Course IMSc, Chennai, India

January-March 2018

The cellular ansatz:
bijective combinatorics and quadratic algebra

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Chapter 3

Tableaux for the PASEP quadratic algebra

Ch3a

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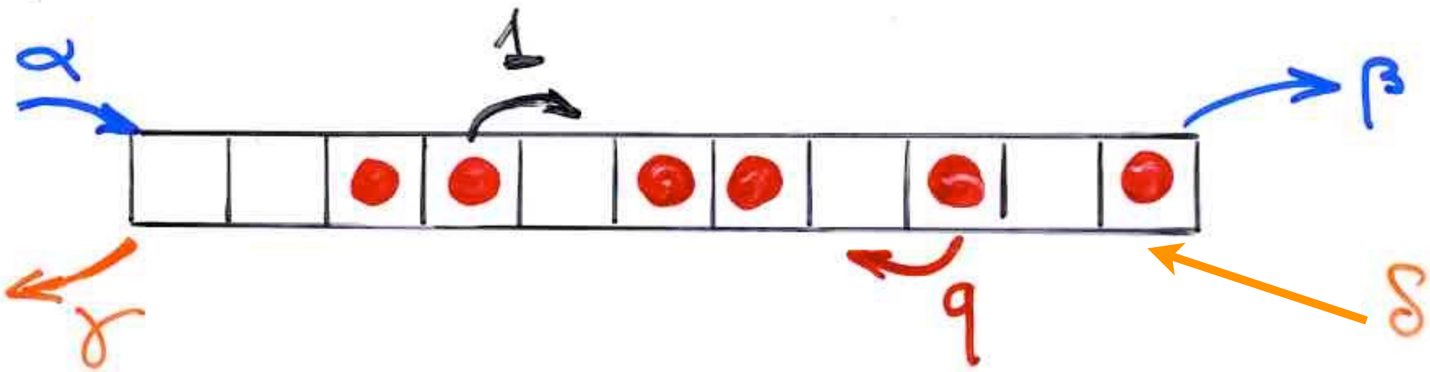
The PASEP

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



computation of the
"stationary probabilities"

molecular biology

molecular diffusion
linear array of enzymes
bio polymers ---

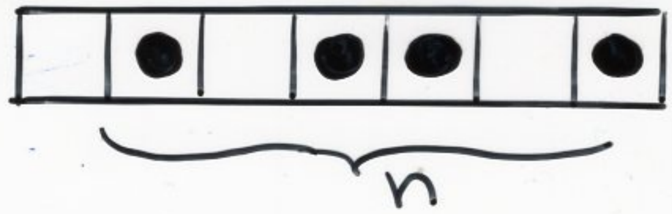
traffic flow --

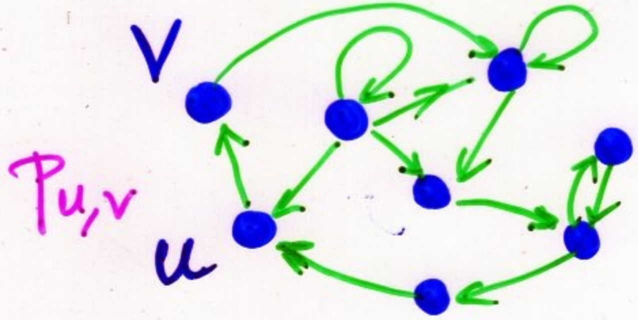
formation of shocks ---

phase transitions

Markov chain

2^n states





graph

$p_{u,v}$ probability

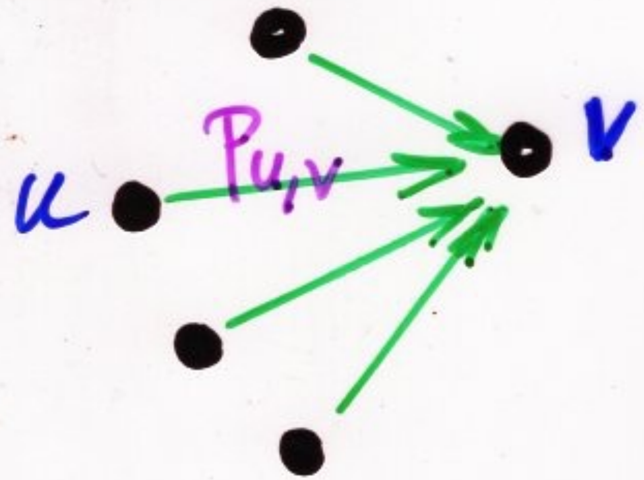
S set of states
(vertices of the graph)

$T = (p_{u,v})_{u,v \in S}$

(stochastic)
transition matrix

time t $\qquad V_t = (p_u^t, \dots)_{u \in S}$ probability vector at time t

time $t+1$ $\qquad V_{t+1} = V_t T$



$$p_v^{(t+1)} = \sum_u p_u^{(t)} p_{u,v}$$

time $(t+1)$ $\qquad$ time t

$$V_t = V_{t+1}$$

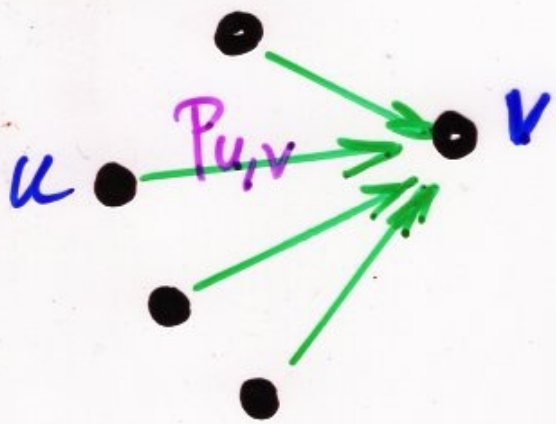
$$V = \left(p_u^{\infty}, \dots\right)_{u \in S}$$

$$V = V\,T$$

eigenvector
of T^{T}
eigenvalue 1
unique

stationnary
probabilities

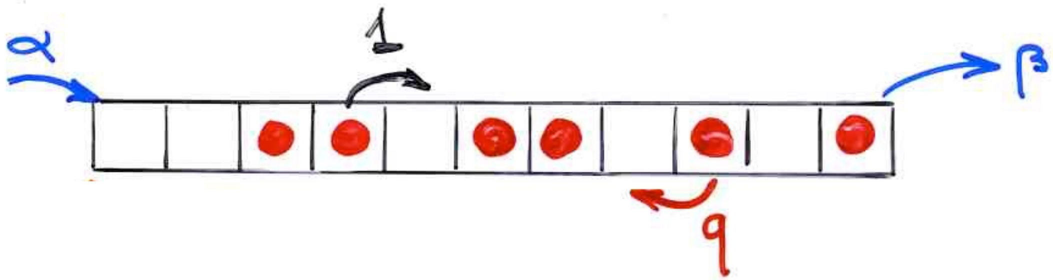
time $\longrightarrow \infty$



$$p_v^{\infty} = \sum_{u \in S} p_u^{\infty}\, p_{u,v}$$

PASEP with 3 parameters

$\gamma = \delta = 0$ q, α, β

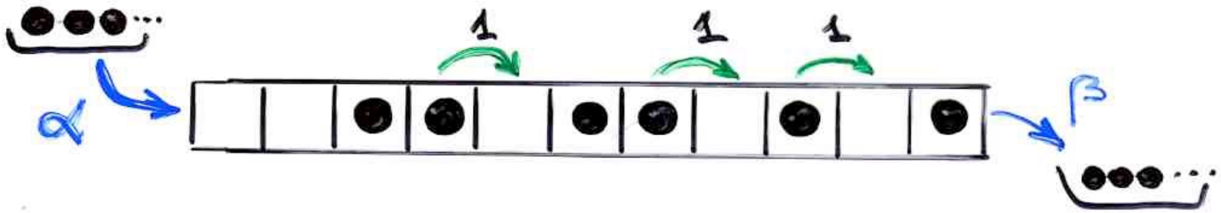


PASEP

$q = 0$ TASEP (α, β)

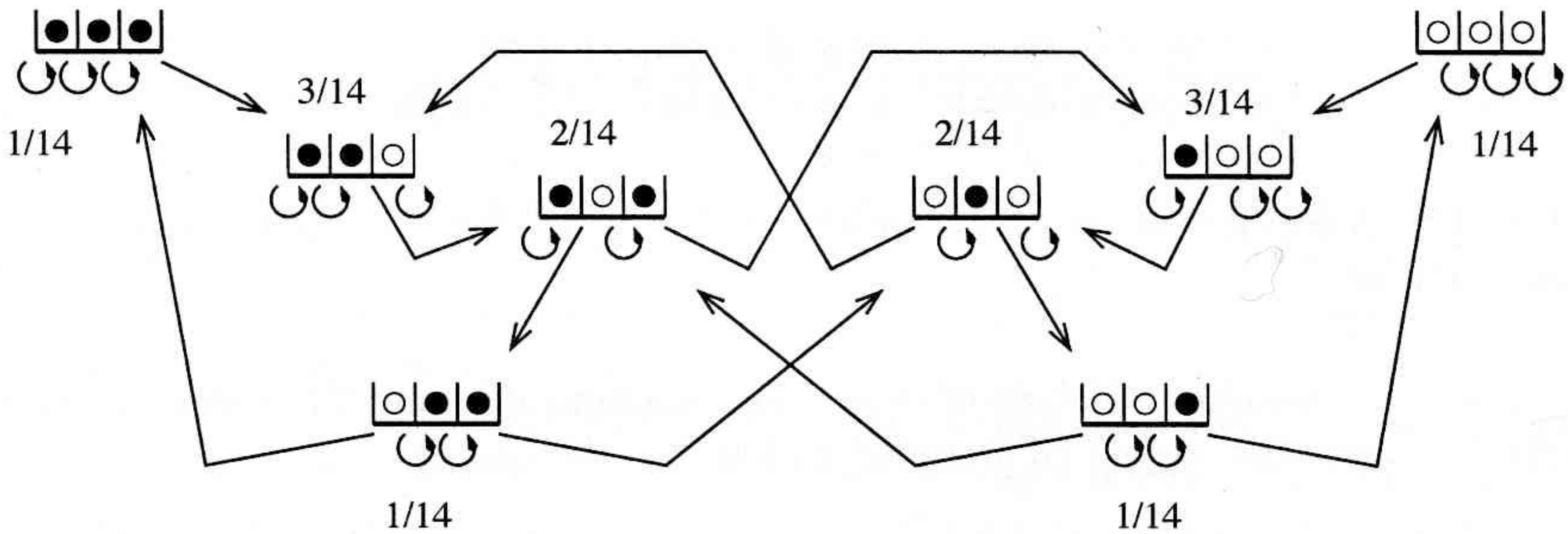
TASEP

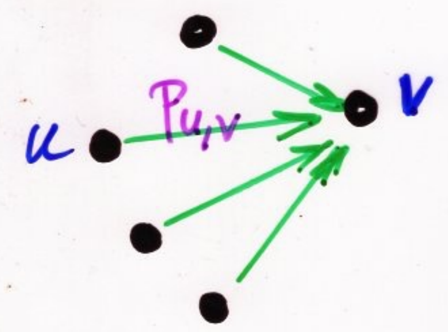
"totally asymmetric exclusion process"



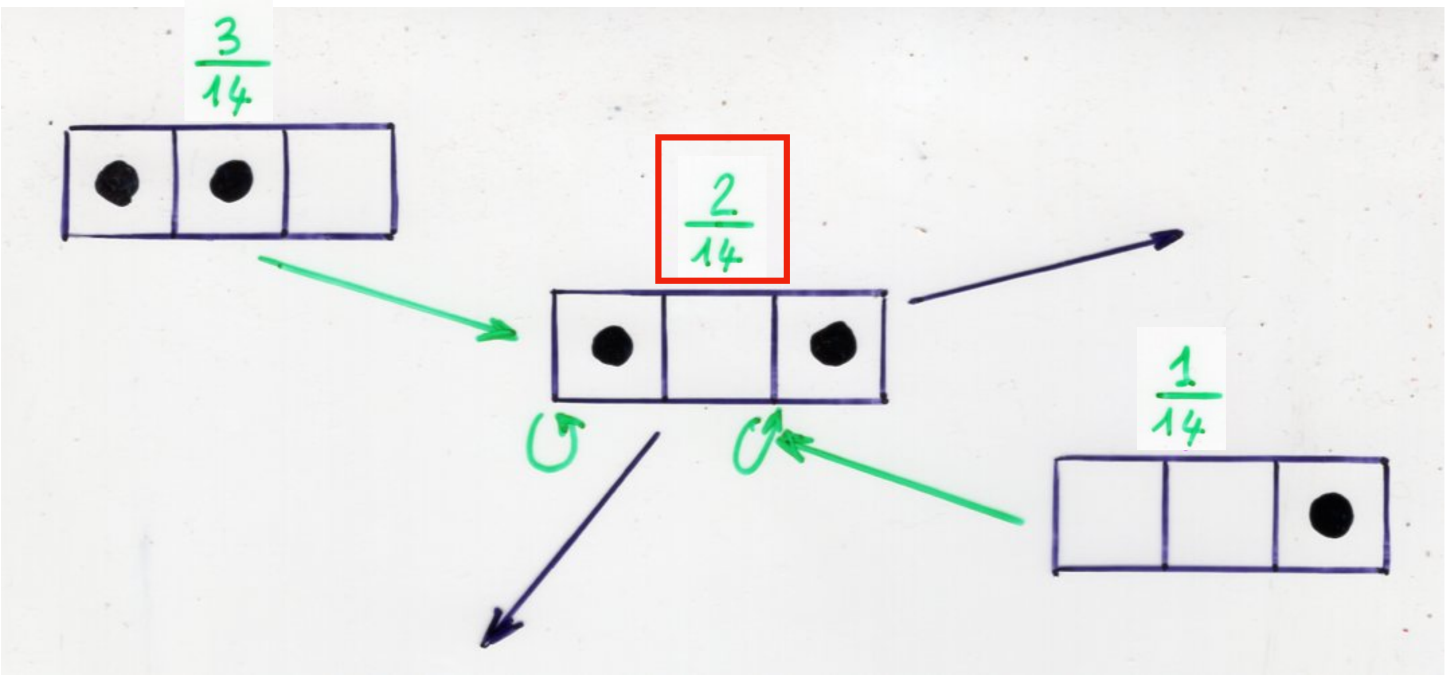
$q=0$ TASEP (α,β)

