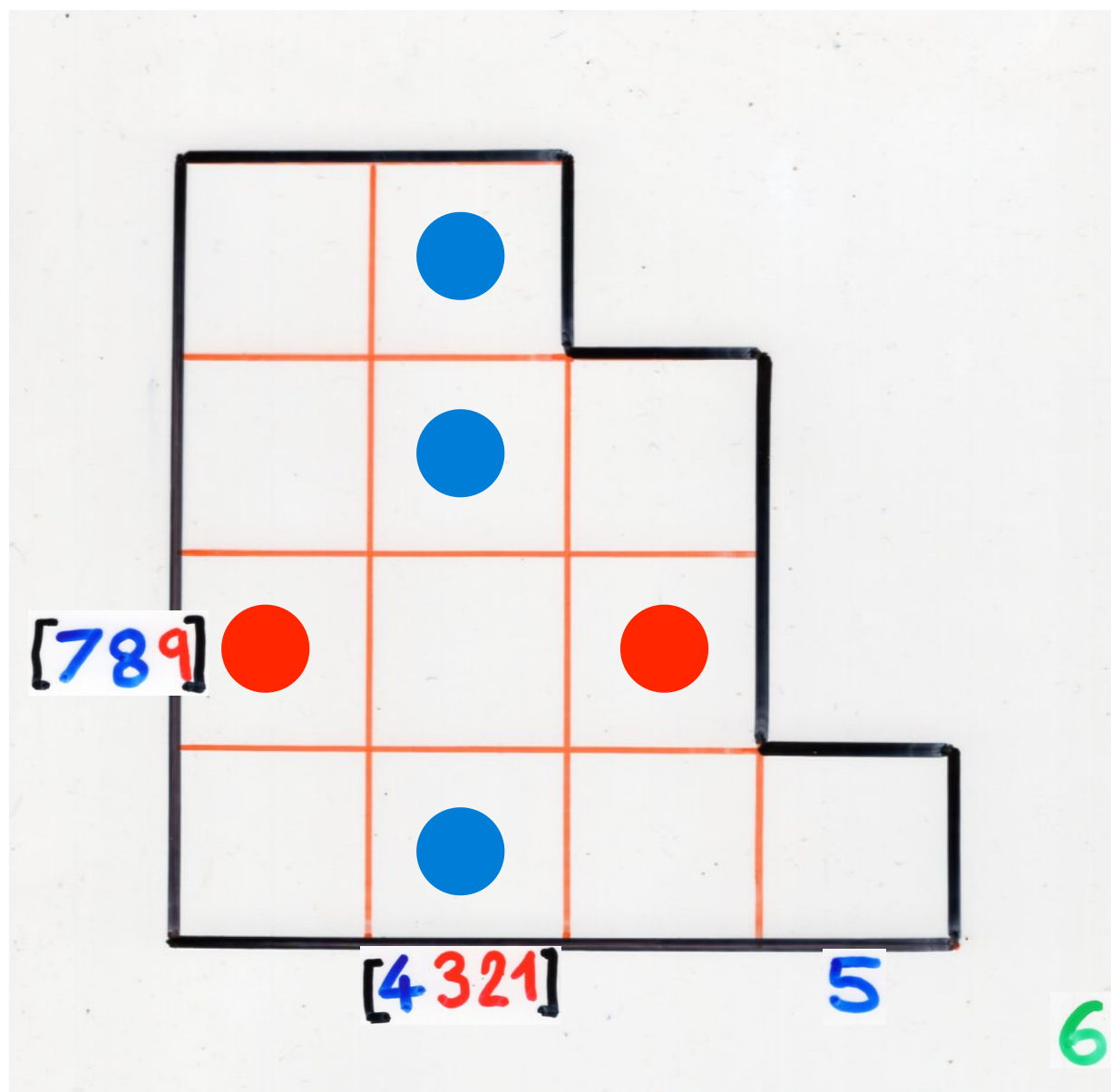


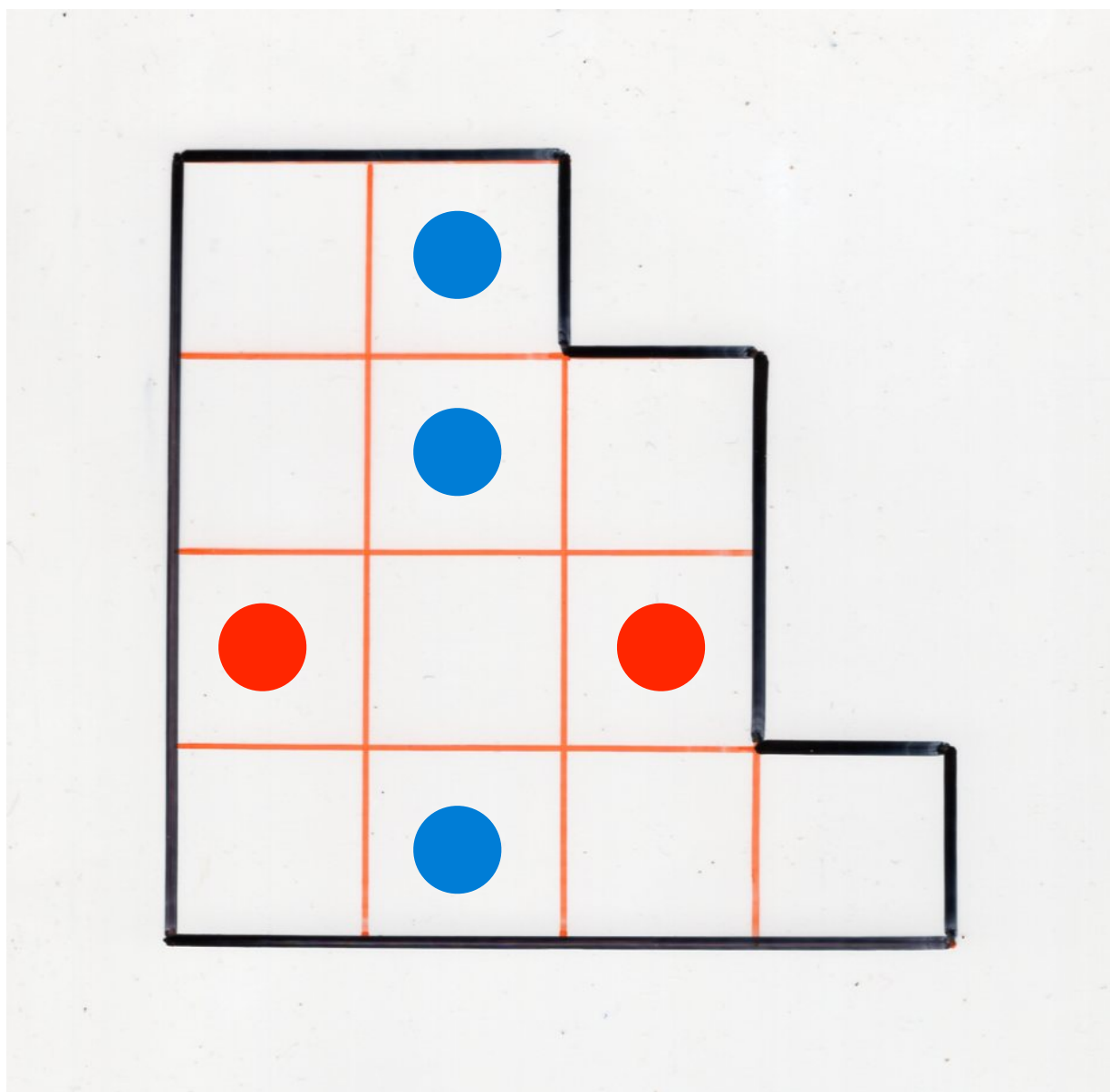








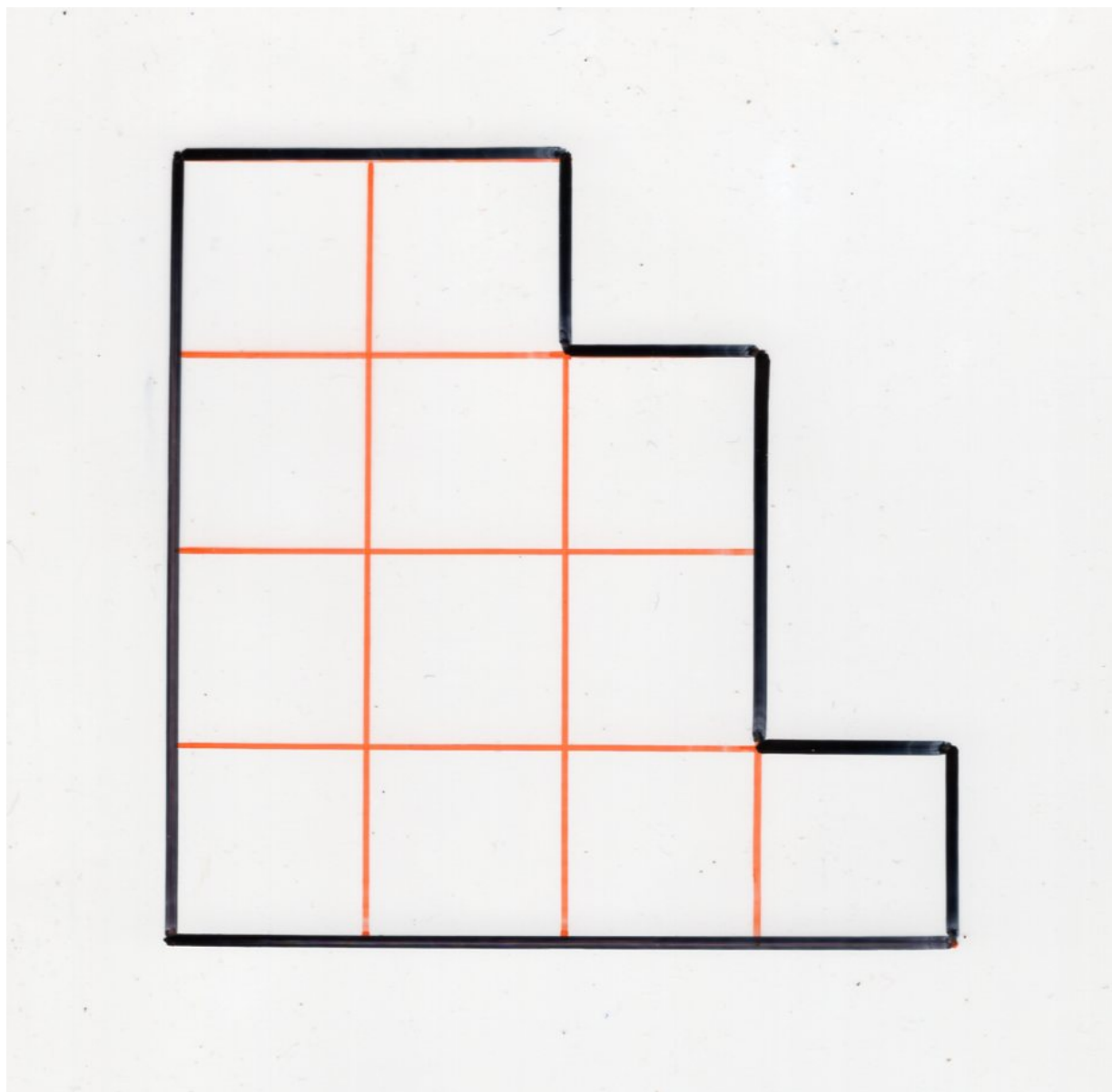


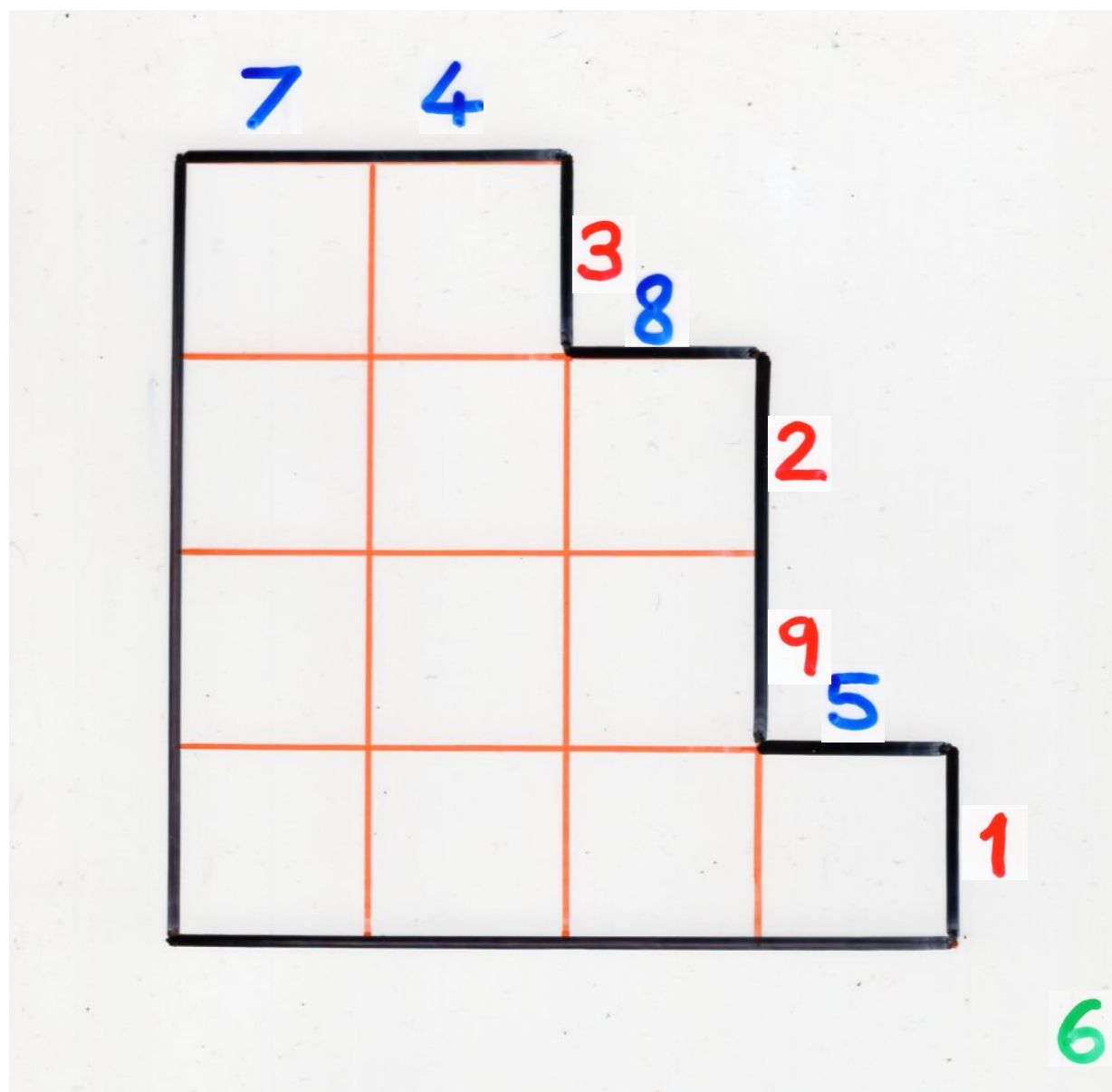


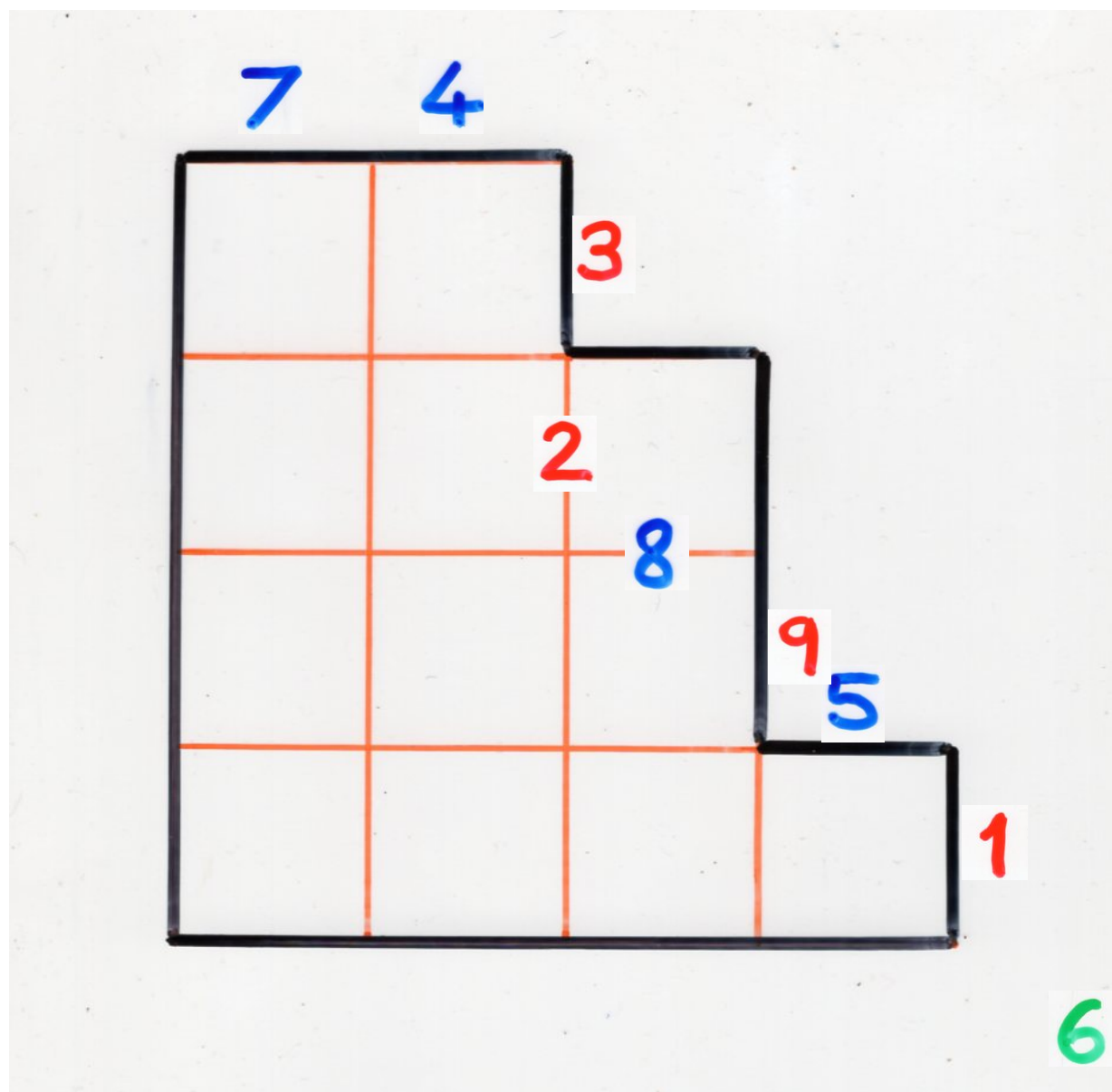
Reminding Ch3b

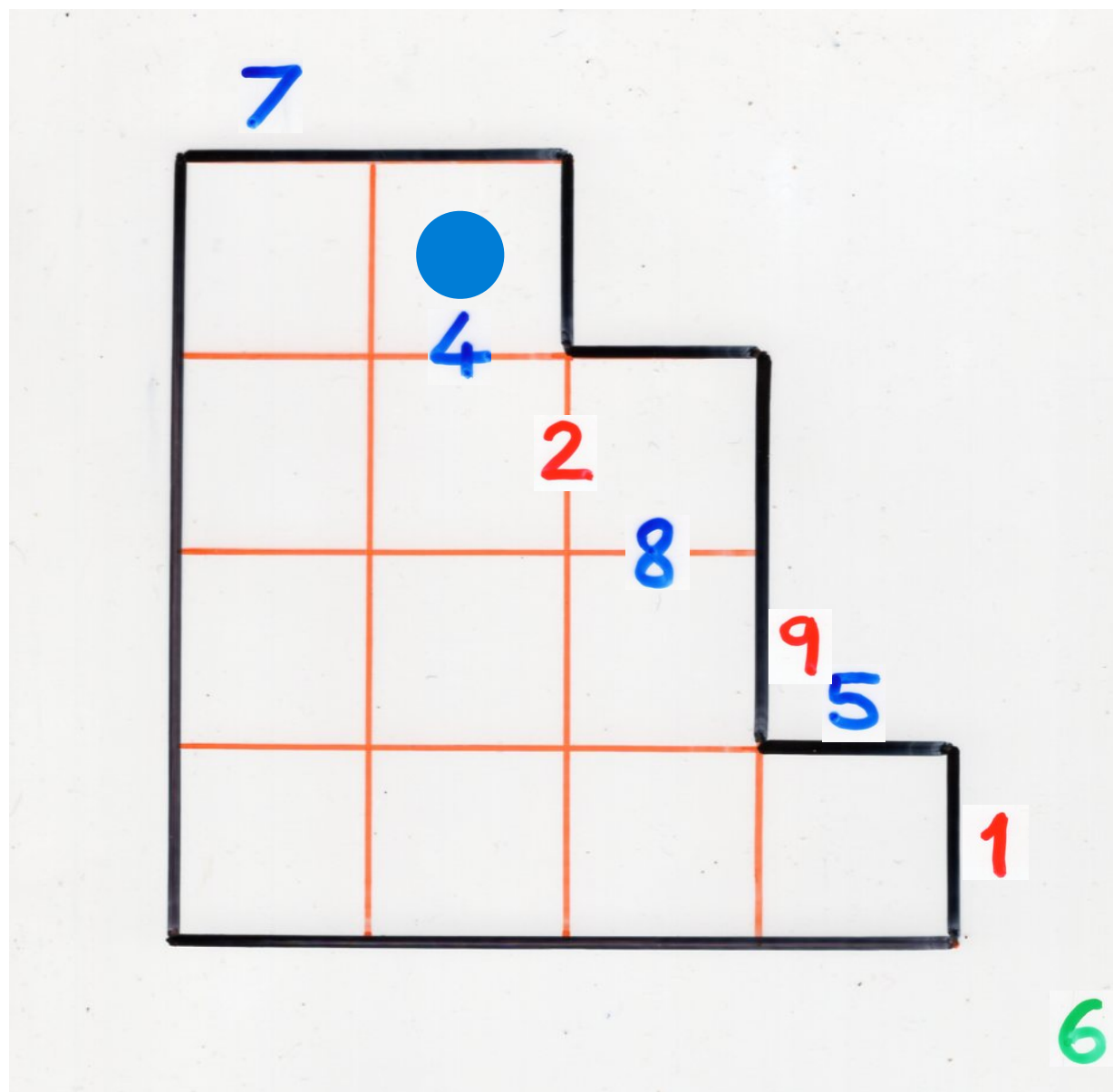
The “exchange-delete” algorithm

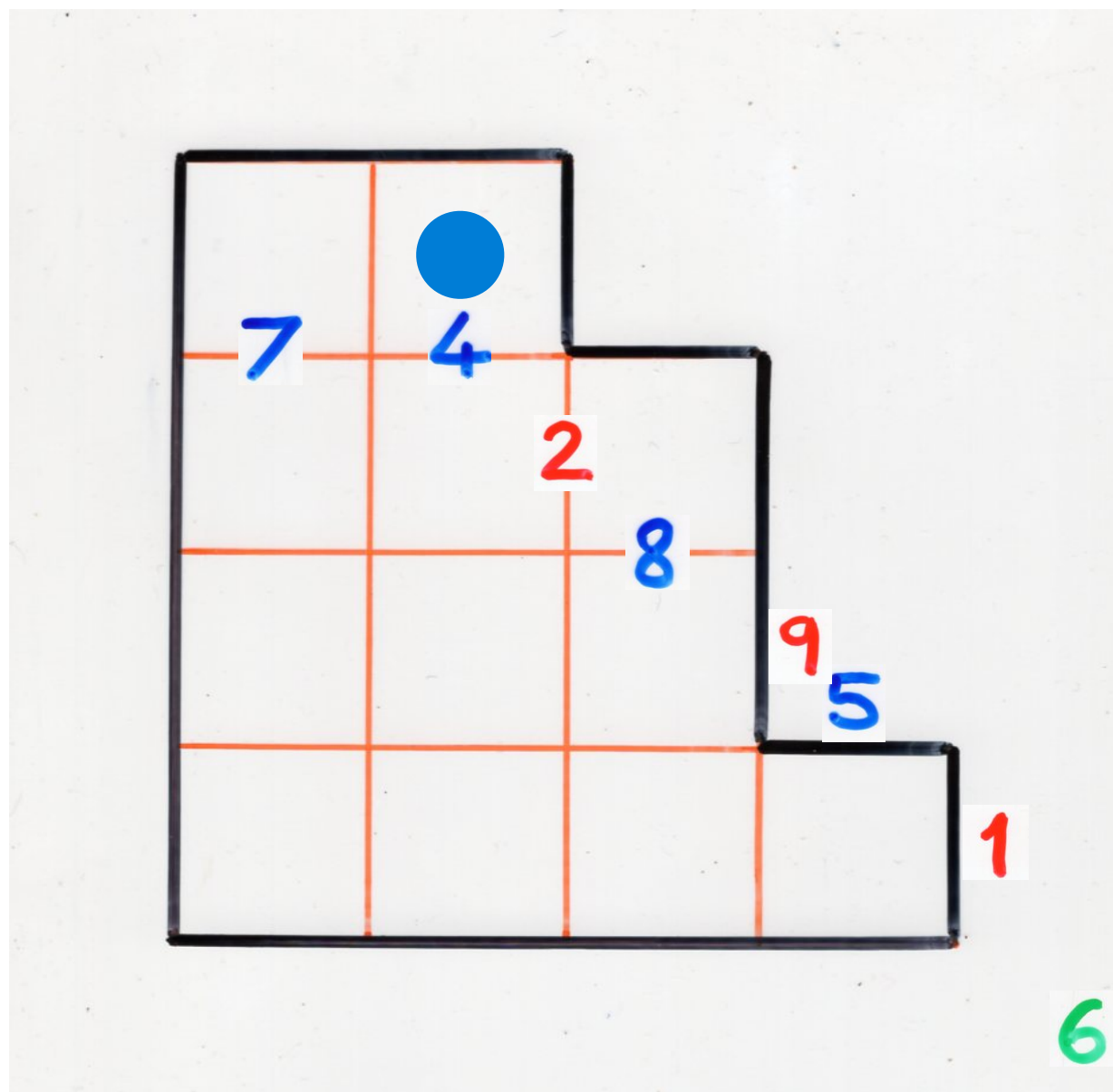
7 4 3 8 2 9 5 1 6

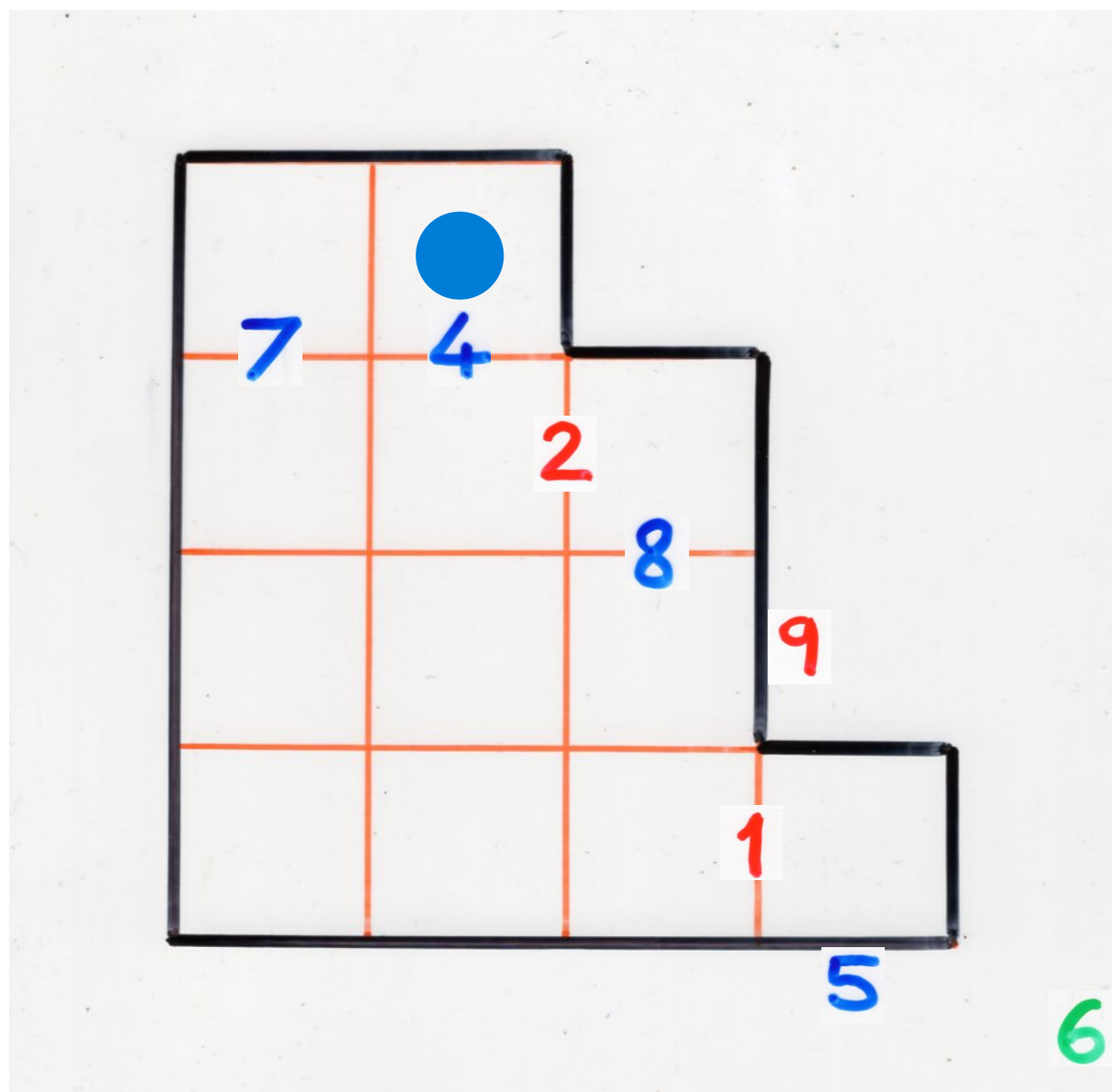


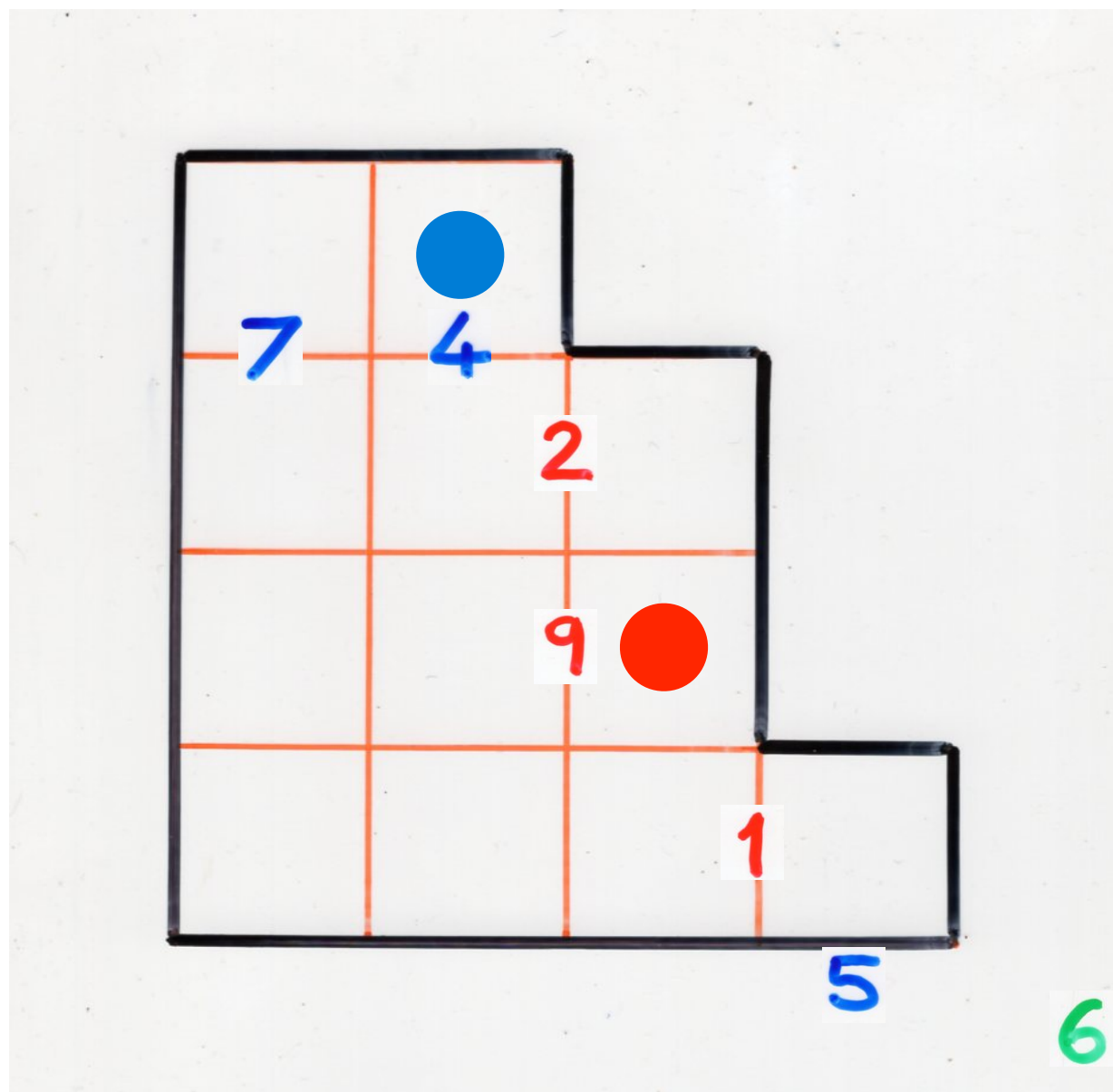


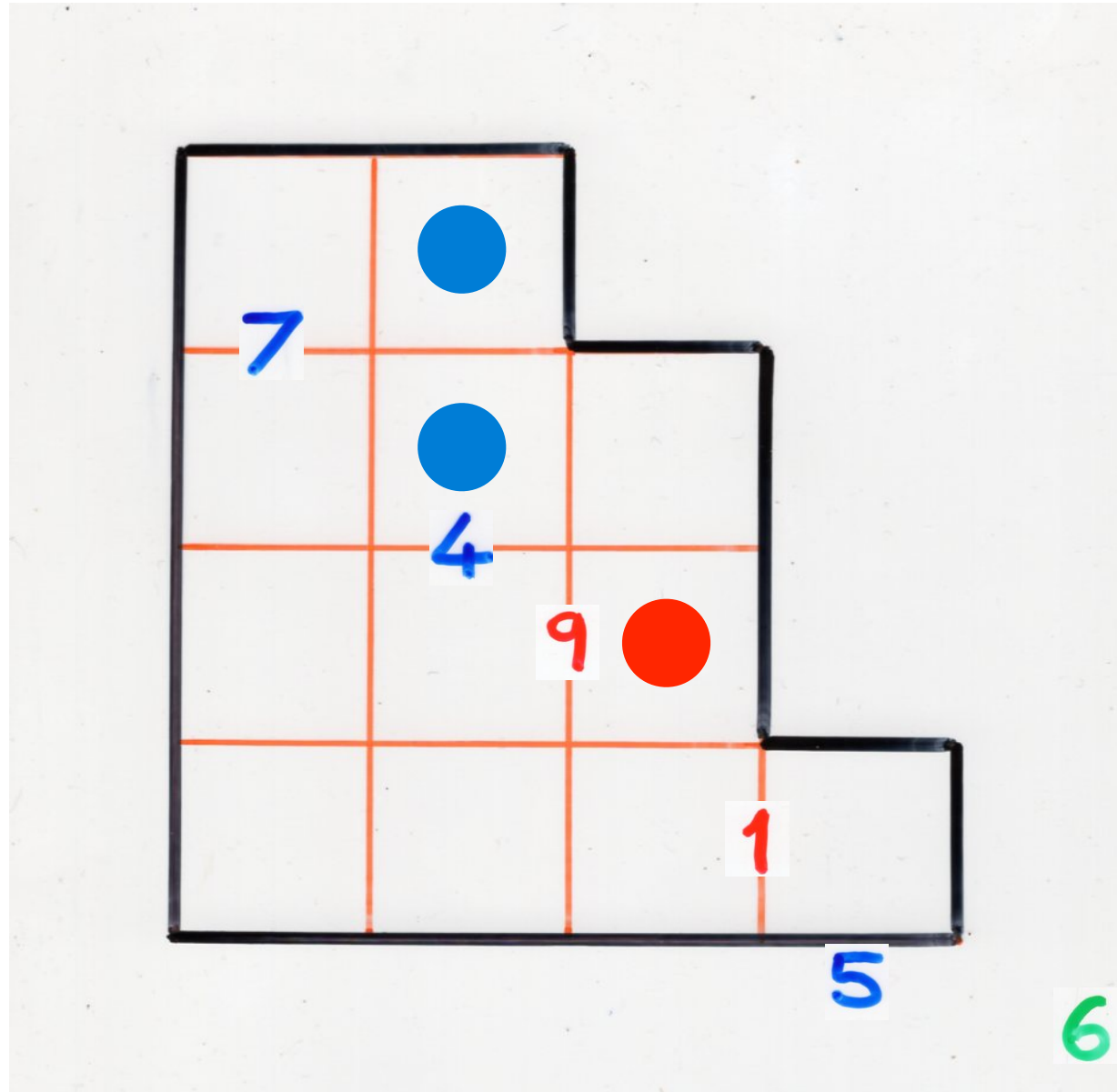


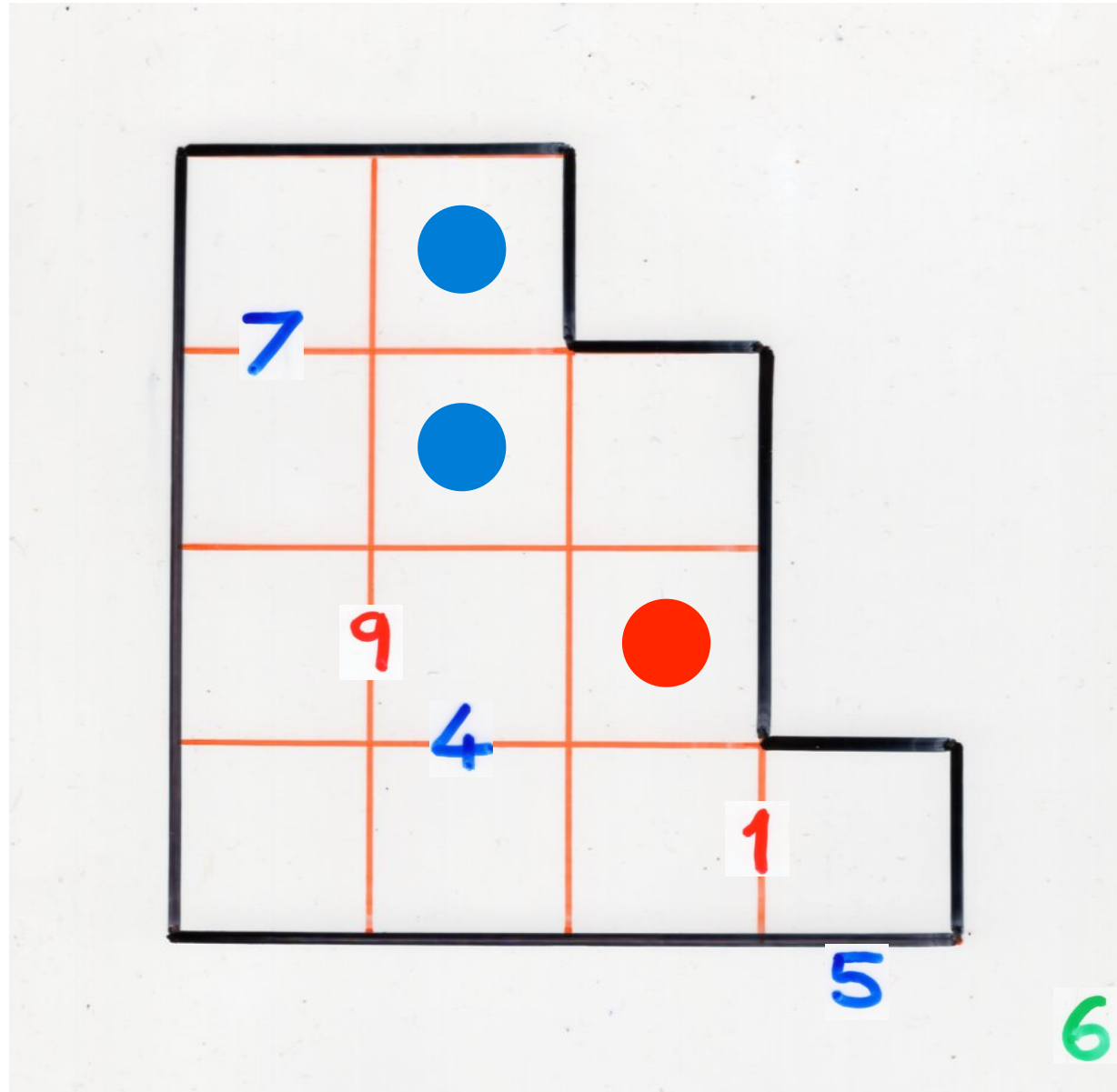


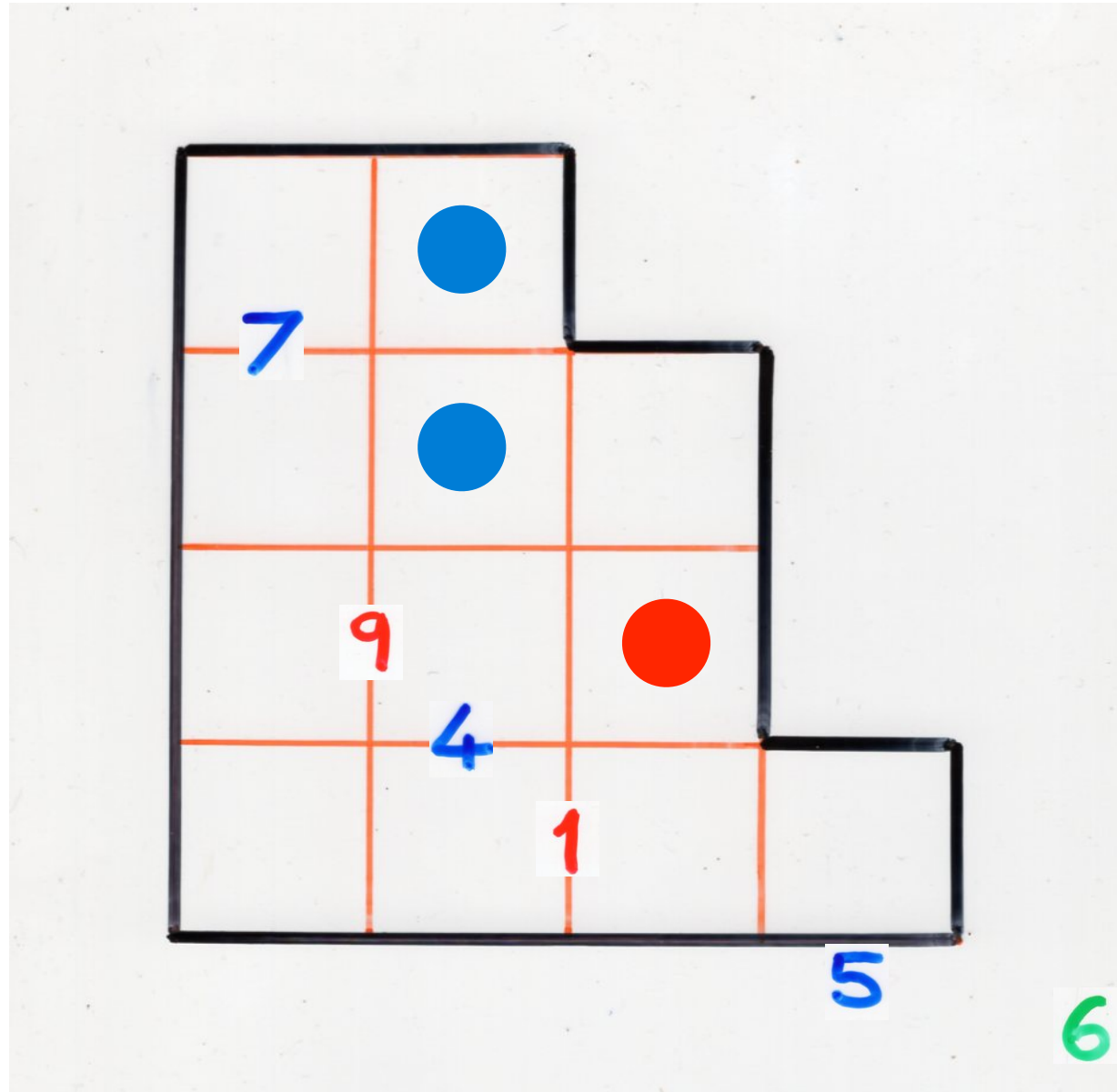


