Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 2
Quadratic algebra, Q-tableaux and planar automata

Ch2c

IMSc, Chennai
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From Ch 2b

Duplication of equations

in quadratic algebras

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$X_2 Y_2 = Y_3 X_3$$

$$\cdots\cdots$$

$$X_i Y_i = Y_{i+1} X_{i+1}$$

$$\cdots\cdots$$

$$U Y_i = Y_i U$$

$$X_j U = U X_j$$

$$\cdots\cdots$$

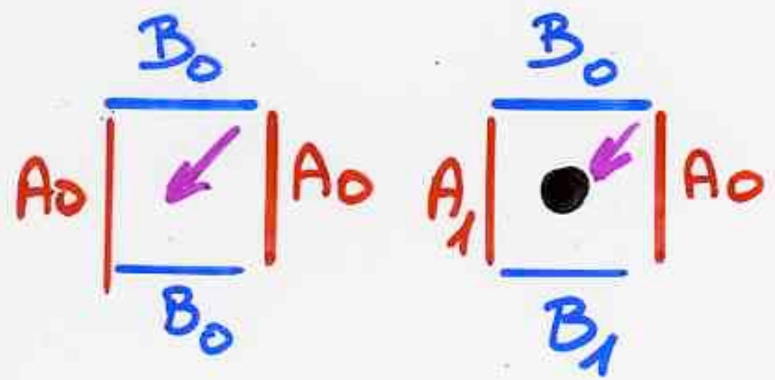
The "RSK planar automaton"

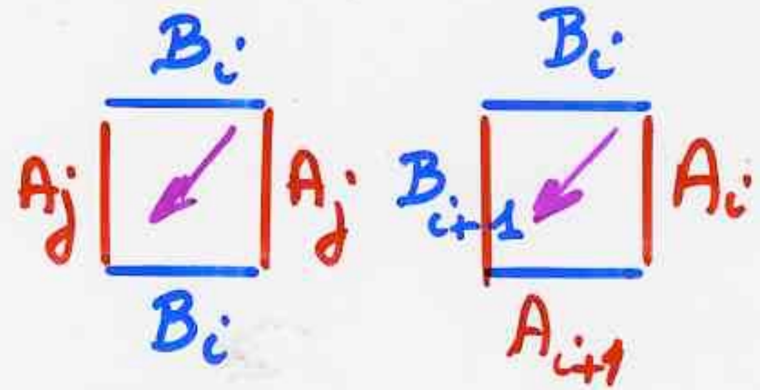
$\mathcal{B} = \{B_0, B_1, \dots B_k\}$

$\mathcal{A} = \{A_0 \;\; A_1 \;\; \;\; A_k\}$

$w \in \{B_0, A_0\}^*$

$S = \{\square, \boxdot\}$

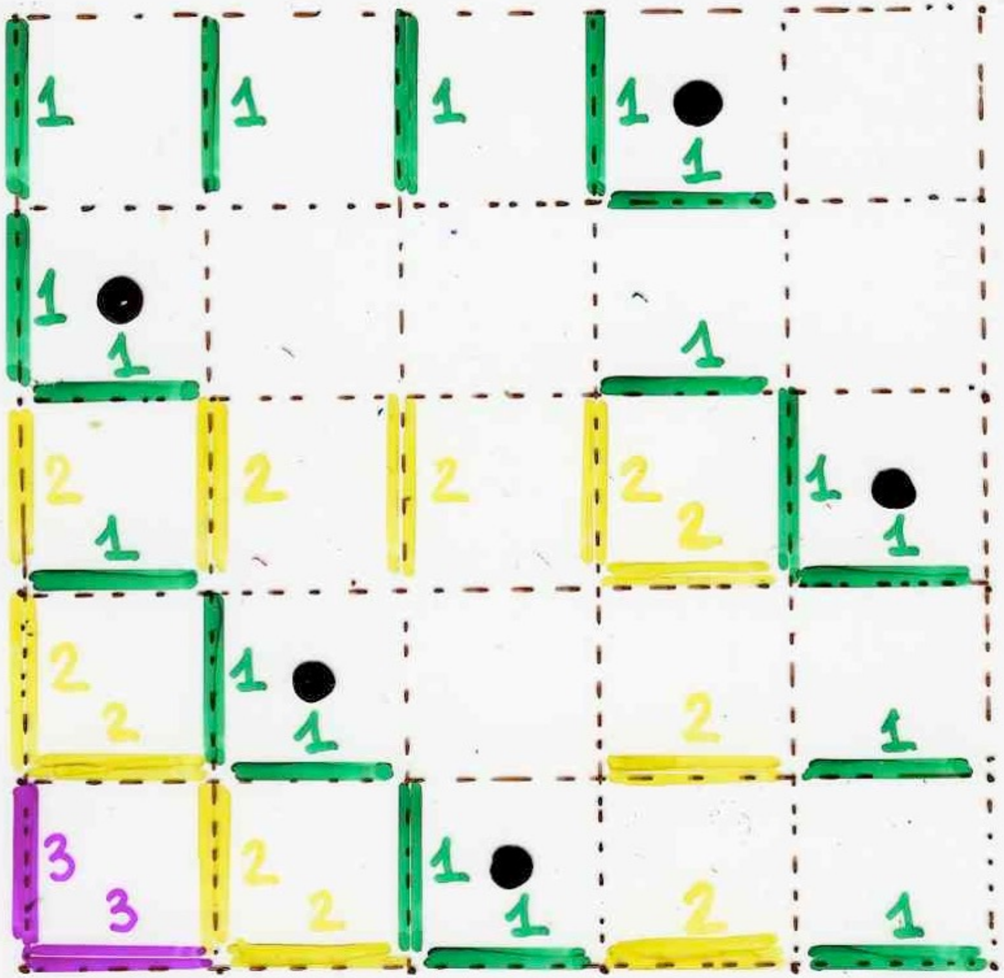




$i \neq j$

$$\begin{cases} U = B_0 \\ X_i = B_i \end{cases} \qquad \begin{cases} D = A_0 \\ Y_i = A_i \end{cases}$$

$i \geq 1 \qquad\qquad i \geq 1$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2
2
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2
1
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1
3
3
2
2
1
1
2
1

$$\begin{cases} U D = D U + \boxed{Y X} \\ U Y = Y U \\ X U = U X \\ X Y = \boxed{Y X} \end{cases}$$

another "duplication" of the commutation relations of the algebra Q

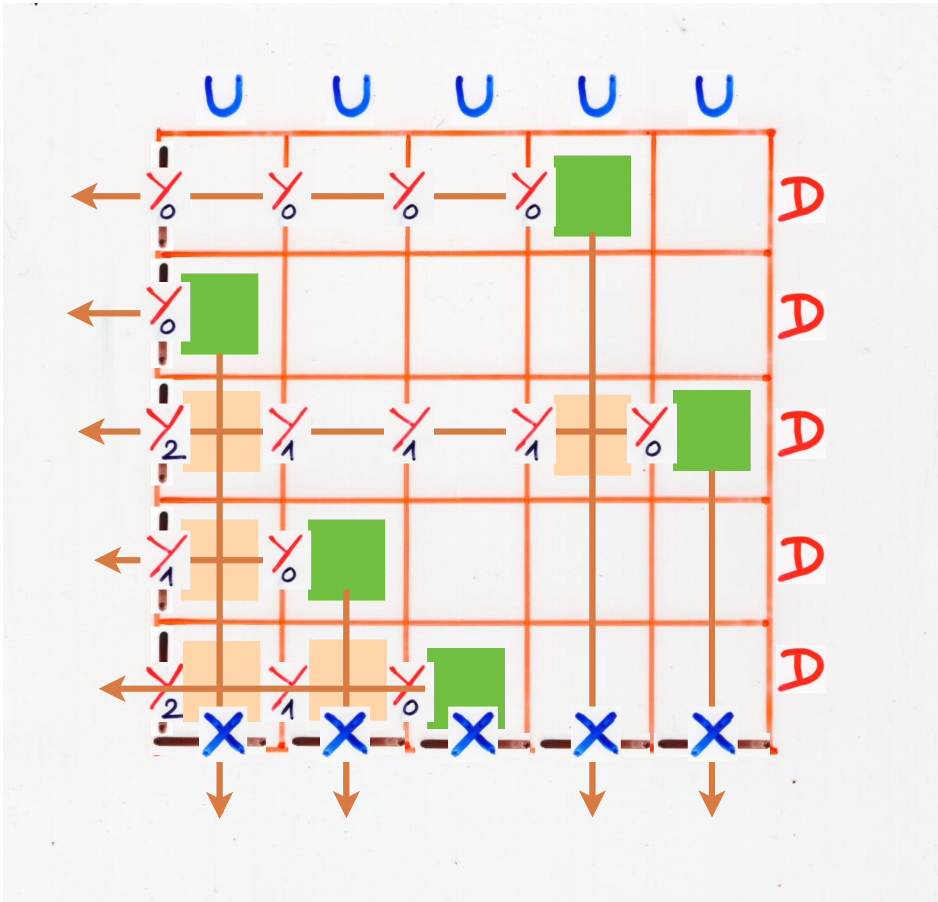
$$U D = D U + Y_0 X$$

$$\begin{cases} X Y_0 = Y_1 X \\ X Y_1 = Y_2 X \\ X Y_2 = Y_3 X \\ \cdots\cdots\cdots \\ X Y_i \quad Y_{i+1} X \\ \cdots\cdots\cdots \end{cases}$$

$$X U = U X$$
$$U Y_i = Y_i U$$

$$UD = DU + Y_0 X$$

$$\begin{cases} X Y_0 = Y_1 X \\ X Y_1 = Y_2 X \\ X Y_2 = Y_3 X \\ \cdots \\ X Y_i \quad Y_{i+1} X \\ \cdots \end{cases}$$



demultiplication
In the PASEP algebra

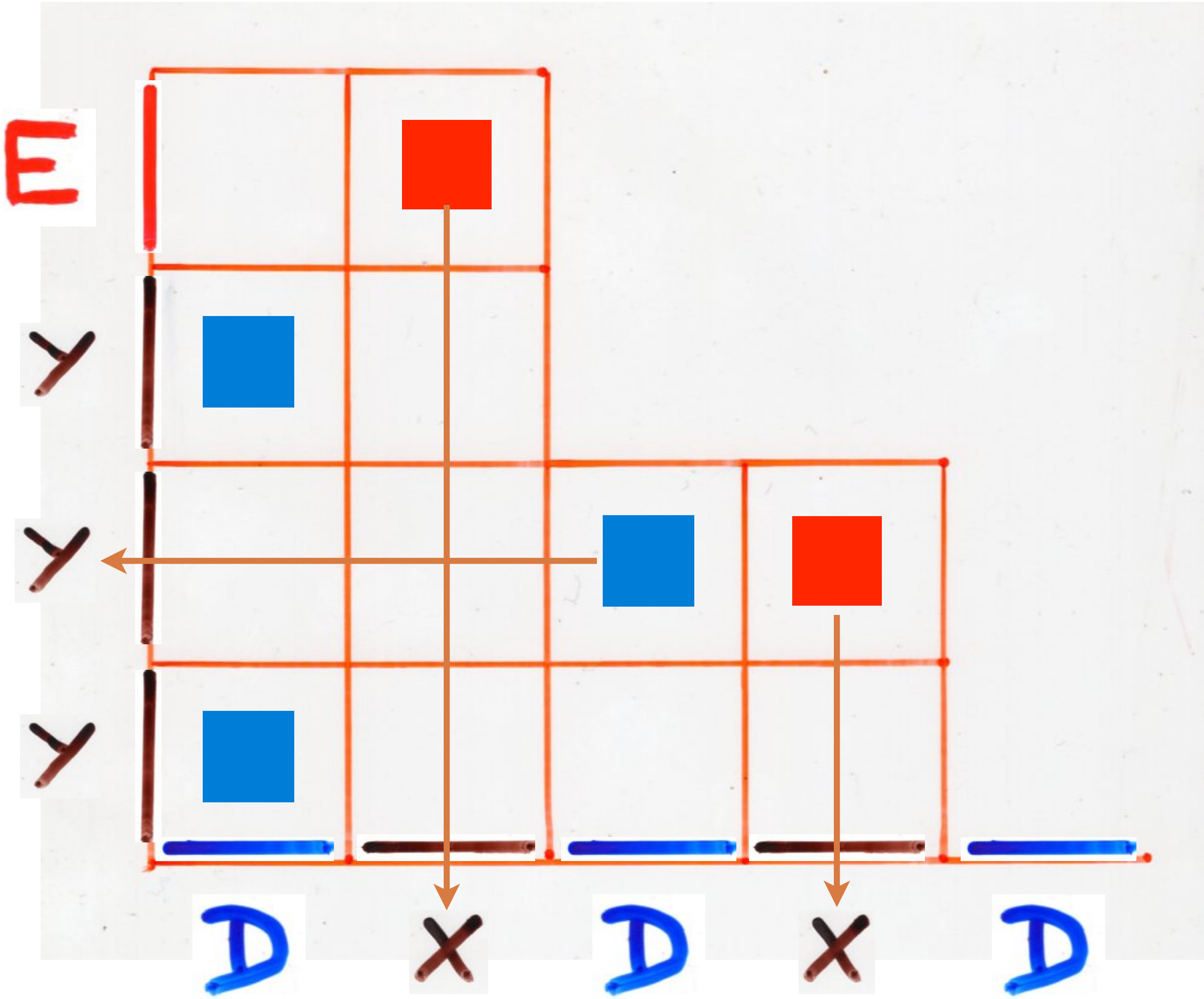
PASEP algebra

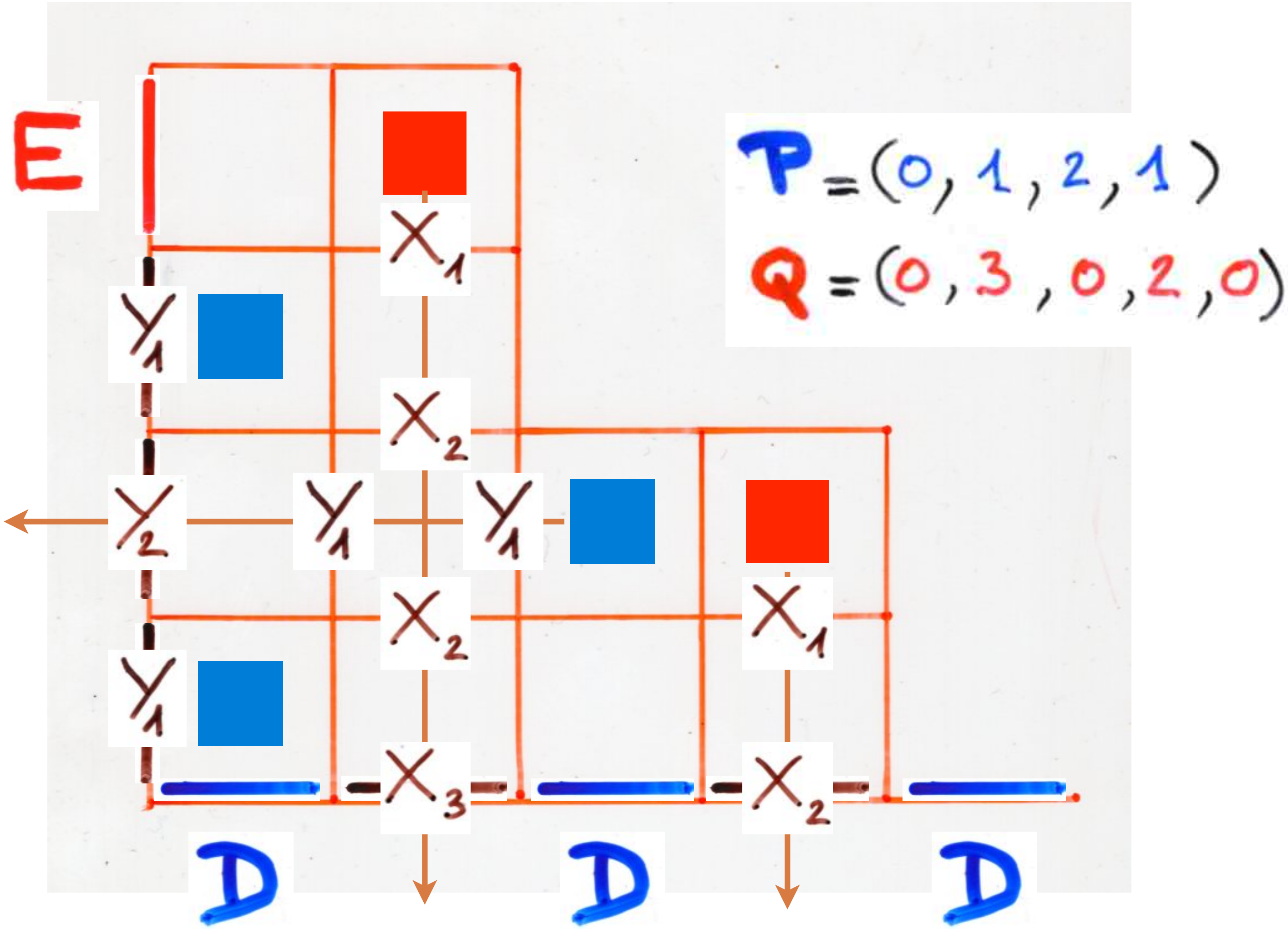
$$Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

$$DE = ED + EX_1 + Y_1 D$$

$$\begin{cases} X_1 E = E X_2 \\ \cdots \\ X_i E = E X_{i+1} \\ \cdots \end{cases} \qquad \begin{cases} D Y_1 = Y_2 D \\ \cdots \\ D Y_i = Y_{i+1} D \\ \cdots \end{cases}$$

$$X_i Y_j = Y_j X_i$$


E
Y
Y
Y
D
X
D
X
D



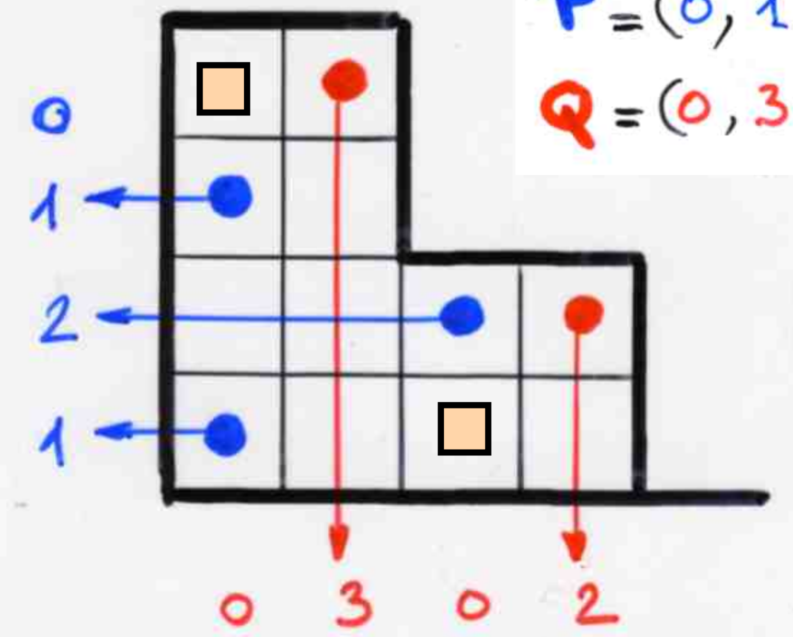
E
P = (0, 1, 2, 1)
Q = (0, 3, 0, 2, 0)
X1
X2
X2
X3
X1
X2
Y1
Y2
Y1
Y1
Y1
D
D
D

Adela bijection

$$\text{Adela}(T) = (P, Q)$$

$$P(T) = (a_1, a_2, \dots, a_k)$$

$$Q(T) = (b_1, \dots, b_\ell)$$



$P = (0, 1, 2, 1)$

$Q = (0, 3, 0, 2, 0)$

$a_i = \begin{cases} 0 & \text{if no } \bullet \text{ in row } i \\ 1 + \text{number of cells} & \text{in row } i \end{cases}$

$b_j = \begin{cases} 0 & \text{if no } \bullet \text{ in the } j^{th} \text{ column} \\ 1 + \text{number of cells} & \text{in the } j^{th} \text{ column} \end{cases}$

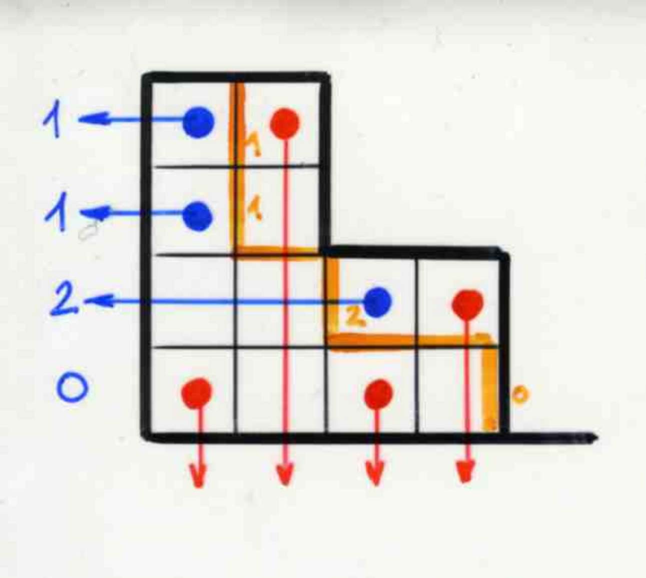
Research problem

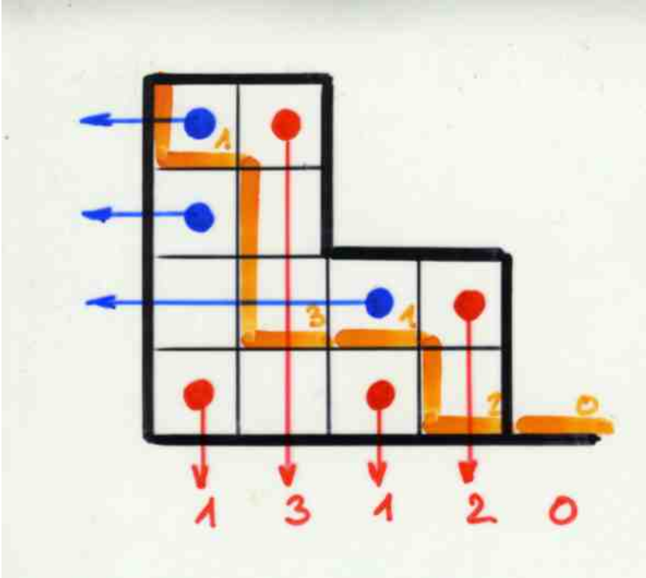
- T alternative tableau size n Adela $(T) = (P, Q)$

- give a characterization of the pairs (P, Q)

- give a bijection between such pairs and the $(n+1)!$ permutations of $\mathfrak{S}_{n+1}$

Adela (T) = (P , Q)





Catalan alternative tableaux $\longleftrightarrow$ Pair of paths

see Ch4, this course BJC3

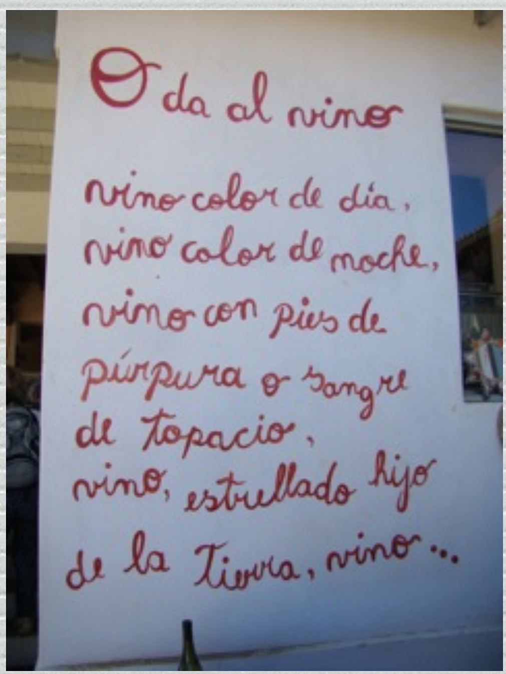
The "Adela duality" P (T) $\longleftrightarrow$ Q (T)

Why Adela bijection ?

