Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 2

Quadratic algebra, Q-tableaux and planar automata

Ch2b

IMSc, Chennai
February 1, 2018

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"The cellular ansatz"

quadratic algebra Q

UD = DU + Id

Physics

$DE = qED + E + D$

commutations

rewriting rules

planarization

Q-tableaux

combinatorial objects on a 2D lattice

permutations

towers placements

alternative tableaux

ASM
alternating sign matrices

"planar automata"

representation of Q by combinatorial operators

bijections

RSK

pairs of Young tableaux

Reminding Ch 2a

quadratic algebra $\mathbf{Q}$

generators

$$\mathcal{B} = \{B_j\}_{j \in J}$$

$$\mathcal{A} = \{A_i\}_{i \in I}$$

commutations

$$B_j A_i = \sum_{k\,l} c_{ij}^{kl} A_k B_l$$

Lemma In $\mathbf{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

$$c(u, v; w) = \sum_{T} wgt(T)$$

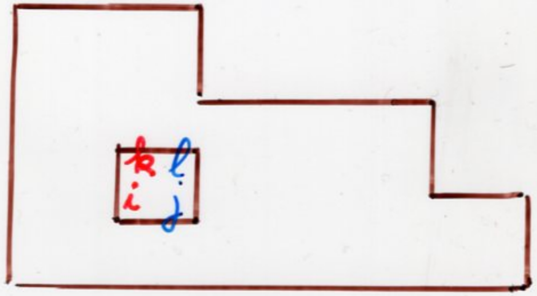
complete $\mathbf{Q}$-tableau

$uwb(T) = w$

$lwb(T) = uv$

Definition complete Q-tableau

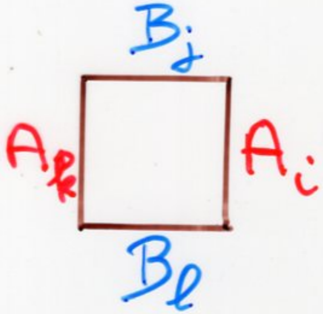
Ferrers diagram F where each cell is labeled by the set R of rewriting rules with "compatibility" condition



$$R = \left\{ \boxed{\begin{matrix} k & l \\ i & j \end{matrix}},\ i, k \in I,\ j, l \in J \right\}$$

$$B_j A_i \rightarrow c_{i\,j}^{k\,l} A_k B_l$$

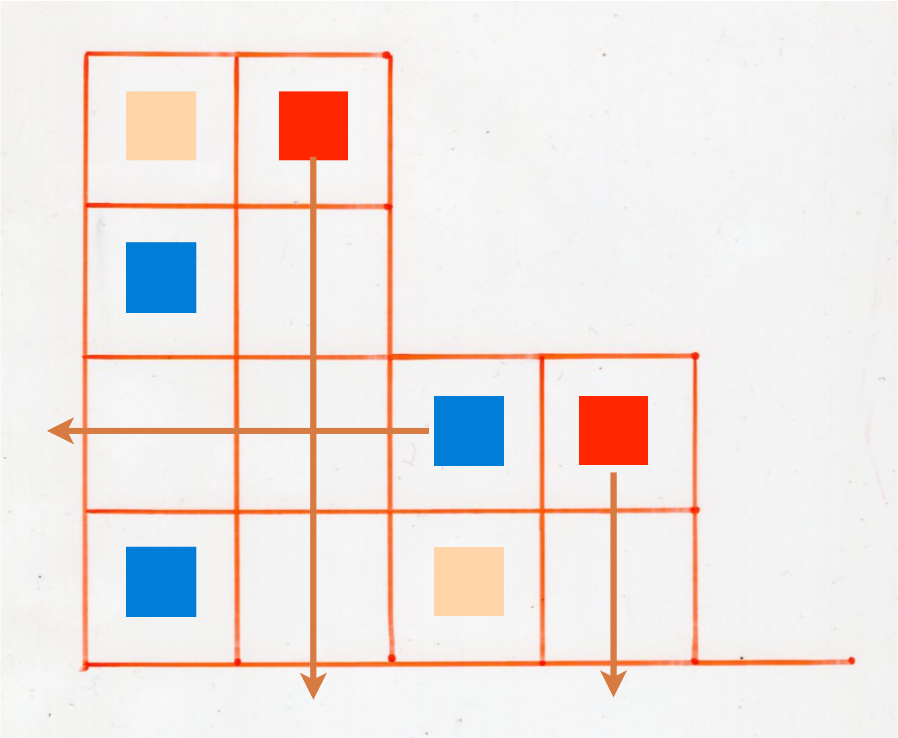
or



The PASEP algebra

$$DE = qED + E + D$$

$$DE = qED + EI_h + I_vD$$
$$DI_v = I_vD$$
$$I_hE = EI_h$$
$$I_hI_v = I_vI_h$$



The PASEP algebra

$$DE = qED + E + D$$

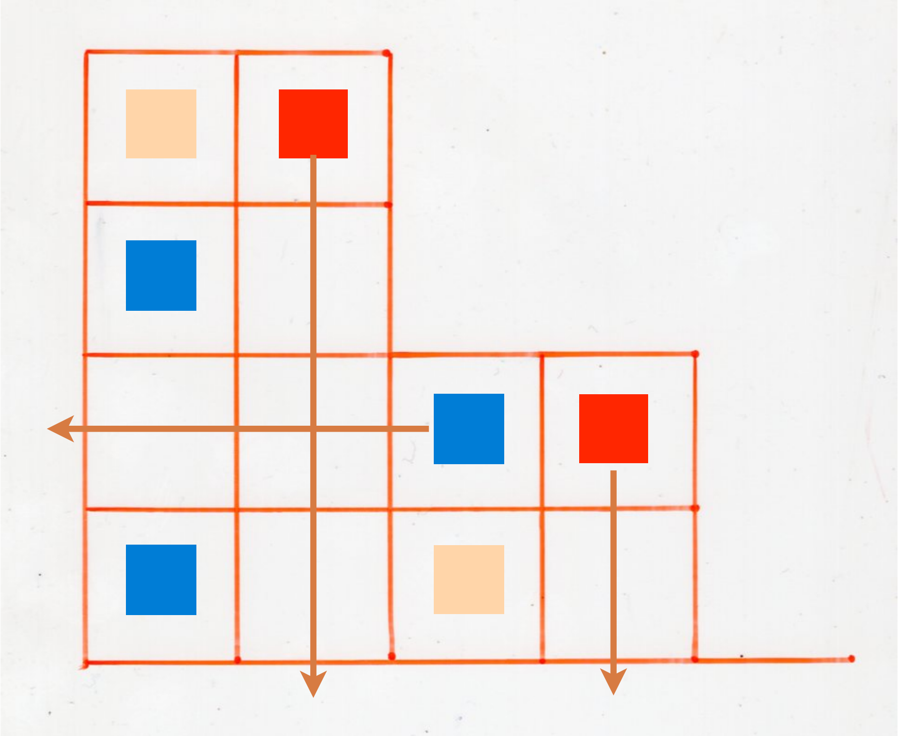
$$DE = \underset{q}{\square}ED + \underset{\blacksquare}{E}I_h + I_v\underset{\blacksquare}{D}$$

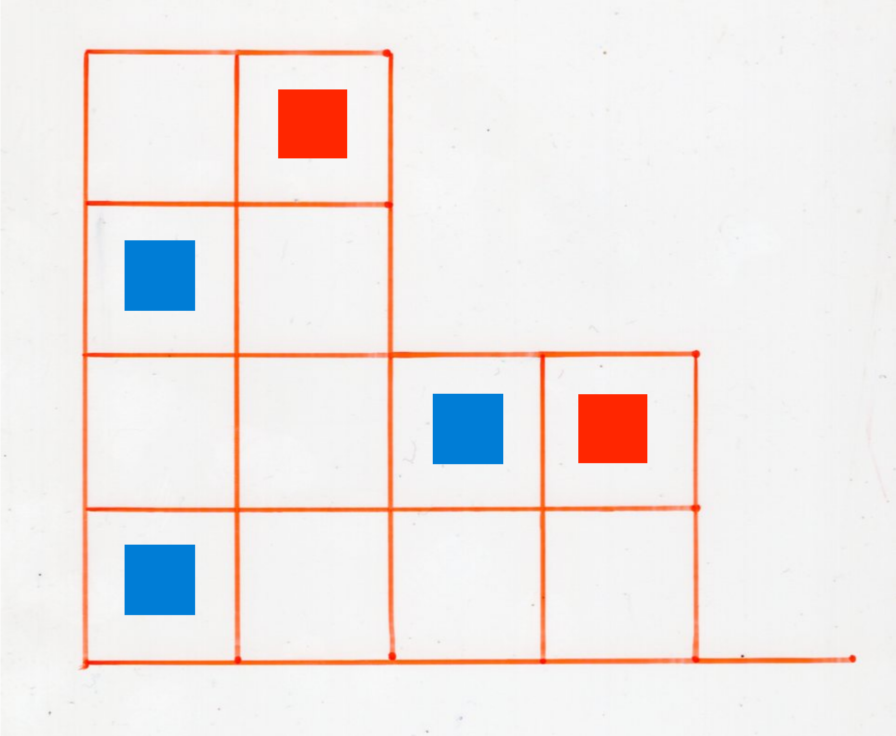
$$DI_v = \square I_v D$$

$$I_h E = \square E I_h$$

$$I_h I_v = \square I_v I_h$$

Q-tableaux





The PASEP algebra

$$DE = qED + E + D$$

$$DE = \underset{q}{\square}ED + \underset{\color{red}\blacksquare}{E}I_h + I_v\underset{\color{blue}\blacksquare}{D}$$

$$DI_v = \square I_v D$$

$$I_h E = \square E I_h$$

$$I_h I_v = \square I_v I_h$$

L set of "labels"

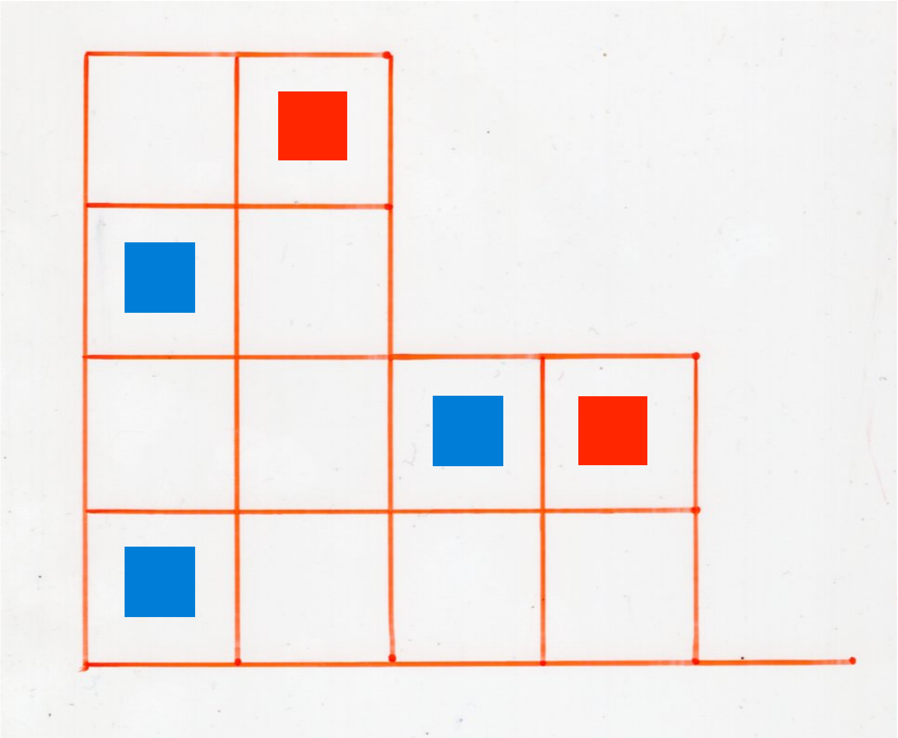
$$\varphi : \left\{ \boxed{\begin{smallmatrix} k & l \\ i & j \end{smallmatrix}} \right\} = R \longrightarrow L$$

set of rewriting rules

$$B_j A_i \longrightarrow c_{ij}^{kl} A_k B_l$$

$(*)$

Q-tableaux



Lemma In Q every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

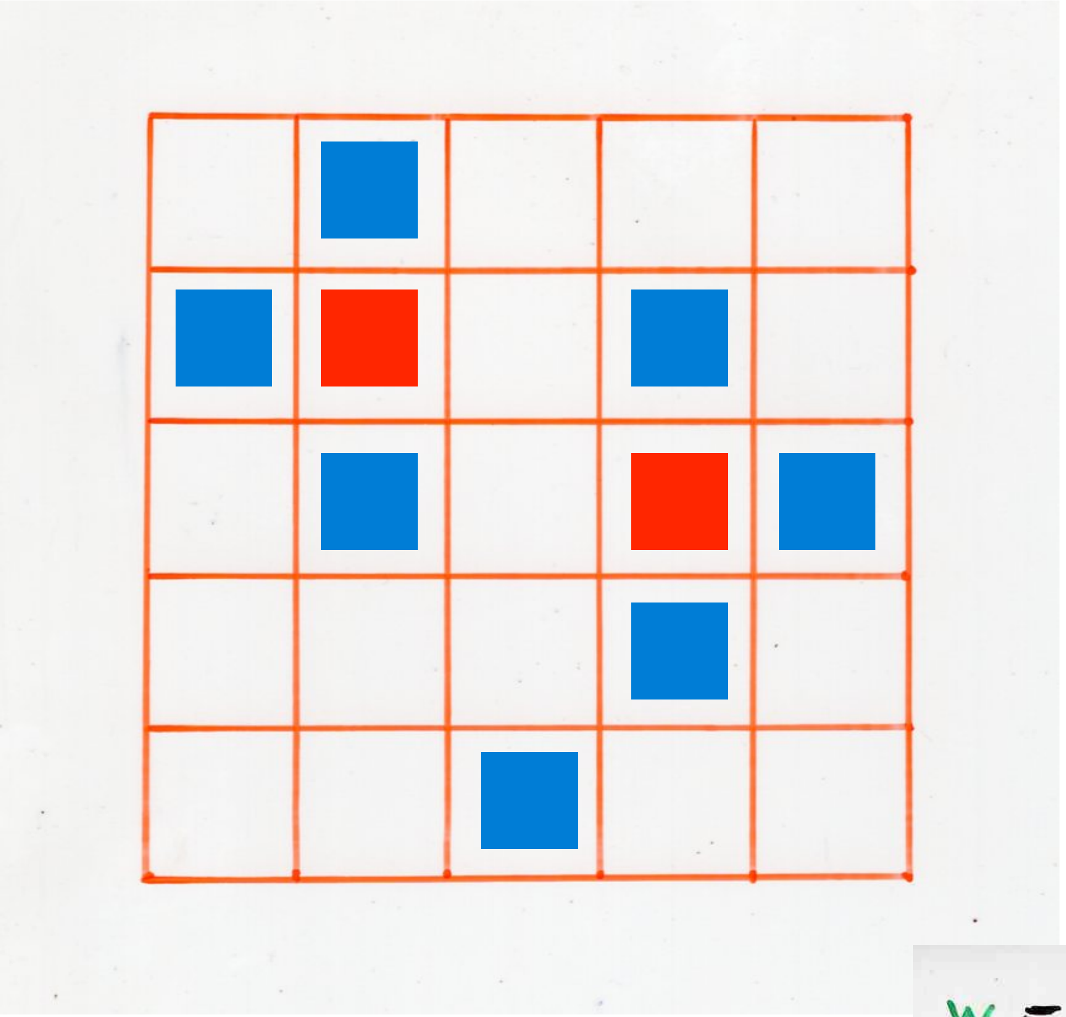
$$c(u, v; w) = \sum_{T} \mathrm{wgt}(T)$$

T complete Q-tableau
$\mathrm{uwb}(T) = w$
$\mathrm{lwb}(T) = u v$

Q-tableaux

Alternating sign matrices

ASM



A , A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

$w = B^n A^n$ $uv = A'^n B'^n$

$c(u,v;w) =$ number of ASM $_{n \times n}$

Planar automaton

Def. **planar automaton** $\mathcal{P}$

- 3 finite sets
  - • $\mathcal{B}$ horizontal alphabet
  - • $\mathcal{A}$ vertical alphabet
  - • L planar labels

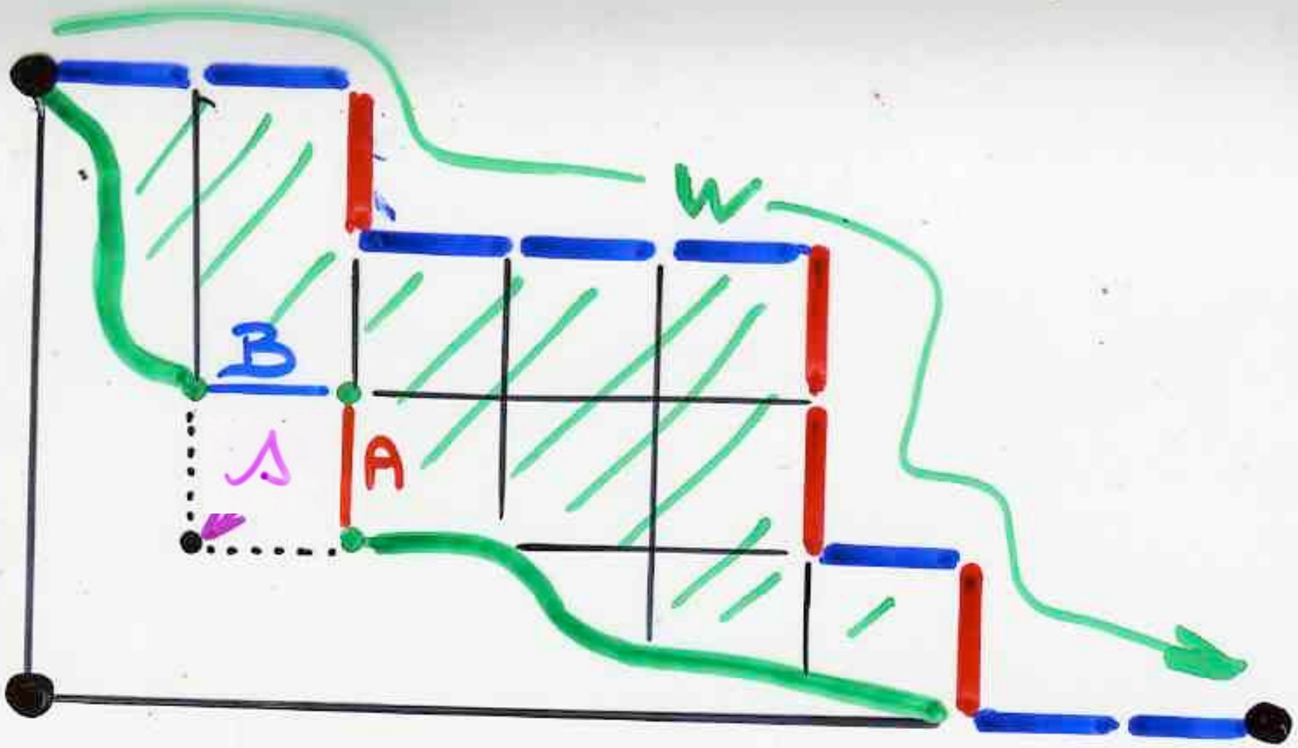
- θ (partial) transition function

  $(s, B, A) \xrightarrow{\theta} (B', A')$ or $\emptyset$

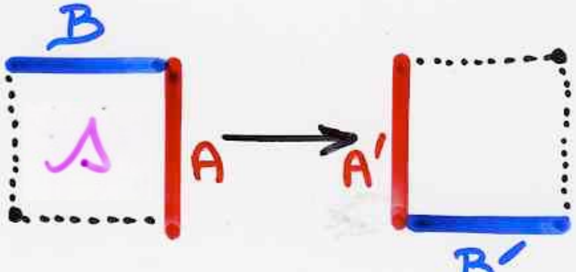
  $s \in L$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word
- uv, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

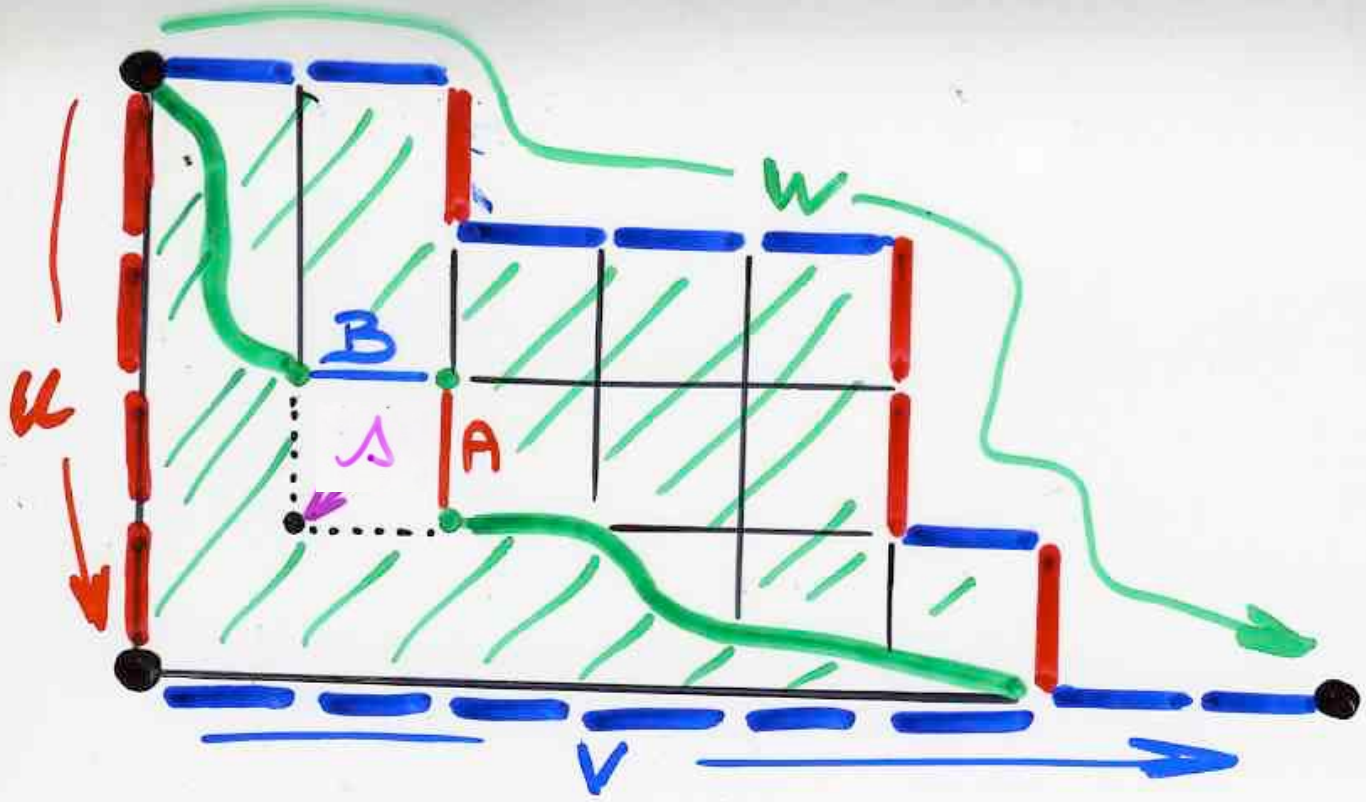
Def. tableau T accepted by a planar automaton $P = (L, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



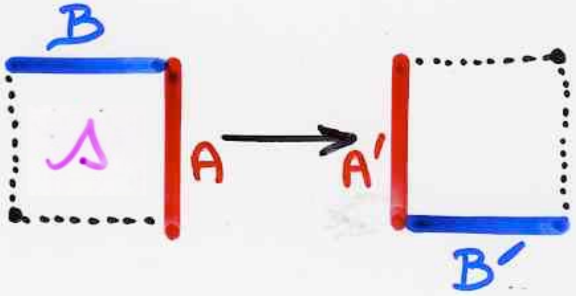
planar rewriting



Def. tableau T accepted by a planar automaton $\mathcal{P} = (\mathcal{L}, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



planar rewriting



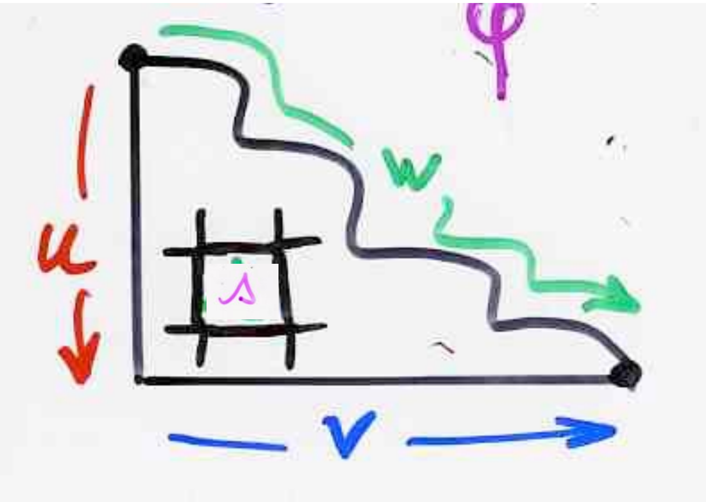
Q- tableaux ↔ tableaux accepted by a planar automaton

Q quadratic algebra

$$\mathcal{P} = (L, \mathcal{B}, \mathcal{A}, \theta, w, uv)$$

with $\mathcal{P}$ satisfying

$$\theta(s, B, A) = \theta(t, B, A) \Rightarrow s = t$$

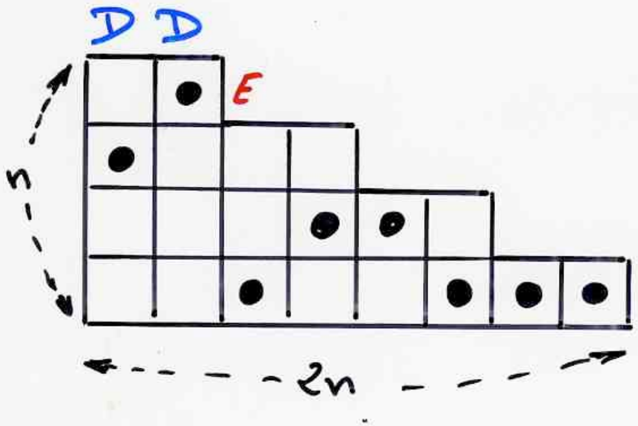


$$BA = \sum_{s \in L} A'B' \qquad (B', A') = \theta(s, B, A)$$

Planar automata

examples

example surjective pistol



Genocchi numbers

G_{2n+2}

- In each column one and only one cell ⊡

- In each row at least one cell ⊡ (surjectivity)

Dumont (1972)

$$G_{2n} = 2(2^{2n}-1)\, B_{2n}$$

Genocchi numbers

Bernoulli numbers

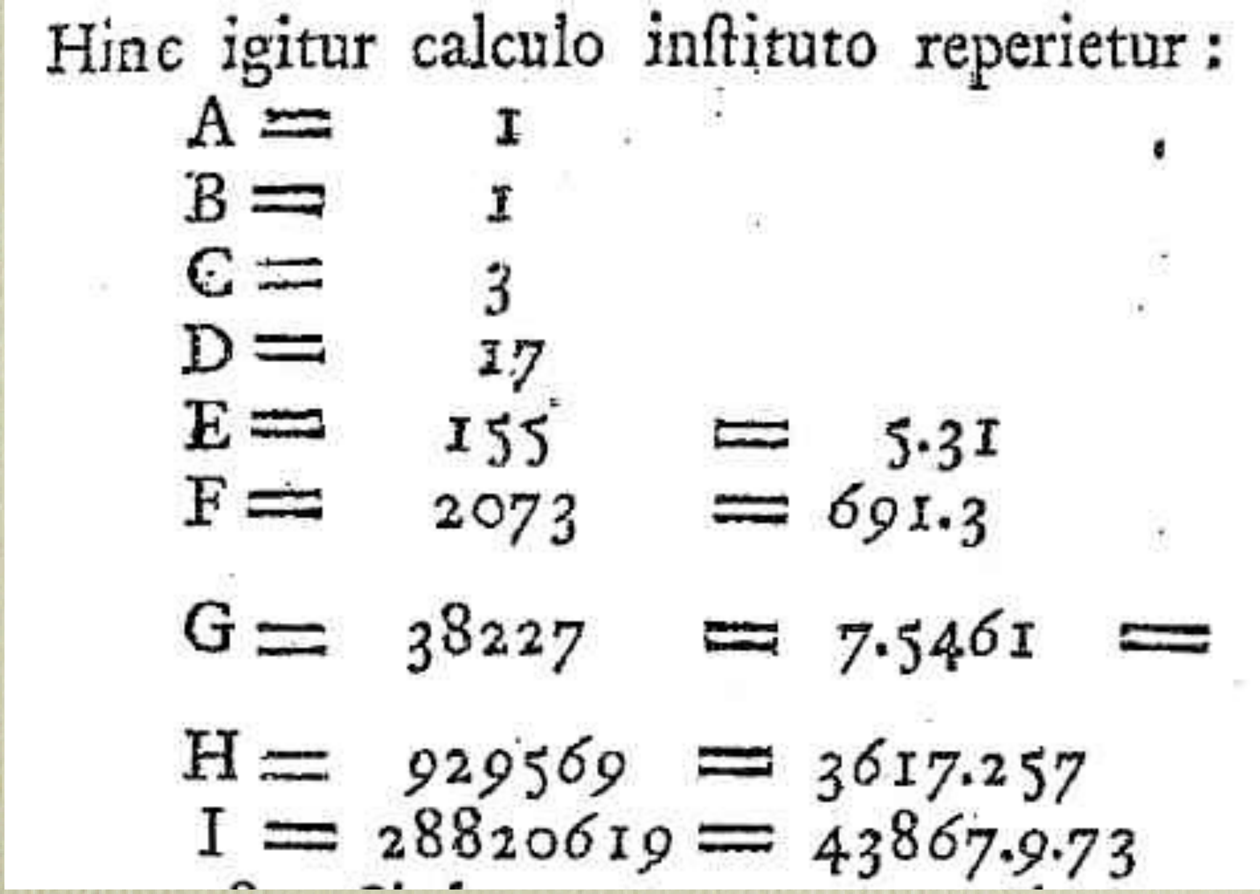
$$2^{2n} G_{2n+2} = (n+1)\, T_{2n+1}$$

Tangent numbers

BJC 1

(bijective course, Part I)
Ch.3b, p.63-65

Angelo Genocchi
(1817-1889)