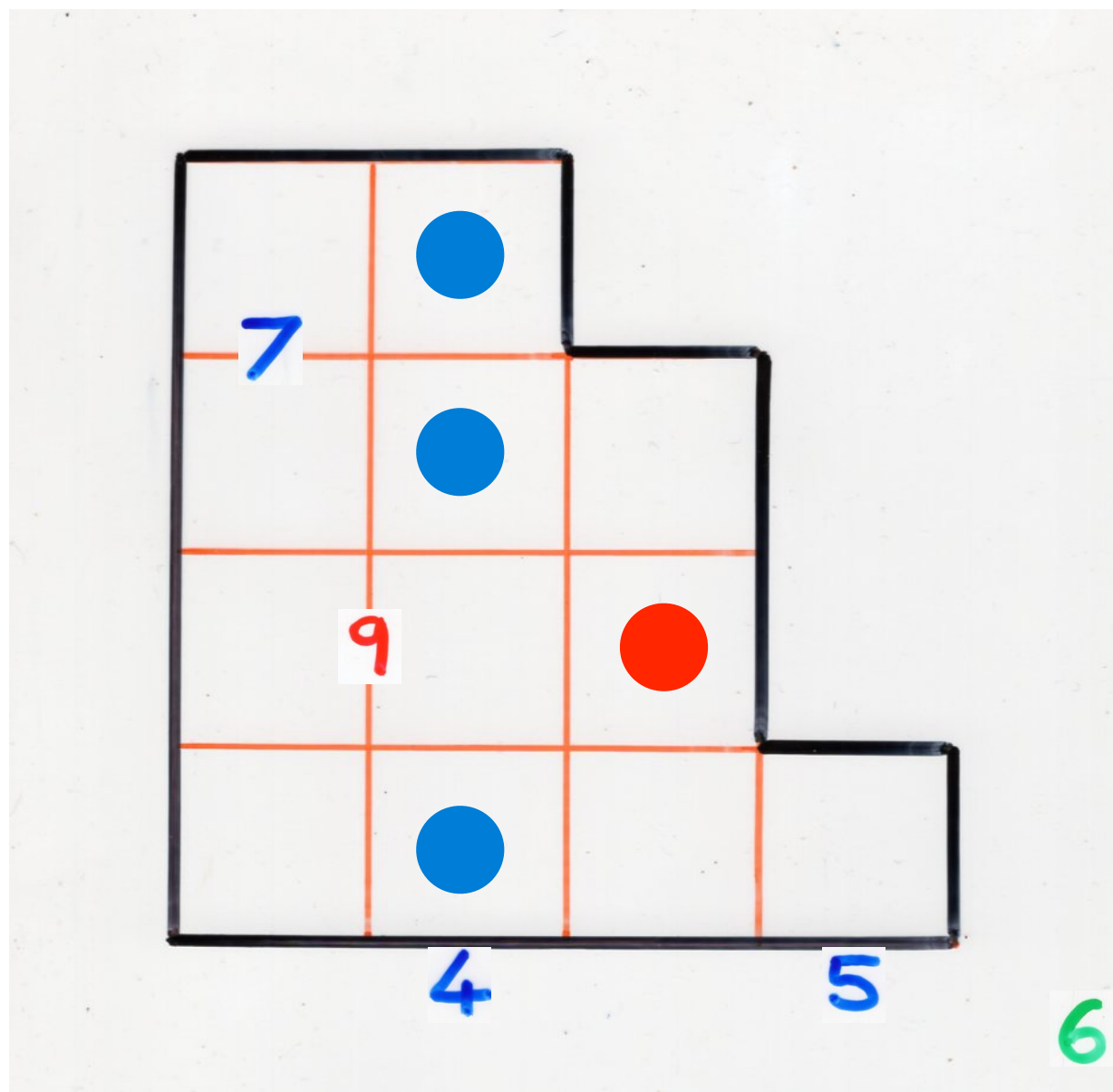


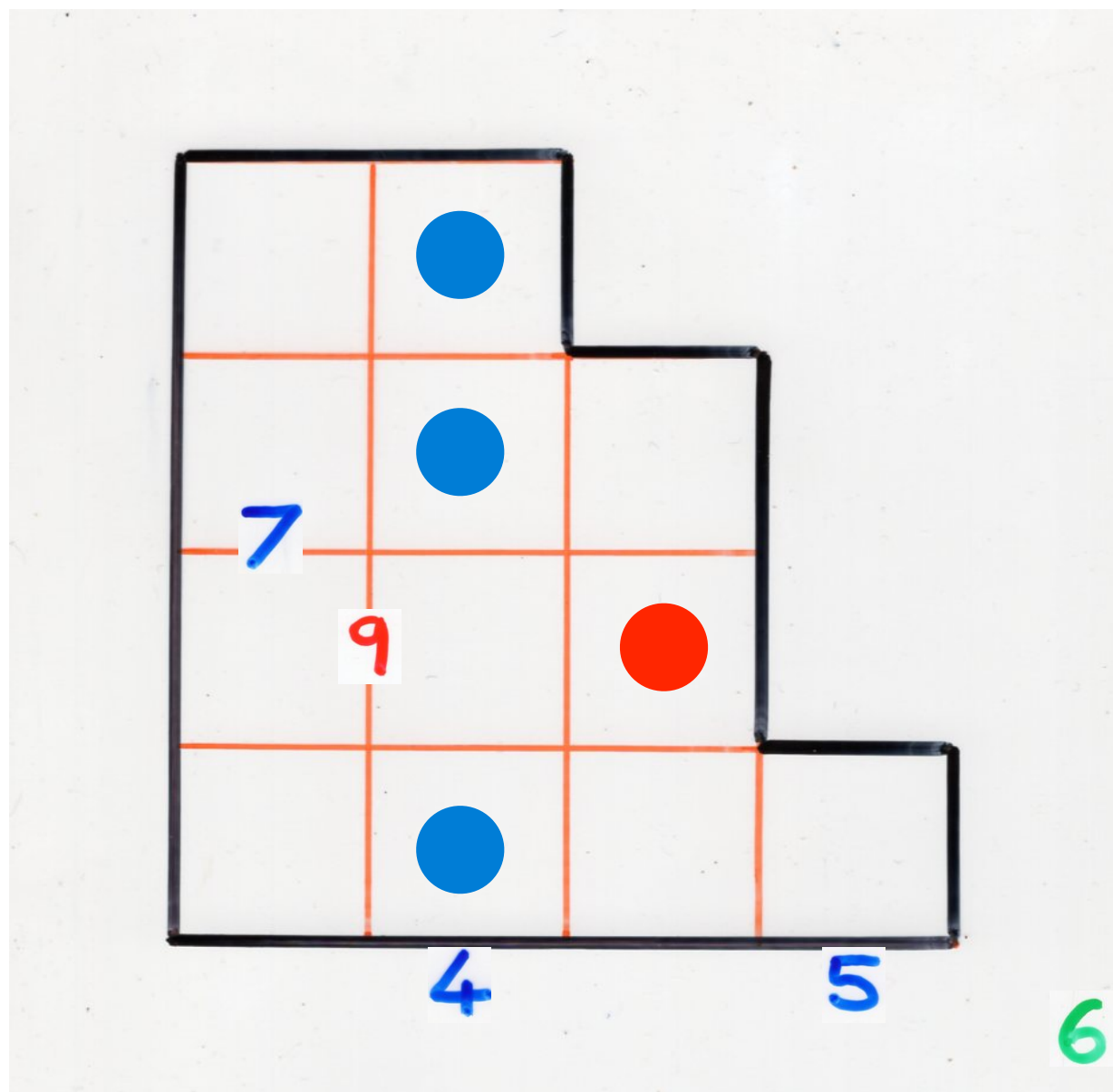


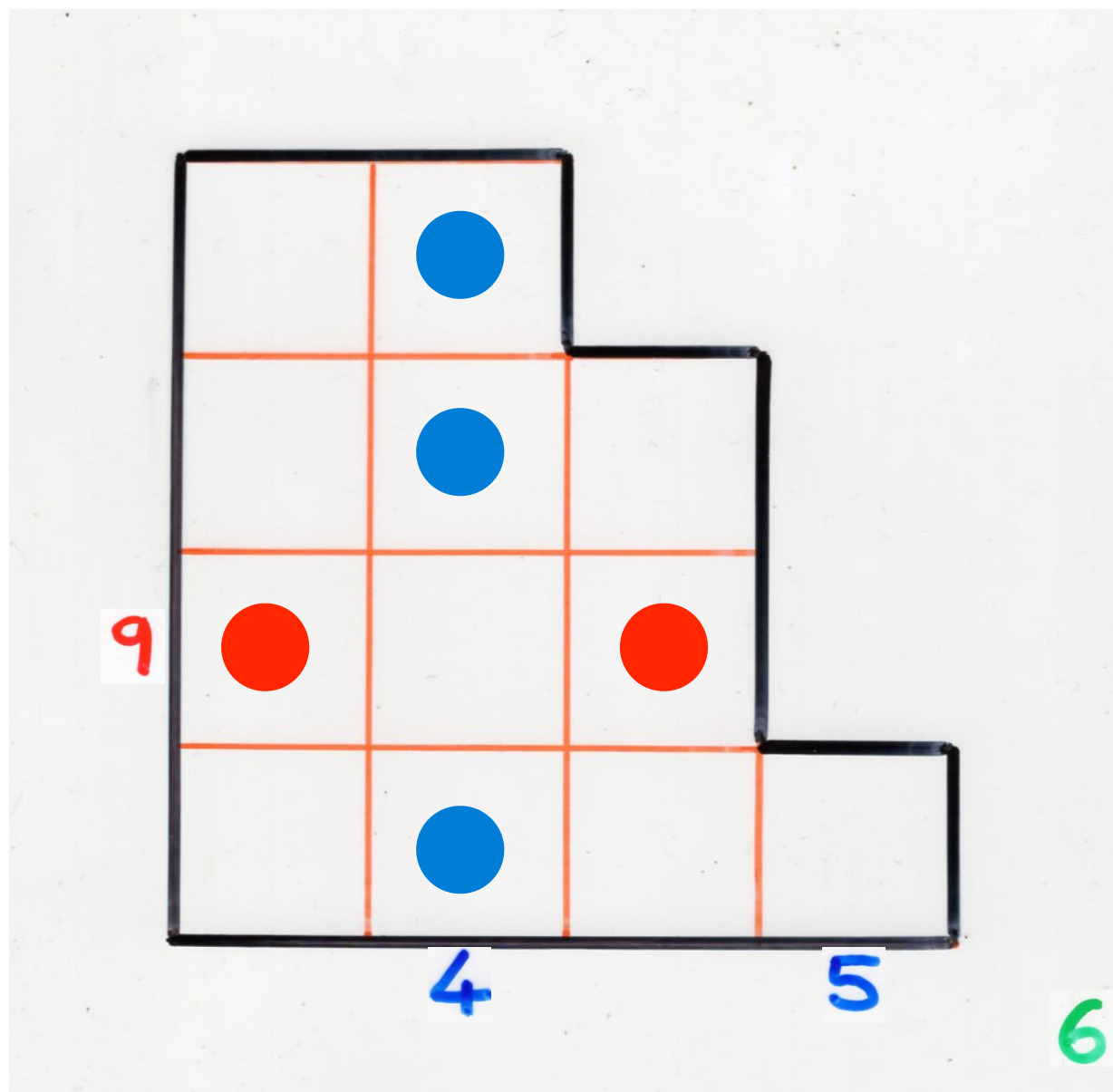


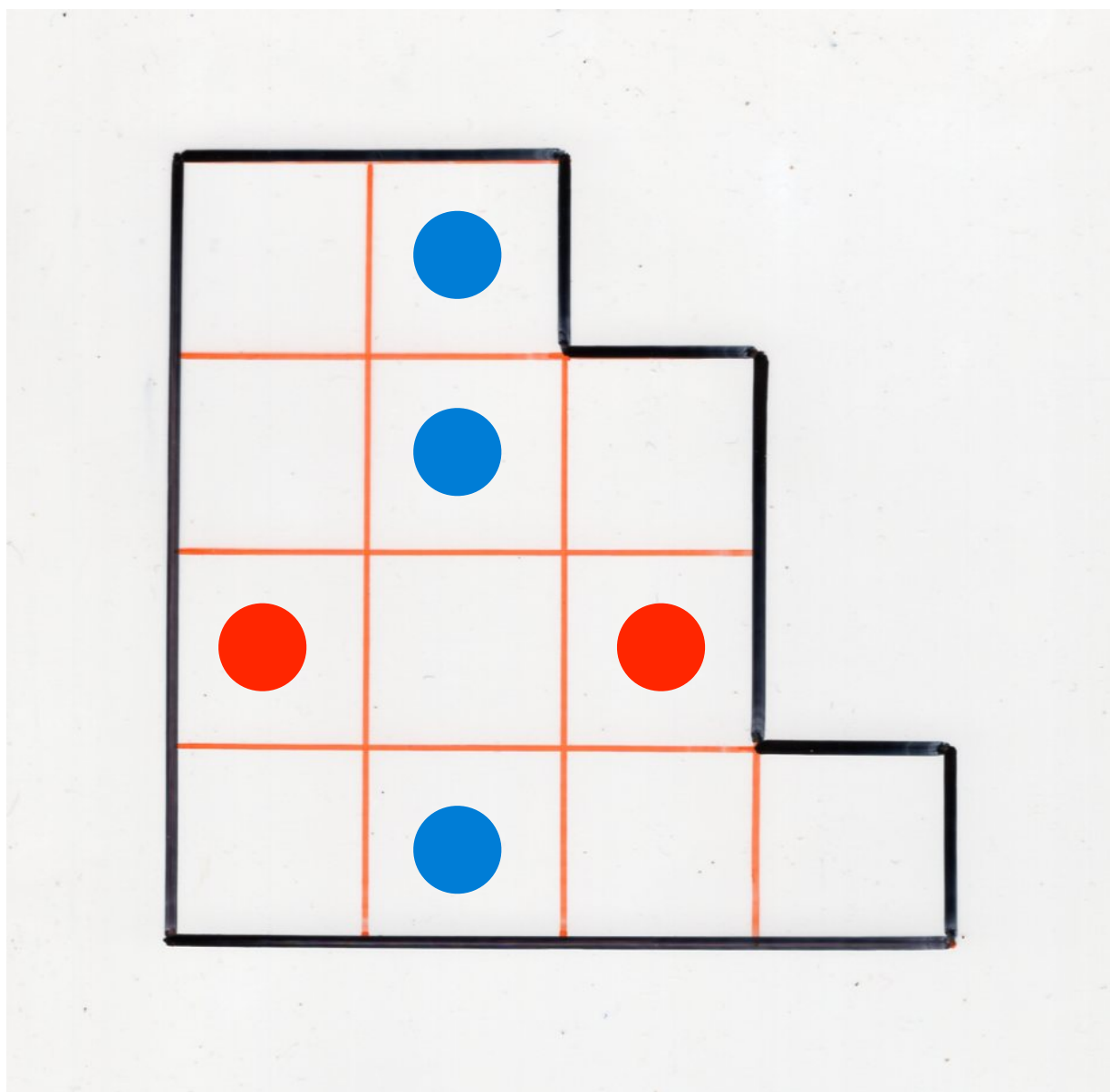










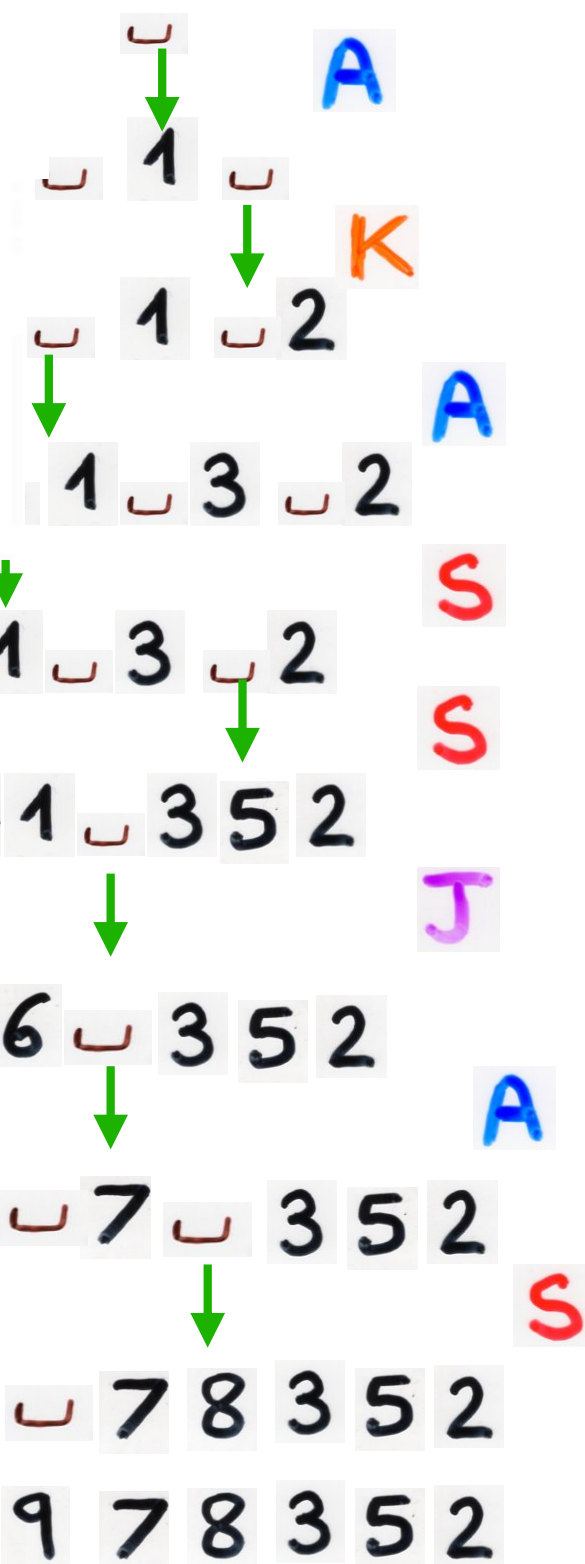
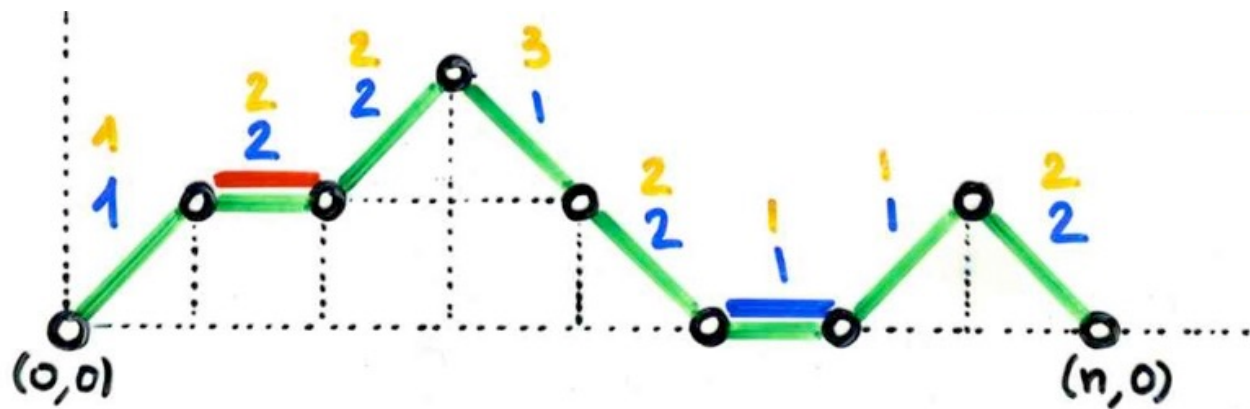


Reminding Ch3b

bijection

Laguerre histories $\longrightarrow$ permutations

description with operators



Reminding Ch3b

equivalence

local rules
(commutation diagrams)
and Laguerre histories

exchange-fusion
(or exchange-delete)
algorithm

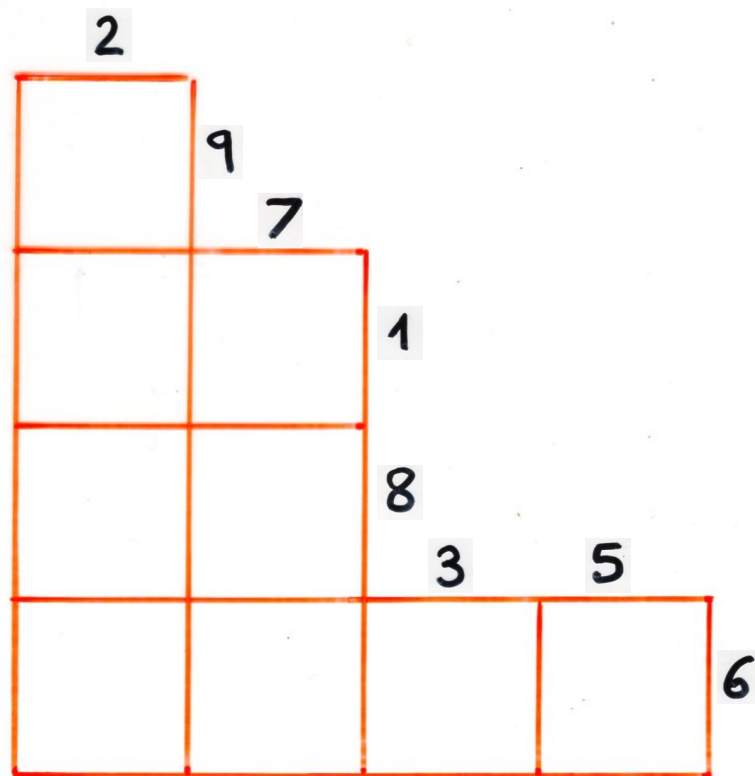
