Chapter 2
Quadratic algebra, Q-tableaux and planar automata

Ch2d
Commplements

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February 8, 2018

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complements:

ASM

TSSCPP

DPP

FPL

RS

……

The beautiful garden
of some jewels of combinatorics …

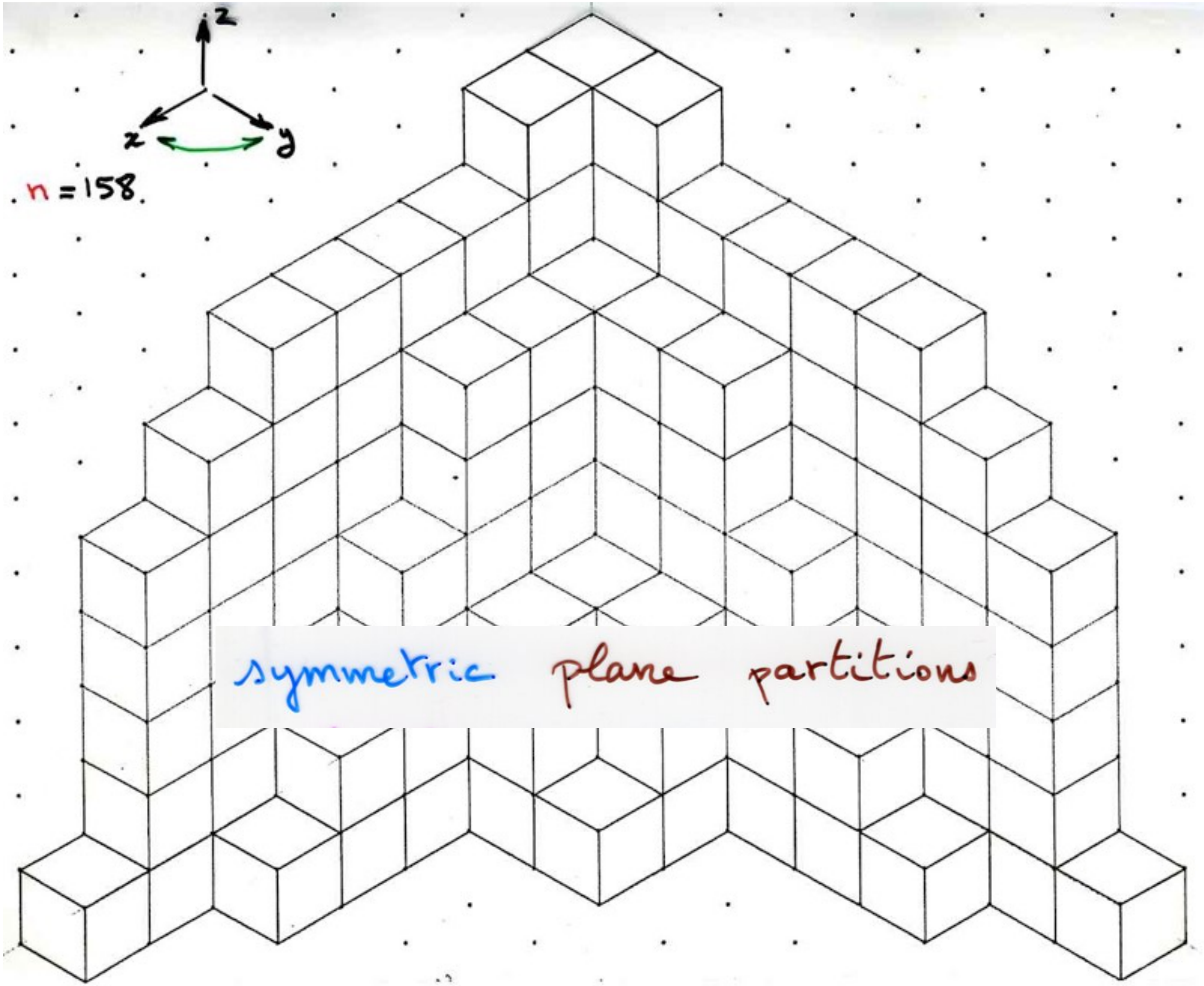
« deep combinatorics »

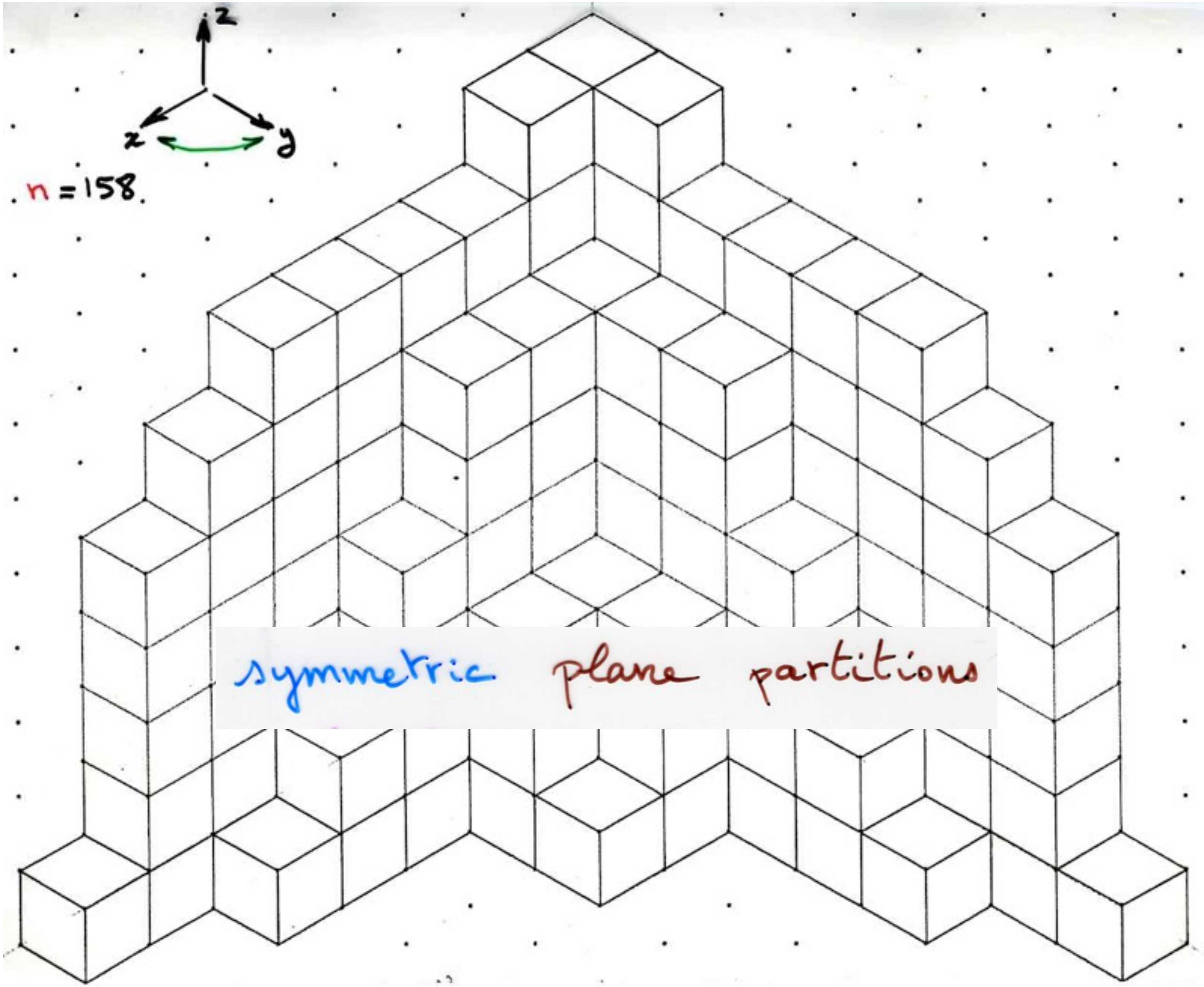
ASM (as "monotone triangle")
is the completion (as lattice)
of the Bruhat order
(strong order) on permutations

Lascoux, Schützenberger (1995)

Schubert and Grothendieck polynomials
defined from ASM

symmetric plane partitions


z
x
y
n = 158
symmetric plane partitions


z
x
y
n = 158
symmetric plane partitions

symmetric plane partitions

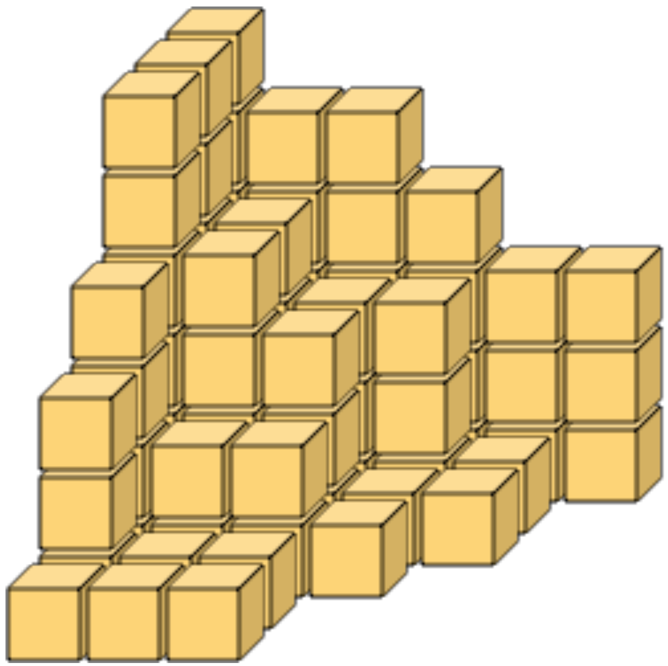
MacMahon conjecture

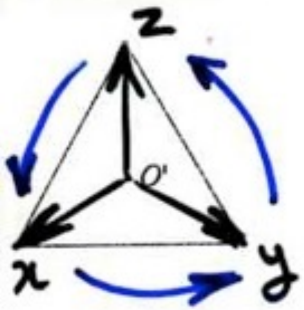
proof: G. Andrews (1978)
I. Macdonald (1979)

