Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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mirror website

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Chapter 3
Tableaux for the PASEP quadratic algebra

Ch3b

Ch3b (2nd part)

IMSc, Chennai
February 15, 2018

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A variation of
the “exchange-fusion” algorithm:

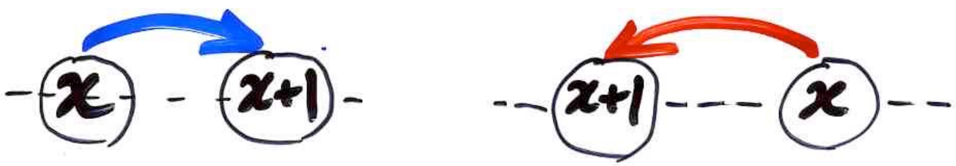
The “exchange-delete” algorithm

Def- Permutation $\sigma = \sigma(1) \cdots \sigma(n)$

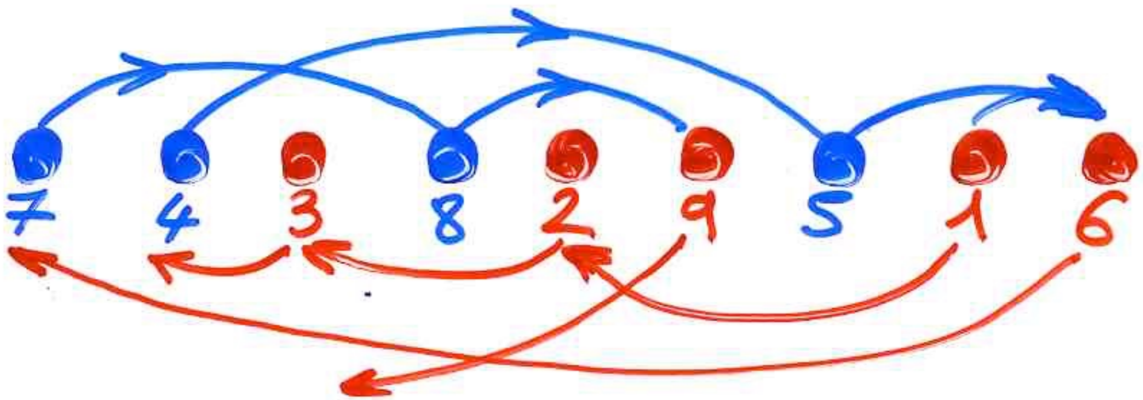
$x = \sigma(i)$, $1 \leq x < n$

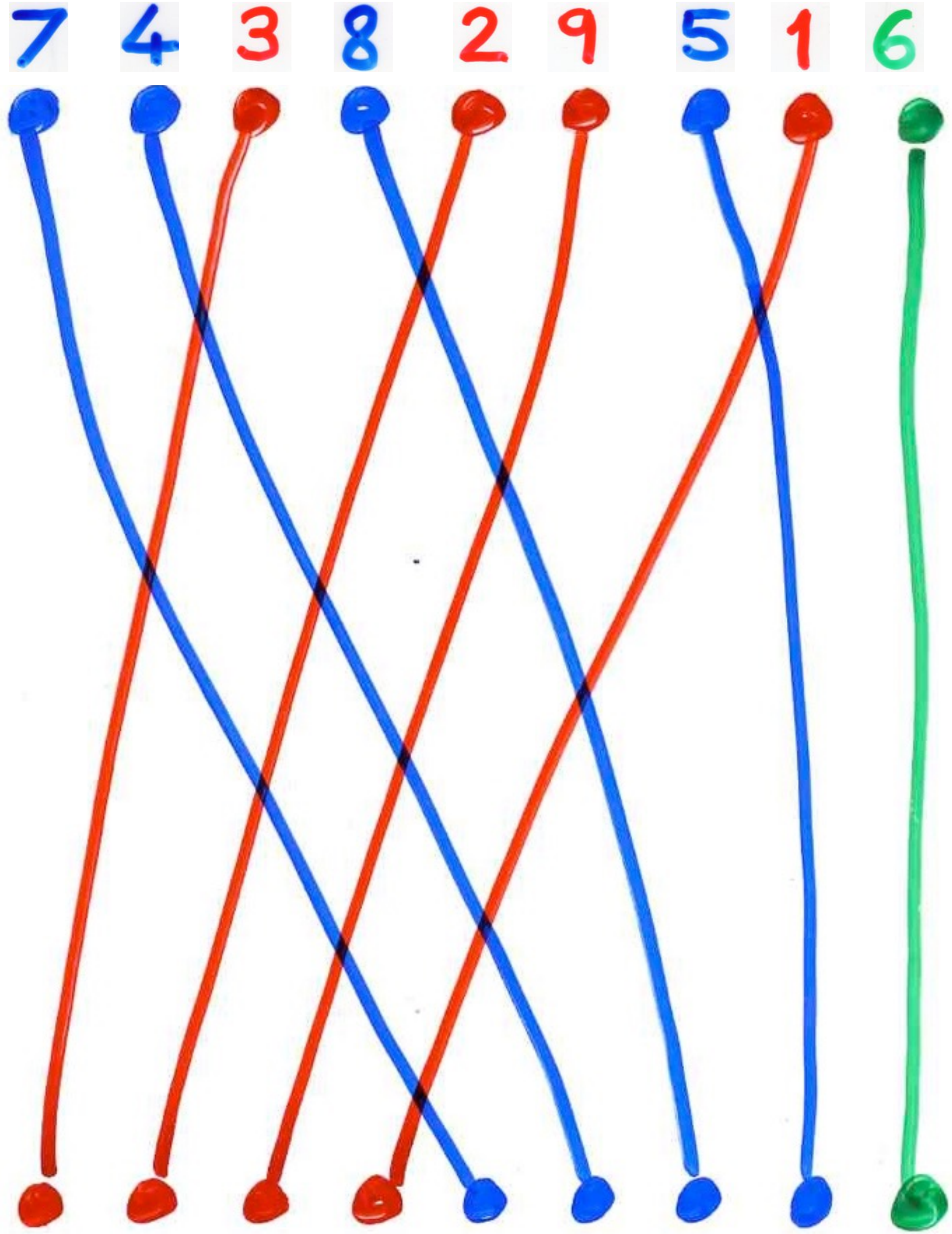
(valeur) x $\begin{cases} \text{avance} \\ \text{recul} \end{cases}$ $x+1 = \sigma(j)$, $\begin{cases} i < j \\ j < i \end{cases}$

- convention $x = n$ est un recul



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

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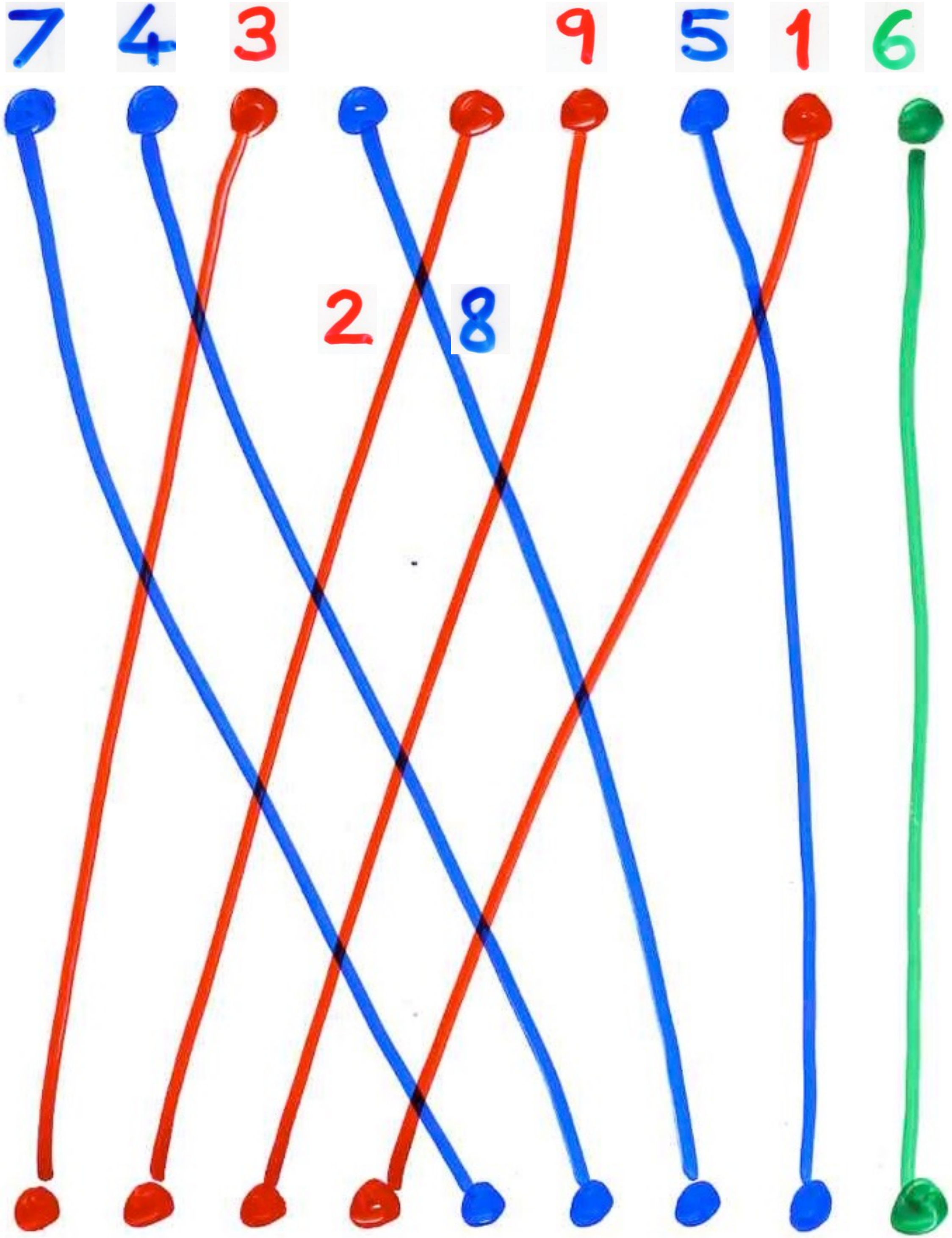
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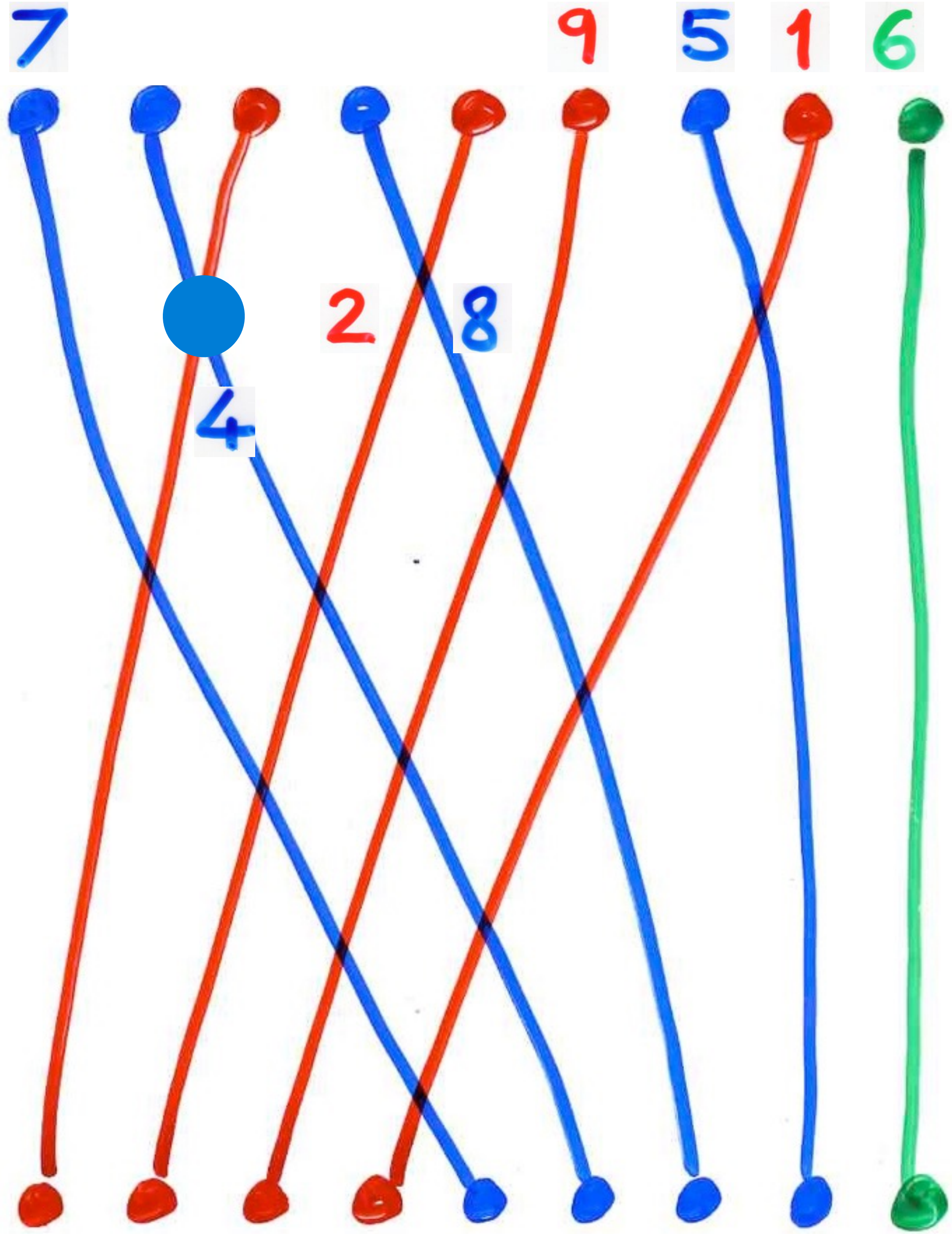
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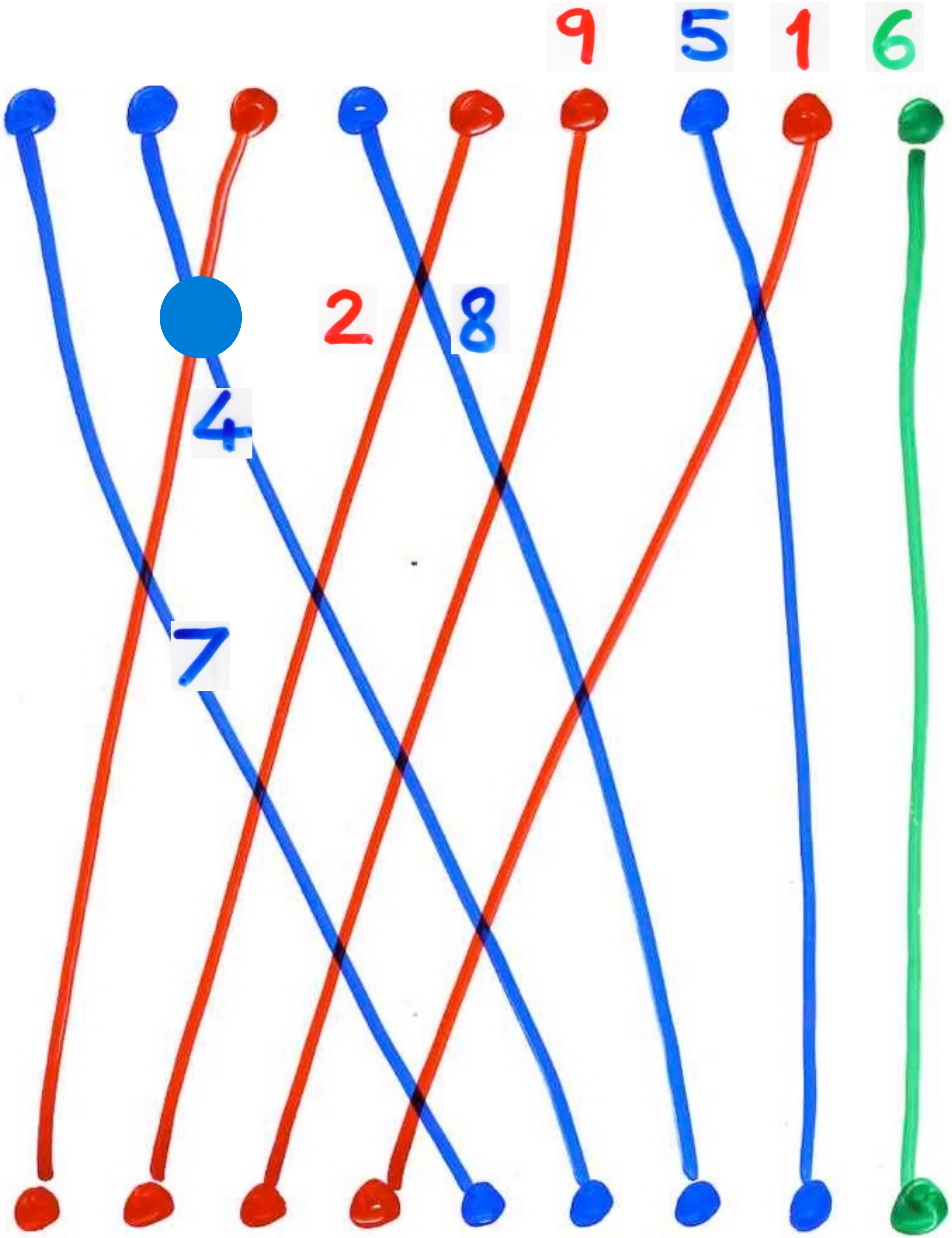

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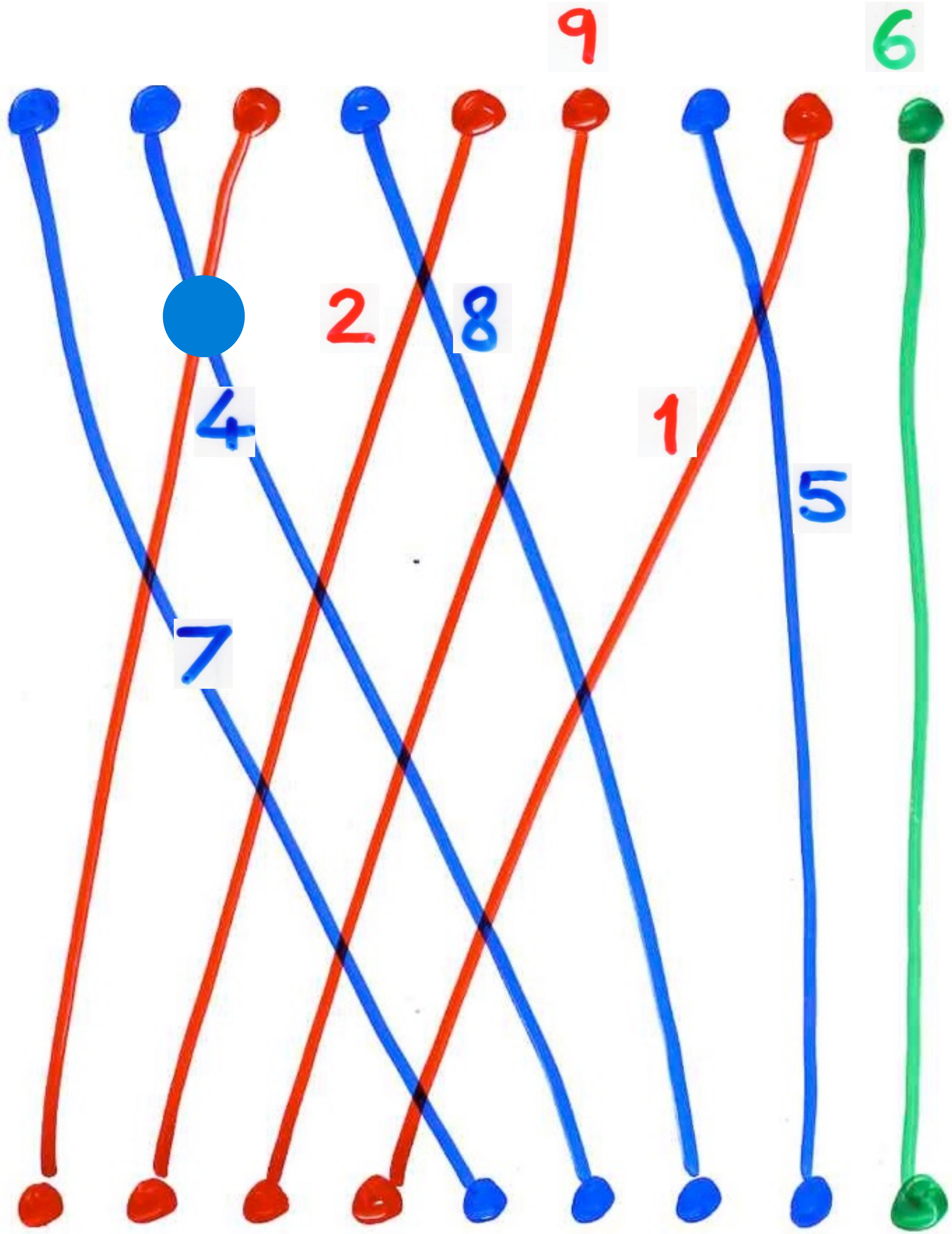
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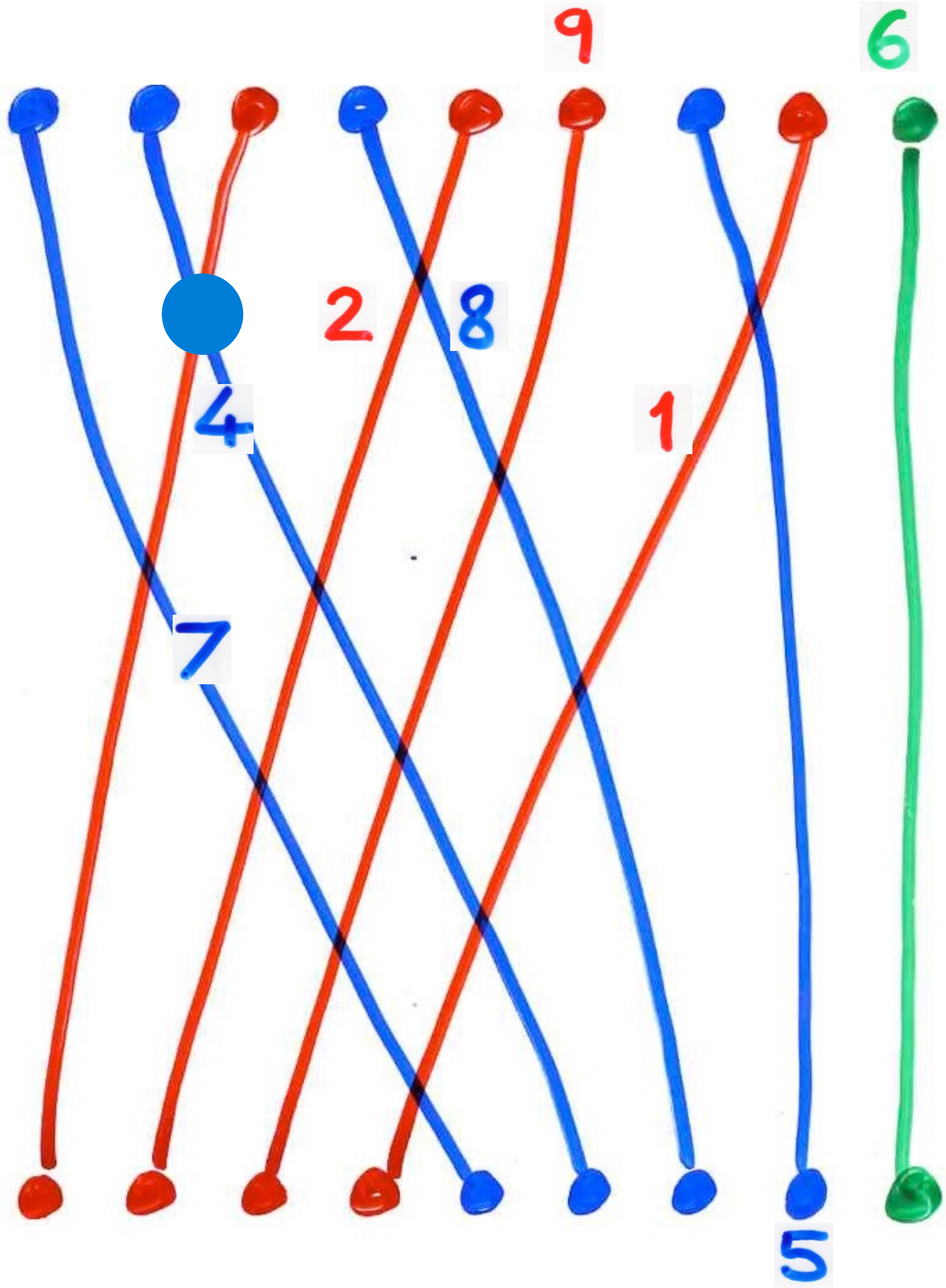