Hinc igitur calculo inſtituto reperietur:

A = 1

B = 1

C = 3

D = 17

E = 155 = 5·31

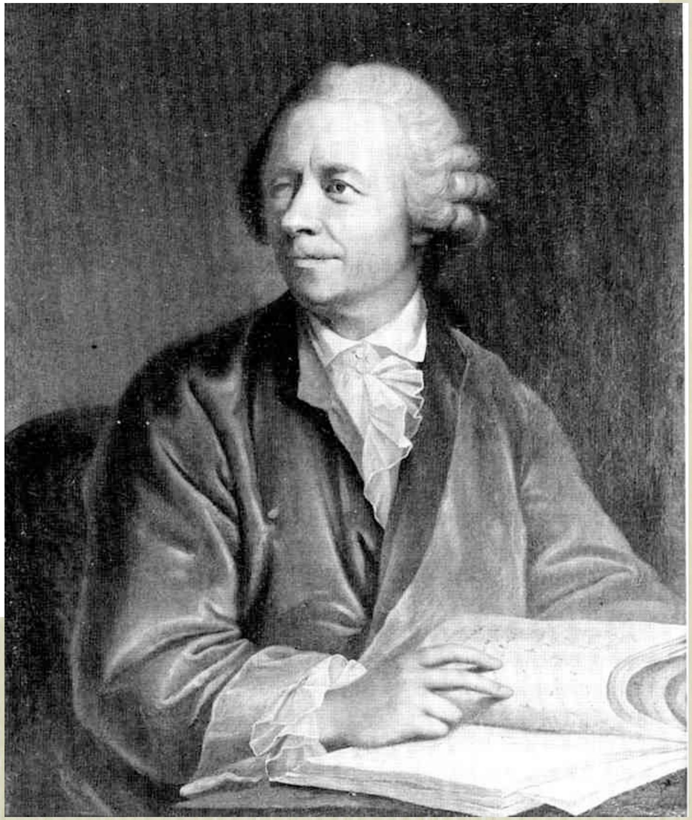
F = 2073 = 691.3

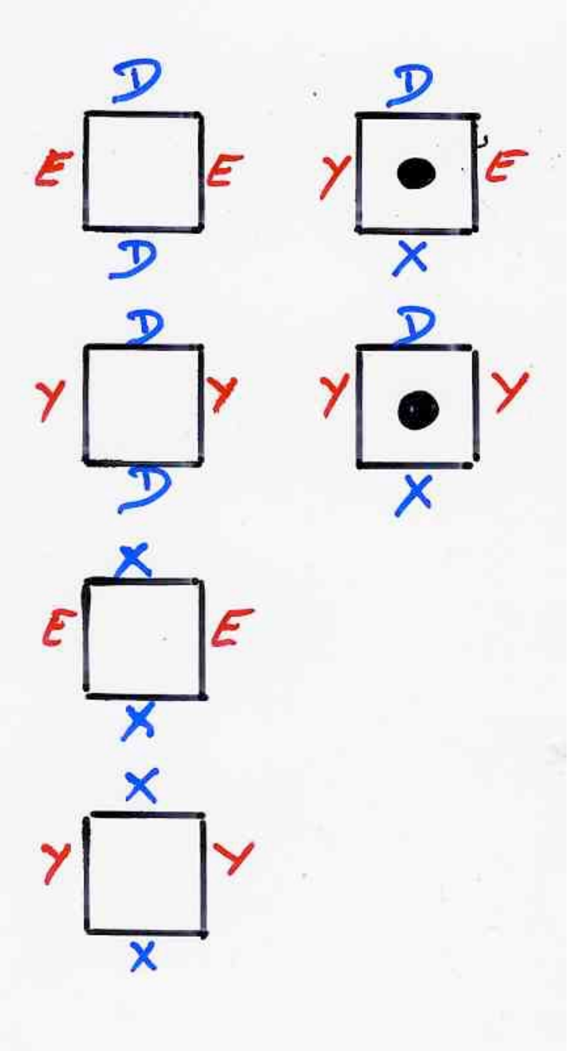
G = 38227 = 7·5461 =

H = 929569 = 3617.257

I = 28820619 = 43867·9·73

Leonhard Euler
(1707- 1783)




D
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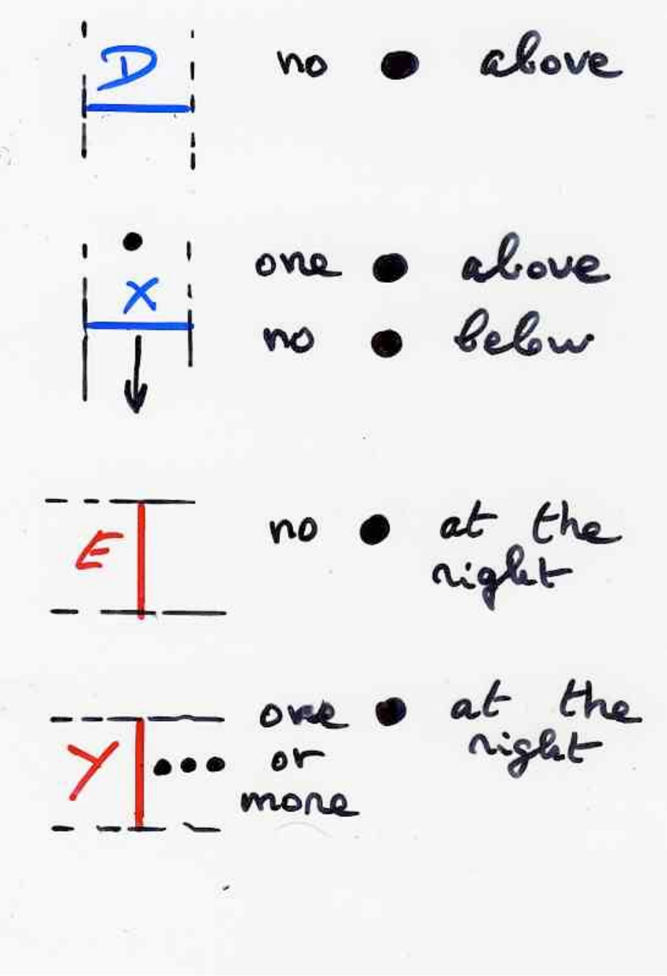
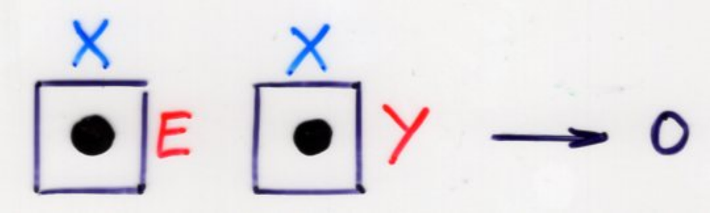
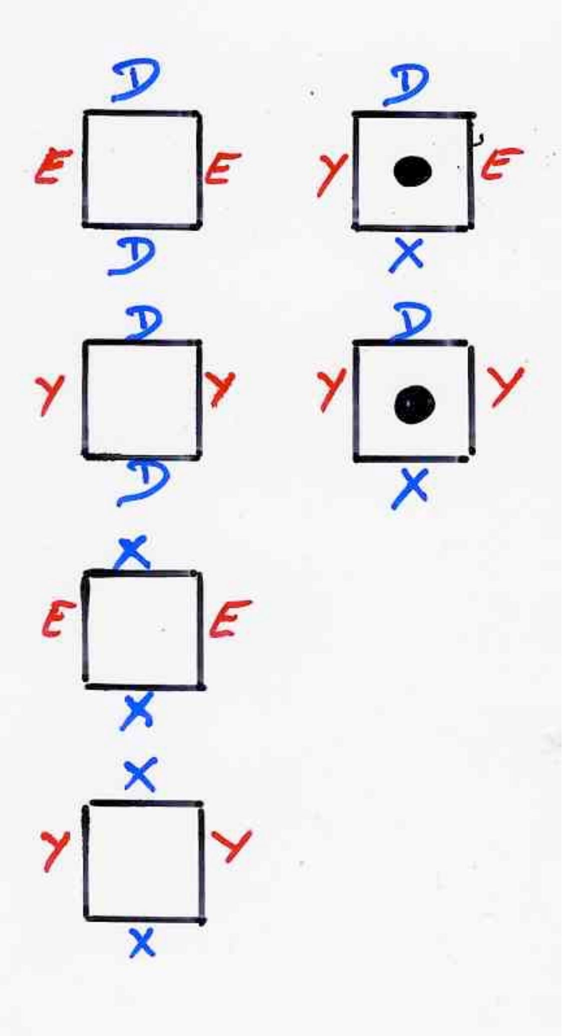
X

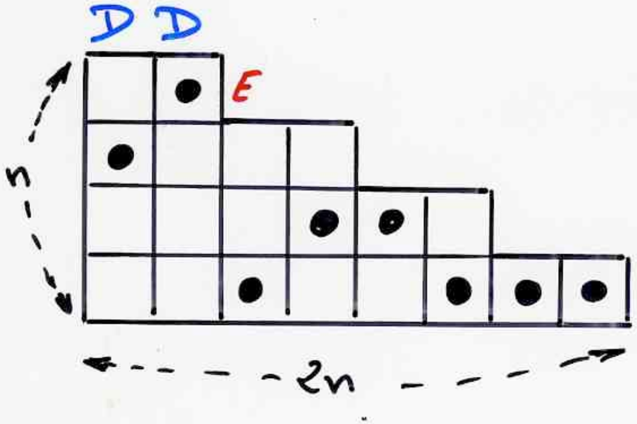

D
no ● above
X
one ● above
no ● below
E
no ● at the right
Y
one ● or more
at the right


X
E
X
Y
0



$$\begin{cases} DE = ED + YX \\ DY = YD + YX \\ XE = EX \\ XY = YX \end{cases}$$

example surjective pistol

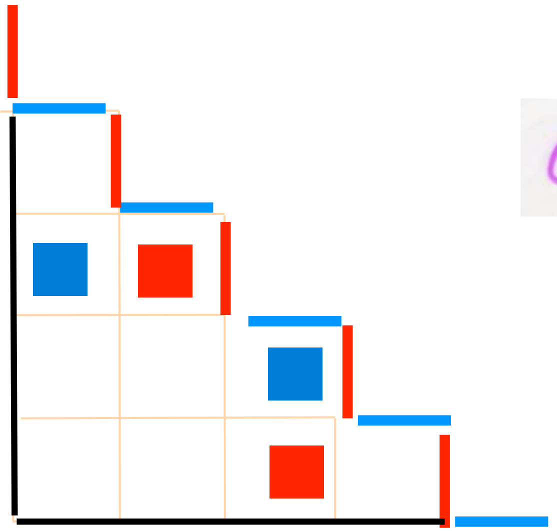


Genocchi numbers

G_{2n+2}

$$\begin{cases} DE = ED + YX \\ DY = YD + YX \\ XE = EX \\ XY = YX \end{cases}$$

$$c(\underset{u}{Y^n}, \underset{v}{X^{2n}}; \underset{w}{(D^2E)^n}) = G_{2n+2}$$

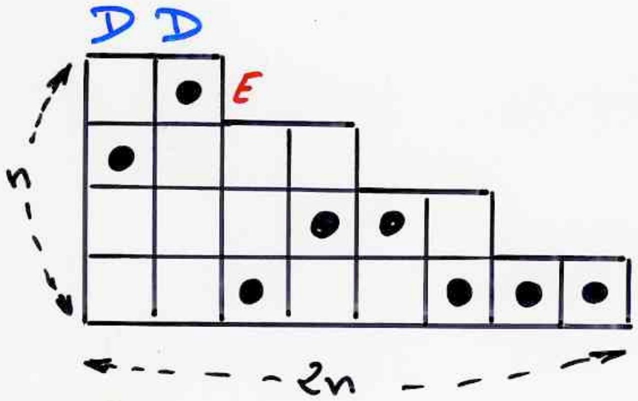


alternating shape

Genocchi numbers

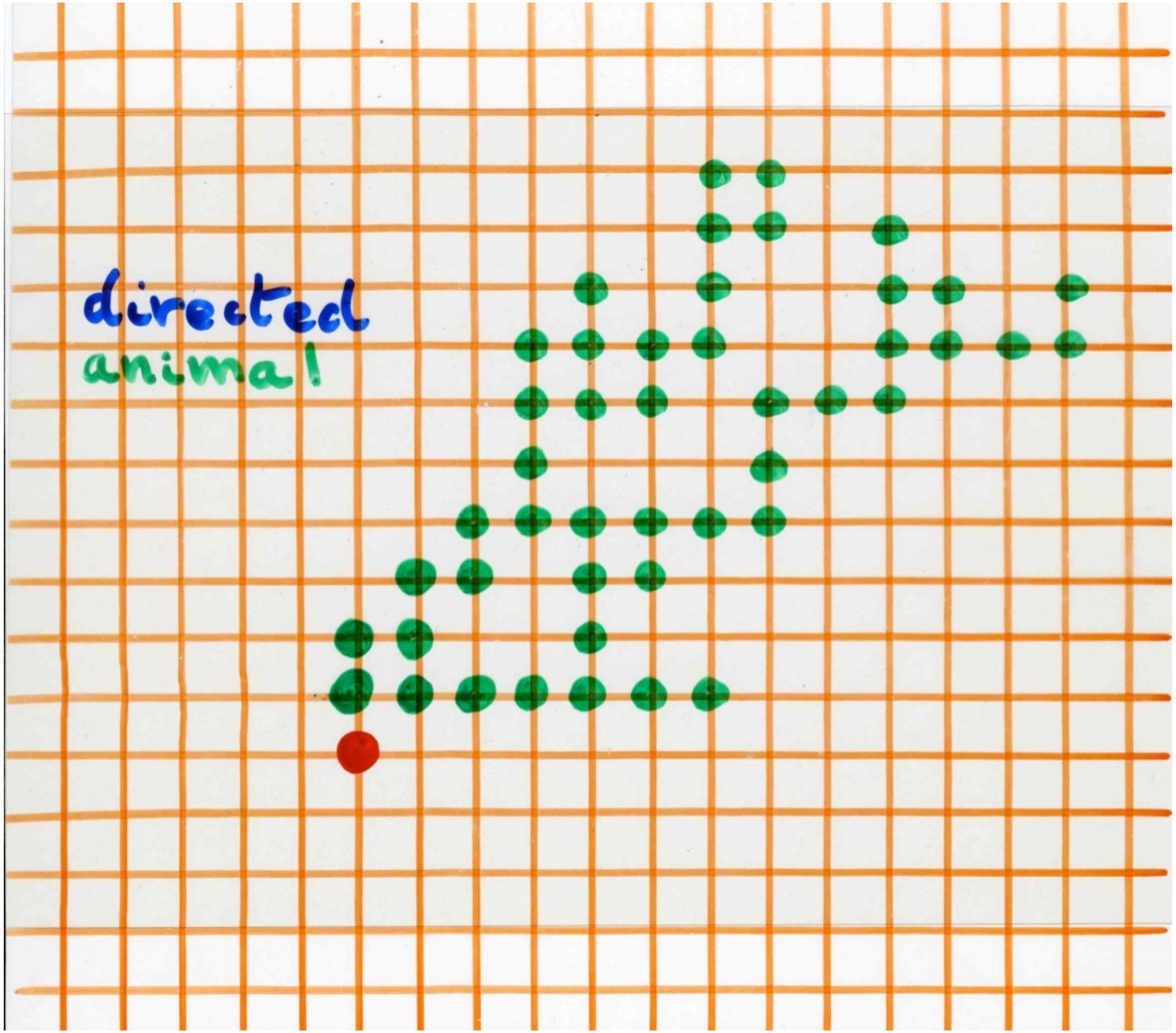
Josuat-Vergès
(2010)

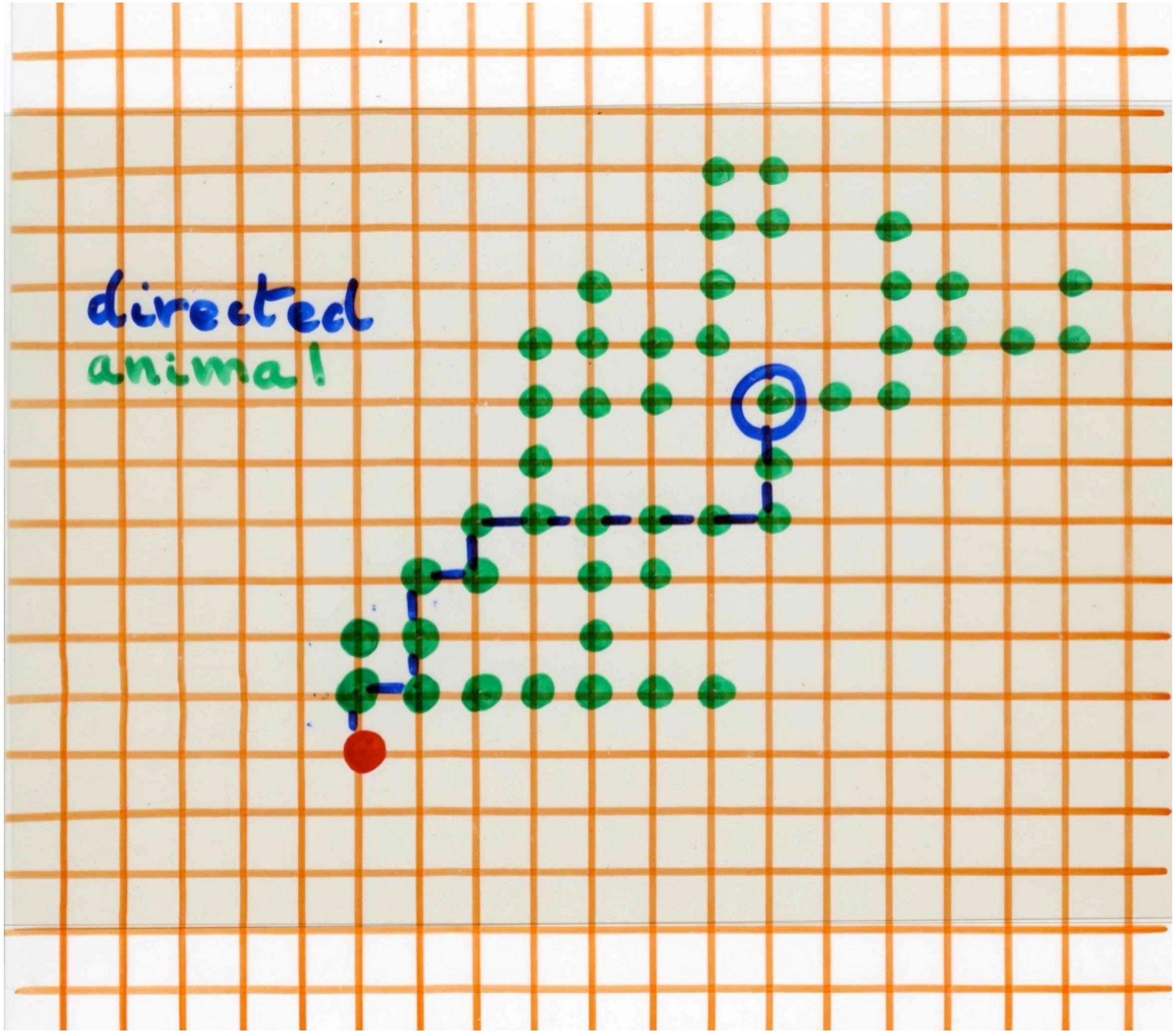
example surjective pistol



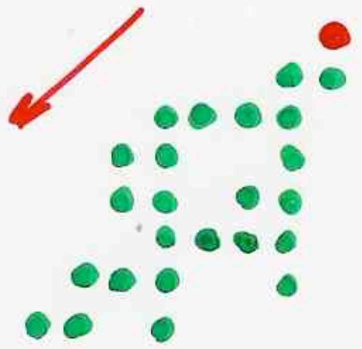
Genocchi numbers

G_{2n+2}


directed
animal


directed
animal

example - directed animal

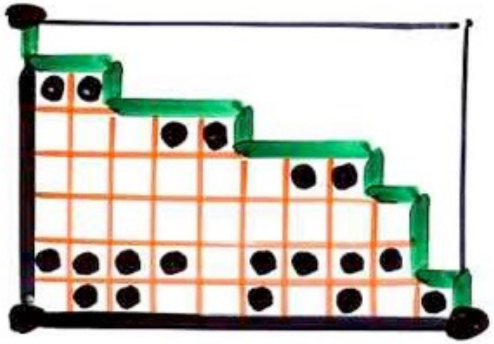


exercise find a planar automaton accepting the following "picture" (or "tableau")

and write the equations of the corresponding quadratic algebra Q

exercise

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

☐ = 0 ⊡ = 1

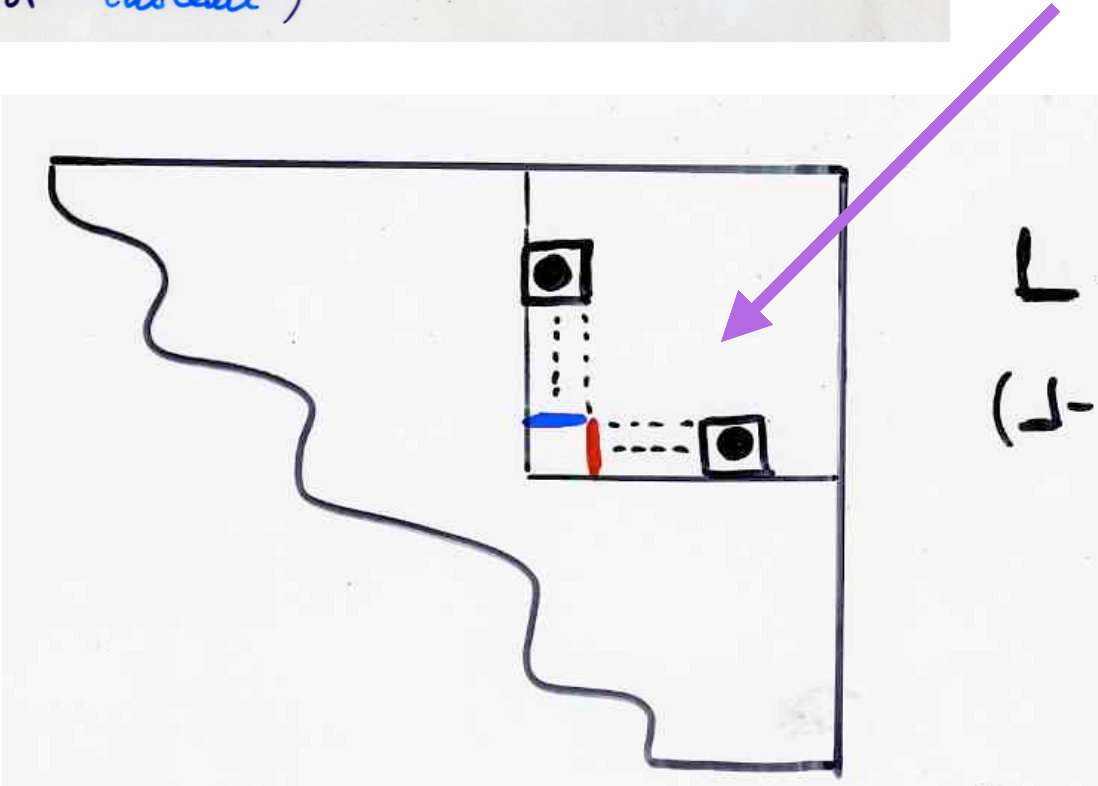
(ii)

forbidden

⅃-diagram

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian
E. Steingrímsson, L. Williams (2005)

exercise find a planar automaton accepting the following "picture" (or "tableau")



L-diagrams

(⅃-, ꓶ-)

and write the equations of the corresponding quadratic algebra Q

The RSK planar automaton

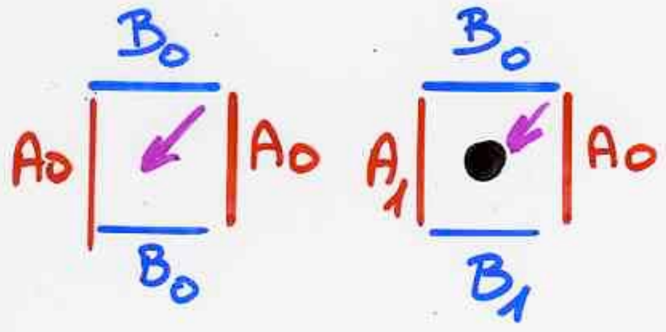
The "RSK planar automaton"

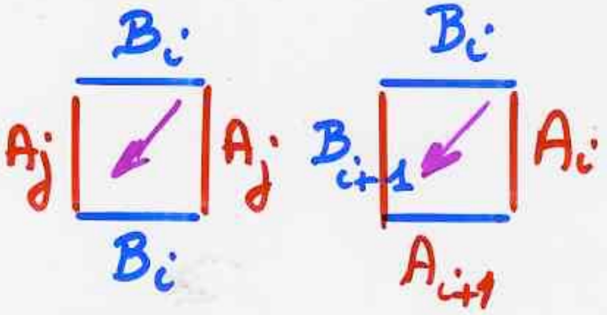
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \quad A_k\}$