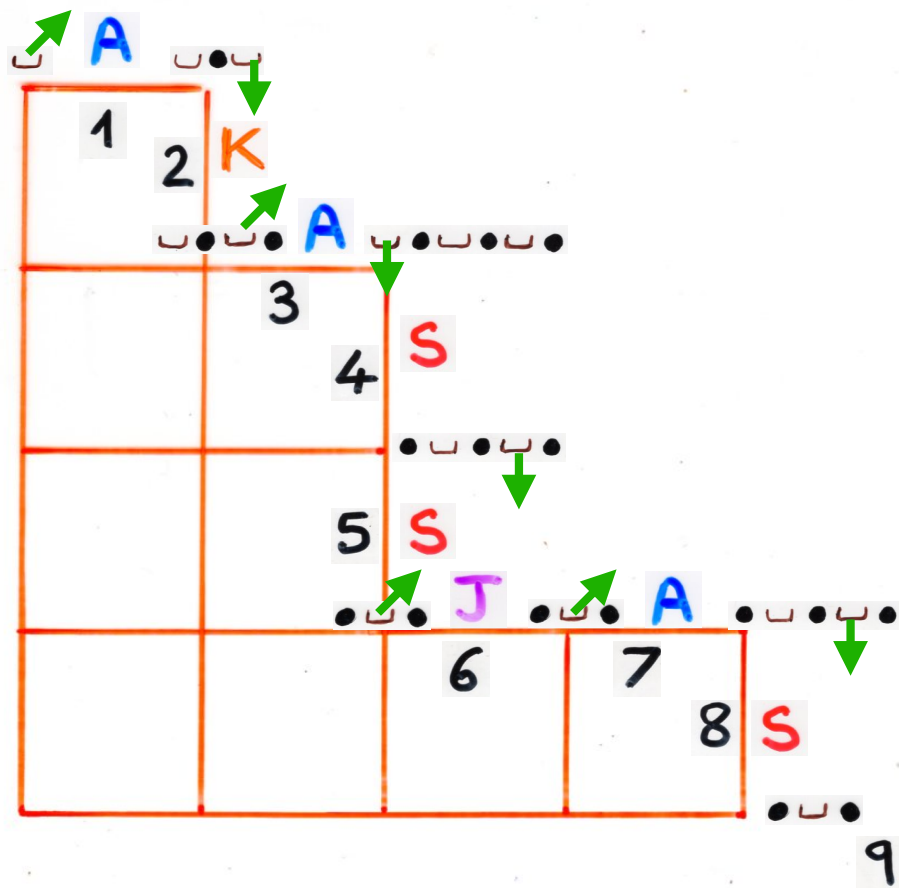
4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4

$$\sigma = \tau^{-1}$$

4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4



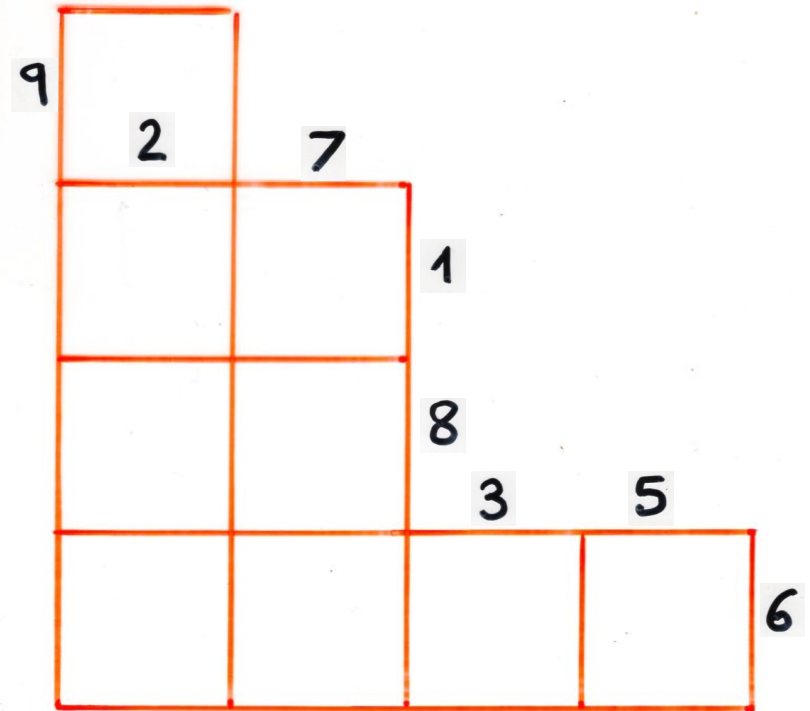
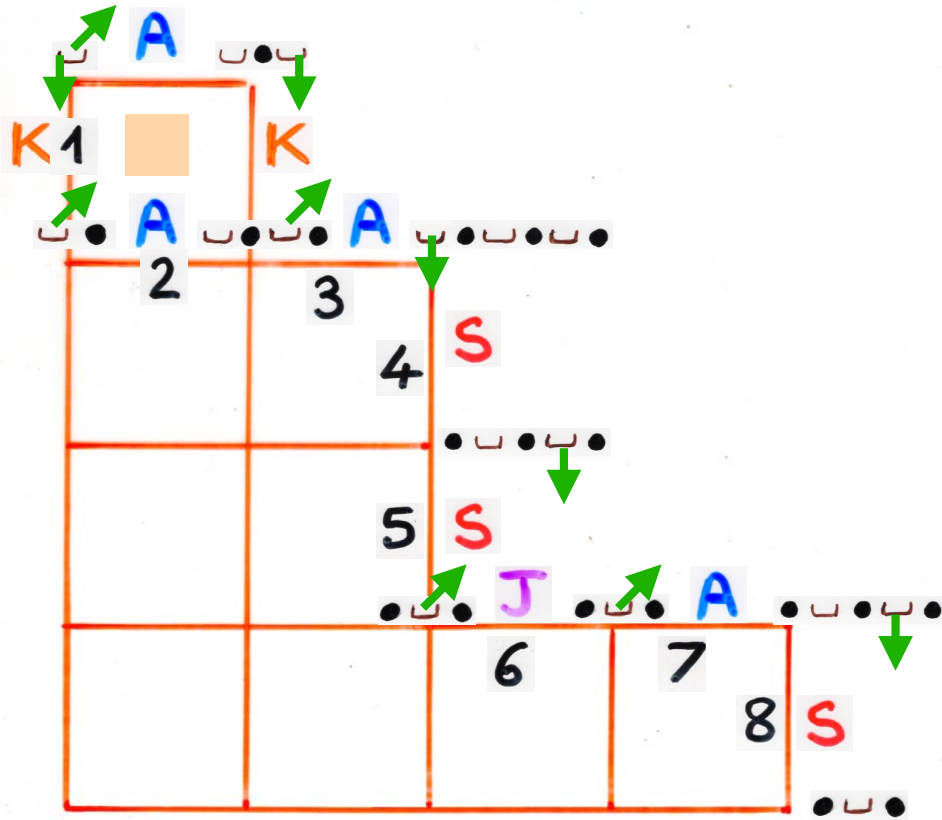
4

4 1 6 9 7 8 3 5 2

4 2 6 9 7 8 3 5 1

2 9 7 1 8 3 5 6 4

9 2 7 1 8 3 5 6 4

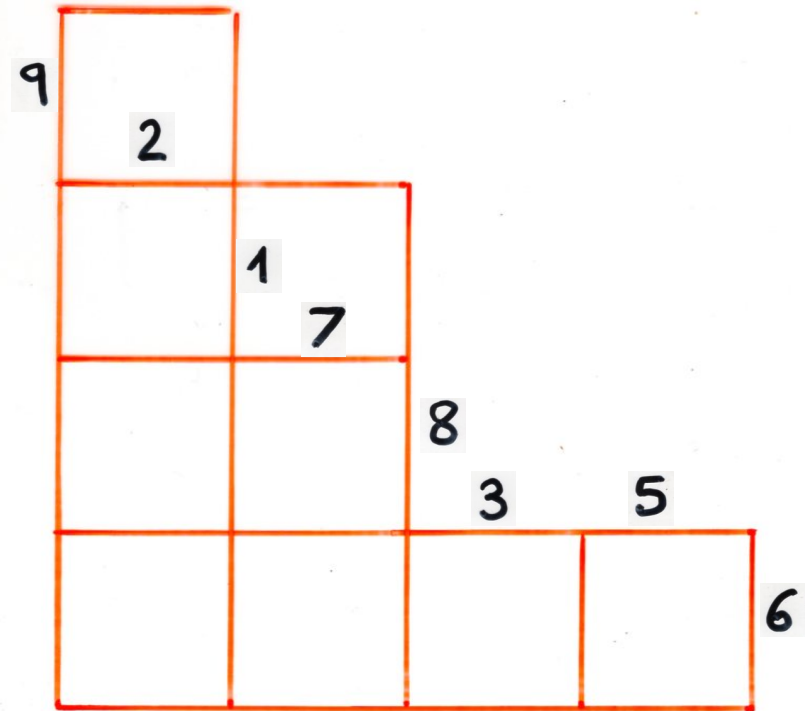
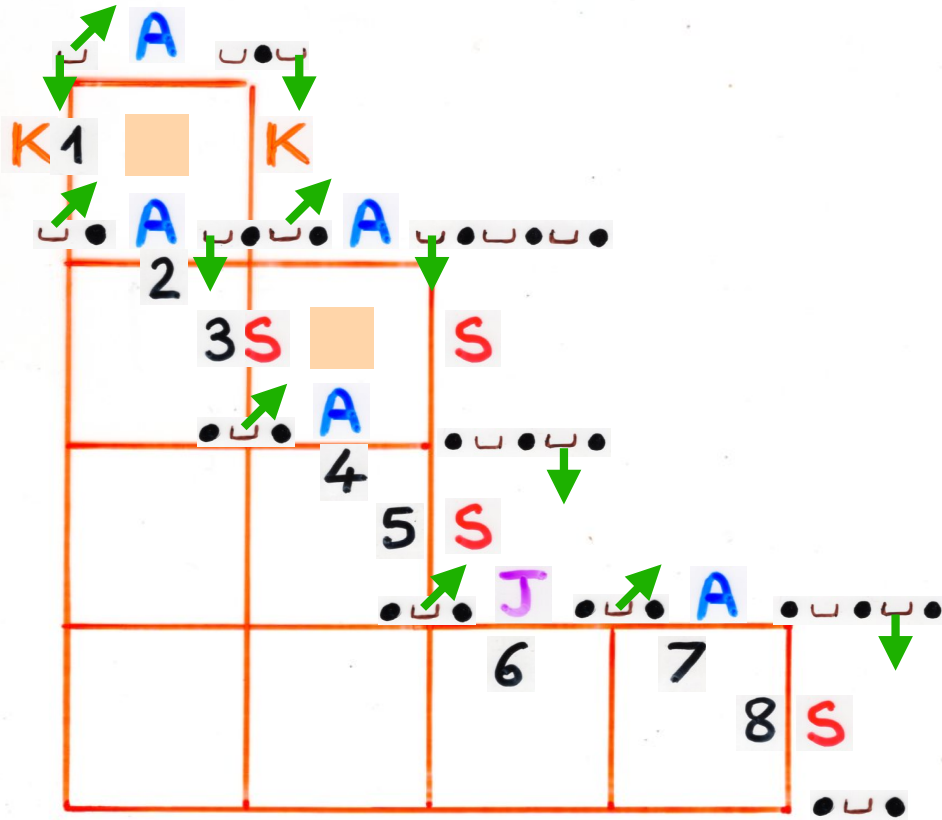


4 2 6 9 7 8 3 5 1

3 2 6 9 7 8 4 5 1

9 2 7 1 8 3 5 6 4

9 2 1 7 8 3 5 6 4

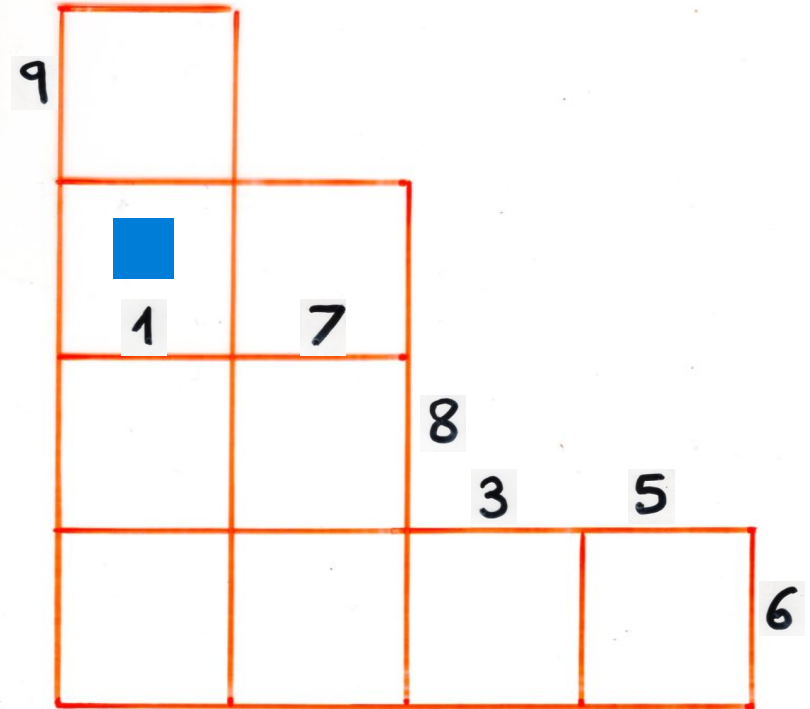
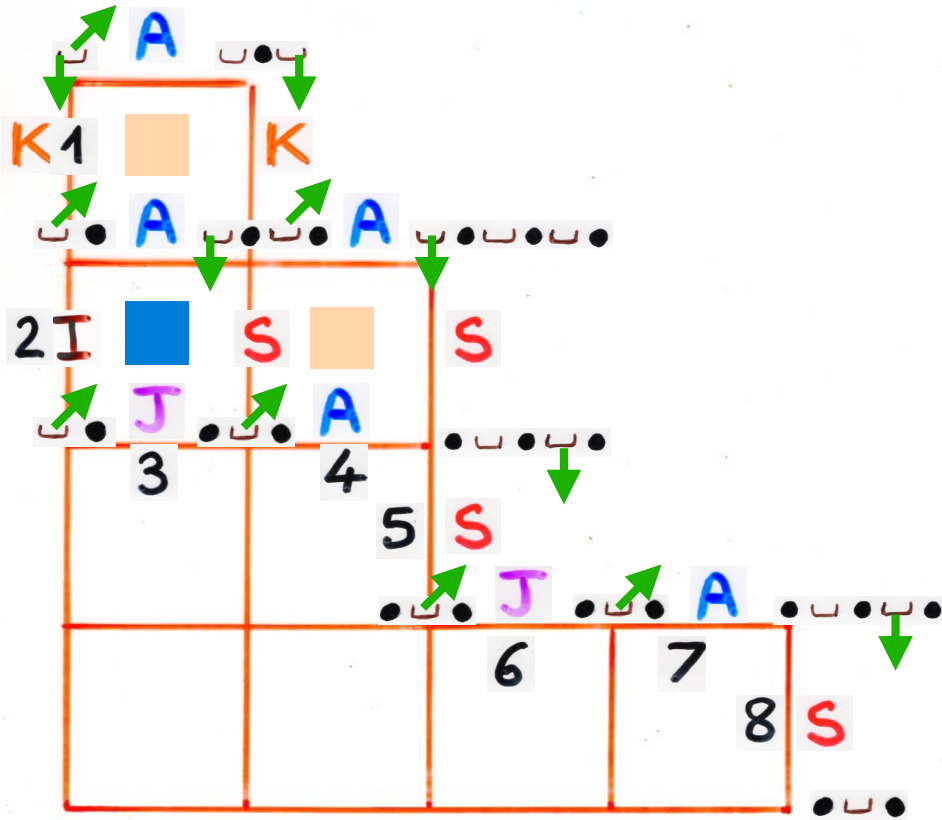