q- series

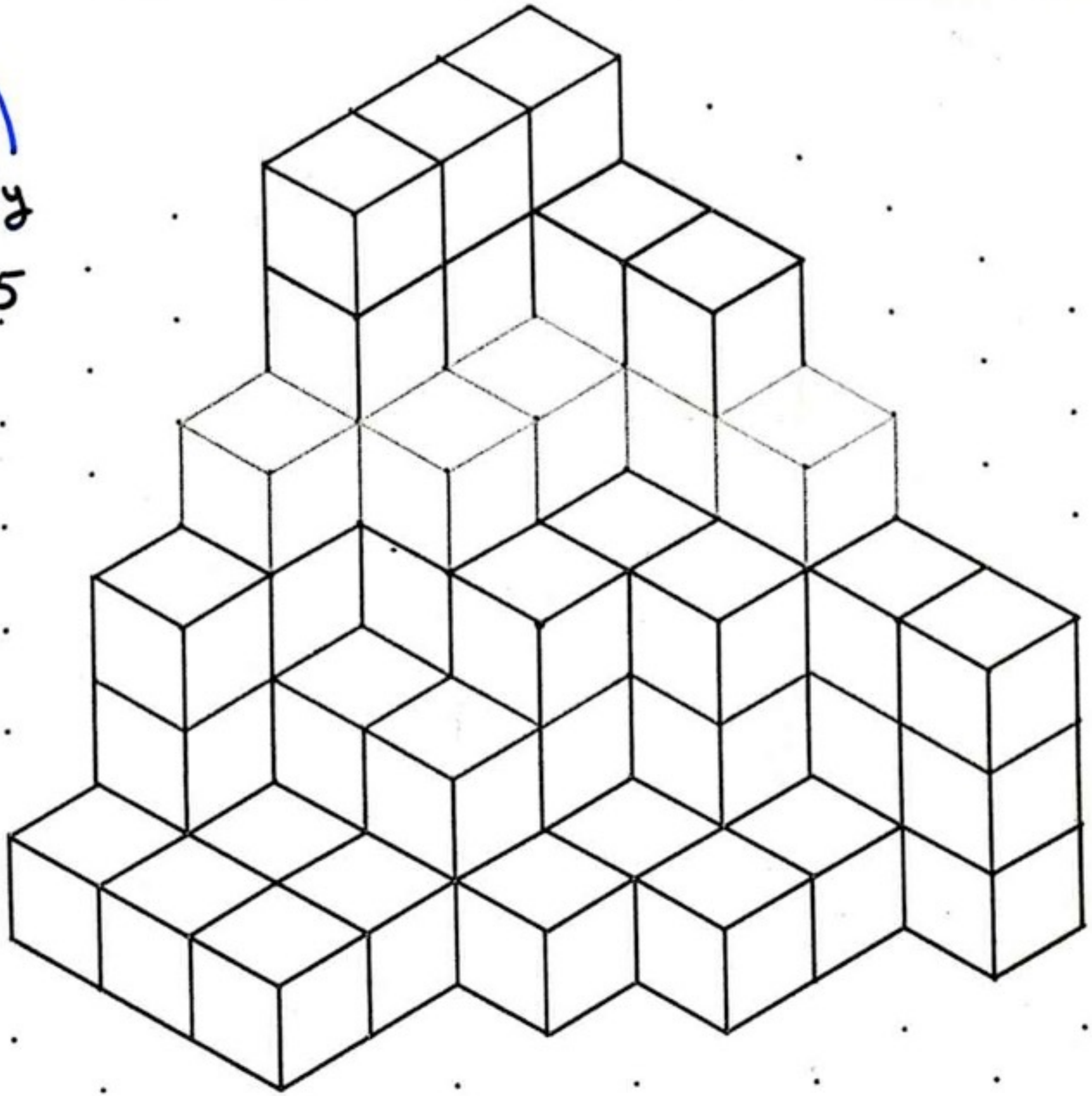
cyclicaly symmetric plane partitions

cyclicaly symmetric
plane partitions





$n = 75$



cyclicaly symmetric
plane partitions

Macdonald conjecture

proof: $(q=1)$ G. Andrews (1979)

W. Mills, D. Robbins, H. Rumsey (1982)

The ASM conjecture

C. I. Dodgson (1866)

Condensation of determinants

$$\det(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

$1, 2, 7, 42, 429, \ldots$

$$\frac{1!\ 4!}{n!\ (n+1)!} \qquad \frac{n-2)!}{(n+n-1)!}$$

alternating sign matrices conjecture

W. Mills, D. Robbins, H. Rumsey (1982)

Robbins

The Mathematical Intelligencer (1991)

“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

Descending plane partitions

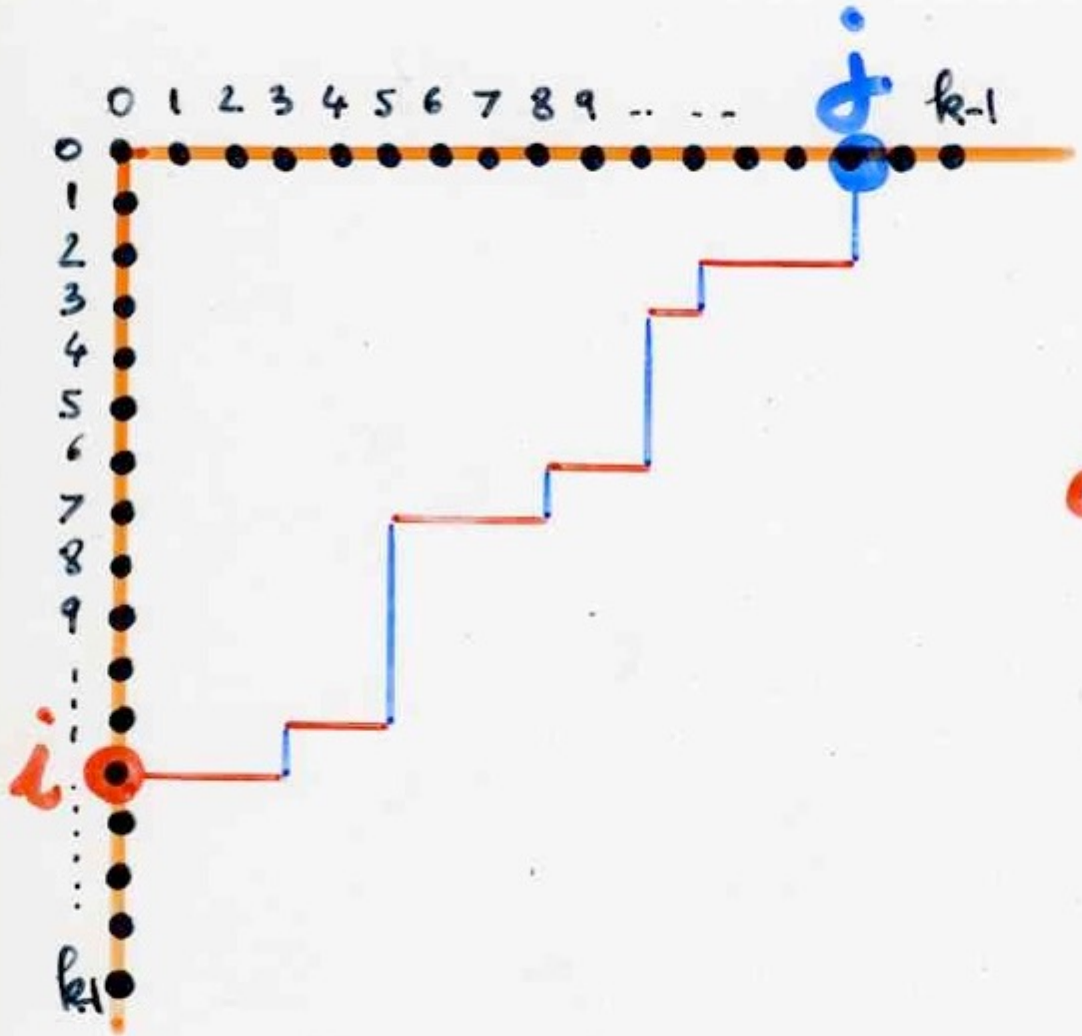
cyclicaly symmetric
plane partitions

Macdonald conjecture

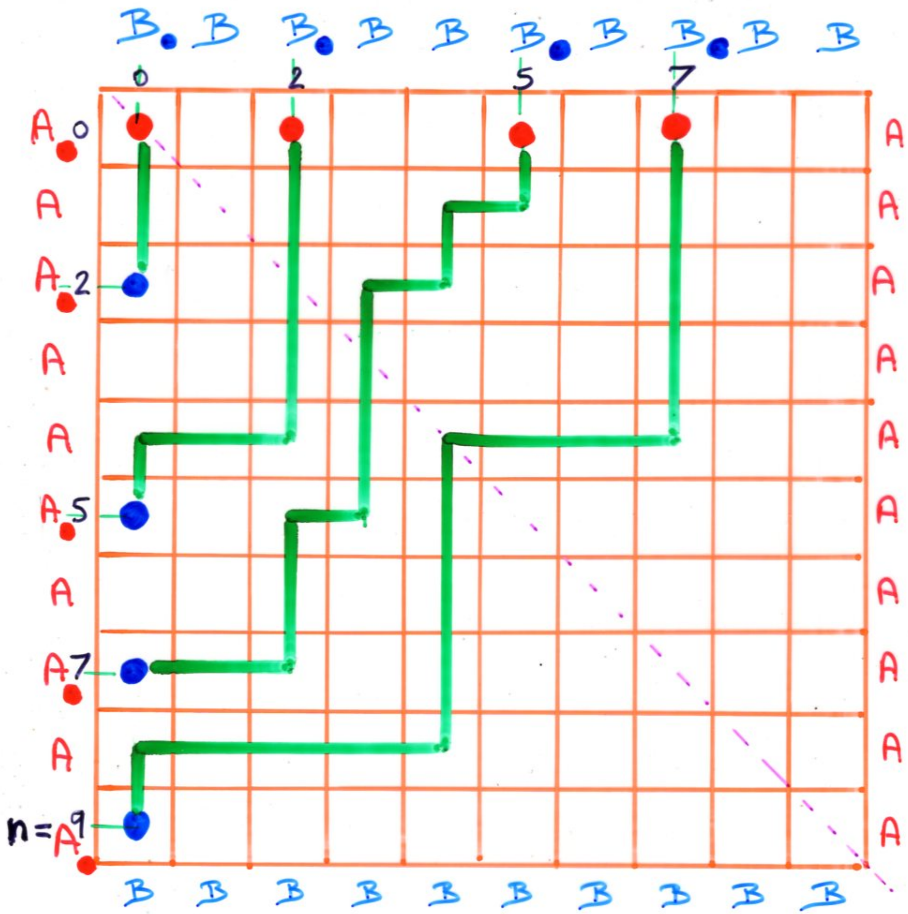
descending
plane partition

DPP

proof: ($q = 1$) G. Andrews (1979)
W. Mills, D. Robbins, H. Rumsey (1982)