$\gamma=\delta=0$ $\alpha=\beta=1$





$$p_v^{\infty} = \sum_{u\in S} p_u^{\infty}\, p_{u,v}$$



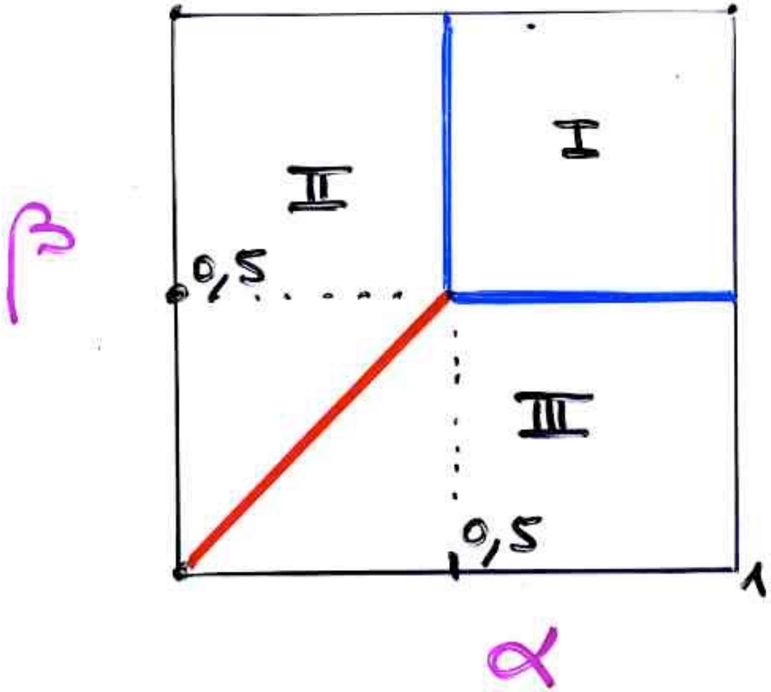
$$\frac{3}{14}\,\frac{1}{4}+\frac{1}{14}\,\frac{1}{4}+2\,\frac{2}{14}\,\frac{1}{4}$$

$$=\frac{3+1+4}{14}\times\frac{1}{4}$$

$$=\frac{8}{14}\times\frac{1}{4}$$

$$=\frac{2}{14}$$

$$P_v^{\infty}=\sum_{u\in S}P_u^{\infty}\,P_{u,v}$$



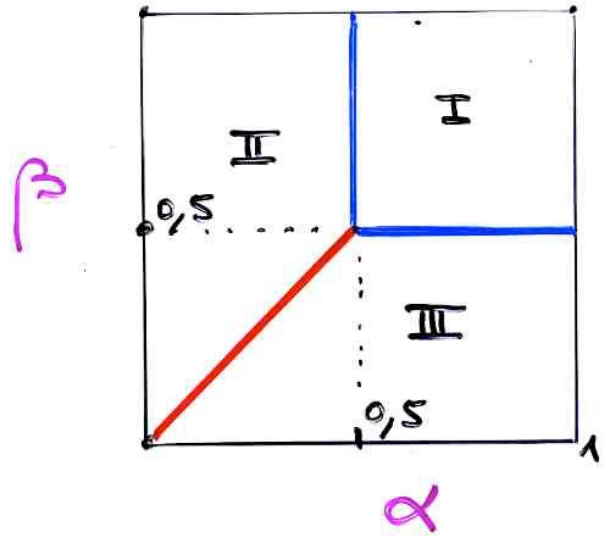
$n \to \infty$

$\rho = \langle \tau_i \rangle$ average occupation

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1 - \beta$



$n \to \infty$ average occupation

$\rho = \langle \tau_i \rangle$

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1 - \beta$

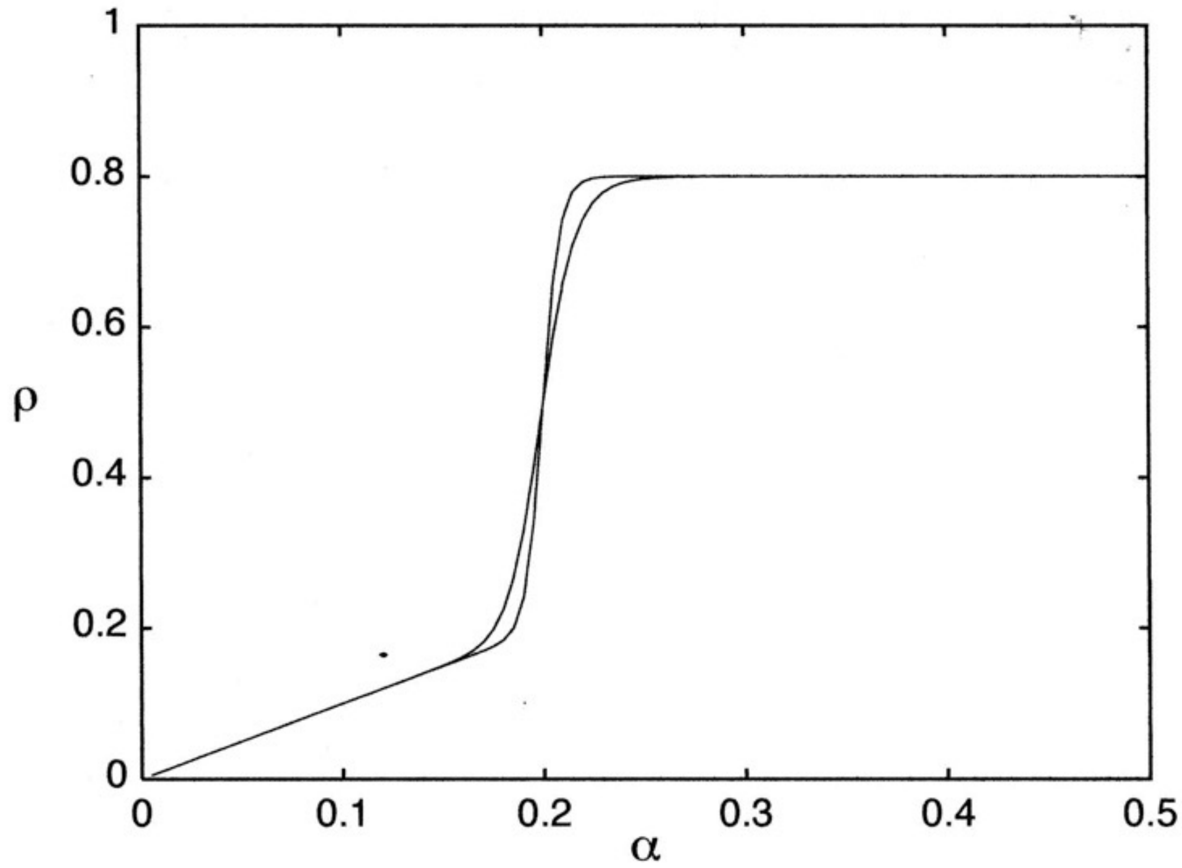


Figure 2: The average occupation $\rho = \langle \tau_{(N+1)/2} \rangle$ of the central site versus α for $N = 61$ and $N = 121$ when $\beta = .2$.

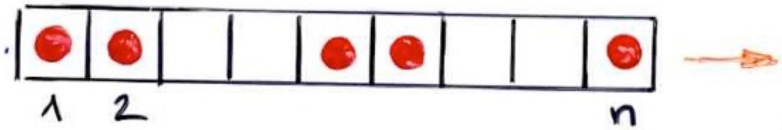
combinatorics

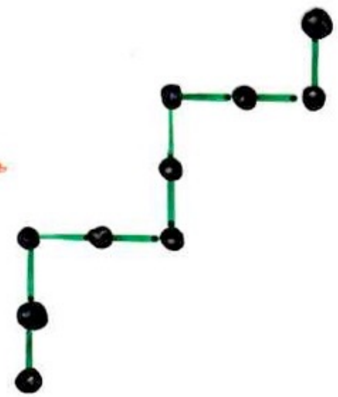
$q = 0$ TASEP (α, β)

$\gamma = \delta = 0$ $\alpha = \beta = 1$

Shapiro, Zeilberger (1982)

state $s = (\tau_1, \dots, \tau_n)$





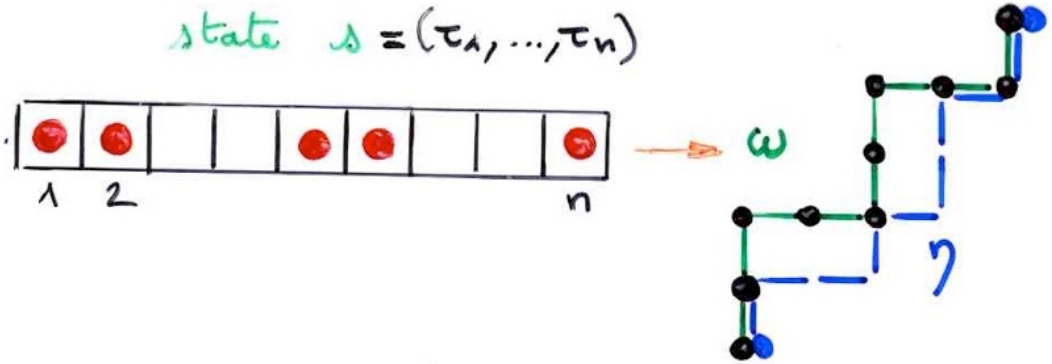
$$\mathcal{P}_n(s) =$$

$q = 0$ TASEP (α, β)

$\gamma = \delta = 0$ $\alpha = \beta = 1$

Shapiro, Zeilberger (1982)

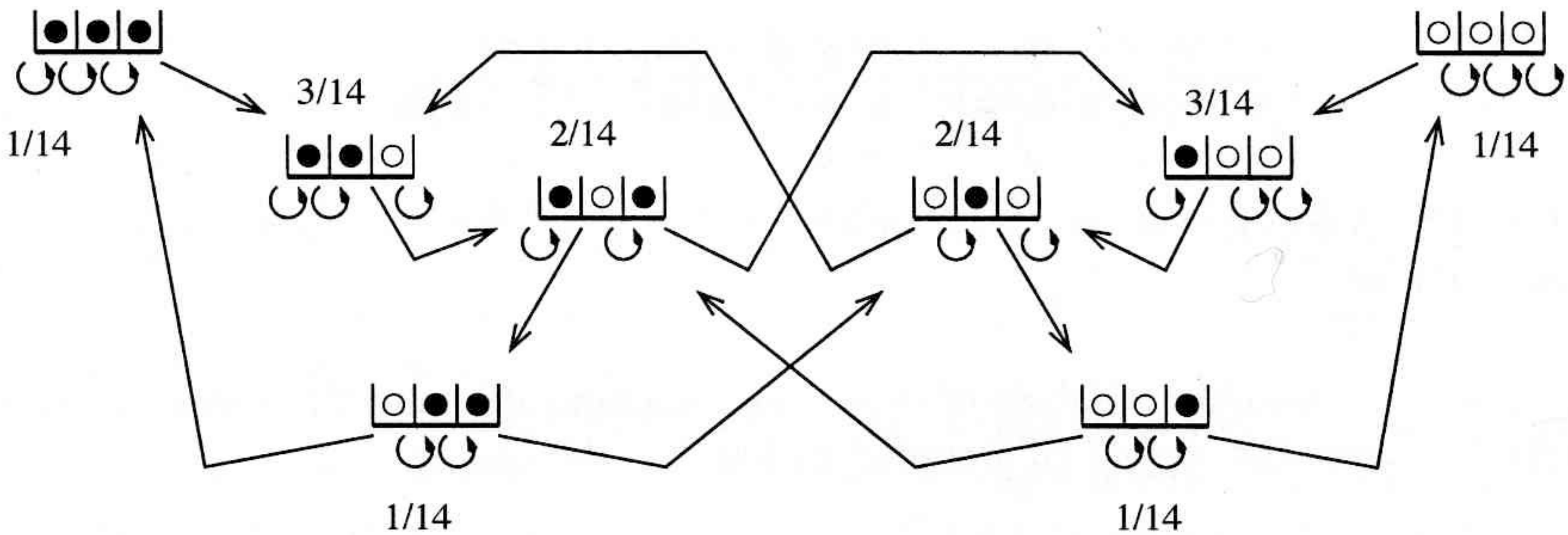
state $s = (\tau_1, \dots, \tau_n)$



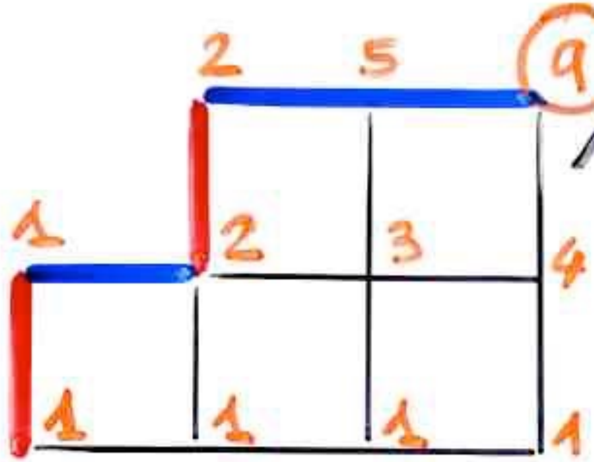
$$\mathbb{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

$q = 0$ TASEP (α, β)

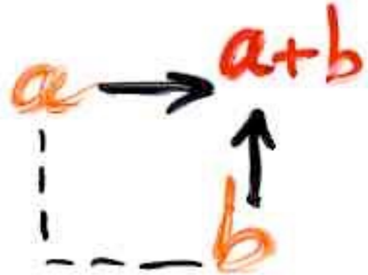
$\gamma = \delta = 0$ $\alpha = \beta = 1$



$s = (1, 0, 1, 0, 0)$ $\lambda = (1, 2, 2)$



$9 / C_6 = P(1,0,1,0,0)$



TASEP

Brak, Essam (2003) Duchi, Schaeffer (2004)
Angel (2005) X.V (2007)

PASEP

M. Josuat-Vergès (2007)

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006, 7, 8) X.V. (2008)
Corteel, Stanton (2011), Stanley, Williams (2010)
Dasse-Hartaut (2011, 12, ...)
Aval, Boussicault, Nadeau (2013)
Corteel, Williams, Mandelshtam, X.V. (2015)