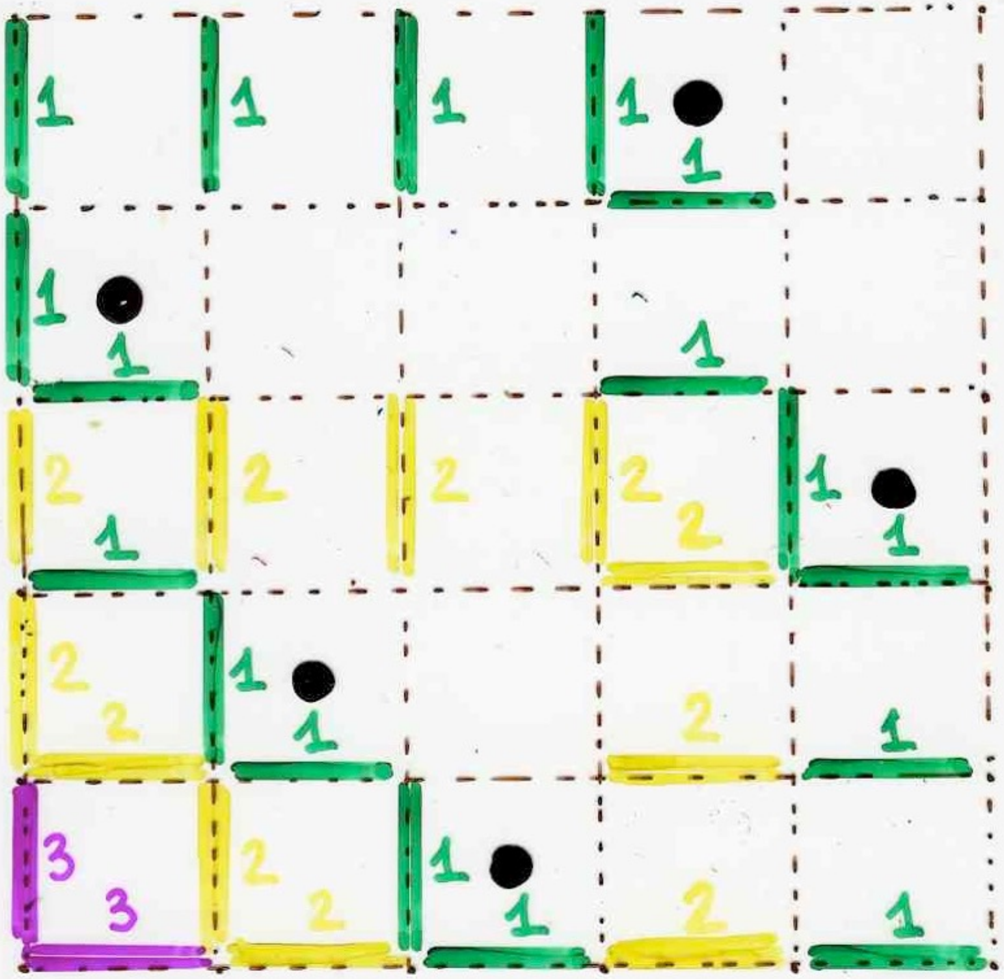
set of labels

L = {□, ⊡}





$i \neq j$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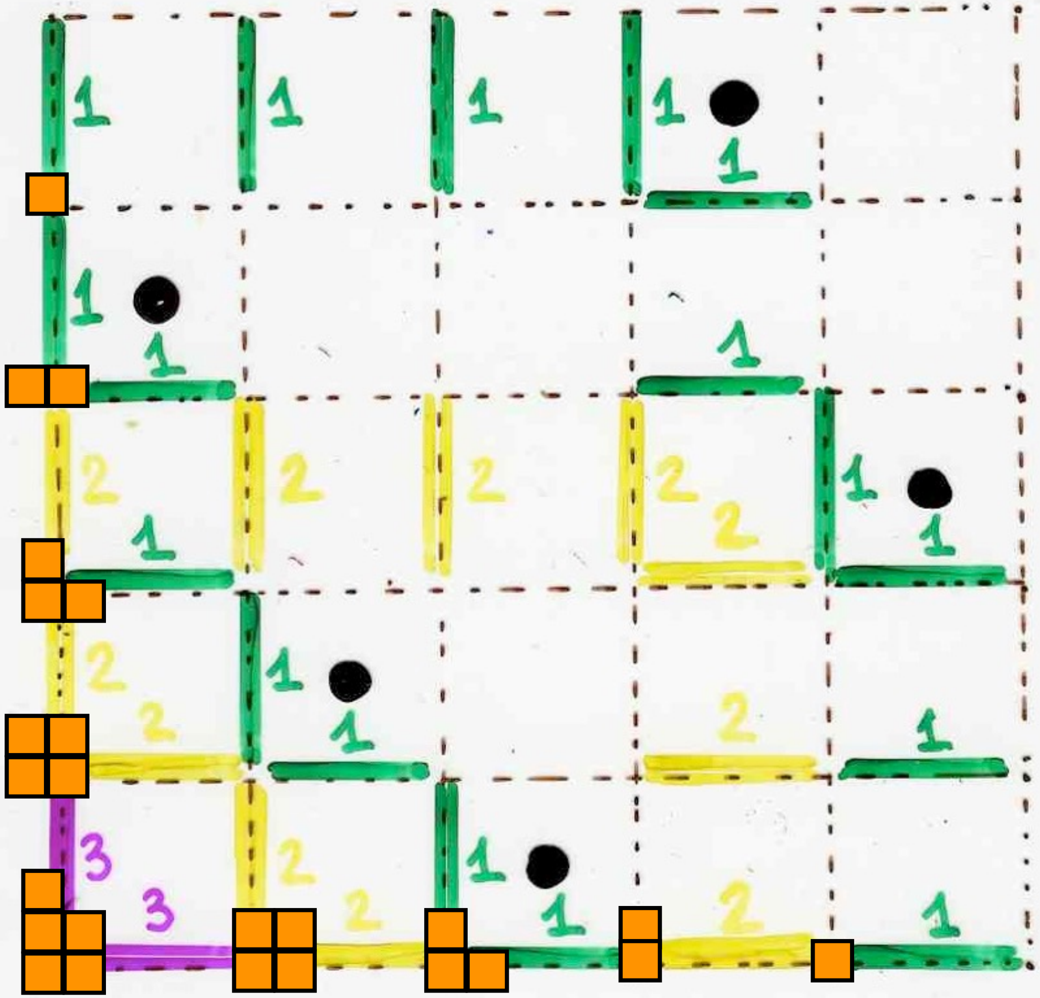
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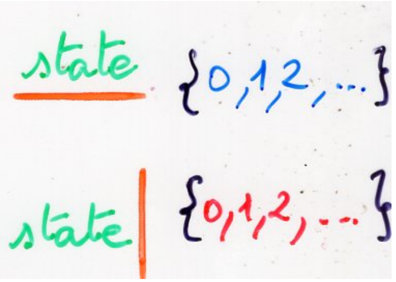
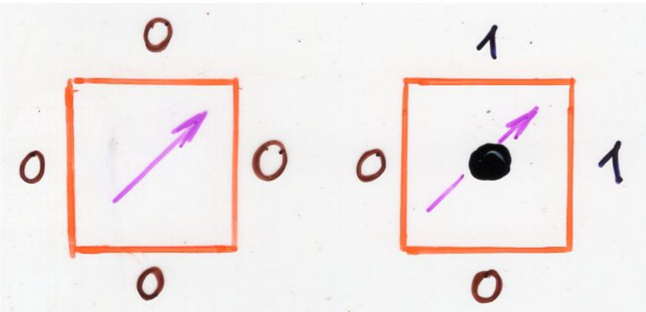
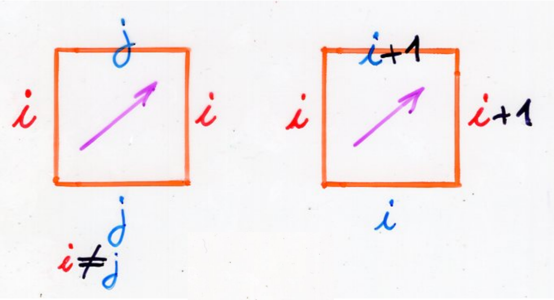
"local rules"
on the edges

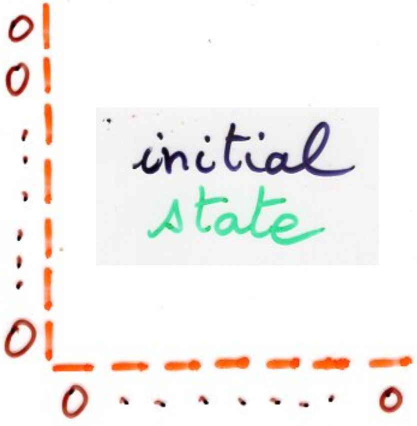


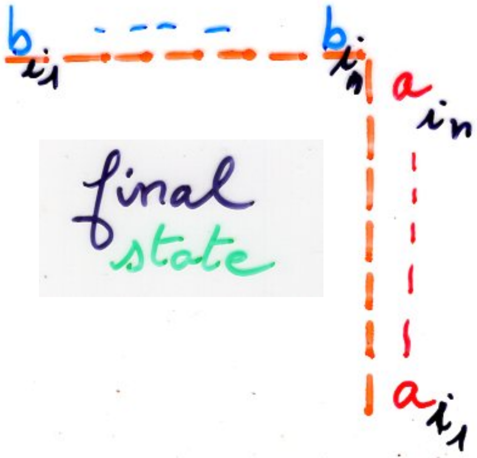
set of labels
$L = \{\square, \boxdot\}$



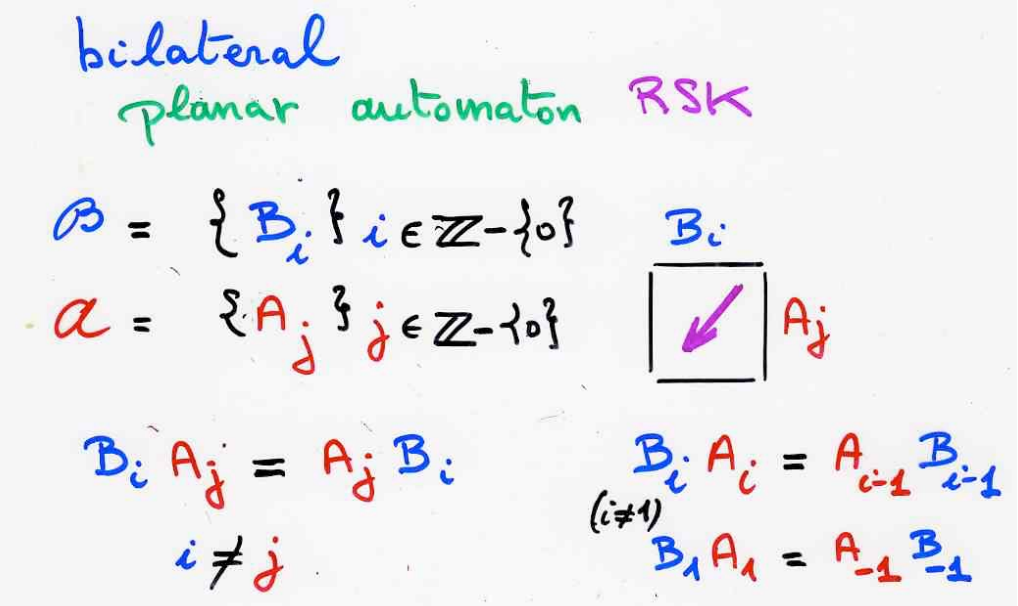


The RSK (reverse) planar automaton





Ch 1b, p109



bilateral

planar automaton RSK

$\mathcal{B} = \{B_i\}_{i\in\mathbb{Z}-\{0\}}$

$\mathcal{A} = \{A_j\}_{j\in\mathbb{Z}-\{0\}}$

$B_i A_j = A_j B_i$

$i \neq j$

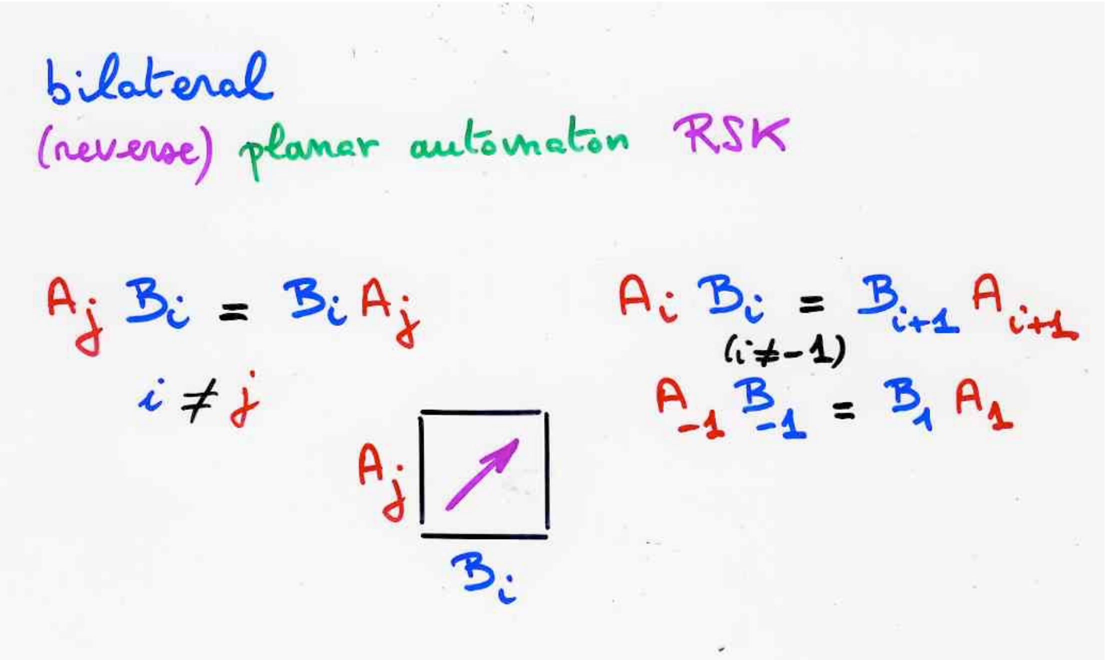
$B_i A_i = A_{i-1} B_{i-1}$

$(i\neq 1)$

$B_1 A_1 = A_{-1} B_{-1}$

set of labels

$L = \{\square, \quad\}$



bilateral

(reverse) planar automaton RSK

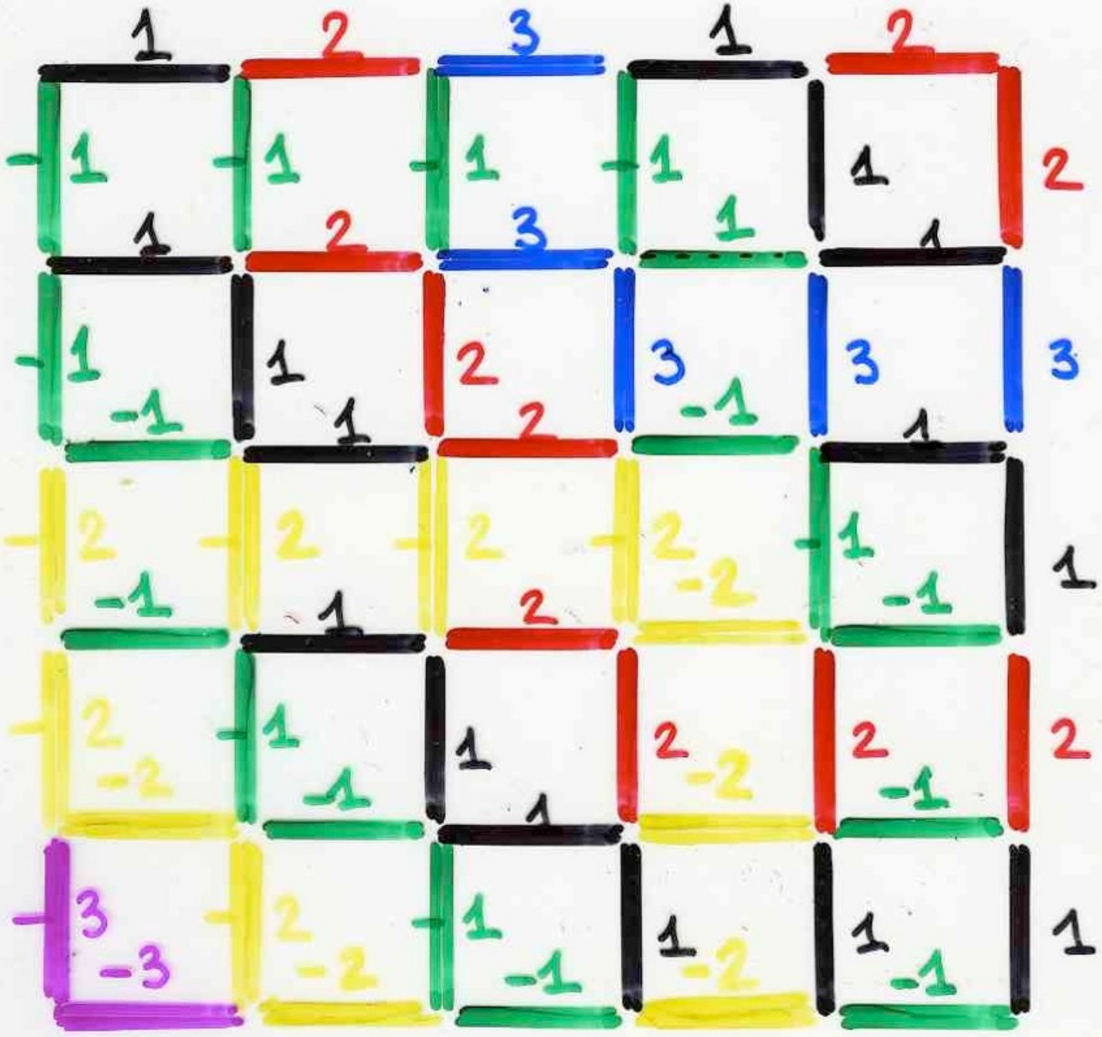
$A_j B_i = B_i A_j$

$i \neq j$

$A_i B_i = B_{i+1} A_{i+1}$

$(i\neq -1)$

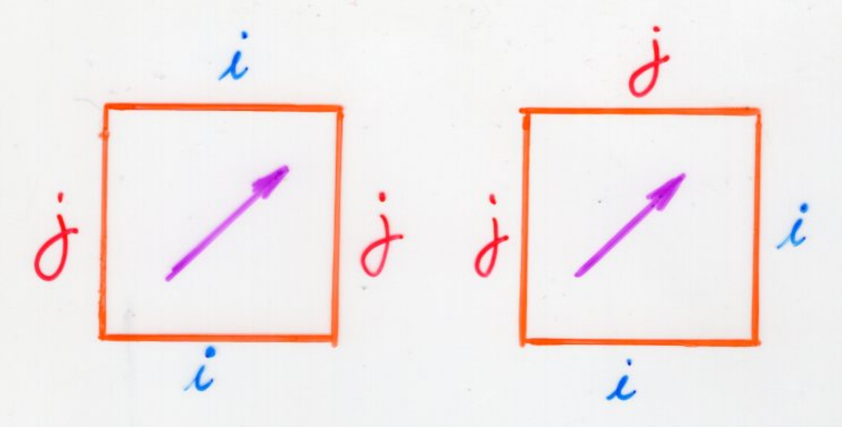
$A_{-1} B_{-1} = B_1 A_1$


1 2 3 1 2
-1 -1 -1 -1 1 1 2
1 2 3 1 1
-1 -1 1 2 2 3 -1 3 3 1
-2 -1 -2 1 -2 2 -2 -2 -1 -1 1
-2 -2 -1 -1 1 1 2 -2 2 -1 2
-3 -3 -2 -2 -1 -1 1 -2 1 -1 1

Ch 1d, p102

jeu de taquin
local rules on edges

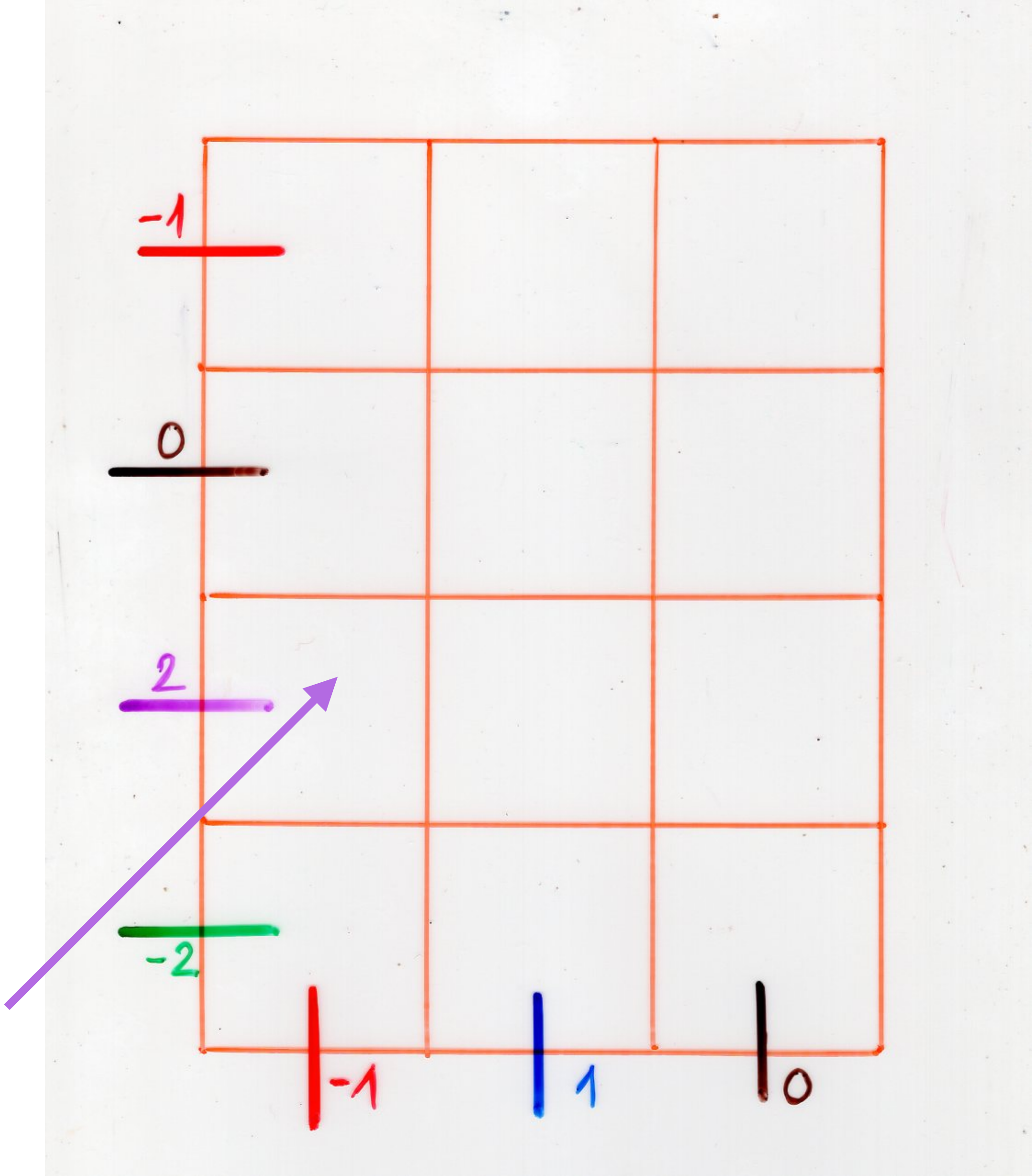
set of labels
$L = \{ \square , \quad \}$



$i, j \in \mathbb{Z}$

$|i - j| \geqslant 2$

$|i - j| \leqslant 1$


-1
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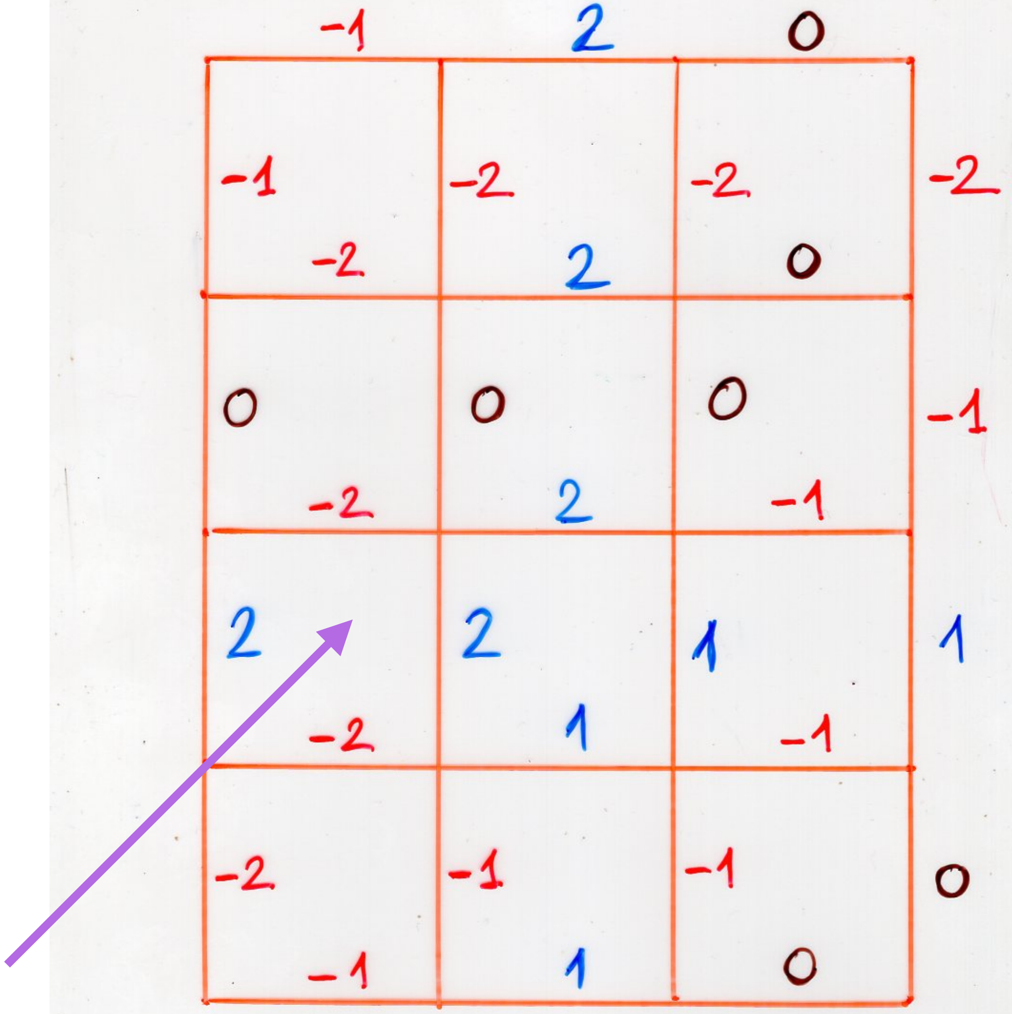
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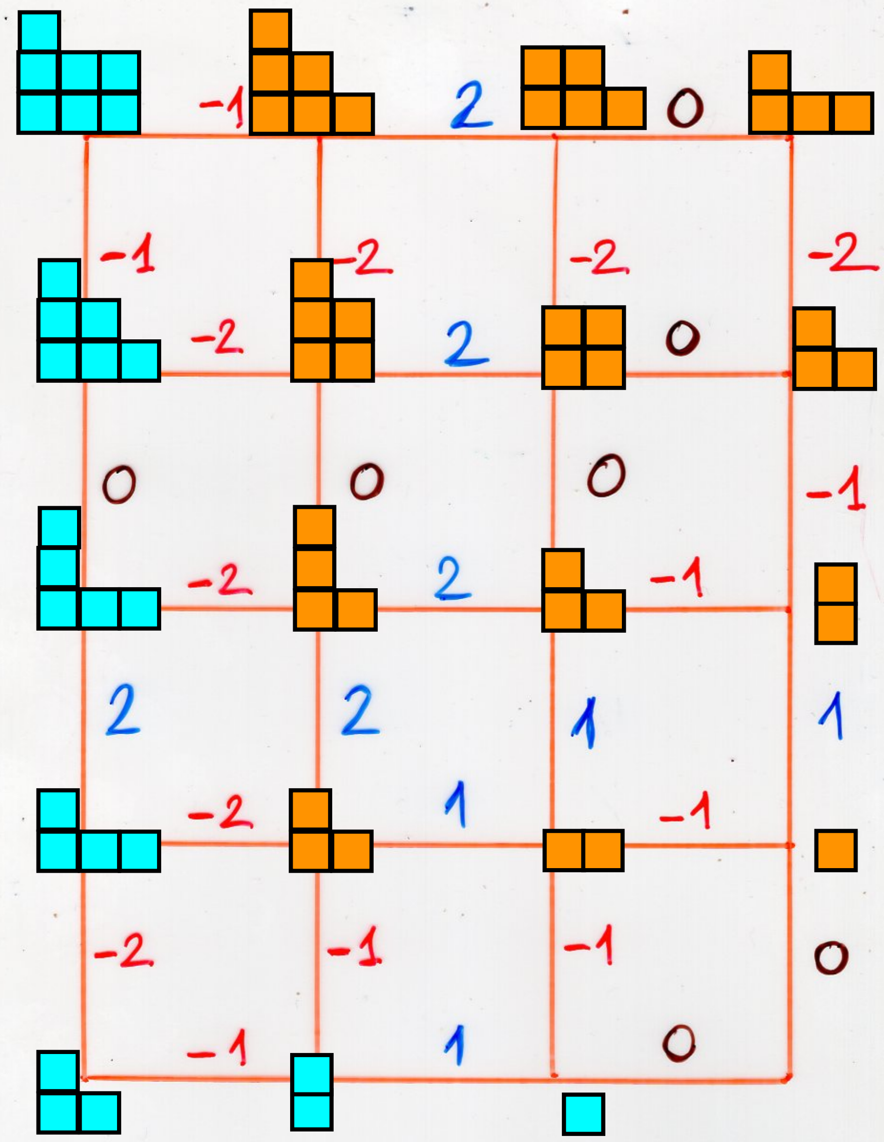
0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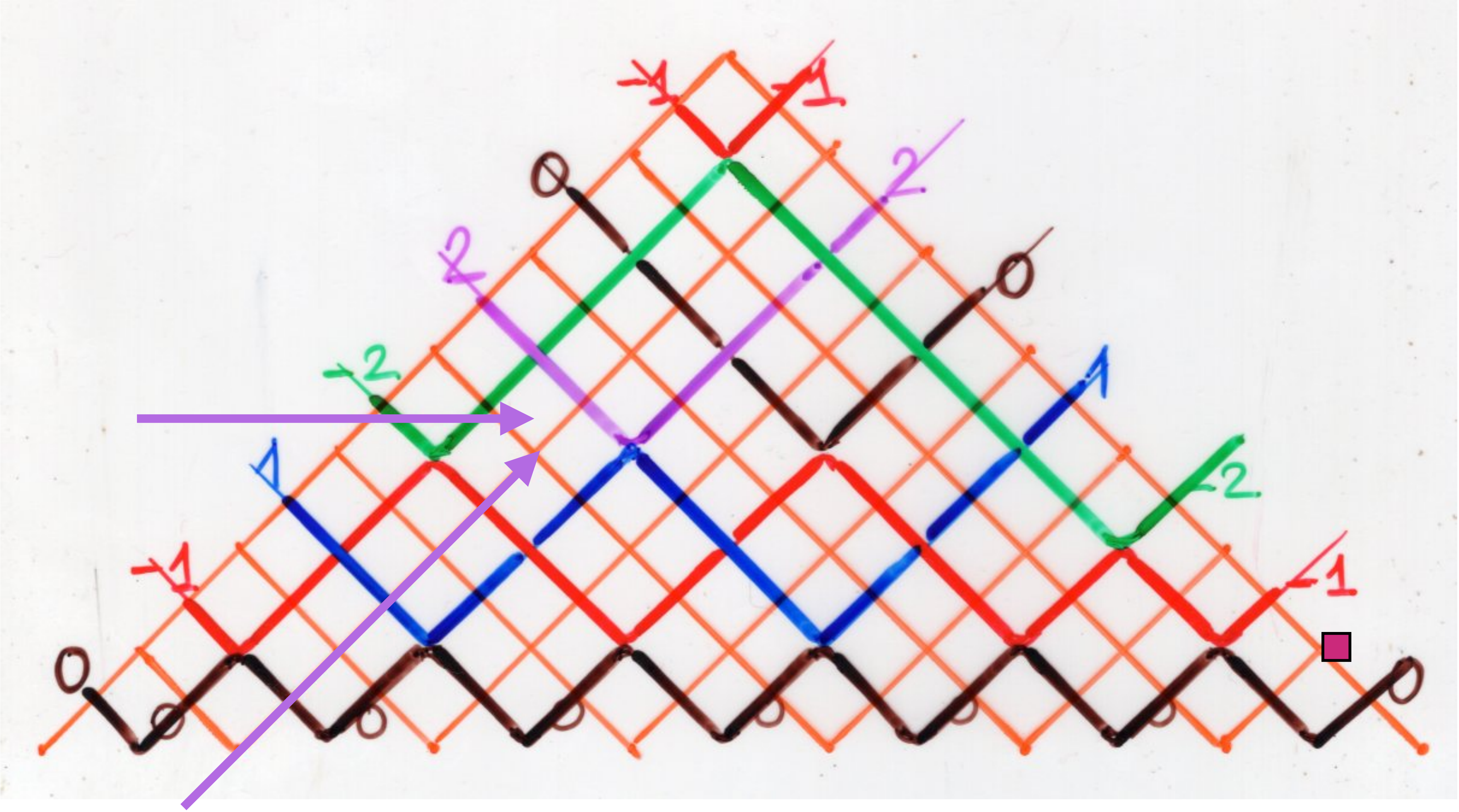
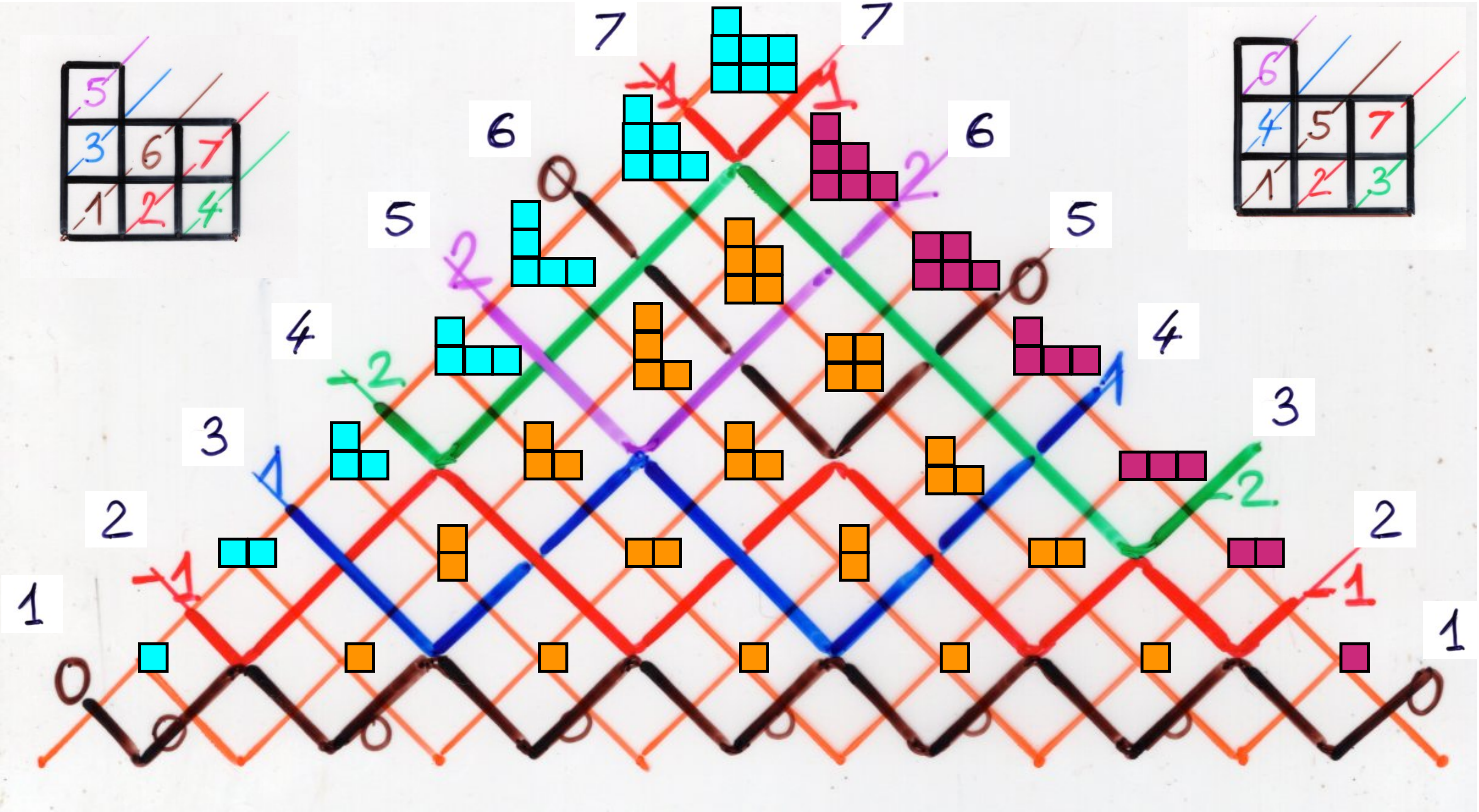
0
-1
1
0