Isla Negra
Pablo Neruda

The names «Adela bijection» and «Adela duality» is in honour of my friend Adela where part of this research was done in her house in Isla Negra, Chile, inspiring place where Pablo Neruda spent many years in his house in front of the Pacific Ocean.

Isla Negra
Pablo Neruda



The 8-vertex algebra

(or XYZ - algebra)

(or Z - algebra)

The quadratic algebra Z

4 generators $B_\bullet \quad A_\bullet \quad B \quad A$

8 parameters $q_{xy},\ t_{xy} \quad \begin{cases} x = \bullet, \circ \\ y = \bullet, \circ \end{cases}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet \end{cases}$$

The Z-quadratic algebra

XYZ- quadratic algebra

XXZ- spin chain

Spin chains and combinatorics

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Institute for High Energy Physics
142284 Protvino, Moscow region, Russia

(J. R. R. Tolkien)

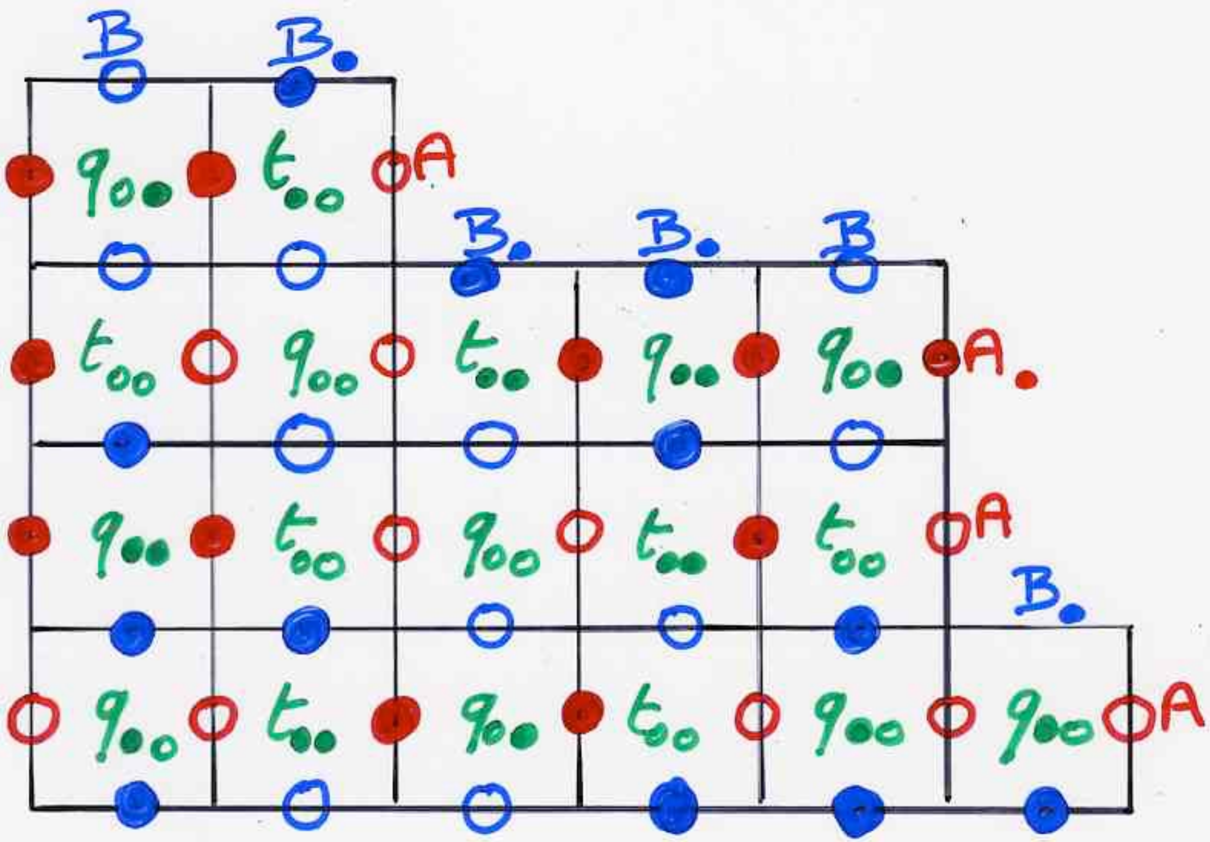
The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

(XY)Z-tableaux
and
B.A.BA configurations
(or XYZ- configurations)

complete Z- tableaux

XYZ- tableaux



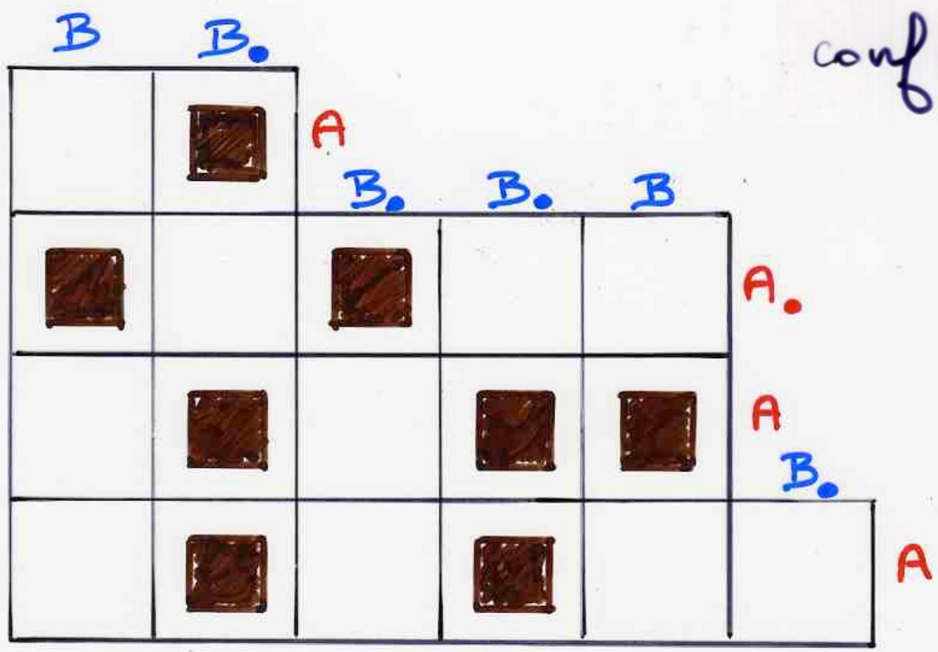
$$
\begin{cases}
B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\
B_\bullet\,A_\bullet = q_{\bullet\bullet}\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\
B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + t_{\bullet\circ}\,A_\bullet\,B \\
B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + t_{\circ\bullet}\,A\,B_\bullet
\end{cases}
$$

$$\begin{cases} B\,A = \square\; A\,B + \blacksquare\; A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = \square\; A_\bullet\, B_\bullet + \blacksquare\; A\,B \\ B_\bullet\, A = \square\; A\, B_\bullet + \blacksquare\; A_\bullet\, B \\ B\, A_\bullet = \square\; A_\bullet\, B + \blacksquare\; A\, B_\bullet \end{cases}$$

$w \in \{B, B_\bullet, A, A_\bullet\}^*$ $F(w)$

$w = w_1 \cdots w_n$

i^{th} step of the upper border of $F(w)$ $= \begin{cases} \text{—} & \text{if } w_i = B \text{ or } B_\bullet \\ | & \text{if } w_i = A \text{ or } A_\bullet \end{cases}$



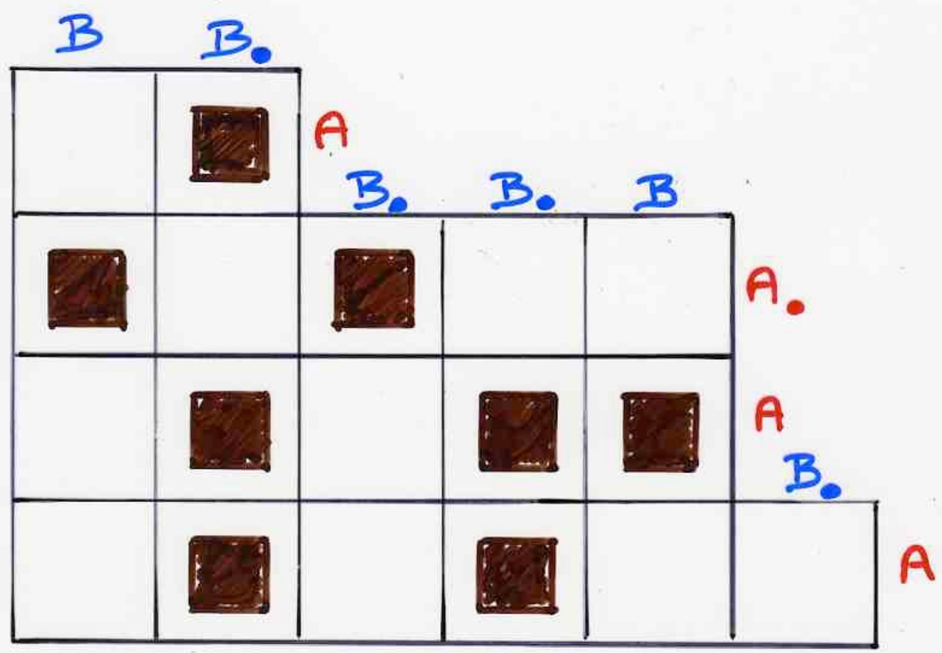
configurations B. A. B A

bijection (s)

$(w, C) \longleftrightarrow T$

B. A. B A configuration in the diagram F(w)

complete Z-tableau with diagram F(w)



B. A. B A configuration

$C \subseteq F(w)$ $\rightarrow$ complete Z-tableau T

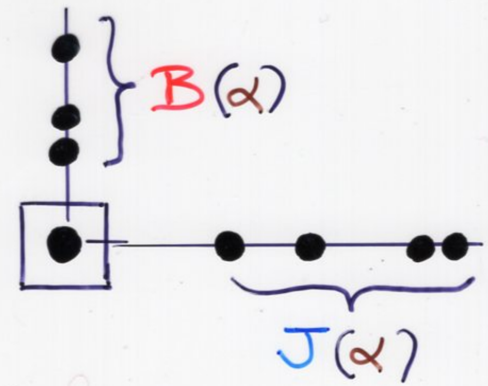
$w \in \{ B_\circ, B_\bullet, A_\circ, A_\bullet \}^*$, with $B_\circ = B$, $A_\circ = A$

The label of α in T is
- if $\alpha \in C$, t_{xy}
- if $\alpha \notin C$, q_{xy}

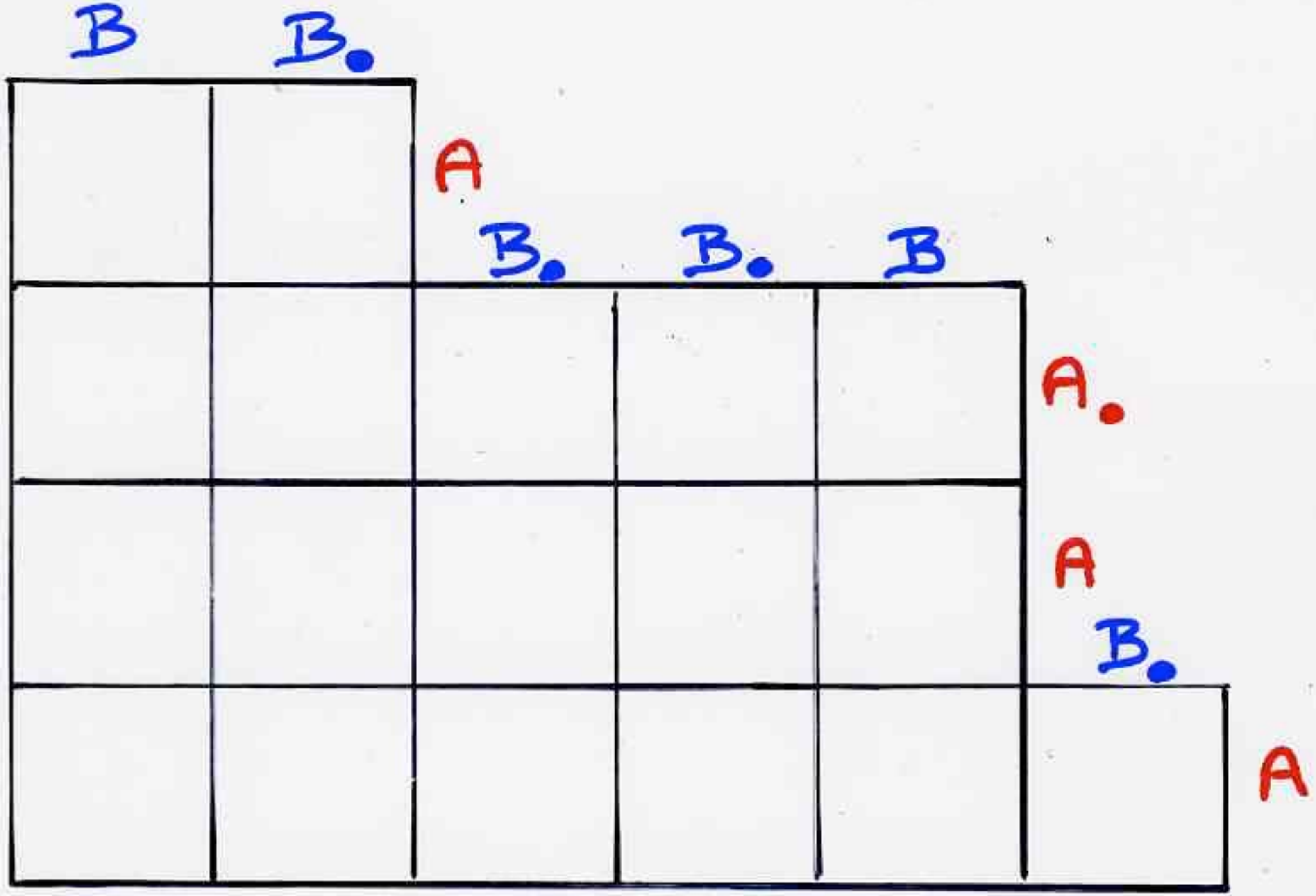
$$x = \begin{cases} \bullet & J(\alpha) \text{ odd and } A_\circ \\ \circ & J(\alpha) \text{ even and } A_\circ \end{cases}$$

$$\begin{cases} \circ & J(\alpha) \text{ odd and } A_\bullet \\ \bullet & J(\alpha) \text{ even and } A_\bullet \end{cases}$$