Phys

Derrida, Mallick, Golinelli, ...
Cantini (2015)

orthogonal polynomials

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

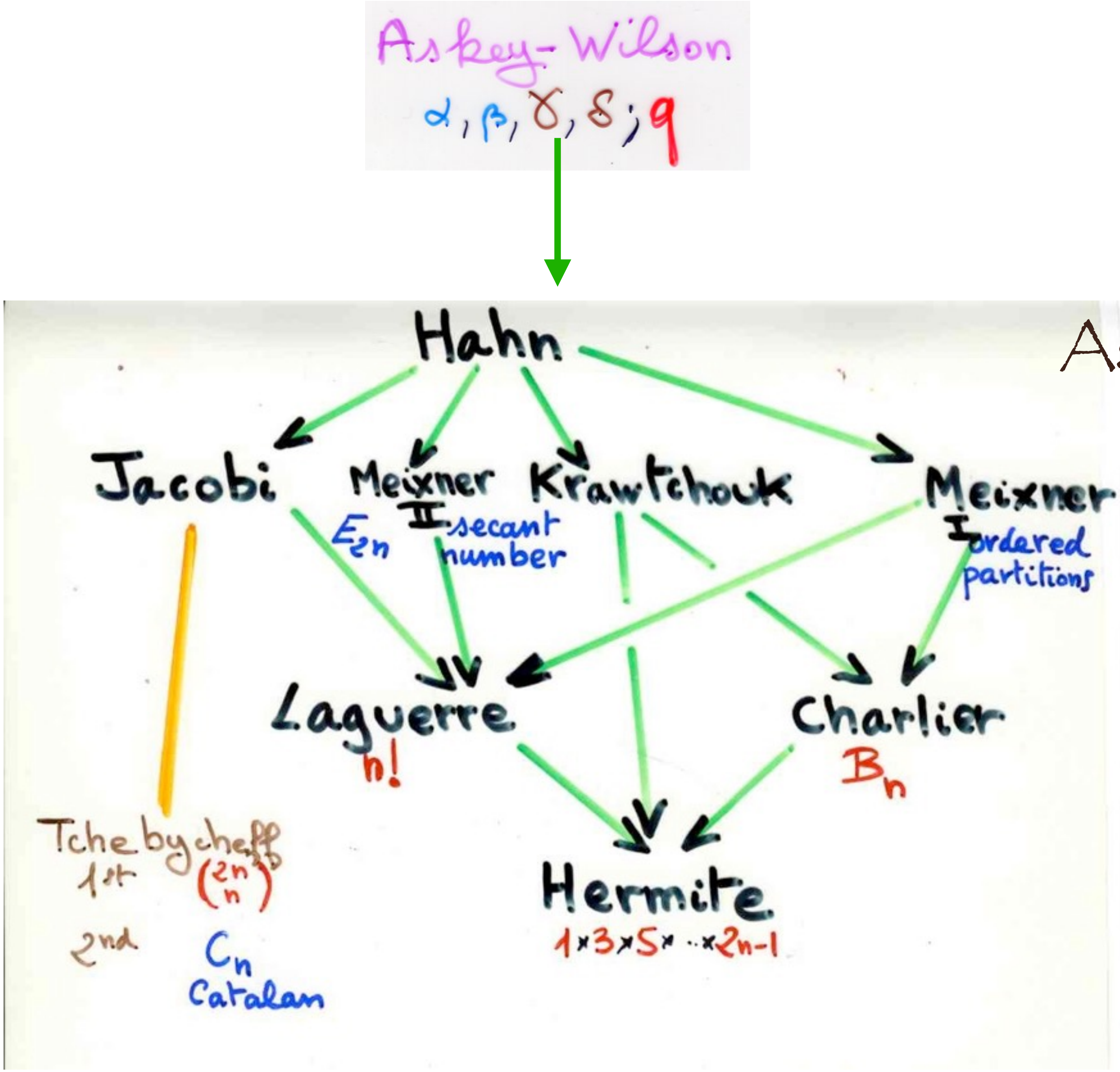
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

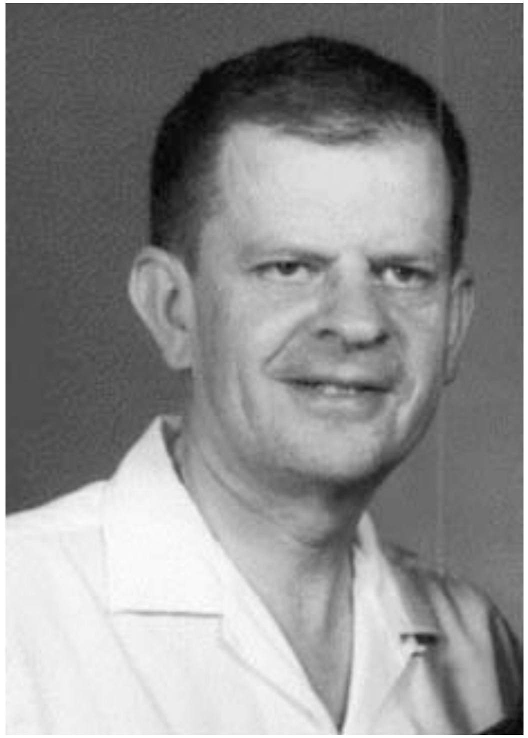
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

Askey tableau





The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

colum vector V

row vector W

$$\begin{cases} DE = qED + E + D \\ \langle W| (\alpha E - \gamma D) = \langle W| \\ (\beta D - \delta E)|V\rangle = |V\rangle \end{cases}$$

Then

$$\tau = (\tau_1, \ldots, \tau_n) \qquad \tau_i = \begin{cases} 0 & \square \\ 1 & \boxdot \end{cases}$$

$$\mathbb{P}_n(\tau_1, \ldots, \tau_n) = \frac{1}{Z_n} f_n(\tau_1, \ldots, \tau_n)$$

partition function

$$Z_n = \sum_{\tau} f_n(\tau_1, \ldots, \tau_n)$$

$$f_n(\tau_1, \ldots, \tau_n) = \langle W | \prod_{1 \leq i \leq n} \left(\tau_i D + (1-\tau_i) E \right) | V \rangle$$

$$\tau = 0,1,0,1,1,0\ 0 \qquad \langle W | E D E D D E E | V \rangle$$

$$f_n(\tau_1, \dots, \tau_n) =$$

$$\langle W | \prod_{1 \le i \le n} (\tau_i D + (1-\tau_i) E) | V \rangle$$

PASEP with 3 parameters q, α, β

$\gamma = \delta = 0$

$$\begin{cases} DE = qED + E + D \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

$\bar{\beta} = \frac{1}{\beta}$

$\bar{\alpha} = \frac{1}{\alpha}$

$q=0$ TASEP (α,β)

$$\begin{cases} DE = D+E \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

examples

$$D = \begin{bmatrix} 0 & \bar\beta & 0 & \cdots \\ & 0 & \bar\beta & \ddots \\ & 0 & \ddots & \ddots \\ & & & 0 \end{bmatrix} \qquad E = \begin{bmatrix} \bar\alpha & & 0 & \\ \bar\alpha\beta & 1 & & \\ \bar\alpha\beta^2 & \beta & 1 & \\ \bar\alpha\beta^3 & \beta^2 & \beta & 1 \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

$$\bar\beta = \frac{1}{\beta}, \quad \bar\alpha = \frac{1}{\alpha}$$

$$\langle W | = (1, 0, \ldots, 0, \ldots) \qquad | V \rangle = (1, 1, \ldots, 1, \ldots)^T$$

(infinite matrices)

examples

$$D = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & 0 & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\beta} & & & \\ \bar{\beta} & 1 & & \\ \vdots & \vdots & 1 & \\ \bar{\beta} & 1 & \cdots & 1 \end{bmatrix}$$

$$\bar{\beta} = \frac{1}{\beta}, \quad \bar{\alpha} = \frac{1}{\alpha}$$

$$\langle W | = (1, 0, \cdots) \qquad | V \rangle = (1, \bar{\alpha}, \bar{\alpha}^2, \cdots)^T$$

(infinite matrices)

examples

$$D = \begin{bmatrix} \bar\beta & \kappa & & & \\ 0 & 1 & 1 & & \\ & 0 & 1 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \qquad E = \begin{bmatrix} \bar\alpha & 0 & & & \\ \kappa & 1 & 0 & & \\ & 1 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\langle W| = (1, 0, \cdots 0) \qquad |V\rangle = (1, 0, \cdots 0)$$

$$\bar\alpha = \frac{1}{\alpha} \qquad \bar\beta = \frac{1}{\beta} \qquad \kappa^2 = \bar\alpha + \bar\beta - \bar\alpha\bar\beta$$

(infinite matrices)

The PASEP algebra

$$DE = qED + E + D$$

The PASEP algebra

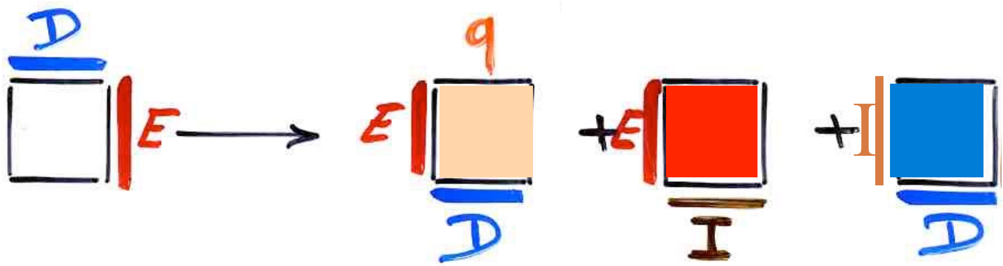
$$DE = qED + E + D$$

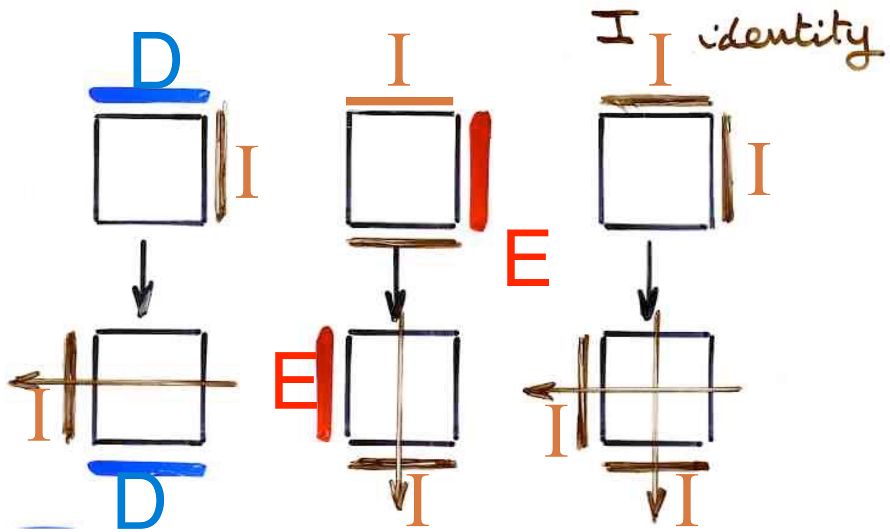
$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

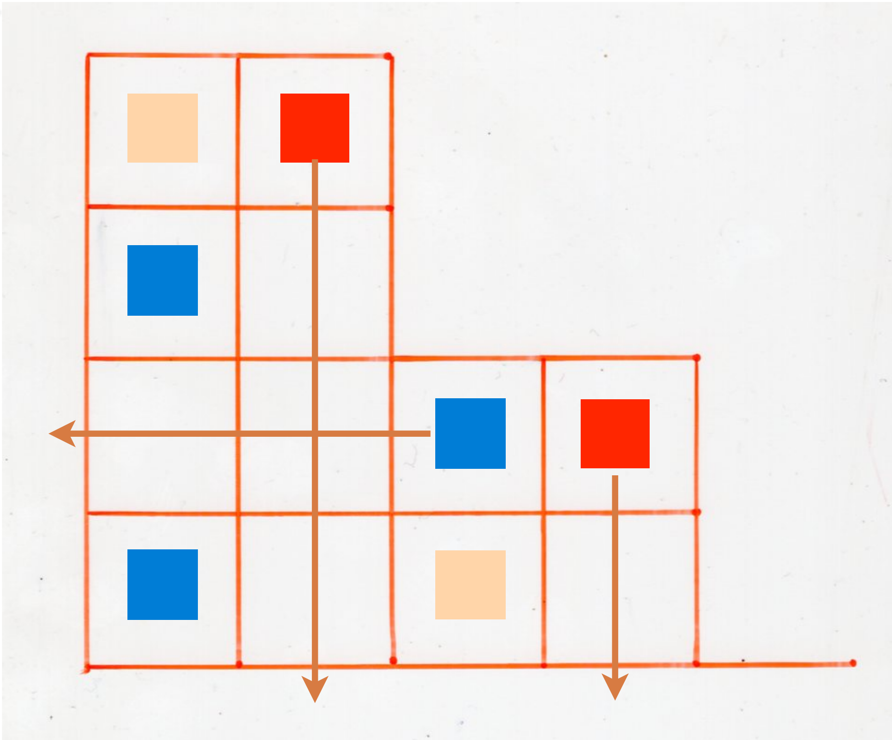
$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$





complete Q-tableau

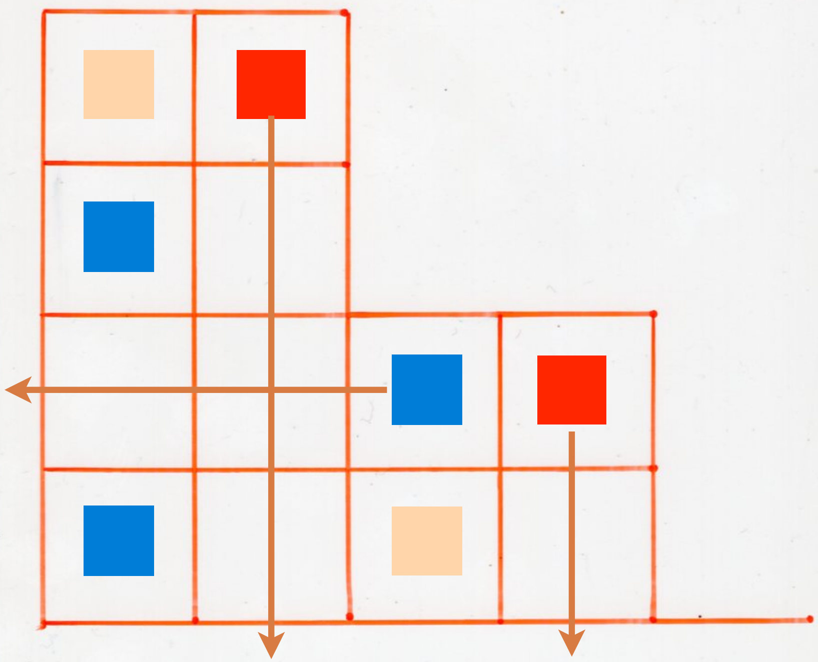


The PASEP algebra

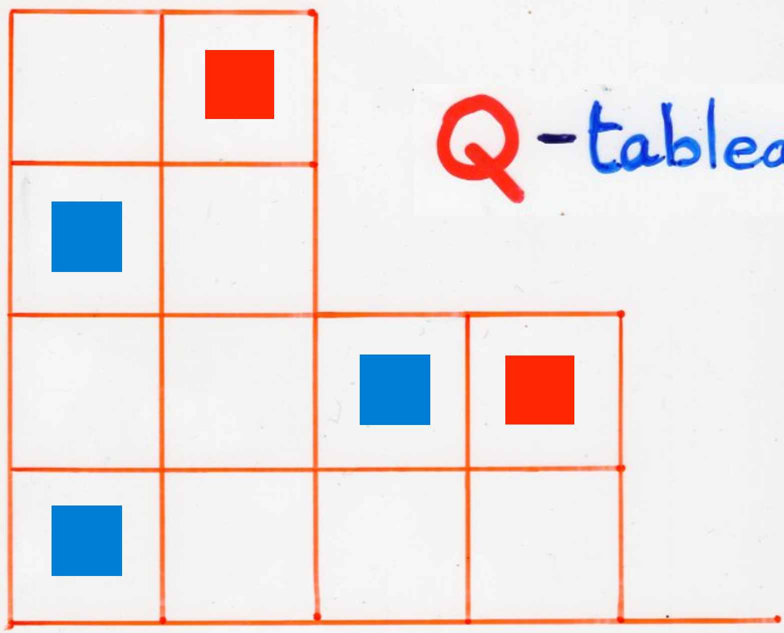
$$DE = qED + E + D$$

$$\begin{aligned}
DE &= \underset{q}{\square}\, ED + E\,I_h + I_v\,D \\
D\,I_v &= \square\, I_v\,D \\
I_h\,E &= \square\, E\,I_h \\
I_h\,I_v &= \square\, I_v\,I_h
\end{aligned}$$