3 2 6 9 7 8 4 5 1

3 6 9 7 8 4 5 1

9 2 1 7 8 3 5 6 4

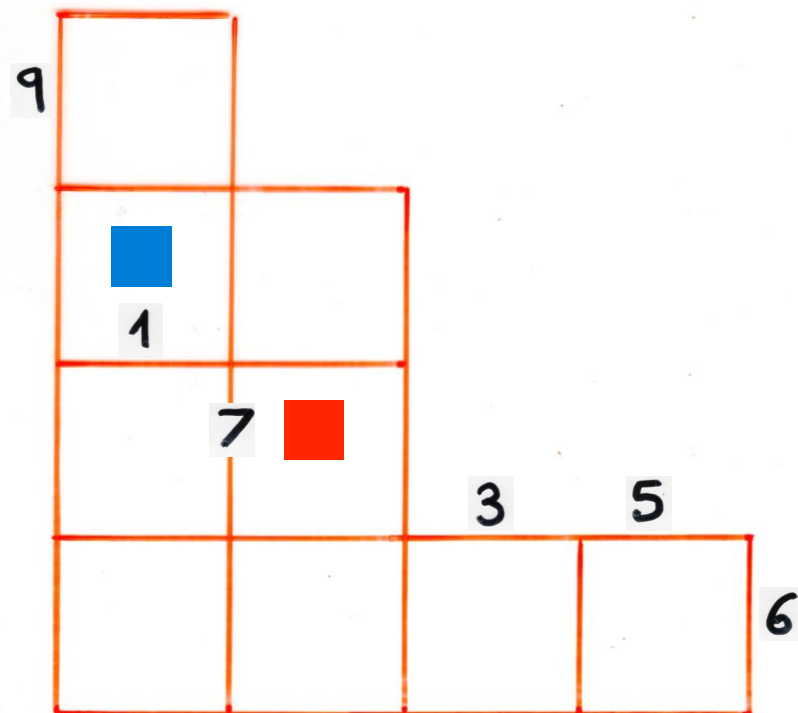
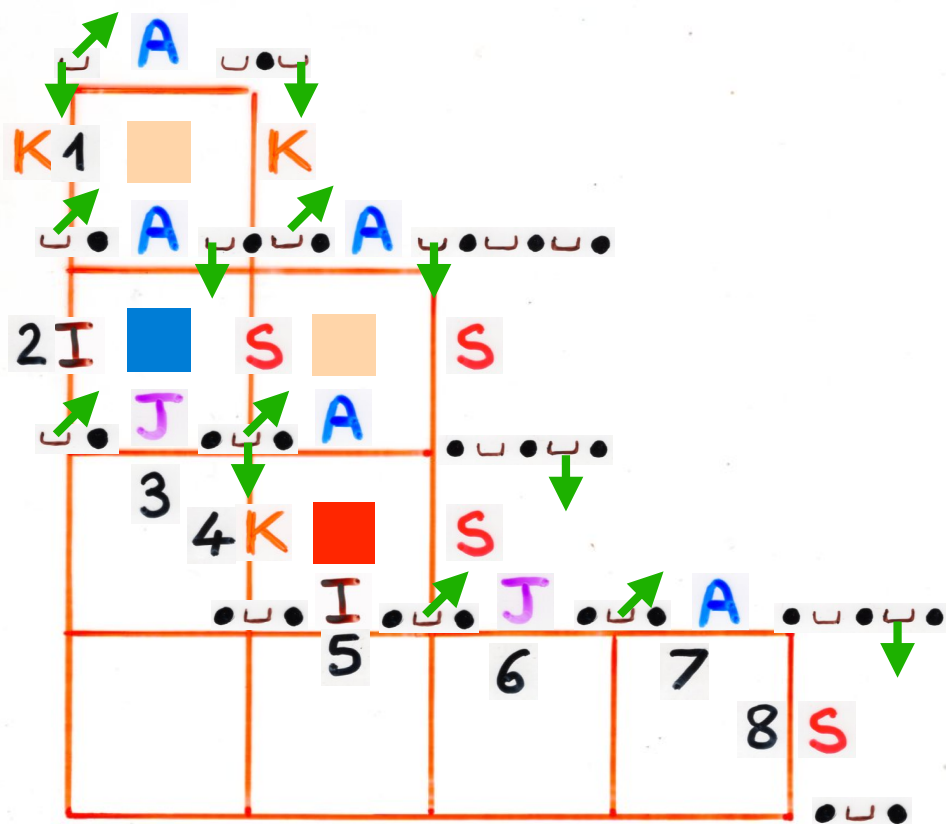
9 1 7 8 3 5 6 4



4

3 6 9 7 8 4 1

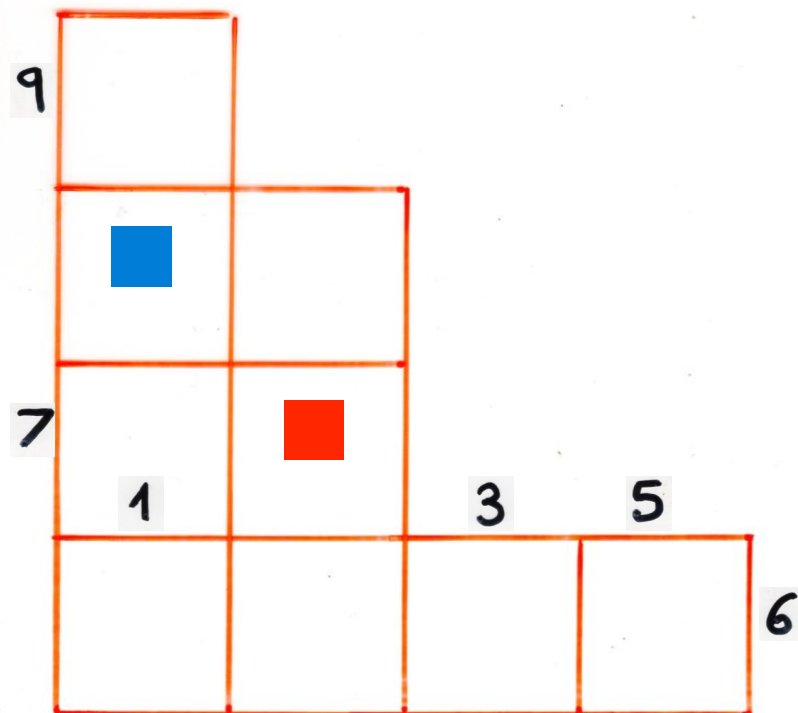
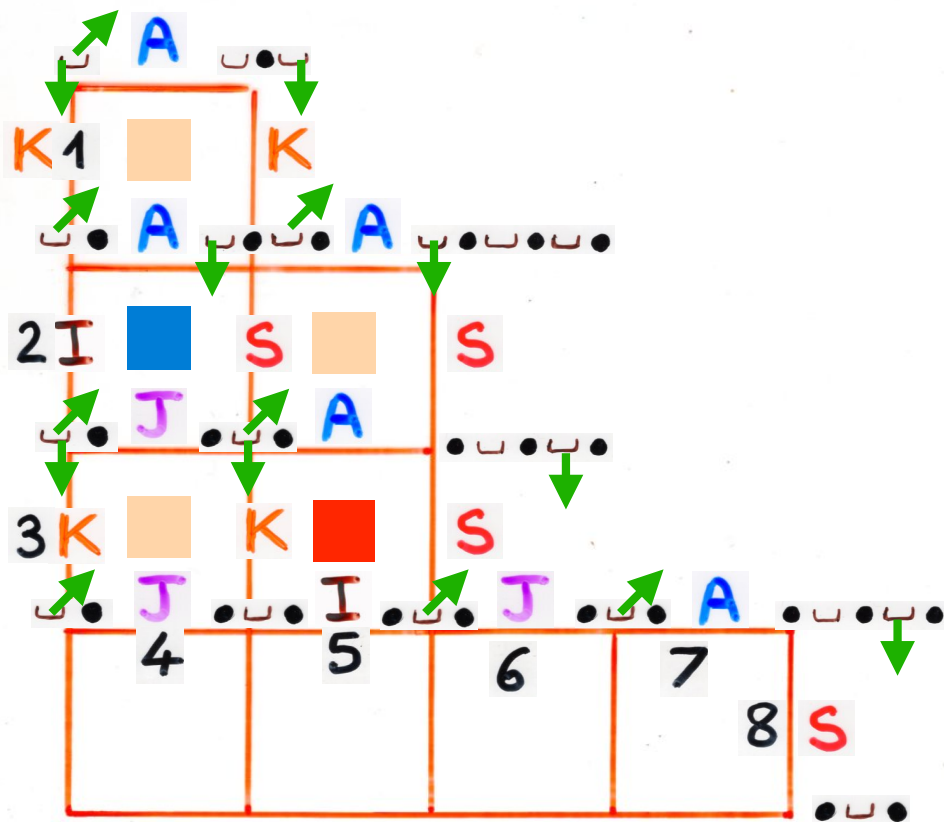
9 17 3564



4

4 6 9 7 8 3 1

9 71 3564



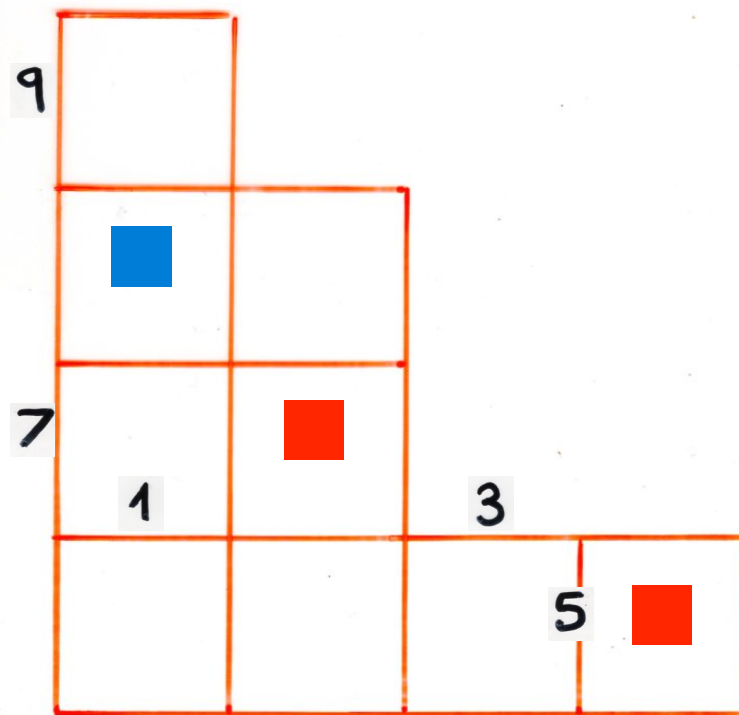
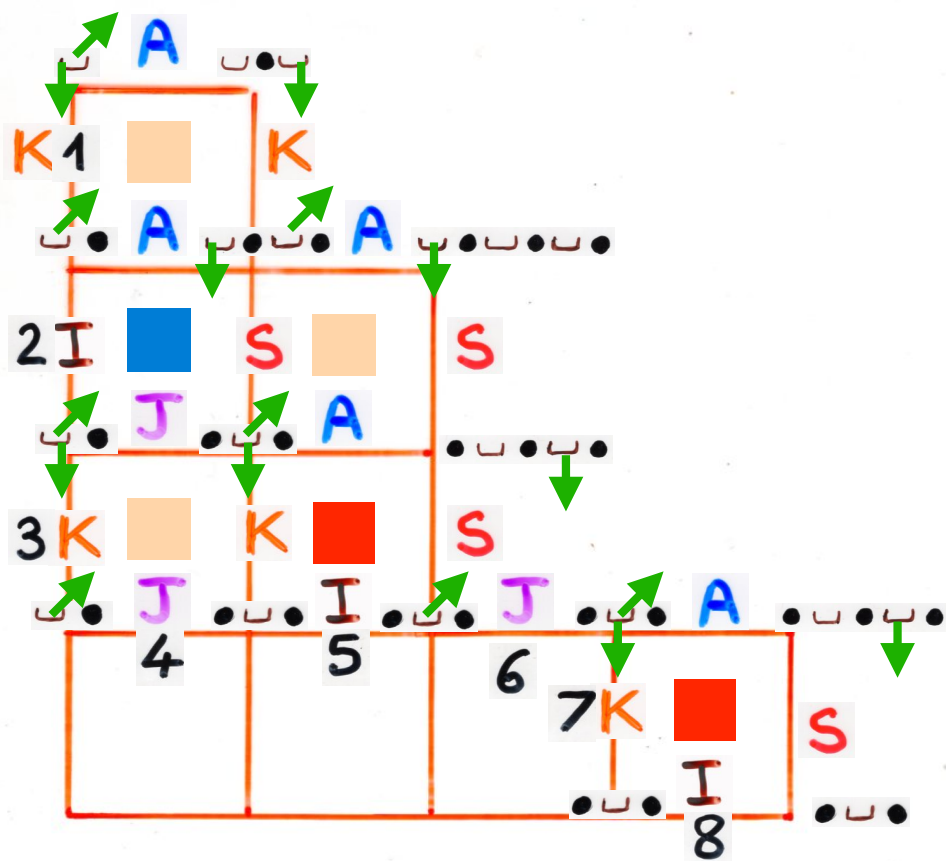
4

4 6 9 7 8 3 1

4 6 9 7 3 1

9 7 1 3 5 6 4

9 7 1 3 5 4

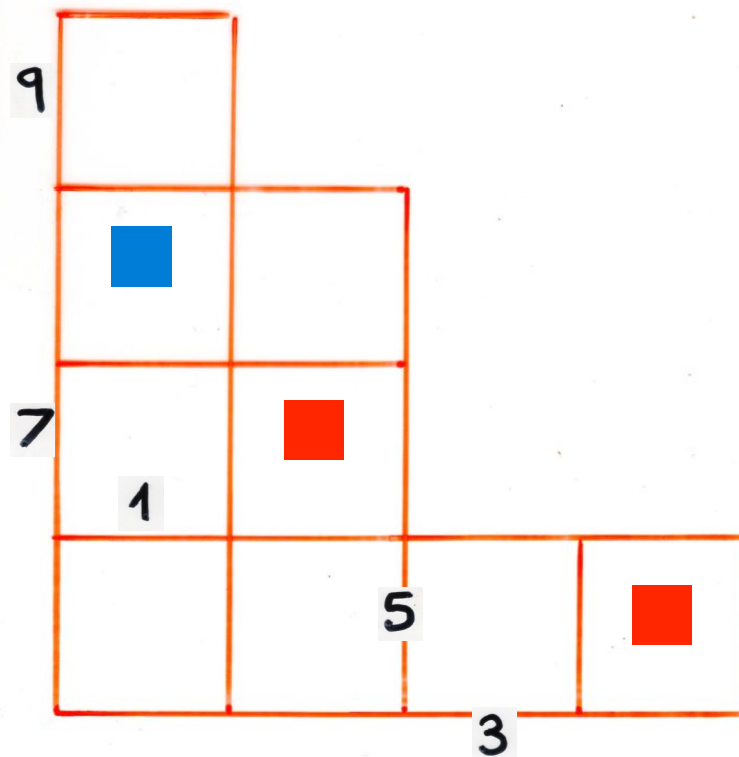
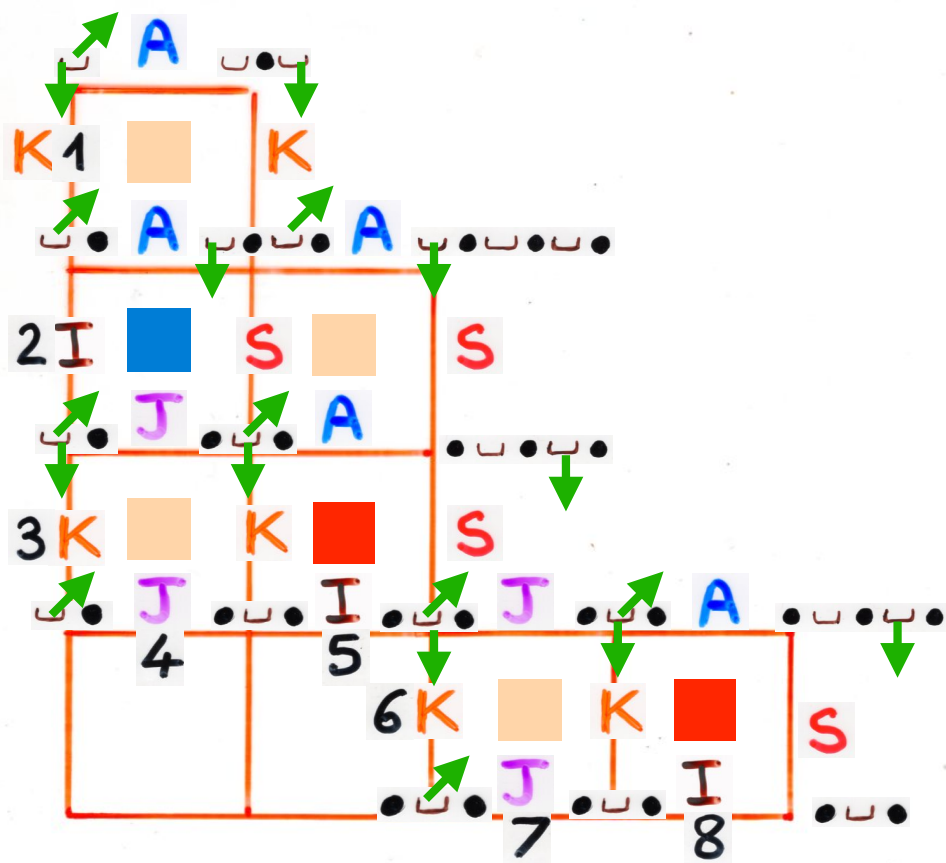


4 6 9 7 3 1

4 7 9 6 3 1

9 7 1 3 5 4

9 7 1 5 3 4



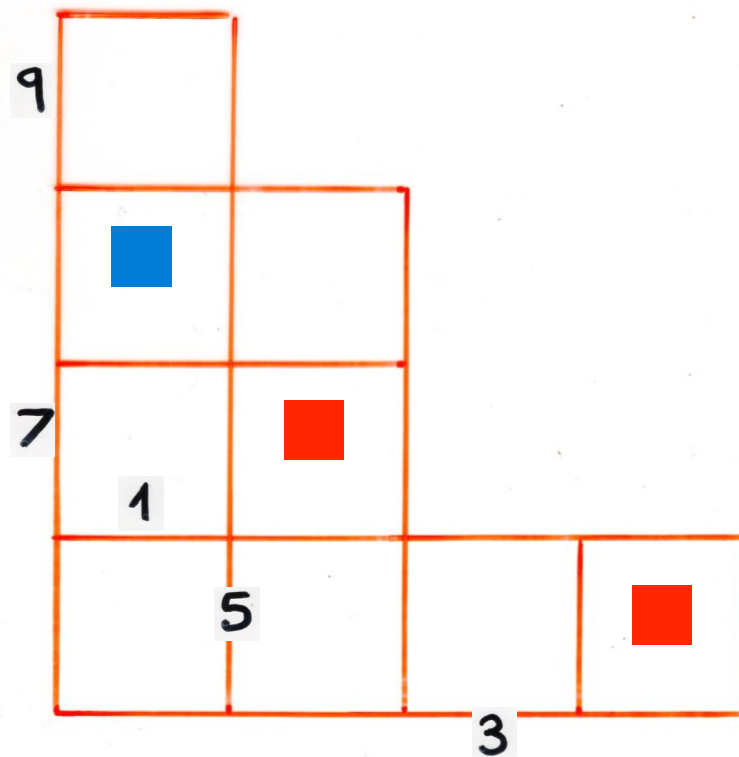
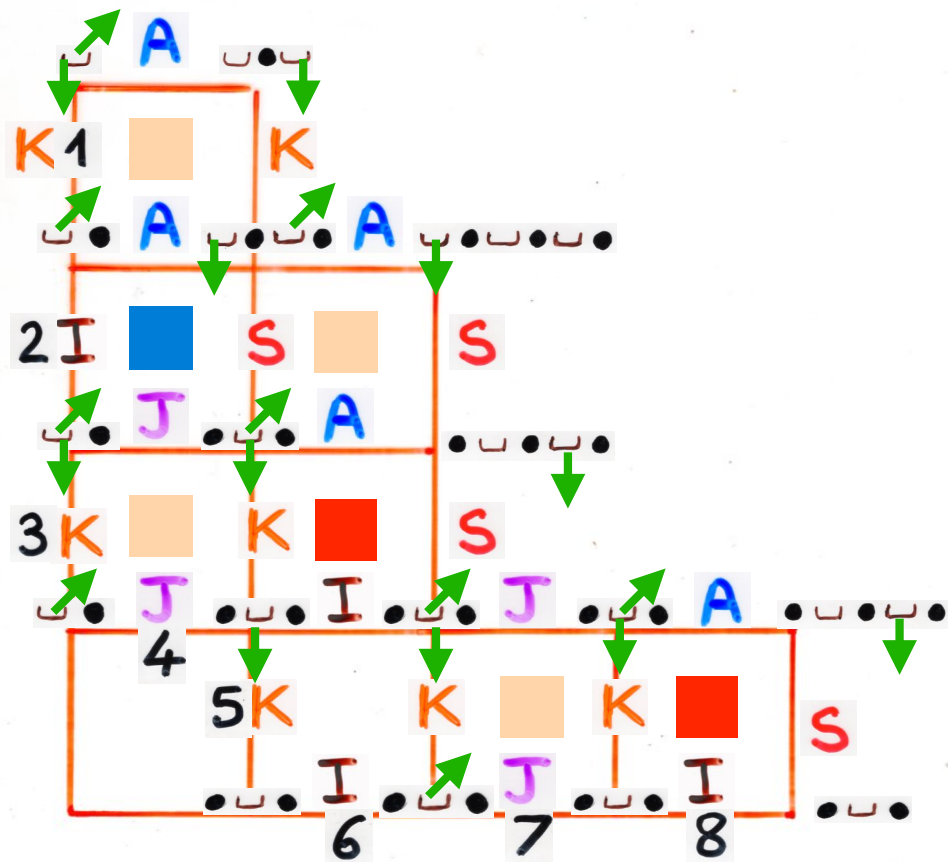
4

4 7 9 6 3 1

4 7 9 5 3 1

9 7 1 5 3 4

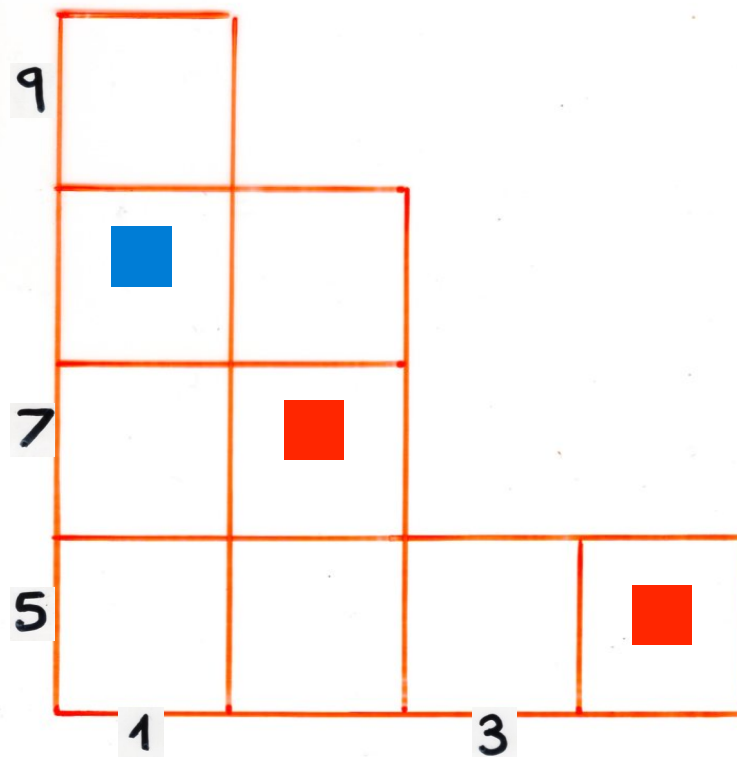
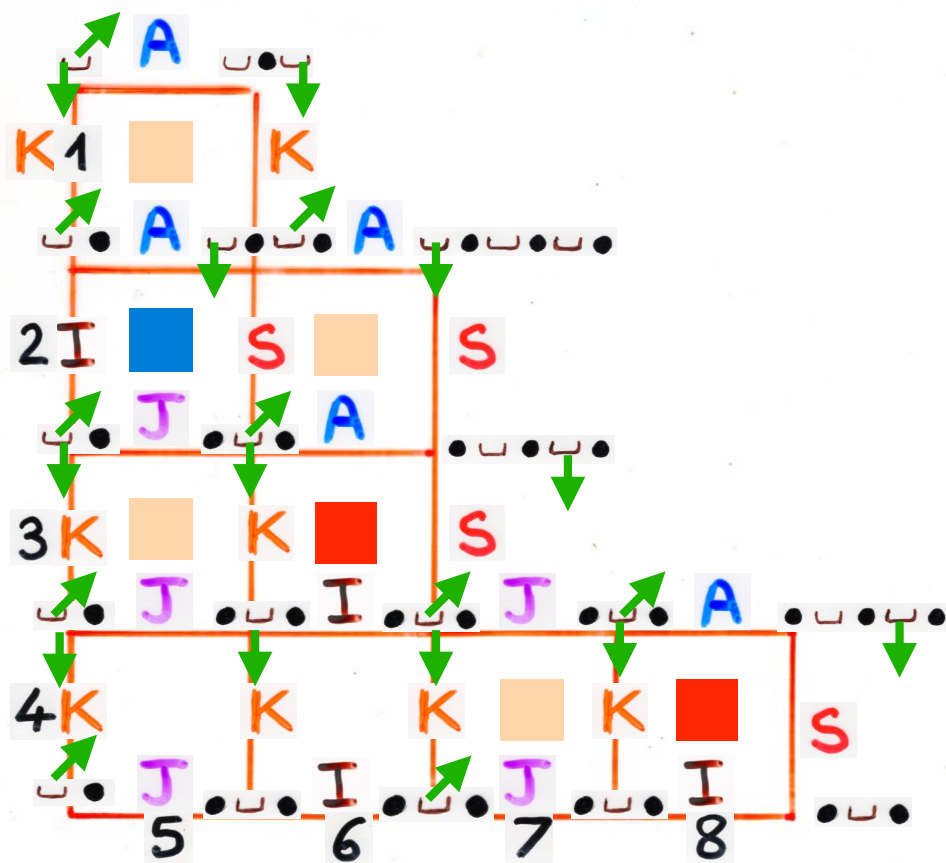
9 7 1 5 3 4



4

5 794 3 1

9 751 3 4



4

Two bijections

- from a combinatorial representation of the **PASEP** algebra (X.V., 2008)

equivalent to a bijection
Corteel, Nadeau (2007)

(with permutation tableaux)

Steingrímsson, Williams
(2005, 2007)

Postnikov

- direct bijection (with tree-like tableaux)
Aval, Bousicault, Nadeau (2011)

tableaux size $(n+1)$ $\longleftrightarrow$ (tableaux size n $\triangleright 1 \leq i \leq n+1$)

$(n+1)!$

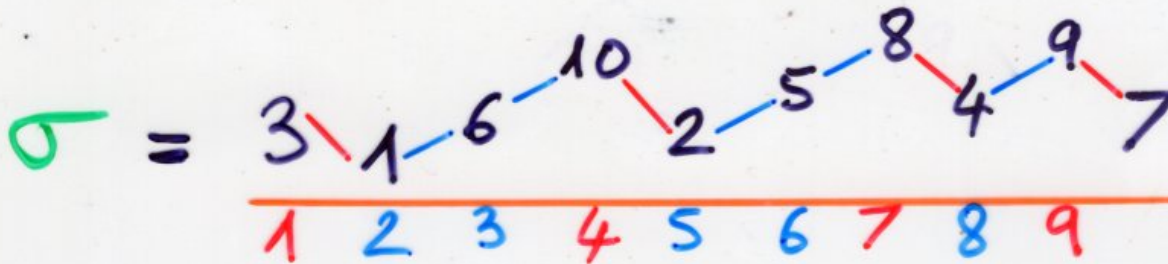
Genocchi sequence
of a permutation

$$\sigma \in \mathcal{G}_n$$

i is a descent of σ iff

$$\sigma(i) > \sigma(i+1)$$

rise $\sigma(i) < \sigma(i+1)$ (or ascent)



up-down sequence
of a permutation

$$w \in \{ \backslash, / \}^*$$

$$|w| = n-1$$

$$\{ -, + \}^*$$

D. André (1880)

6-2-9-7-8-4-5-1-3

alternating
sequence

alternating permutations

up-down sequence
of a permutation

T_{2n+1} = number of alternating permutations
on $[1, 2n+1]$

BJC 1

(bijective course, Part I)