cell $\alpha \in C$



same for y with $B(\alpha)$ and $B_\circ$ or $B_\bullet$


B
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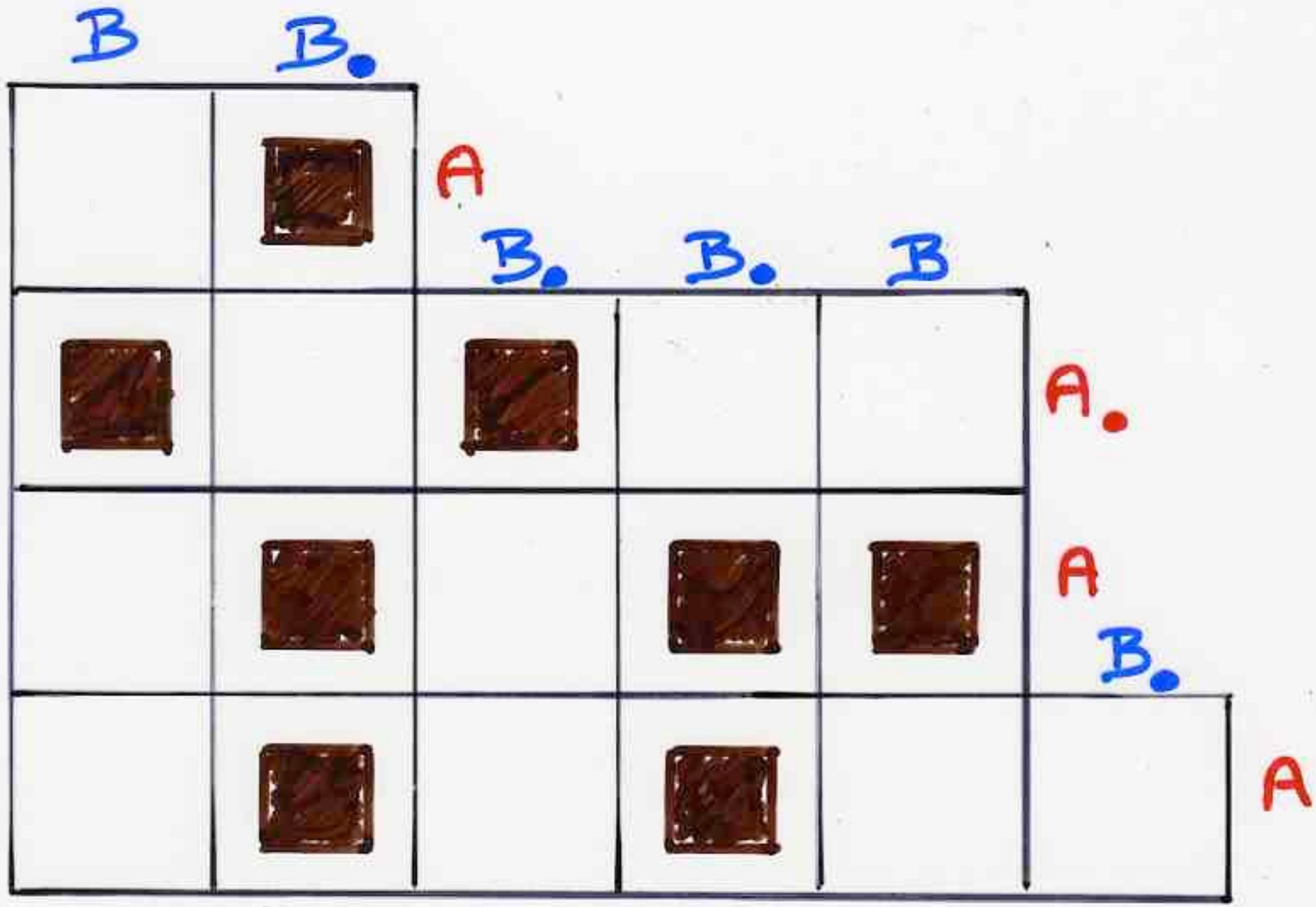
A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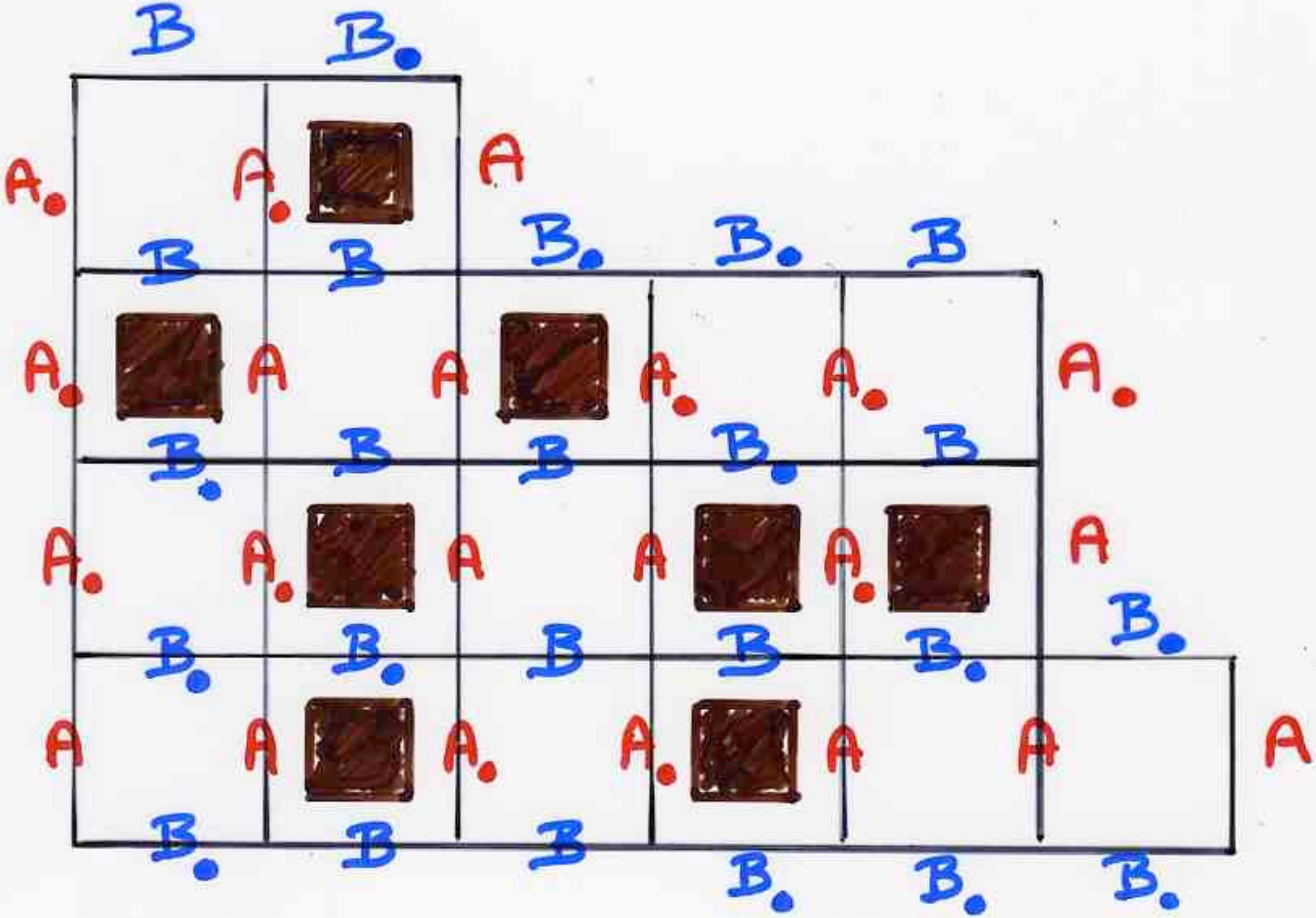
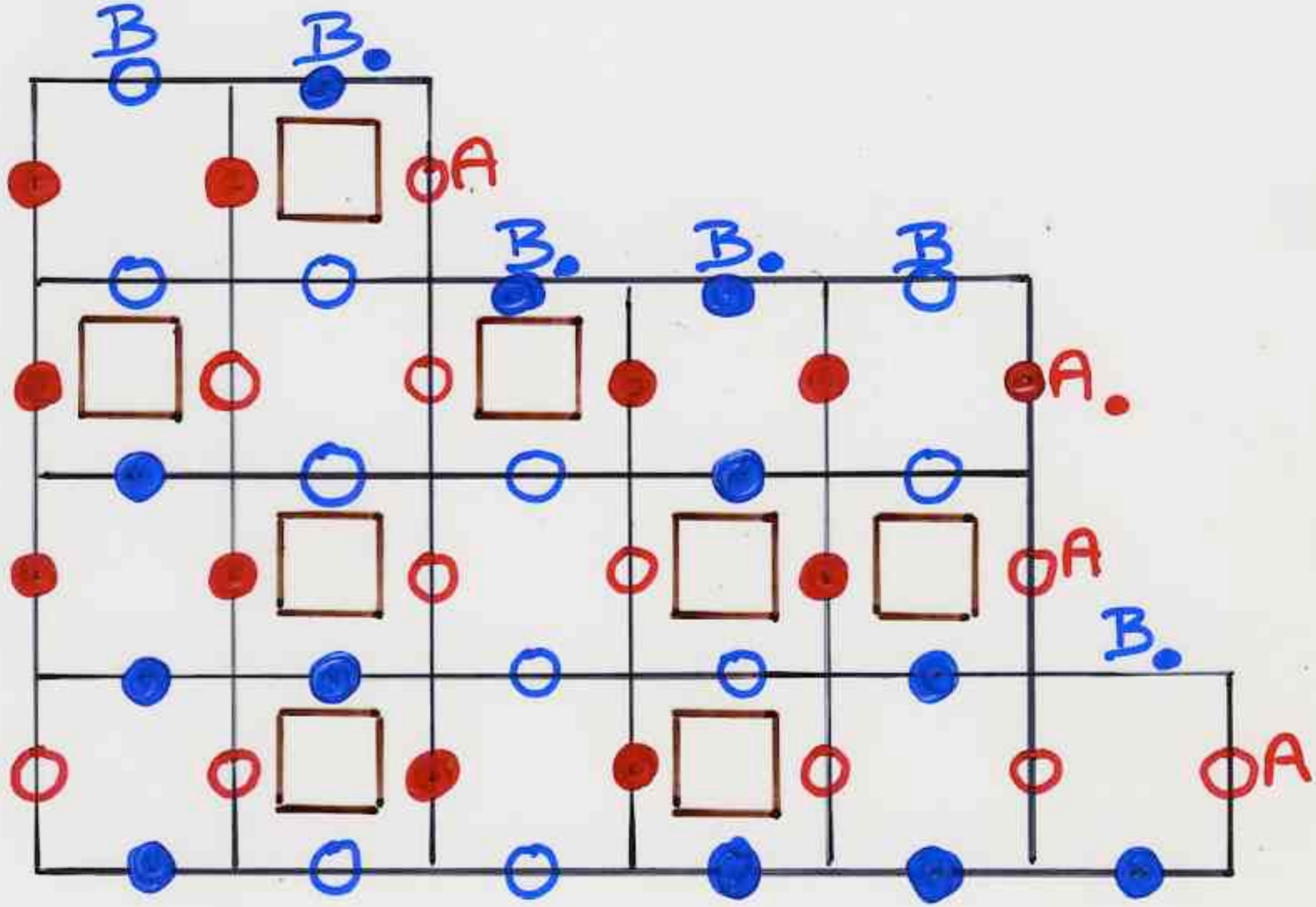
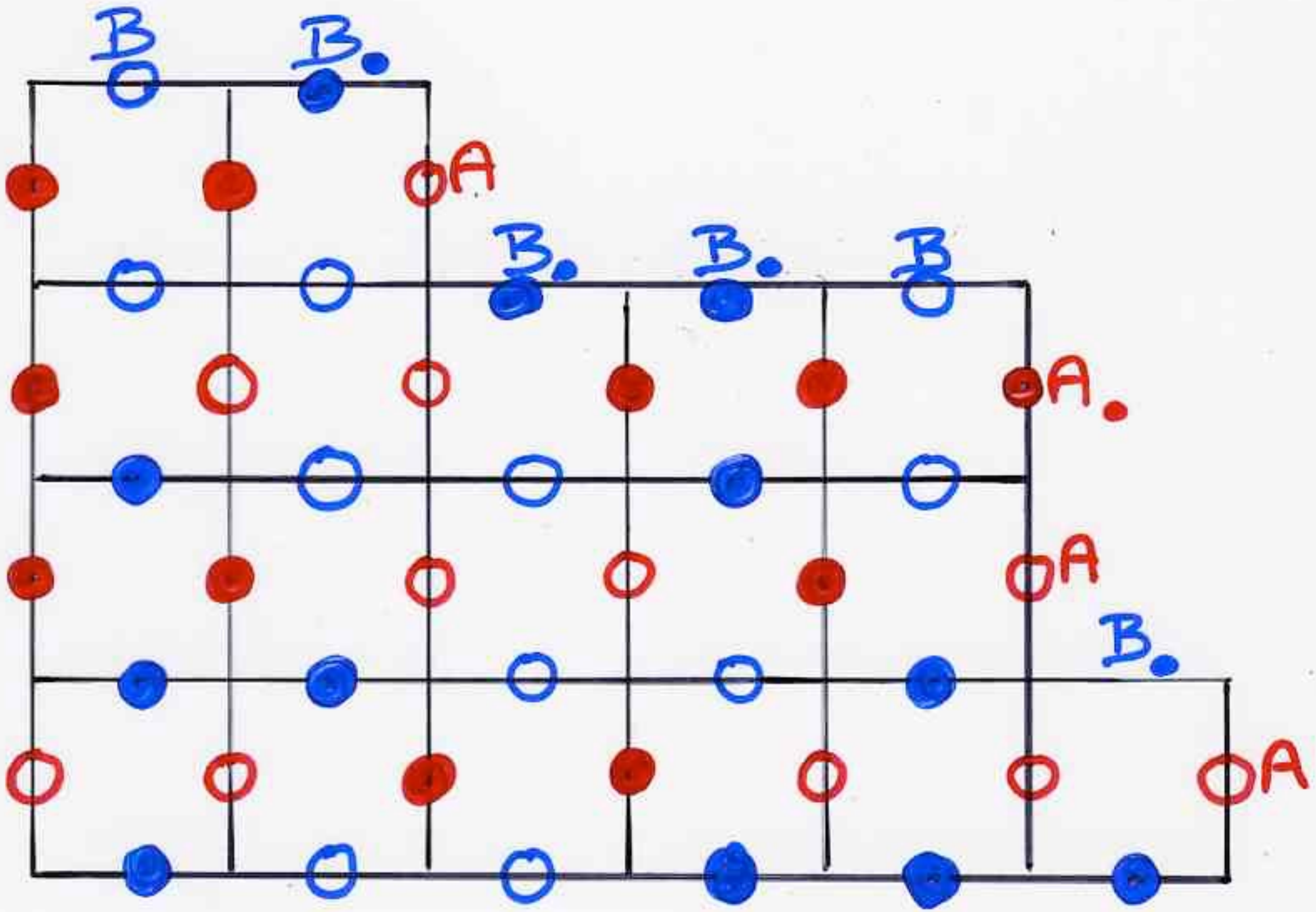

A
B



B
B
A
B
B
B
A
A
B
A


B
B.
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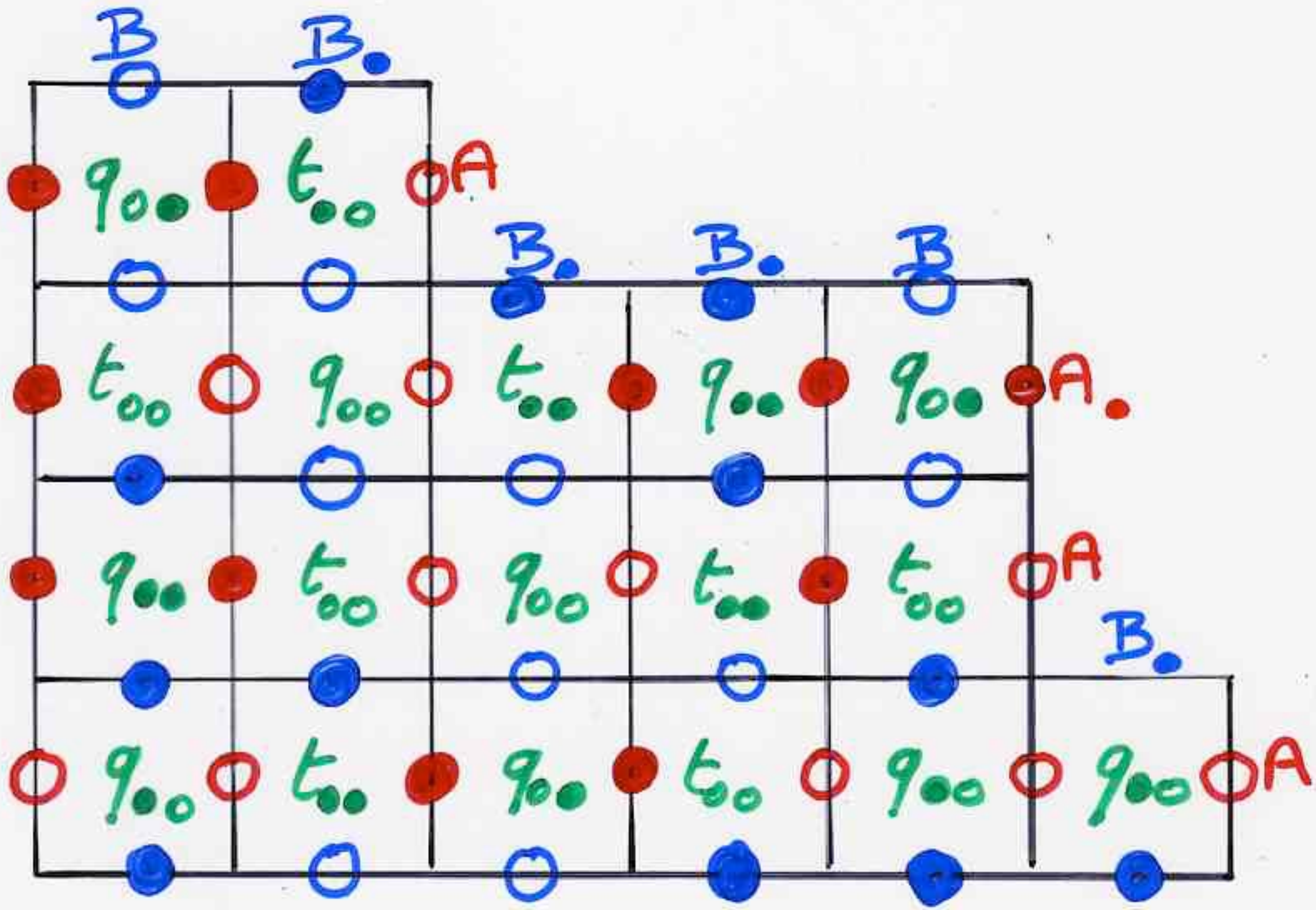
B
A.
A
B.
A


B B
q00 t00 A
B B B
t00 q00 t00 q00 q00 A
q00 t00 q00 t00 t00 A
B
q00 t00 q00 t00 q00 q00 A

bijection(s)

$(w, C) \longleftrightarrow T$

B. A. BA configuration in the diagram $F(w)$

complete Z-tableau with diagram $F(w)$

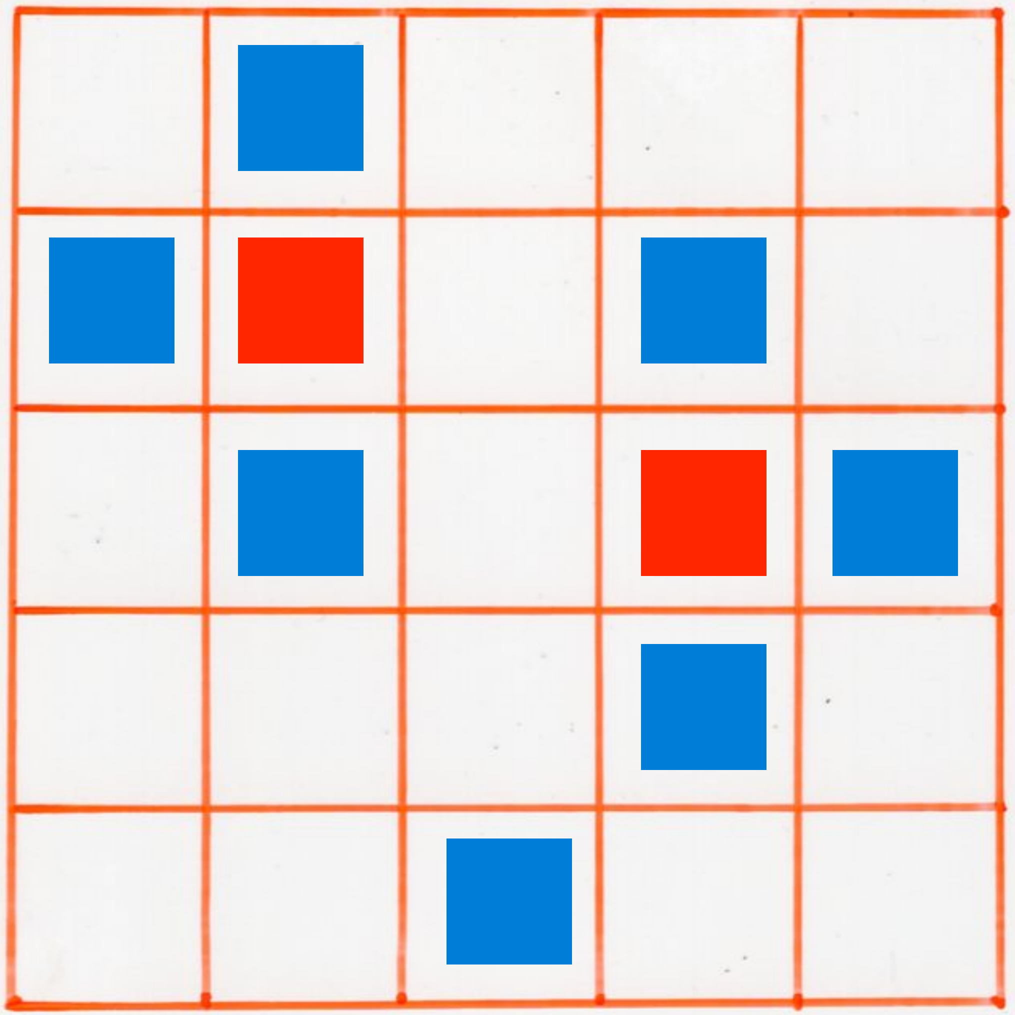
F Ferrers diagram $C \subseteq F(w)$

- for each cell α of F
- for each pair BA, $B_\bullet A_\bullet$, $B_\bullet A$, $BA_\bullet$

we fix a rule for the labeling of the cell by q_{xy} or t_{xy} (with $x = \bullet$ or $\circ$, $y = \bullet$ or $\circ$) according to $\alpha \in C$ or $\alpha \notin C$

alternating sign matrices (ASM)

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

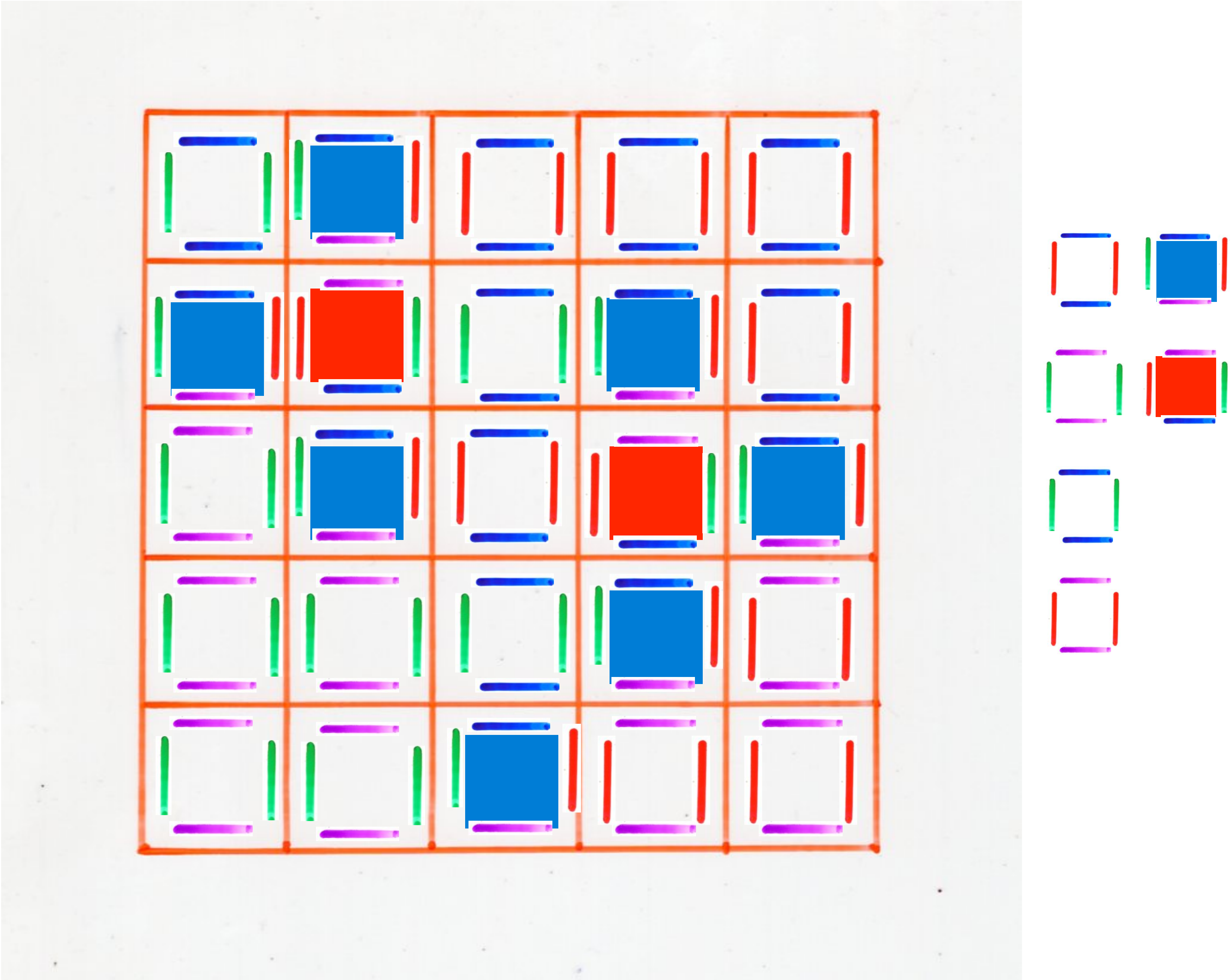


$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

$$\begin{cases} B A = q_{\circ\circ}\, A B + t_{\circ\circ}\, A_{\bullet} B_{\bullet} \\ B_{\bullet} A_{\bullet} = q_{\bullet\bullet}\, A_{\bullet} B_{\bullet} + t_{\bullet\bullet}\, A B \\ B_{\bullet} A = q_{\bullet\circ}\, A B_{\bullet} + 0\, A_{\bullet} B \\ B A_{\bullet} = q_{\circ\bullet}\, A_{\bullet} B + 0\, A B_{\bullet} \end{cases}$$

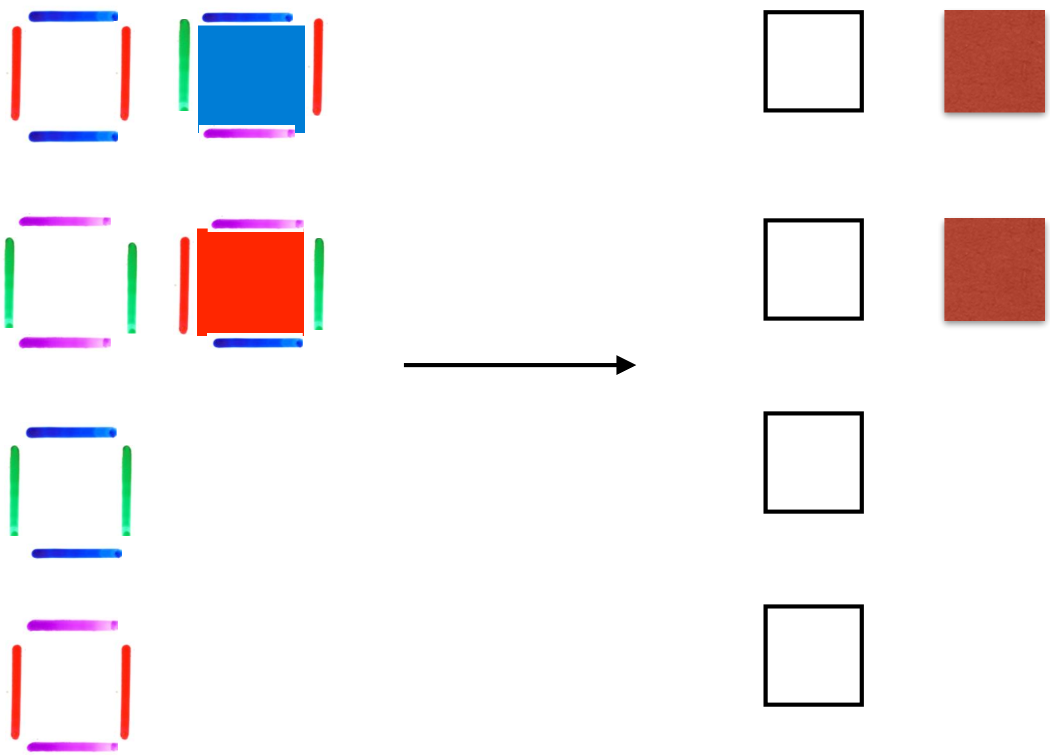
$$w = B^n A^n \qquad uv = A_{\bullet}^n\, B_{\bullet}^n$$

