Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 2
Quadratic algebra, Q-tableaux and planar automata

Ch2a

IMSc, Chennai

January 29, 2018

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"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

RSK ⟷

pairs of Young tableaux

$UD = DU + Id$

permutations

towers placements

Physics

$DE = qED + E + D$

commutations

rewriting rules

planarization

$UD = DU + Id$ commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering in physics

$UD \to DU$ $UD \to Id$ rewriting rules

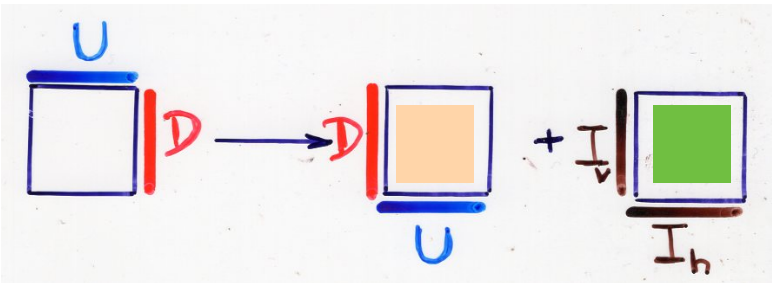
$$\begin{aligned} UUDD &= UDUD + UD \\ &= DUUD + 2UD \\ &= (DUDU + DU) + 2(DU + Id) \\ &= (DDUU + 2DU) + 2(DU + Id) \\ &= DDUU + 4DU + 2\,Id \end{aligned}$$

$UD = DU + Id$ commutations

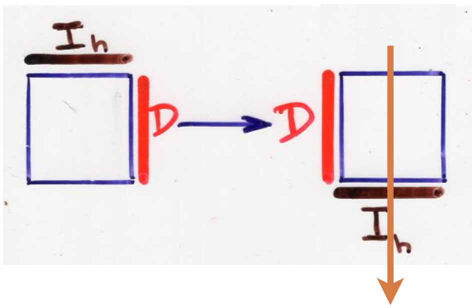
$UD \to DU$ $UD \to Id$ rewriting rules

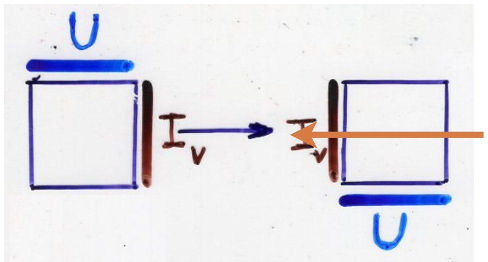
planarization of the rewriting rules

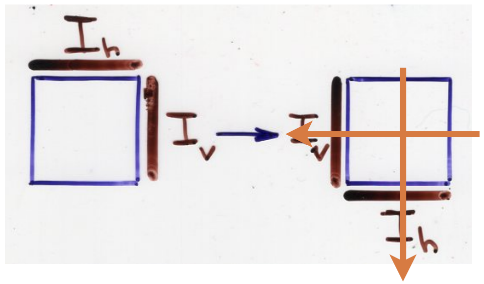
$UD =_q DU + I$

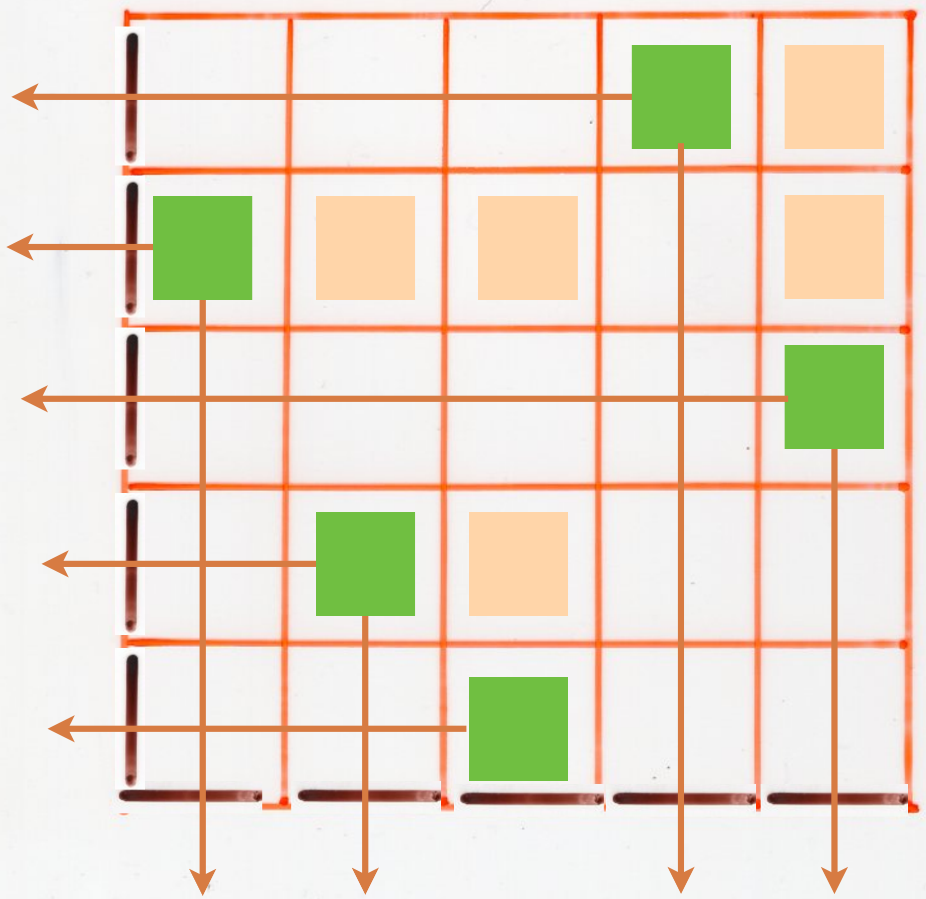


$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$







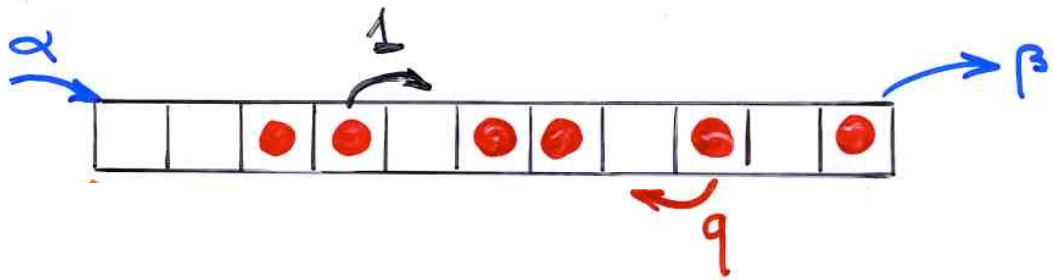


"complete"
Q- tableau

The PASEP

toy model in the physics of
dynamical systems far from equilibrium

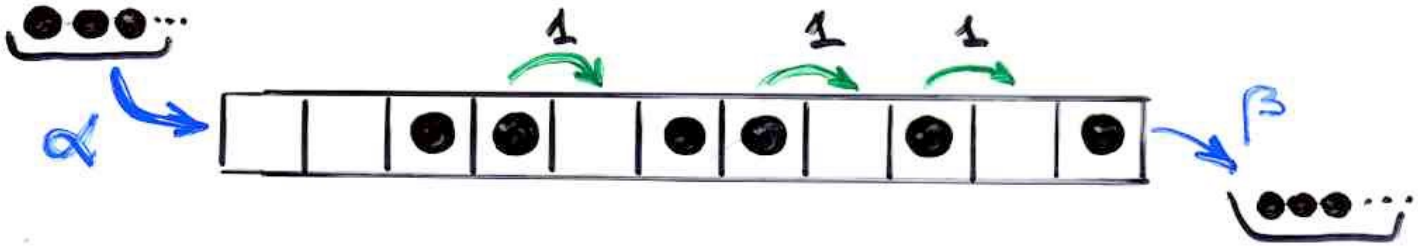
ASEP
TASEP
PASEP

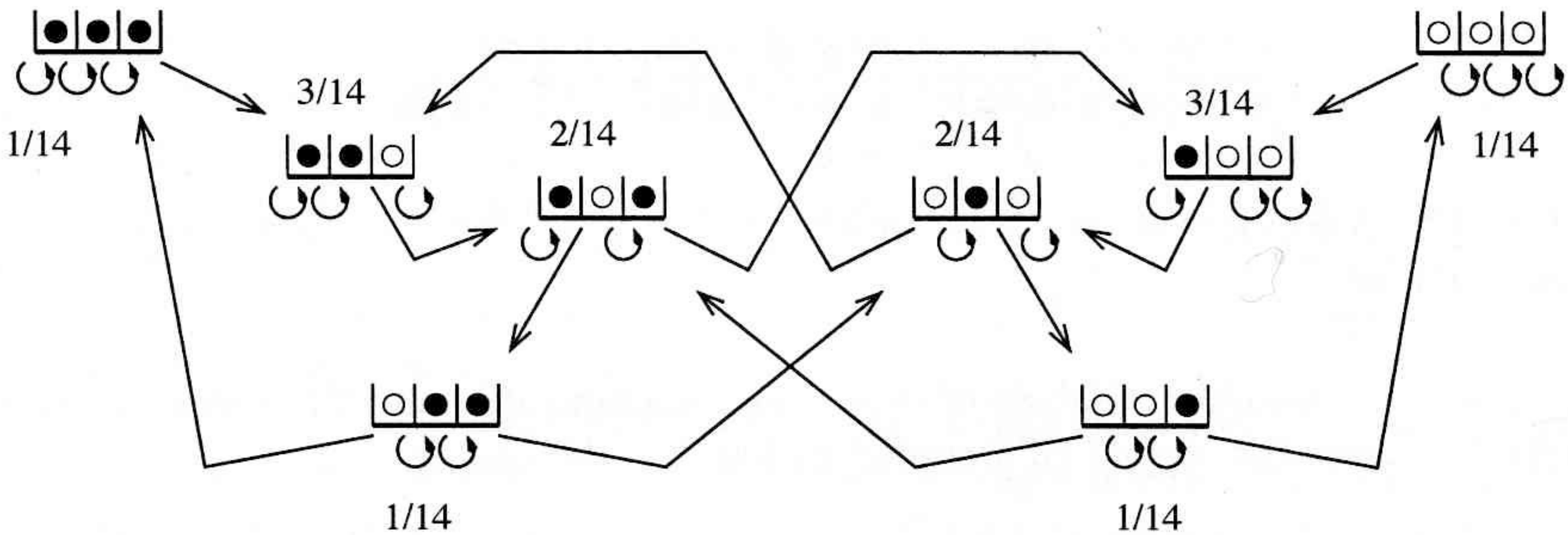


computation of the
"stationary probabilities"

TASEP

"totally asymmetric exclusion process"





stationary probabilities

The PASEP algebra

$$DE = qED + E + D$$

The PASEP algebra

$$DE = qED + E + D$$

$$\begin{tikzcd}[column sep=small]
 & DDE(DE)EDE \arrow[dl] \arrow[d] \arrow[dr] & \\
q \; DDE(ED)EDE & DDE(E)EDE & DDE(D)EDE
\end{tikzcd}$$

The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

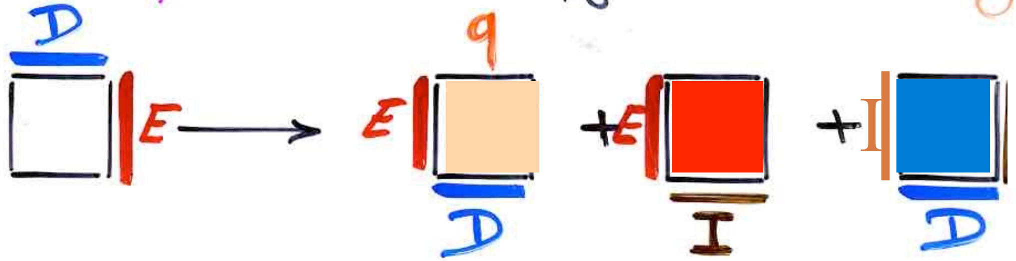
tableau

unique

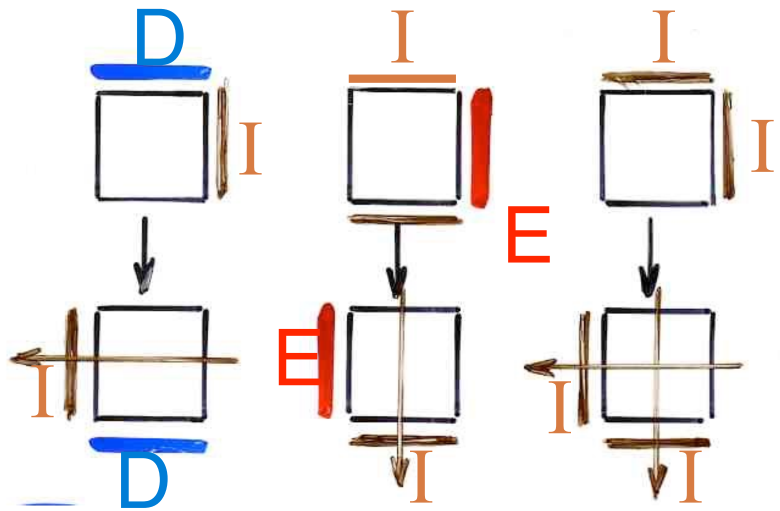
analog of the normal ordering

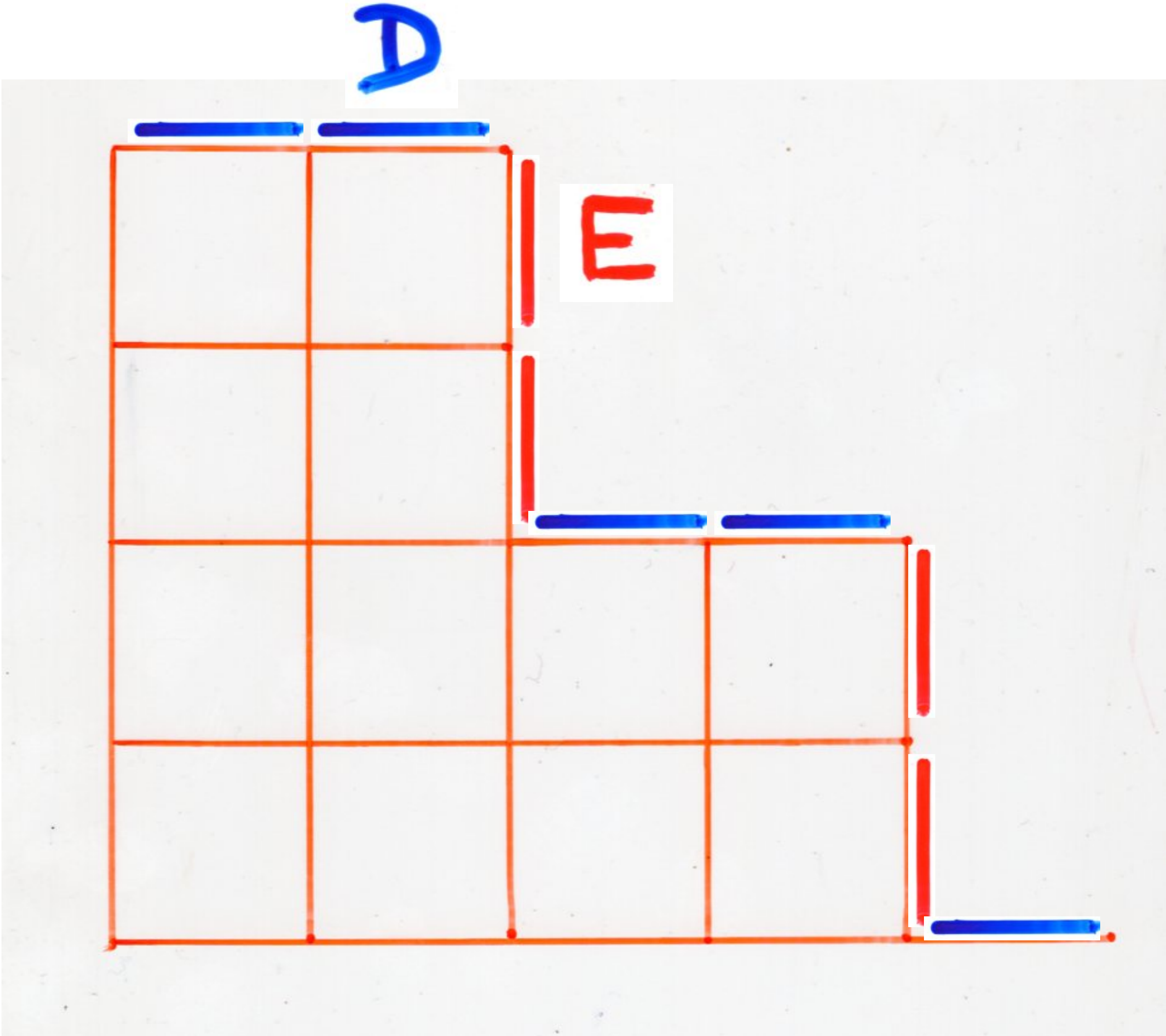
Tableaux for the PASEP algebra

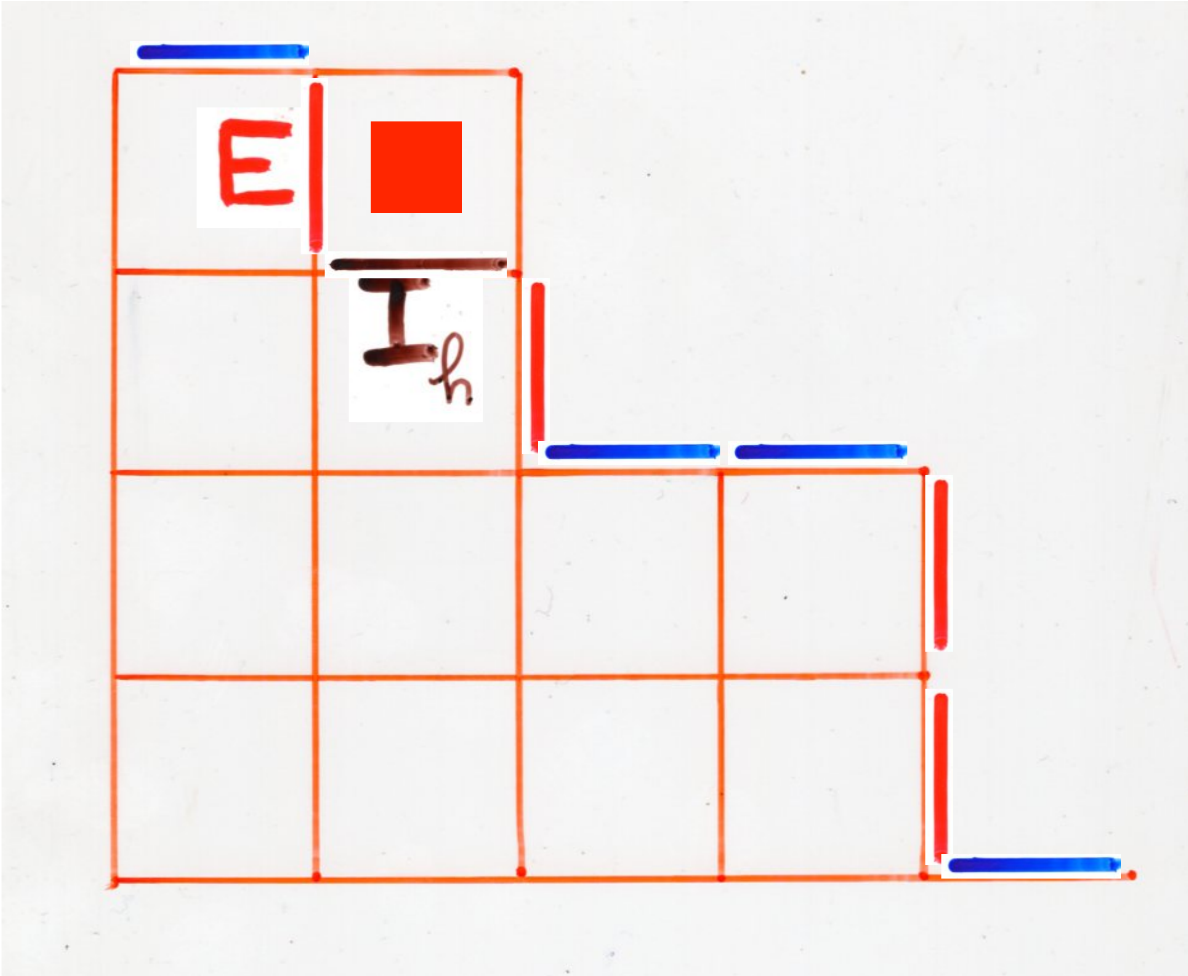
Proof: "planarization" of the rewriting rules