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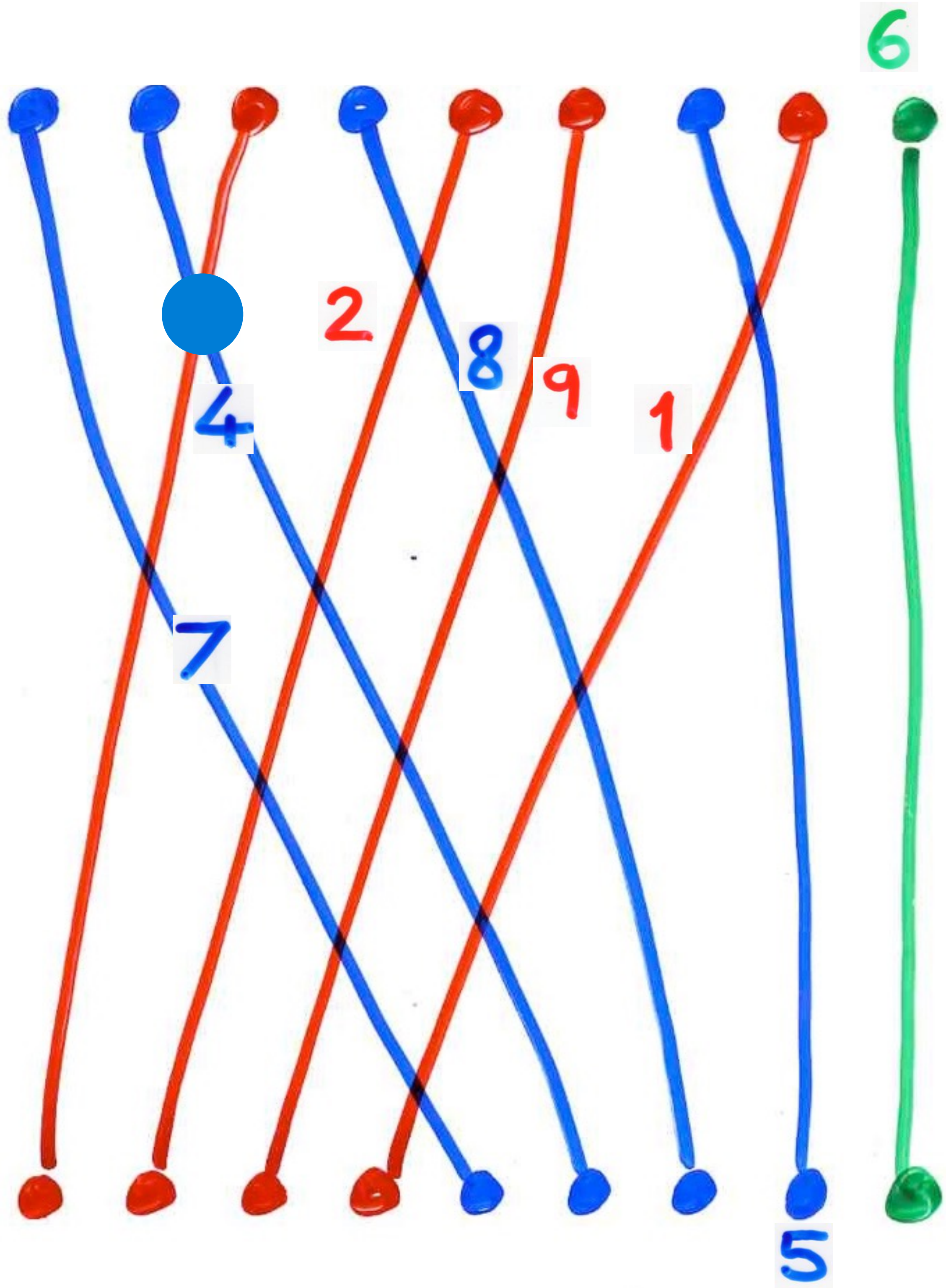
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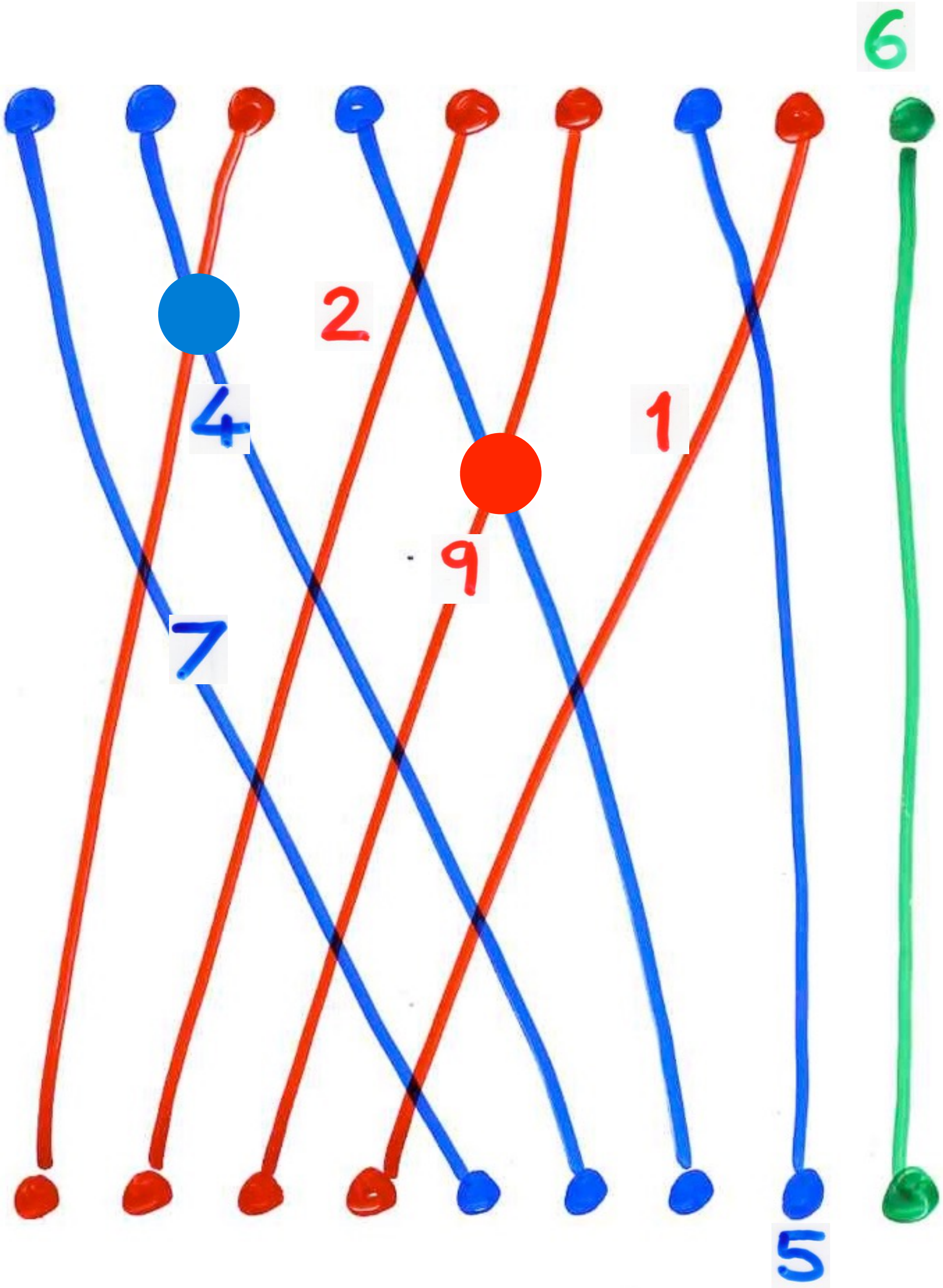
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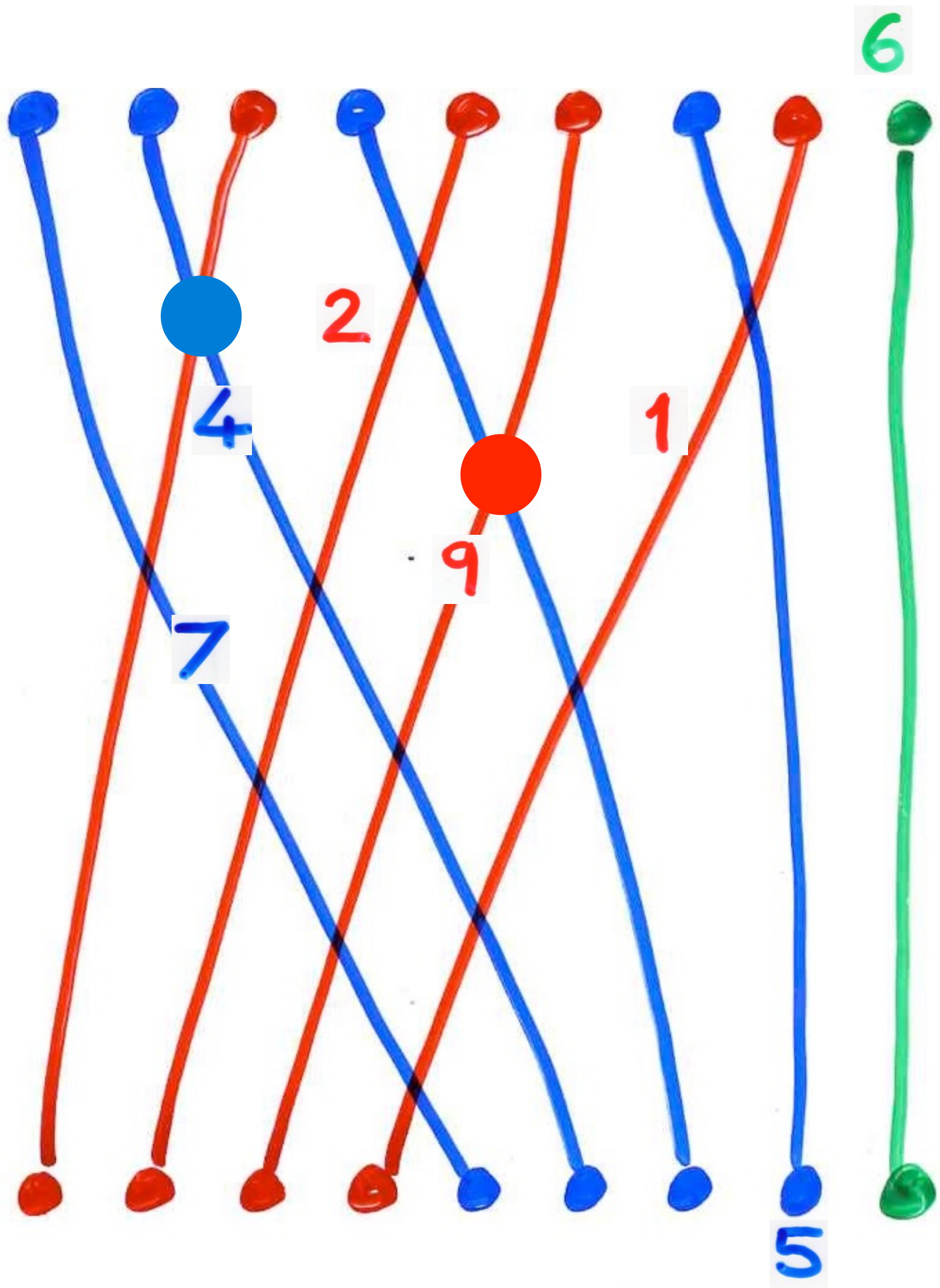
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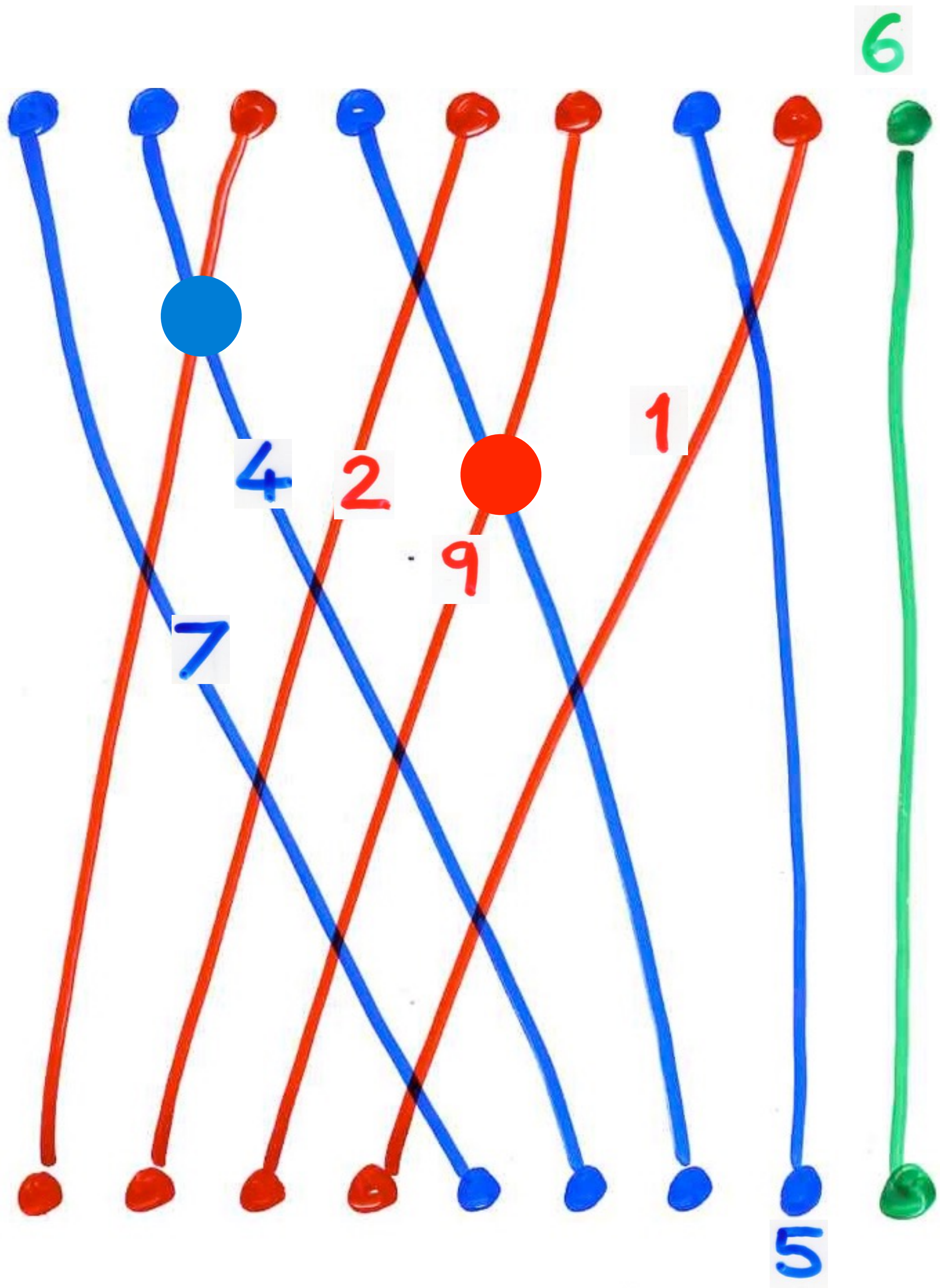
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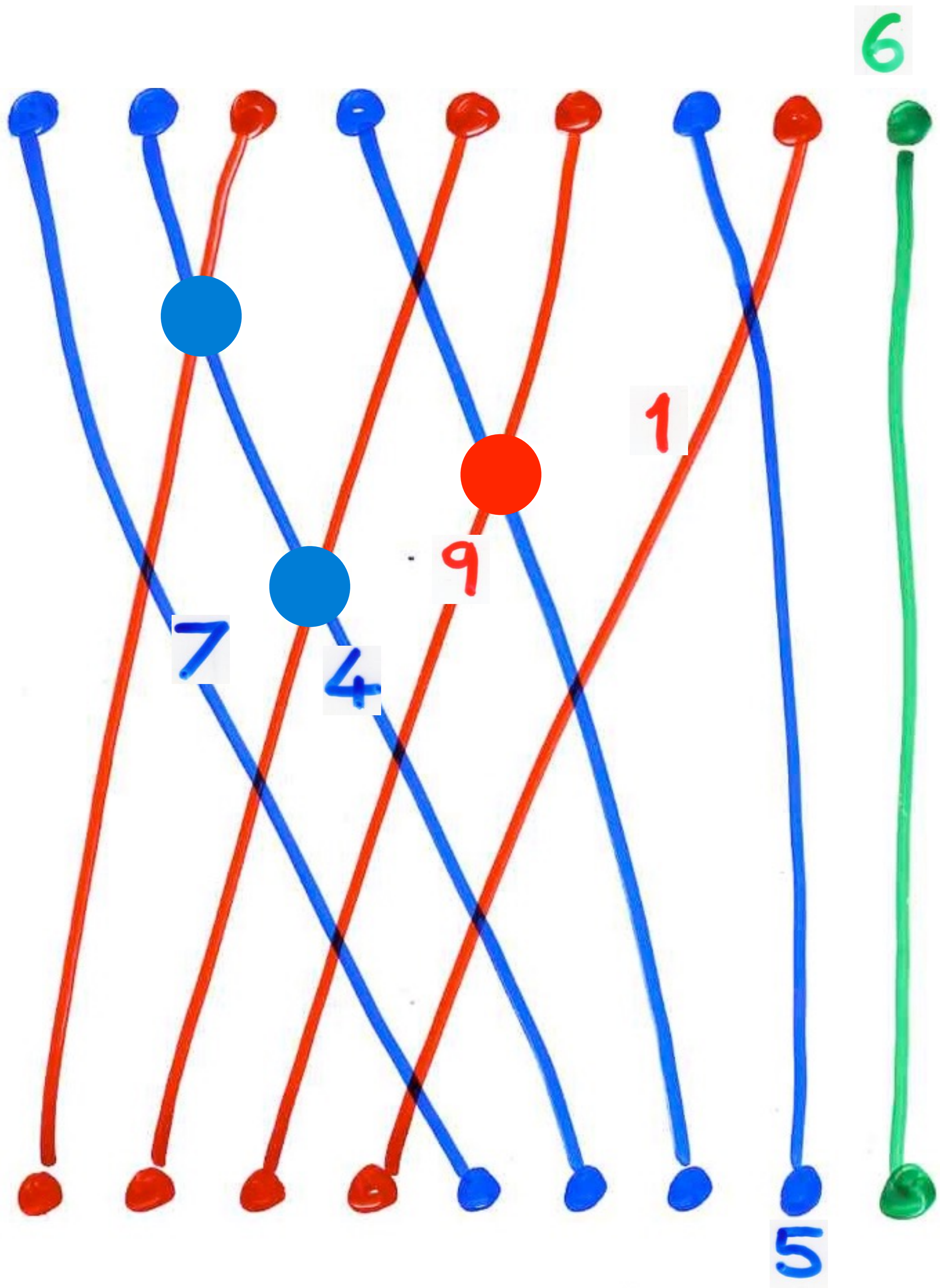
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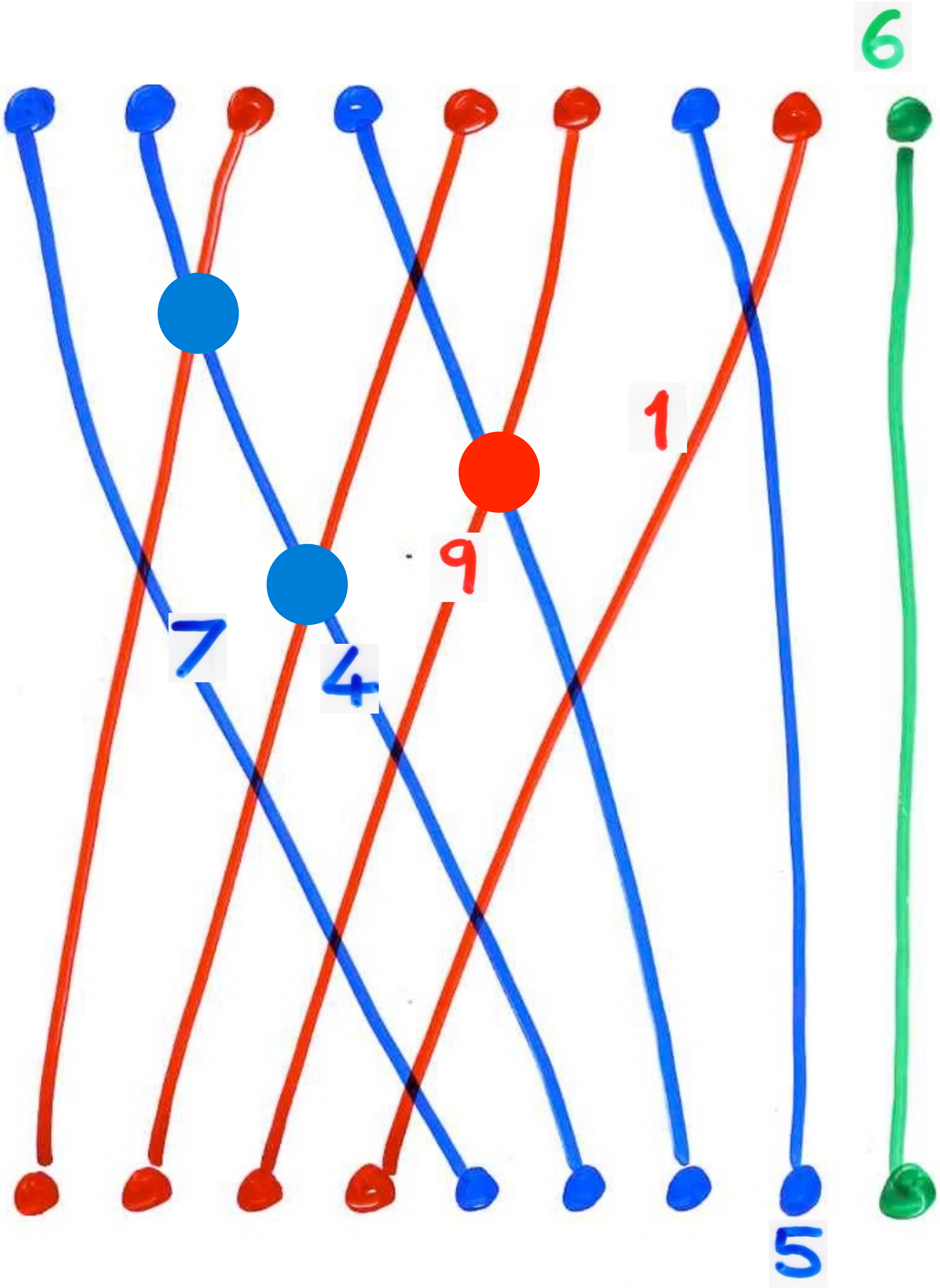
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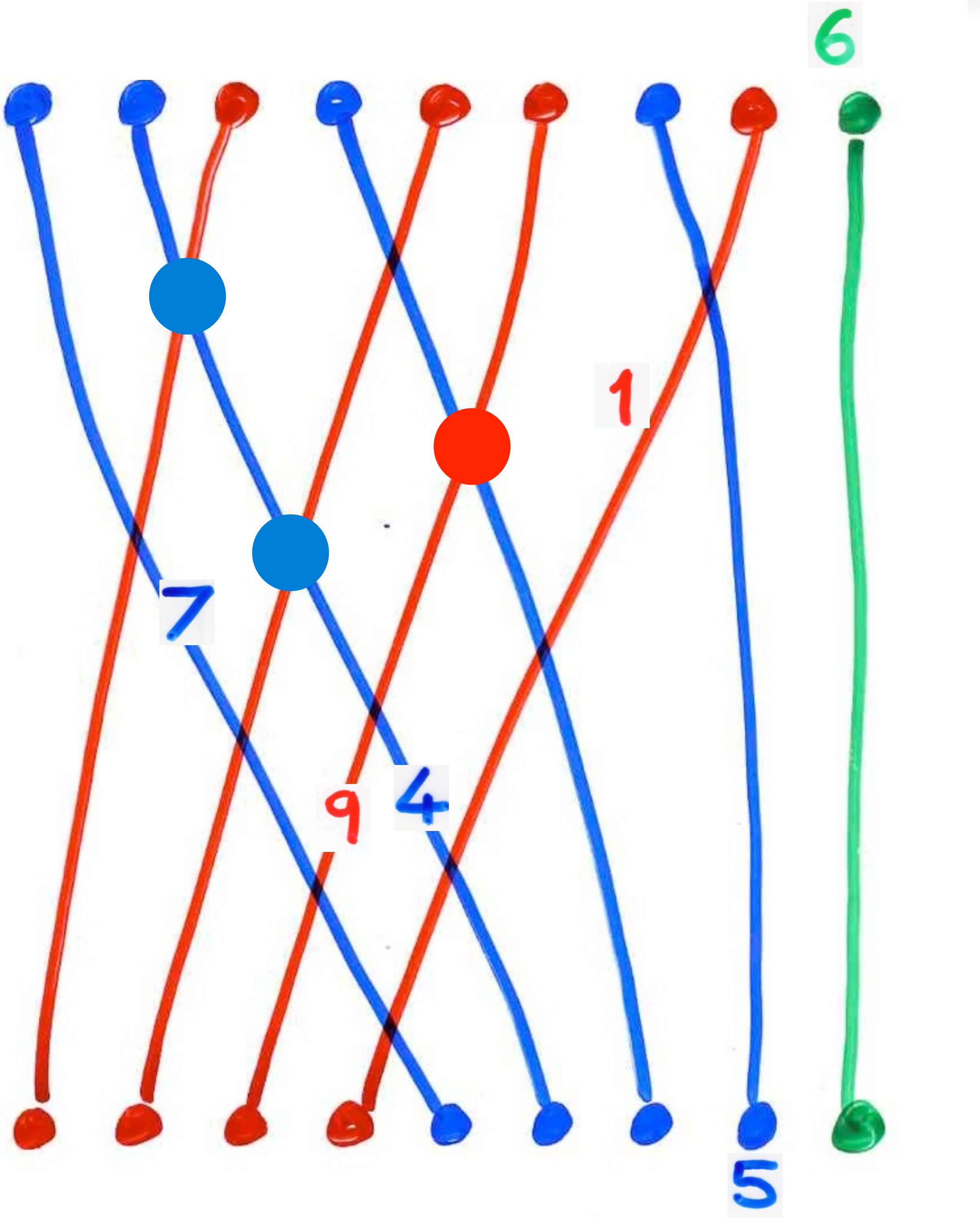
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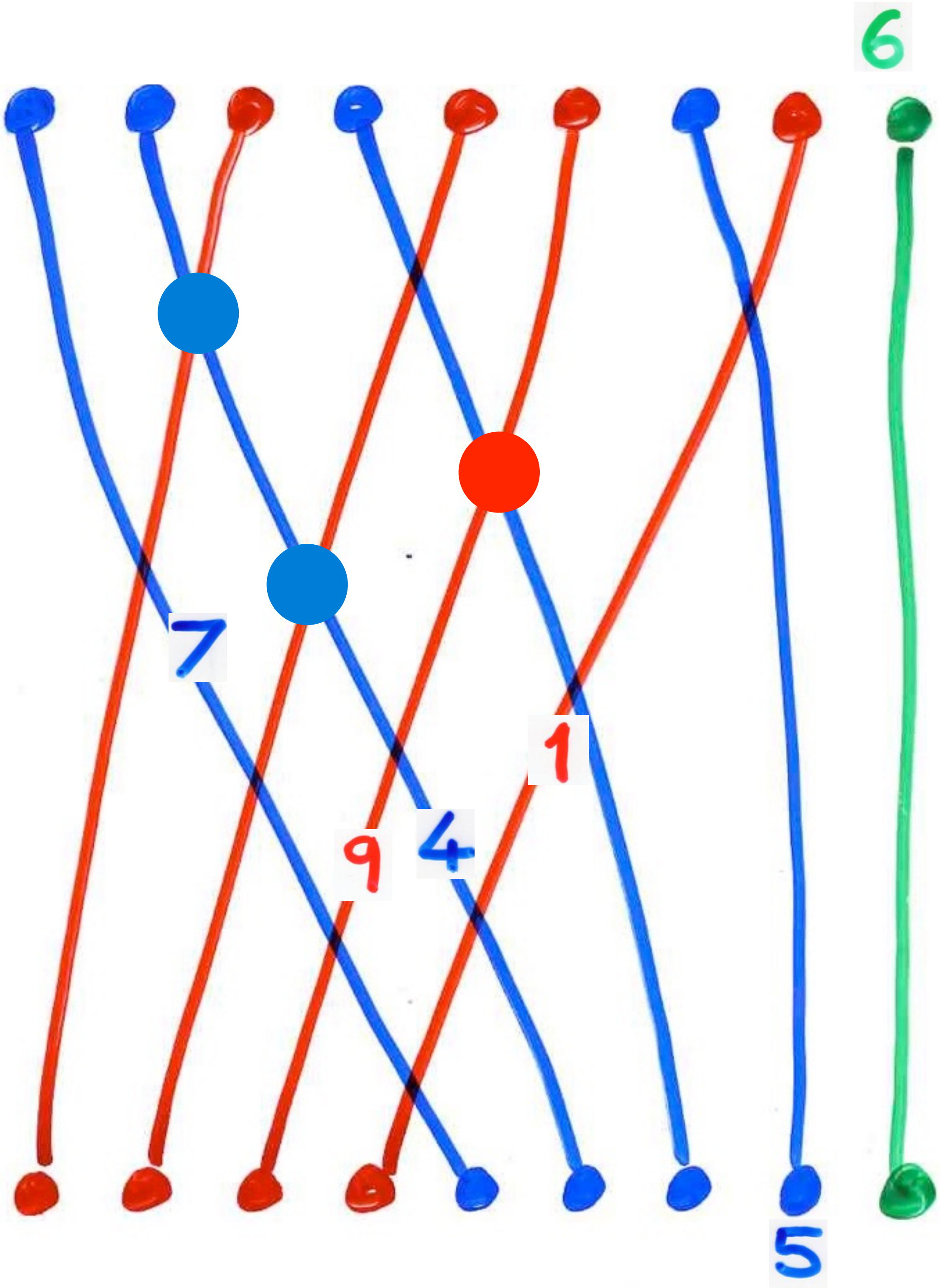
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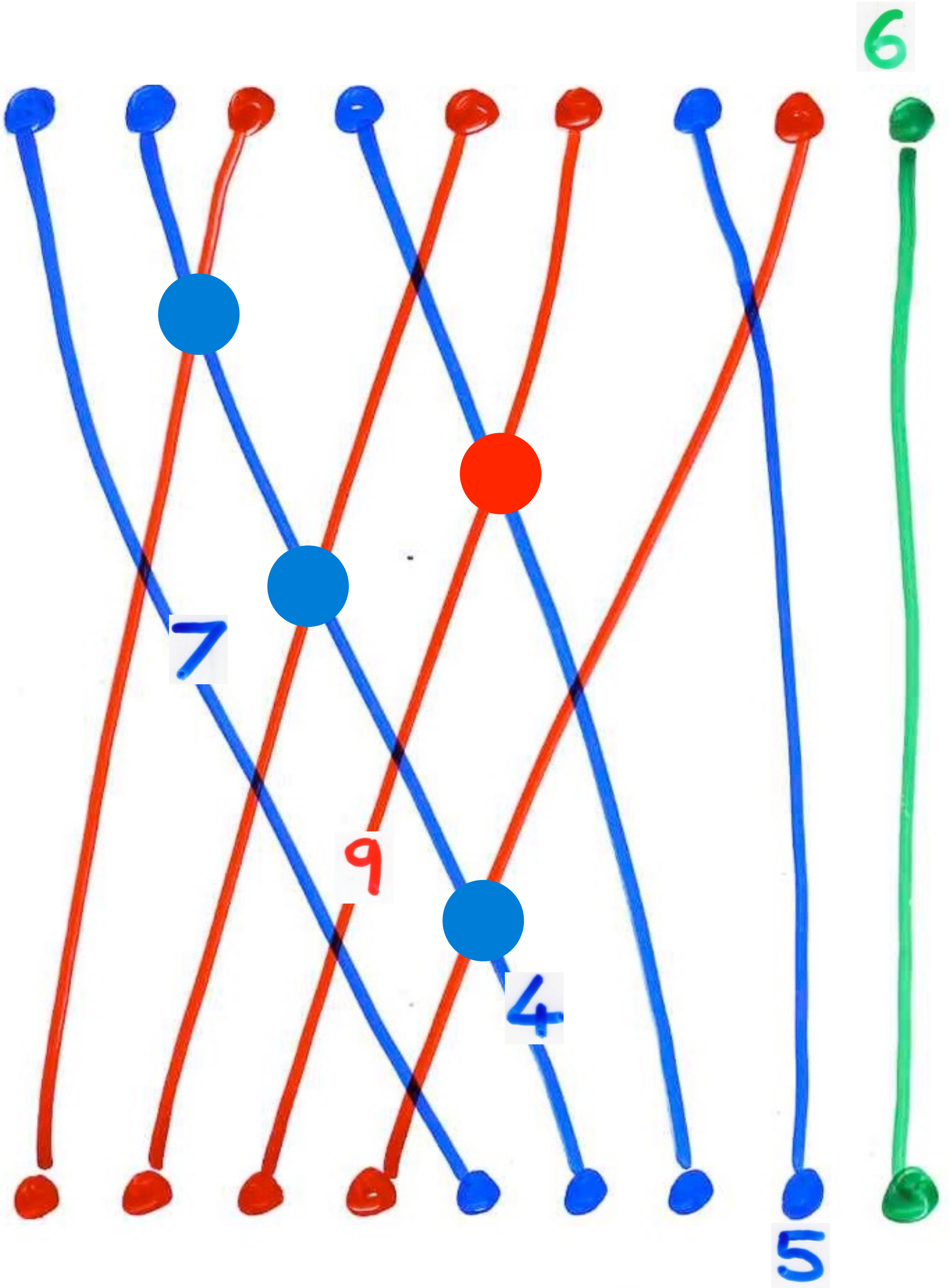
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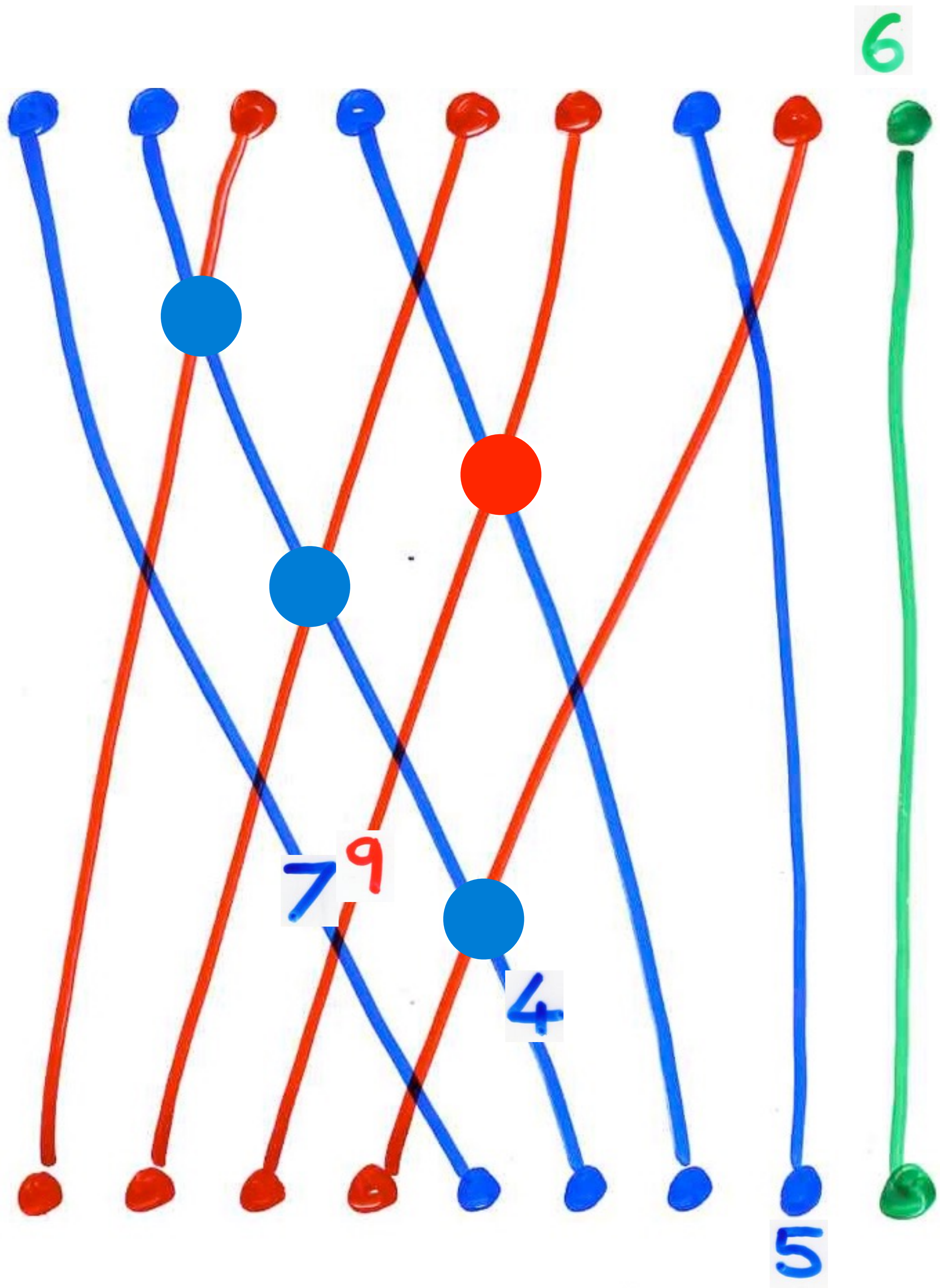
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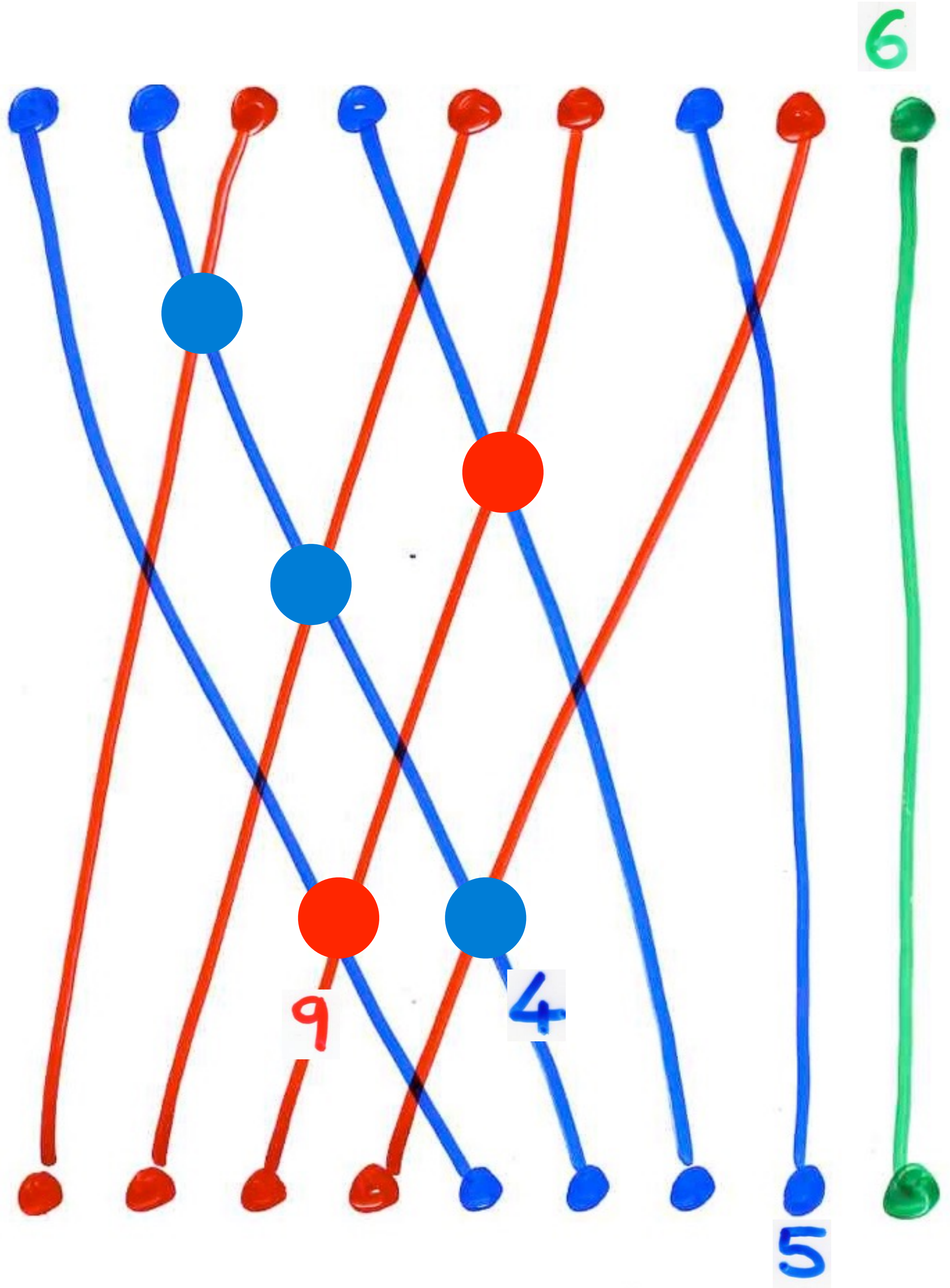
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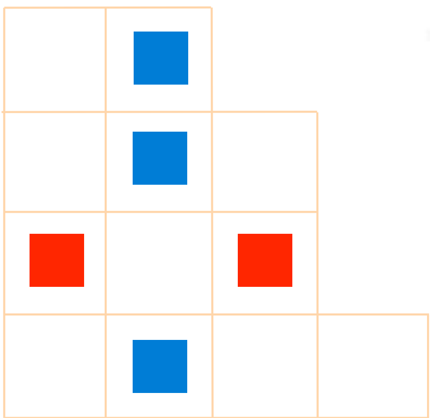
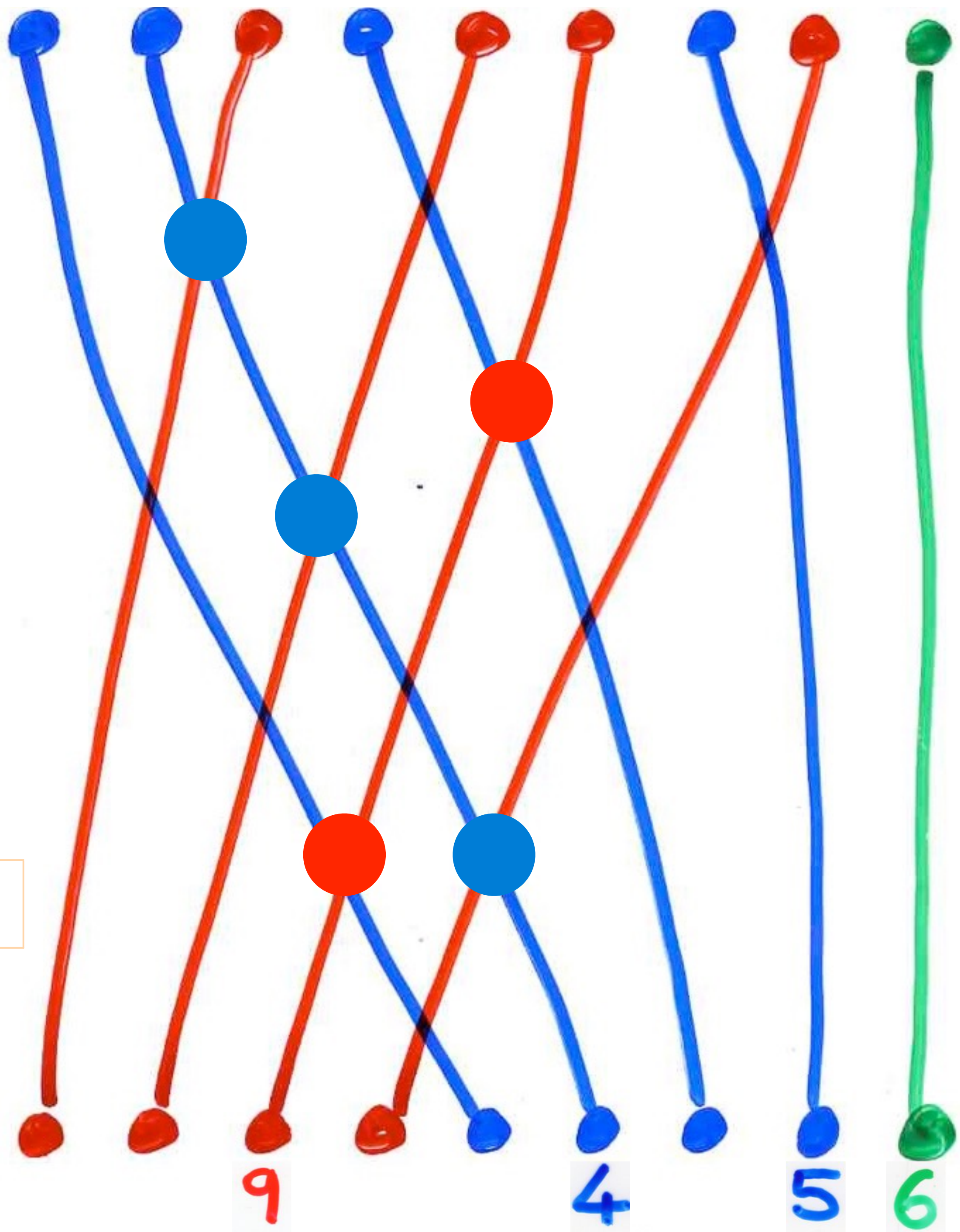
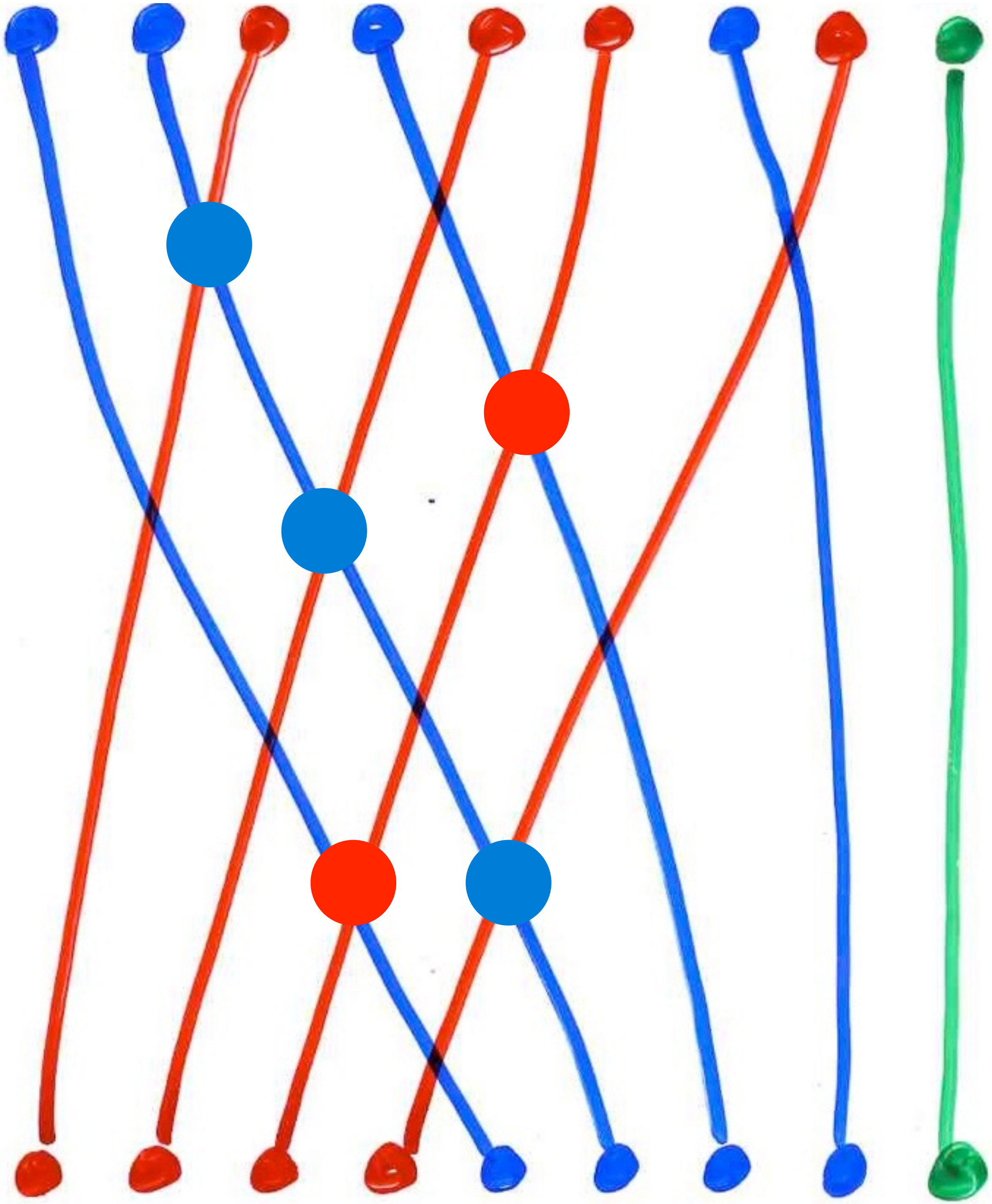
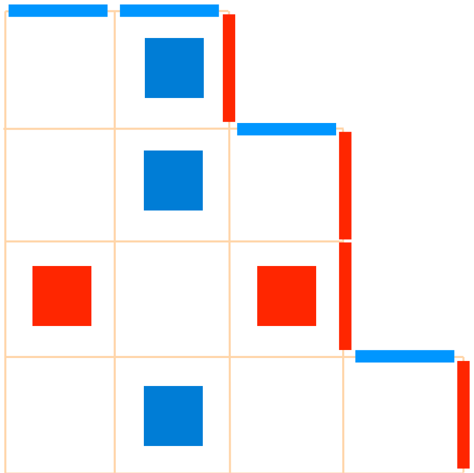
“exchange-deletion” algorithm

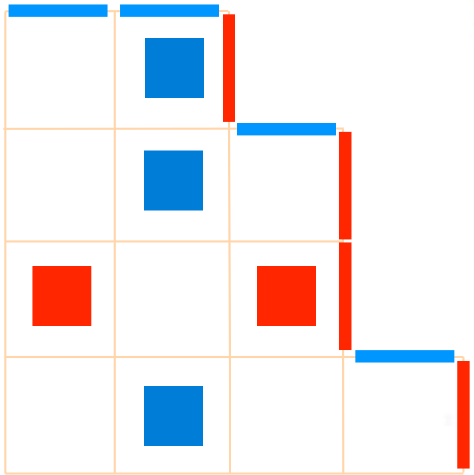


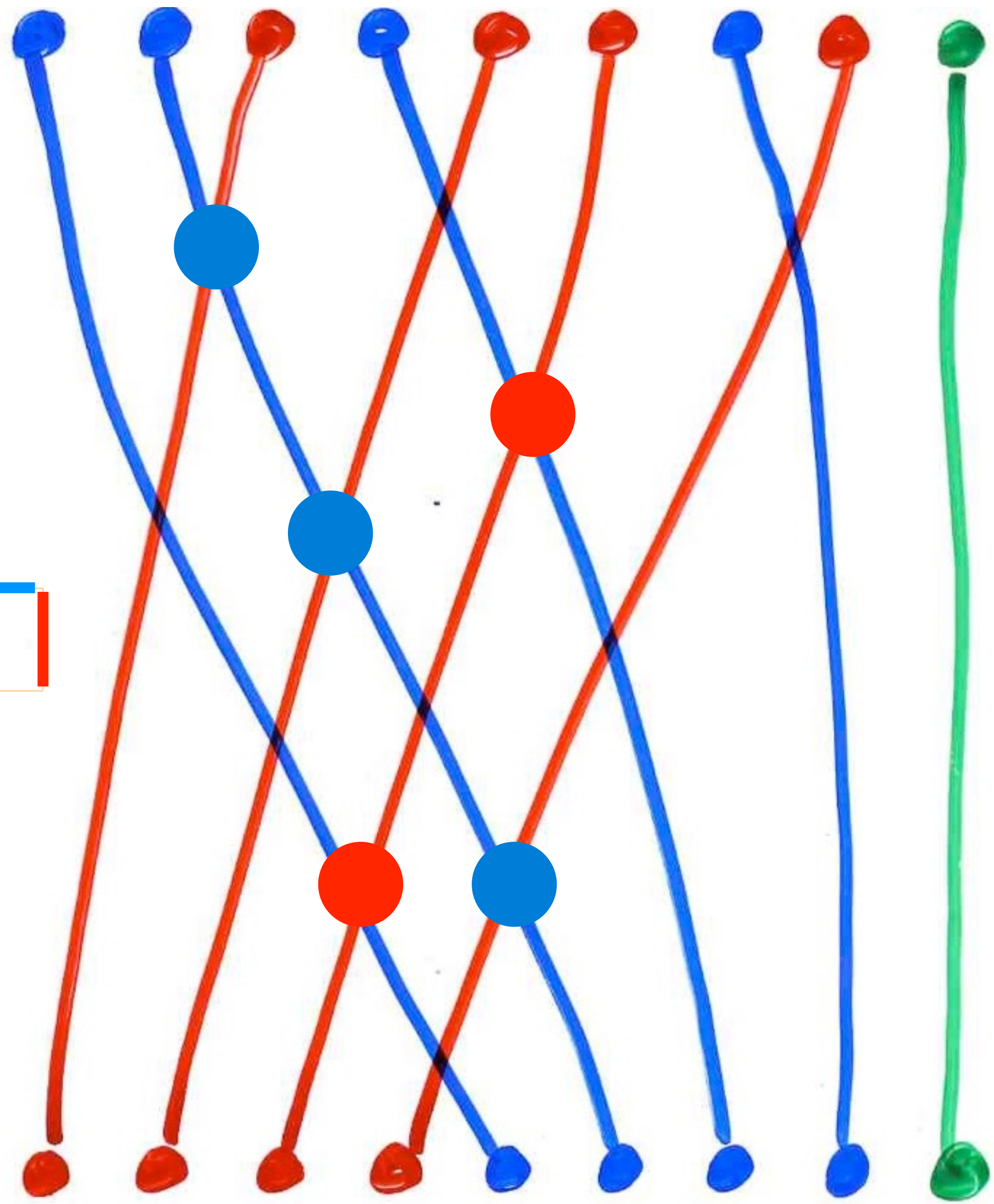


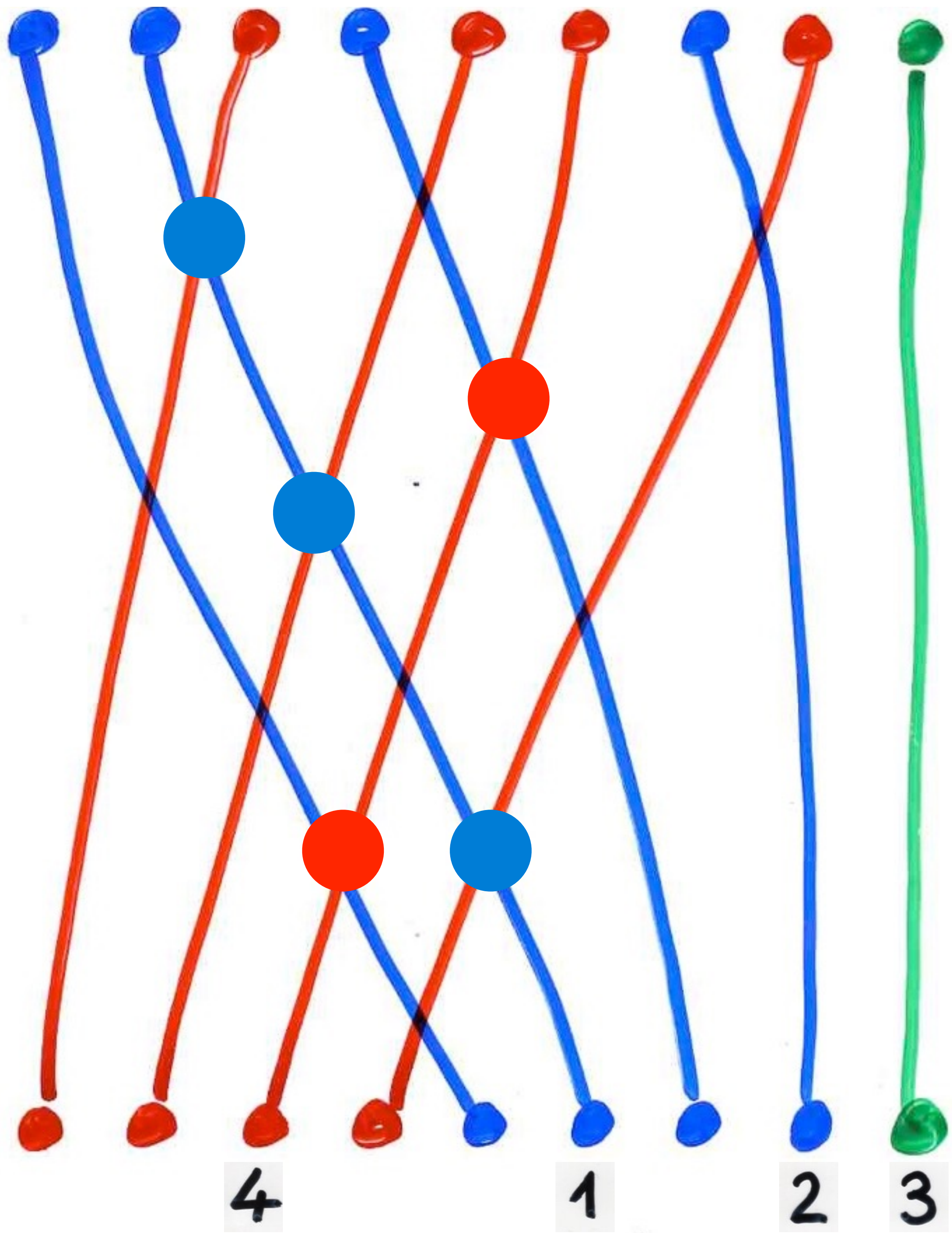


The inverse "exchange-delete" algorithm





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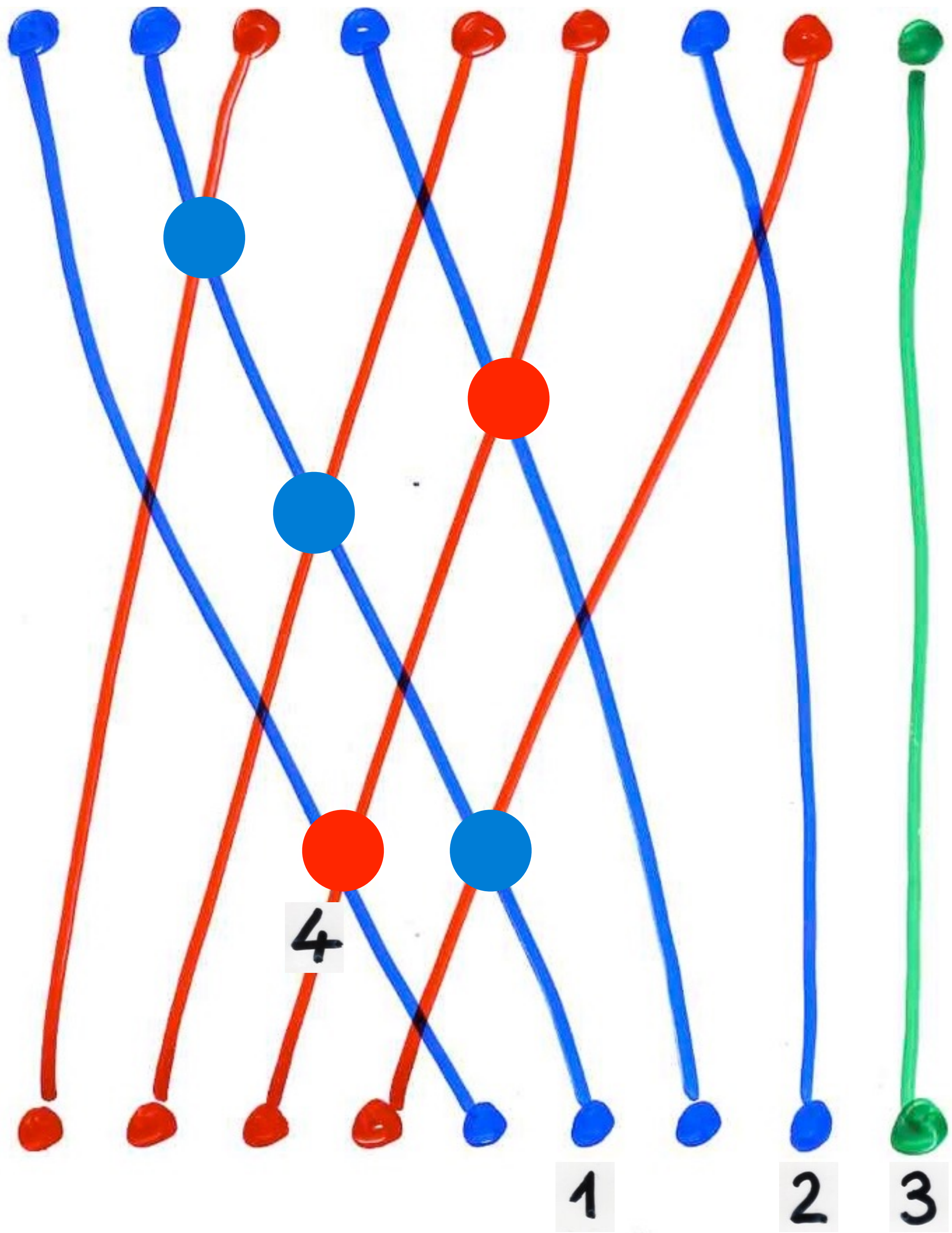
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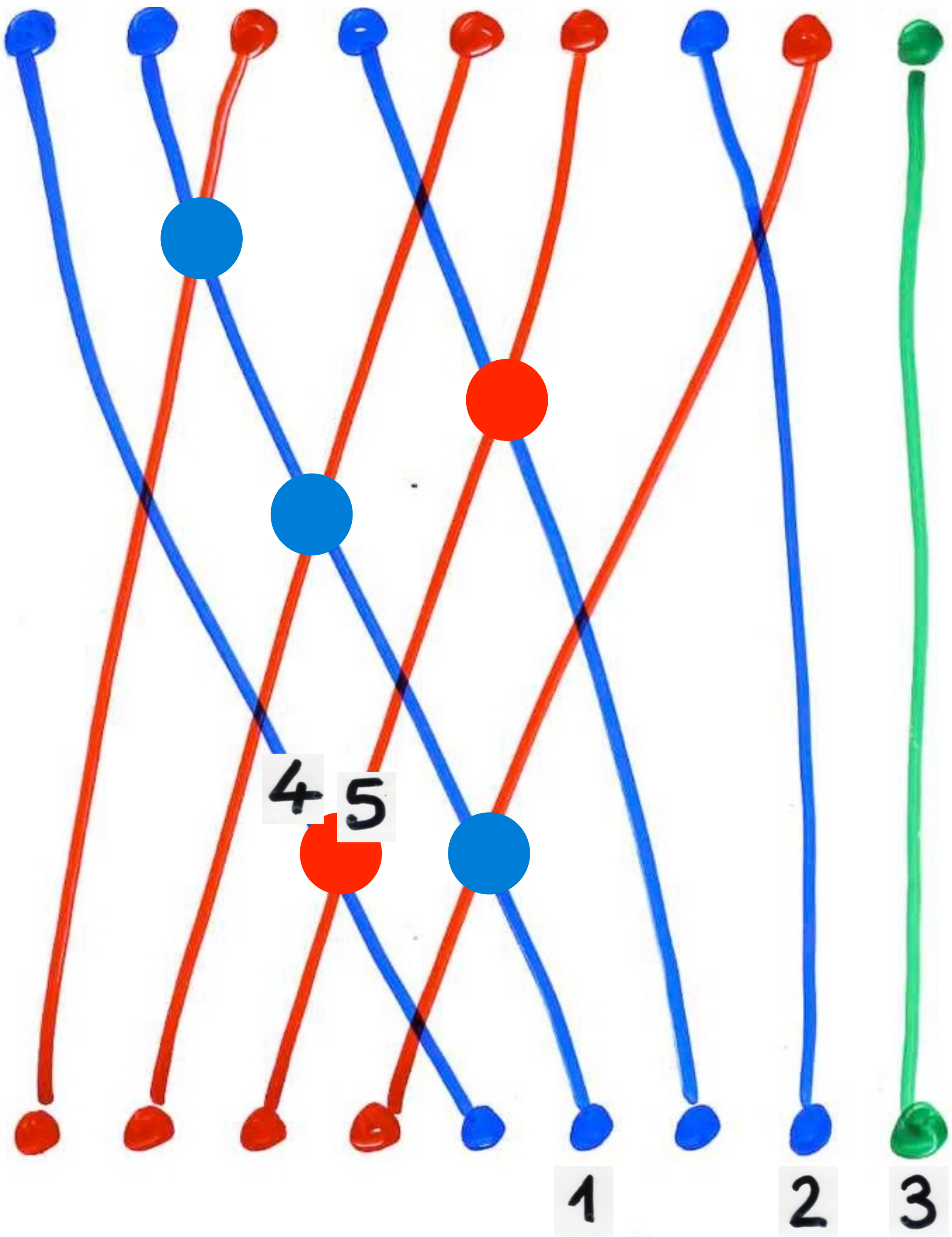
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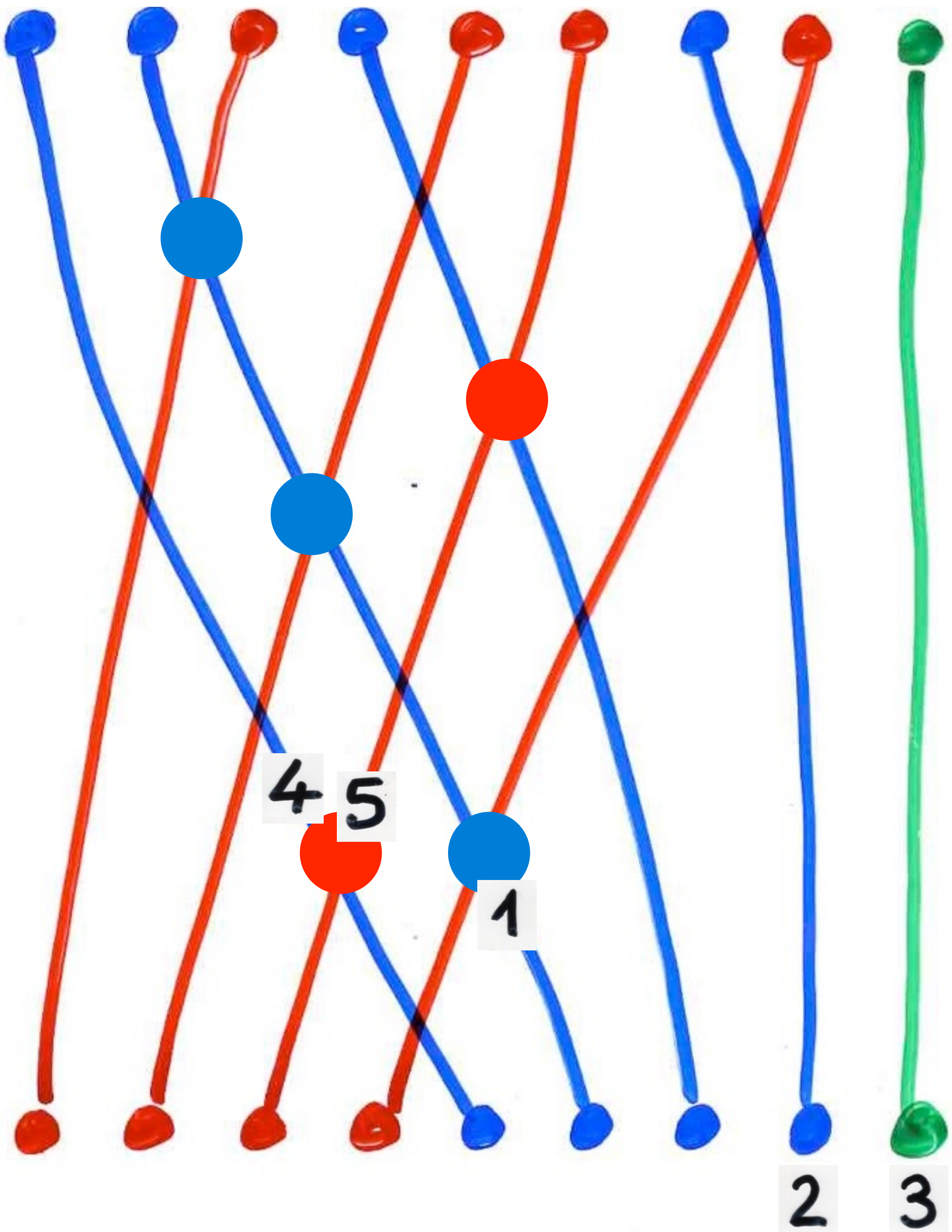