Ch 3b, p 63-65

$$\tan t = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

Definition Genocchi sequence
of a permutation

$$\sigma = \sigma(1) \dots \sigma(n)$$

$$G(\sigma) = z_1 \dots z_{n-1} \in \{a, d\}^*$$

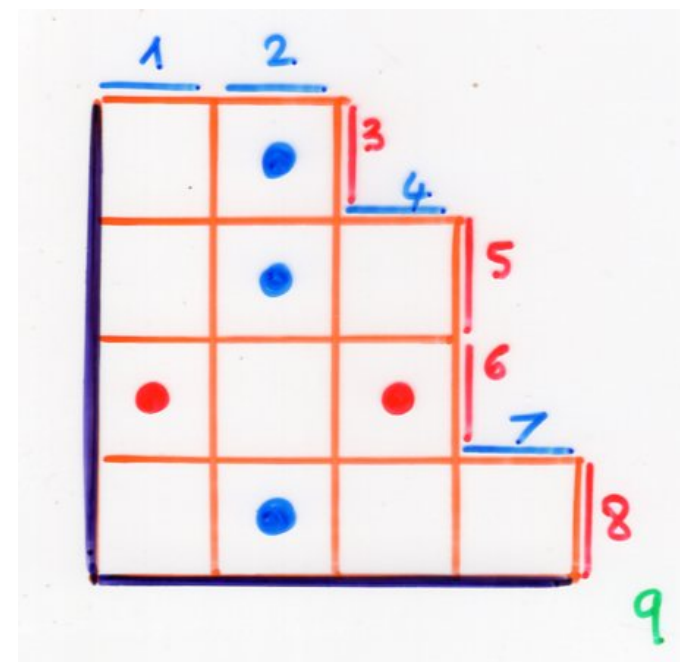
$$z_x = \begin{cases} a \\ d \end{cases} \text{ iff } \underset{\text{"value"}}{x} = \begin{cases} \sigma(i) < \sigma(i+1) \\ \sigma(i) > \sigma(i+1) \end{cases}$$

↓
"index"

Proposition $\sigma \in \mathcal{G}_{n+1} \xrightarrow{ExF} T$ alternative
tableau
size n
 "Exchange-Fusion"
algorithm

$$G(\sigma^{-1}) = \underset{\text{word}}{w} \in \{a, d\}^* \quad |w| = n$$

The profile of T is the word $f(w)$
 obtained from w by replacing
 $a \rightarrow D$ (East step) D
 $d \rightarrow E$ (South step) E



example

$$G(\sigma^{-1}) = a a d a d d a d$$

1 2 3 4 5 6 7 8 9

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

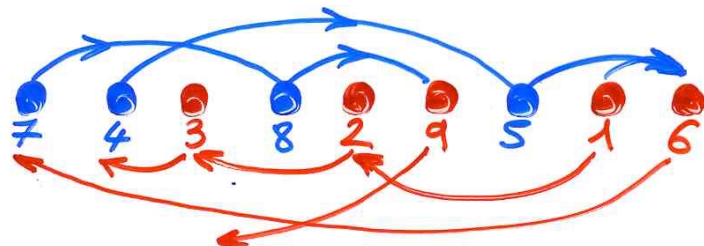


(value) x $\begin{cases} \text{advance} \\ \text{backward} \end{cases}$ in σ ("avance")
 ("recul")

iff (the index) x is $\begin{cases} \text{rise} \\ \text{descent} \end{cases}$ in σ^{-1}

$$\begin{aligned} \sigma^{-1}(x) &< \sigma^{-1}(x+1) \\ \sigma^{-1}(x) &> \sigma^{-1}(x+1) \end{aligned}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$



$$\sigma^{-1} = \begin{pmatrix} 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$



Remark The **Genocchi** sequence of the permutation σ^{-1} is also the word $w = z_1 \dots z_{n-1}$ defined by

$z_i = \begin{cases} a & \text{iff there is an advance ("avance")} \\ d & \text{a backward ("recul")} \end{cases}$
 "at the index i "

$$\begin{aligned} \sigma(i) &< \sigma(i+1) \\ \sigma(i) &> \sigma(i+1) \end{aligned}$$

$$\sigma^{-1} = \begin{pmatrix} 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

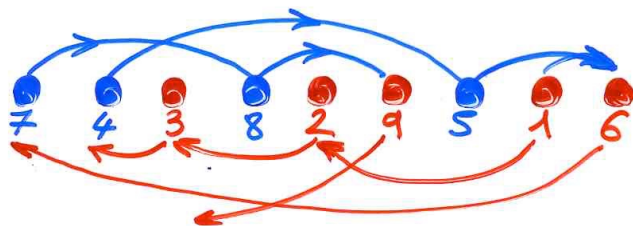
example

$$G(\sigma^{-1}) = a a d a d d a d$$

1 2 3 4 5 6 7 8 9

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$

convention $\sigma(n+1) = 0$
 ($\sigma(n)$ is a descent)



alternating
sequence

d a d a ... a d a

Proposition Dumont (1974)

The number of permutations on $[1, 2n]$
having an alternating Genocchi sequence
is the Genocchi number G_{2n+2}

$$G_{2n} = 2(2^{2n} - 1) B_{2n}$$
$$2^{2n} G_{2n+2} = (n+1) T_{2n+1}$$

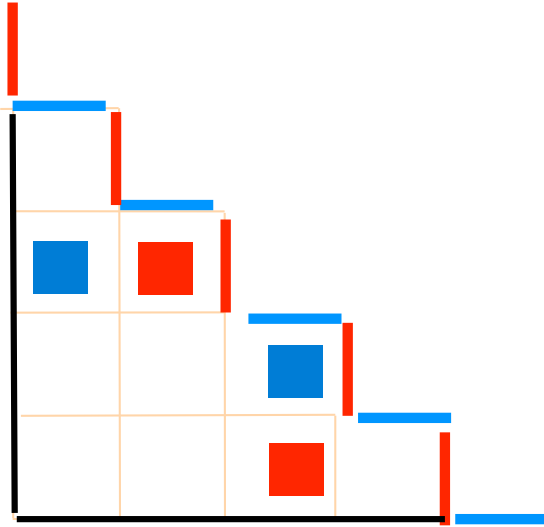
Bernoulli
numbers

Tangent
numbers

Genocchi
numbers

Angelo Genocchi
(1817-1889)





Genocchi numbers

$$G_{2n+2}$$

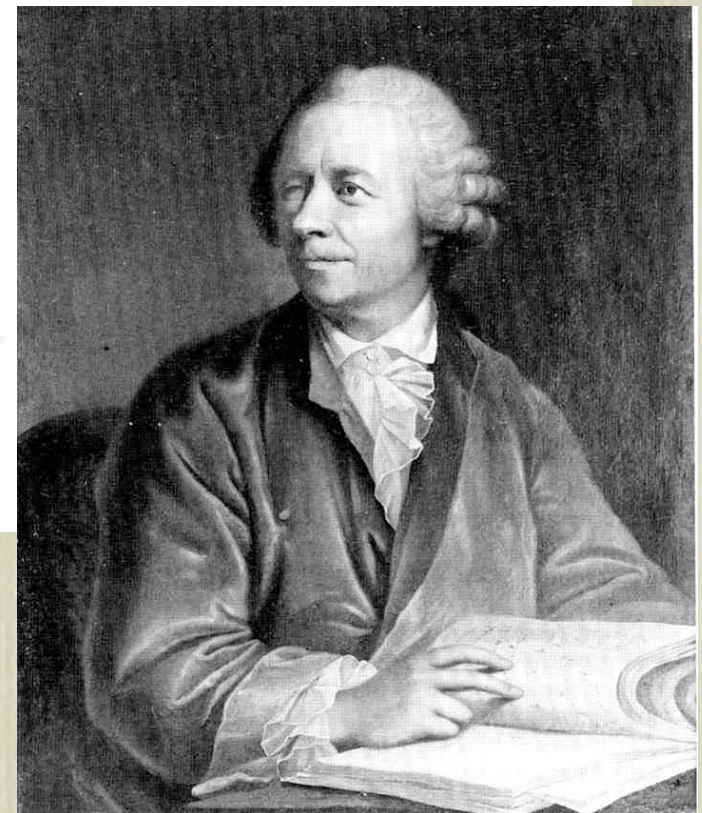
alternating shape

Josuat-Vergès
(2010)

Hinc igitur calculo instituto reperietur :

$$\begin{aligned}
 A &= 1 \\
 B &= 1 \\
 C &= 3 \\
 D &= 17 \\
 E &= 155 = 5.31 \\
 F &= 2073 = 691.3 \\
 G &= 38227 = 7.5461 = \\
 H &= 929569 = 3617.257 \\
 I &= 28820619 = 43867.9.73
 \end{aligned}$$

Leonhard Euler
(1707-1783)



E_{2n} = number of alternating permutations on $[1, 2n]$

$$\frac{1}{\cos t} = 1 + 5 \frac{t^2}{2!} + 61 \frac{t^4}{4!} + 1385 \frac{t^6}{6!} + \dots$$

BJC 1

(bijective course, Part I)

Ch 3b, p 63-65

BORDEAUX 1. Le professeur Donald Knuth consacre sa vie à la programmation informatique, considérée comme un art. Il vient d'être sacré docteur honoris causa à Bordeaux

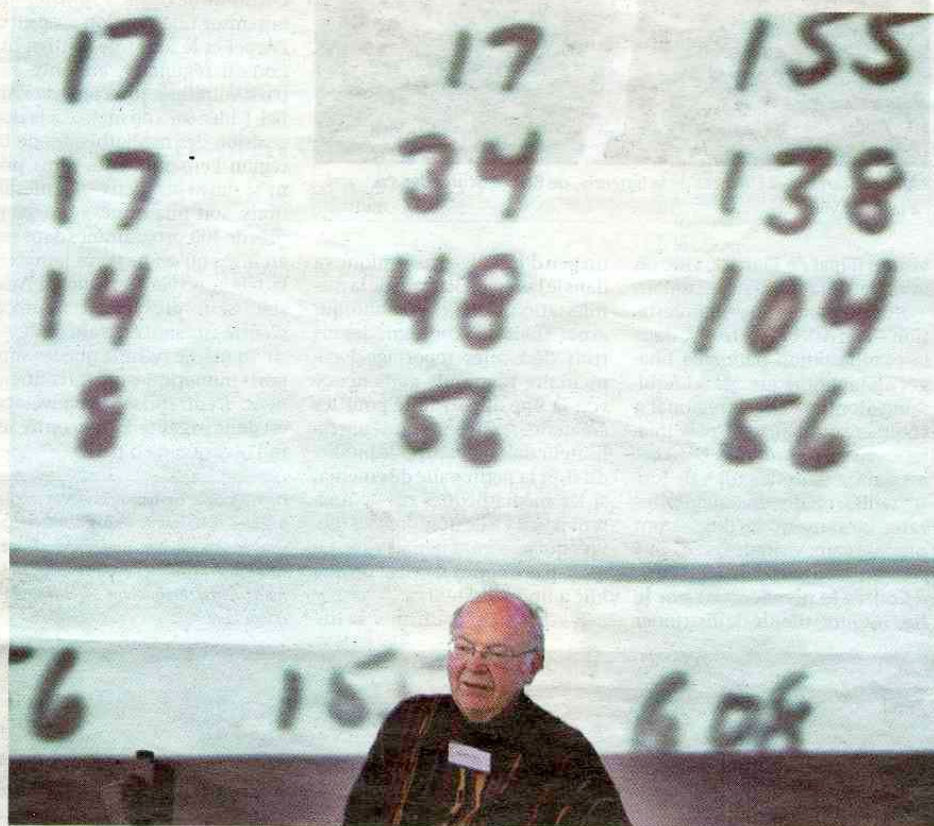
L'ermite de l'informatique

de Bernard Broustet

Une sommité de l'informatique mondiale a séjourné en Gironde ces derniers jours. Donald Knuth, 69 ans, a été sacré mardi docteur honoris causa de l'université Bordeaux 1, après avoir été lundi au centre d'une journée d'échanges qui réunissait une bonne partie du gratin français et européen de la recherche en informatique (1).

Depuis son premier contact, il y a un demi-siècle, avec un monumental et dinosaure IBM 650, Donald Knuth n'a cessé d'être habité par la passion de l'informatique. Physicien, puis mathématicien de formation, ce géant affable et modeste a voué sa vie à ce qu'il appelle « l'art de la programmation informatique ». Car, à ses yeux, plus qu'une technique, c'est une forme d'activité qui requiert à la fois rigueur, intuition et sens esthétique. Les programmes informatiques réussis ont une sorte de beauté à laquelle même les non-spécialistes peuvent être sensibles.

Une encyclopédie. Au long de sa carrière académique (pour l'essentiel à l'université californienne de Stanford), Donald Knuth a fait preuve d'une grande fécondité, en jouant notamment un rôle essentiel dans le développement de langages toujours utilisés par la communauté des mathématiciens. Mais, à 55 ans, le professeur Knuth a décidé de prendre sa retraite de Stanford. Il trouve que les fonctions administratives sont trop absorbantes pour lui permettre de mener à bien l'œuvre entamée à la fin des années 60 sous le titre de « Art of computer programming », sorte d'encyclopédie de l'algorithmique et de la programmation informati-



Donald Knuth, à Bordeaux, le 29 octobre. À 69 ans, il animait une journée d'échanges avec le gratin européen de la recherche en informatique

PHOTO LAURENT THEILLET

que. Donald Knuth a publié, il y a quelque temps déjà, les trois premiers volumes de cette gigantesque somme, traduite en russe, en japonais, en polonais, etc. mais pas en français. Le quatrième tome est pour bientôt. Et Donald Knuth se dit décidé à poursuivre sa tâche tant qu'il en aura la force. Ses ouvrages, dont les ventes cumulées au fil des ans approchent le million d'exemplaires, visent essentiellement les informaticiens et créateurs de programmes. Une communauté cer-

tes minoritaire à travers le monde, mais qui se trouve investie d'une mission considérable. En quelques décennies, l'écriture informatique a aidé à résoudre d'innombrables problèmes. « Mais il y en a tant d'autres qui attendent des solutions, notamment dans le domaine médical », affirme le professeur émérite de Stanford.

Un chèque de 2,56 dollars. Pour mener à bien sa tâche, Donald Knuth s'est imposé une vie

d'ermite. D'ordinaire, sa journée débute par la bibliothèque ou la piscine. Après quoi, il passe tout le reste de son temps à sa table de travail, dimanche compris. Il n'a plus d'e-mail depuis le début des années 90, considérant que le courrier électronique représente une perte de temps, dès lors qu'on veut aller au fond des choses et non pas rester à leur surface. Une secrétaire lui fait passer les messages considérés comme les plus urgents. Pour le reste, Donald Knuth demande qu'on lui

écrive par courrier ordinaire ou par fax, dont il prend parfois connaissance avec des mois de retard. Il s'oblige, en revanche, à tenir aussi scrupuleusement que possible sa promesse d'envoyer un chèque de 2,56 dollars à tout lecteur ayant détecté une erreur dans un de ses livres. Par ailleurs, pour se détendre, il pratique l'orgue, appris dans sa prime jeunesse auprès de son père qui partagea sa vie entre la musique et l'enseignement.

L'orgue de Sainte-Croix. Donald Knuth n'est pas fermé aux choses de ce monde. Sur son site Internet, à la rubrique « Questions qui ne me sont pas fréquemment posées », il demande entre autres : « Pourquoi mon pays a-t-il le droit d'occuper l'Irak ? », « Pourquoi mon pays ne soutient-il pas une Cour internationale de justice ? » Mais cet homme de conscience ne se veut pas militant, pas plus qu'il n'aspire au vedettariat et à la richesse. « Beaucoup de gens, dit-il, ont tendance à considérer que l'informatique, c'est surtout des histoires de business, d'entreprise. Ce n'est pas mon cas. » Sortant de sa semi-réclusion, Donald Knuth s'est donc laissé convaincre d'accepter les hommages de l'université de Bordeaux, après celles de Harvard, d'Oxford, de Tübingen. Il a eu le coup de foudre pour la beauté et l'agrément de la ville. Et il n'oubliera sans doute pas de sitôt l'orgue illustre de l'église Sainte-Croix (2), sur lequel il a eu le bonheur d'exercer son talent.

(1) Ces journées étaient organisées par le Laboratoire bordelais de recherche en informatique (Labri).

(2) Thierry Semenou, professeur d'orgue au conservatoire de Bordeaux, a joué dans ce domaine un rôle de cicérone auprès de Donald Knuth.

some parameters

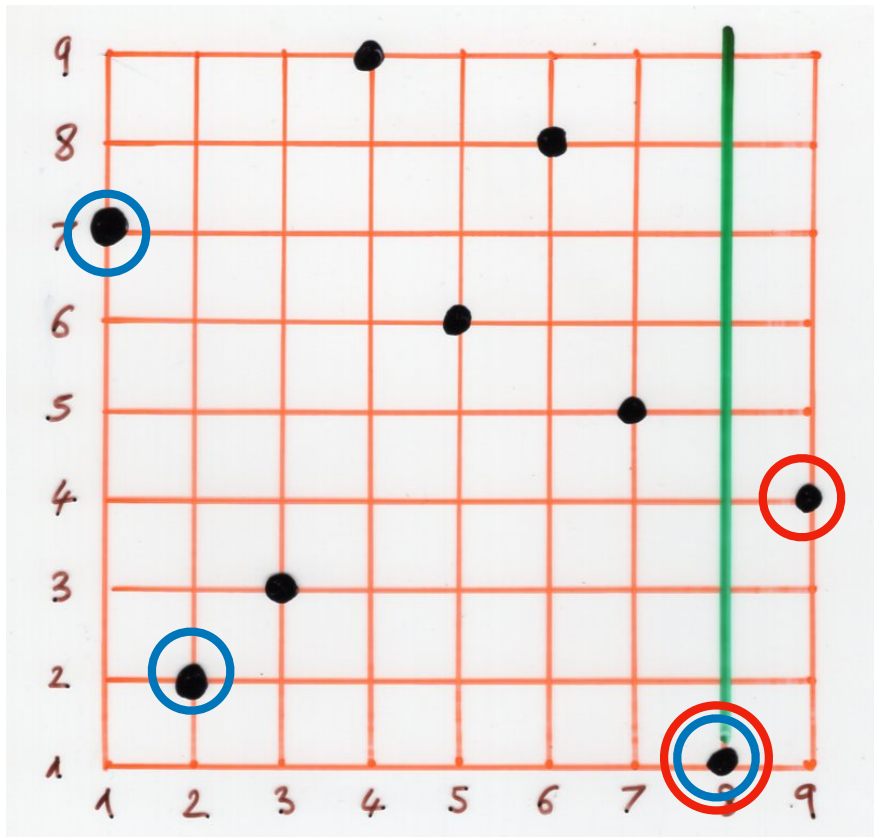
$\tau = 6\ 8\ 5\ 2\ 3\ 9\ 1\ 4\ 7$

$w = x_1 x_2 \dots x_n$
word with distinct letters

lr-min

left to right minimum element

$$x_i = \min(x_1, x_2, \dots, x_i)$$



$$\sigma = 723968514$$

permutation word

left-to-right
right-to-left

minimum

elements

Definition

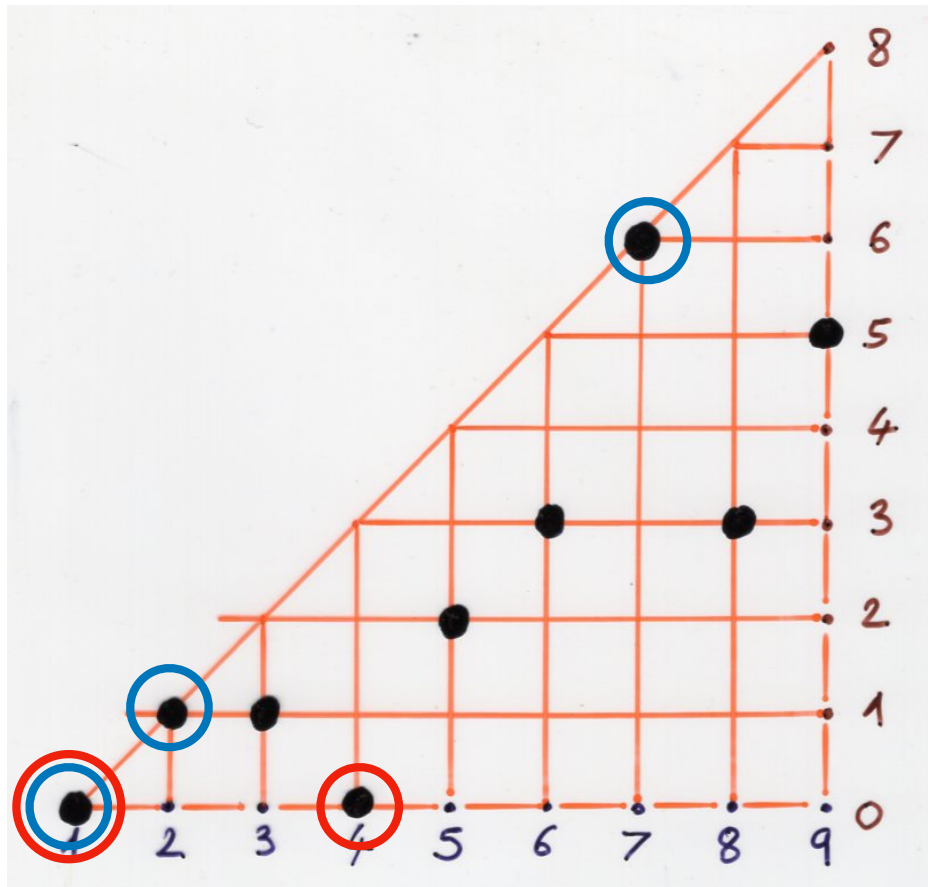
sub-excedant functions

$$f: [1, n] \rightarrow [0, n-1]$$

pour tout $1 \leq i \leq n$, $0 \leq f(i) < i$

$\mathcal{F}_n$ set of sub-excedante functions

$$|\mathcal{F}_n| = n!$$



generating - polynomial:

$$xy(x+y)(x+1+y) \dots (x+n-1+y)$$

$$\sigma = 723968514$$

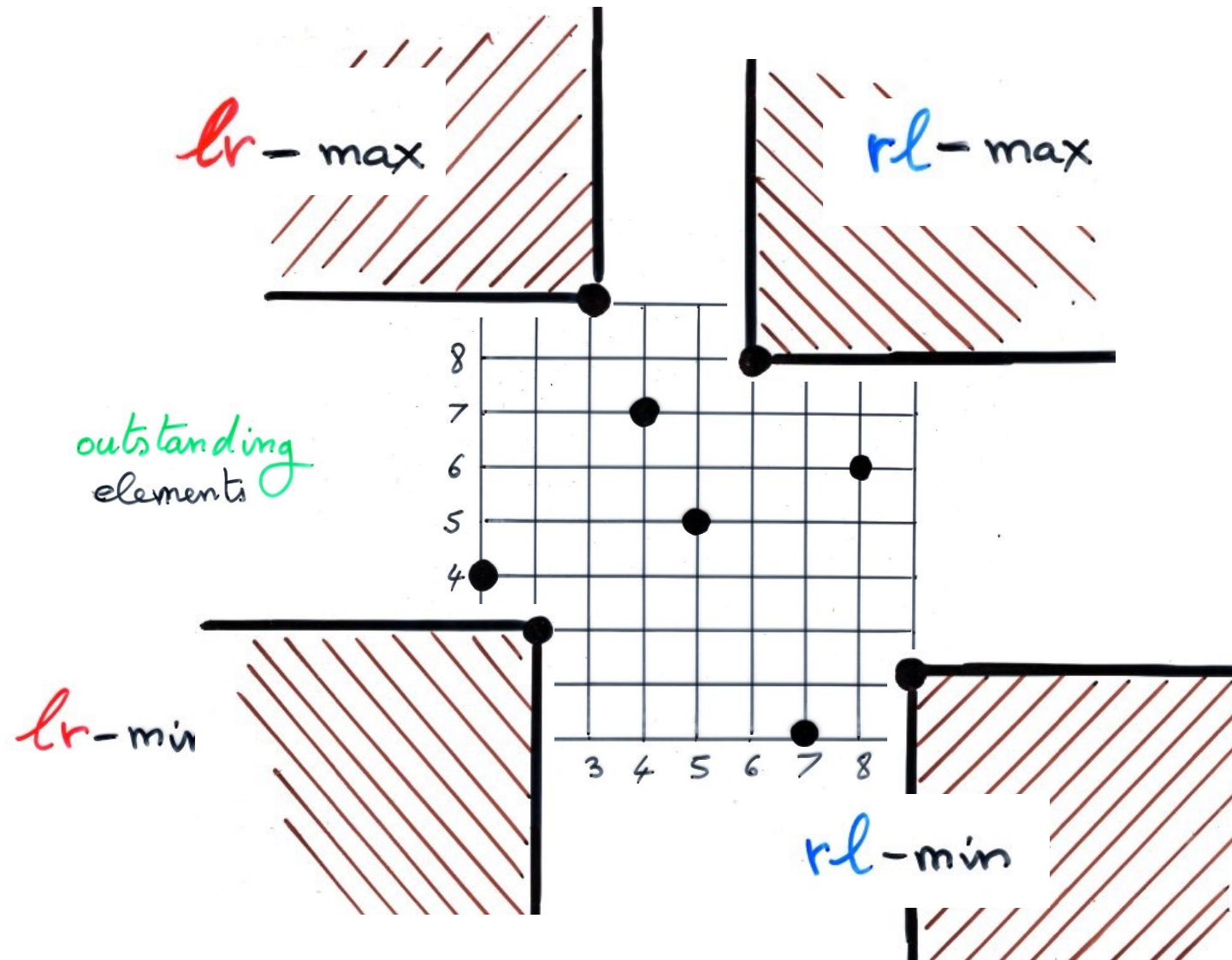
permutation word

left-to-right
right-to-left

minimum

maximal

elements



left-to-right
right-to-left

minimum

maximal

elements

Proposition

The **maximum** letter of the blocks of letters reaching the ground level are:

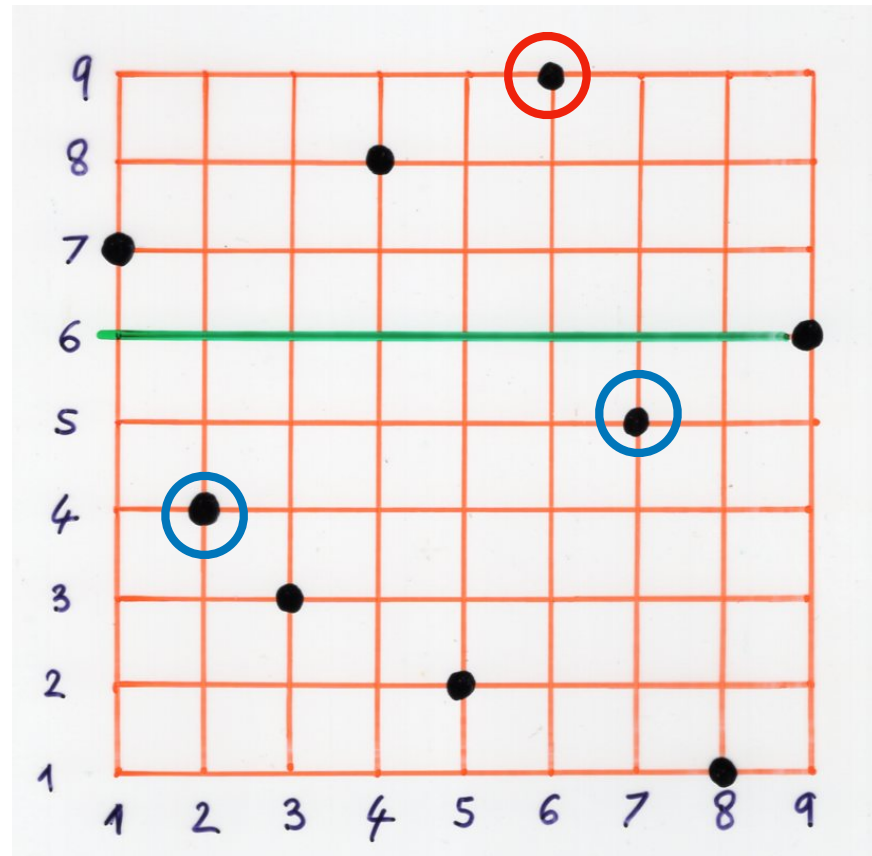
- for the **columns** of T , the left-to-right **maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) < \sigma(n+1)$