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} =$$



DPP

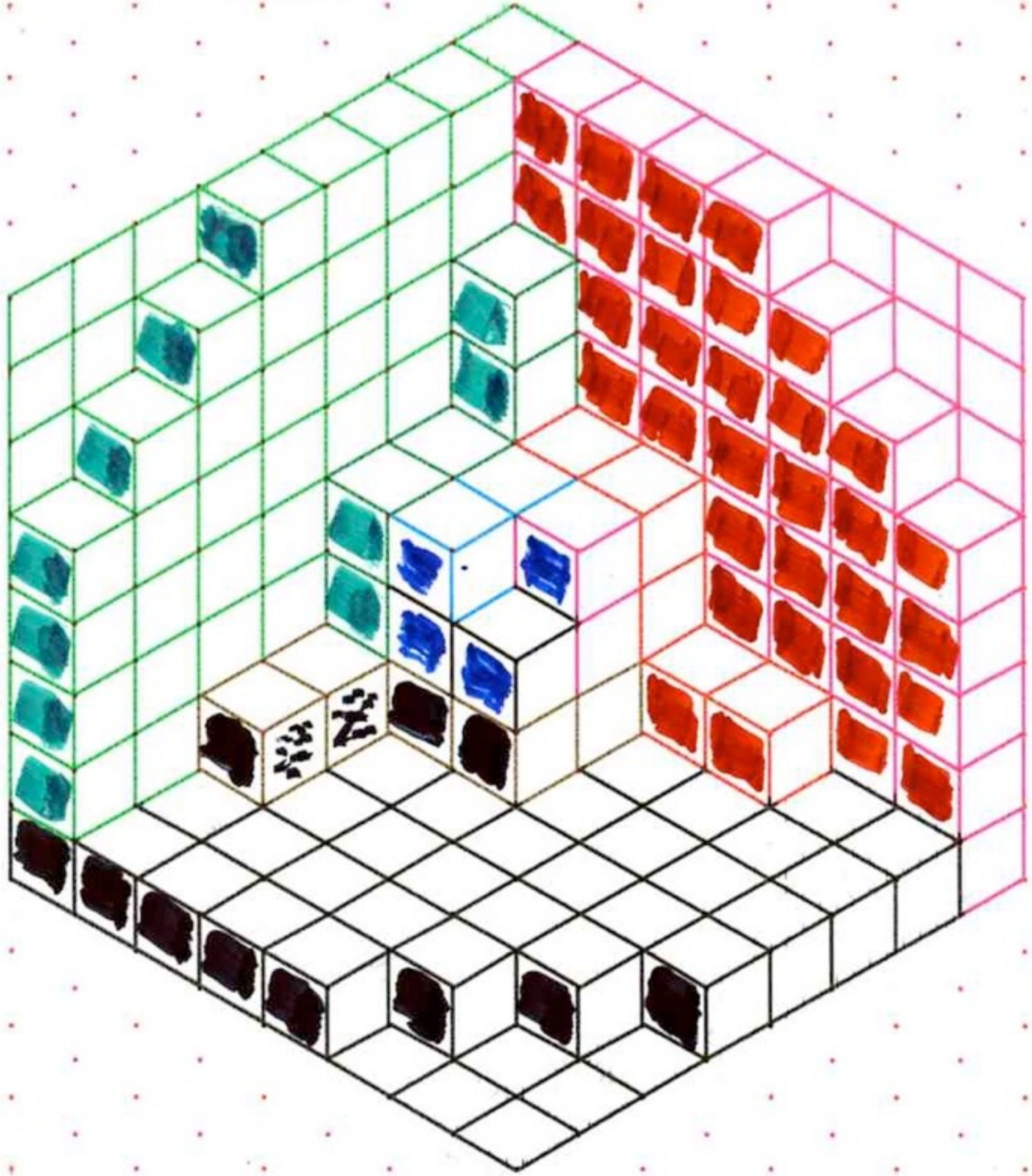
descending plane partition

Number A_n of ASM of size n

= number of DPP "size" n

no bijections !

totally symmetric plane partitions



Ten formulae

complement of
a plane partition

3D Ferrers diagrams
up to symmetries

10 formulae

$$\frac{(\text{product})}{(\text{product})}$$

Stanley (1985)

Table 2. Symmetry classes of plane partitions.

1. No restrictions

$$P_1(n) = \prod_{1 \le i,j,k \le n} \frac{i+j+k-1}{i+j+k-2}$$

2. Symmetric (exchange of x and y axes)

$$P_2(n) = \prod_{\substack{1 \le i,j,k \le n \\ i \le j}} \frac{i+j+k-1}{i+j+k-2}$$

3. Cyclically symmetric (cyclic permutations of the three coordinate axes)

$$P_3(n) = \prod_{1 \le i < j,k \le n} \frac{i+j+k-1}{i+j+k-2} \prod_{1 \le i \le j \le n} \frac{2i+j-1}{2i+j-2}$$

4*. Totally symmetric (all six permutations of the coordinate axes)

$$P_4(n) = \prod_{1 \le i \le j \le k \le n} \frac{i+j+k-1}{i+j+k-2}$$

5. Self-complementary

$$P_5(2n) = P_1(n)^2$$

6. Complement = mirror image

$$P_6(2n) = \binom{3n-1}{2n-1} \prod_{1 \le i \le j \le 2n-2} \frac{2n+i+j+1}{i+j+1}$$

7. Symmetric and self-complementary

$$P_7(2n) = P_1(n)$$

8. Cyclically symmetric and complement = mirror image

$$P_8(2n) = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i+1)!(4i)!}$$

9*. Cyclically symmetric and self-complementary

$$P_9(2n) = P_{10}(2n)^2$$

10*. Totally symmetric and self-complementary

$$P_{10}(2n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Remarks: $P_k(n)$ gives the formula for the number of plane partitions from the k-th symmetry class whose Ferrers graph fits in an n-by-n-by-n box. There are no plane partitions for odd n in the self-complementary symmetry classes. For those symmetry classes marked with an asterisk the given formula has not been proved.

totally symmetric
self-complementary
plane partition

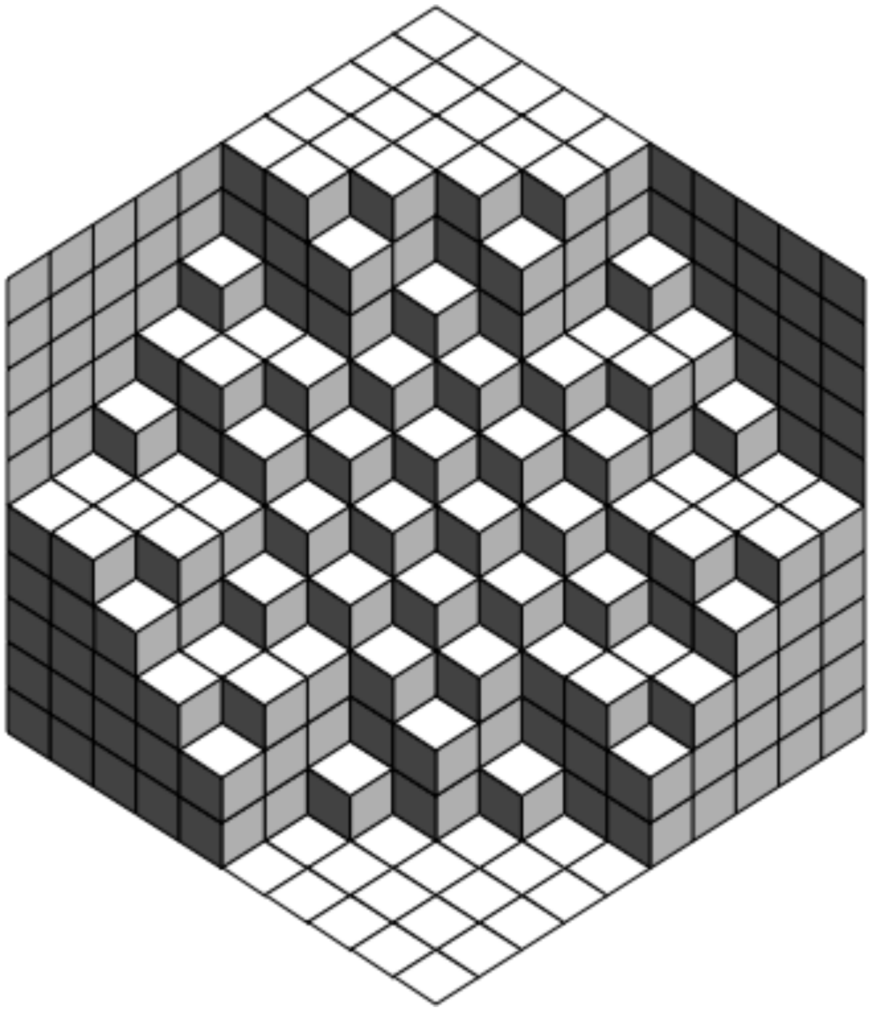
Number A_n of ASM of size n

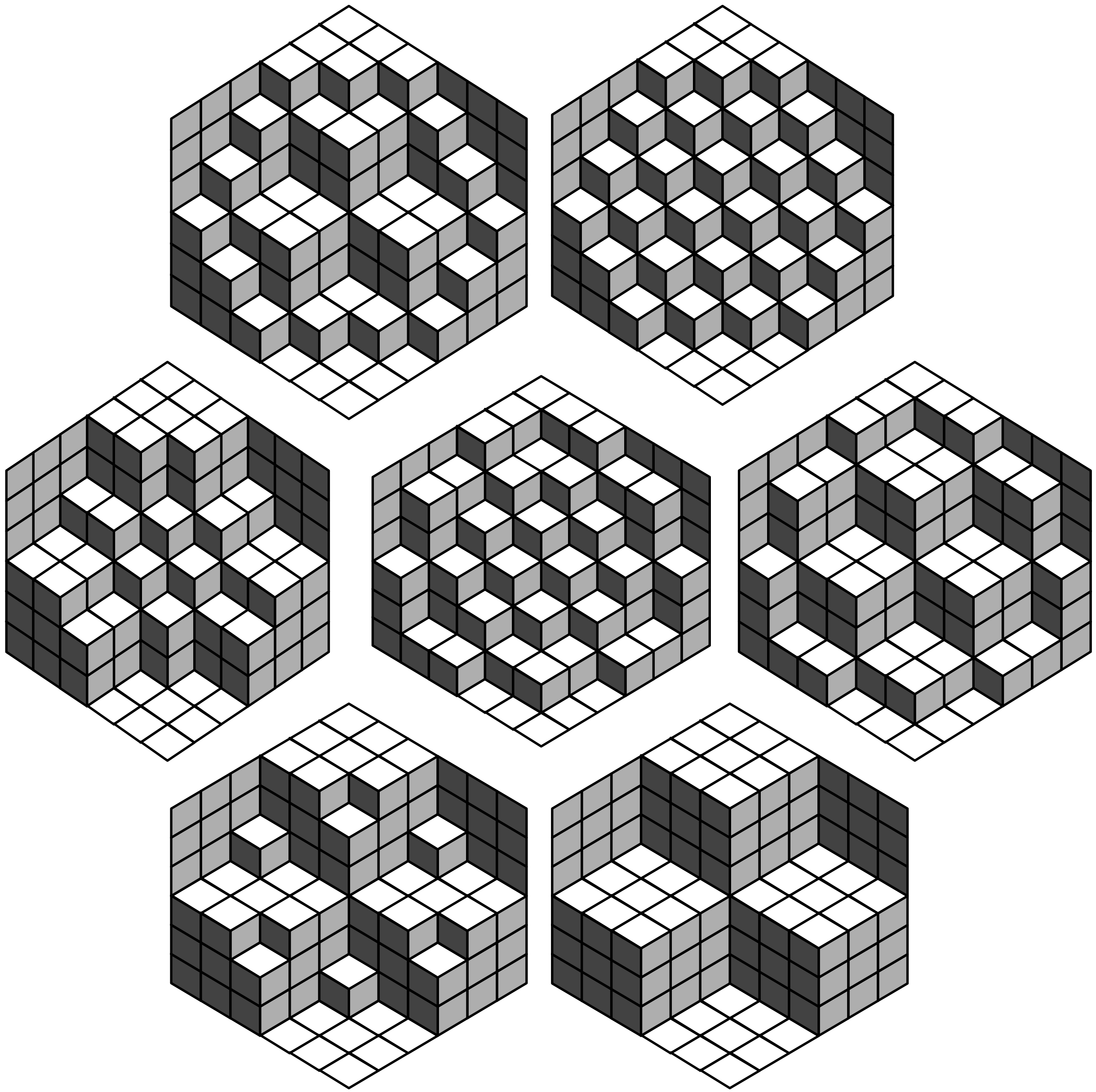
= number of TSSCPP
in a $[2n] \times [2n] \times [2n]$ box

no bijections !