$c(u, v; w) =$ nb of ASM $n \times n$



A alternating sign matrix

$\varphi(A)$ B.A.BA configuration



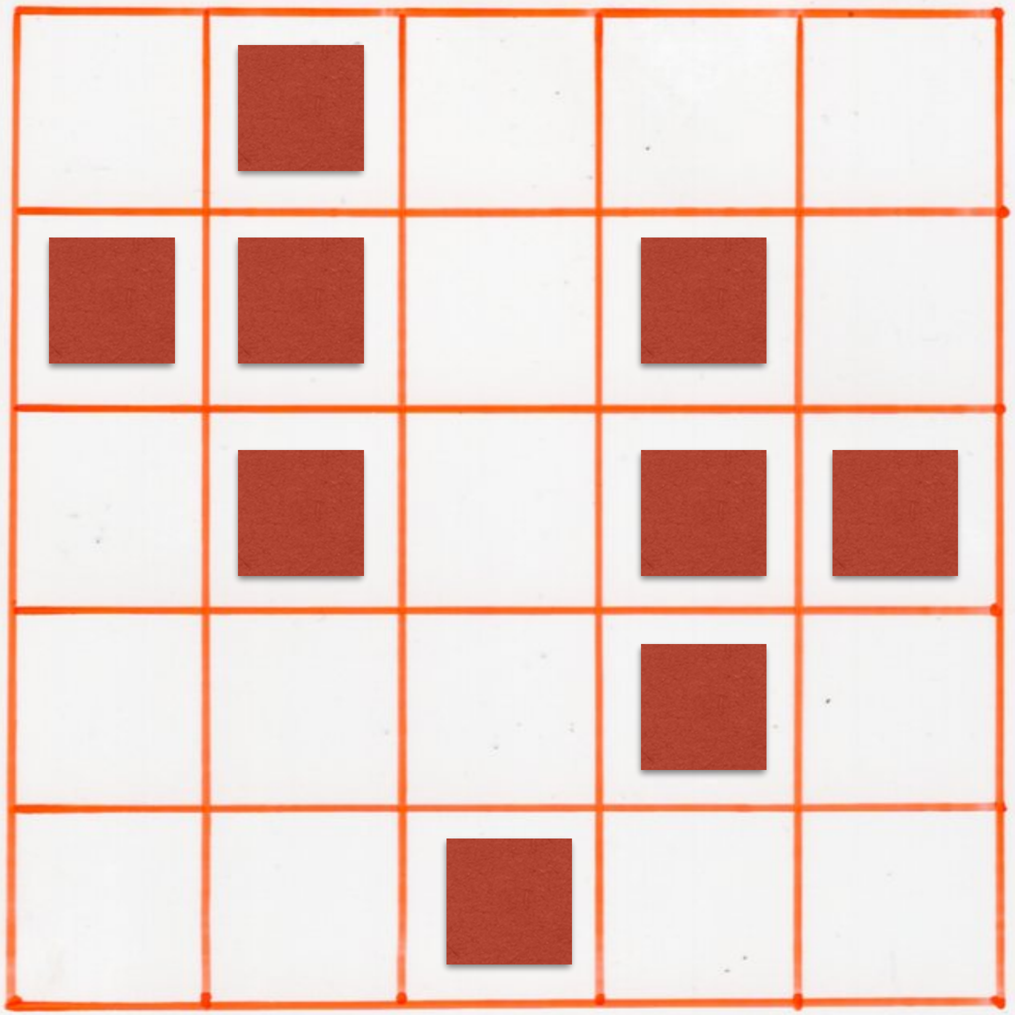
A alternating sign matrix

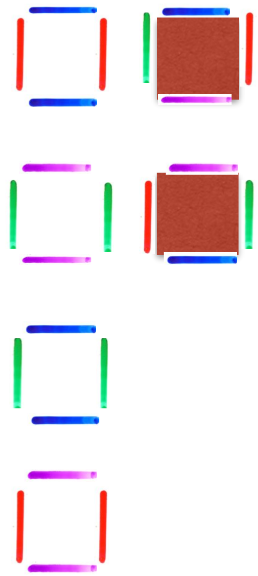
$\varphi(A)$ $B_\bullet A_\bullet BA$ configuration

$$\begin{cases} BA = \square\, AB + \blacksquare\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \square\, A_\bullet B_\bullet + \blacksquare\, AB \\ B_\bullet A = \square\, AB_\bullet + \bigcirc\, \cancel{A_\bullet B} \\ BA_\bullet = \square\, A_\bullet B + \bigcirc\, \cancel{AB_\bullet} \end{cases}$$

A alternating sign matrix

φ (A) B.A.BA configuration





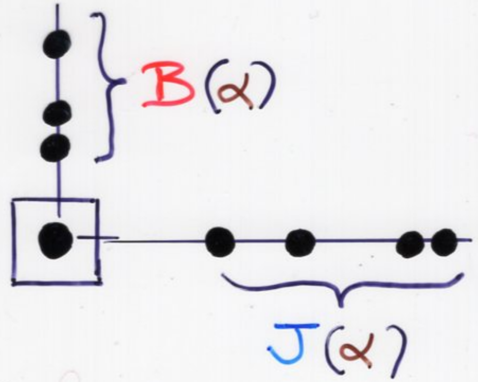
A alternating sign matrix

$\varphi(A)$ B.A.BA configuration

exercise

characterisation

cell $\alpha \in C$



(i) – for any cell $\alpha \in C$, $|B(\alpha)|$ and $|J(\alpha)|$ same parity

and

(ii) – in each row and each column, the number of cells in C is odd

correlations functions in XXZ spin chains

LPENSL-TH-05

Exact results for the σ^z two-point function of the XXZ chain at $\Delta = 1/2$

arXiv:hep-th/0506114 v1 14 Jun 2005

N. Kitanine[1], **J. M. Maillet**[2], **N. A. Slavnov**[3], **V. Terras**[4]

Abstract

We propose a new multiple integral representation for the correlation function $\langle\sigma_1^z\sigma_{m+1}^z\rangle$ of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the disordered regime. We show that for $\Delta = 1/2$ the integrals can be separated and computed exactly. As an example we give the explicit results up to the lattice distance $m = 8$. It turns out that the answer is given as integer numbers divided by $2^{(m+1)^2}$.

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[3]Steklov Mathematical Institute, Moscow, Russia, nslavnov@mi.ras.ru
[4]LPTA, UMR 5207 du CNRS, Montpellier, France, terras@lpta.univ-montp2.fr

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m=9$:

$$
\begin{aligned}
&P_1(\kappa) = 1+\kappa,\\
&P_2(\kappa) = 2+12\kappa+2\kappa^2,\\
&P_3(\kappa) = 7+249\kappa+249\kappa^2+7\kappa^3,\\
&P_4(\kappa) = 42+10004\kappa+45444\kappa^2+10004\kappa^3+42\kappa^4,\\
&P_5(\kappa) = 429+738174\kappa+16038613\kappa^2+16038613\kappa^3+738174\kappa^4+429\kappa^5,\\
&P_6(\kappa) = 7436+96289380\kappa+11424474588\kappa^2+45677933928\kappa^3+11424474588\kappa^4\\
&+96289380\kappa^5+7436\kappa^6,\\
&P_7(\kappa) = 218348+21798199390\kappa+15663567546585\kappa^2+265789610746333\kappa^3\\
&+265789610746333\kappa^4+15663567546585\kappa^5+21798199390\kappa^6+218348\kappa^7,\\
&P_8(\kappa) = 10850216+8485108350684\kappa+39461894378292782\kappa^2\\
&+3224112384882251896\kappa^3+11919578544950060460\kappa^4+3224112384882251896\kappa^5\\
&+39461894378292782\kappa^6+8485108350684\kappa^7+10850216\kappa^8\\
&P_9(\kappa) = 911835460+5649499685353257\kappa+177662495637443158524\kappa^2\\
&+77990624578576910368767\kappa^3+1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

intergers ?

positivity ?

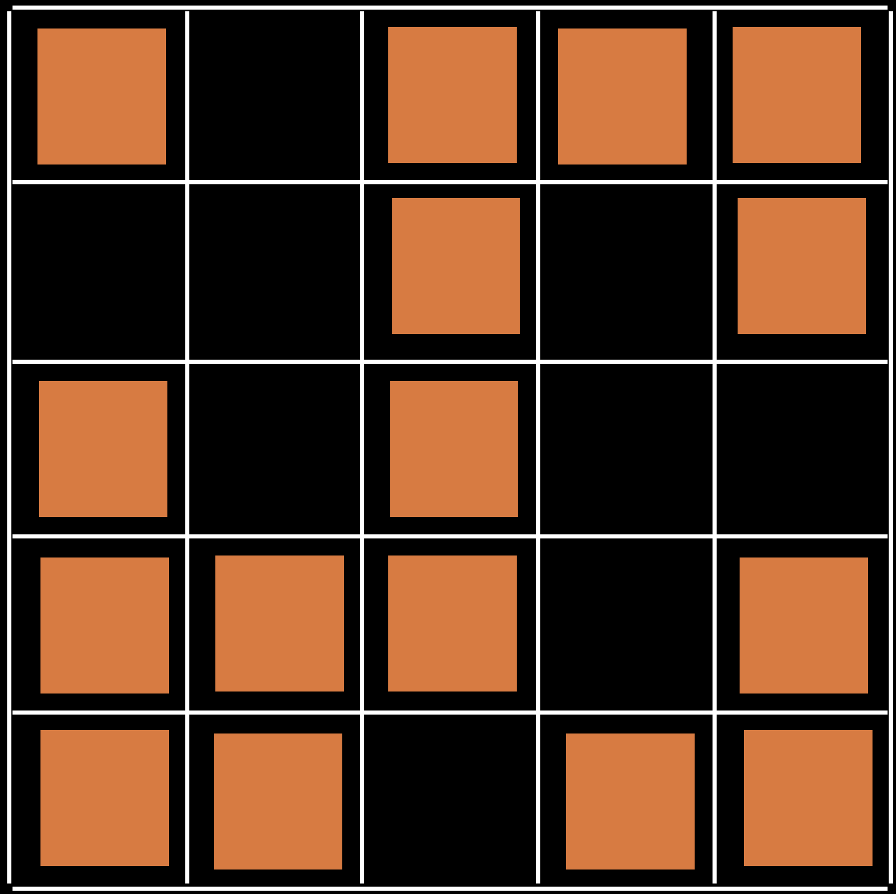
combinatorial interpretation

?

ASM

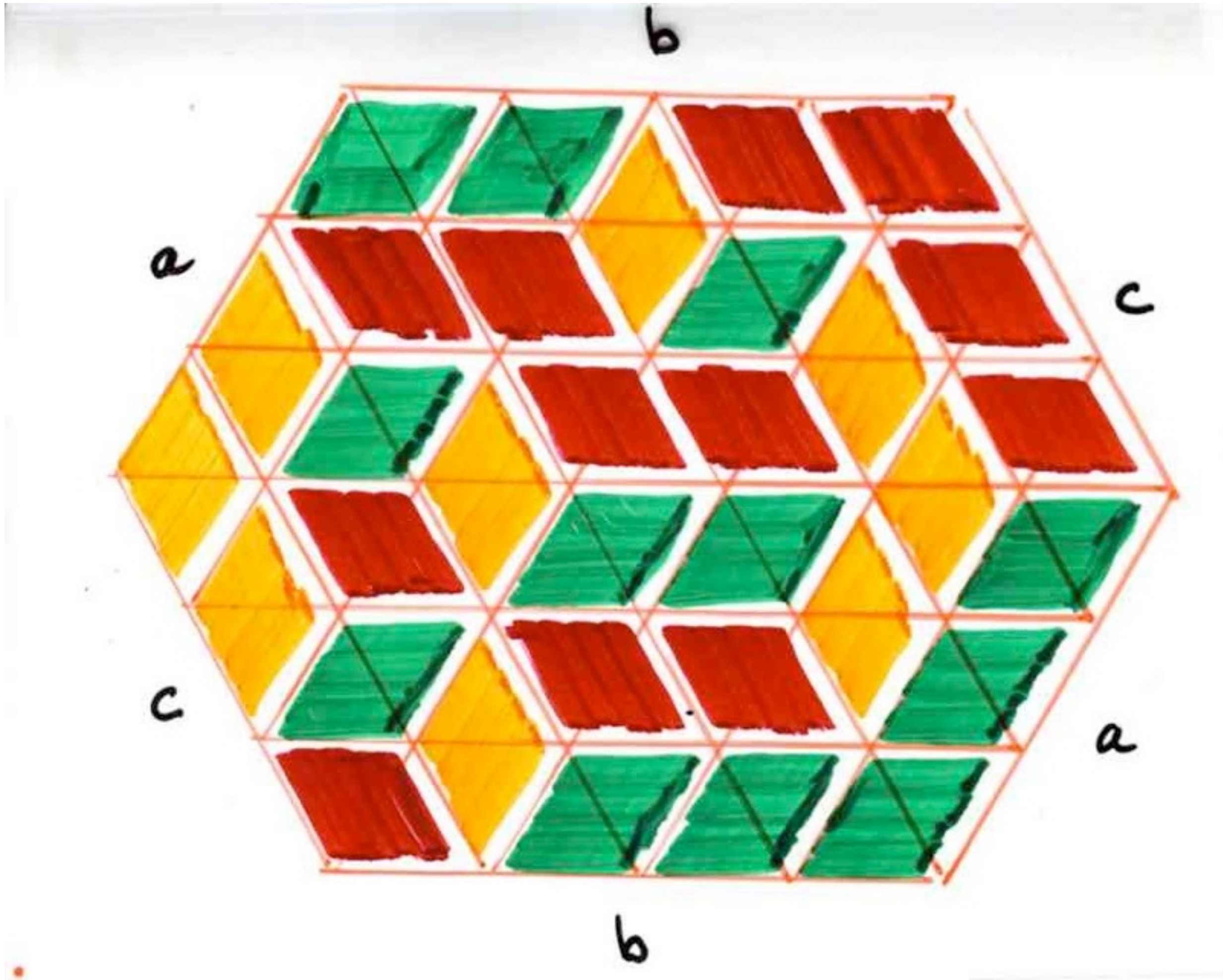
$$
\begin{aligned}
P_1(\kappa) &= 1 + \kappa,\\
P_2(\kappa) &= 2 + 12\kappa + 2\kappa^2,\\
P_3(\kappa) &= 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,\\
P_4(\kappa) &= 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4,\\
P_5(\kappa) &= 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,\\
P_6(\kappa) &= \underline{7436} + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4\\
&\quad +96289380\kappa^5 + \underline{7436}\kappa^6,\\
P_7(\kappa) &= \underline{218348} + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3\\
&\quad +265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + \underline{218348}\kappa^7,\\
P_8(\kappa) &= \underline{10850216} + 8485108350684\kappa + 39461894378292782\kappa^2\\
&\quad +3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5\\
&\quad +39461894378292782\kappa^6 + 8485108350684\kappa^7 + \underline{10850216}\kappa^8\\
P_9(\kappa) &= 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2\\
&\quad +77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

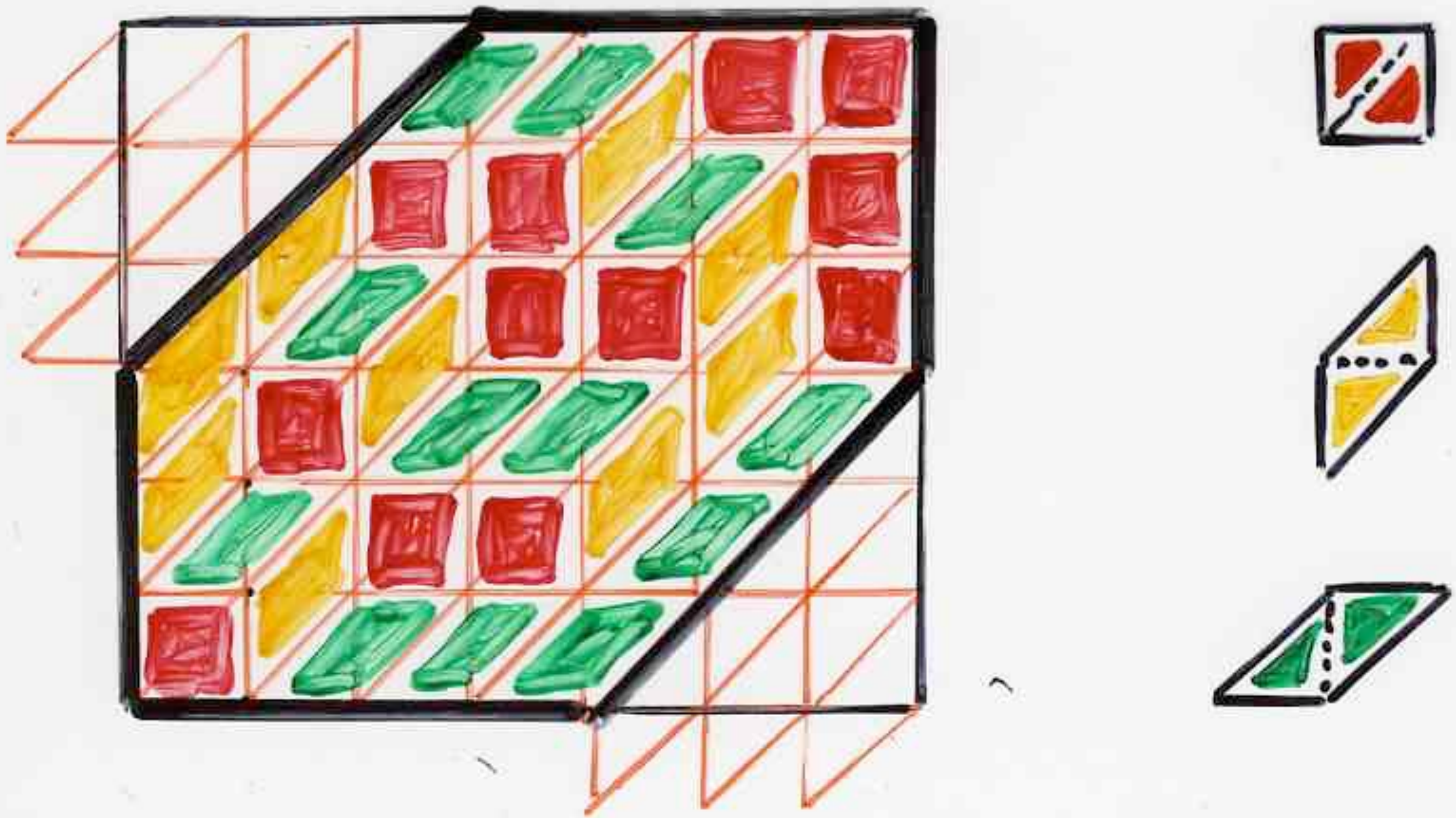
The number of B. A. BA configurations in the grid $[n] \times [n]$ is