- for the **rows** of T , the right-to-left **maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) > \sigma(i+1)$

example

$\sigma = 7 \ 4 \ 3 \ 8 \ 2 \ 9 \ 5 \ 1 \ 6$
7 8 9 ← 6
→ 4 3 2 5 1 6

7 4 3 8 2 9 5 1 6

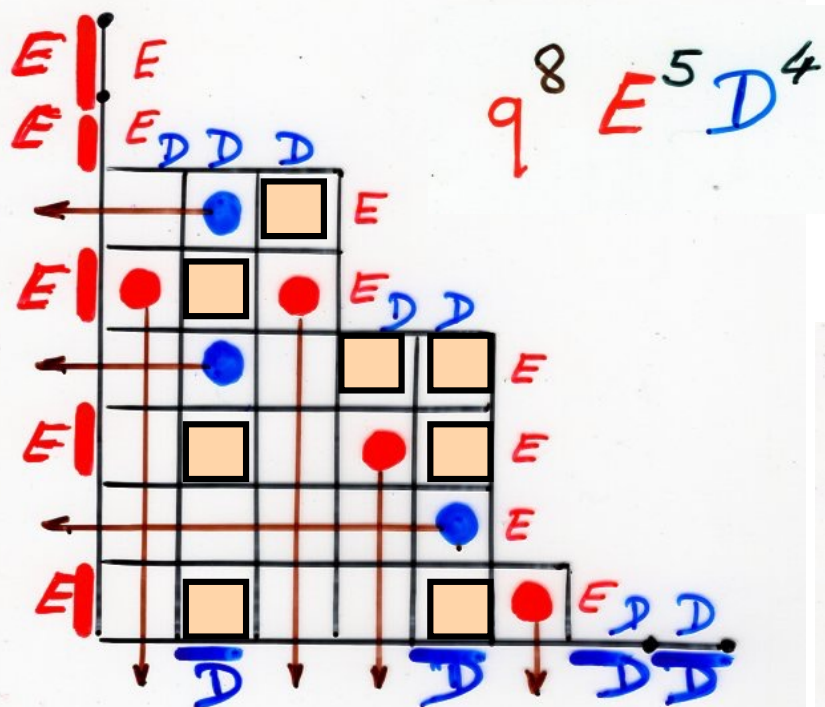


- for the **columns** of T , the **left-to-right maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) < \sigma(n+1)$

- for the **rows** of T , the **right-to-left maximum** elements of the **subsequence** of the **permutation** σ formed with the values $\sigma(i) > \sigma(i+1)$

example

$\sigma = 7 \ 4 \ 3 \ 8 \ 2 \ 9 \ 5 \ 1 \ 6$
 $\begin{matrix} & 7 & & 8 & & 9 & & 6 \\ & & & & & \leftarrow & & \\ \rightarrow & 4 & 3 & 2 & 5 & 1 & 6 \end{matrix}$



- interpretation of the two parameters on *alternative tableaux*
- number of "open" columns (i.e. columns without a red cell)
 - number of "open" rows (i.e. rows without a blue cell)

$$i(T) = \text{nb of rows without } \bullet$$

$$j(T) = \text{nb of columns without } \bullet$$

Corollary for T alternative tableau (size n)
 The double distribution $(i(T), j(T))$
 of parameters

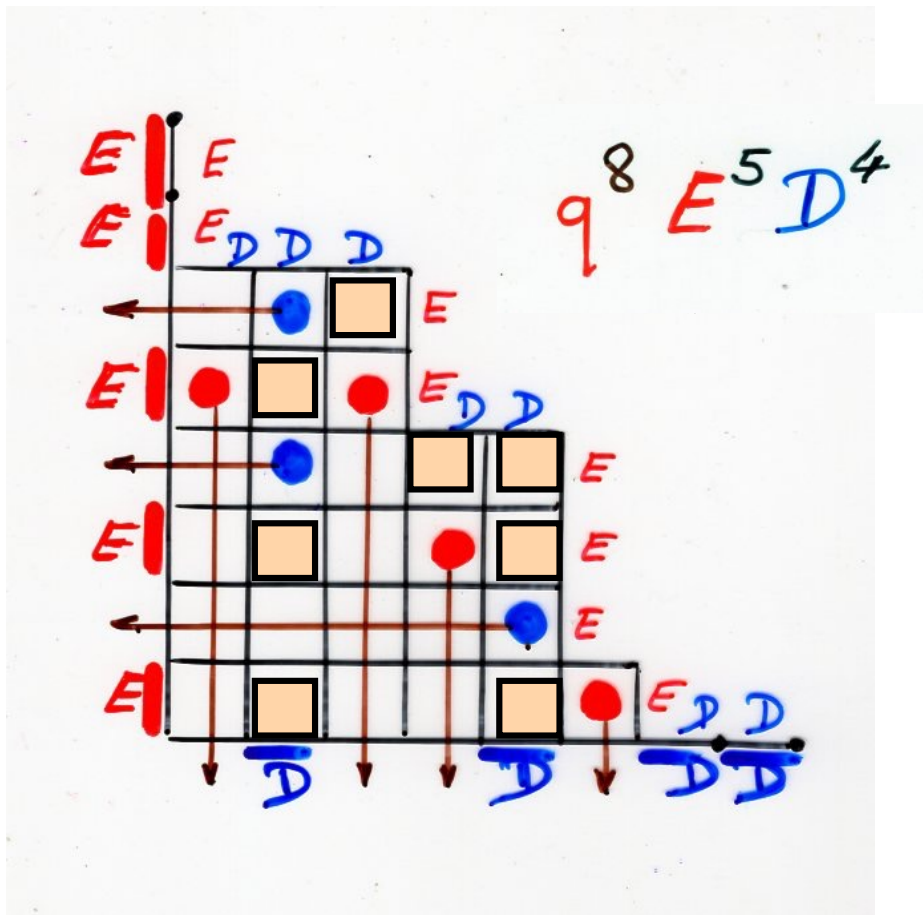
$\nearrow$ number of open rows
 $\nwarrow$ number of open column

is the same as the double distribution
 of permutations of S_{n+1} according
 to the parameters

(left-to-right minimum elements of σ
 right-to-left

generating polynomial:

$$xy(x+y)(x+1+y) \dots (x+n-1+y)$$



cell 

crossing of T

exchange in the "Ex-Fu" algorithm

$k(T) = \text{nb of cells}$ 

"exchange-fusion" algorithm

the parameter q

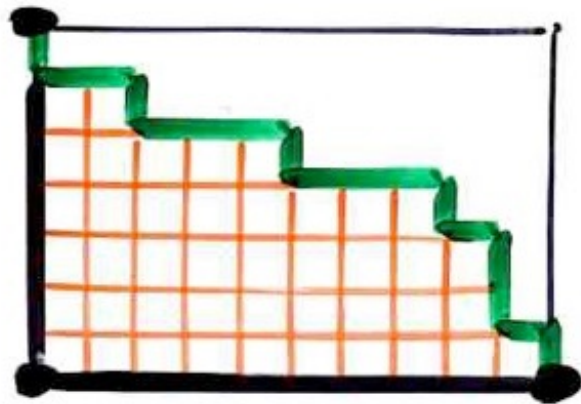
this is another story !

.... related to q -analogue of Laguerre polynomials

permutation
tableaux

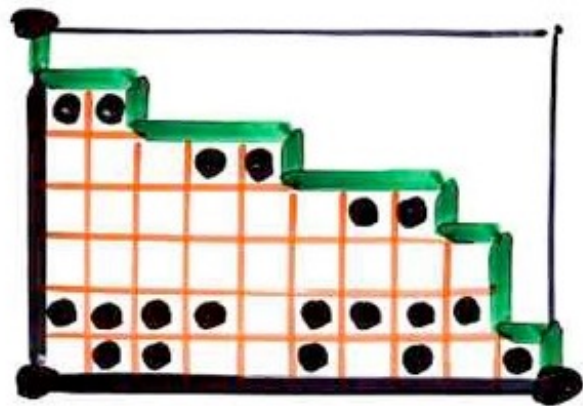
Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells
with 0 and 1

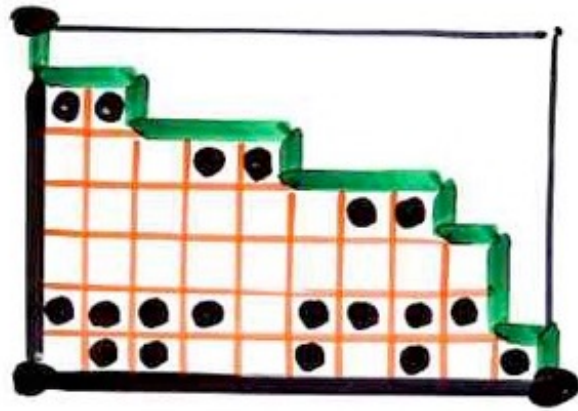
(i)

$\square = 0$ $\square \bullet = 1$

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



$\square = 0$ $\blacksquare = 1$

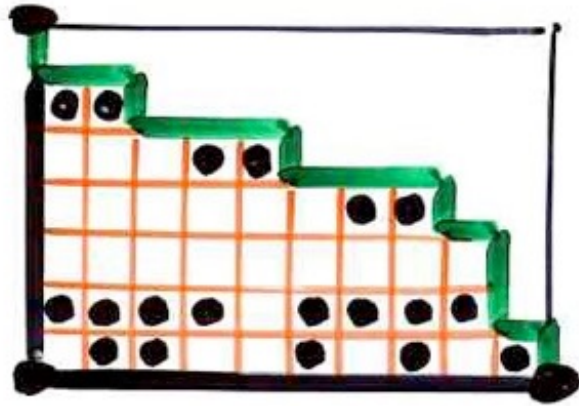
filling of the cells
with 0 and 1

(i) in each column:
at least one 1

(ii)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



$\square = 0$ $\square \bullet = 1$

filling of the cells
with 0 and 1

(i) in each column:
at least one 1

(ii)  forbidden

permutation tableaux

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

Corteel, Williams (2006) PASEP

Partially Asymmetric Exclusion Process

The total number of permutation tableaux (n fixed, $1 \leq k \leq n$) is

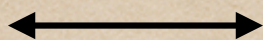
$n!$

bijection $\longleftrightarrow$ permutation tableaux

- Postnikov, Steingrímsson, Williams (2005)
- Corteel (2006)
- Corteel, Nadeau (2007)