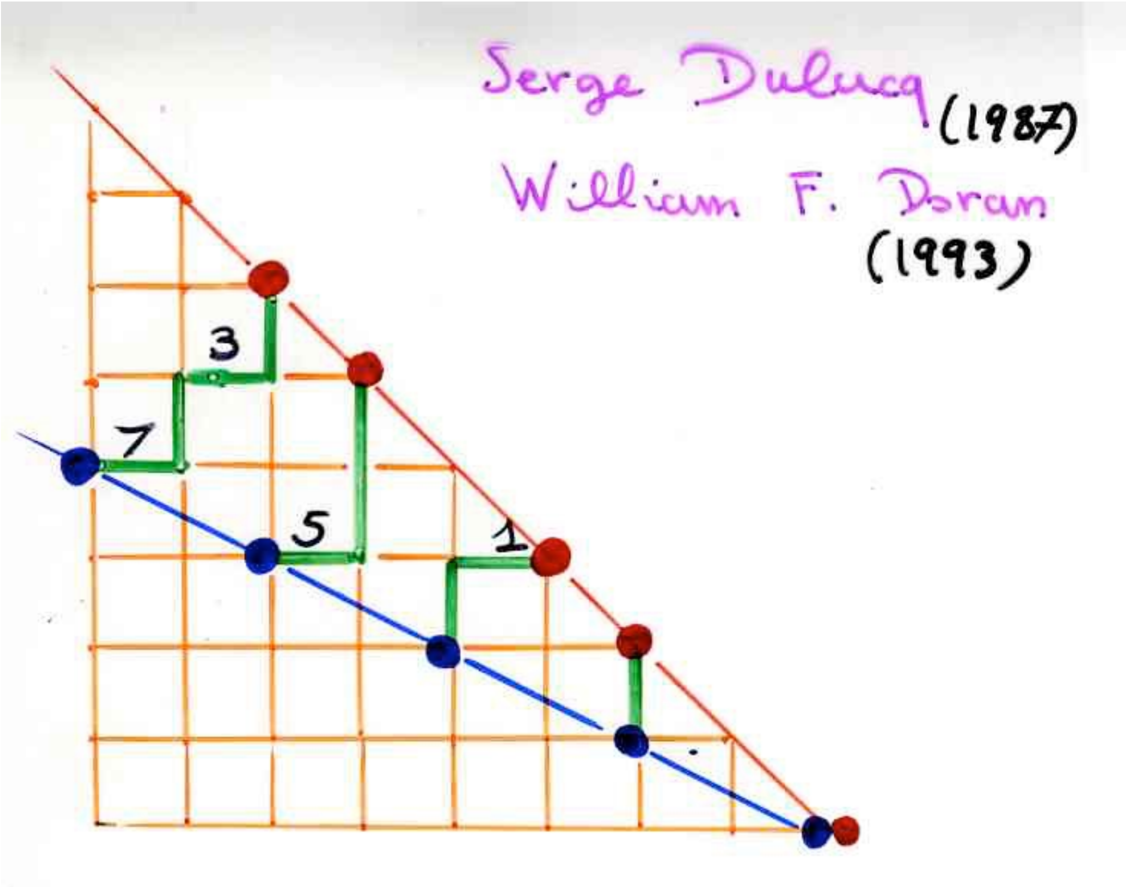
TSSCPP

totaly symmetric
self complementary
plane partitions

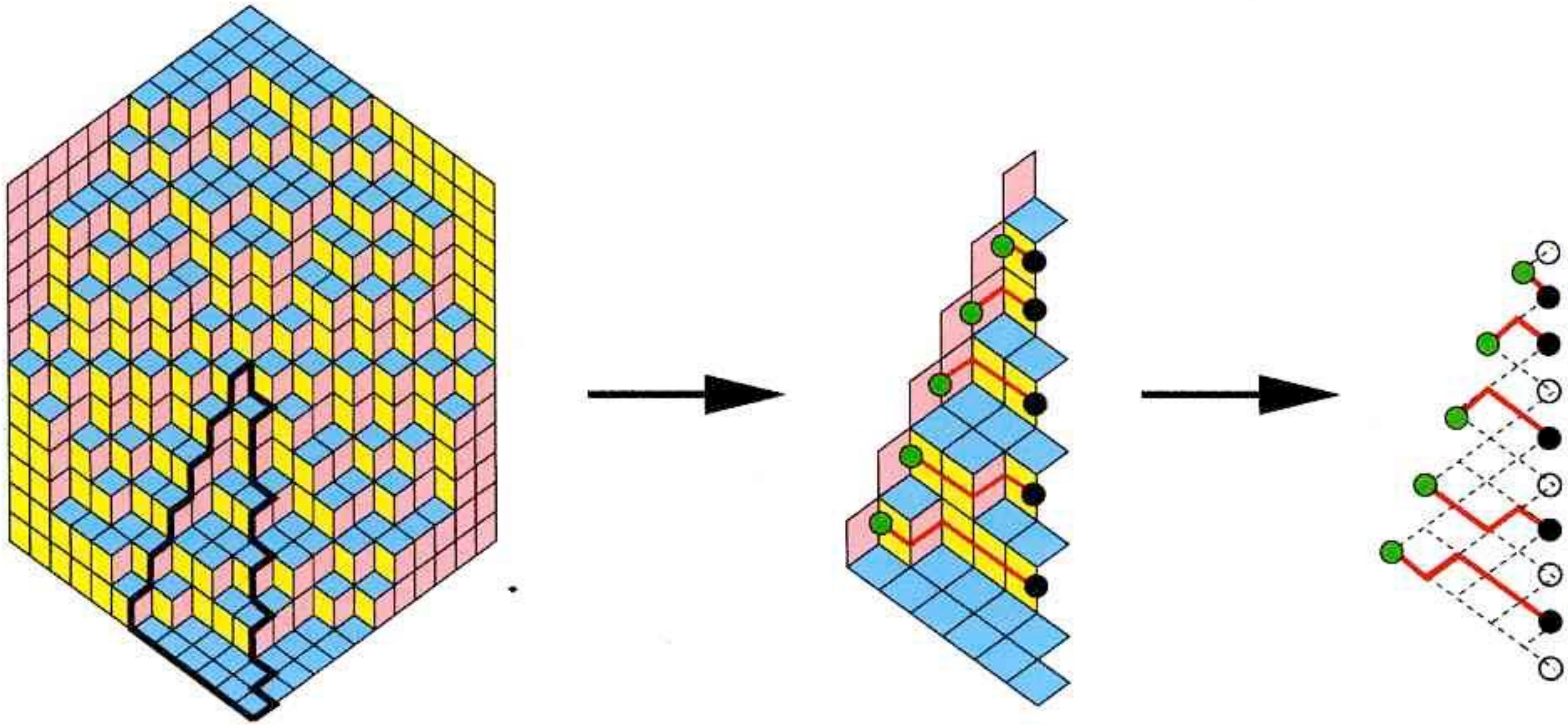




$$M = \begin{cases} \binom{j}{i-j} & 0 \leq j \leq i \leq 2j \\ 0 & \text{sinon} \end{cases}$$



$$\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3 & 4 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 6 & 5 & 1 \end{matrix}$$



Di Francesco (2006)

The last of the ten conjectures

TSSCPP

"tour de force" G. Andrews (1994)

TSSCPP

↓

configuration of non-intersecting paths

↓

Pfaffian

↓

determinant

↓

calculus

hypergeometric series

WZ methodology

The Proof of the AMS conjecture

D. Zeilberger (1992- 1995)

(+ 90 "checkers")

1st Proof of the ASM conjecture

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Two stones build two houses. Three build six houses. Four build four and twenty houses. Five build hundred and twenty houses. Six build Seven hundreds and twenty houses. Seven build five thousands and forty houses. From now on, [exit and] ponder what the mouth cannot speak and the ear cannot hear.

(Sepher Yetsira IV,12)

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\dots(3n-2)!]/[n!(n+1)!\dots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

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1 To appear in Electronic J. of Combinatorics (Foata's 60th Birthday issue). Version of July 31, 1995; original version written December 1992. The Maple package ROBBINS, accompanying this paper, can be downloaded from the www address in footnote 2 below.

2 Department of Mathematics, Temple University, Philadelphia, PA 19122, USA.
E-mail:`zeilberg@math.temple.edu`. WWW:`http://www.math.temple.edu/~zeilberg`. Anon. ftp: `ftp.math.temple.edu`, directory `/pub/zeilberg`. Supported in part by the NSF.

3 See the Exodion for affiliations, attribution, and short bios.

Subsublemma 1.1.3:

$$\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi\left[\frac{x_1x_2^2\dots x_k^k}{(1-x_k)(1-x_kx_{k-1})\cdots(1-x_kx_{k-1}\dots x_1)}\right]=\frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)} \quad . \tag{$Issai$}$$

[Type '`S113(k);`' in ROBBINS, for specific k.]

Proof : See [PS], problem VII.47. Alternatively, (Issai) is easily seen to be equivalent to Schur's identity that sums all the Schur functions ([Ma], ex I.5.4, p. 45). This takes care of subsublemma 1.1.3. □

Inserting $(Issai)$ into $(Stanley)$, expanding $\prod_{1\le i<j\le k}(x_j-x_i)$ by Vandermonde's expansion,

$$\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\cdots x_k^{k-1}) \quad ,$$

using the antisymmetry of Δ_k once again, and employing crucial fact $\aleph_4$, we get the following string of equalities:

$$b_k(n)=\frac{1}{k!}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n}x_i^{n+k-1}}\left(\frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)}\right)\right\}$$

$$=\frac{1}{k!}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\dots x_k^{k-1})\right)\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\prod_{i=1}^k x_i^{i-1}\right)\right]\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right]\right\}$$

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$=\frac{1}{k!}\left(\sum_{\pi\in\mathcal{S}_k}1\right)CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$=CT_{x_1,\dots,x_k}\left\{\frac{\Delta_k(x_1,\dots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}\ , \tag{$George'''$}$$

where in the last equality we have used Levi Ben Gerson's celebrated result that the number of elements in $\mathcal{S}_k$ (the symmetric group on k elements,) equals $k!$. The extreme right of $(George''')$ is exactly the right side of $(MagogTotal)$. This completes the proof of sublemma 1.1. □

2^{nd} proof: G. Kuperberg (1995)

6-vertex model

with "domain wall"

boundary condition

fonction de partition

Gaudin

Korepin, Bogoliubov, Izergin

"Quantum Inverse Scattering Method and Correlation Functions" (1993)

$$Z_n(\vec{x}, \vec{y}; a) = \frac{\prod_{i=1}^{n} x_i/y_i \prod_{1 \le i,j \le n} (x_i/y_j)(ax_i/y_j)}{\prod_{1 \le i < j \le n} (x_i/x_j)(y_j/y_i)} \det(M)$$

$$M = \frac{1}{(x_i/y_j)(ax_i/y_j)}$$

équation Yang-Baxter

Number A_n of ASM of size n

= number of DPP "size" n

= number of TSSC PP in a $[2n] \times [2n] \times [2n]$ box

no bijections !