dual of a tableau



Schützenberger involution

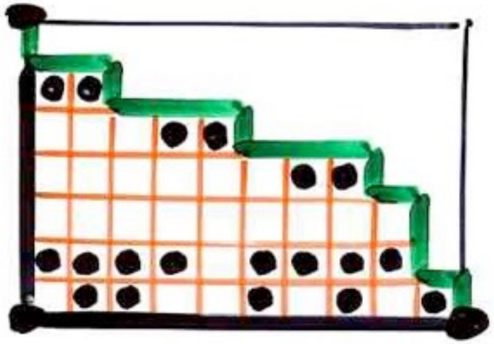
dual of a tableau



Schützenberger involution

Bijections for «tableaux» accepted by planar automata ?

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells
with 0 and 1

☐ = 0   ▣ = 1

(ii)

forbidden

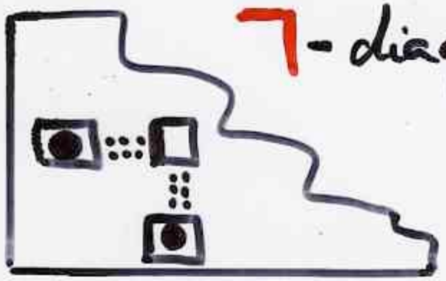
⅃-diagram

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian
E. Steingrímsson, L. Williams (2005)

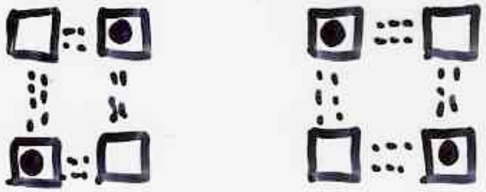
Bijections between pattern-avoiding fillings of Young diagrams

Josuat-Vergès (2008)

⅂-diagrams



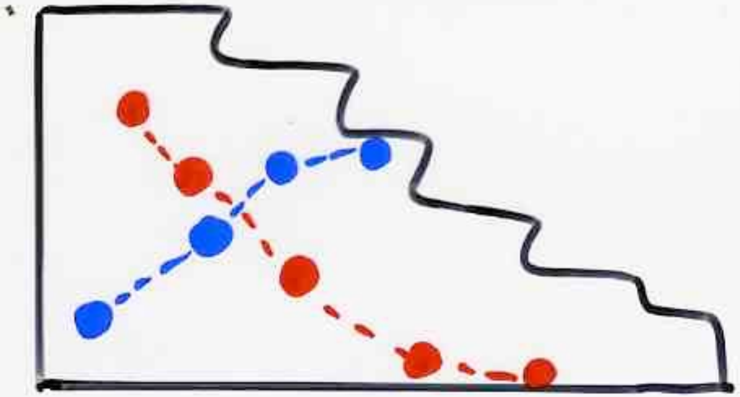
X-diagrams



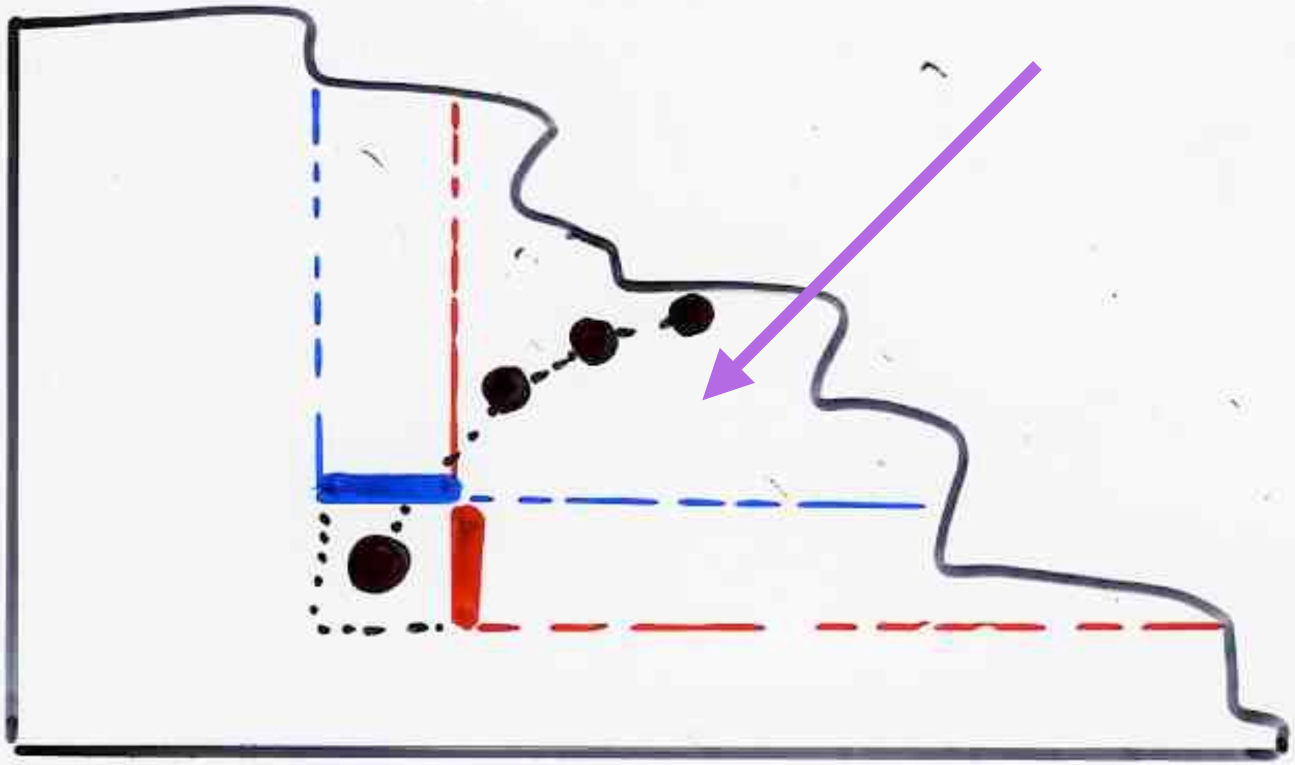
?

increasing
decreasing
chains in fillings of Ferrers shapes

(Jonsson, 2005) (Krattenthaler, 2006)
(Backelin, West, Xin, 2005) (Bousquet-Mélou, Steingrímsson, 2005) ....
... . - - - —

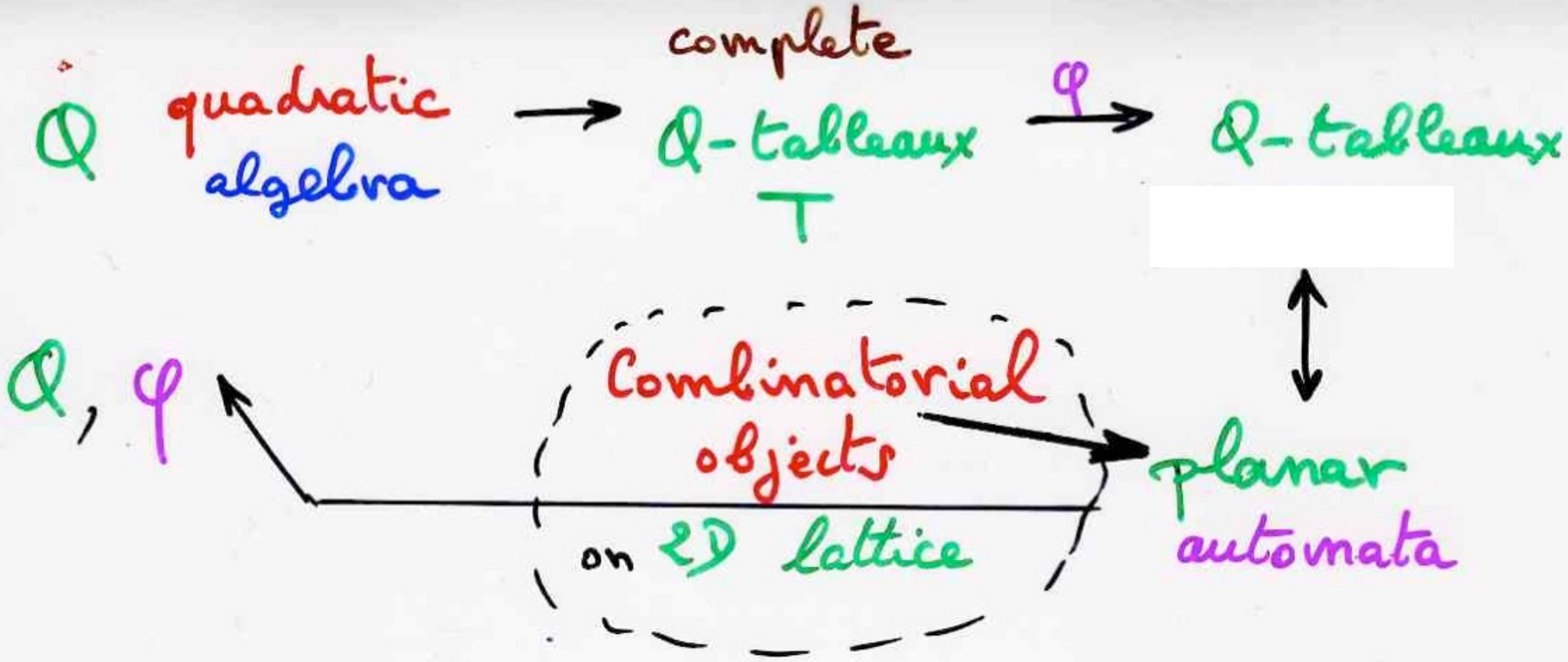


increasing
decreasing
subsequences
(chains)



Summary

Q-tableaux and planar automata


Q quadratic algebra
complete
Q-tableaux
T
φ
Q-tableaux
Q, φ
Combinatorial objects
on 2D lattice
planar automata

- formula for $c(u, v; w)$?

Q determinant ?

- or at least efficient procedure for computing $c(u, v; w)$?

- generating function ?

Dual Q-tableaux

after the video: change of vocabulary
« dual Q-tableaux » should be called

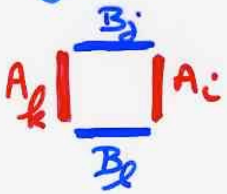
« reverse Q-tableaux »

$$B_j A_i = \sum_{k l} c_{ij}^{kl} A_k B_l$$

Def- dual Q-tableau

R set of "rewriting rules" $B_j A_i \to c_{ij}^{kl} A_k B_l$

i.e. set of labels $\begin{pmatrix} k & l \\ i & j \end{pmatrix}$ or



$\varphi : R \to S$ such that

$$\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix} \Rightarrow (k,l) \neq (k',l')$$

if $\begin{pmatrix} k & l \\ i & j \end{pmatrix} \neq \begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$

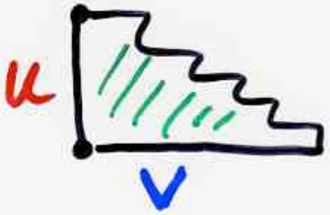
T dual Q-tableau. "image" by φ of a complete Q-tableau

dual Q-tableau = reverse Q-tableau

bijection

= reverse Q-tableau

complete
Q-tableau



⟷

dual
Q-tableau

$$(T\,,\ uv)$$

$$\begin{tikzcd} u \arrow[d] & v \arrow[d] \\ \in \mathcal{A}^{*} & \in \mathcal{B}^{*} \end{tikzcd}$$

Dual Q-tableaux

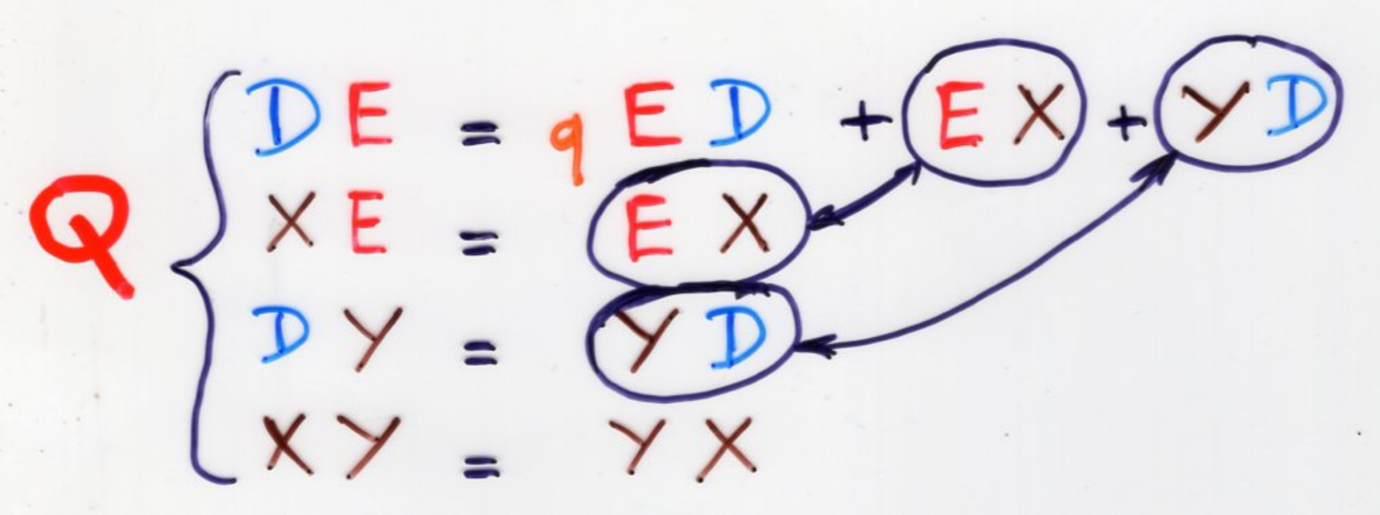
= reverse Q-tableau

example with the PASEP algebra

PASEP algebra

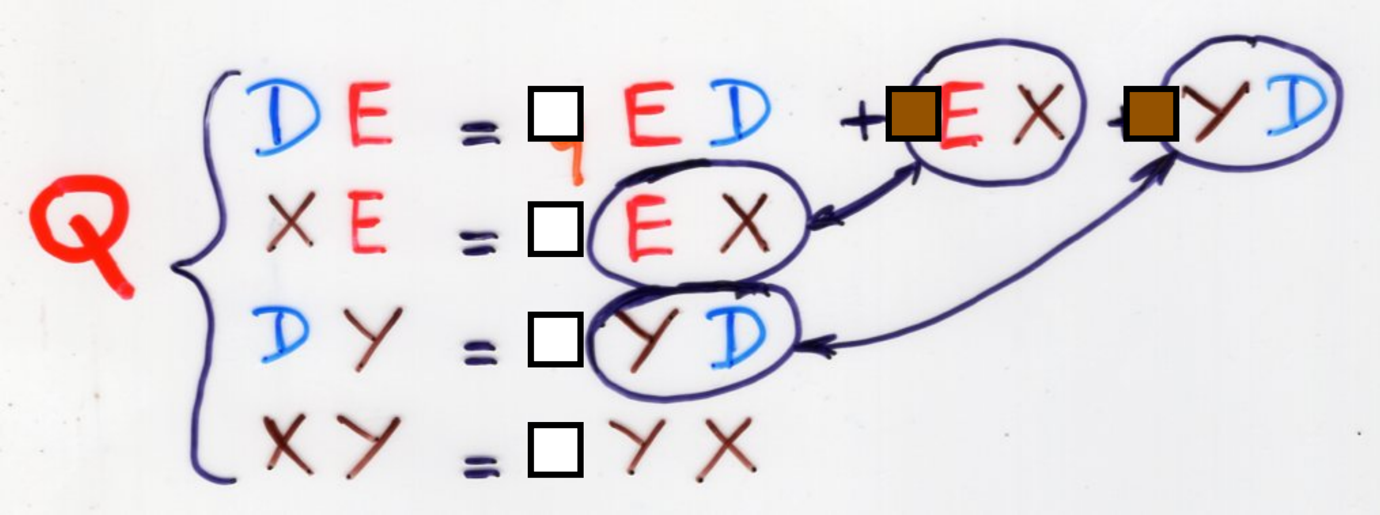
$$Q \begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

PASEP algebra



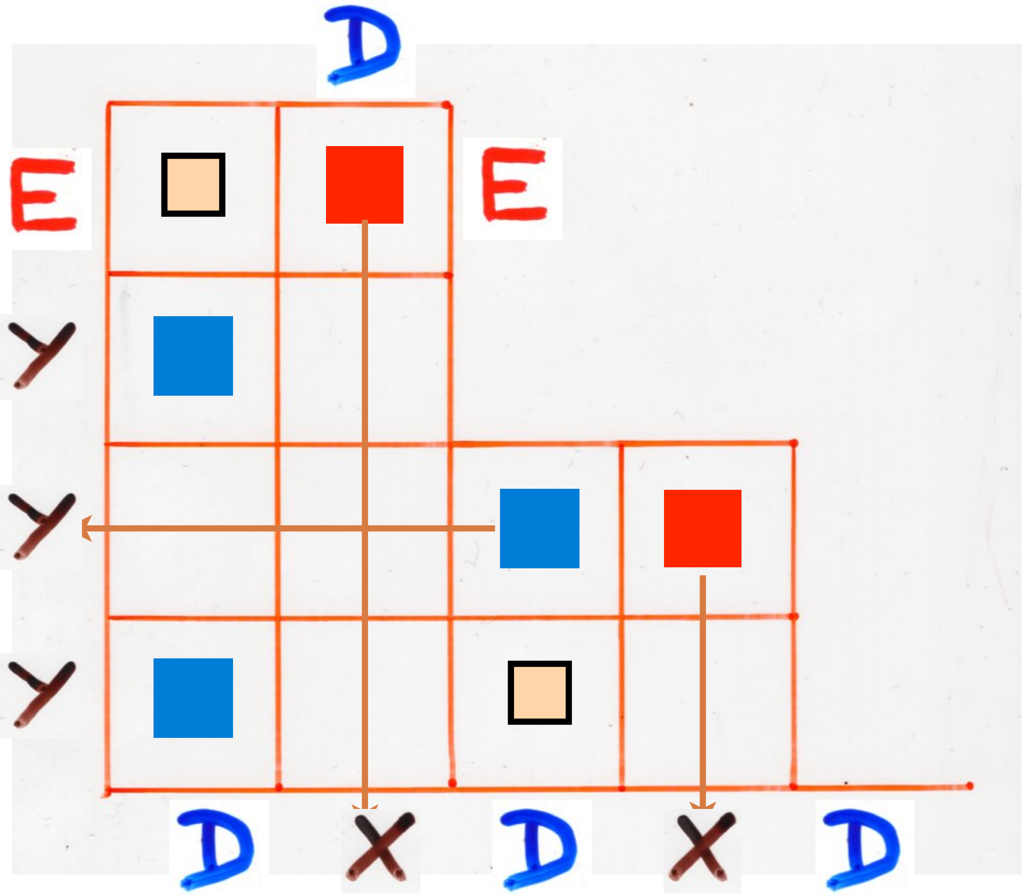
$$Q\begin{cases} DE = qED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

PASEP algebra



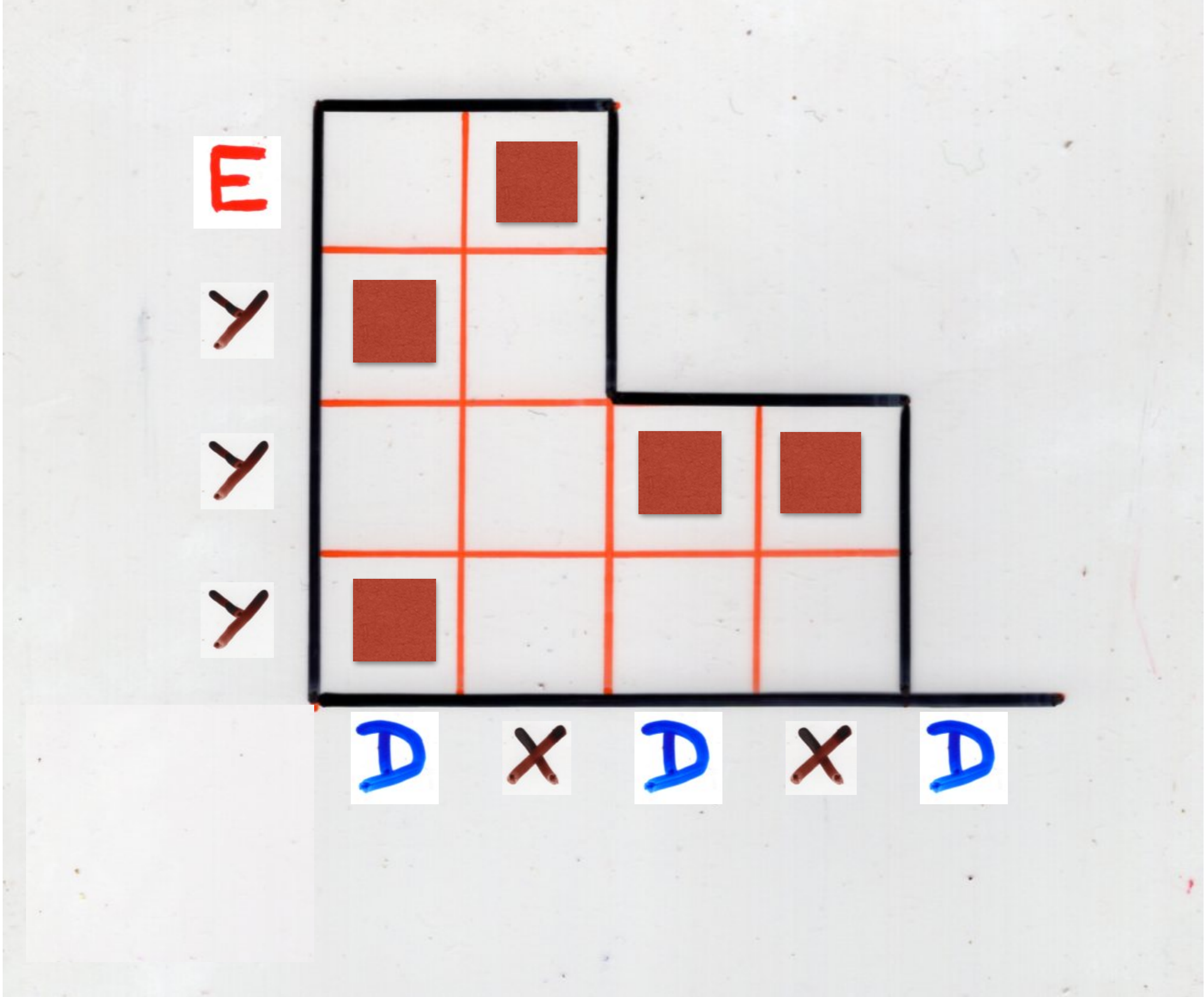
$$Q\begin{cases} DE = \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\ XE = \square\, EX \\ DY = \square\, YD \\ XY = \square\, YX \end{cases}$$