Bijection

permutation
tableaux



alternative tableaux

bijection



alternative

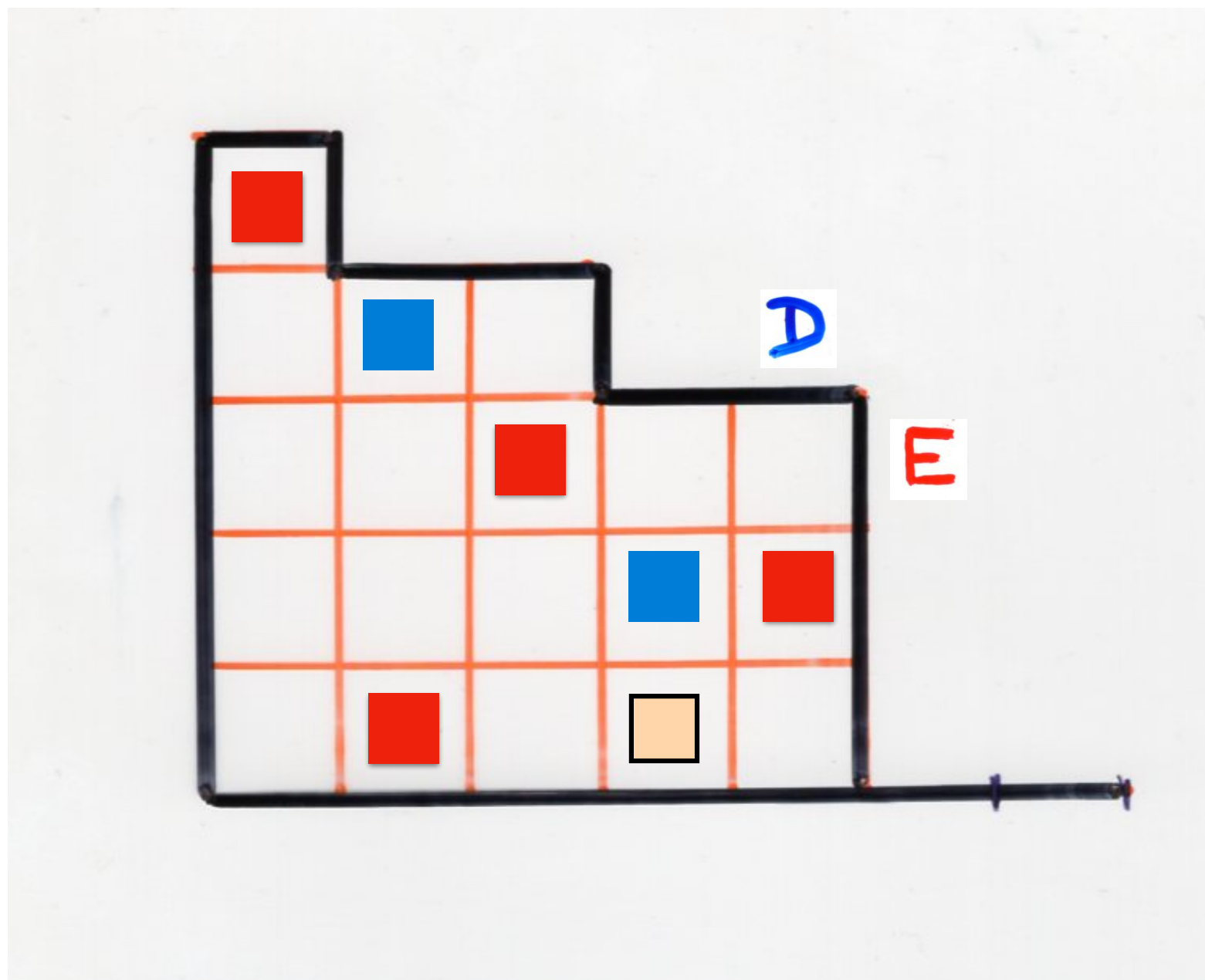
tableaux

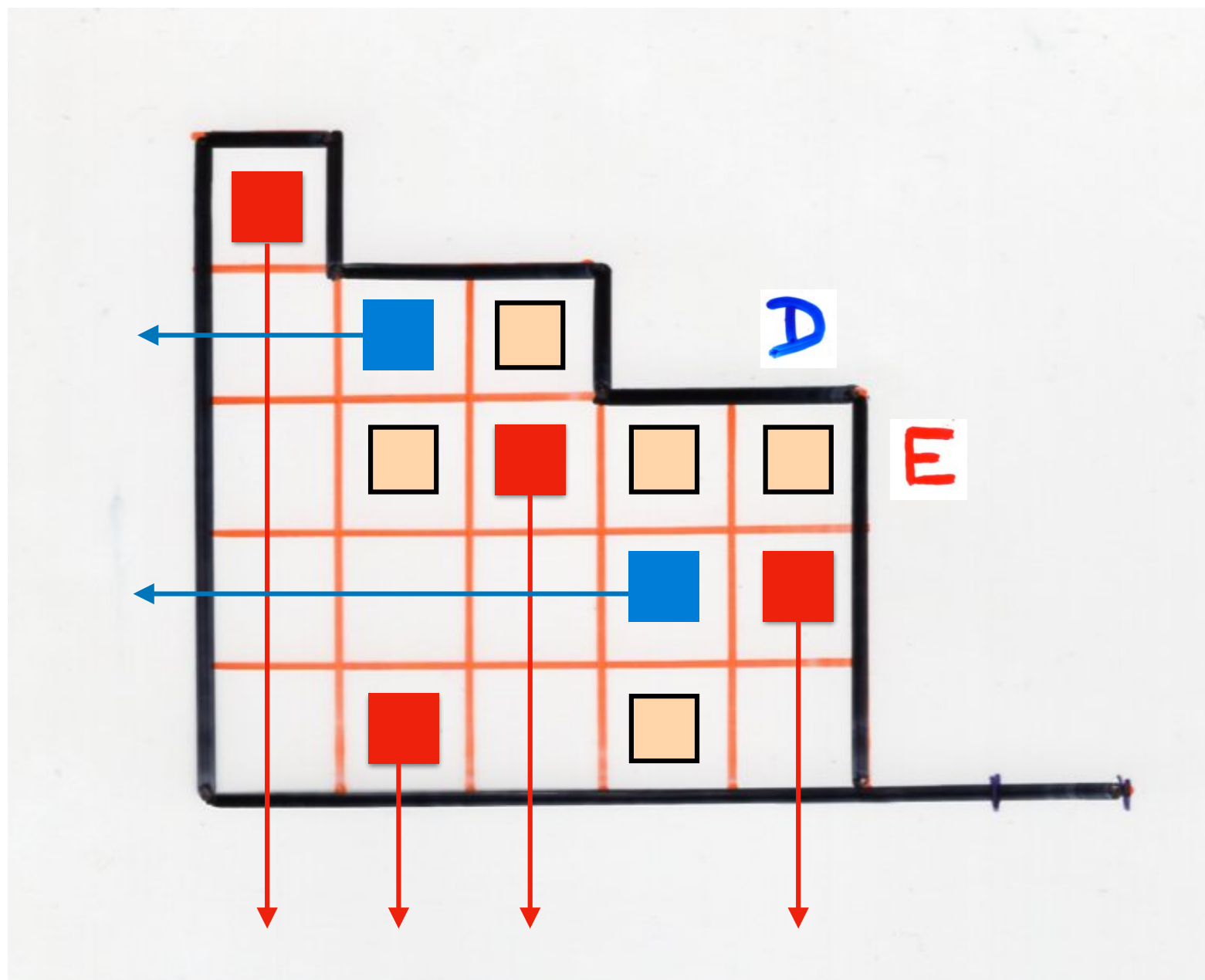
size n

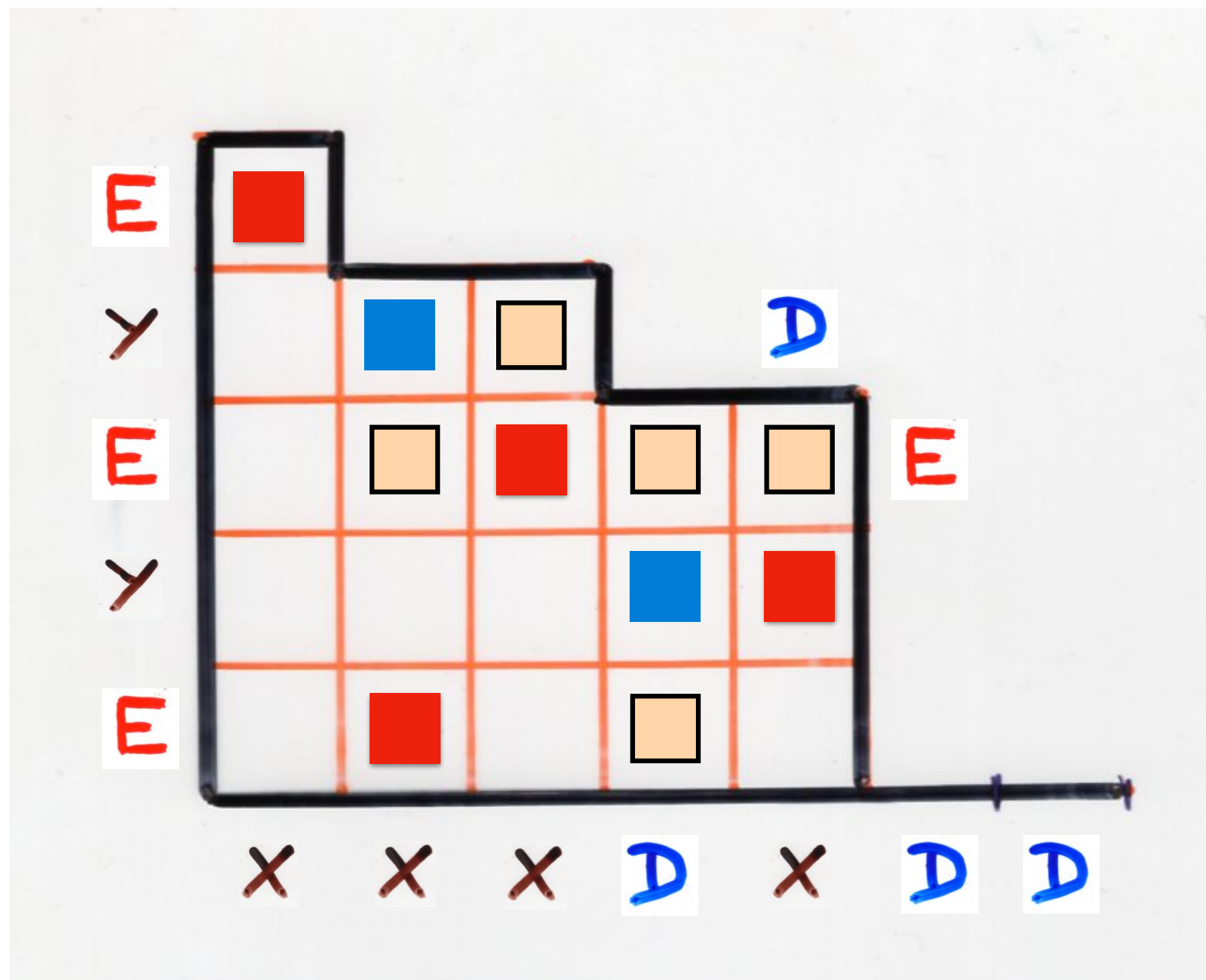
permutation

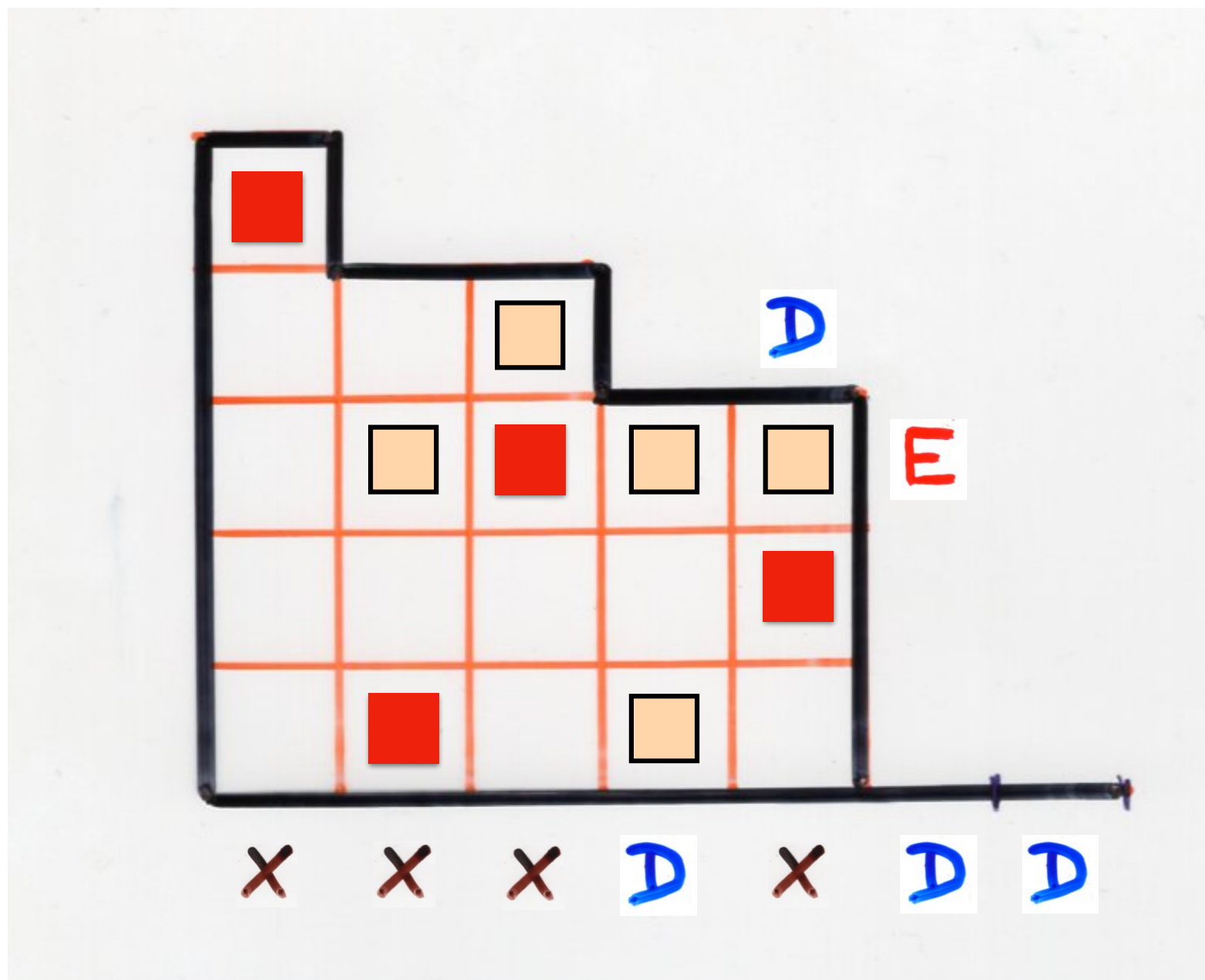
tableaux

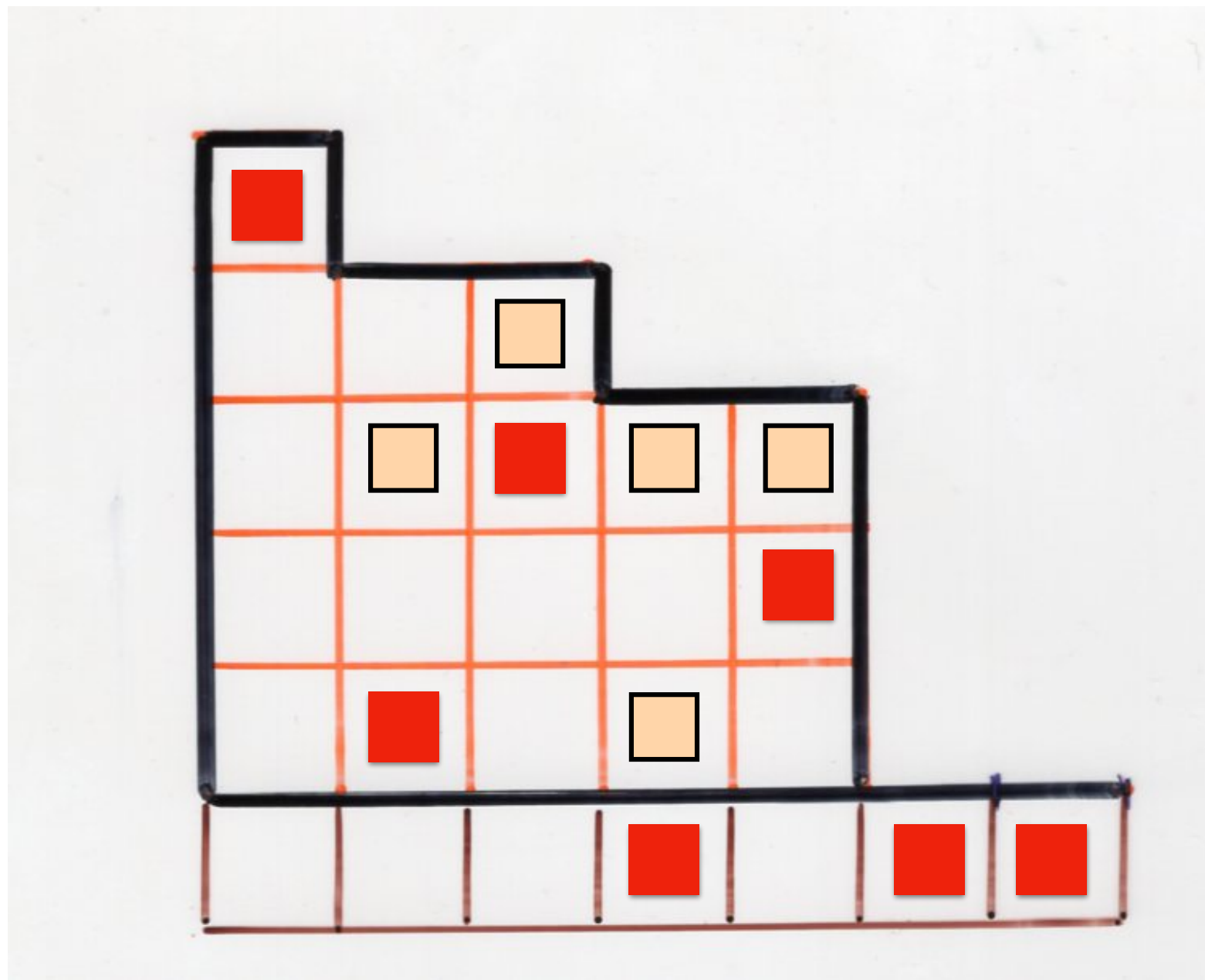
size $(n+1)$

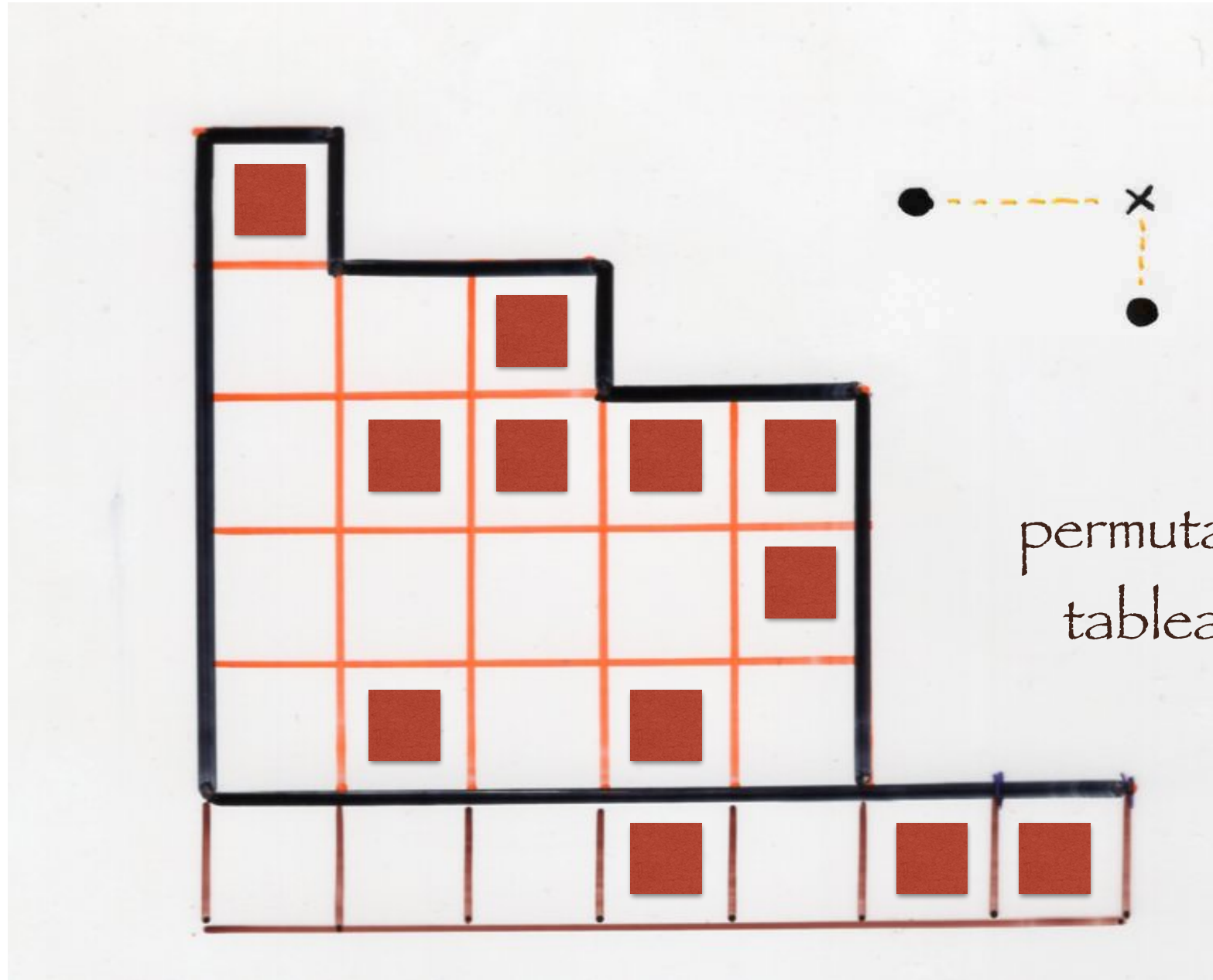




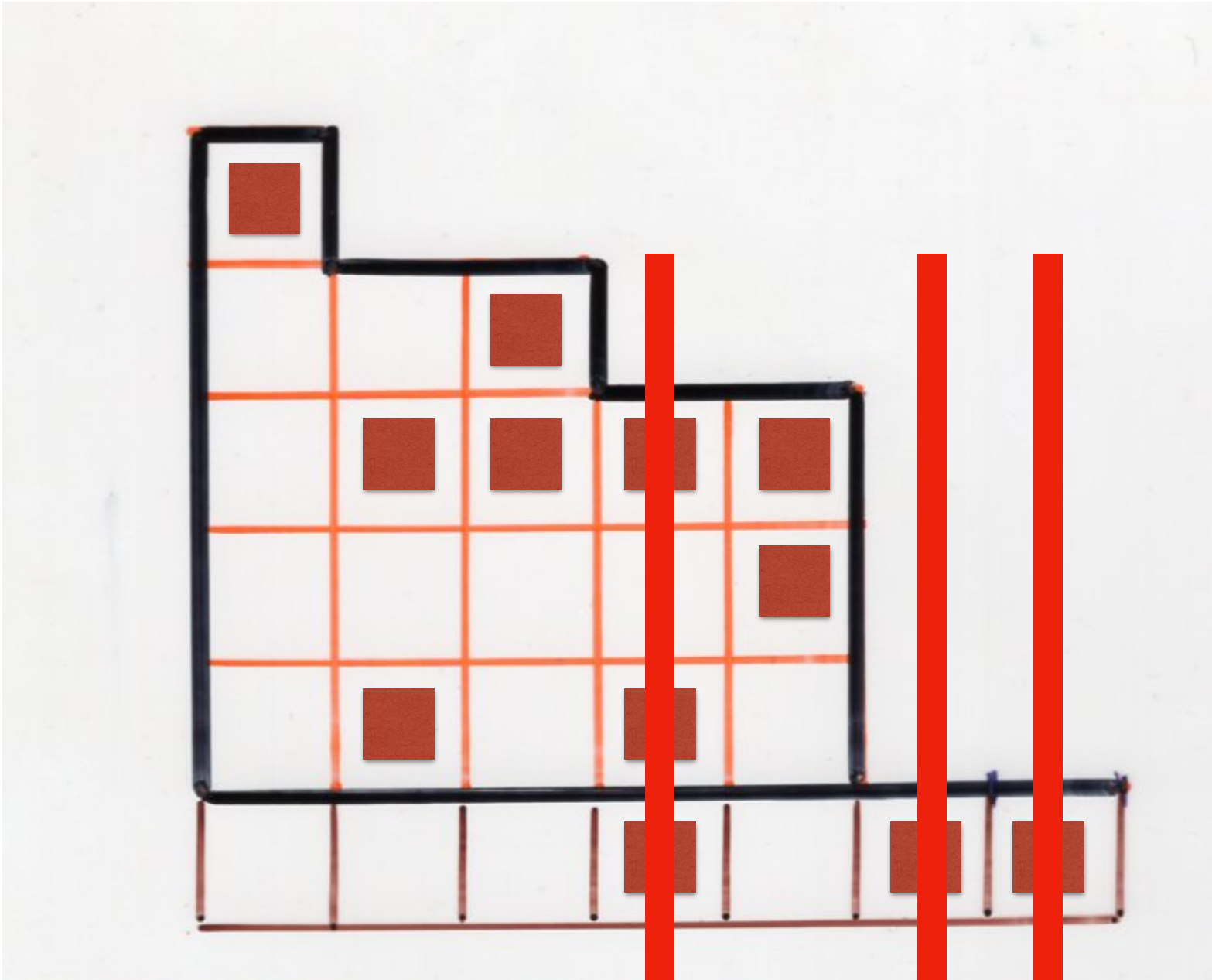


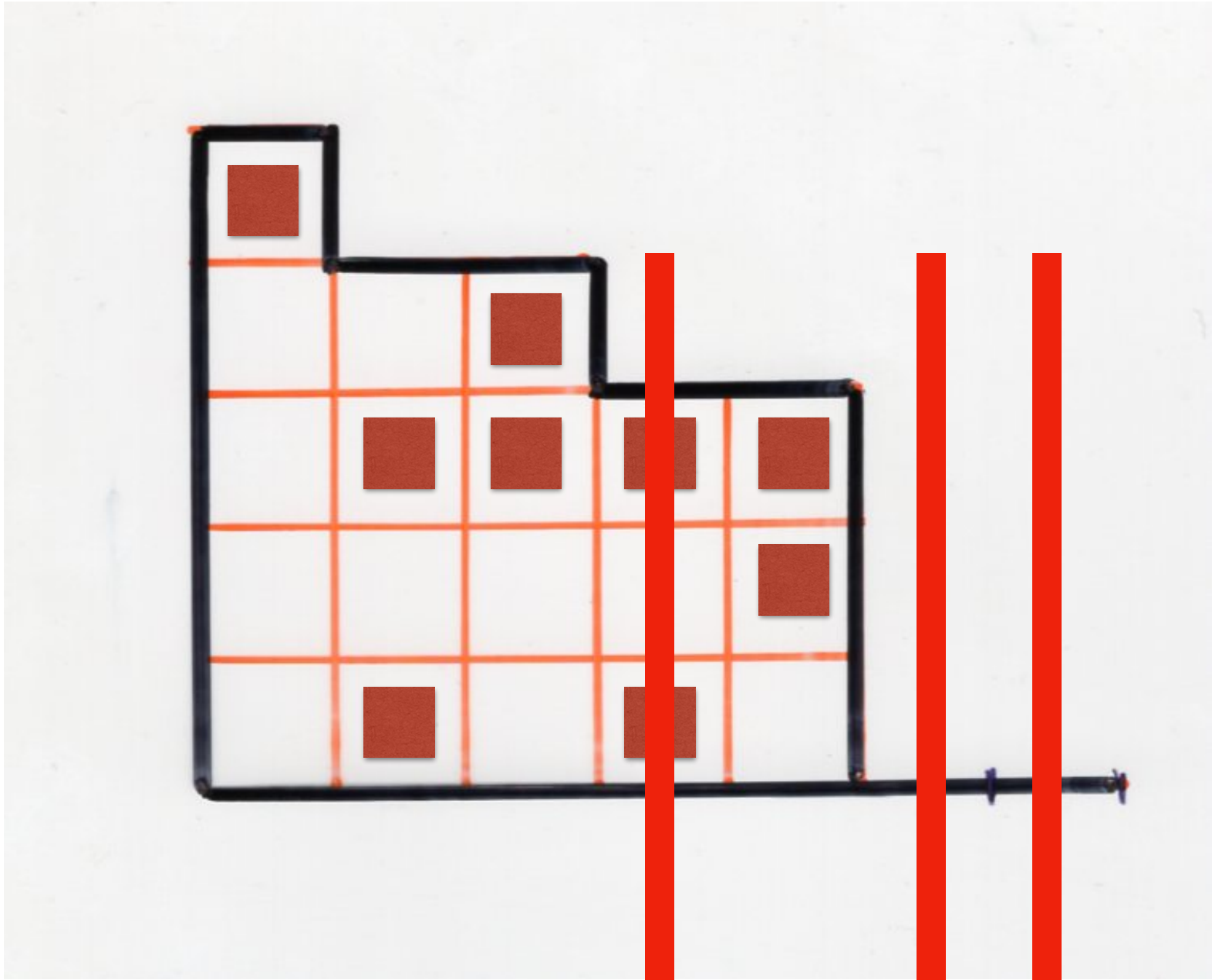


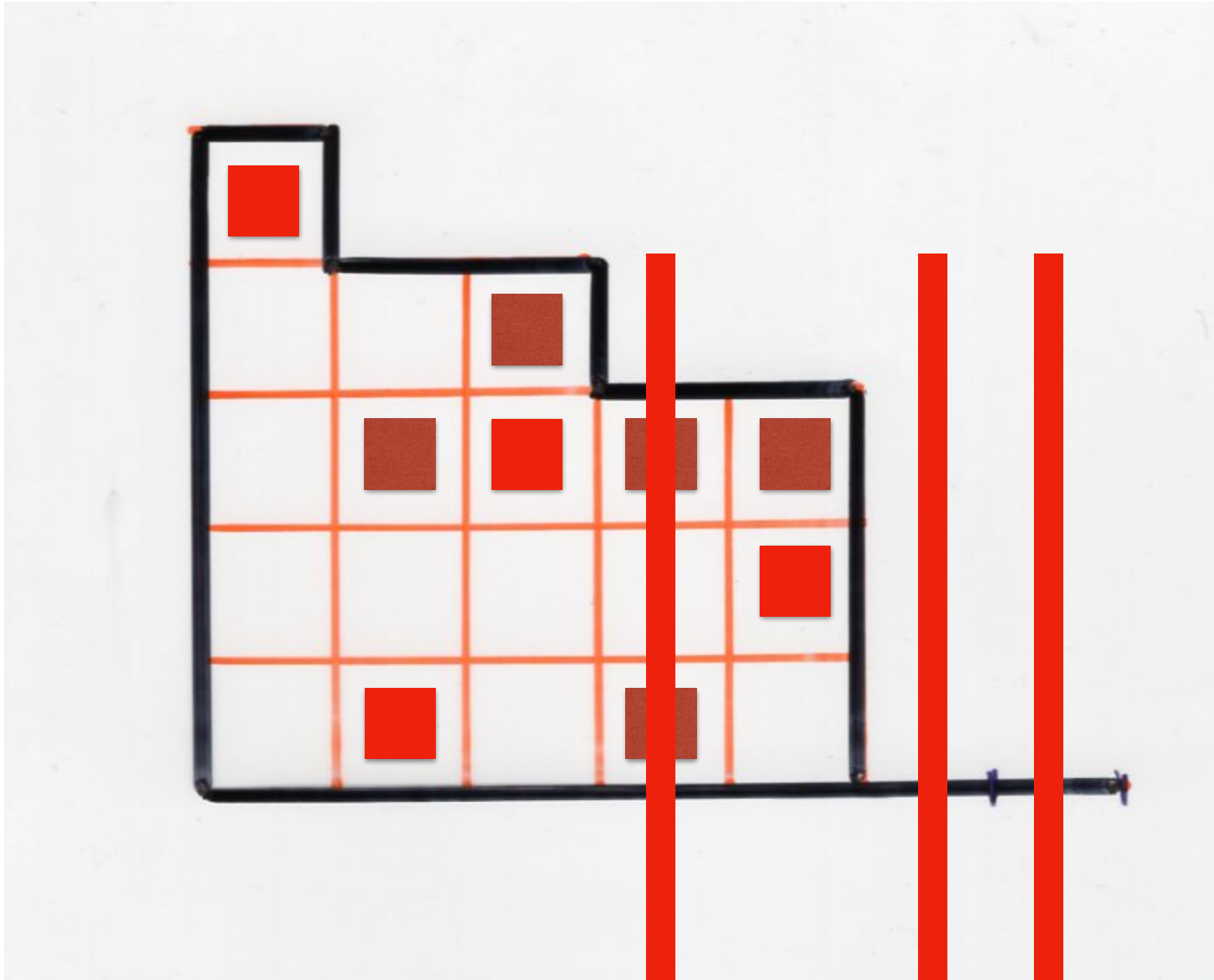


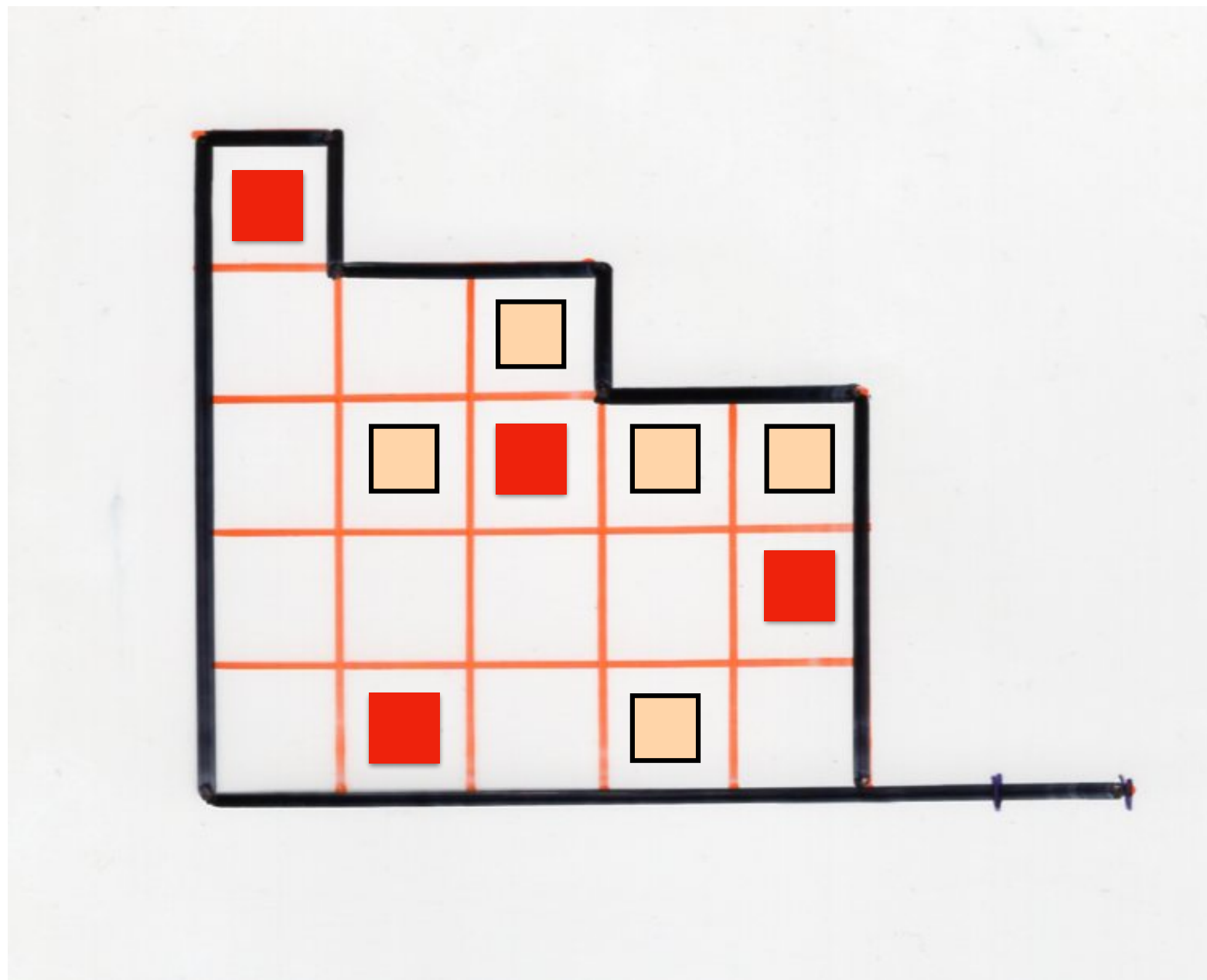


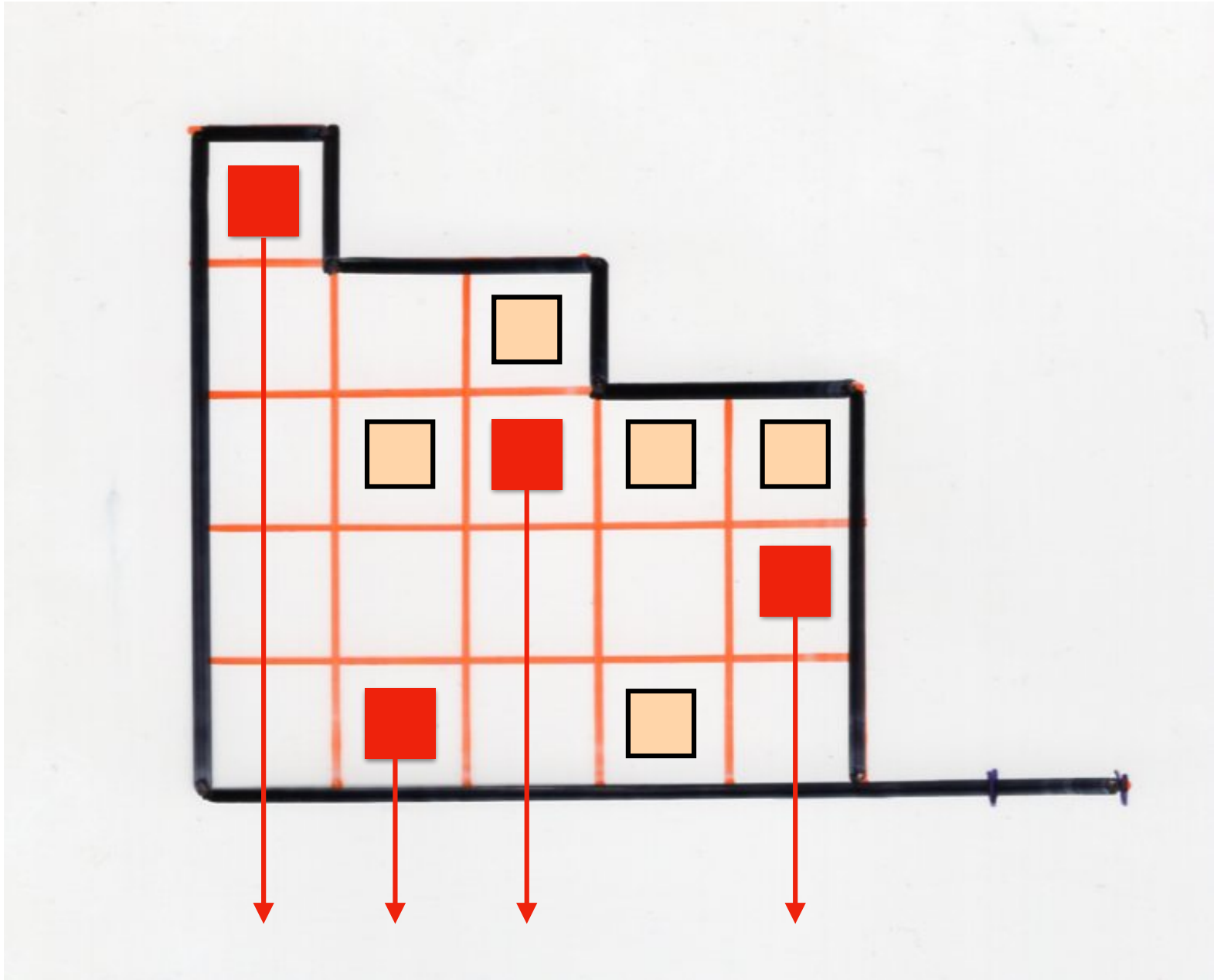
permutation
tableaux

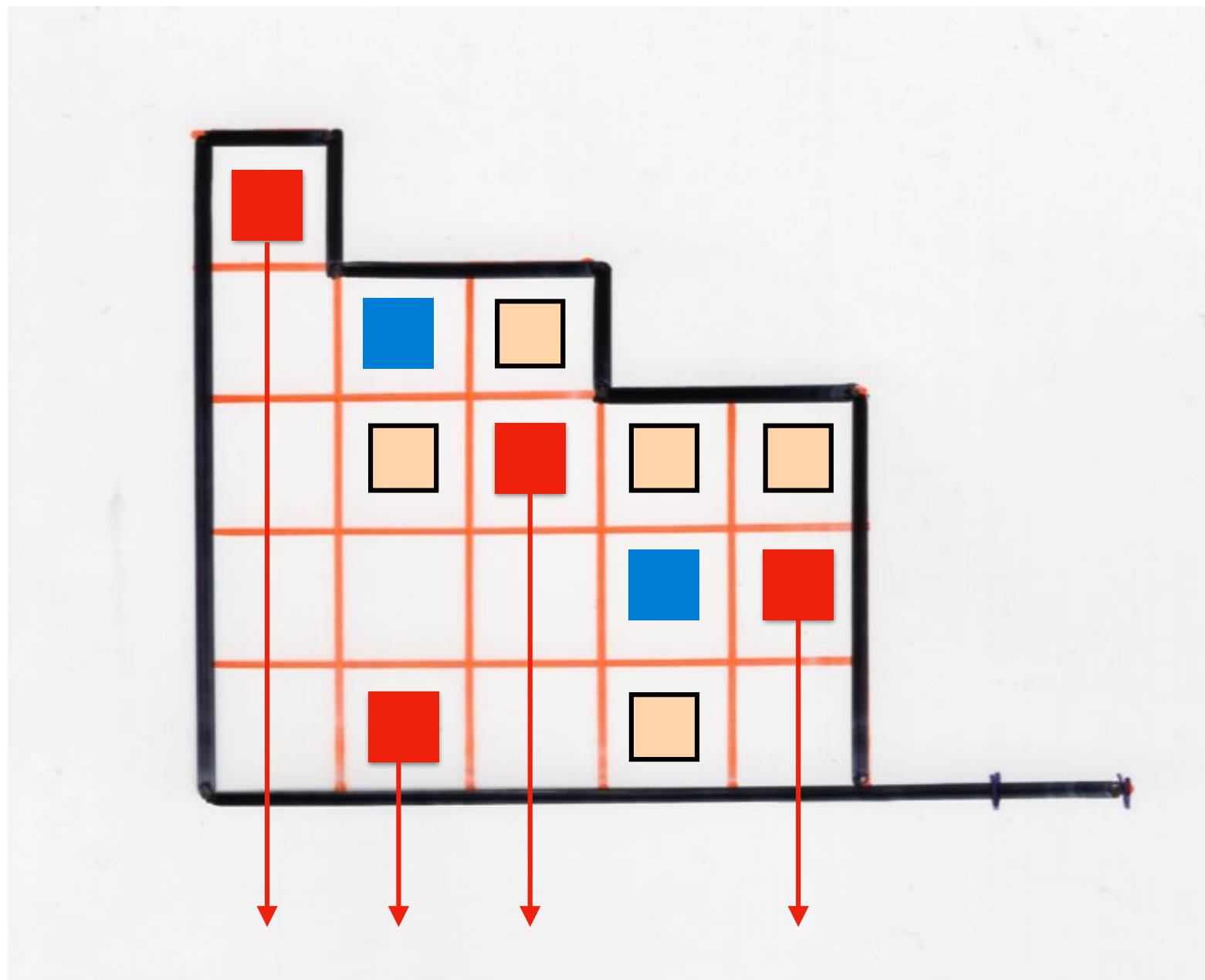


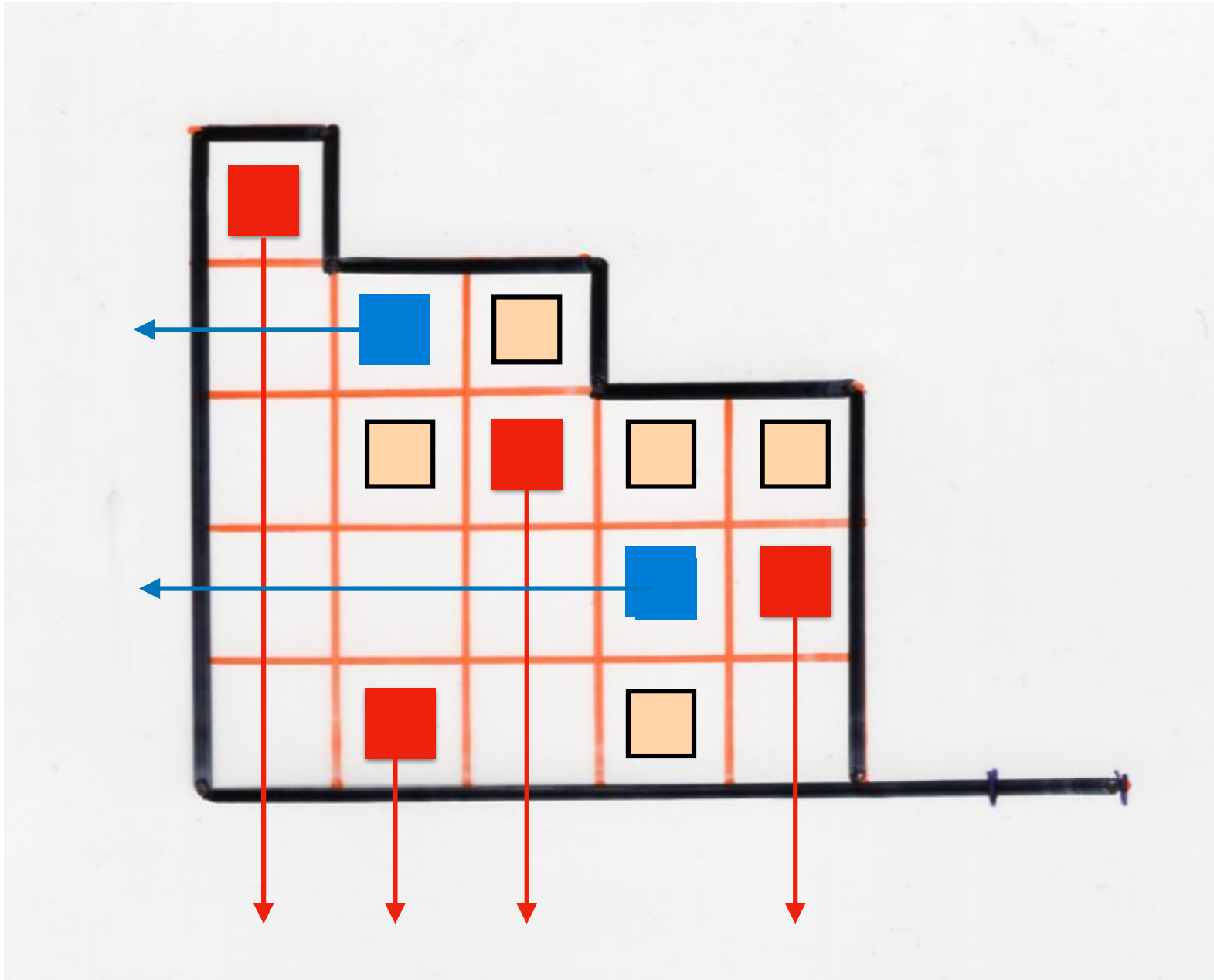












$$\begin{aligned}
 Q \left\{ \begin{aligned}
 DE &= \boxed{} ED - \boxed{} EX + \boxed{} \cancel{Y} D \\
 XE &= \boxed{} EX \\
 DY &= \boxed{} \cancel{Y} D \\
 XY &= \boxed{} \cancel{Y} X
 \end{aligned} \right.
 \end{aligned}$$

$\frac{D}{X}$

$\cancel{Y} \mid E$

permutation
tableaux

$$\begin{aligned}
 Q^+ \left\{ \begin{aligned}
 ED &= \boxed{} DE \\
 EX &= \boxed{} XE + \boxed{} DE \\
 YD &= \boxed{} DY \quad \boxed{} DE \\
 YX &= \boxed{} XY
 \end{aligned} \right.
 \end{aligned}$$

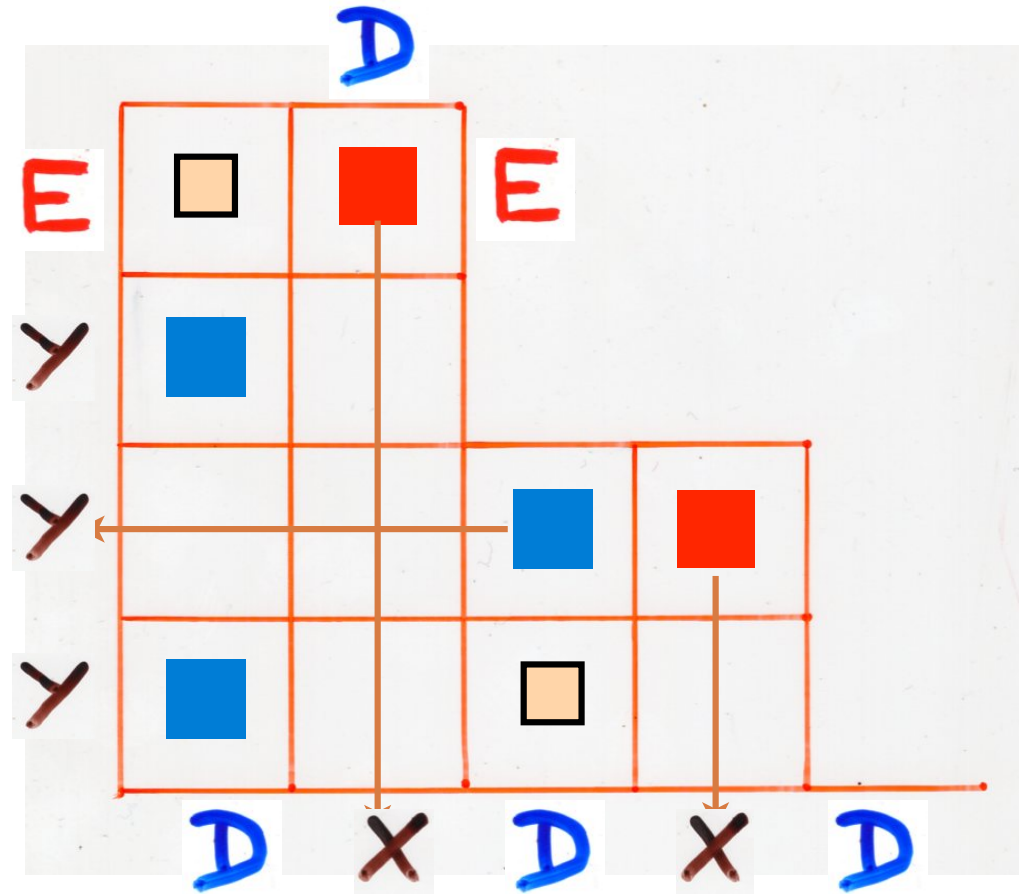
$$\begin{array}{l}
 Q \left\{ \begin{array}{l}
 DE = \square ED - \blacksquare EX + \blacksquare YD \\
 XE = \square EX \\
 DY = \square YD \\
 XY = \square YX
 \end{array} \right.
 \end{array}
 \quad
 \begin{array}{c}
 \frac{D}{X} \quad Y|E
 \end{array}$$

alternative

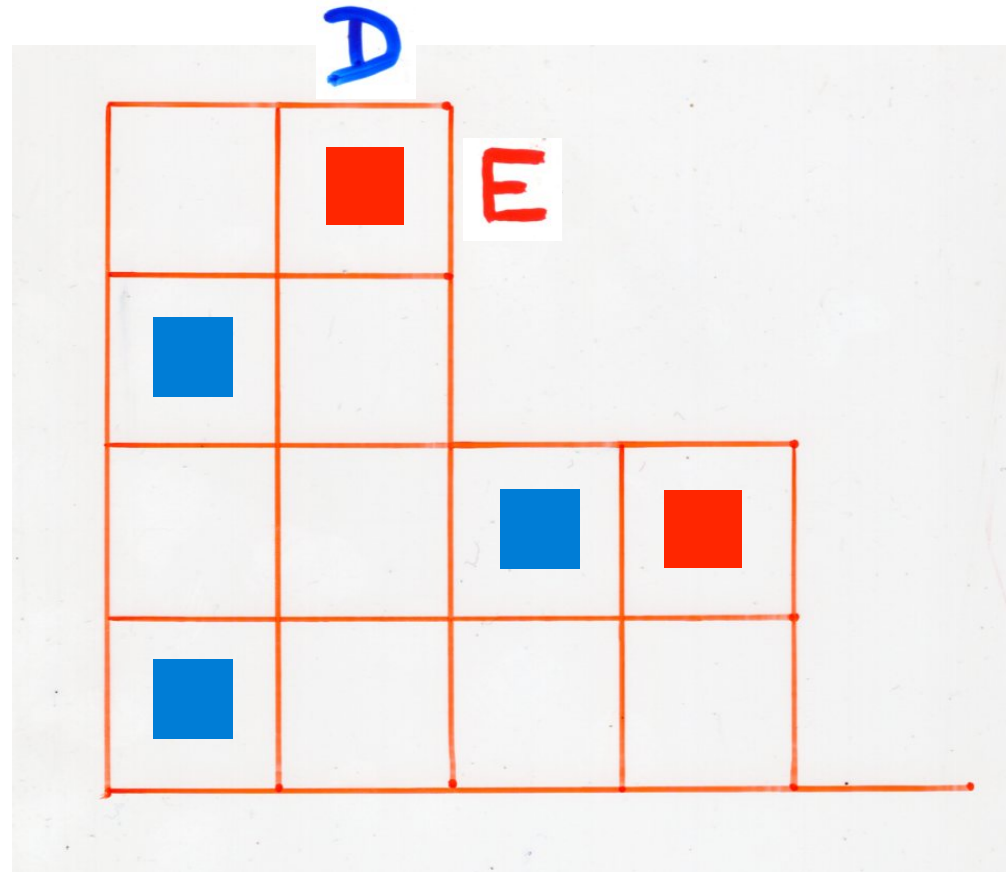
tableaux

$$\begin{array}{l}
 Q^+ \left\{ \begin{array}{l}
 ED = \square DE \\
 EX = \square XE + \blacksquare DE \\
 YD = \square DY + \blacksquare DE \\
 YX = \square XY
 \end{array} \right.
 \end{array}$$

alternative
tableaux



alternative
tableaux

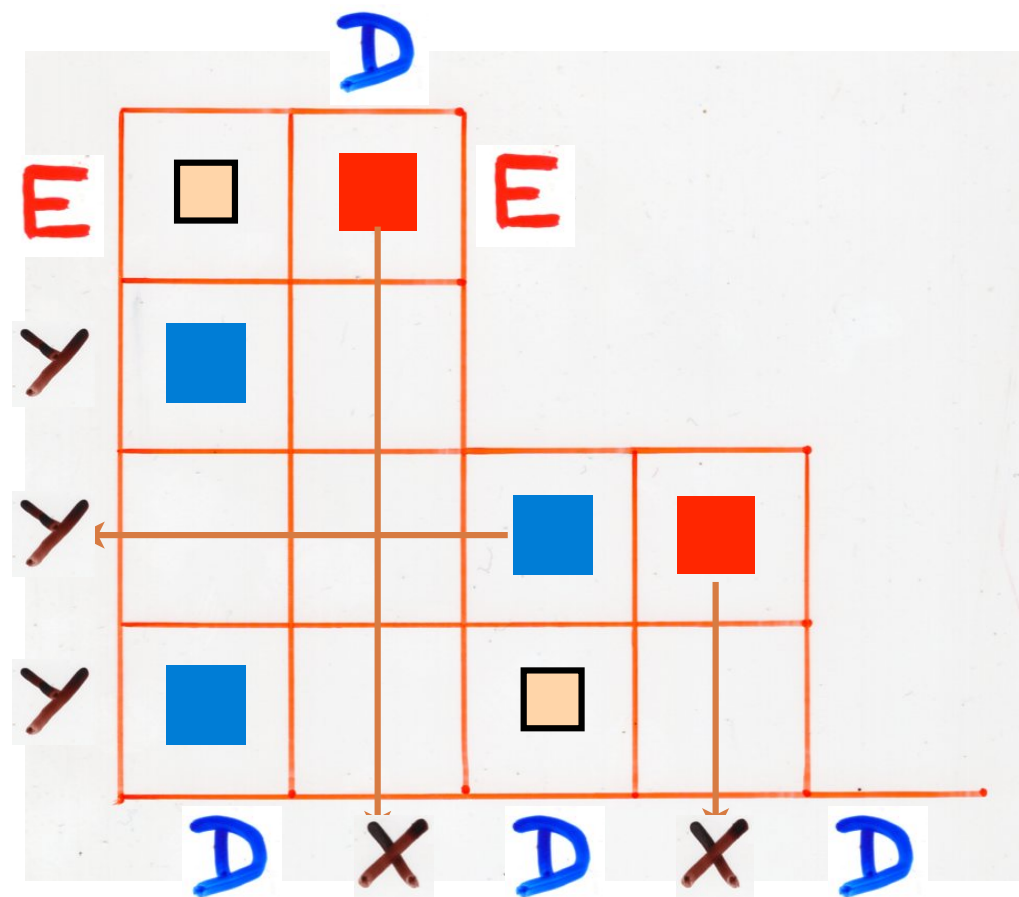


$$\begin{array}{l}
 Q \left\{ \begin{array}{l}
 DE = \square ED - \blacksquare EX + \blacksquare YD \\