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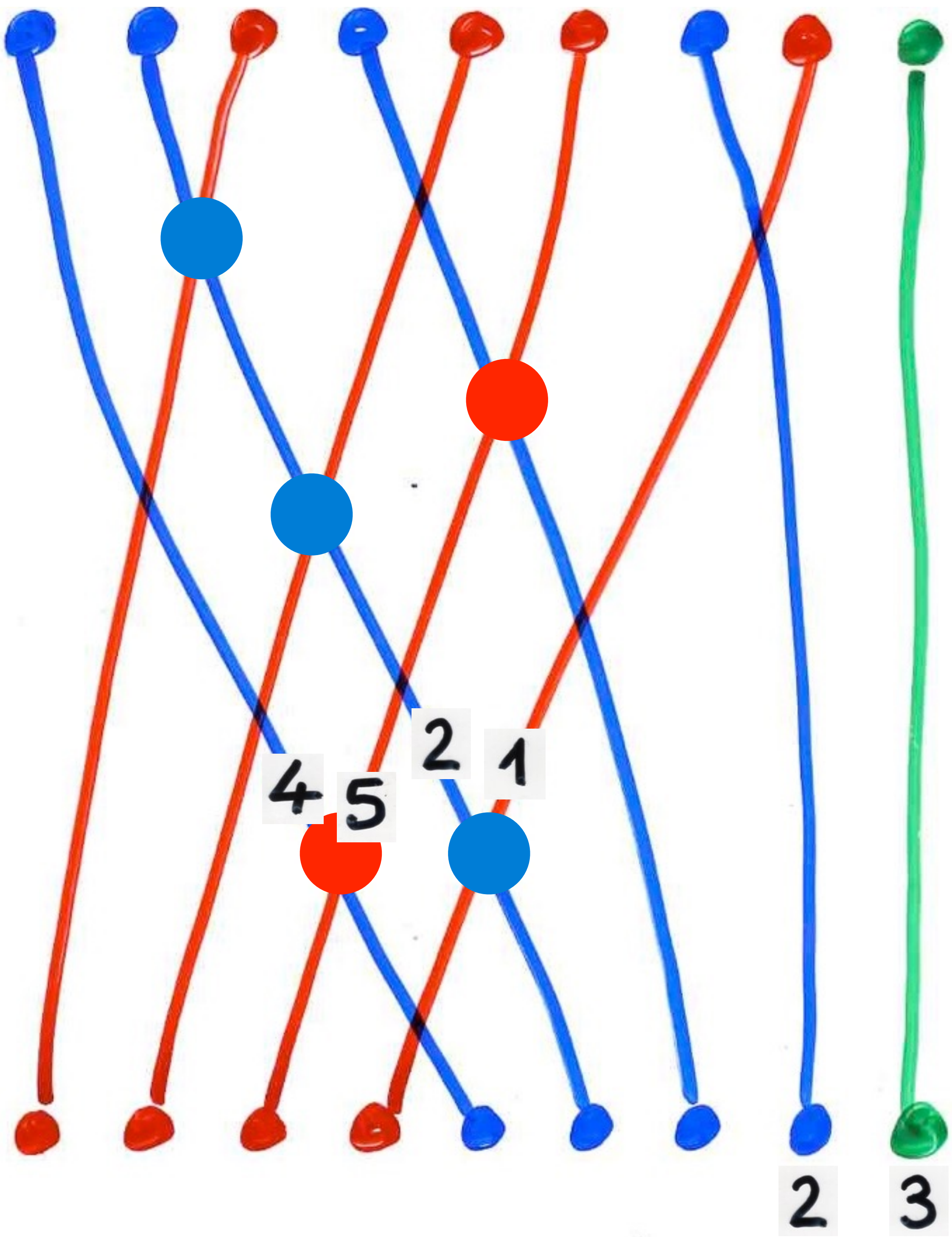
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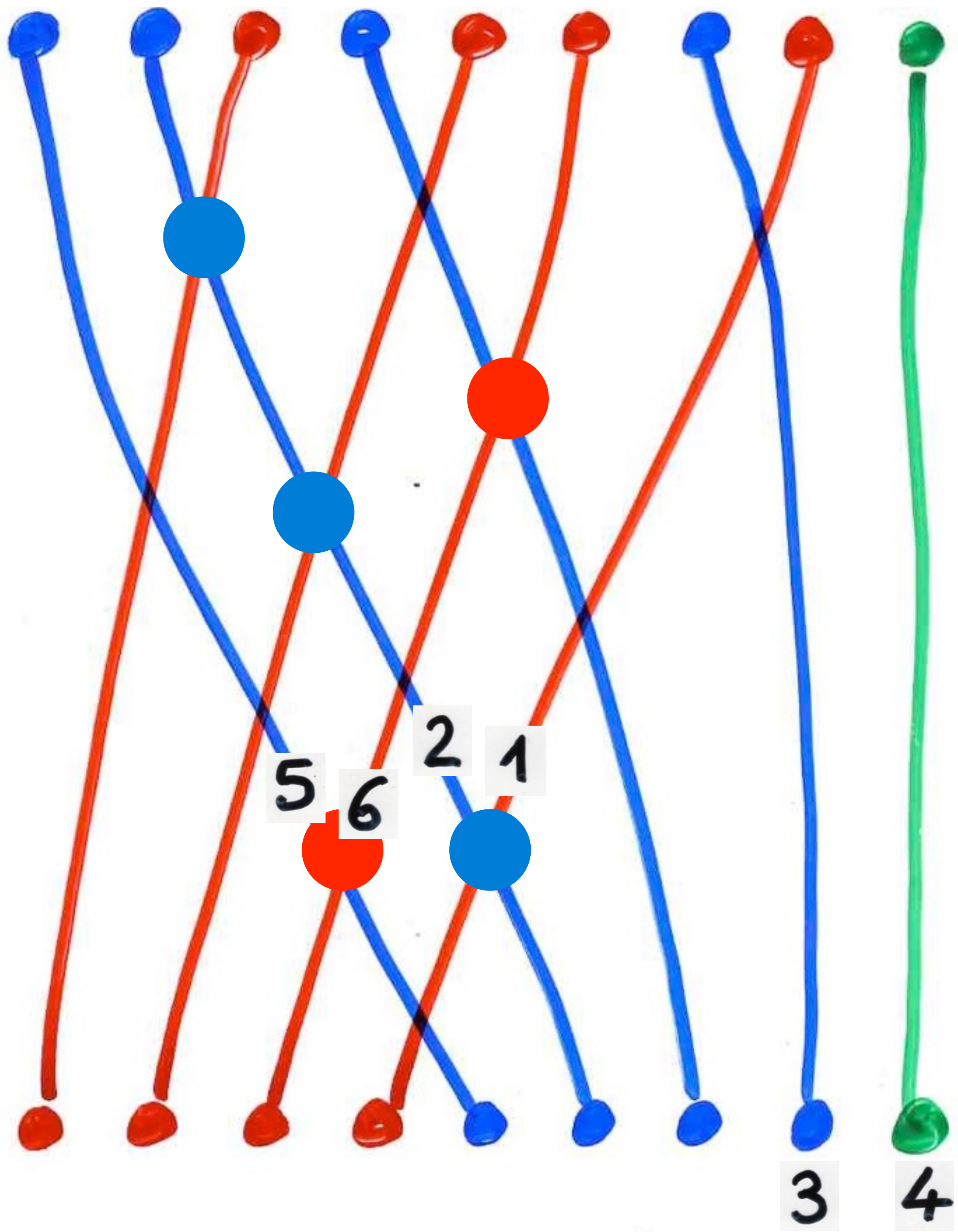

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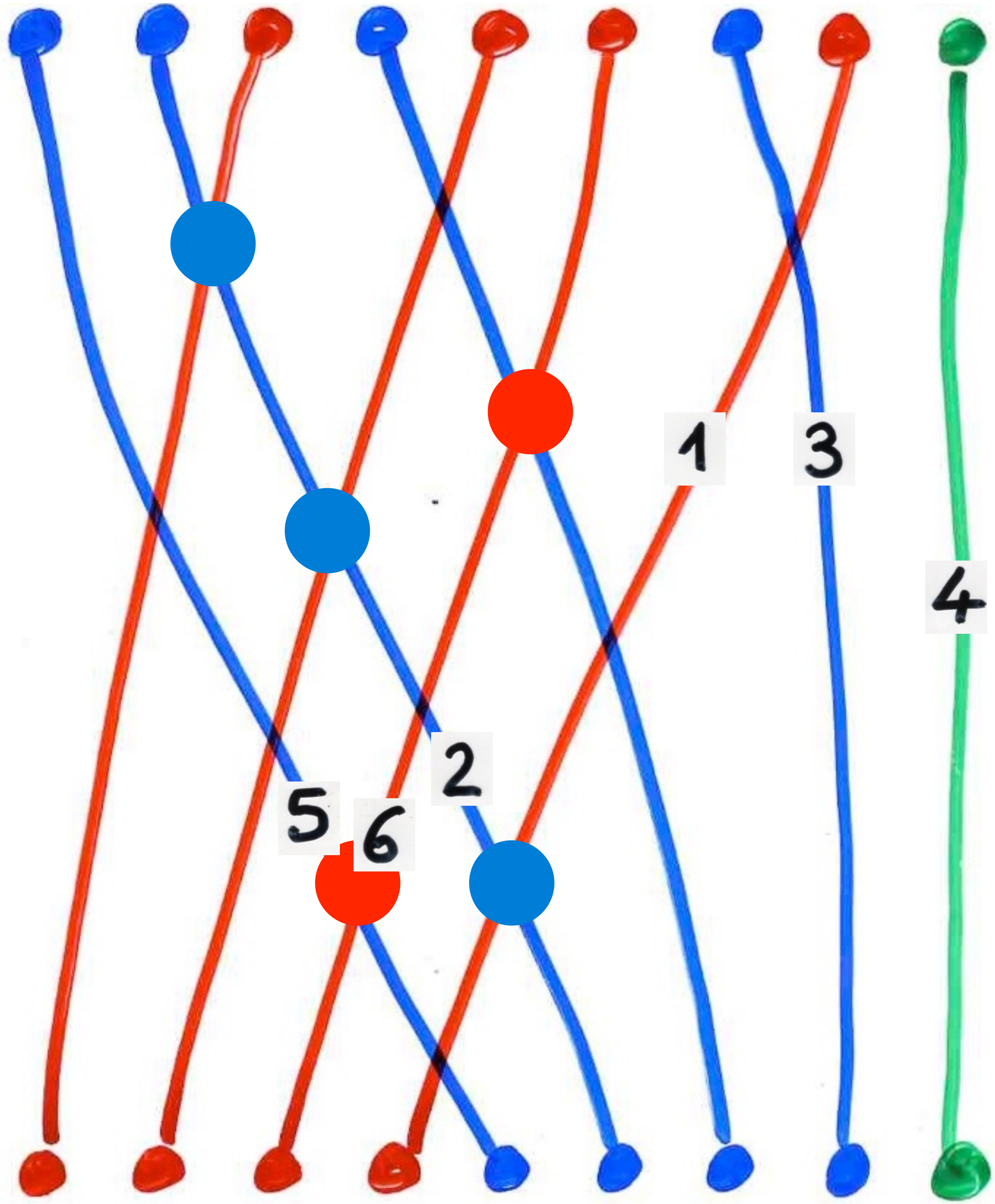

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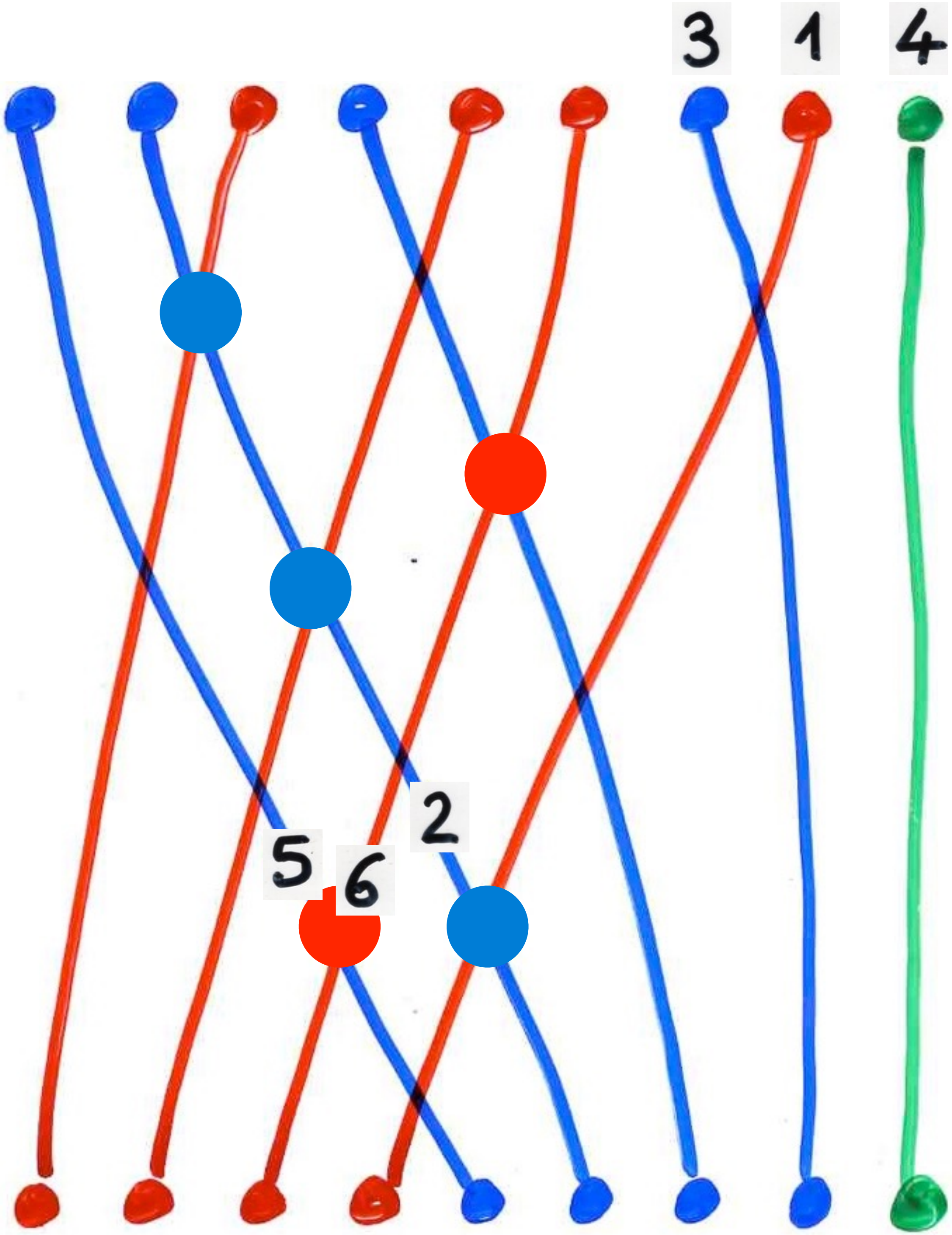
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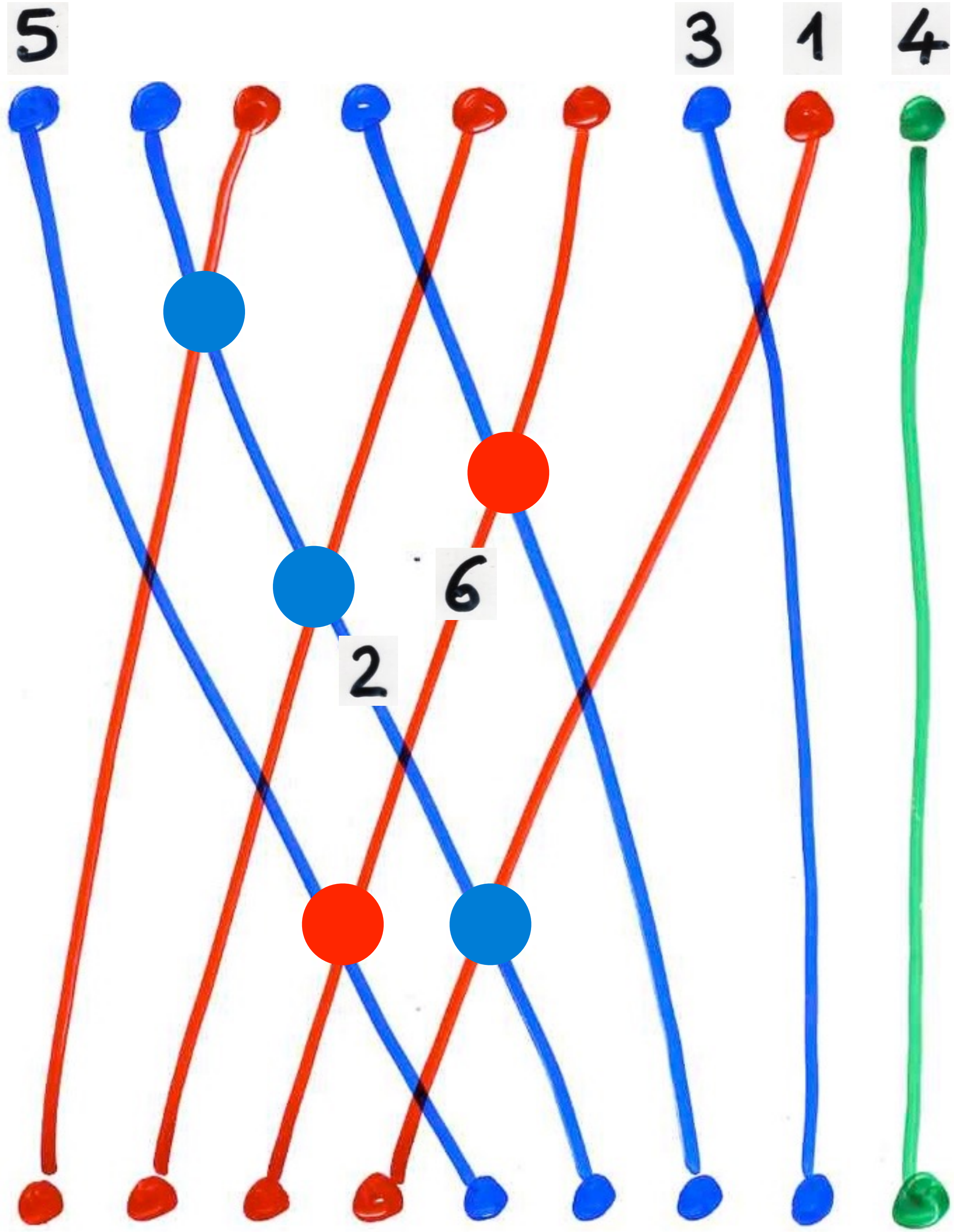
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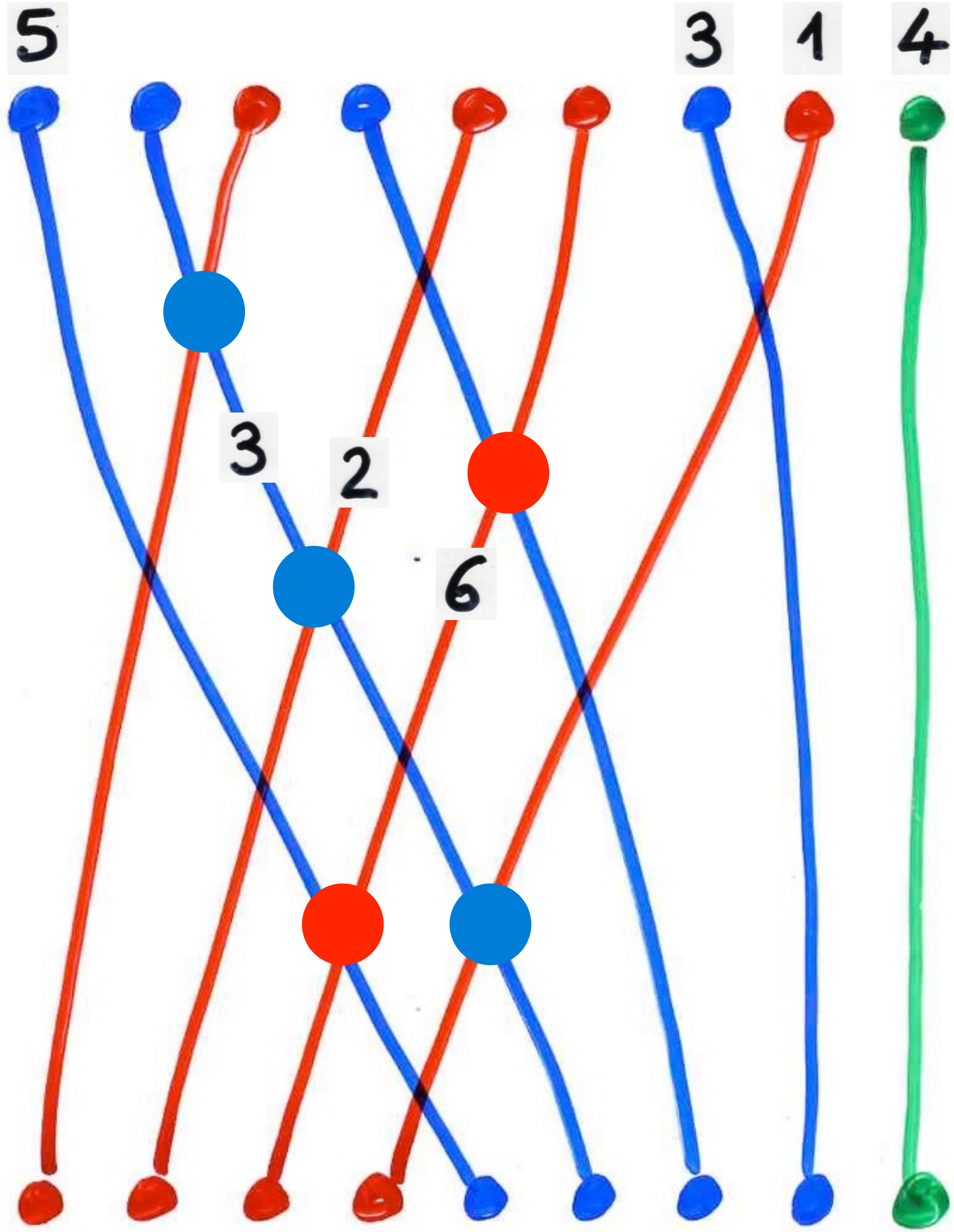
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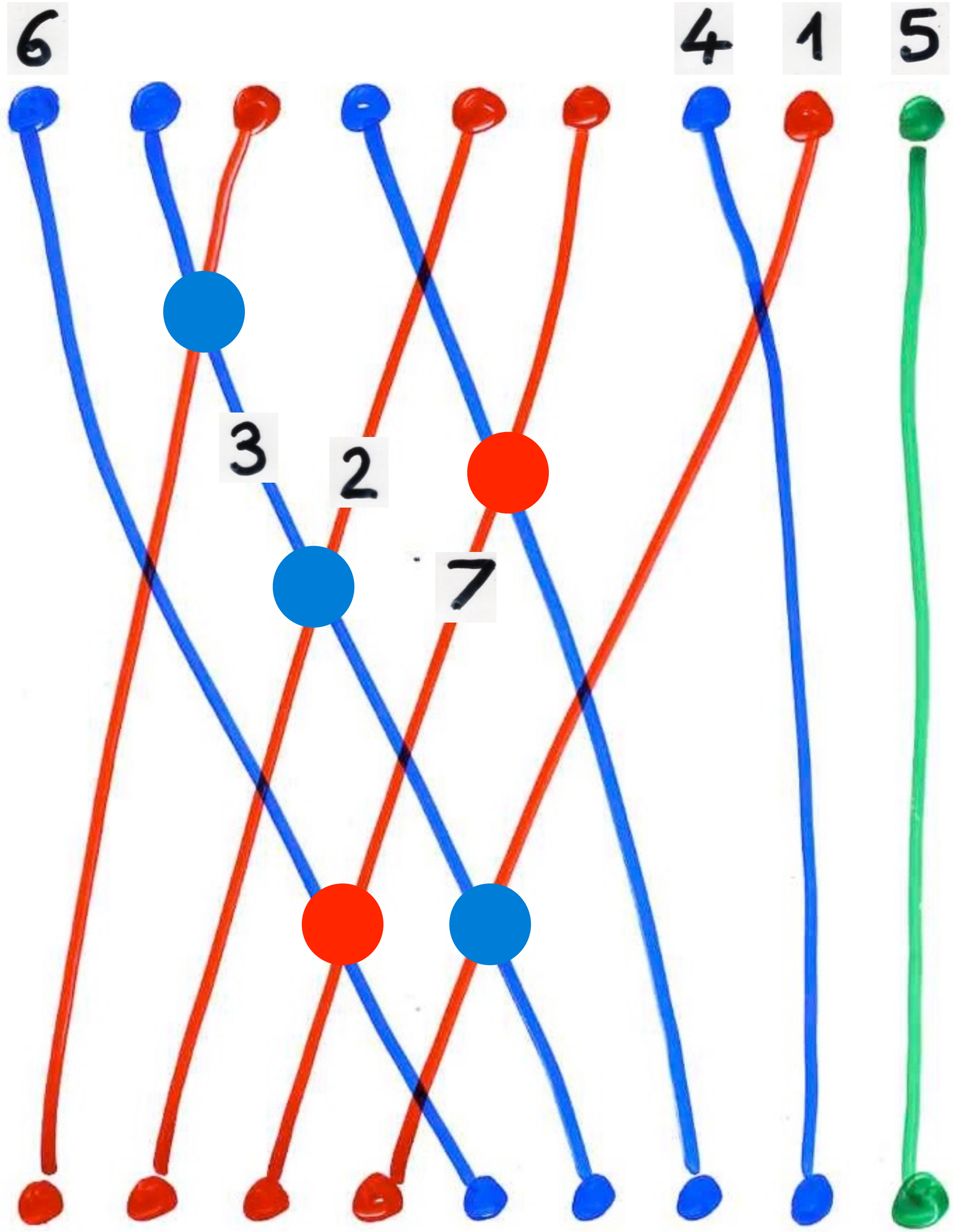

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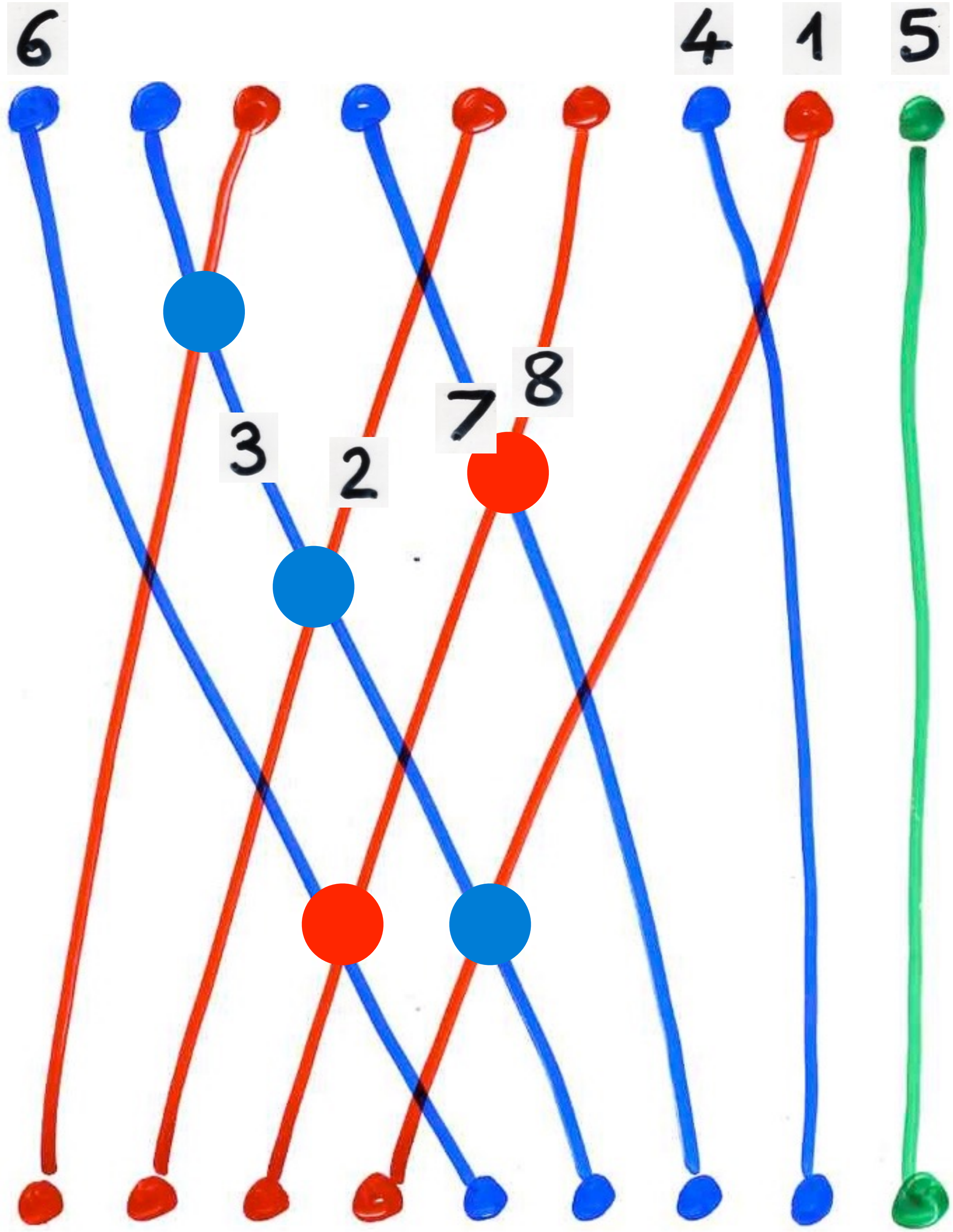
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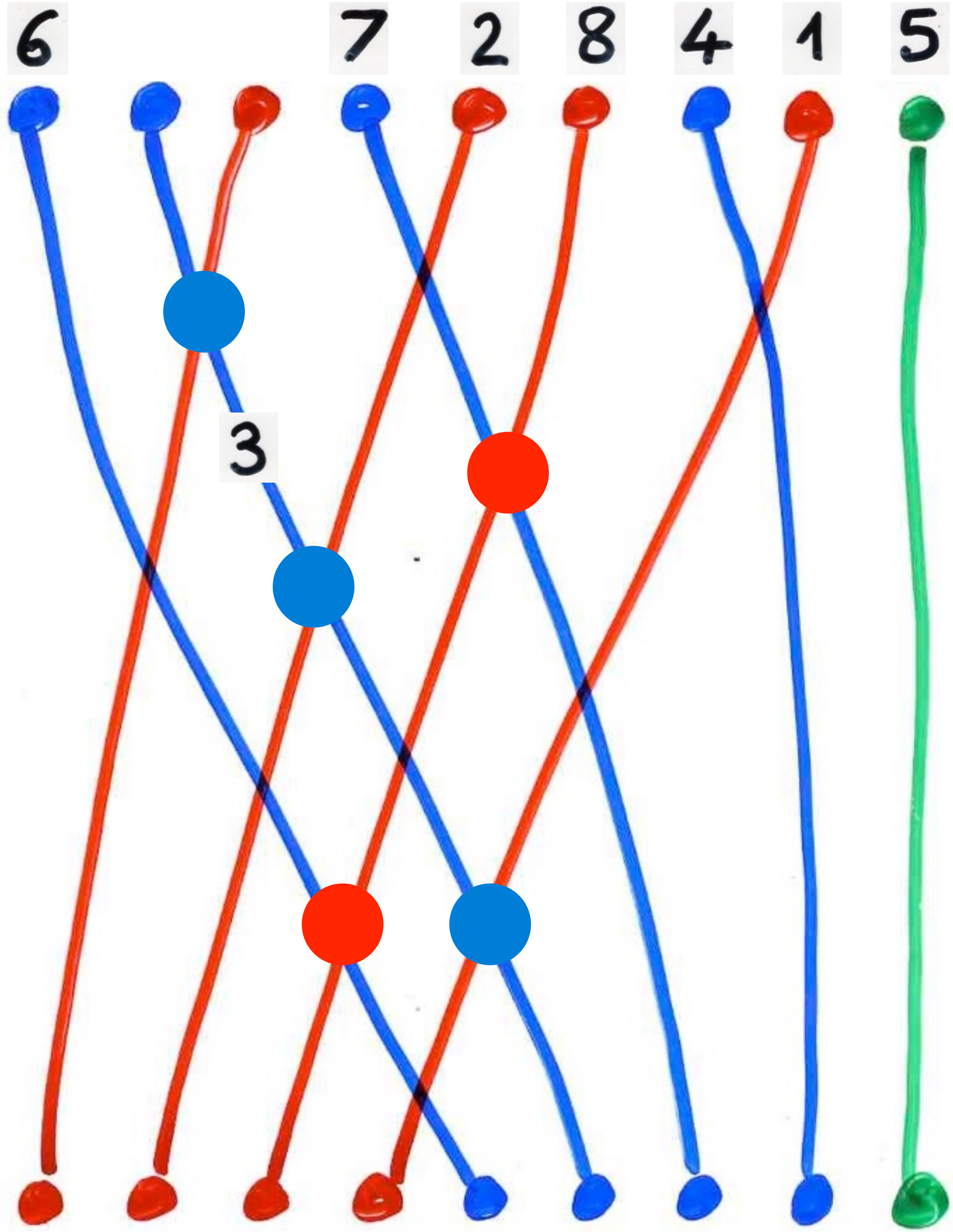
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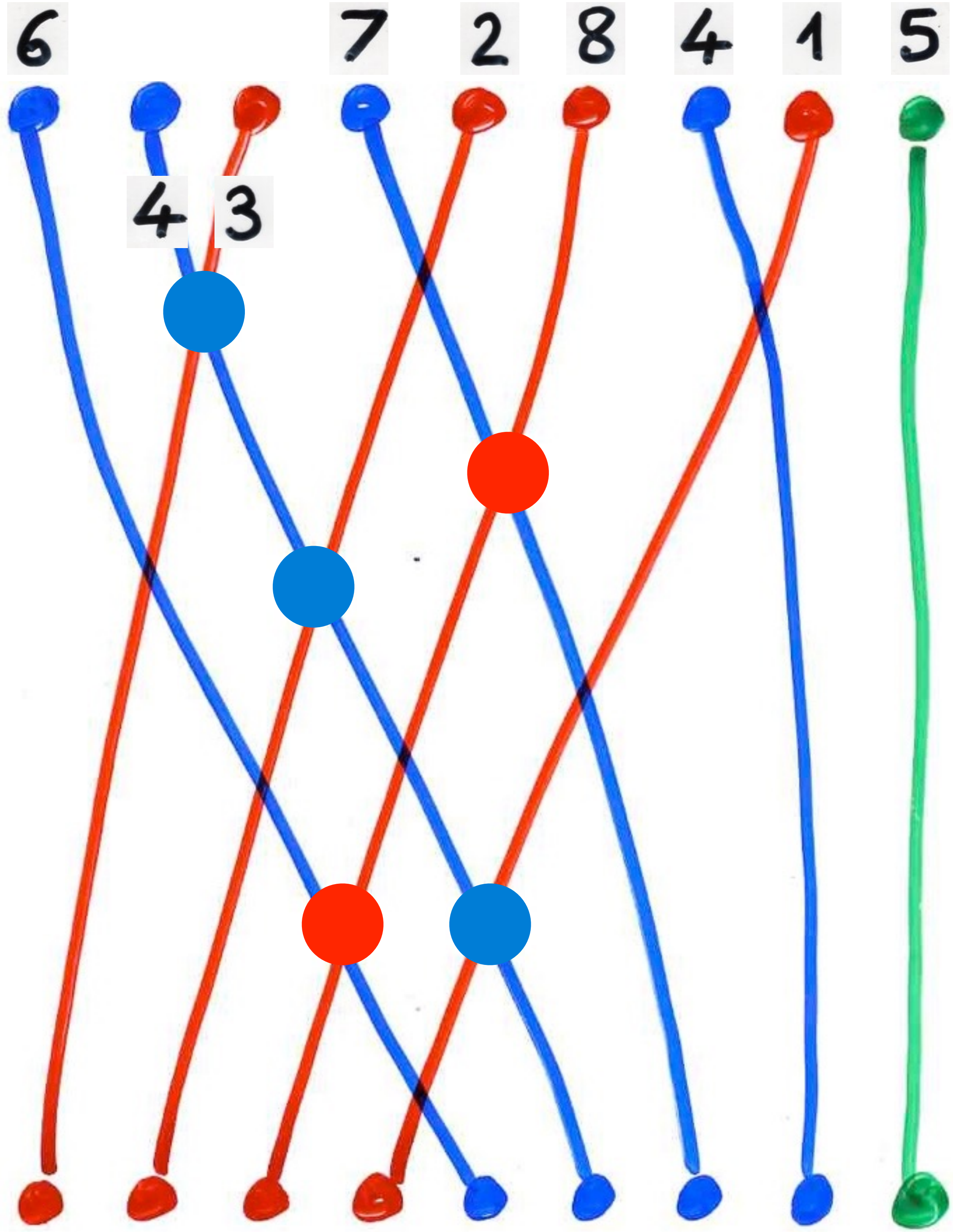
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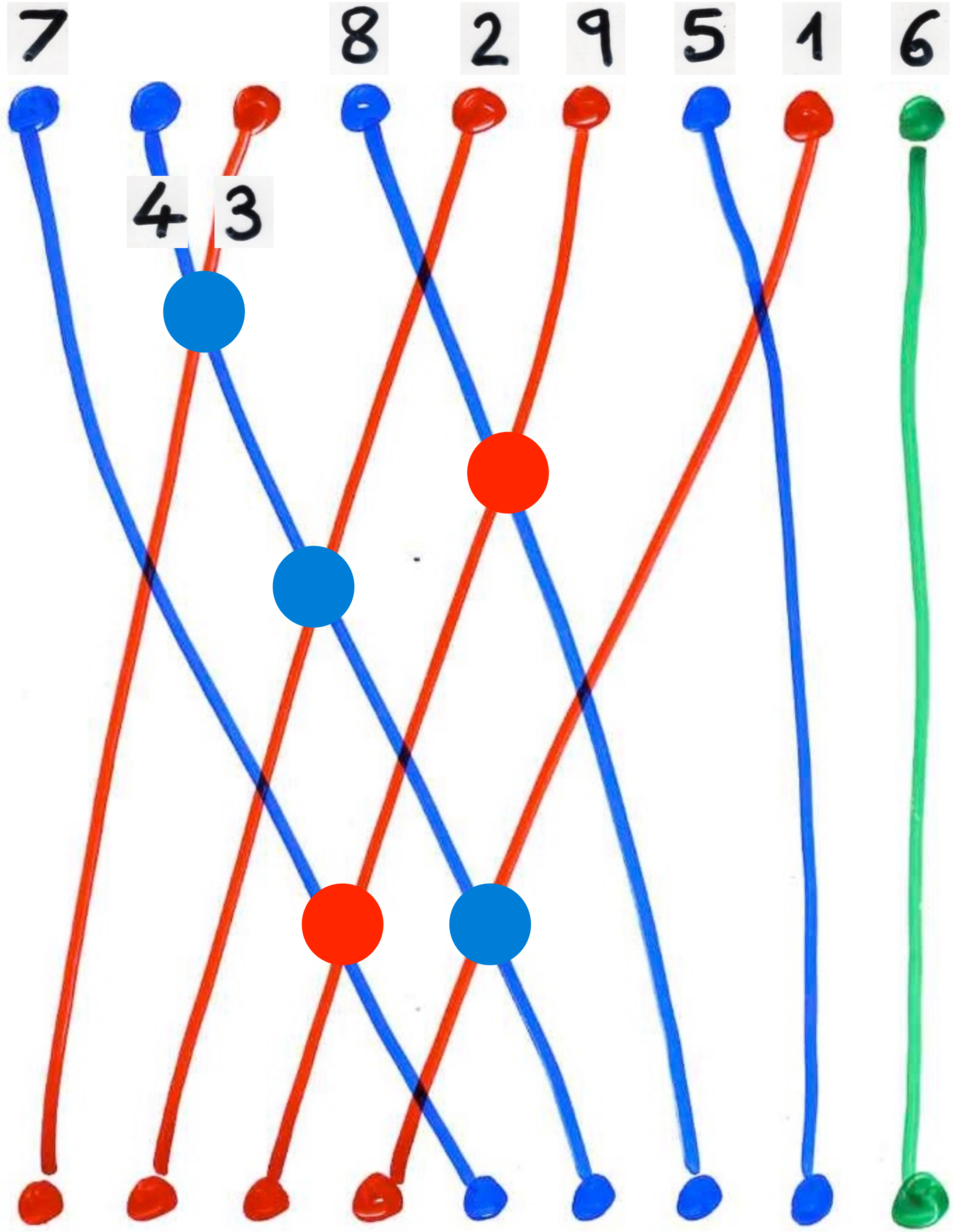
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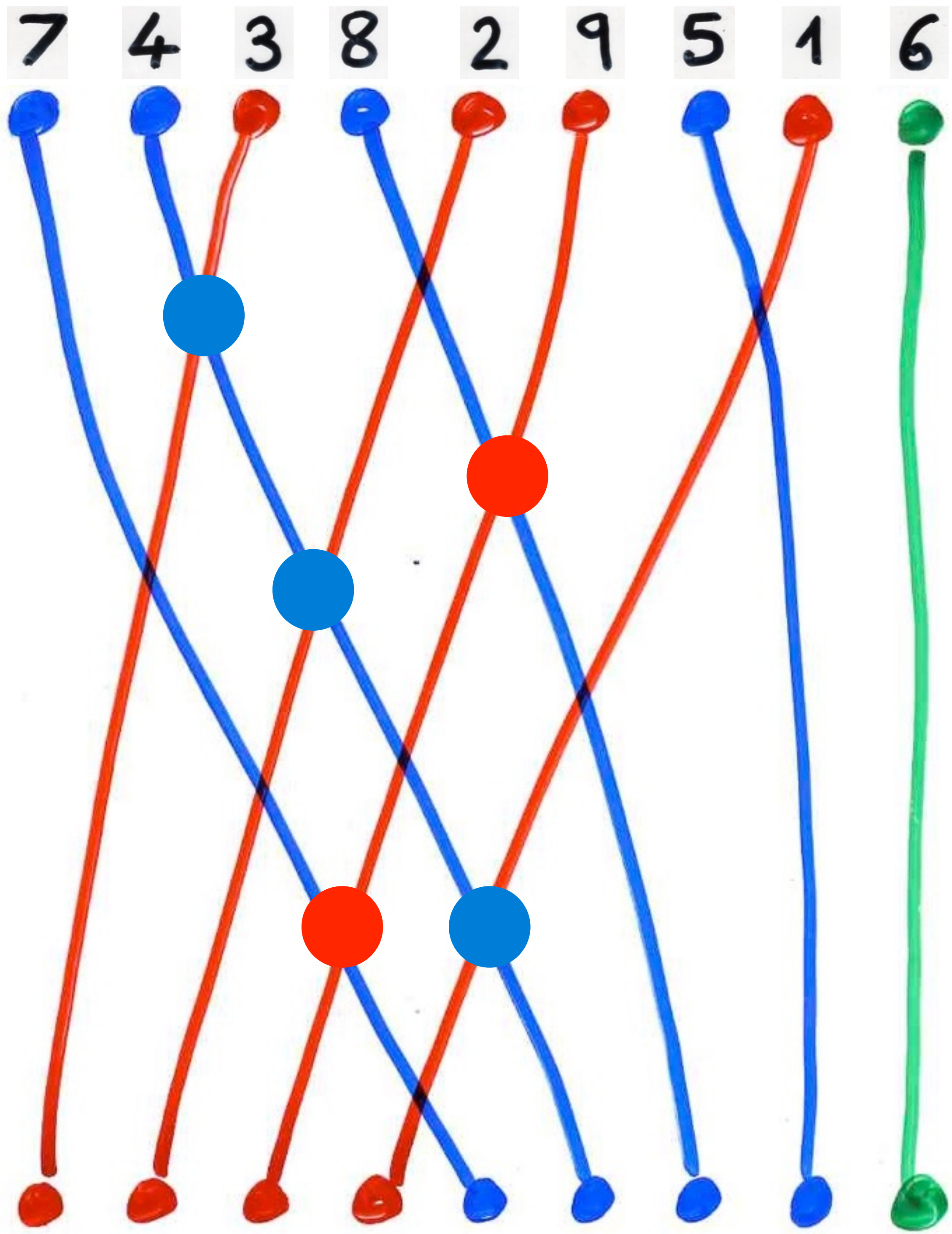