alternative
tableaux



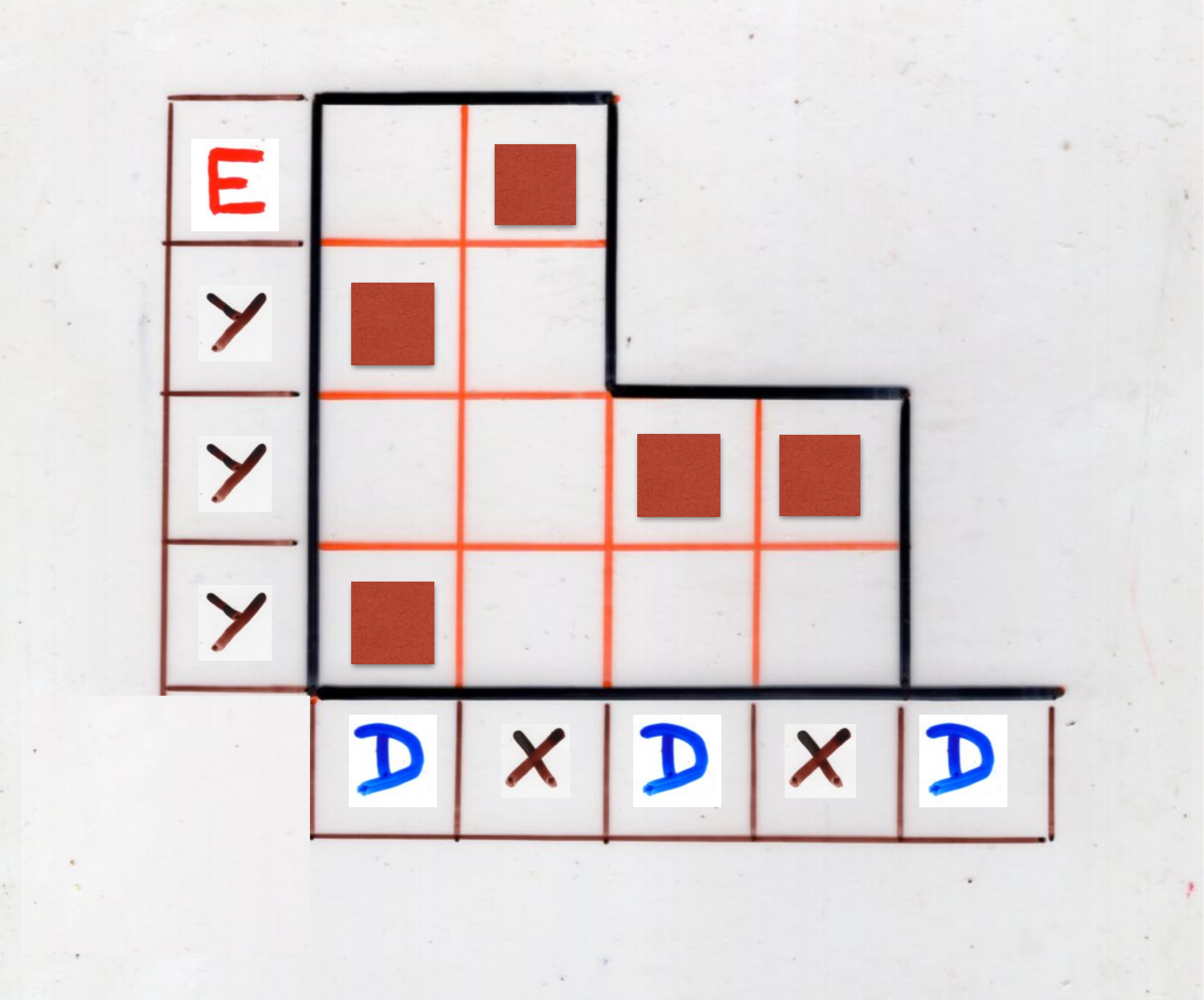
= reverse Q-tableau

dual Q-tableau



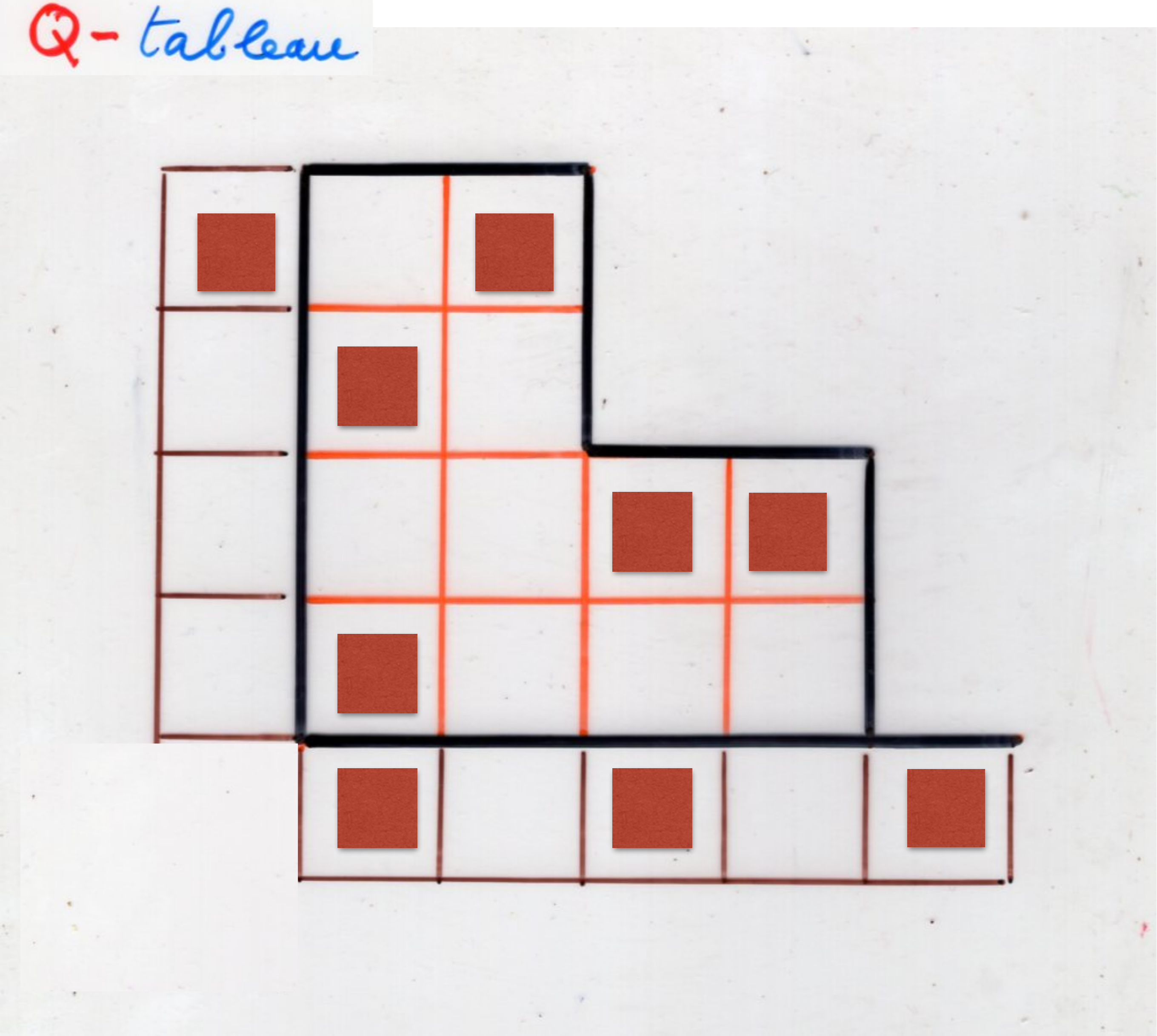
= reverse Q-tableau

dual Q-tableau



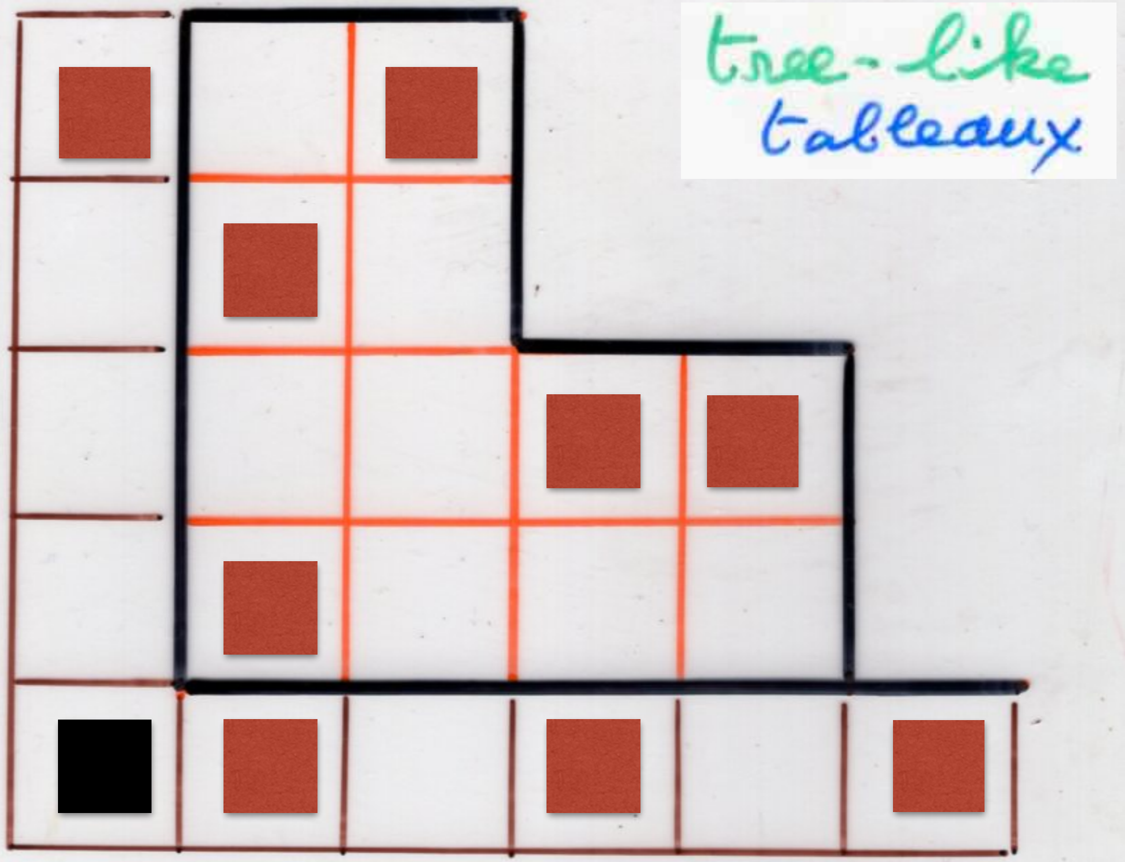
= reverse Q-tableau

dual Q-tableau



= reverse Q-tableau

dual Q-tableau



Definition Tree-like tableaux

Aval, Boussicault, Nadeau (2011)

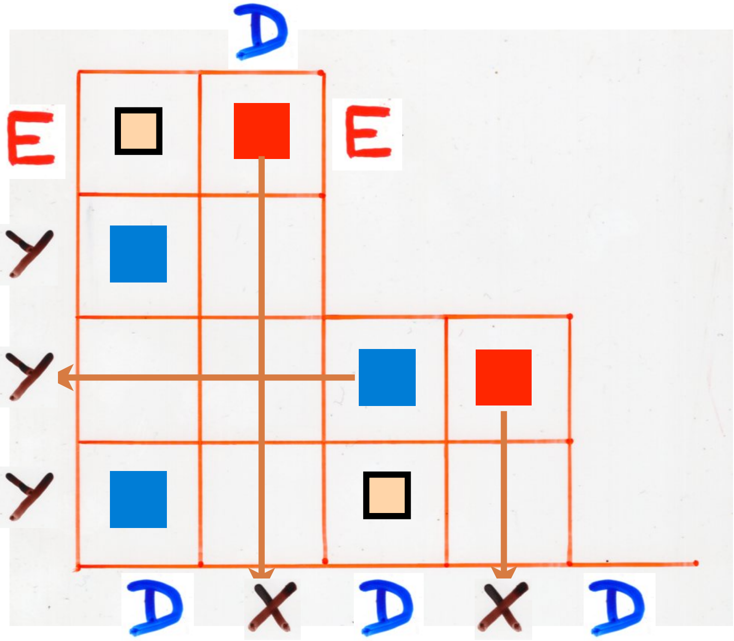
Ferrers diagram F with cell

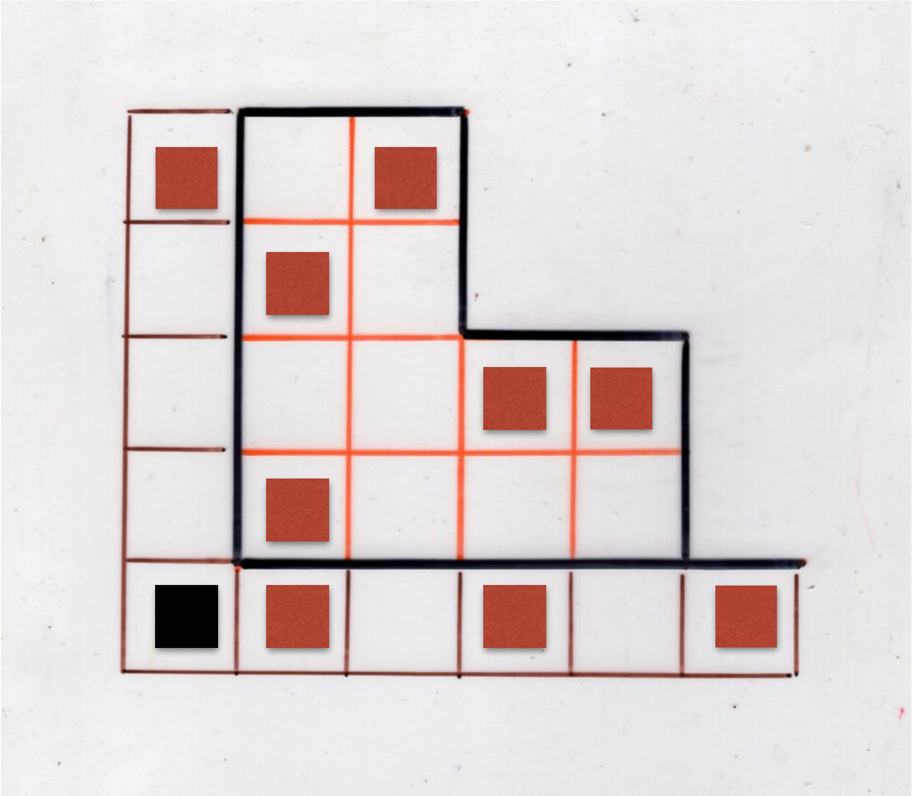
□ empty cell ⊡ pointed cell

(i) the botom left cell is pointed
(called the root cell)

(ii) for every non-root pointed cell c,
there exist a pointed cell below c
in the same column, or a pointed cell
to its left in the same row,
but not both

(iii) every column and every row
possesses at least one pointed cell


D
E
E
Y
Y
Y
D
X
D
X
D



Q

DE = □ ED + 🟥 EX + 🟦 YD

XE = □ EX

DY = □ YD

XY = □ YX

alternative tableaux

Q

DE = □ ED + 🟫 EX + 🟫 YD

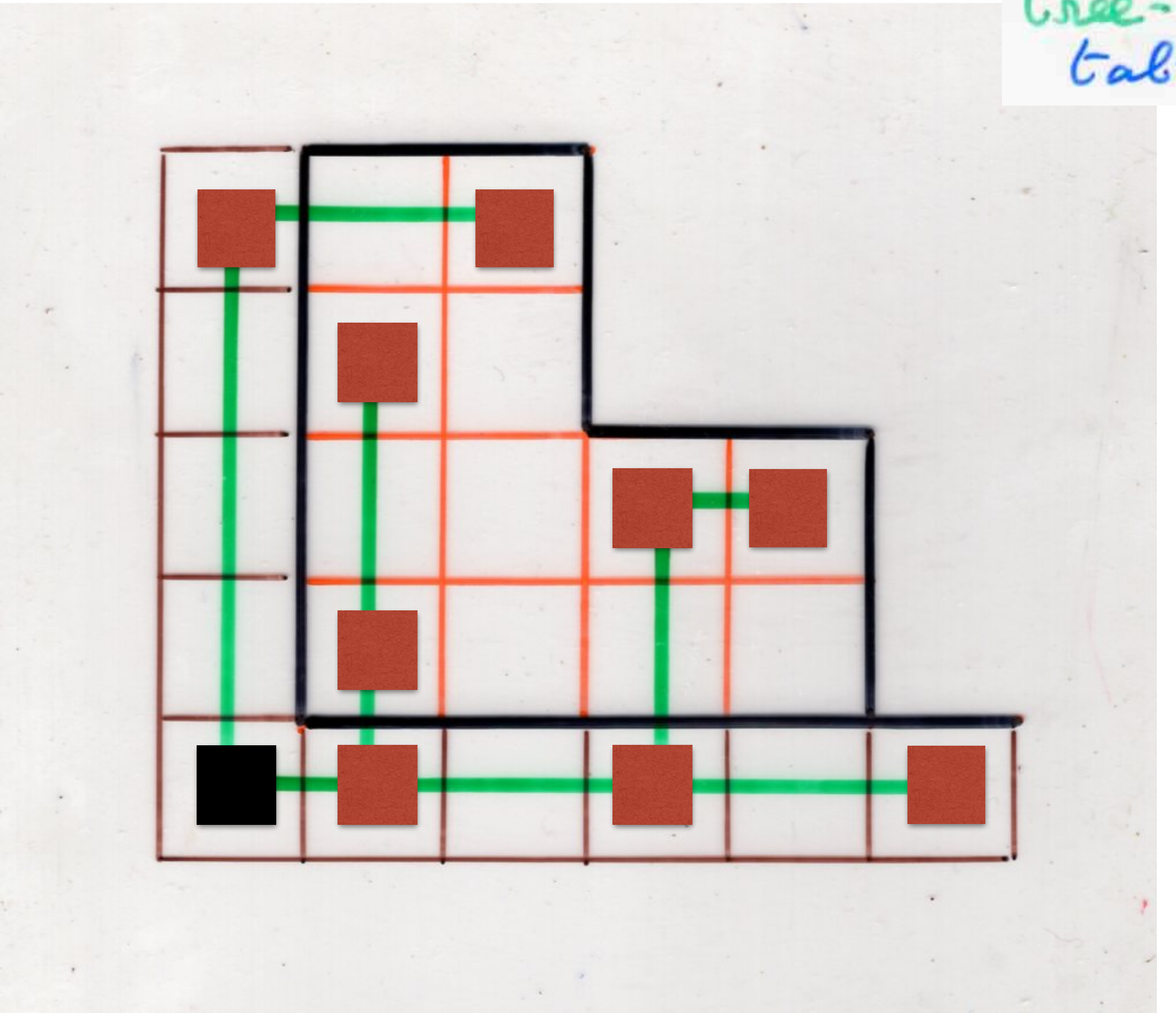
XE = □ EX

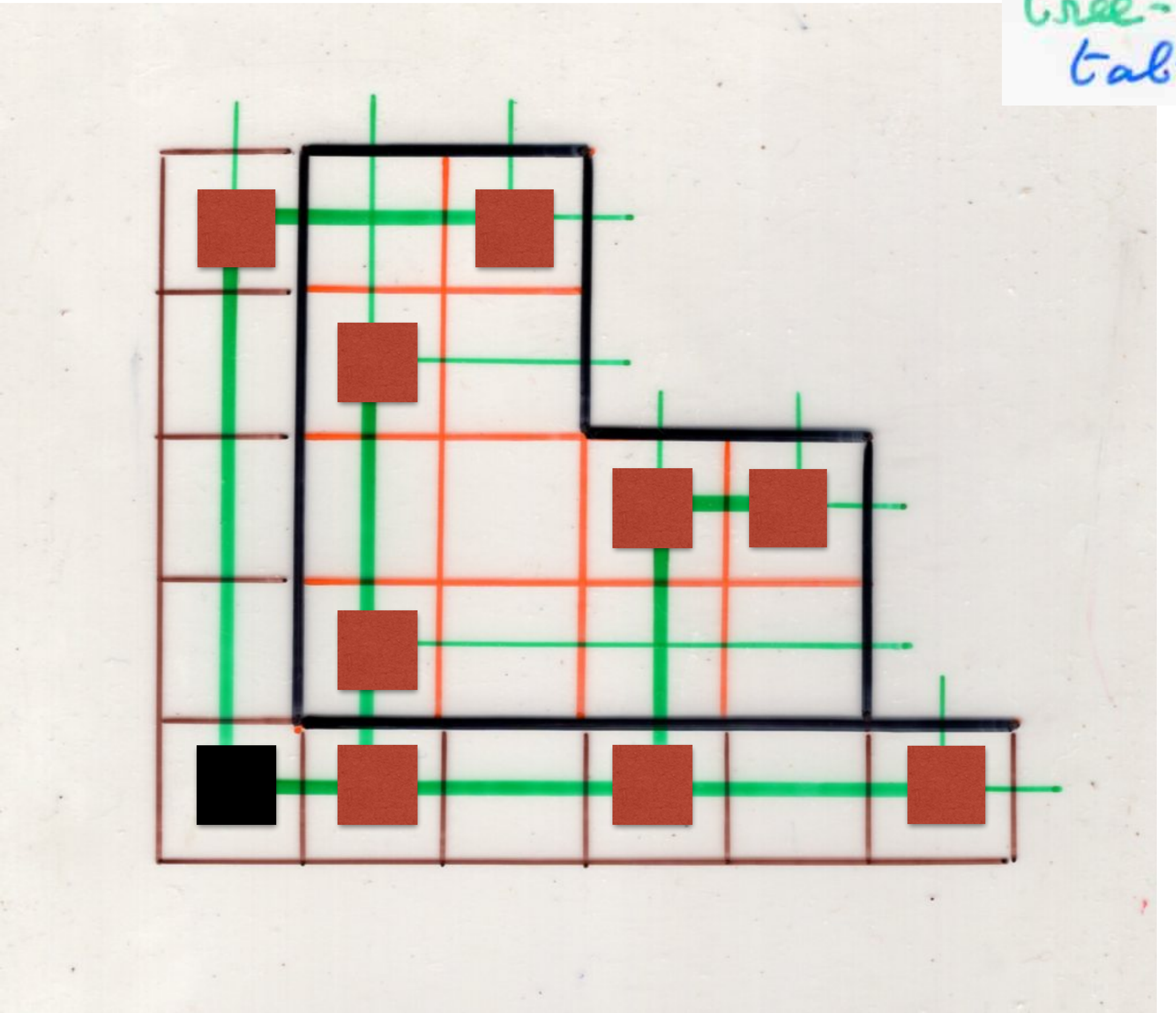
DY = □ YD

XY = □ YX

tree-like tableaux

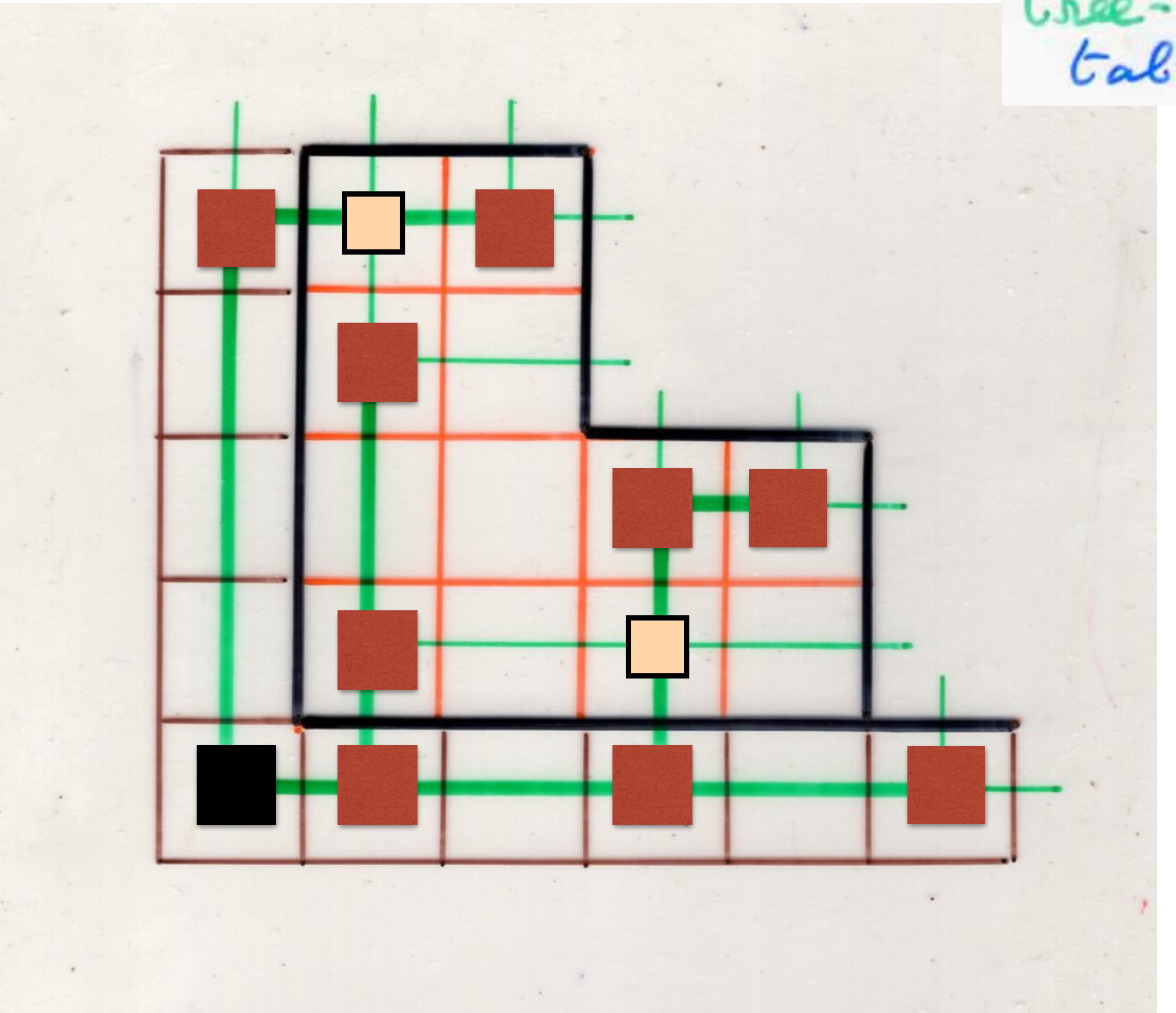
tree-like tableaux





tree-like
tableaux

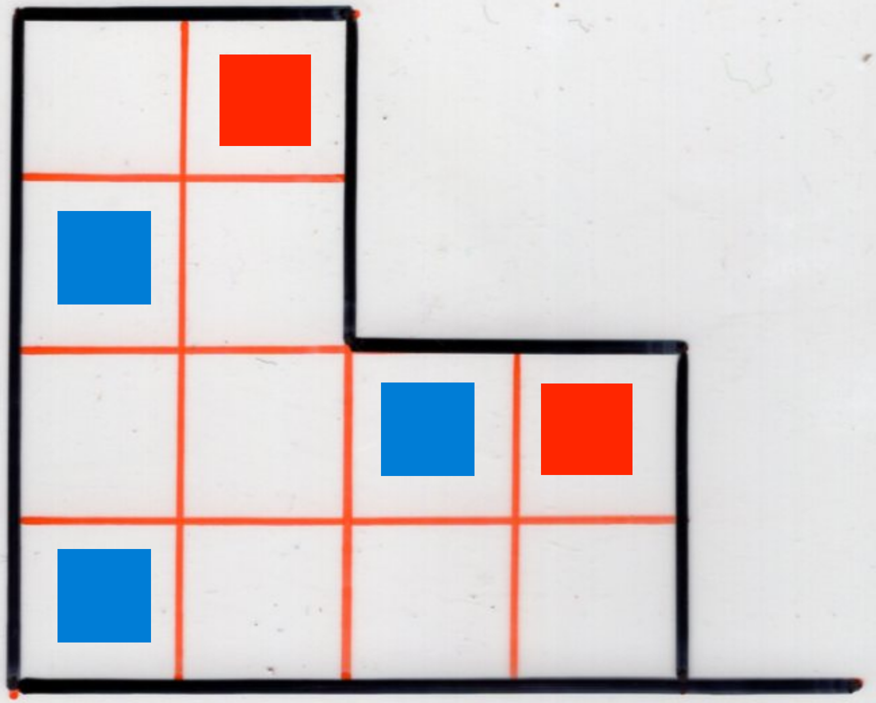
tree-like
tableaux



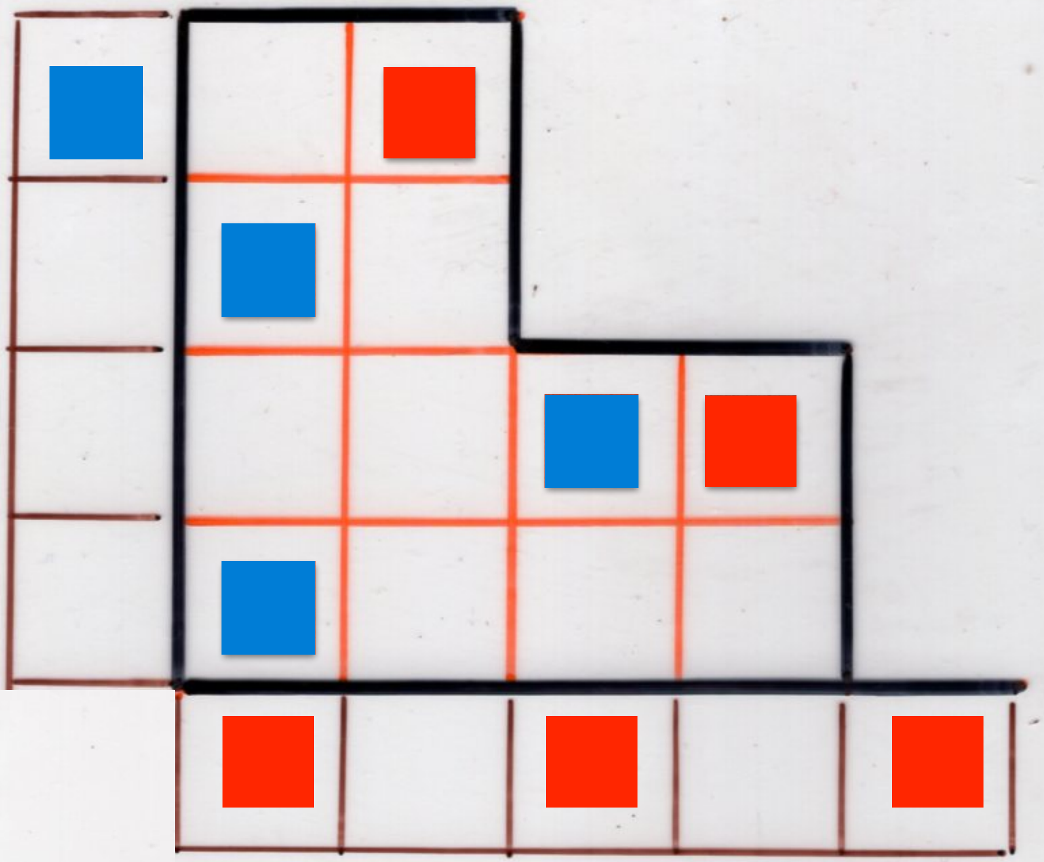
In fact, for the PASEP algebra, tree-like tableaux are the dual of alternative tableaux = reverse Q-tableau