 XE = \square EX \\
 DY = \square YD \\
 XY = \square YX
 \end{array} \right.
 \end{array}
 \quad
 \begin{array}{c}
 \frac{D}{X} \quad Y|E
 \end{array}$$

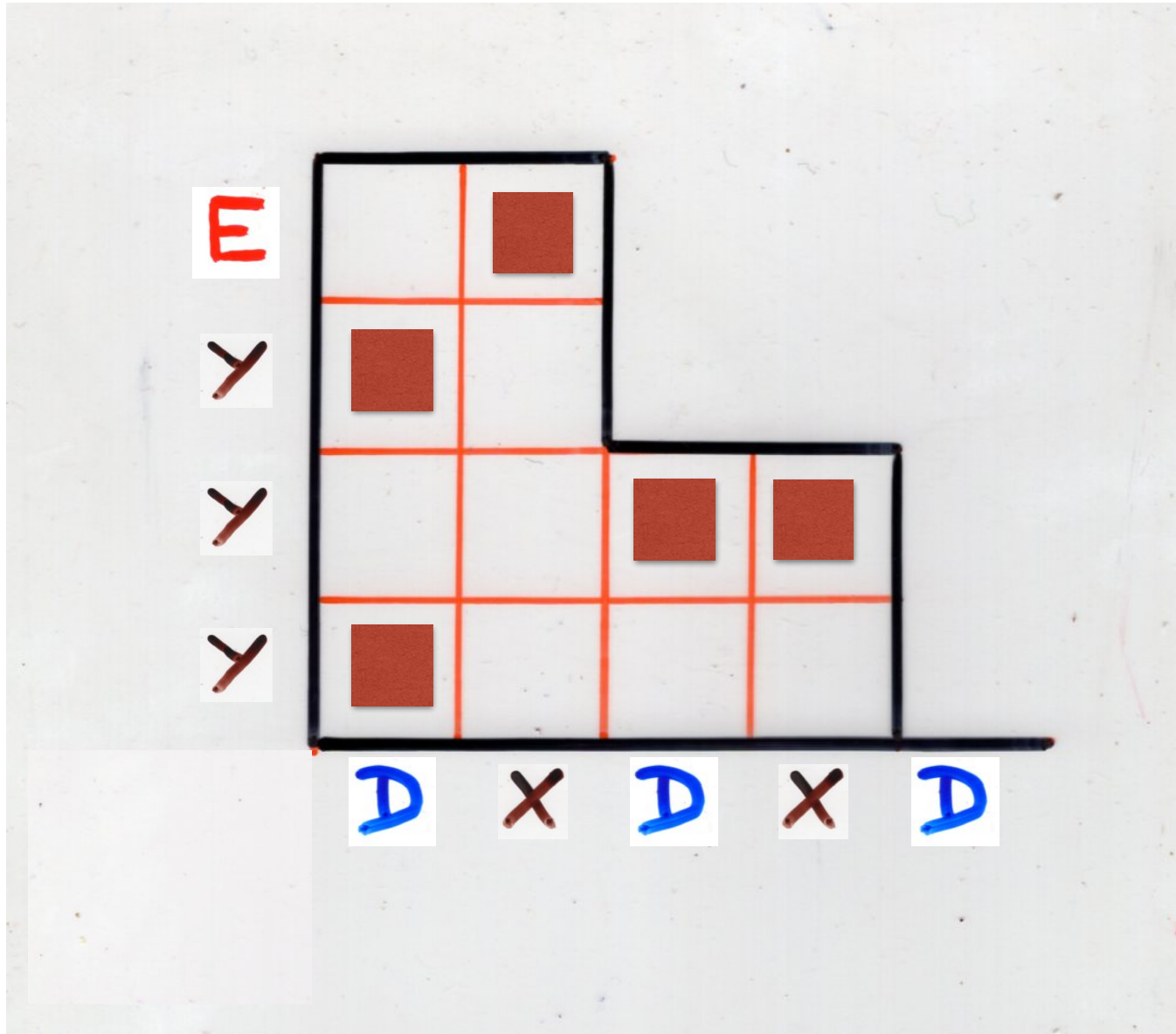
tree-like tableaux

$$\begin{array}{l}
 Q^+ \left\{ \begin{array}{l}
 ED = \square DE \\
 EX = \square XE - \blacksquare DE \\
 YD = \square DY + \blacksquare DE \\
 YX = \square XY
 \end{array} \right.
 \end{array}$$

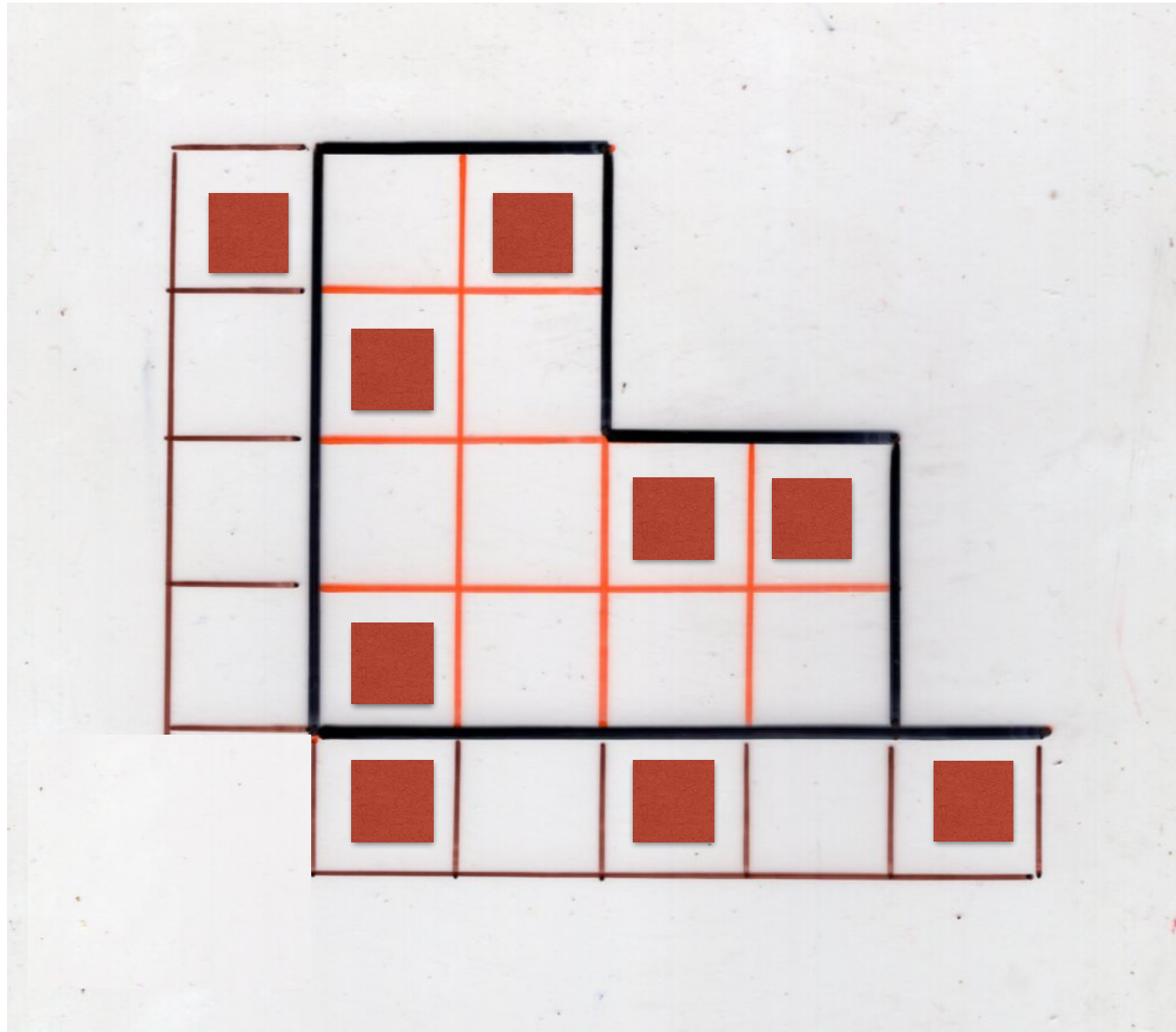
alternative
tableaux



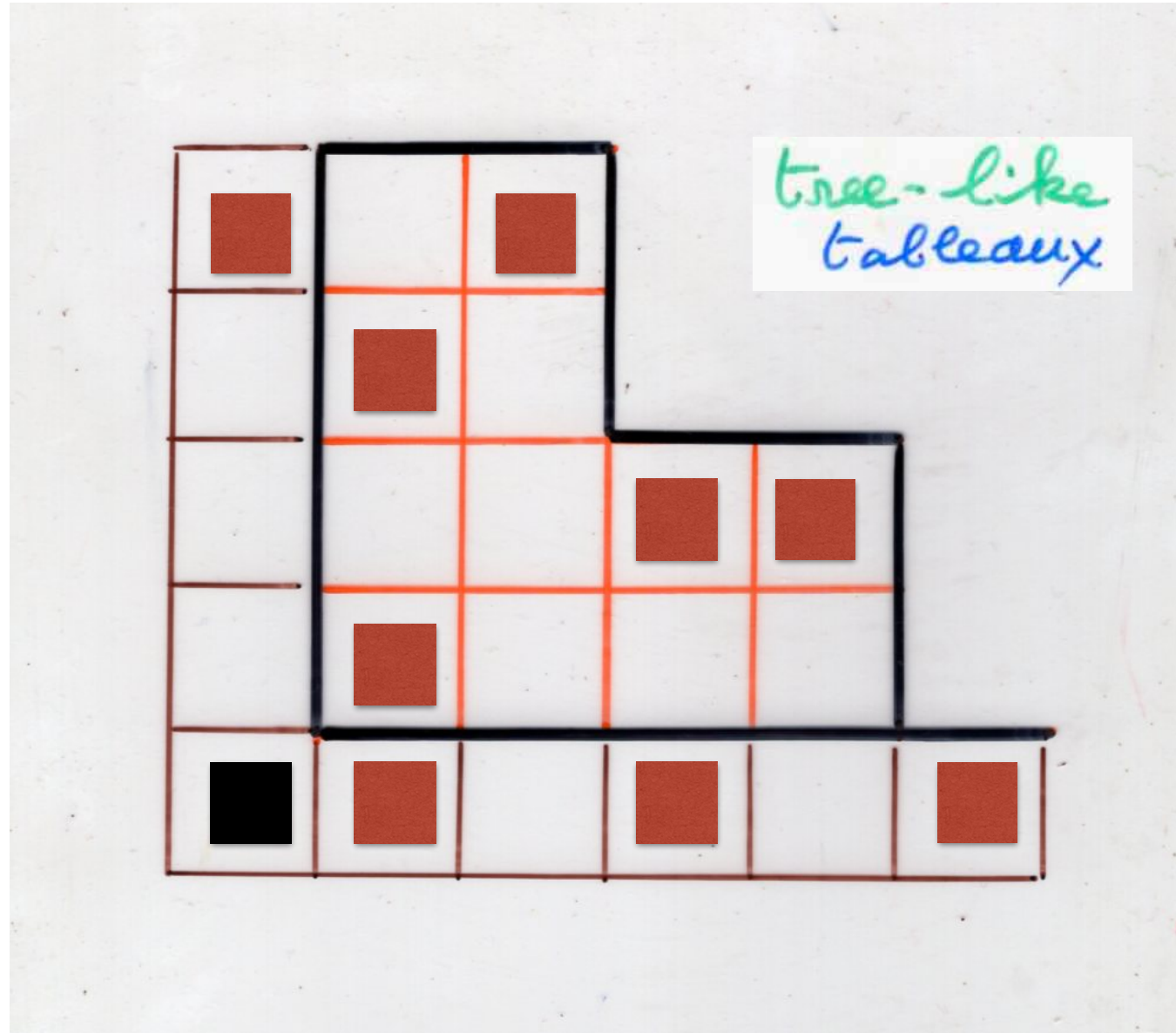
≈ reverse Q-tableau



$\approx$ reverse Q-tableau



= reverse Q-tableau



Definition Tree-like tableaux
Aval, Bousicault, Nadeau (2011)

Ferret diagram F with cell



empty cell

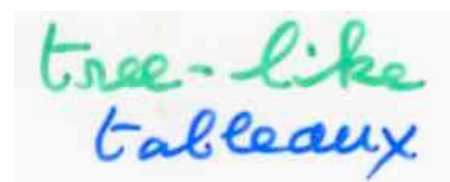


pointed cell

(i) the bottom left cell is pointed
(called the root cell)

(ii) for every non-root pointed cell c ,
there exist a pointed cell below c
in the same column, or a pointed cell
to its left in the same row,
but not both

(iii) every column and every row
possesses at least one pointed cell



$$Q \left\{ \begin{array}{l} DE = \square ED - \blacksquare EX + \blacksquare YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$

$$\frac{D}{X} \quad Y|E$$

alternative tableaux

$$Q \left\{ \begin{array}{l} DE = \square ED - \blacksquare EX + \blacksquare YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$

$$\frac{D}{X} \quad Y|E$$

tree-like tableaux

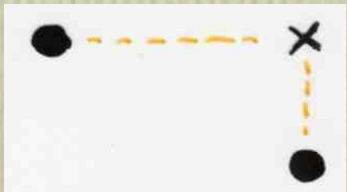
$$Q \left\{ \begin{array}{l} DE = \blacksquare ED - \blacksquare EX + \square YD \\ XE = \square EX \\ DY = \square YD \\ XY = \square YX \end{array} \right.$$

$$\frac{D}{X} \quad Y|E$$

permutation tableaux

$n!$

permutations



permutation
tableaux

tree-like
tableaux

Aval, Bousicault, Nadeau (2013)

alternative
tableaux

X.V. (2008)



J-diagrams

decorated
permutations

Steingrímsson, Williams (2007)

Postnikov (2006)

totally non-negative part
of the type A Grassmannian

