Course IMSc, Chennai, India

January-March 2018

The cellular ansatz:
bijective combinatorics and quadratic algebra

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Chapter 2
Quadratic algebra, Q-tableaux and planar automata

Ch2d

IMSc, Chennai
February 8, 2018

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Reminding Ch 2c

The quadratic algebra $\mathbb{Z}$

4 generators $B_\bullet \quad A_\bullet \quad B \quad A$

8 parameters $q_{xy},\ t_{xy}$ $\begin{cases} x = \bullet, \circ \\ y = \bullet, \circ \end{cases}$

$$
\begin{cases}
B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\
B_\bullet\, A_\bullet = q_{\bullet\bullet}\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\
B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + t_{\bullet\circ}\, A_\bullet\, B \\
B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + t_{\circ\bullet}\, A\, B_\bullet
\end{cases}
$$

The $\mathbb{Z}$-quadratic algebra

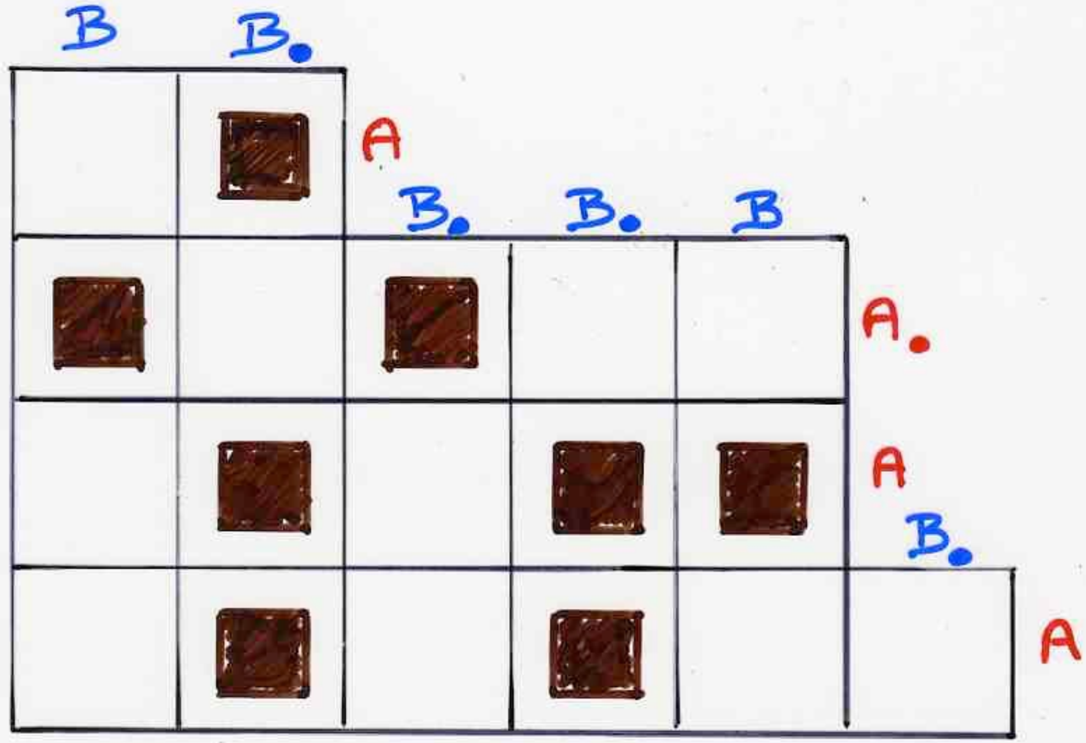
XYZ-quadratic algebra

$$
\begin{cases}
B\,A = \square\; A\,B + \blacksquare\; A_{\bullet}\,B_{\bullet} \\
B_{\bullet}\,A_{\bullet} = \square\; A_{\bullet}\,B_{\bullet} + \blacksquare\; A\,B \\
B_{\bullet}\,A = \square\; A\,B_{\bullet} + \blacksquare\; A_{\bullet}\,B \\
B\,A_{\bullet} = \square\; A_{\bullet}\,B + \blacksquare\; A\,B_{\bullet}
\end{cases}
$$

B.A.BA configuration

Z-tableau

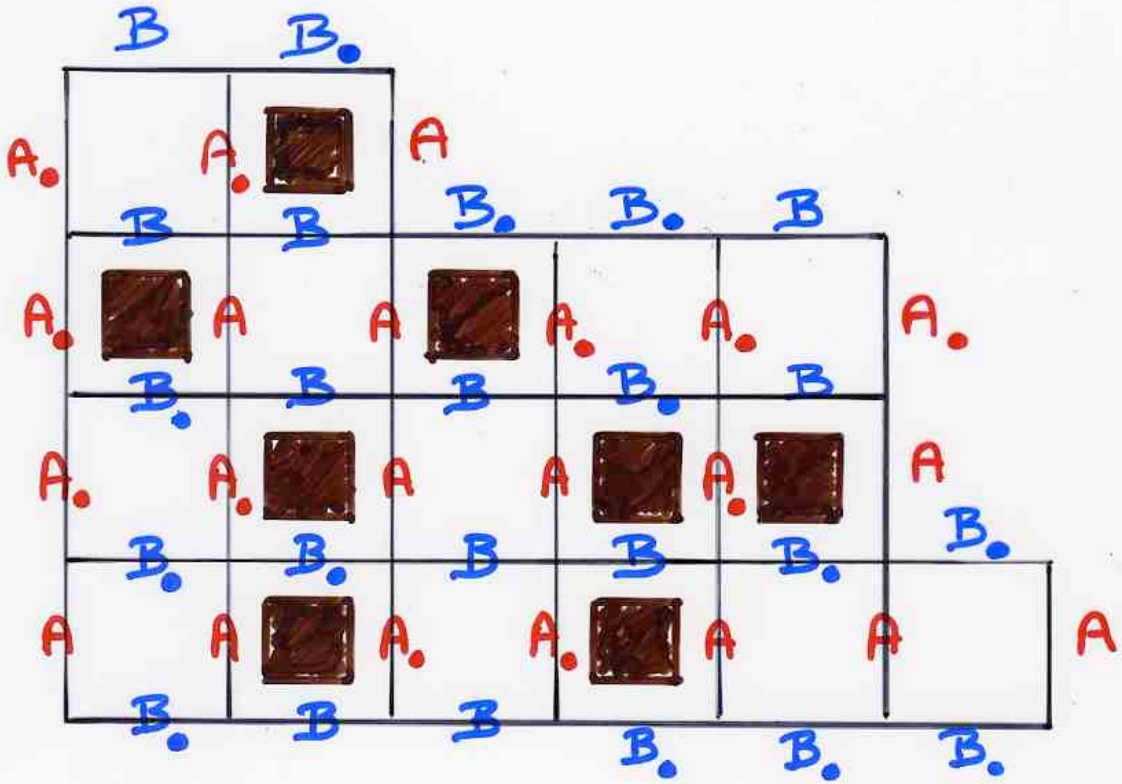
XYZ-tableau



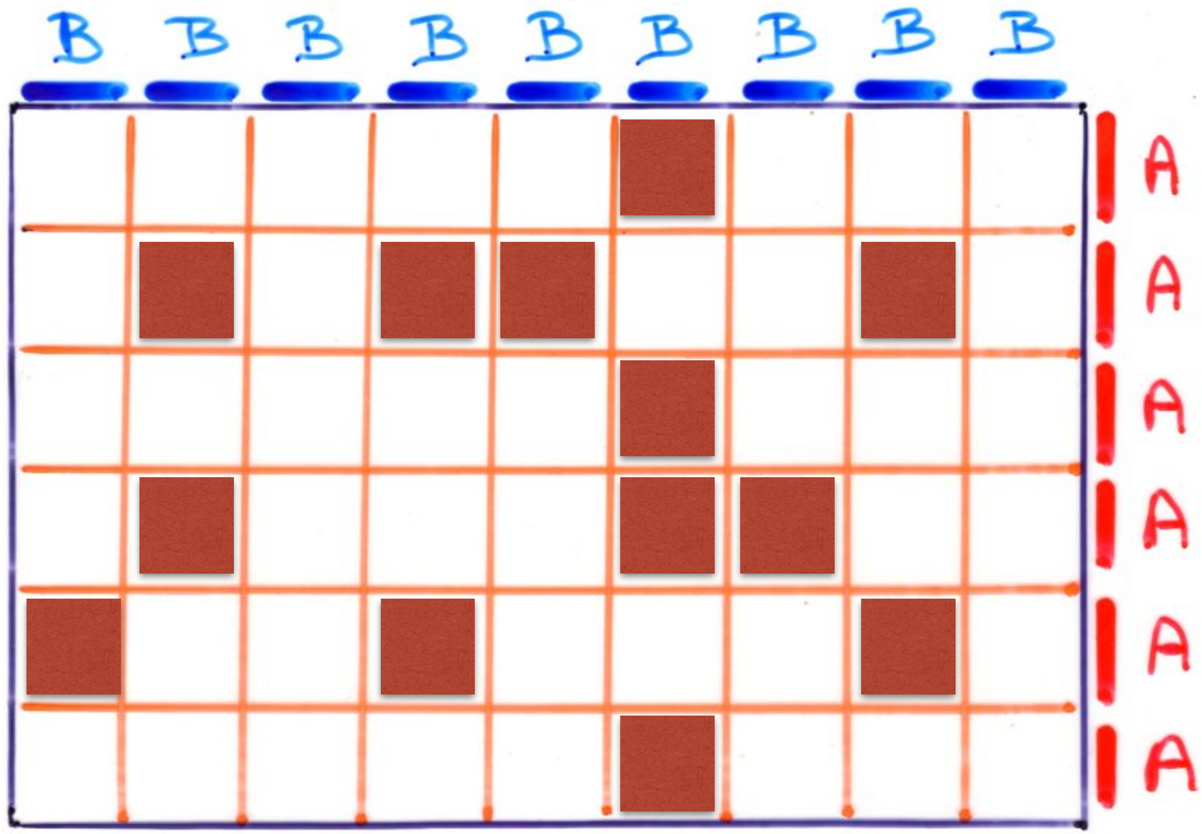
complete

Z- tableau

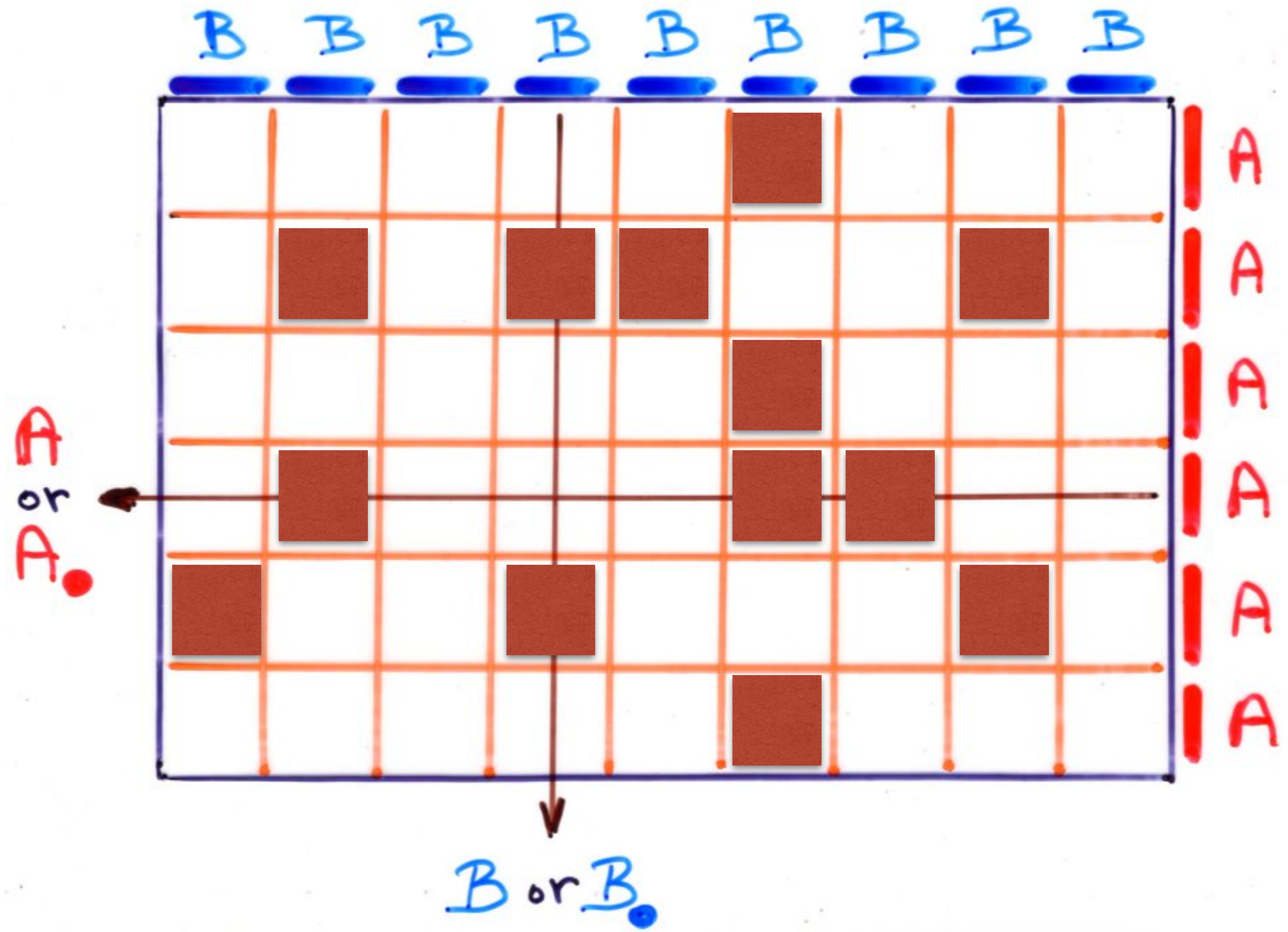
XYZ- tableau

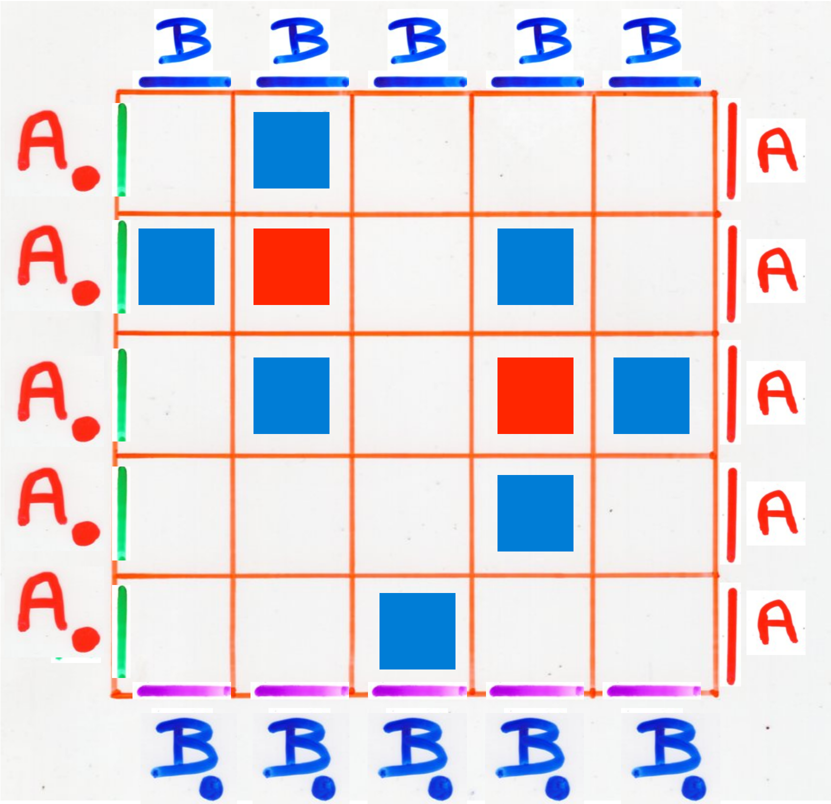


B. A. BA configuration



B. A. BA configuration

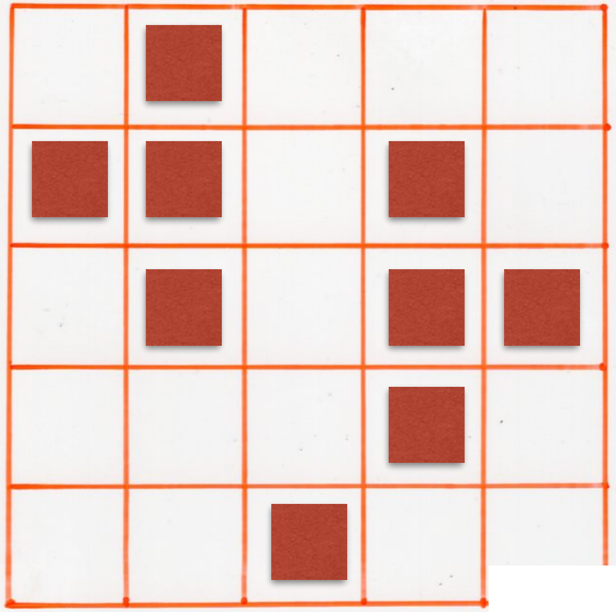




A alternating sign matrix

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + \bigcirc\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, AB_\bullet \end{cases}$$



A alternating sign matrix

$\varphi(A)$ B.A.BA configuration

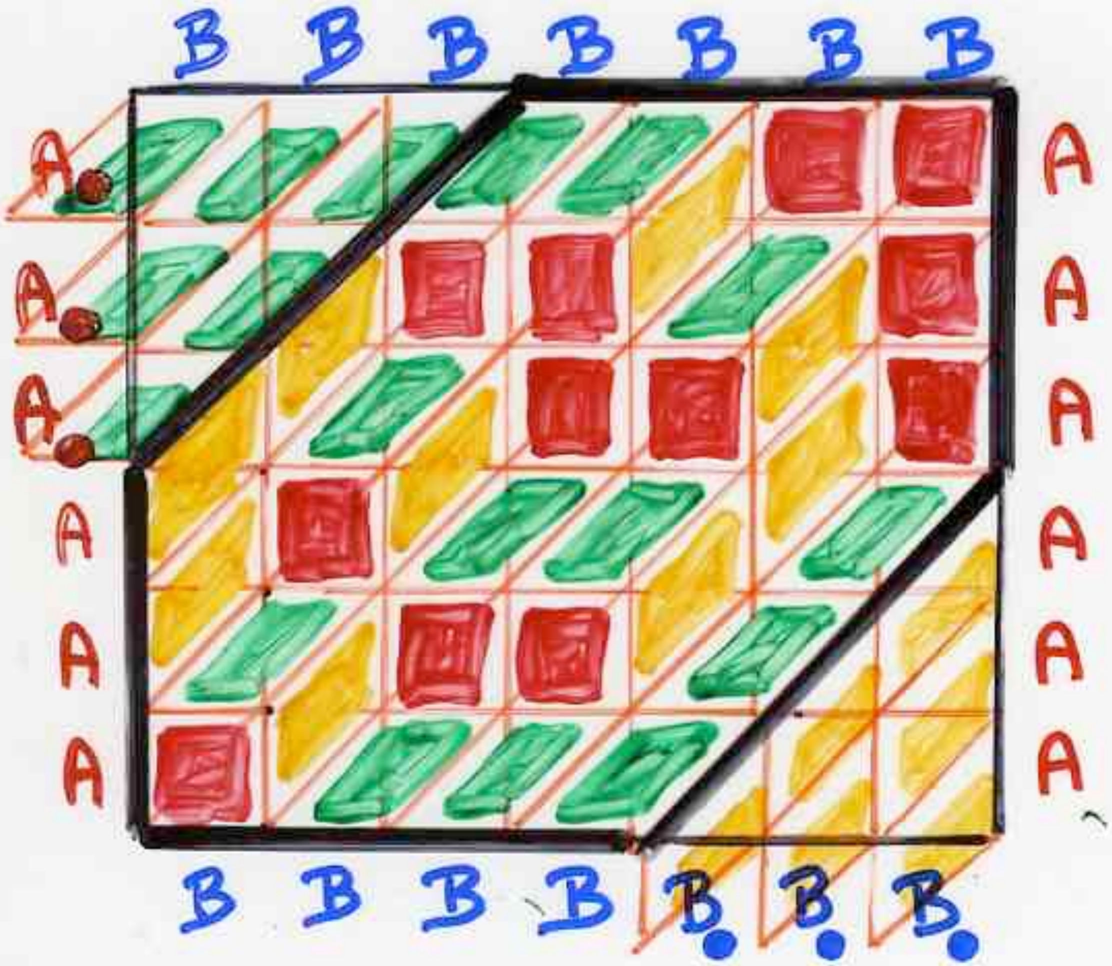
$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases}$$

Rhombus tilings

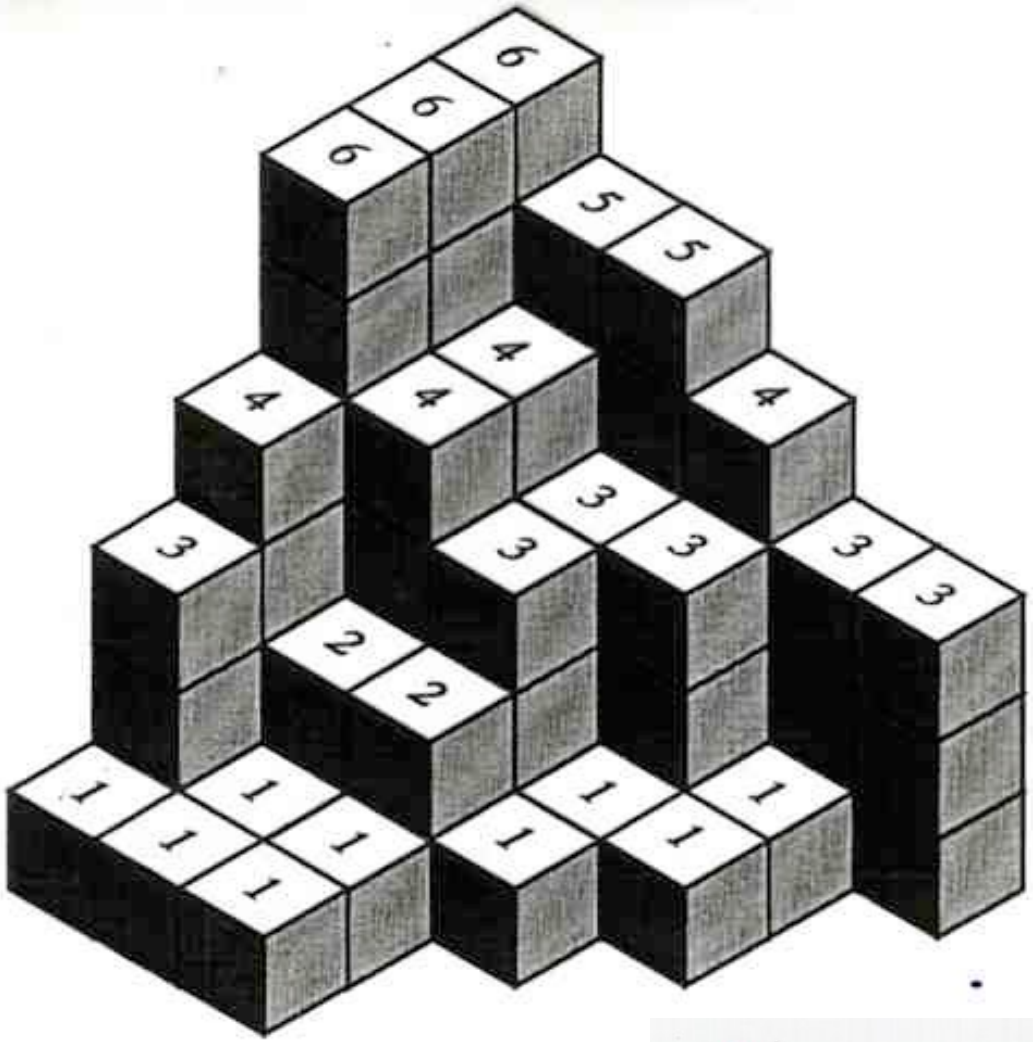
$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_{\bullet}\,B_{\bullet} \\ B_{\bullet}\,A_{\bullet} = \bigcirc\,A_{\bullet}\,B_{\bullet} + t_{\bullet\bullet}\,A\,B \\ B_{\bullet}\,A = q_{\bullet\circ}\,A\,B_{\bullet} + \bigcirc\,A_{\bullet}\,B \\ B\,A_{\bullet} = q_{\circ\bullet}\,A_{\bullet}\,B + \bigcirc\,A\,B_{\bullet} \end{cases}$$

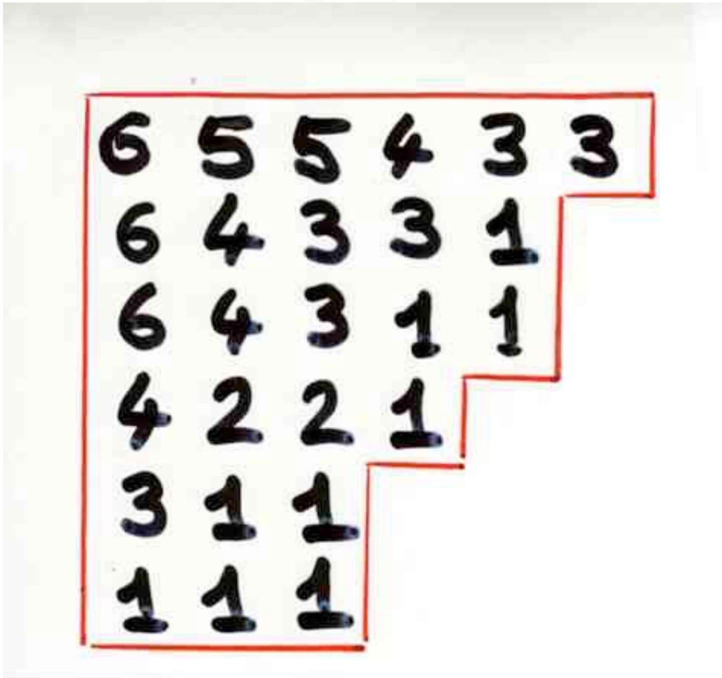


$$w = B^{c} B^{a} A^{b} A^{c}$$

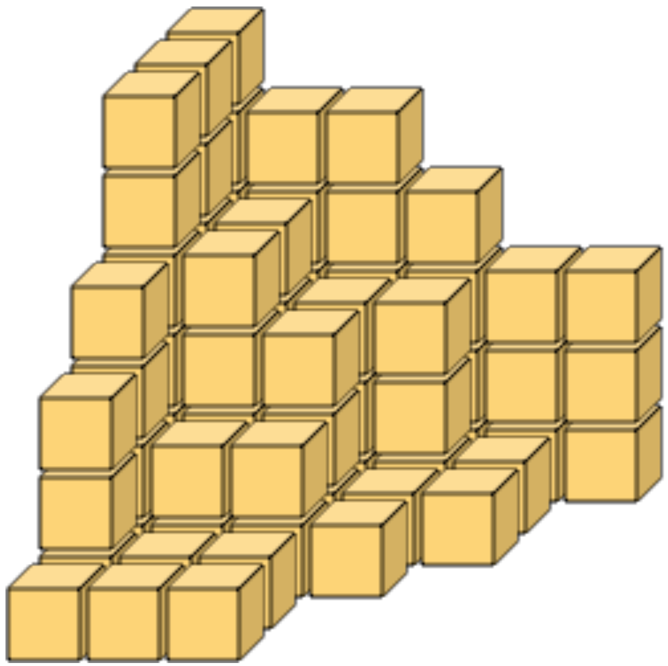
$$u = A^{c}_{\bullet} \; A^{b}$$
$$v = B^{a} \; B^{c}_{\bullet}$$

$$c\,(u, v; w)$$



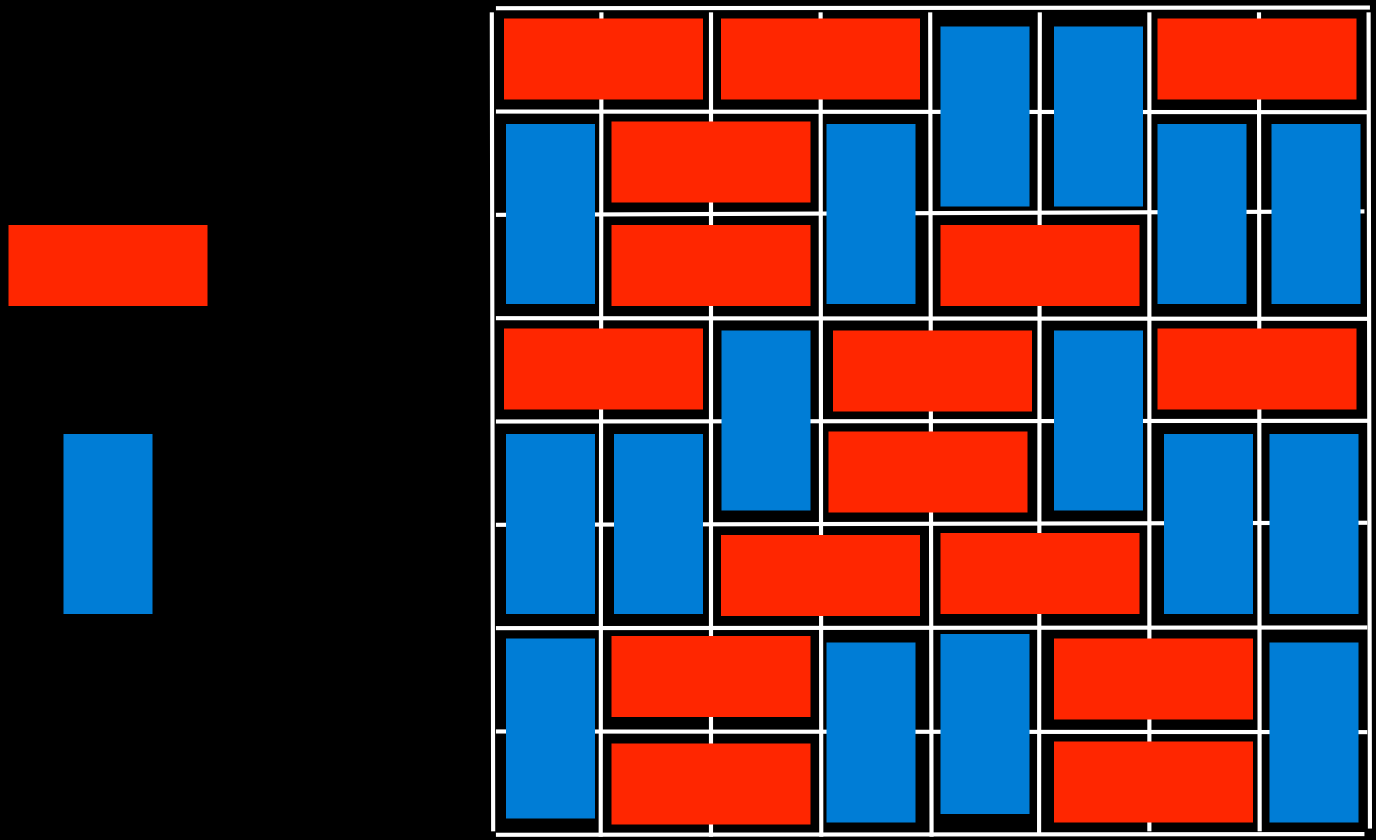


plane partitions



Proof of MacMahon formula, see:
BJC 1, Ch 5a, p105

a tiling (of a rectangle) on the square lattice



$$\begin{cases} B A = A_{\bullet} B + A B_{\bullet} \\ B_{\bullet} A = A B \\ B A_{\bullet} = A B \\ B_{\bullet} A_{\bullet} = 0 \end{cases}$$

not a subalgebra of the $\mathbb{Z}$-algebra

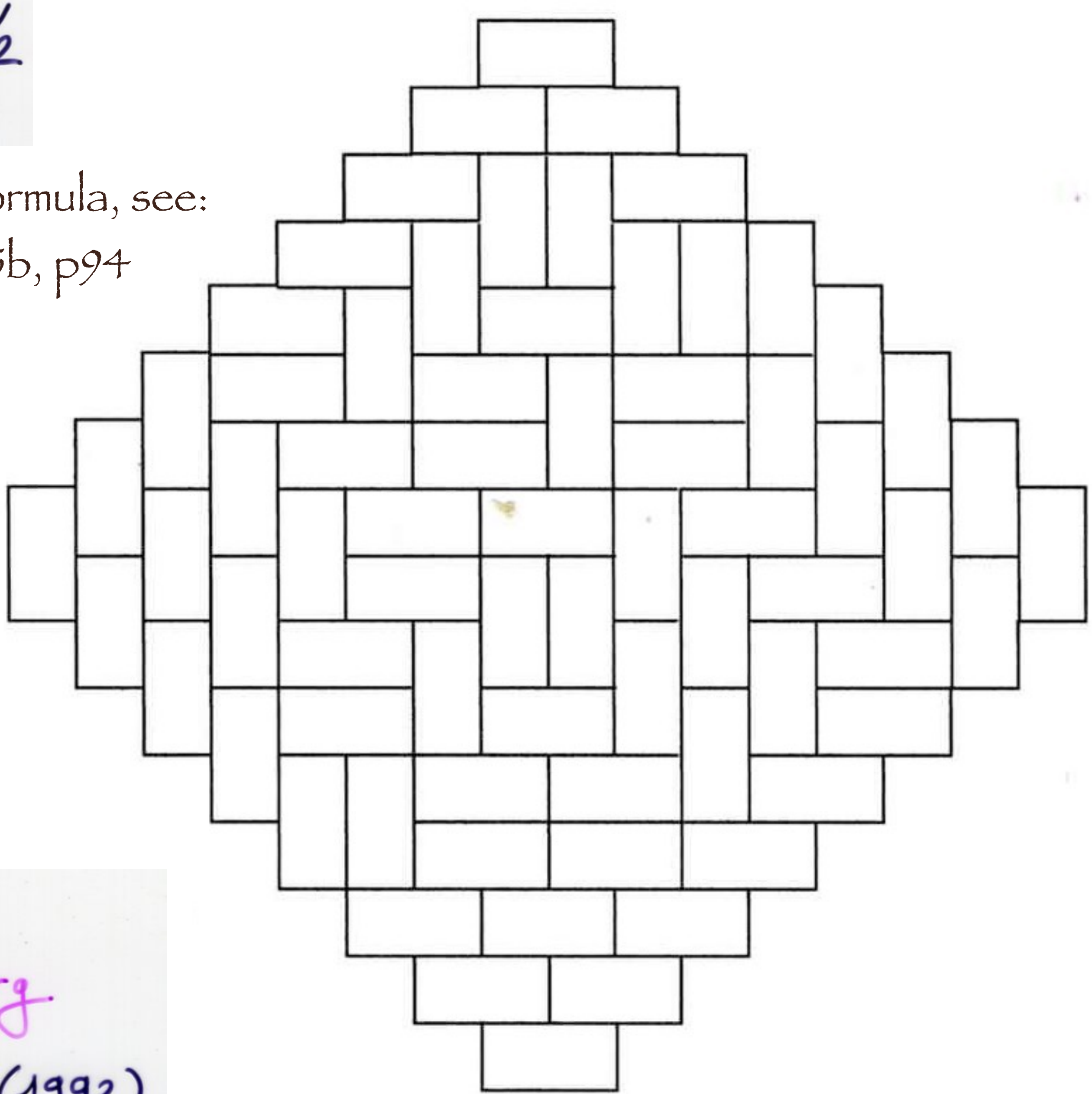
Correction to the video:

$$\begin{cases} B A = q_{00} A B + t_{00} A_{\bullet} B_{\bullet} \\ B_{\bullet} A_{\bullet} = \bigcirc A_{\bullet} B_{\bullet} + \bigcirc A B \\ B_{\bullet} A = q_{\bullet 0} A B_{\bullet} + \bigcirc A_{\bullet} B \\ B A_{\bullet} = q_{0 \bullet} A_{\bullet} B + \bigcirc A B_{\bullet} \end{cases}$$

Formula for the number of tilings of an m x n rectangle, see BJC 1, Ch 5b, p66-67 (without proof)

$2^{n(n-1)/2}$

Proof of this formula, see:
BJC 1, Ch 5b, p94



Elkies
Kuperberg
Larsen
Propp (1992)