D. Bressoud's book (1999)

quantum mechanics:
spin chain model

Razumov-Stroganov
(ex-) conjecture (2000)

Spin chains and combinatorics

A. V. Razumov, Yu. G. Stroganov

Institute for High Energy Physics
142284 Protvino, Moscow region, Russia

(J. R. R. Tolkien)

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N=3: \quad \psi_{001}=1;$$
$$N=5: \quad \psi_{00011}=1, \quad \psi_{00101}$$
$$N=7: \quad \psi_{0000111}=1, \quad \psi_{000} \quad 3, \quad \psi_{0010011}=4 \quad \psi_{0010101}=7.$$

?

All components not included in t btained by shifting. Notice that the components of the ground state a cordance with the Perron–Frobenius theorem.

Let us continue the list. For ponents of the eigenvector with the energy $-27/2$ and $S_z=-1/2$ are

$$\psi_{000001111}=1, \quad \psi_{000010111}=4, \quad \psi_{000011011}=6, \quad \psi_{000100111}=7,$$
$$\psi_{000101011}=17, \quad \psi_{000101101}=14, \quad \psi_{000110011}=12, \quad \psi_{001001011}=21,$$
$$\psi_{001010011}=25, \quad \psi_{001010101}=42.$$

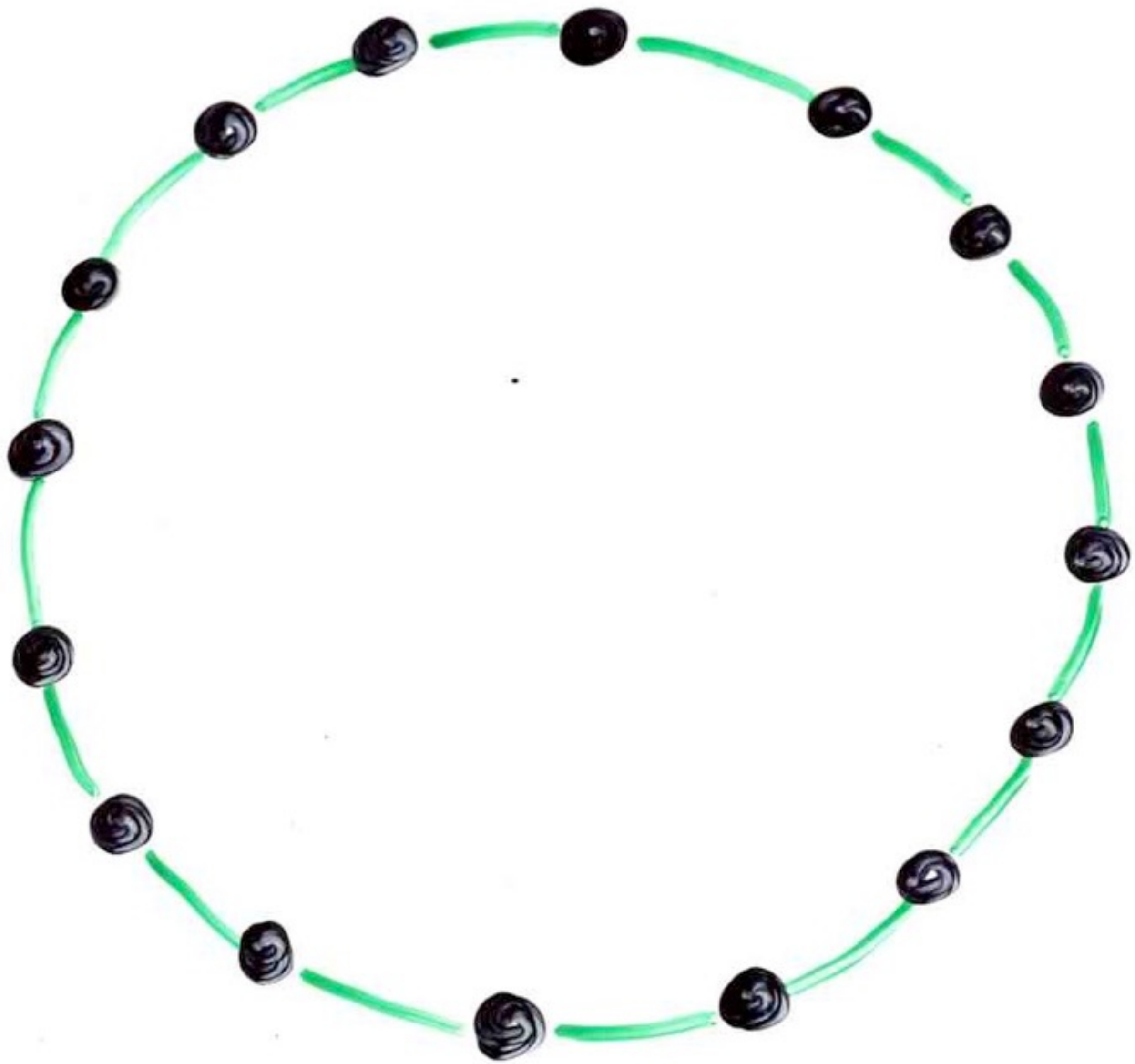
Let us co 1, 2, 7, 42, 429, ... vector with the energy $-27/2$

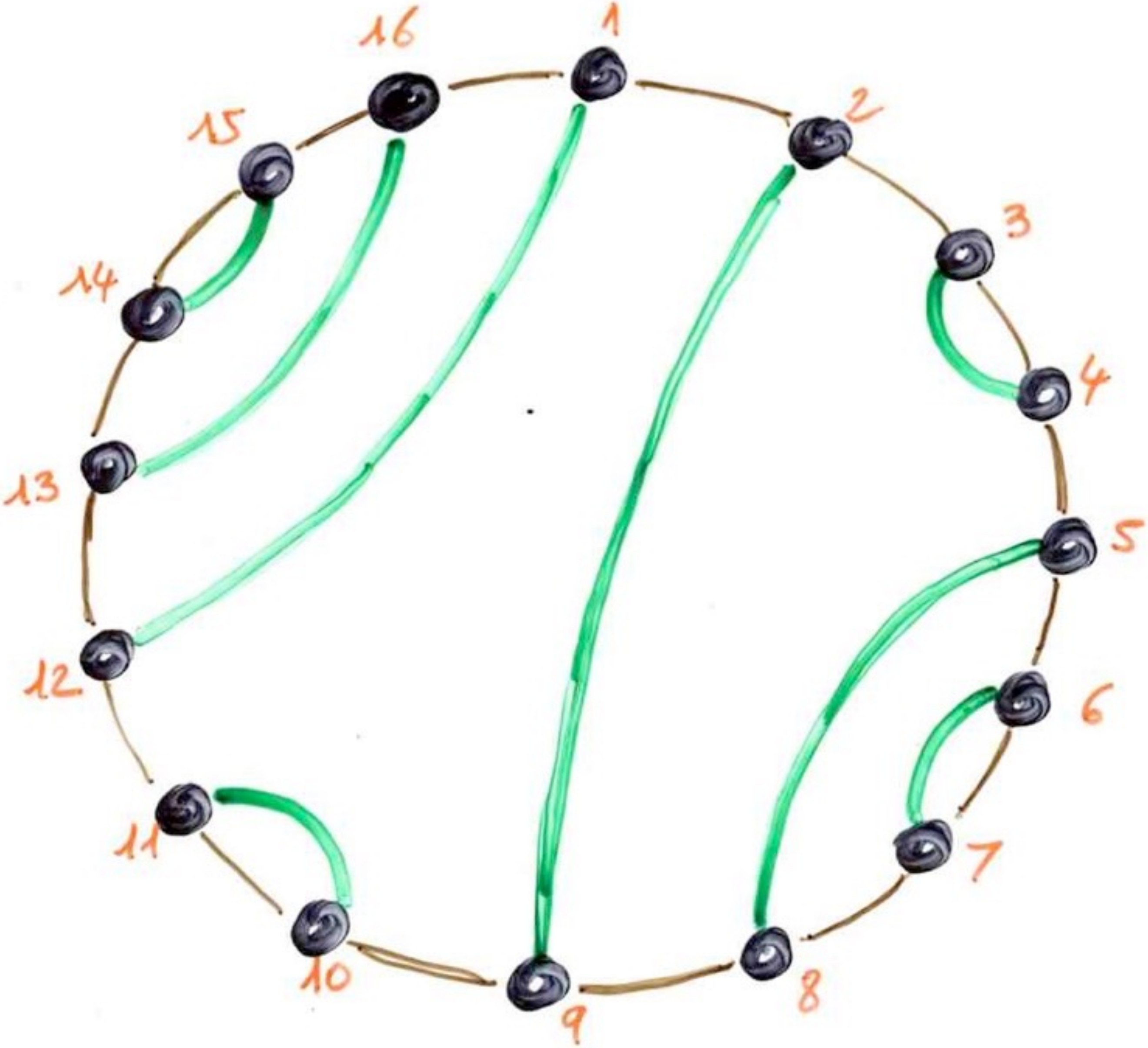
$$\psi_{000001111}=1, \quad \psi_{000010111}=4, \quad \psi_{000011011}=6, \quad \psi_{000100111}=7,$$
$$\psi_{000101011}=17, \quad \psi_{000101101}=14, \quad \psi_{000110011}=12, \quad \psi_{001001011}=21,$$
$$\psi_{001010011}=25, \quad \psi_{001010101}=42.$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101}=\psi_{000010111}=4.$$