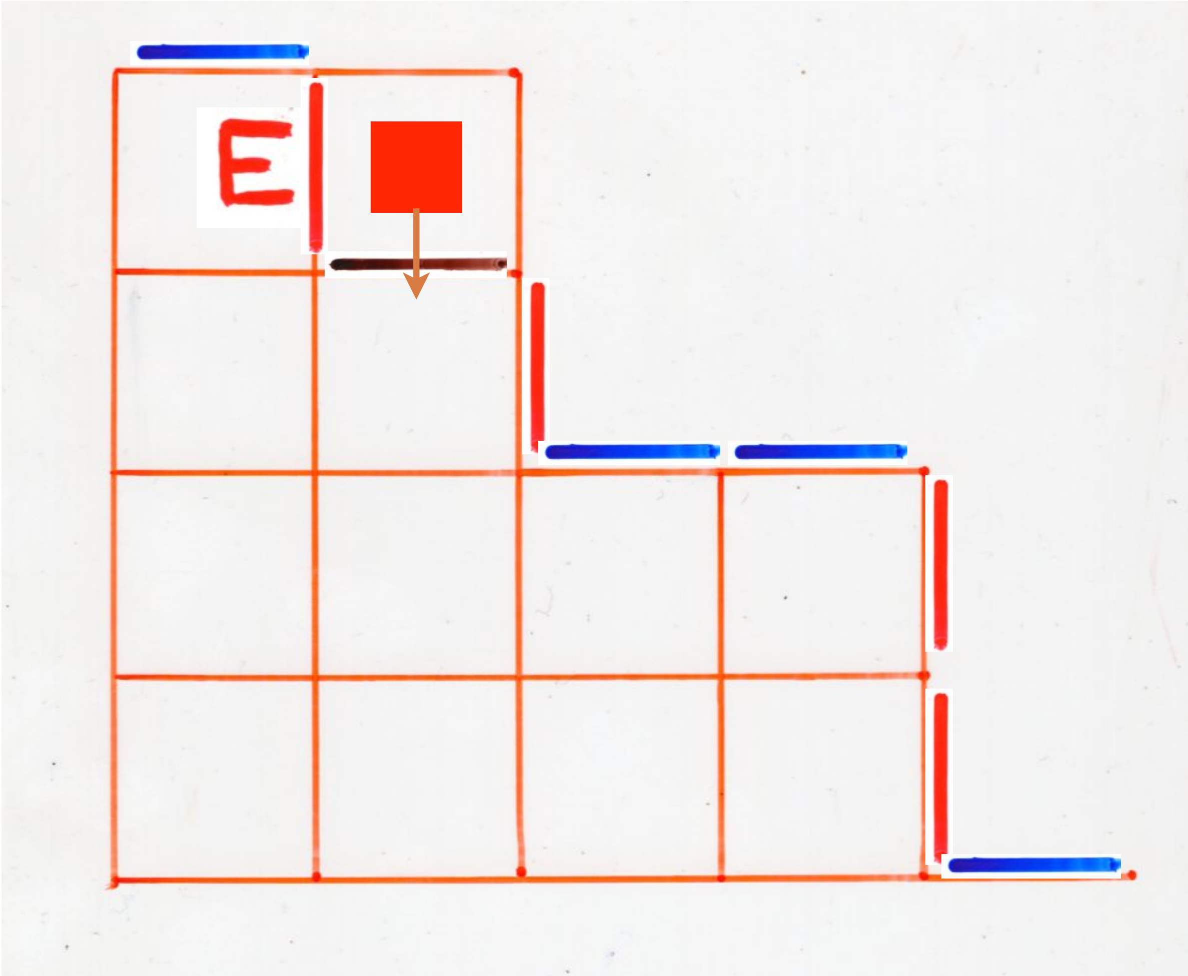


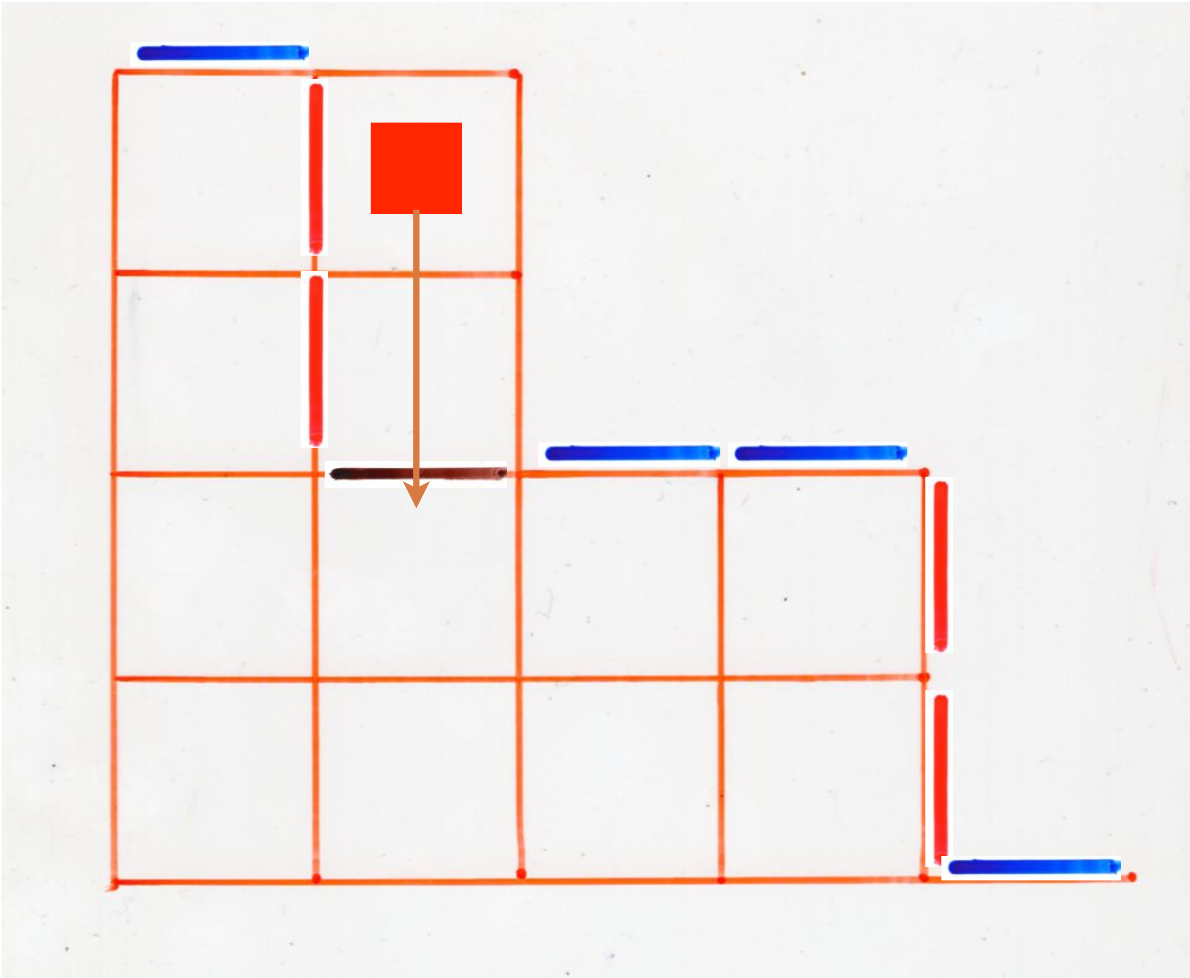
I identity

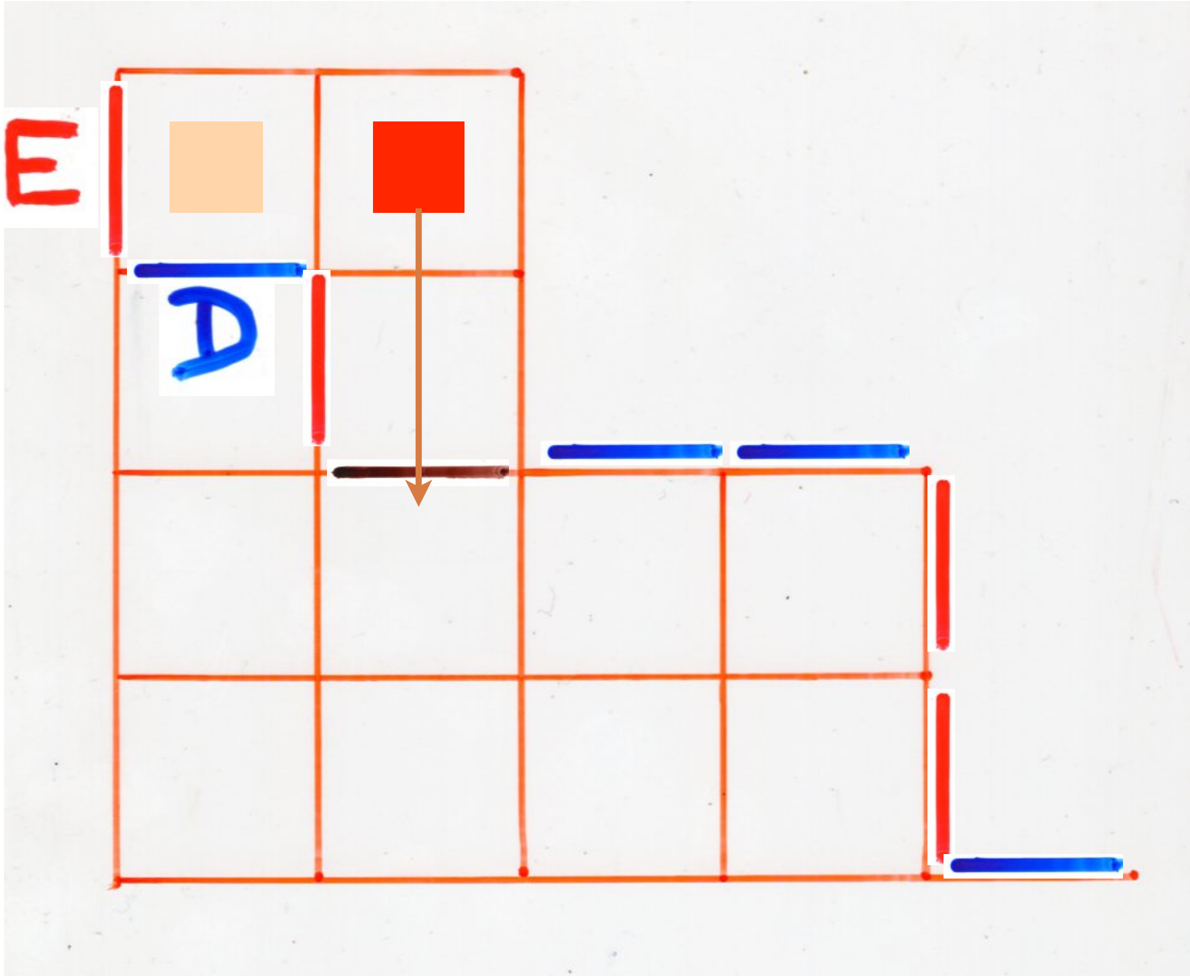


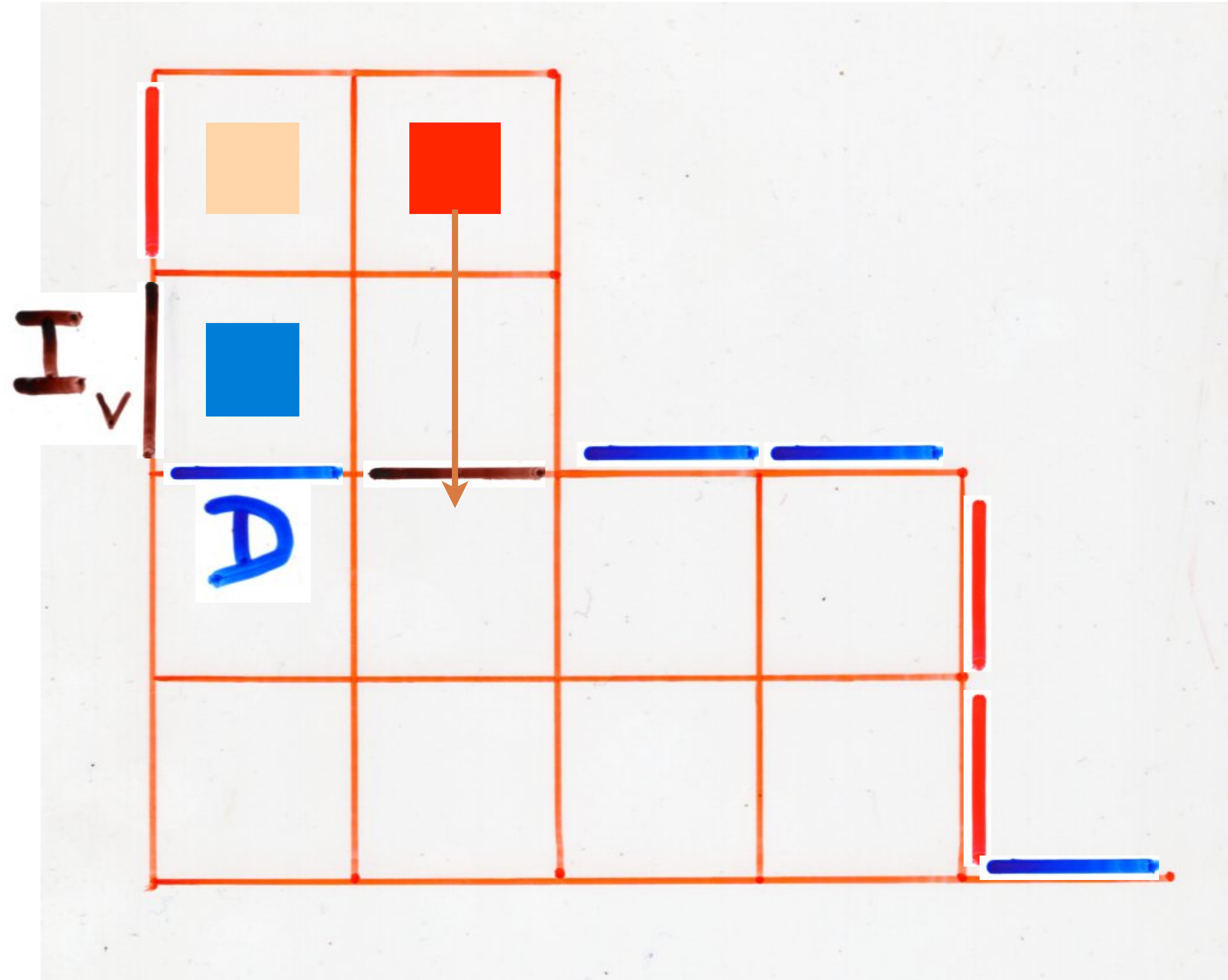

D
E


E
I_h

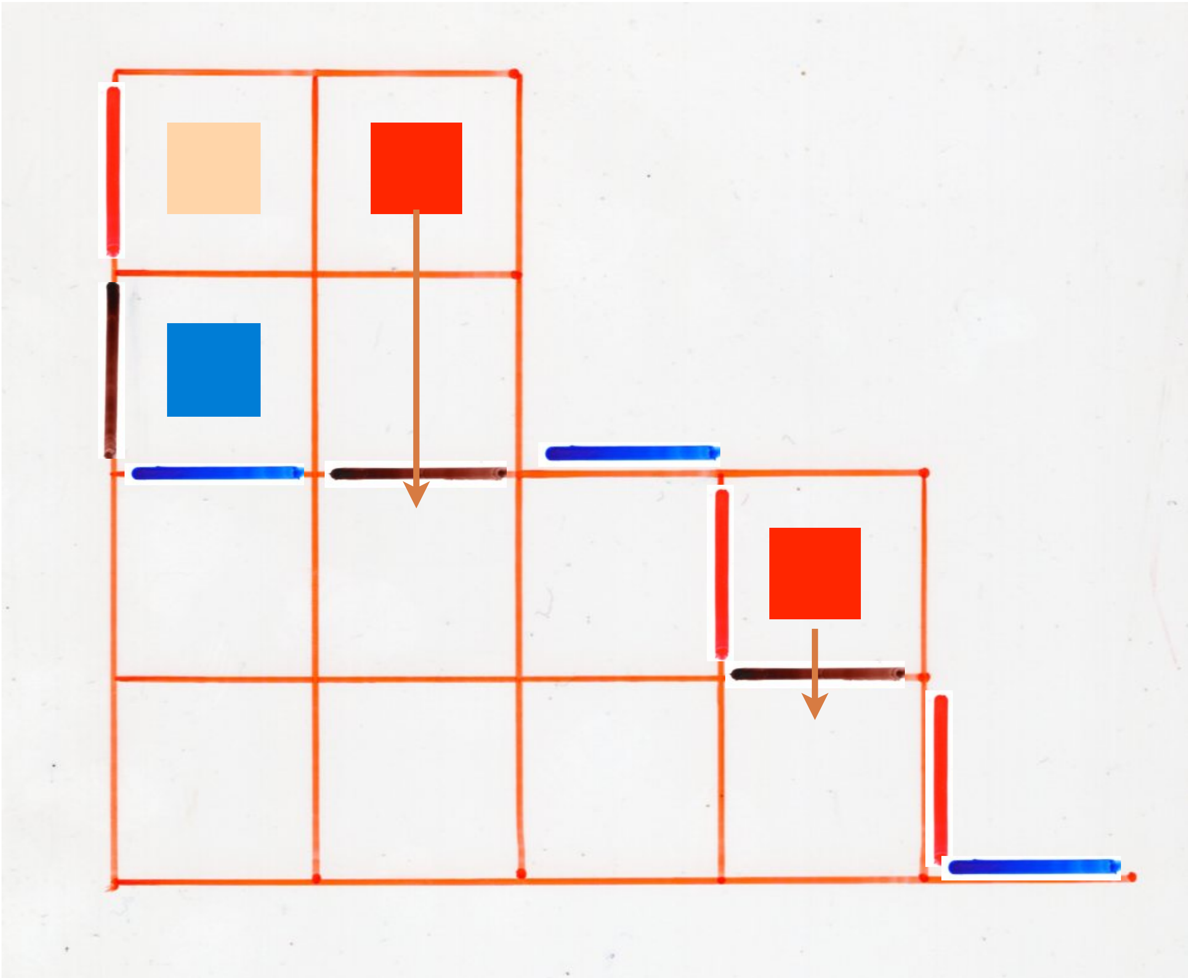

E

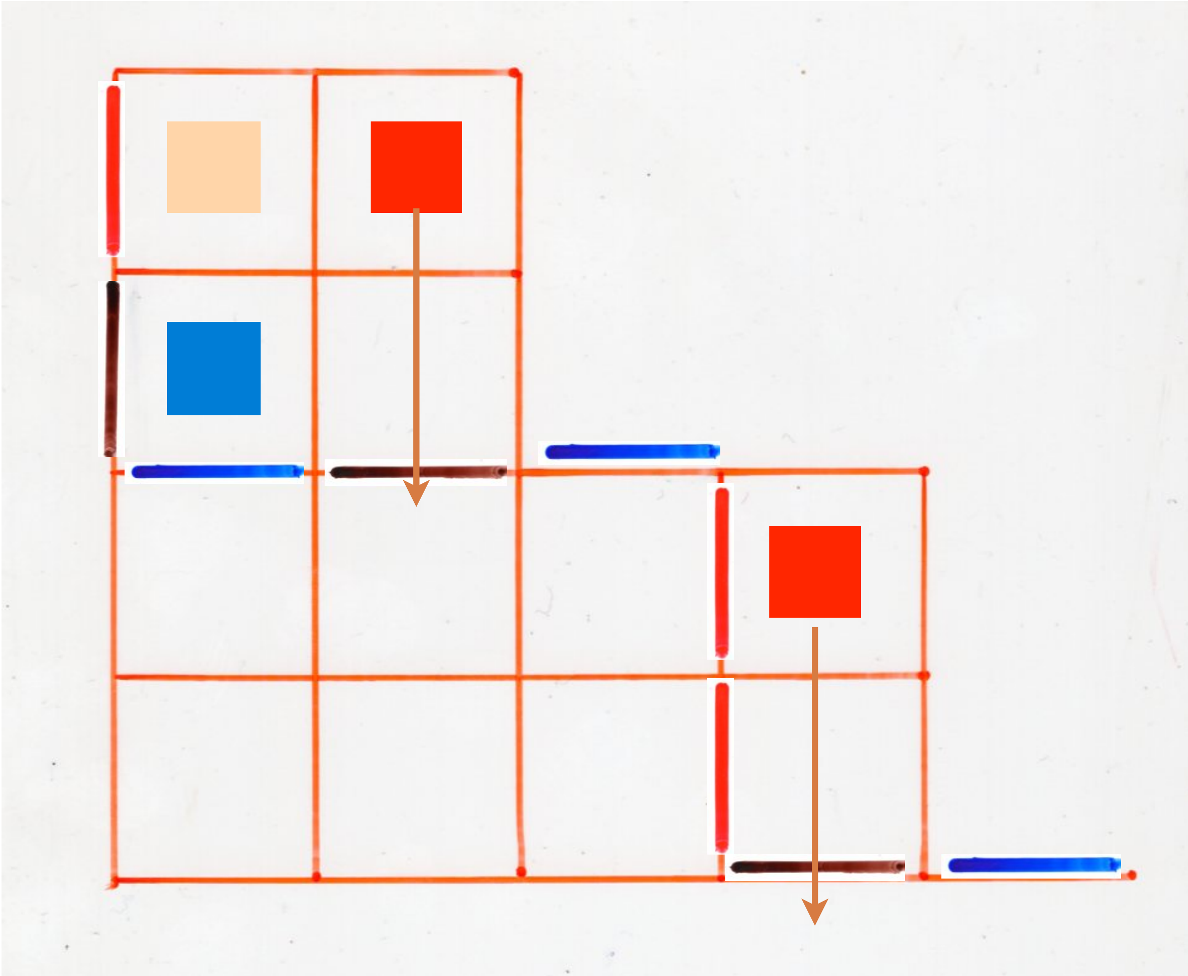


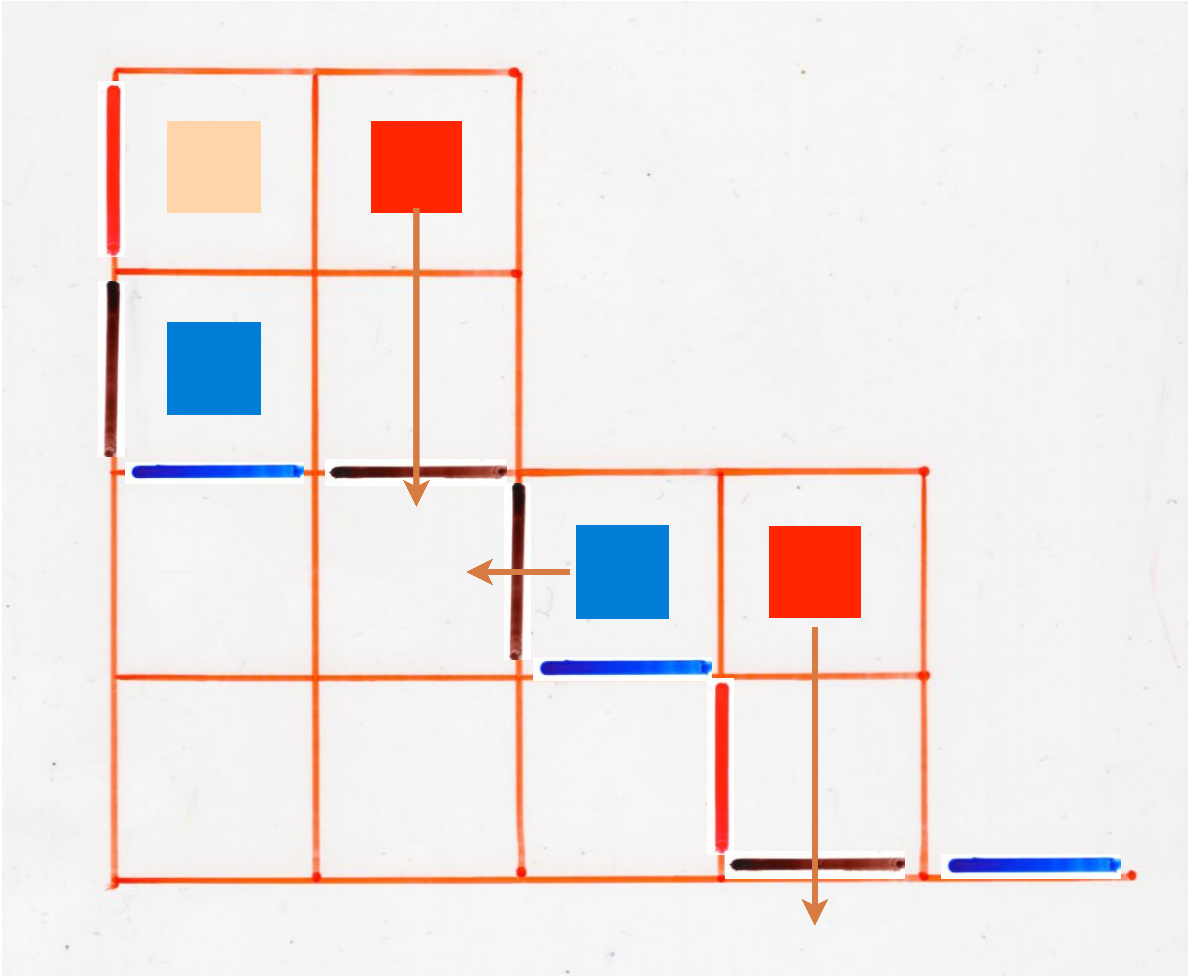

E
D

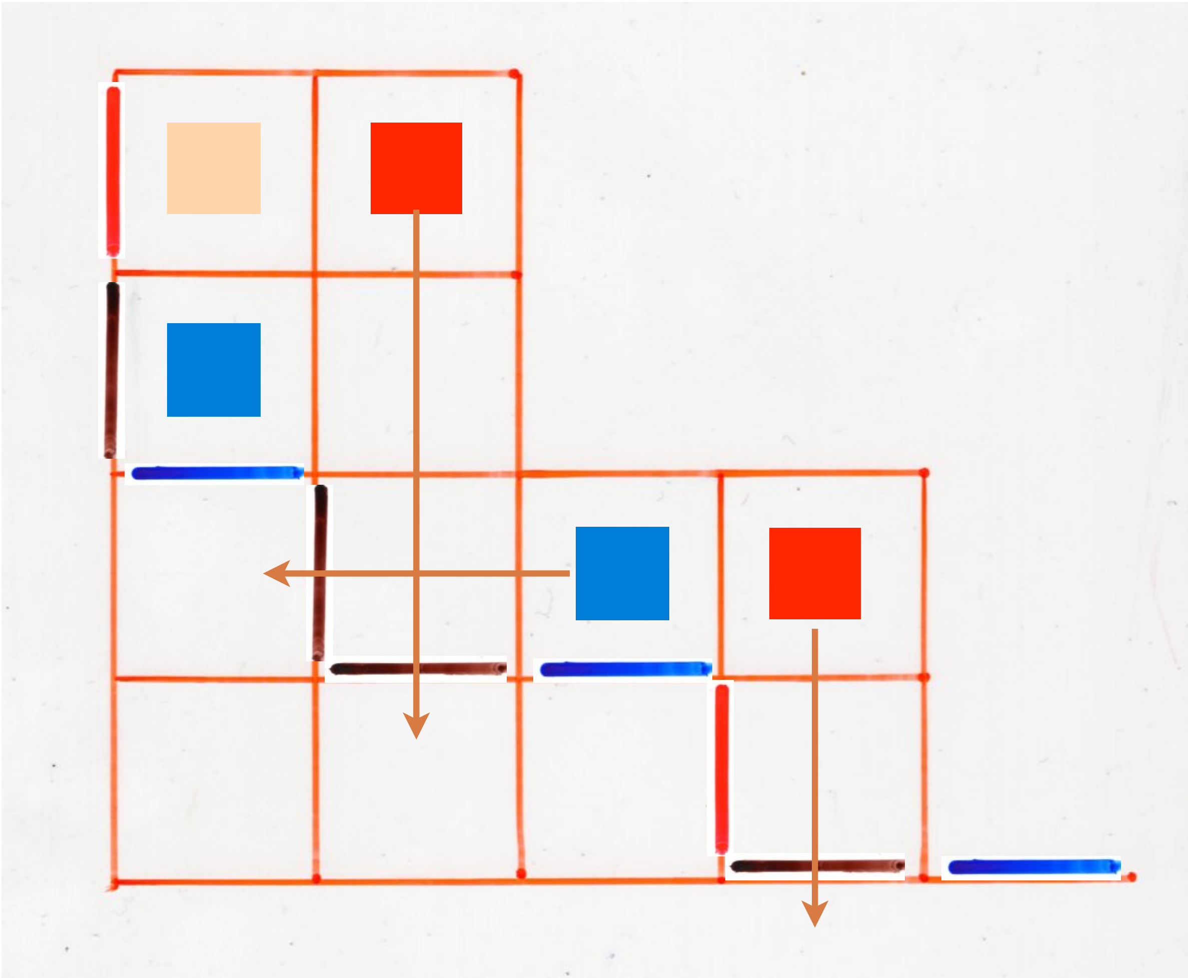


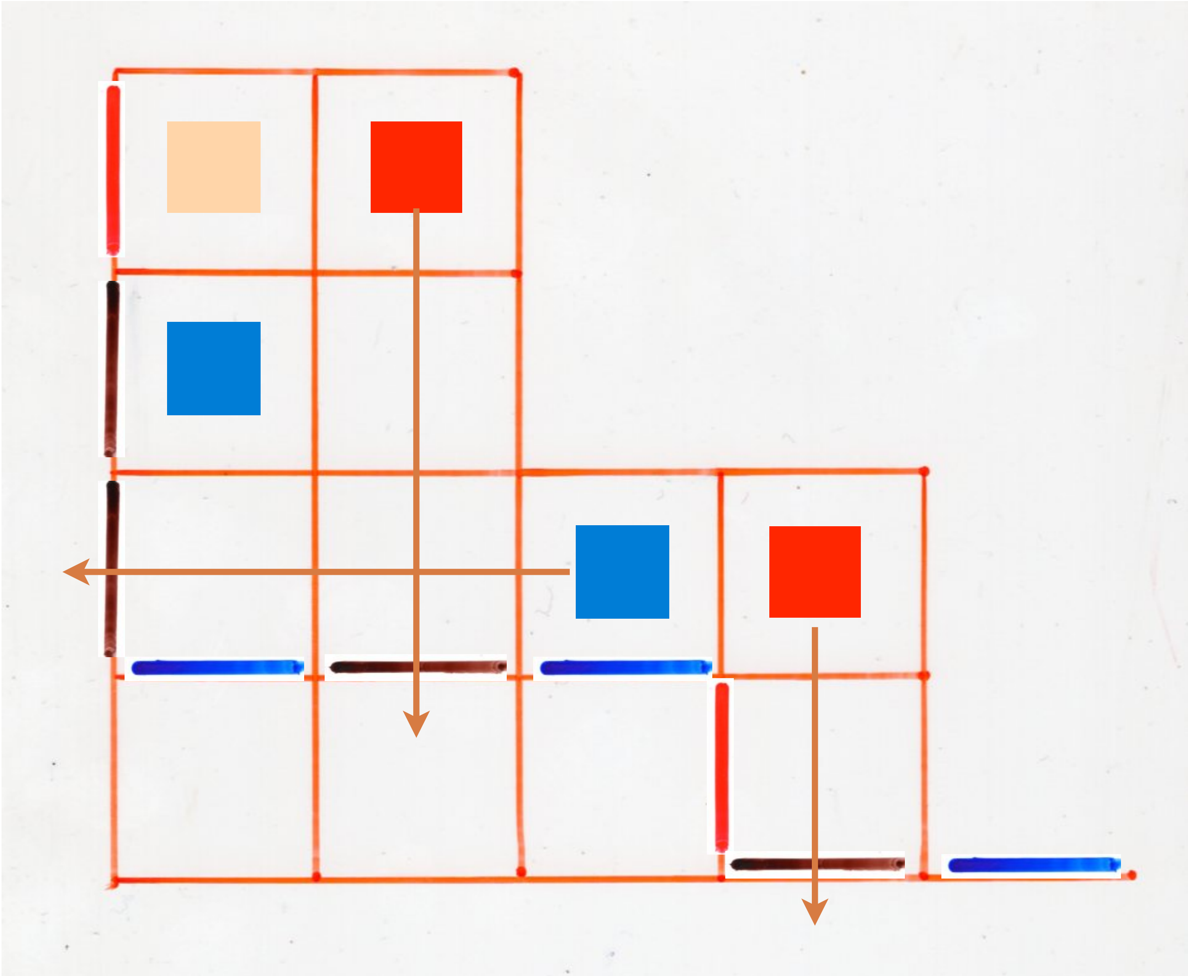
I_v
D

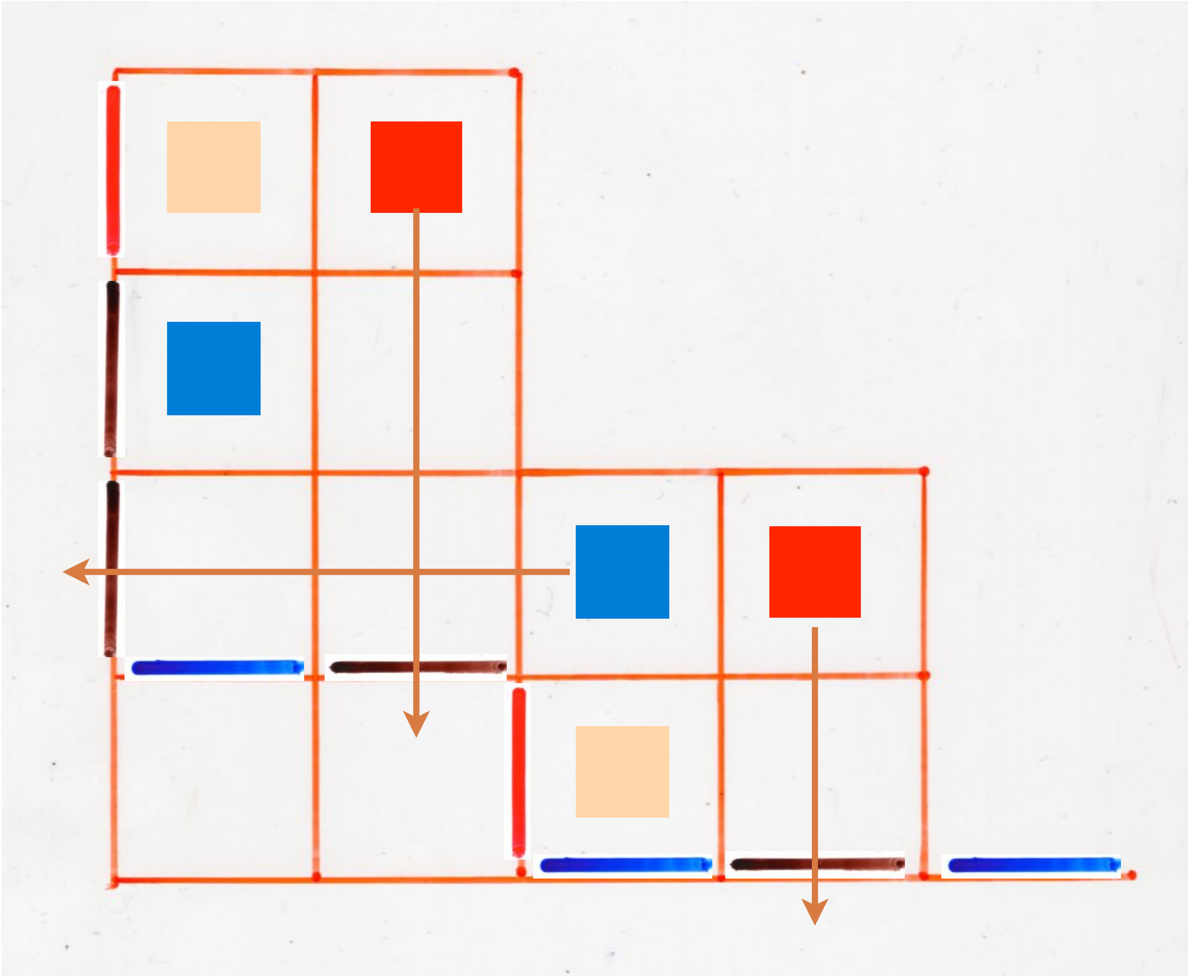


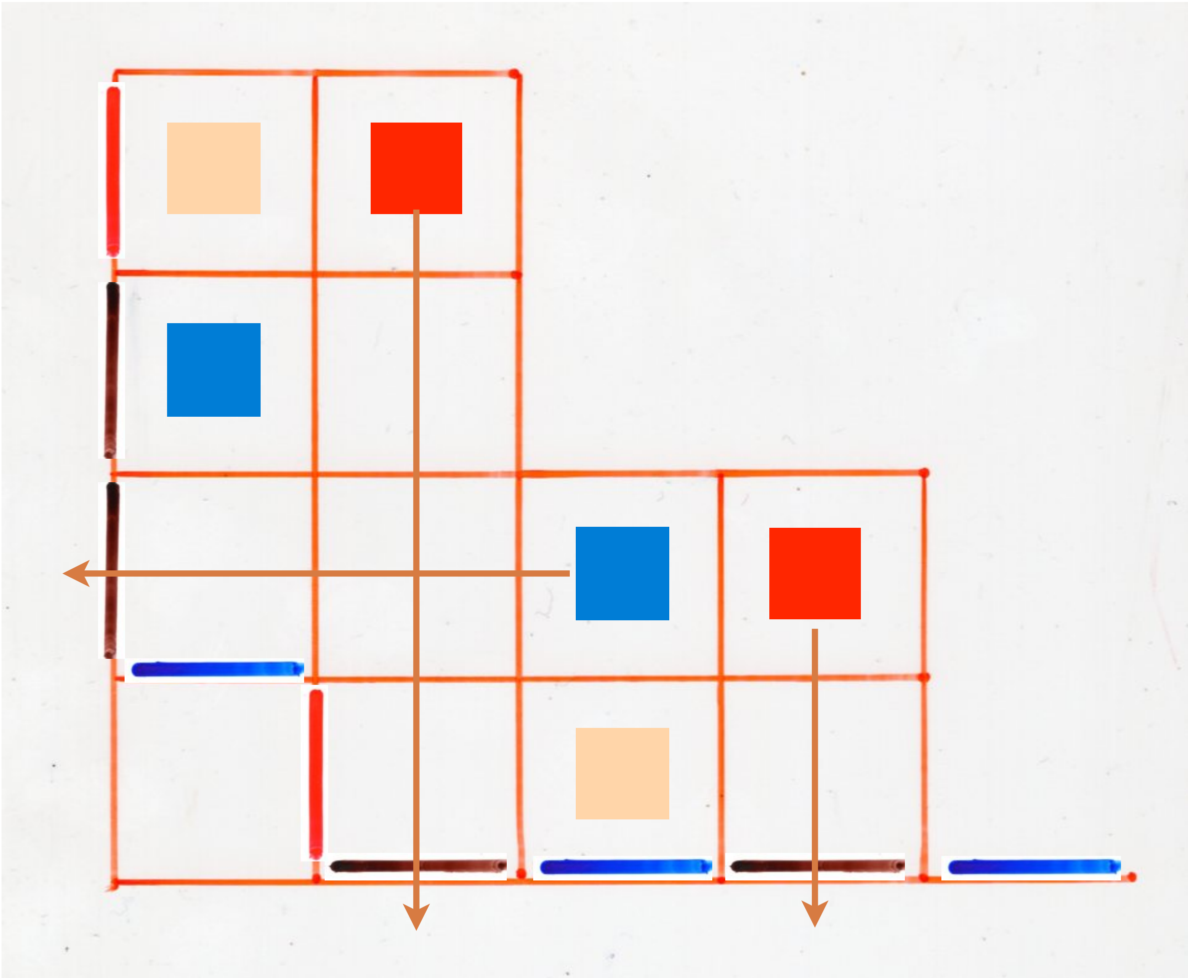


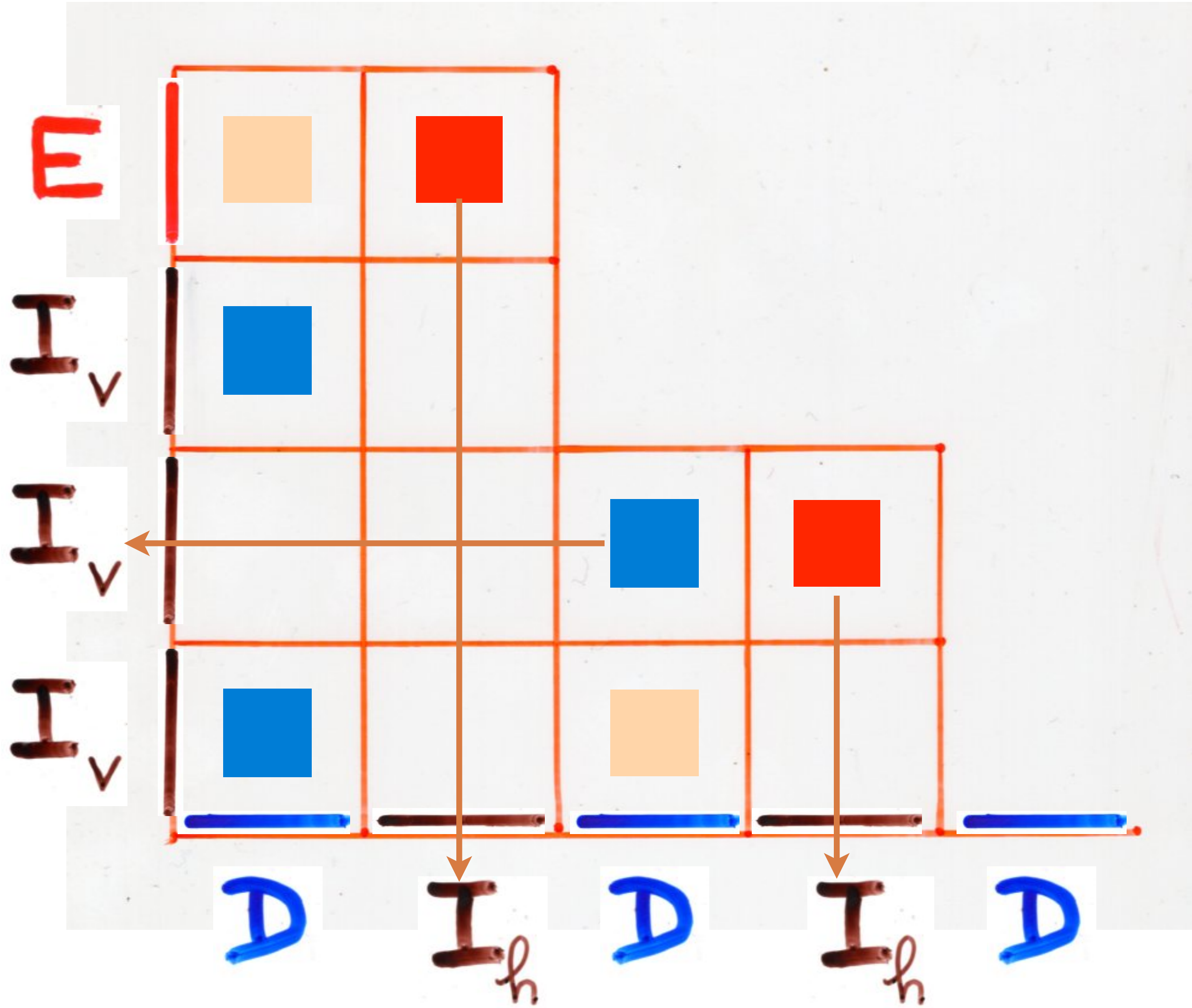











E
I_v
I_v
I_v
D
I_h
D
I_h
D

The PASEP algebra

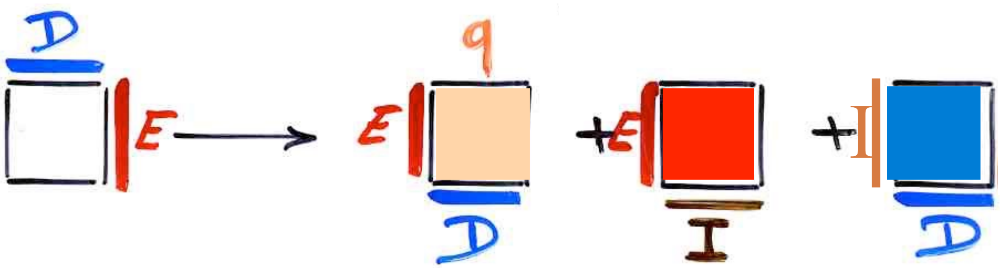
$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

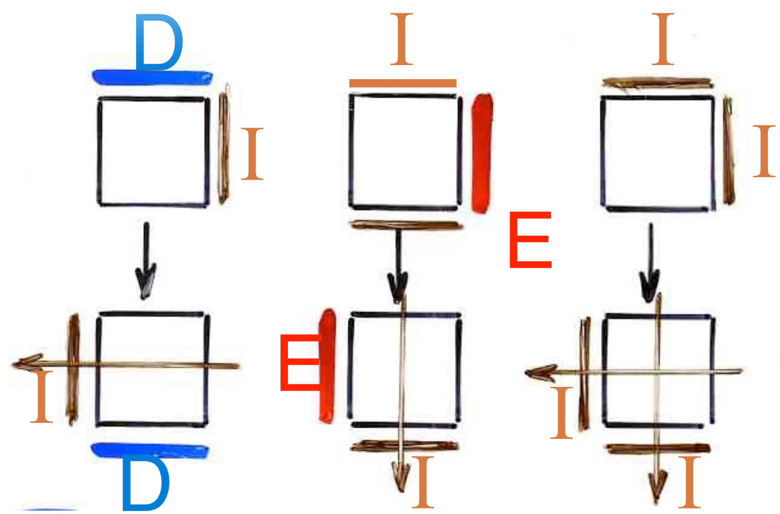
word

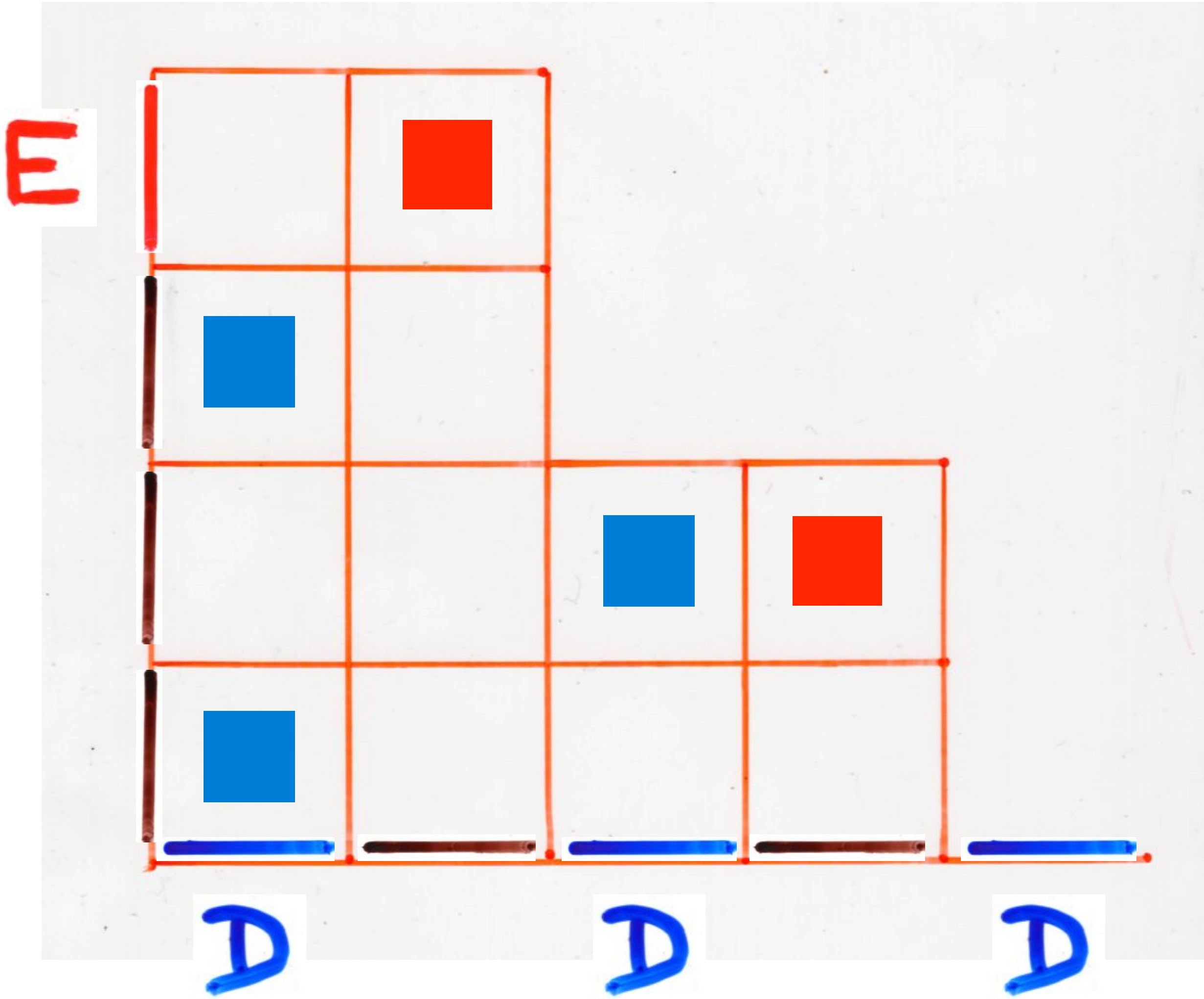
tableau

unique



I identity

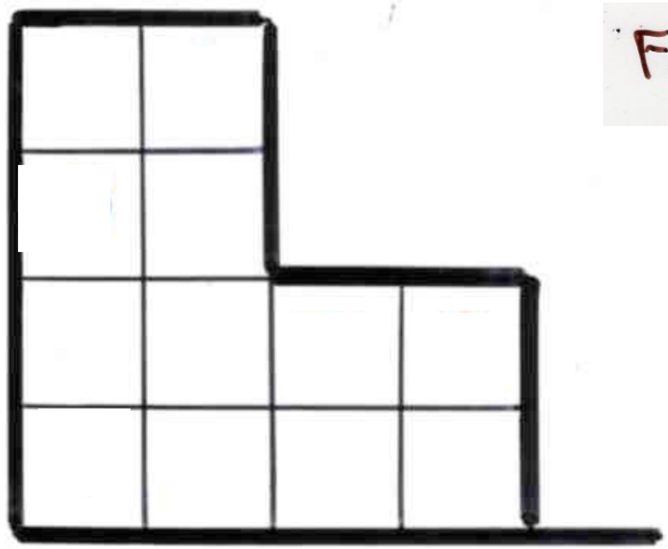



E
D
D
D

alternative tableaux

alternative tableau

Definition



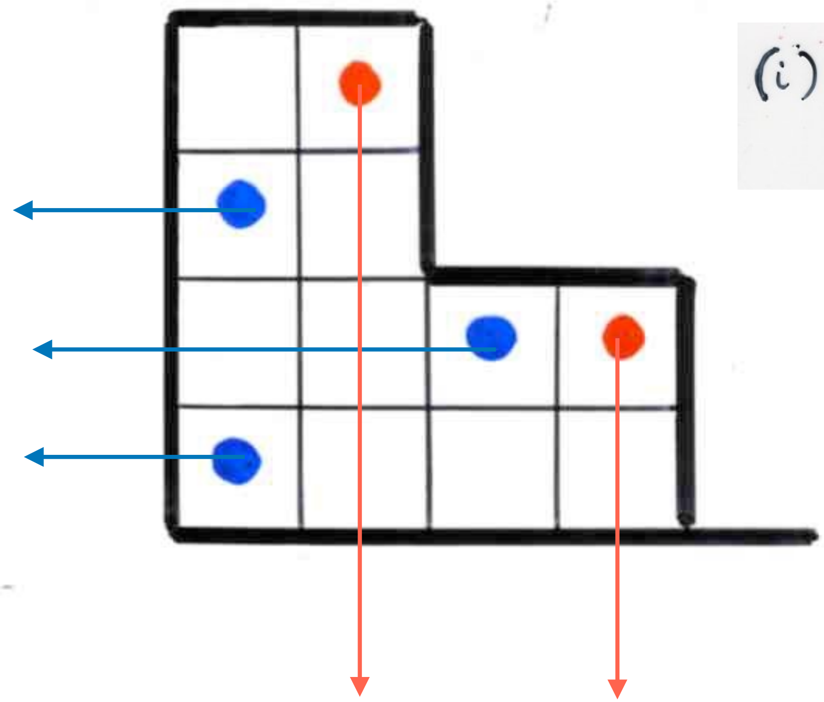
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