Aztec tilings

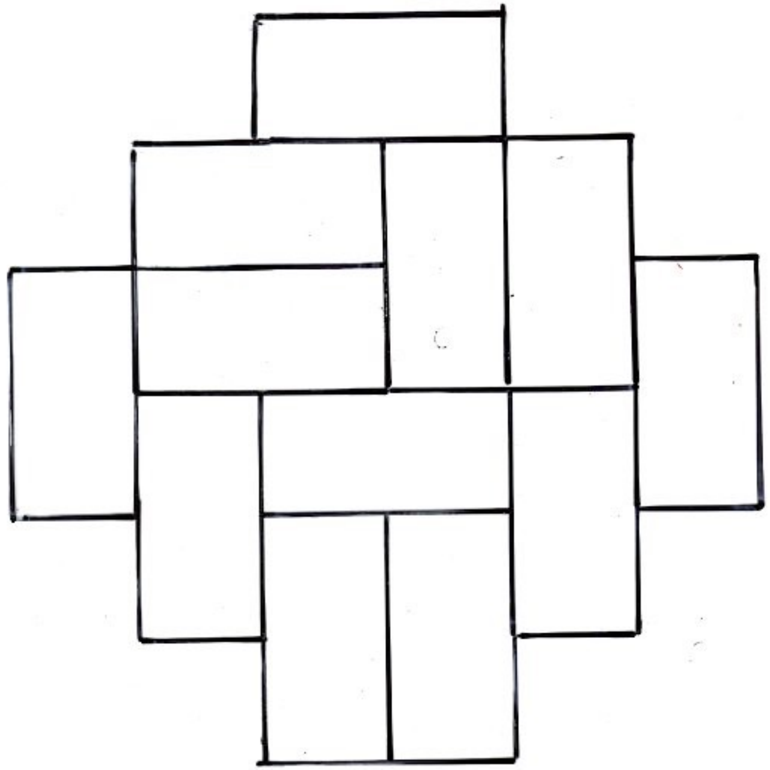
$$t_{\bullet\bullet} = t_{\circ\circ} = 0$$

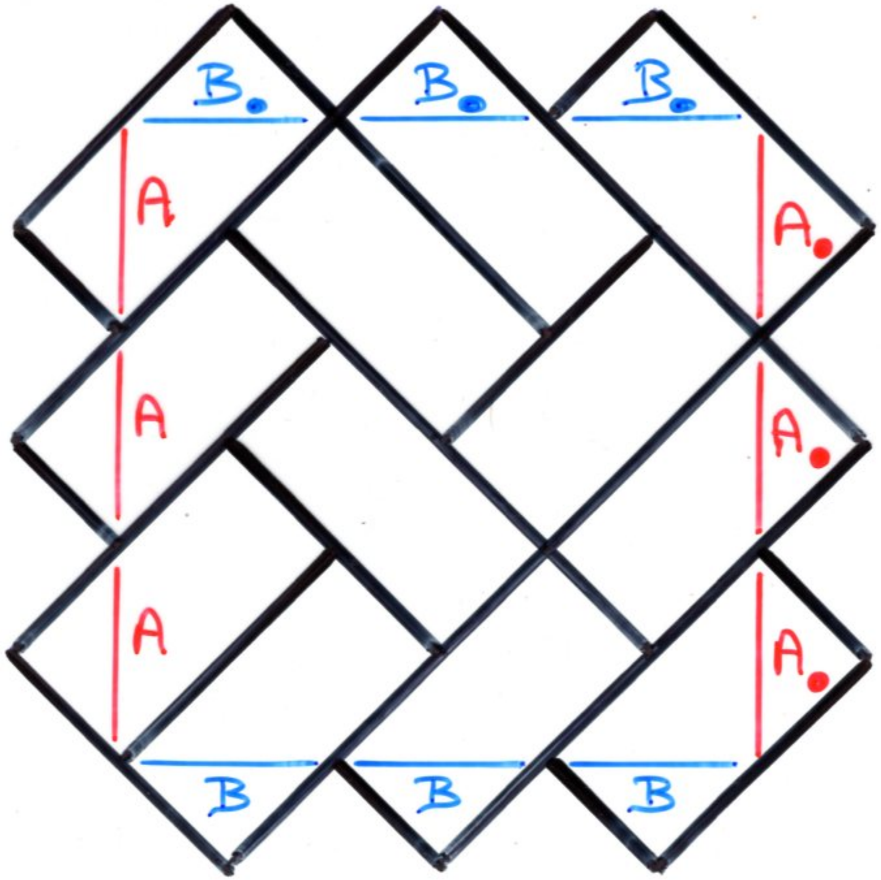
$$t_{\bullet\bullet} = 2$$

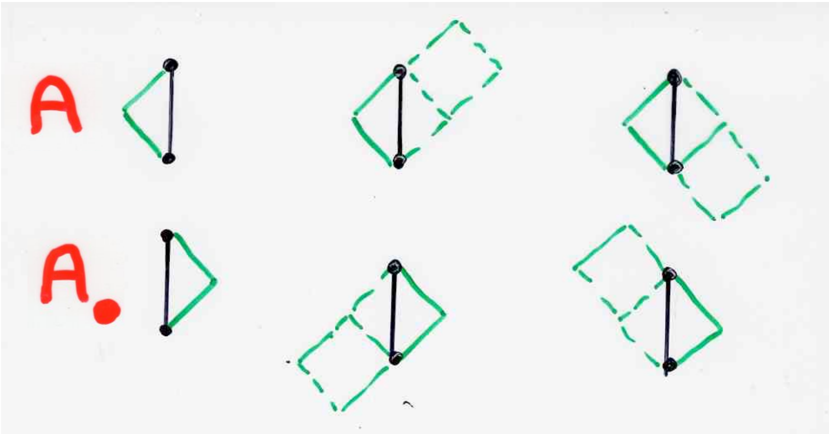
$A_n(x)$ enumeration of ASM according to the number of (−1)

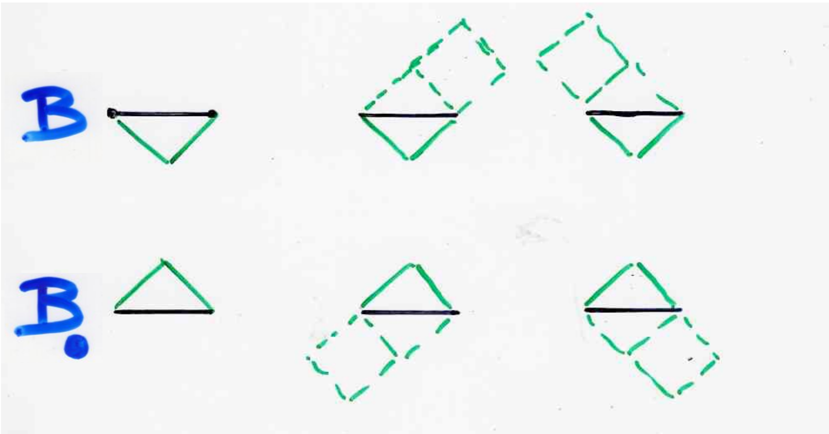
$$A_n(2) = 2^{n(n-1)/2}$$

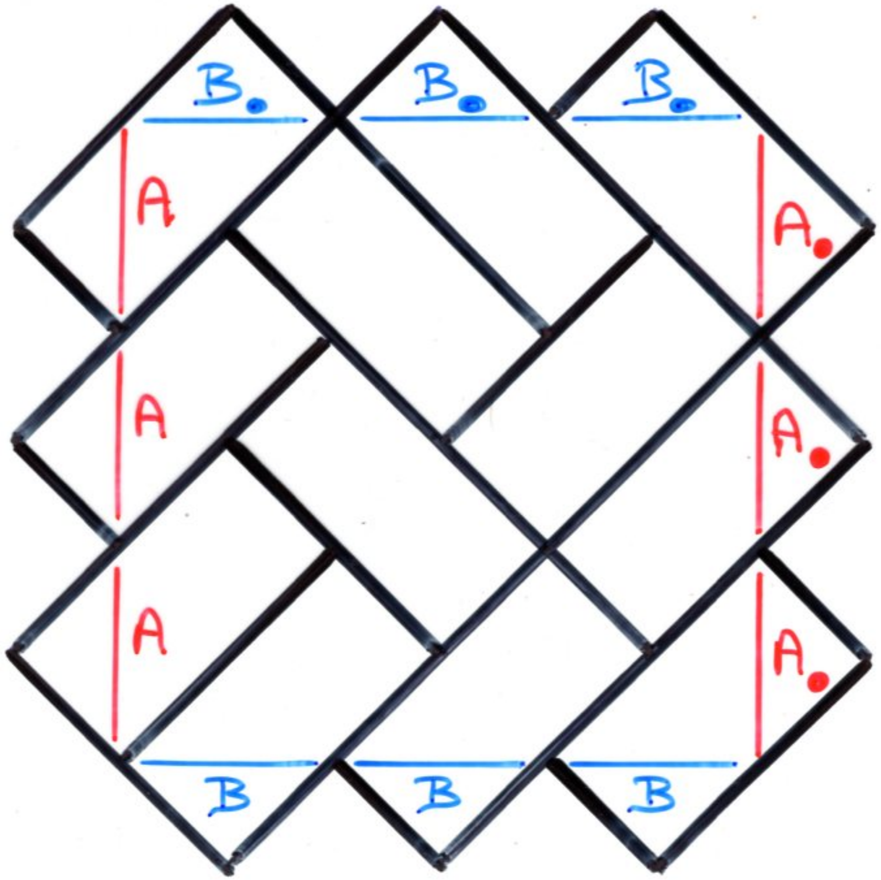
$$\begin{cases} BA = q_{\circ\circ} AB + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} A_\bullet B_\bullet + 2\, AB \\ B_\bullet A = q_{\bullet\circ} AB_\bullet + \bigcirc\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc\, AB_\bullet \end{cases}$$

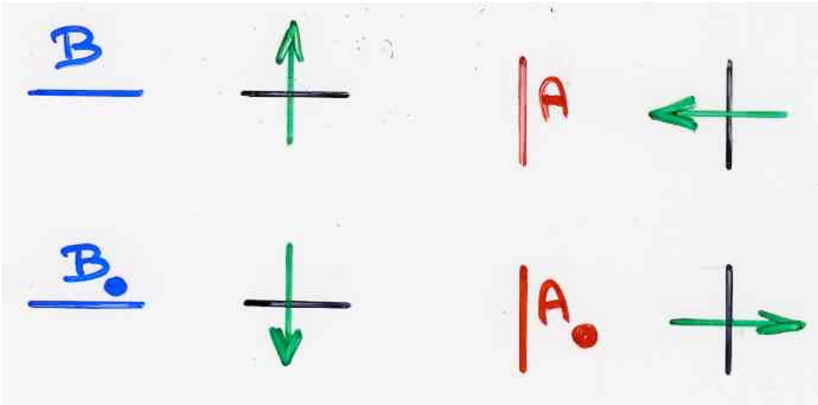



B
B
B
A
A
A
A
A
A
B
B
B


A
A

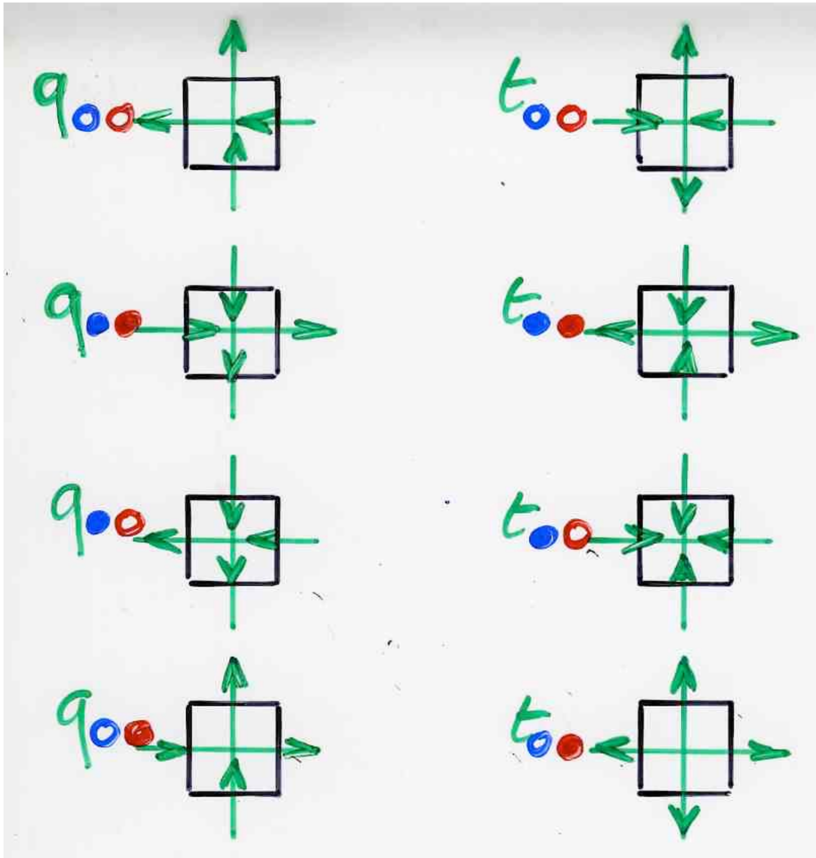

B
B


B B B
A A
A A
A A
B B B



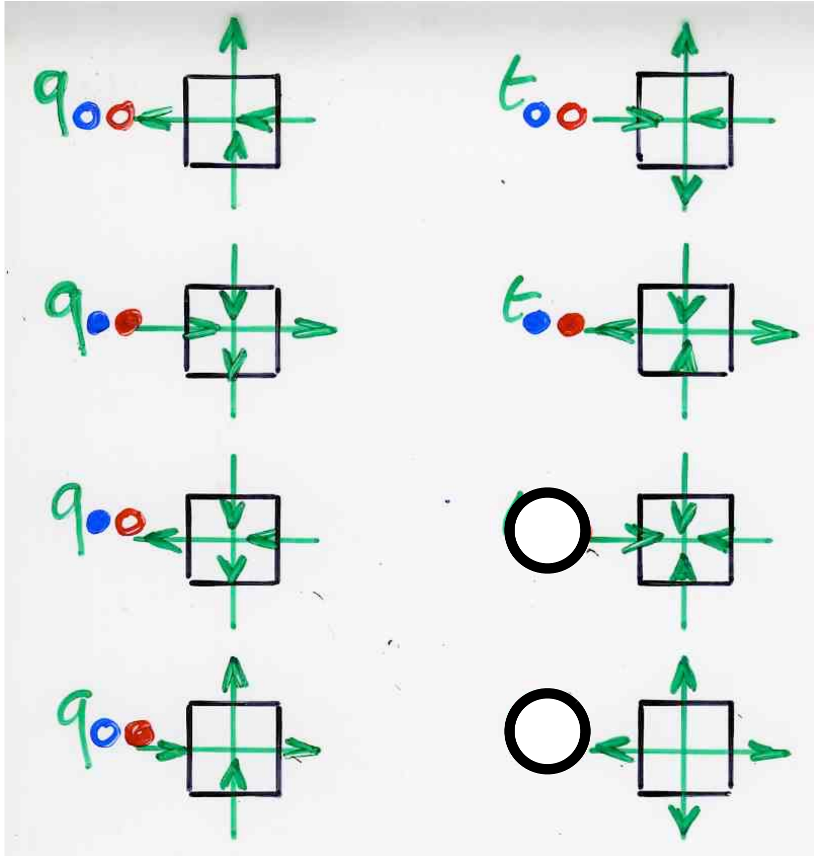
The 8-vertex model

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = q_{\bullet\bullet}\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + t_{\bullet\circ}\, A_\bullet\, B \\ B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + t_{\circ\bullet}\, A\, B_\bullet \end{cases}$$



The 6-vertex model

$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = q_{\bullet\bullet}\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + \bigcirc\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + \bigcirc\,A\,B_\bullet \end{cases}$$

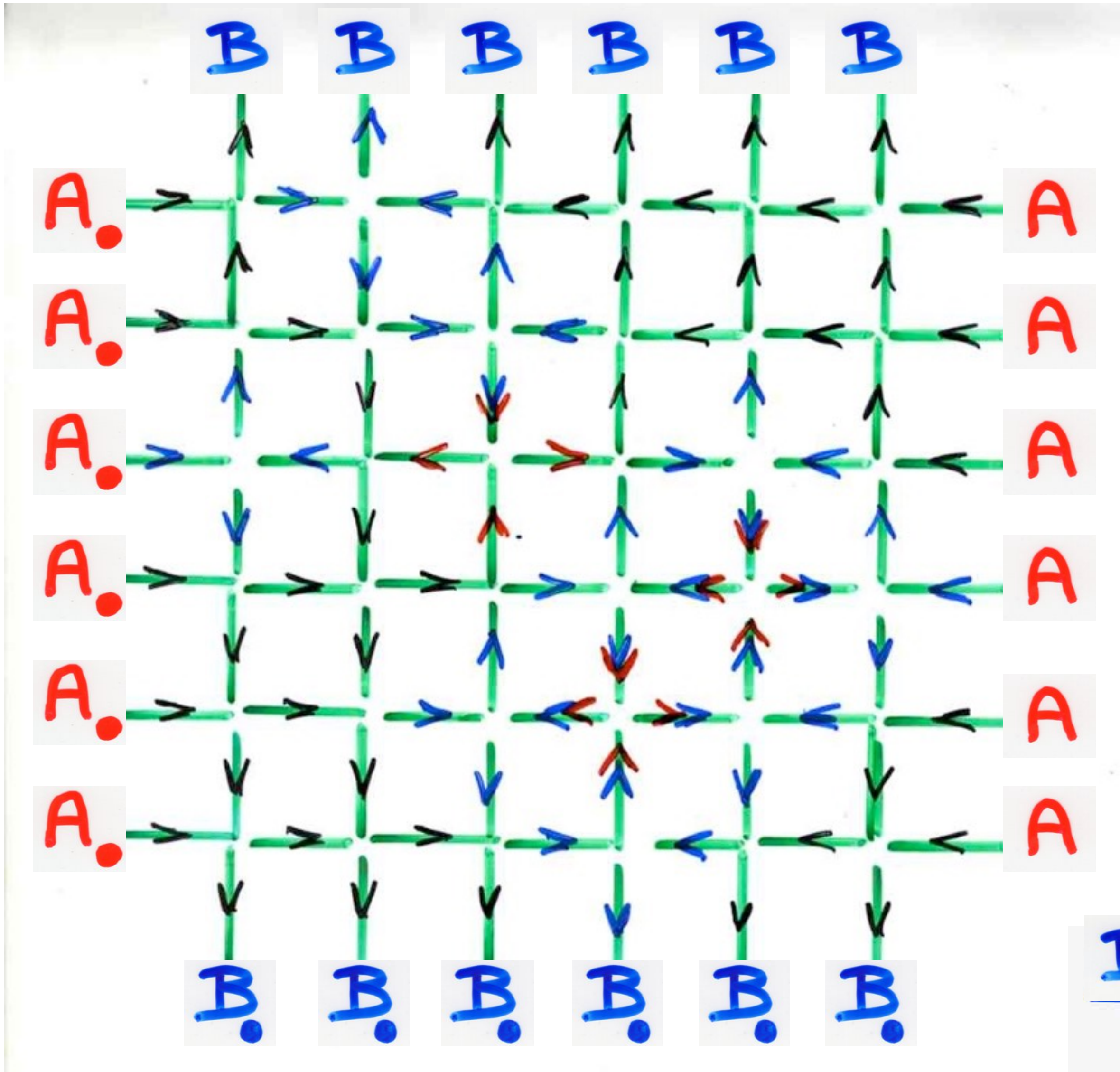


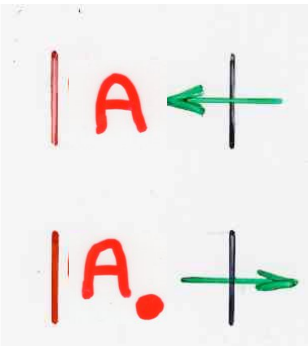
The 6-vertex model

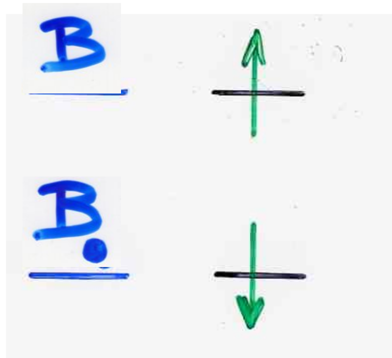
domain wall condition

complete XXZ-tableau

configurations of the 6-vertex model







The 6-vertex model

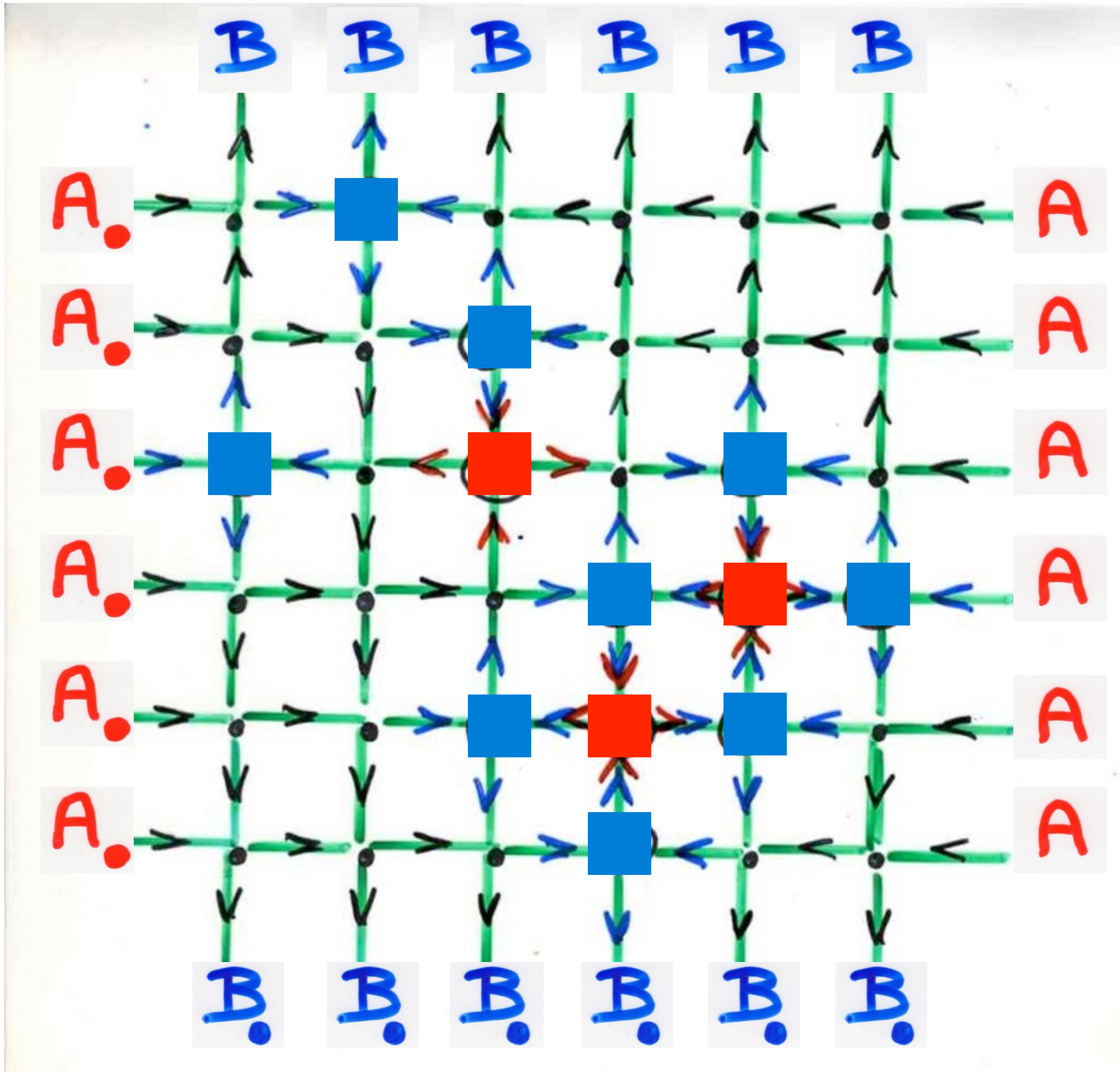
domain wall condition

bijection

configurations of the 6-vertex model

↕

ASM alternating sign matrices

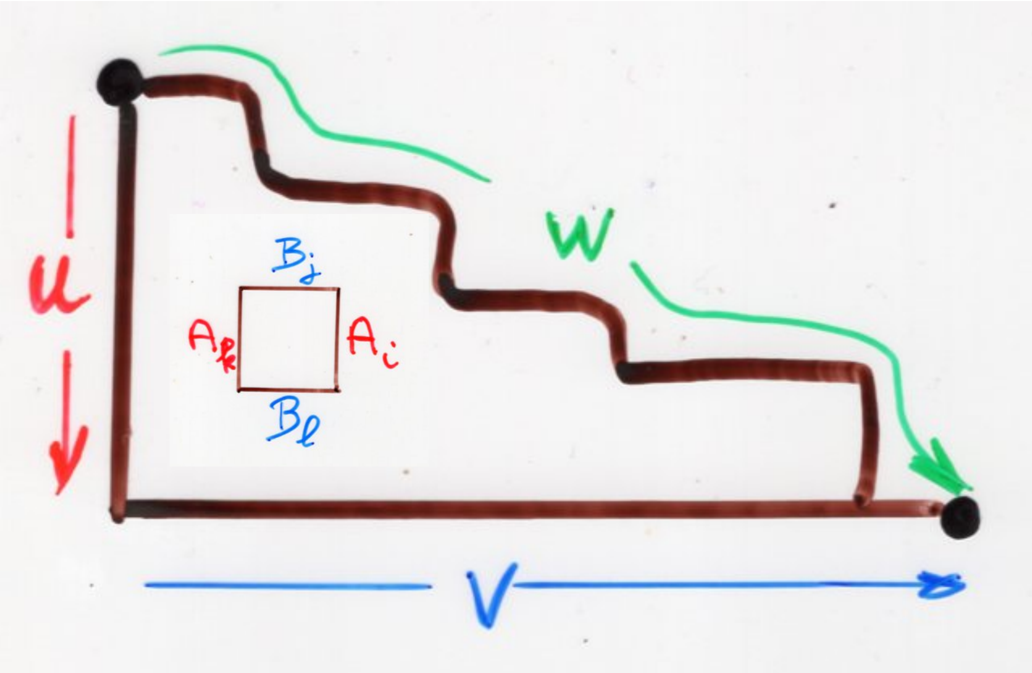


quadratic algebra Q

$$B_j A_i = \sum_{k\,l} c_{ij}^{kl} A_k B_l$$

$$c(u,v;w) = \sum_{T} wgt(T)$$

complete Q-tableau
$uwb(T) = w$
$lwb(T) = uv$



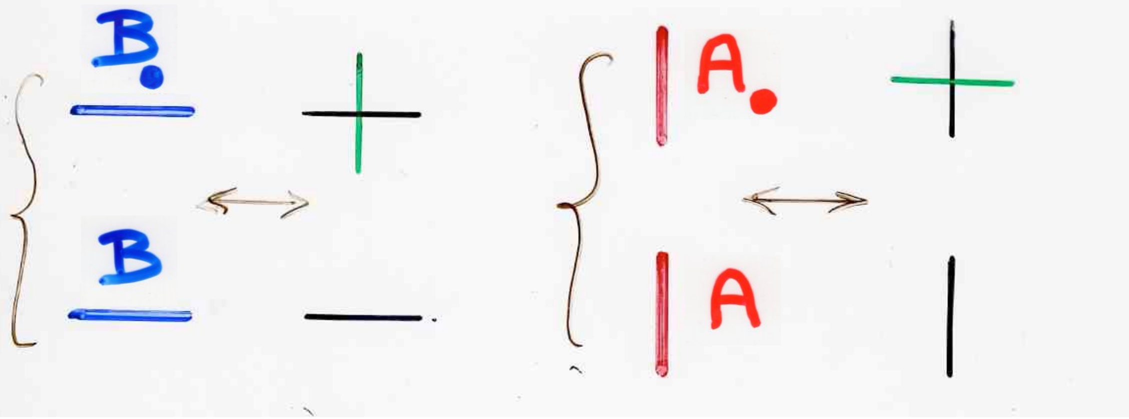
many examples

in general $F(w)$ is a rectangle

2nd geometric interpretation
of
XYZ-tableaux:

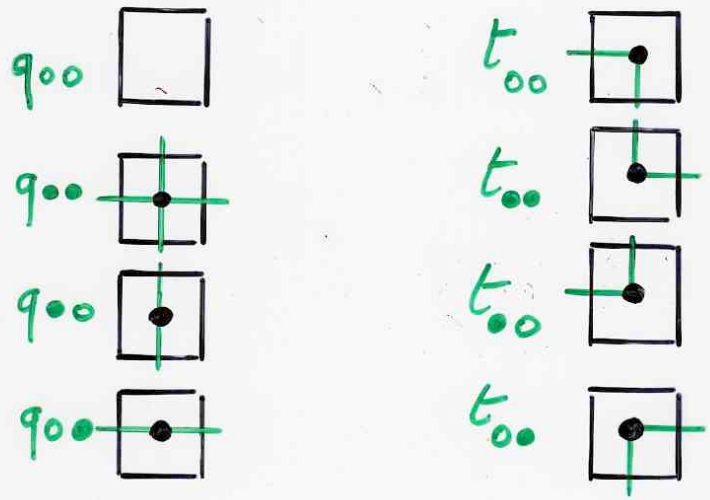
Paths, loops, ...

geometric interpretations of Z-tableaux

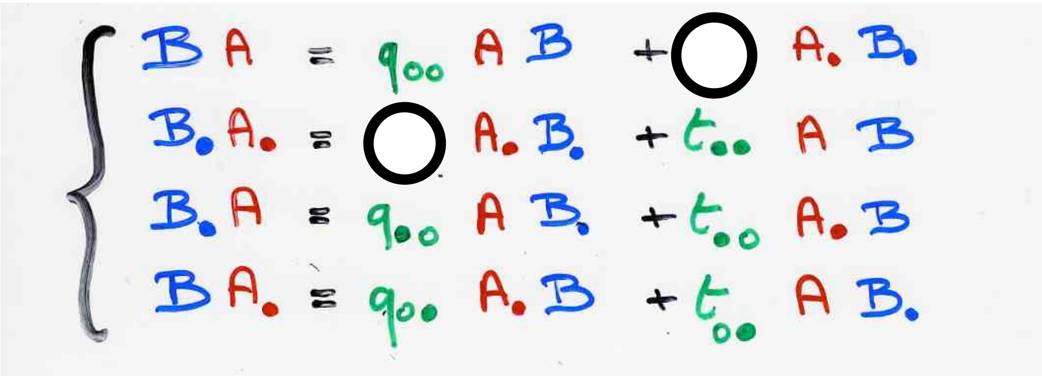


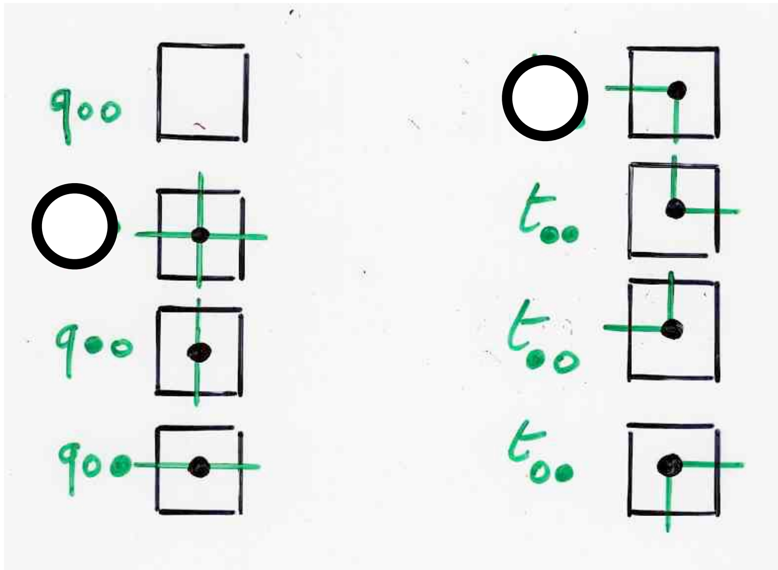
8-vertex model

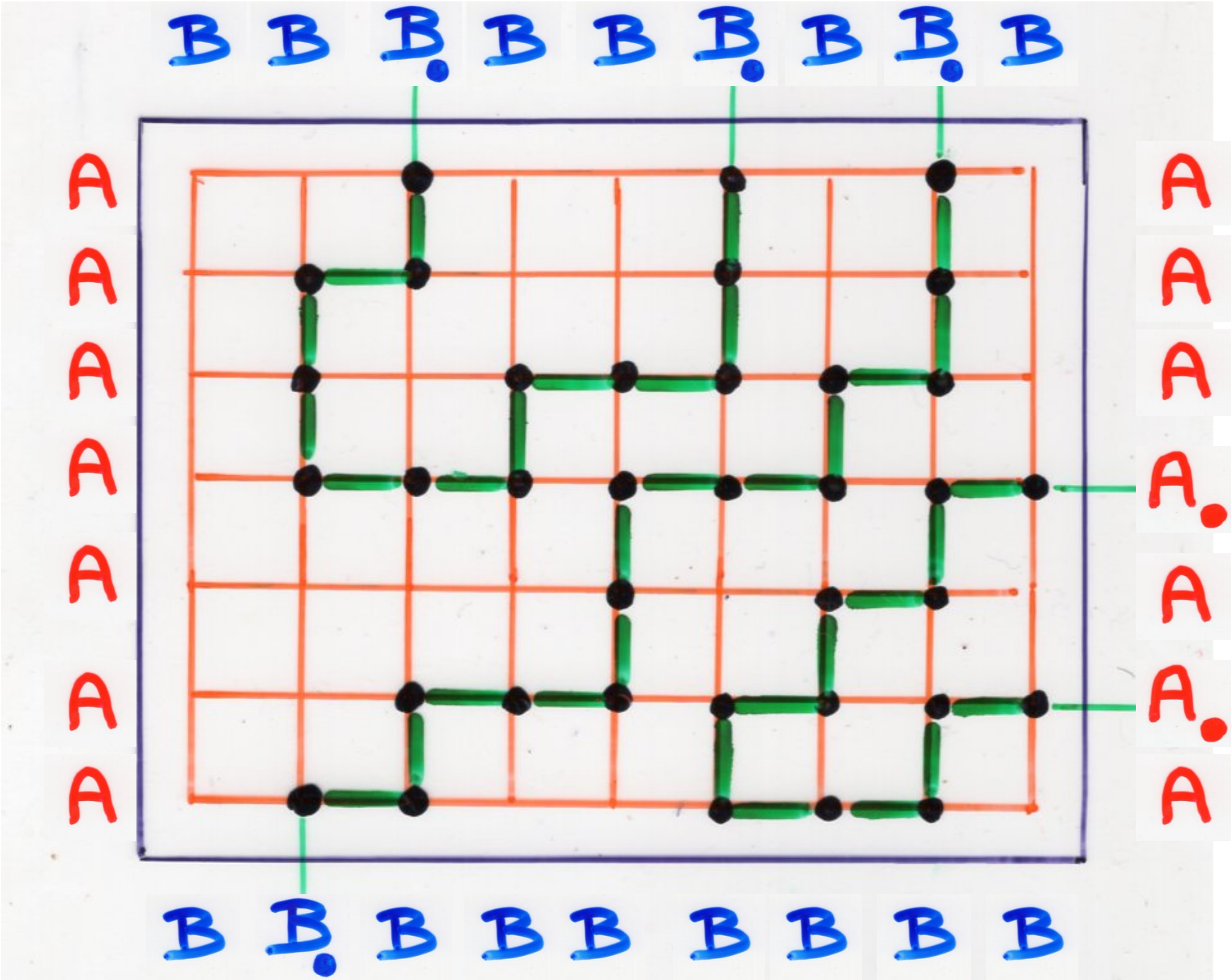
$$
\begin{cases}
B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\
B_\bullet\, A_\bullet = q_{\bullet\bullet}\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\
B_\bullet\, A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet\, B \\
B\,A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + t_{\circ\bullet}\, A\,B_\bullet
\end{cases}
$$



non-intersecting paths

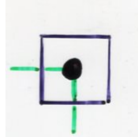






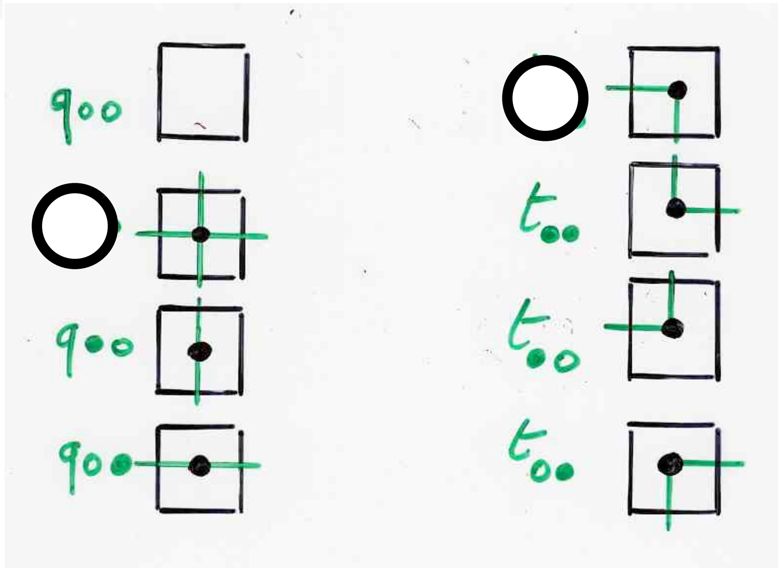
The figure on the video is wrong.