complete Q-tableau



Q-tableaux



The PASEP algebra

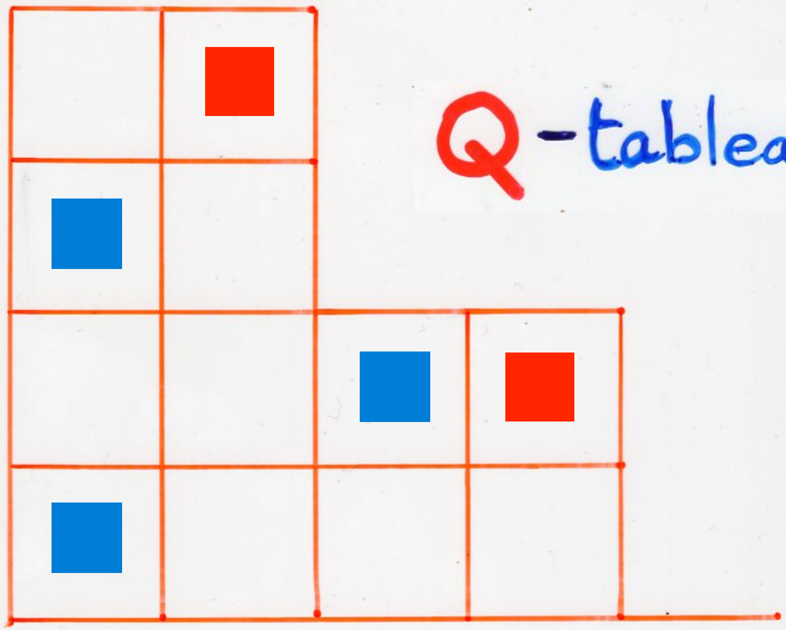
$$DE = qED + E + D$$

$$DE = \square ED + E I_h + I_v D$$

$$D I_v = \square I_v D$$

$$I_h E = \square E I_h$$

$$I_h I_v = \square I_v I_h$$



Q-tableaux

L set of "labels"

$$\varphi : \left\{ \begin{array}{|cc|} \hline k & l \\ i & j \\ \hline \end{array} \right\} = R \longrightarrow L$$

set of rewriting rules

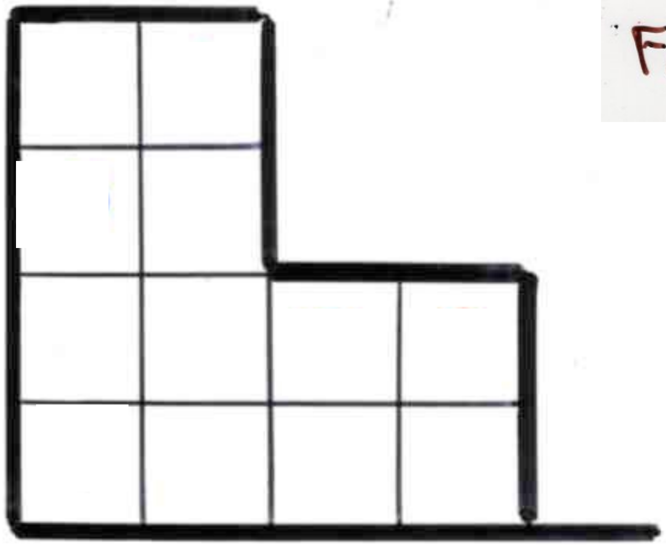
$$B_j A_i \longrightarrow C_{ij}^{kl} A_k B_l$$

$(*)$

alternative tableaux

alternative tableau

Definition



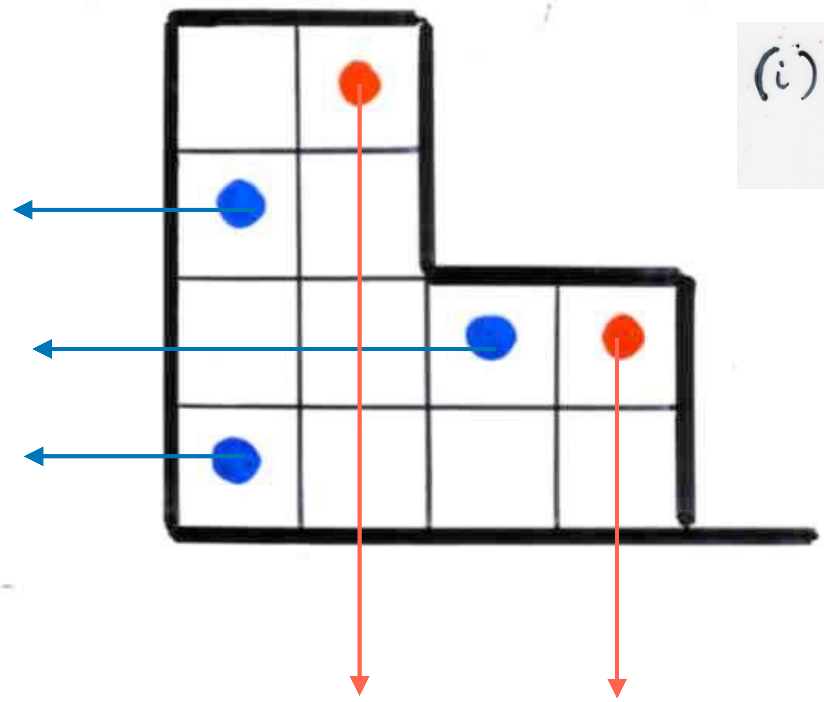
Ferrers diagram $\mathbf{F}$

with possibly empty rows or columns

size of $\mathbf{F}$

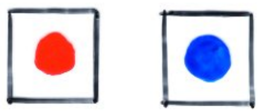
$\mathbf{n} = \left(\begin{array}{c}\text{number of}\\ \text{rows}\end{array}\right) + \left(\begin{array}{c}\text{number of}\\ \text{columns}\end{array}\right)$