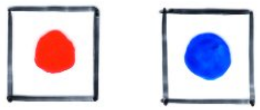
n = (number of rows) + (number of columns)

alternative tableau

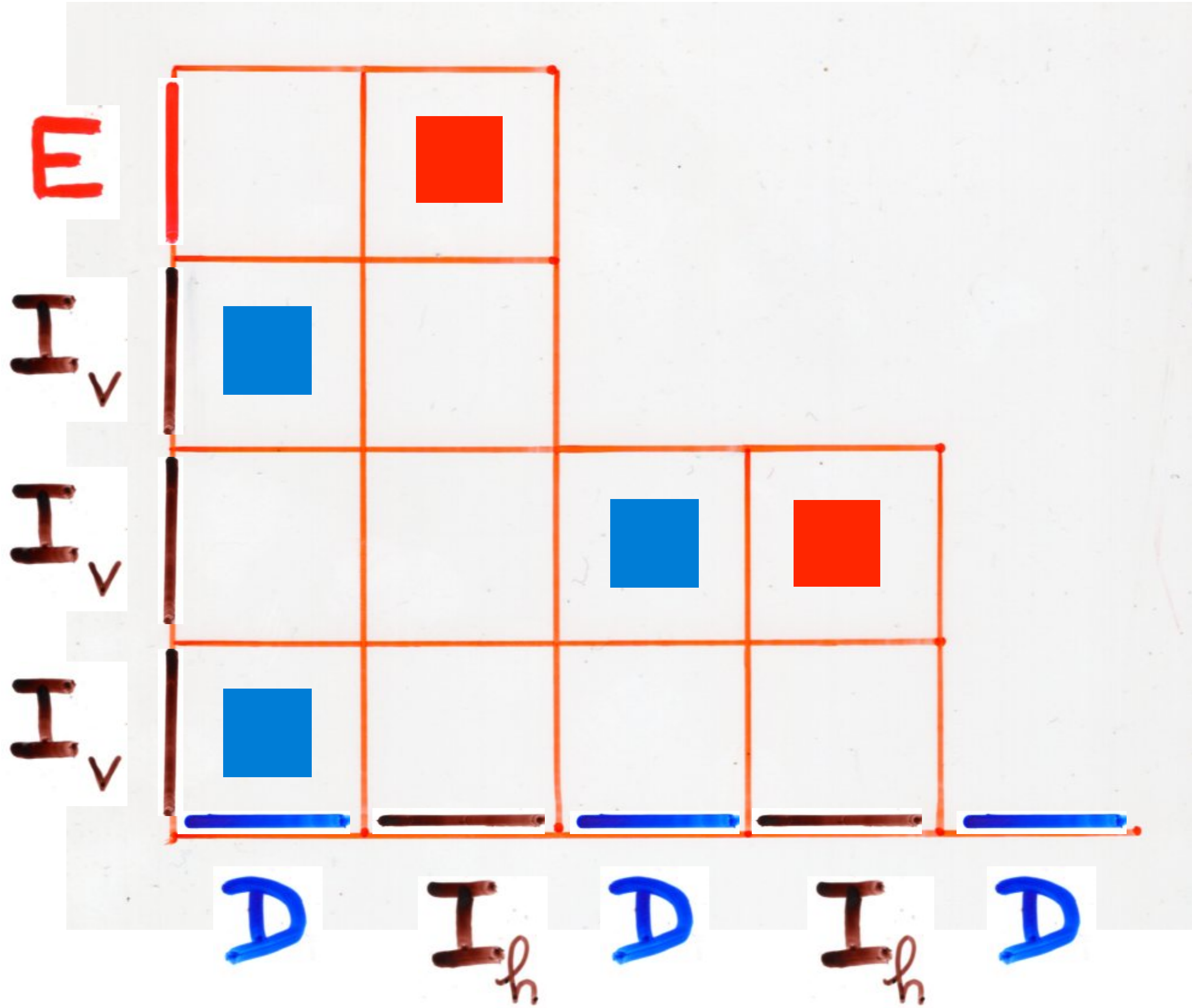


Definition

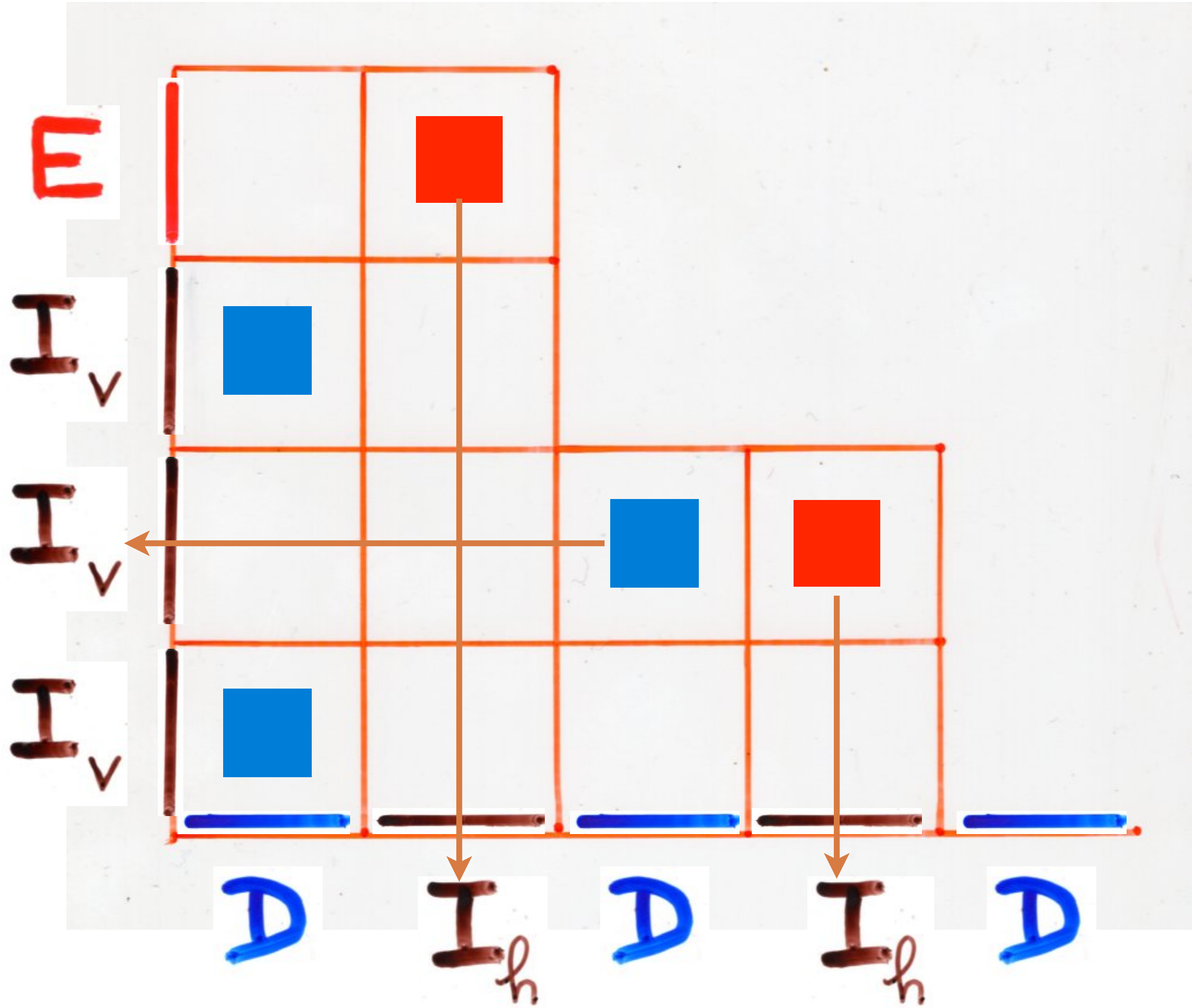
(i) some cells are coloured red or blue

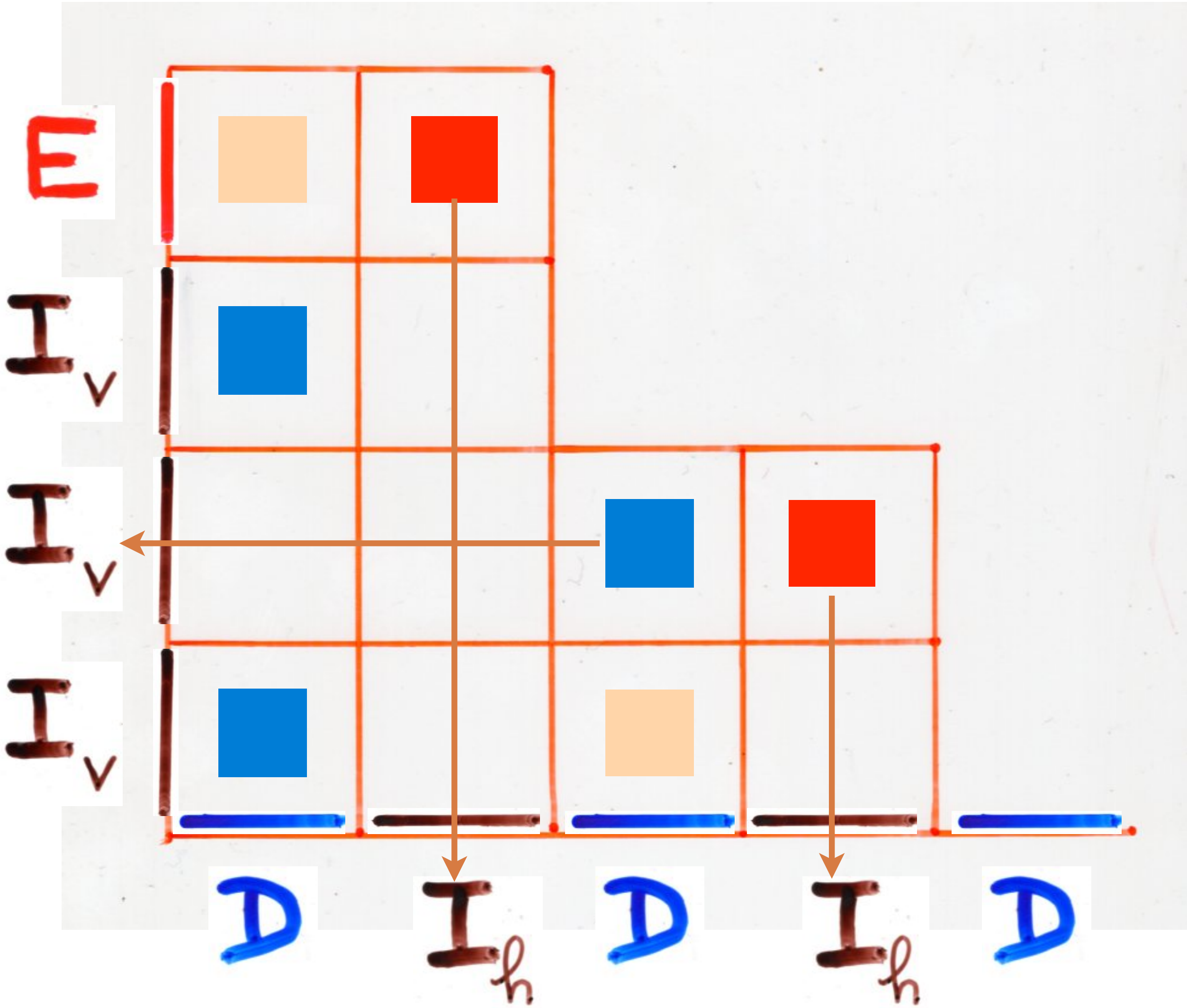


(ii) • no coloured cell at the left of a blue cell
• no coloured cell below a red cell



E
I v
I v
I v
D
I h
D
I h
D


E
I_v
I_v
I_v
D
I_h
D
I_h
D



E
I v
I v
I v
D
I h
D
I h
D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

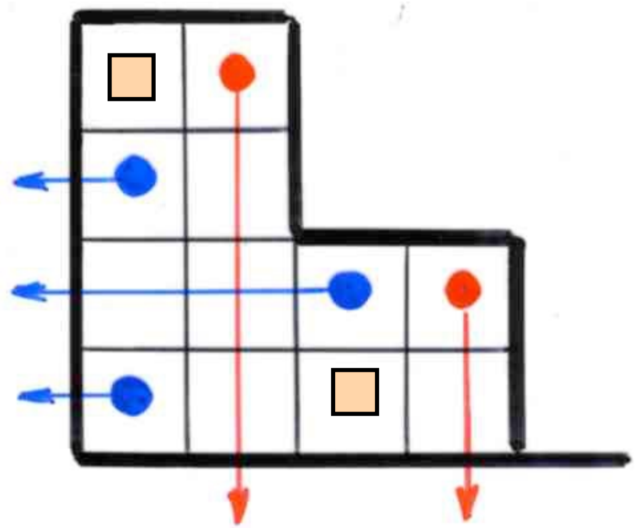
tableau

unique

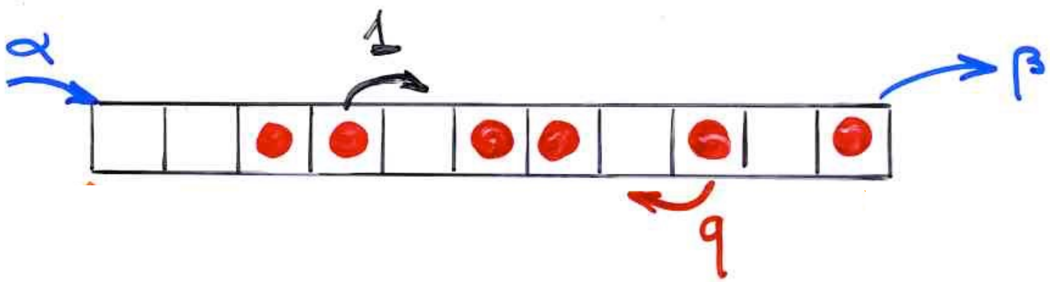
$k(T)$ = nb of cells

$i(T)$ = nb of rows without

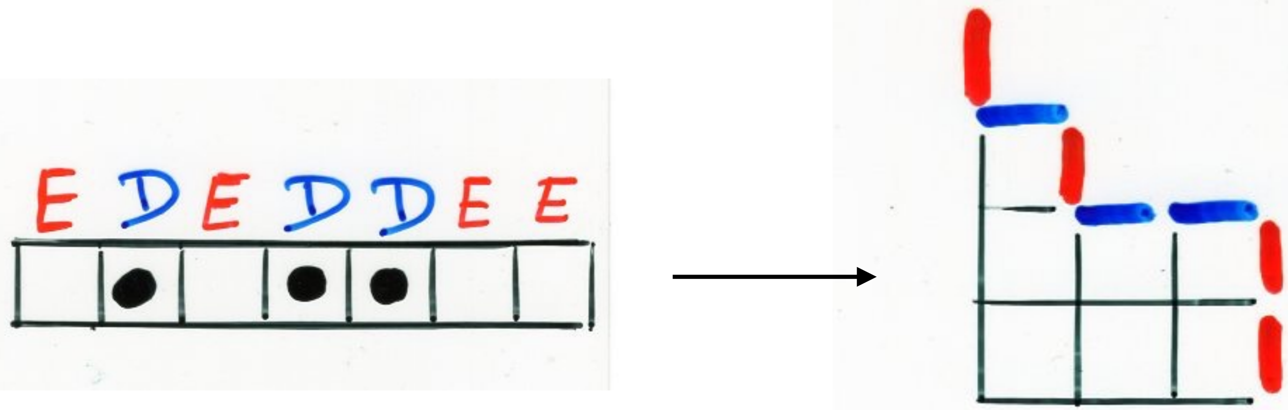
$j(T)$ = nb of columns without



ASEP
TASEP
PASEP



computation of the
"stationary probabilities"



Def- profile of an alternative tableau

word $w \in \{E, D\}^*$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

$k(T)$ = nb of cells

$i(T)$ = nb of rows without

$j(T)$ = nb of columns without

"The cellular ansatz"

representation of Q by combinatorial operators

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permutations

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alternative tableaux

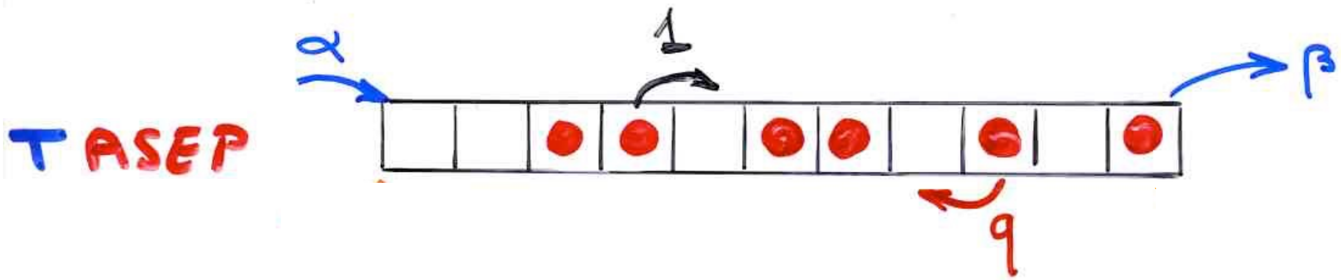
commutations

rewriting rules

planarization

$q = 0$

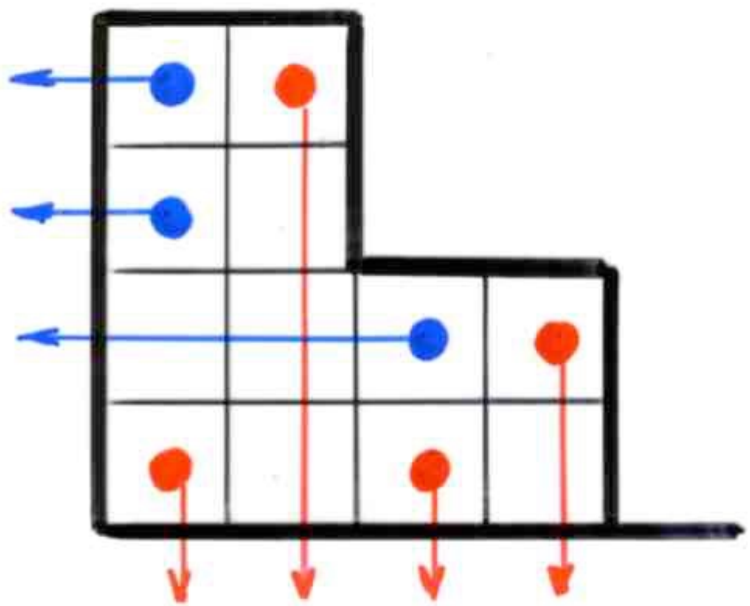
The PASEP algebra $\quad DE = \quad E + D$



Definition Catalan alternative tableau

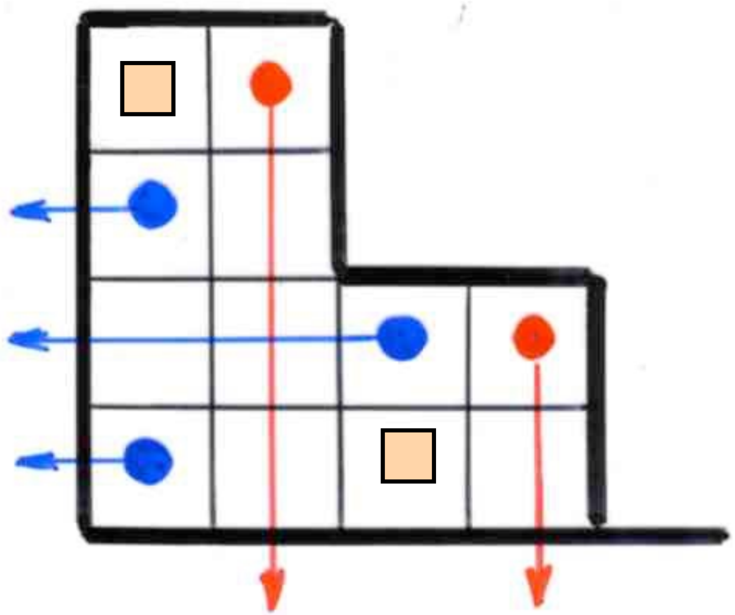
alternative tableau T without cells $\square$

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



Prop. The number of alternative tableaux of size n is $(n+1)!$

alternating sign matrices (ASM)
and a quadratic algebra

Def- **ASM** alternating sign matrix

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

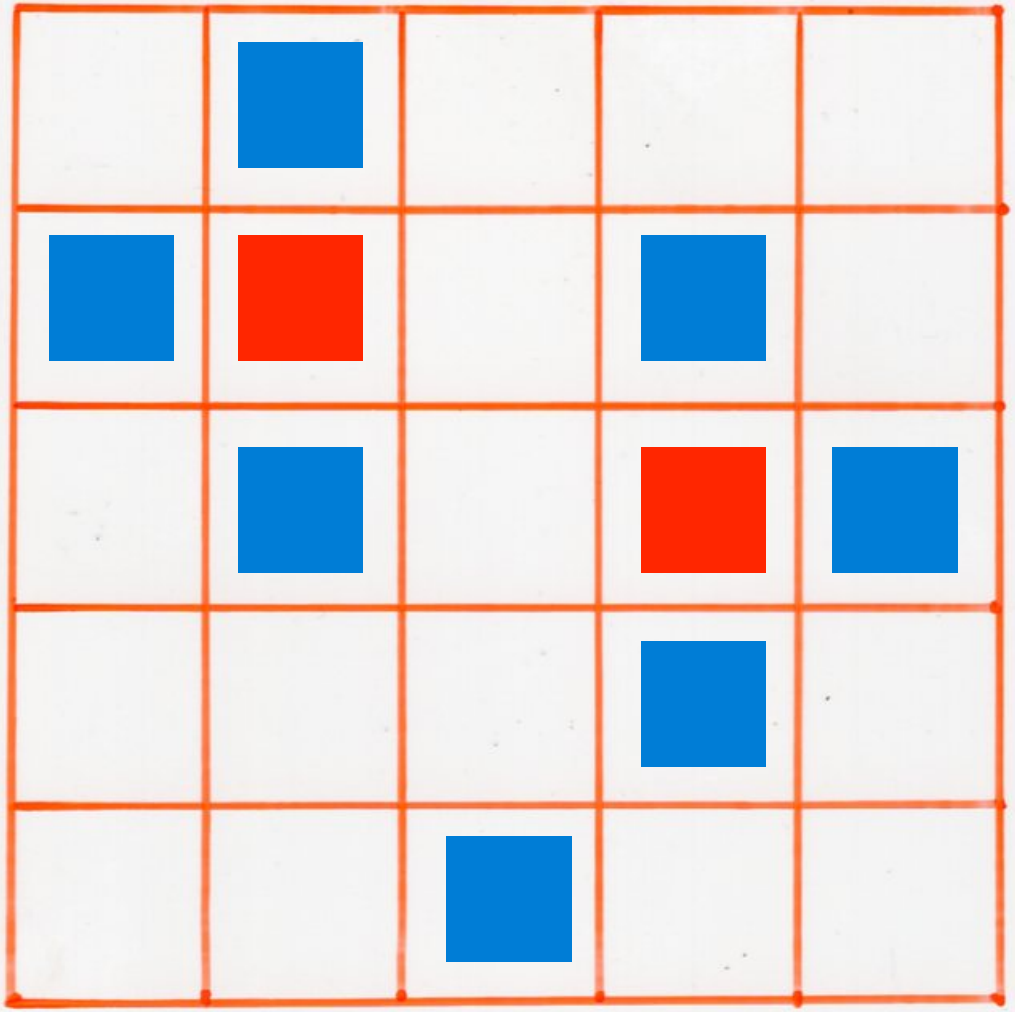
(i) entries: 0, 1, -1

(ii) sum of entries in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right.$ = 1

(iii) non-zero entries alternate in each $\left\{\begin{array}{l}\text{row}\\ \text{column}\end{array}\right.$

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0

0	1	0	0	0
1	-1	0	1	0
0	1	0	-1	1
0	0	0	1	0
0	0	1	0	0



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \underset{\text{permutations}}{6}$$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

1 , 2 , 7 , 42 , 429 , ...

$$\frac{1!\ 4! \quad ? \quad (3n-2)!}{n!\ (n+1 \quad ? \quad (n+n-1)!}$$

alternating sign matrix

(ex-) conjecture

$A\,,\ A'\,,\ B\,,\ B'\,,$

commutations

$$\begin{cases} B\,A = A\,B + A'\,B' \\ B'\,A' = A'\,B' + A\,B \end{cases}$$

$$\begin{cases} B'\,A = A\,B' \\ B\,A' = A'\,B \end{cases}$$

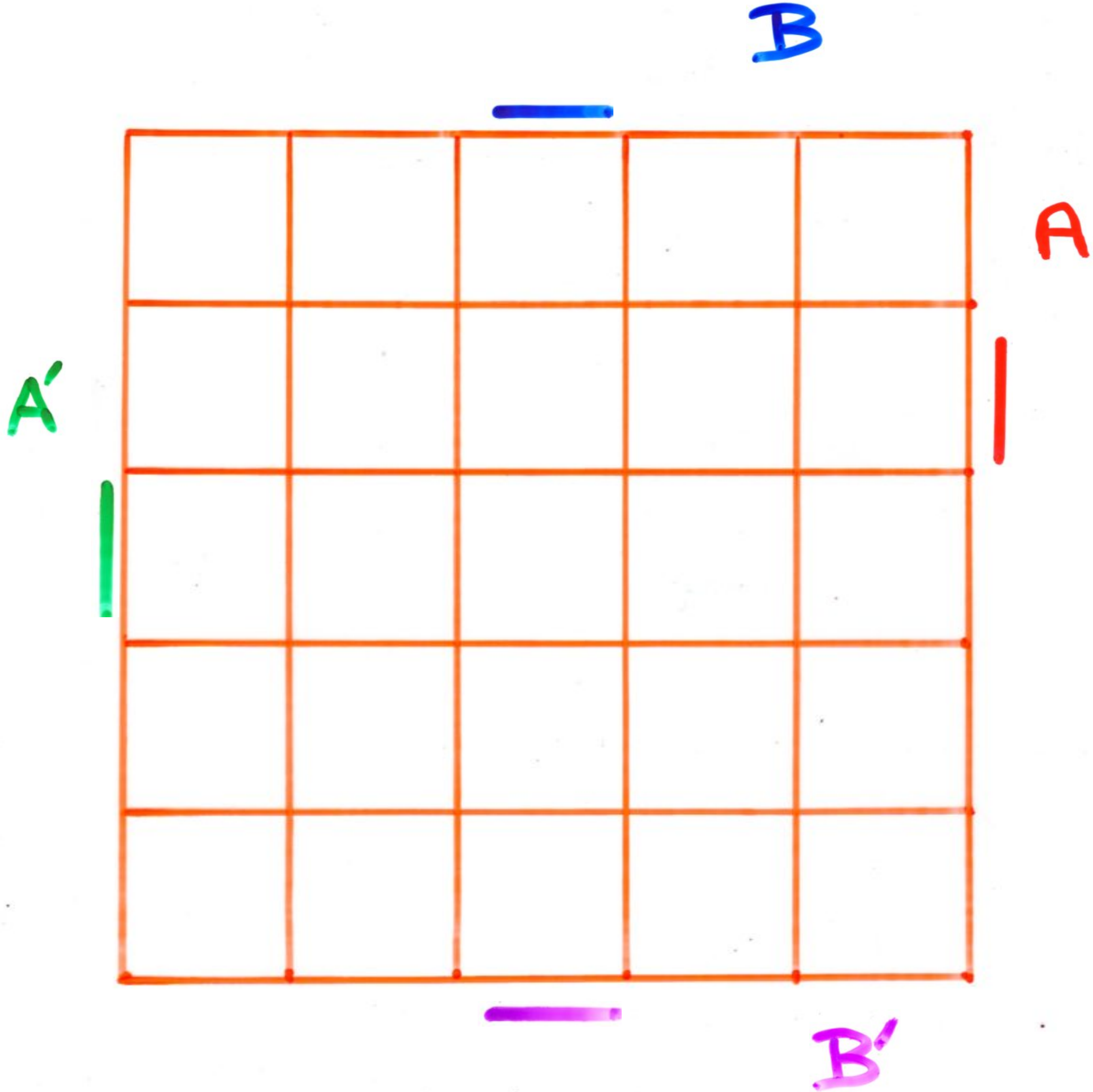
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

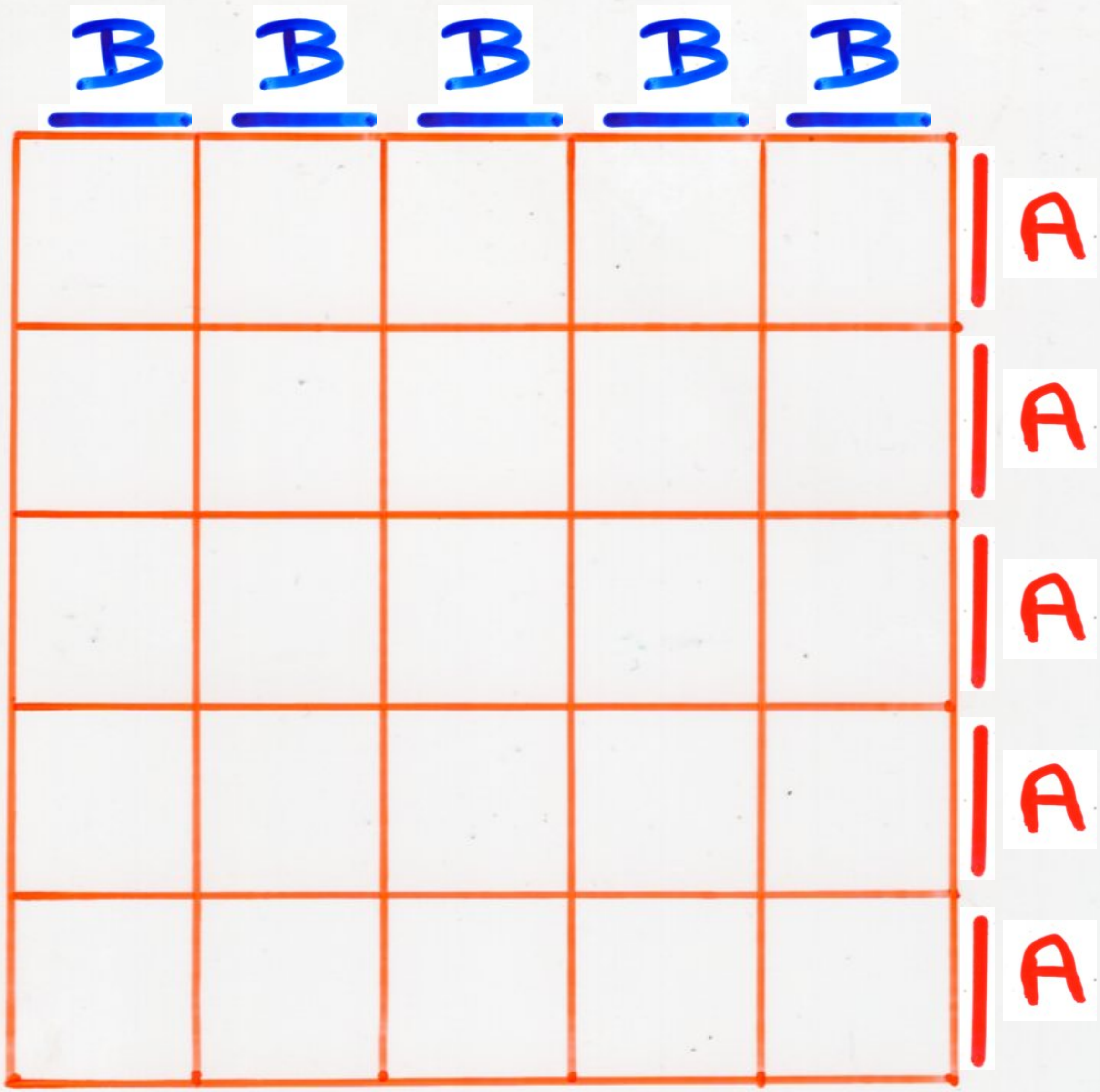
$$\sum c(u,v;w)\ \underbrace{u(A,A')}_{\substack{\text{word}\\ \text{in } A,\, A'}}\ \underbrace{v(B,B')}_{\substack{\text{word}\\ \text{in } B,\, B'}}$$