The fordidden pattern appears twice.

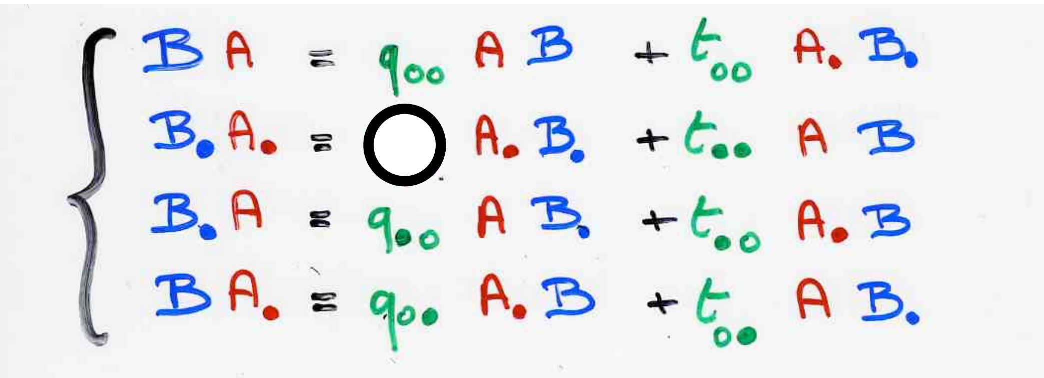


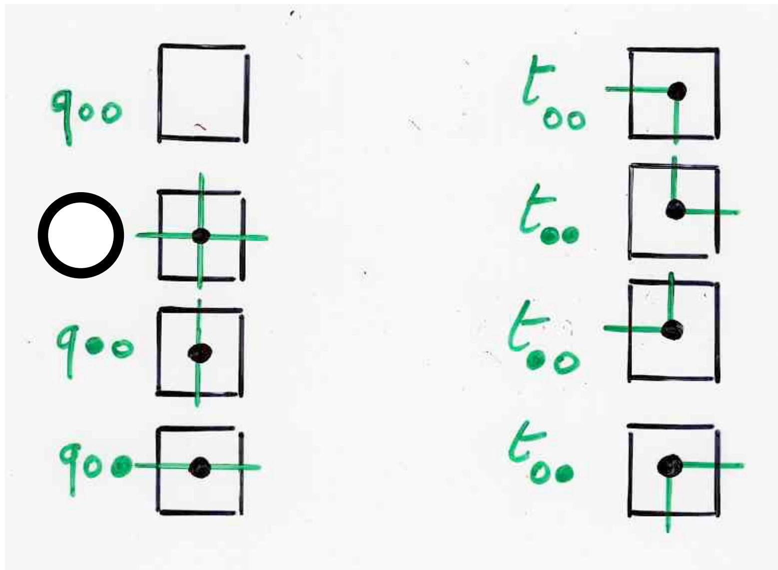
(remark of a student during the class)

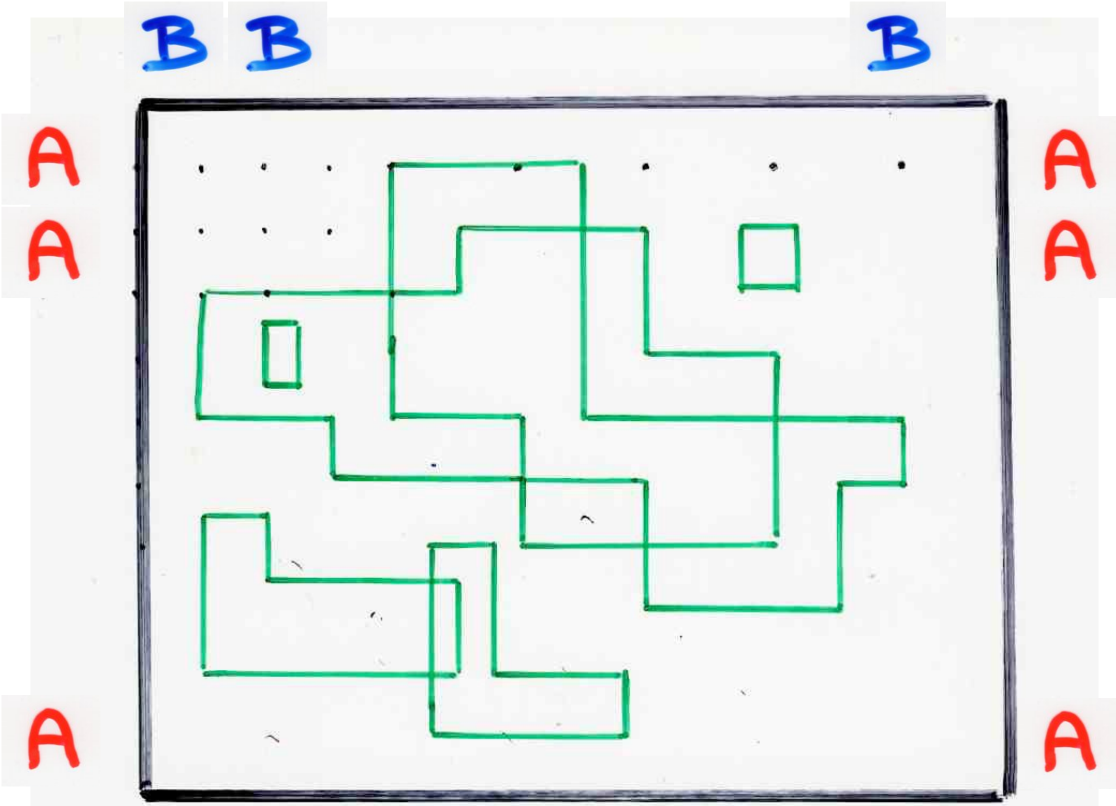
corrected figure after the class



non-intersecting loops and paths



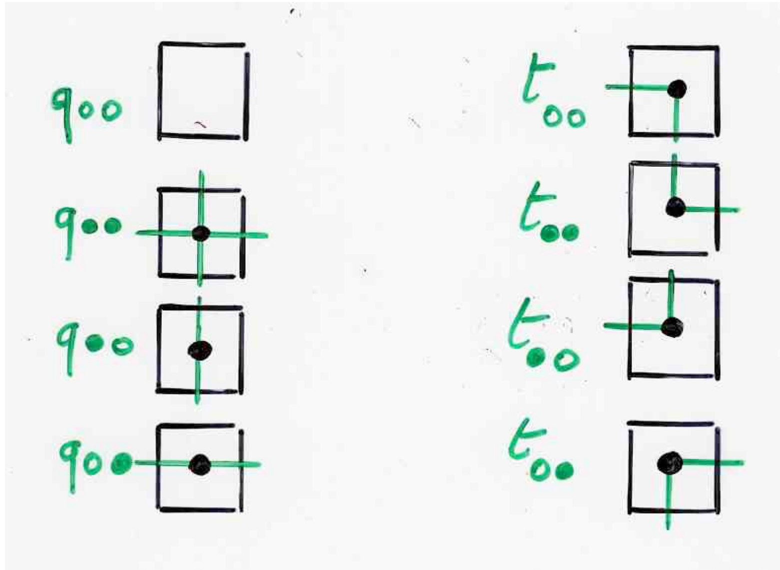




"closed" graph

Ising model

8-vertex model



$$
\begin{cases}
B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\
B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\
B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet
\end{cases}
$$

2nd geometric interpretation

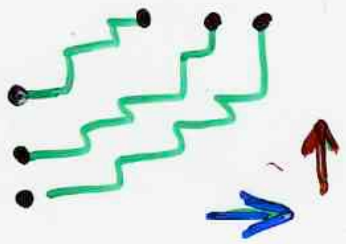
of

XYZ-tableaux:

non-intersecting paths

and

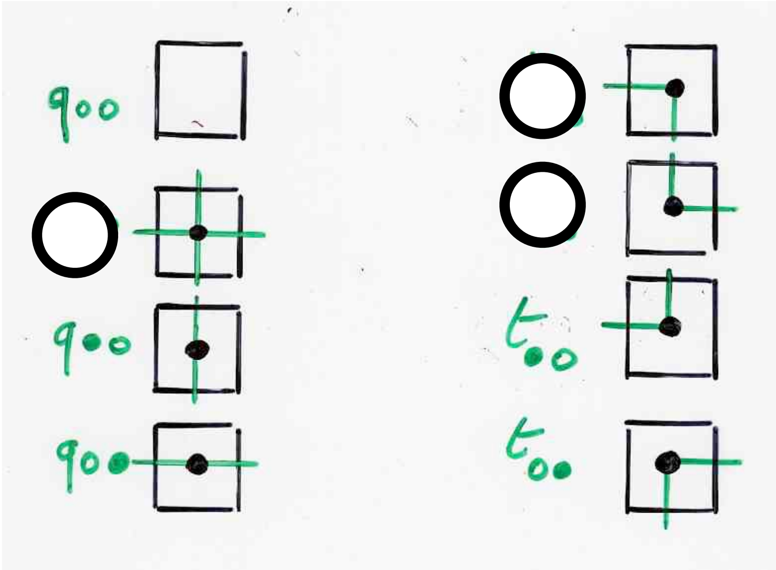
determinants

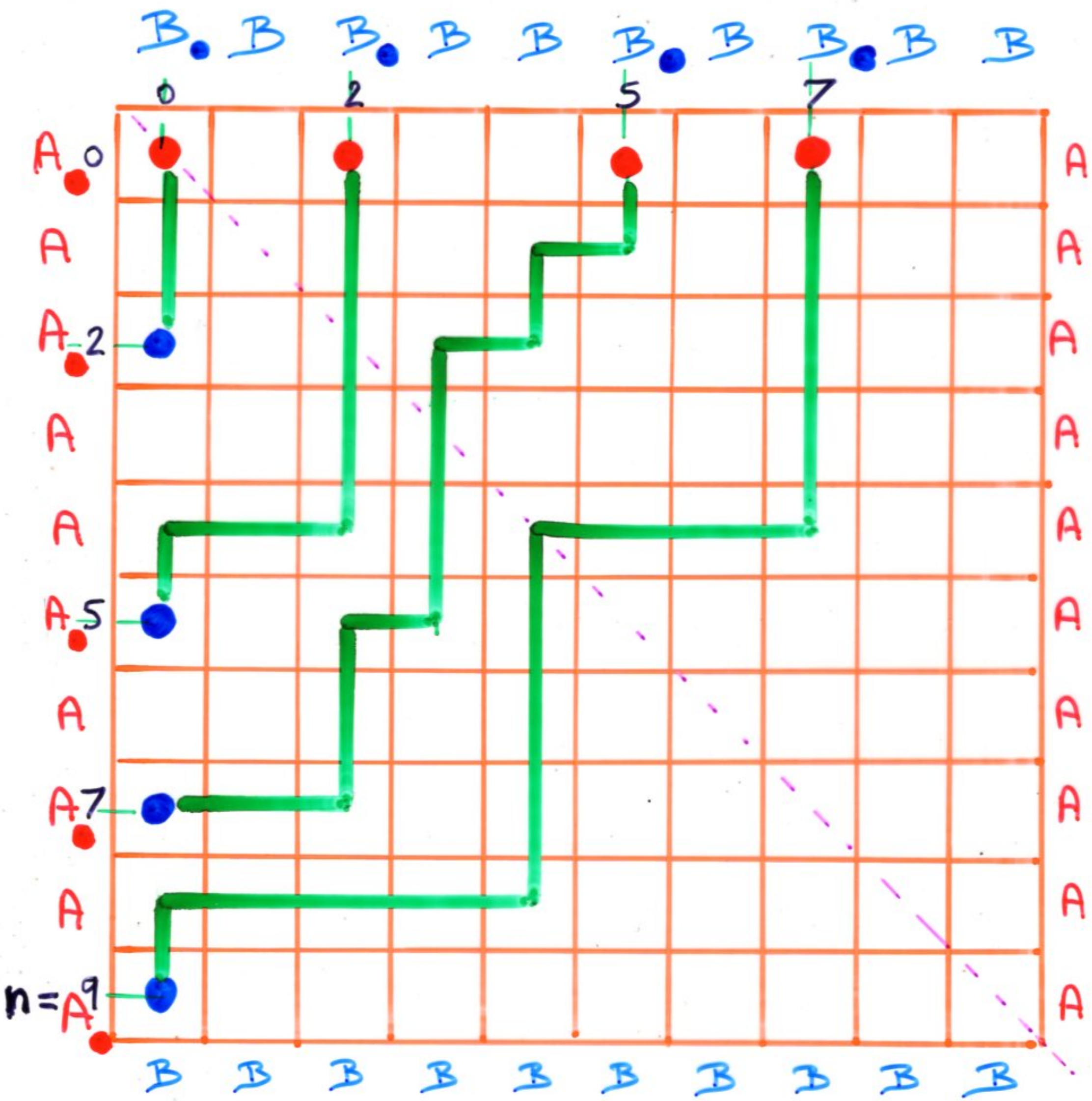


$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

non-intersecting paths

$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + \quad A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = \quad A_\bullet\,B_\bullet + \quad A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + t_{\bullet\circ}\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + t_{\circ\bullet}\,A\,B_\bullet \end{cases}$$





B B B B B B B B B B
0 2 5 7
A 0
A 2
A 5
A 7
n = 9

The LGV Lemma

see. BJC 1, Ch 5a

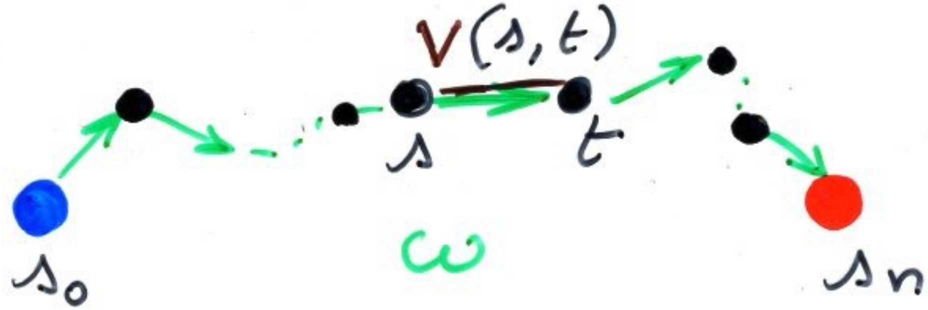

non-intersecting configuration of paths
determinant

Path $\omega = (s_0, s_1, \dots, s_n)$ $s_i \in S$

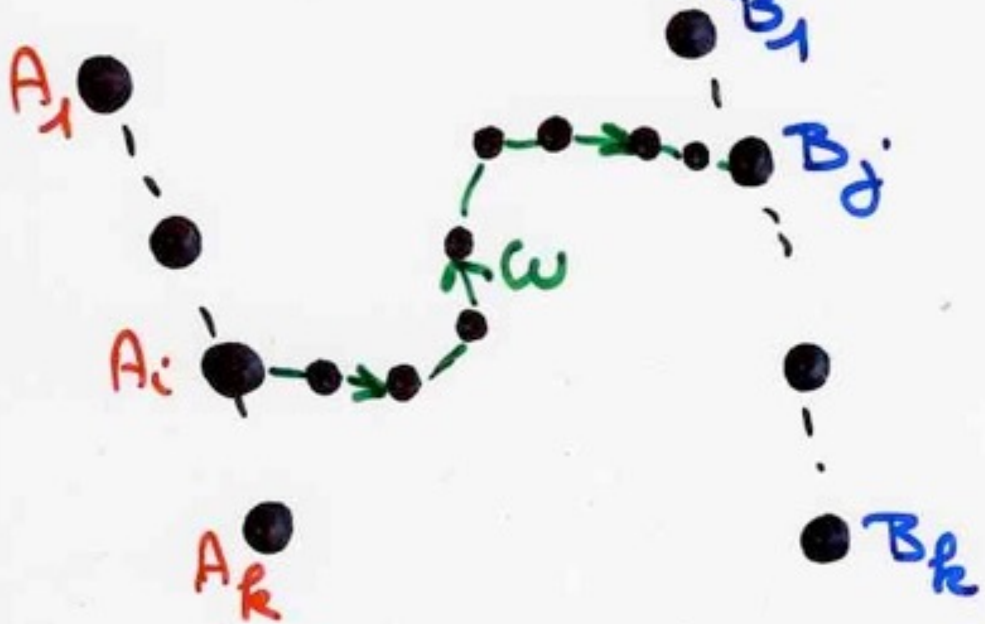
<u>notation</u> $s_0 \overset{\omega}{\rightsquigarrow} s_n$

valuation $v : S \times S \longrightarrow \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \dots v(s_{n-1}, s_n)$$



weighted path



$A_1, \dots, A_k$

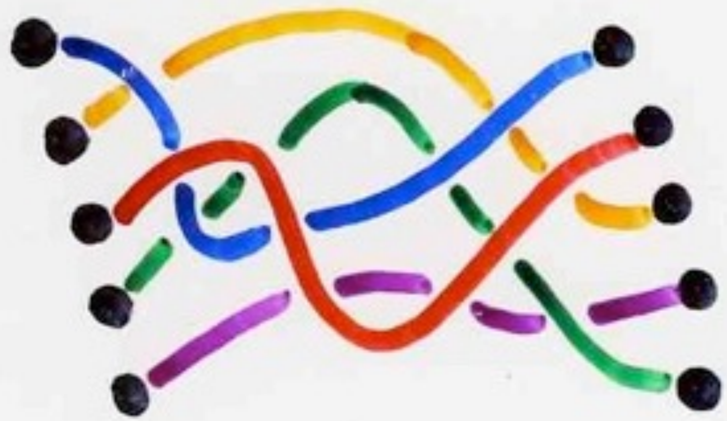
$B_1, \dots, B_k$

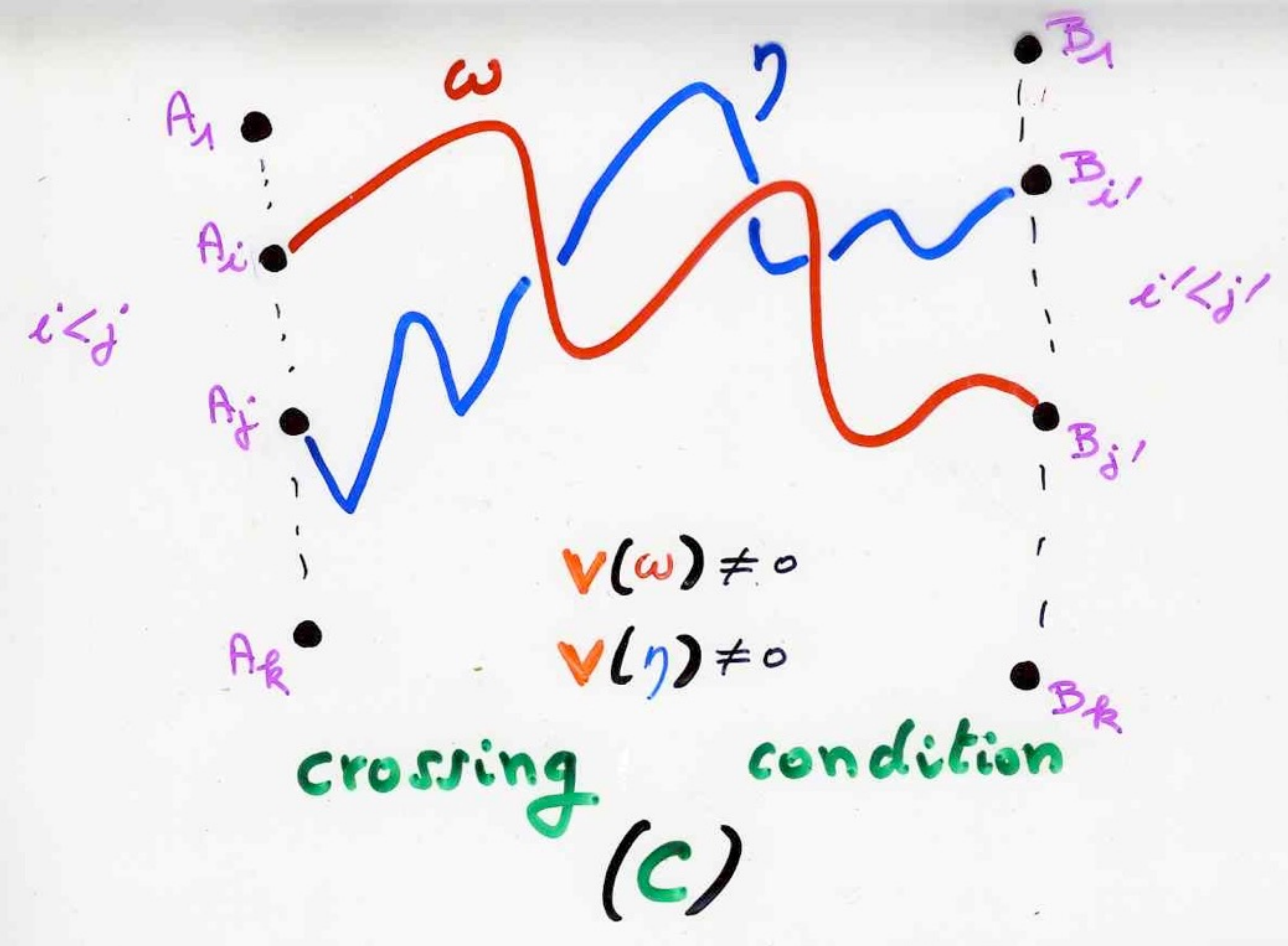
$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\,\omega_1,\ldots,\omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1)\ldots(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$





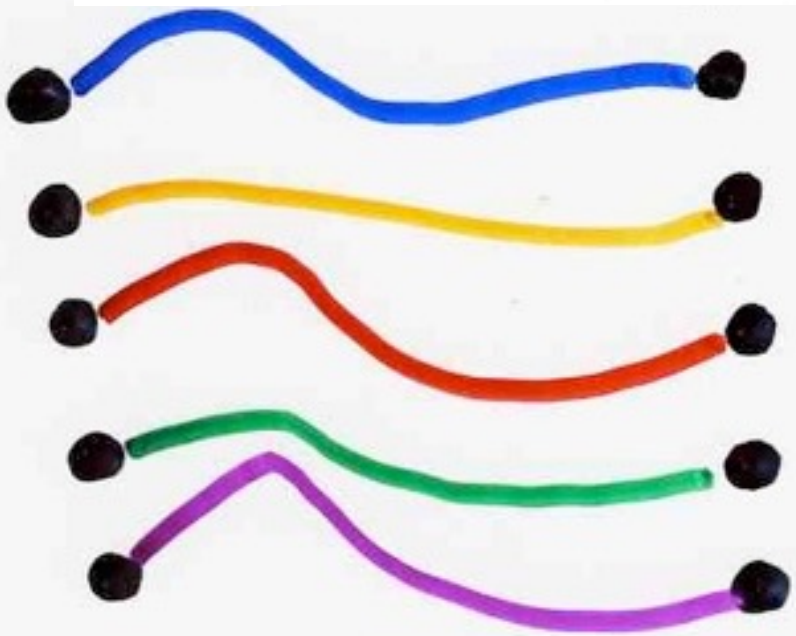
ω
η
A1
Ai
Aj
Ak
i<j
B1
Bi'
Bj'
Bk
i'<j'
V(ω)≠0
V(η)≠0
crossing condition
(C)

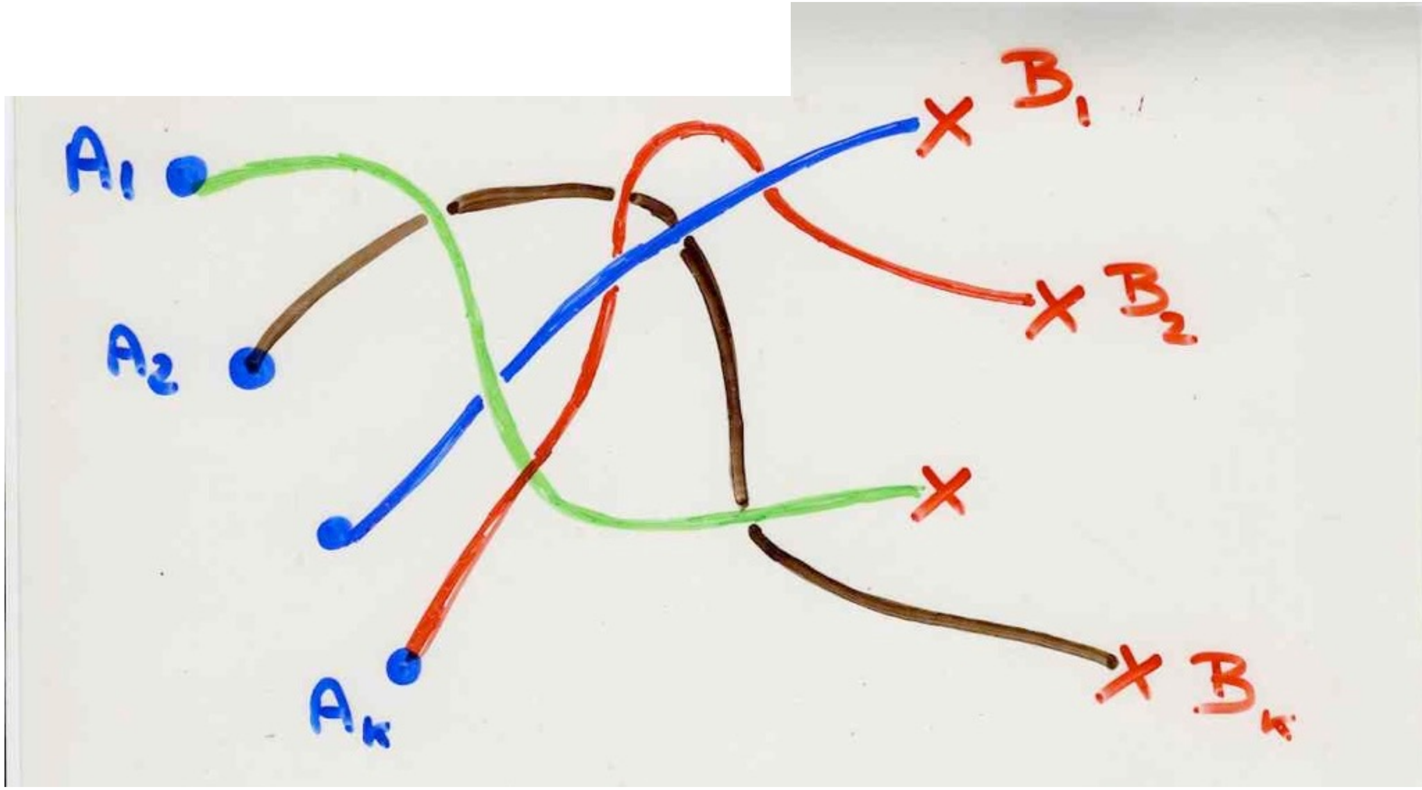
Proposition (LGV Lemma) (C) crossing condition

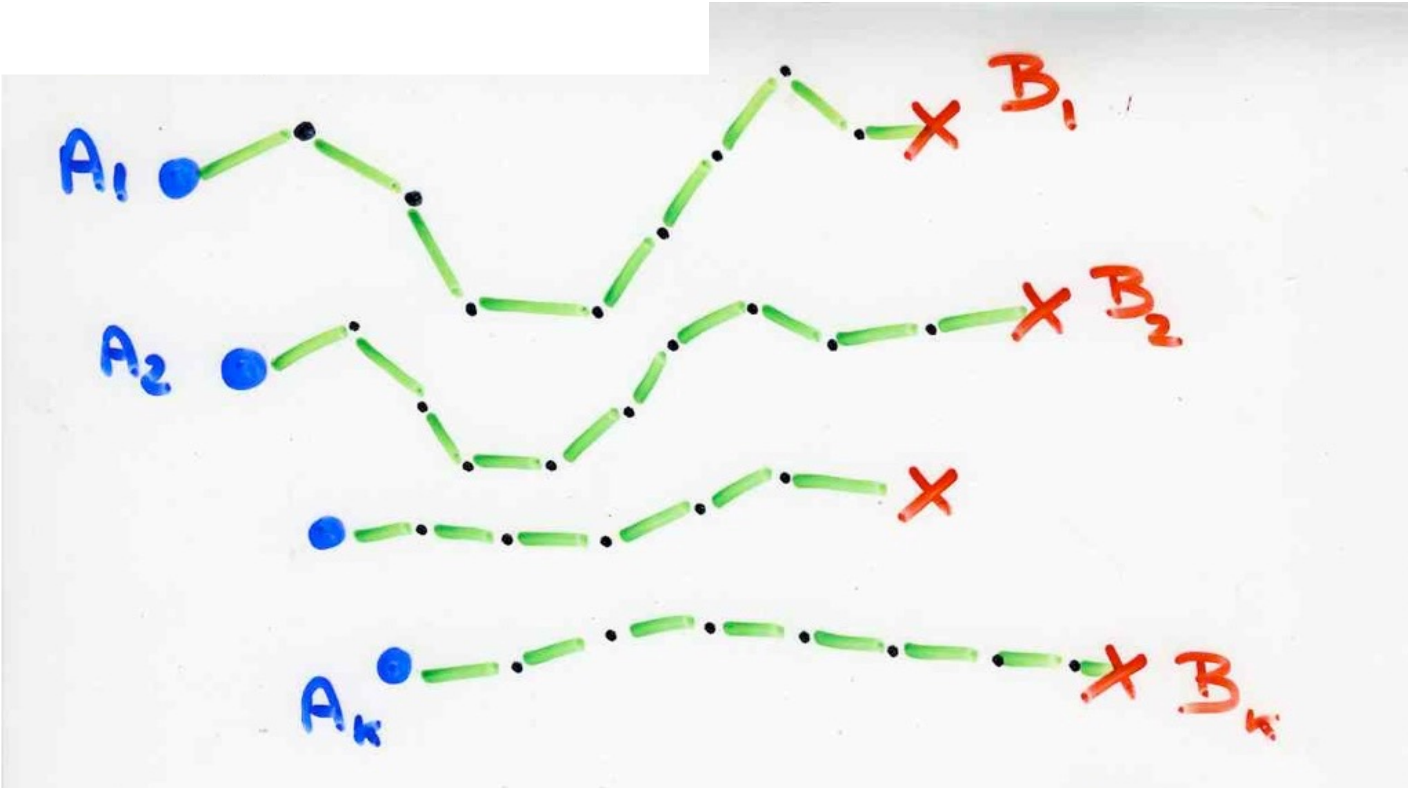
$$\det(a_{ij}) = \sum_{(\omega_1, \dots, \omega_k)} v(\omega_1) \dots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