alternative tableau



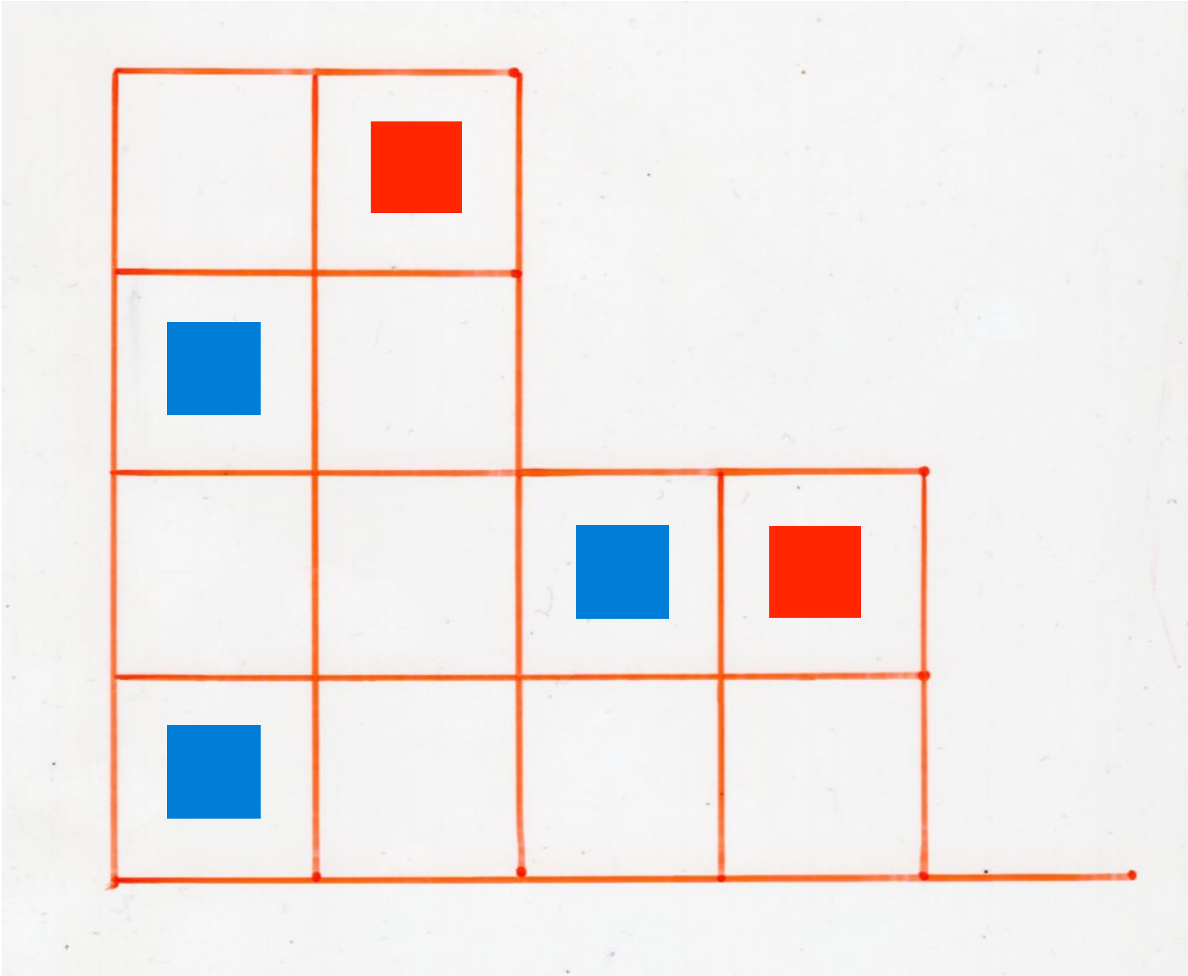
Definition

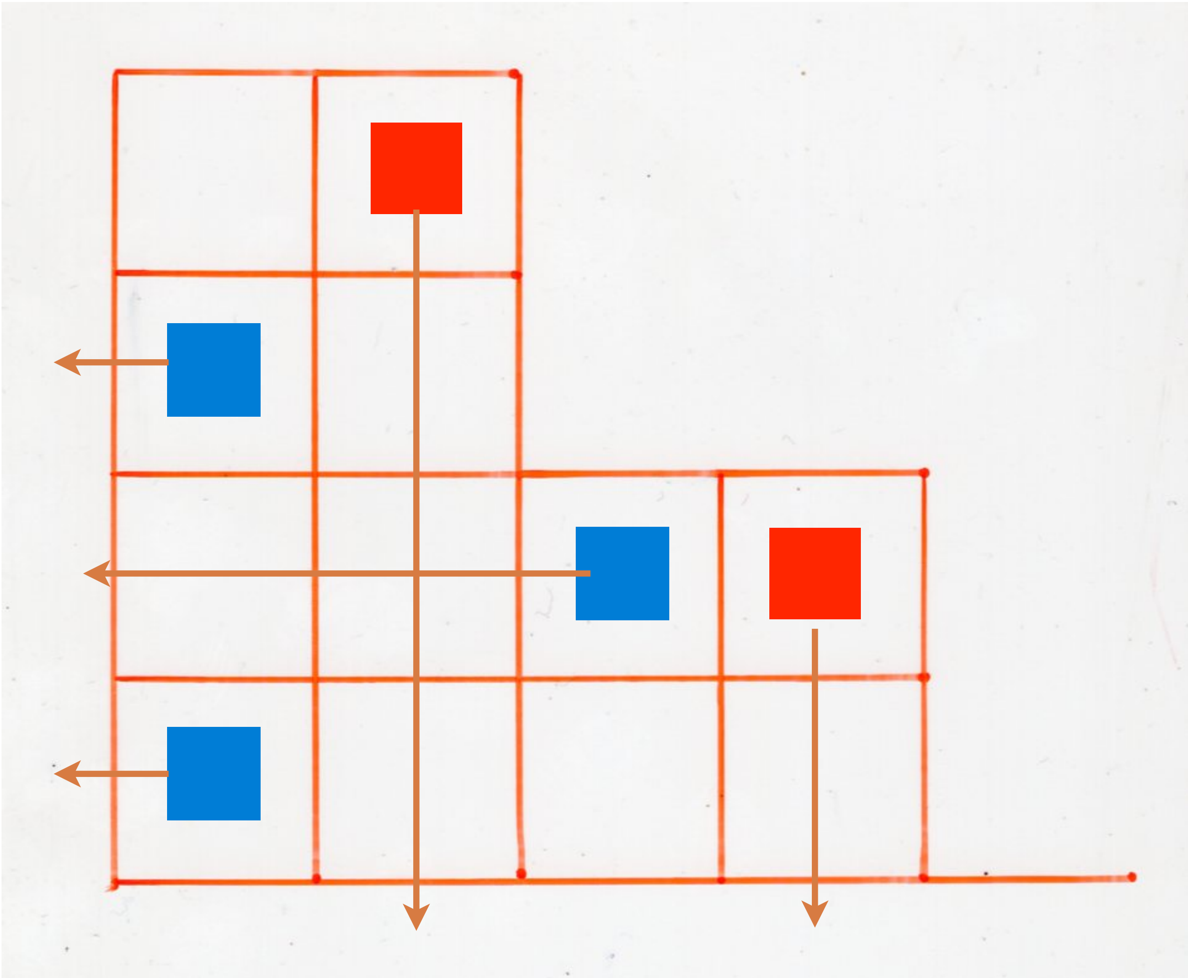
(i) some cells are coloured red or blue

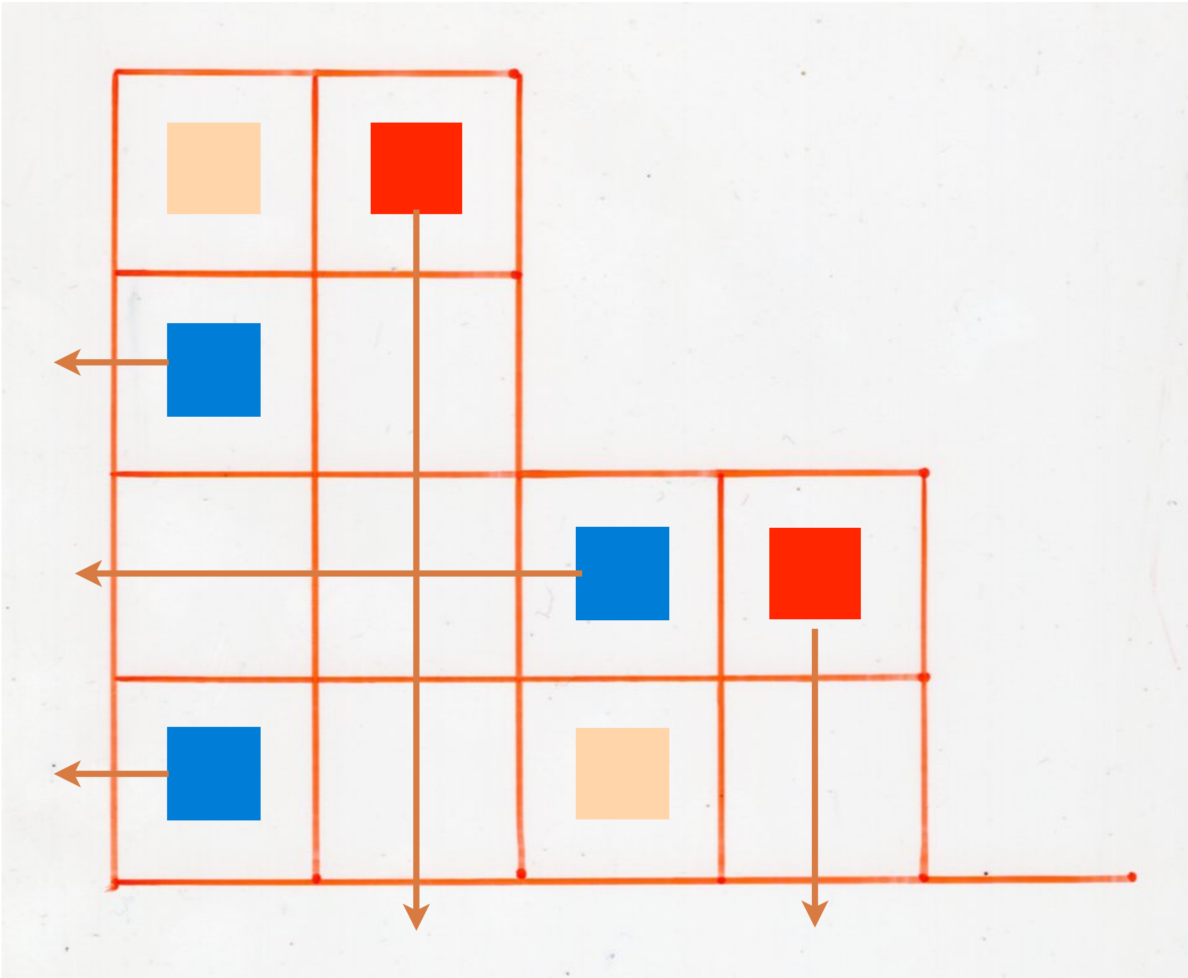


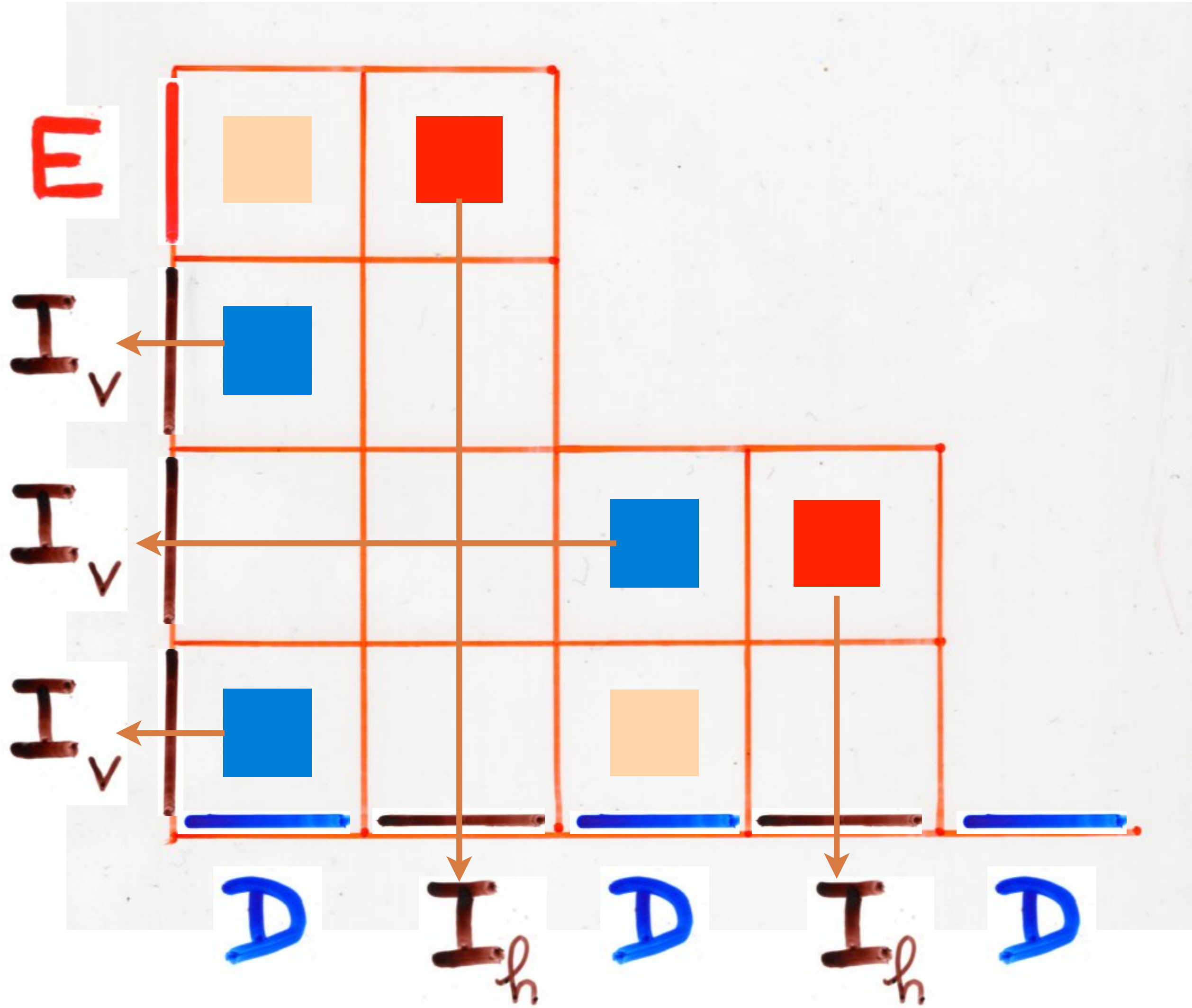
(ii) • no coloured cell at the left of a blue cell