Markov chain
on
chord diagrams

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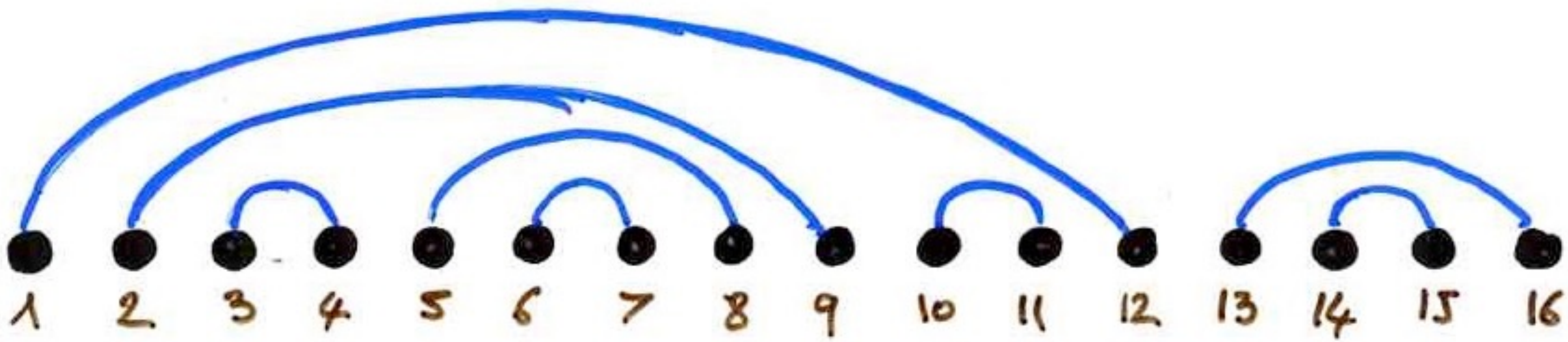
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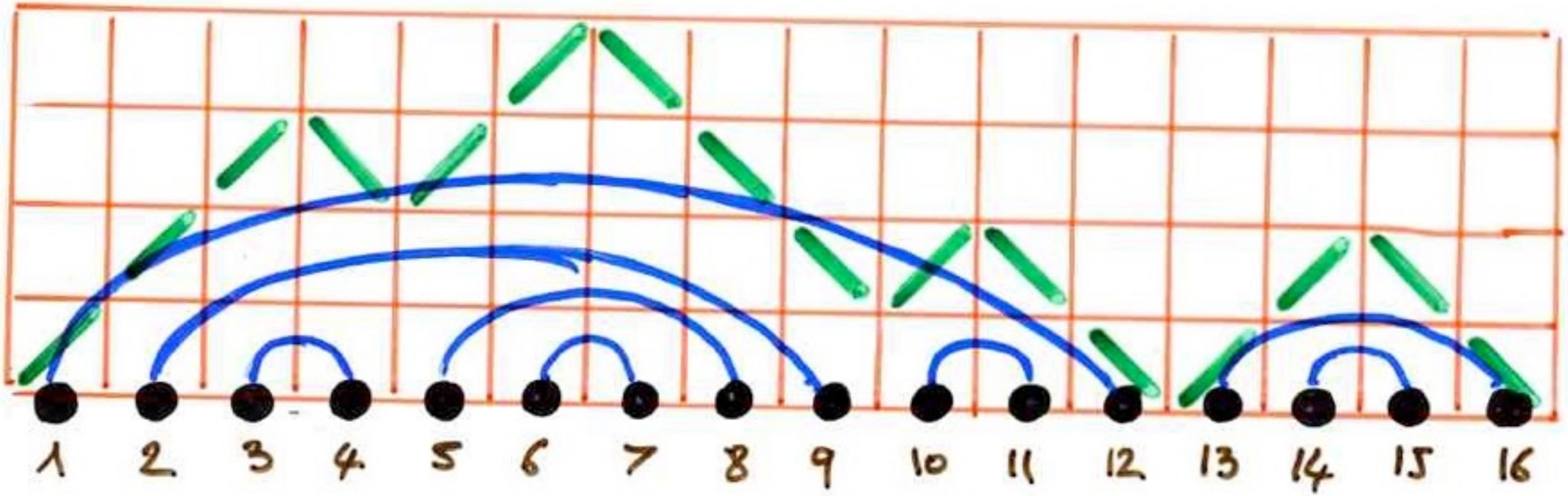
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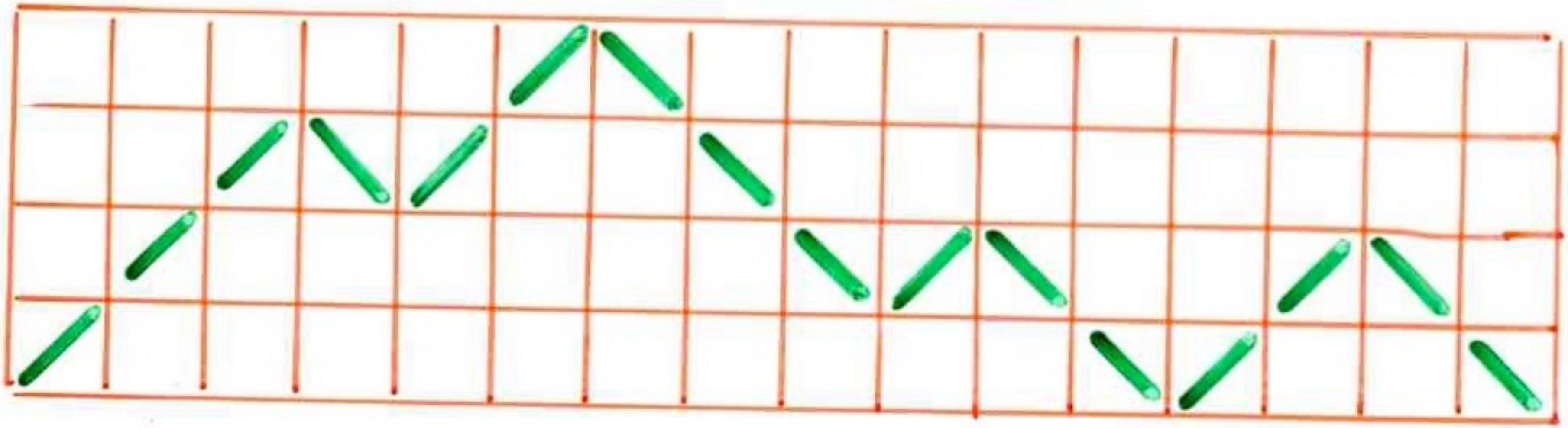