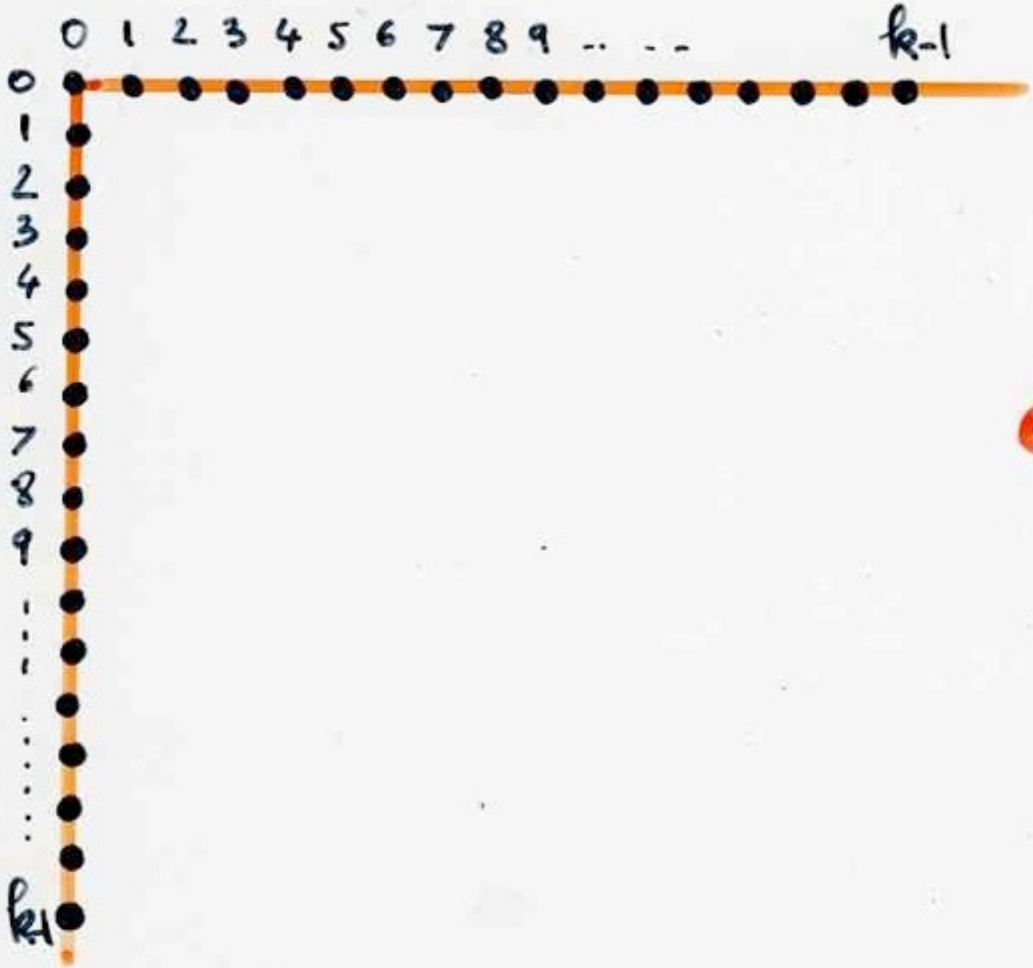
non-intersecting



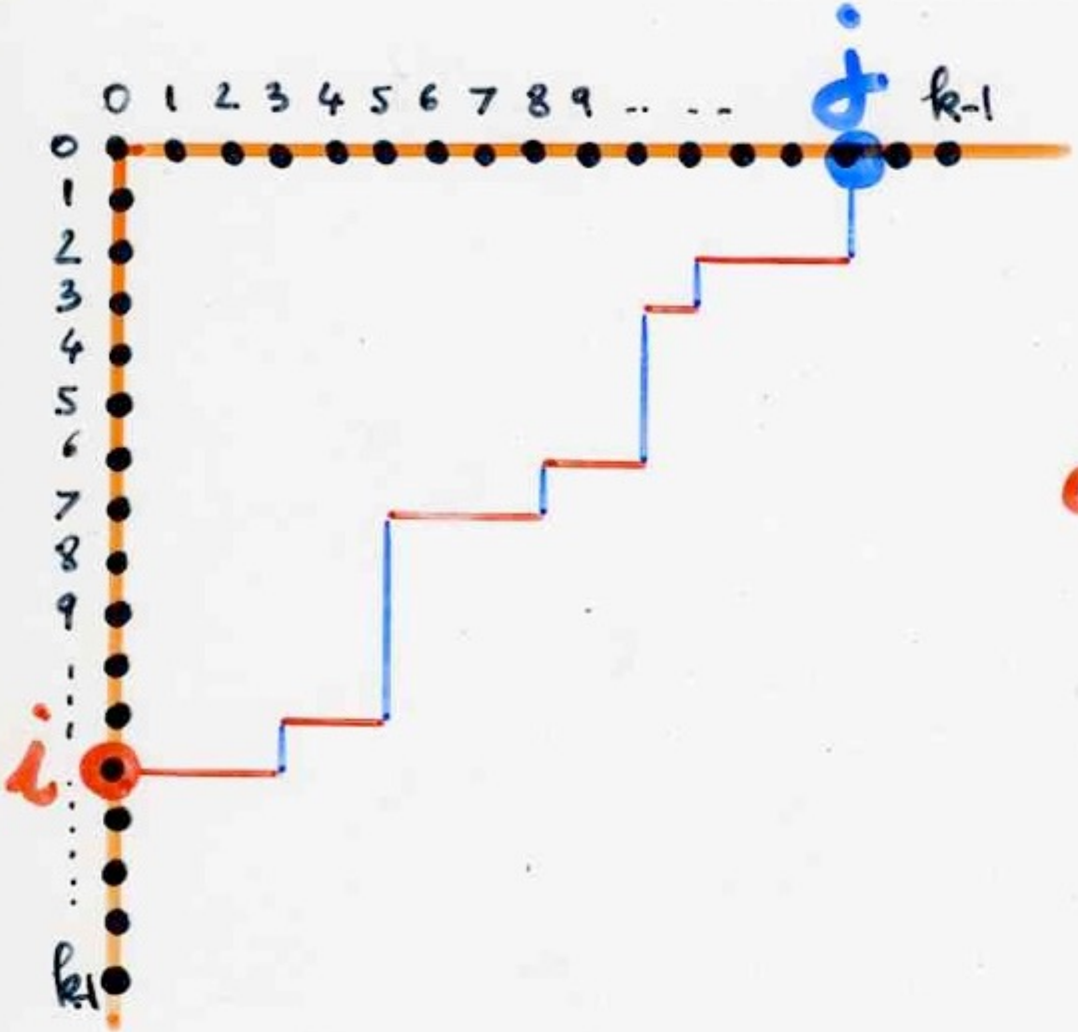

A1
A2
Ak
B1
B2
Bk


A1
B1
A2
B2
Ak
Bk

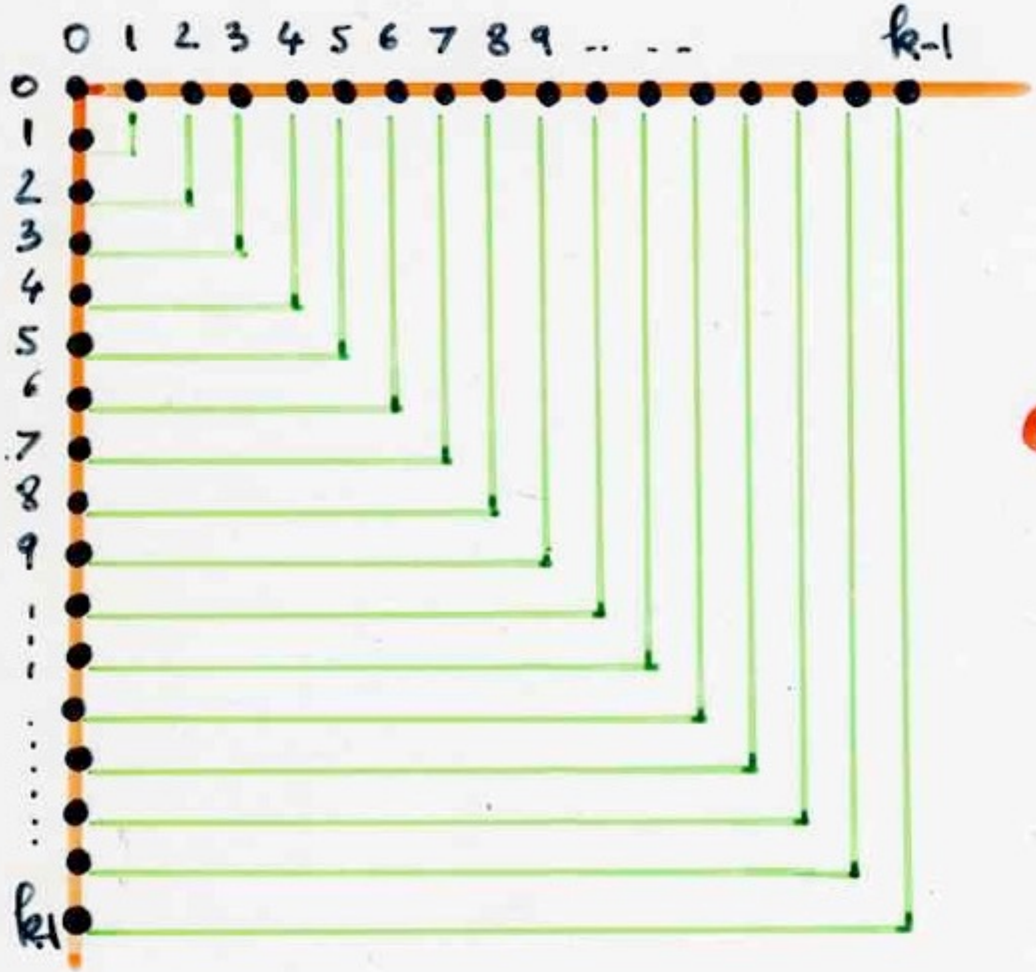
a simple example



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \\ & & & & & & \end{bmatrix}_{k \times k} =$$

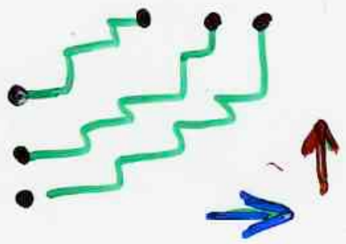


$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k \times k} =$$



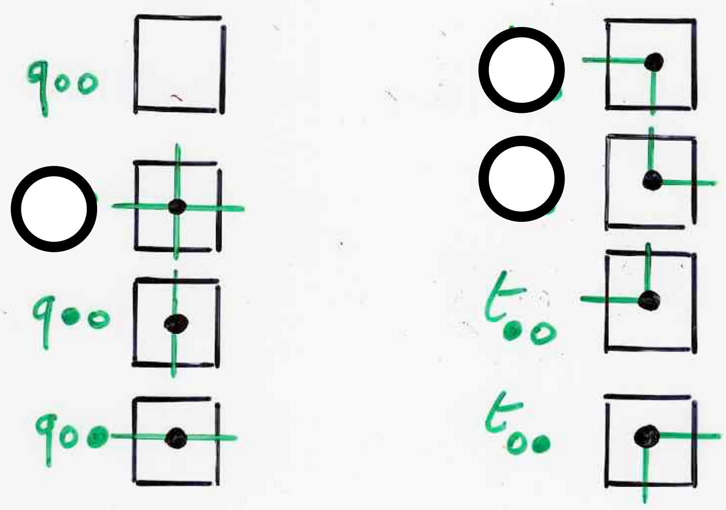
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & \binom{i+j}{i} & & \\ 1 & 5 & & & & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

non-intersecting paths
and
Binomial determinants



$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + \bigcirc\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = \bigcirc\, A_\bullet\, B_\bullet + \bigcirc\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet\, B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + t_{\circ\bullet}\, A\,B_\bullet \end{cases}$$



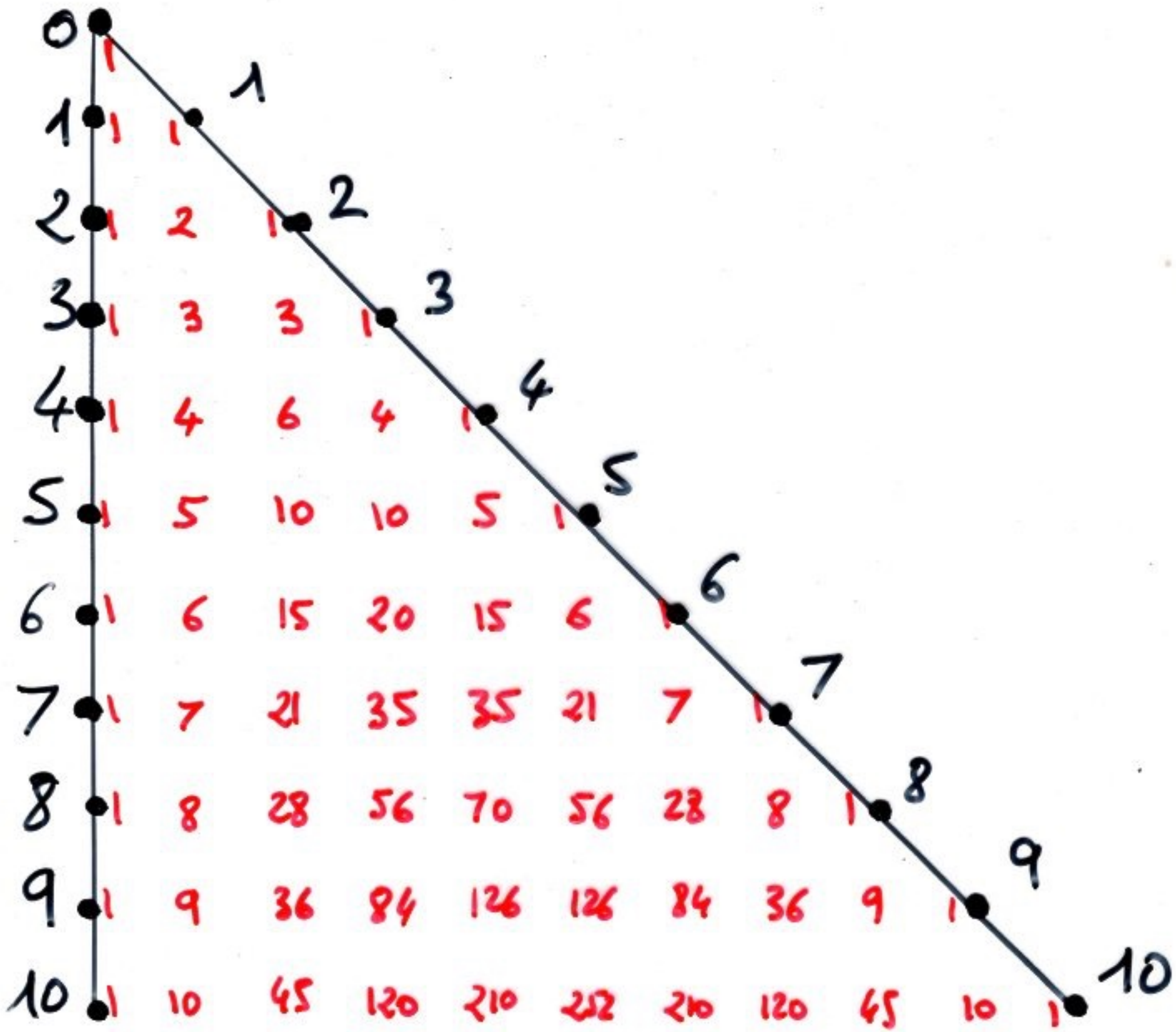
binomial determinant

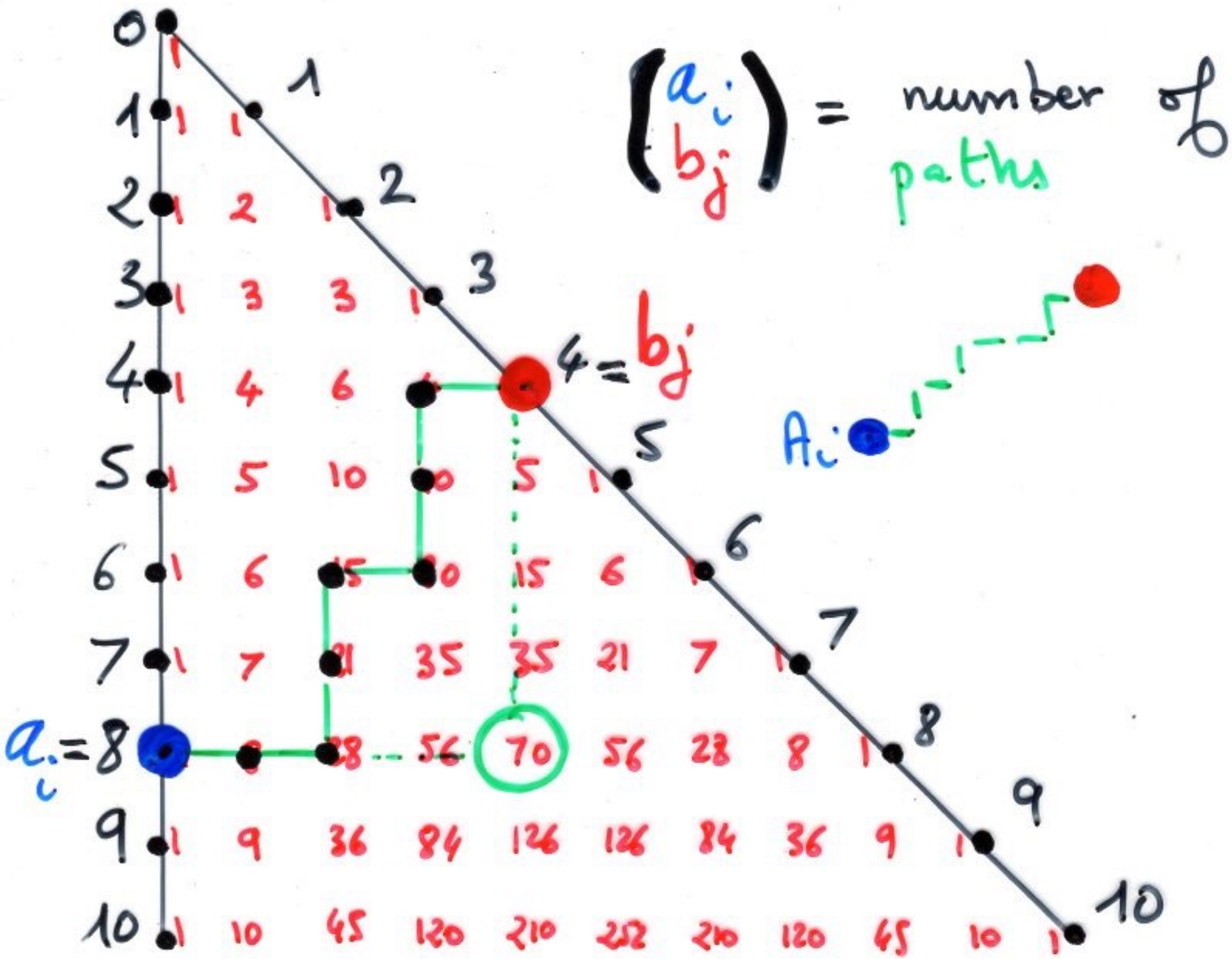
$0 \leq a_1 < \cdots < a_k$

$0 \leq b_1 < \cdots < b_k$

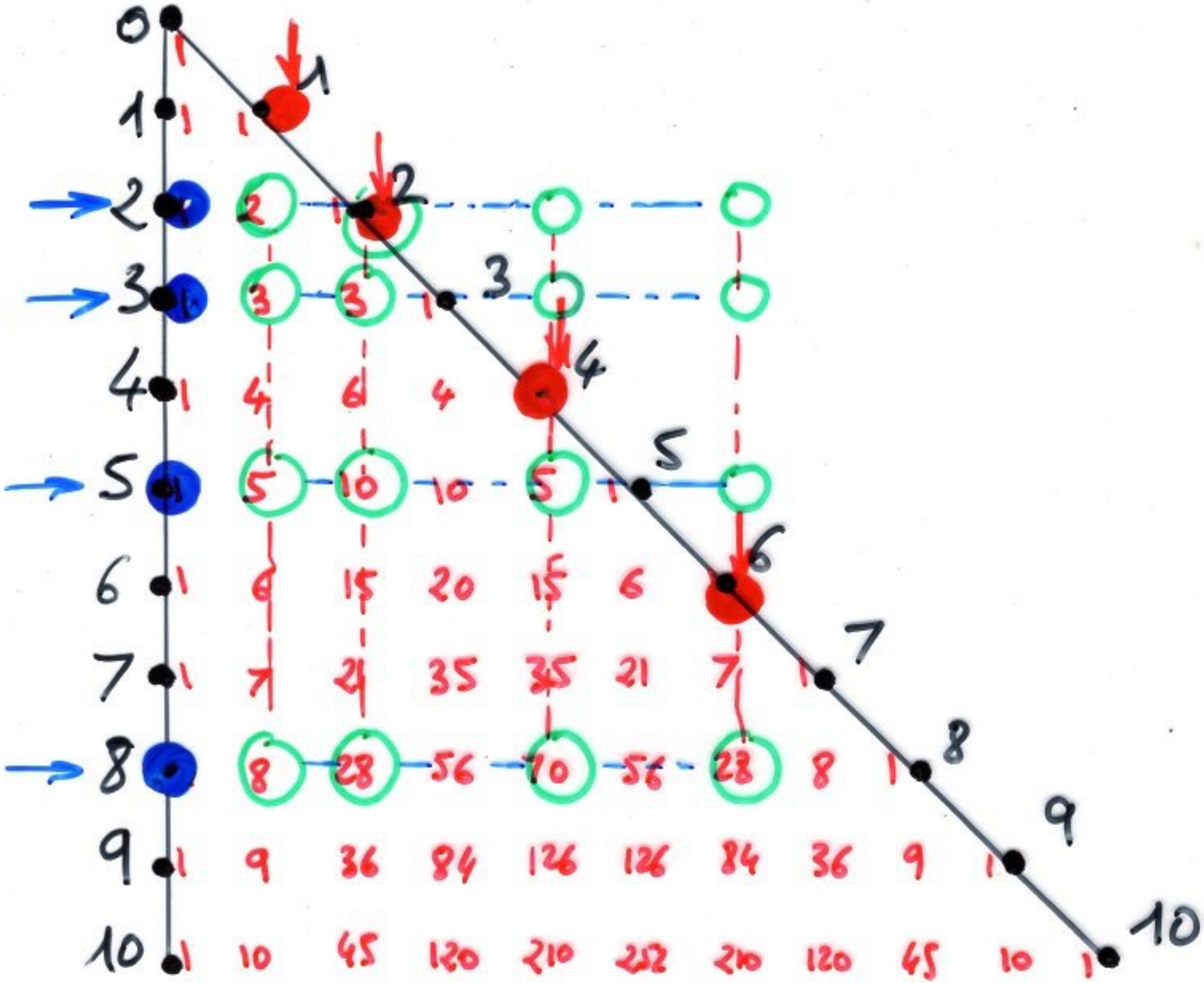
$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \det\left(\binom{a_i}{b_j}\right)_{1 \leq i,j \leq k}$$

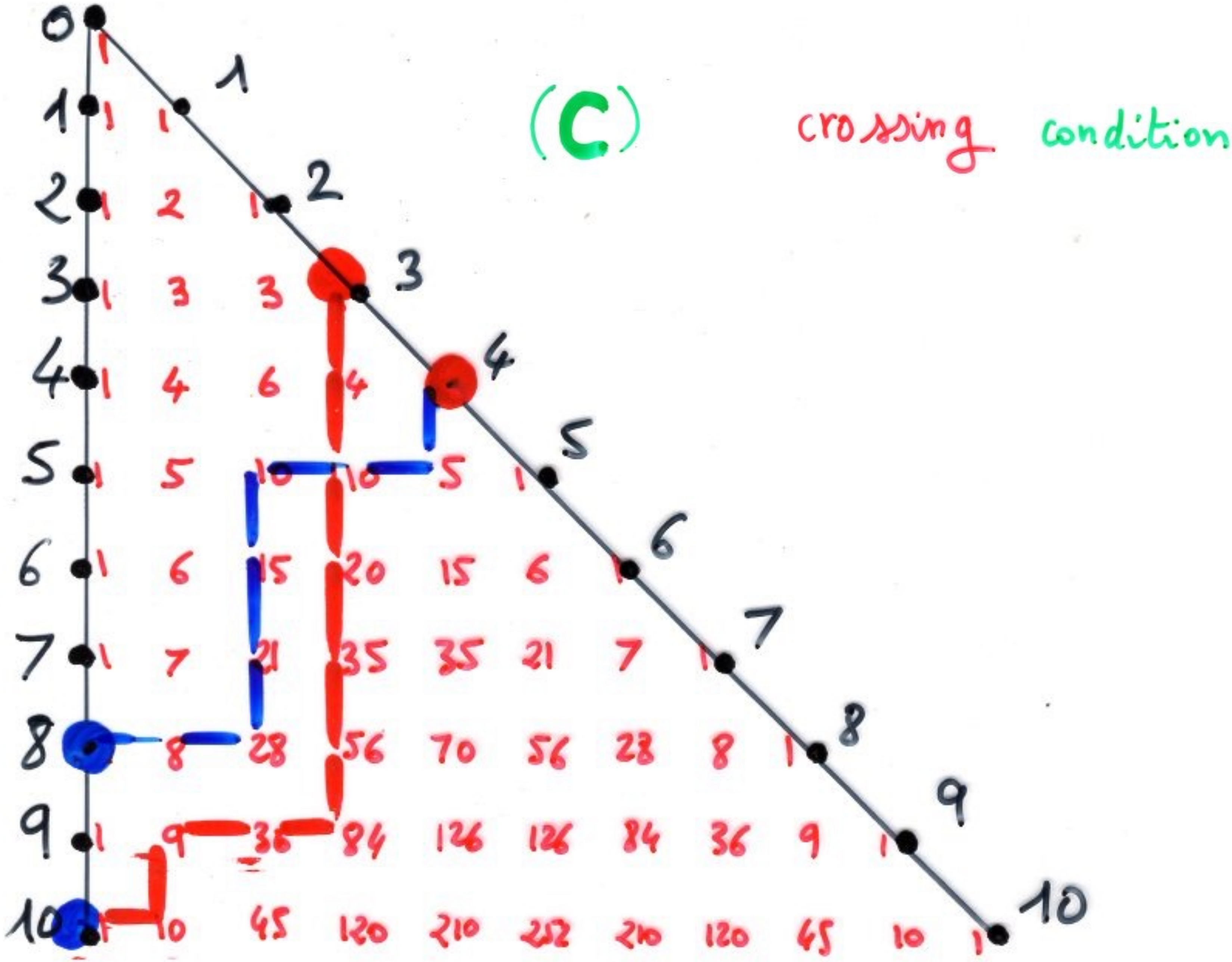
Binomial determinants:
see BJC1, Ch 5a, p29

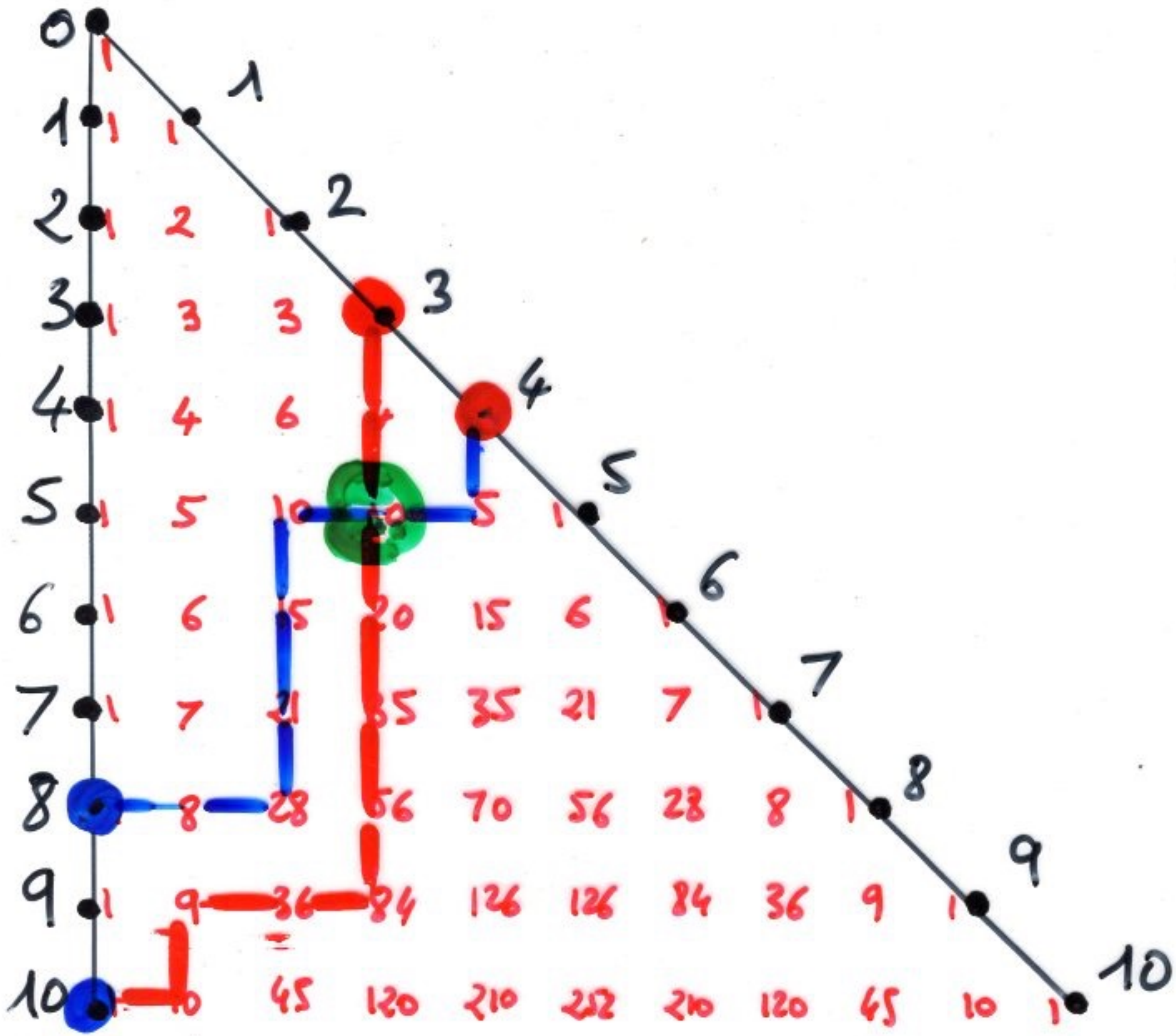

0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



(a_i b_j) = number of paths
A_i
a_i = 8
4 = b_j
0 1 2 3 4 5 6 7 8 9 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1


0 1
1 1 1 1
2 1 2 1 2
3 1 3 3 1 3
4 1 4 6 4 1 4
5 1 5 10 10 5 1 5
6 1 6 15 20 15 6 1 6
7 1 7 21 35 35 21 7 1 7
8 1 8 28 56 70 56 28 8 1 8
9 1 9 36 84 126 126 84 36 9 1 9
10 1 10 45 120 210 252 210 120 45 10 1 10


(C)
crossing condition



0
1
2
3
4
5
6
7
8
9
10
1
1 1
1 2 1
1 3 3
1 4 6
1 5 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
45 120 210 252 210 120 45 10 1

Proposition The binomial determinant

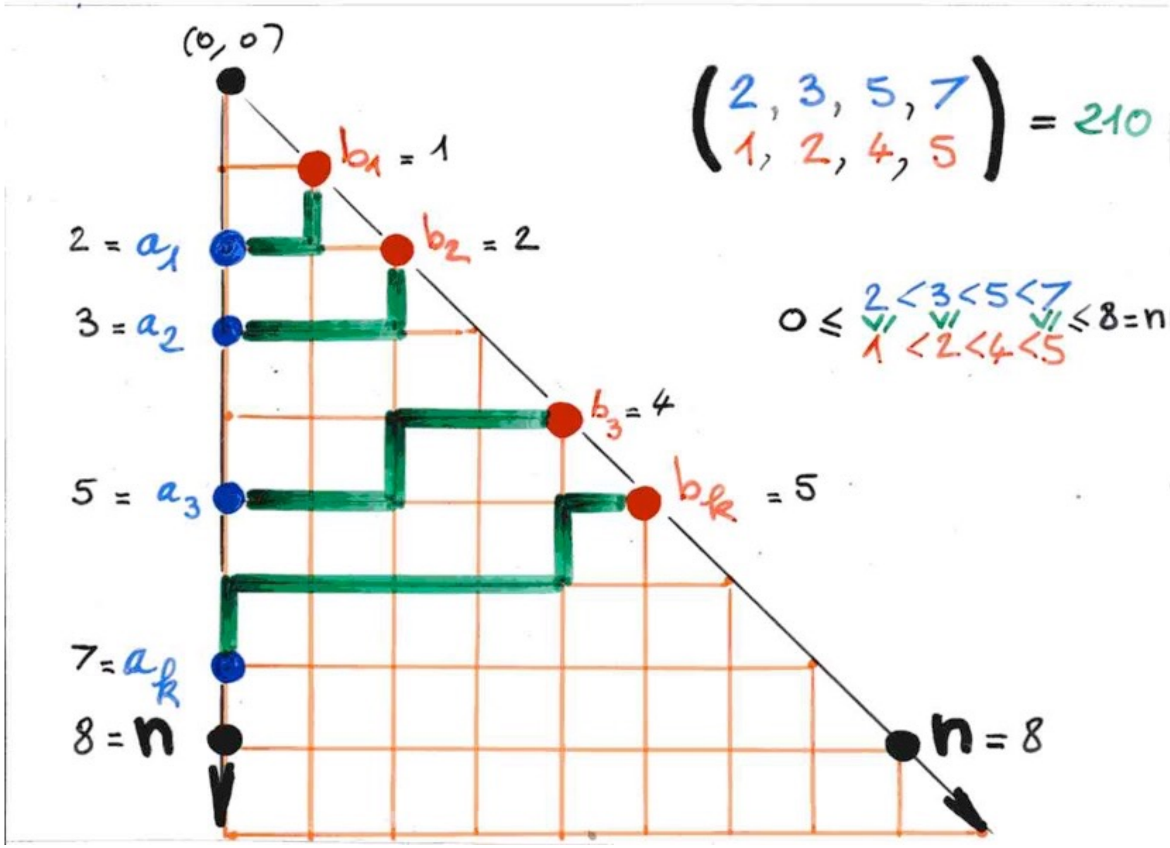
$$\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix}$$ is the number of

configurations of non-intersecting paths

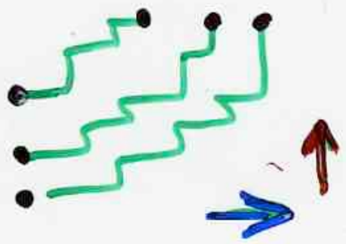
$(\omega_1, \dots, \omega_k)$, $\omega_i : A_i \rightsquigarrow B_j$,

$A_i = (0, a_i)$, $B_j = (b_j, b_j)$

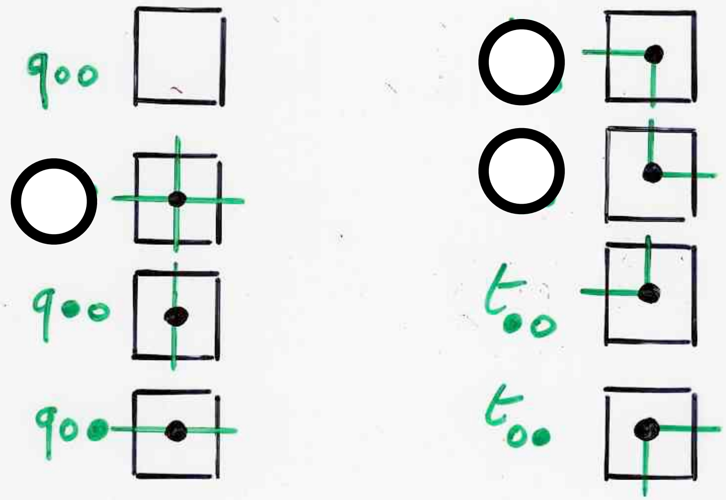
with elementary steps $\uparrow N$, $\xrightarrow{E}$



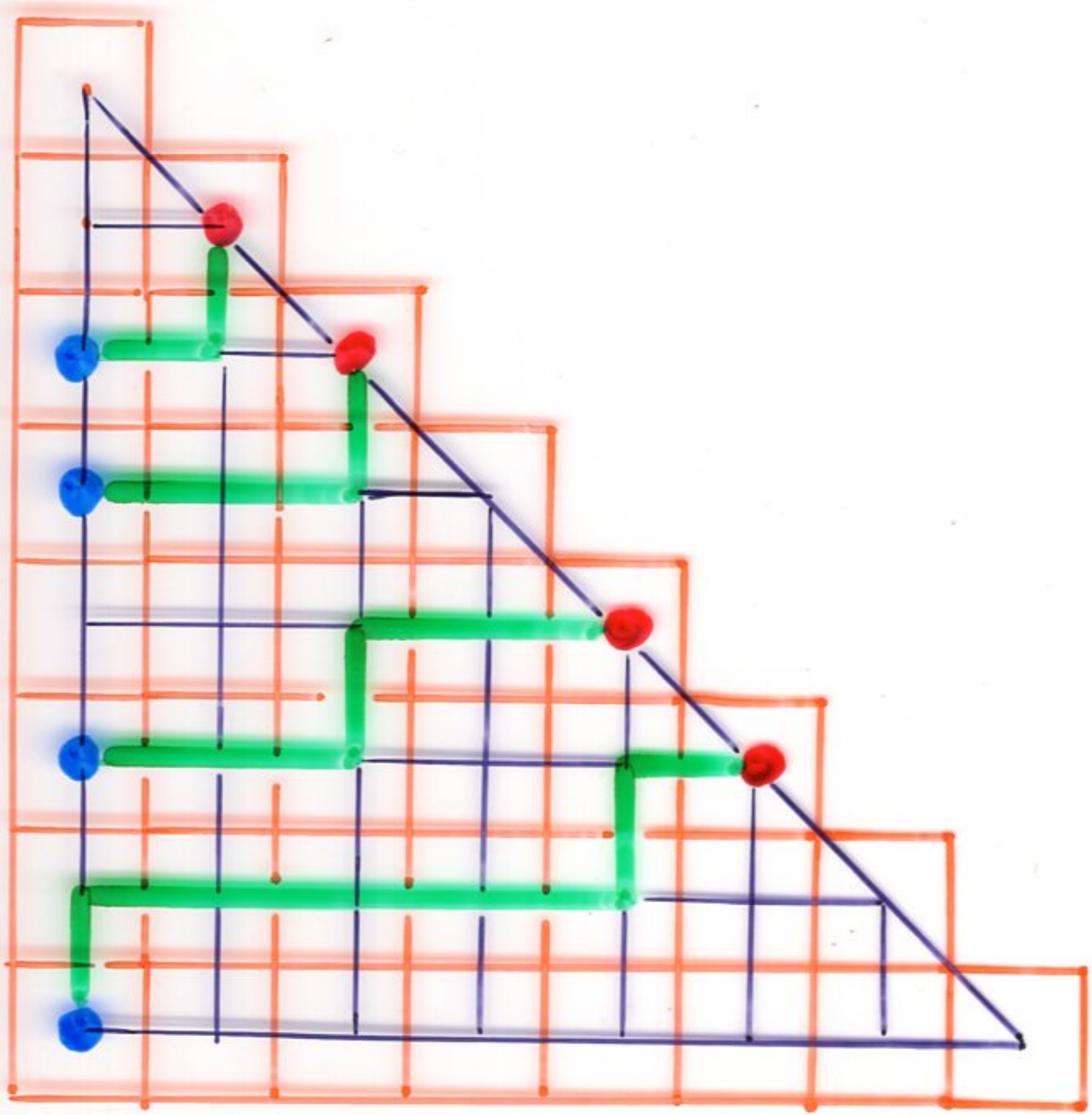
(0, 0)
$b_1 = 1$
$2 = a_1$
$b_2 = 2$
$3 = a_2$
$b_3 = 4$
$5 = a_3$
$b_k = 5$
$7 = a_k$
$8 = n$
$n = 8$
$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$
$0 \leq 2 < 3 < 5 < 7 \leq 8 = n$
$1 < 2 < 4 < 5$