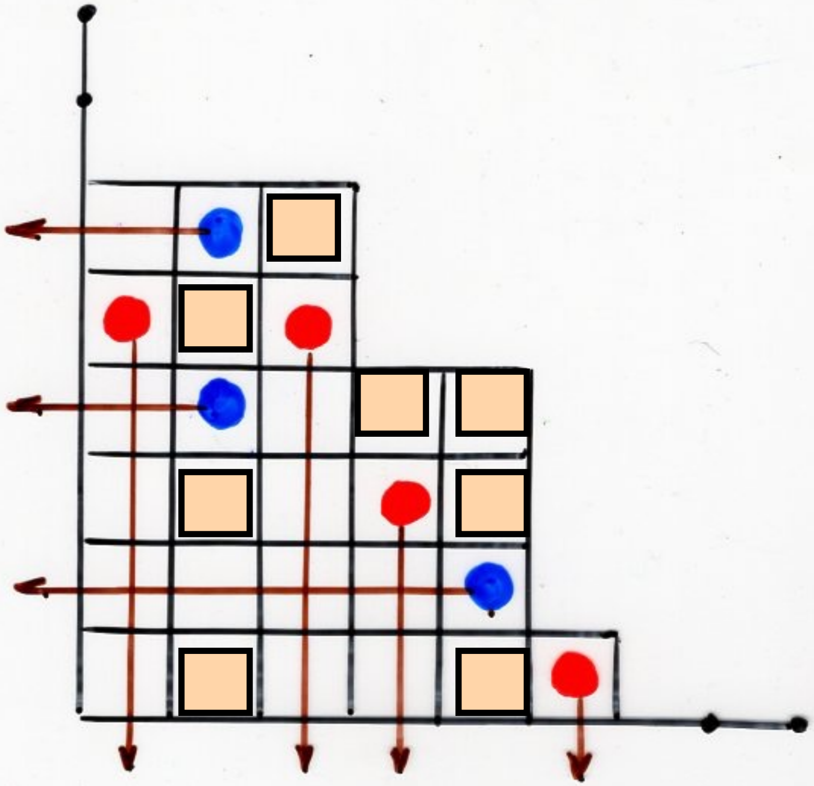
• no coloured cell below a red cell







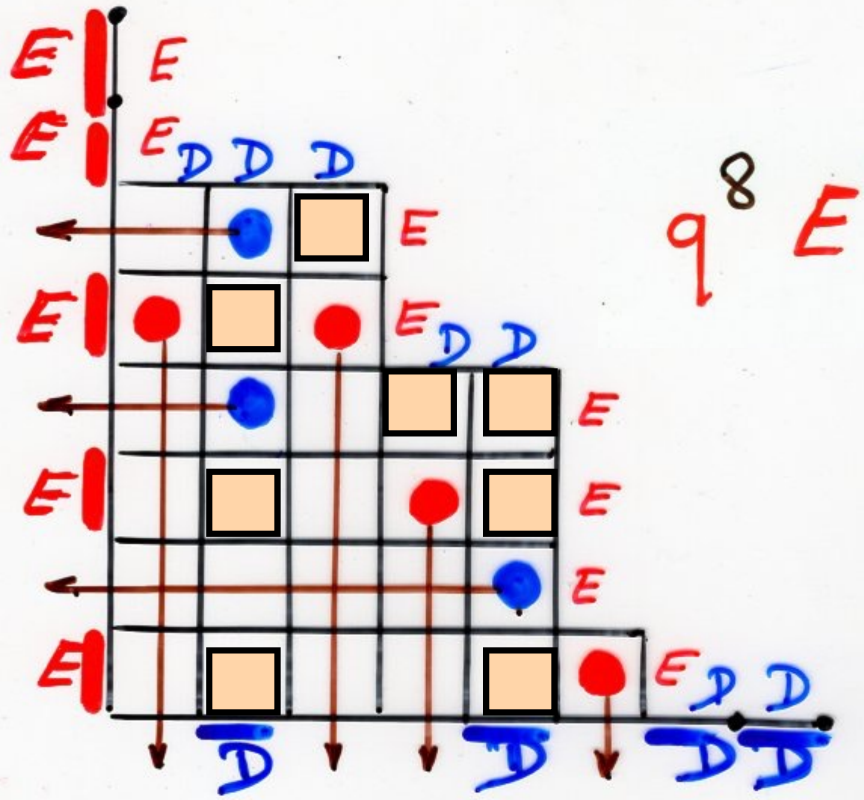

E
I_v
I_v
I_v
D
I_h
D
I_h
D



$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without



$q^8 E^5 D^4$

$k(T)$ = nb of cells $\square$

$i(T)$ = nb of rows without $\bullet$

$j(T)$ = nb of columns without $\bullet$

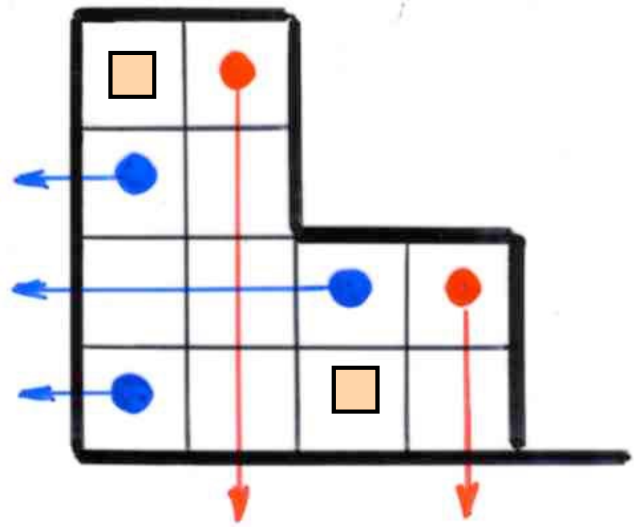
$DE = qED + E + D$

In the **PASEP** algebra

any word $w(E,D)$ can be uniquely written

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

alternative tableaux profile w



$k(T)$ = nb of cells ▢

$i(T)$ = nb of rows without ●

$j(T)$ = nb of columns without ●

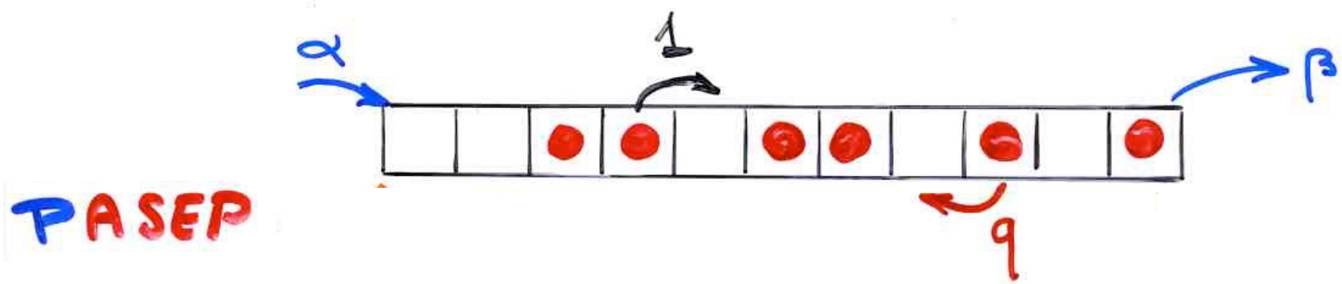
Def- profile of an alternative tableau

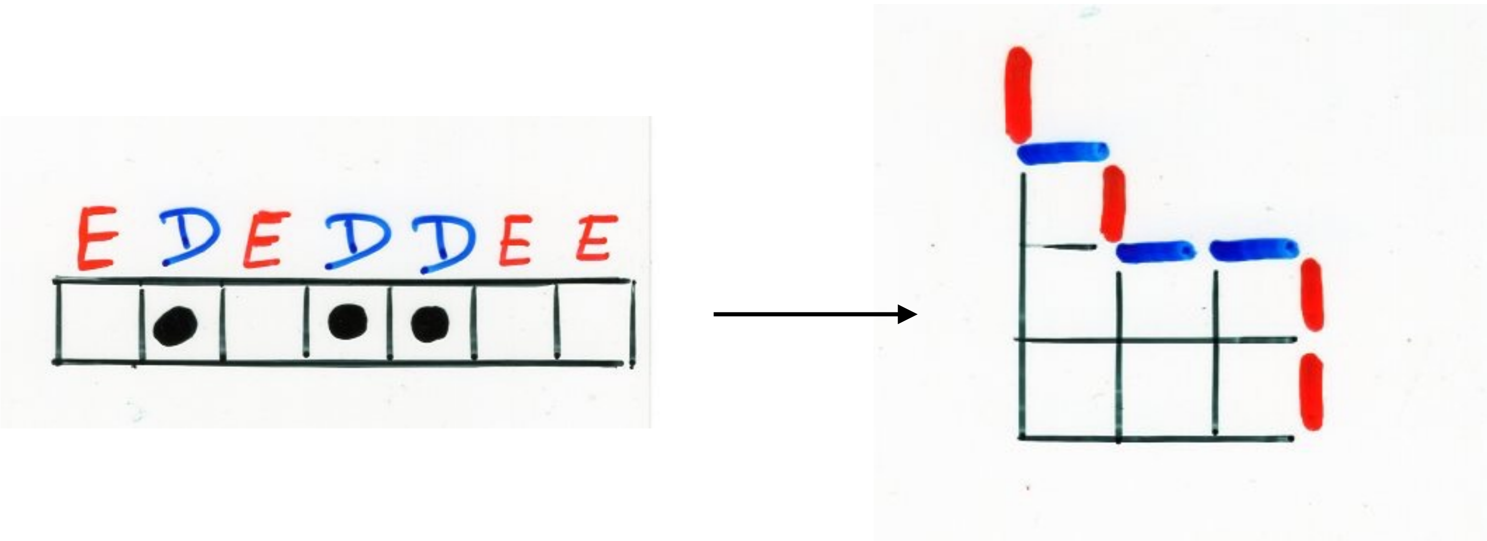
word $w \in \{E,D\}^*$

stationary probabilities for the PASEP

computation of the
"stationary probabilities"

PASEP with 3 parameters q, α, β





computation of the "stationary probabilities"

$$\begin{cases} DE = qED + E + D \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

$$\bar{\beta} = \frac{1}{\beta}$$

$$\bar{\alpha} = \frac{1}{\alpha}$$

any word $w(E,D)$ can be uniquely written

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

alternative tableaux profile ~~w~~ w

$$\langle W| E^i D^j |V\rangle = \bar{\alpha}^i \bar{\beta}^j \underbrace{\langle W|V\rangle}_{=1}$$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, .., \tau_n)$ is

$$\mathrm{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without

alternative tableau

X.V. (2008)

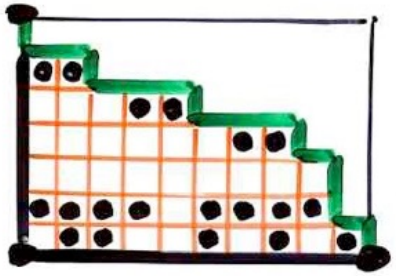
permutation tableau

S. Corteel, L. Williams

(2007, 2008, 2009)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii)  forbidden

□ = 0 ⊡ = 1

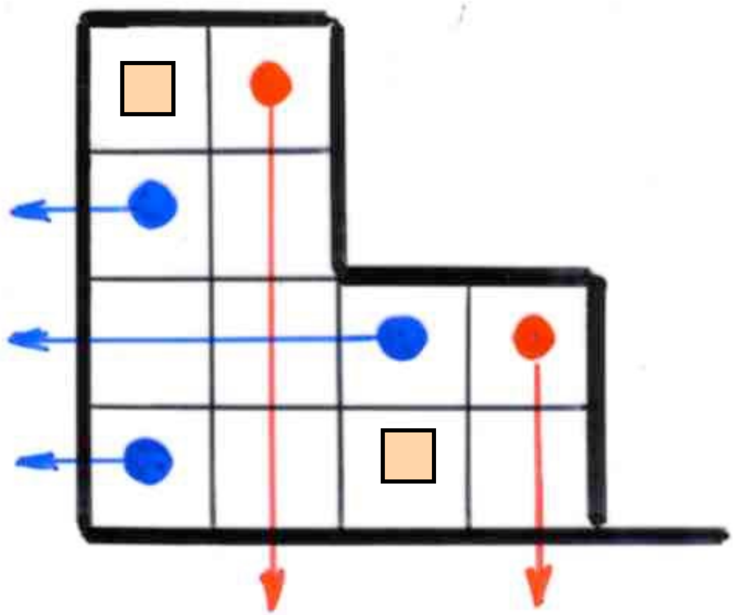
permutation tableau

A. Postnikov (2001, ...)

totally nonnegative part of the Grassmannian

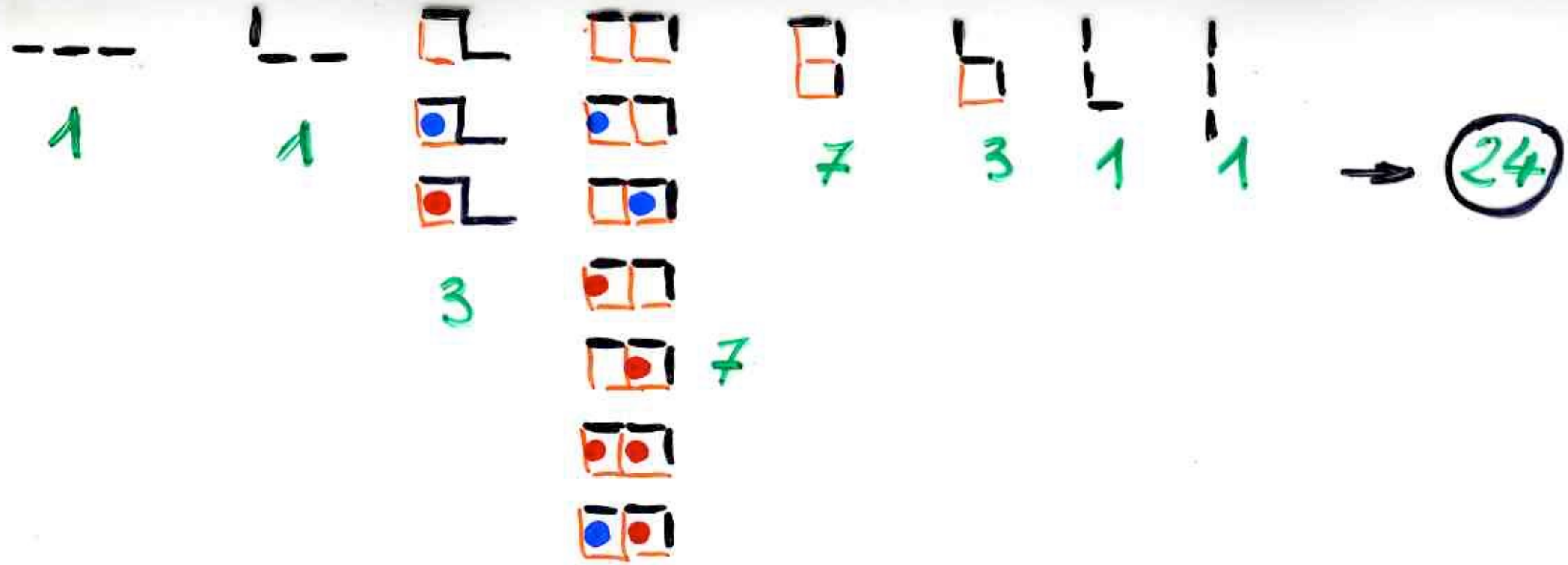
E. Steingrímsson, L. Williams (2005)

Enumeration of alternative tableaux



Prop. The number of alternative tableaux of size n is $(n+1)!$

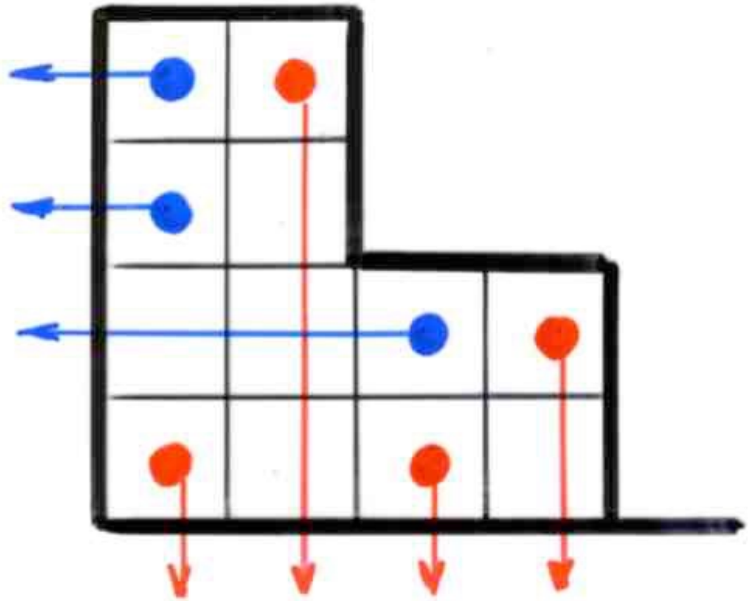
ex: n=2



Definition Catalan alternative tableau

alternative tableau T without cells $\square$

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

$q=0$

TASEP

(α,β)

Two bijections

- from a combinatorial representation of the PASEP algebra (X.V., 2008)

  equivalent to a bijection Corteel, Nadeau (2007)

  (with permutation tableaux)

  Steingrímsson, Williams (2005, 2007)

  Postnikov

- direct bijection (with tree-like tableaux) Aval, Boussicault, Nadeau (2011)

$$\text{tableaux size } (n+1) \longleftrightarrow \left(\text{tableaux size } n, \ 1 \leq i \leq n+1\right)$$

$$(n+1)!$$

"The cellular ansatz"

(i) first step

(ii) second step

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

$UD = qDU + Id$

permutations

RSK ⟷

pairs of Young tableaux

towers placements

Physics

$DE = qED + E + D$

alternative tableaux

EXF ⟷

"Laguerre histories"

permutations

commutations

data structures "histories"

orthogonal polynomials

rewriting rules

ASM alternating sign matrices

tilings

non-crossing paths

planarization

"planar automata"

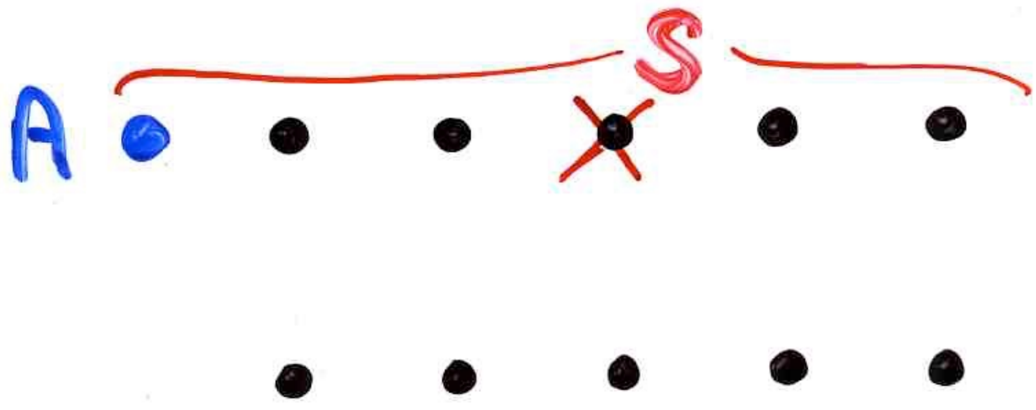
8-vertex model

⟷ ?