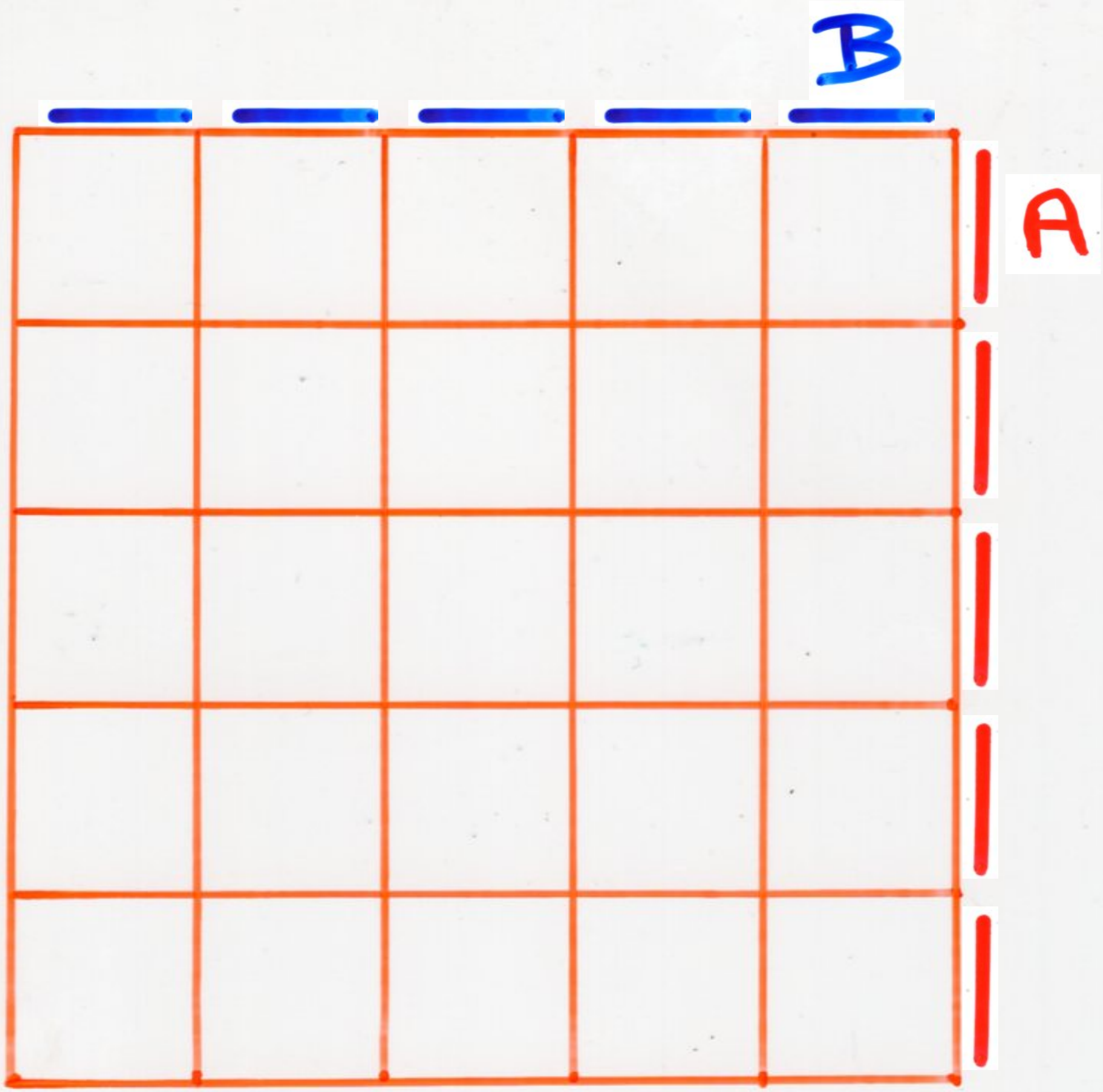
Prop. For $w = B^n A^n$

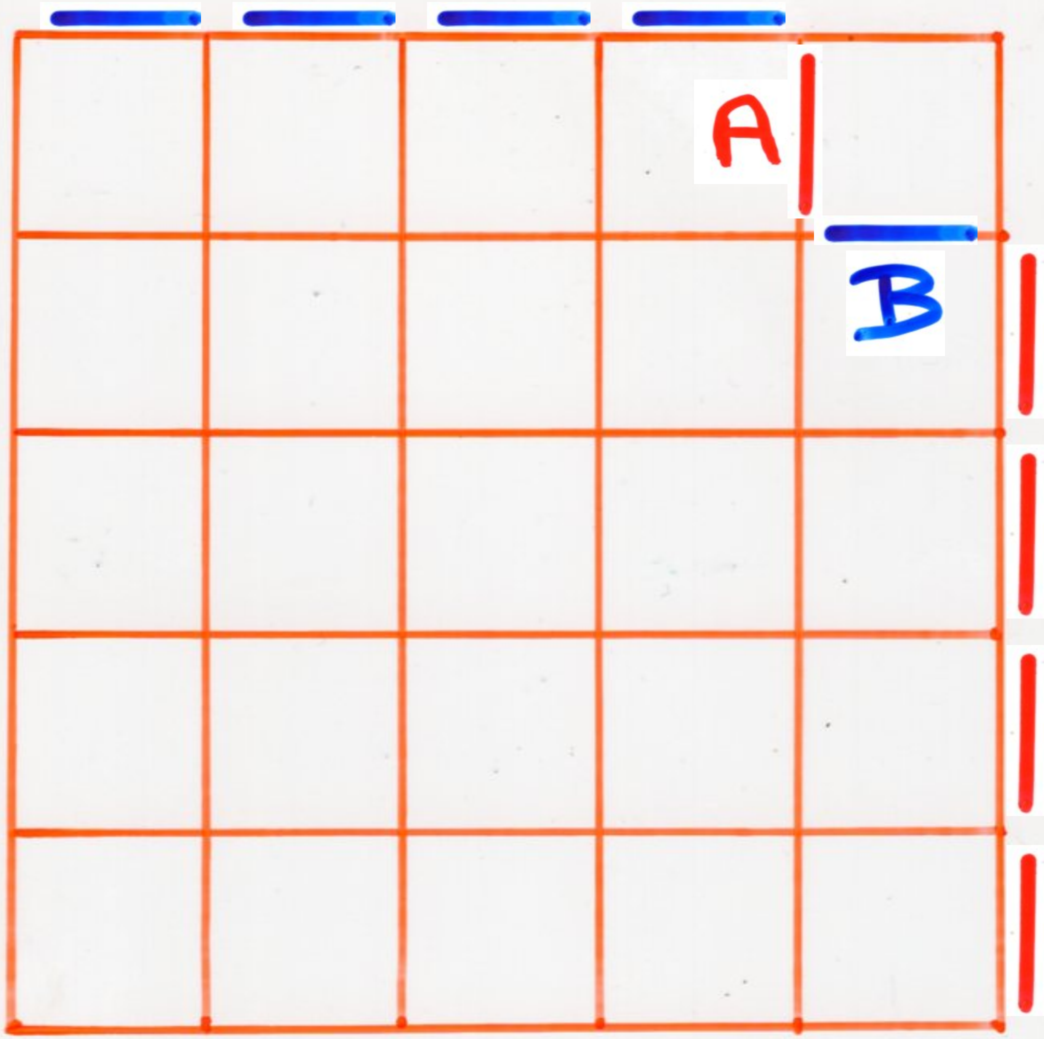
$u = A'^n$, $v = B'^n$

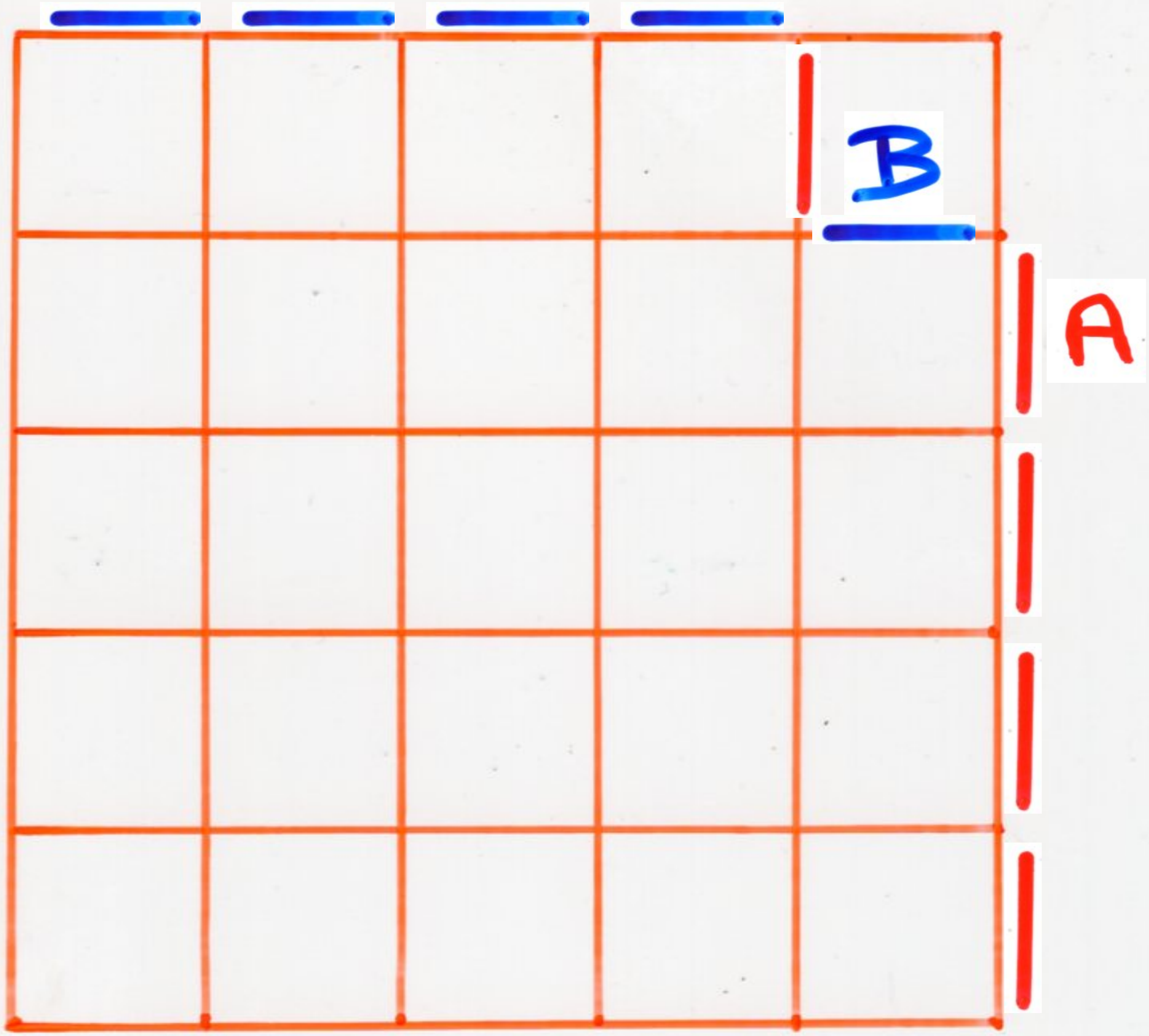
$c(u,v;w) =$ the number of $n \times n$ ASM (alternating sign matrices)

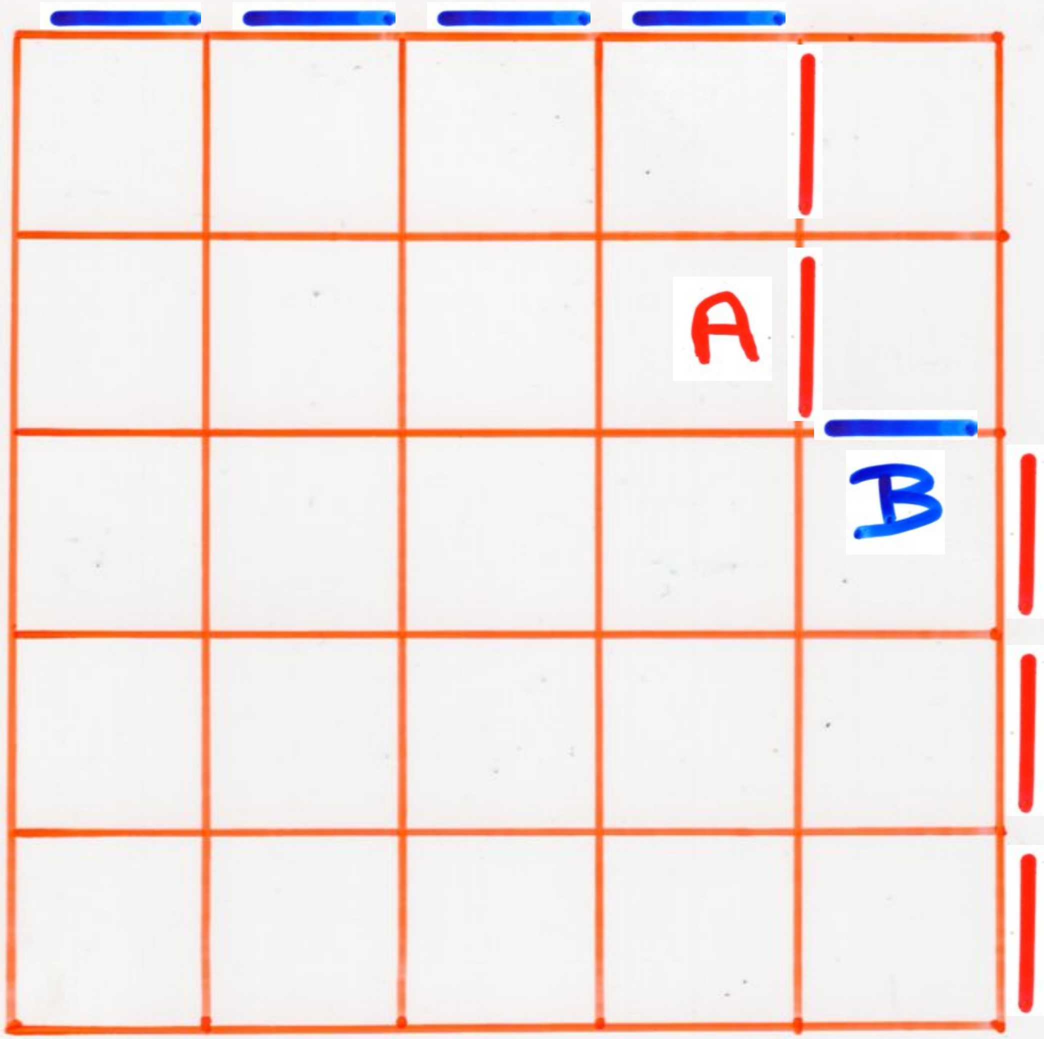

B
A
A′
B′

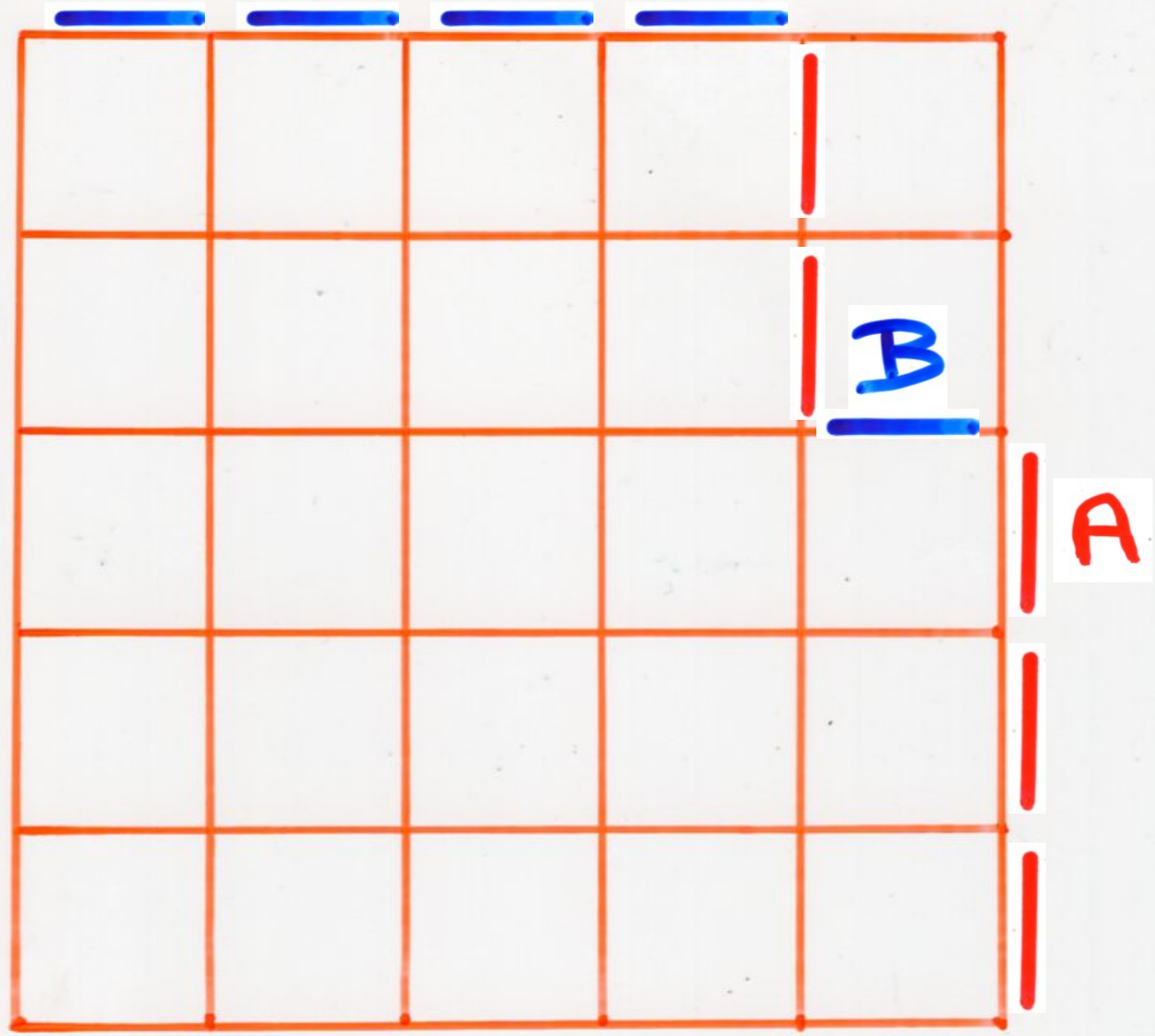

B
B
B
B
B
A
A
A
A
A

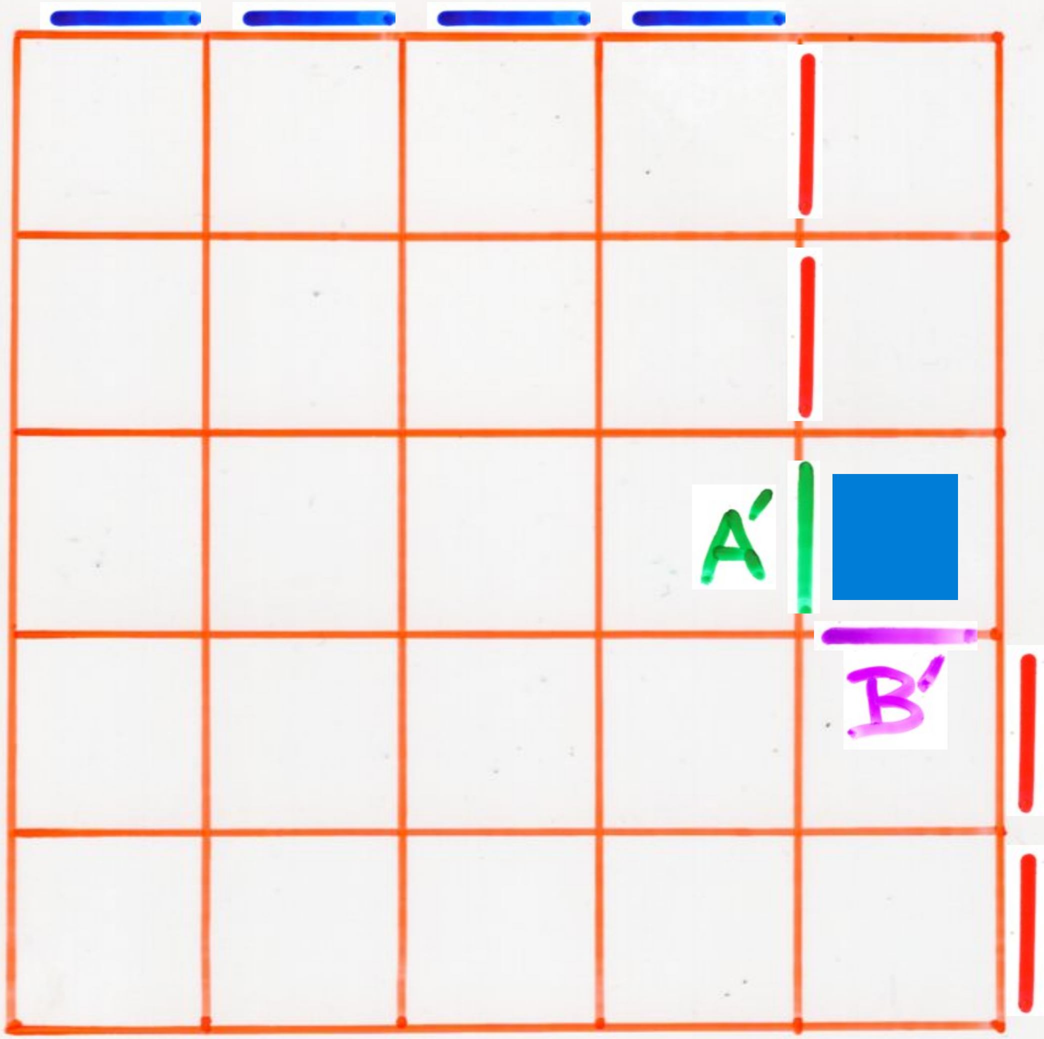

B
A


A
B

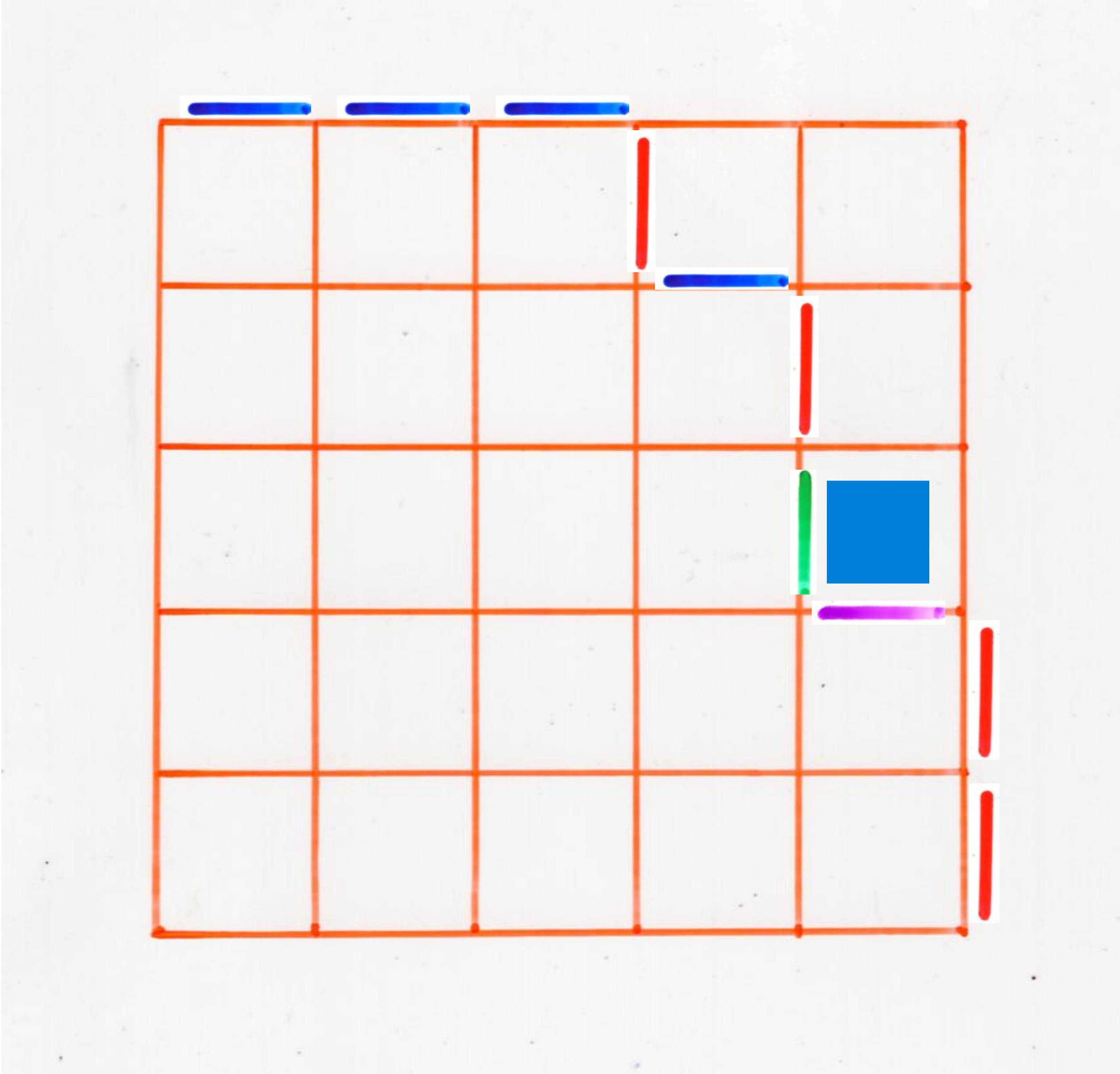

B
A

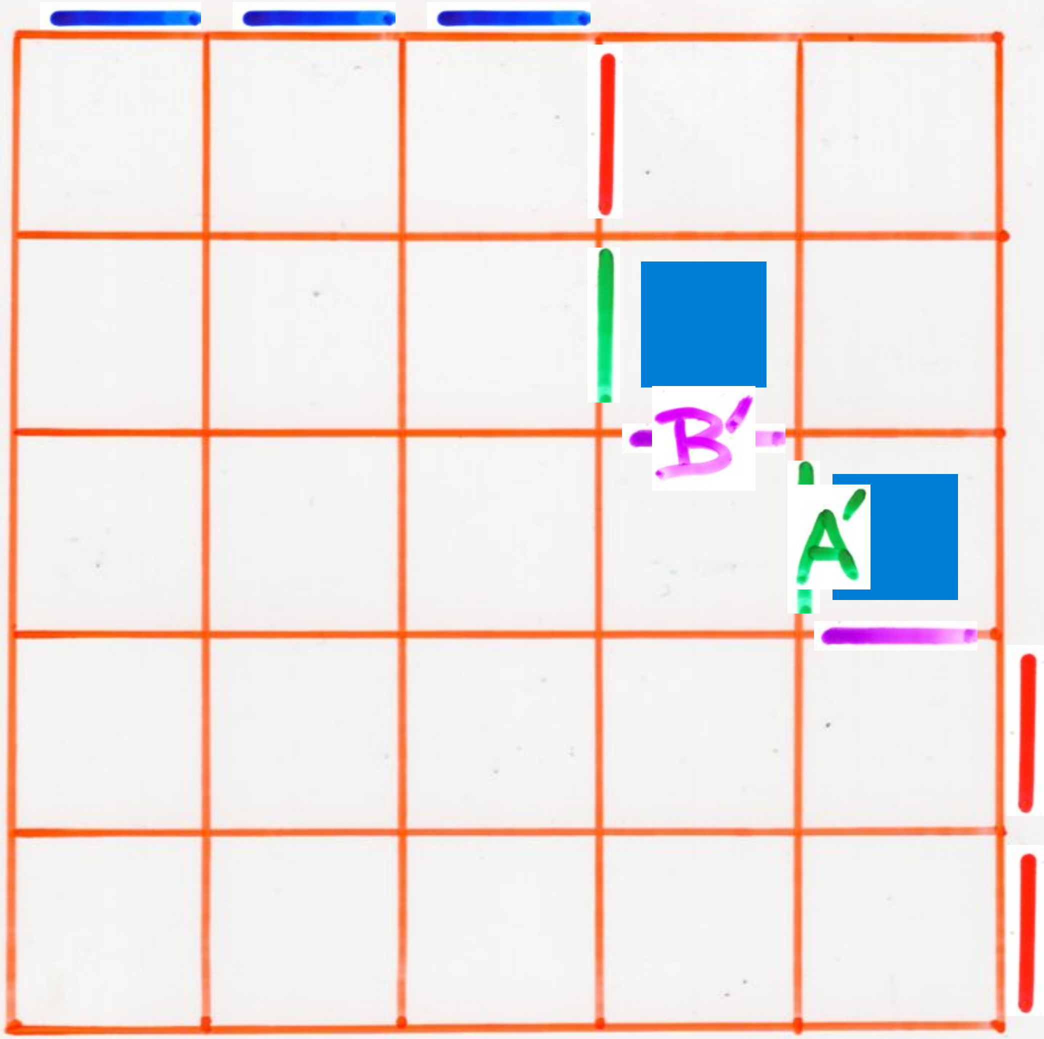

A
B

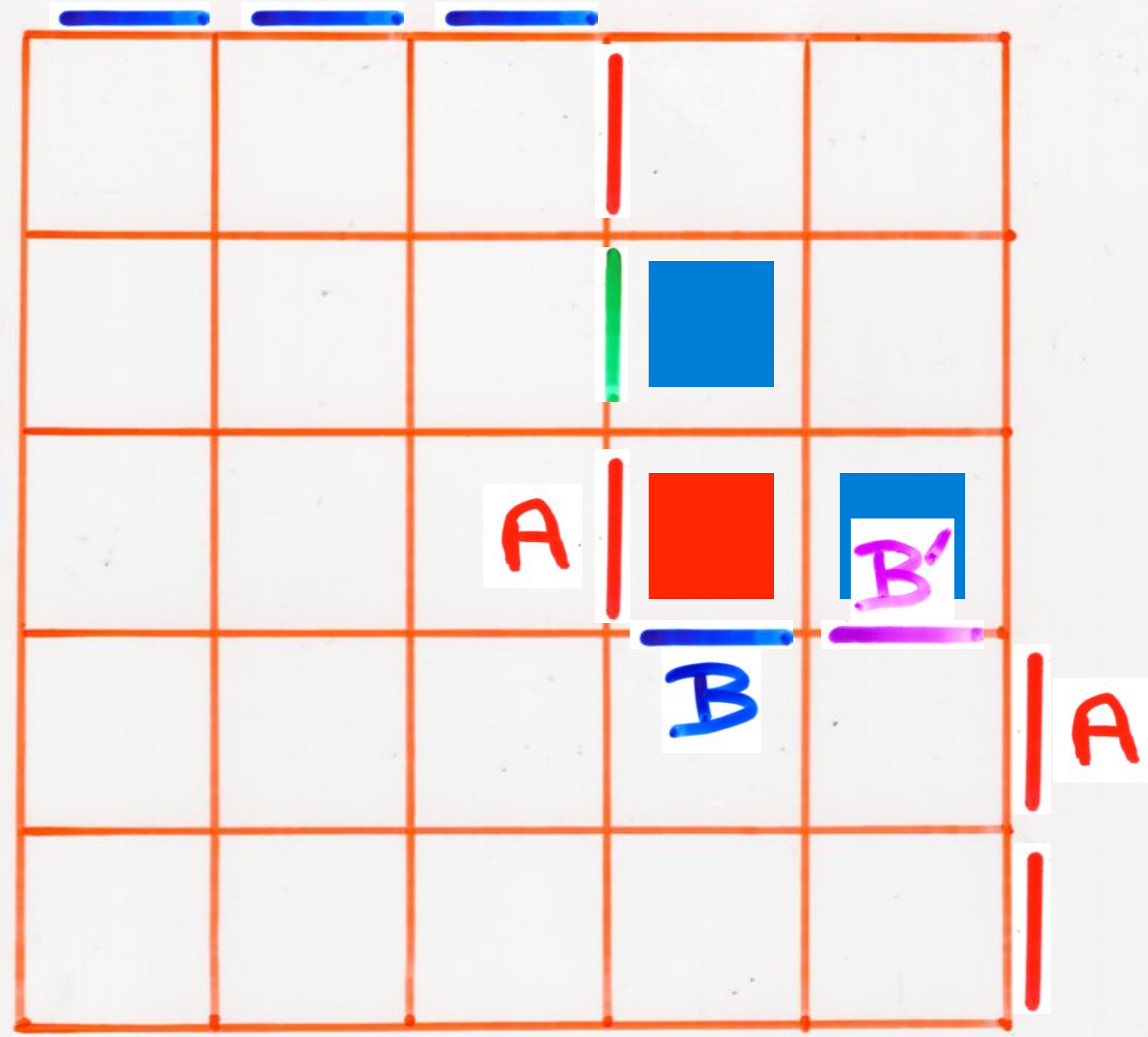

B
A

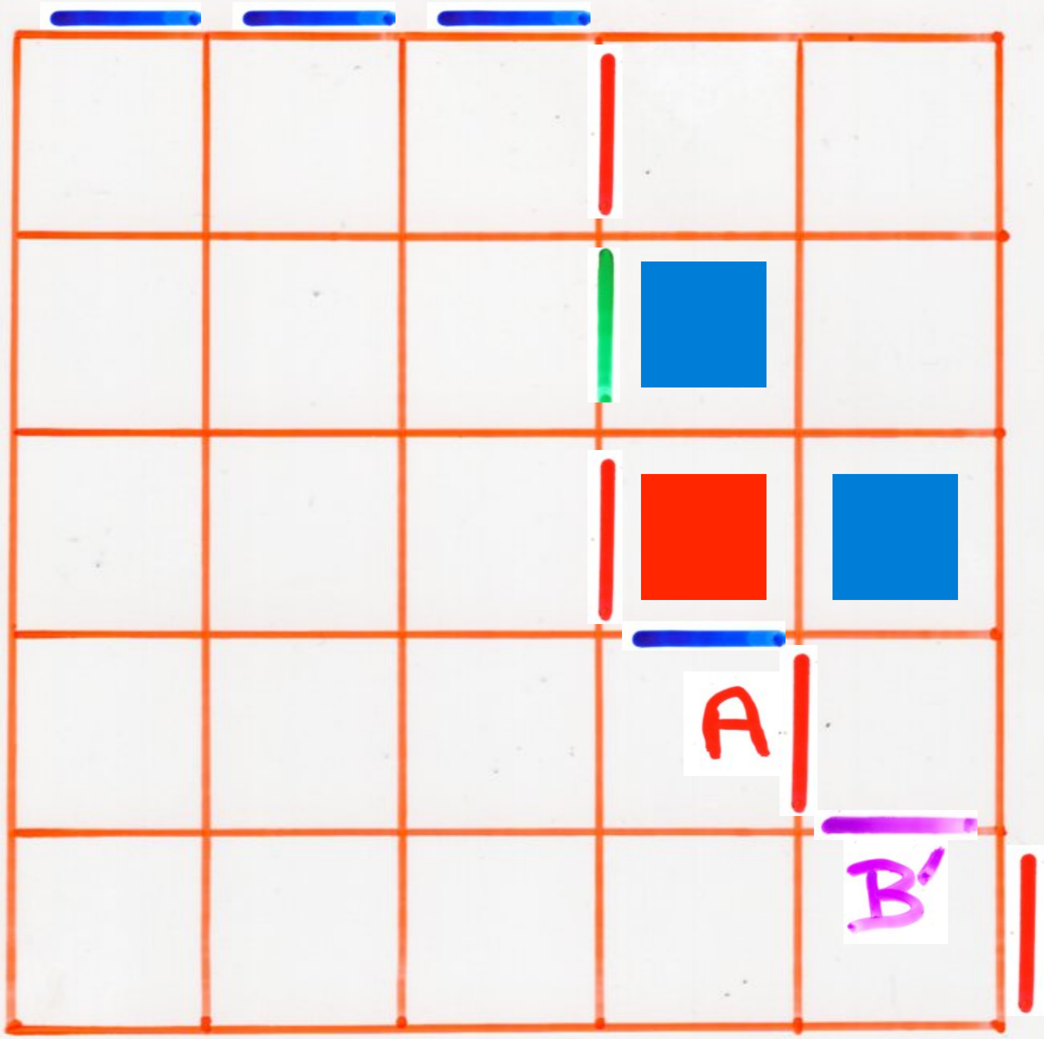


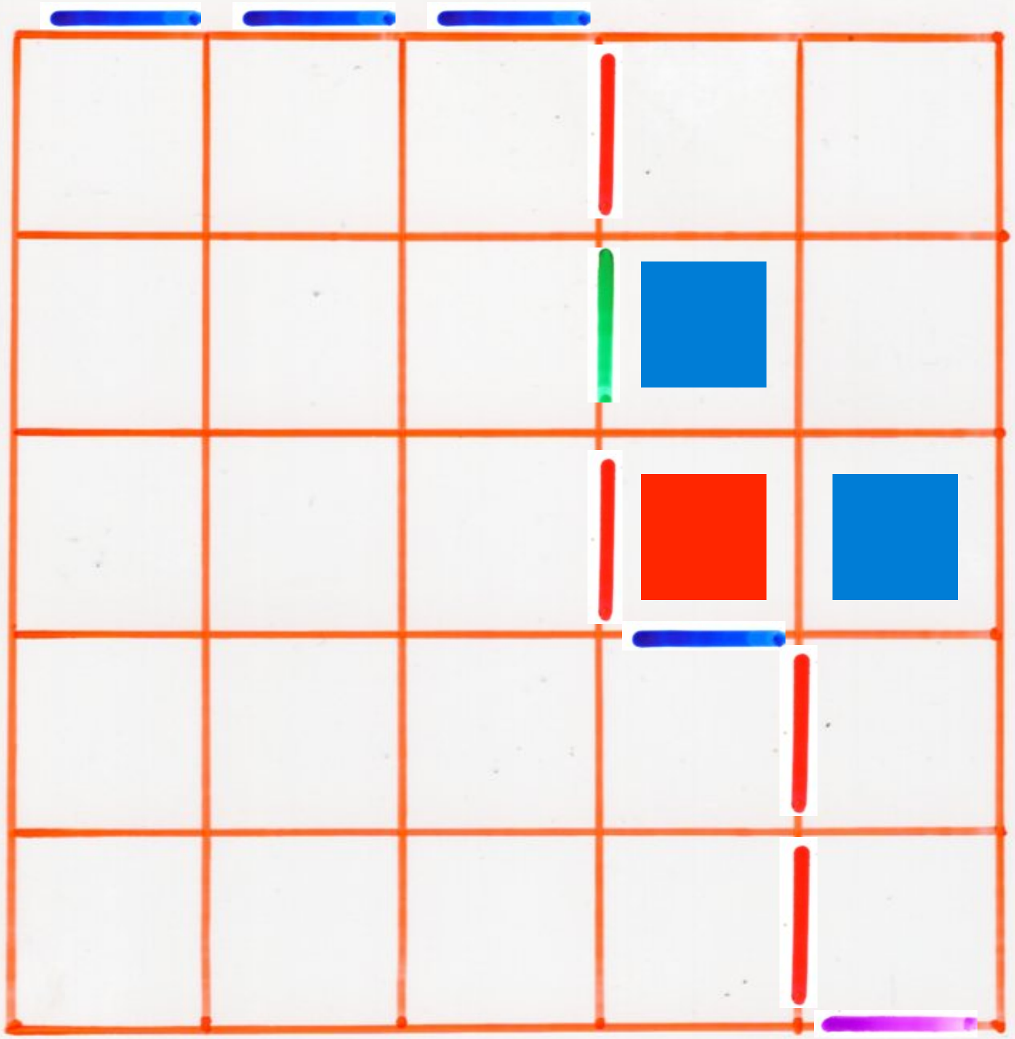
A'
B'