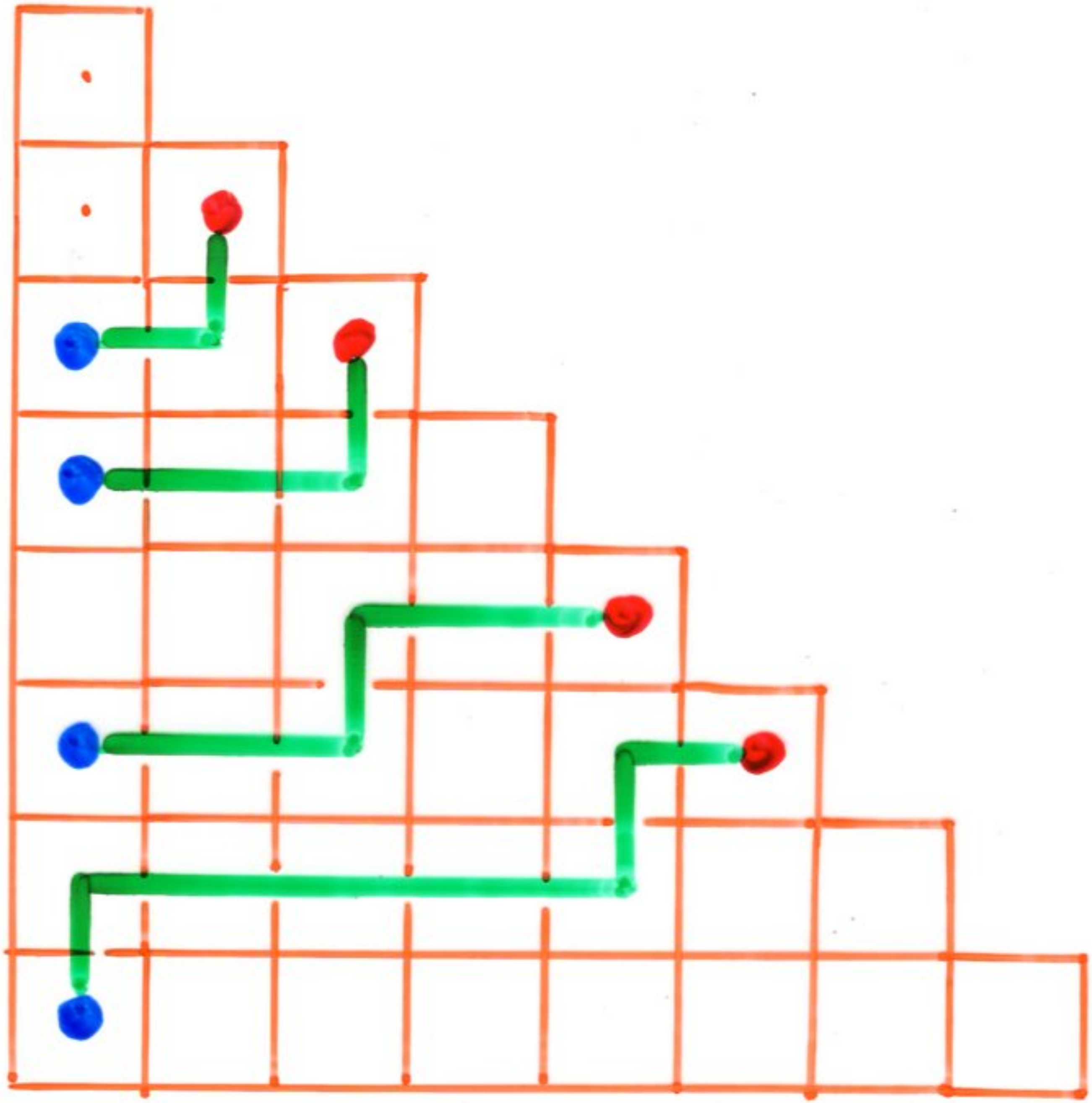


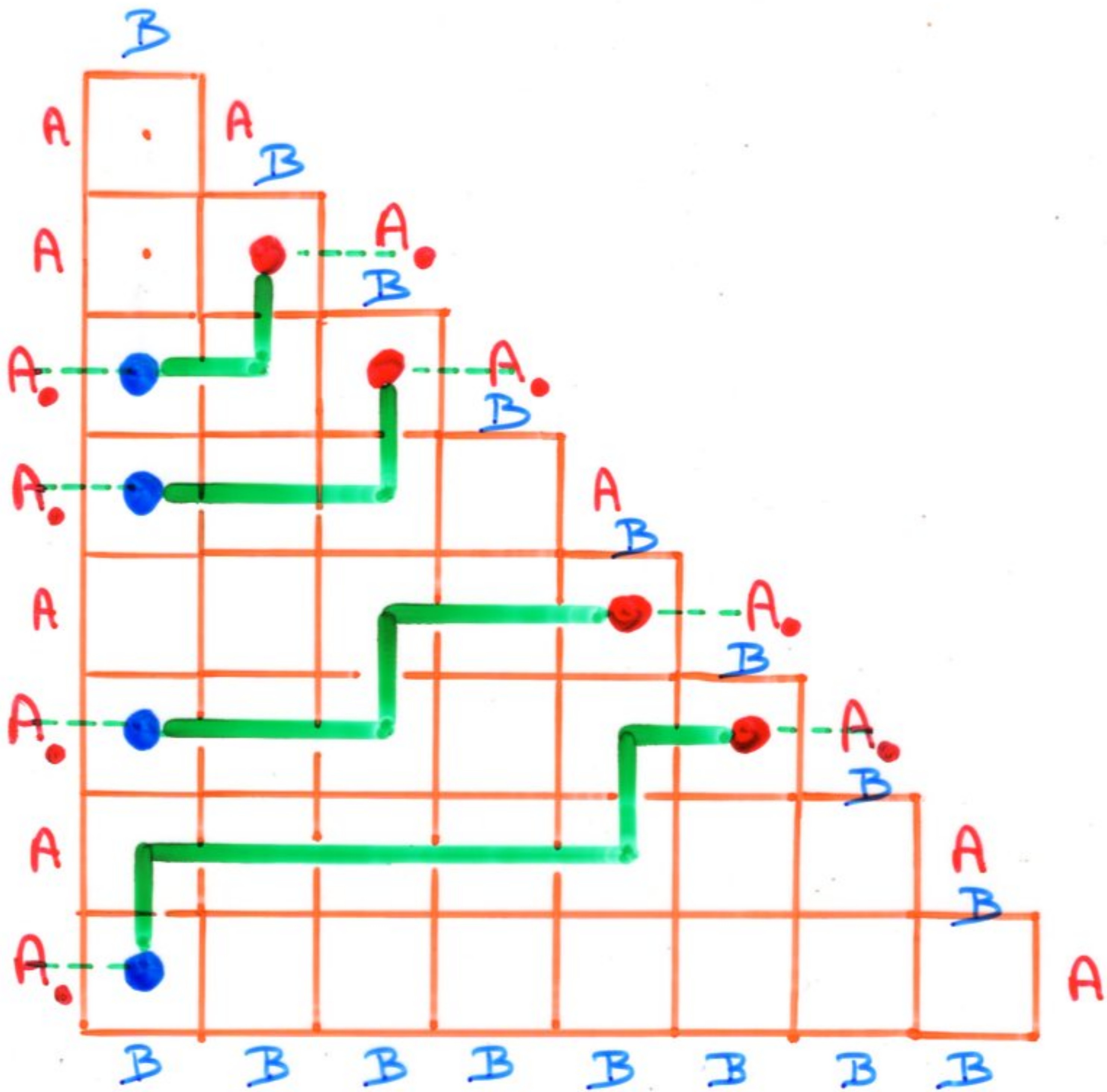
$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$



$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + \bigcirc\,A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = \bigcirc\,A_\bullet\,B_\bullet + \bigcirc\,A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + t_{\bullet\circ}\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + t_{\circ\bullet}\,A\,B_\bullet \end{cases}$$



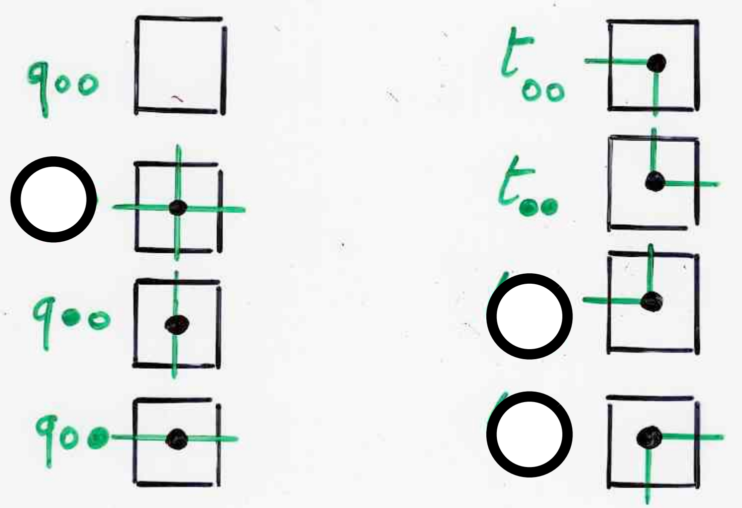



B
A
A
B
A
A
B
A
A
B
A
A
B
A
A
B
A
A
B
A
A
B
A
B B B B B B B B

non-intersecting paths,
rhombus tilings,
and
plane partitions

non intersecting paths

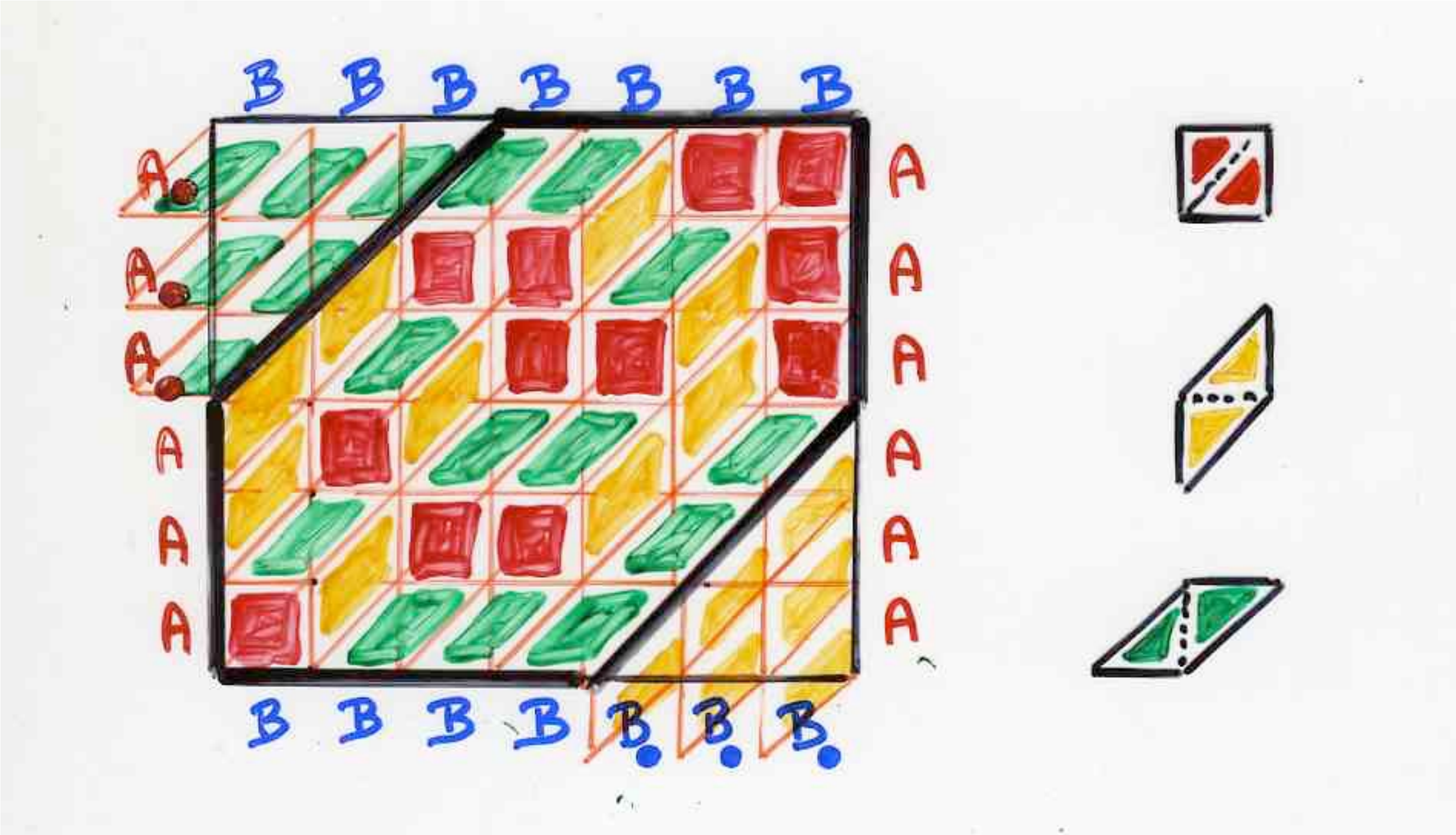
$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$



Rhombus tilings

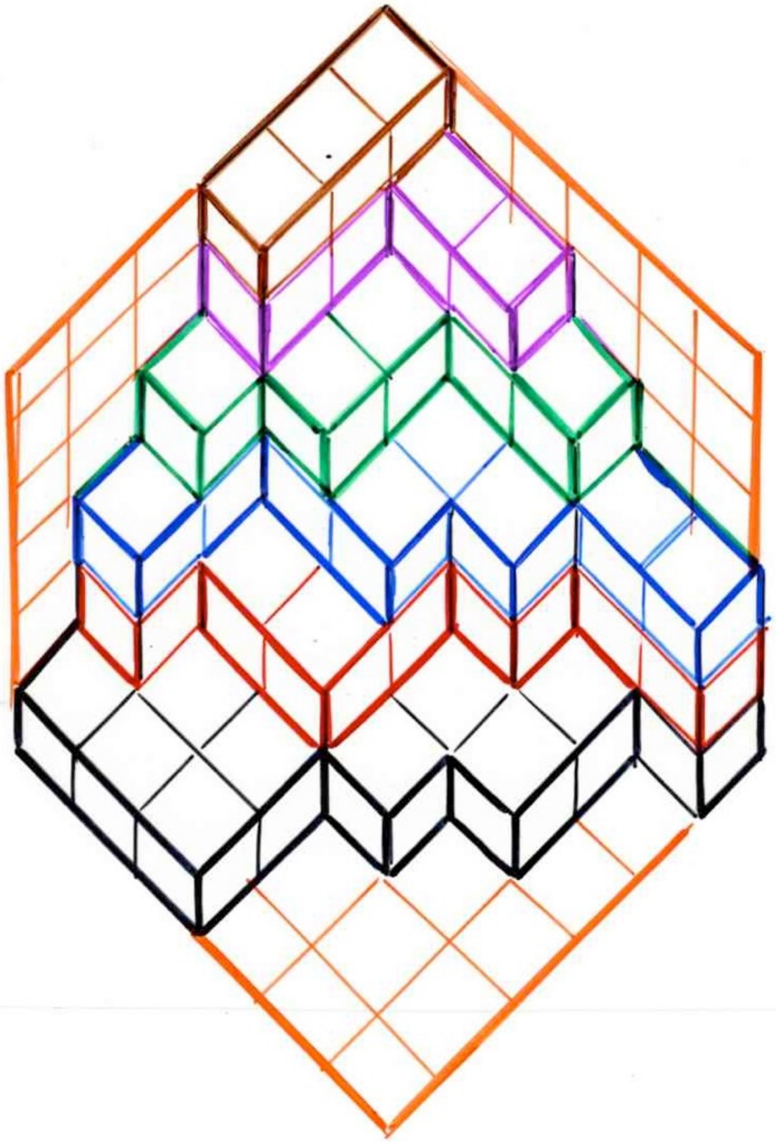
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = 0\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + 0\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + 0\, A B_\bullet \end{cases}$$

Rhombus tilings



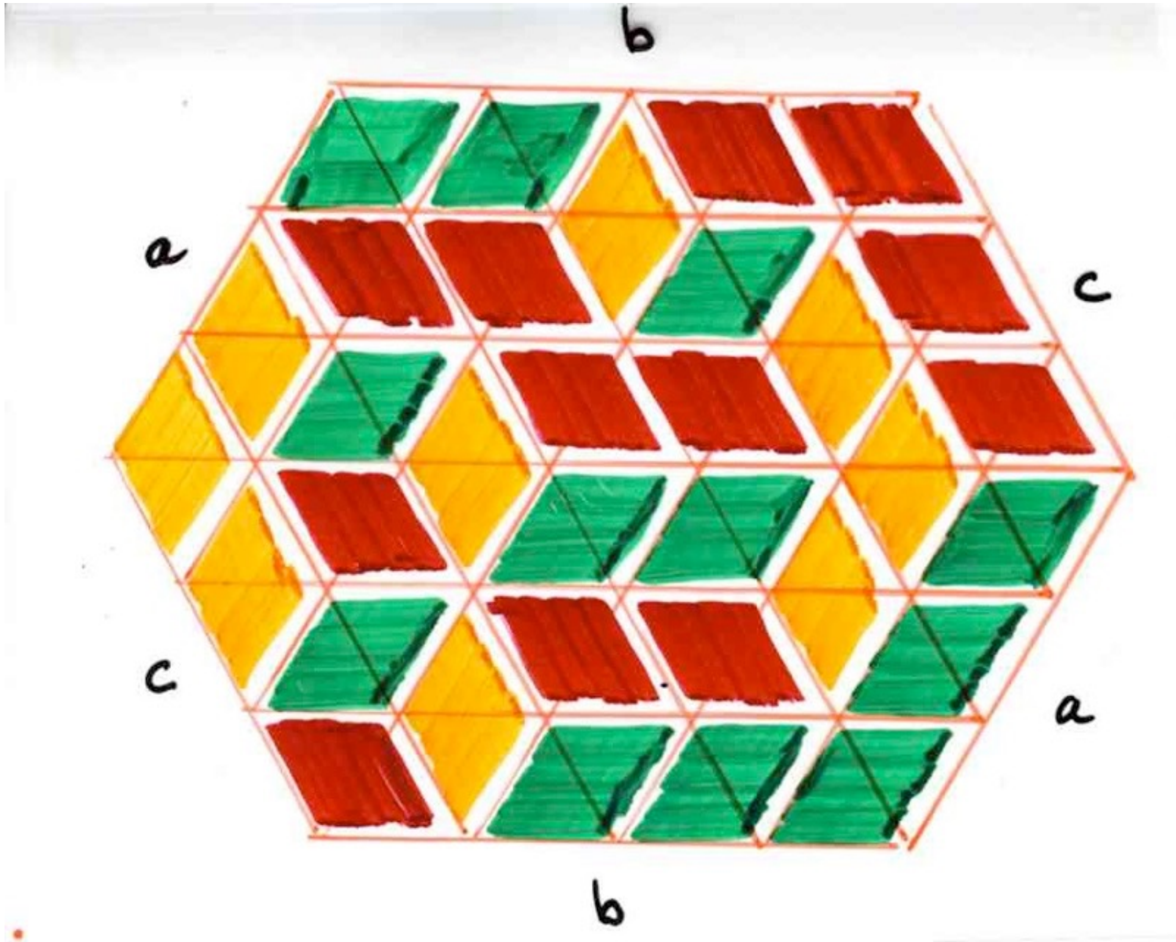
$$\begin{cases} B\,A = q_{00}\, A\,B + t_{00}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = \bigcirc\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet\, A = q_{\bullet 0}\, A\, B_\bullet + \bigcirc\, A_\bullet\, B \\ B\, A_\bullet = q_{0\bullet}\, A_\bullet\, B + \bigcirc\, A\, B_\bullet \end{cases}$$

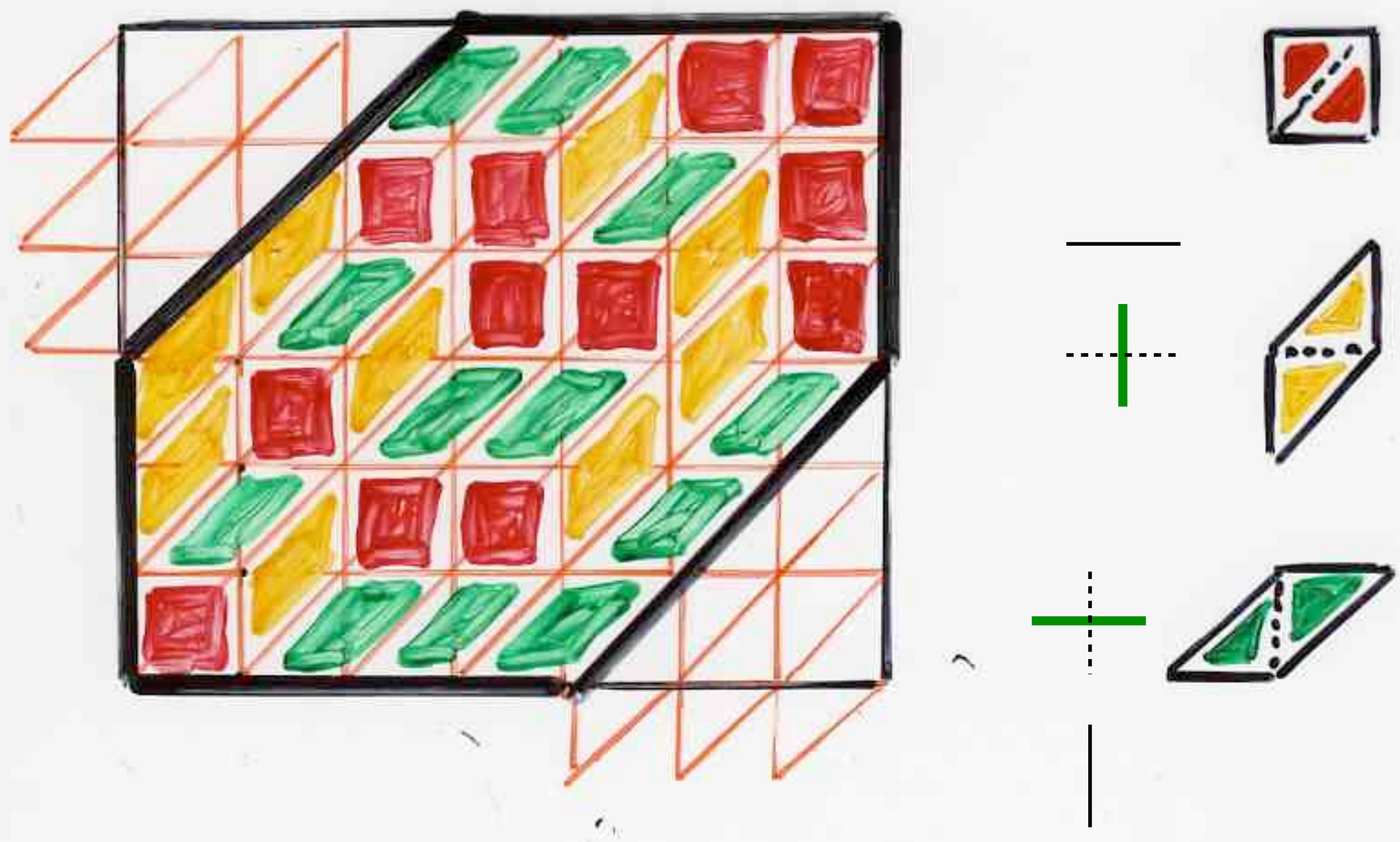
plane partitions
in a box (a, b, c)

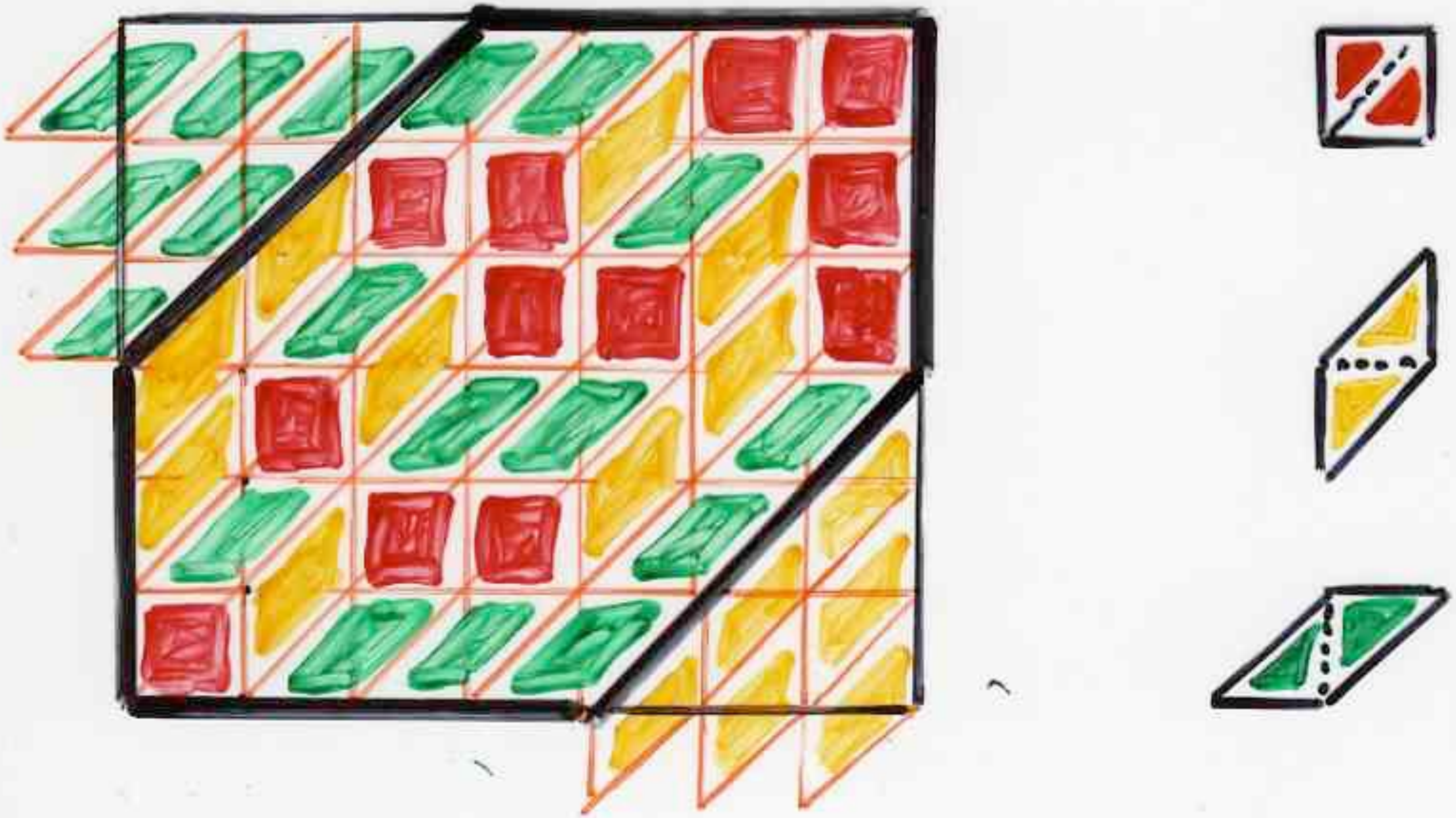


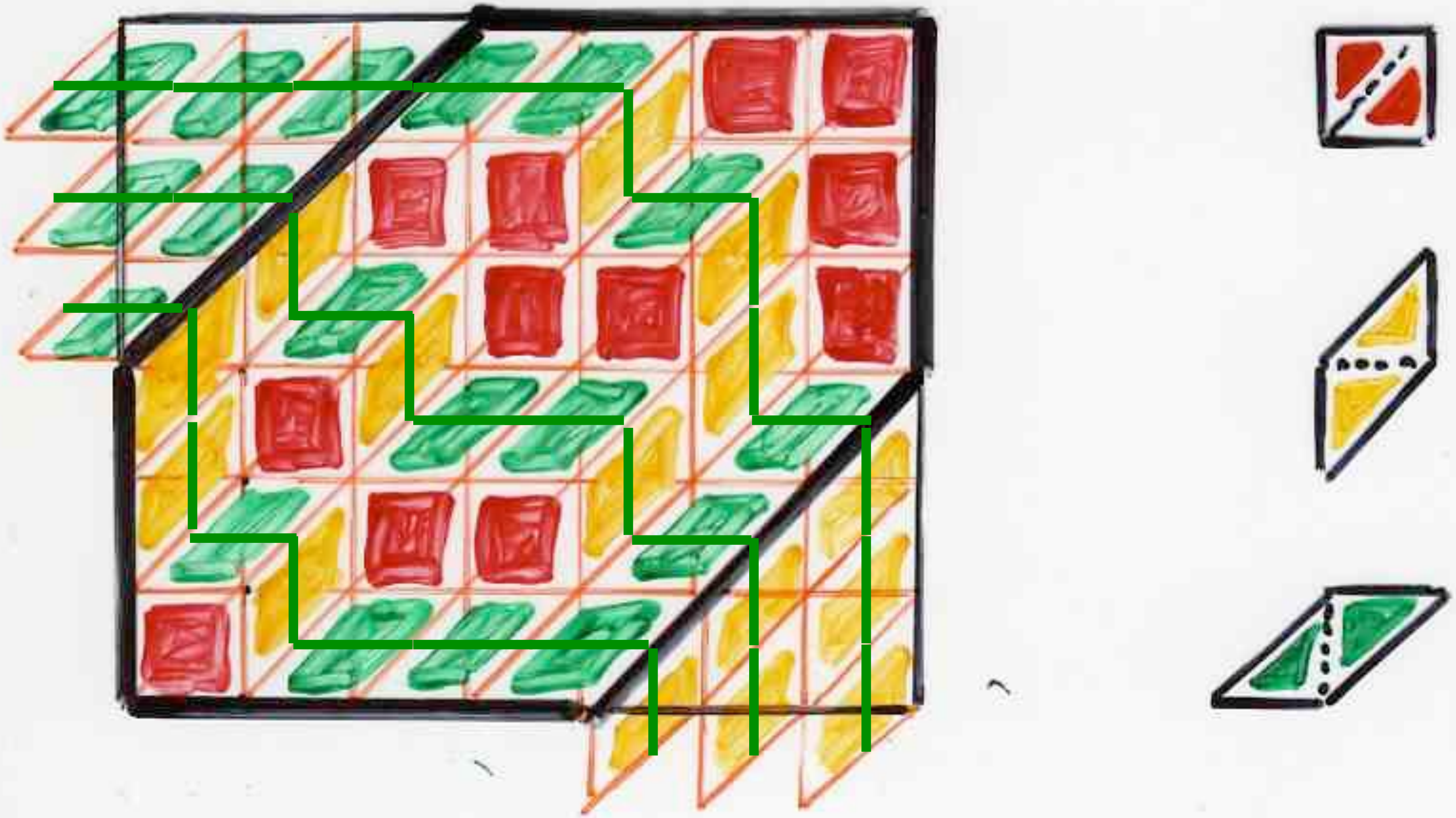
$$\begin{cases} B A = q_{00} \, A B + t_{00} \, A_{\bullet} B_{\bullet} \\ B_{\bullet} A_{\bullet} = \bigcirc \, A_{\bullet} B_{\bullet} + t_{\bullet\bullet} \, A B \\ B_{\bullet} A = q_{\bullet 0} \, A B_{\bullet} + \bigcirc \, A_{\bullet} B \\ B A_{\bullet} = q_{0 \bullet} \, A_{\bullet} B + \bigcirc \, A B_{\bullet} \end{cases}$$

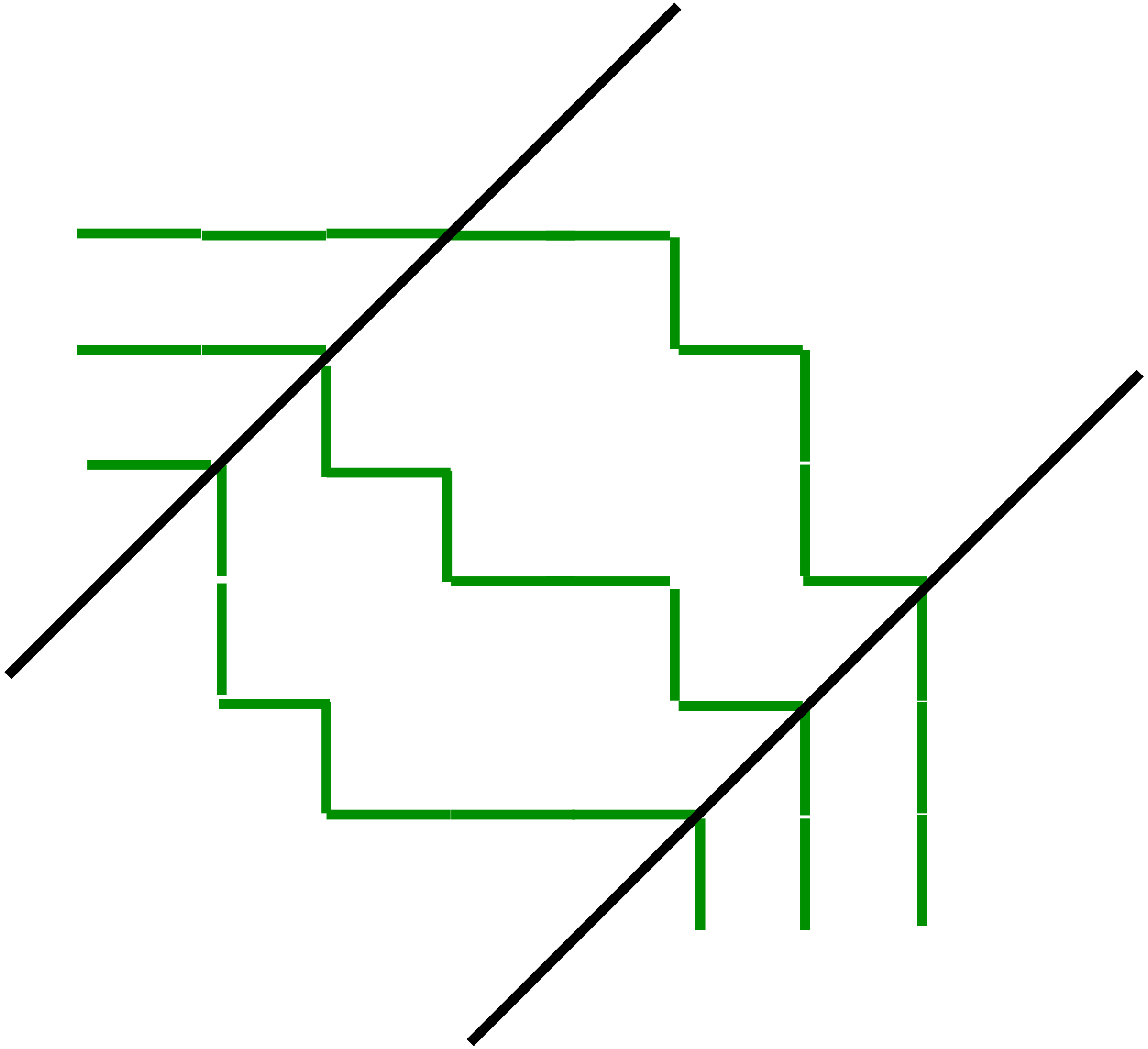
bijection
rhombus tilings
non-intersecting paths