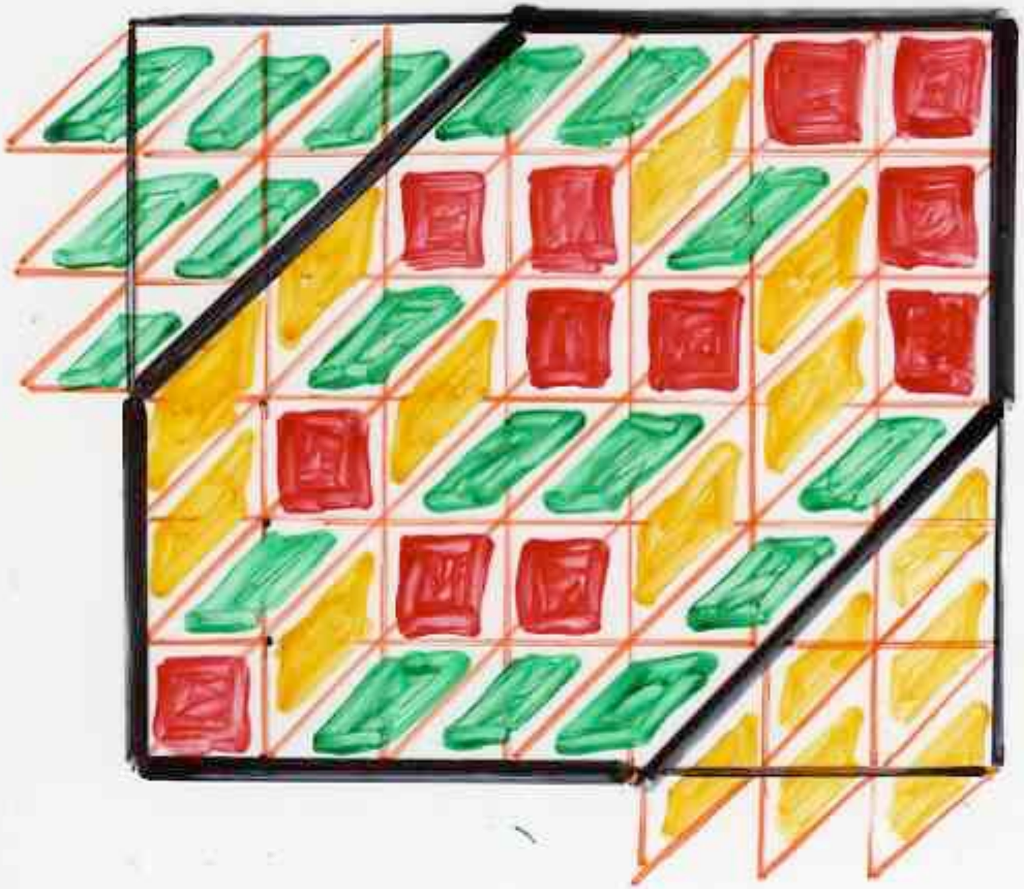


$$2^{(n^2)}$$

rhombus tilings

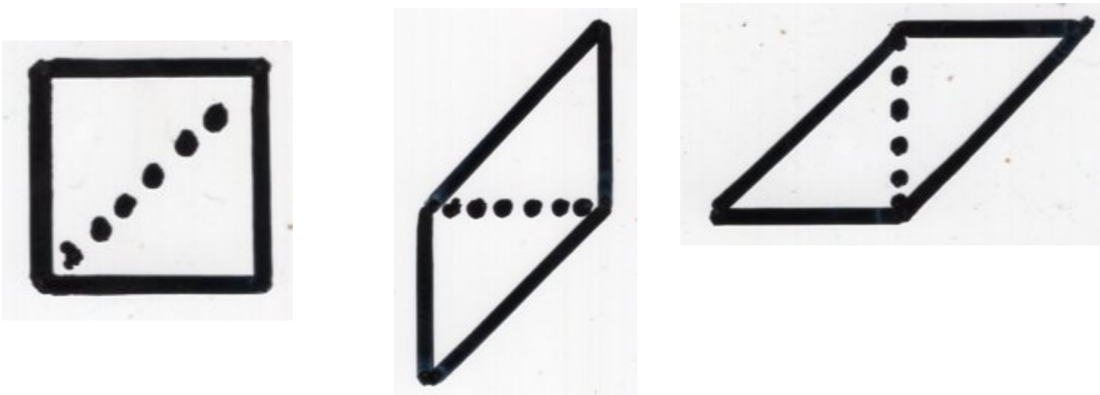

b
a
c
c
a
b






coding of the edges
for a tiling
of the triangular lattice

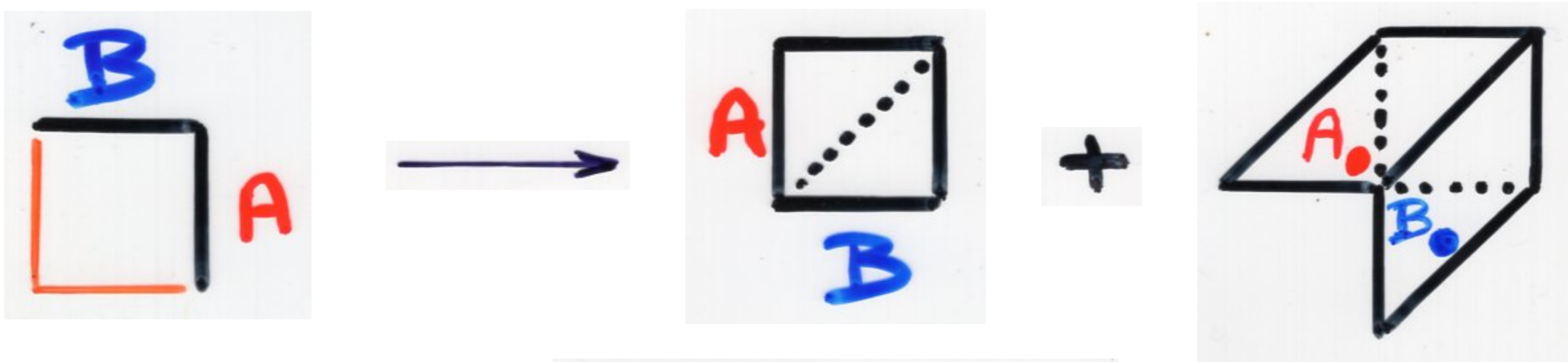
3 types of tiles



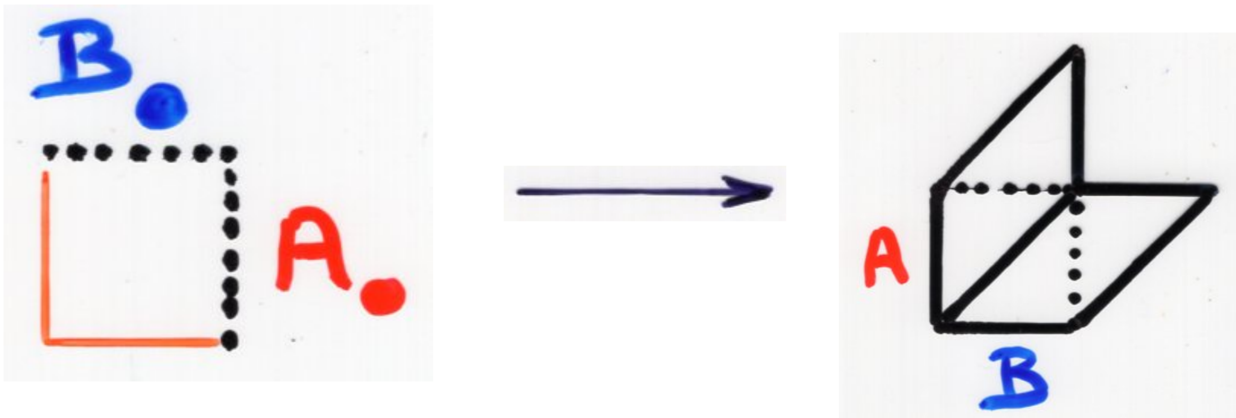
border
of a tile

inside
a tile

rewriting rules for tilings of the triangular lattice

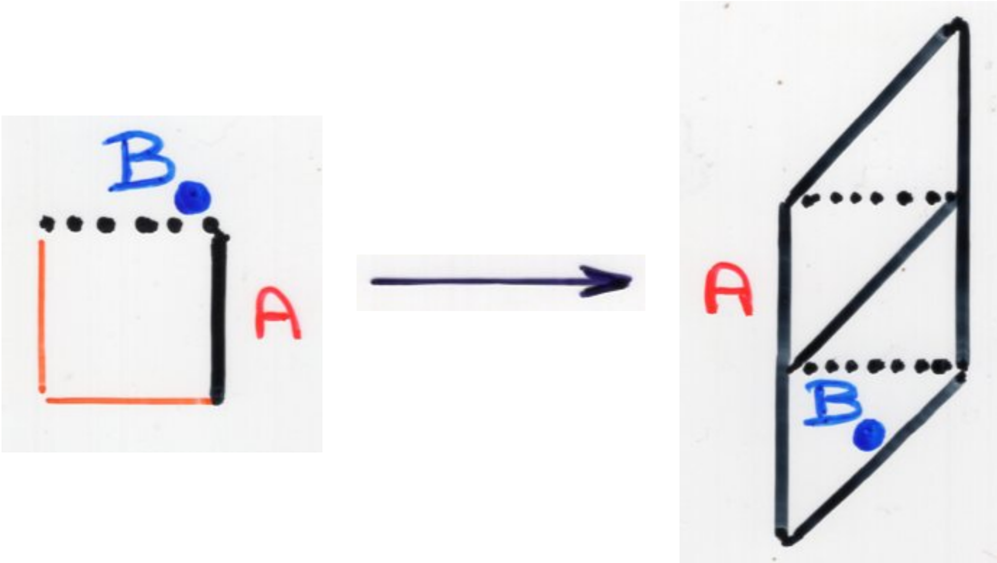


$$BA = AB + A_{\bullet} B_{\bullet}$$

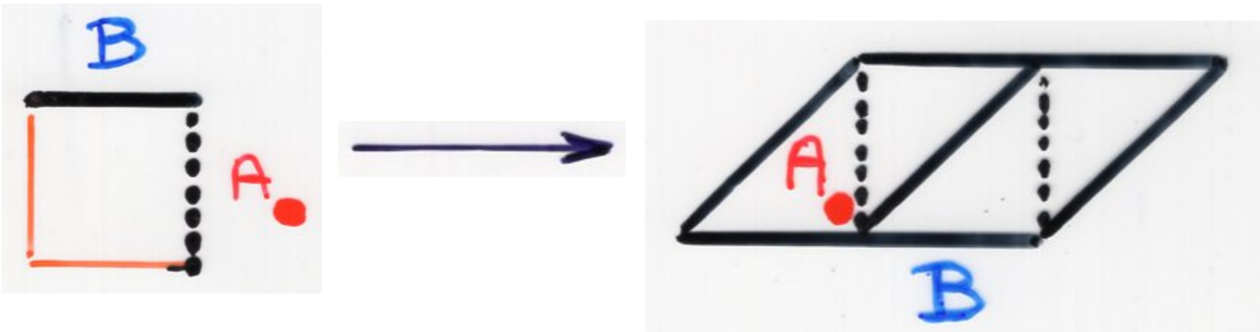


$$B_{\bullet} A_{\bullet} = AB$$

rewriting rules for tilings of the triangular lattice



$$B_\bullet\, A = A\, B_\bullet$$



$$B\, A_\bullet = A_\bullet\, B$$

rewriting rules for tilings of the triangular lattice

$$BA = AB + A_{\bullet} B_{\bullet}$$

$$B_{\bullet} A_{\bullet} = AB$$

$$B A_{\bullet} = A_{\bullet} B$$

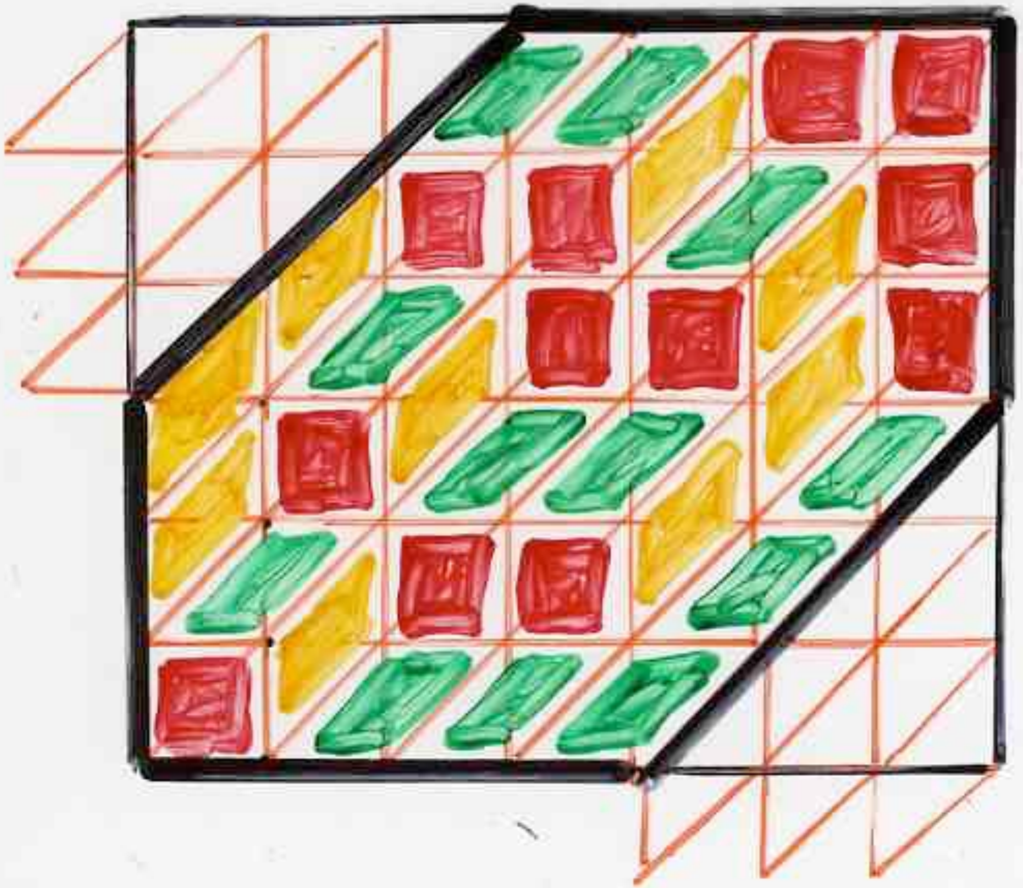
$$B_{\bullet} A = A B_{\bullet}$$

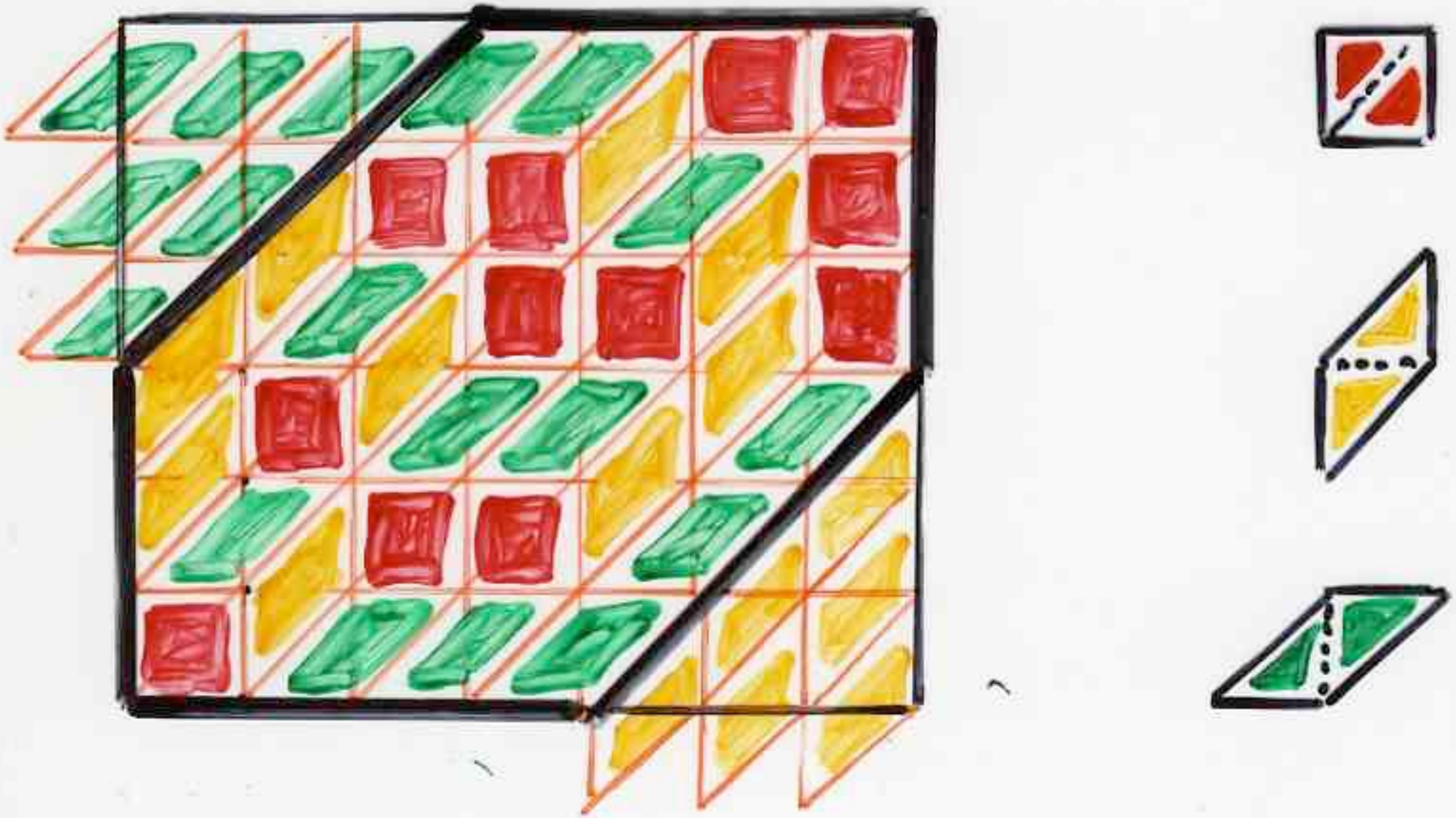
$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases}$$

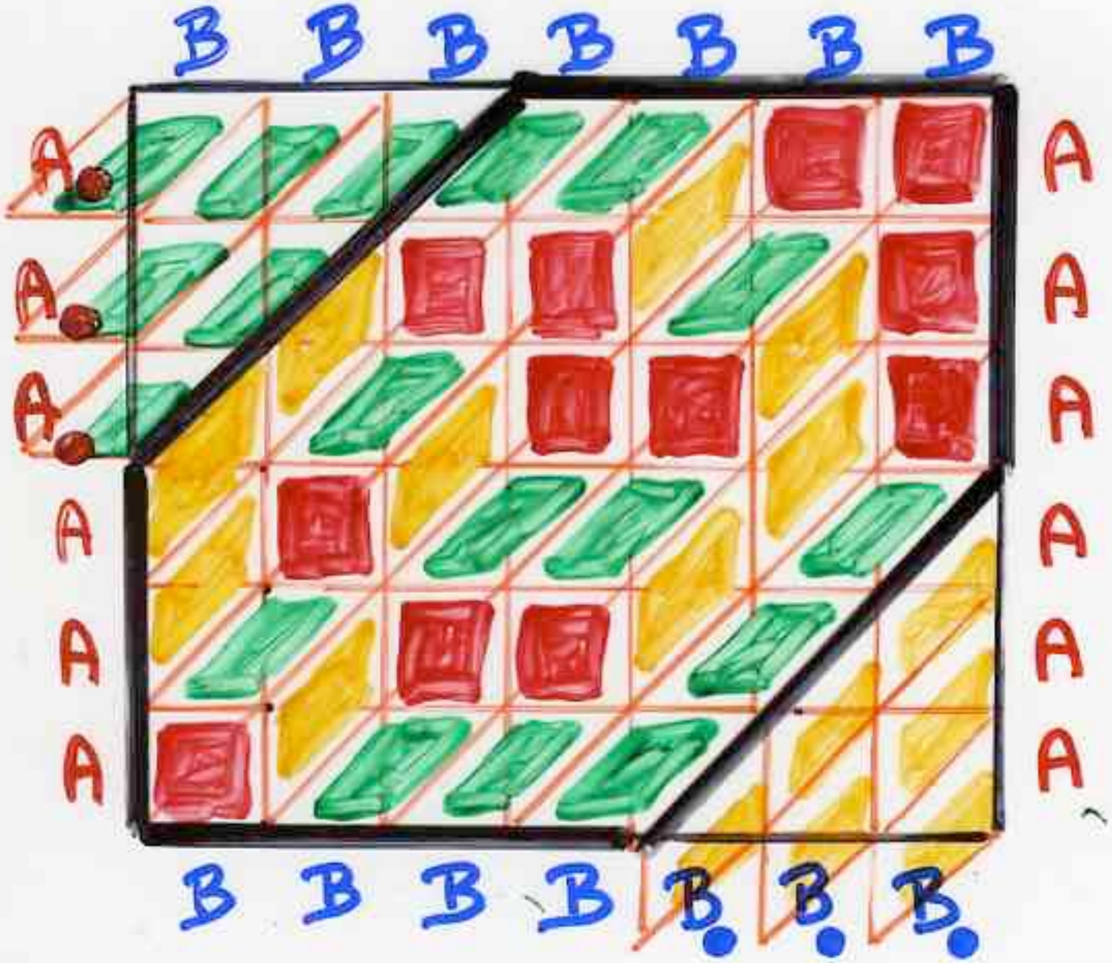
(ASM)

Rhombus tilings

$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = 0\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + 0\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + 0\,A\,B_\bullet \end{cases}$$



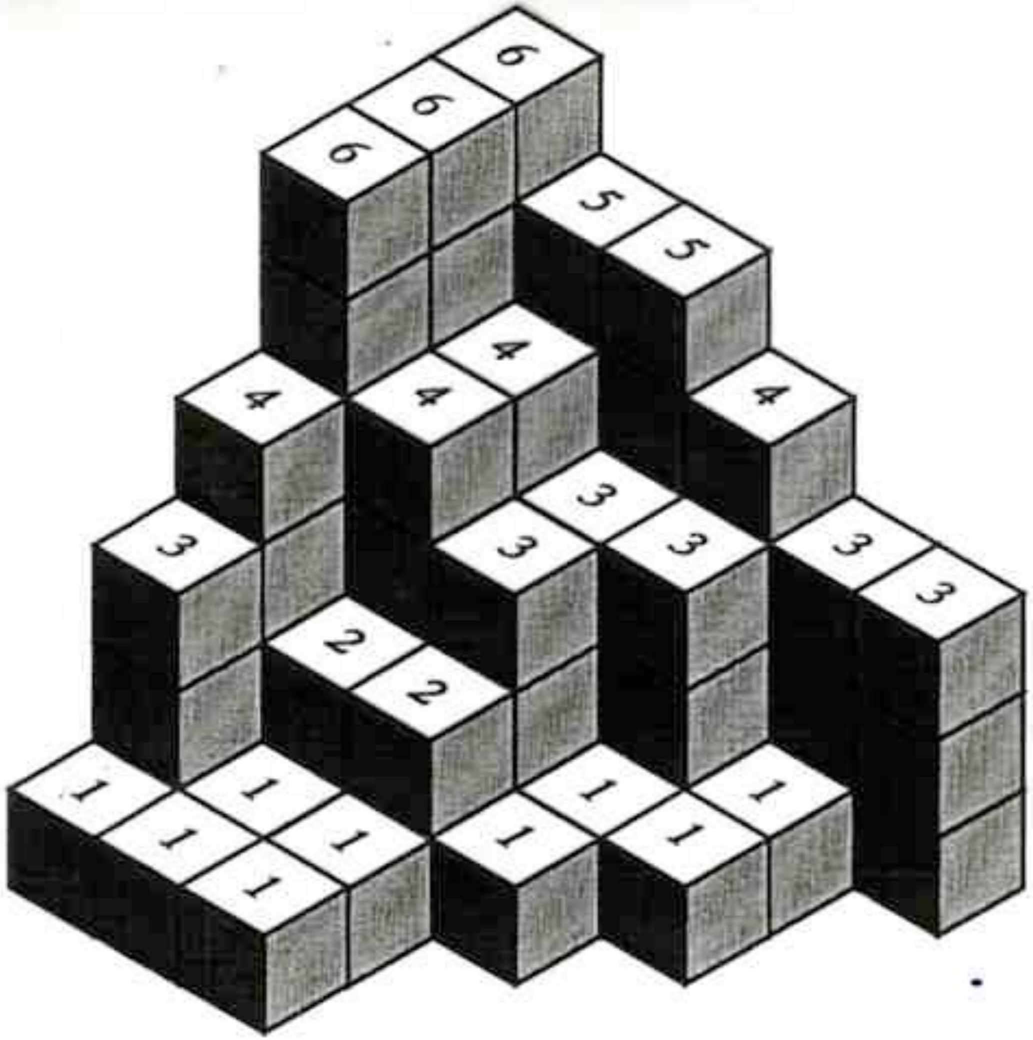




$$w = B^{c} B^{a} A^{b} A^{c}$$

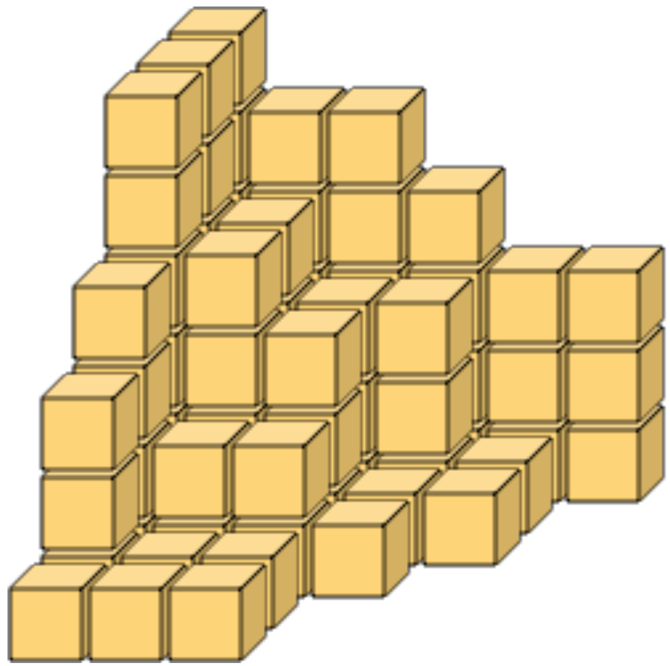
$$u = A^{c}_{\bullet}\ A^{b}$$
$$v = B^{a}\ B^{c}_{\bullet}$$