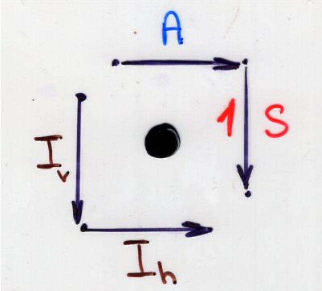
A combinatorial representation of the PASEP algebra

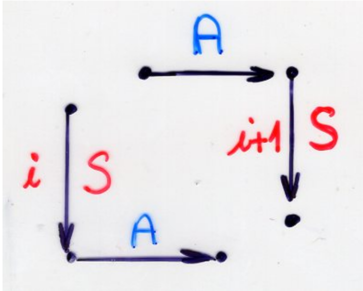
Polya urn

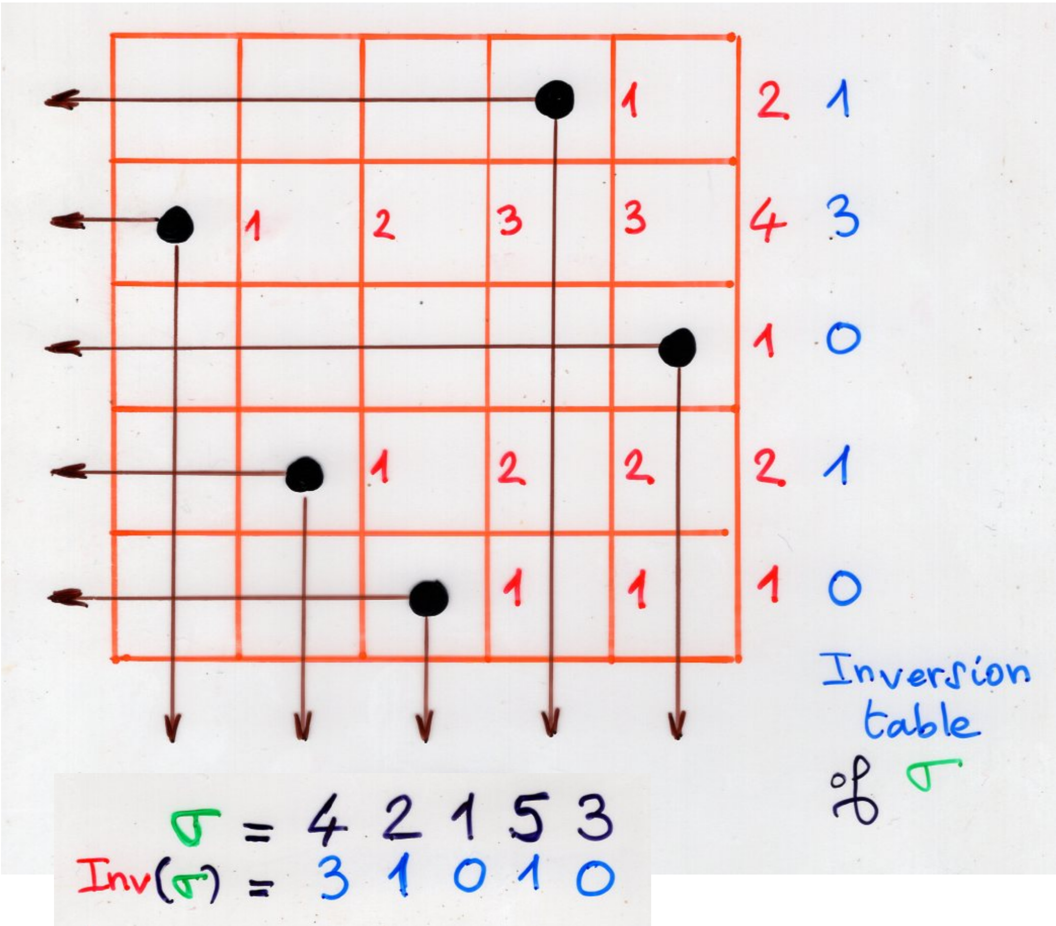
$$A\,|k\rangle = |(k+1)\rangle$$
$$S\,|k\rangle = k\,|(k-1)\rangle$$



$$AS - SA = I$$

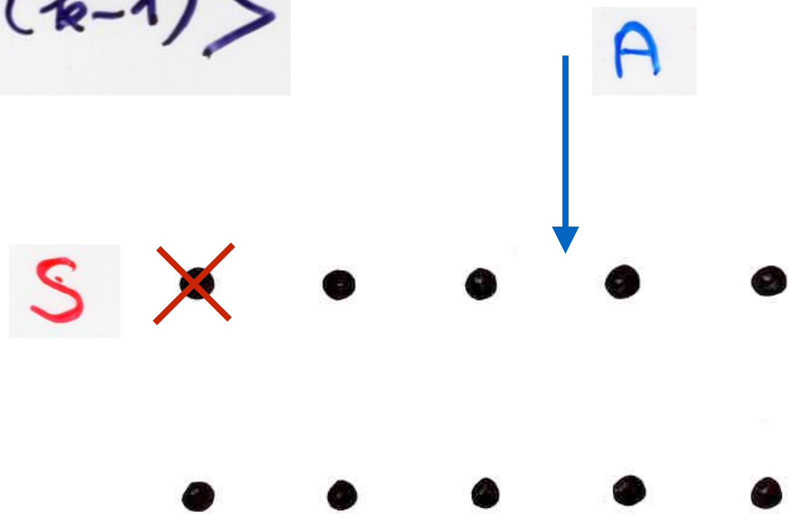

A
1
S
I_v
I_h


A
i
S
i+1
S
A


1 2 1
1 2 3 3 4 3
1 0
1 2 2 2 1
1 1 1 0
Inversion table of σ
$\sigma = 4\ 2\ 1\ 5\ 3$
$\mathrm{Inv}(\sigma) = 3\ 1\ 0\ 1\ 0$

Priority queue

$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$
$$S\,|k\rangle = |(k-1)\rangle$$