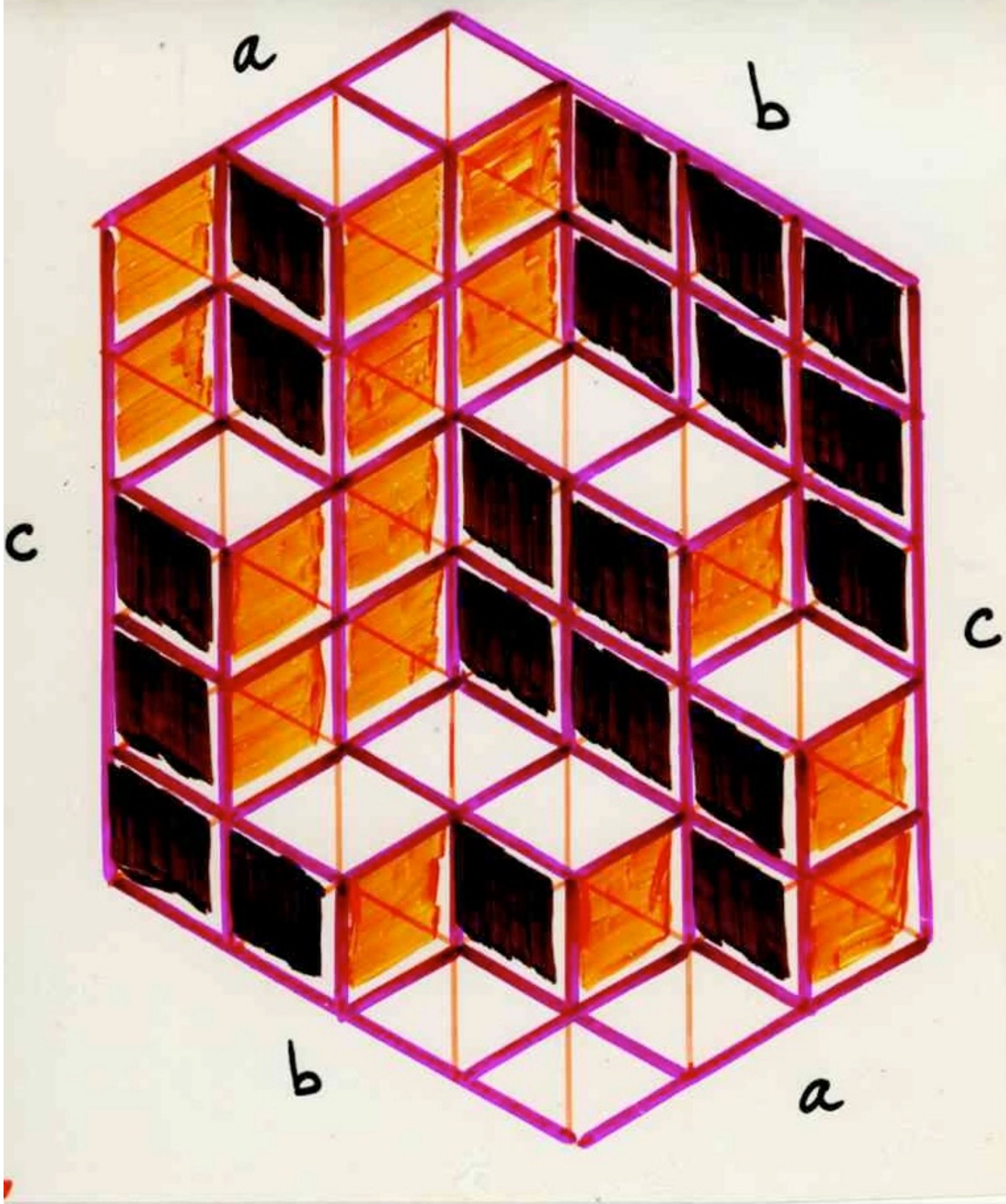

b
a
c
c
a
b

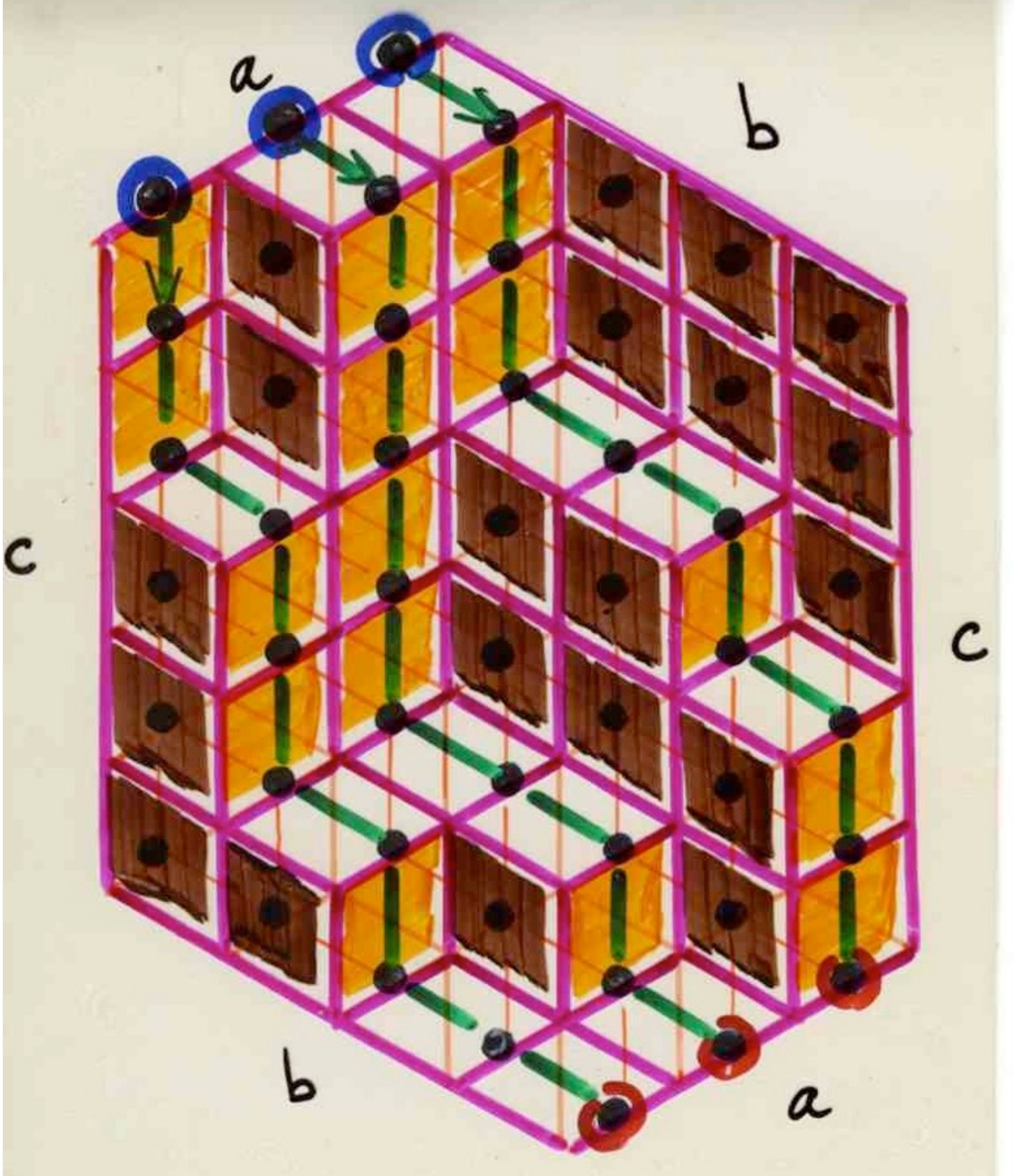






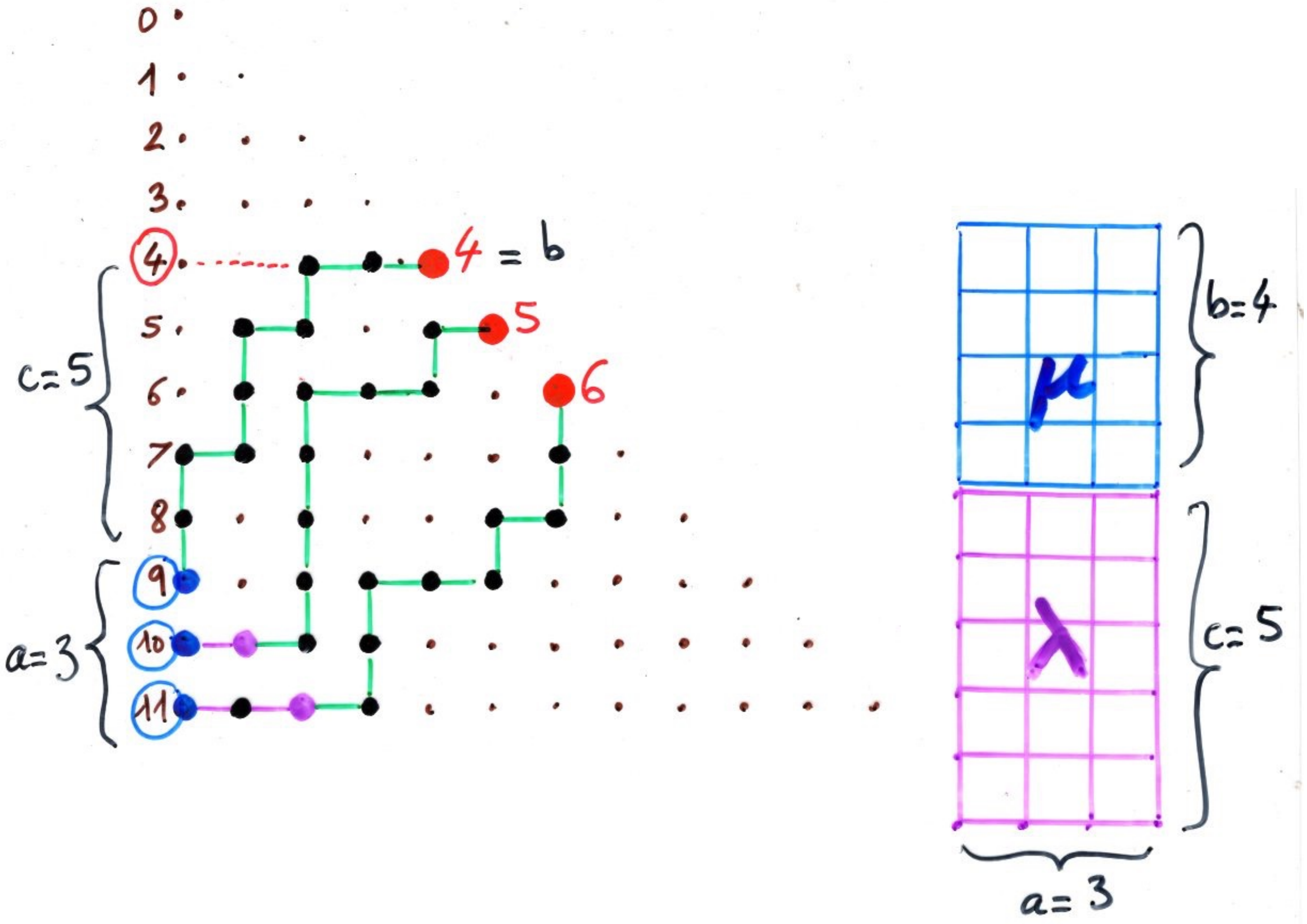



a
b
c
c
b
a


a
b
c
c
b
a

Proof of MacMahon formula, see:

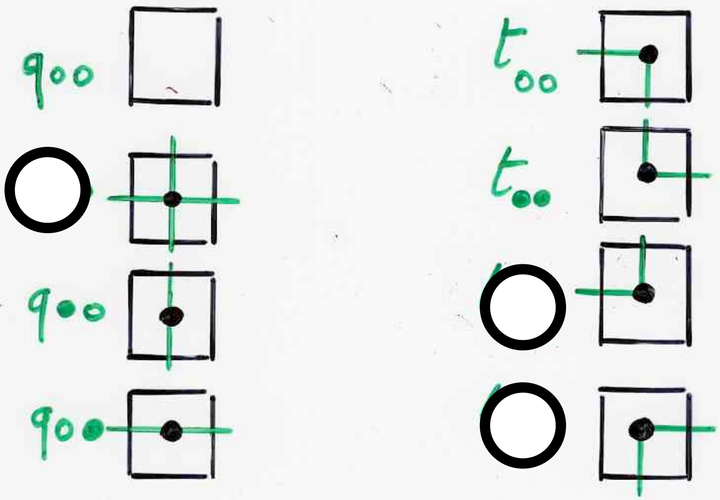
BJC 1, Ch 5a, p105



Osculating paths

non intersecting paths

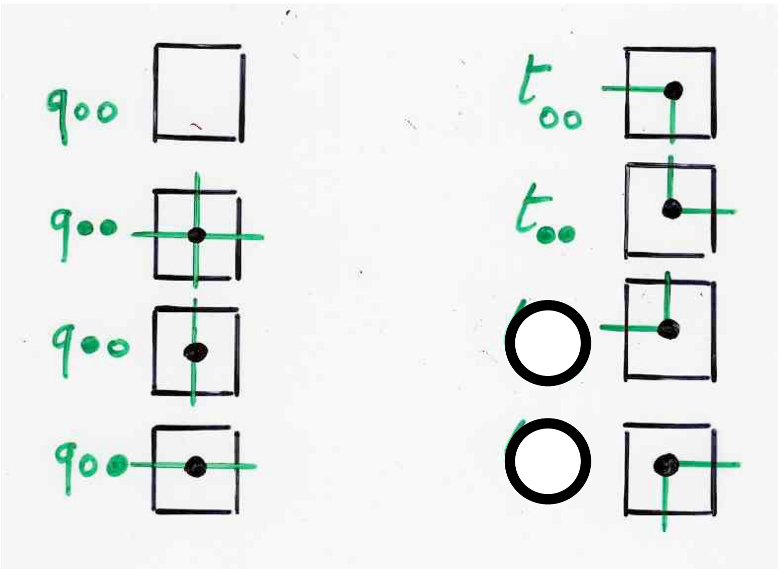
$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$



$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = \bigcirc\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + \bigcirc\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + \bigcirc\,A\,B_\bullet \end{cases}$$

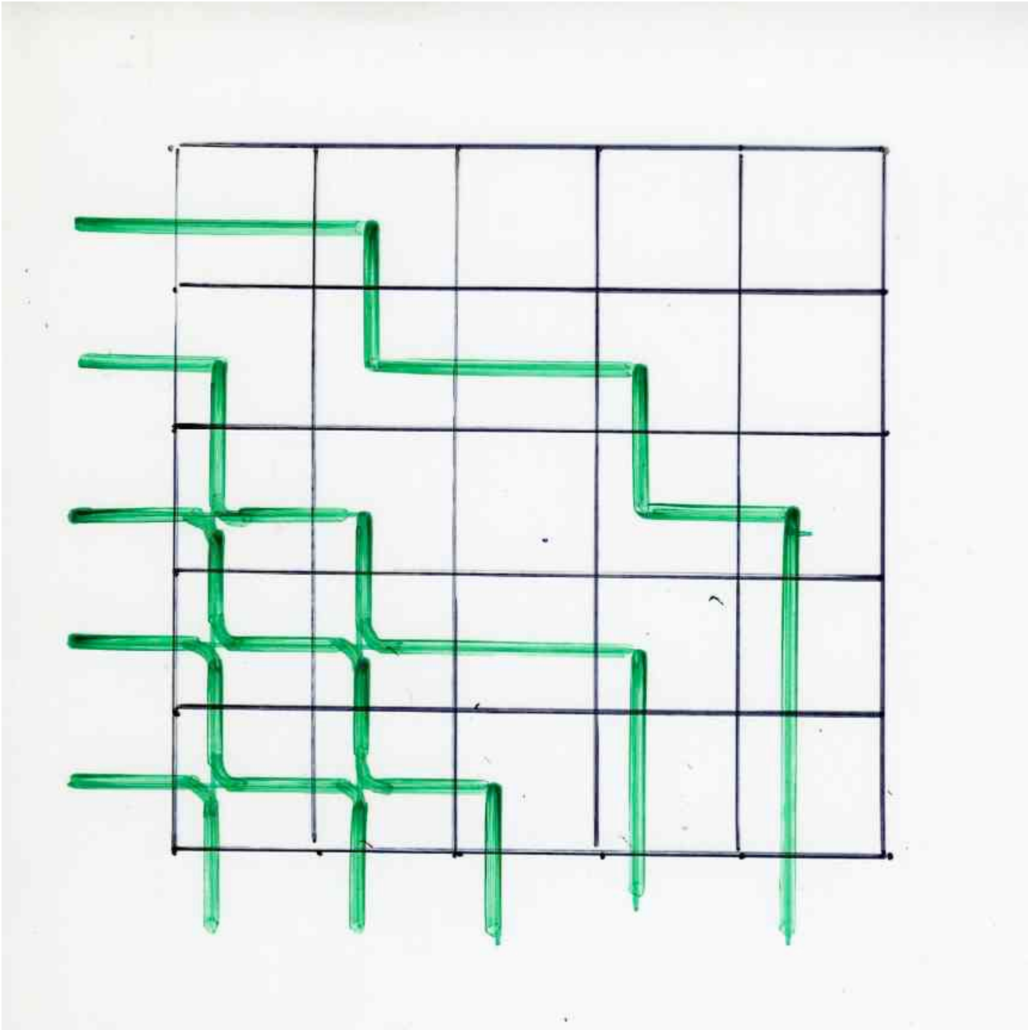
osculating paths

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$



$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_{\bullet}\,B_{\bullet} \\ B_{\bullet}\,A_{\bullet} = q_{\bullet\bullet}\,A_{\bullet}\,B_{\bullet} + t_{\bullet\bullet}\,A\,B \\ B_{\bullet}\,A = q_{\bullet\circ}\,A\,B_{\bullet} + \bigcirc\,A_{\bullet}\,B \\ B\,A_{\bullet} = q_{\circ\bullet}\,A_{\bullet}\,B + \bigcirc\,A\,B_{\bullet} \end{cases}$$

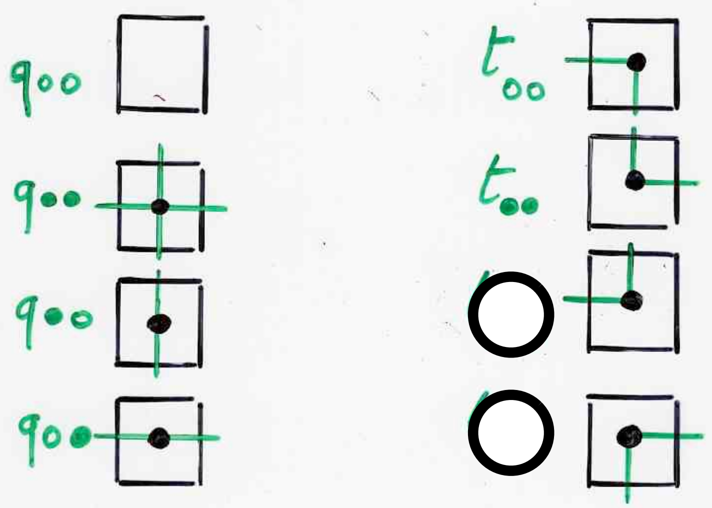
osculating paths



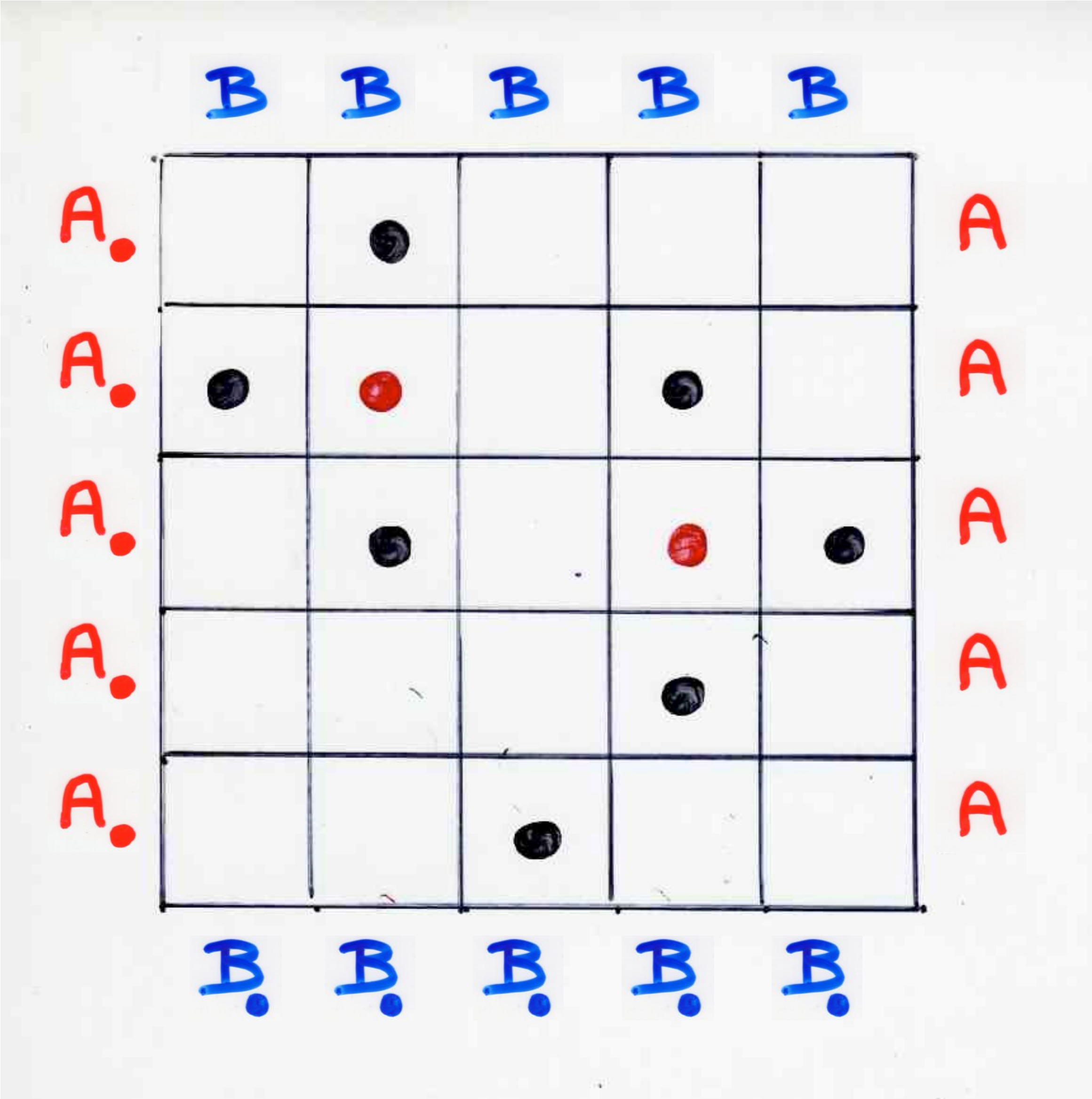
osculating paths

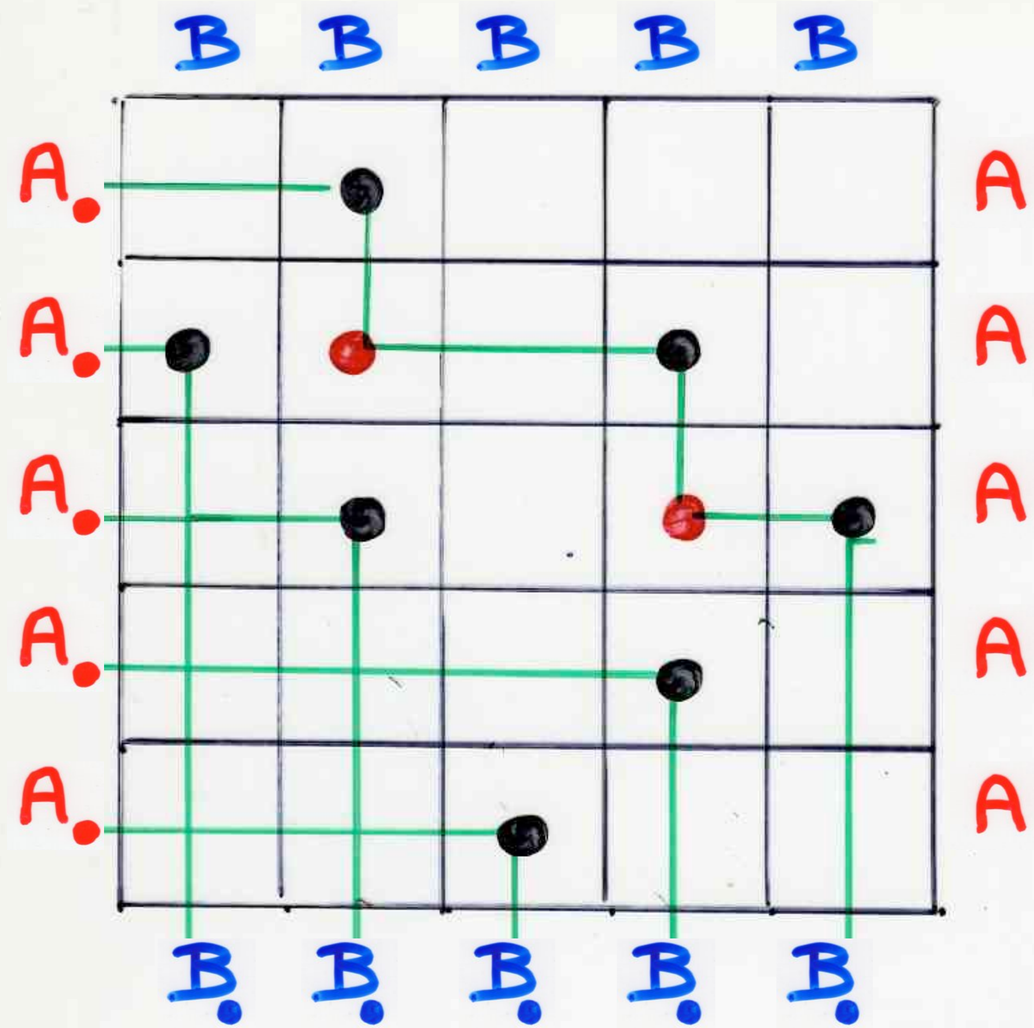
$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

The 6-vertex model



$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\,A_\bullet B_\bullet + t_{\bullet\bullet}\,A\,B \\ B_\bullet A = q_{\bullet\circ}\,A\,B_\bullet + \bigcirc\,A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet B + \bigcirc\,A\,B_\bullet \end{cases}$$

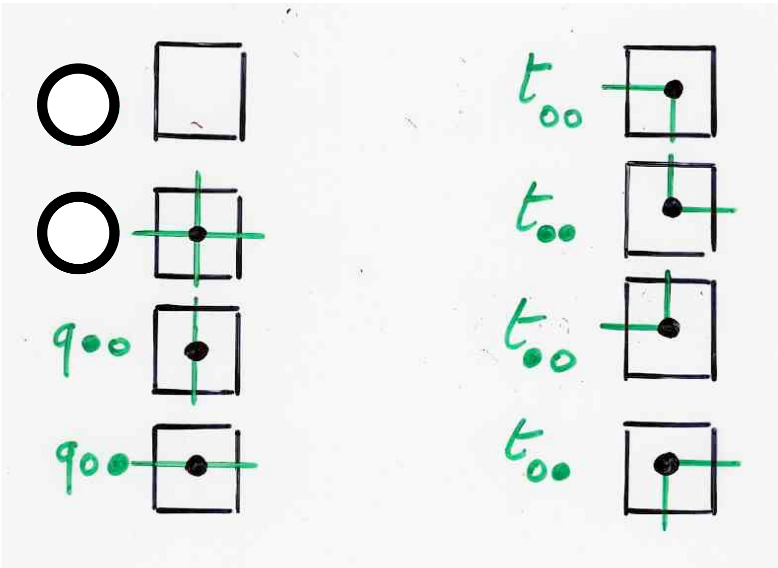

B B B B B
A A A A A
A A A A A
B B B B B



B B B B B
A A A A A
A A A A A
B B B B B

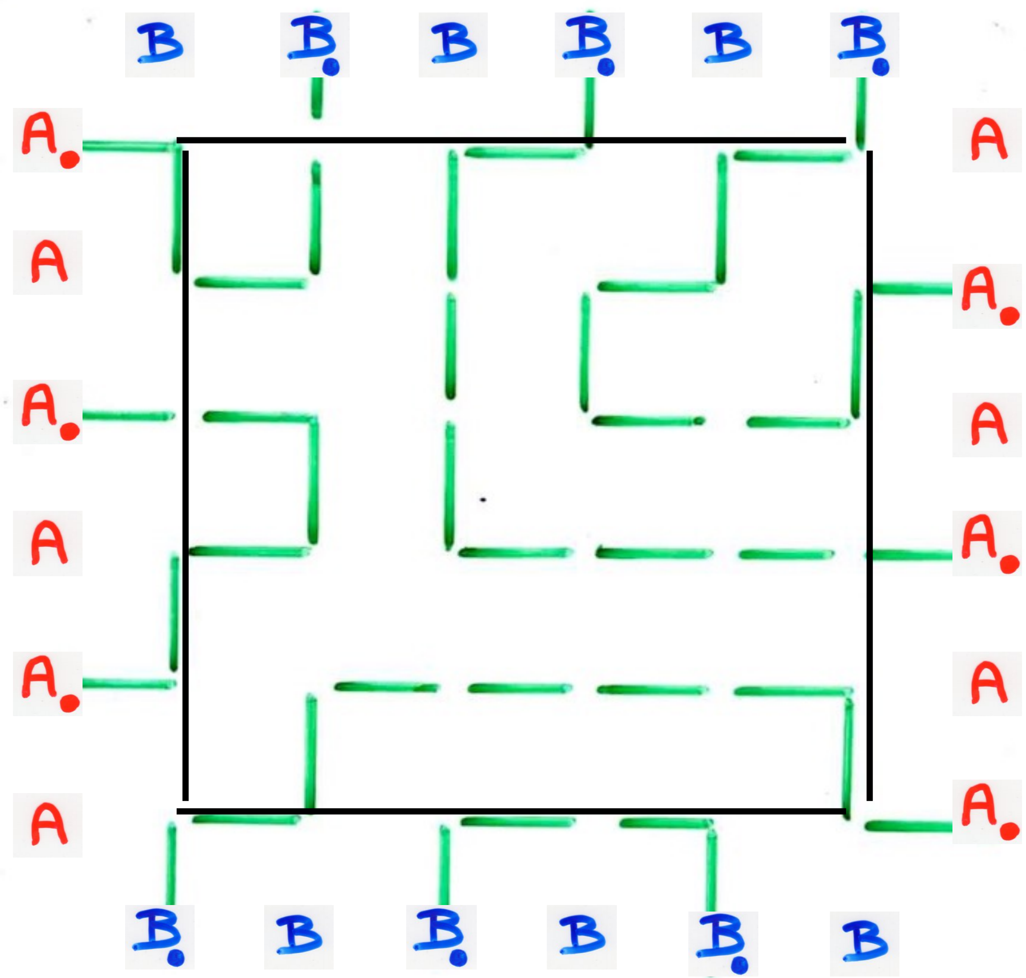
FPL
fully packed loops

Fully packed loops



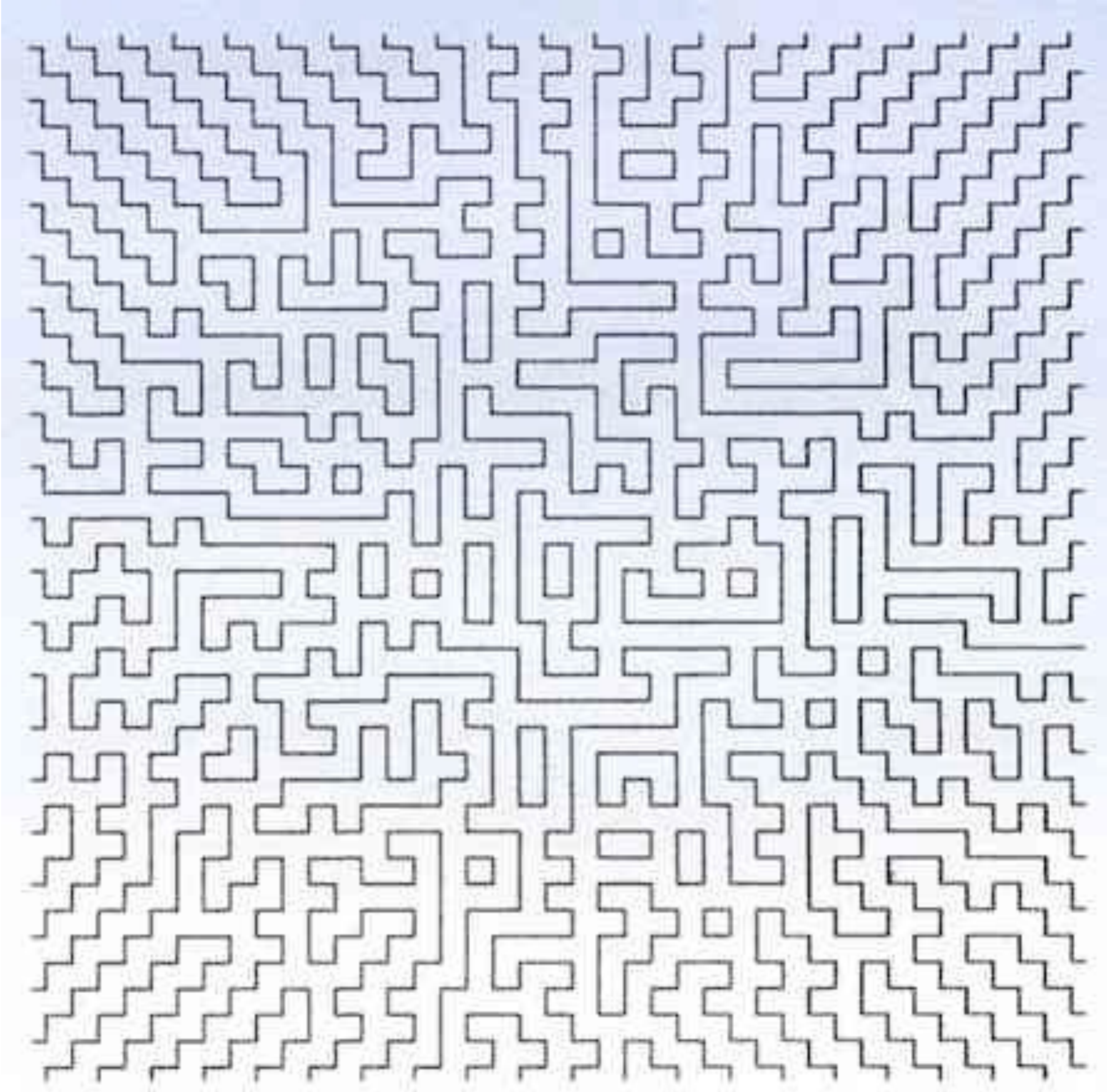
$$\begin{cases} B\,A = \bigcirc\, A\,B + t_{oo}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = \bigcirc\, A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\, B \\ B_\bullet\, A = q_{\bullet o}\, A\, B_\bullet + t_{\bullet o}\, A_\bullet\, B \\ B\, A_\bullet = q_{o\bullet}\, A_\bullet\, B + t_{o\bullet}\, A\, B_\bullet \end{cases}$$