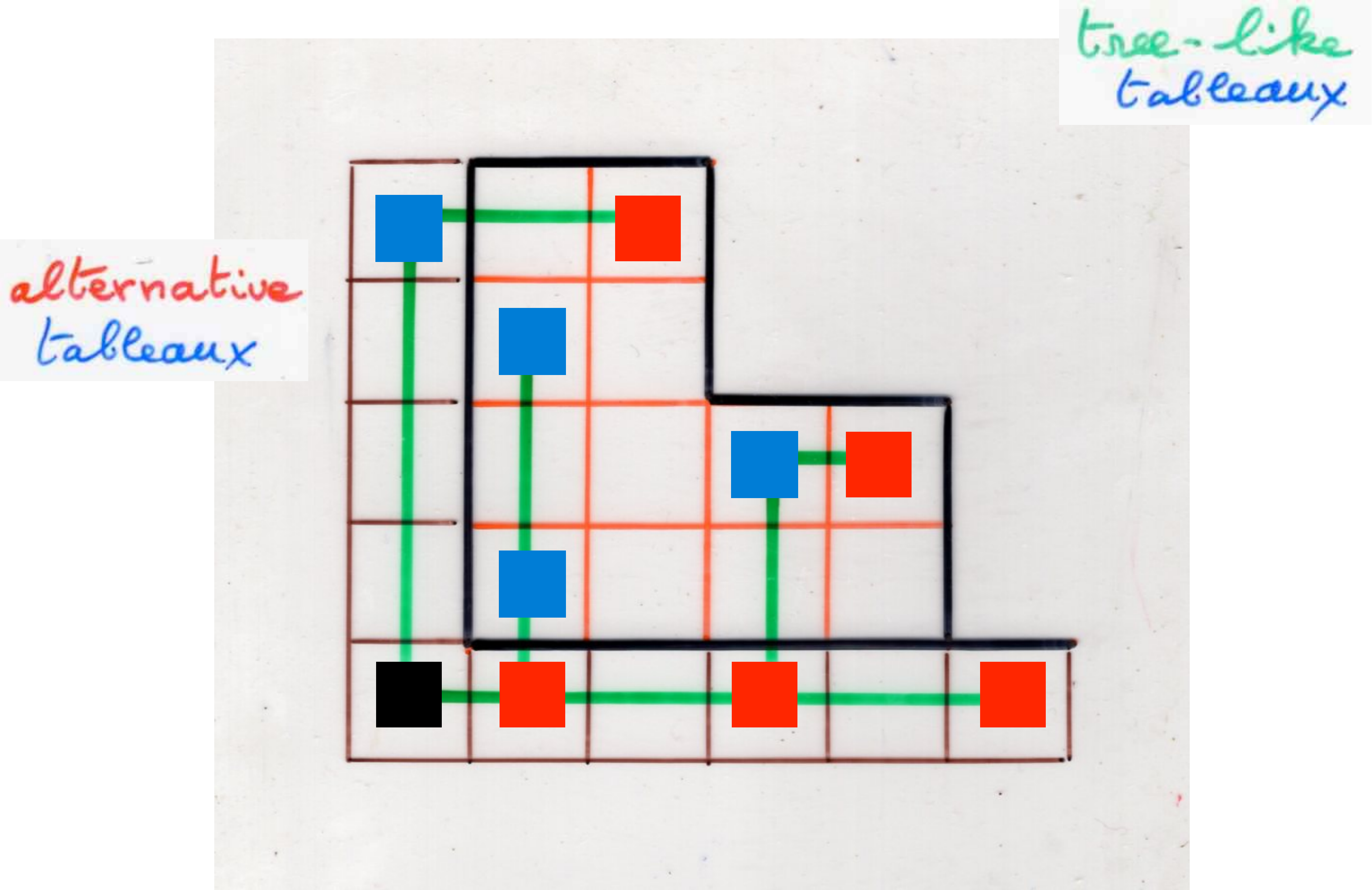
bijection alternative tableaux
tree-like tableaux

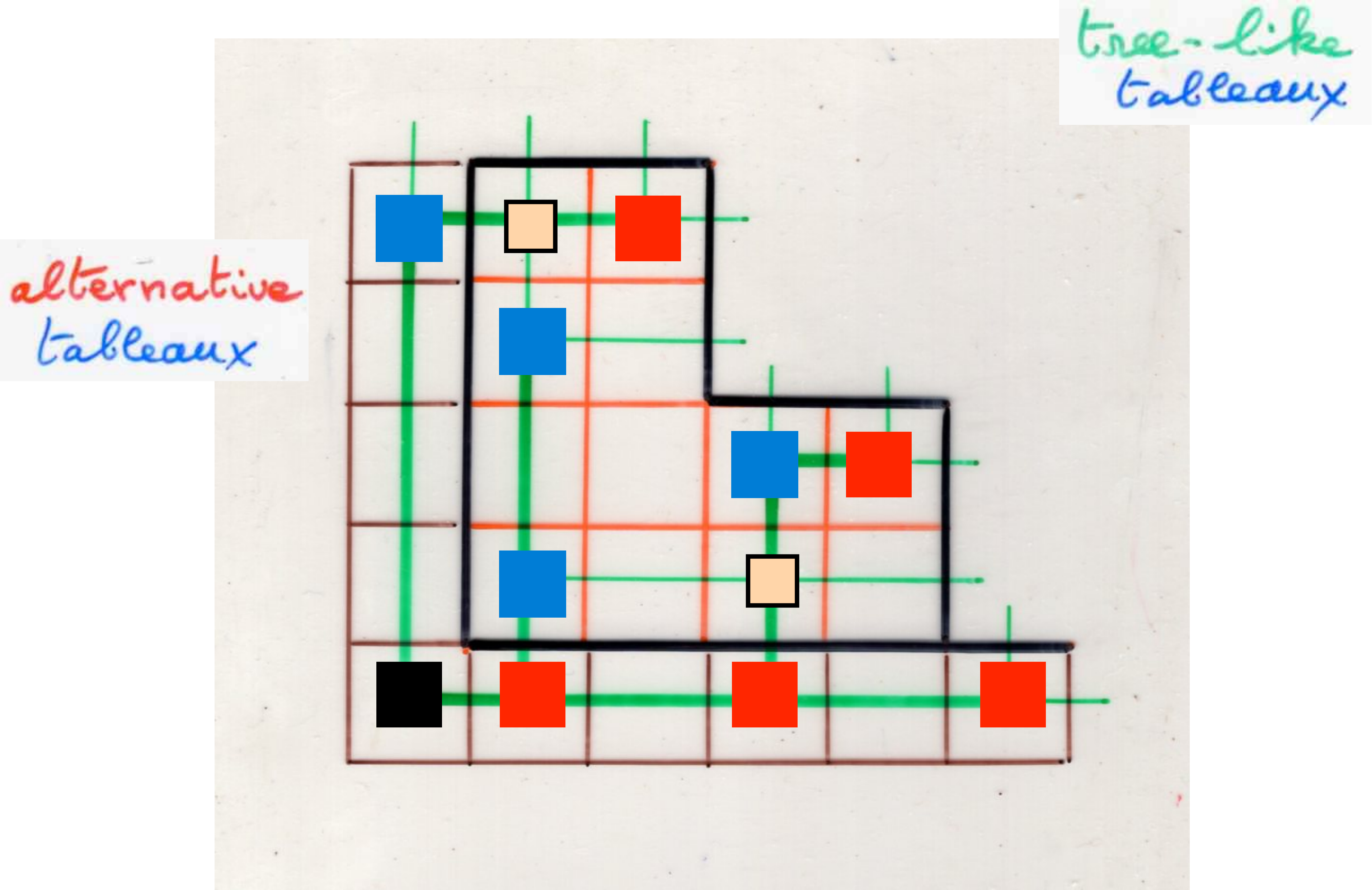
alternative
tableaux



alternative tableaux




tree-like
tableaux
alternative
tableaux


alternative
tableaux
tree-like
tableaux

Dual Q-tableaux

= reverse Q-tableau

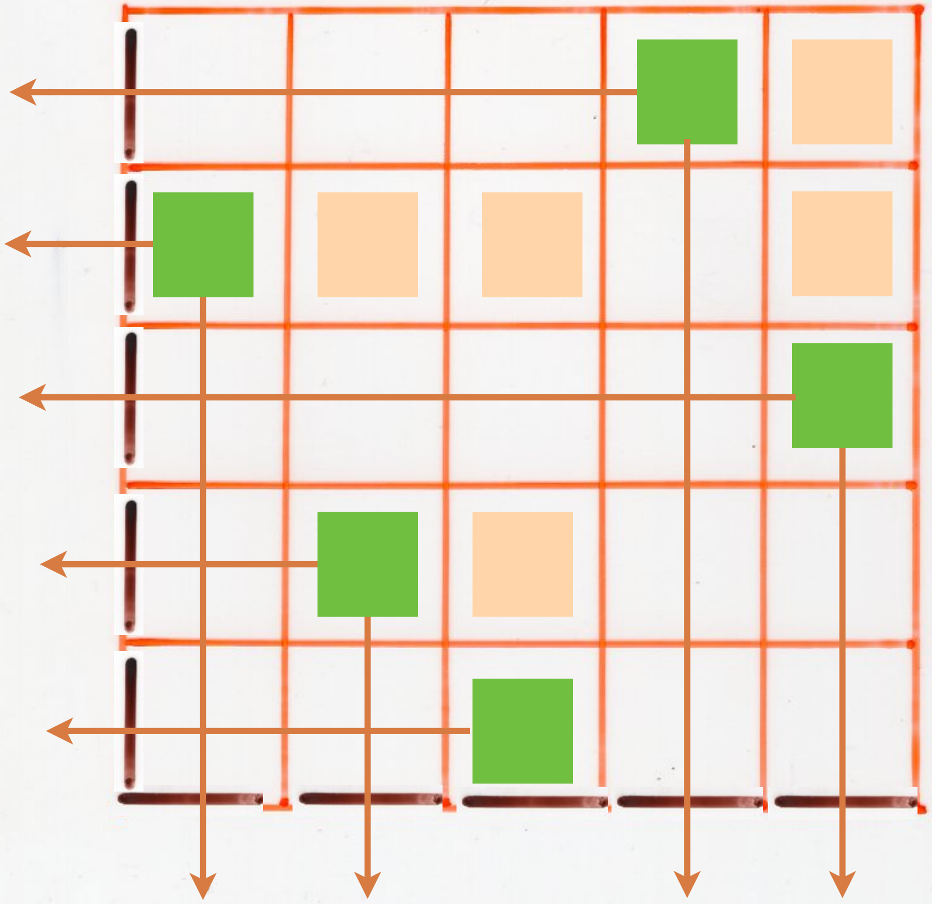
example with the Weyl-Heisenberg algebra

$$UD = DU + Id$$

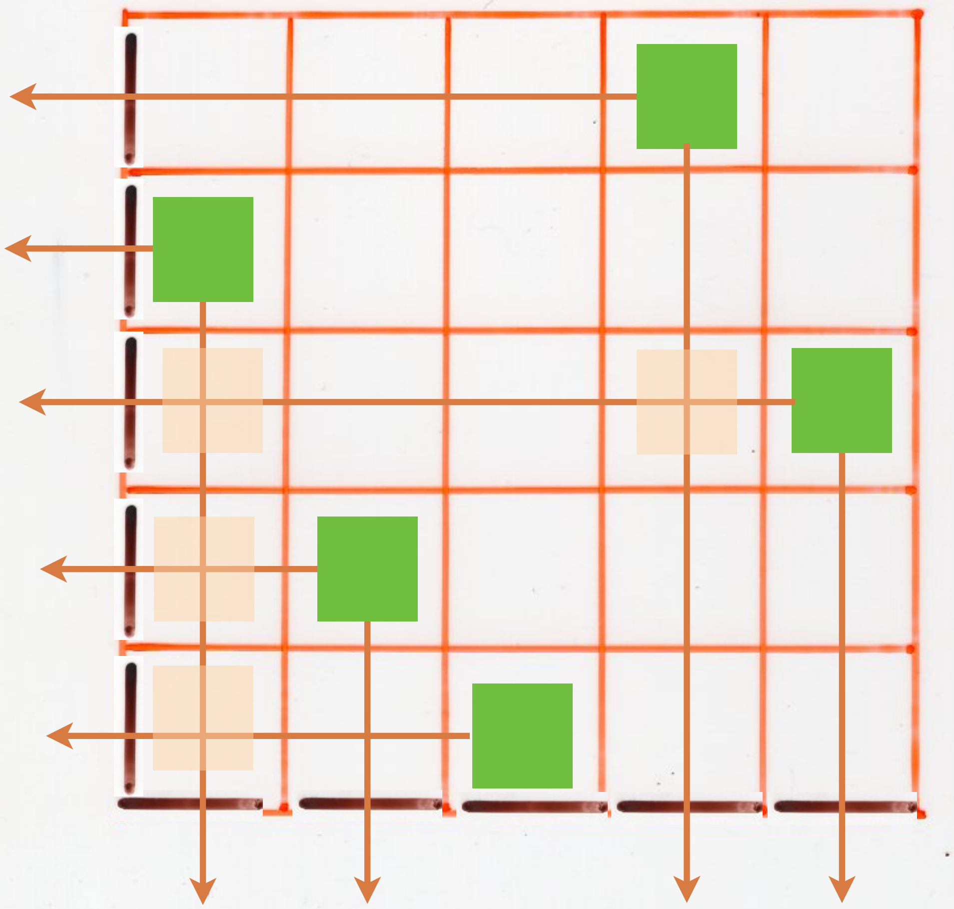
Weil-Heisenberg algebra

$$Q \begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$

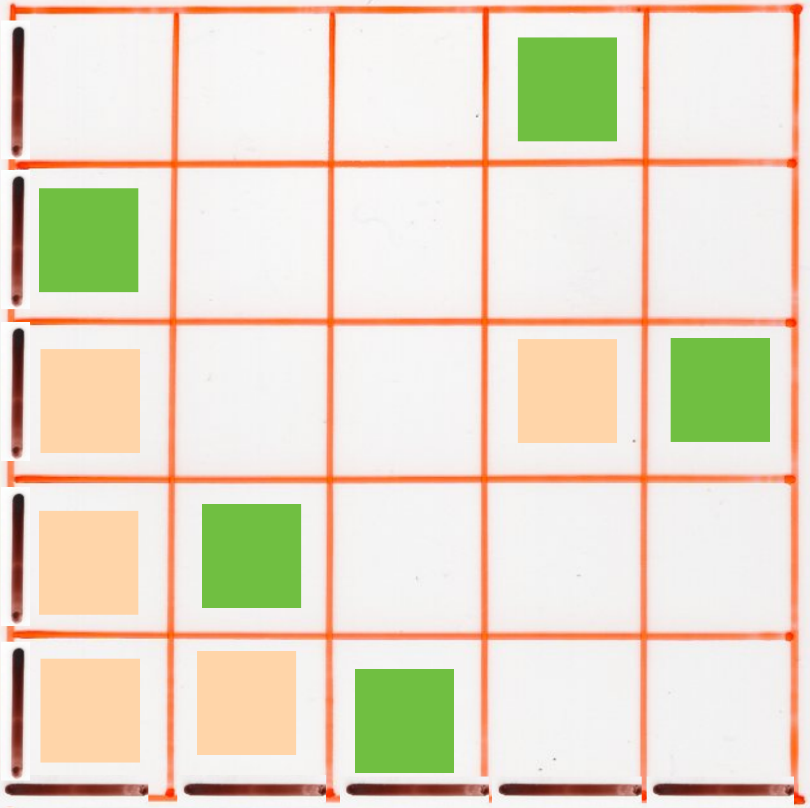
$$Q\begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$



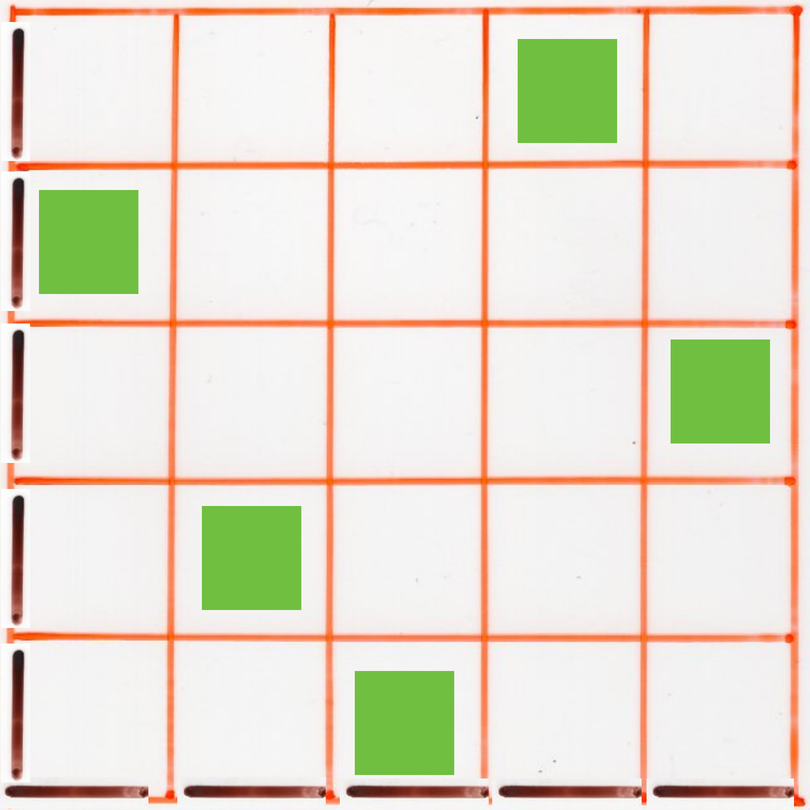
$$Q\begin{cases} UD = q_1\, DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$



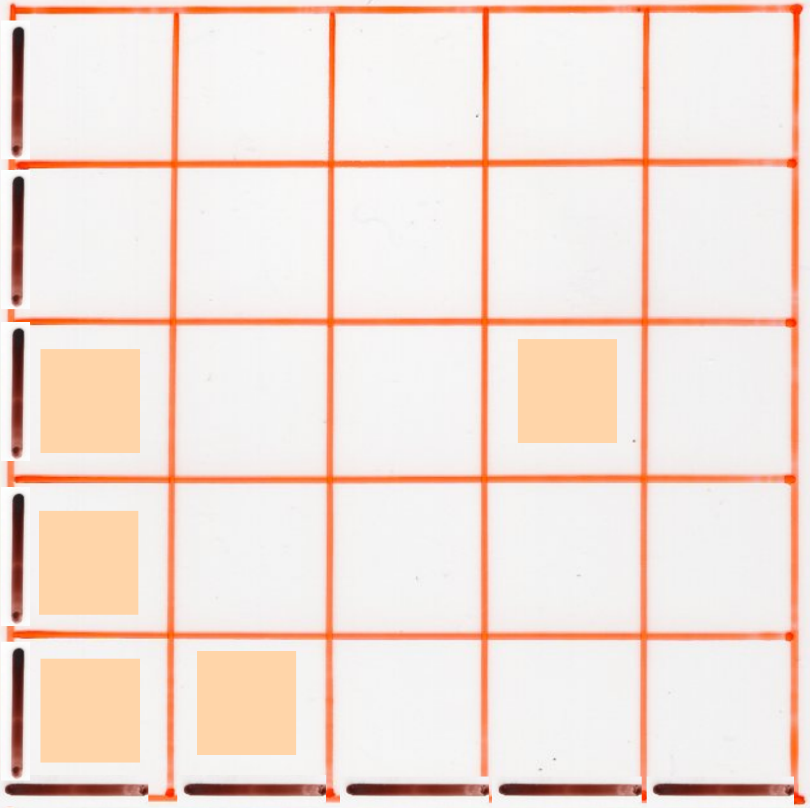
$$Q\begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$



$$
Q \begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}
$$



$$Q\begin{cases} UD = q_1 DU + t\,\boxed{YX} \\ UY = YU \\ XD = DX \\ XY = q_2\,\boxed{YX} \end{cases}$$



reverse quadratic algebra

reverse
(or dual ?)

quadratic algebra

$$\mathbf{Q} : \left\{ B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \right.$$

$\forall\, i \in I, \forall\, j \in J$

$$\mathbf{Q}^{\dagger} \left\{ A_k B_l = \sum_{i,j} c_{ij}^{kl} B_j A_i \right.$$

$\forall\, k \in I,\ l \in J$

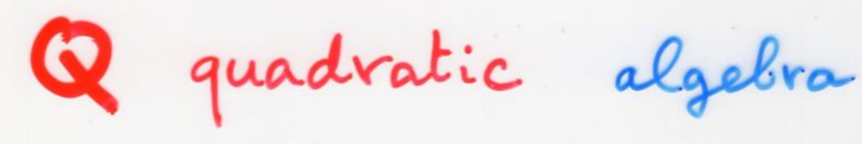
(possibly 0)

PASEP algebra

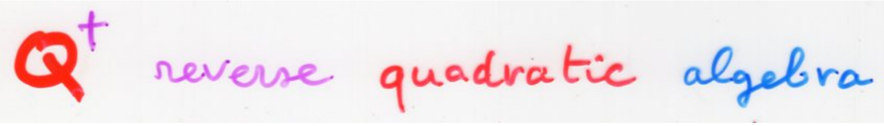
$$\mathbf{Q}\begin{cases} DE = q\,ED + EX + YD \\ XE = EX \\ DY = YD \\ XY = YX \end{cases}$$

reverse PASEP algebra (dual)

$$\mathbf{Q}^{\dagger}\begin{cases} ED = q\,DE \\ EX = XE + DE \\ YD = DY \quad DE \\ YX = XY \end{cases}$$

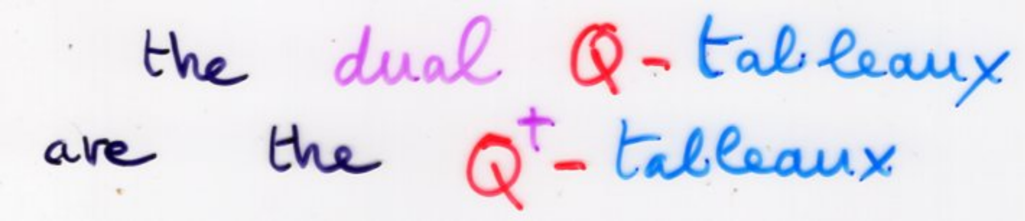


Q quadratic algebra



$Q^{\dagger}$ reverse quadratic algebra

= reverse Q-tableau



the dual Q-tableaux are the $Q^{\dagger}$-tableaux

Q

DE = □ ED + 🟥 EX + 🟦 YD

XE = □ EX

DY = □ YD

XY = □ YX

alternative tableaux

$Q^{\dagger}$

ED = □ DE

EX = □ XE + 🟥 DE

YD = □ DY 🟦 DE

YX = □ XY

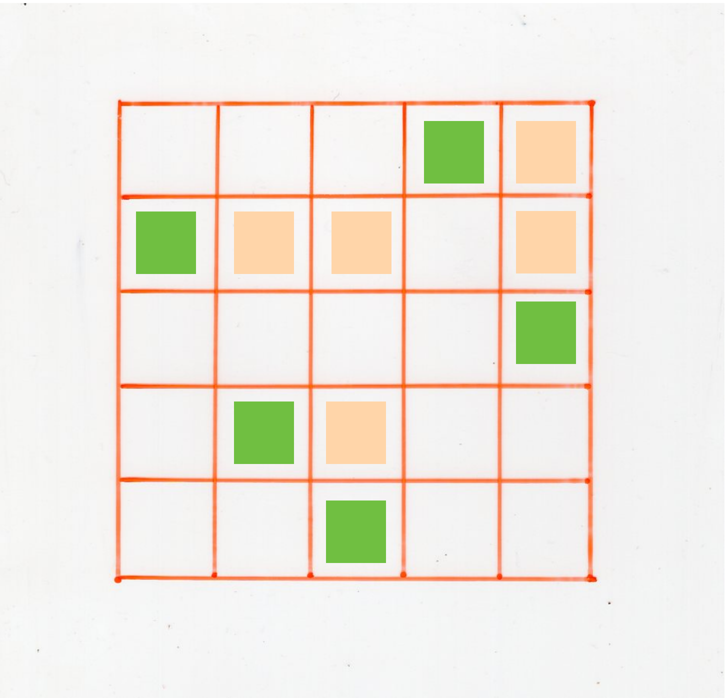
$$Q\left\{\begin{aligned} DE &= \square\, ED + \blacksquare\, EX + \blacksquare\, YD \\ XE &= \square\, EX \\ DY &= \square\, YD \\ XY &= \square\, YX \end{aligned}\right.$$

$$Q^{\dagger}\left\{\begin{aligned} ED &= \square\, DE \\ EX &= \square\, XE + \blacksquare\, DE \\ YD &= \square\, DY \quad \blacksquare\, DE \\ YX &= \square\, XY \end{aligned}\right.$$

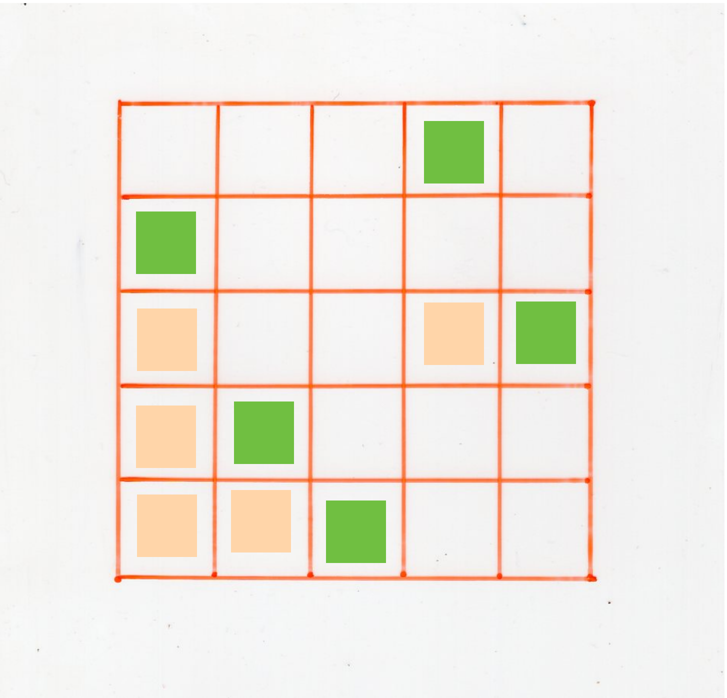
tree-like
tableaux

Weyl-Heisenberg algebra

$$\mathbf{Q}\begin{cases} UD = q_1 DU + t\,YX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$

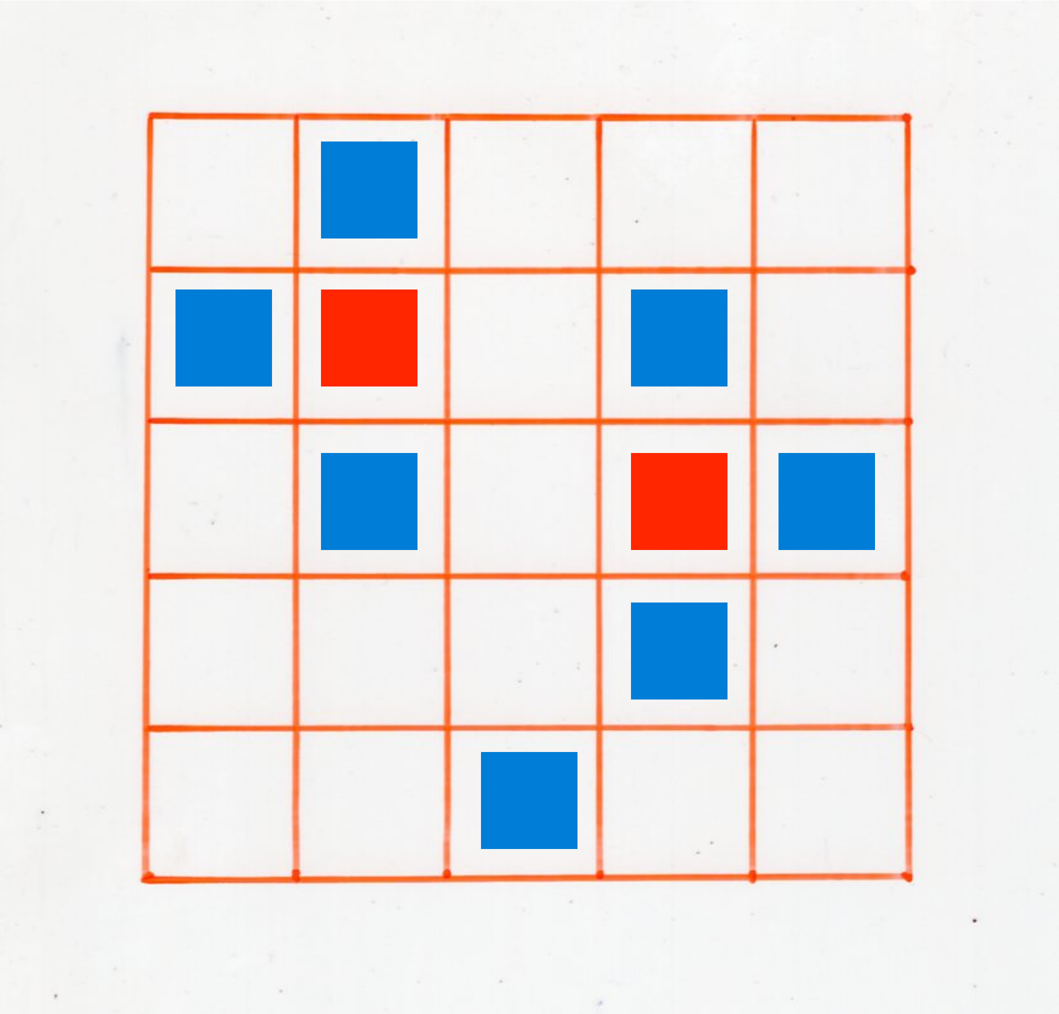


$$\mathbf{Q}^{+}\begin{cases} YX = q_2\, XY + t\, XY \\ YU = UY \\ DX = XD \\ DU = q_1\, UD \end{cases}$$