7 4 3 8 2 9 5 1 6

commutation diagrams

$$AS = SA + I_v J + K I_h$$

$$AK = KA + I_v A$$

$$JS = SJ + S I_h$$

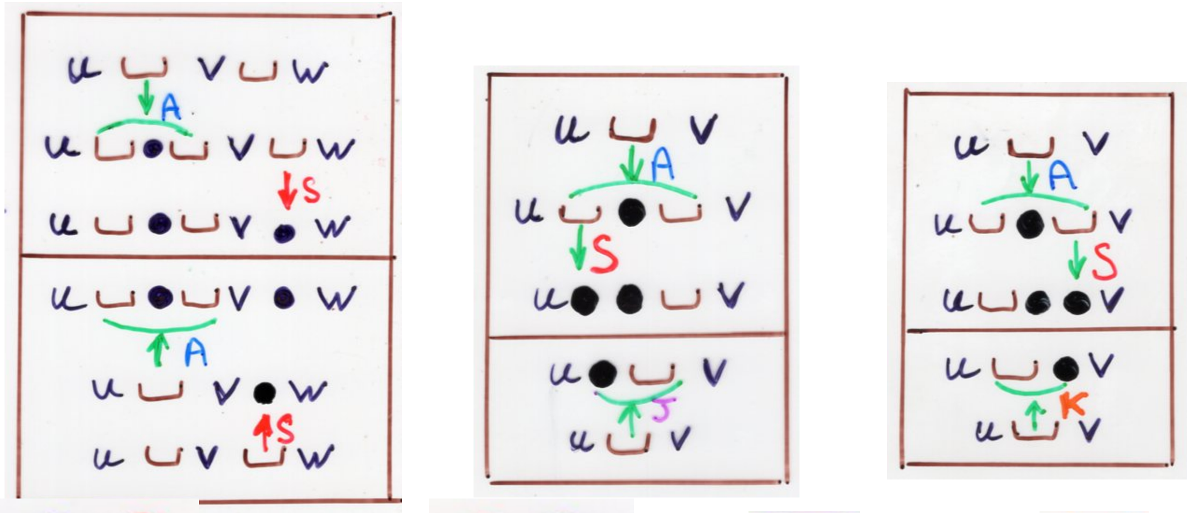
$$JK = KJ$$

$$A I_v = I_v A$$

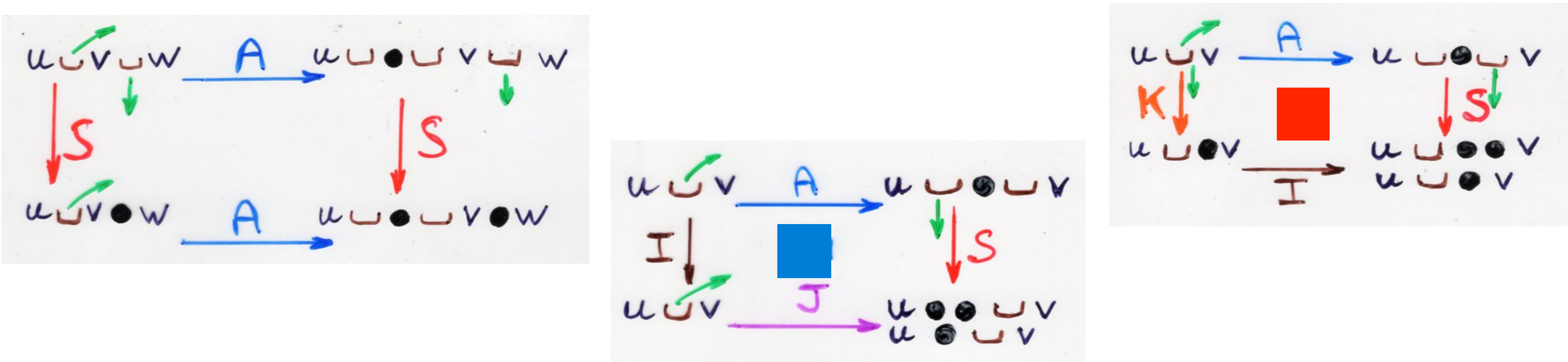
$$J I_v = I_v J$$

$$I_h S = S I_h$$

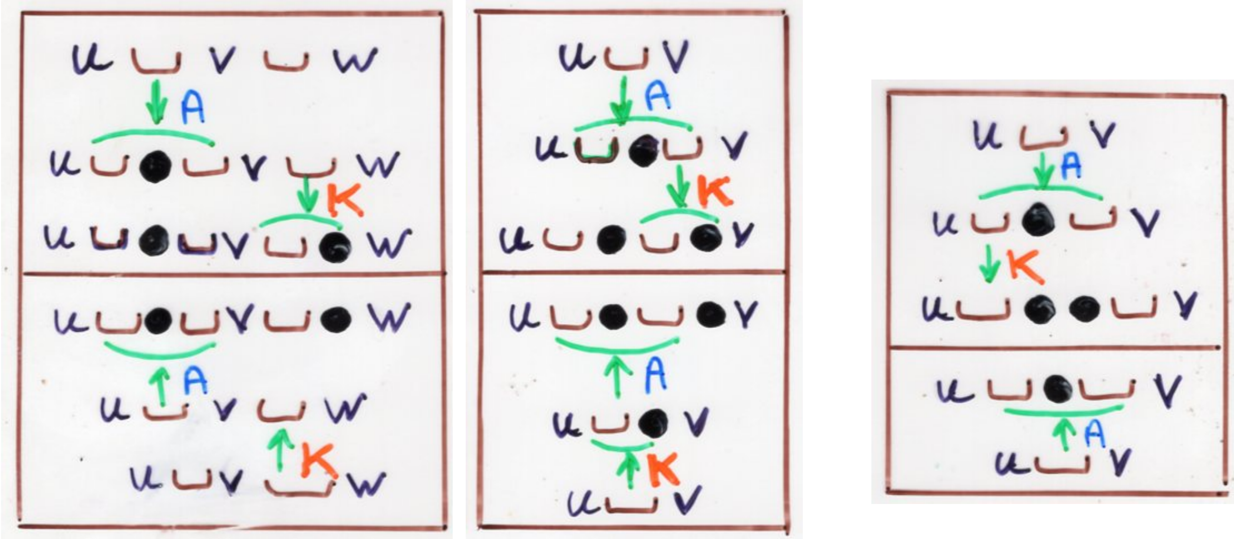
$$I_h K = K I_h$$



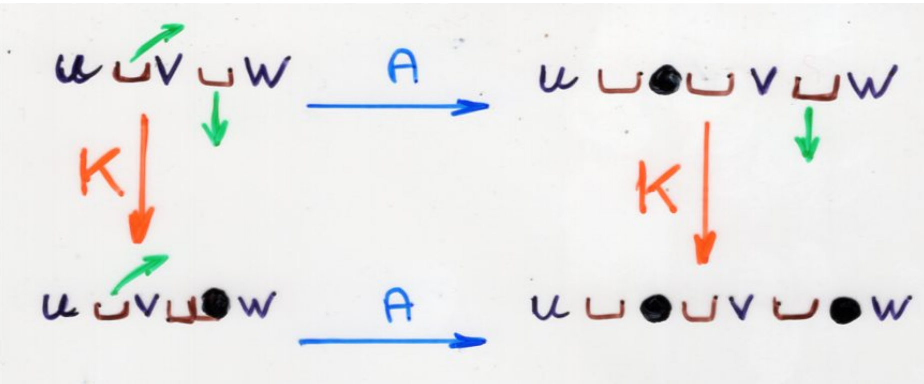
$$AS = SA + J + K$$

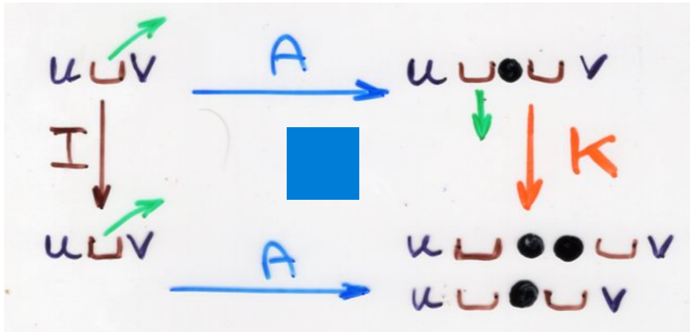


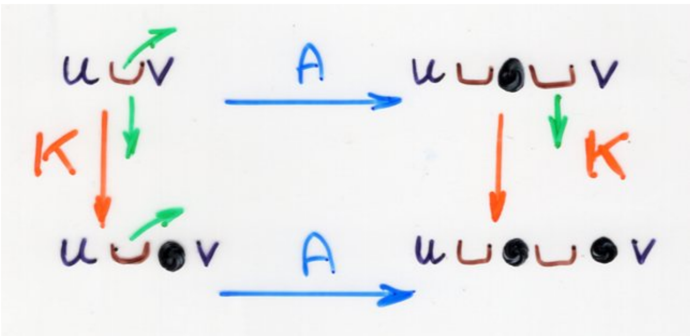
$$AS = SA + I_v J + K I_h$$



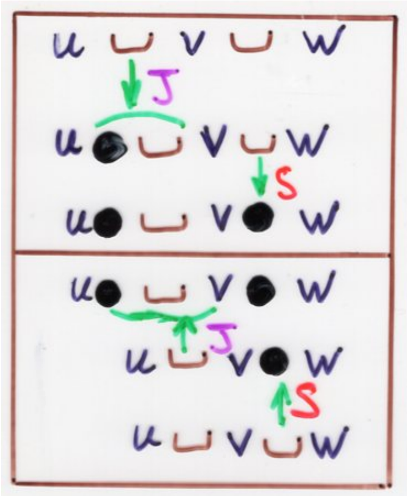
$$AK = KA + A$$

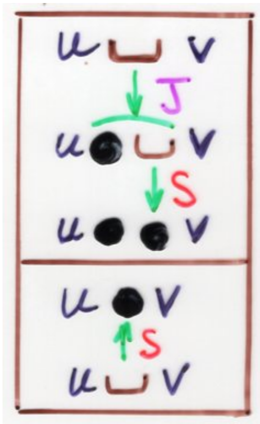




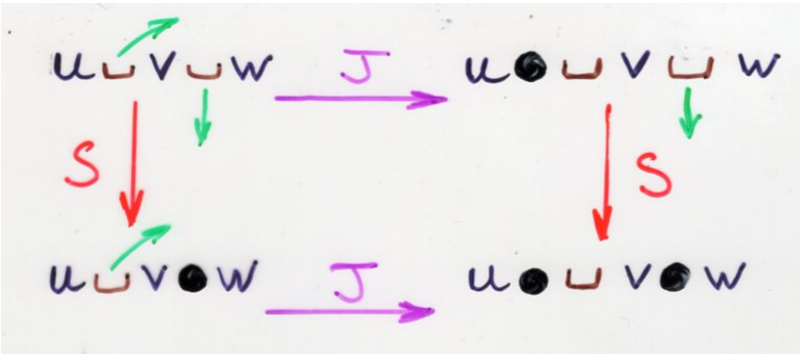


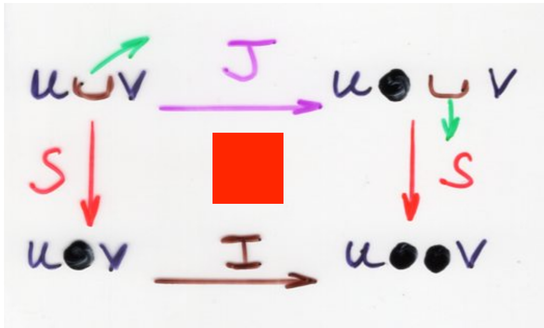
$$AK = KA + I_v A$$



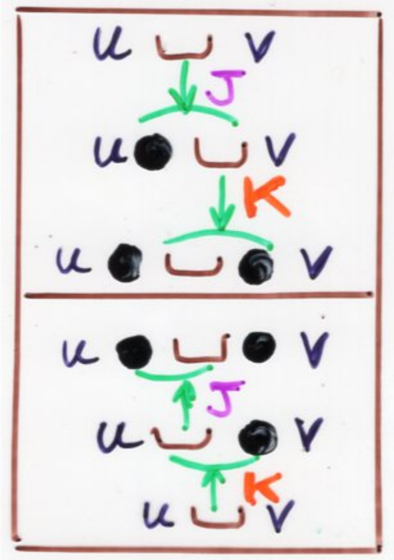


$$JS = SJ + S$$

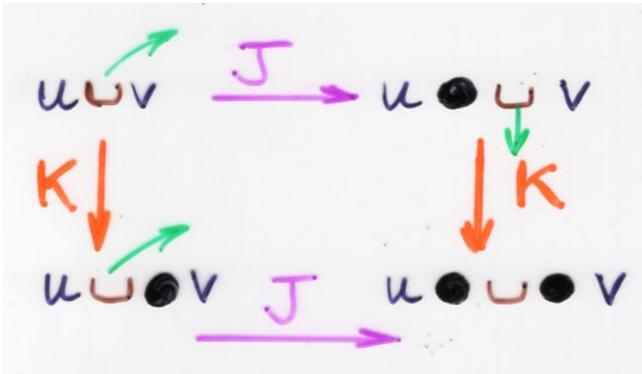




$$JS = SJ + SI_h$$



$$JK = KJ$$



$$JK = KJ$$

$$A\, I_v = I_v\, A$$

$$J\, I_v = I_v\, J$$

$$I_h\, S = S\, I_h$$

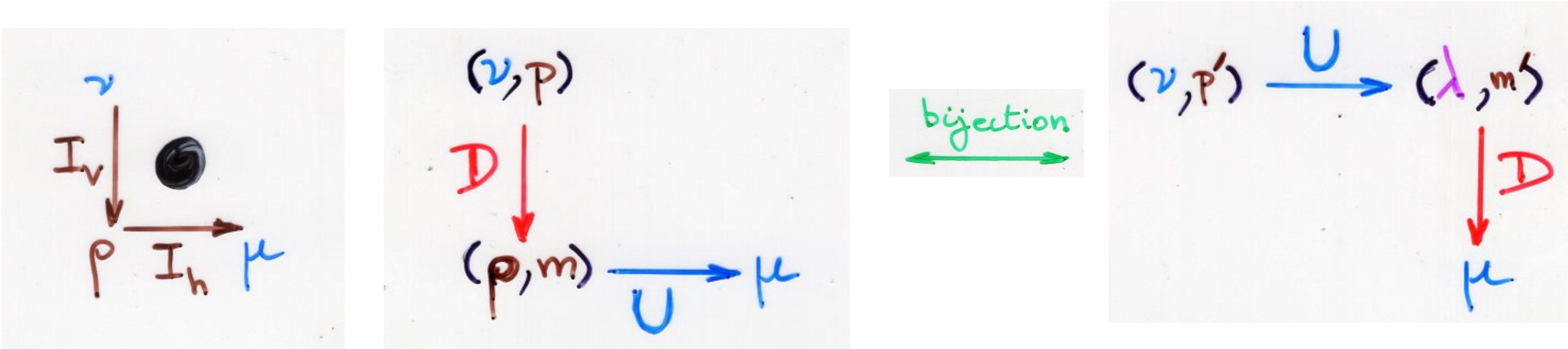
$$I_h\, K = K\, I_h$$

commutation diagrams bijections

analogy with commutation diagrams bijection
for the representation of the Weyl-Heisenberg algebra
(Ch2)

$$UD = DU + I_v I_h$$

"commutation diagrams"



p, m, p′, m′ are "positions"

in ν, ρ, ν, λ respectively

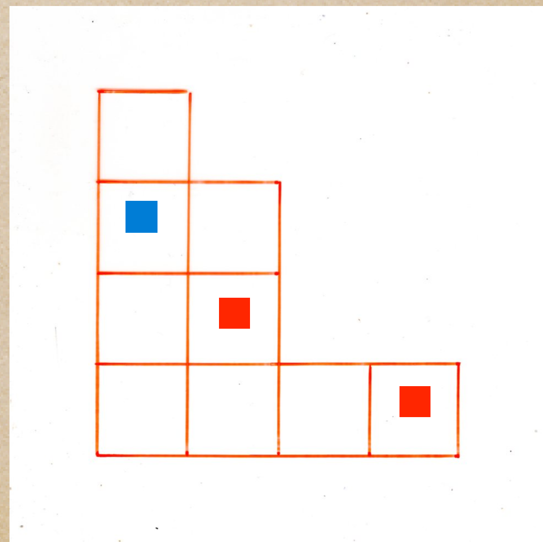
the bijection

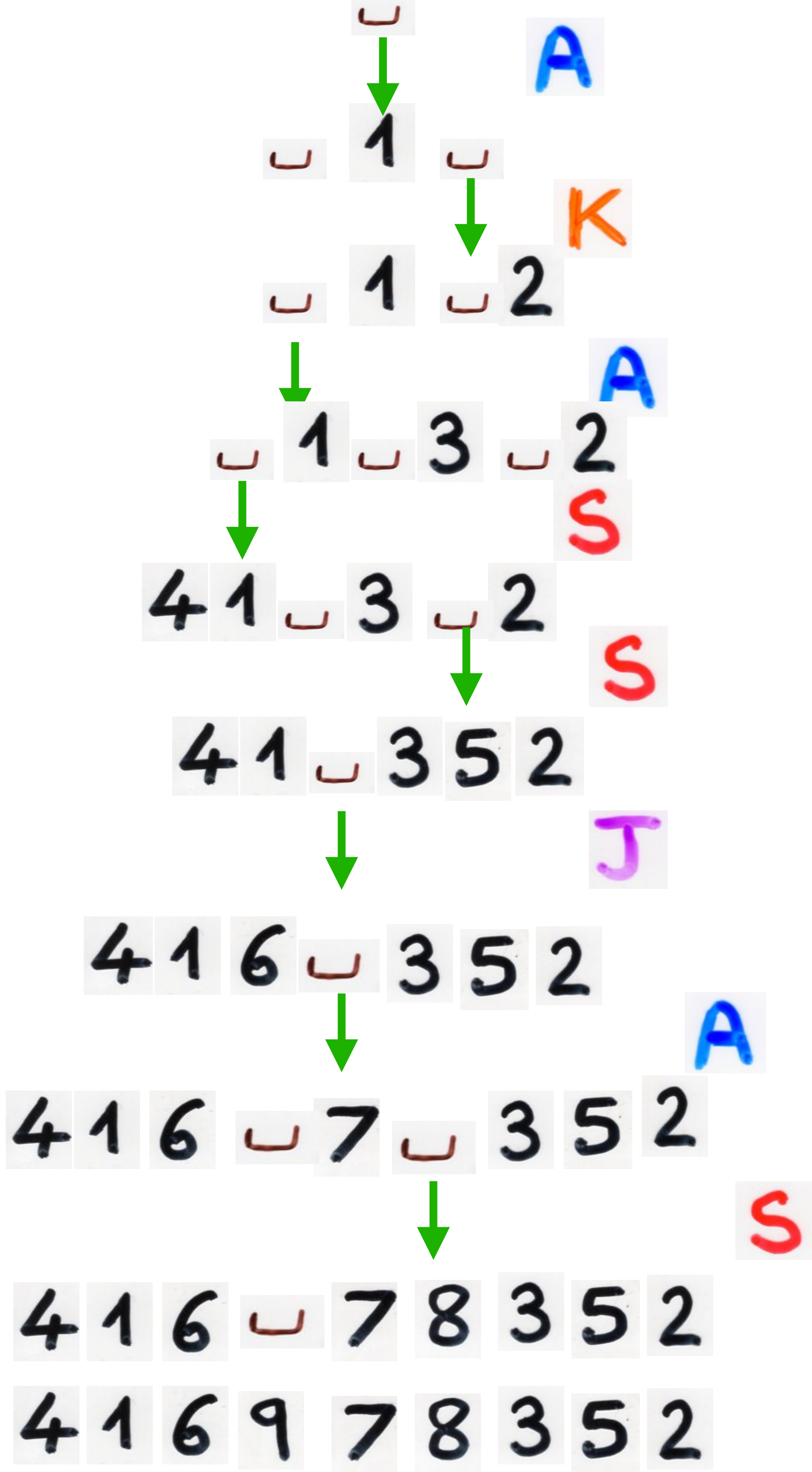
permutations — alternative tableaux

(Laguerre histories)

with local rules
(commutation diagrams)

4 1 6 9 7 8 3 5 2 →




␣
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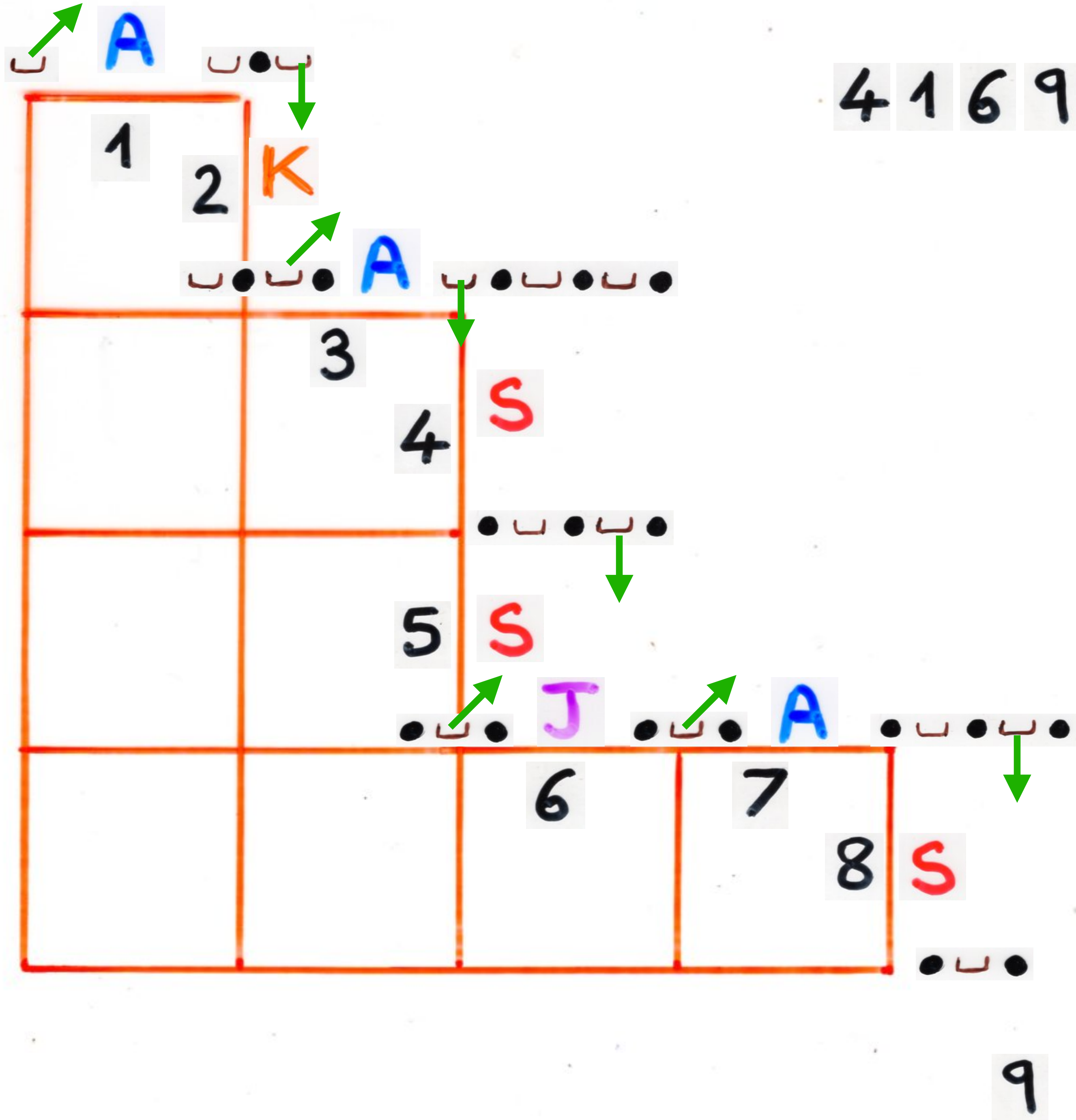
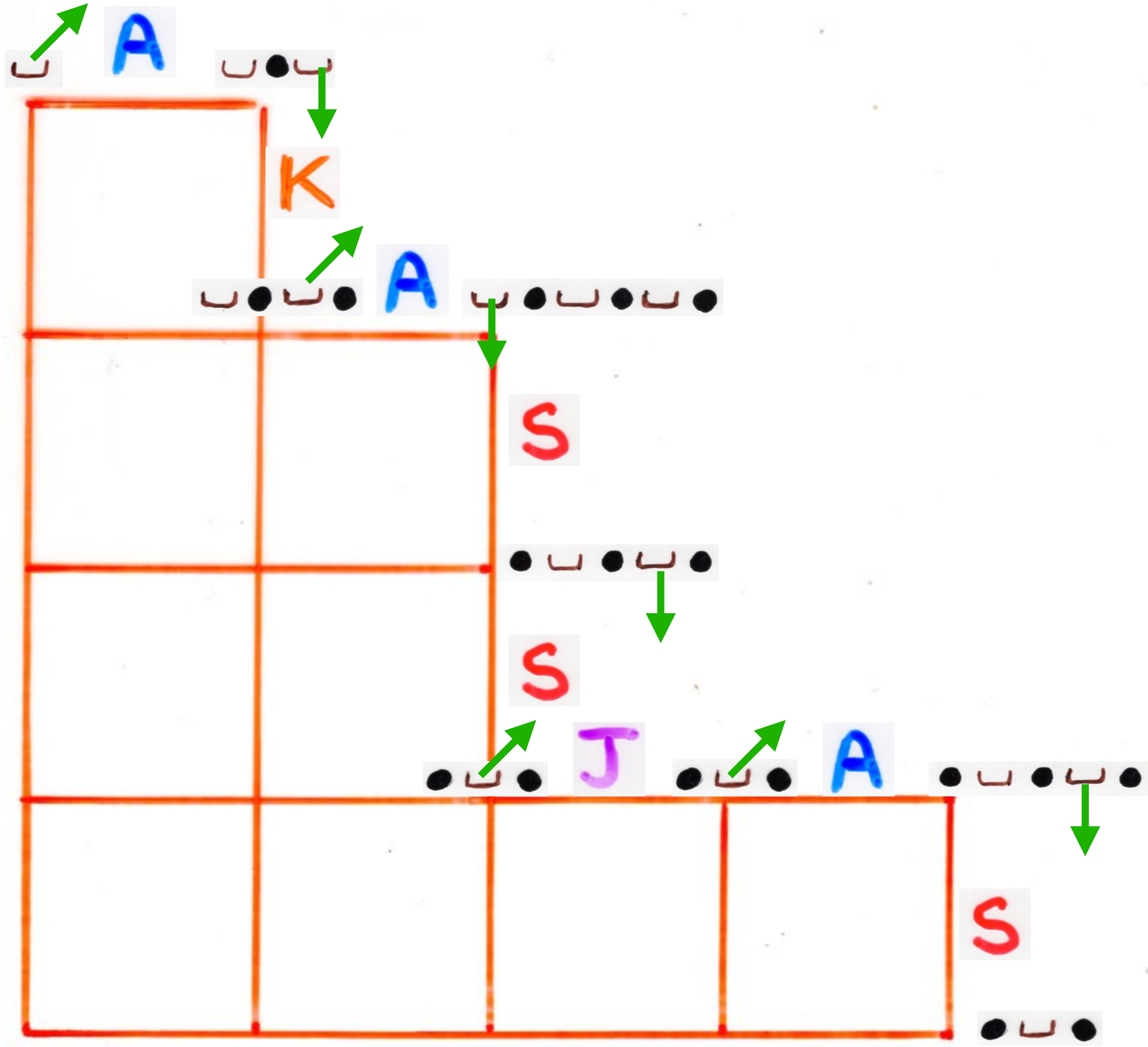
␣ 1 ␣
K
␣ 1 ␣ 2
A
␣ 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 ␣ 2
S
4 1 ␣ 3 5 2
J
4 1 6 ␣ 3 5 2
A
4 1 6 ␣ 7 ␣ 3 5 2
S
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2

4 1 6 9 7 8 3 5 2

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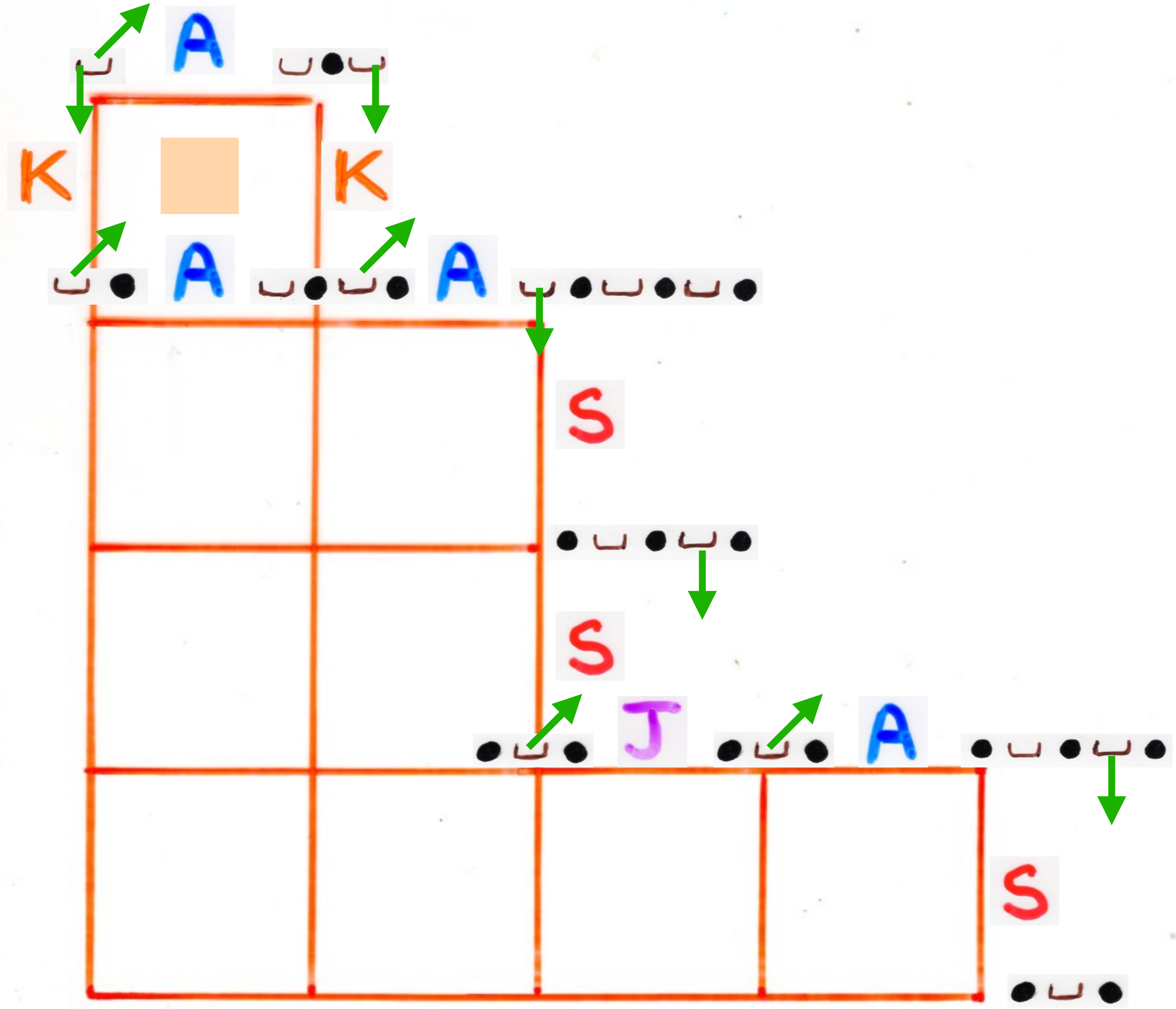
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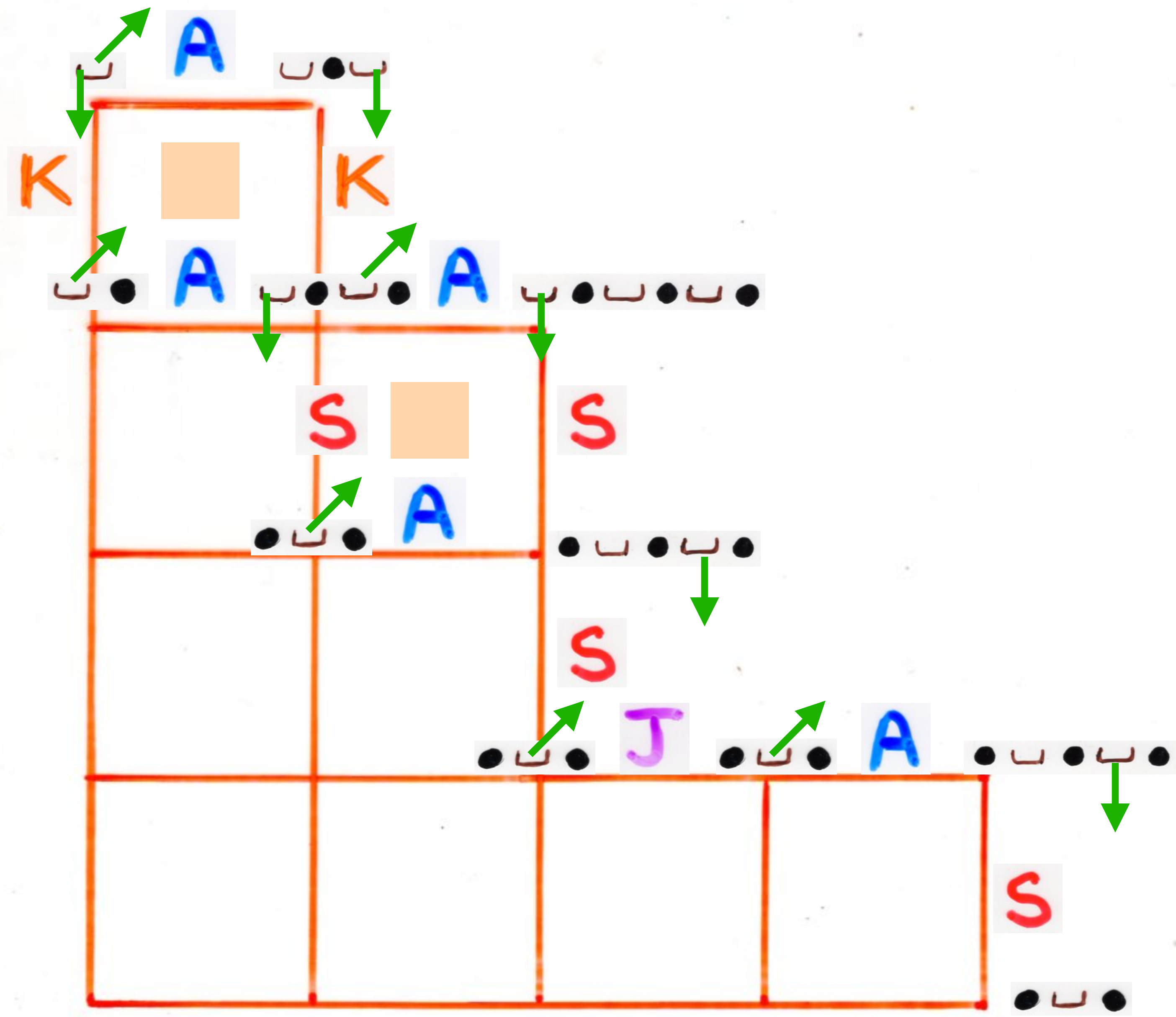
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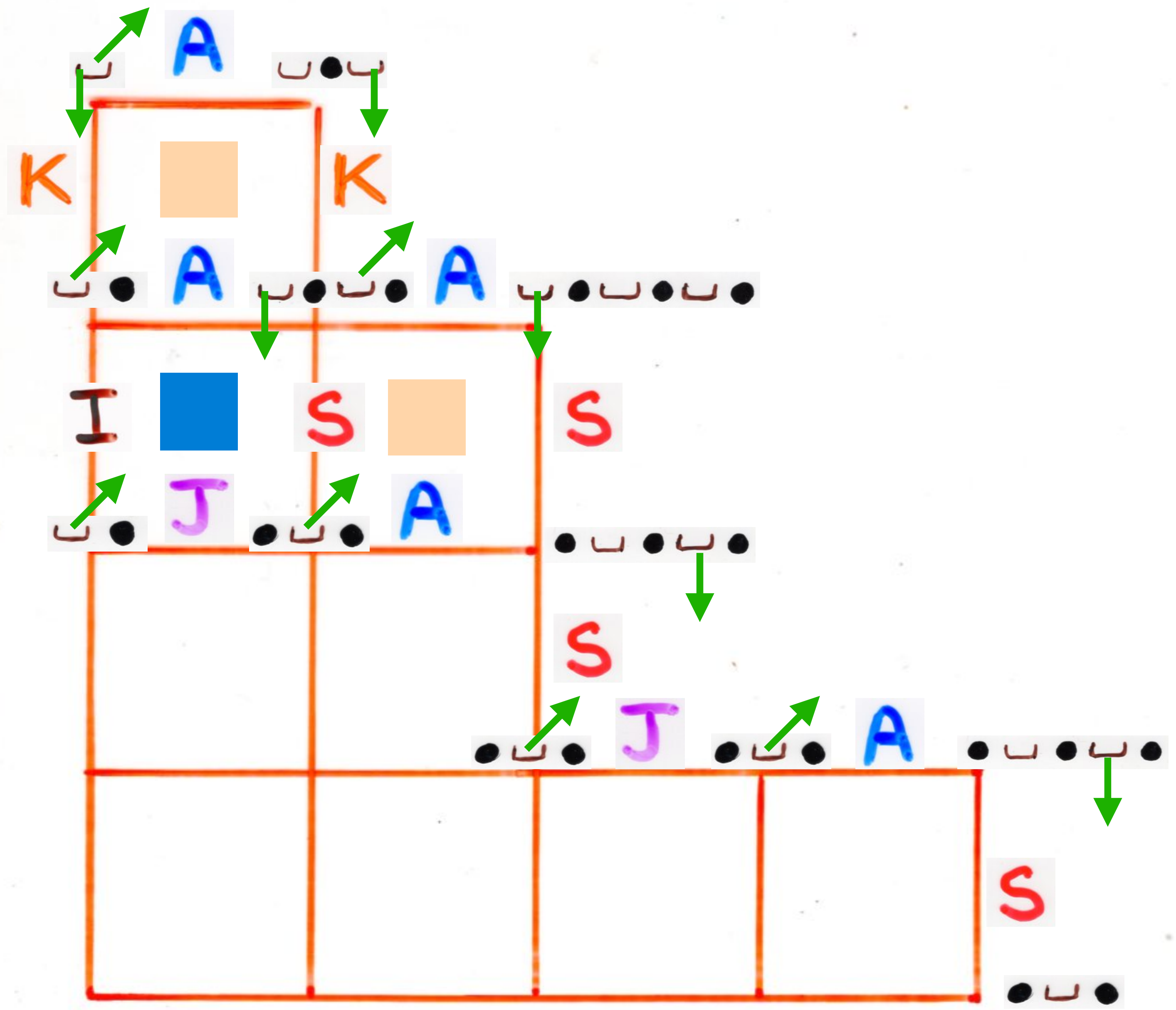
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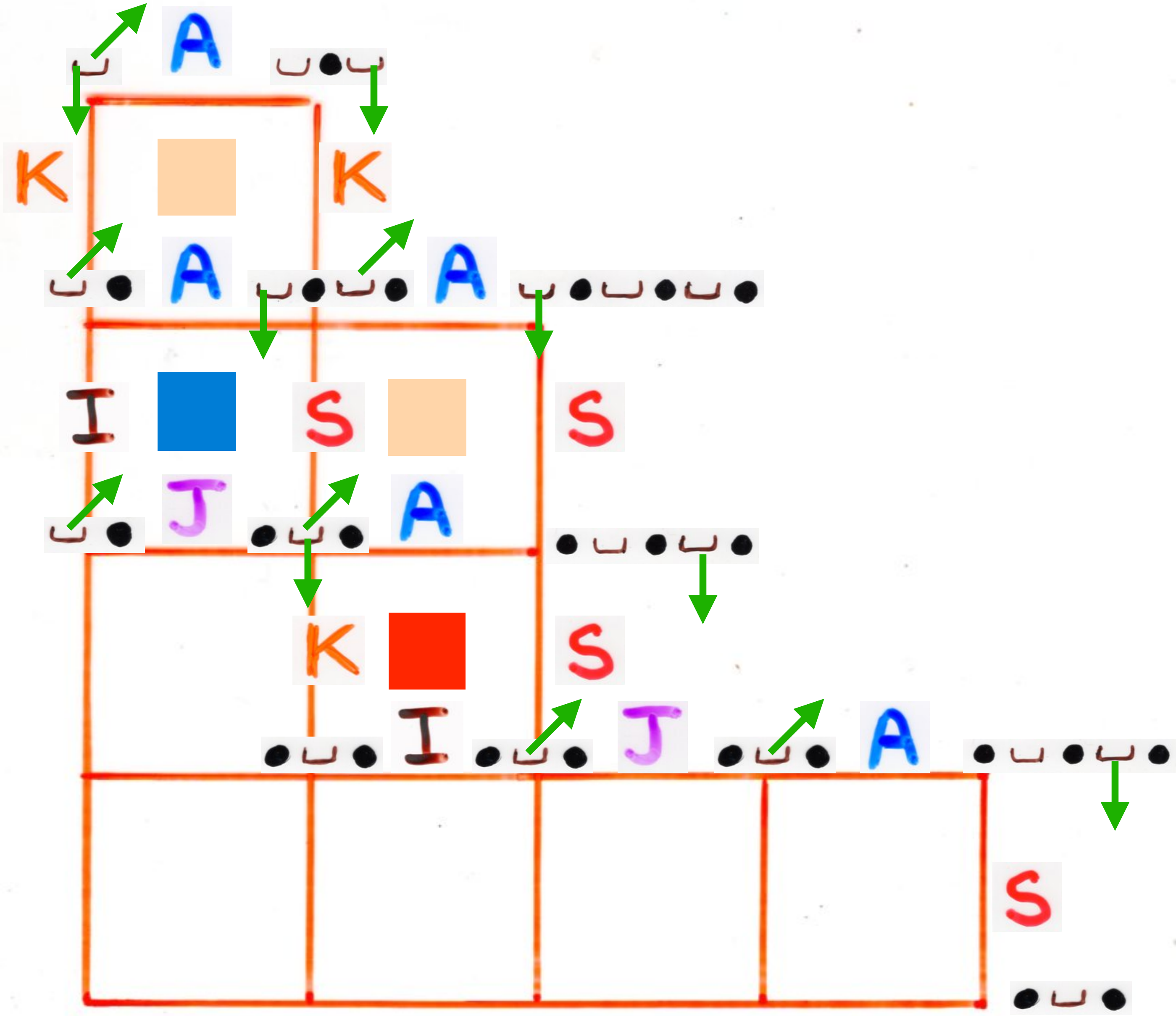
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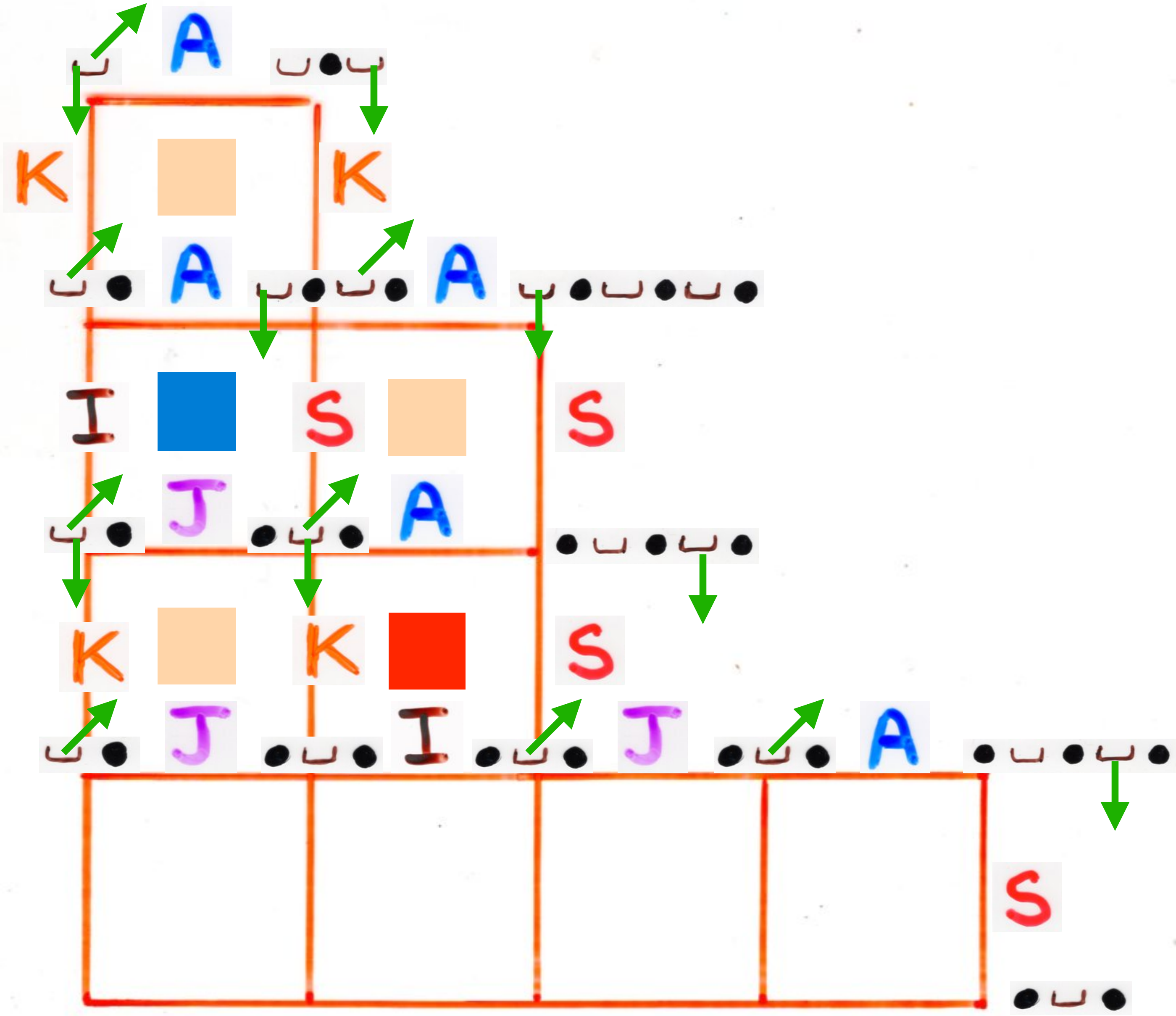