Fully packed loops

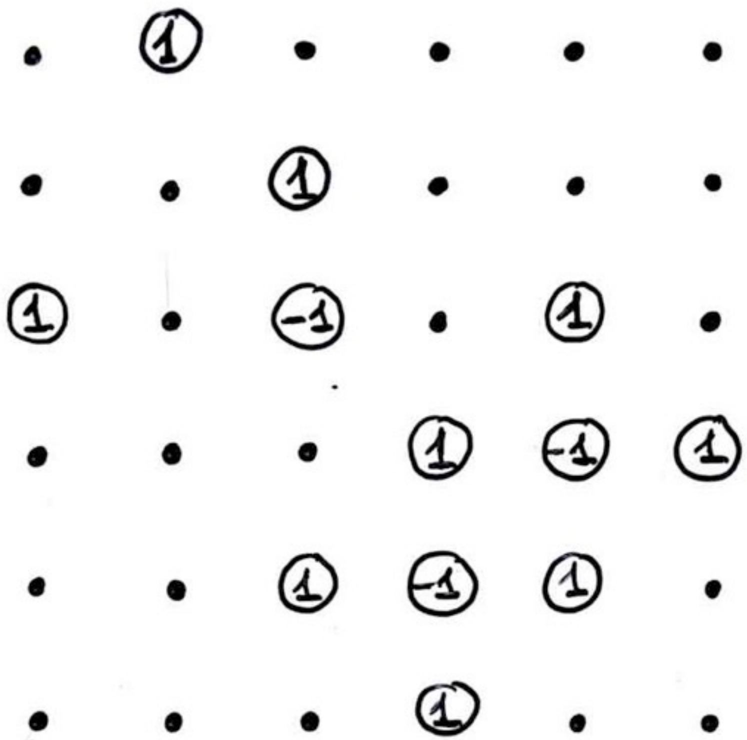


random FPL

P. Duchon

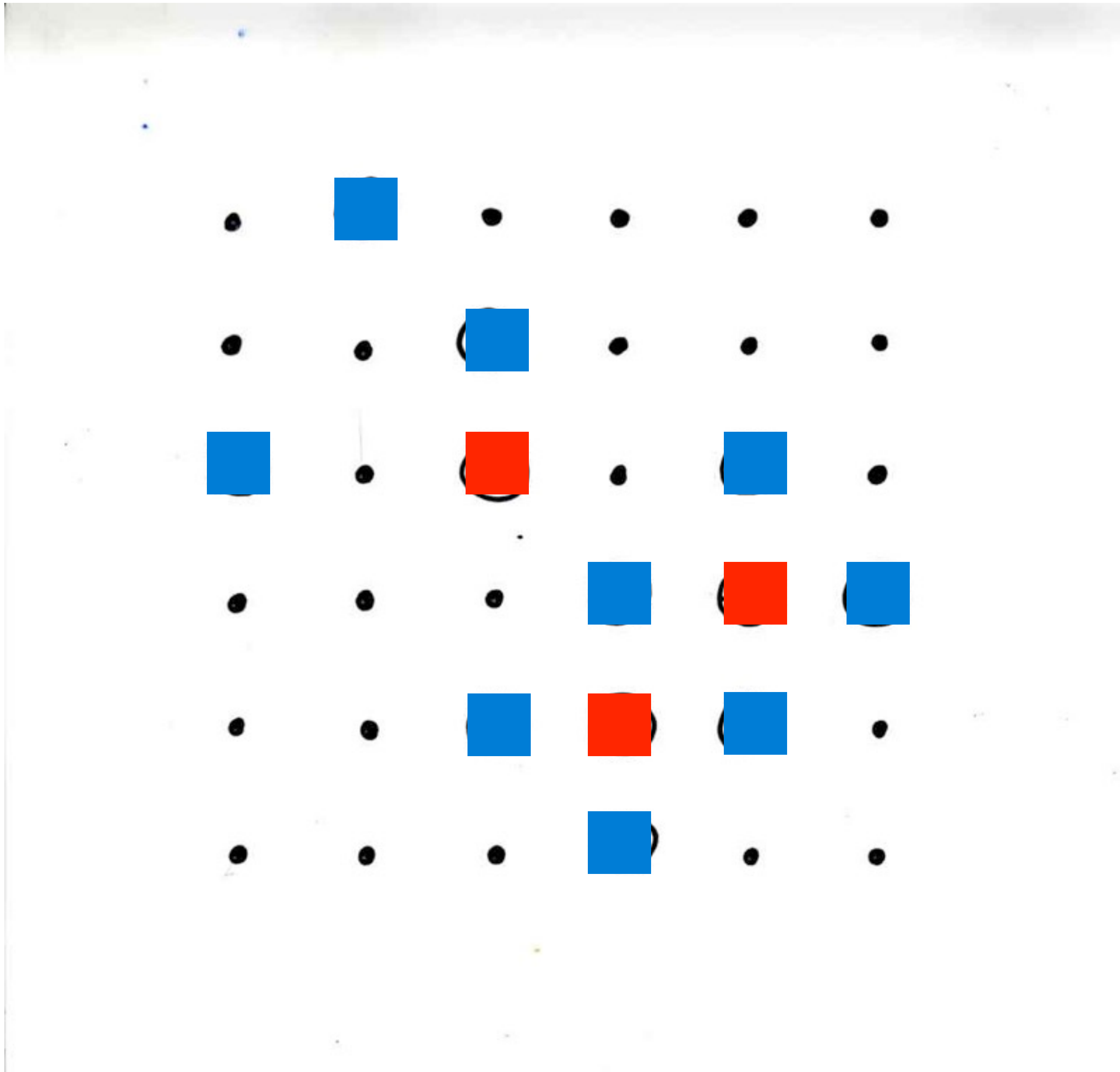


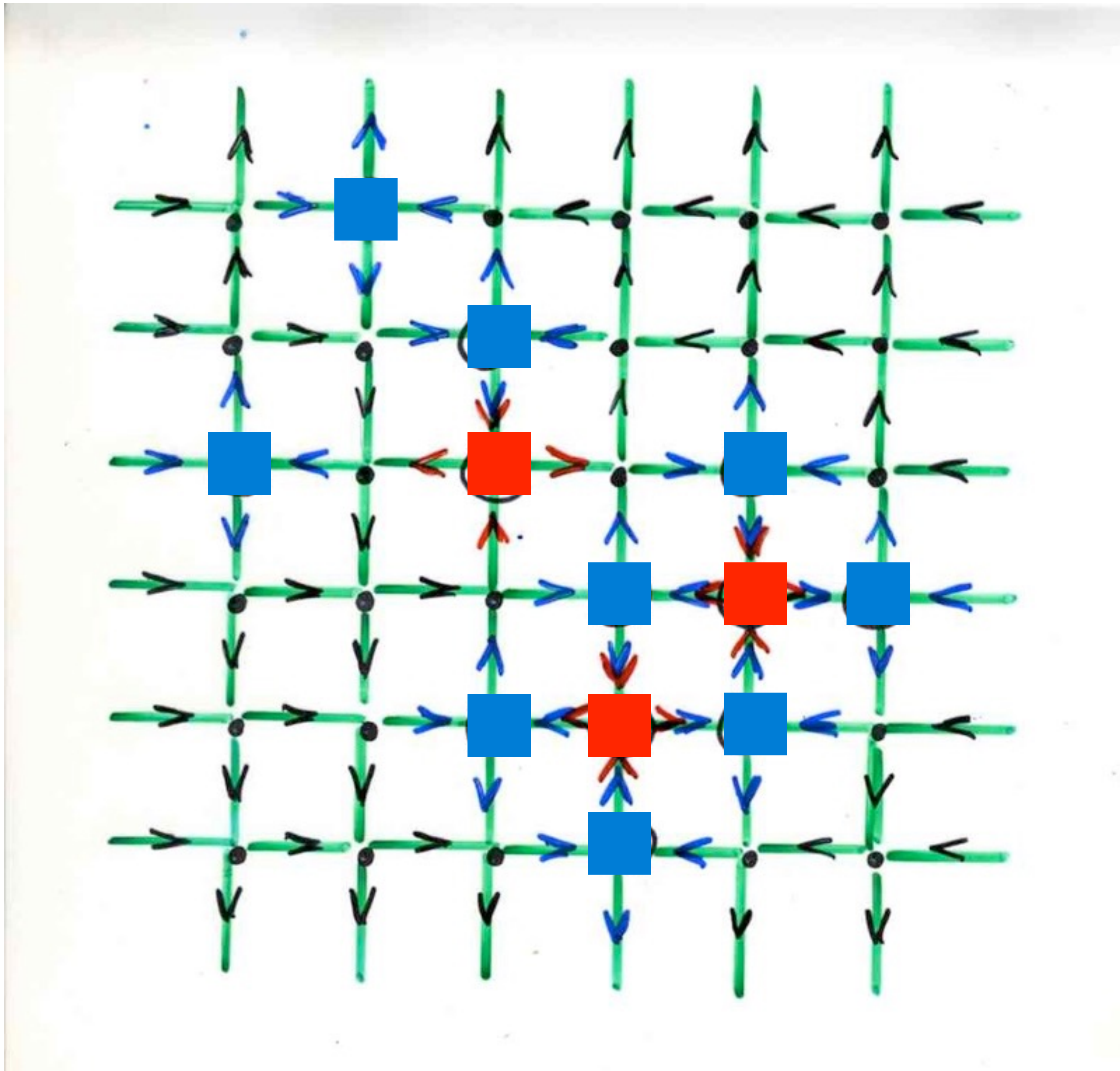
The bijection
ASM ⟷ FPL

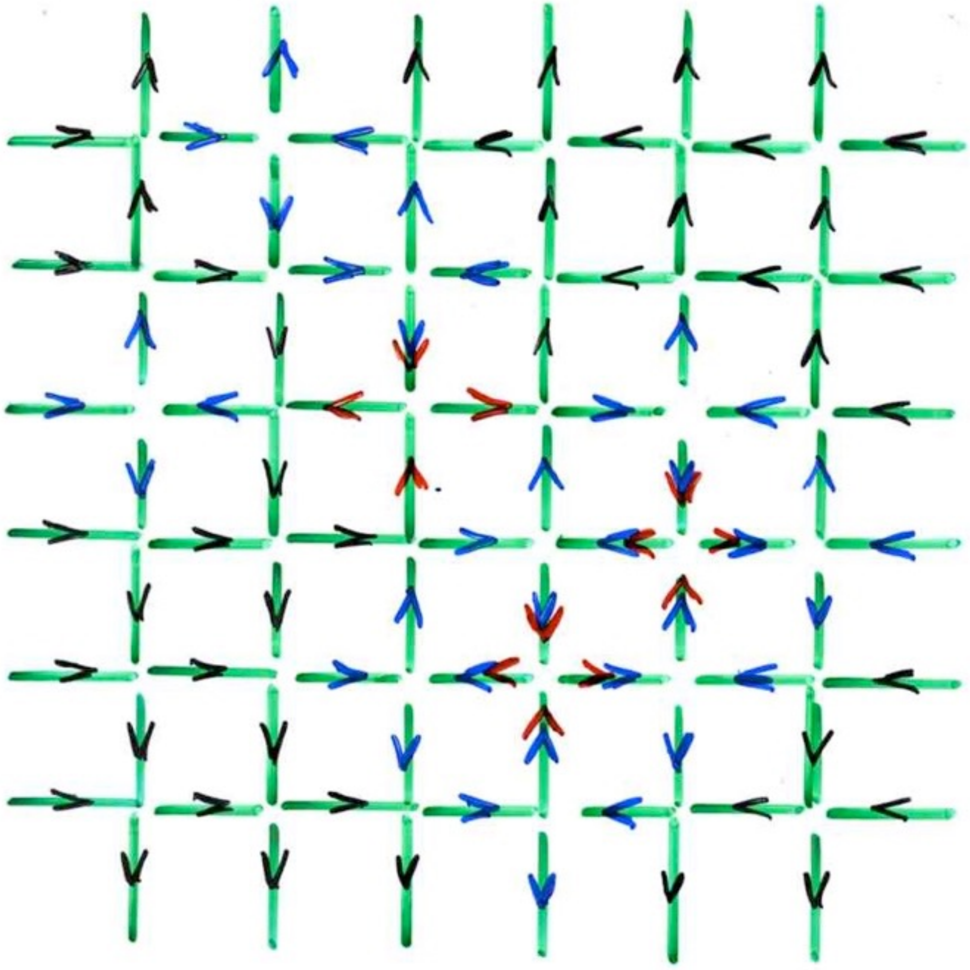


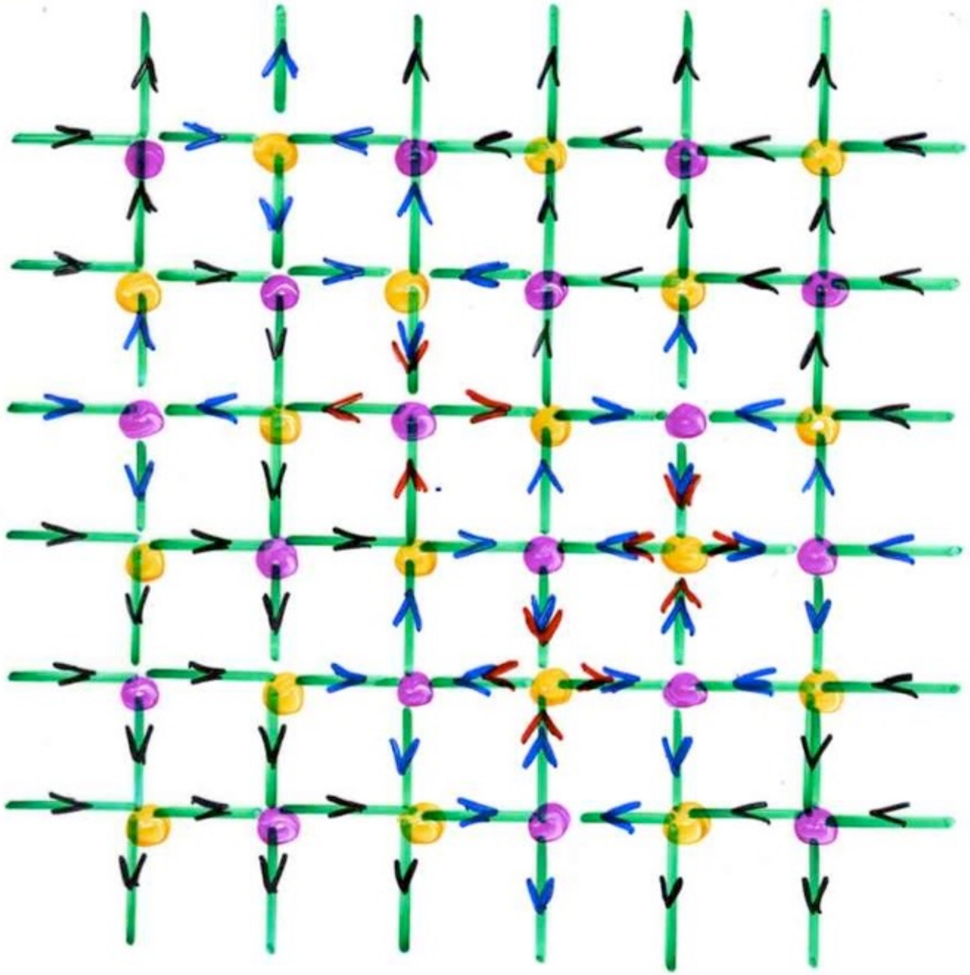
The bijection

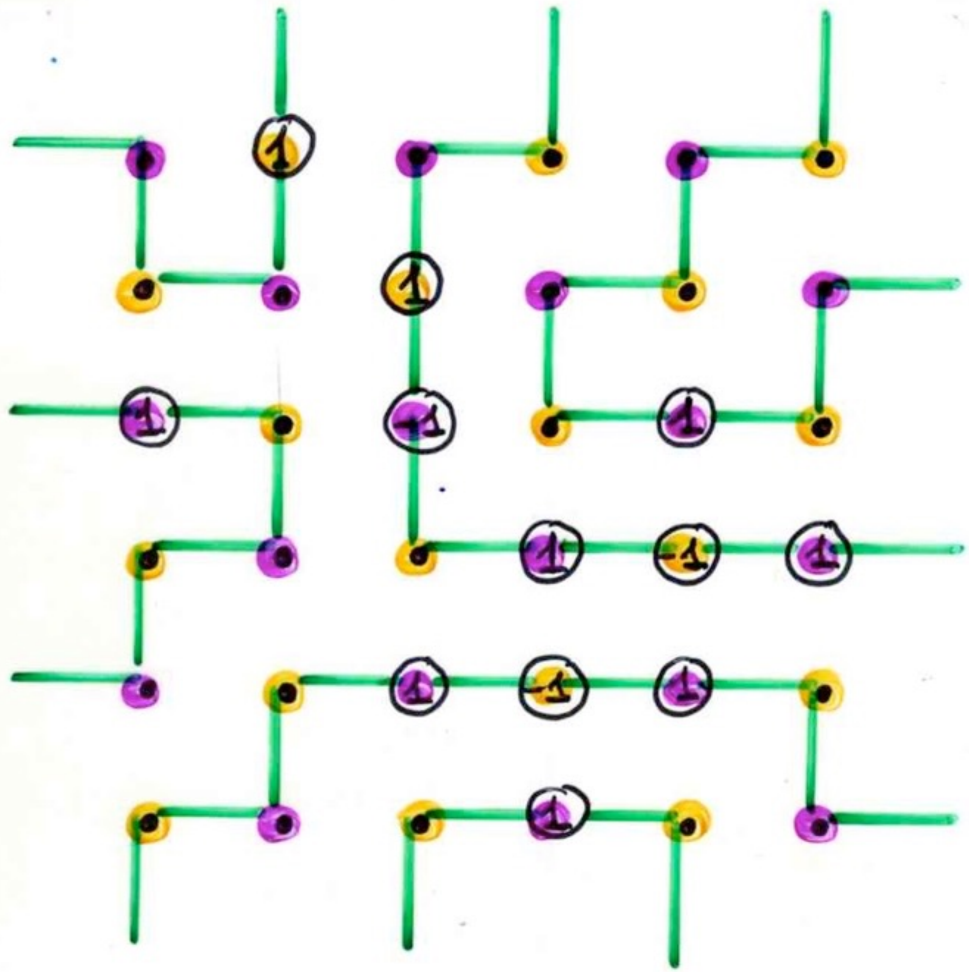
ASM ⟷ FPL



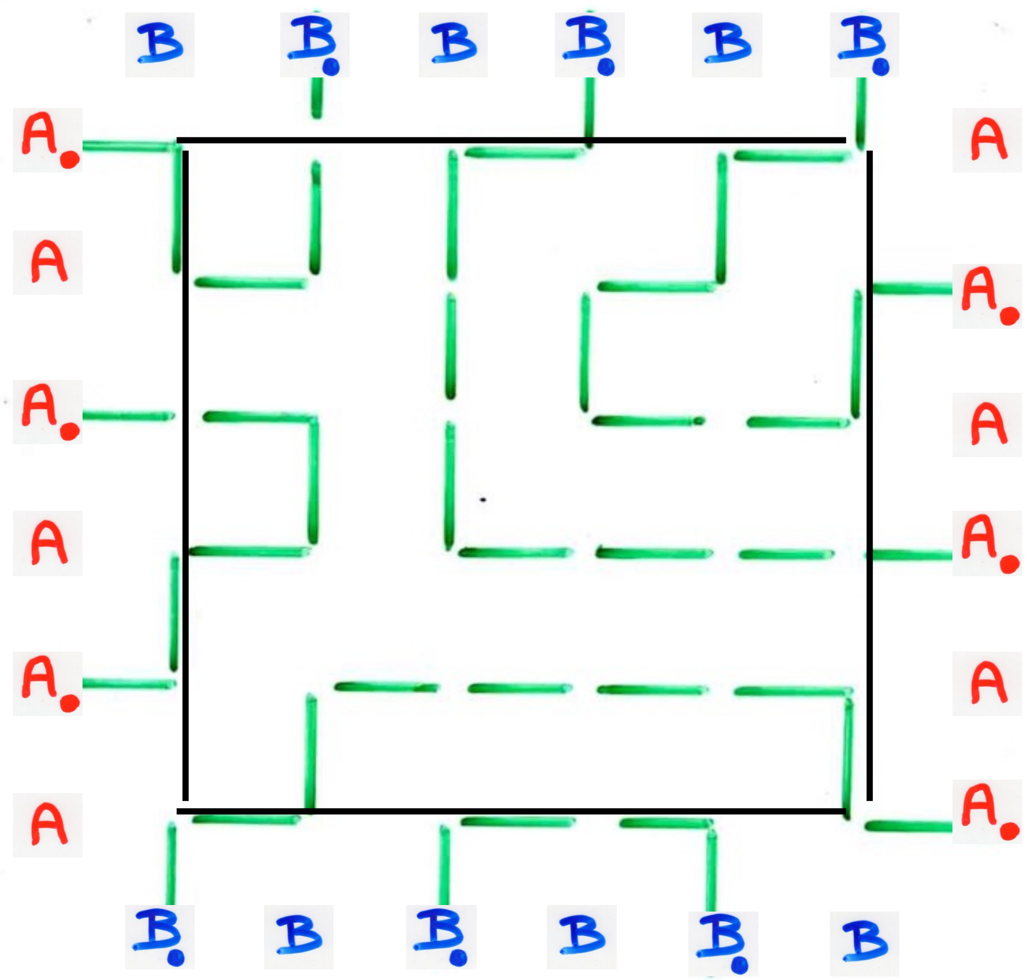


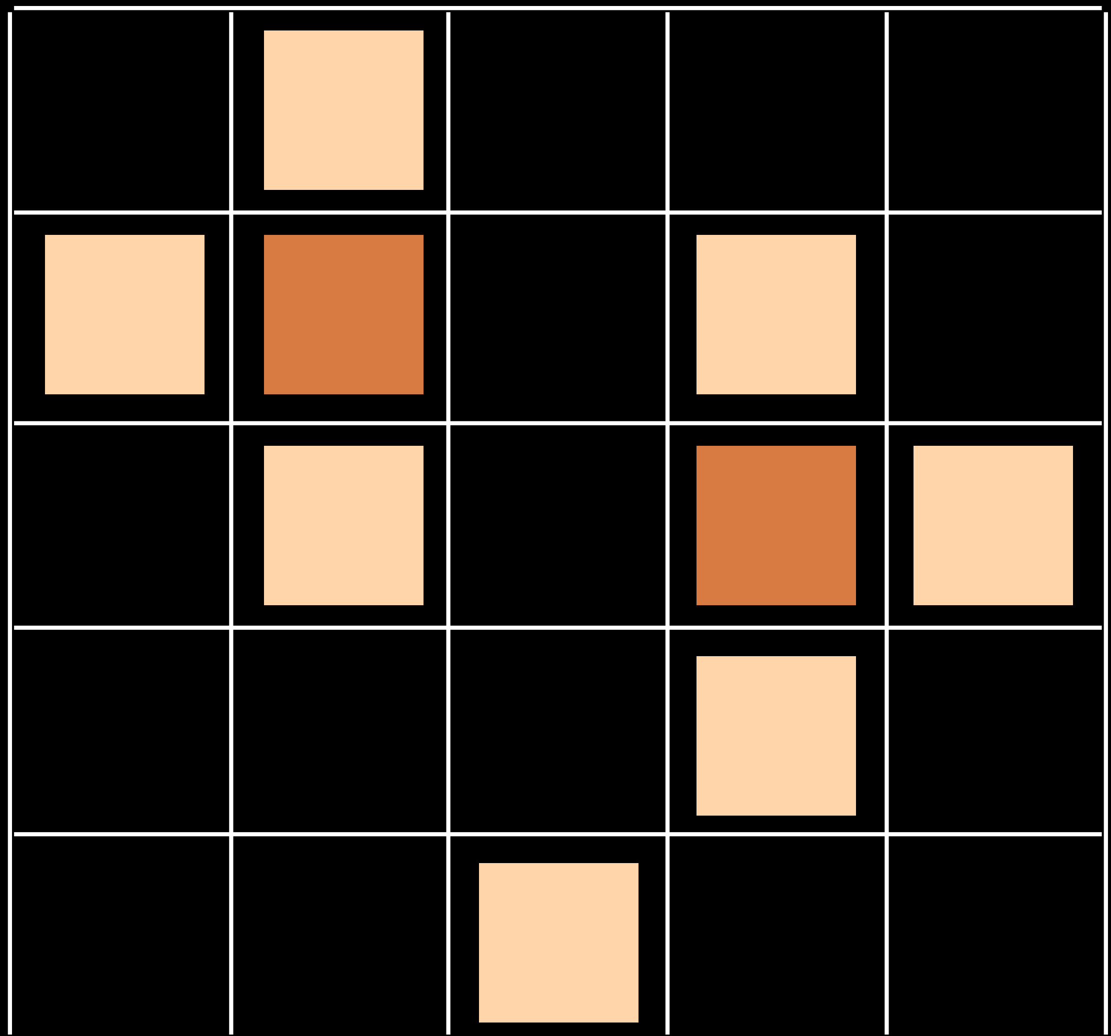


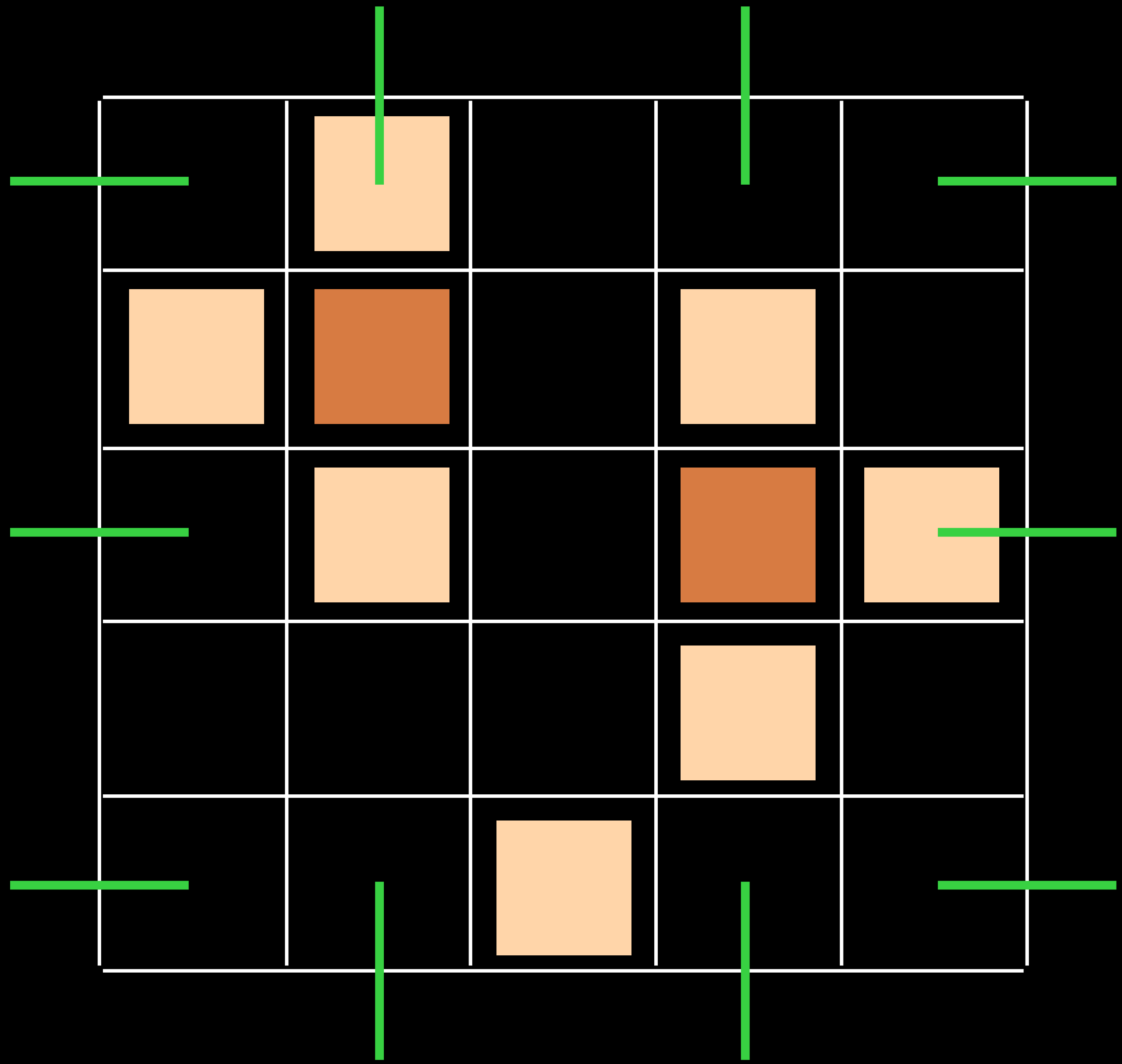


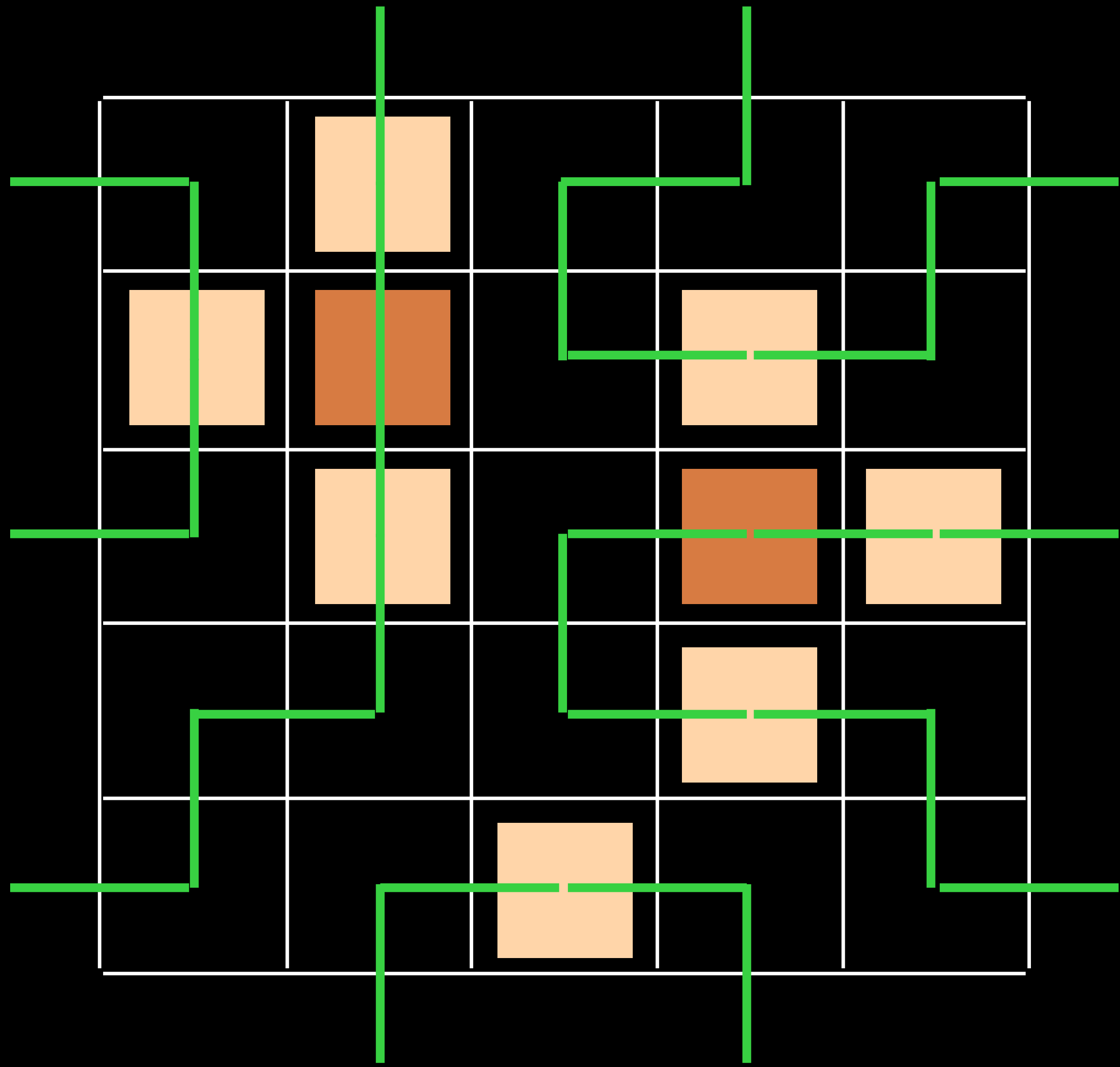


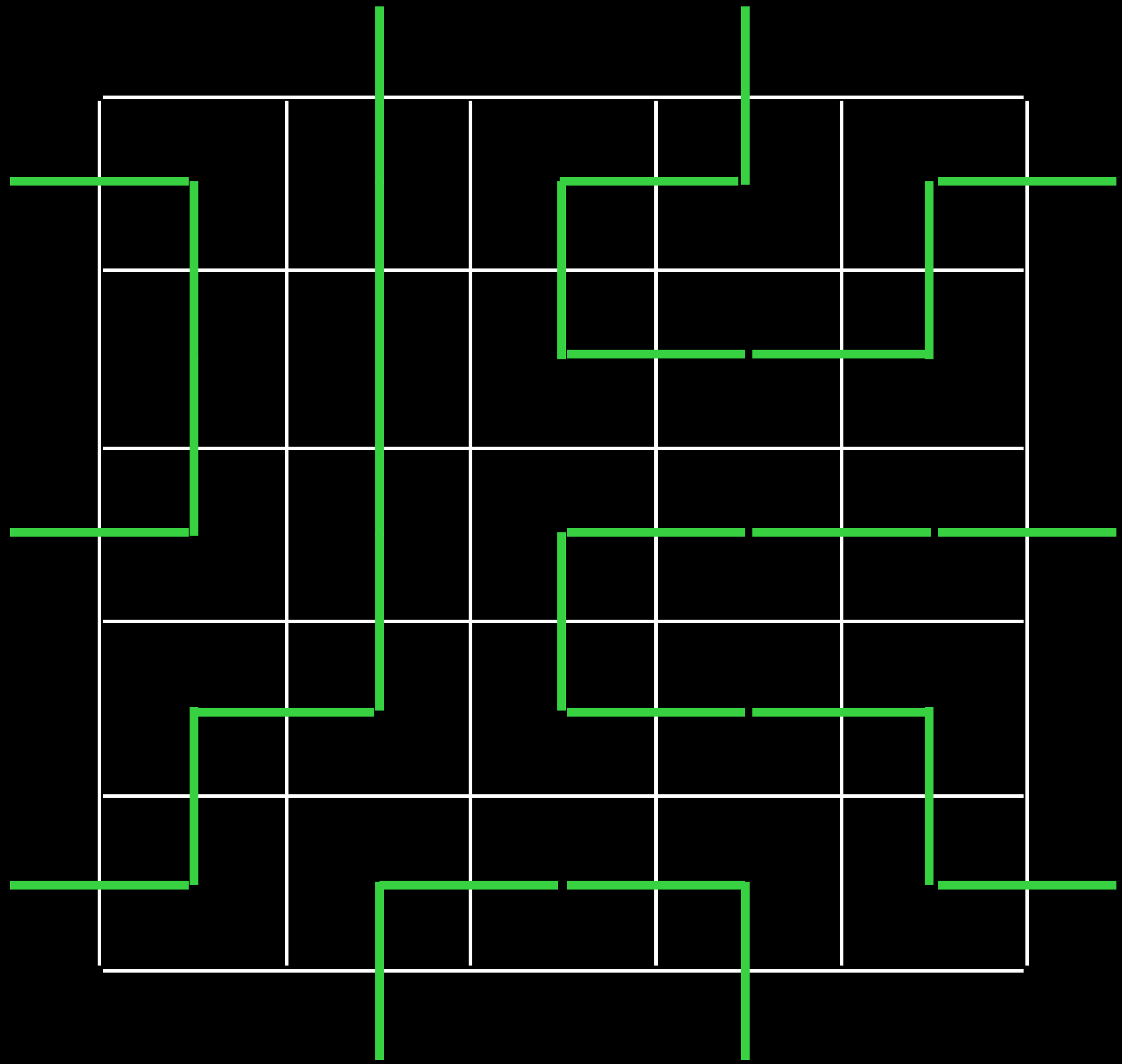
Fully packed loops



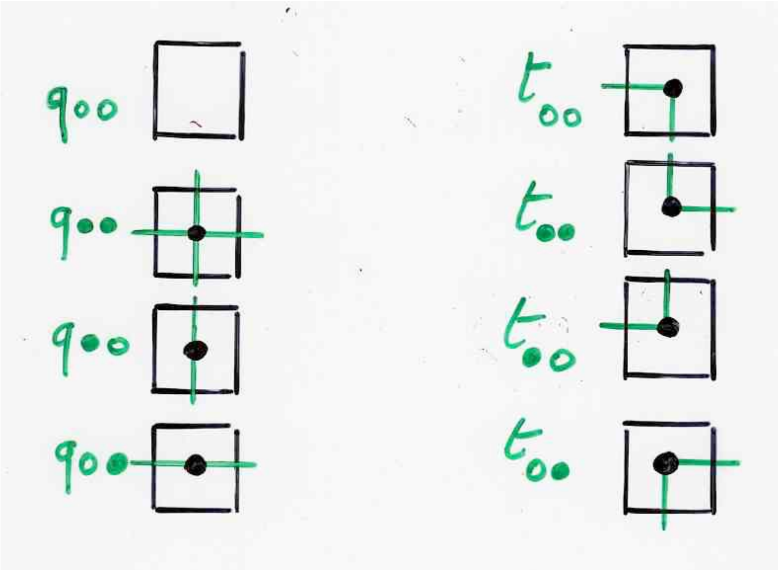


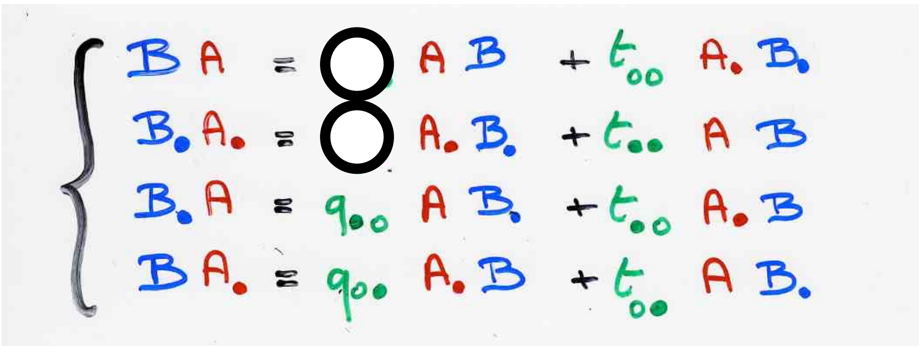


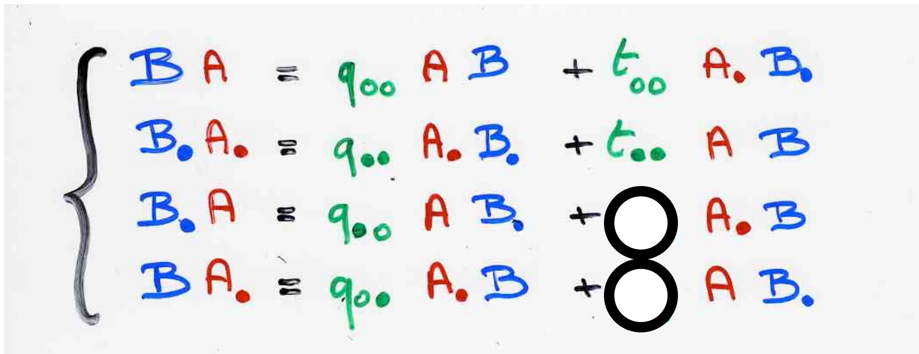


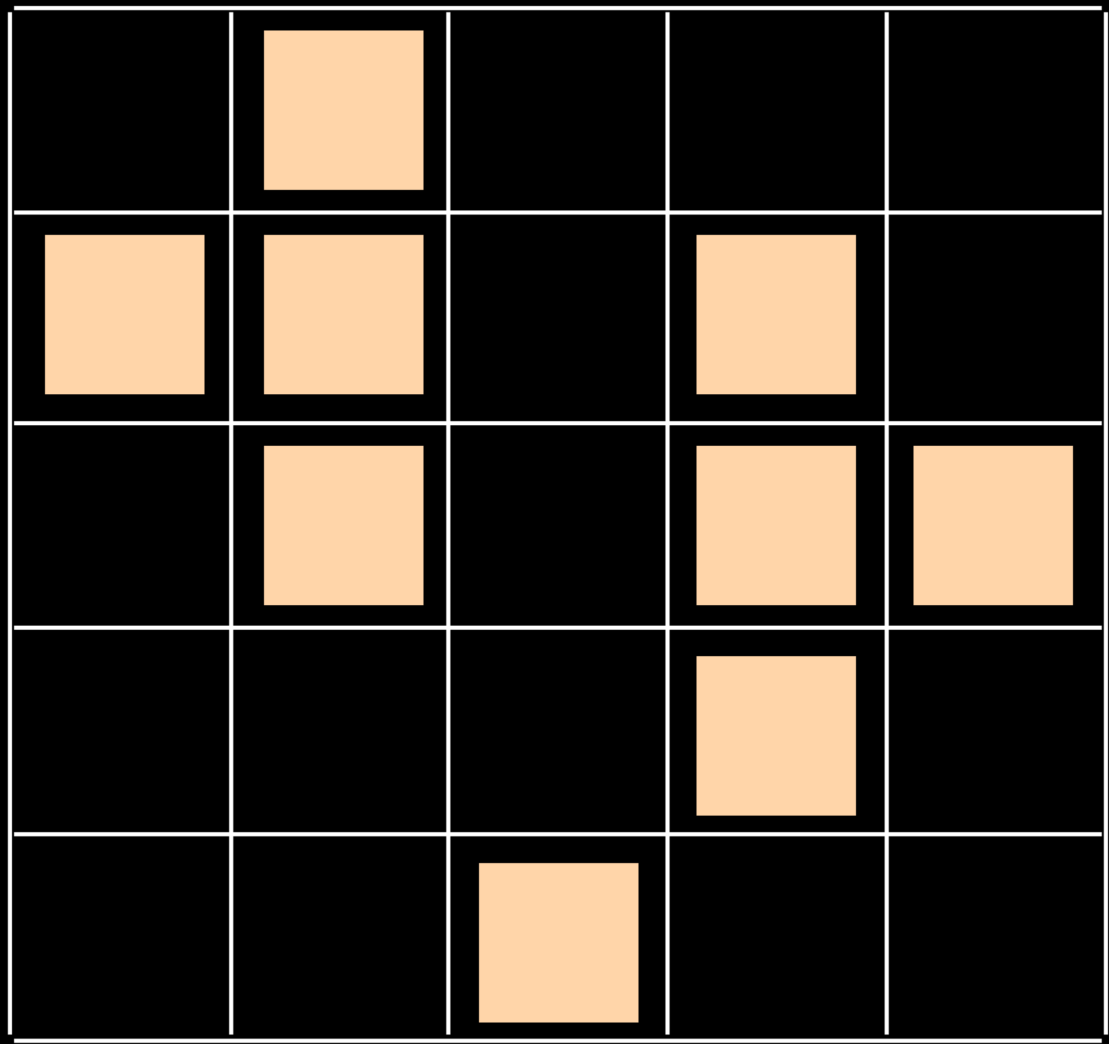


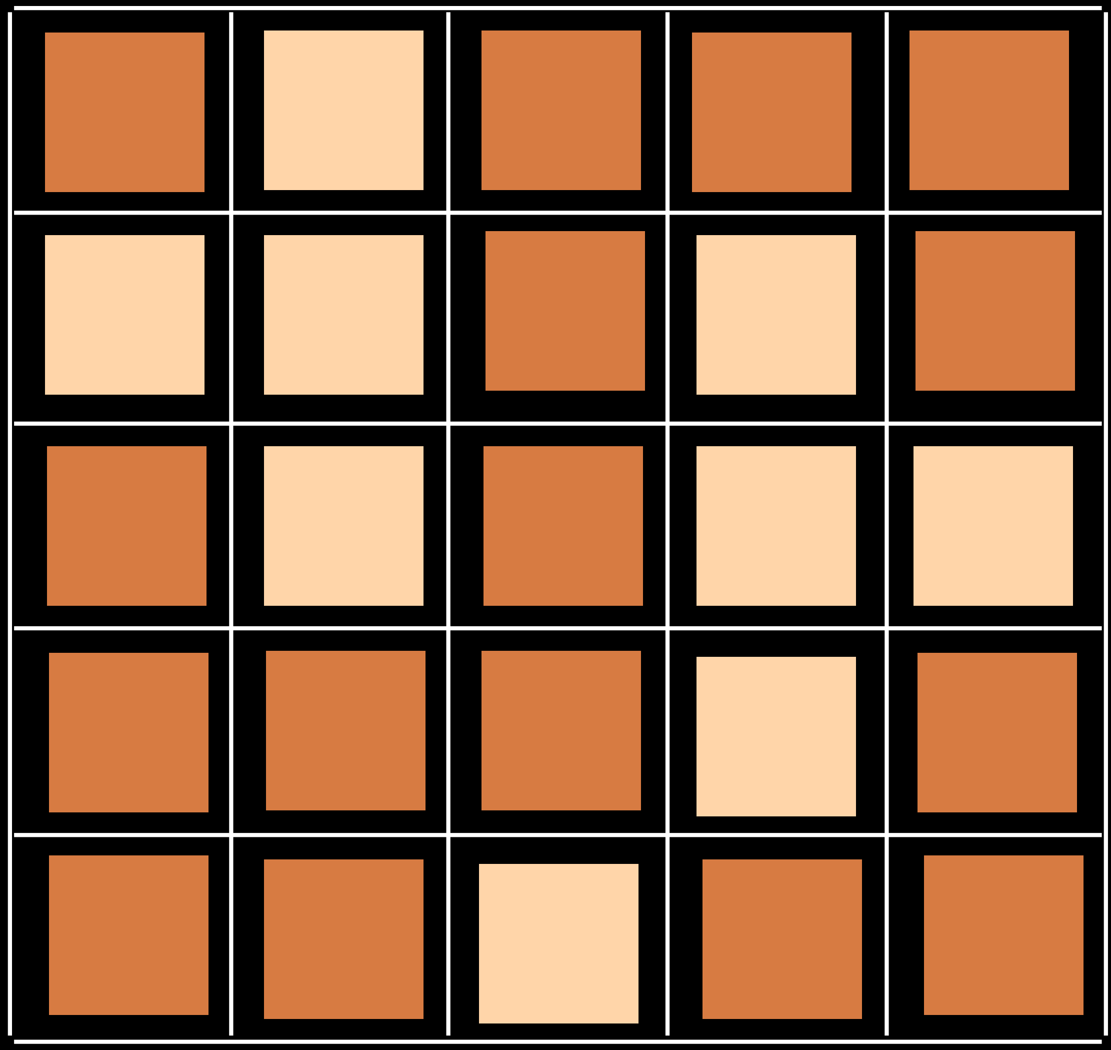
About the bijection ASM — FPL











$$w = B^n A^n$$

configuration $C \subseteq [n] \times [n]$

$$\begin{cases} B\,A = \square\, A\,B + \blacksquare\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \square\, A_\bullet B_\bullet + \blacksquare\, A\,B \\ B_\bullet A = \square\, A\,B_\bullet + \blacksquare\, A_\bullet B \\ B\,A_\bullet = \square\, A_\bullet B + \blacksquare\, A\,B_\bullet \end{cases}$$

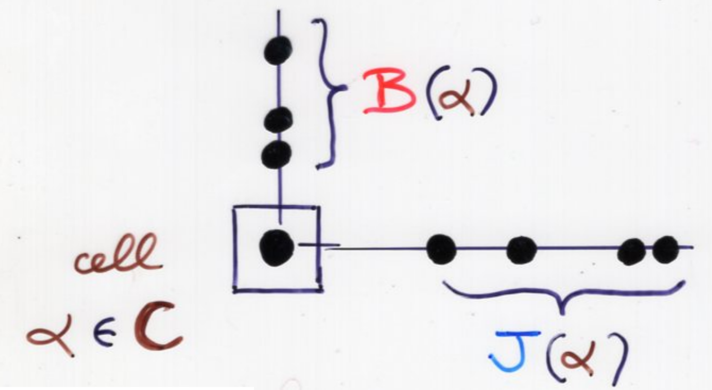
$w = B^n A^n$

configuration $C \subseteq [n] \times [n]$

The label of α in T is

- if $\alpha \in C$, t_{xy}
- if $\alpha \notin C$, q_{xy}

$$x = \begin{cases} \bullet & J(\alpha) \text{ odd} \\ \circ & J(\alpha) \text{ even} \end{cases}$$



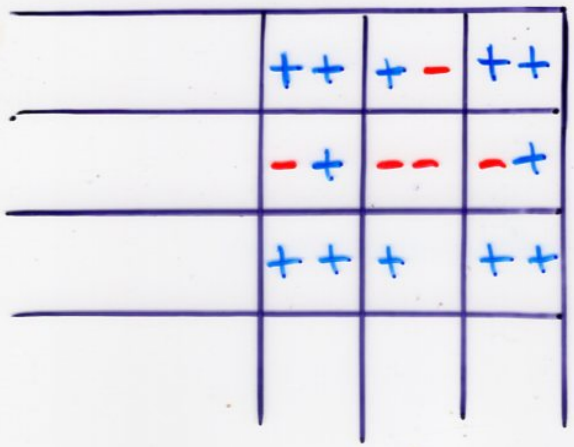
same for y with $B(\alpha)$

$w = B^n A^n$

configuration $C \subseteq [n] \times [n]$

change the rules for the complement

q_{xy}



$++ \quad q_{xy} \leftrightarrow t_{xy}$

$-- \quad q_{xy} \leftrightarrow t_{\bar{x}\bar{y}}$

$+- \quad q_{xy} \leftrightarrow t_{x\bar{y}}$

$-+ \quad q_{xy} \leftrightarrow t_{\bar{x}y}$

$x = \begin{cases} \bullet \\ \circ \end{cases} \quad \bar{x} = \begin{cases} \circ \\ \bullet \end{cases}$ same for y, $\bar{y}$

ASM $\quad w = B^n A^n$

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

complement: $\Rightarrow$ $q_{\bullet\circ}$, $q_{\circ\bullet}$ forbidden on cells $(++)(--)$

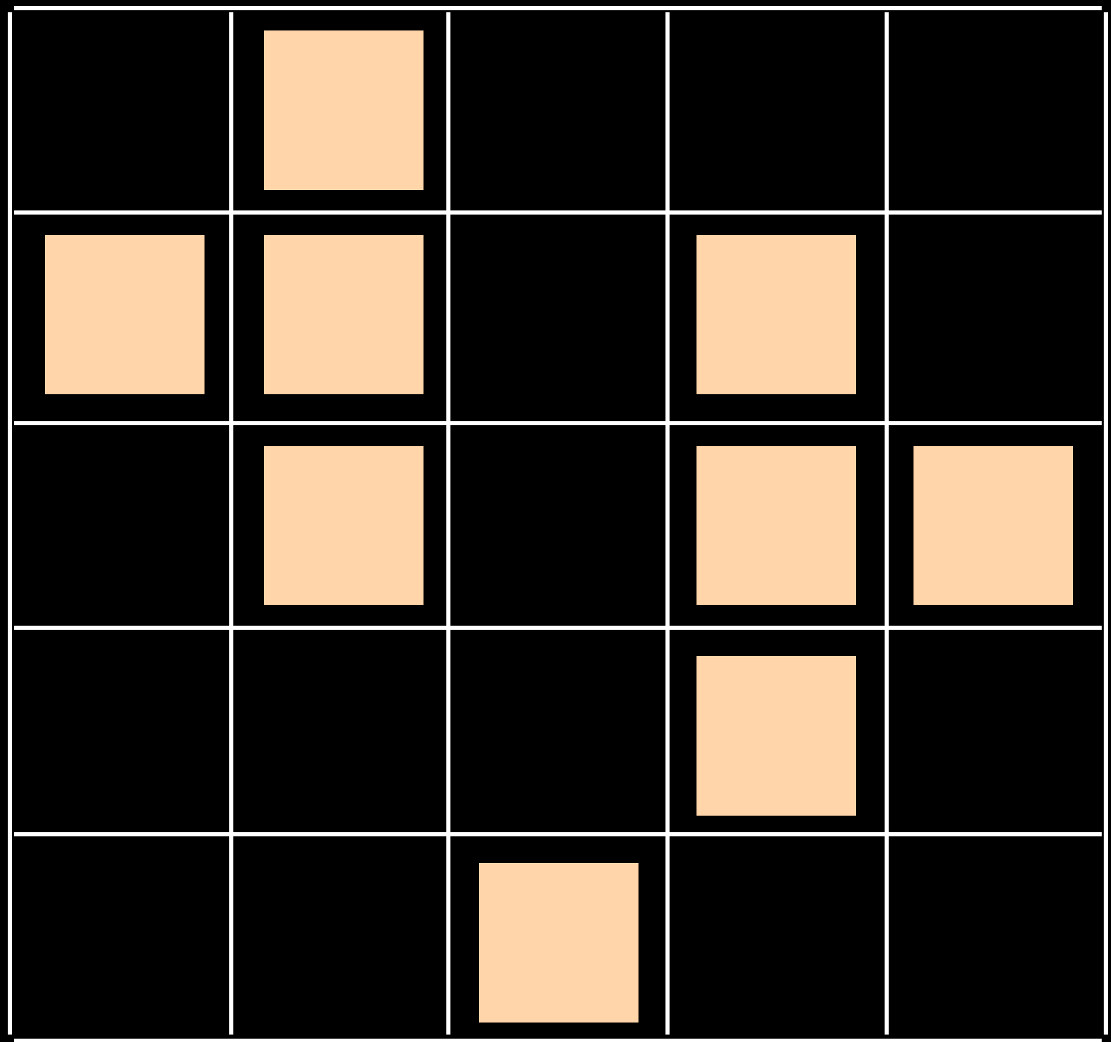
$q_{\bullet\bullet}$, $q_{\circ\circ}$ forbidden on cells $(+-)(-+)$

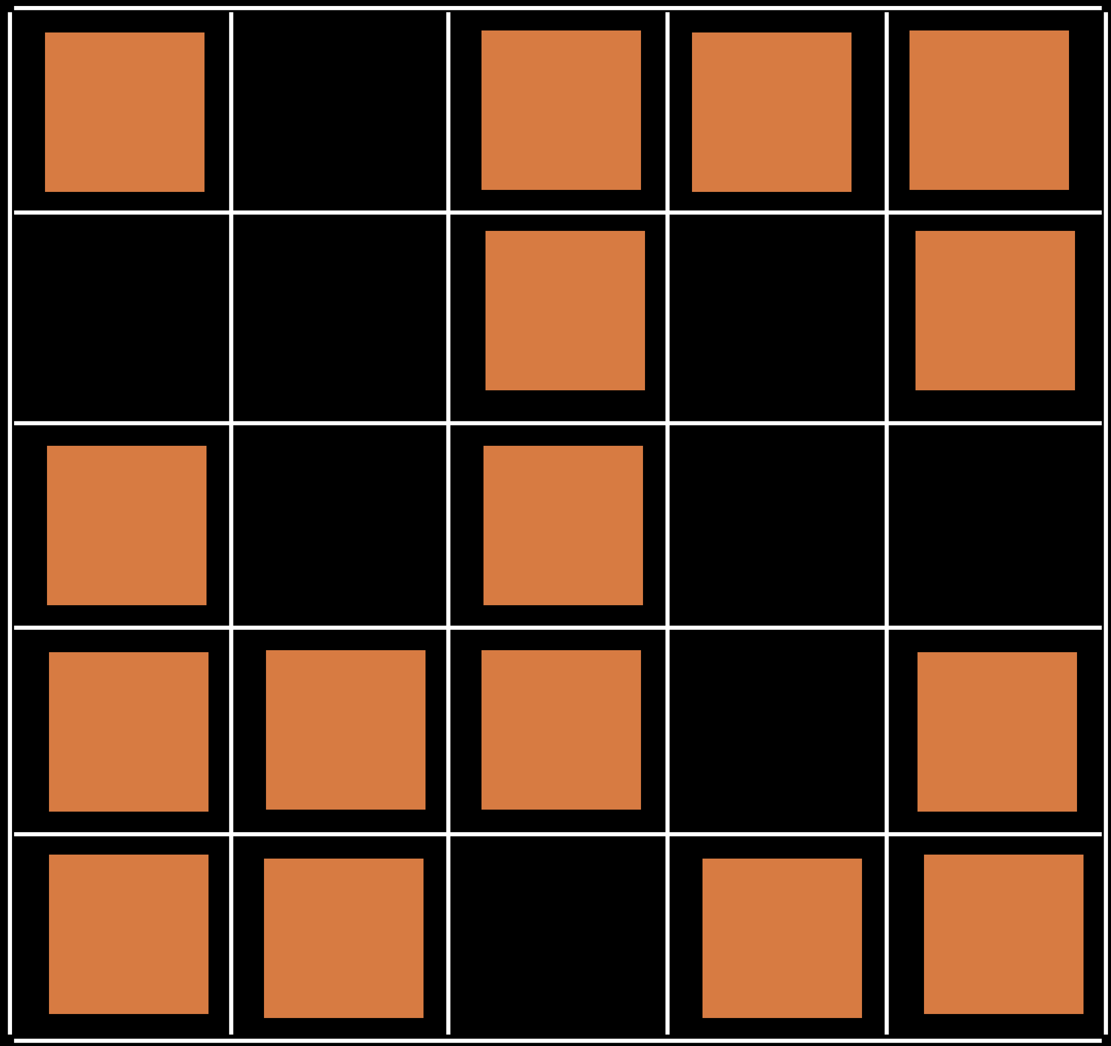
if $w = \underbrace{B_{\times} \cdots B_{\bullet} B_{\circ} B_{\bullet}}_{n} \underbrace{A_{\circ} A_{\bullet} A_{\circ} \cdots}_{n}$

$q_{\bullet\bullet}$ and $q_{\circ\circ}$ forbidden for every cells

$\rightarrow$ bijection ASM $\leftrightarrow$ FPL

same with $\underbrace{\cdots B_{\circ} B_{\bullet} B_{\circ}}_{n} \underbrace{A_{\bullet} A_{\circ} A_{\bullet} \cdots}_{n}$

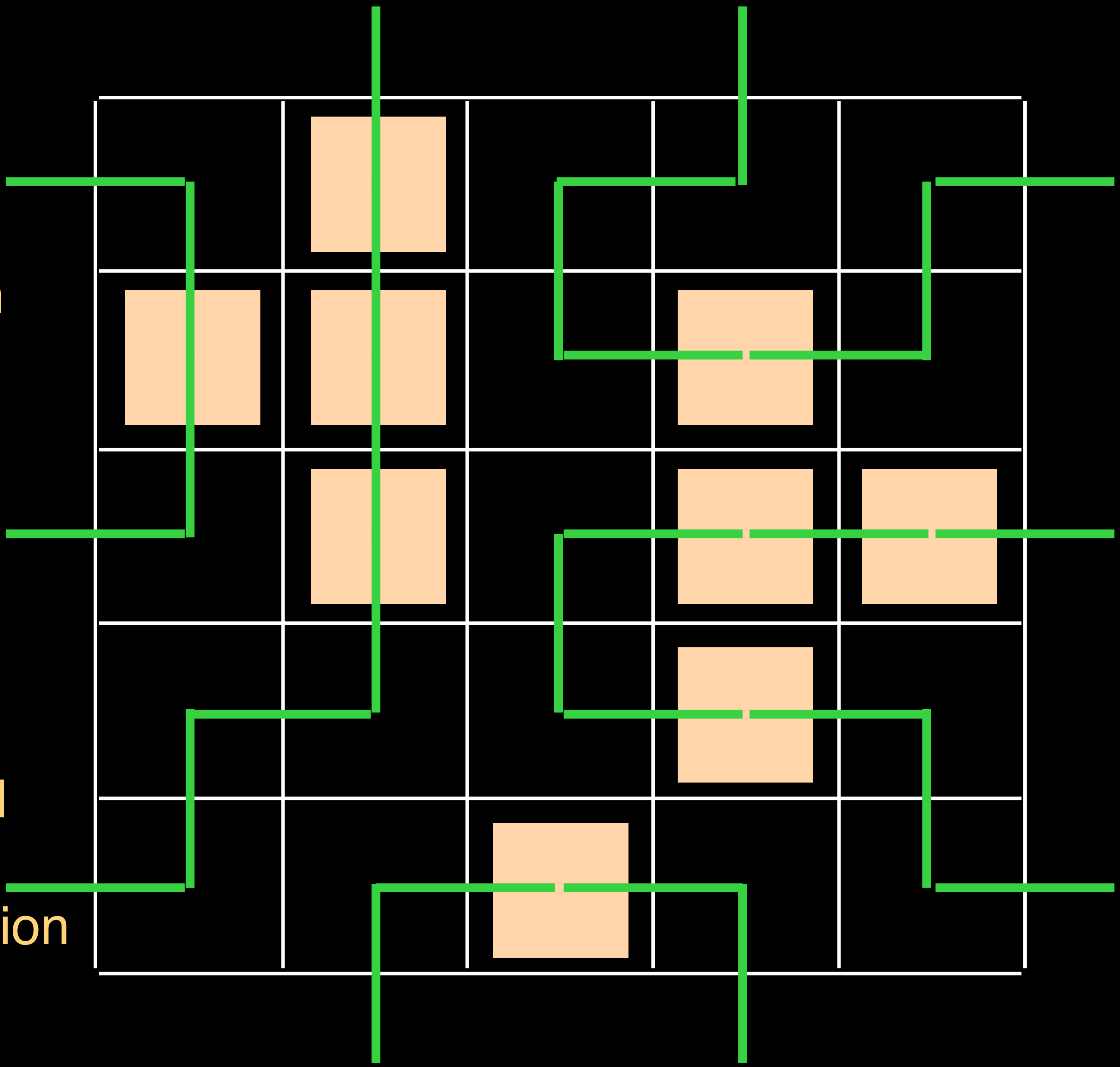


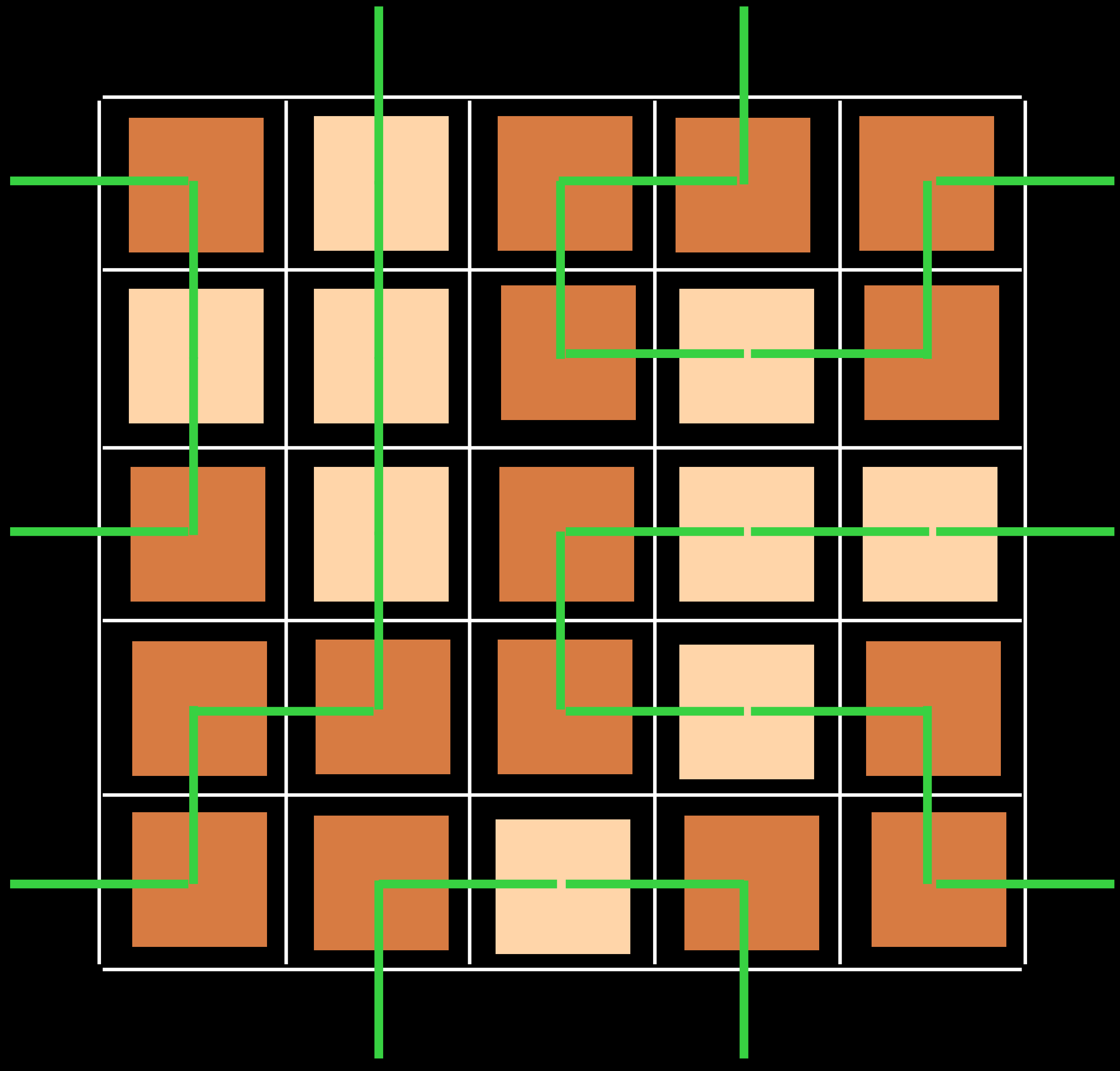


bijection

ASM

FPL
“Fully
Packed
Loop”
configuration





correlations functions in XXZ spin chains

A research Problem

LPENSL-TH-05

Exact results for the σ^z two-point function of the XXZ chain at $\Delta = 1/2$

N. Kitanine[1], **J. M. Maillet**[2], **N. A. Slavnov**[3], **V. Terras**[4]

Abstract

We propose a new multiple integral representation for the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the disordered regime. We show that for $\Delta = 1/2$ the integrals can be separated and computed exactly. As an example we give the explicit results up to the lattice distance $m = 8$. It turns out that the answer is given as integer numbers divided by $2^{(m+1)^2}$.

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e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m) \rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m) \rangle$ up to $m = 9$:

$$
\begin{aligned}
&P_1(\kappa) = 1 + \kappa, \\