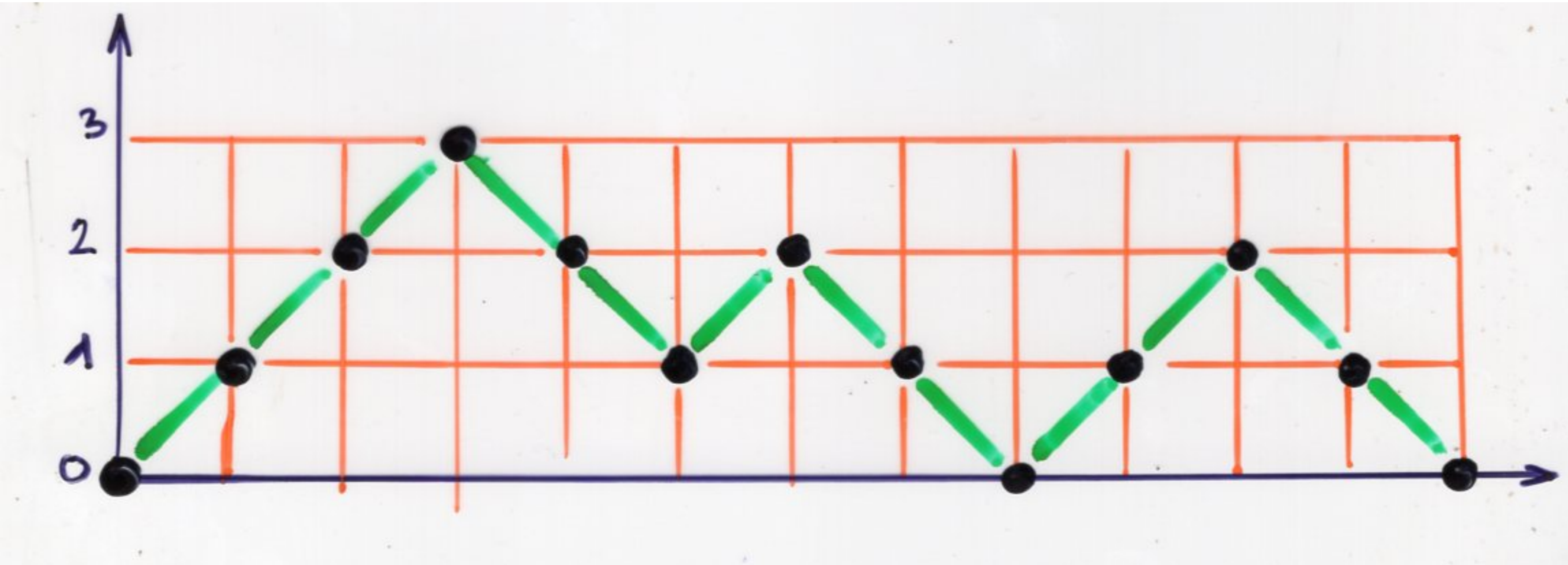


data structures

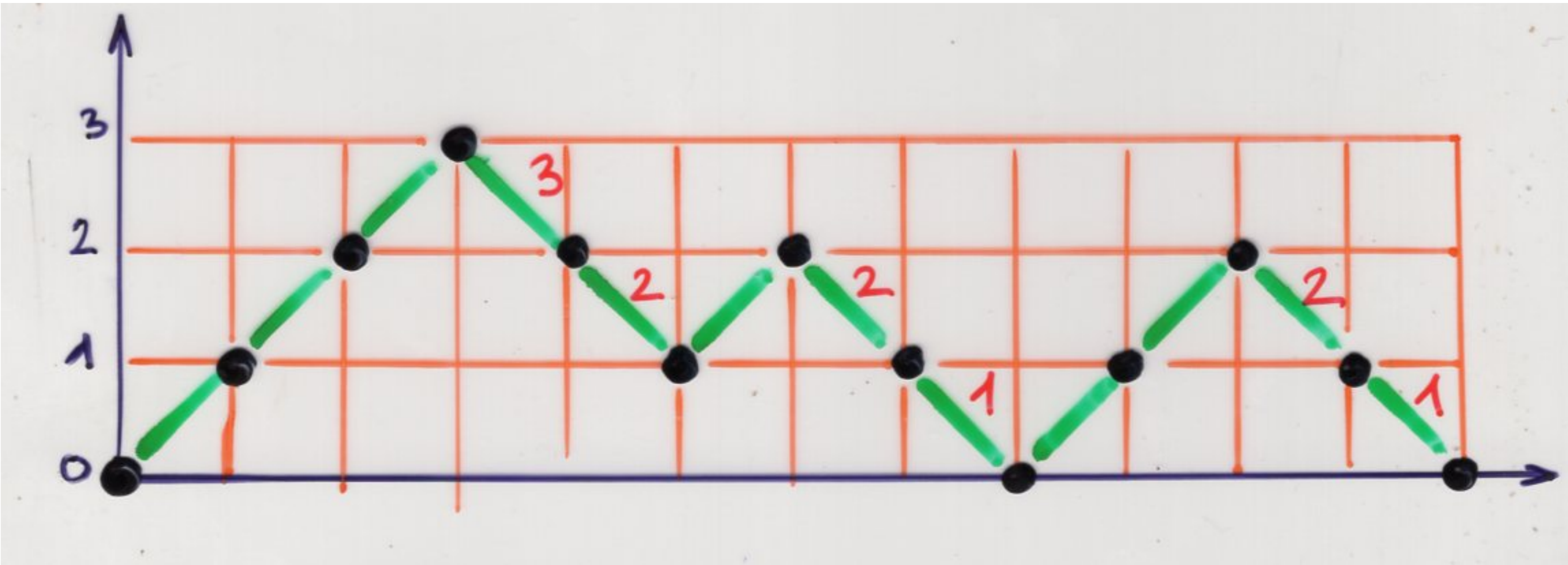
Computer Science

$$AS - SA = I$$

Dyck path



Dyck path



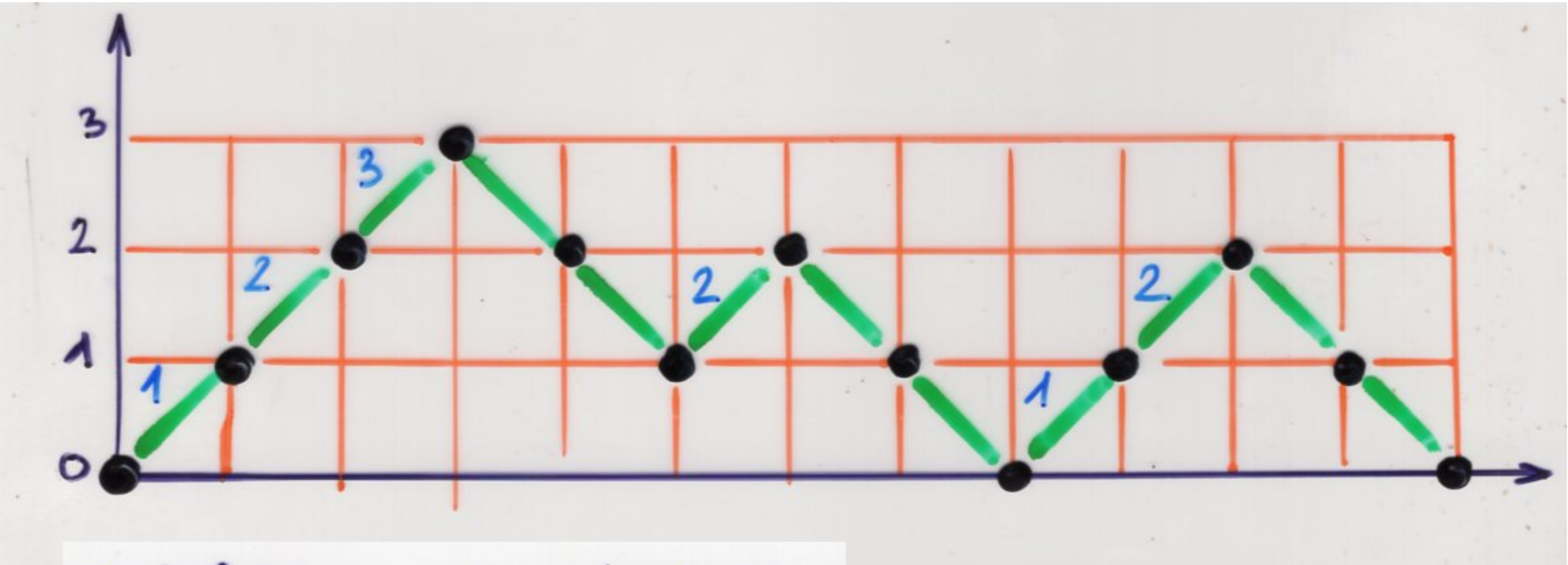
Polya urn

$$A \,|k\rangle = |(k+1)\rangle$$

$$S \,|k\rangle = k\,|(k-1)\rangle$$

Priority queue

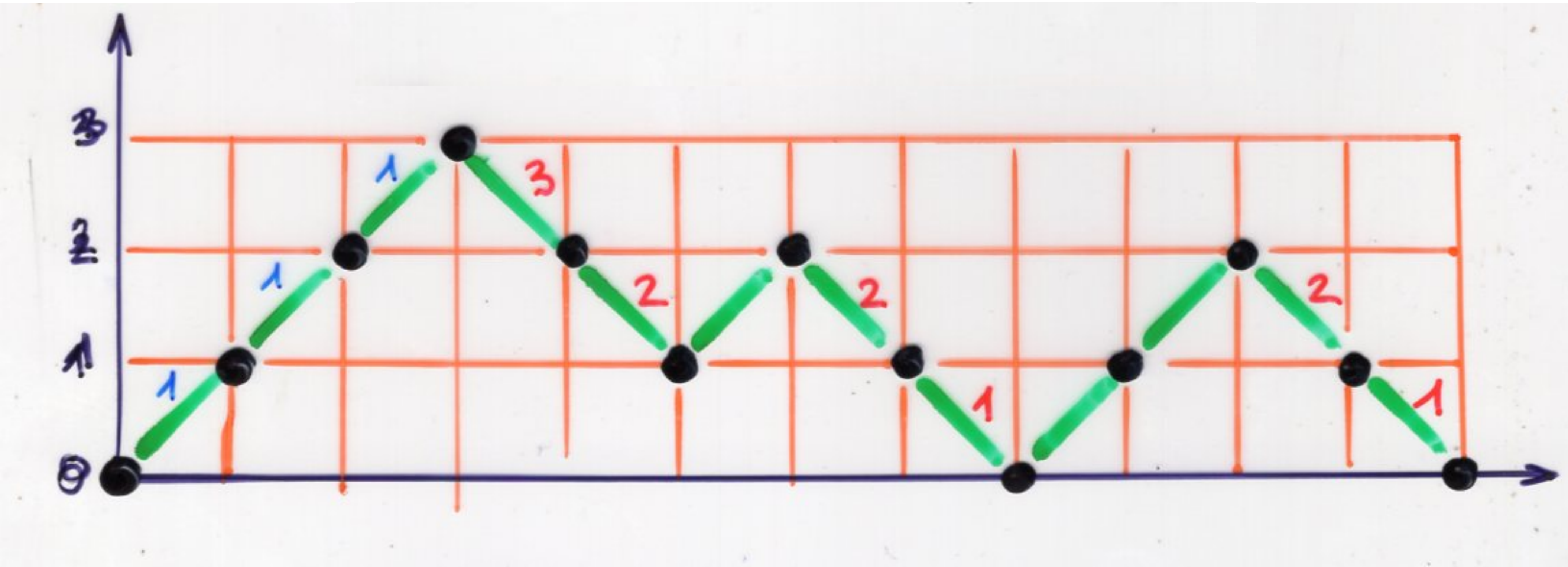
Dyck path



$$A \mid k \rangle = (k+1) \mid (k+1) \rangle$$

$$S \mid k \rangle = \mid (k-1) \rangle$$

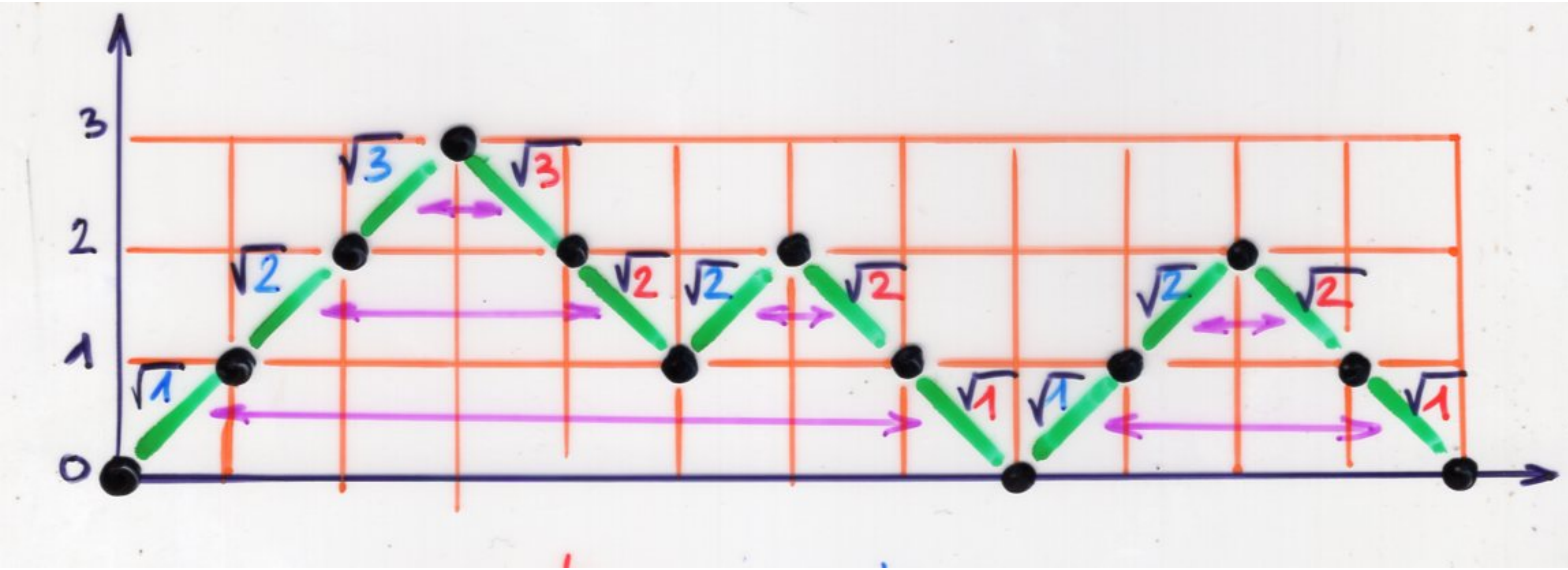
Dyck path



Dyck path

a annihilation D

$a^{\dagger}$ creation U



quantum mechanics

$$a\,|n\rangle = \sqrt{n}\,|(n-1)\rangle$$
$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|(n+1)\rangle$$

quantum mechanics

a annihilation D

$a^{\dagger}$ creation U

$$[a, a^{\dagger}] = 1$$

n number of particles

$\mathcal{H}$ Hilbert space $\quad \langle m | n \rangle = \delta_{m,n}$

$\{n\}$ basis of $\mathcal{H}$

Fock space

bosons

$$a|n\rangle = \sqrt{n}\,|(n-1)\rangle$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|(n+1)\rangle$$

$$N|n\rangle = n|n\rangle$$

$$N = a^{\dagger} a$$

computer science

data structure

integrated cost

Priority queue

data structure history

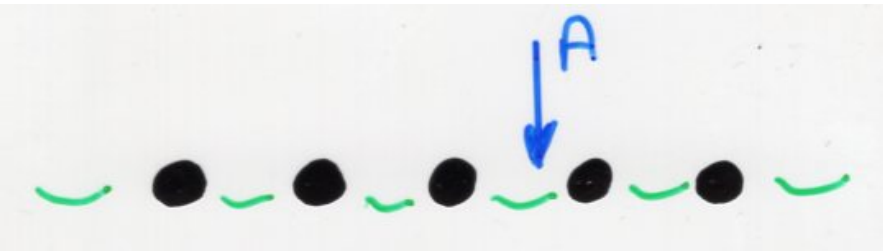
Françon (1976)

Françon, Flajolet, Vuillemin, (1980)

dictionnary data structure

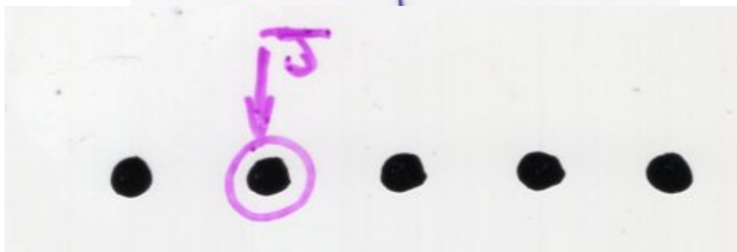
add or delete any element

$$N|n\rangle = n|n\rangle \qquad N = a^{\dagger} a$$



ask questions

J positive



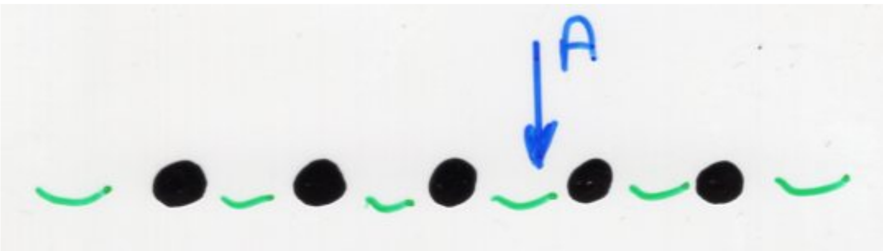
dictionnary data structure

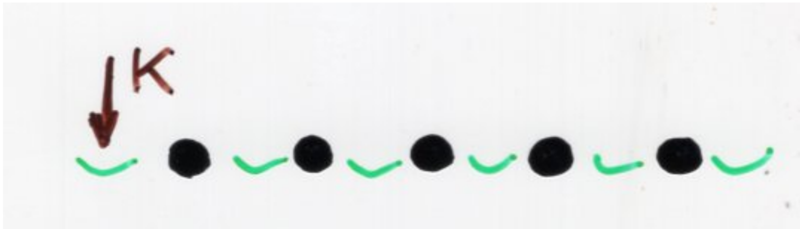
add or delete any element

ask questions

J positive

K negative



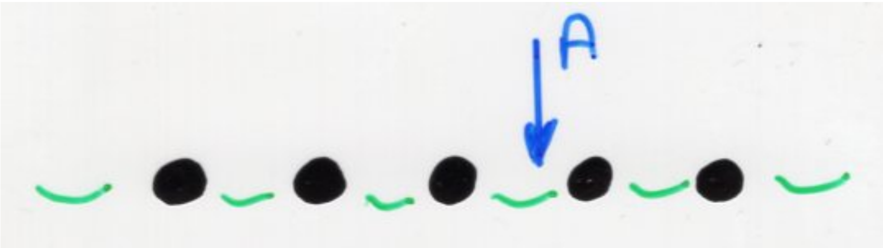


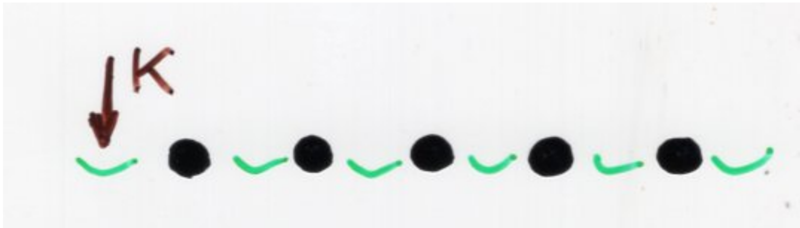
$$A\,|k\rangle = (k+1)\,|(k+1)\rangle$$

$$S\,|k\rangle = k\,|(k-1)\rangle$$

$$J\,|k\rangle = k\,|k\rangle$$

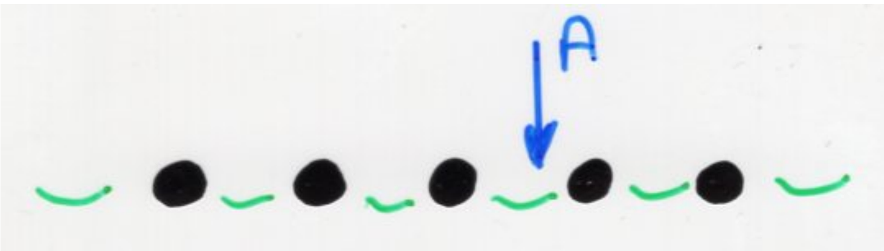
$$K\,|k\rangle = (k+1)\,|k\rangle$$

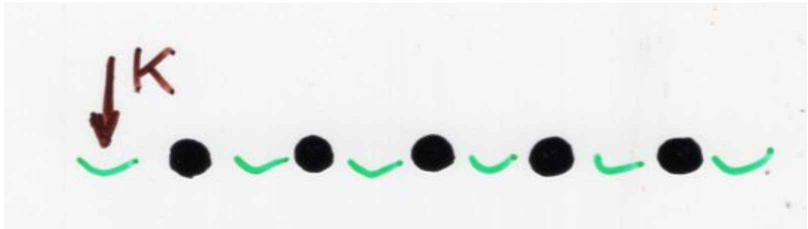


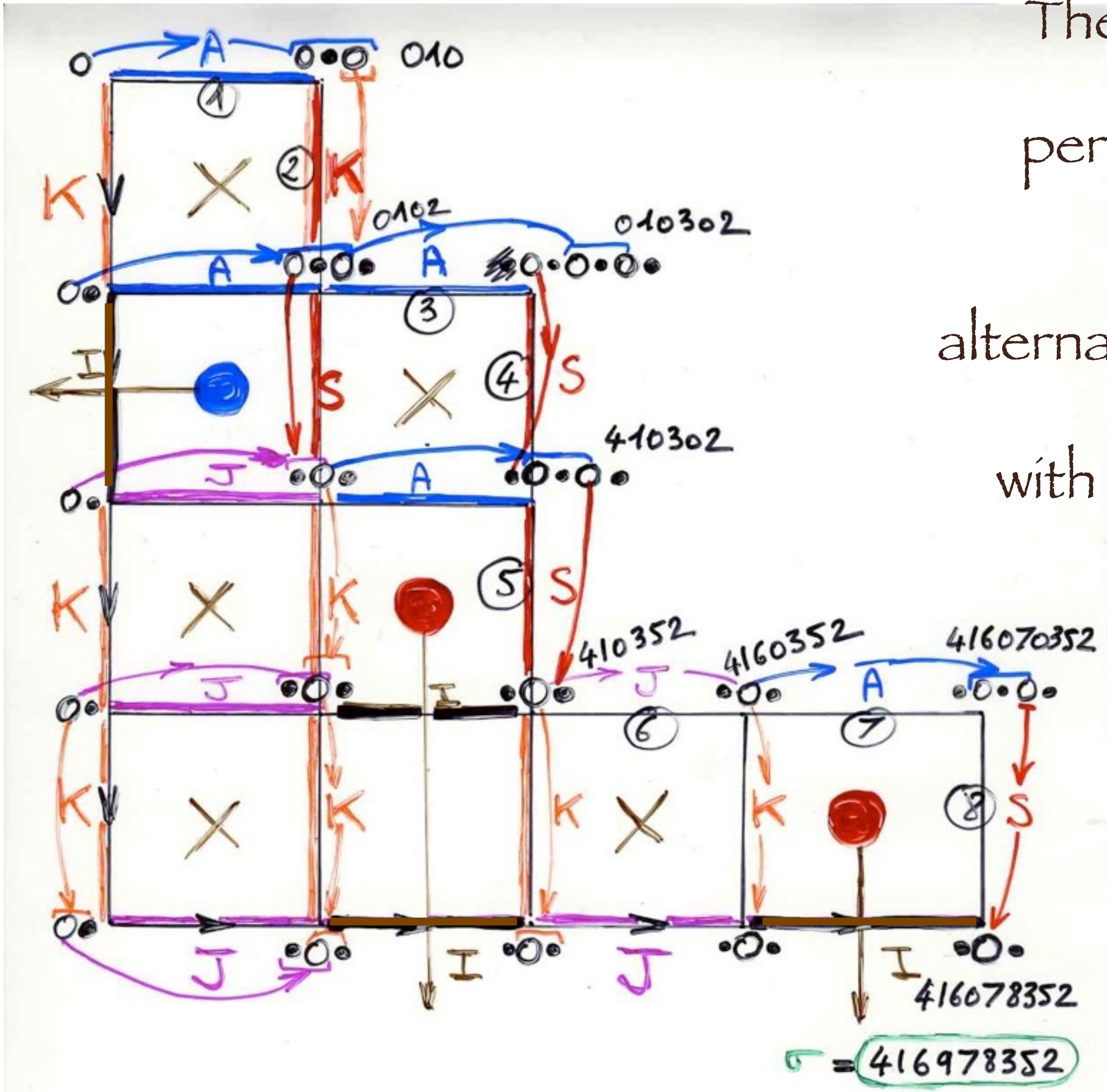


$$\begin{cases} D = A + K \\ E = S + J \end{cases}$$

$$DE = ED + E + D$$







The bijection

permutations

↕

alternating tableaux

with « local rules »

(next class)