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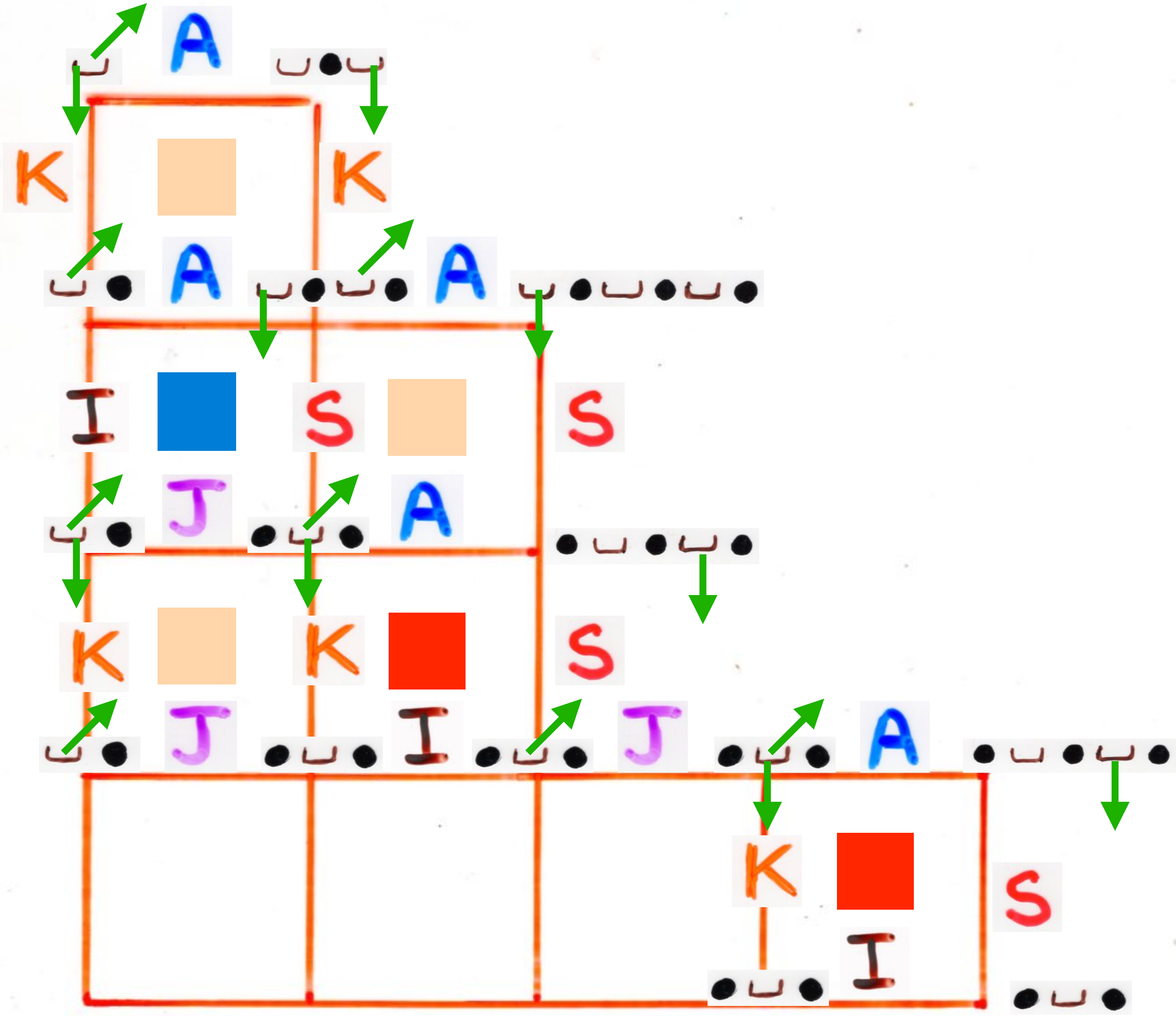
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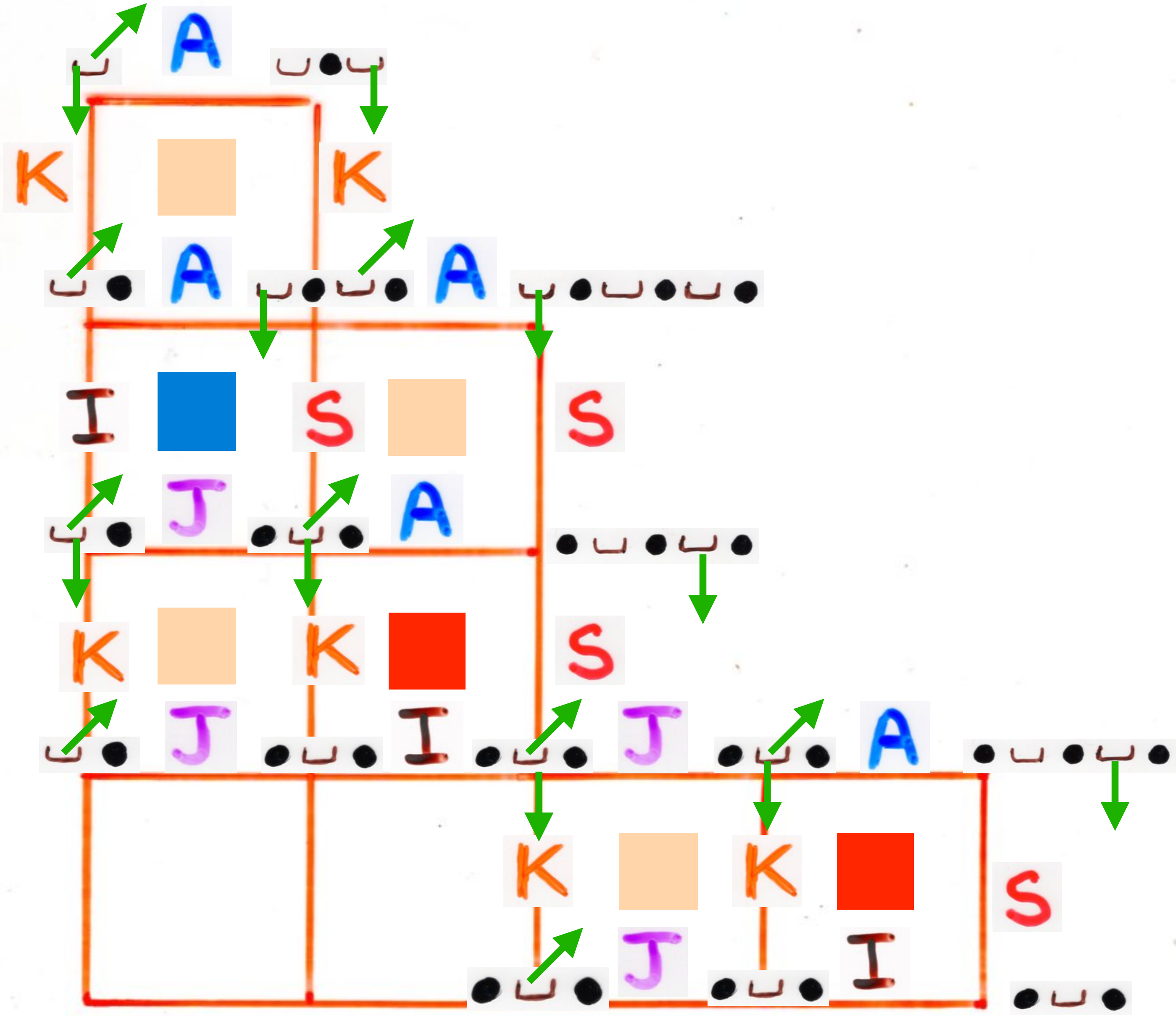

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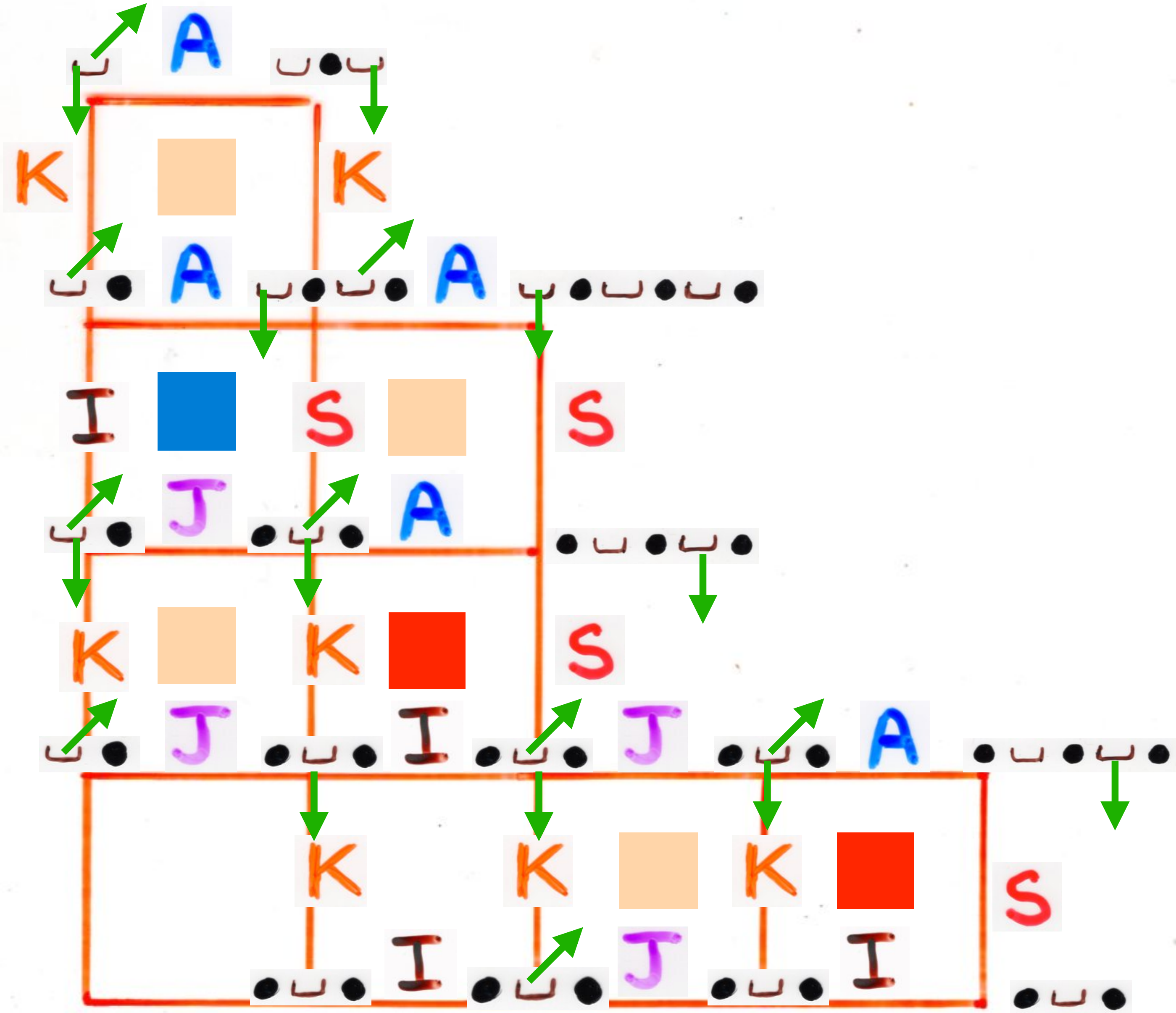

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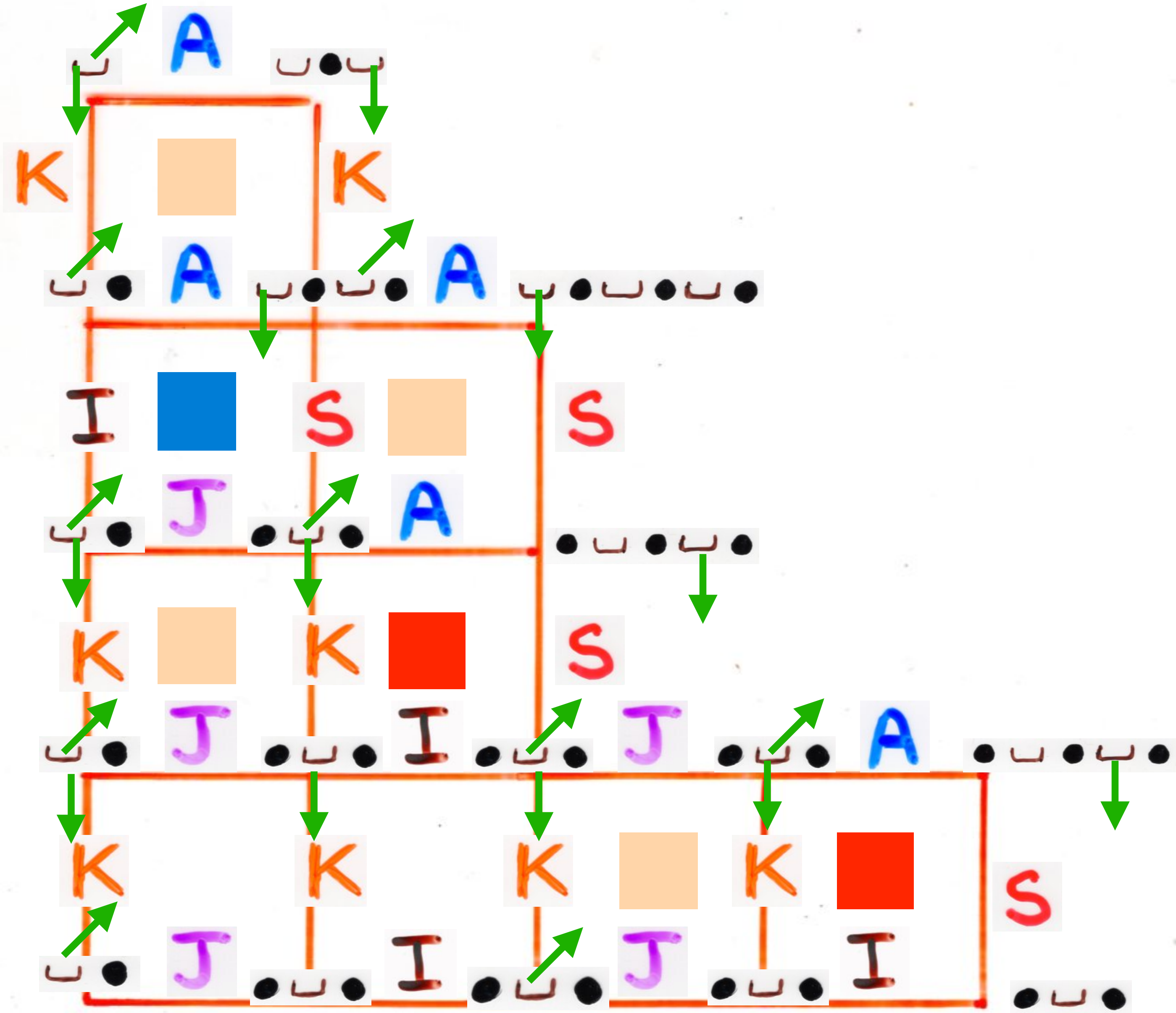
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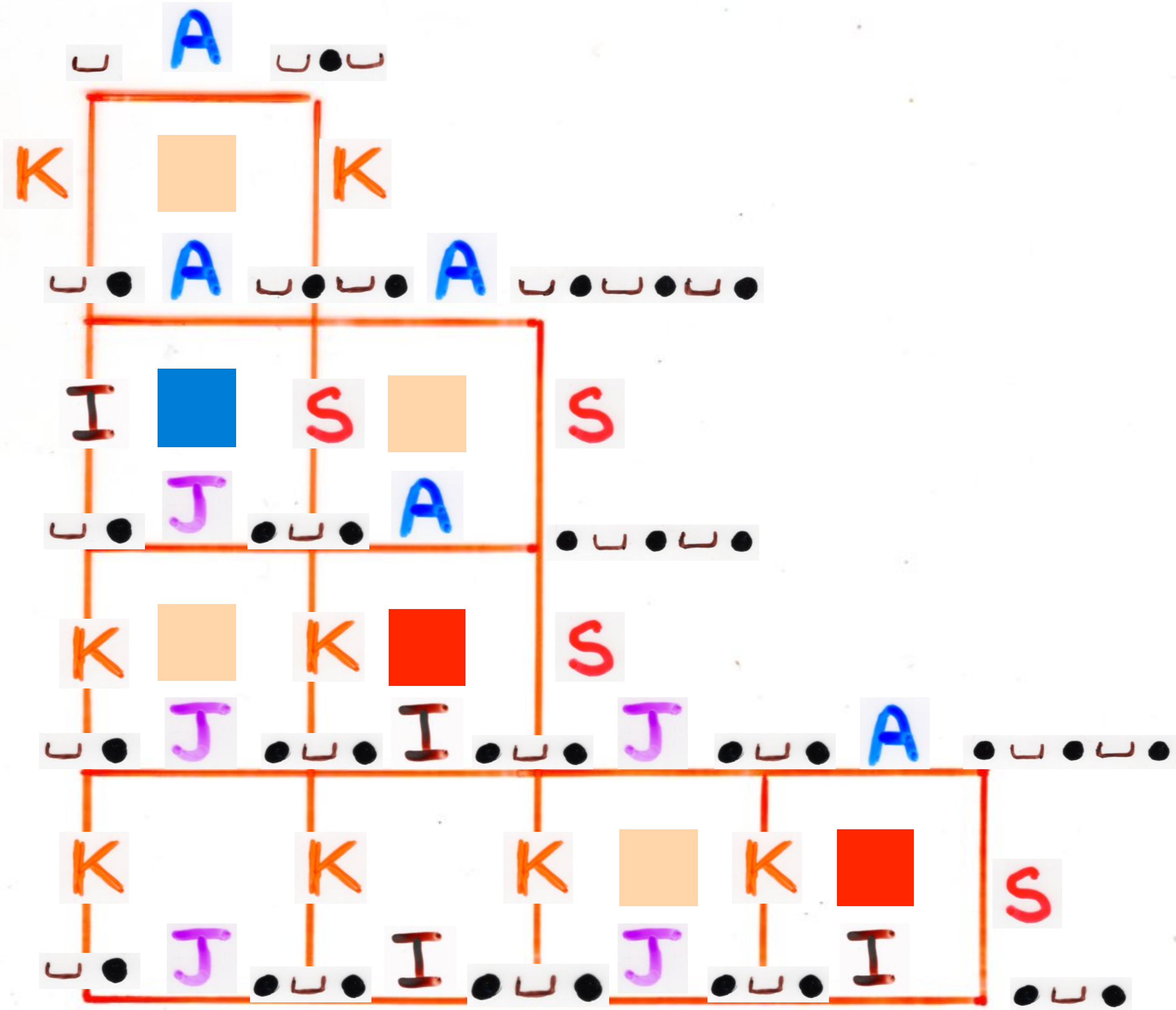
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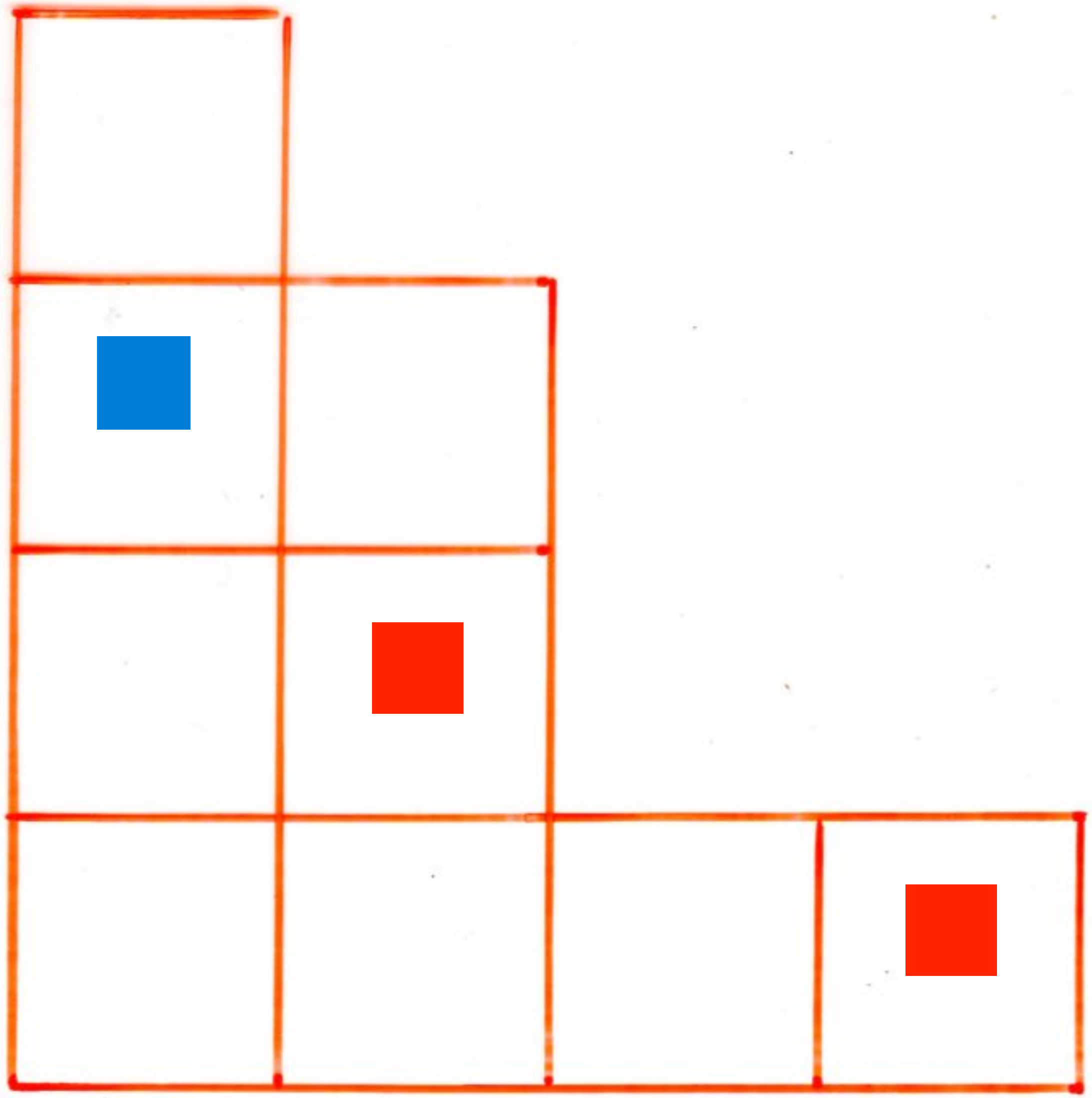


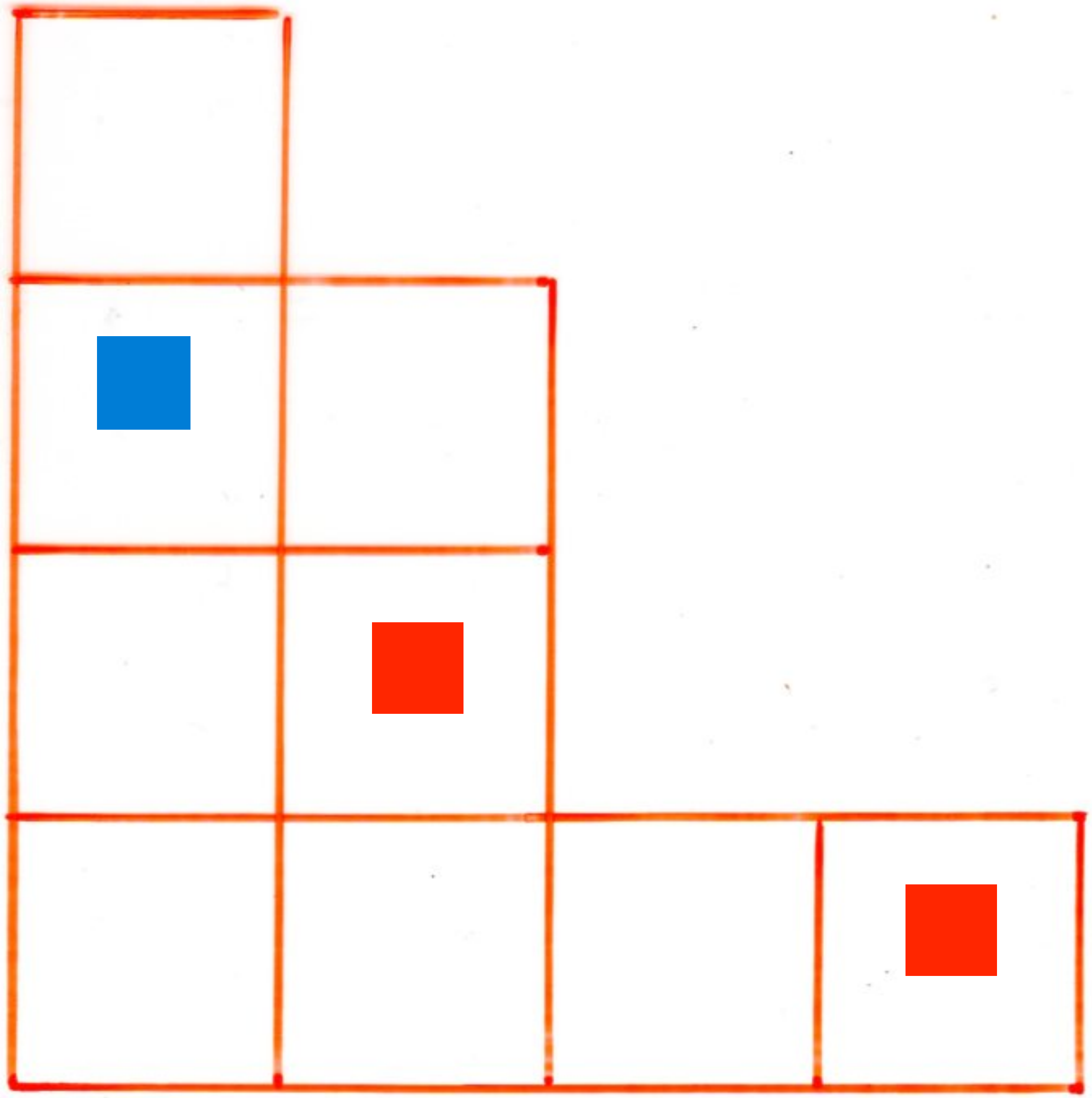
the reverse bijection

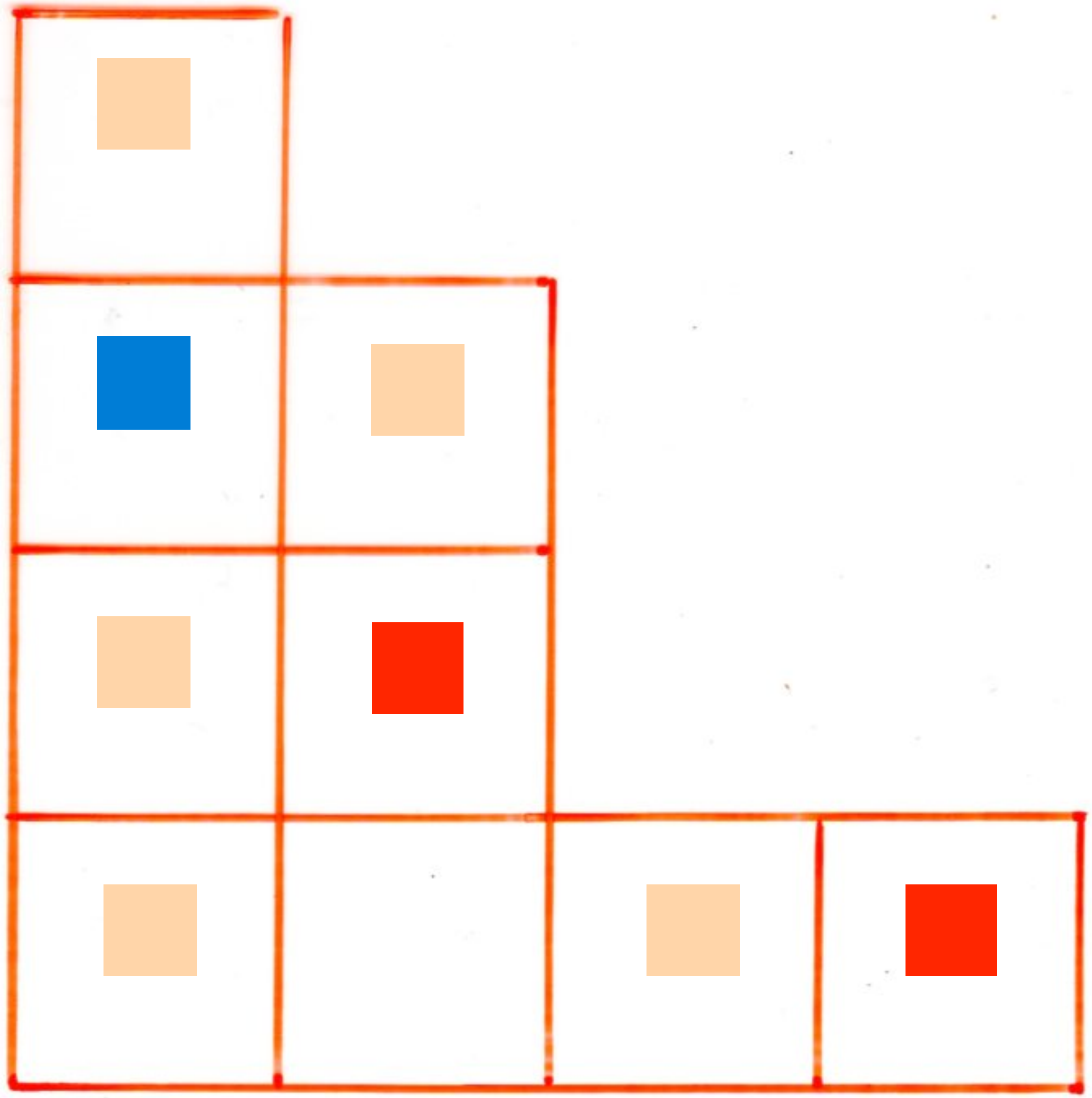
permutations — alternative tableaux

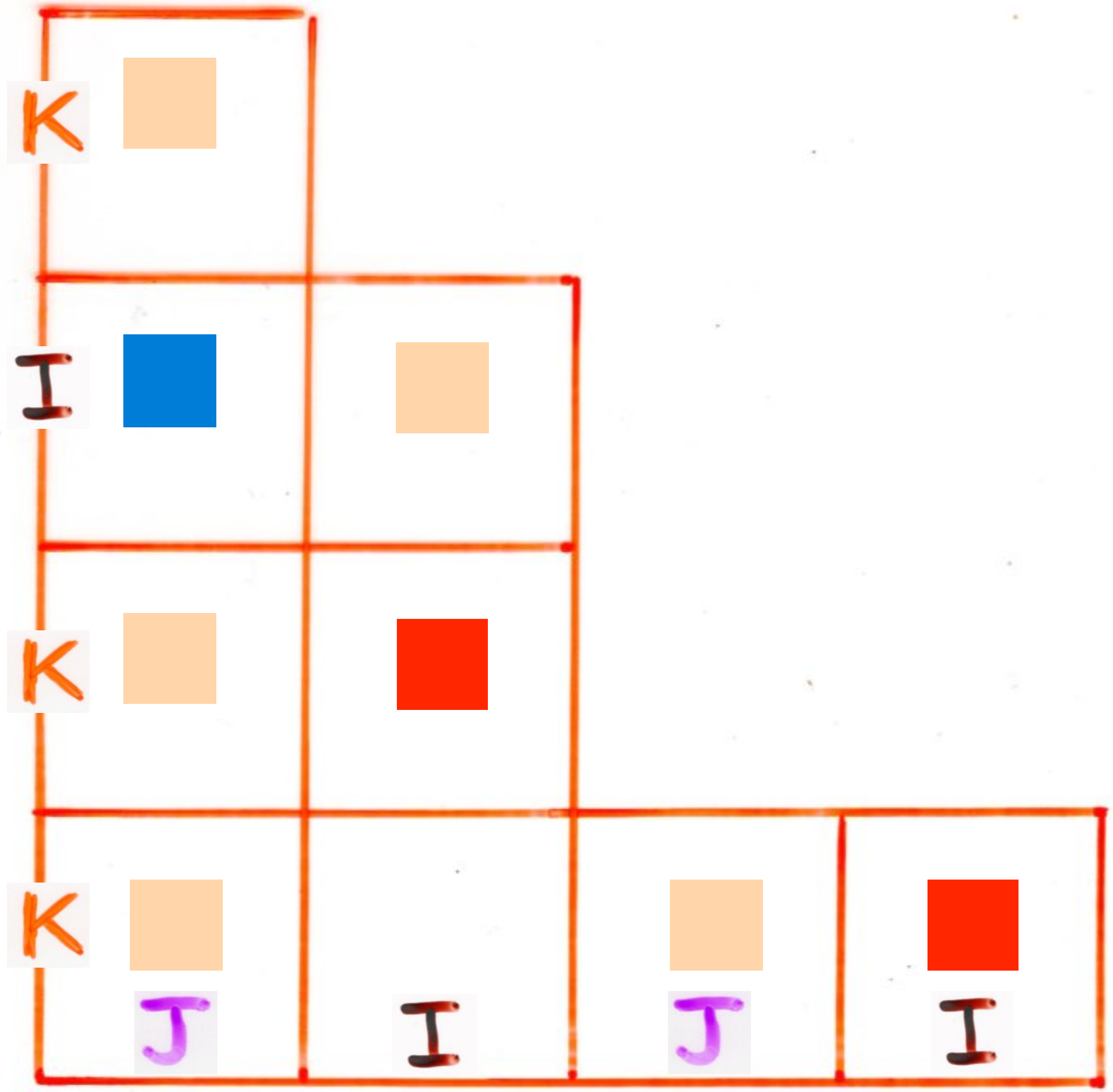
(Laguerre histories)

local rules

(commutation diagrams)



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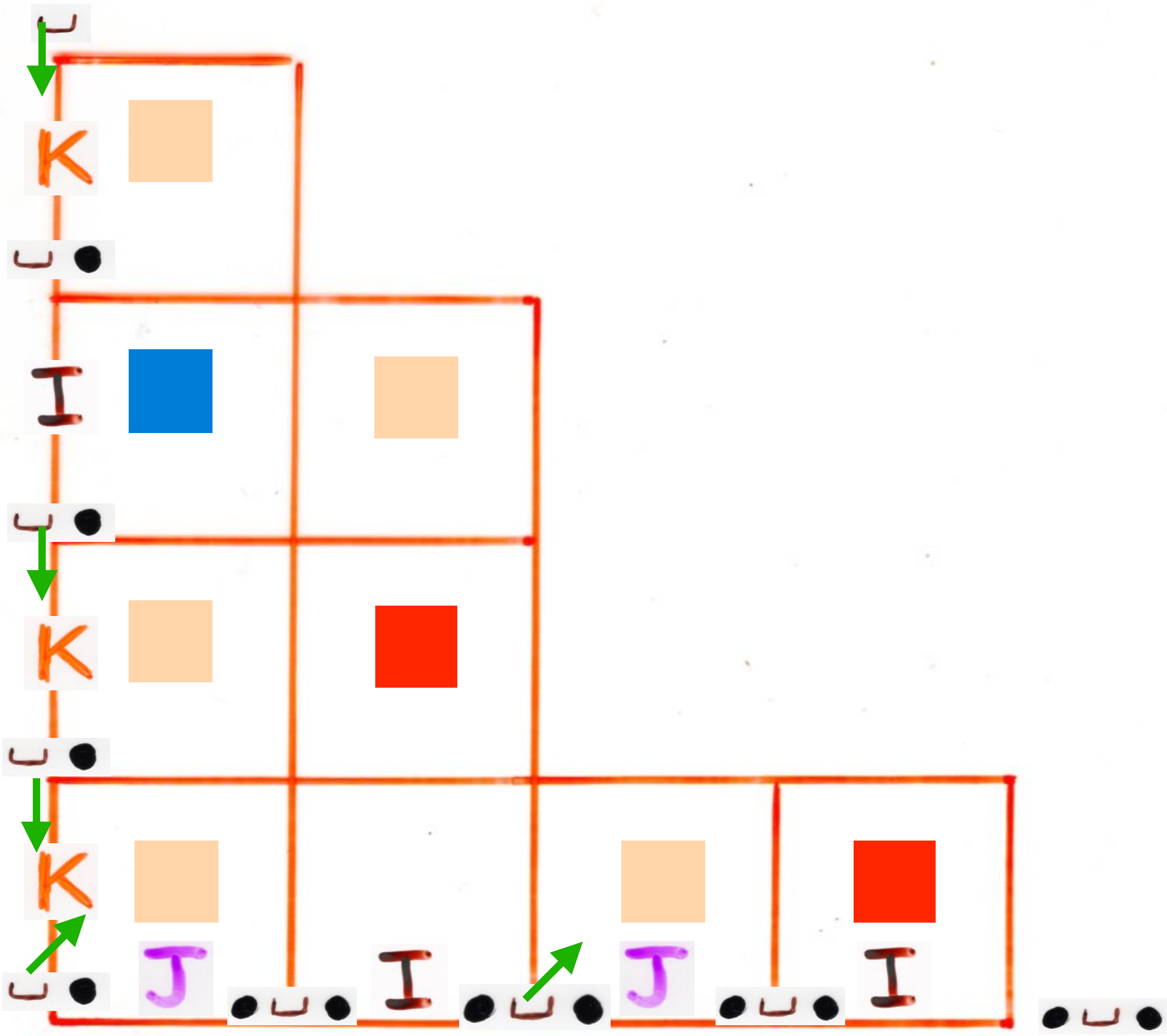

