Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 1
RSK
The Robinson-Schensted-correspondence
(Ch1d)

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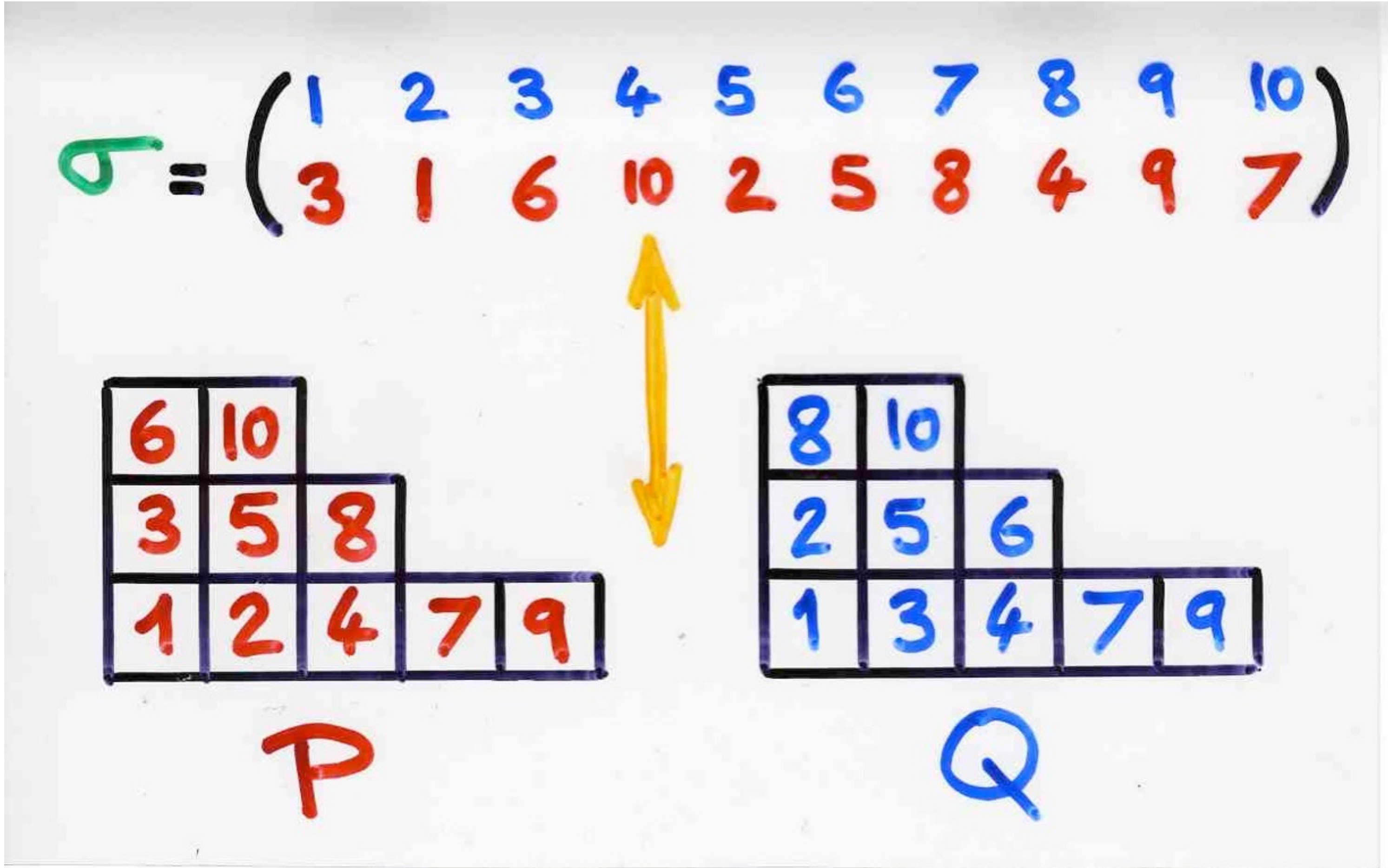
From Ch 1a, 1b,1c