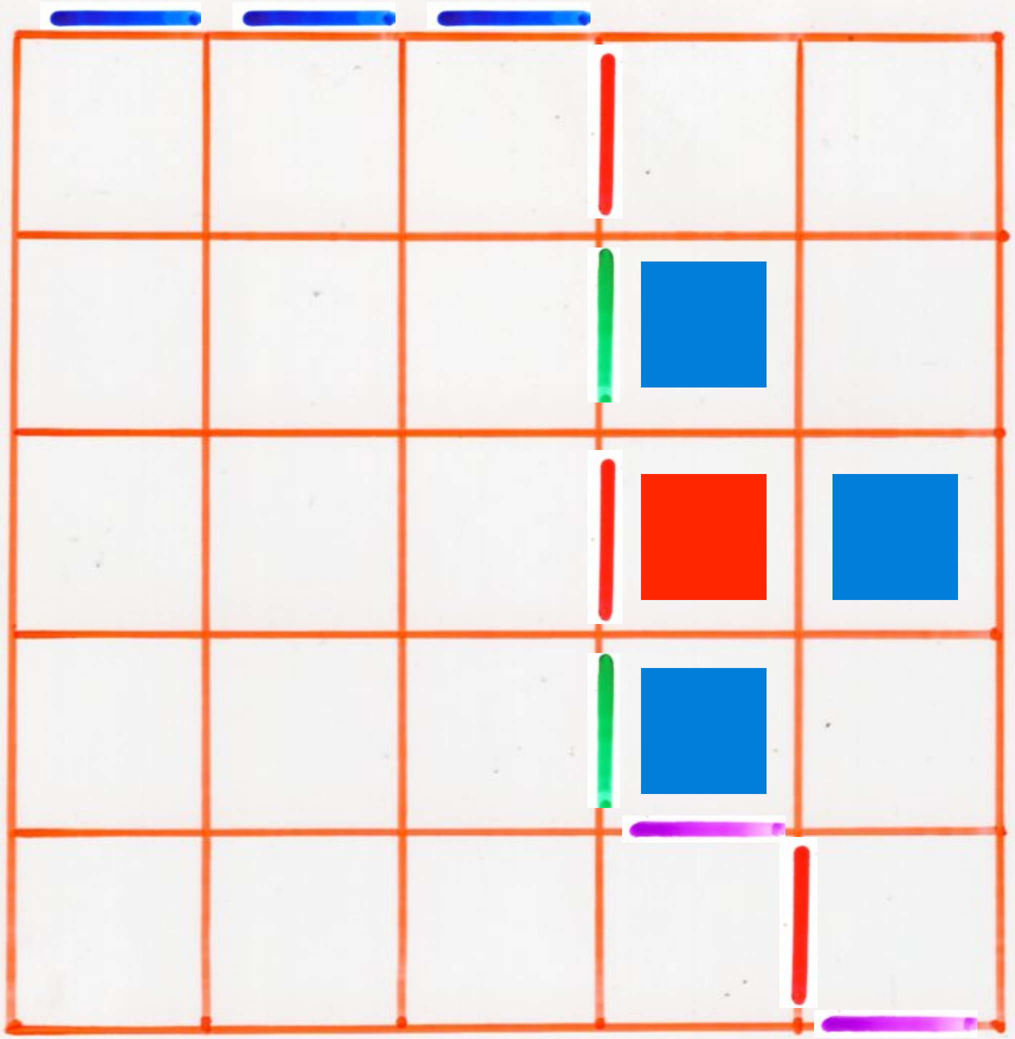


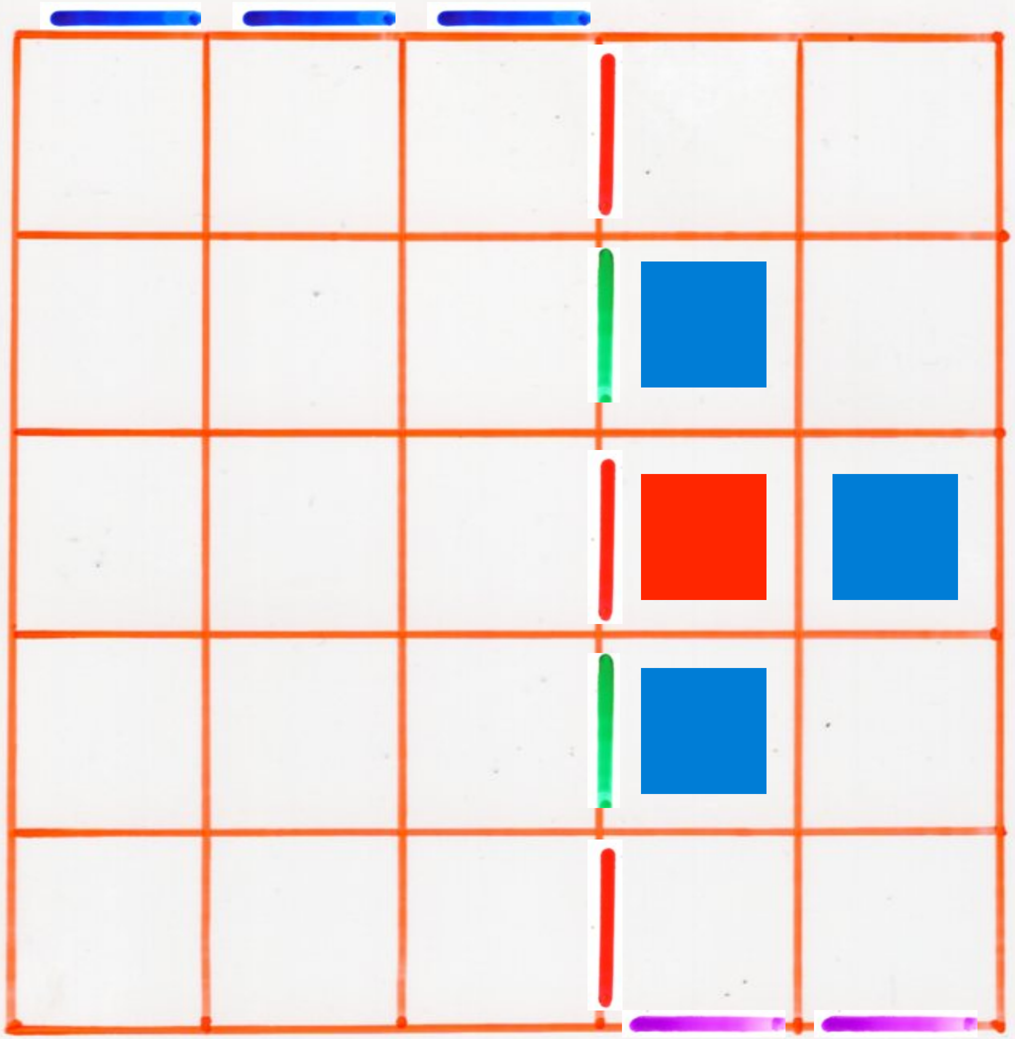

B′
A′

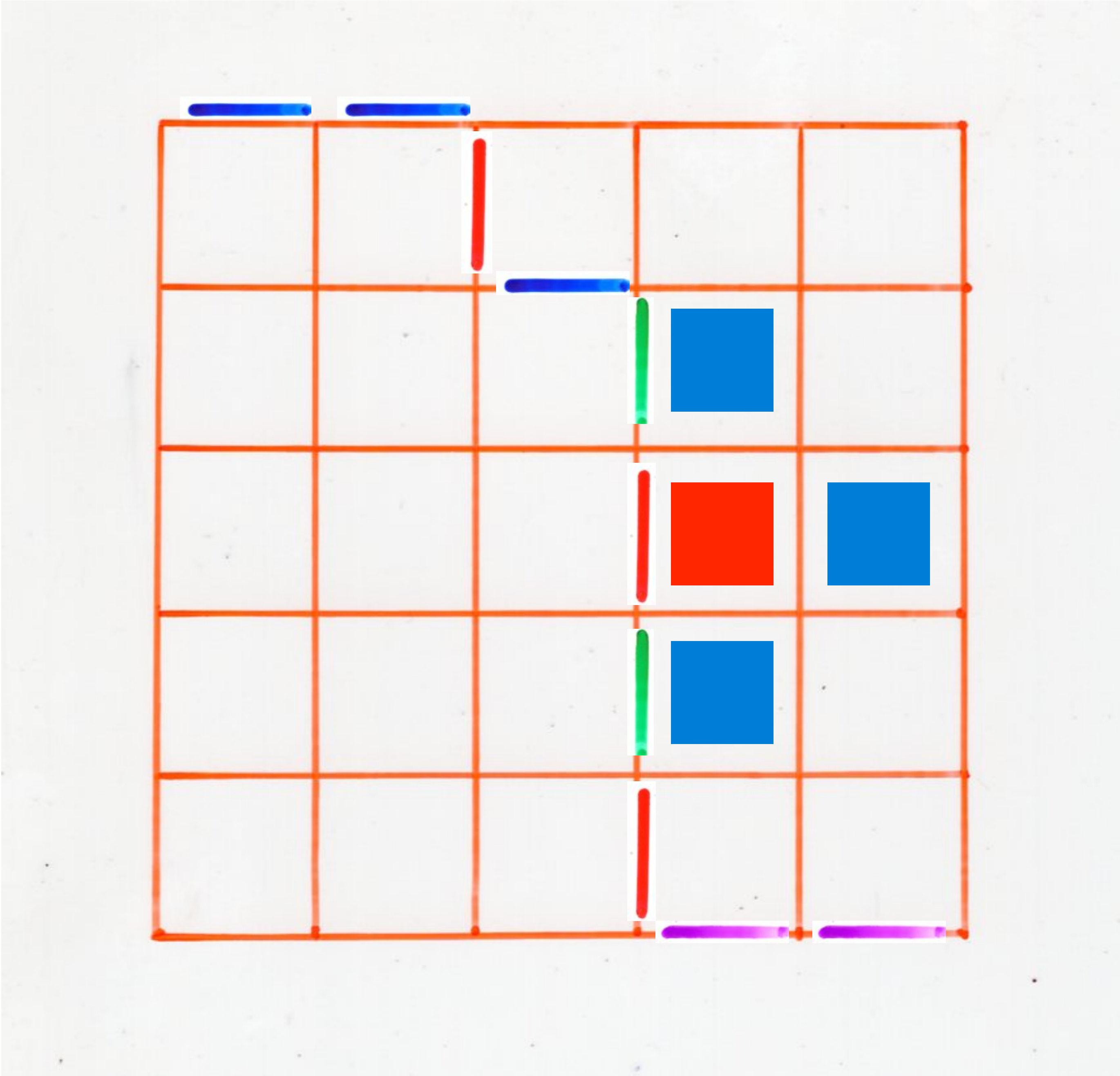

A
B'
B
A

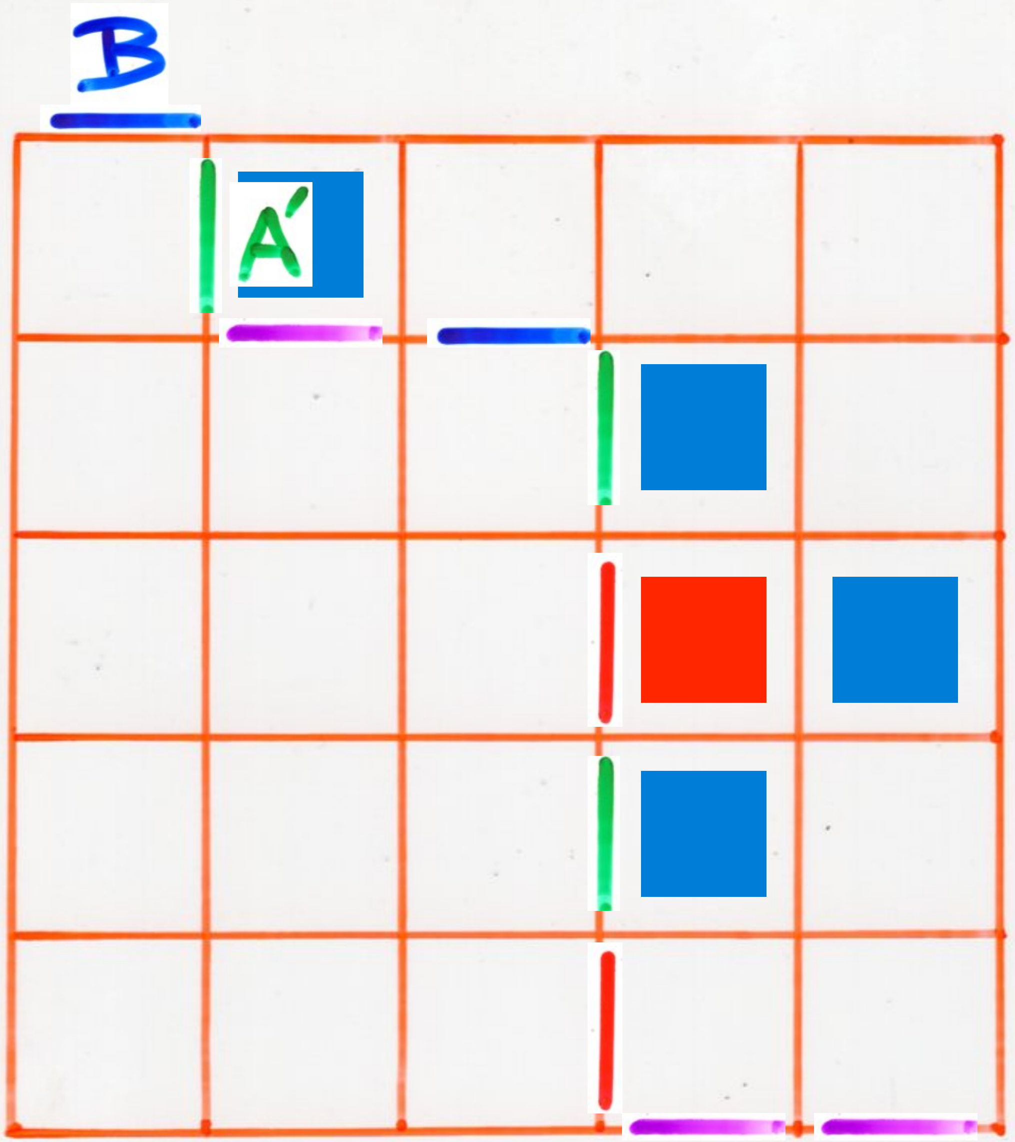

A
B'



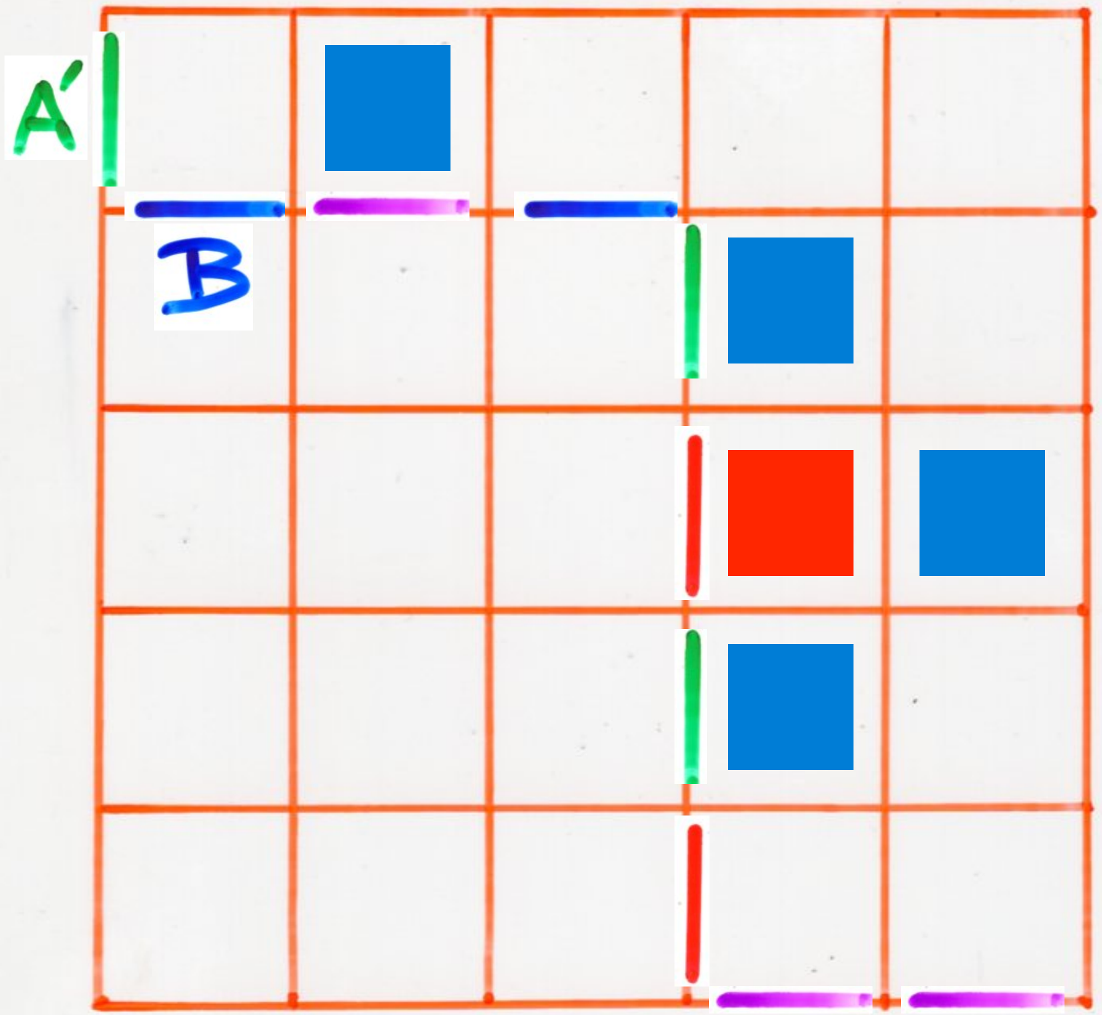




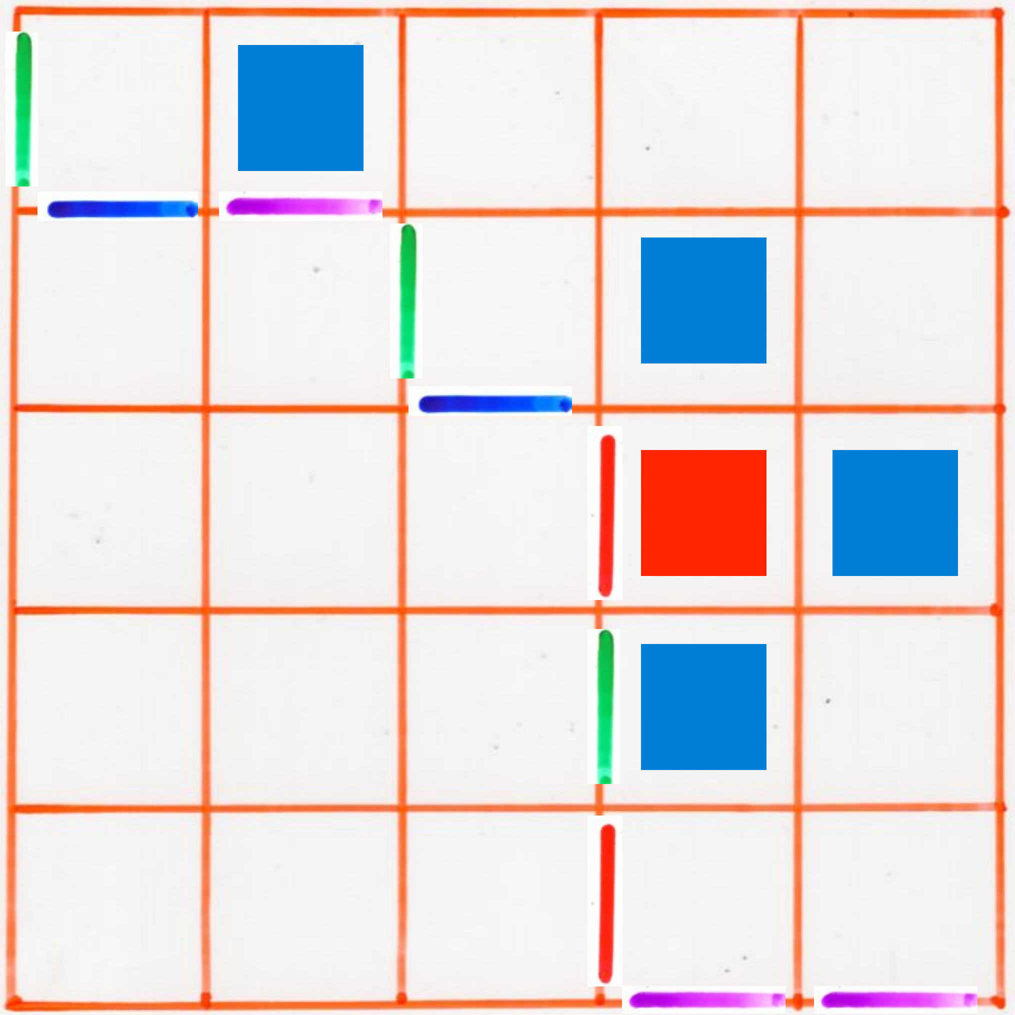


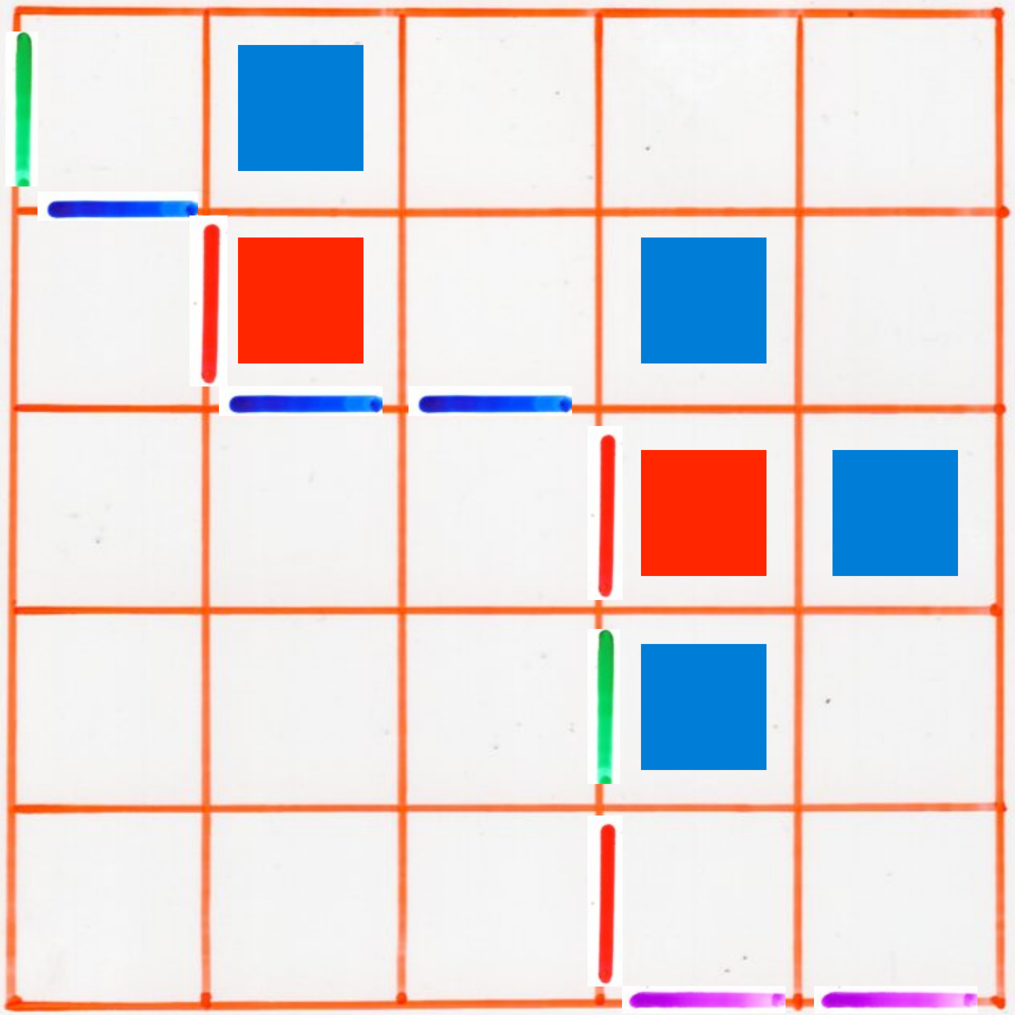


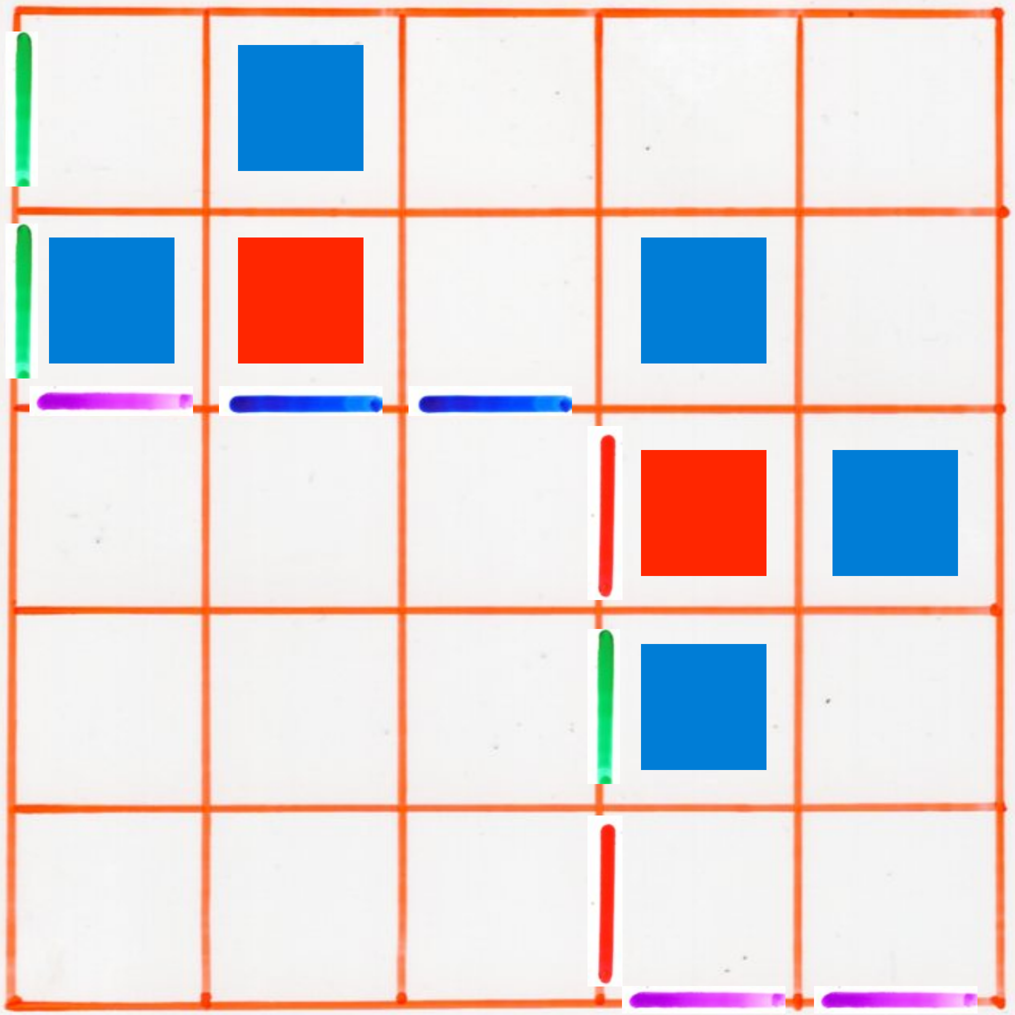
B
A'