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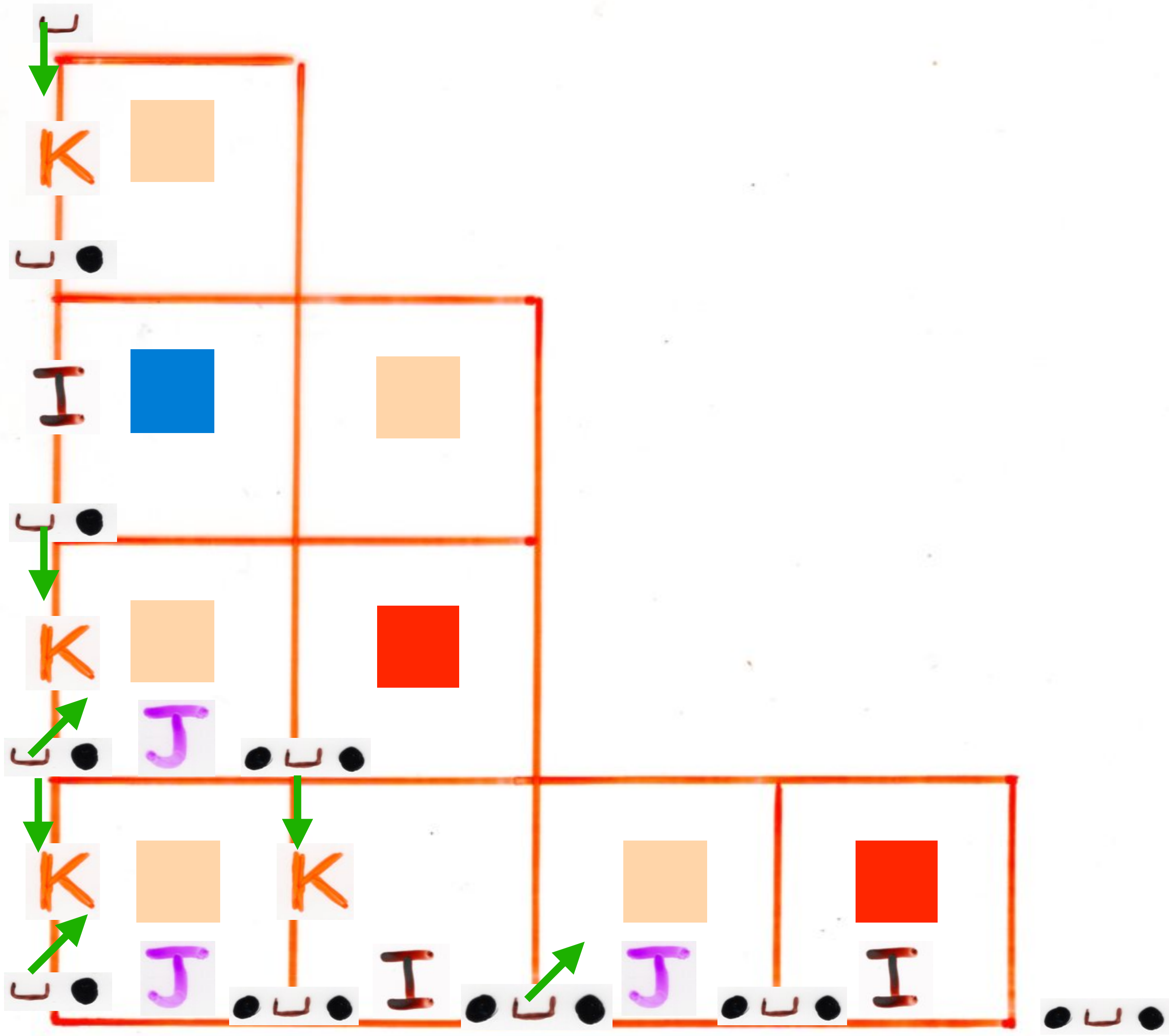
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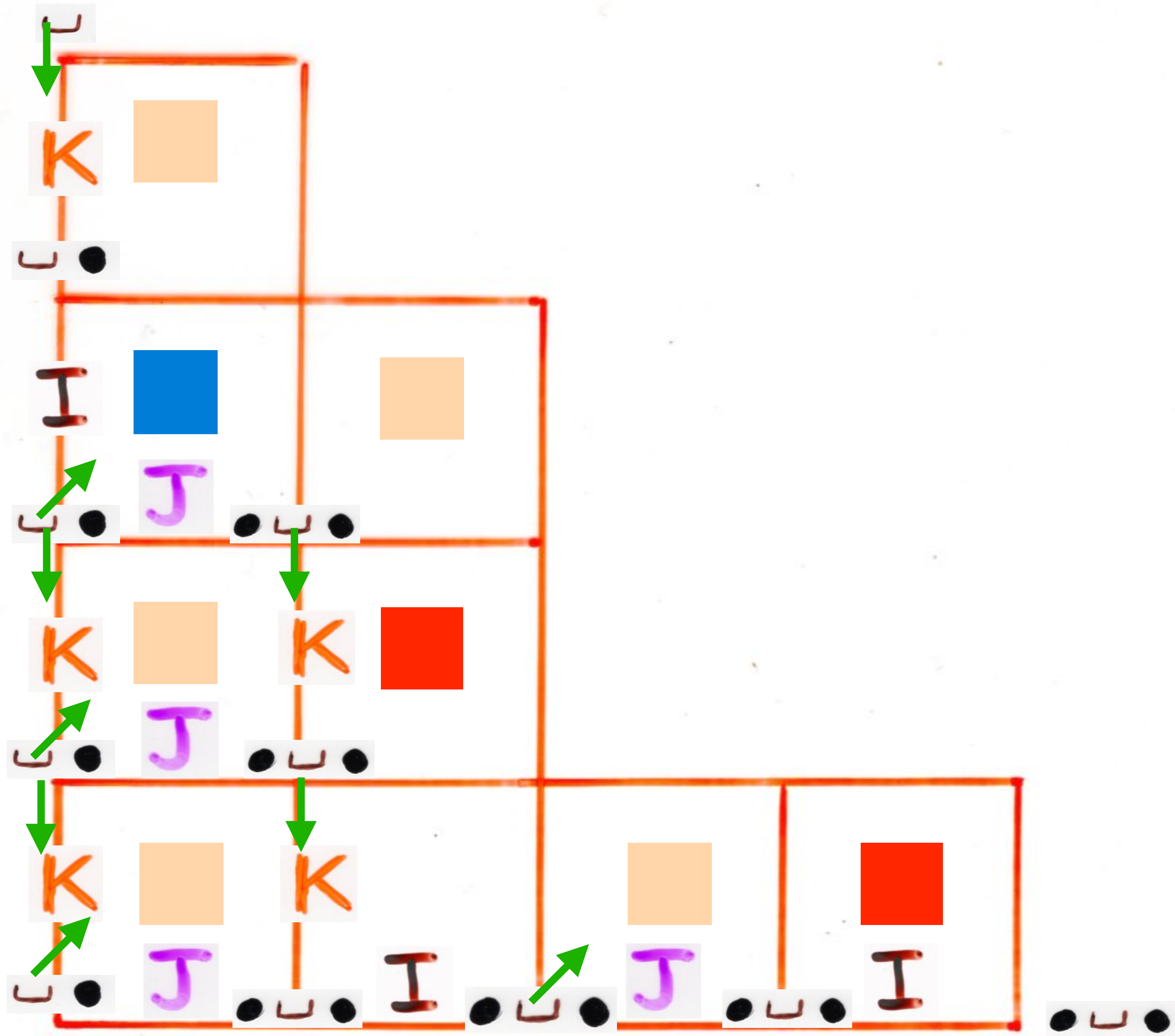
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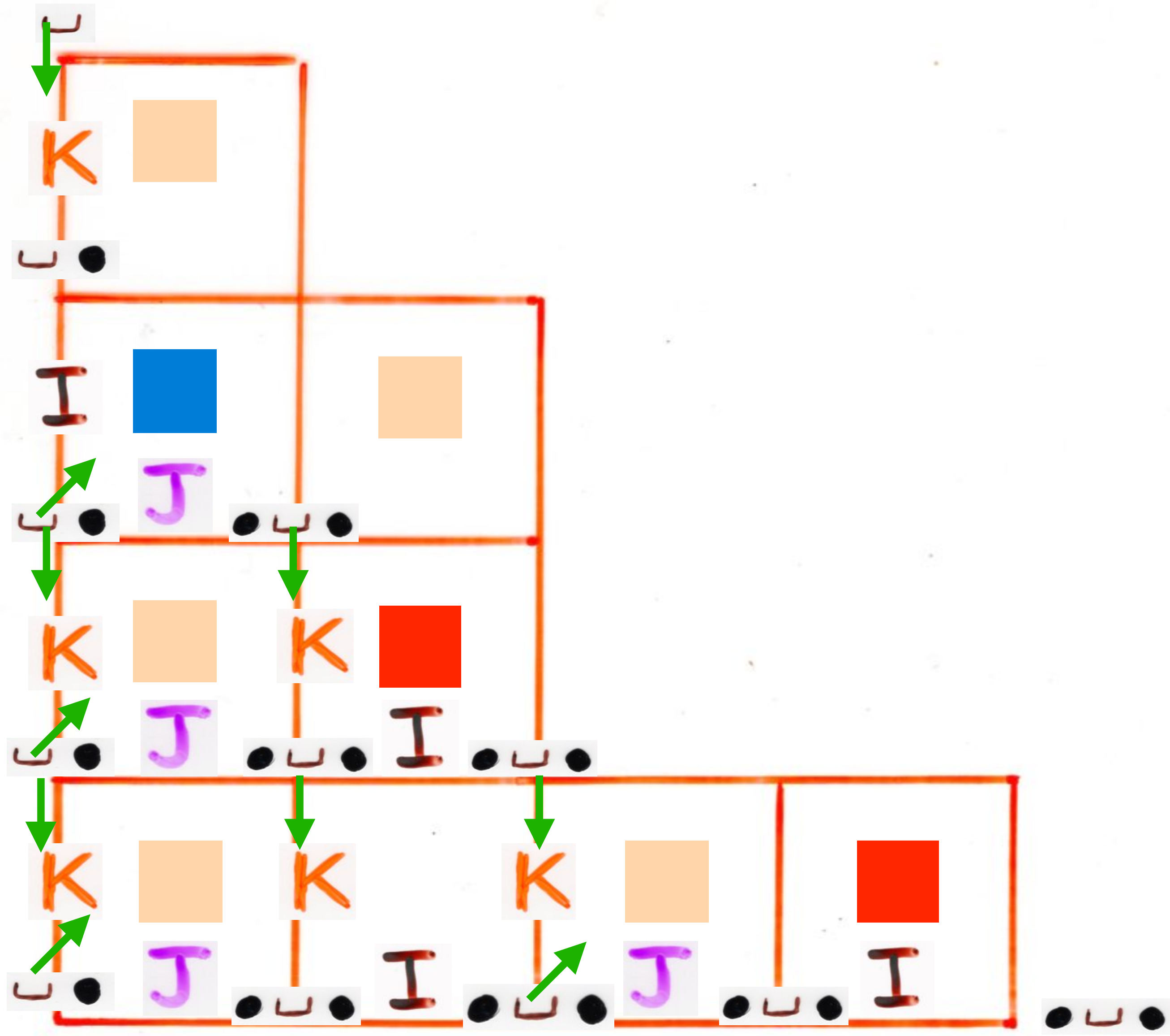

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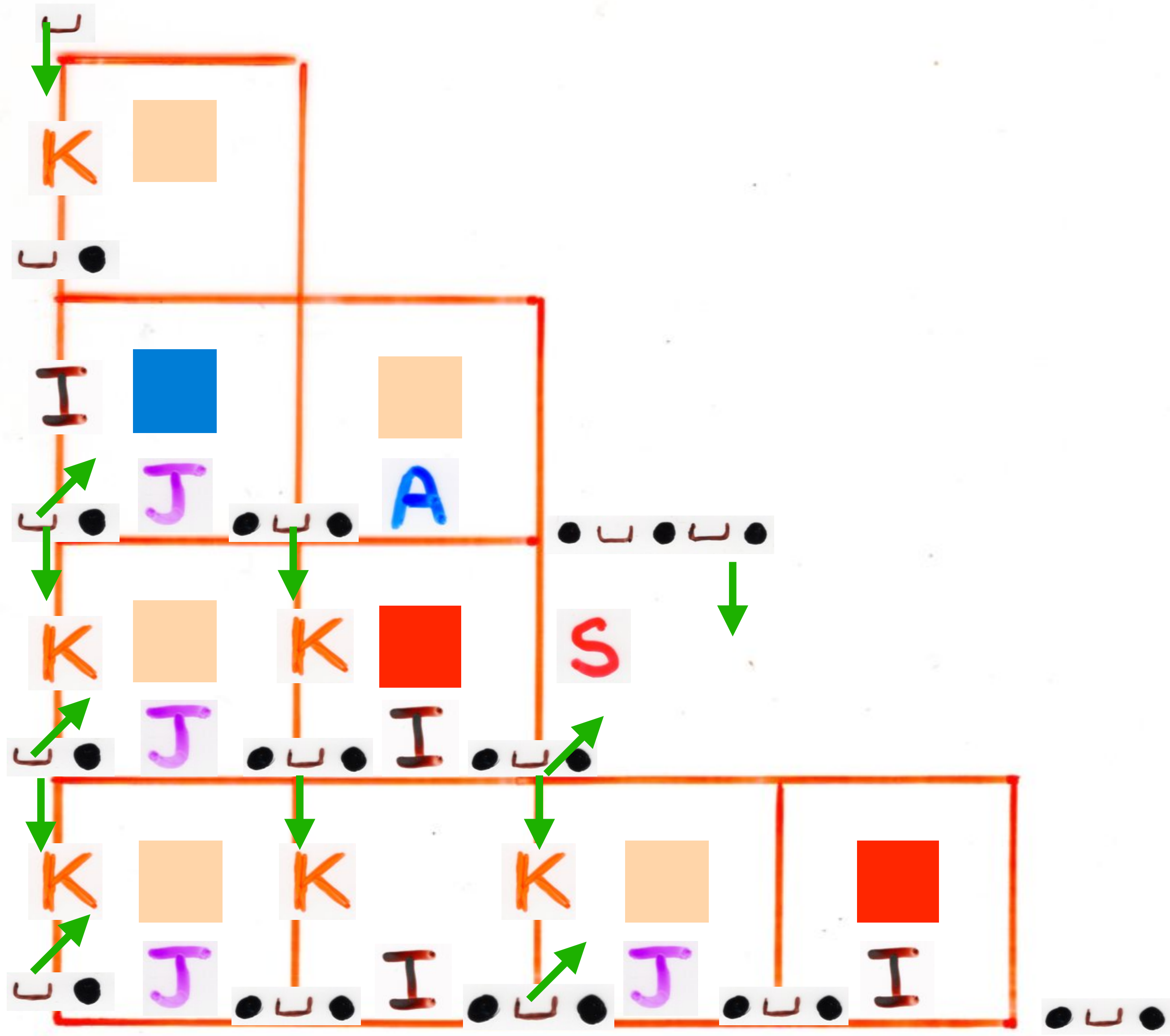
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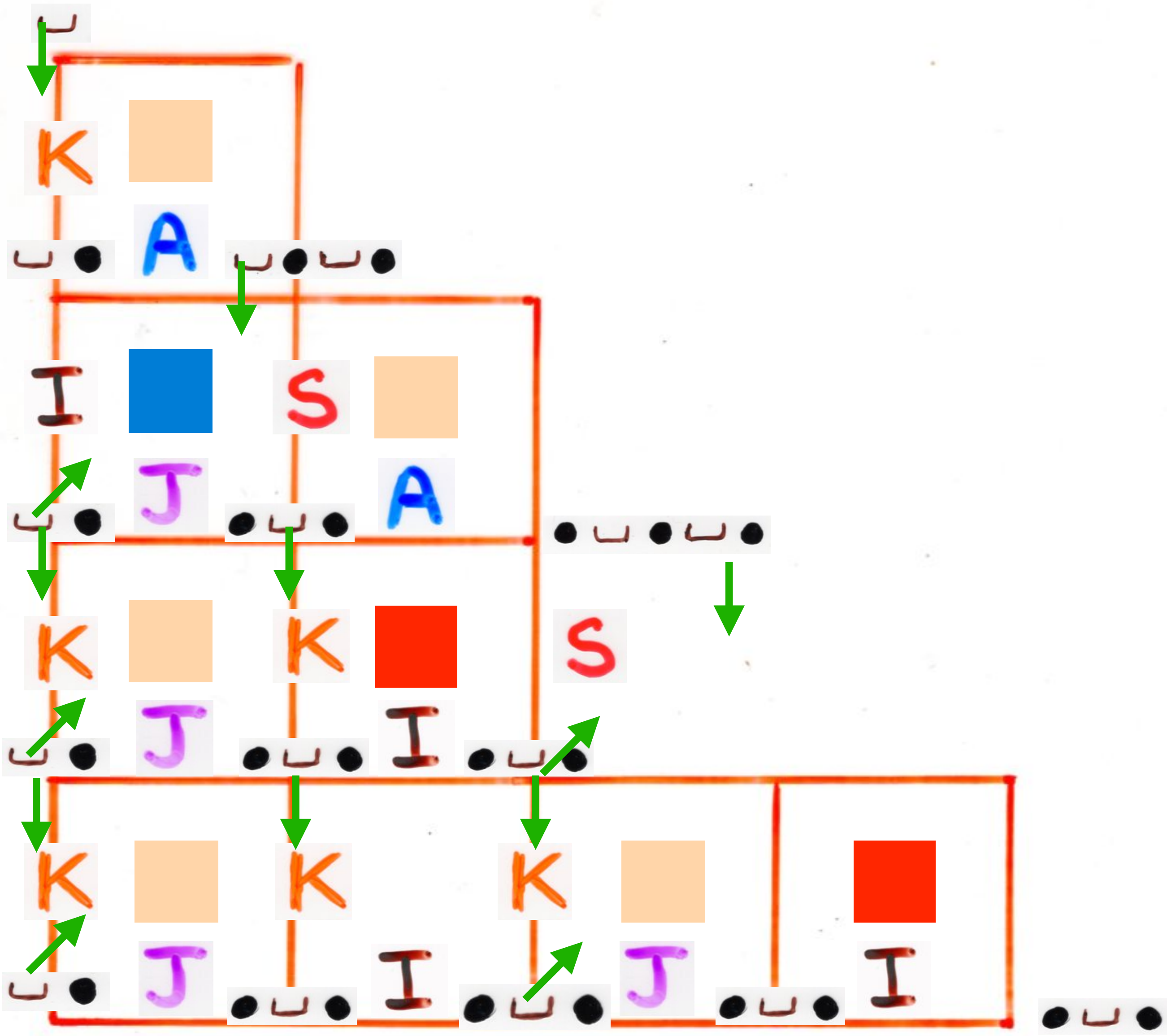

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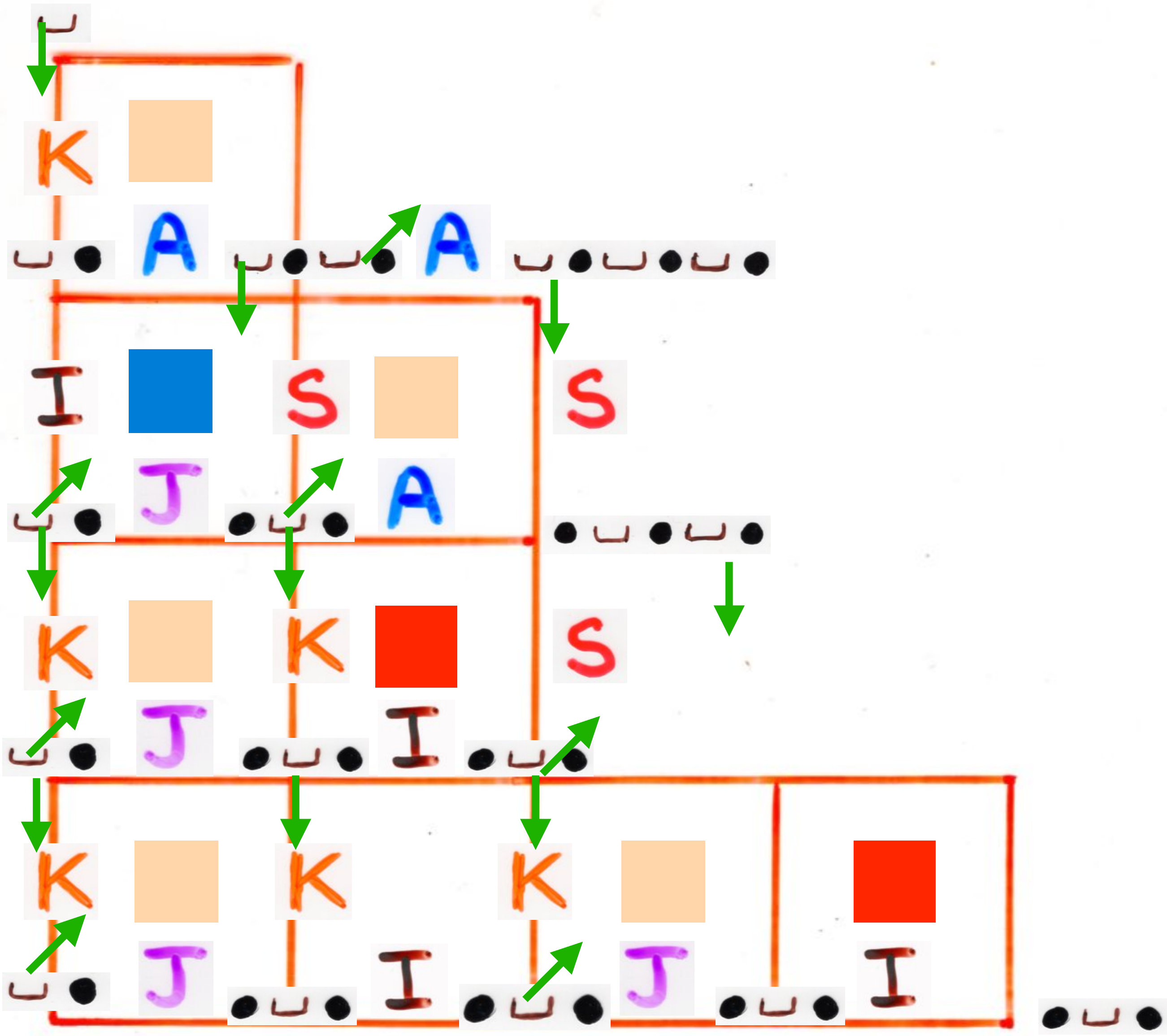

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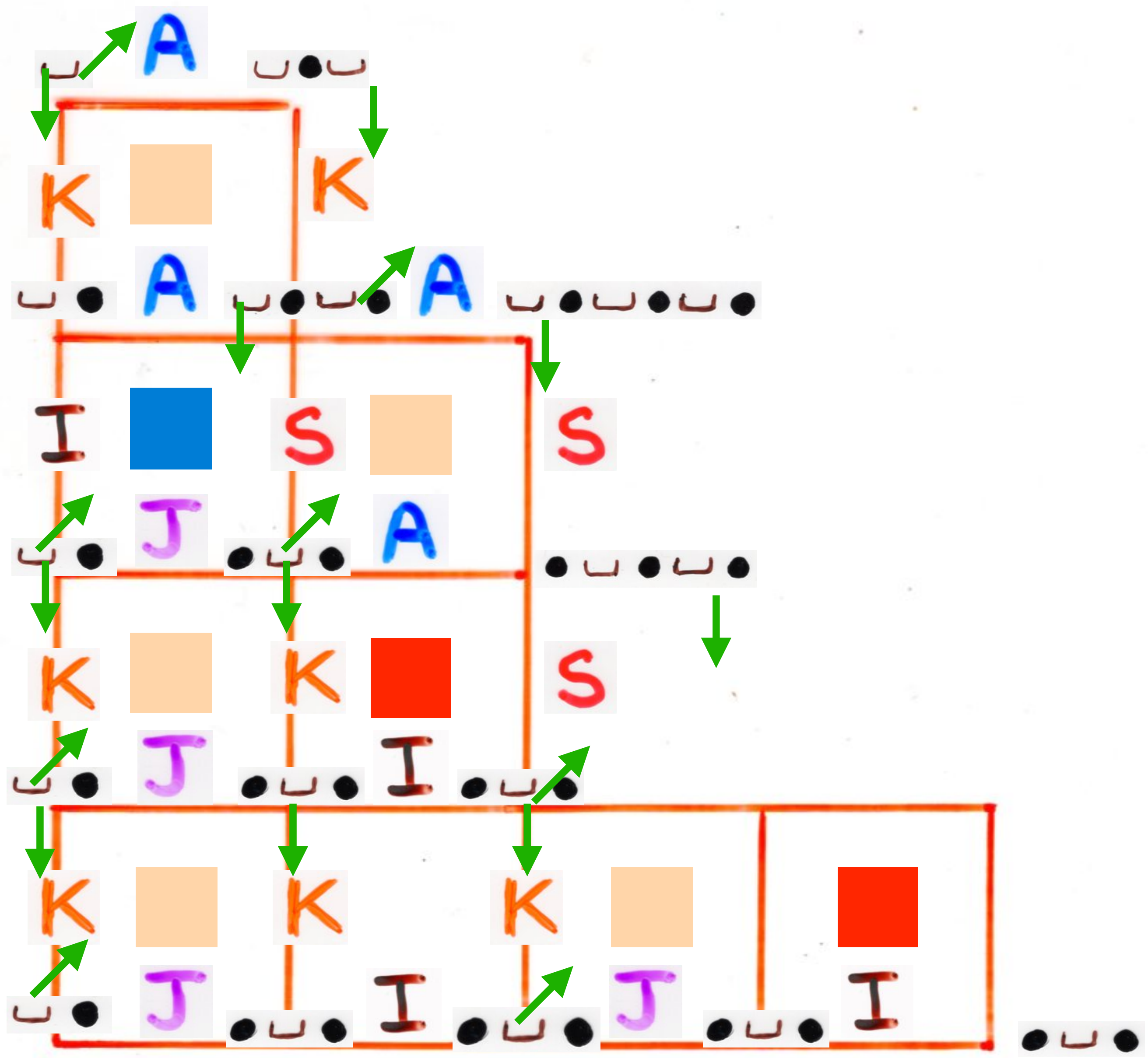

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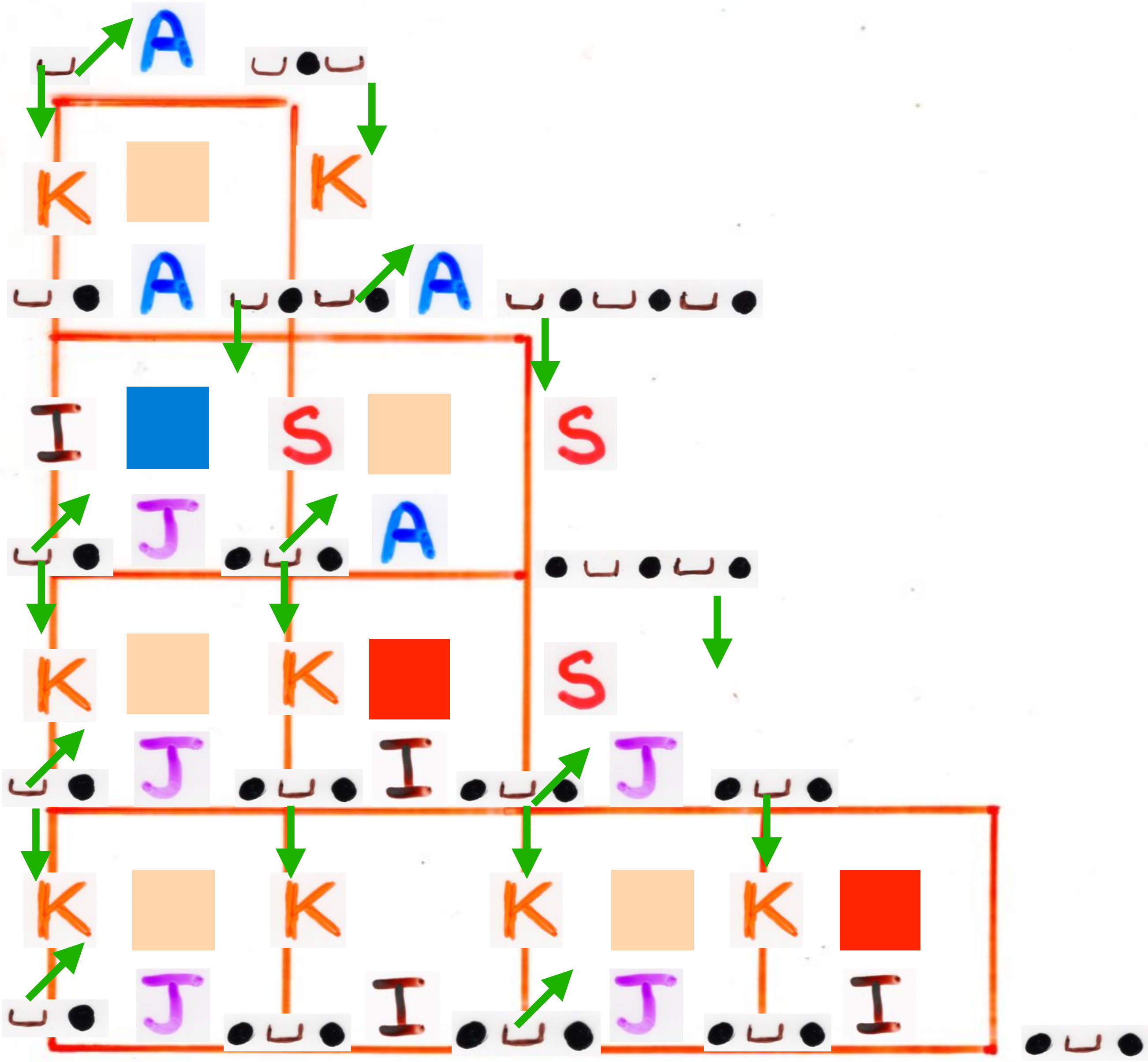

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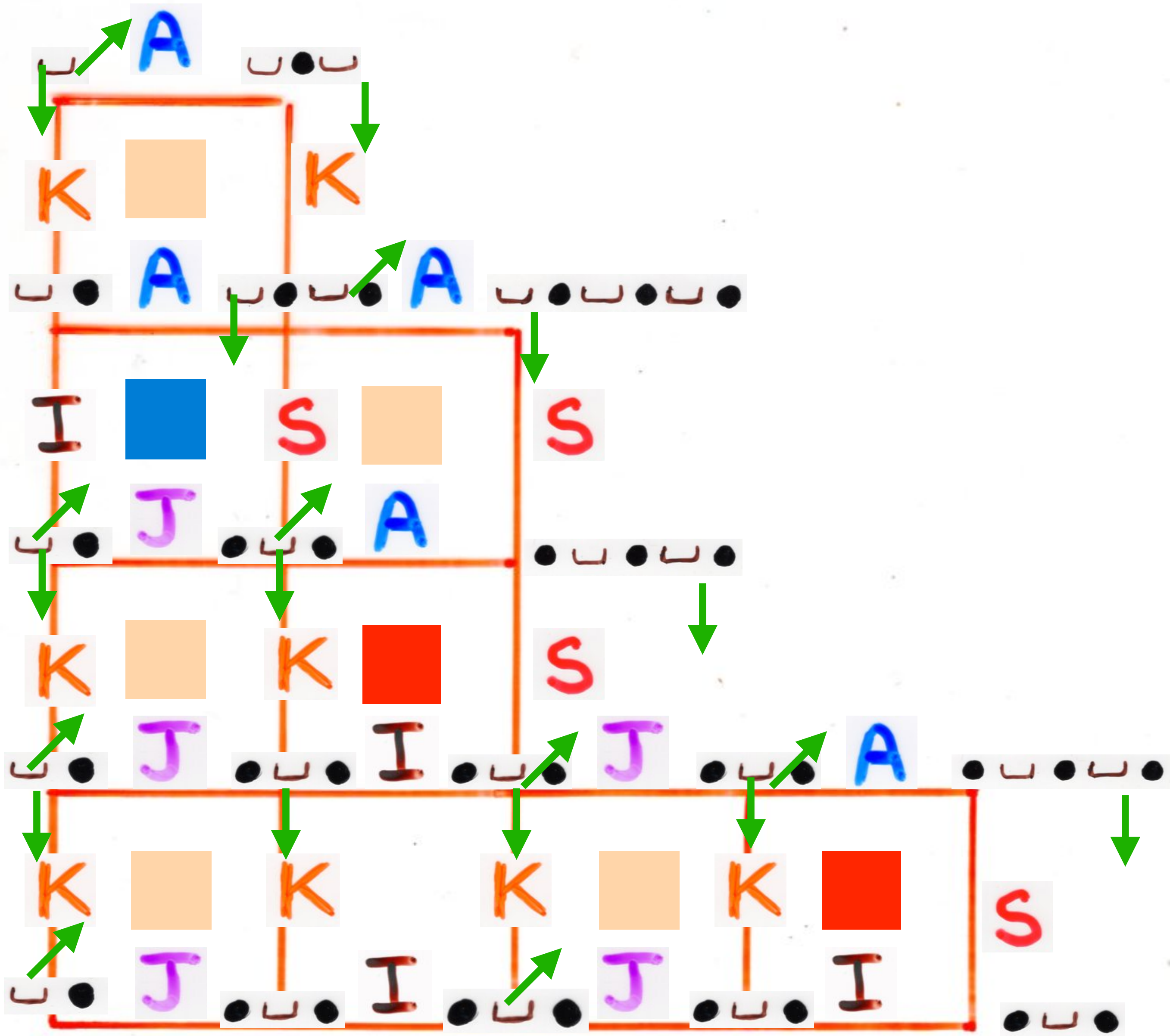
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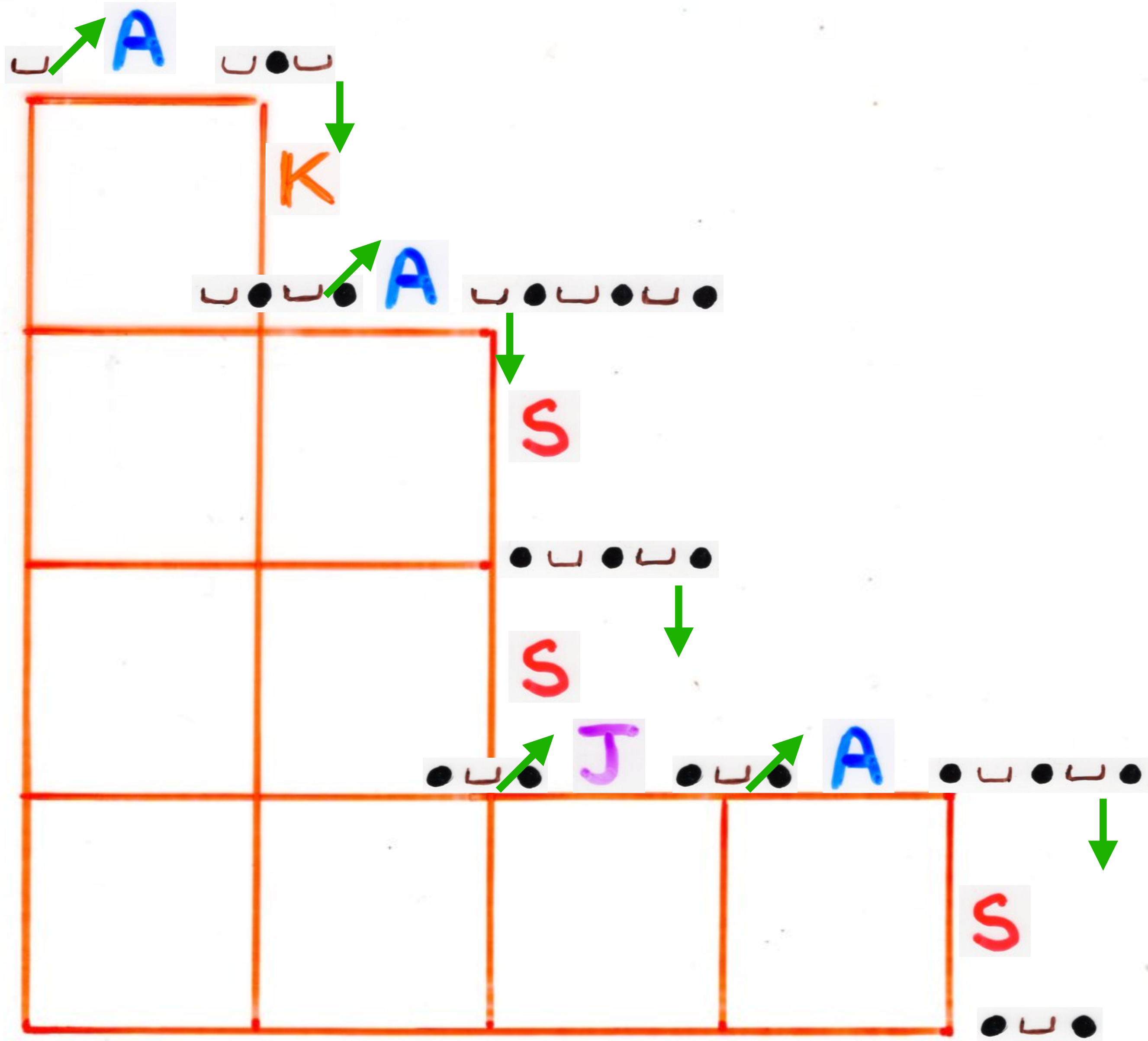
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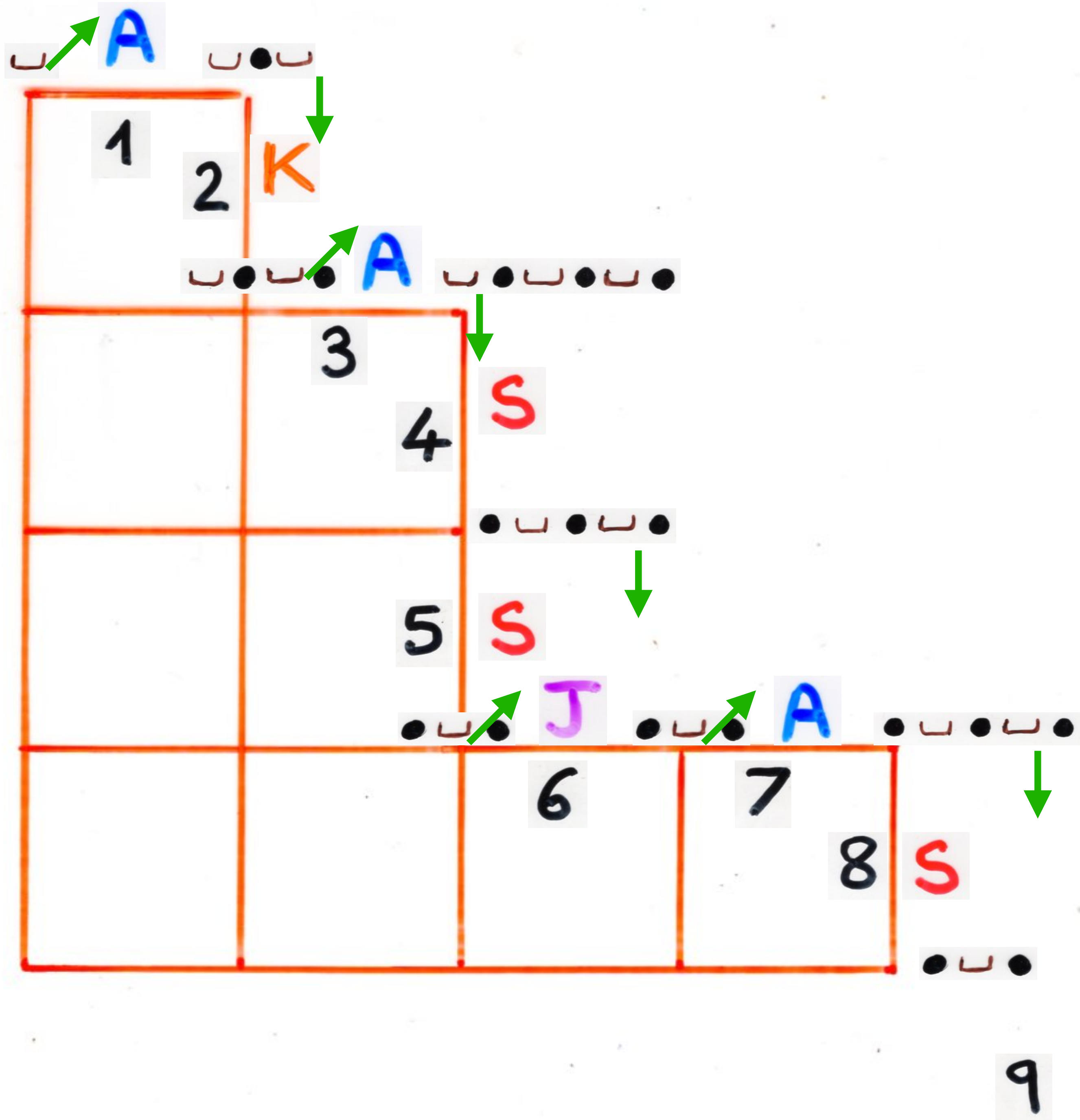
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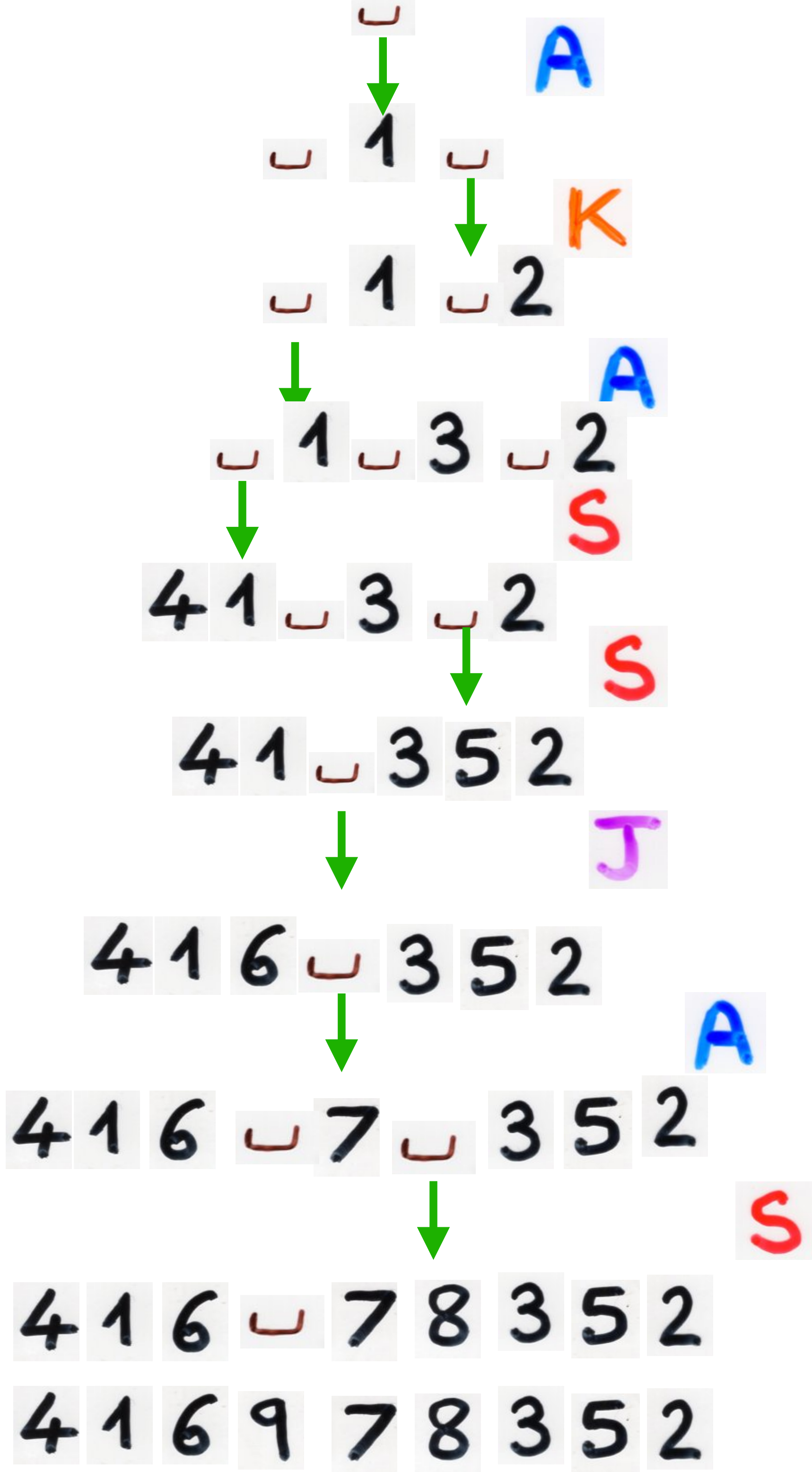
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4 1 6 7 8 3 5 2
4 1 6 9 7 8 3 5 2

two bijections
one theorem

Prop.