The Robinson-Shensted correspondence

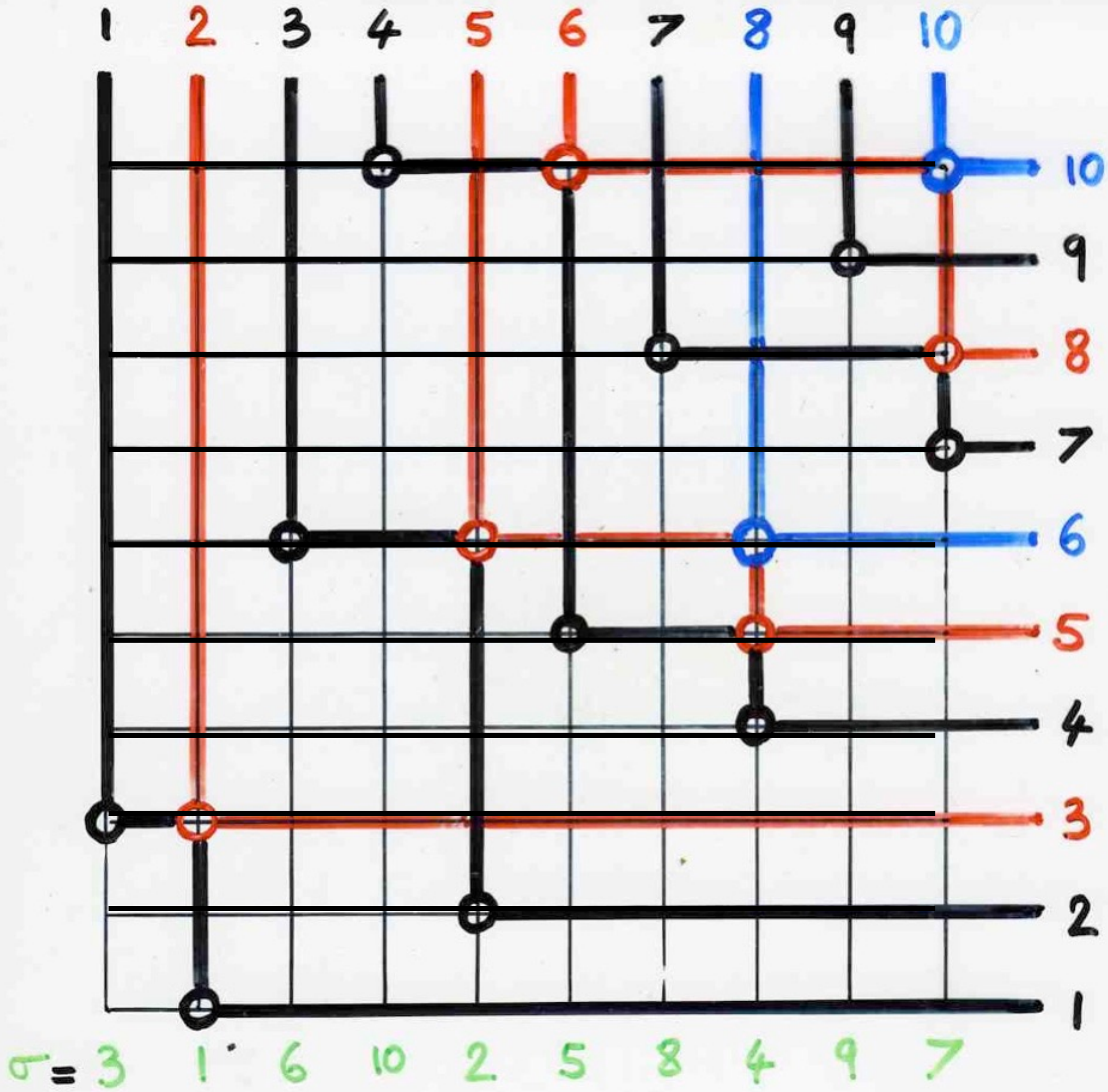
Ch 1a
- Schensted's insertions
- geometric version with "shadow lines »

Ch 1b
- Fomin "local rules" or "growth diagrams »

Ch 1c
- from a representation of the quadratic algebra $UD=DU+I$, deduce a bijection (P,Q) — Q-tableaux