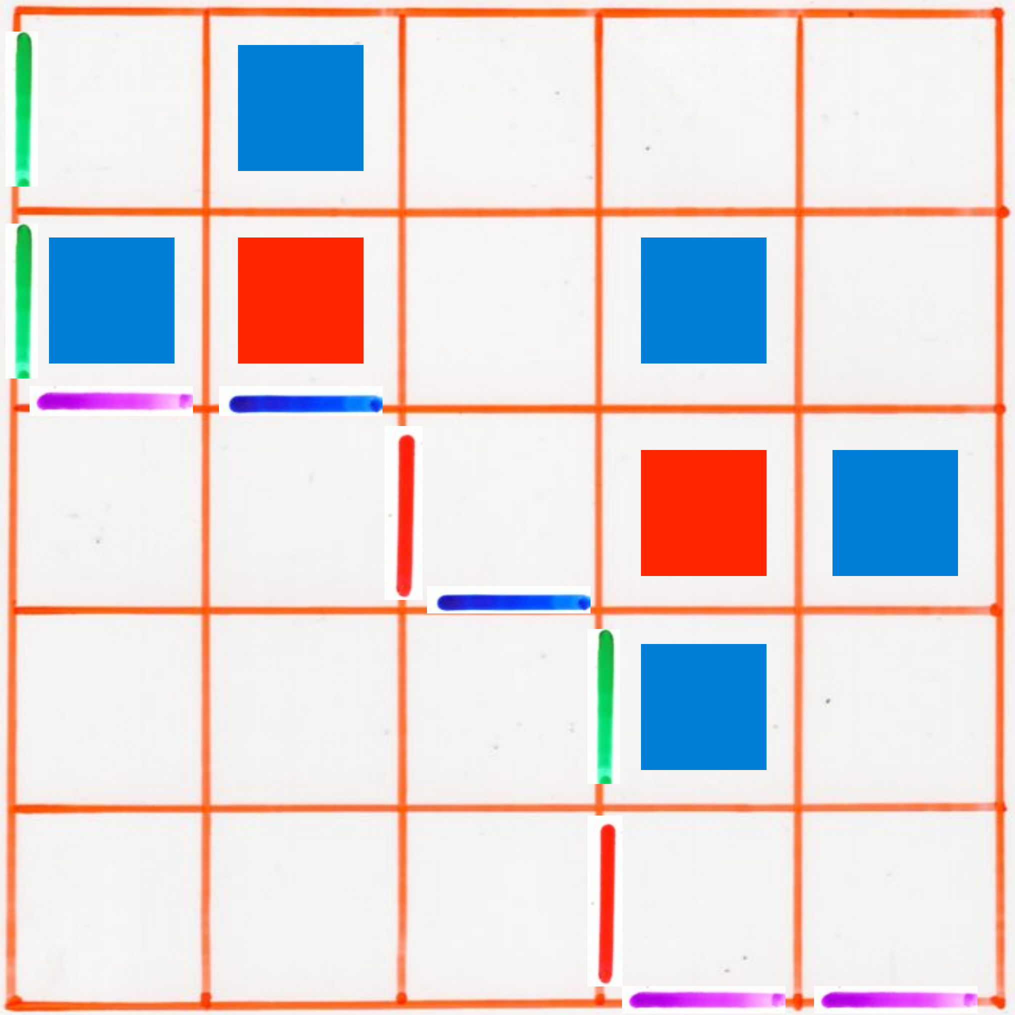


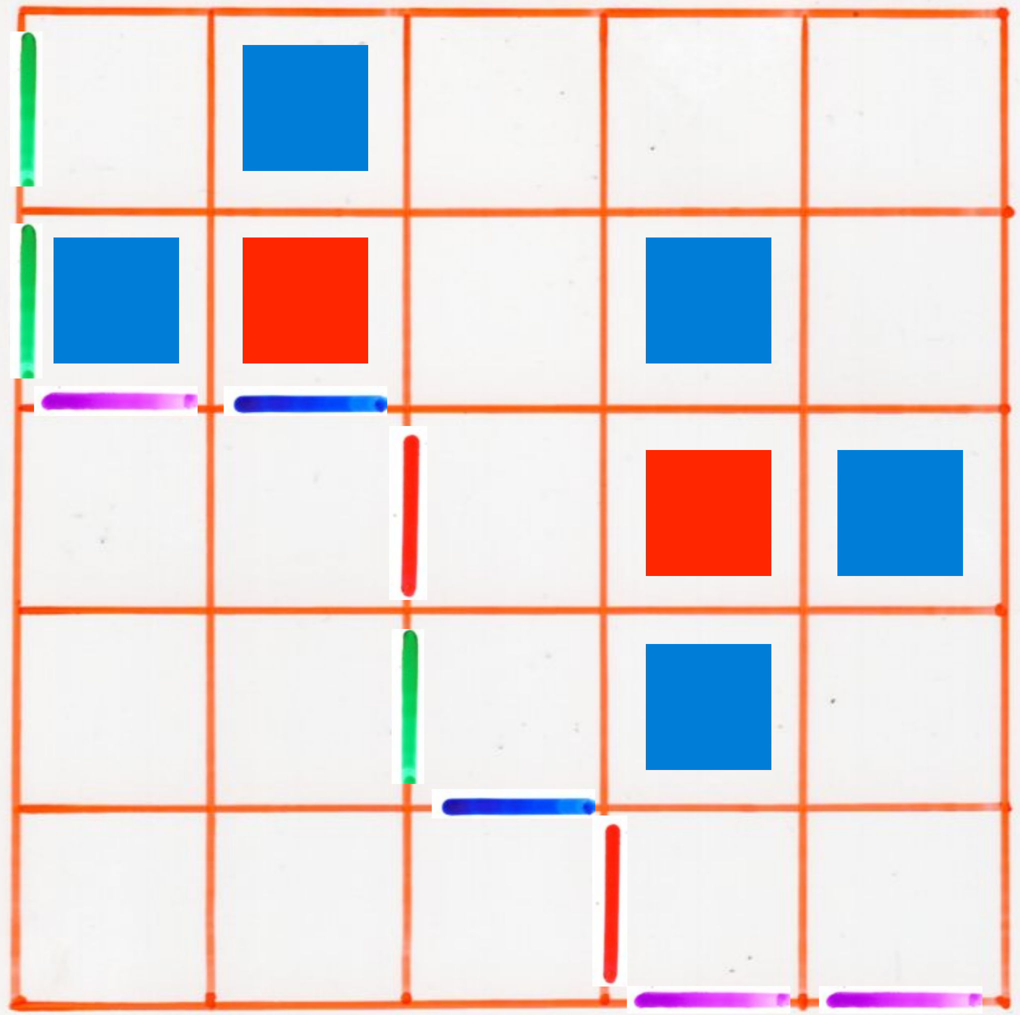
A′
B

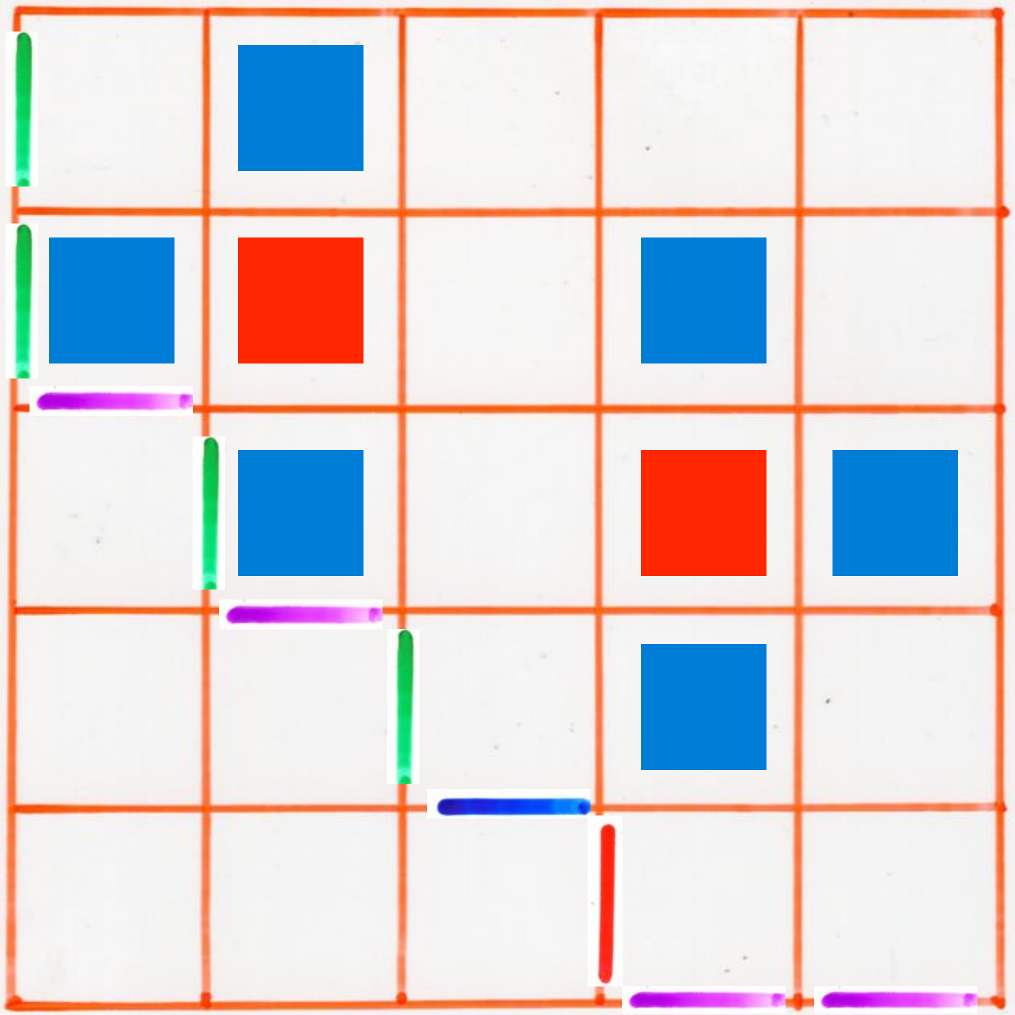


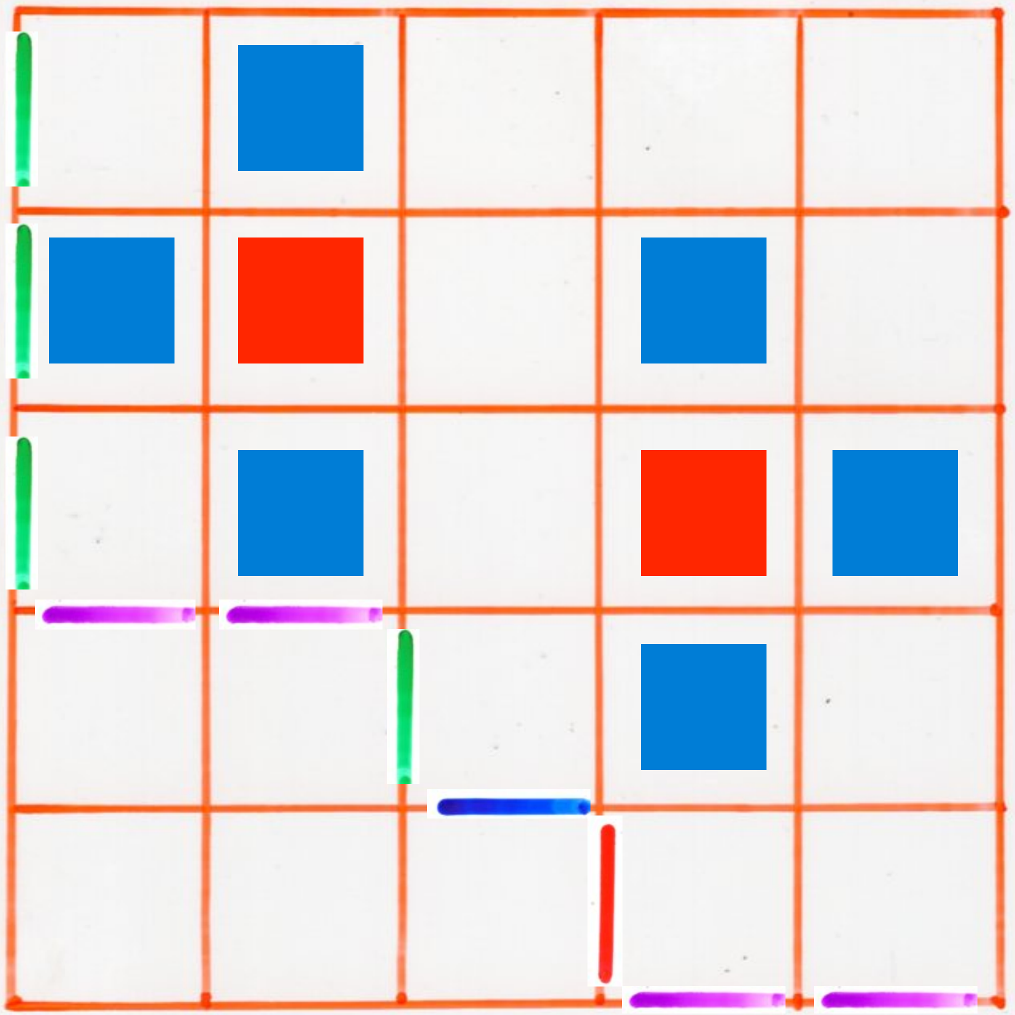


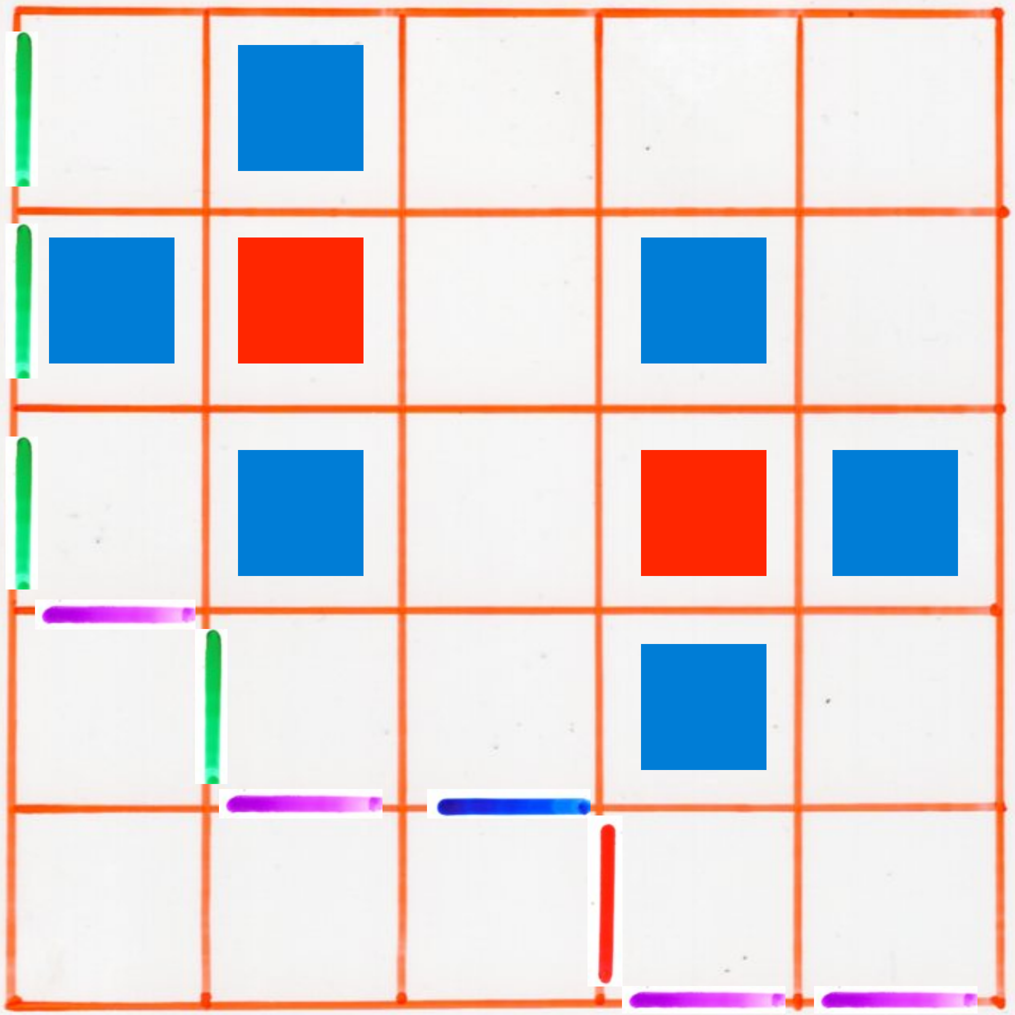


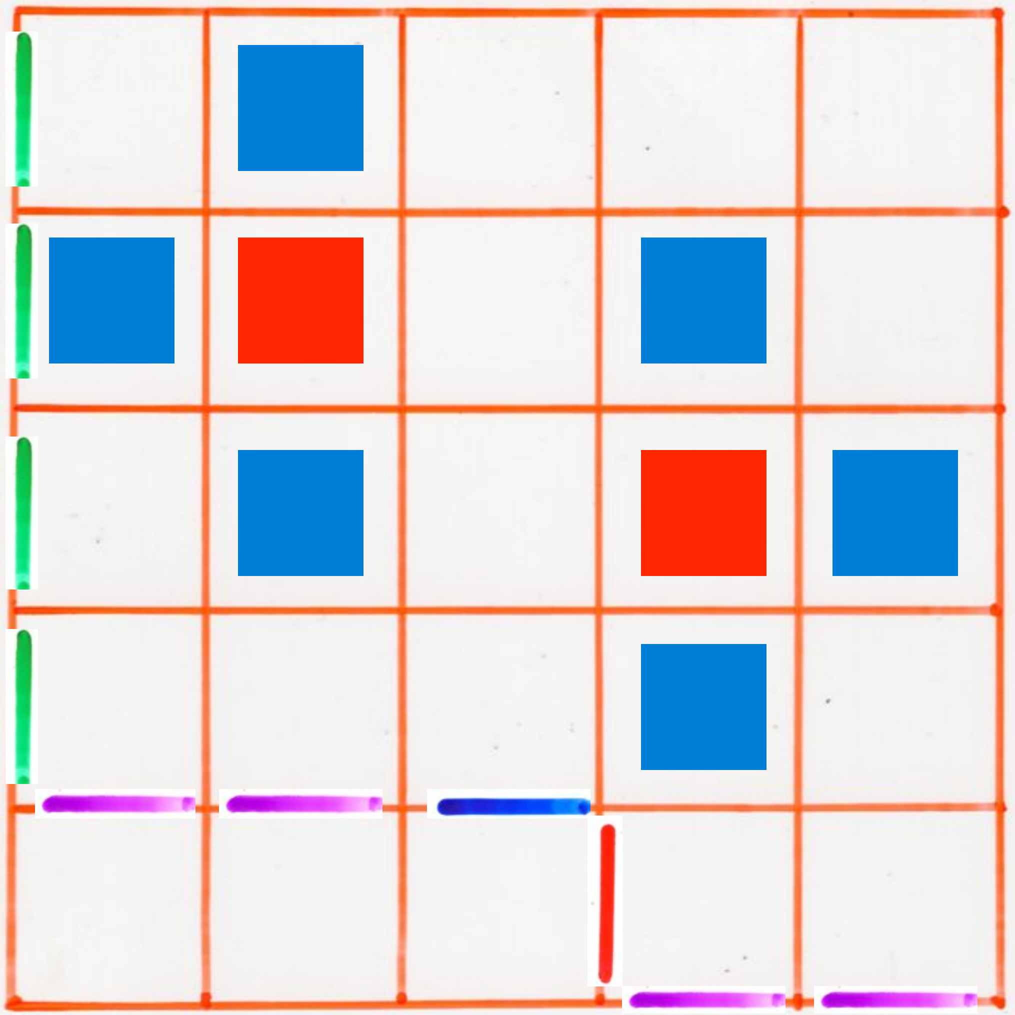


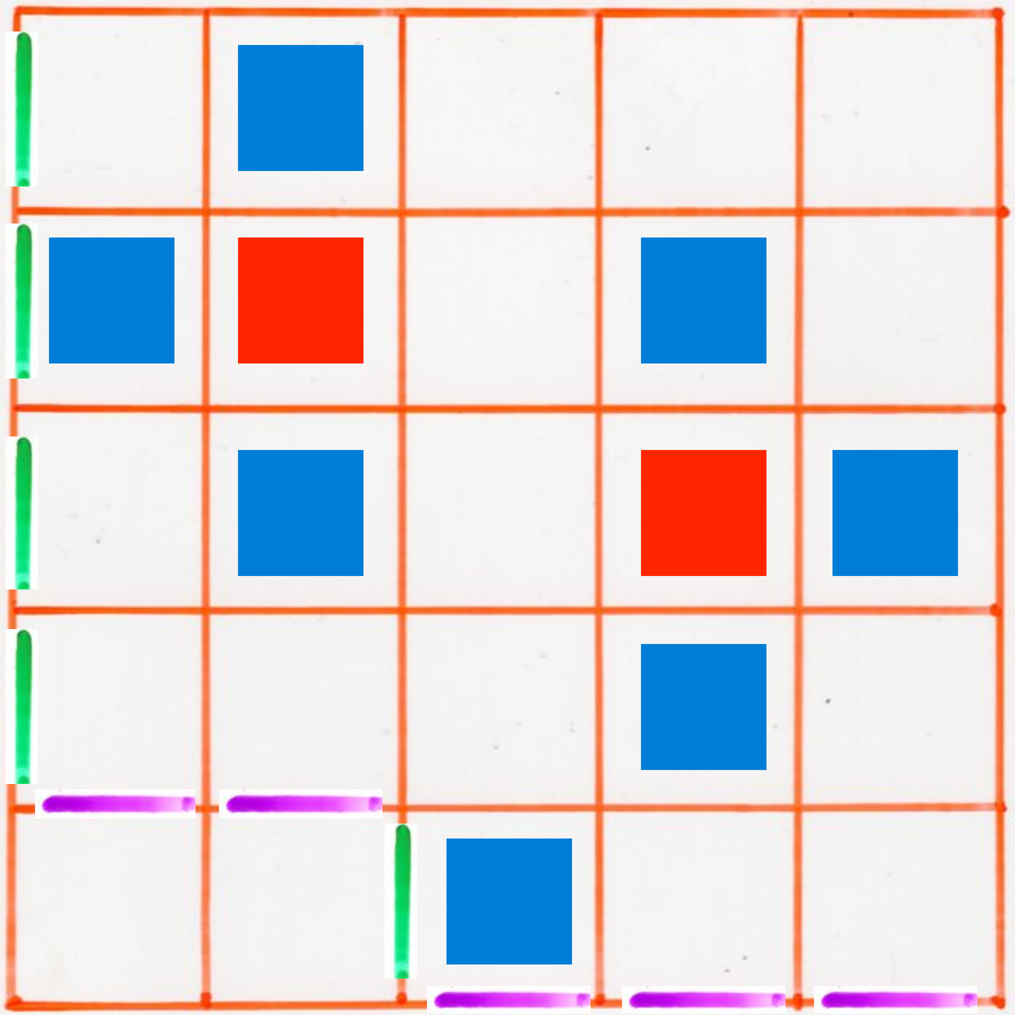


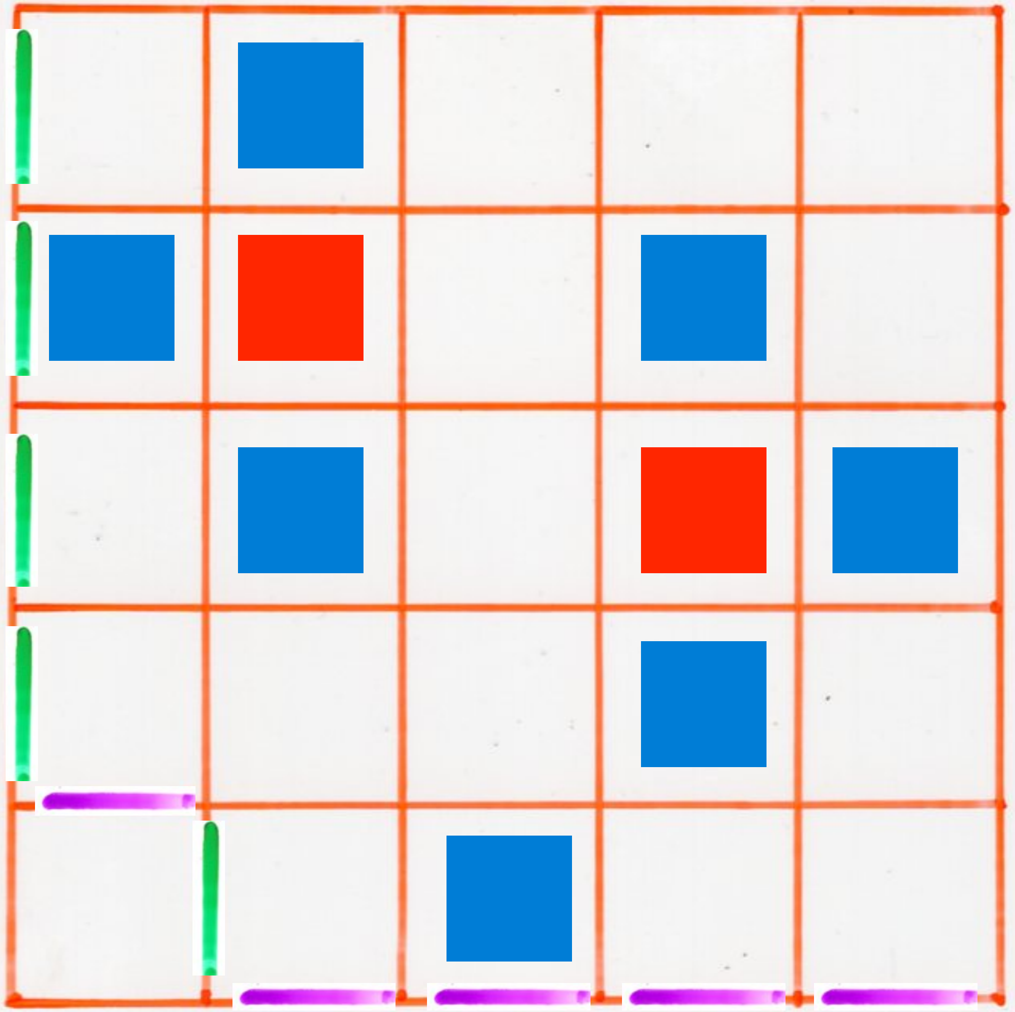


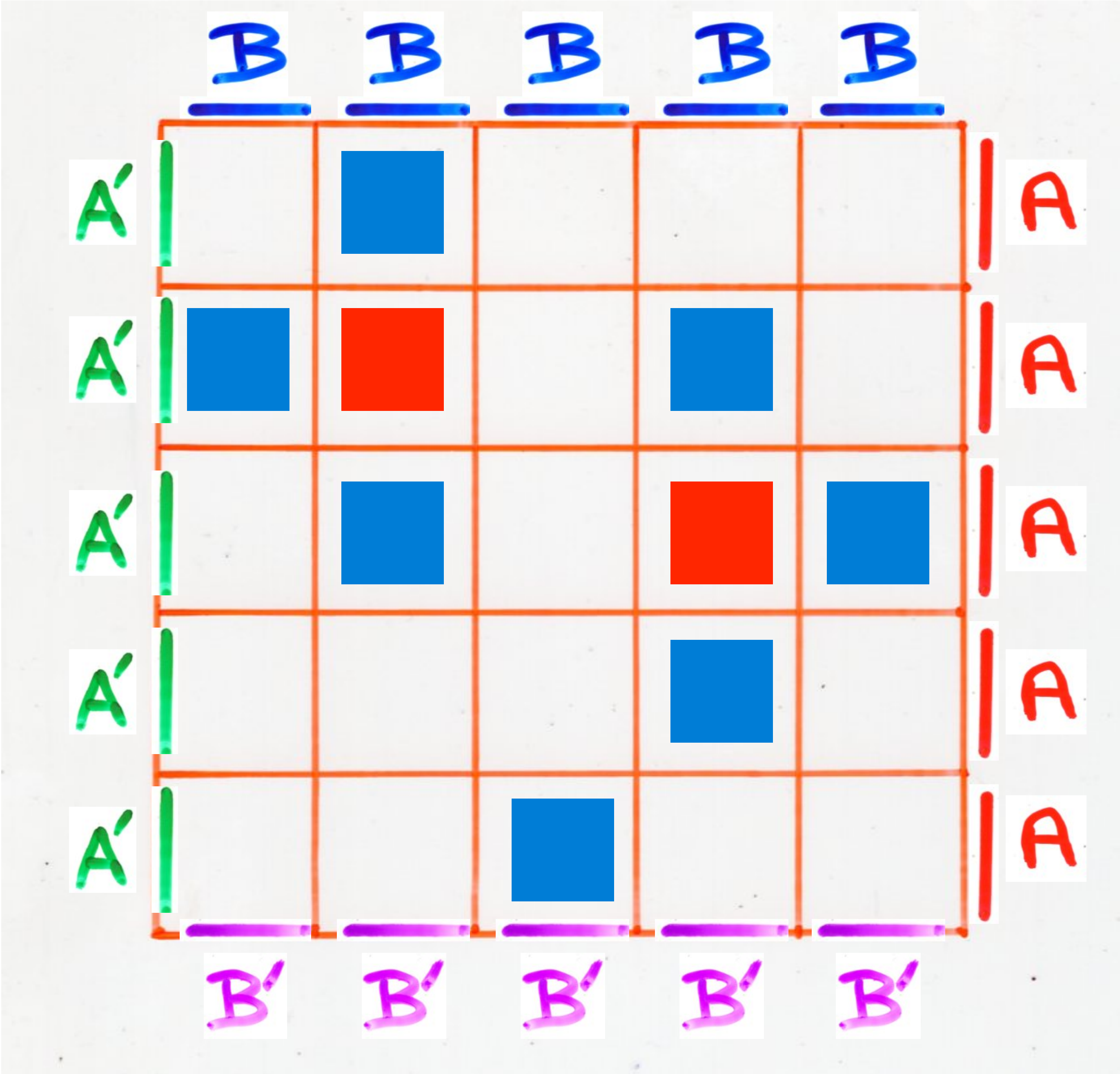











B B B B B
A′ A′ A′ A′ A′
A A A A A
B′ B′ B′ B′ B′

Prop. For $w = B^n A^n$

$u = A'^n$, $v = B'^n$

$c(u, v; w) =$ the number of $n \times n$ ASM (alternating sign matrices)

complete Q-tableaux

quadratic algebra $\mathcal{Q}$

generators

$$\mathcal{B} = \{B_j\}_{j \in J}$$
$$\mathcal{A} = \{A_i\}_{i \in I}$$

for every $i \in I$, $j \in J$

$$\mathcal{A} \cap \mathcal{B} = \emptyset$$

commutations

$$B_j A_i = \sum_{k\ell} c_{ij}^{k\ell} A_k B_\ell$$

Lemma In $\mathbb{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

The monomials $\{uv,\ u \in \mathcal{A}^*, v \in \mathcal{B}^*\}$

form a basis of the algebra $\mathbb{Q} = \mathbb{C}\langle \mathcal{A} + \mathcal{B} \rangle / \mathcal{J}$

non-commutative polynomials with variables $\mathcal{A} + \mathcal{B}$

$(\mathcal{A} \cup \mathcal{B})$

$\mathcal{J}$ ideal generated by the commutations relations

This polynomial can be obtained by successive rewriting rules from w

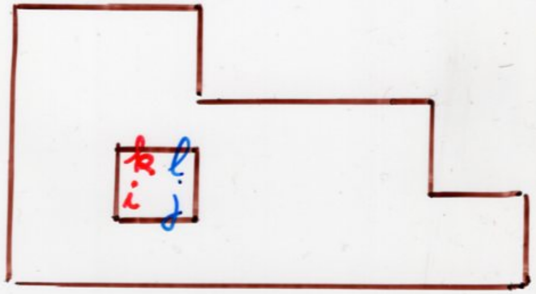
$$B_j A_i \longrightarrow \sum c_{ij}^{kl} A_k B_l$$

until there is no more such occurrence

<u>Lemma</u> This polynomial is independent of the order of rewritings

Definition complete Q-tableau

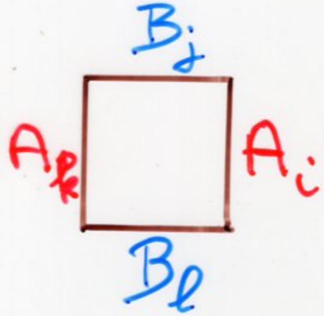
Ferrers diagram F
where each cell is
labeled by the set
R of rewriting rules
with "compatibility" condition

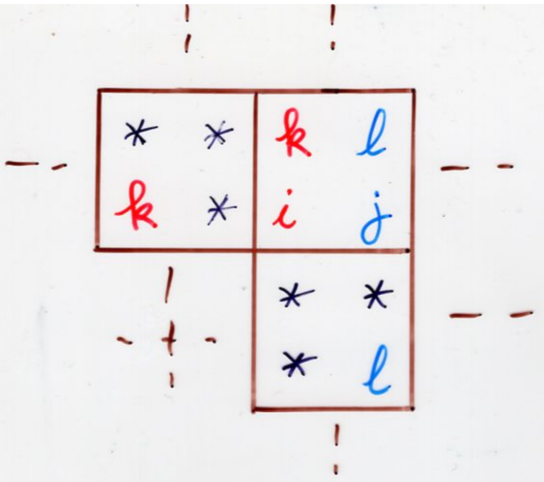


$$R = \left\{ \boxed{\begin{matrix} k & l \\ i & j \end{matrix}} \,,\; i, k \in I,\; j, l \in J \right\}$$

$$B_j \cdot A_i \longrightarrow c^{k\,l}_{i\,j} \cdot A_k B_l$$

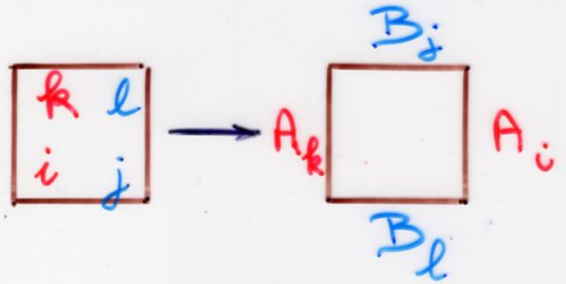
or

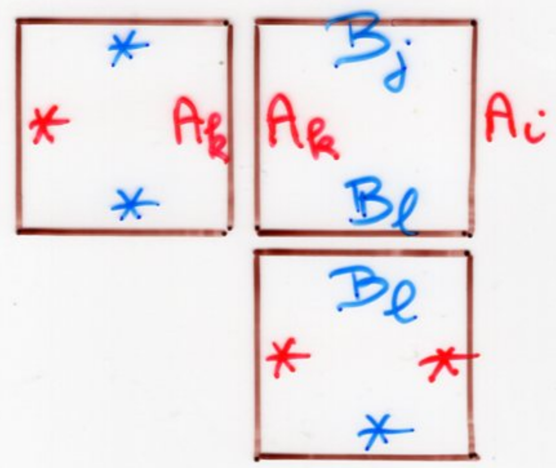




$$B_j A_i \longrightarrow c_{ij}^{k\ell} A_k B_\ell$$

edge labeling of a complete Q-tableau





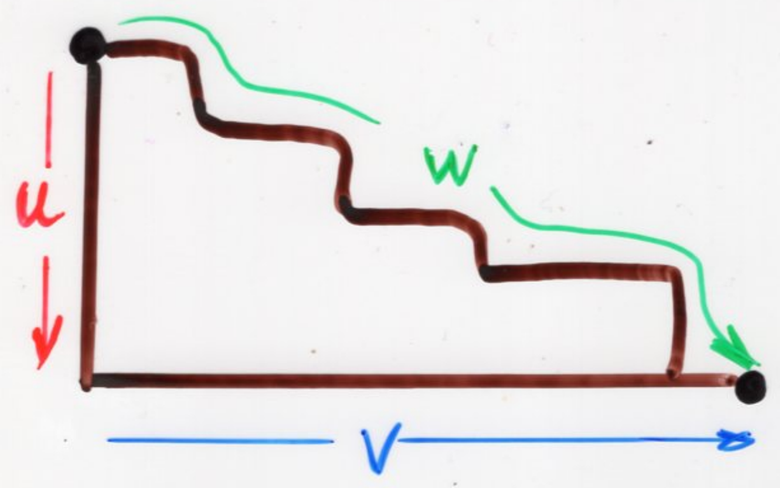
Definition

weight of a complete Q-tableau T

$$\mathrm{wgt}(T) = \prod_{\substack{\text{cells} \\ \text{of } F}} c_{ij}^{kl} \in \mathbb{K}[X]$$

Definition For T complete Q-tableau

$$\mathrm{uwb}(T) = w \in (\alpha \cup \beta)^*$$
$$\mathrm{lwb}(T) = uv, \quad u \in \alpha^*, \; v \in \beta^*$$



is the word obtained by reading the labels of the cells on the NE (resp. SW) border of F going from the NW corner to the SE corner

Lemma In Q every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

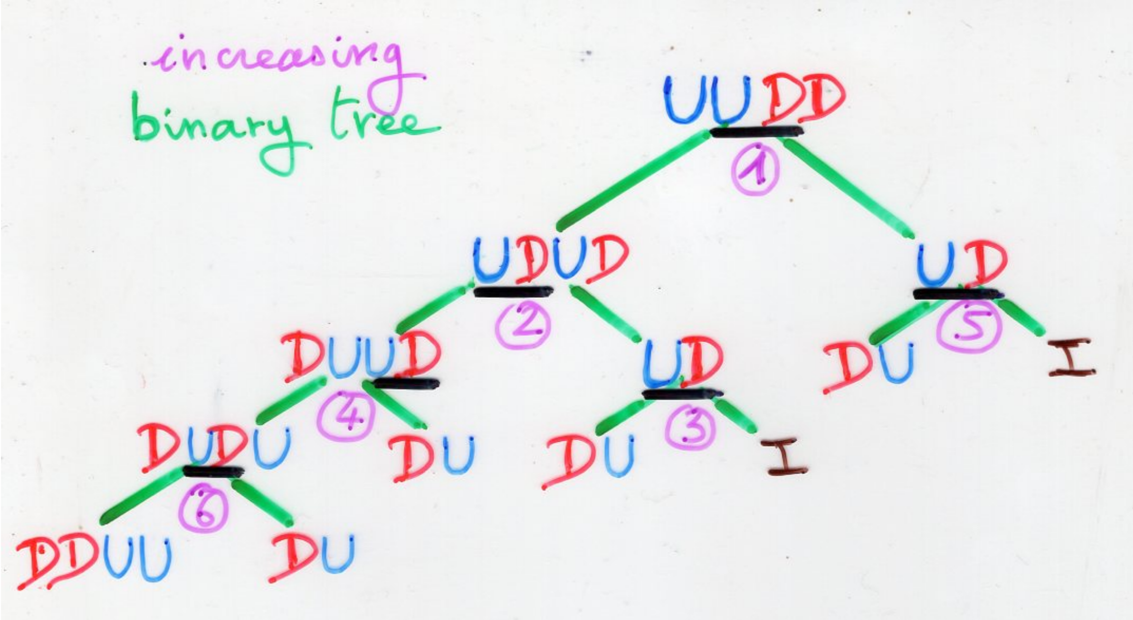
Proposition For any words

$$w \in (\mathcal{A} \cup \mathcal{B})^*, \quad u \in \mathcal{A}^*, \quad v \in \mathcal{B}^*$$

$$c(u, v; w) = \sum_{\substack{T \\ \text{complete } Q\text{-tableau} \\ uwb(T) = w \\ lwb(T) = uv}} wgt(T)$$

proof of the proposition
about «normal ordering»
for the quadratic algebra Q

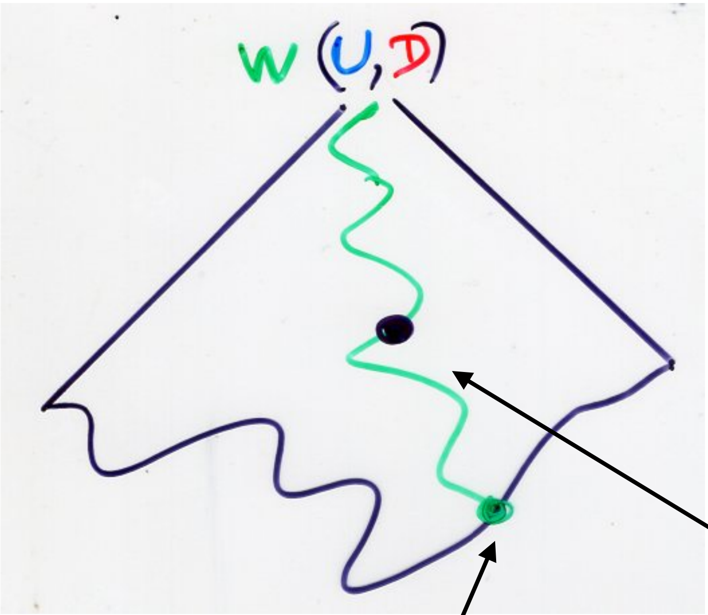
Ch 1c, p115-117



binary tree T associated to a possible rewriting process

$$U^2D^2 = D^2U^2 + 4DU + 2I$$

this polynomial is independant of the order of the substitutions



binary tree T associated to a possible rewriting process

internal vertex

$\updownarrow$

skew-Ferrers

complete skew-Q-tableau

leafs of T

bijection

$\longleftrightarrow$

complete Q-tableaux shape λ

$\lambda = F(w)$

A possible calculus:

after (n-1) steps, starting from $w = w$

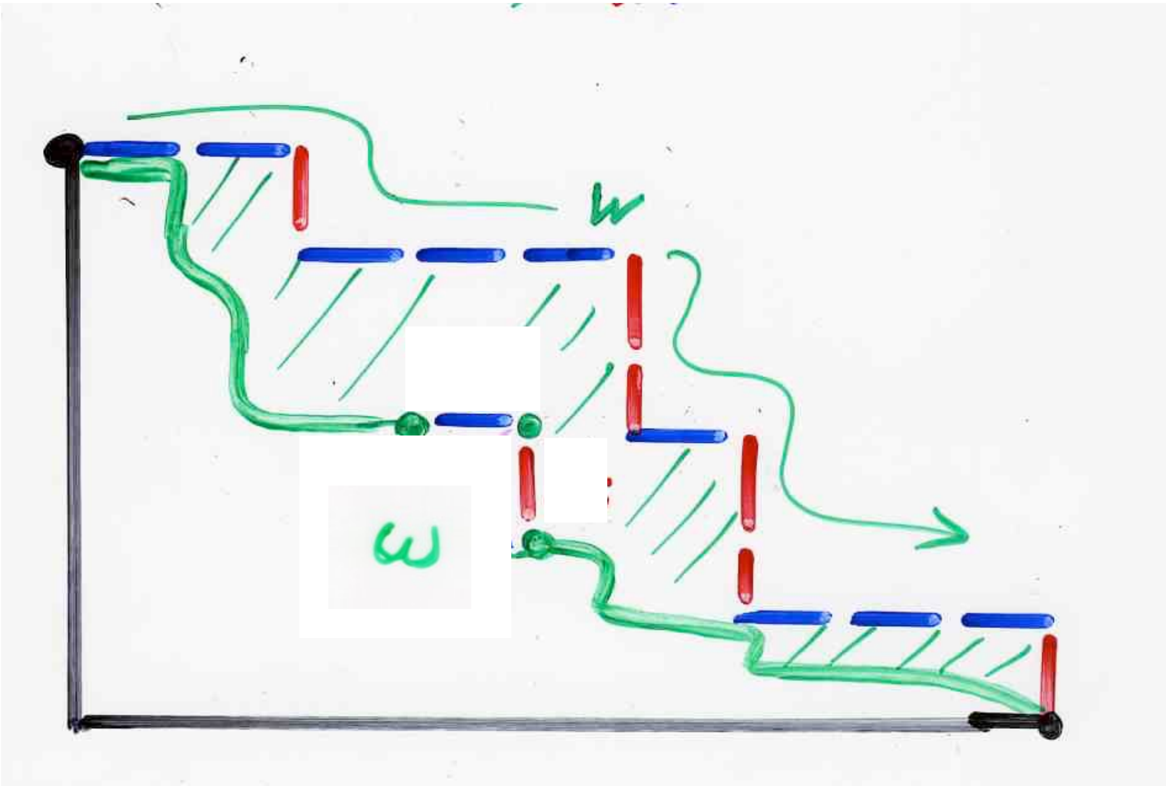
$$w = \sum_{\substack{\omega \\ \text{word} \in (\mathcal{A} \cup \mathcal{B})^*}} c(\omega, w)\, \omega$$

step n: choice of a word ω with a factor $B_j A_i$

rewriting:

$$\omega = \omega' B_j A_i \omega'' \longrightarrow \sum_{k,l} c_{ij}^{kl}\, \omega' A_k B_l \omega''$$

complete skew Q-tableau
between ω and w



we associate a tree A_{n-1}

- vertices are certain complete skew Q-tableaux T with $uwb(T) = w$

- the root is w (as an "empty" Q-tableau between w and w)

- internal vertices (no son) are labeled $1, 2, \dots, (n-1)$

- A_{n-1} is an increasing tree (label of a vertex < labels of its son) step n $A_{n-1} \longrightarrow A_n$

- choice of a leaf T

  T is a skew Q-tableau between ω and w

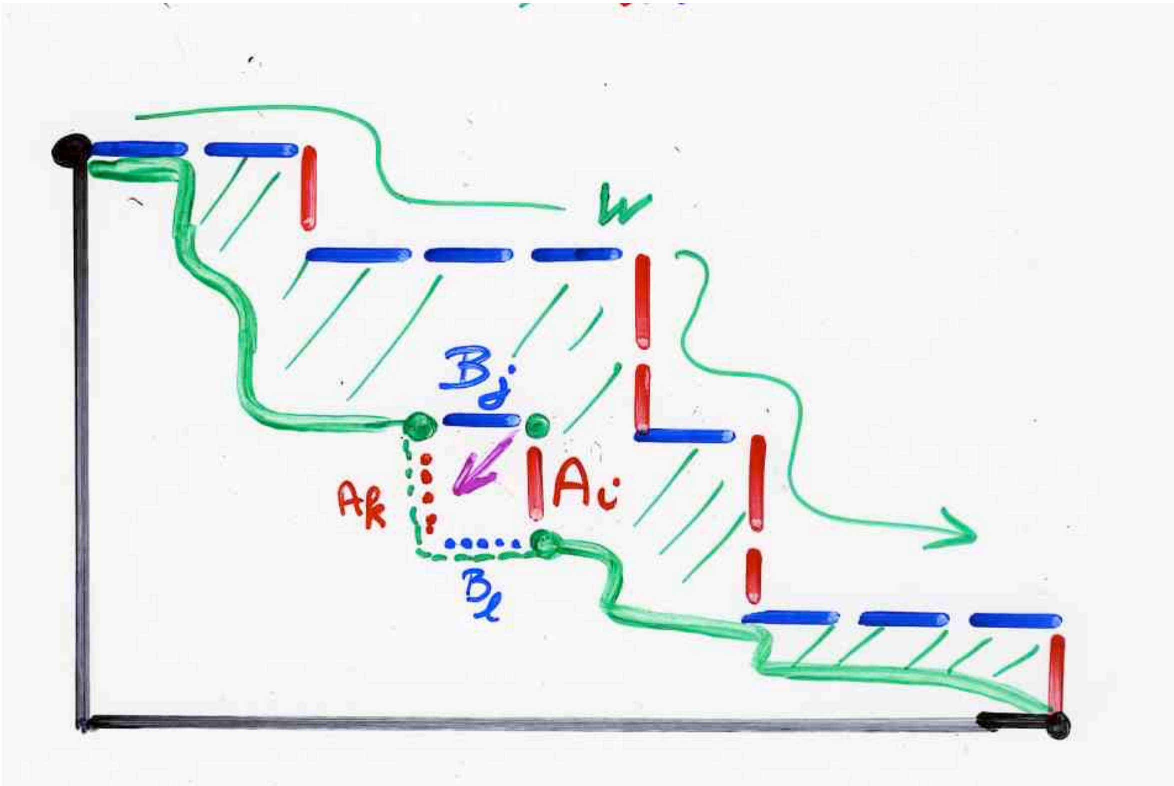
- choice of a factorization

  $$\omega = \omega' B_j A_i \omega''$$

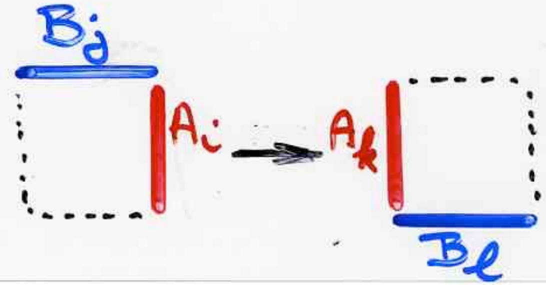
- label the vertex T by n

- the sons of T are the skew Q-tableaux of the form:

complete skew Q-tableau between ω and w



"planar" rewriting



A_n is an increasing tree

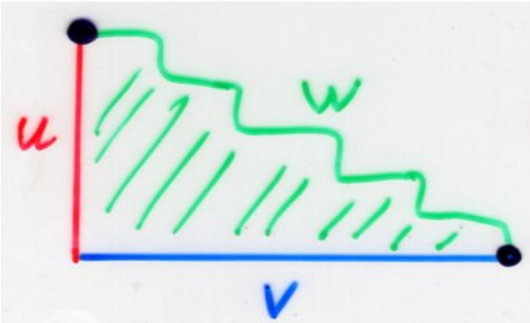
by recurrence :

$$w = \sum_{\substack{\text{leaves } T \\ \text{of } A_n}} wgt(T)\, lwb(T)$$

lower word border

end of calculus at $n = N$

all leaves of A_N are of the form:



$u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

(and thus $N = |F(w)|$ is the common height of the leaves of A_N)

and

$$w = \sum_{\substack{T \\ \text{leaves of } A_N}} \mathrm{wgt}(T)\, \mathrm{lwb}(T)$$

Lemma

complete (straight) Q-tableaux T with $uwb(T) = w$ are in bijection with the leaves of A_N

$\square$

end of the proof

Lemma In Q every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

Proposition For any words

$$w \in (\mathcal{A} \cup \mathcal{B})^*, \quad u \in \mathcal{A}^*, \quad v \in \mathcal{B}^*$$

$$c(u, v; w) = \sum_{\substack{T \\ \text{complete } Q\text{-tableau} \\ uwb(T) = w \\ lwb(T) = uv}} wgt(T)$$

Q-tableau: definition

L set of "labels"

$$\varphi : \left\{ \boxed{\begin{matrix} k & l \\ i & j \end{matrix}} \right\} = R \longrightarrow L$$

set of rewriting rules

$$B_j A_i \longrightarrow C^{kl}_{ij} A_k B_l$$

such that:

if $\begin{pmatrix} k & l \\ i & j \end{pmatrix} \neq \begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$ and $\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix}$ $(*)$

then $(i, j) \neq (i', j')$

φ satisfies $(*)$:

$(*)$ if $\varphi(\alpha \to \beta) = \varphi(\alpha' \to \beta')$
then $\alpha \neq \alpha'$

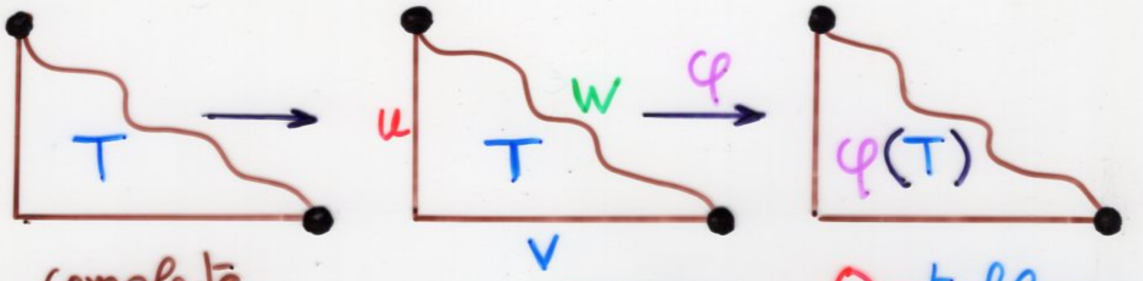
i.e. in a a single commutation equation

$\alpha = \beta_1 + \dots + \beta_r$

all elements $\varphi(\alpha \to \beta_i) \in L$ are $\neq$
set of labels

__Definition__ Q-tableau

is the "image" by φ satisfying (*) of a complete Q-tableau



complete Q-tableau

$w = uwb(T)$
$uV = lwb(T)$

Q-tableau w-compatible

__Proposition__ for $w \in (\mathcal{A} \cup \mathcal{B})^*$ fixed

$$\left\{ \begin{matrix} \text{set of Q-tableaux} \\ w\text{-compatible} \end{matrix} \right\} \overset{\varphi}{\longleftrightarrow} \left\{ \begin{matrix} \text{set of complete} \\ \text{Q-tableaux } T \\ \text{with } uwb(T) = w \end{matrix} \right\}$$

are in bijection by φ

Lemma In Q every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, u v$$

$$c(u, v; w) = \sum_{\substack{T \\ \text{complete } Q\text{-tableau} \\ uwb(T) = w \\ lwb(T) = uv}} wgt(T)$$

Proposition for $w \in (\mathcal{A} \cup \mathcal{B})^*$ fixed

$$\left\{ \begin{matrix} \text{set of } Q\text{-tableaux} \\ w\text{-compatible} \end{matrix} \right\} \overset{\varphi}{\longleftrightarrow} \left\{ \begin{matrix} \text{set of complete} \\ Q\text{-tableaux } T \\ \text{with } uwb(T) = w \end{matrix} \right\}$$

are in bijection by φ

Q-tableaux: example 1

$$UD = DU + Id$$

$$UD = qDU + I$$

$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$w = U^n D^n$$
$$uv = I_v^n I_h^n$$

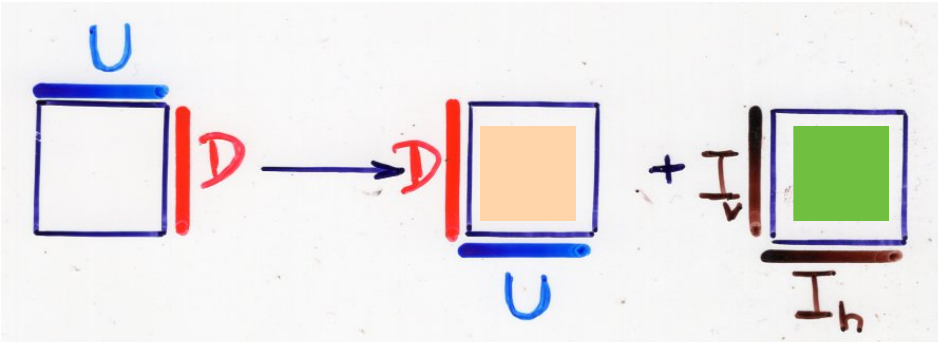
$$c(u, v; w) = n!$$

complete Q-tableau $\longleftrightarrow$ Permutations $\mathfrak{S}_n$

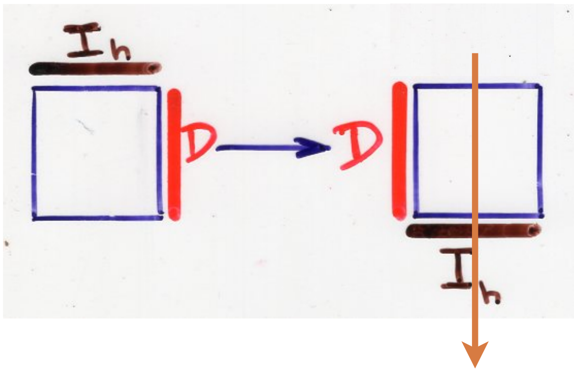
$$uwb(T) = U^n D^n$$
$$lwb(T) = I_v^n I_h^n$$

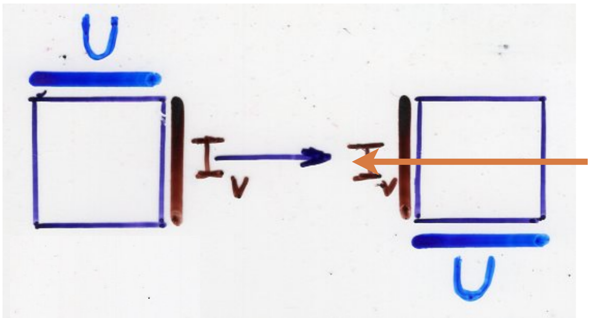
$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

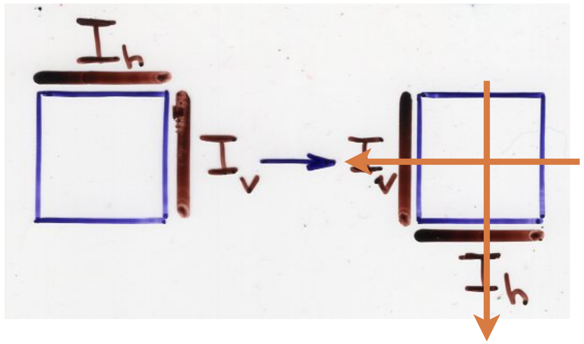
$$UD = qDU + I$$

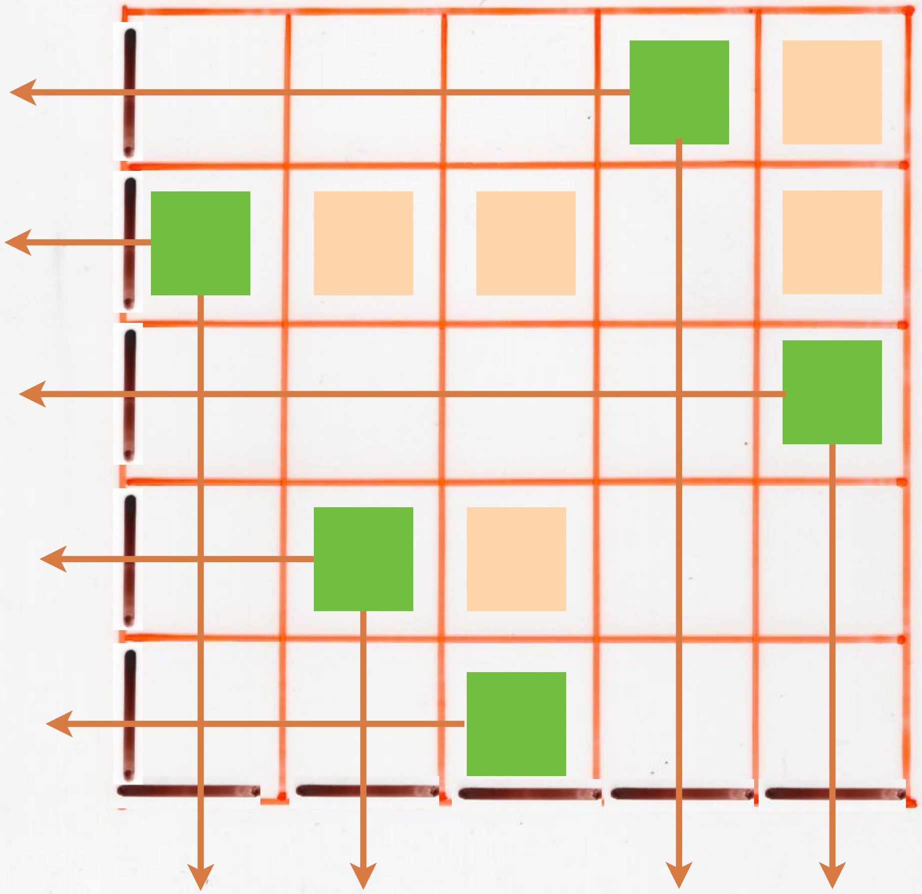


"complete"
Q- tableau









$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

"complete"
Q-tableau

$\varphi : R \longrightarrow L$

map

R = set of rewriting rules of the homogenous system associated to Q

here 5 terms

L a set of "labels" (for the cell of $[n] \times [n]$)

<u>examples</u>

$L = \{\square, \blacksquare\}$

<u>examples</u>

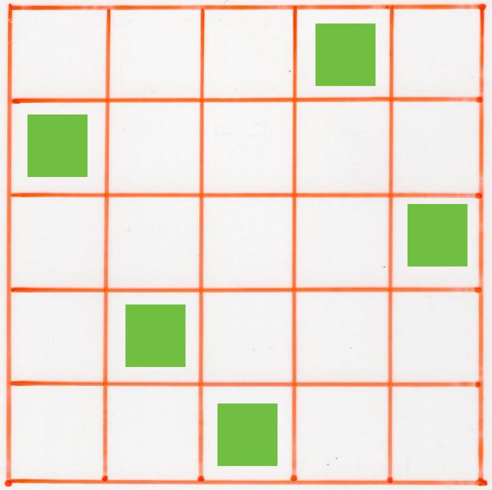
$L = \{\square, \blacksquare\}$

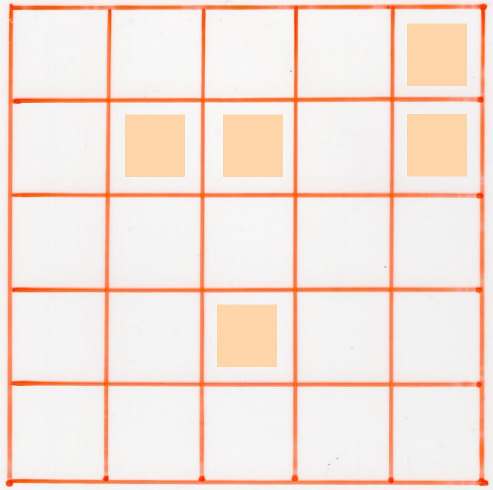
φ: $UD \to DU$, $UI_v \to I_vU$, $I_hD \to DI_h$, $I_hI_v \to I_vI_h$ $\to$ $\square$ empty cell

φ: $UD \to I_vI_h$, $UI_v \to I_vU$, $I_hD \to DI_h$, $I_hI_v \to I_vI_h$ $\to$ $\square$ empty cell

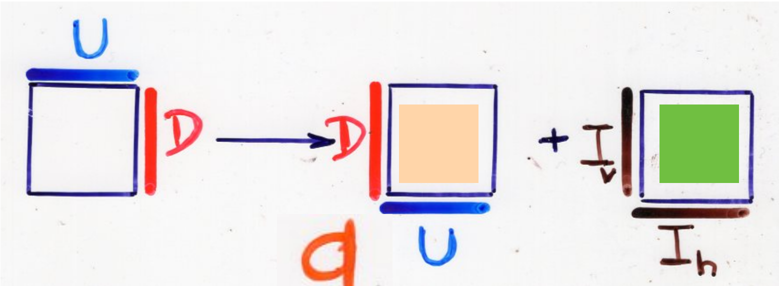
$\varphi(UD \to I_vI_h) = \blacksquare$

$\varphi(UD \to DU) = \blacksquare$





$$
\begin{cases}
UD = DU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$



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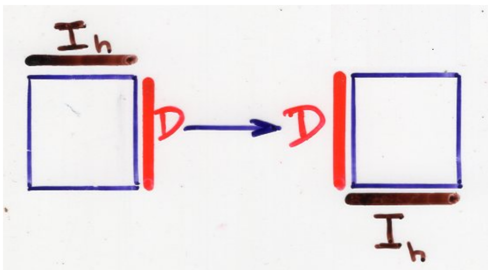
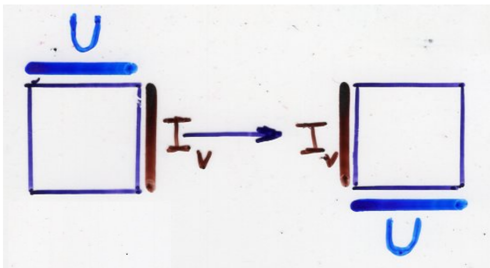
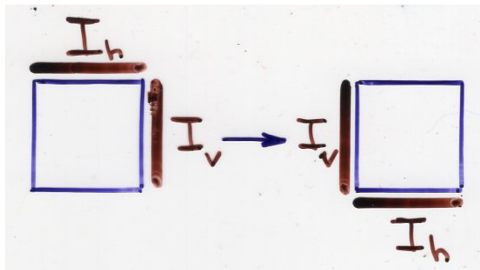
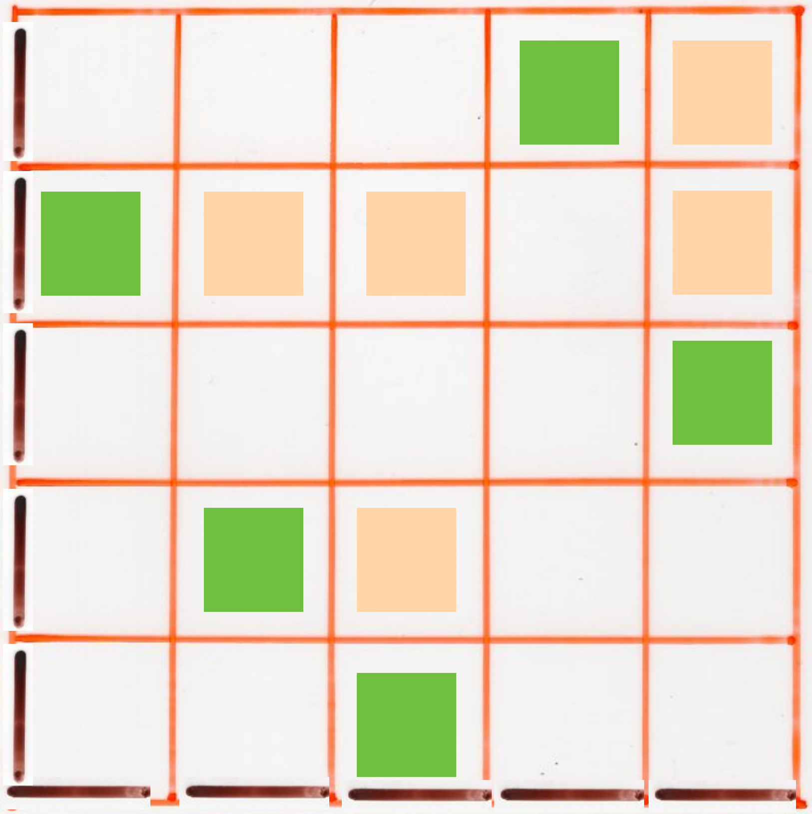






9



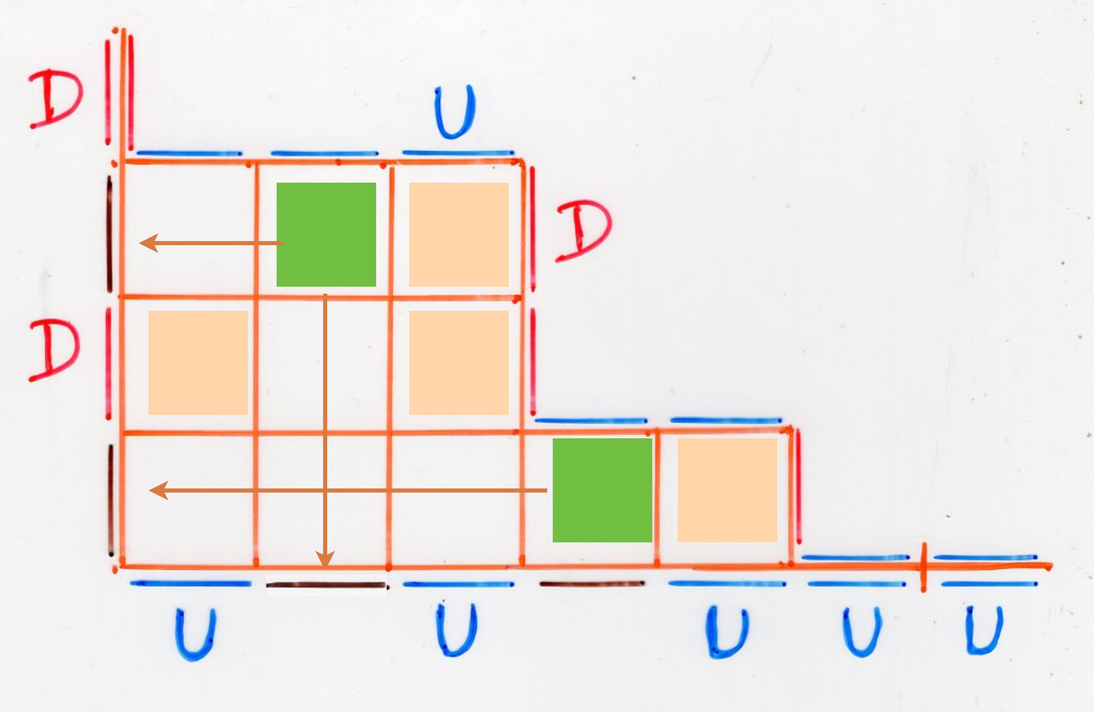
number of inversions of a permutation σ

$w = D\,U^3 D^2 U^2 D\, U^2$

$w \longrightarrow F = F(w)$

F Ferrers diagram

Rooks placement

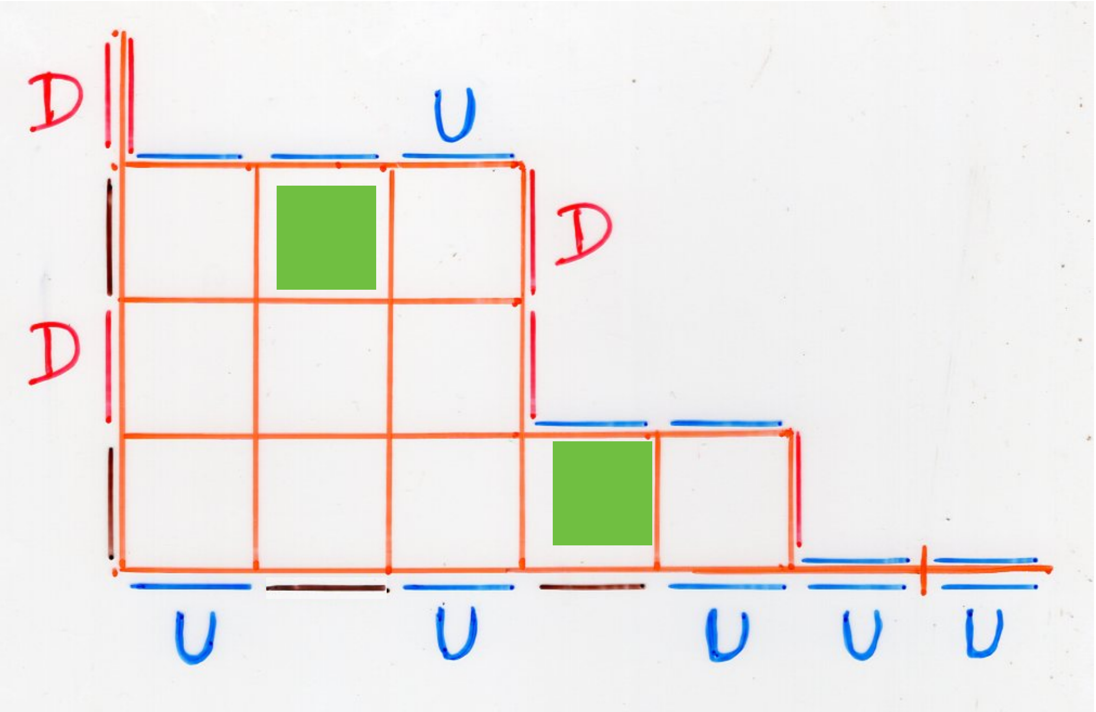


$w = D\,U^3 D^2 U^2 D\,U^2$

$w \longrightarrow F = F(w)$

F Ferrers diagram

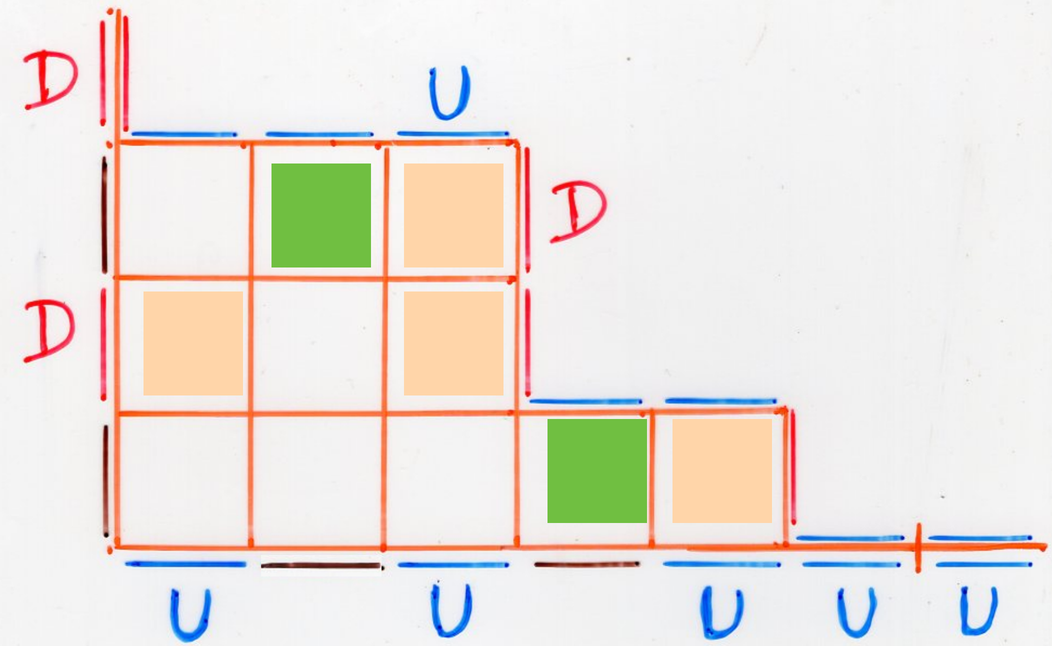
Rooks placement



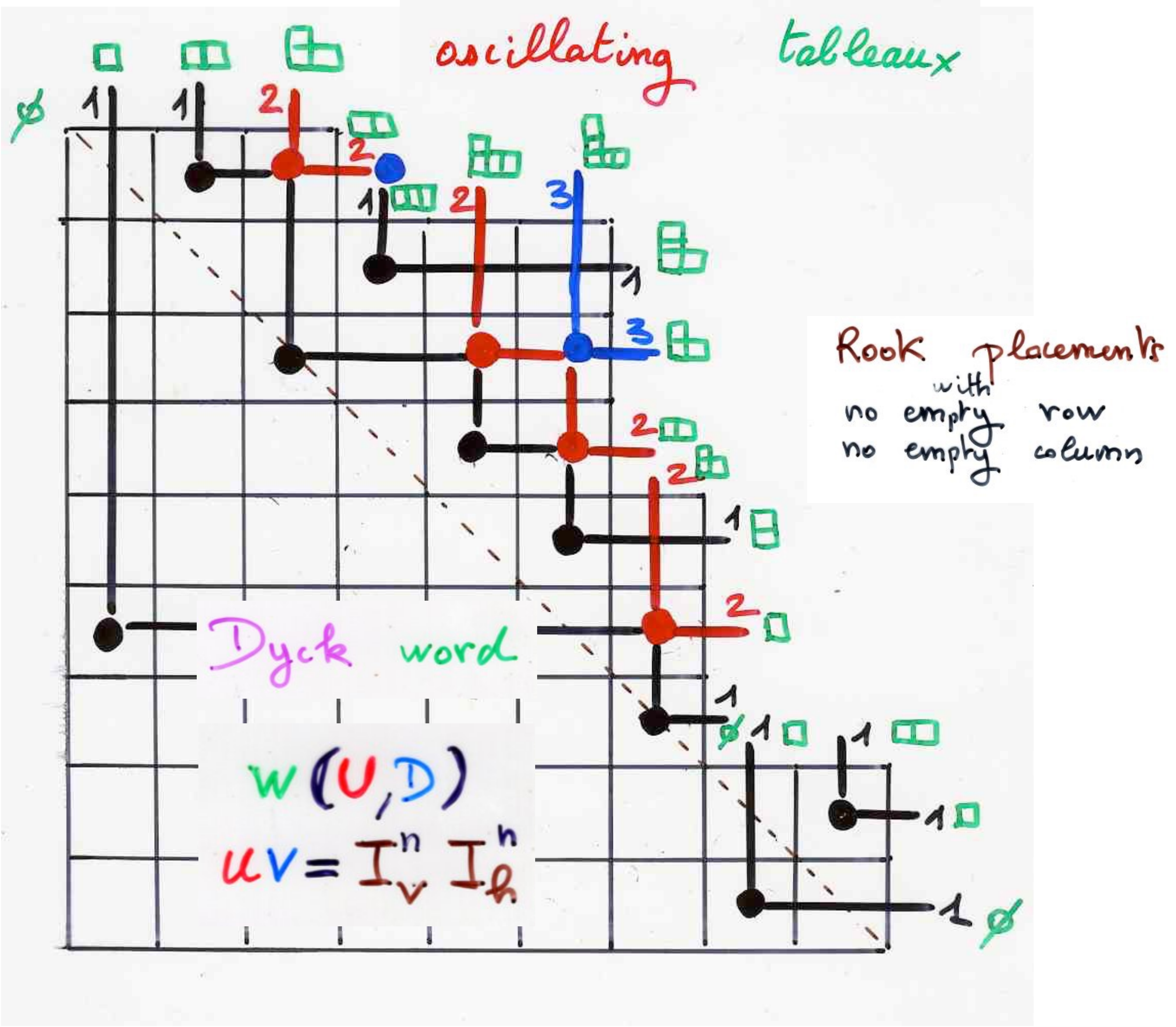
$$w = D\,U^3 D^2 U^2 D\,U^2$$

$$w \longrightarrow F = F(w)$$

F Ferrers diagram



Rooks placement



oscillating tableaux
Rook placements
with
no empty row
no empty column
Dyck word
w(U,D)
UV = I_v^n I_h^n

Q-tableaux: example 2

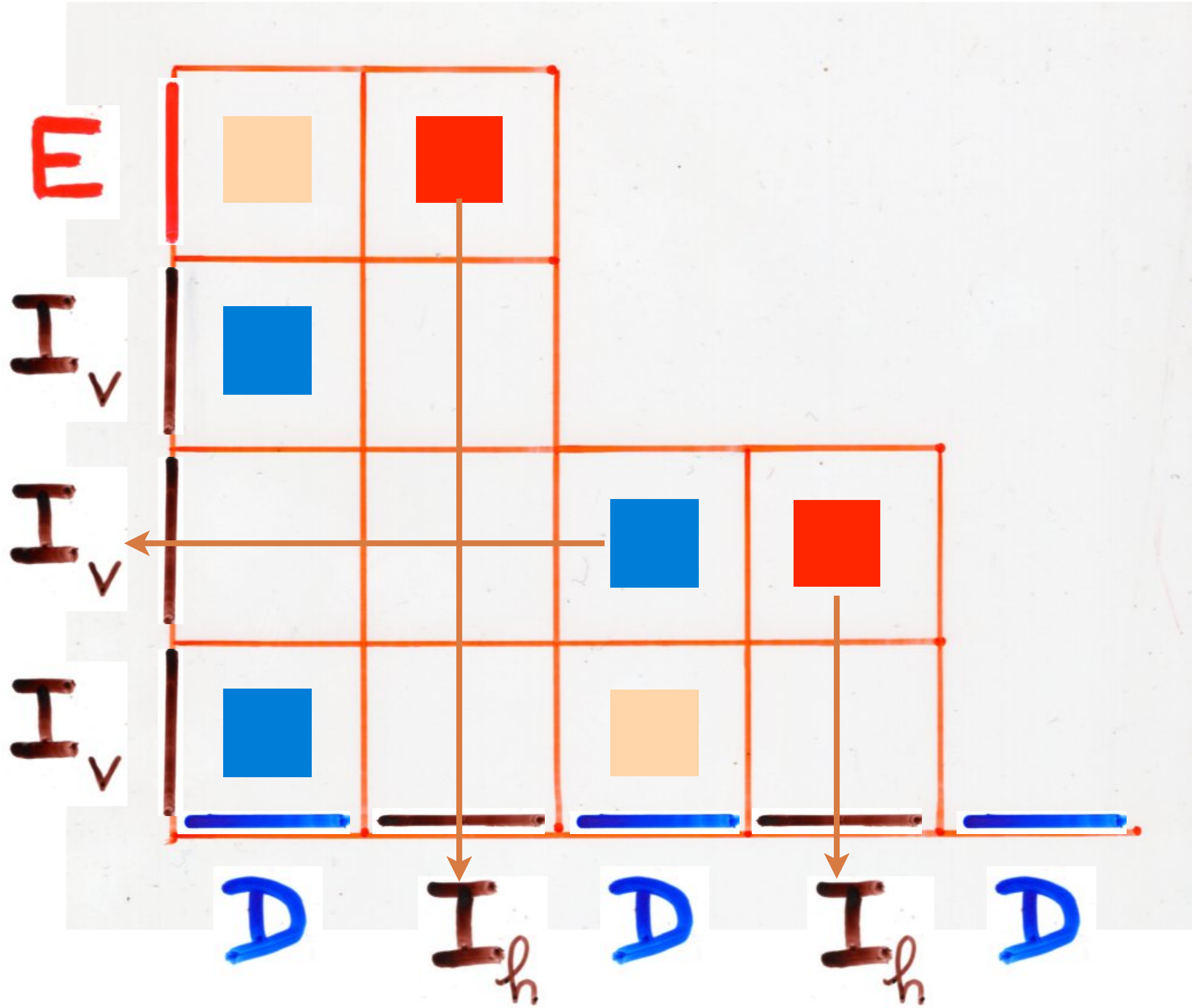
$$DE = qED + E + D$$

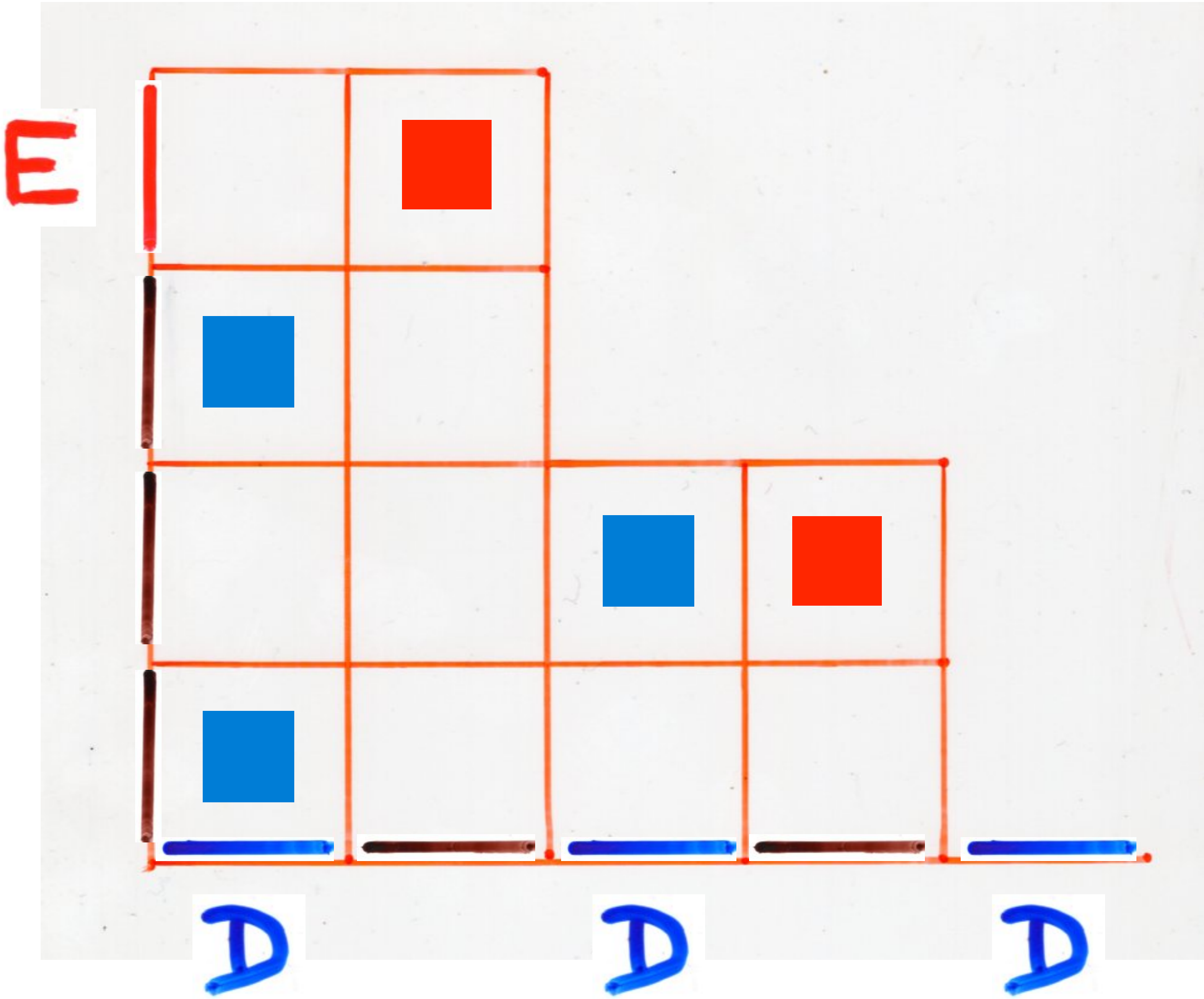
$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$