T alternative tableau $\rightarrow$ σ "exchange-fusion" inverse algorithm

T alternative tableau $\rightarrow$ τ "local" algorithm from $DE = ED + E + D$

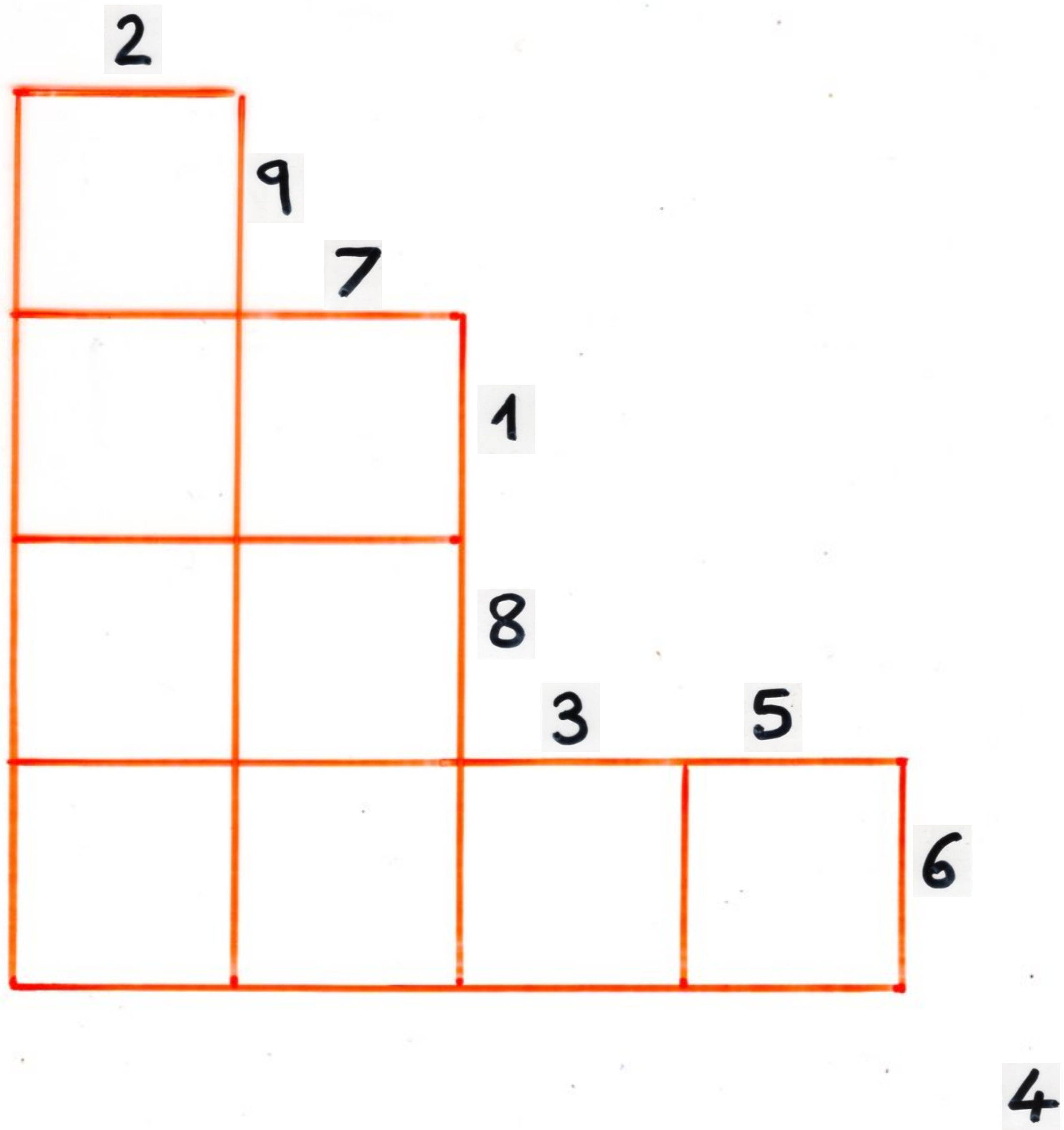
$$\sigma = \tau^{-1}$$

exchange-delelete algorithm

with the inverse permutation

4 1 6 9 7 8 3 5 2 → 2 9 7 1 8 3 5 6 4

(with a variant: keep the min instead of the max)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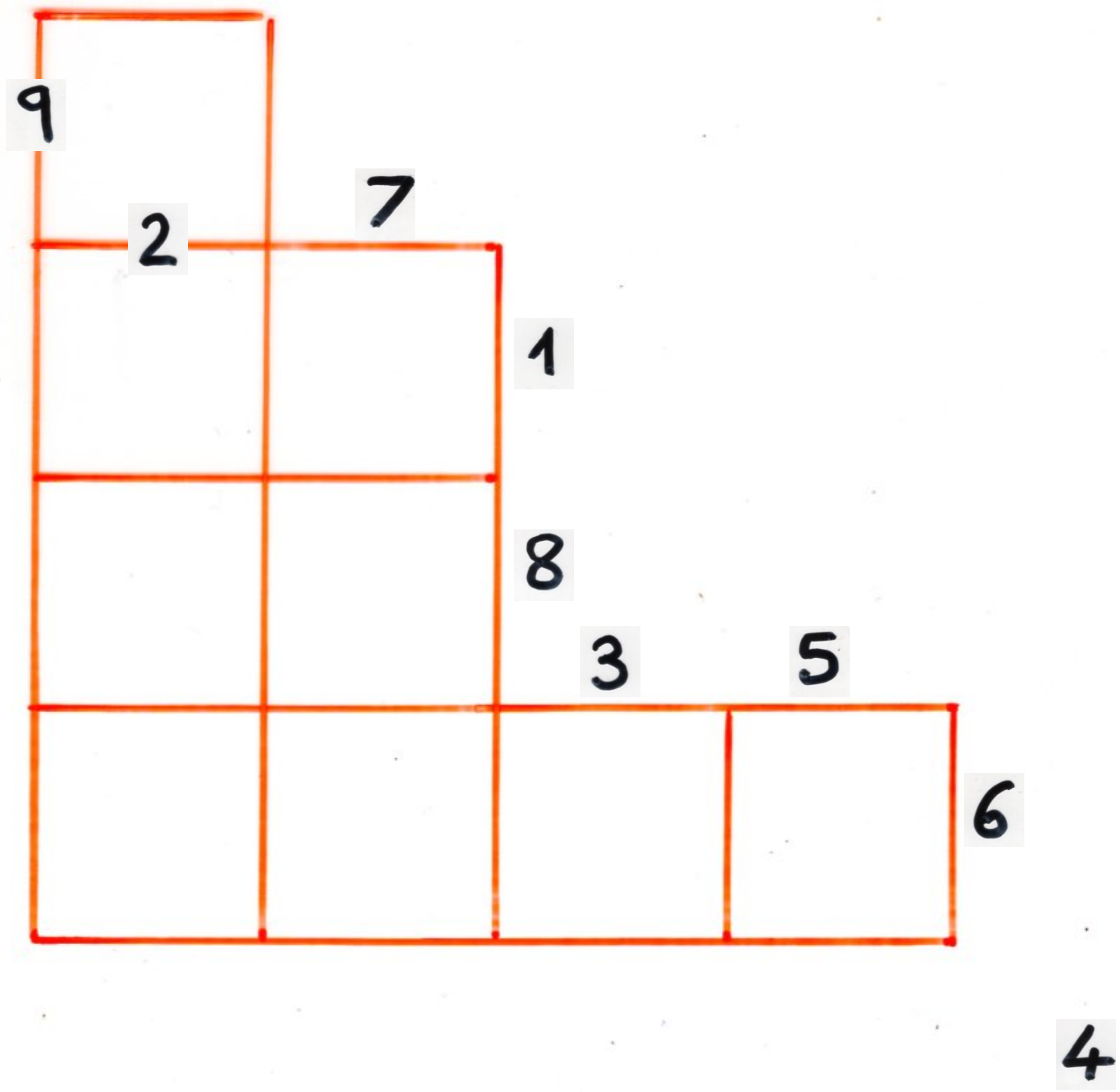
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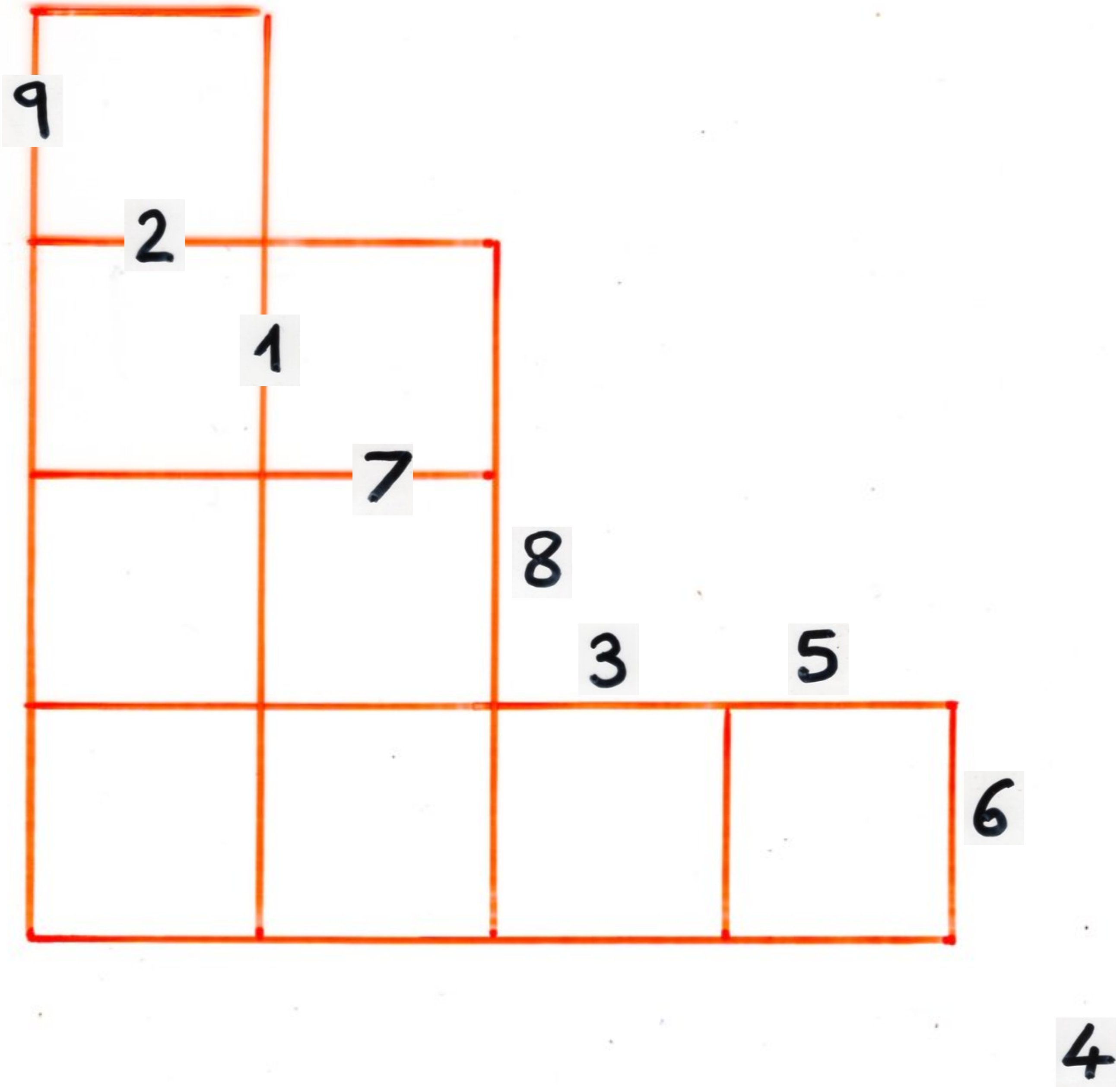
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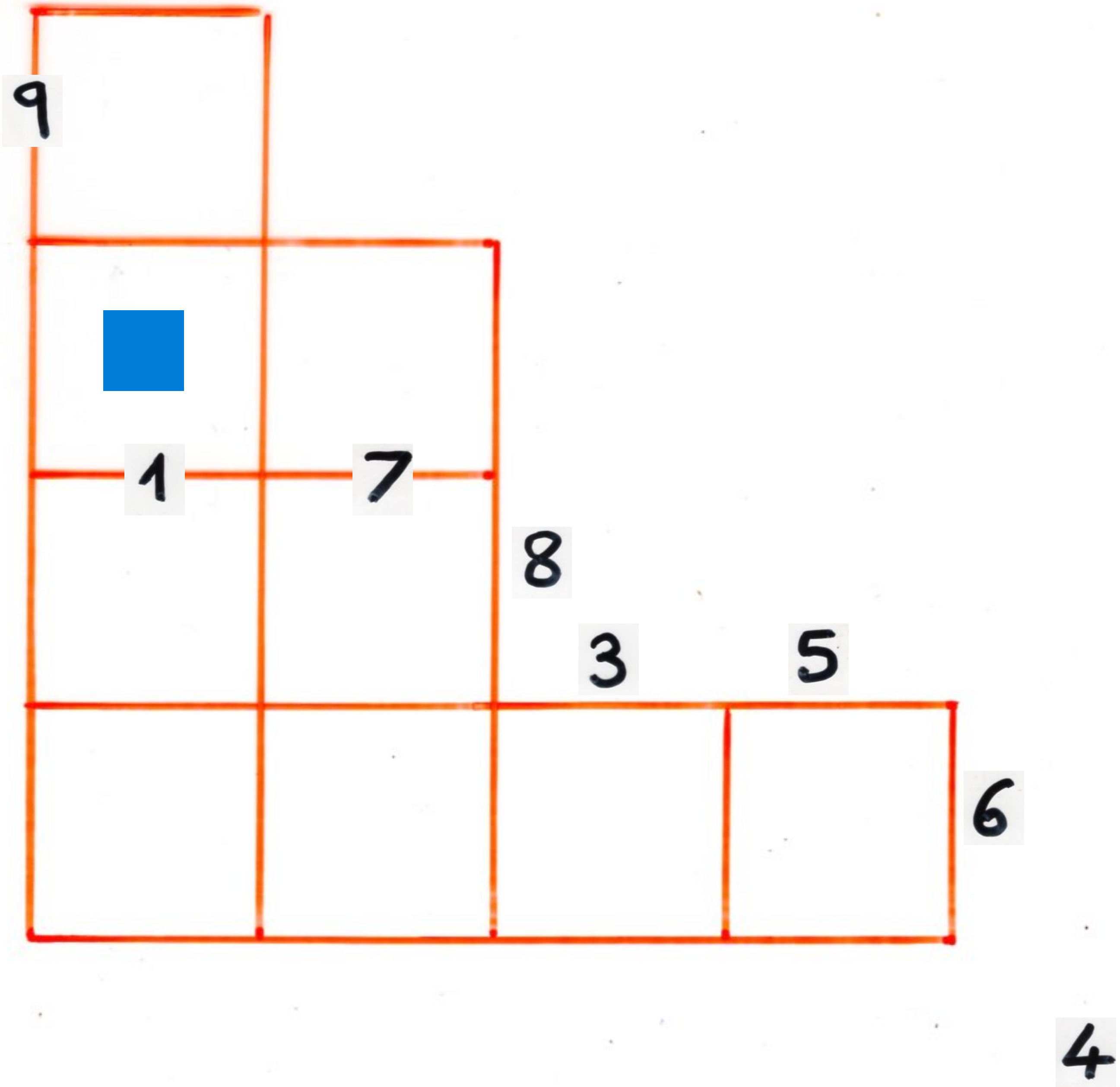
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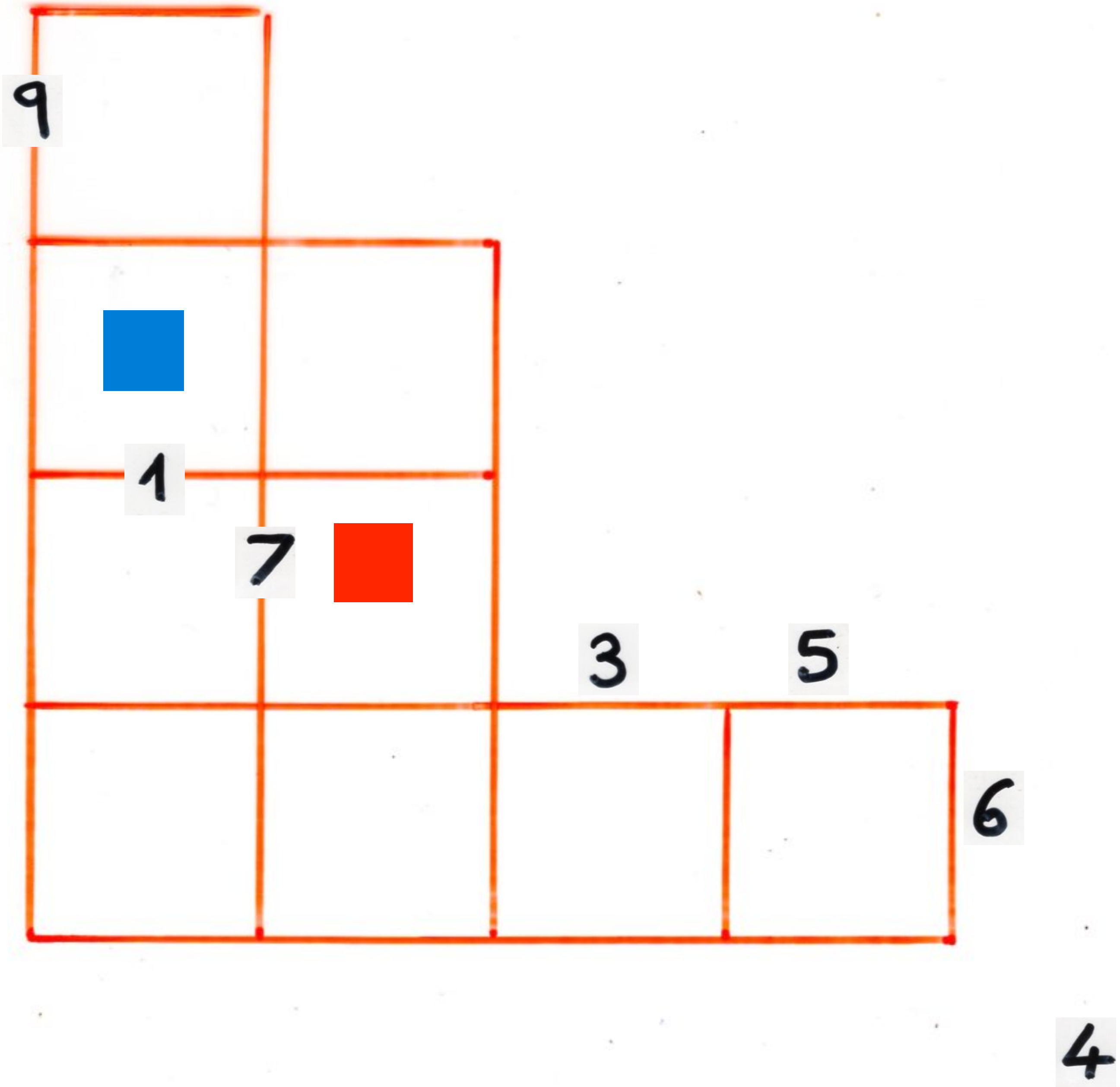
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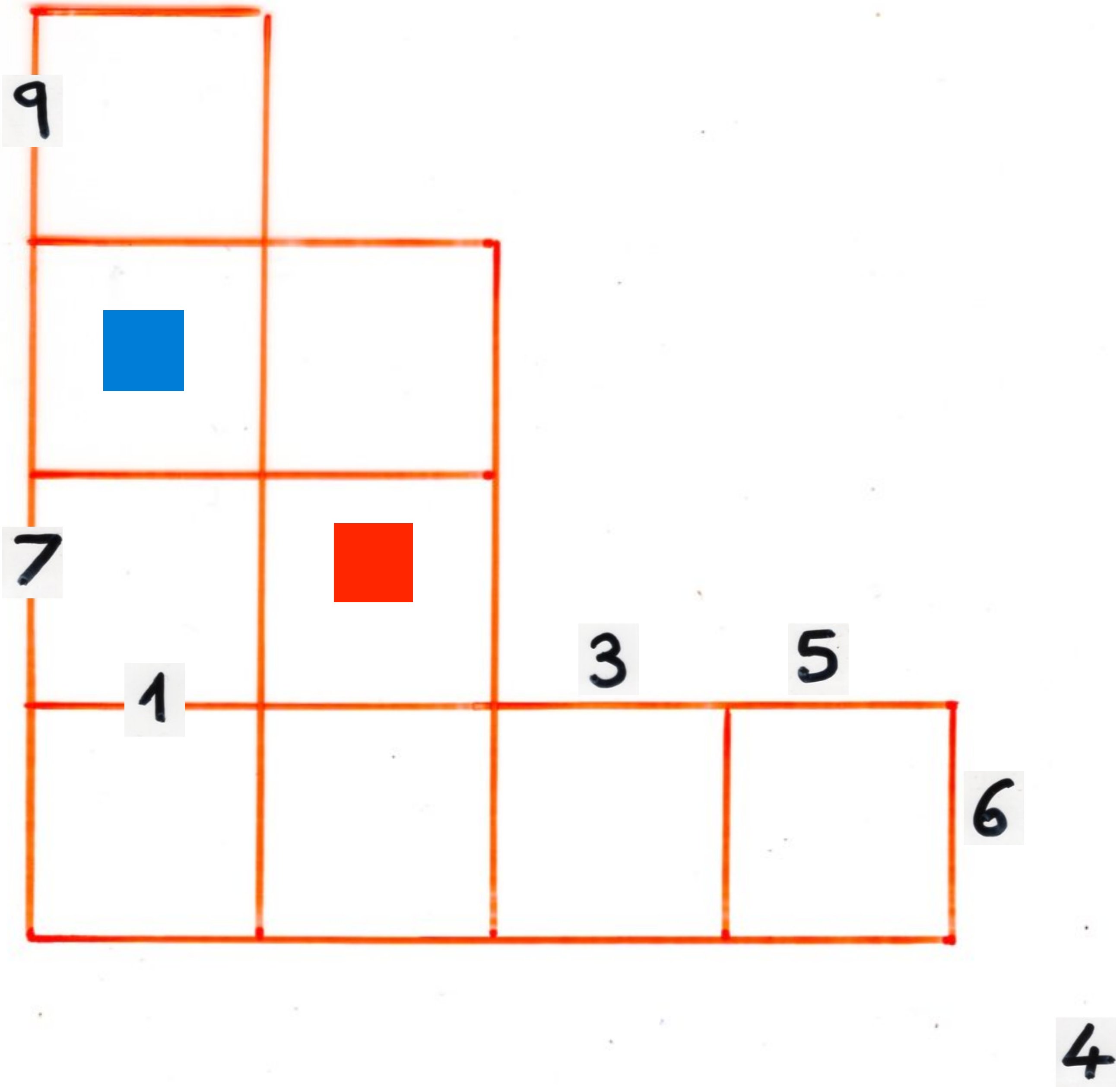
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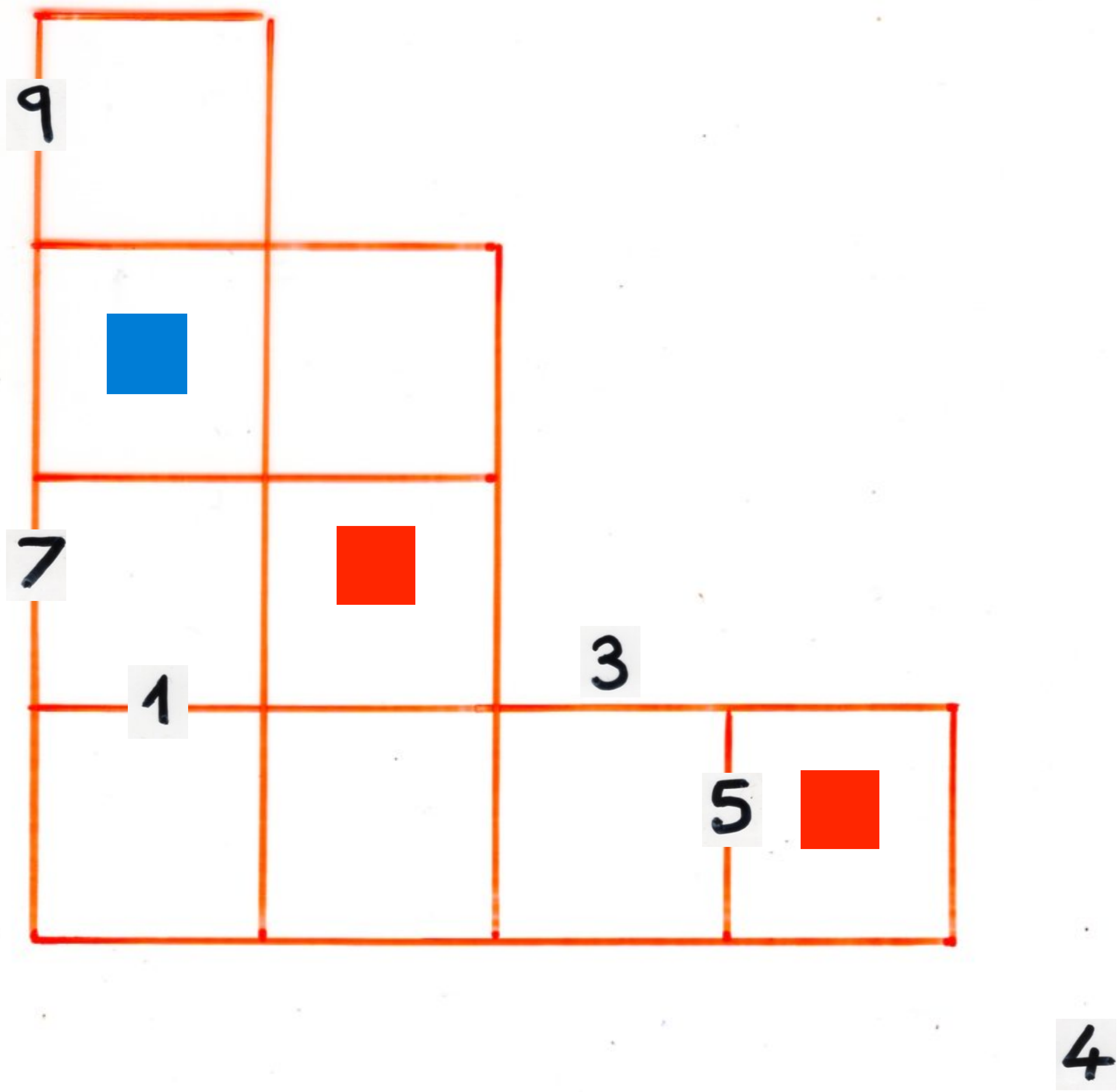

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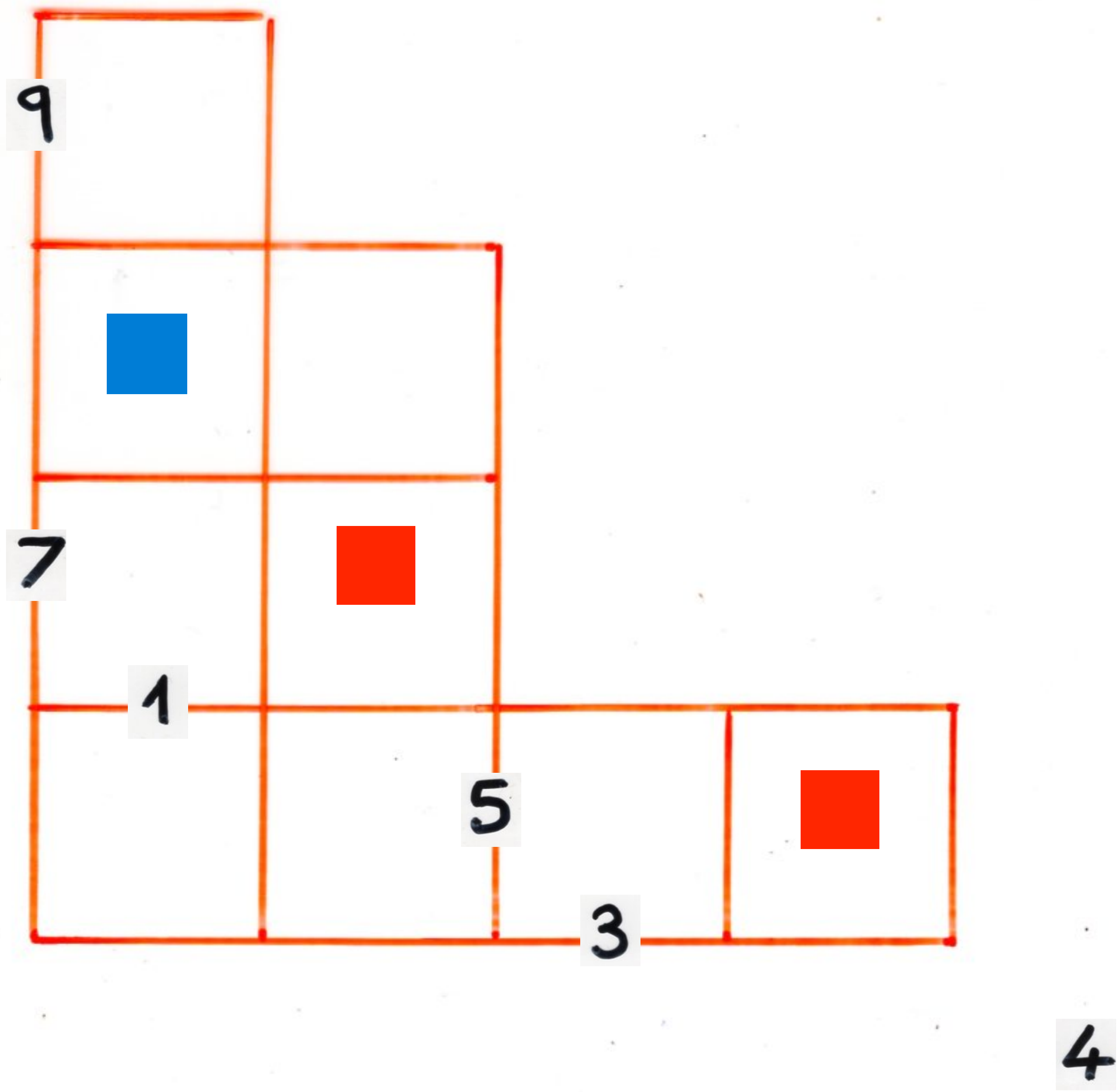
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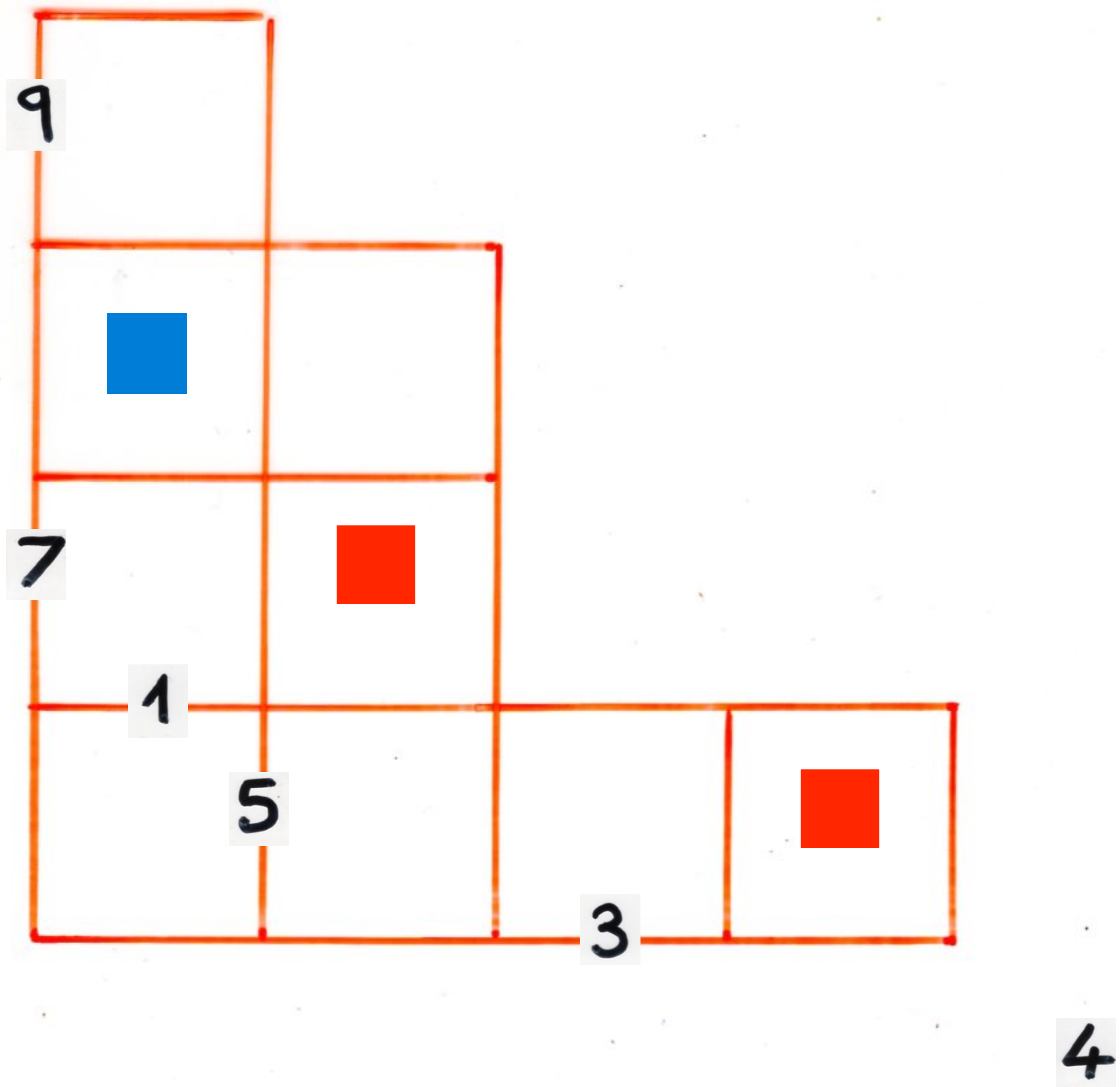
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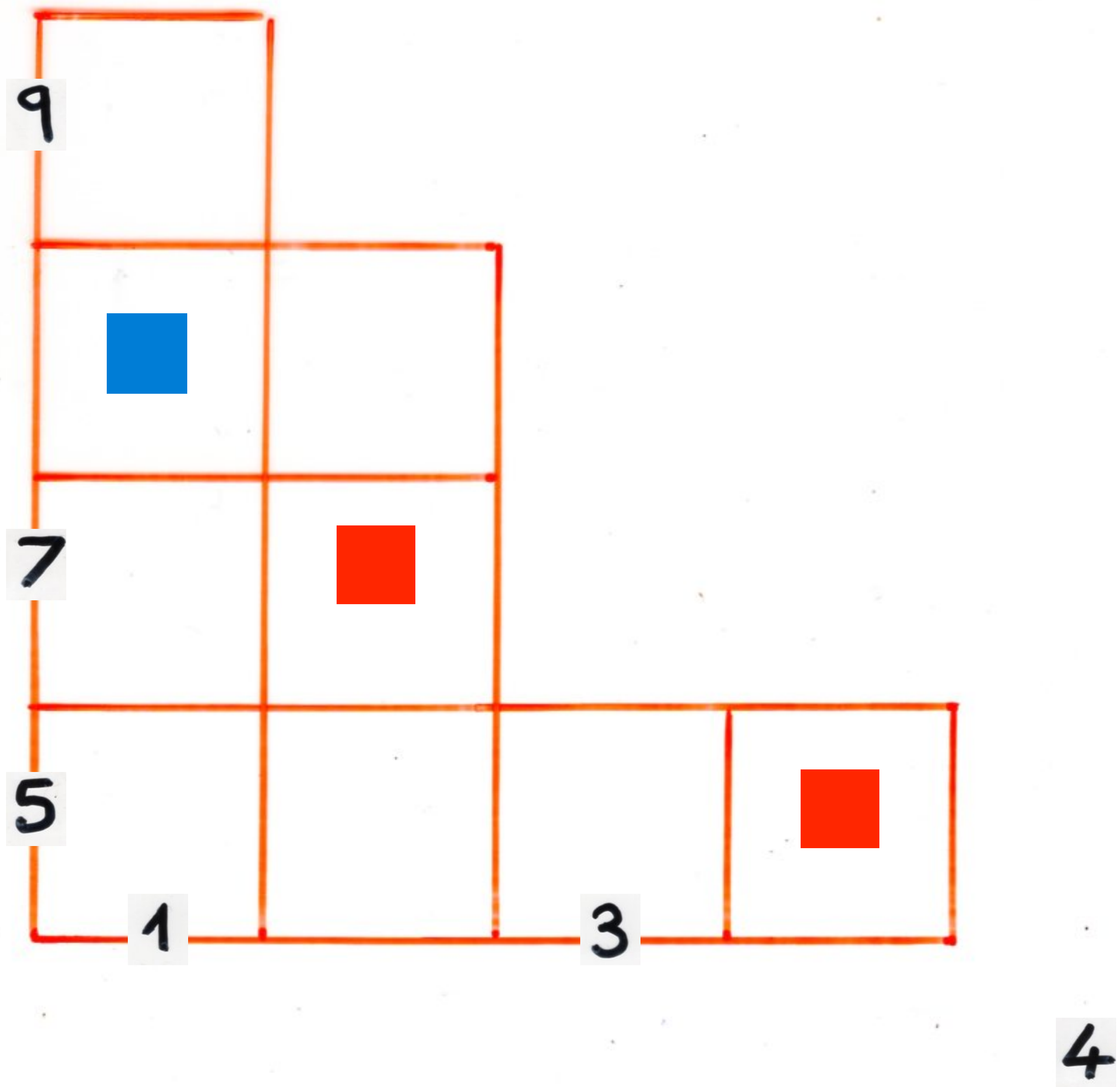
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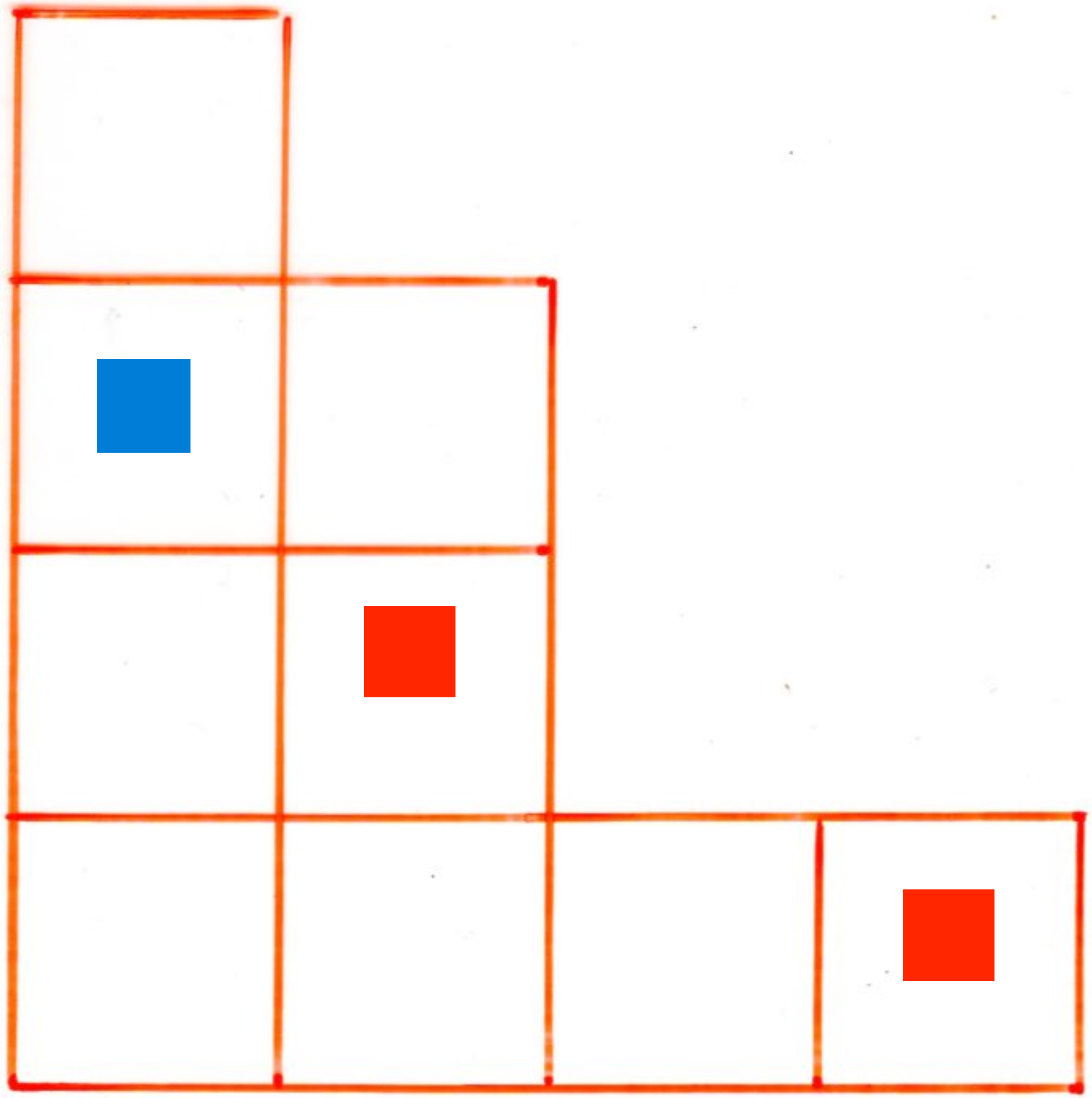
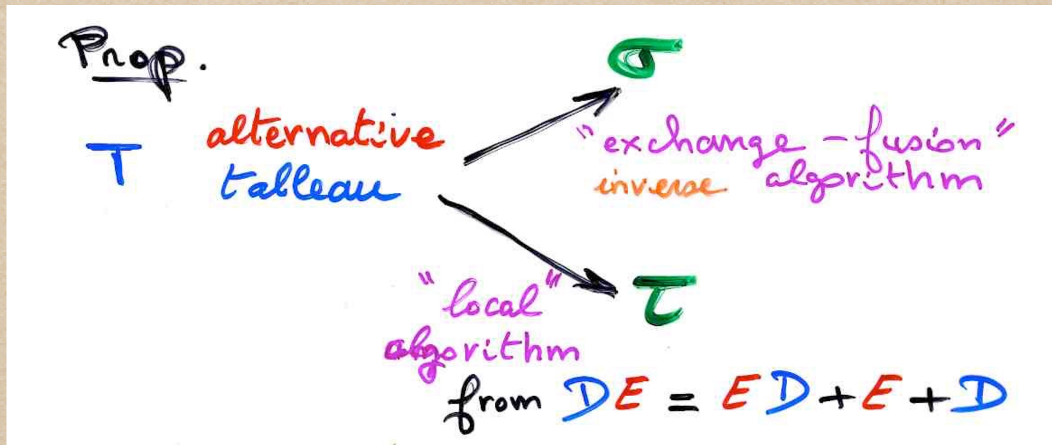