$$c\,(u, v; w)$$

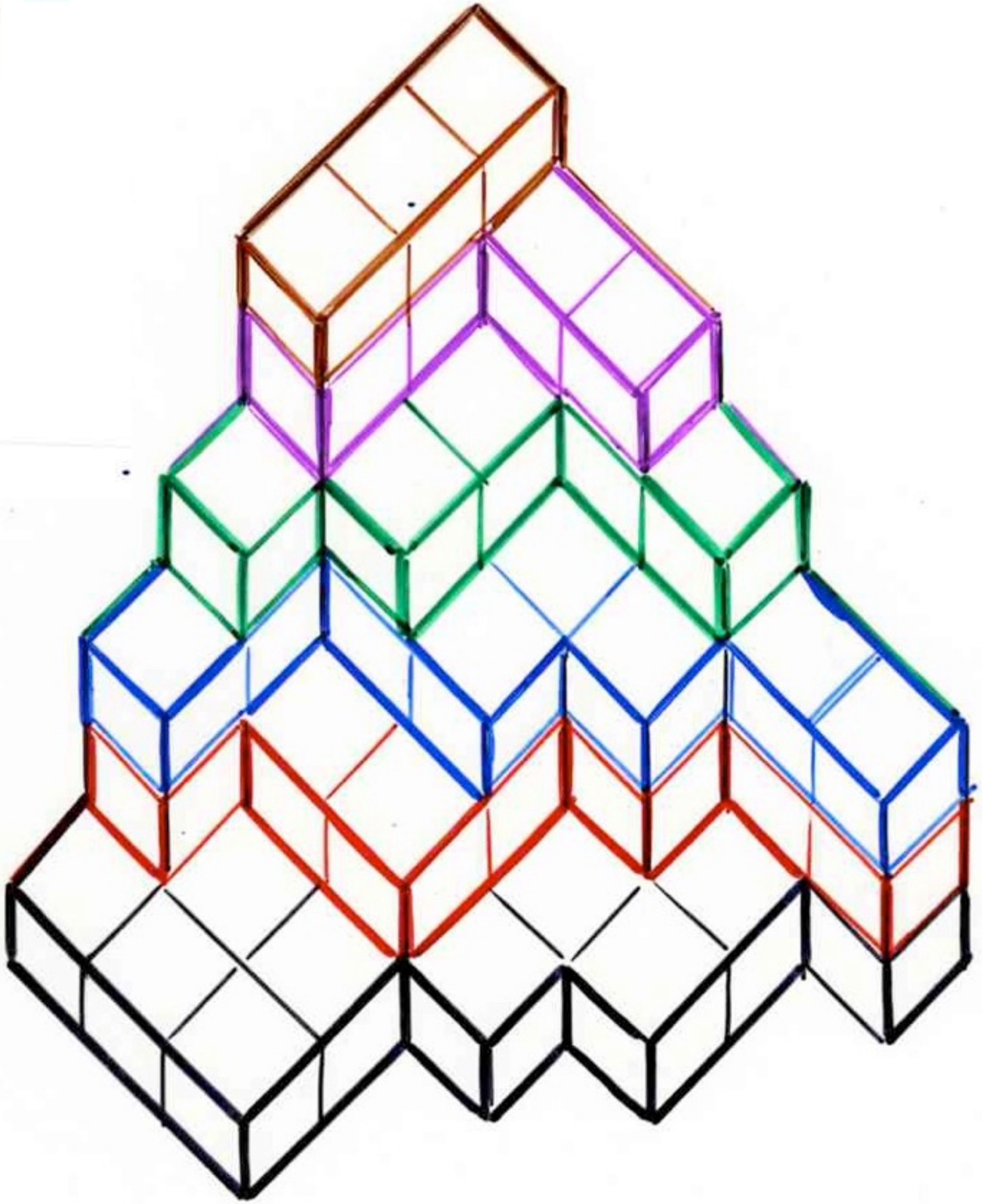


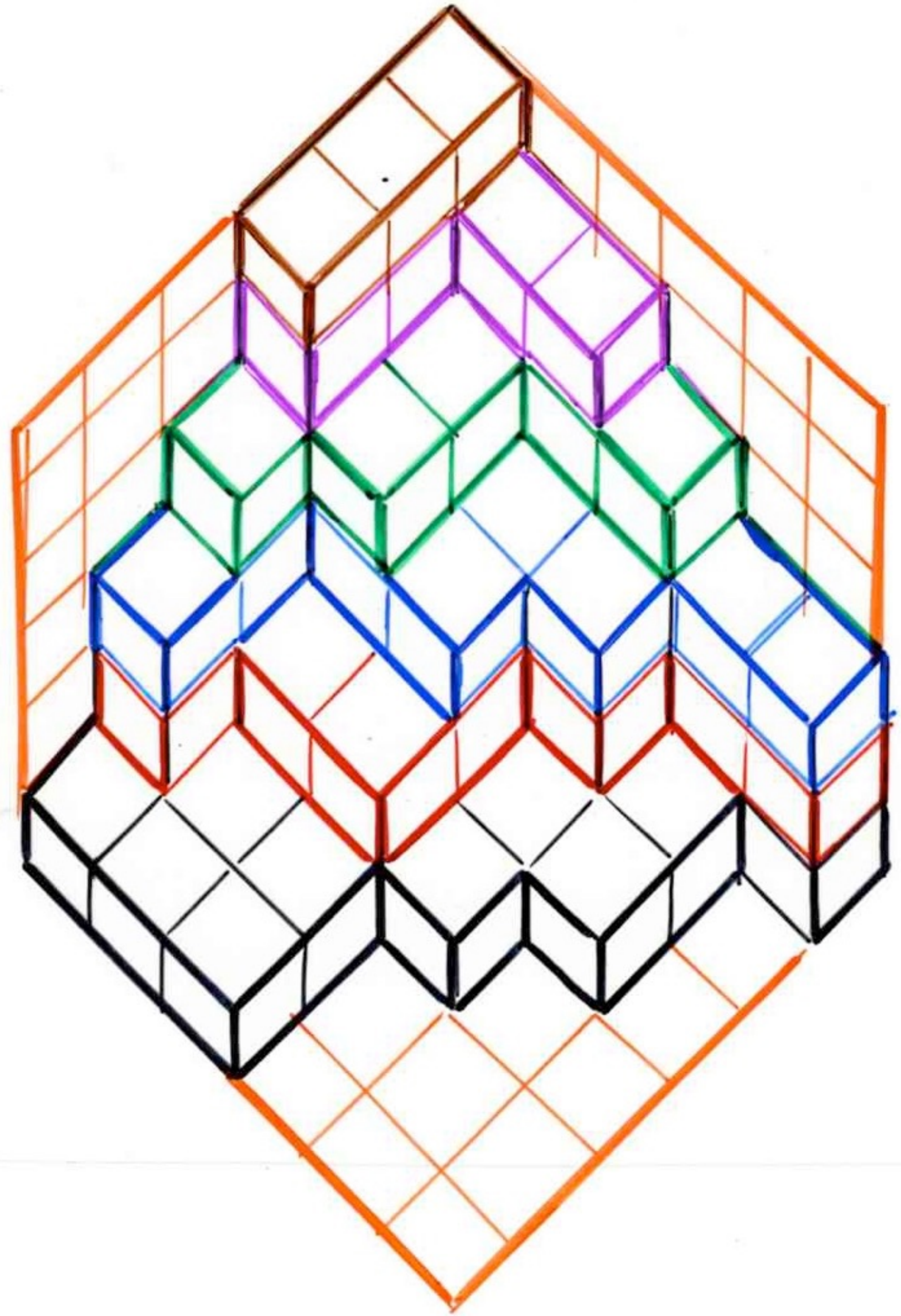
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

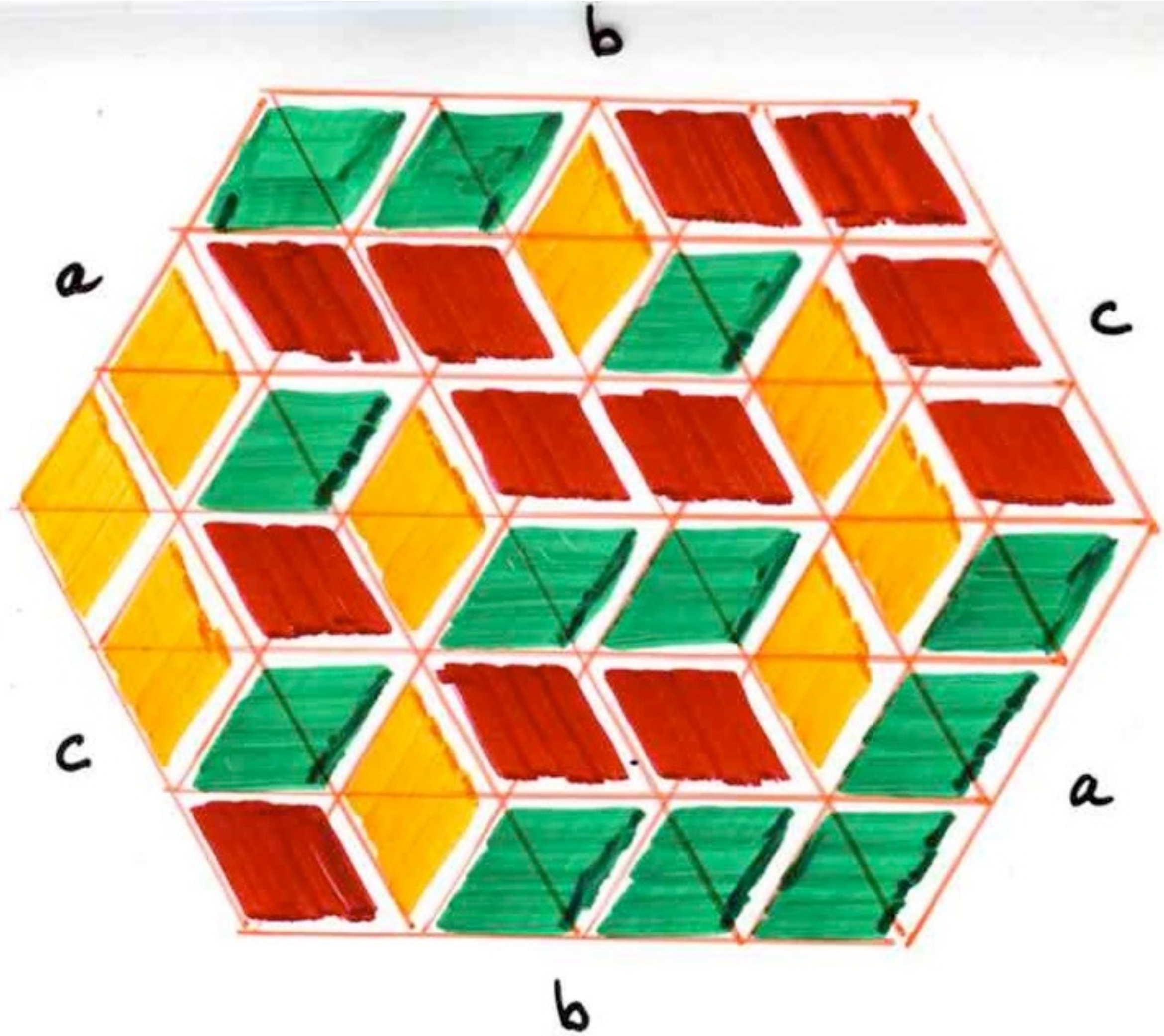
plane partitions



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			






b
a
c
c
a
b

The number of

plane partitions in a box (a, b, c)

$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

MacMahon formula

Proof of MacMahon formula, see:
BJC 1, Ch 5a, p105

number of plane partitions in a box $a \times b \times c$

$$\prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{i+j+k-1}{i+j+k-2} = c(u,v;w)$$

$$w = B^c B^a A^b A^c$$

$$u = A_\bullet^c \, A^b$$

$$v = B^a \, B_\bullet^c$$

$$BA = AB + A_\bullet B_\bullet$$

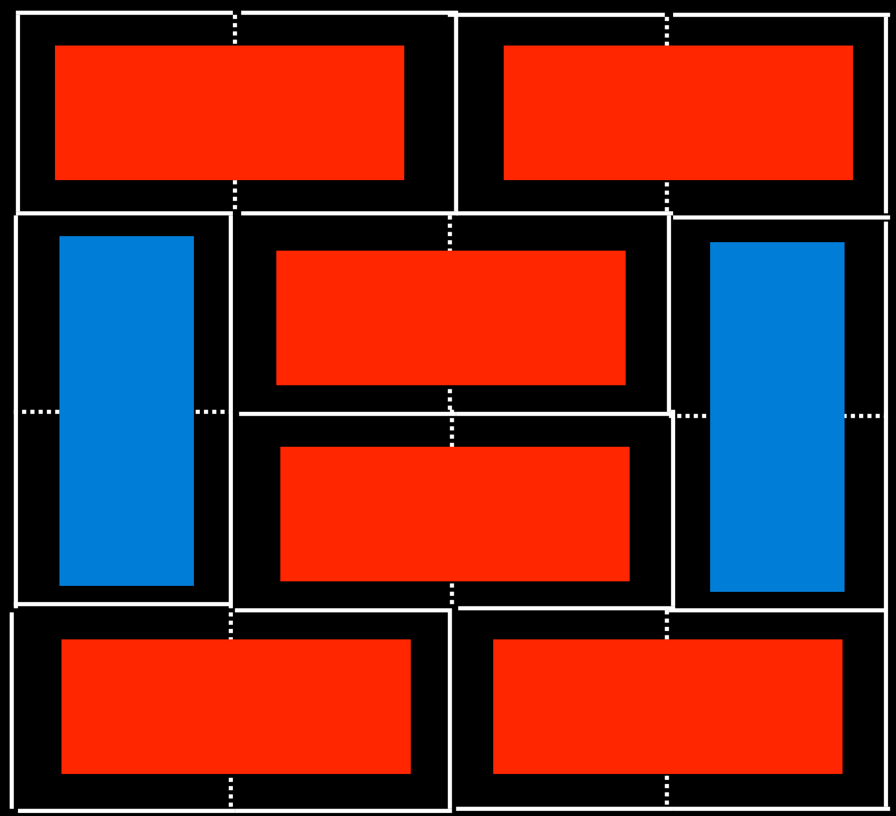
$$B_\bullet A_\bullet = AB$$

$$B A_\bullet = A_\bullet B$$

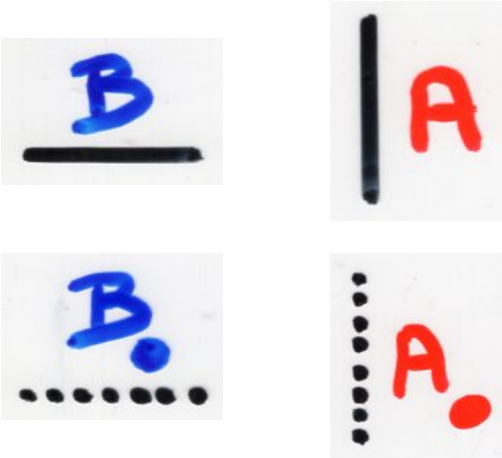
$$B_\bullet A = A B_\bullet$$

dimers tiling
on a square lattice

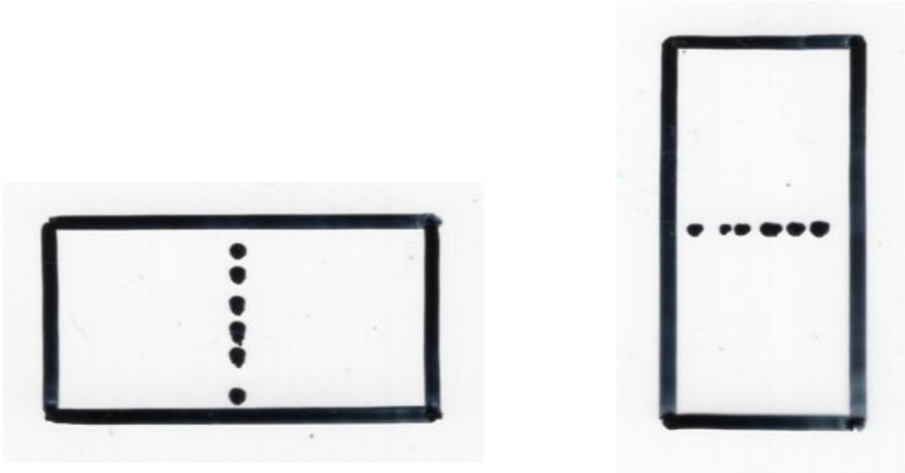
a tiling (of a rectangle) on the square lattice



coding of the edges for tilings on the square lattice



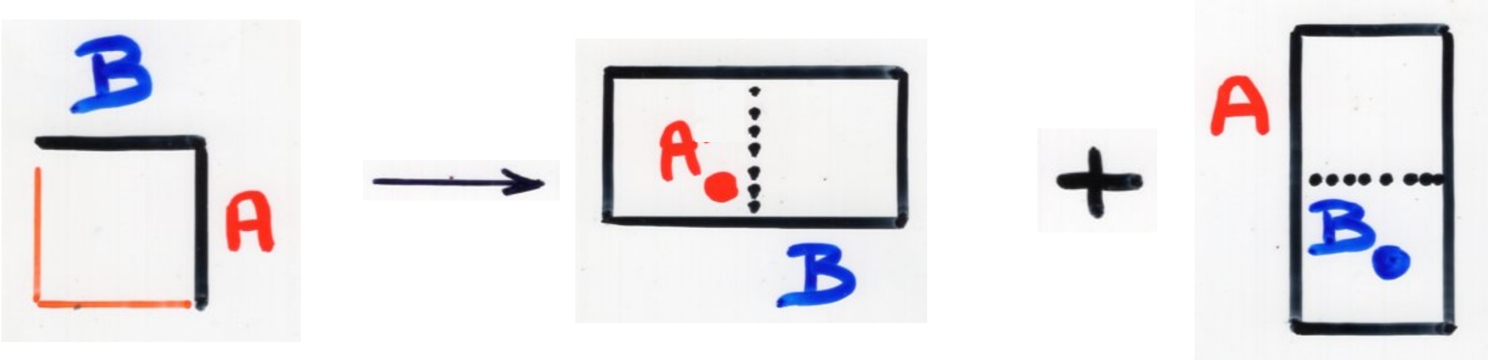
2 types of tiles



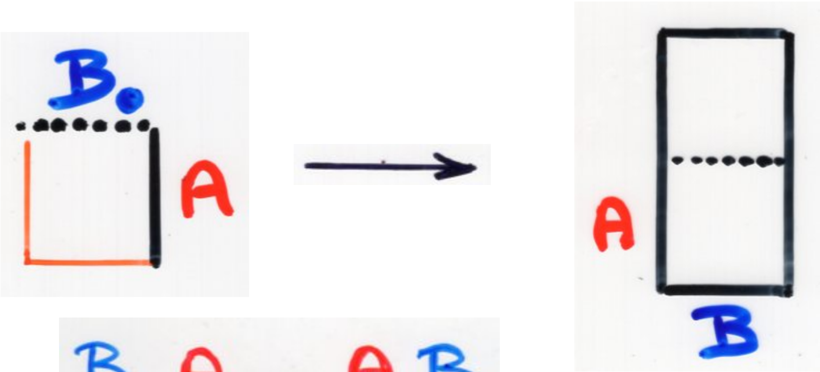
border of a tile

inside a tile

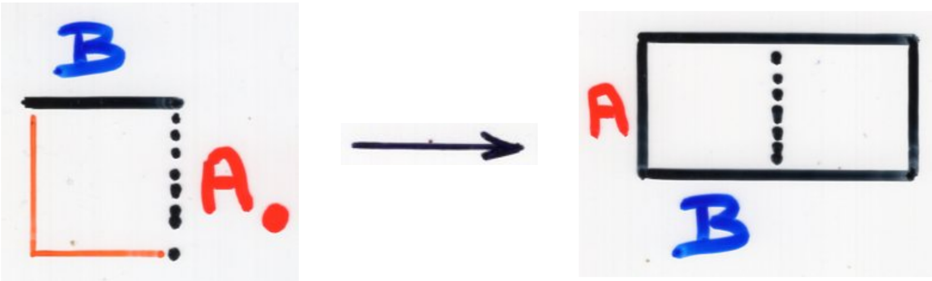
rewriting rules for tilings (square lattice)



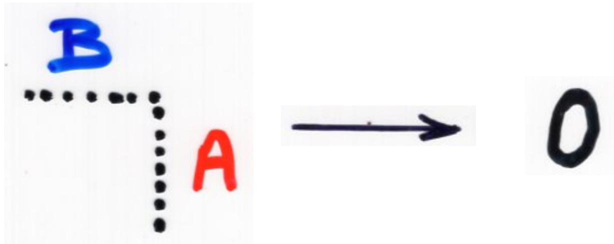
$$B\,A = A_{\bullet}\,B + A\,B_{\bullet}$$



$$B_{\bullet}\,A = A\,B$$



$$B\,A_{\bullet} = A\,B$$



$$B_{\bullet}\,A_{\bullet} = 0$$

$$\begin{cases} B\,A = A_\bullet B + A\,B_\bullet \\ B_\bullet A = A\,B \\ B\,A_\bullet = A\,B \\ B_\bullet A_\bullet = 0 \end{cases}$$

not a subalgebra of the $\mathbb{Z}$-algebra

Correction to the video:

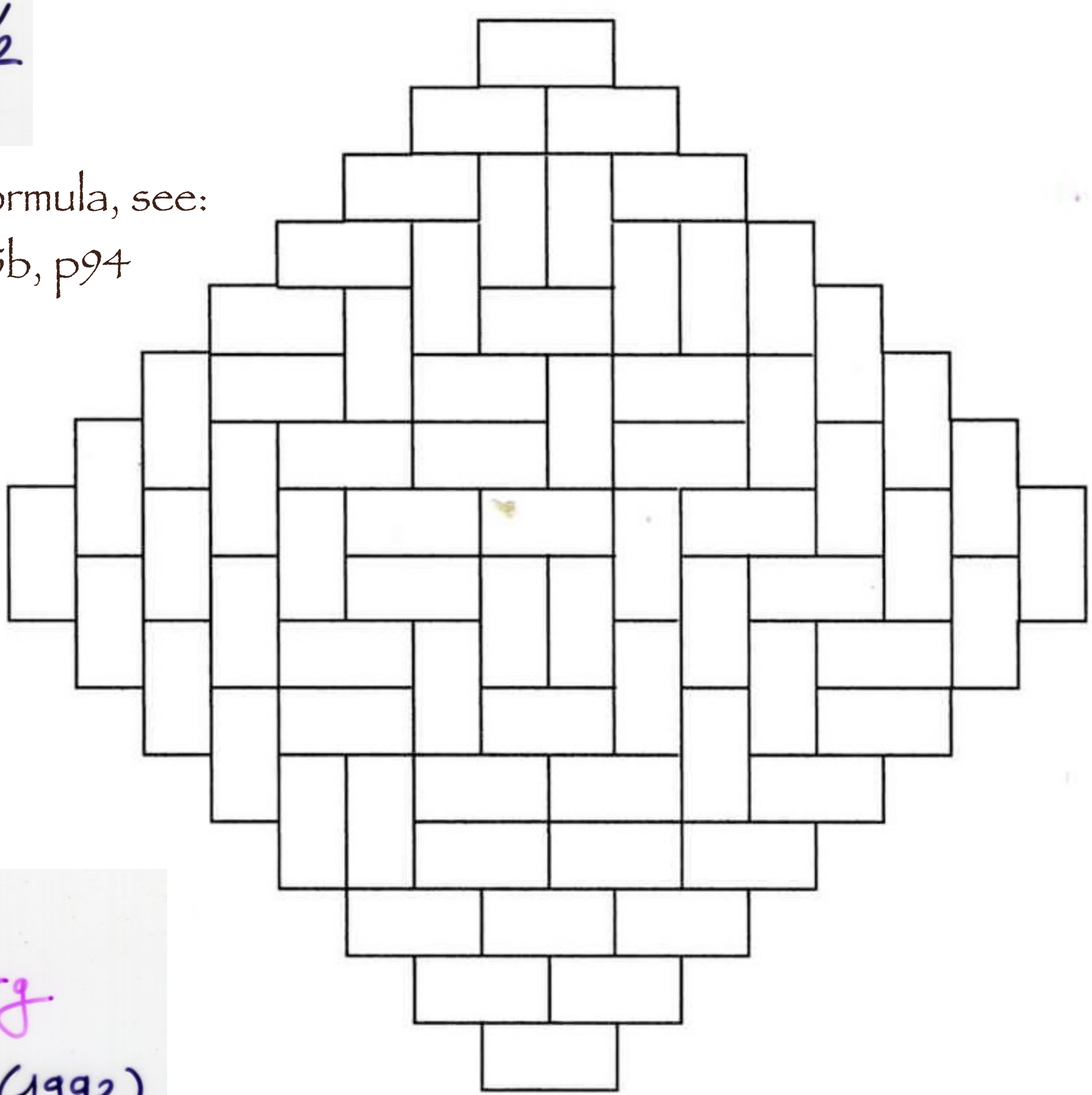
Formula for the number of tilings of an $m \times n$ rectangle, see BJC 1, Ch 5b, p66-67 (without proof)

$$\begin{cases} B\,A = q_{oo}\,A\,B + t_{oo}\,A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\,A_\bullet B_\bullet + \bigcirc\,A\,B \\ B_\bullet A = q_{\bullet o}\,A\,B_\bullet + \bigcirc\,A_\bullet B \\ B\,A_\bullet = q_{o\bullet}\,A_\bullet B + \bigcirc\,A\,B_\bullet \end{cases}$$

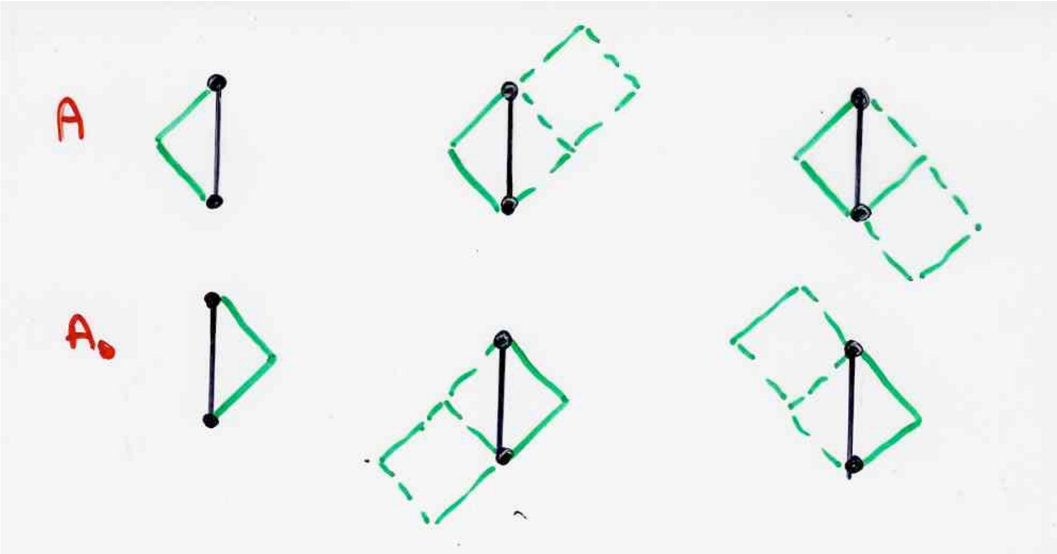
Aztec tilings

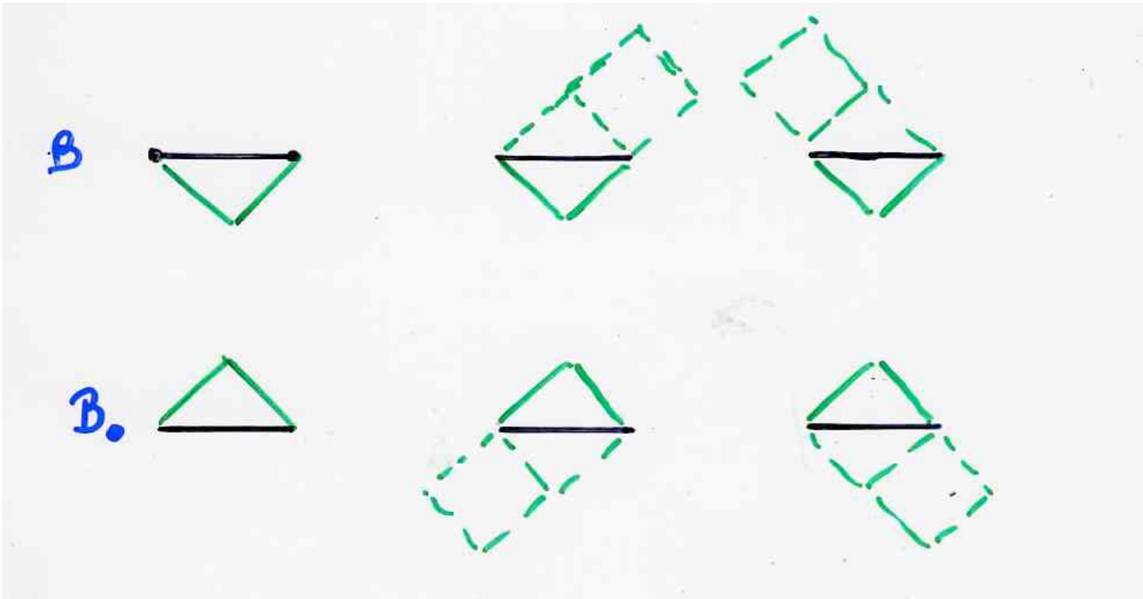
$$2^{n(n-1)/2}$$

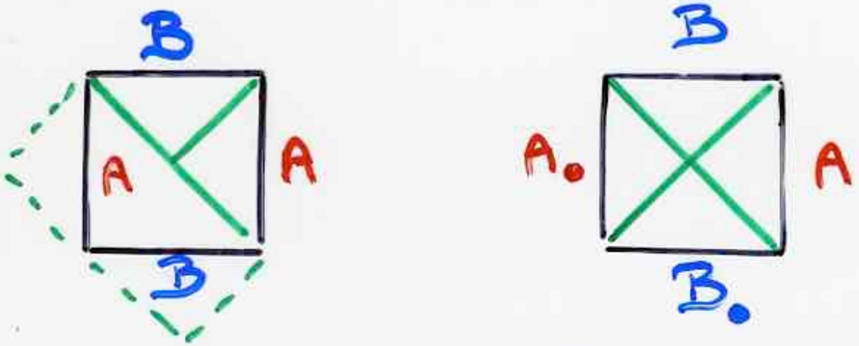
Proof of this formula, see:
BJC 1, Ch 5b, p94



Elkies
Kuperberg
Larsen
Propp (1992)


A
A•

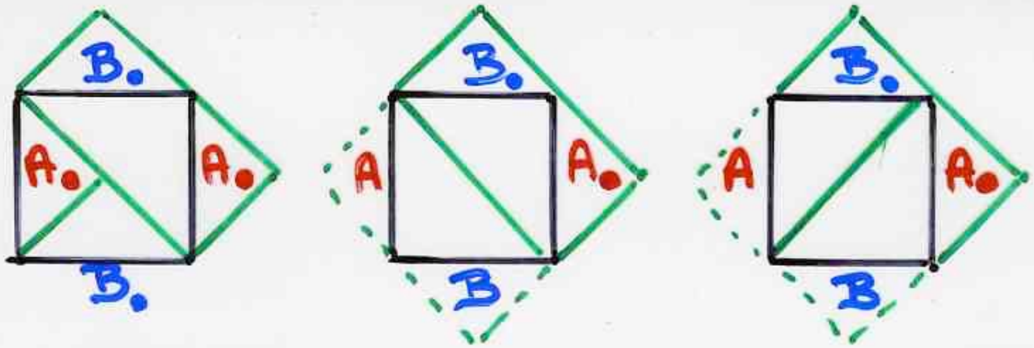

B
B•



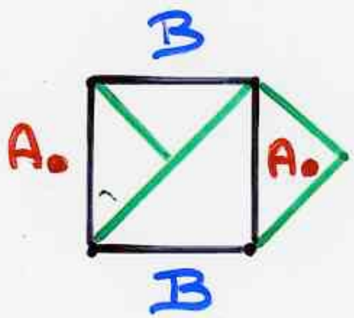
$$BA = AB + A_\bullet B_\bullet$$

rewriting rules for tilings

(Aztec lattice)



$$B_\bullet A_\bullet = A_\bullet B_\bullet + 2\,AB$$



$$B_\bullet A = A B_\bullet \qquad B A_\bullet = A_\bullet B$$

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

$$t_{\bullet\bullet} = 2$$

$A_n(x)$ enumeration of ASM according to the number of (-1)

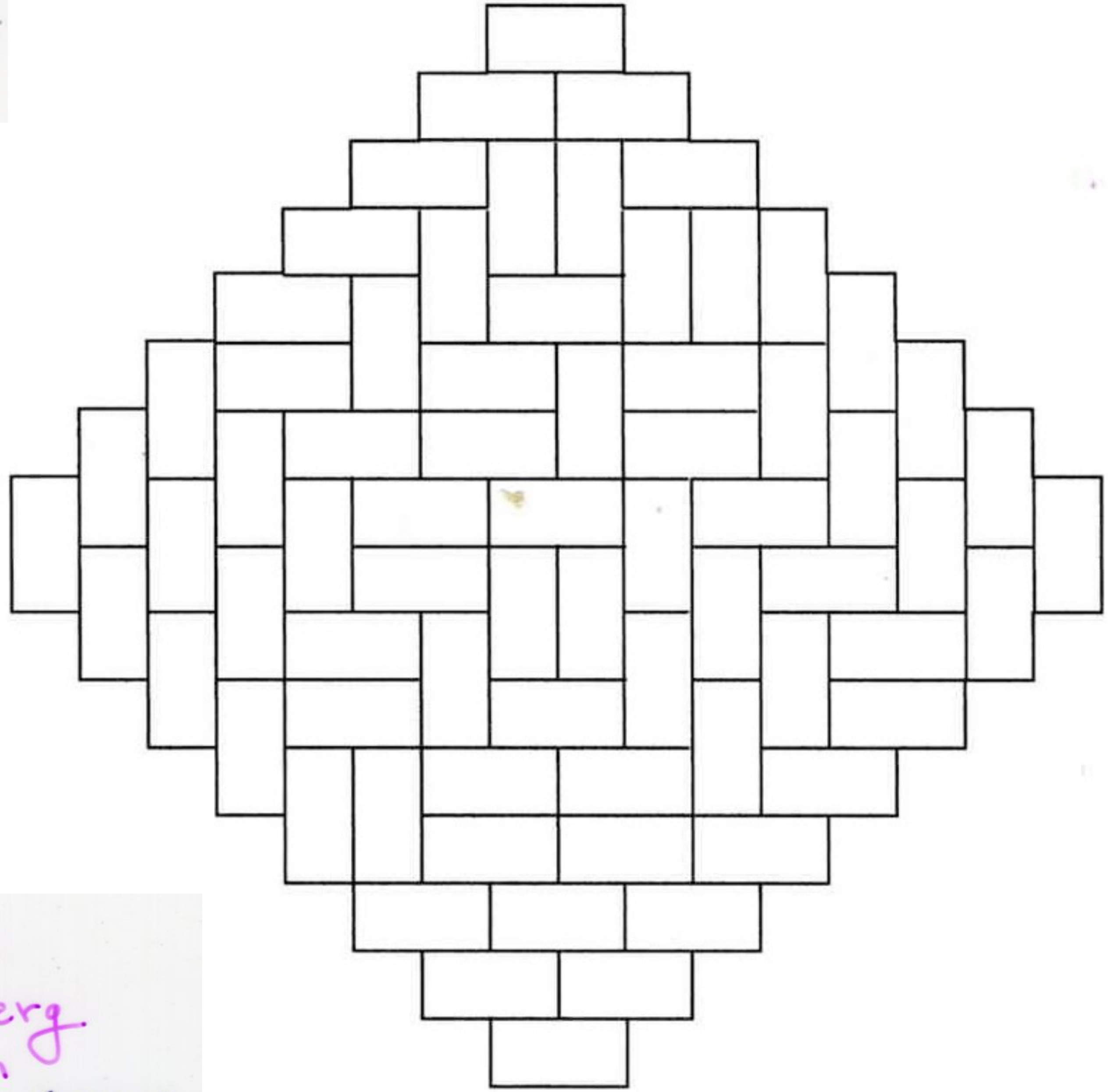
$A_n(2)$

$2^{n(n-1)/2}$

$$\begin{cases} B A = q_{\circ\circ}\, A B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + 2\, A B \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

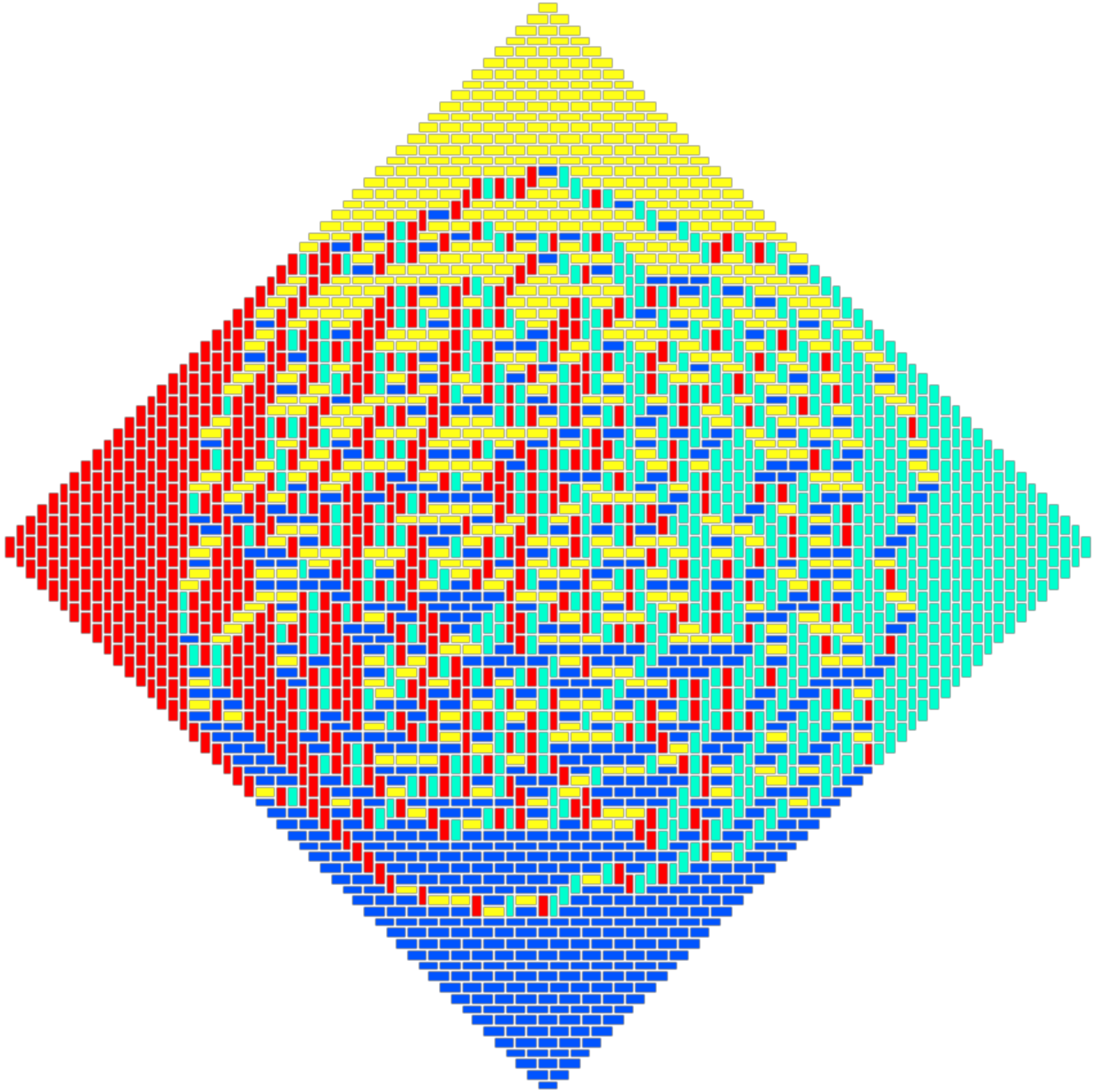
$A_n(2)$

$2^{n(n-1)/2}$

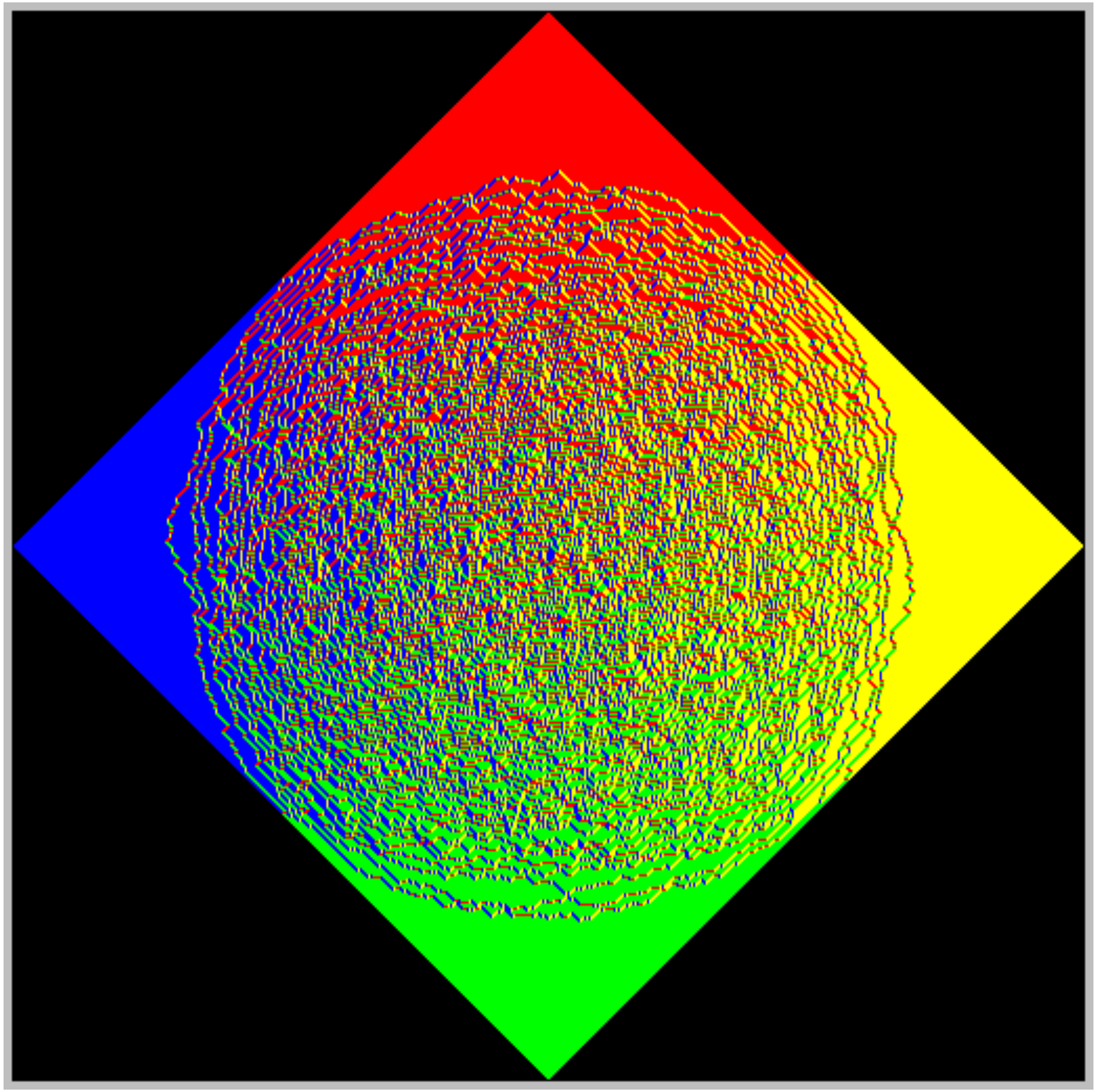


Elkies
Kuperberg
Larsen
Propp (1992)

random
Aztec
tiling



The
"arctic
circle"
theorem

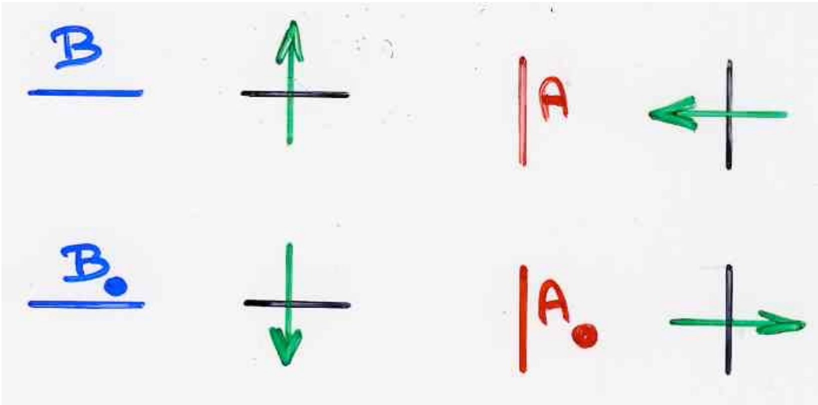


geometric interpretation

of

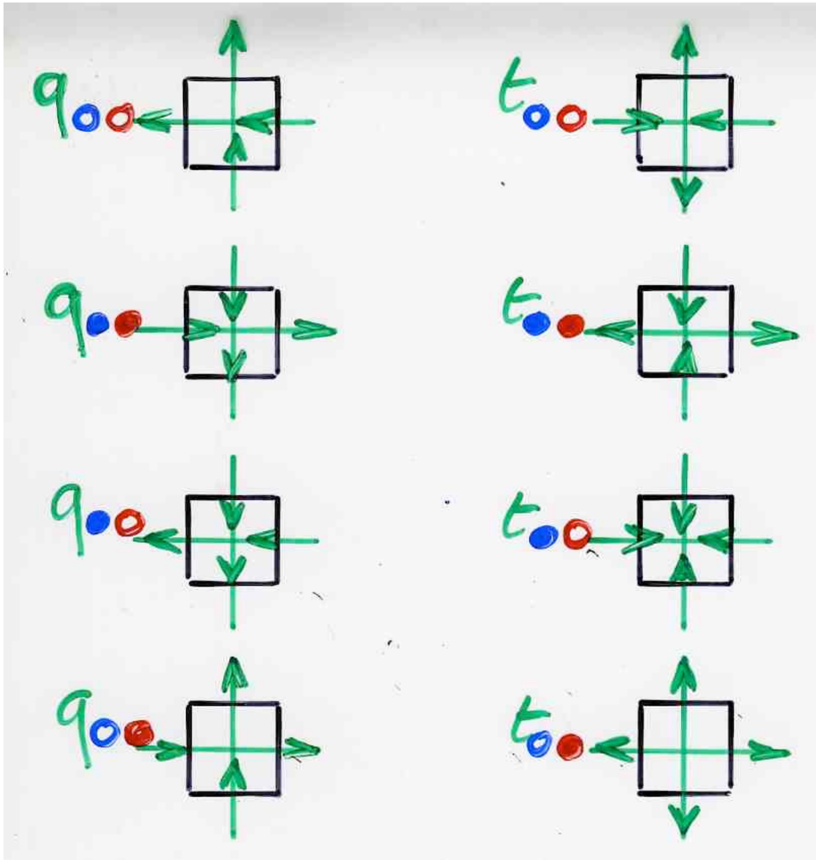
XYZ-tableaux:

6 and 8 vertex models



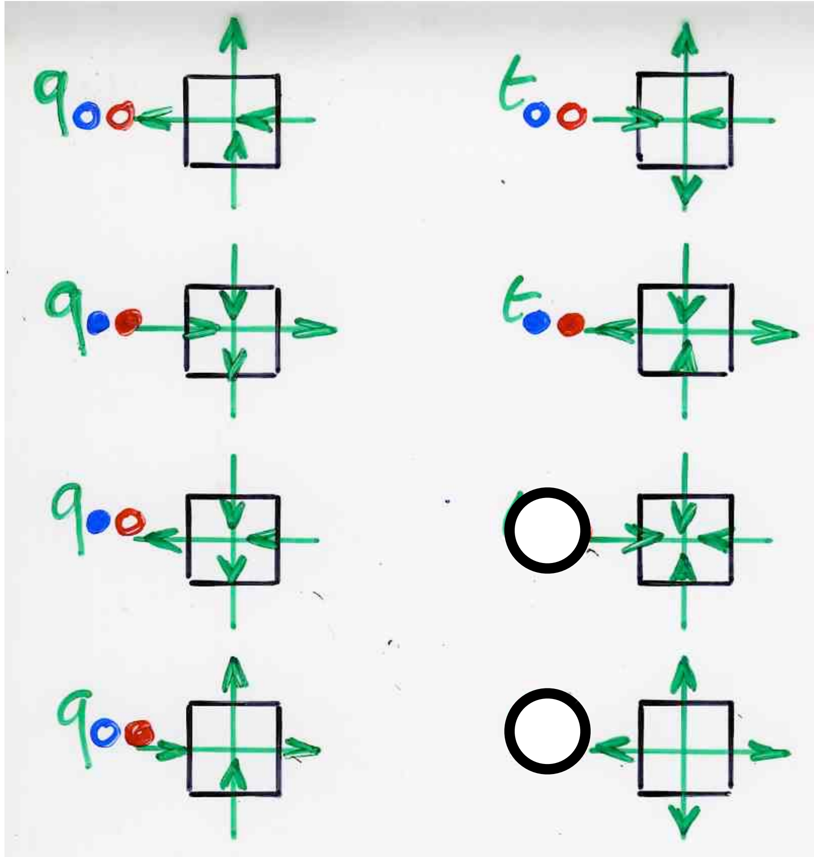
The 8-vertex model

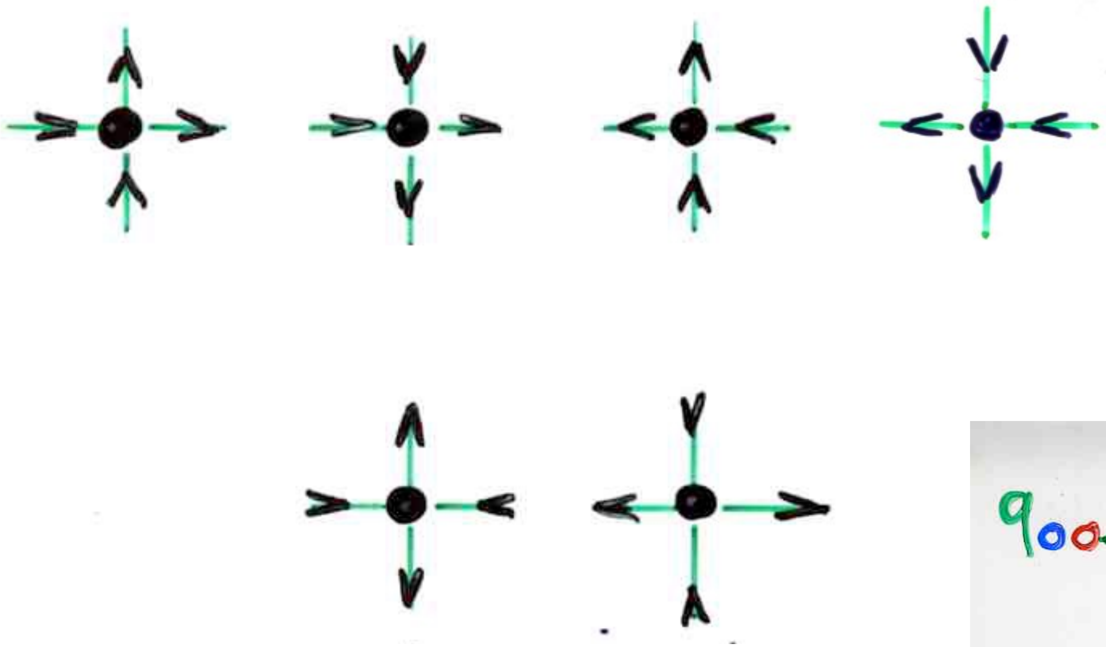
$$\begin{cases} B\,A = q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\ B_\bullet\,A_\bullet = q_{\bullet\bullet}\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\ B_\bullet\,A = q_{\bullet\circ}\,A\,B_\bullet + t_{\bullet\circ}\,A_\bullet\,B \\ B\,A_\bullet = q_{\circ\bullet}\,A_\bullet\,B + t_{\circ\bullet}\,A\,B_\bullet \end{cases}$$



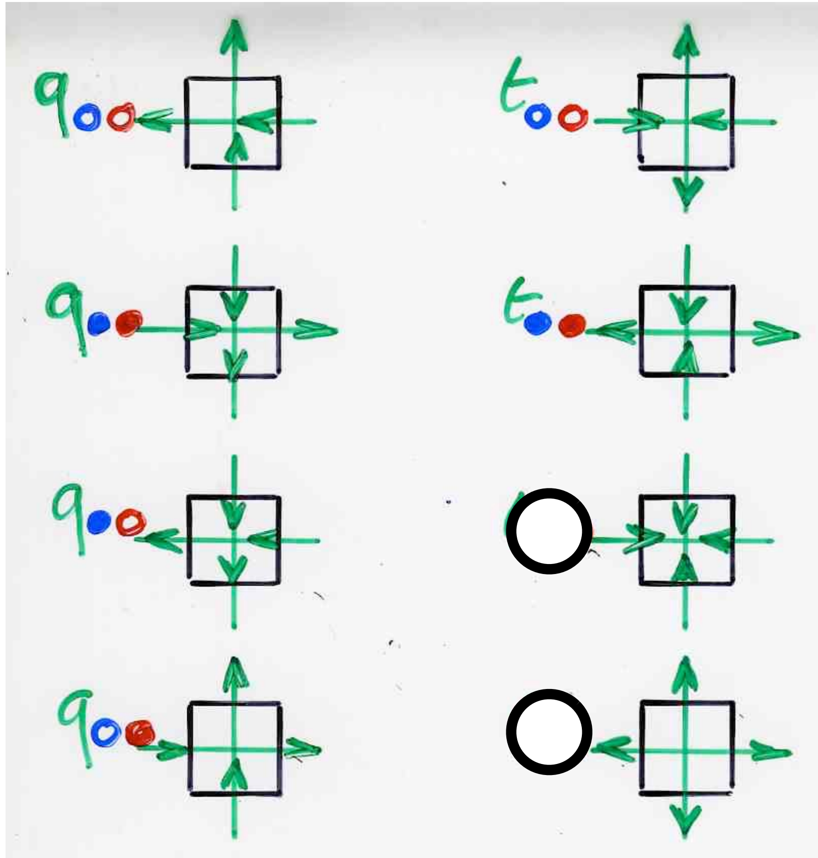
The 6-vertex model

$$
\left\{
\begin{aligned}
B\,A &= q_{\circ\circ}\,A\,B + t_{\circ\circ}\,A_\bullet\,B_\bullet \\
B_\bullet\,A_\bullet &= q_{\bullet\bullet}\,A_\bullet\,B_\bullet + t_{\bullet\bullet}\,A\,B \\
B_\bullet\,A &= q_{\bullet\circ}\,A\,B_\bullet + \bigcirc\,A_\bullet\,B \\
B\,A_\bullet &= q_{\circ\bullet}\,A_\bullet\,B + \bigcirc\,A\,B_\bullet
\end{aligned}
\right.
$$

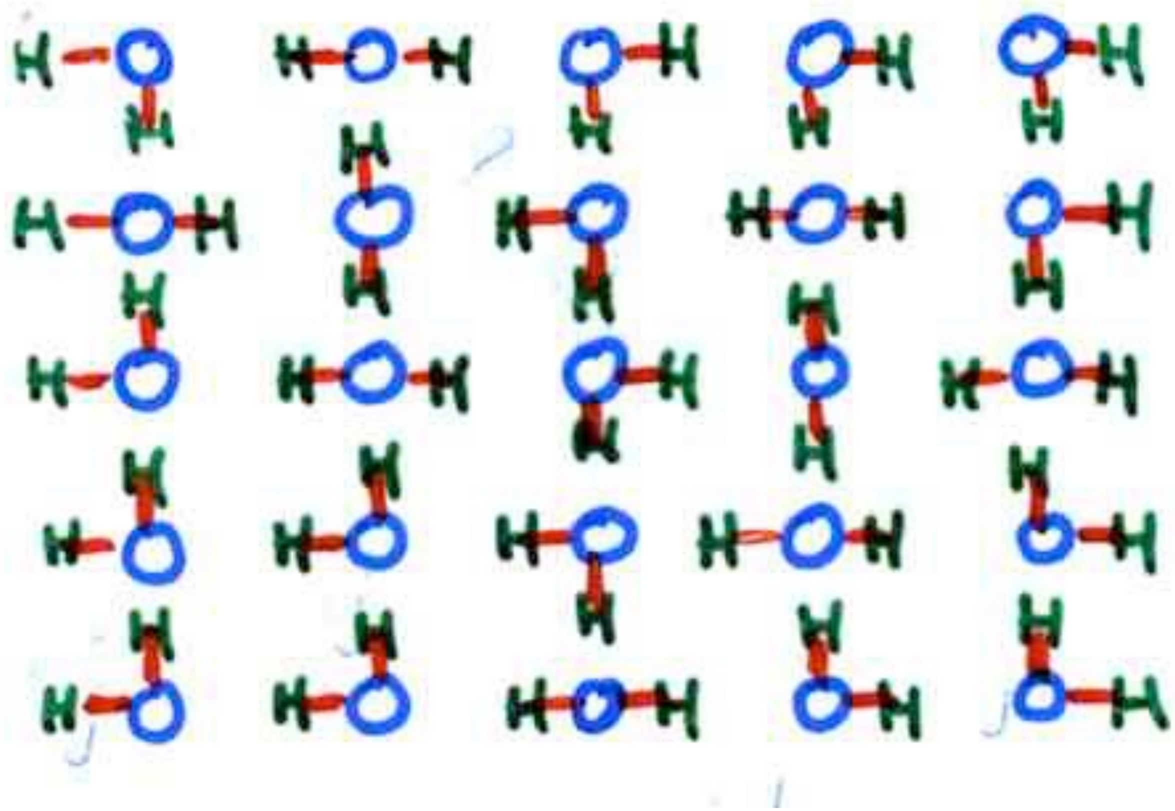




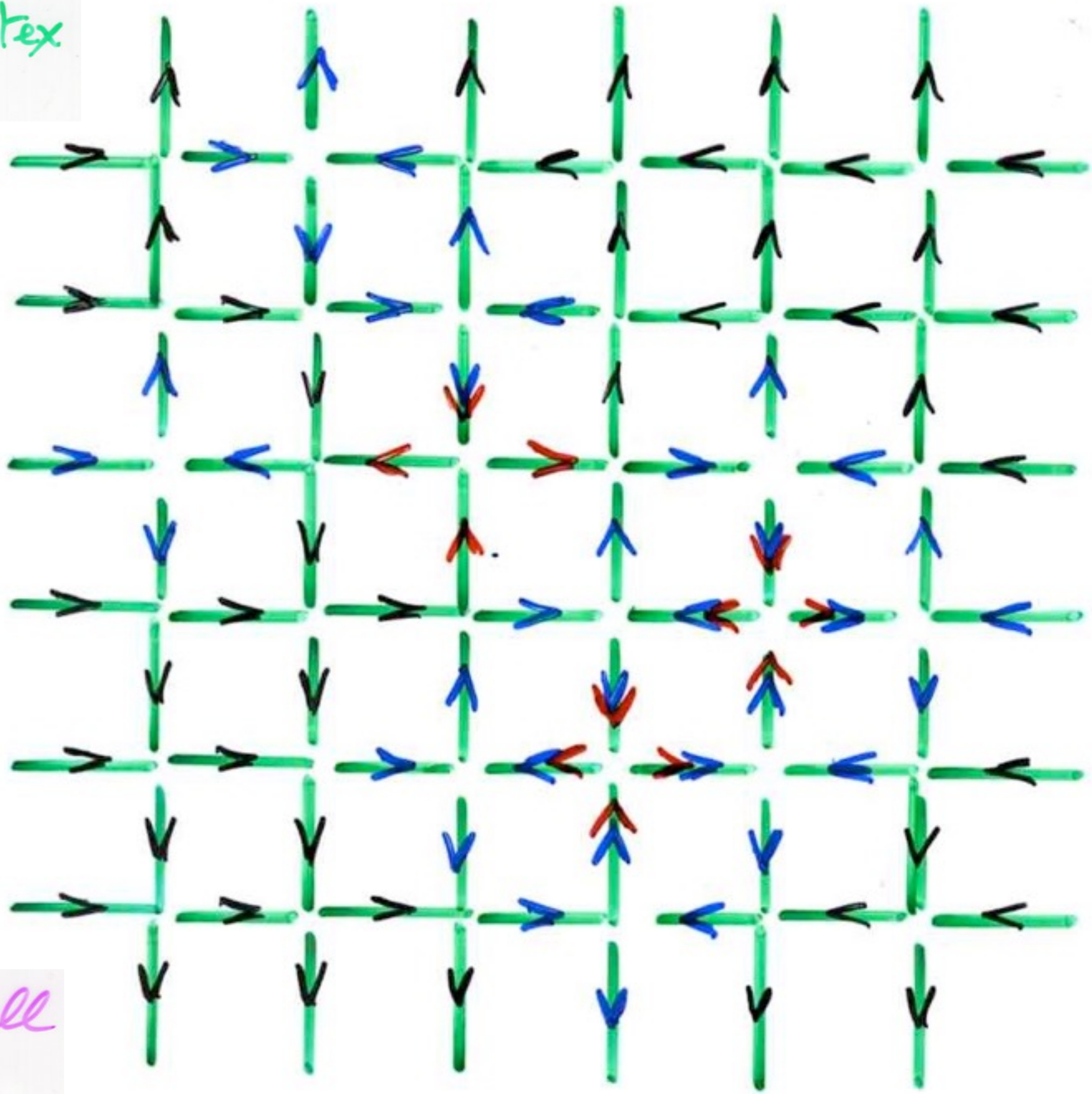
The 6-vertex
model



the ice model



The 6-vertex model

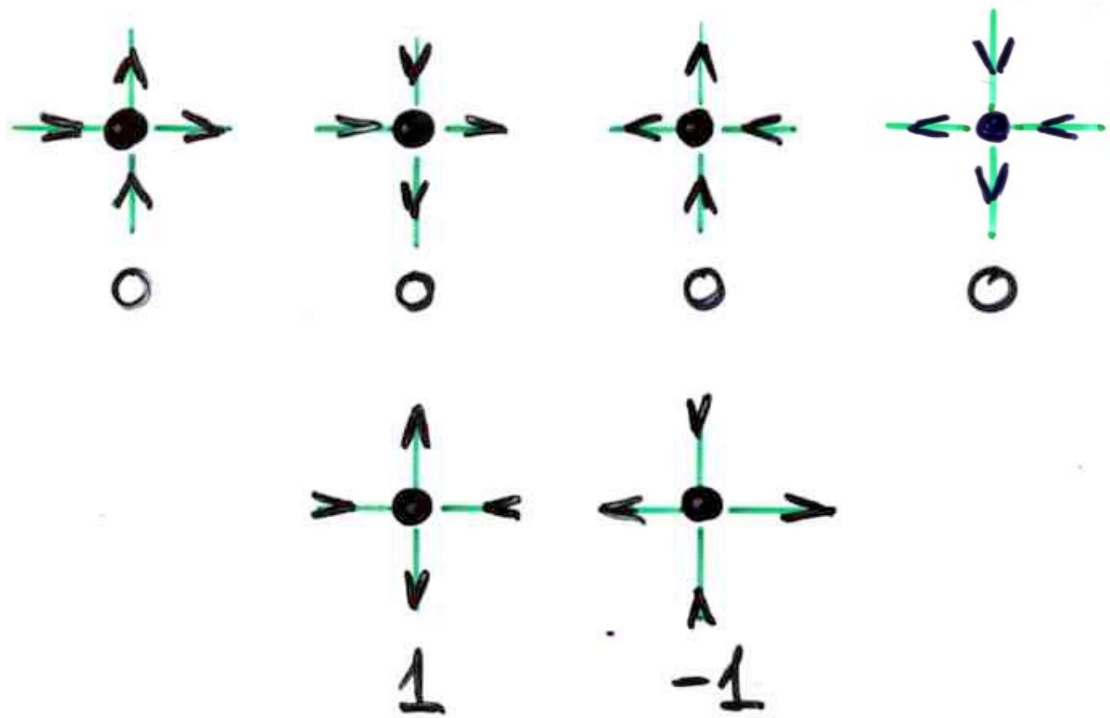


domain wall condition

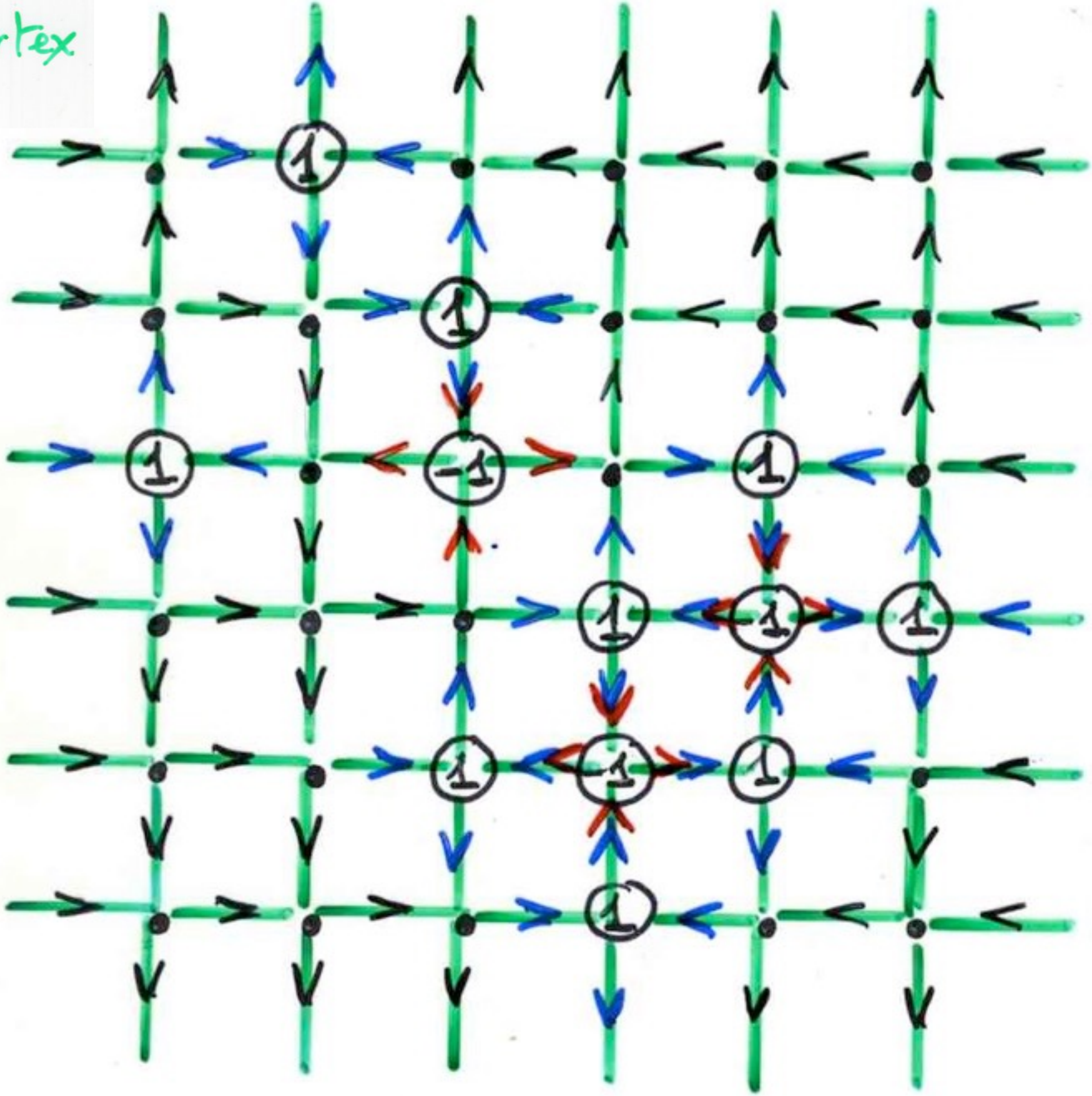
bijection

configurations of the 6-vertex model $\longleftrightarrow$ ASM alternating sign matrices

The 6-vertex model



The 6-vertex model



The 6-vertex model