E
I_v
I_v
I_v
D
I_h
D
I_h
D


E
D
D
D

Q-tableaux: example 3

ASM

A, A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

$w = B^n A^n \qquad uv = A'^n B'^n$

$c(u,v;w)$ = number of ASM$_{n\times n}$

complete

Q- tableau $\xleftrightarrow{\text{bijection}}$ ASM$_{n\times n}$

$Q(B, B', A, A')$

$uwb = B^n A^n$

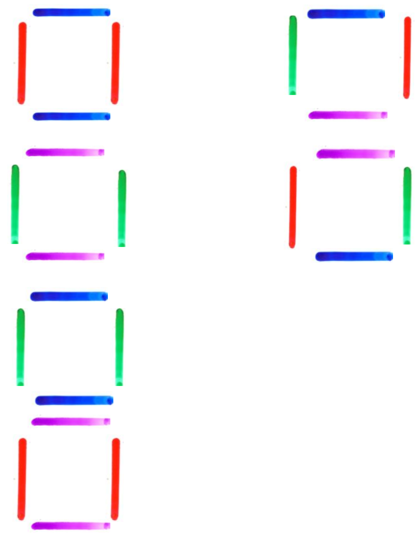
$lwb = A'^n B'^n$

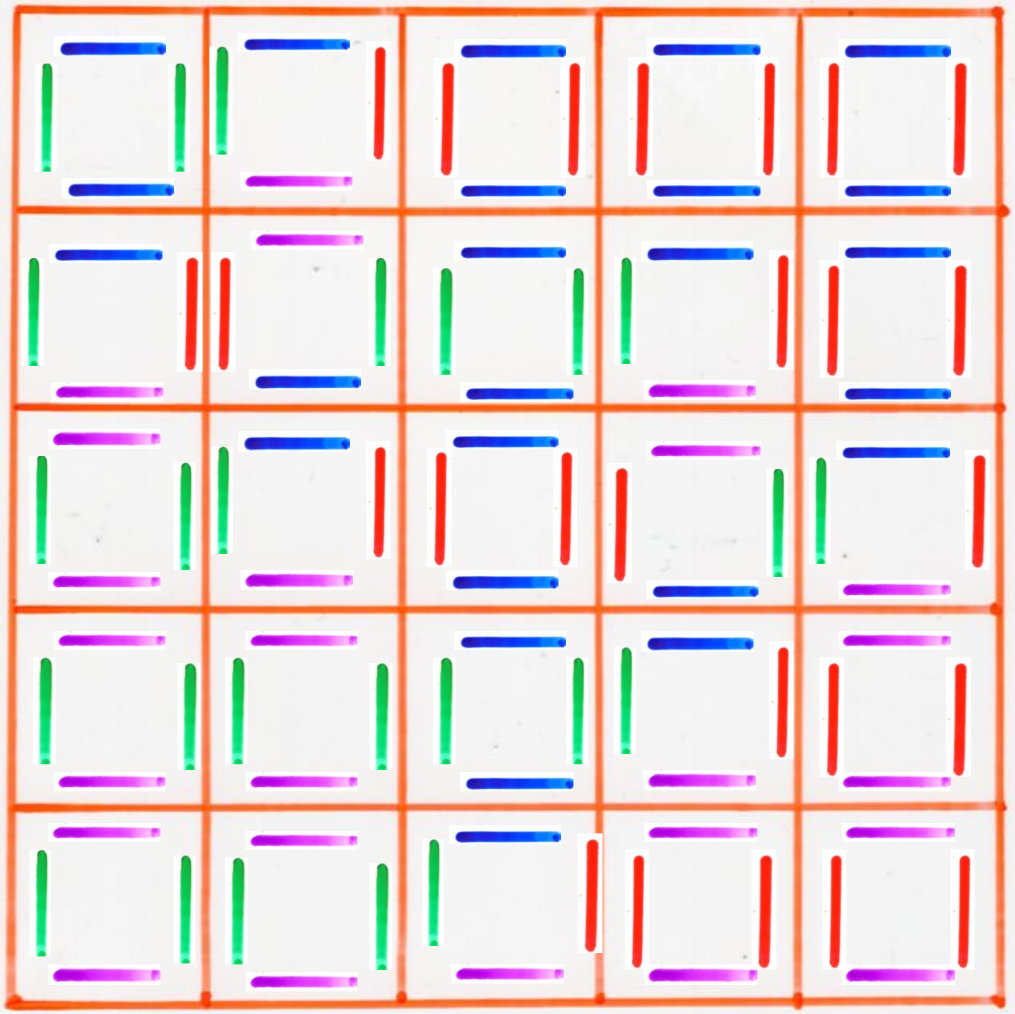
A , A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$



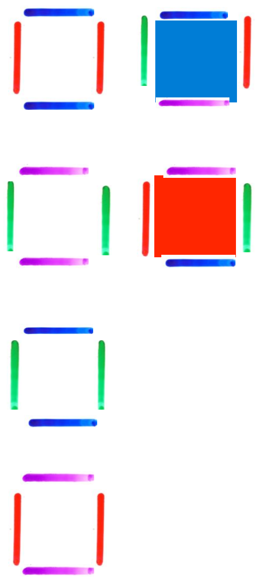


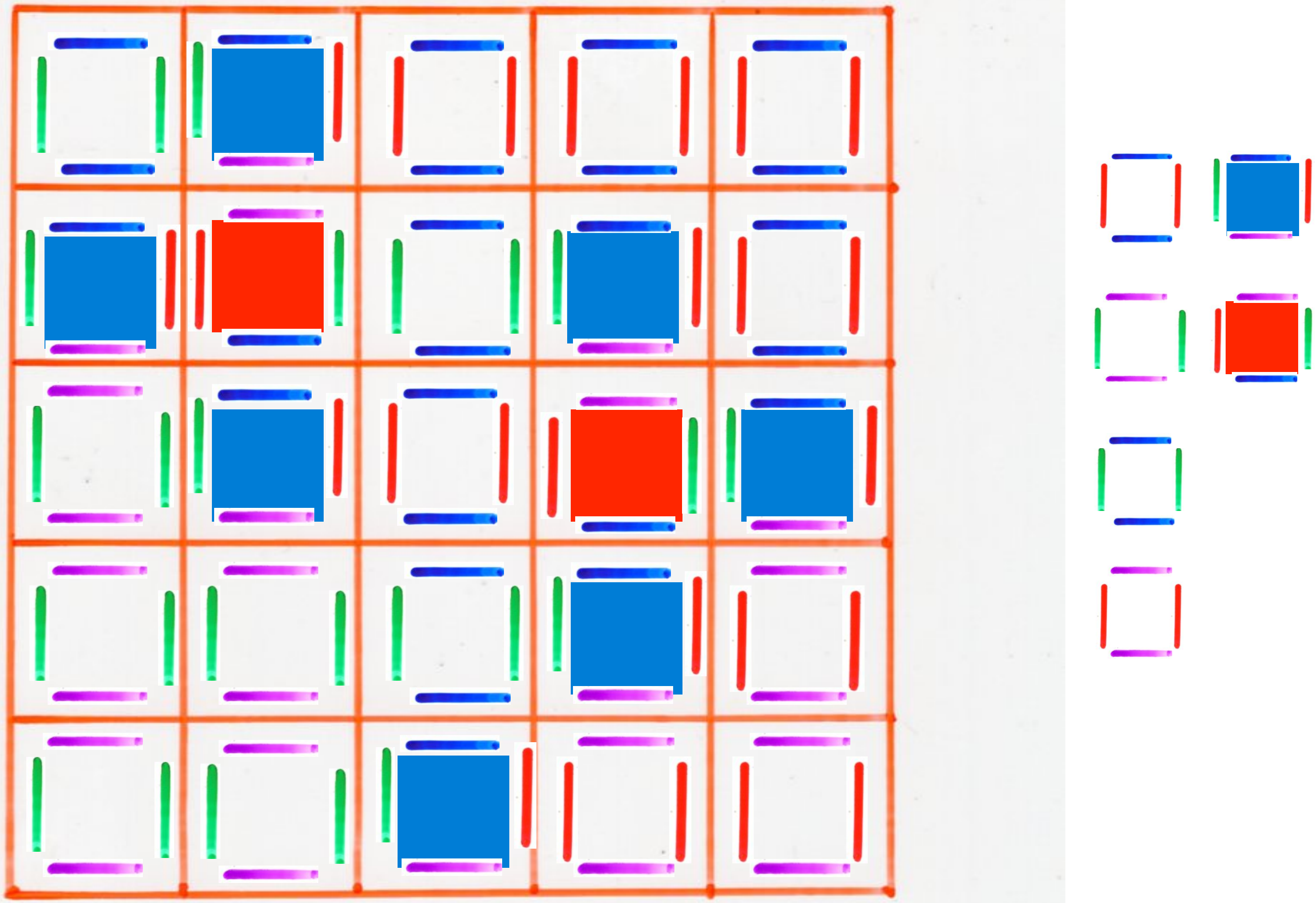
$A, A', B, B',$

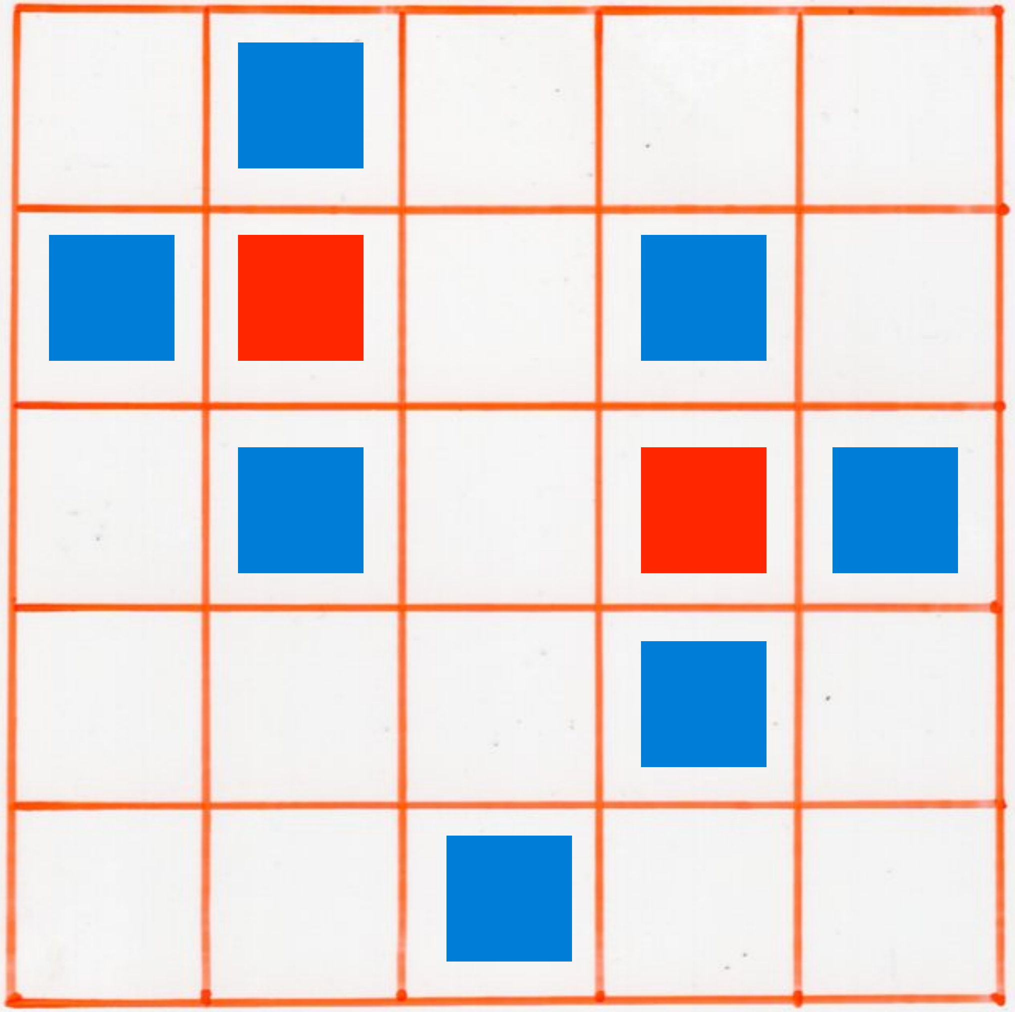
commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

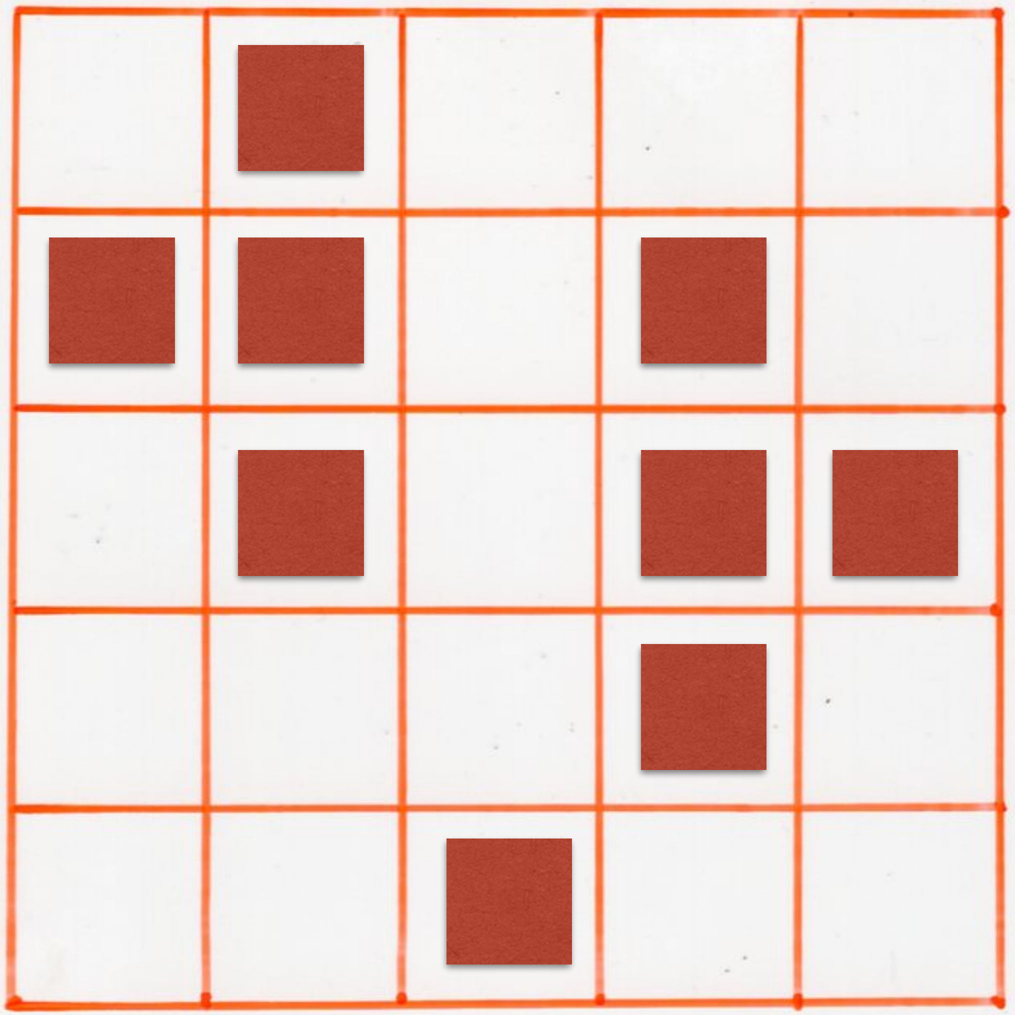
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

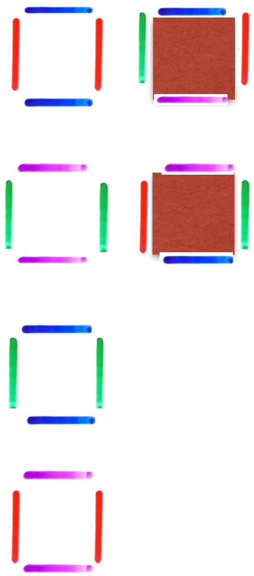




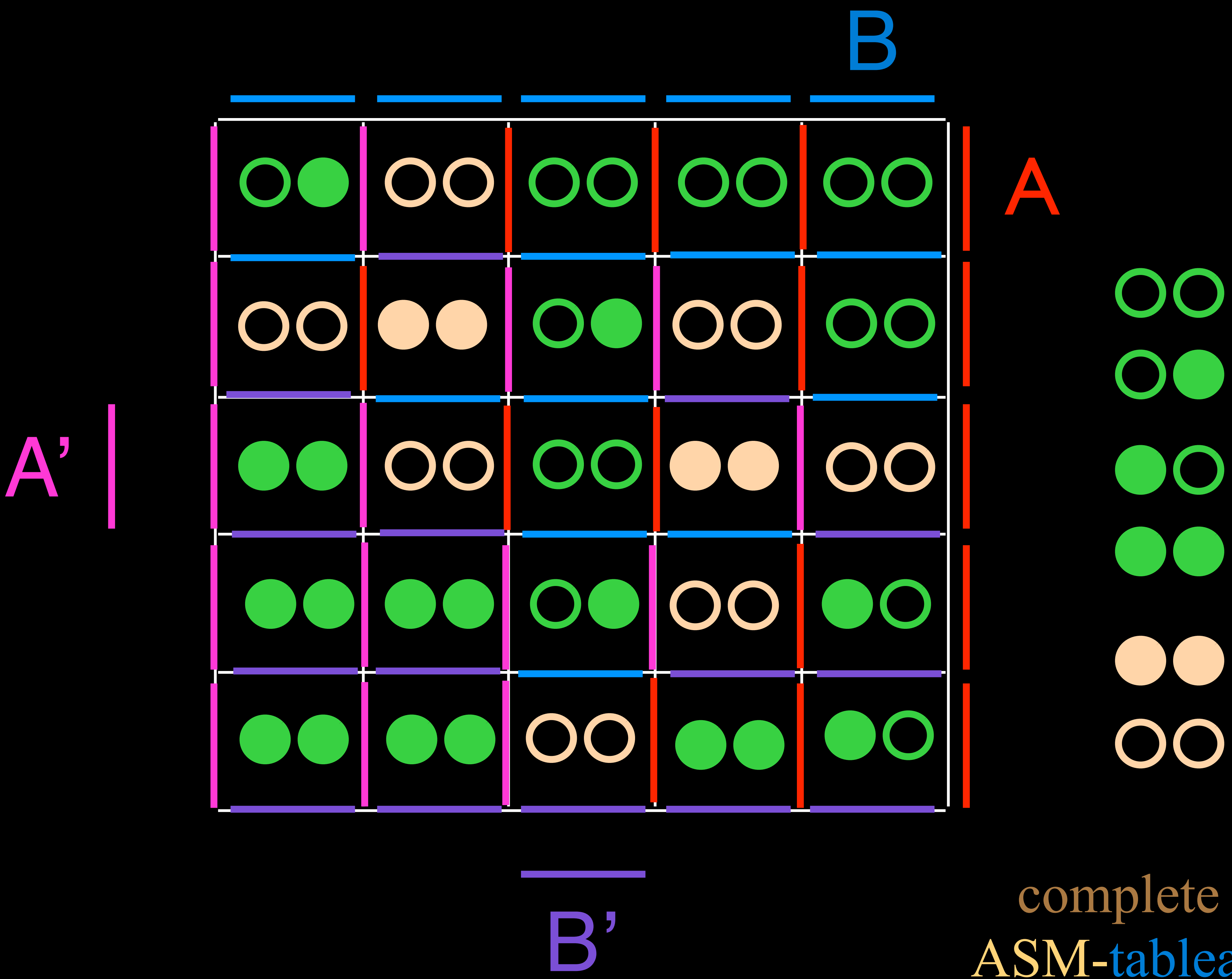


exercise

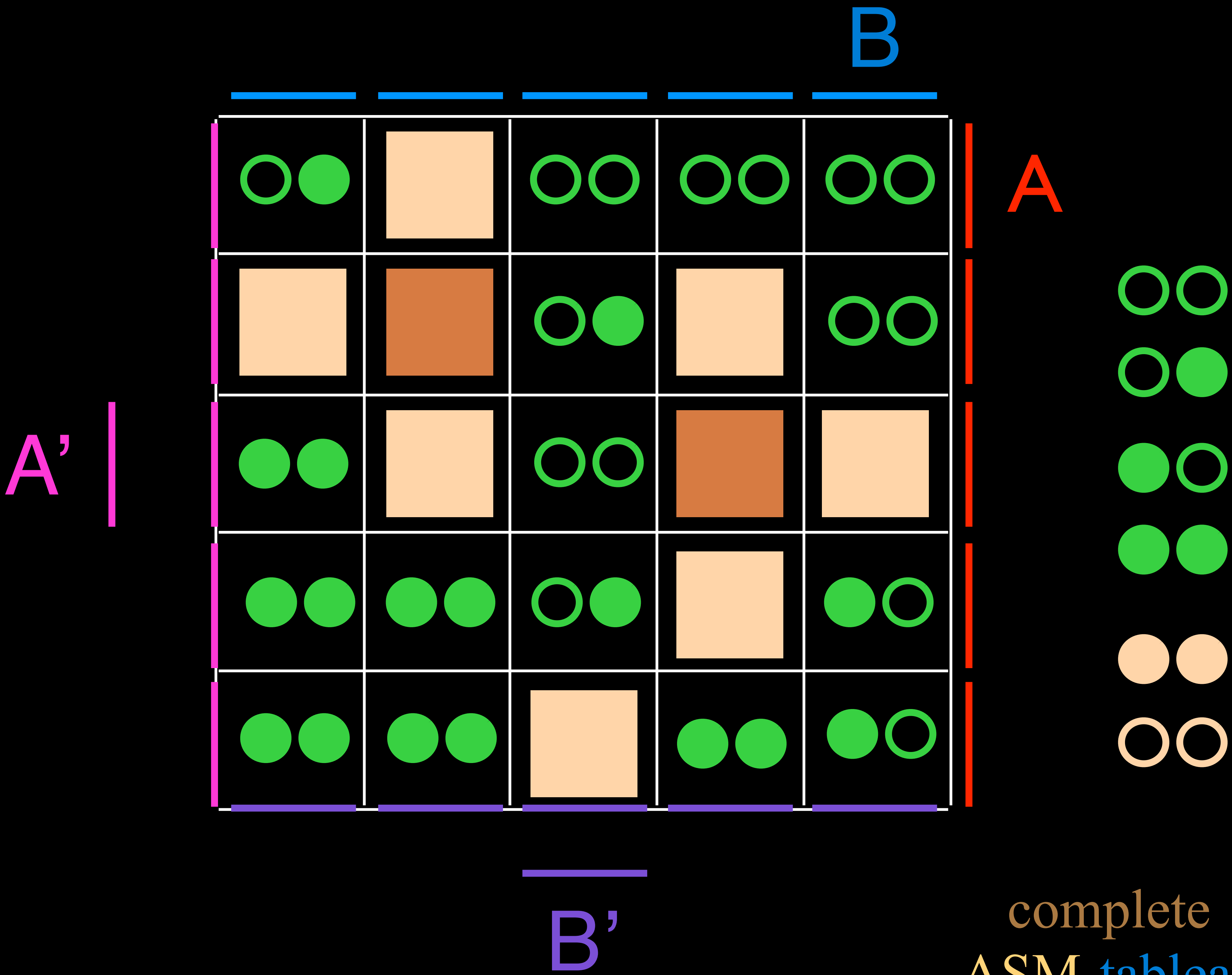




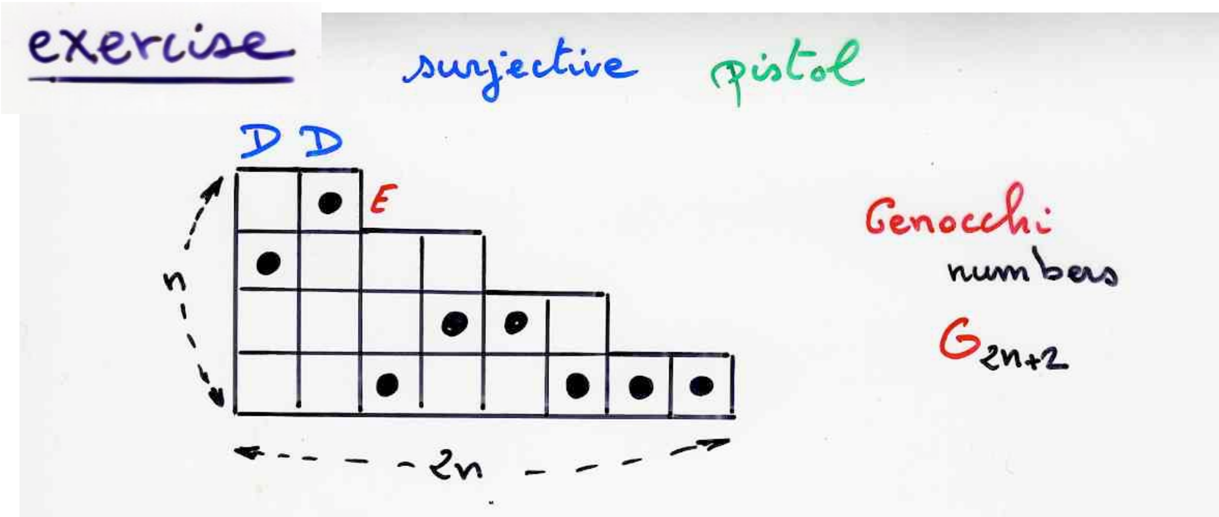
$$B A = g_{\circ\circ} A B + t_{\circ\circ} A' B'$$
$$B' A' = g_{\bullet\bullet} A' B' + t_{\bullet\bullet} A B$$
$$B' A = g_{\bullet\circ} A B',$$
$$B A' = g_{\circ\bullet} A' B,$$



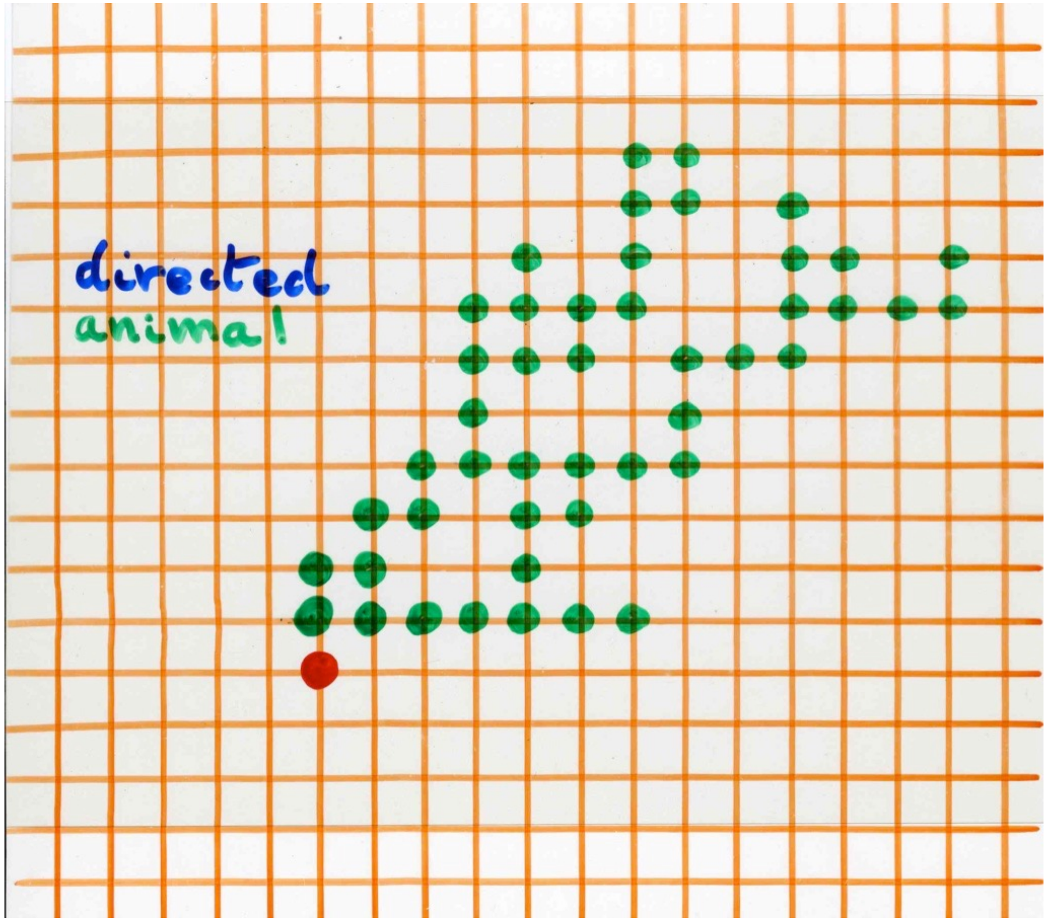
complete
ASM-tableau

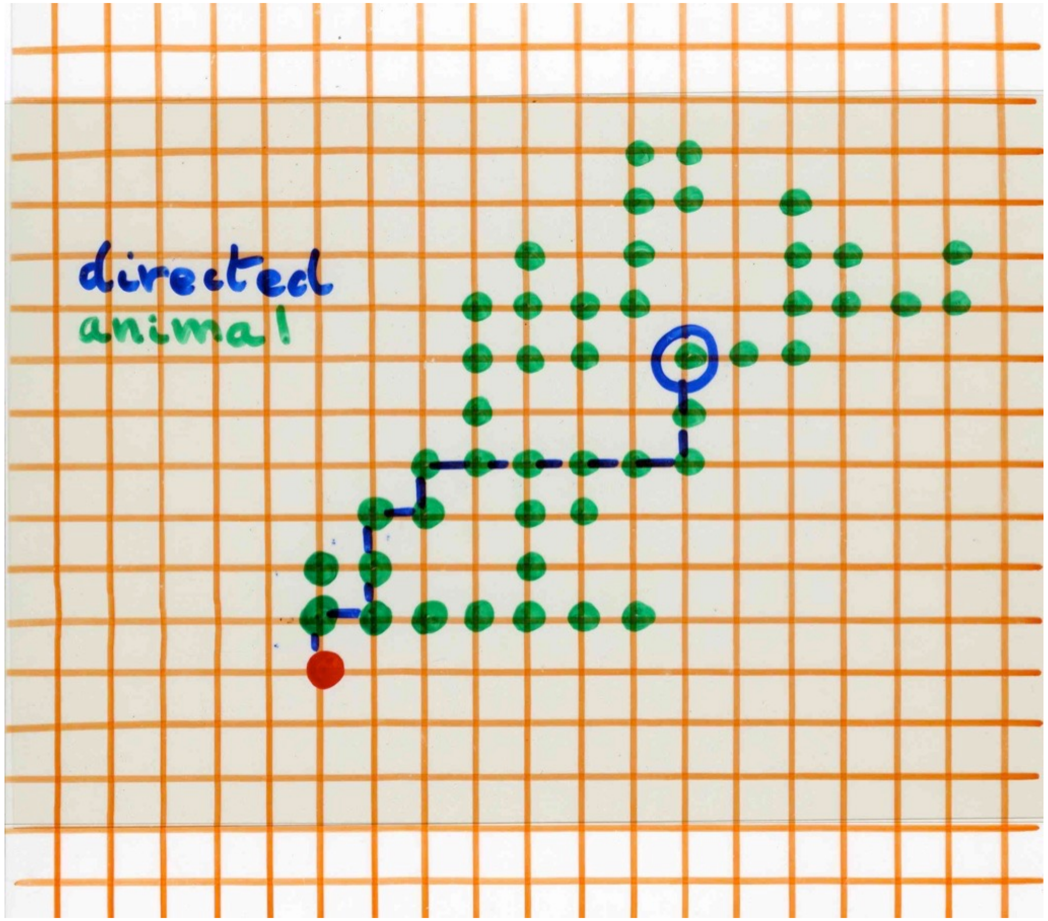


complete
ASM-tableau

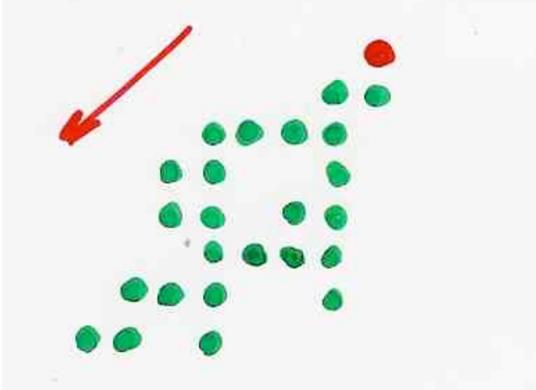


Prove that such « tableaux » are Q-tableaux
for a certain quadratic algebra Q


directed
animal


directed
animal

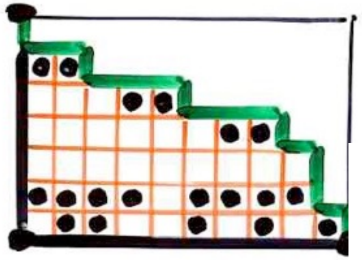
exercise directed animal



Prove that such « tableaux » are Q-tableaux for a certain quadratic algebra Q

exercise

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells with 0 and 1

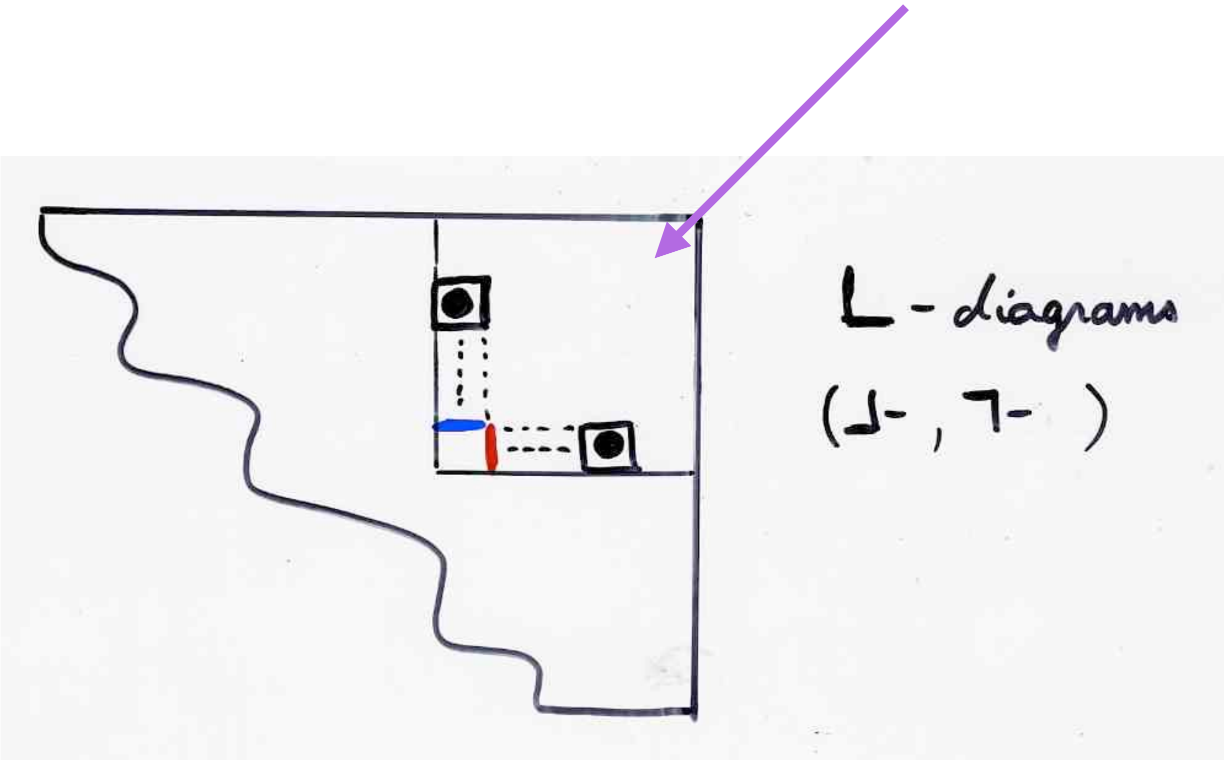
□ = 0   ⊡ = 1

(ii) $\begin{array}{ccc} 1 & \cdots & 0 \\ & & \vdots \\ & & 1 \end{array}$ forbidden

⅃-diagram

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian

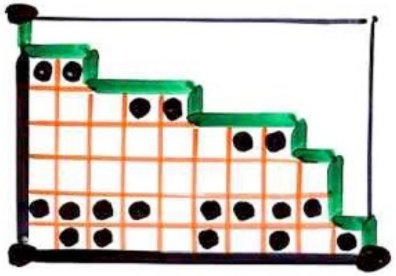
E. Steingrímsson, L. Williams (2005)



Prove that such « tableaux » are Q-tableaux
for a certain quadratic algebra Q

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) $1 \cdots\cdots 0$ forbidden
$\qquad\qquad\;\; \vdots$
$\qquad\qquad\;\; 1$

□ = 0 ⊡ = 1

permutation tableau

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)