Alternating sign matrices

ASM



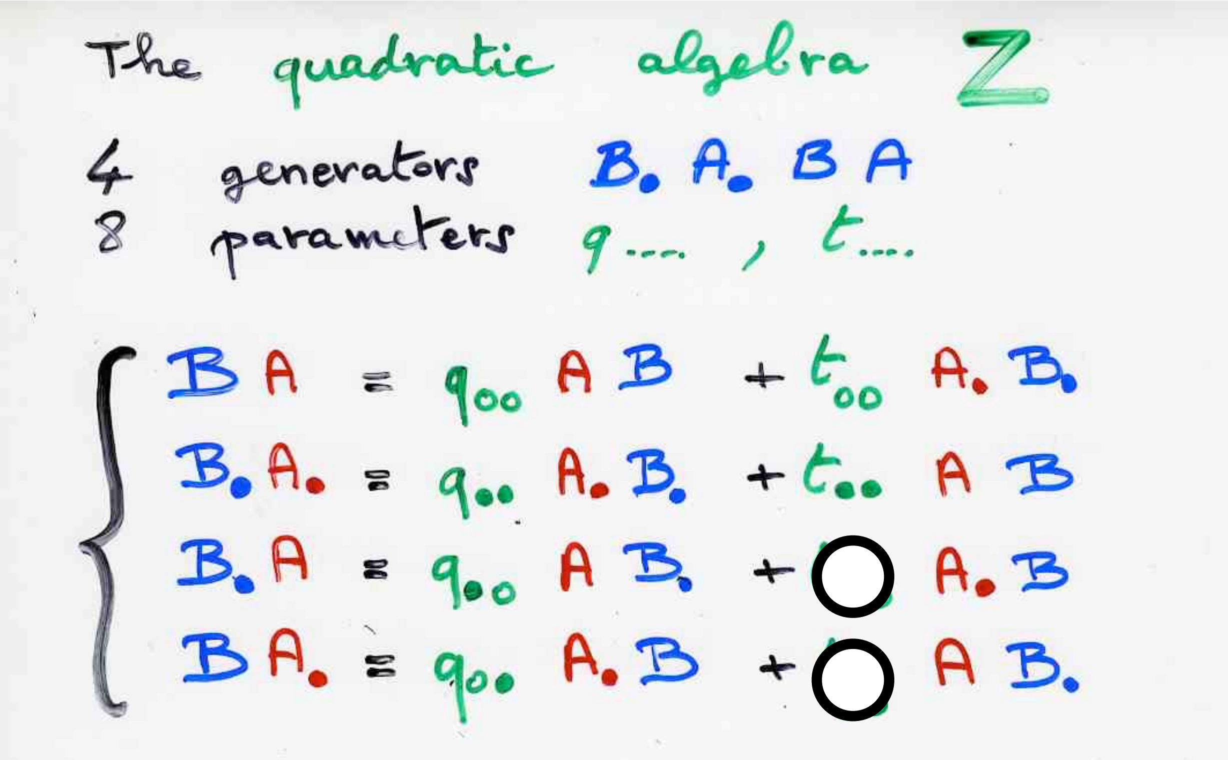
	🟦			
🟦	🟥		🟦	
	🟦		🟥	🟦
			🟦	
		🟦		

A, A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$



The quadratic algebra Z

4 generators $B_\bullet$ $A_\bullet$ B A

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

see ch 2c

XYZ-algebra

The quadratic algebra $\mathbb{Z}$

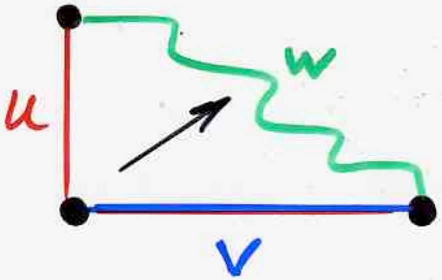
4 generators $B_\bullet \quad A_\bullet \quad B \quad A$
8 parameters $q_{\cdots} \ , \ t_{\cdots}$

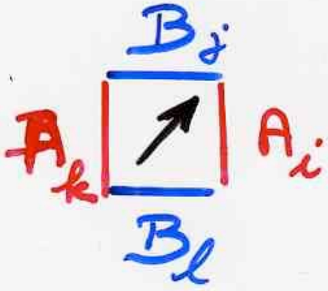
$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\ B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A\,B_\bullet \end{cases}$$

$$\mathbb{Z}^{\dagger} \begin{cases} A\,B = q_{\circ\circ}\, B\,A + t_{\bullet\bullet}\, B_\bullet A_\bullet \\ A_\bullet B_\bullet = q_{\bullet\bullet}\, B_\bullet A_\bullet + t_{\circ\circ}\, B\,A \\ A_\bullet B = q_{\circ\bullet}\, B\,A_\bullet + t_{\bullet\circ}\, B_\bullet A \\ A\,B_\bullet = q_{\bullet\circ}\, B_\bullet A + t_{\circ\bullet}\, B\,A_\bullet \end{cases}$$

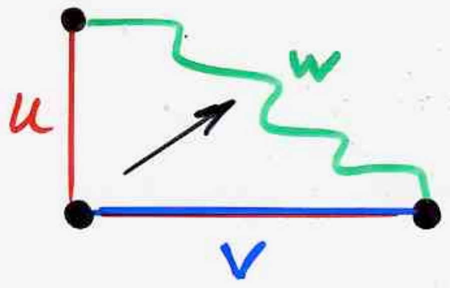
reverse quadratic algebra
reverse planar automata

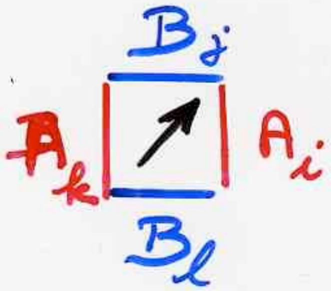
reverse planar automata





reverse planar automata





reverse
(or dual ?)

quadratic algebra

$$Q : \left\{ B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \right.$$

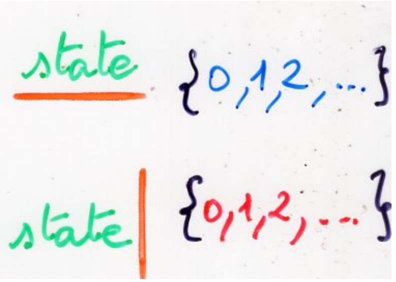
$\forall\, i \in I, \forall\, j \in J$

$$Q^{\dagger} \quad \left\{ A_k B_l = \sum_{i,j} c_{ij}^{kl} B_j A_i \right.$$

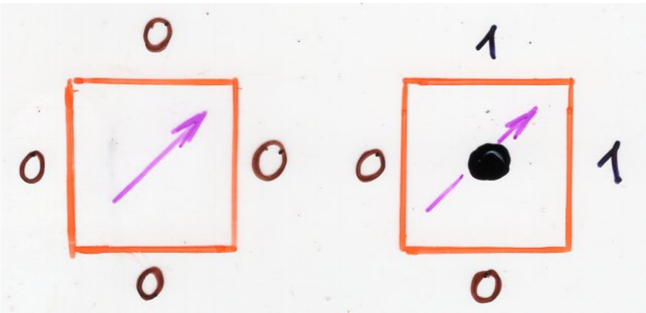
$\forall\, k \in I,\ l \in J$

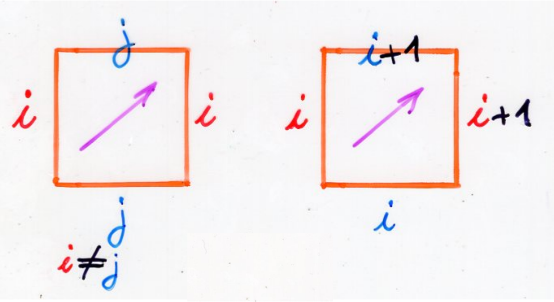
$\underset{0}{\text{(possibly)}}$

"local rules"
on the edges

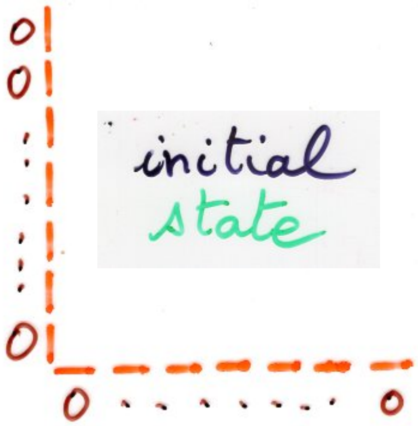


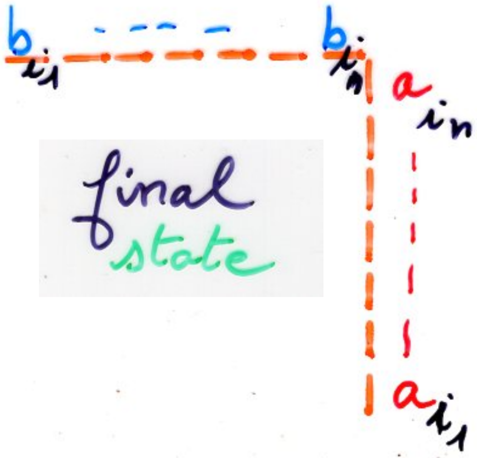
set of labels
$L = \{\square, \boxdot\}$





The RSK (reverse) planar automaton

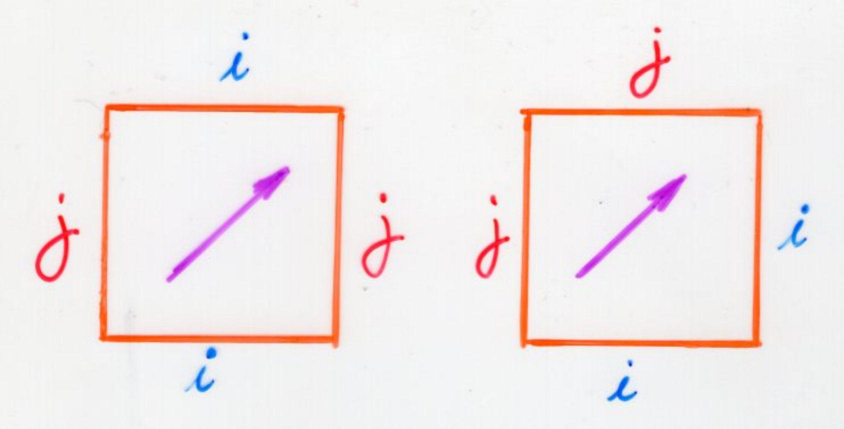




Ch 1d, p102

jeu de taquin
local rules on edges

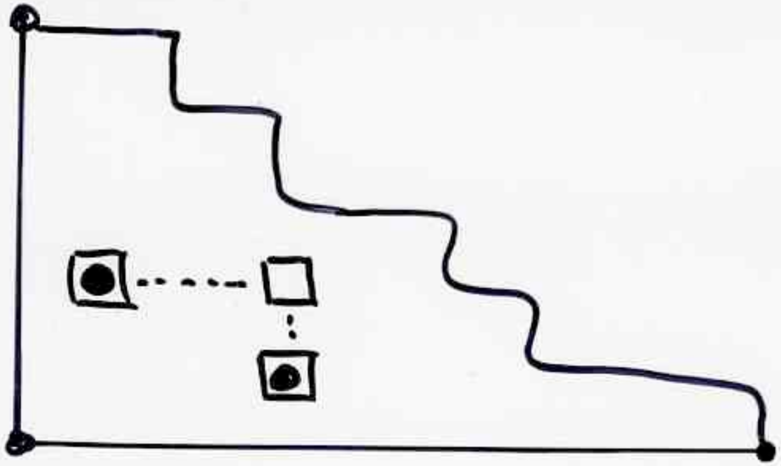
set of labels
$L = \{\square, \quad \}$



$i, j \in \mathbb{Z}$

$|i-j| \geqslant 2$

$|i-j| \leqslant 1$



Postnikov

⅃-diagrams

Le - . . .

Duplication of equations

in quadratic algebras

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication" of the commutation relations defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$\begin{cases} UD = DU + \boxed{YX} \\ UY = YU \\ XU = UX \\ XY = \boxed{YX} \end{cases}$$

"duplication"
of the commutation relations
defining the algebra Q

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"duplication"
of the commutation relations
defining the algebra Q

$$UD = DU + Y_1 X_1$$

$$X_1 Y_1 = Y_2 X_2$$

$$X_2 Y_2 = Y_3 X_3$$

- - - - - - - - - - -

$$X_i Y_i = Y_{i+1} X_{i+1}$$

. . . . - - - - .

$$U Y_i = Y_i U$$

$$X_j U = U X_j$$

- - - - - - -

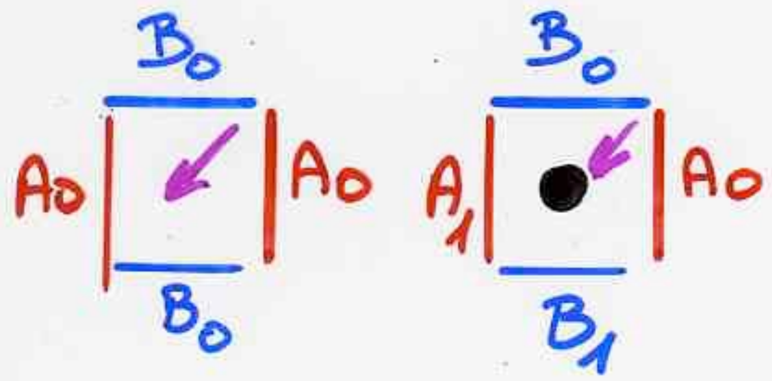
The "RSK planar automaton"

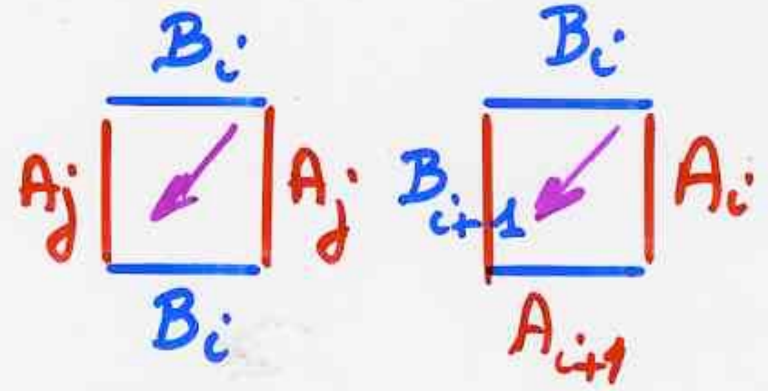
$\mathcal{B} = \{B_0, B_1, \dots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \qquad A_k\}$

$w \in \{B_0, A_0\}^*$

$S = \{\square, \boxdot\}$

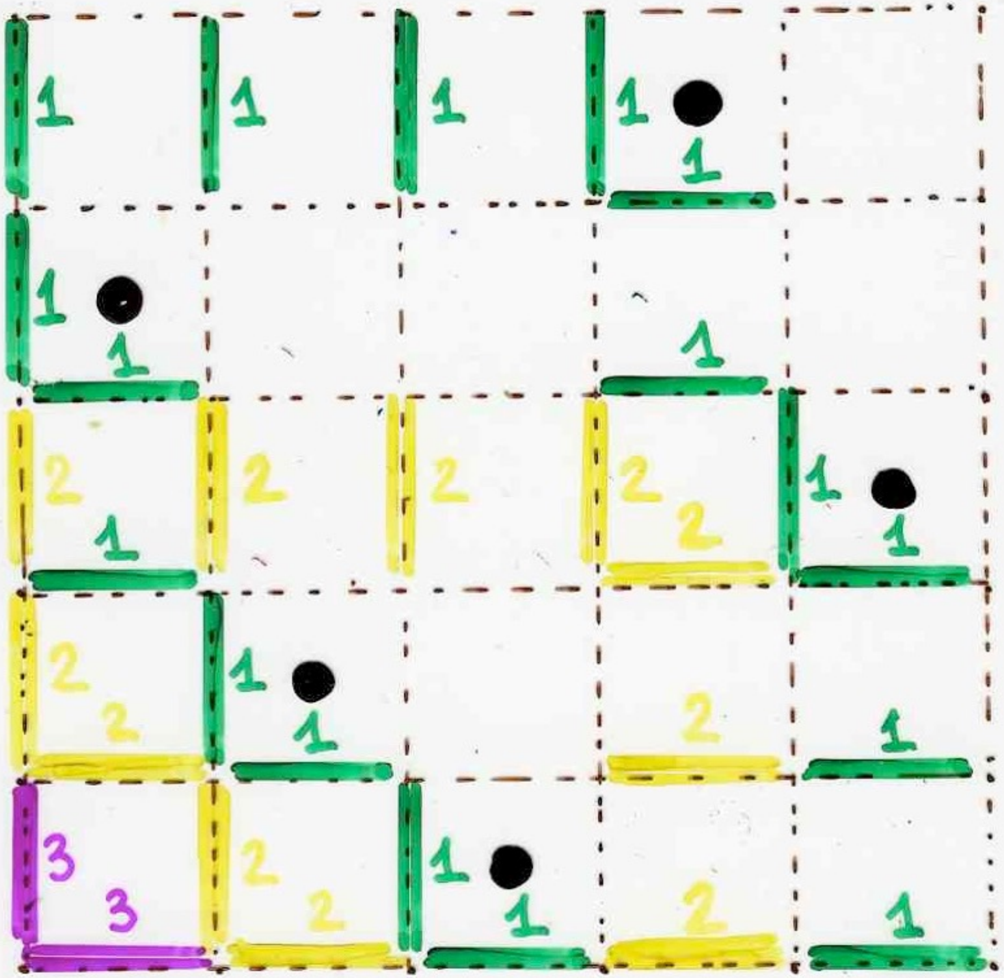




$i \neq j$

$$\begin{cases} U = B_0 \\ X_i = B_i \end{cases} \qquad \begin{cases} D = A_0 \\ Y_i = A_i \end{cases}$$

$i \geq 1 \qquad i \geq 1$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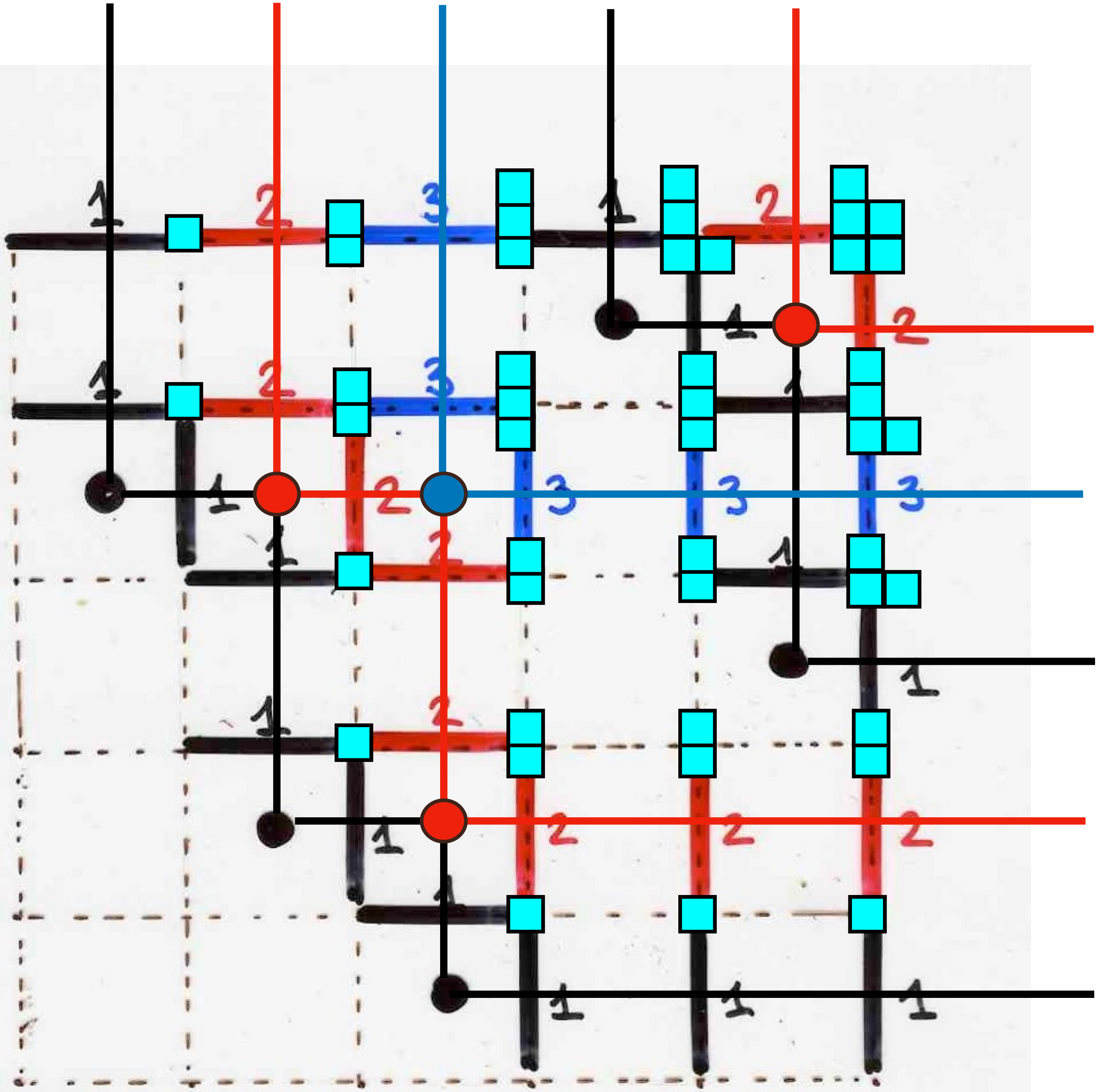
There are two constructions
dual each other :

→ local rules, growth diagrams
geometric RSK , ...
representation of U , D with U_i , D_j ...

from the SW

or from the NE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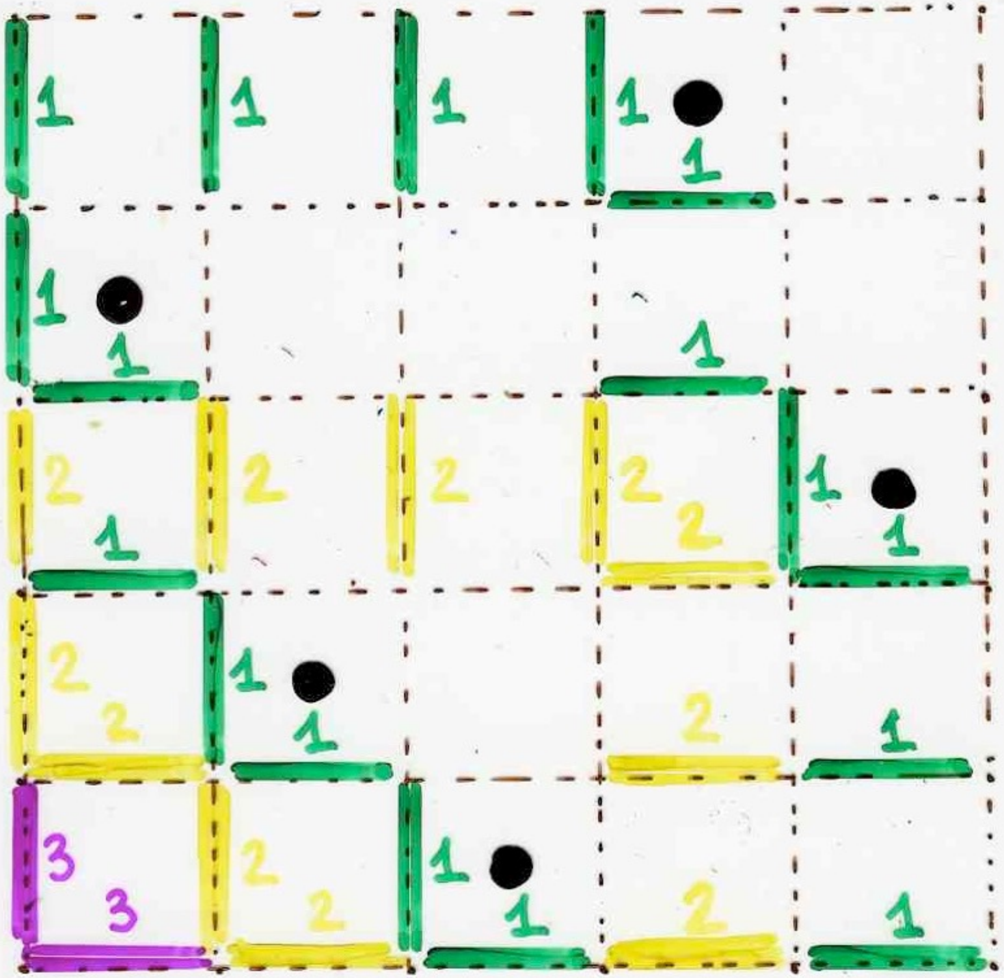

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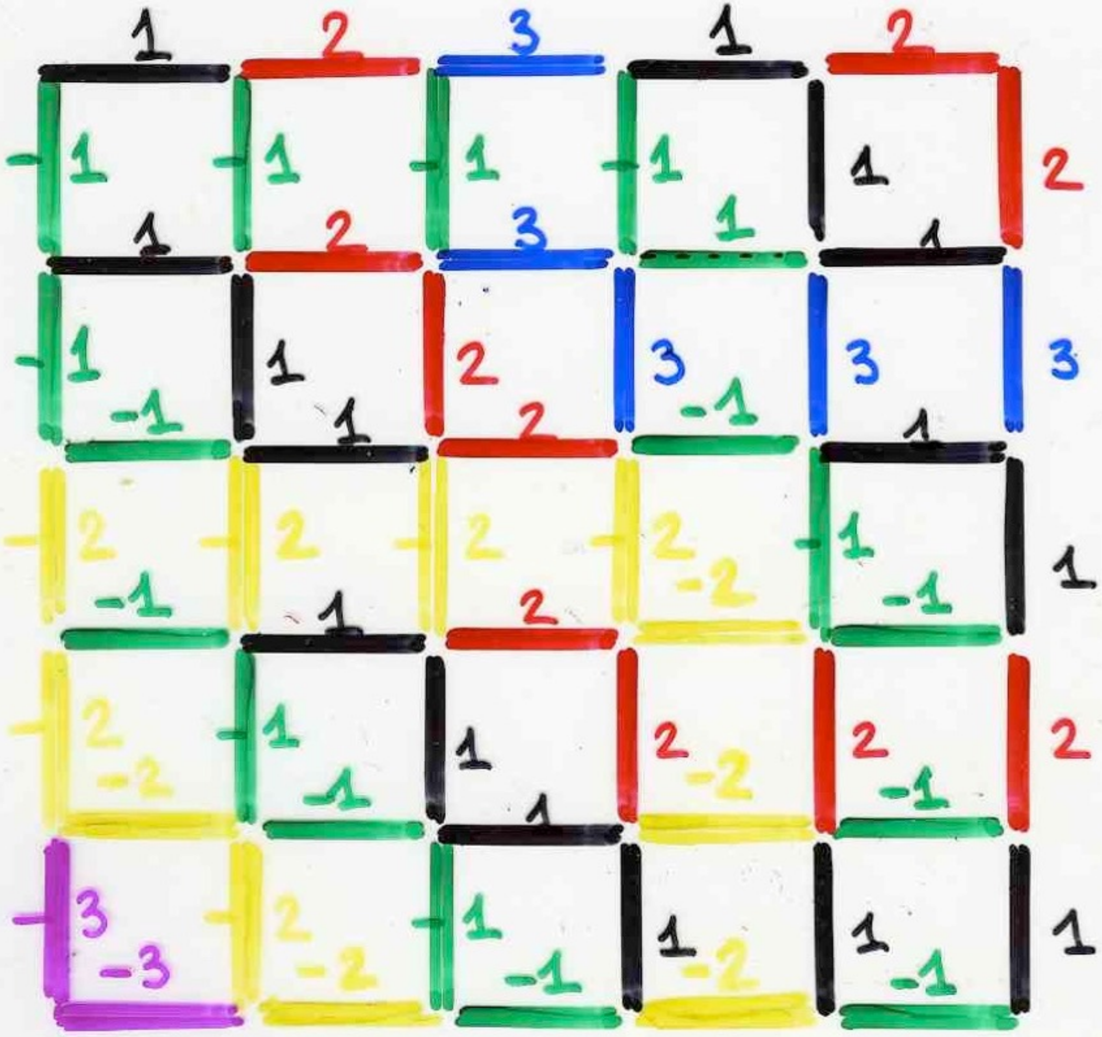
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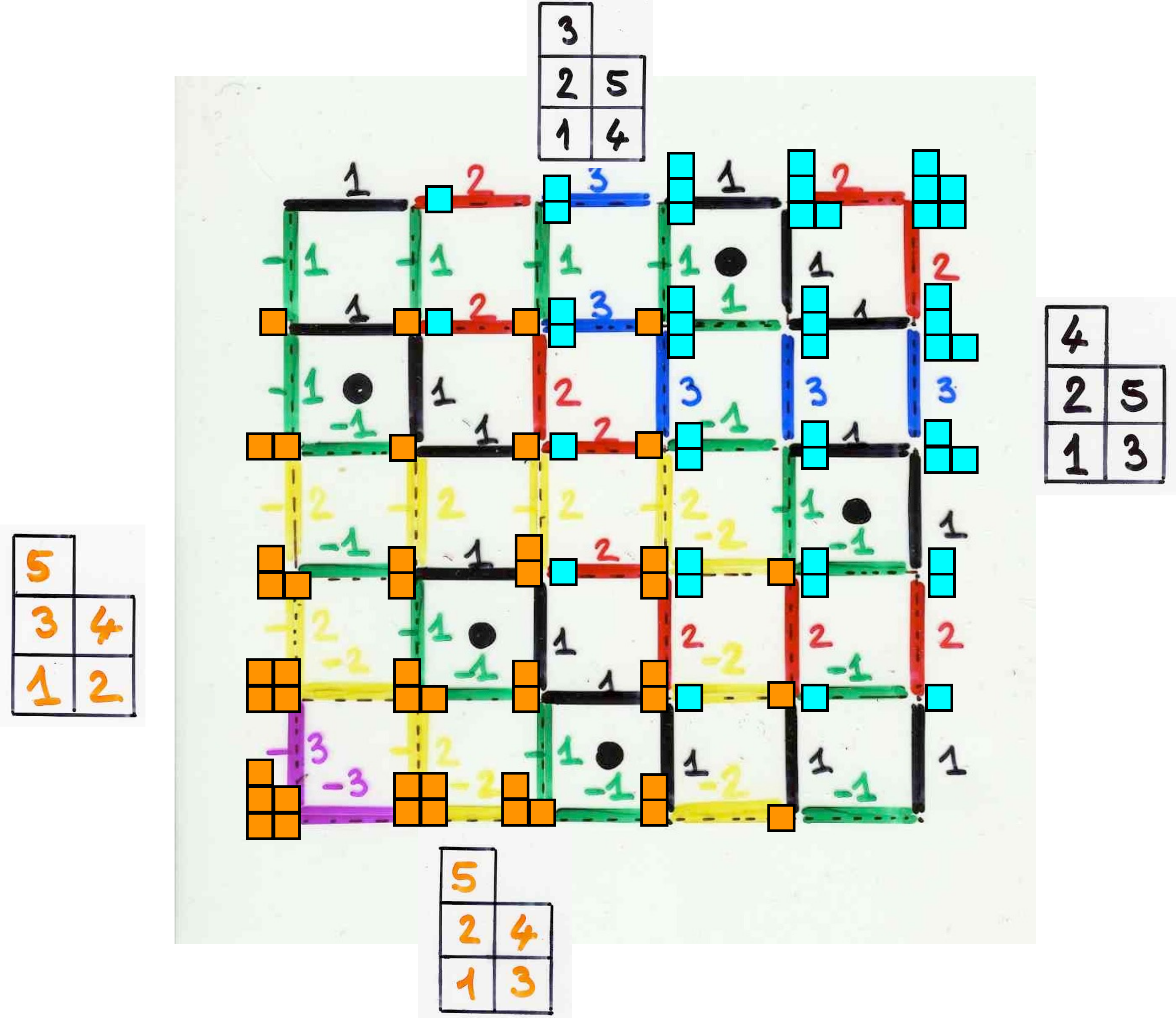
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2 5
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3 4
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Weil-Heisenberg algebra

$$\mathbf{Q}\begin{cases} UD = q_1 DU + t\,YX \\ UY = YU \\ XD = DX \\ XY = q_2 YX \end{cases}$$

$$\mathbf{Q}^{\dagger}\begin{cases} YX = q_2 XY + t\,XY \\ YU = UY \\ DX = XD \\ DU = q_1 UD \end{cases}$$

another « demultiplication »
of the algebra $UD=DU+Id$

$$\left\{\begin{aligned} UD &= DU + \boxed{YX} \\ UY &= YU \\ XU &= UX \\ XY &= \boxed{YX} \end{aligned}\right.$$