The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape



1 2 3 4 5 6 7 8 9 10
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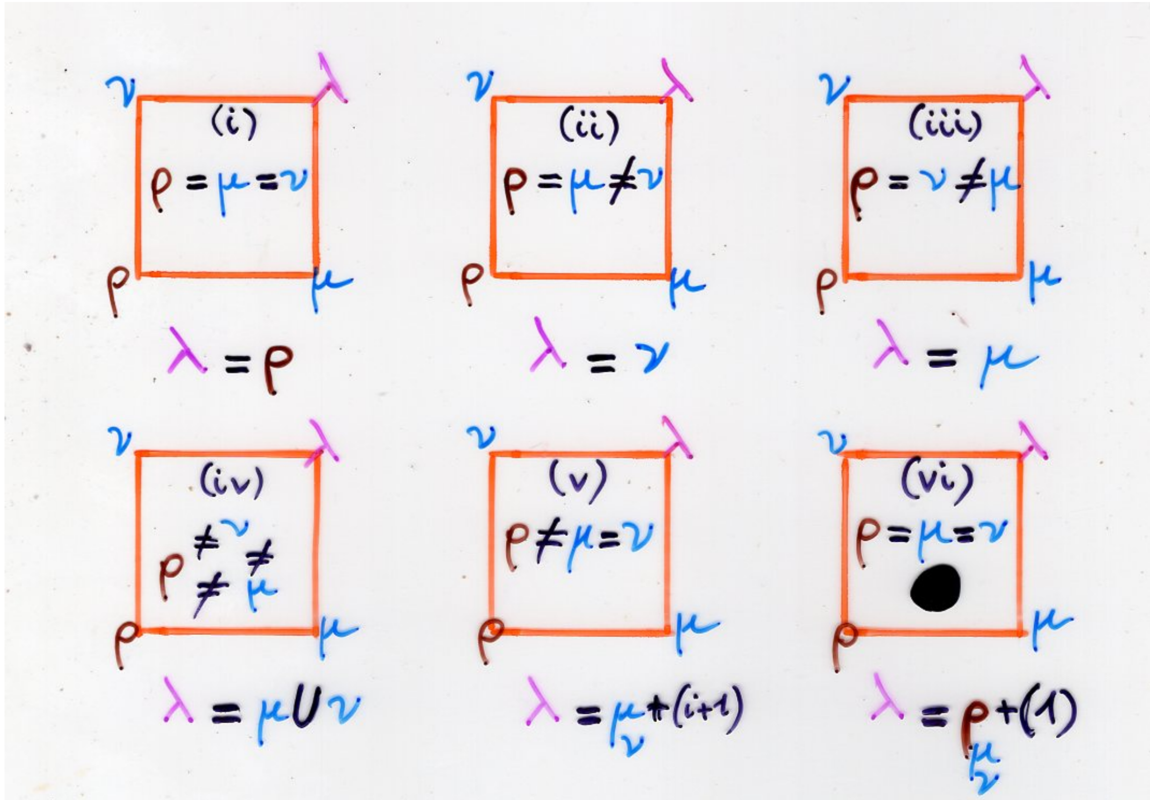
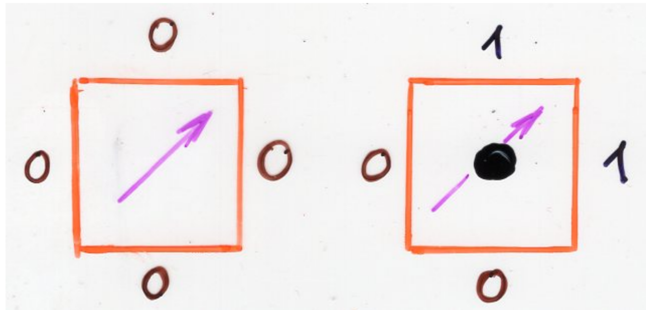
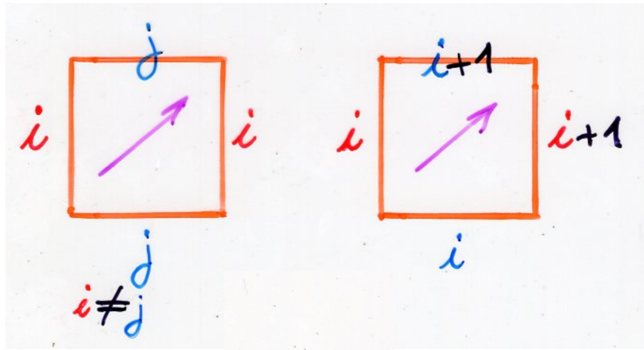
σ= 3 1 6 10 2 5 8 4 9 7