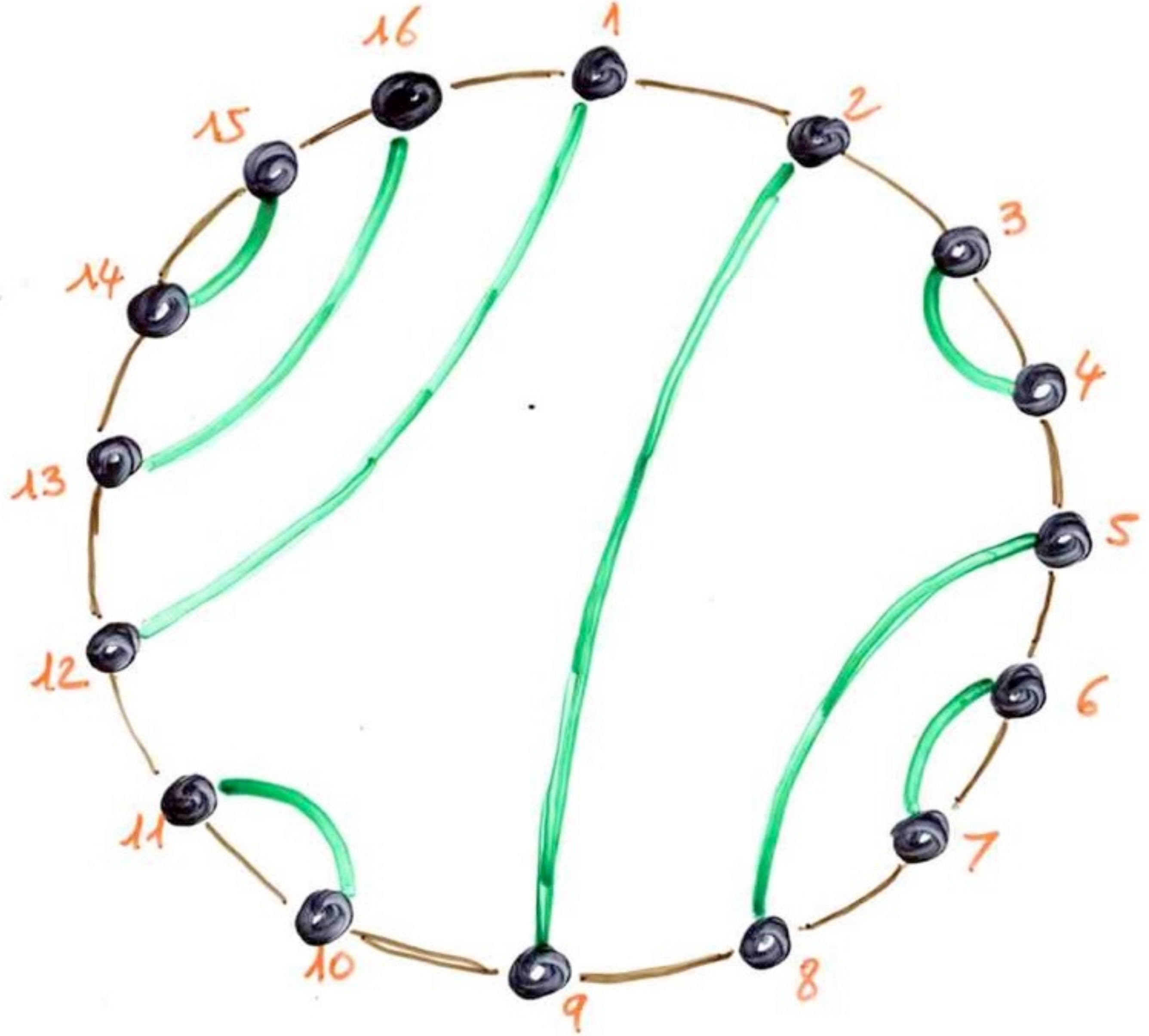
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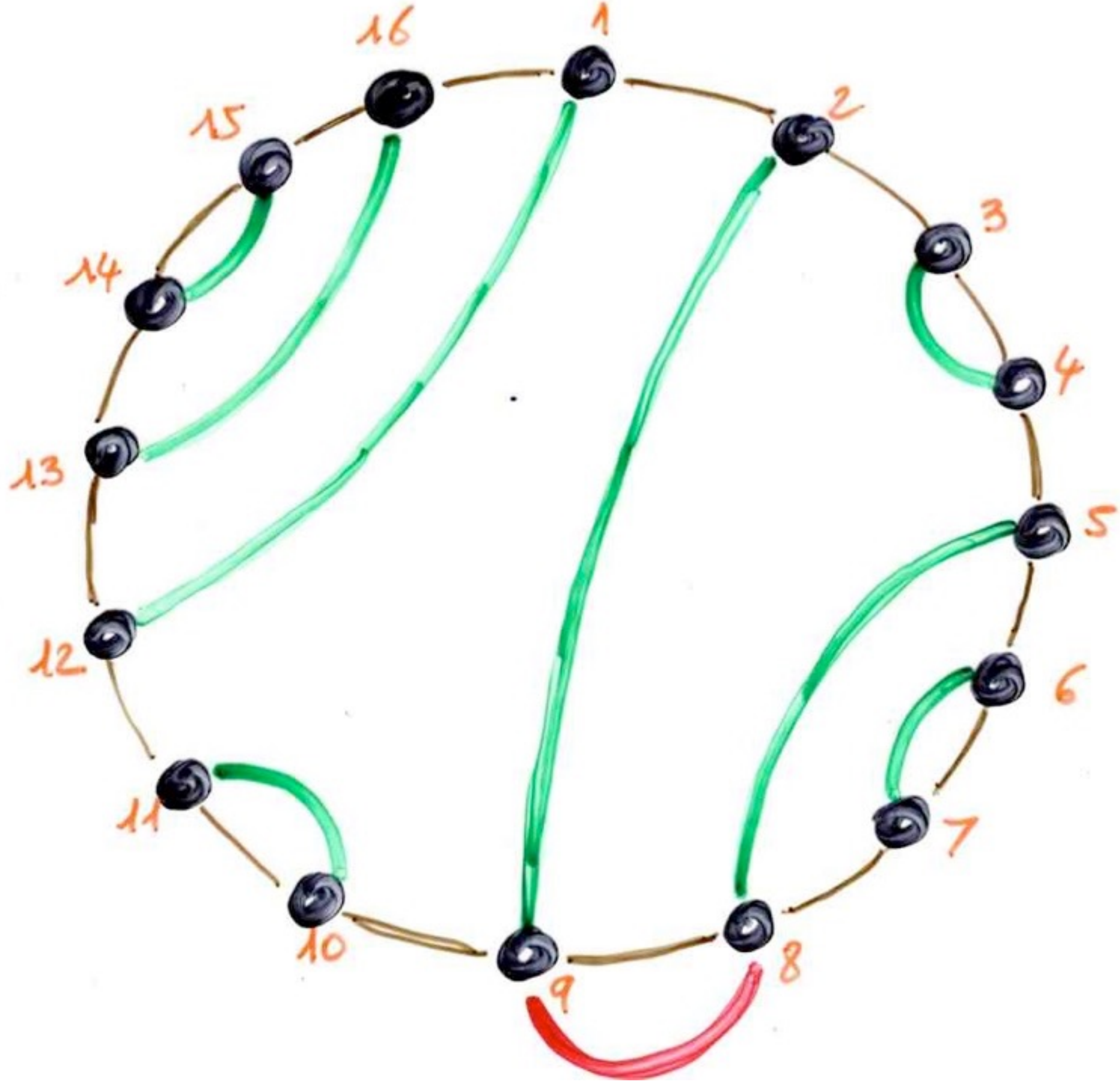
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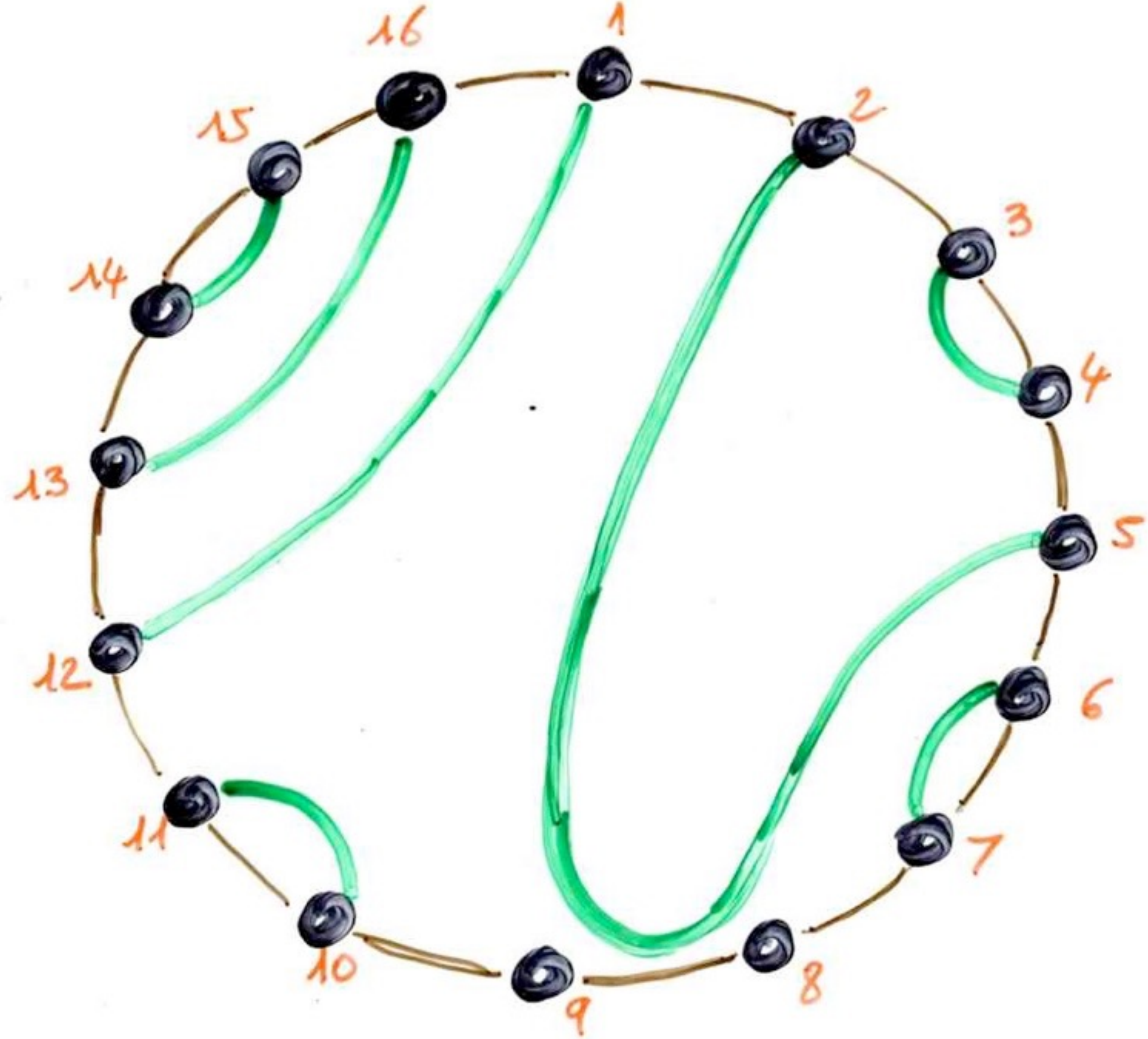
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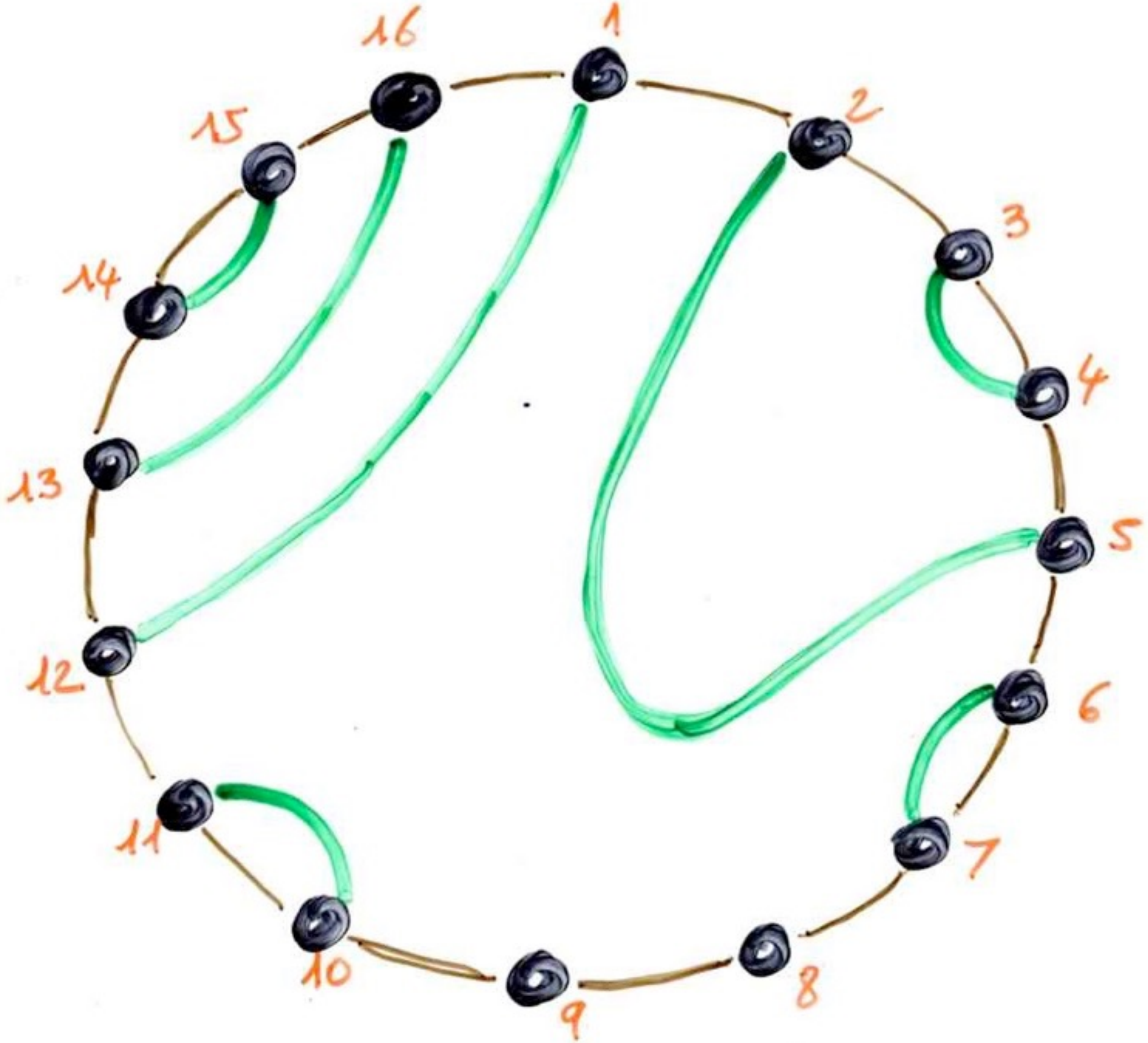
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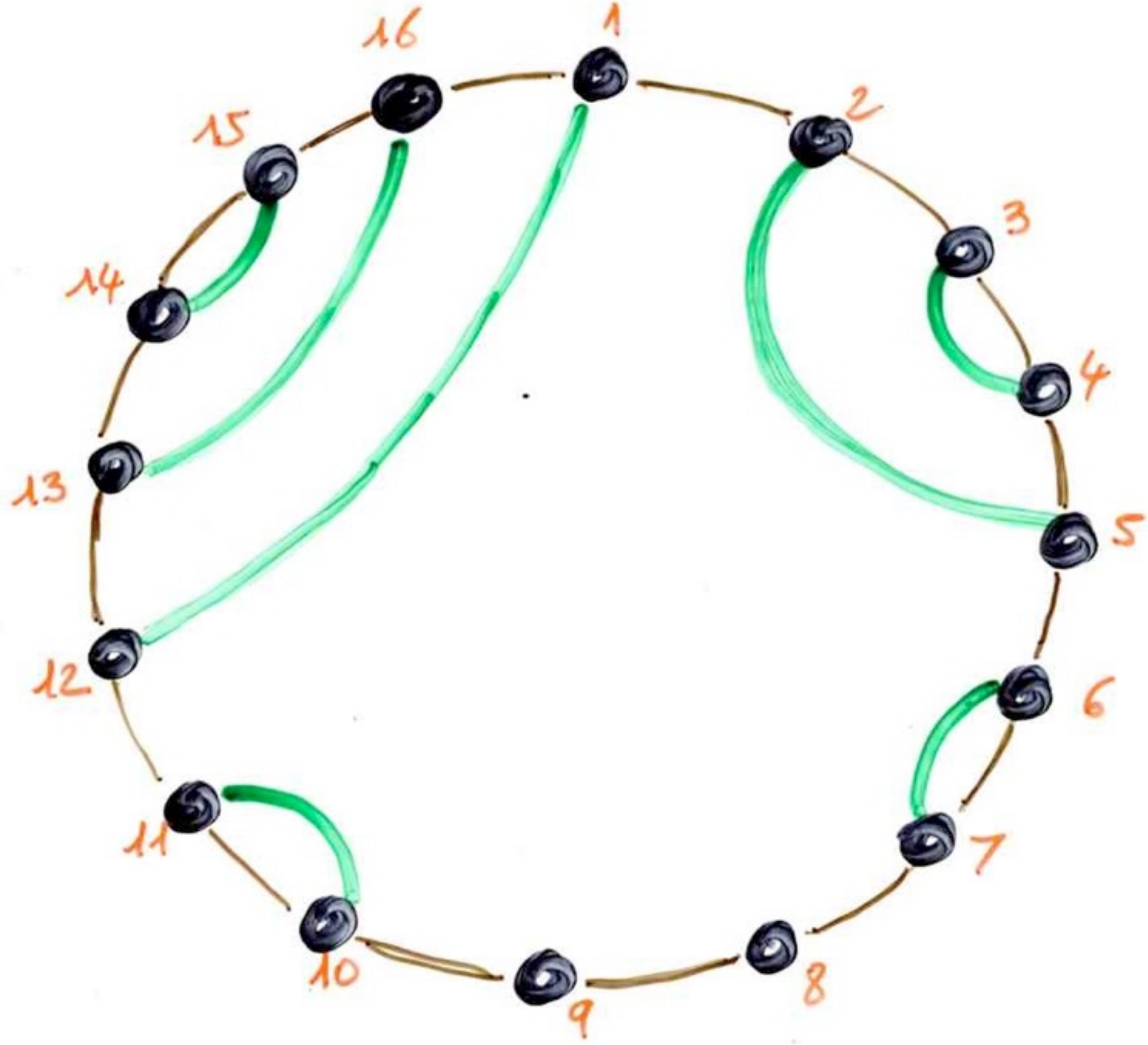
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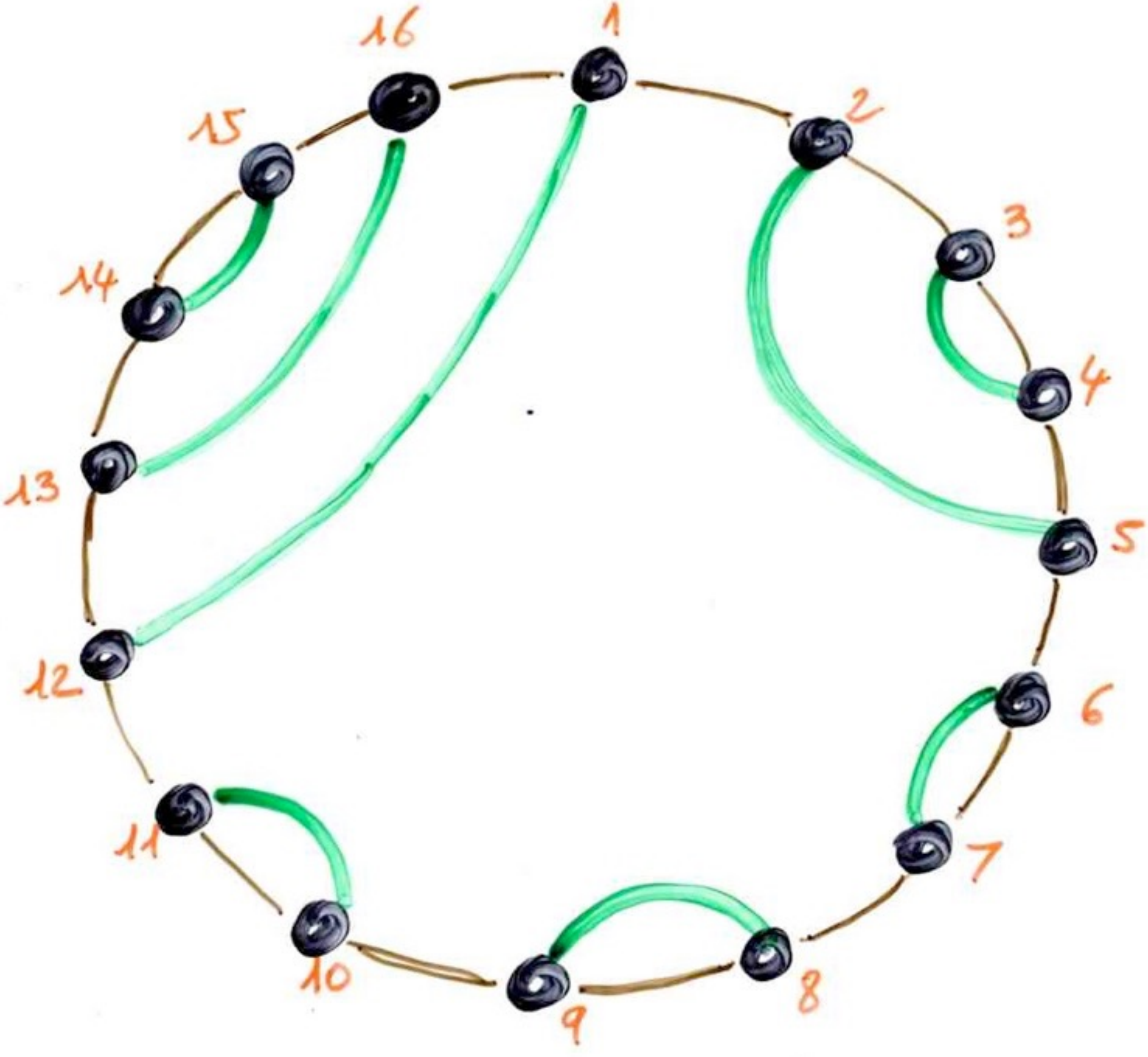
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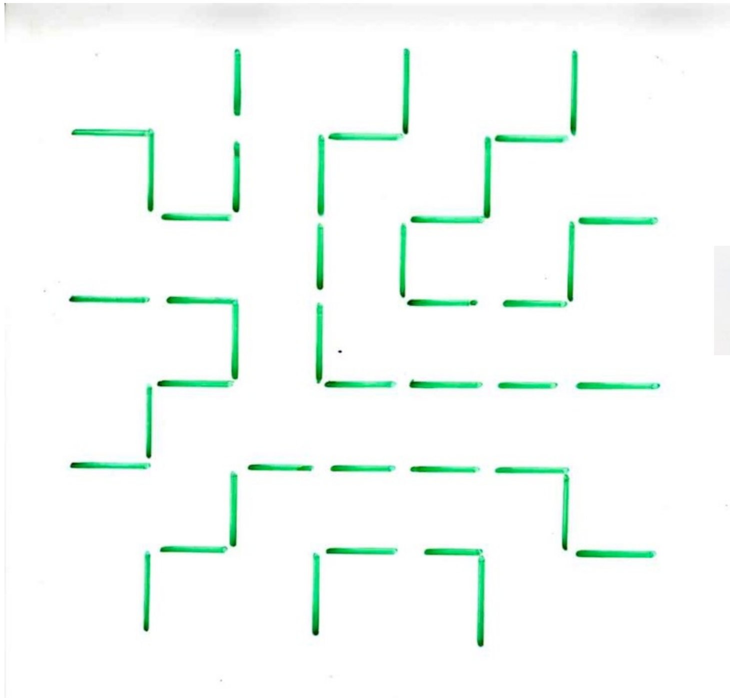
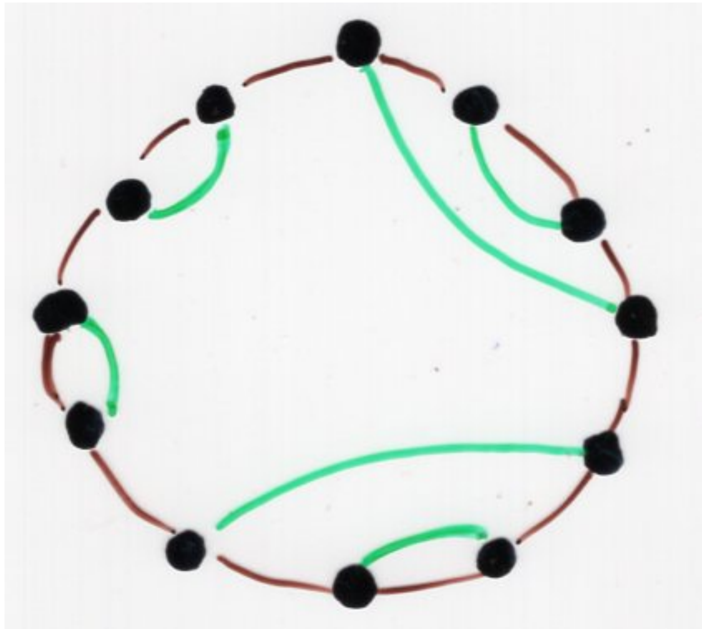
stationary probabilities



FPL $\longrightarrow$ CD



$$\text{proba}(\ \ ) = \frac{1}{A_n}\left(\text{number of FPL giving CD}\right)$$

chord diagram CD

Razumov - Stroganov
(ex) - conjecture

proof by :

L. Cantini and A.Sportiello (March 2010). arXiv: 1003.3376 [math.CO]

based on «Wieland rotation»

completely combinatorial proof

Around the Razumov–Stroganov conjecture

P. Di Francesco, P. Zinn-Justin
(2005–2009)

Knizhnik–Zamolodchikov equation

$$\begin{tikzcd}[column sep=small] & qKZ \arrow[dl] \arrow[dr] & \\ TSSCPP & & ASM \end{tikzcd}$$

1- , 2-, 3- enumeration
formula for $A_n(x)$

Colomo, Pronko (2004)

Hankel determinants

orthogonal polynomials

(continuous) Hahn Meixner-Pollaczek

(continuous) dual Hahn

Ismail, Lin, Roan (2004)

XXZ spin chain and Askey-Wilson operators