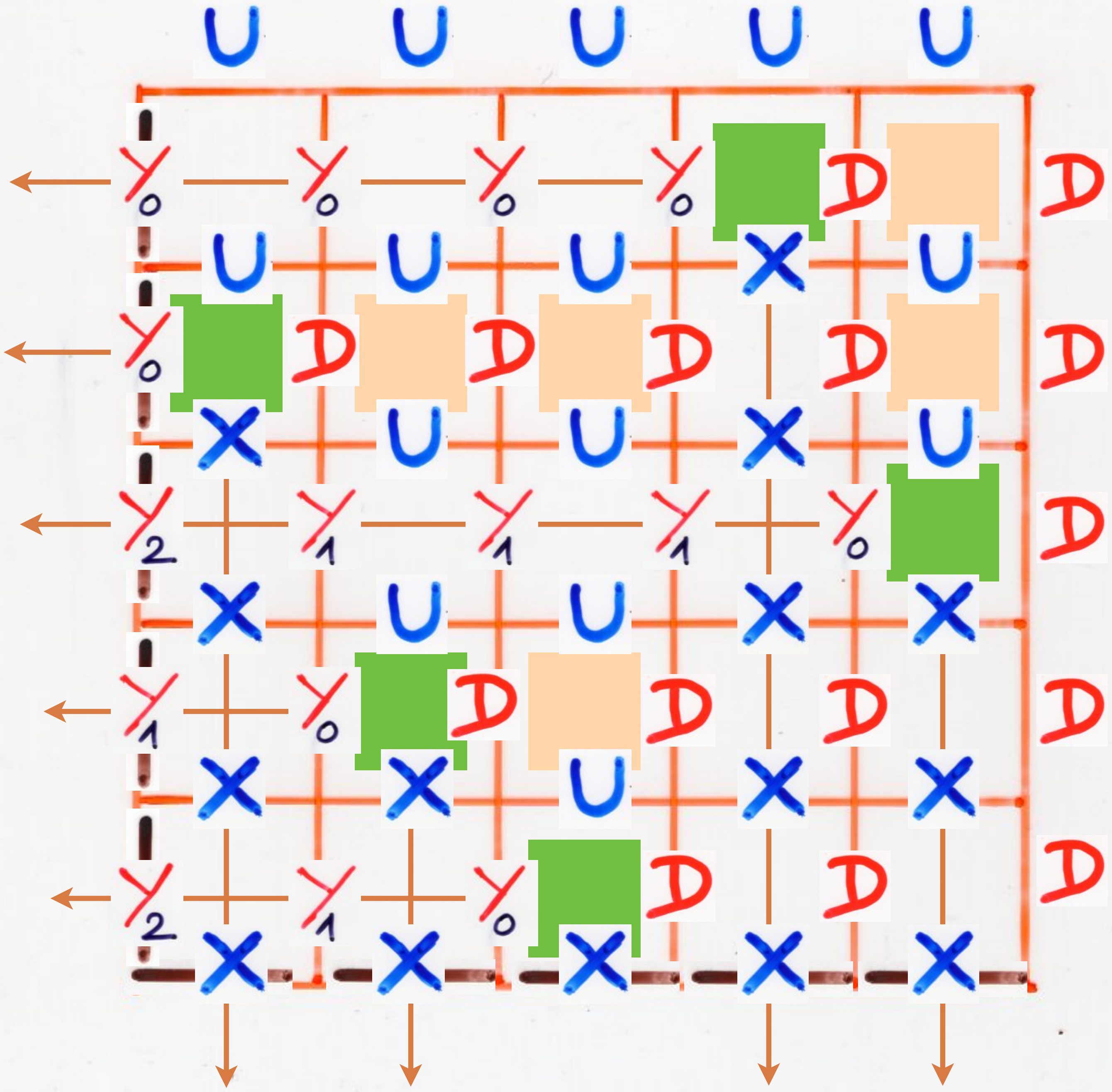
another "duplication" of the commutation relations of the algebra Q

$$UD = DU + Y_0 X$$

$$\big(\; XY_0 = Y_1 X$$

$$XU = UX$$

$$UY_i = Y_i U$$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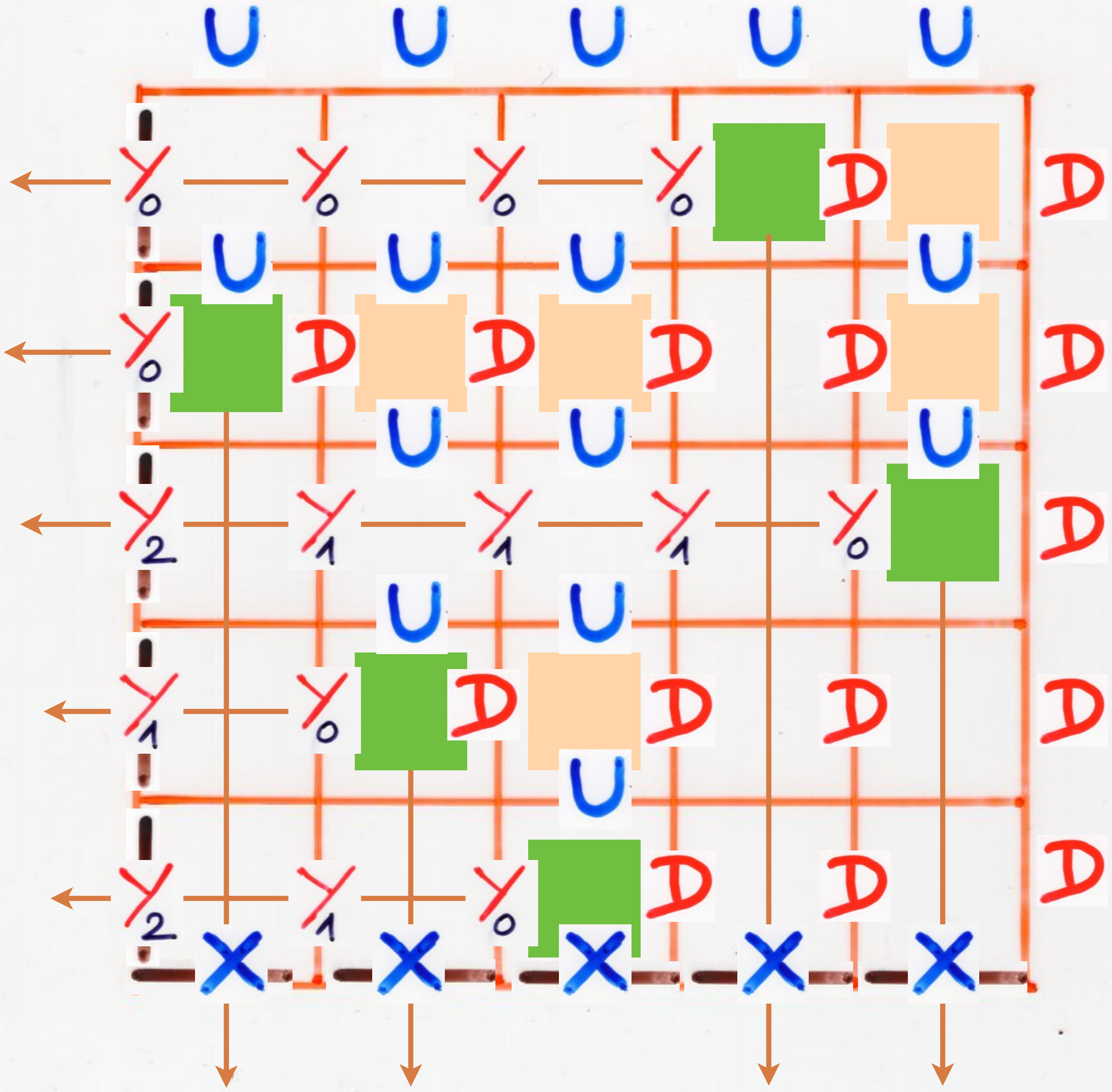
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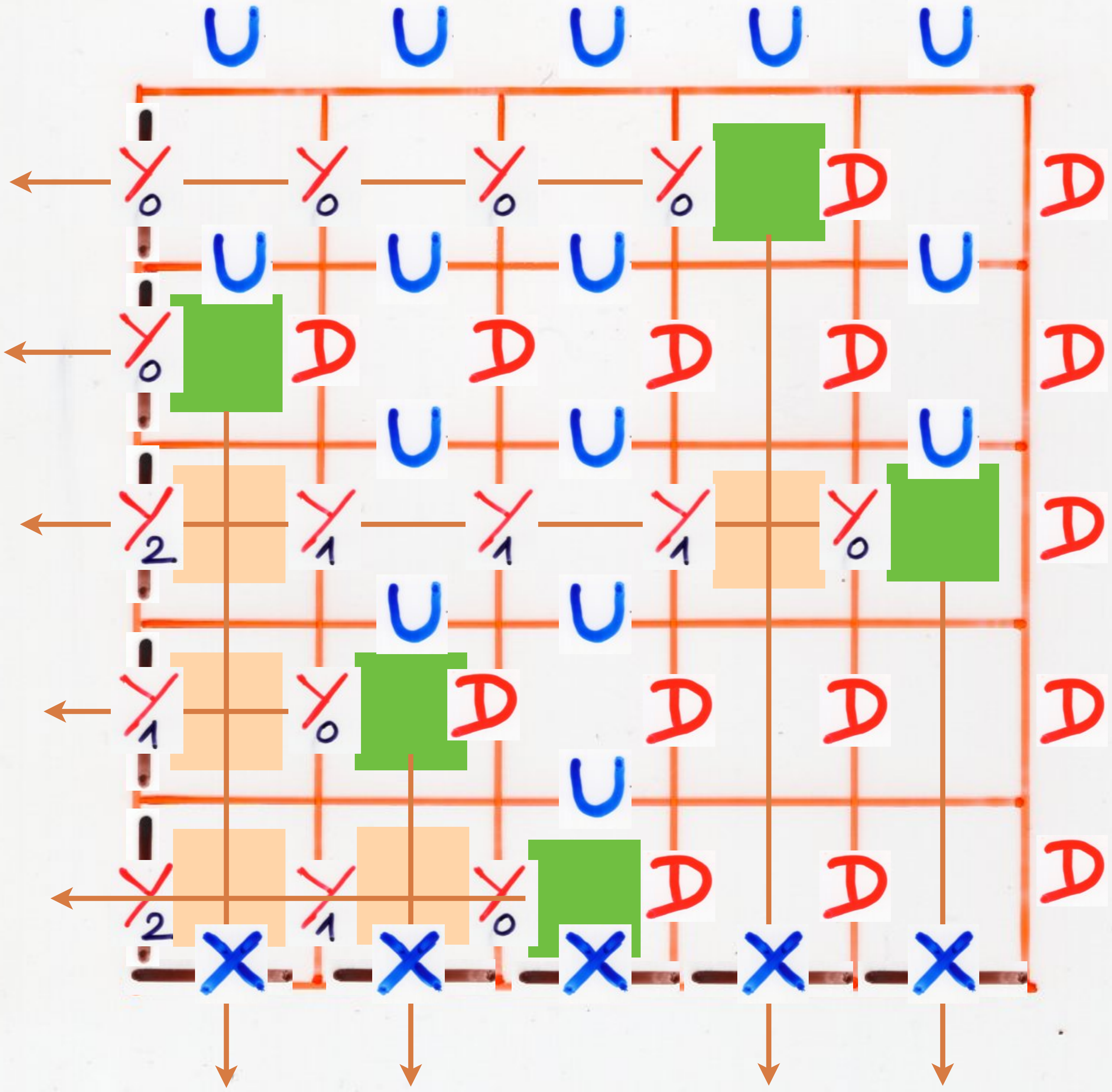
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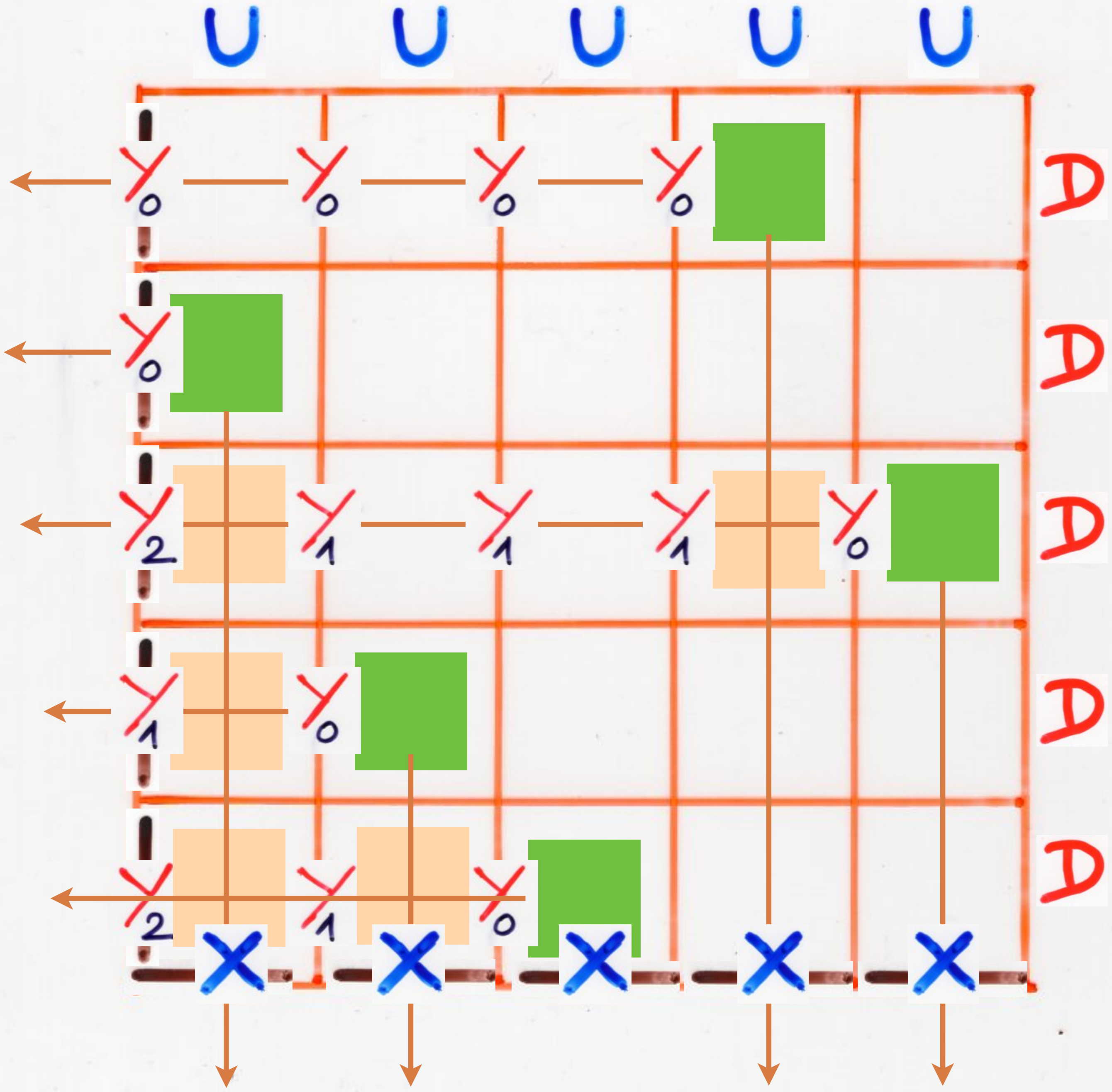
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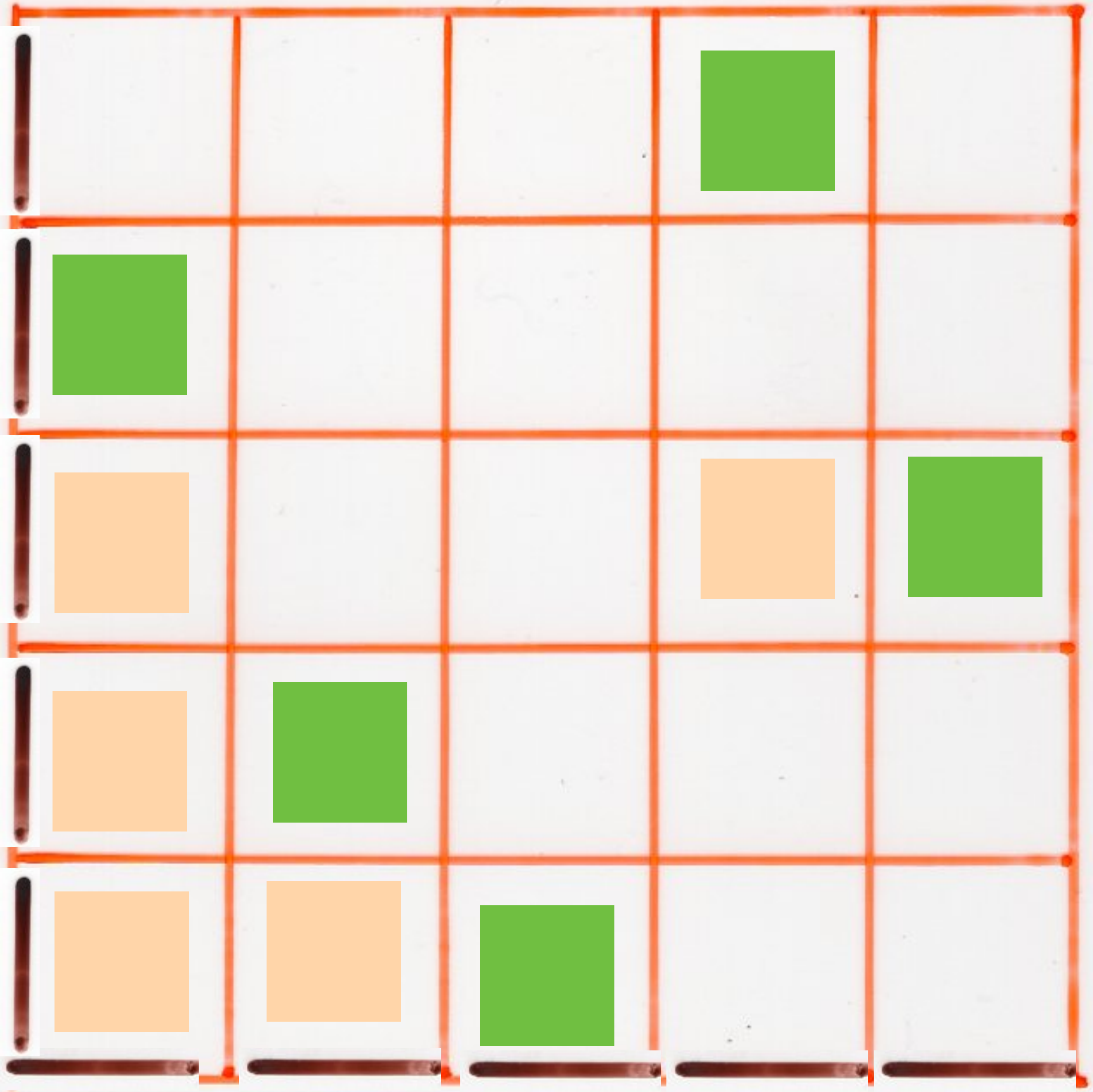
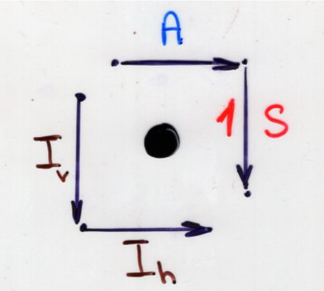
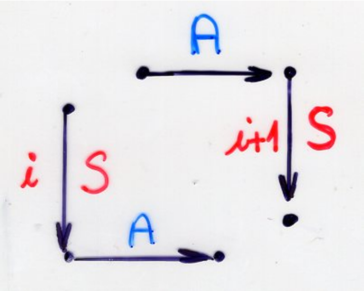
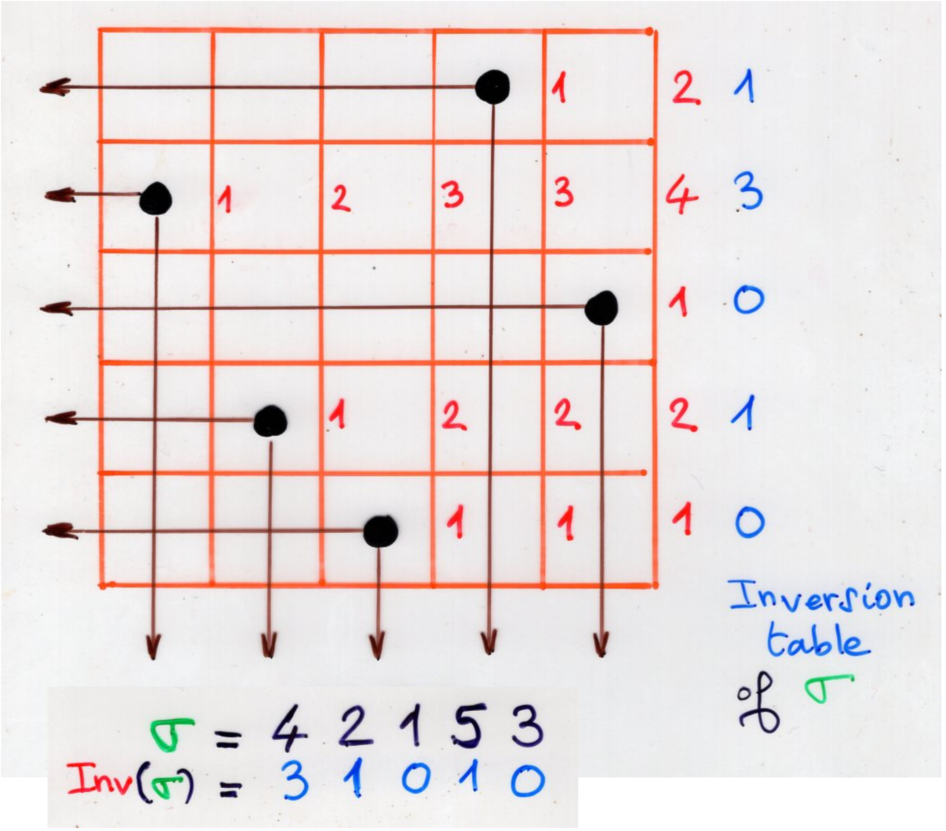






Ch 1c, p124

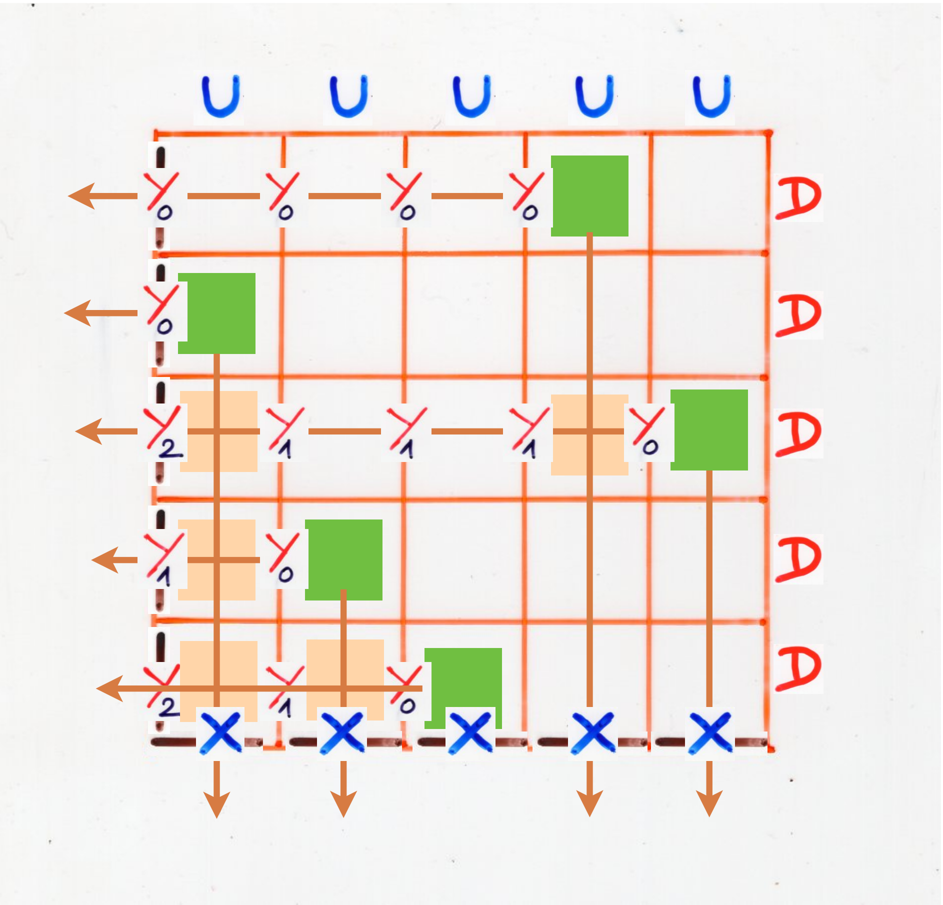
Representation of
UD=DU+Id
with Polya urns



$\sigma = 4\ 2\ 1\ 5\ 3$

$Inv(\sigma) = 3\ 1\ 0\ 1\ 0$

BJC 1, Ch 4a, p23



"The cellular ansatz"

(i) first step

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

$UD = qDU + Id$

permutations

towers placements

Physics

$DE = qED + E + D$

alternative tableaux

bijections

RSK

↔

(ii) second step

representation of Q by combinatorial operators

pairs of Young tableaux

(iii) third step

"duplication"

commutations

rewriting rules

reverse Q-tableaux

planarization

"planar automata"

ASM alternating sign matrices

8-vertex model

next lecture Ch2c