&P_2(\kappa) = 2 + 12\kappa + 2\kappa^2, \\
&P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3, \\
&P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4, \\
&P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5, \\
&P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 \\
&+96289380\kappa^5 + 7436\kappa^6, \\
&P_7(\kappa) = 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3 \\
&+265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7, \\
&P_8(\kappa) = 10850216 + 8485108350684\kappa + 39461894378292782\kappa^2 \\
&+3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5 \\
&+39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8 \\
&P_9(\kappa) = 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2 \\
&+77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

intergers ?

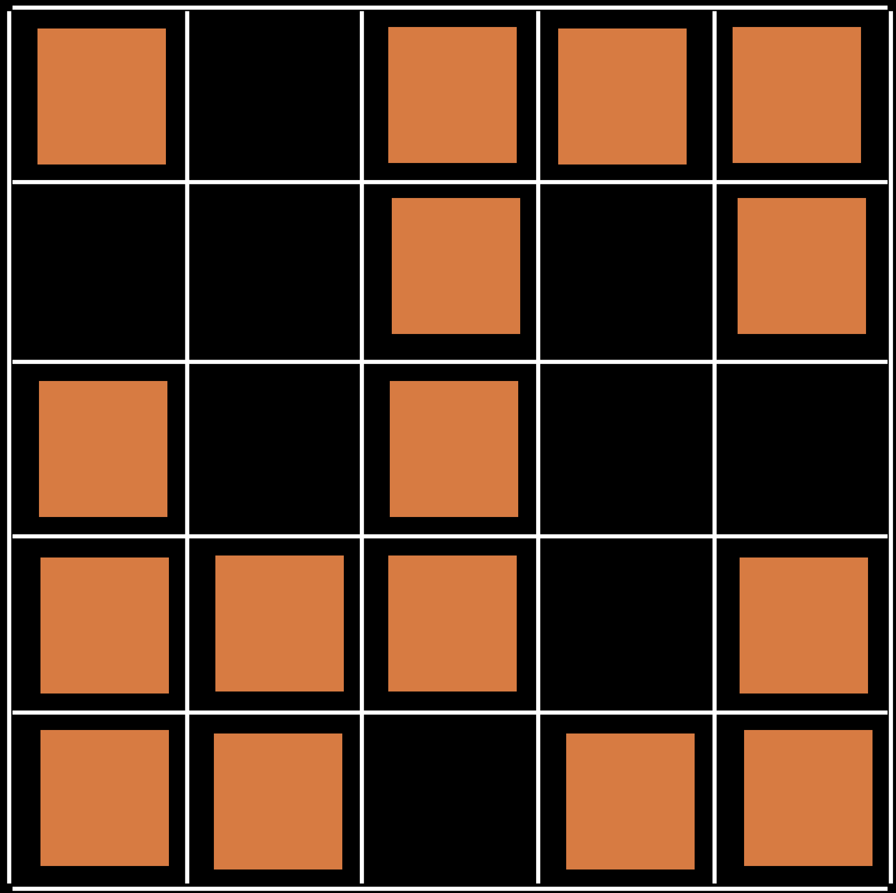
positivity ?

combinatorial interpretation ?

ASM

$$
\begin{aligned}
P_1(\kappa) &= 1+\kappa,\\
P_2(\kappa) &= 2 + 12\kappa + 2\kappa^2,\\
P_3(\kappa) &= 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,\\
P_4(\kappa) &= 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4,\\
P_5(\kappa) &= 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,\\
P_6(\kappa) &= 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4\\
&+96289380\kappa^5 + 7436\kappa^6,\\
P_7(\kappa) &= 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3\\
&+265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7,\\
P_8(\kappa) &= 10850216 + 8485108350684\kappa + 39461894378292782\kappa^2\\
&+3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5\\
&+39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8\\
P_9(\kappa) &= 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2\\
&+77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

The number of B. A. B A configurations in the grid $[n]\times[n]$ is



$$2^{(n^2)}$$

Open questions

Conclusion

quadratic algebra Q

$$B_j A_i = \sum_{k\,l} c_{ij}^{kl} A_k B_l$$

$$c(u, v; w) = \sum_{T} \mathrm{wgt}(T)$$

complete Q-tableau

$uwb(T) = w$

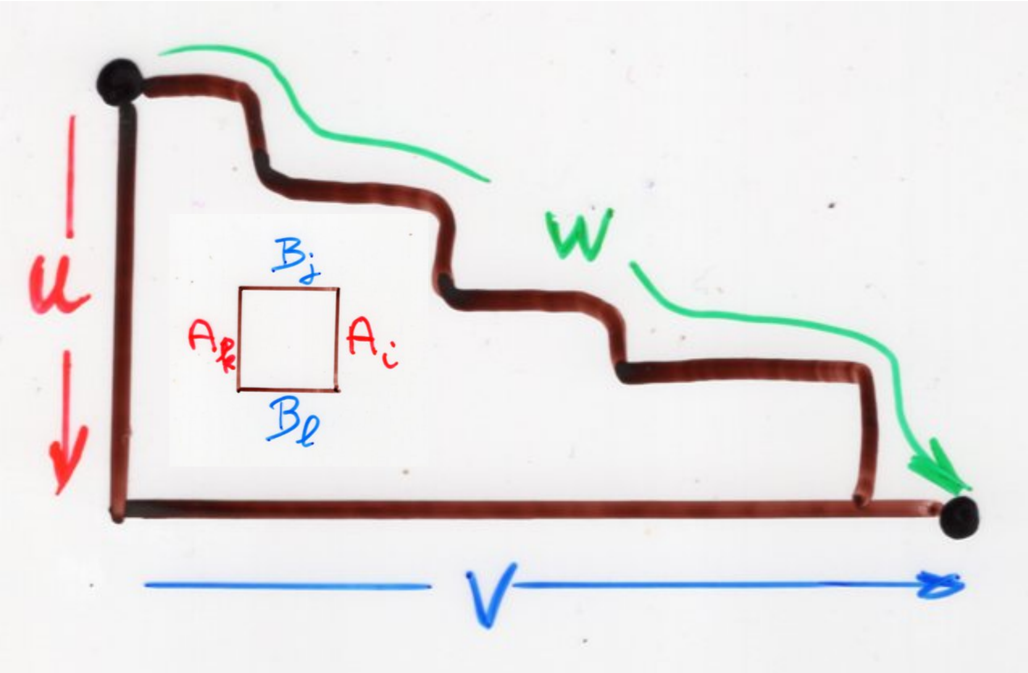
$lwb(T) = uv$

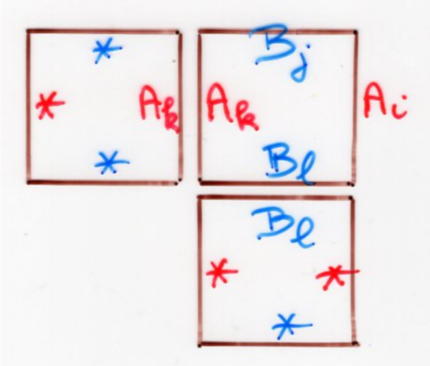
formula for $c(u, v; w)$?

algorithm ?

determinant ?

in general $F(w)$ is a rectangle





- find a "combinatorial representation" of the generators $A, A_\bullet, B, B_\bullet$ of the ASM quadratic algebra of the 8-vertex quadratic algebra?

- demultiplication of equations in the 8-vertex algebra?

- analogue of "local rules" for ASM?

- analogue of RSK?

- direct combinatorial proof for the number A_n of ASM

"The cellular ansatz"

(i) first step

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

UD = qDU + Id

permutations

towers placements

Physics

DE = qED + E + D

alternative tableaux

commutations

rewriting rules

planarization

"planar automata"

ASM alternating sign matrices

tilings

non-crossing paths

8-vertex model

(ii) second step

representation of Q by combinatorial operators

bijections

RSK ⟷

pairs of Young tableaux

(iii) third step

"duplication"

⟷ ?

"The cellular ansatz"

(i) first step

(ii) second step

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

representation of Q by combinatorial operators

bijections

$UD = qDU + Id$

permutations

towers placements

Physics

RSK ⟷

pairs of Young tableaux

$DE = qED + E + D$

alternative tableaux

EXF ⟷

next week

"Laguerre histories"

permutations

data structures "histories"

orthogonal polynomials

commutations

rewriting rules

ASM alternating sign matrices

tilings

planarization

"planar automata"

non-crossing paths

8-vertex model

⟷ ?

complements:

ASM

TSSCPP

DPP

FPL

RS

……

The beautiful garden
of some jewels of combinatorics …

« deep combinatorics »

Go to the second set of slides: Ch 2dc