proof of the main theorem



Proof of the equivalence

local rules
(commutation diagrams)
and Laguerre histories

exchange-fusion
(or exchange-delete)
algorithm

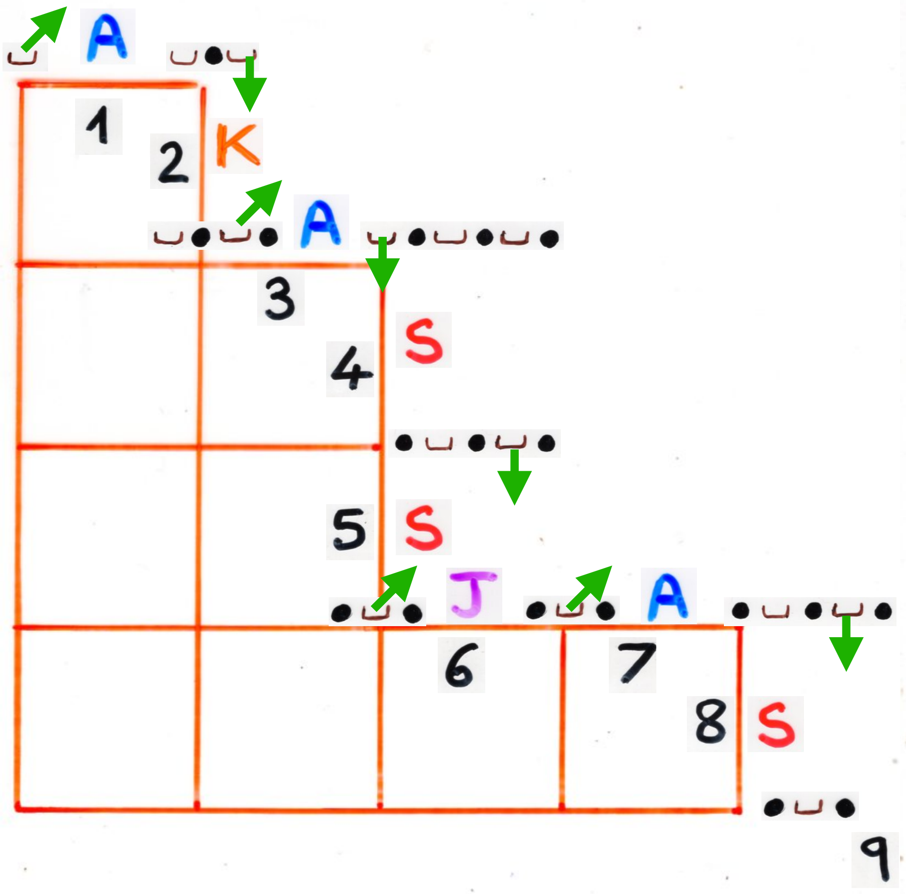
4 1 6 9 7 8 3 5 2

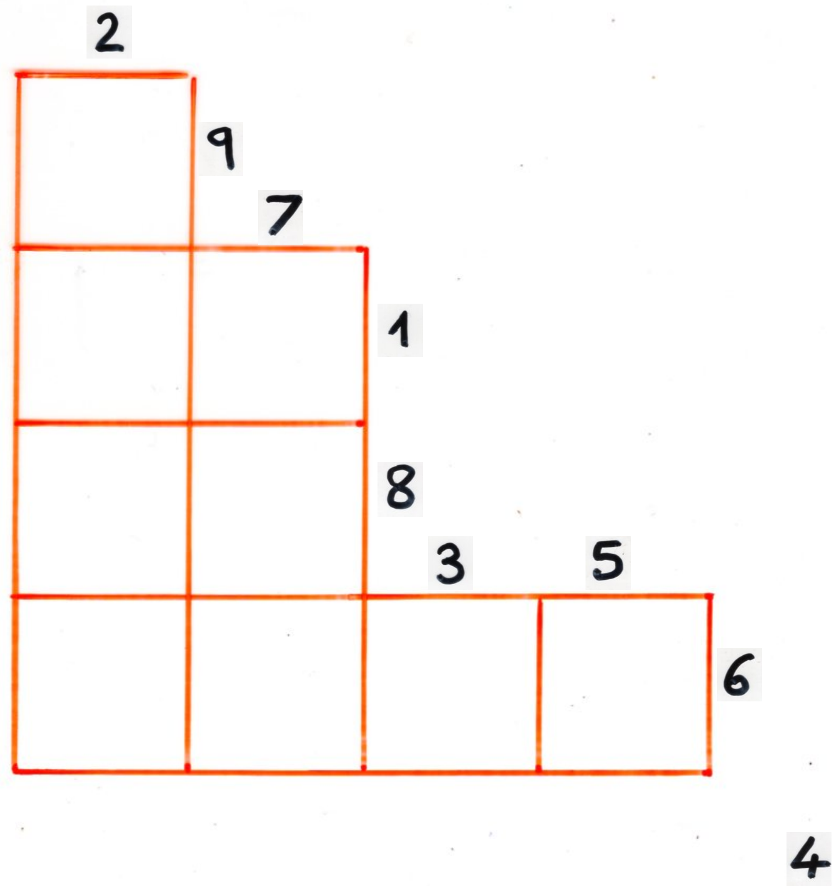
2 9 7 1 8 3 5 6 4

$$\sigma = \tau^{-1}$$

4 1 6 9 7 8 3 5 2

2 9 7 1 8 3 5 6 4



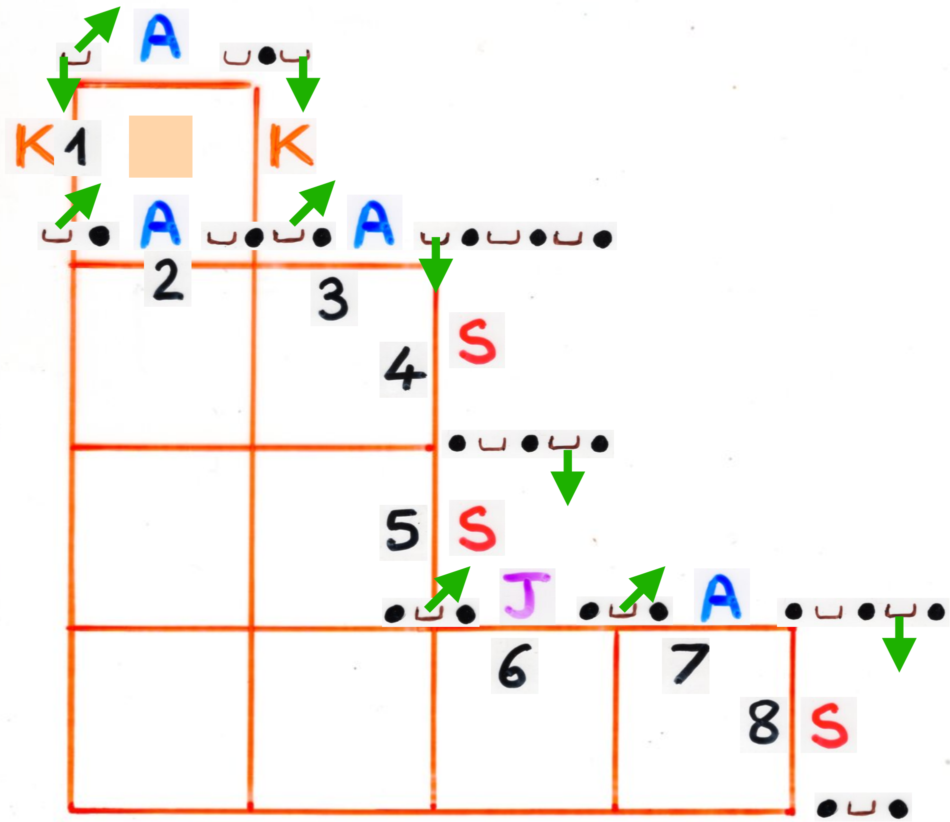


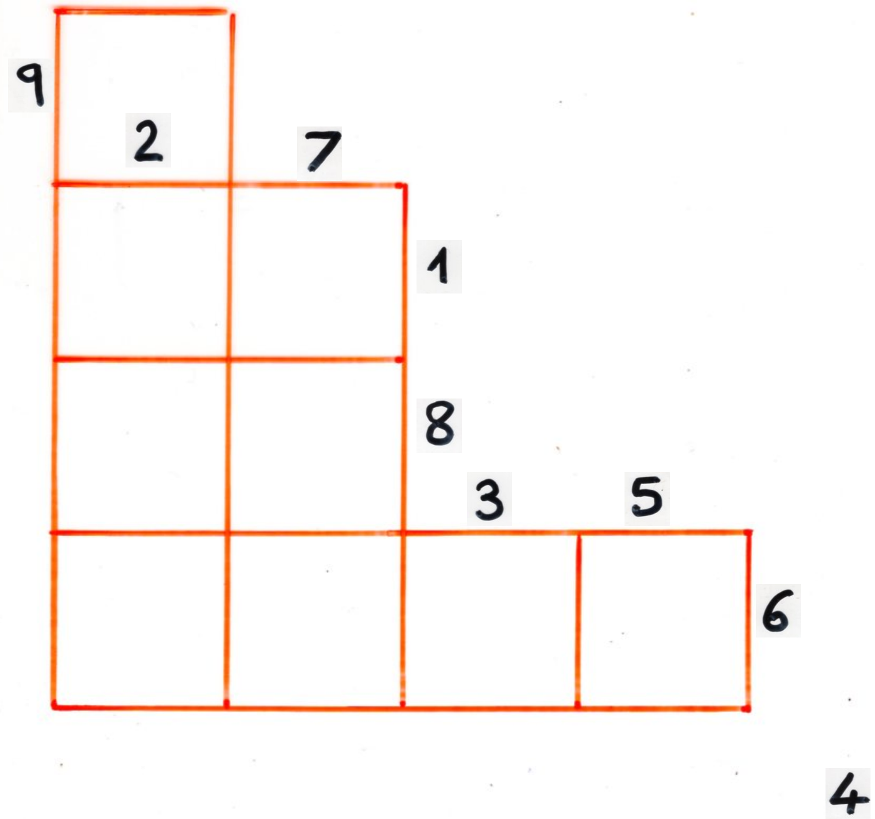
4 1 6 9 7 8 3 5 2

4 2 6 9 7 8 3 5 1

2 9 7 1 8 3 5 6 4

9 2 7 1 8 3 5 6 4





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3 2 6 9 7 8 4 5 1

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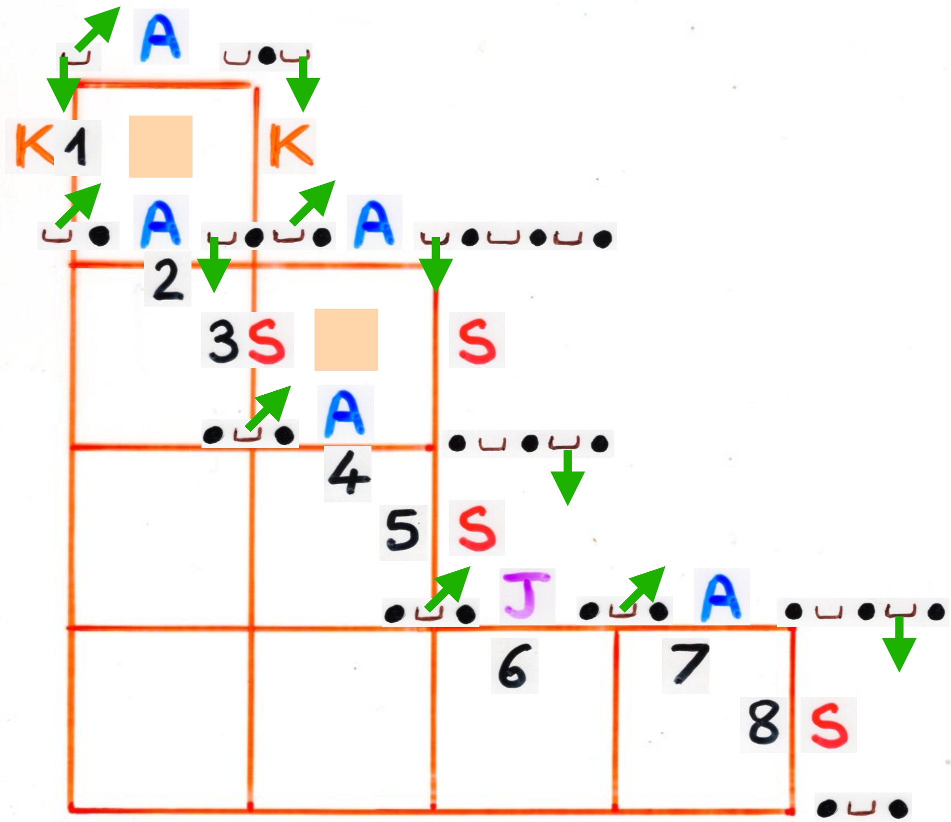

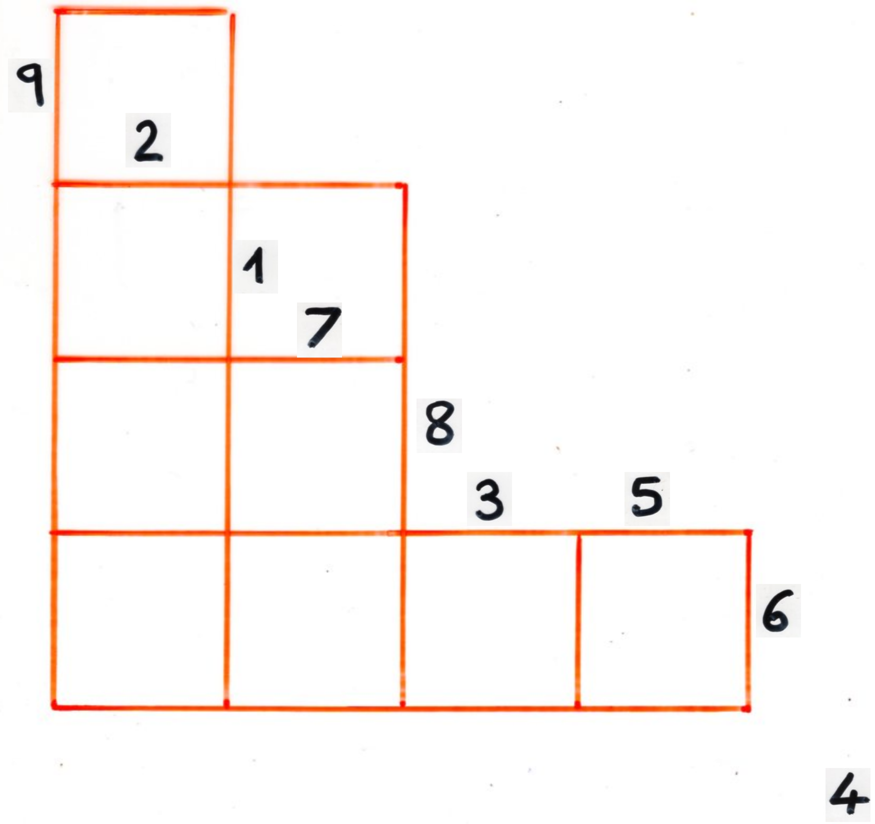

3 2 6 9 7 8 4 5 1

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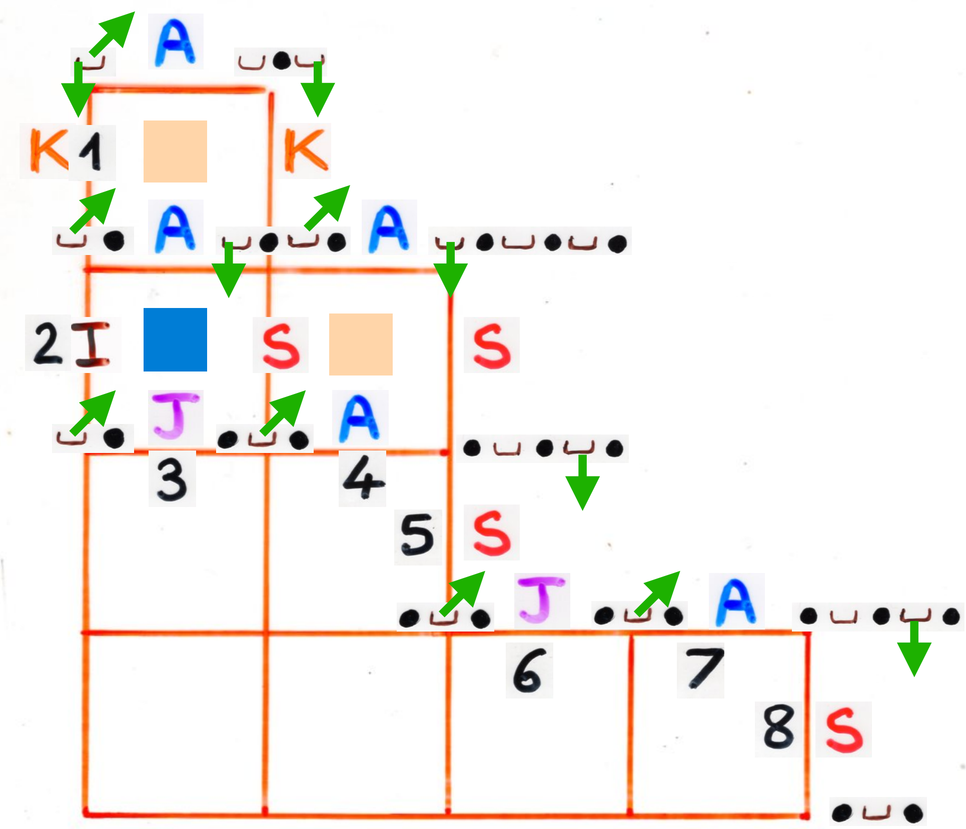
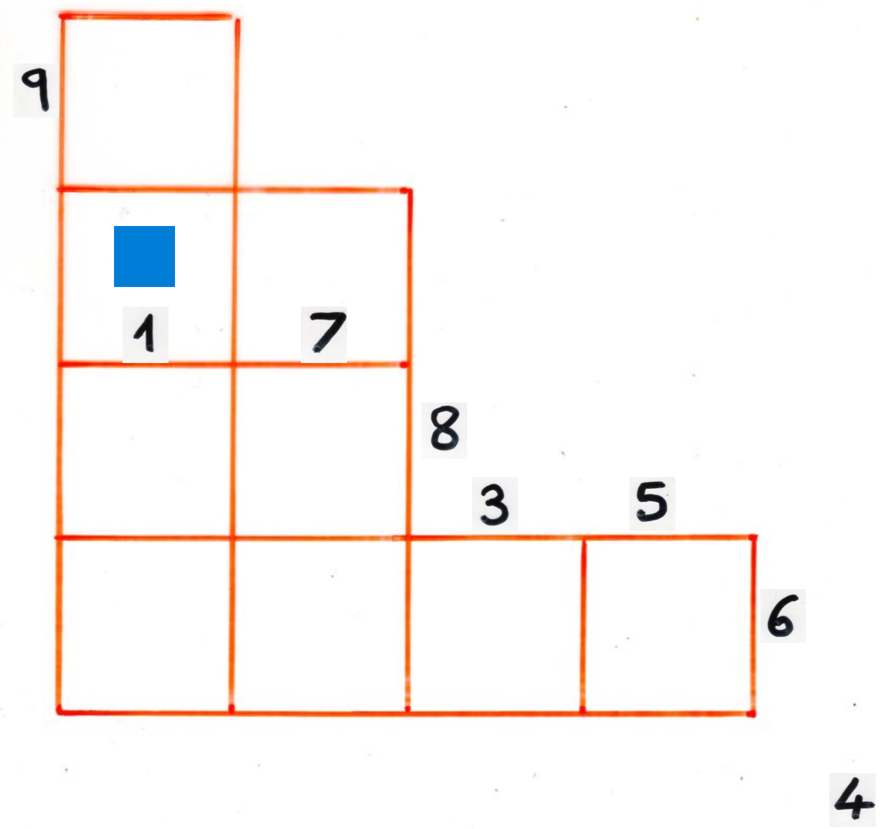
9   1 7 8 3 5 6 4





3		6	9	7	8	4	5	1
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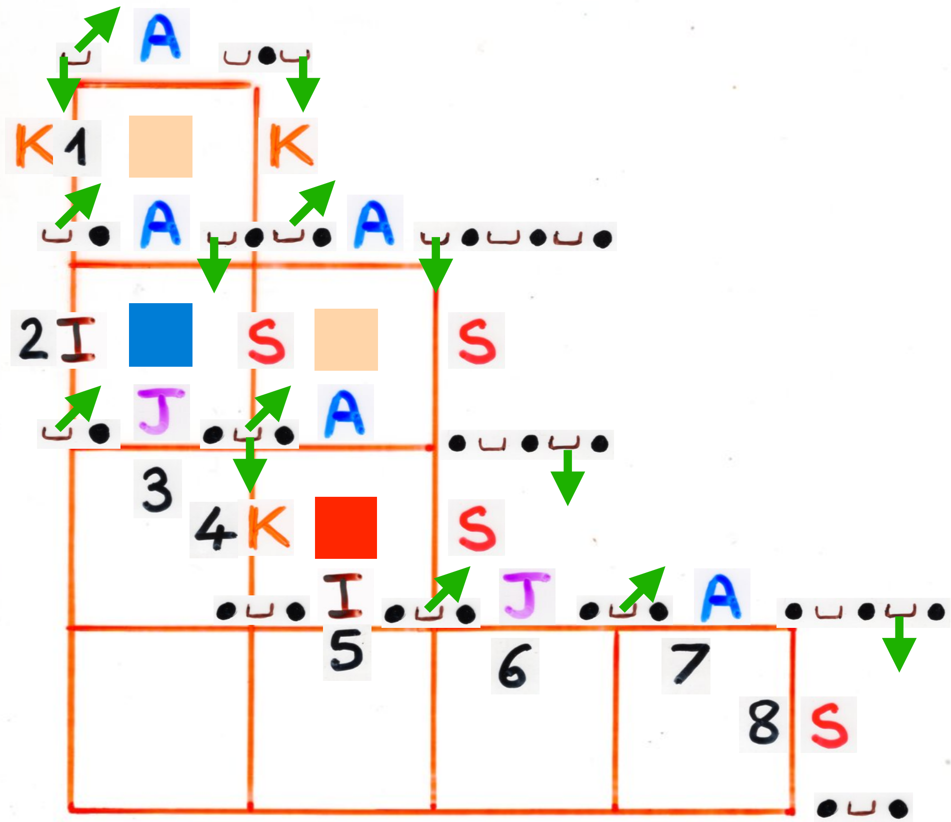

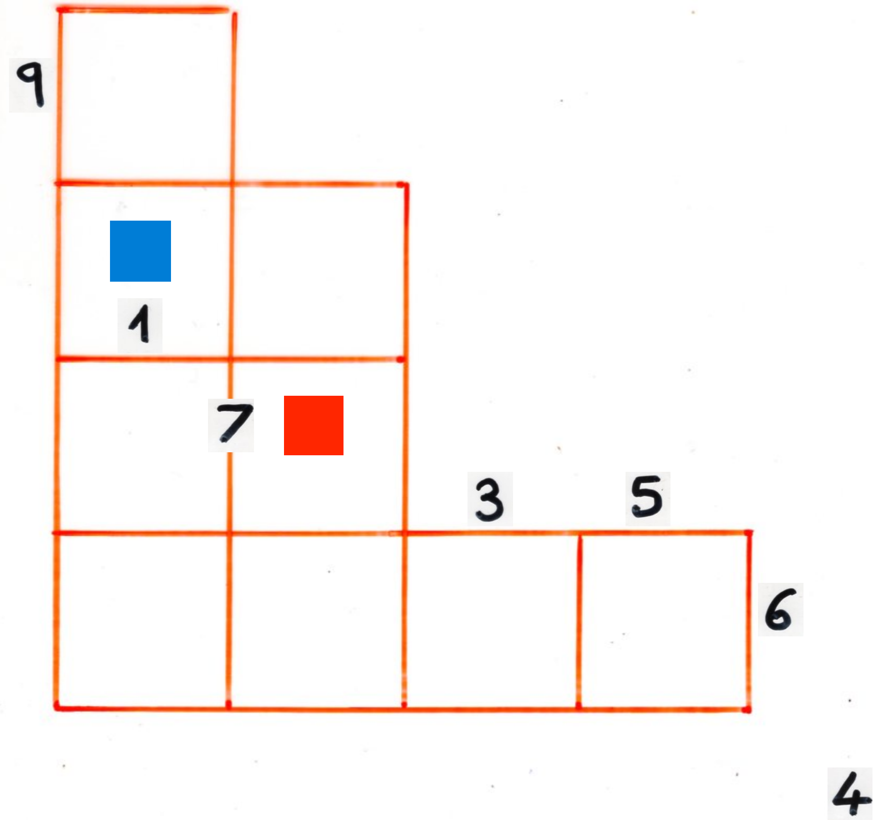

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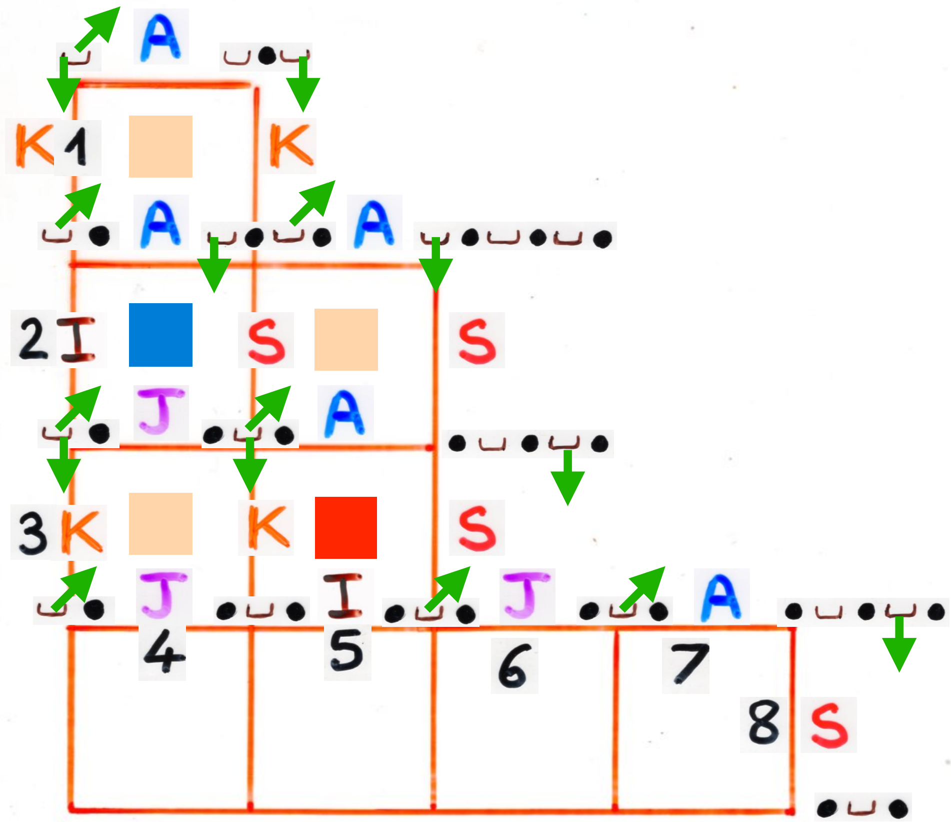

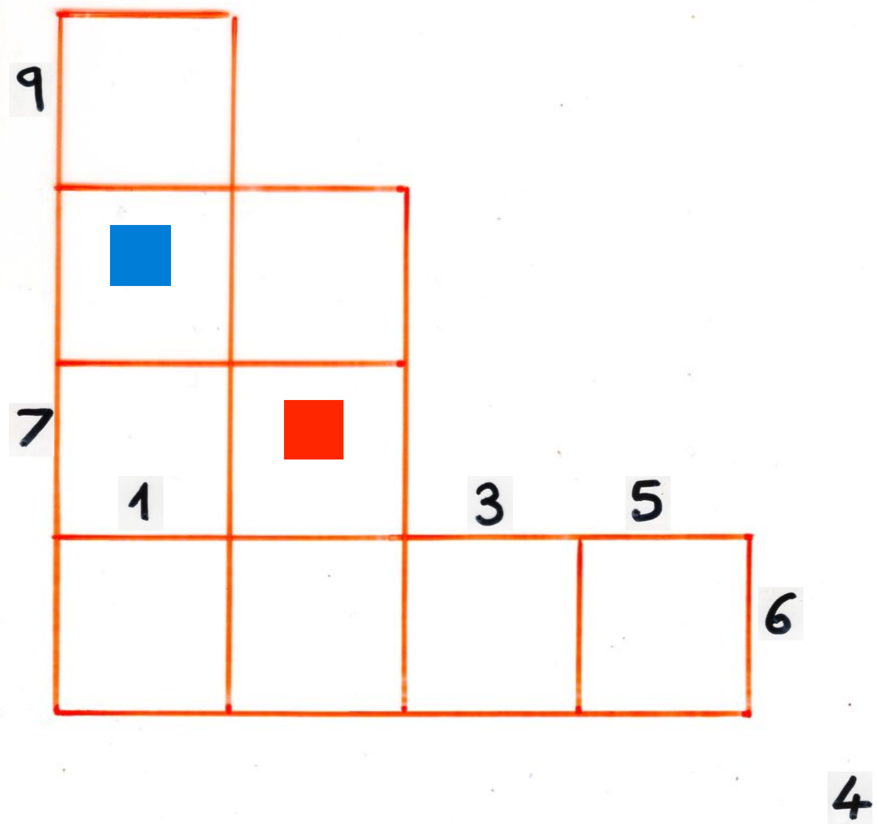

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4 6 9 7 3 1

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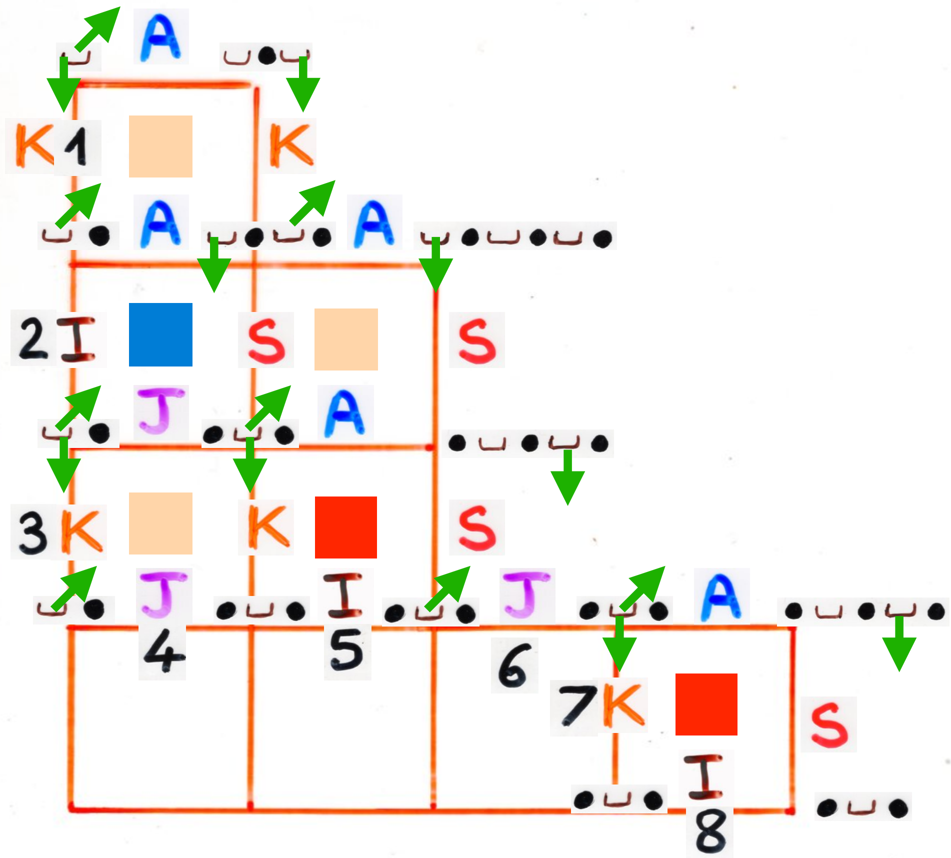

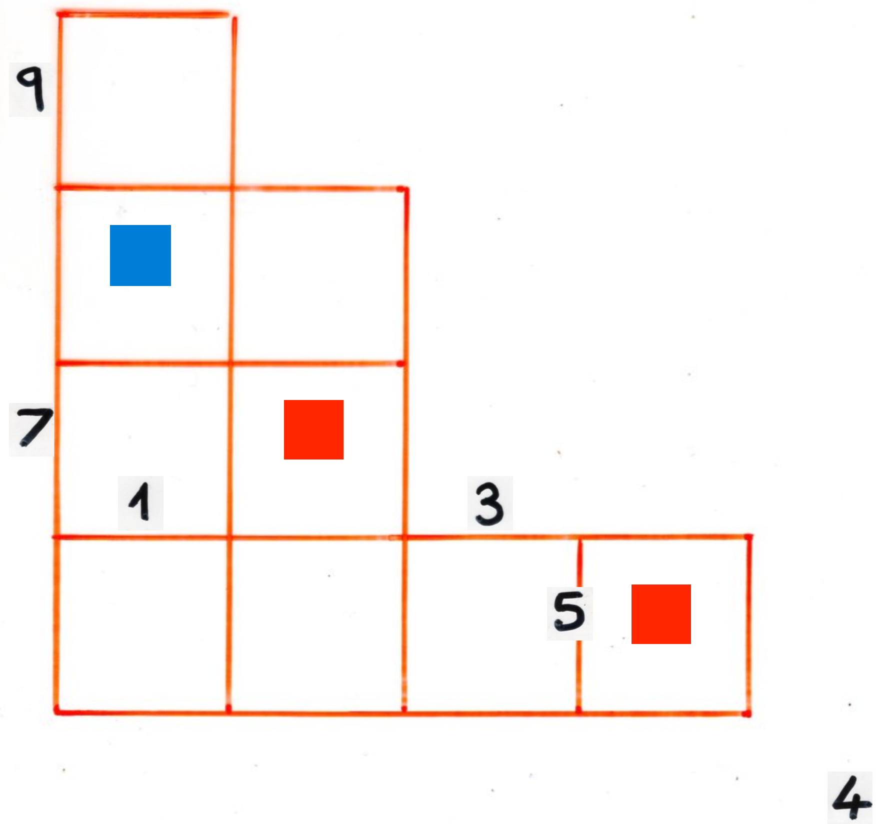

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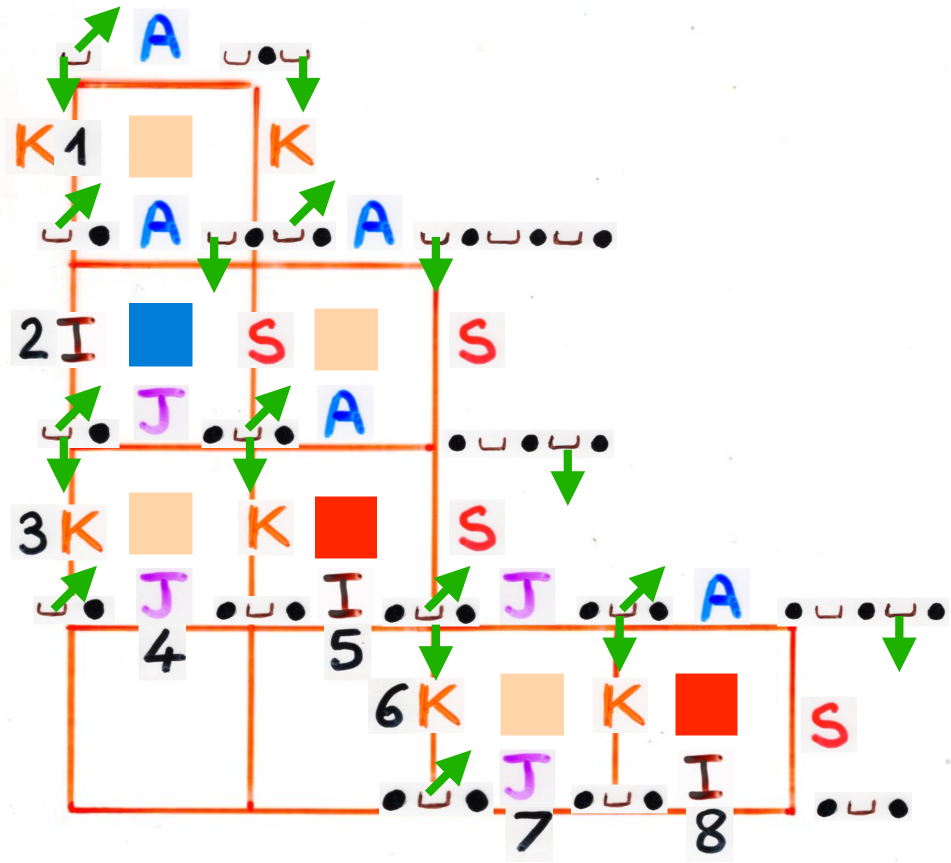
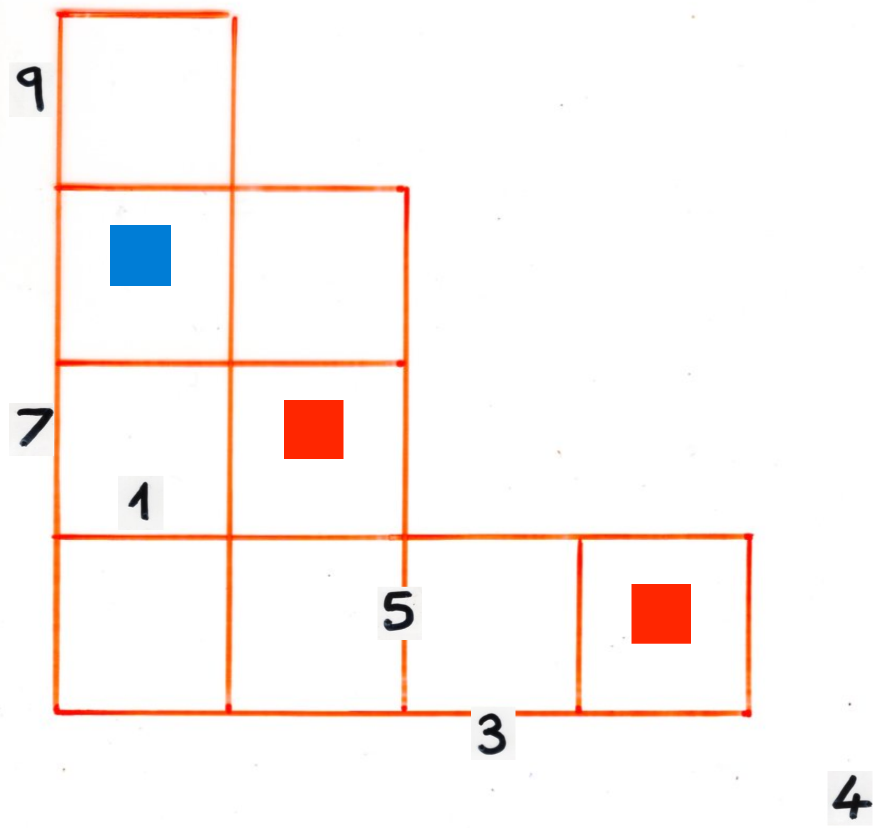
4 796 3 1   9 71 53 4





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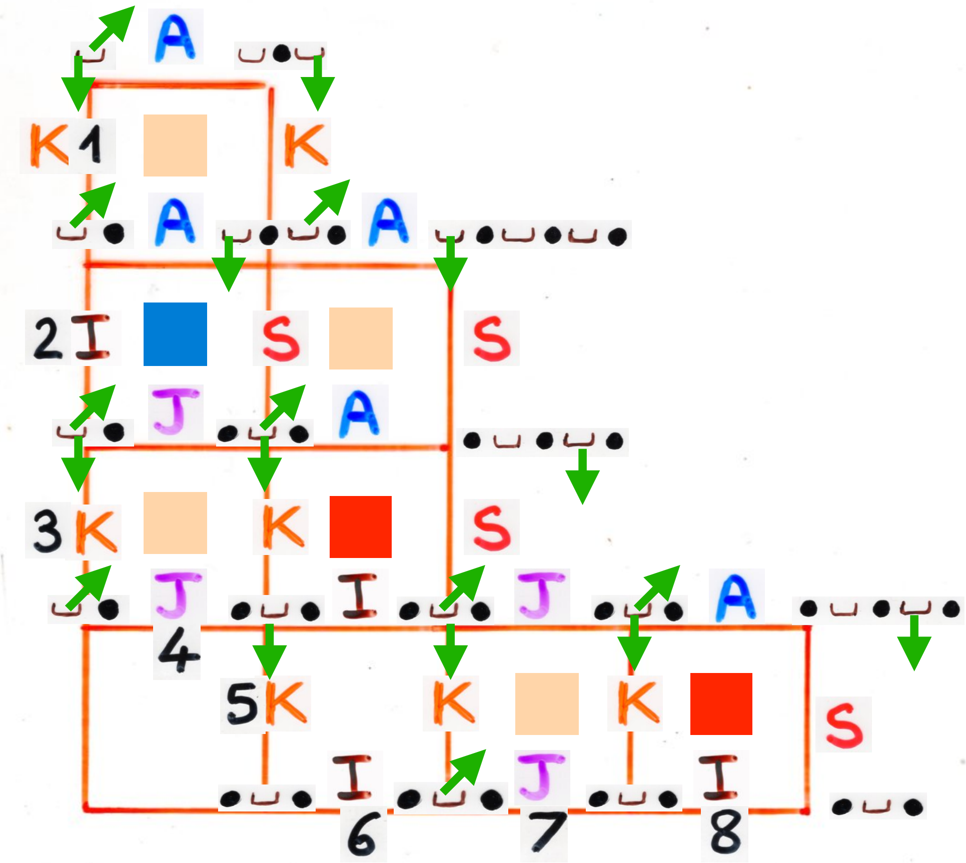

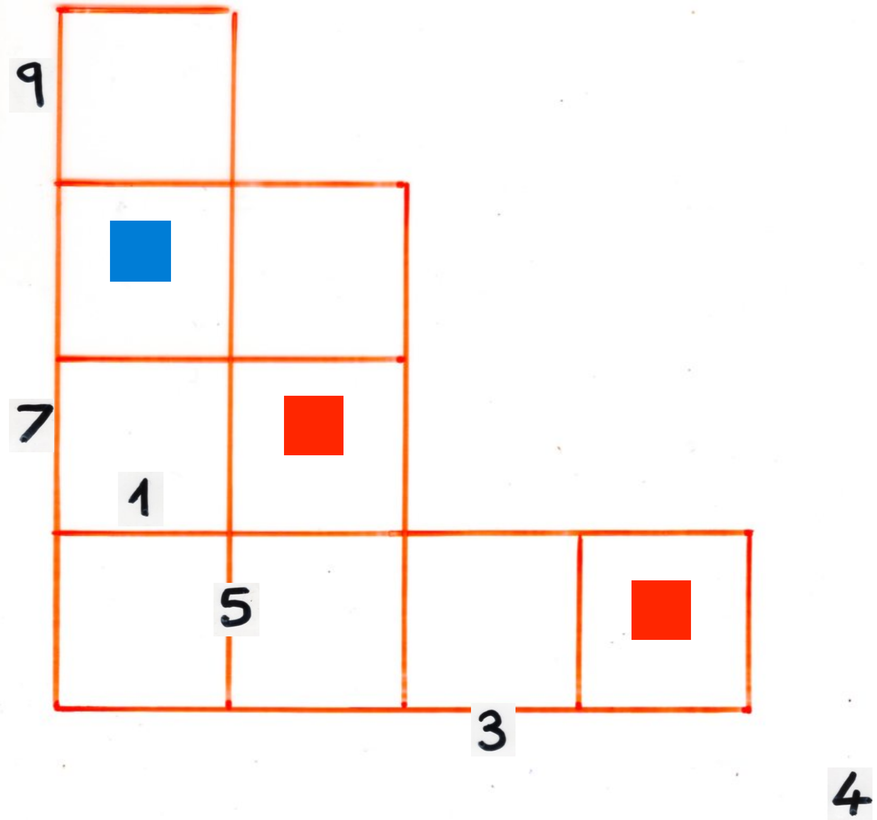

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5 794 3 1   9 751 3 4