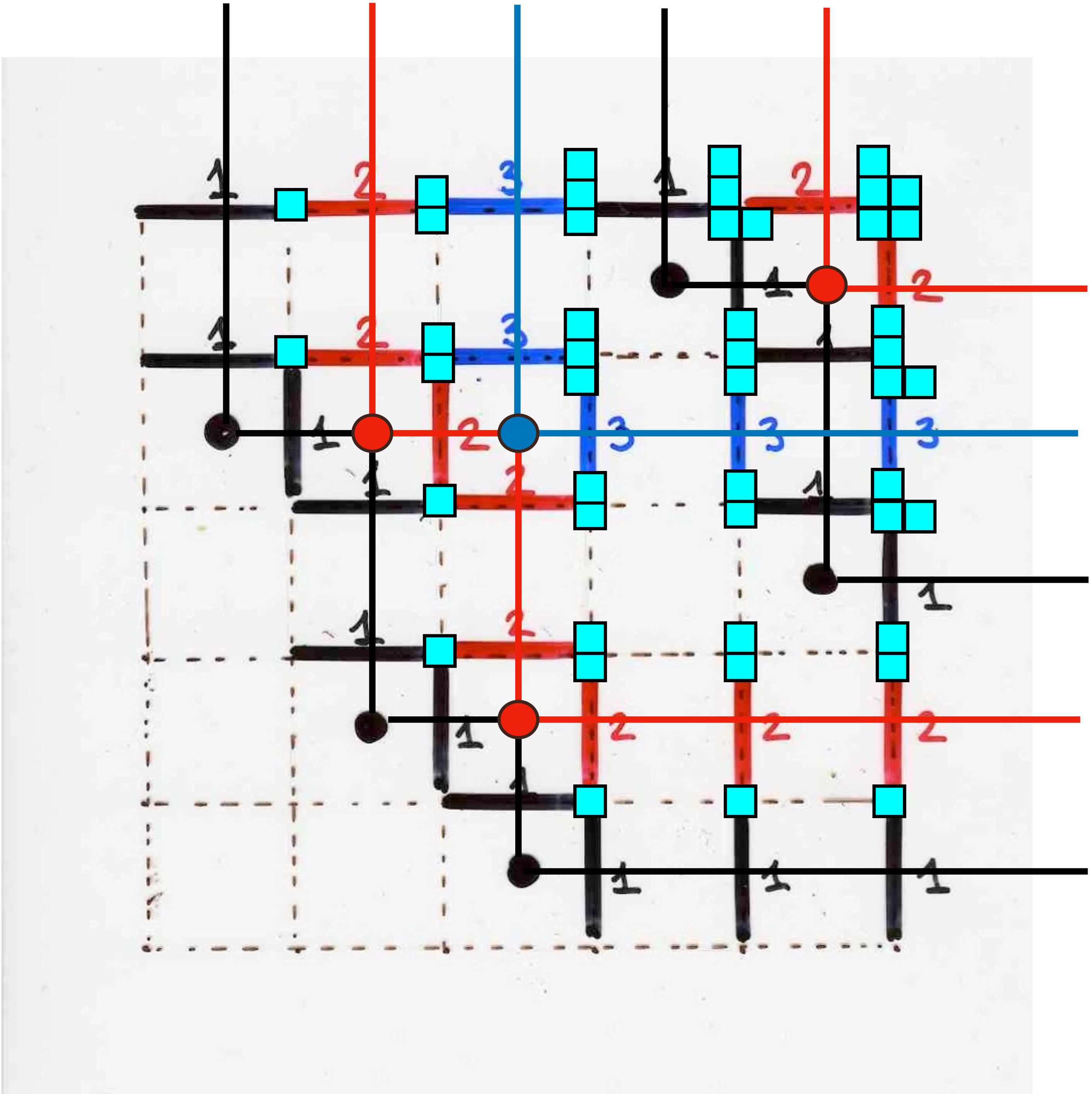
"local rules" on the vertices



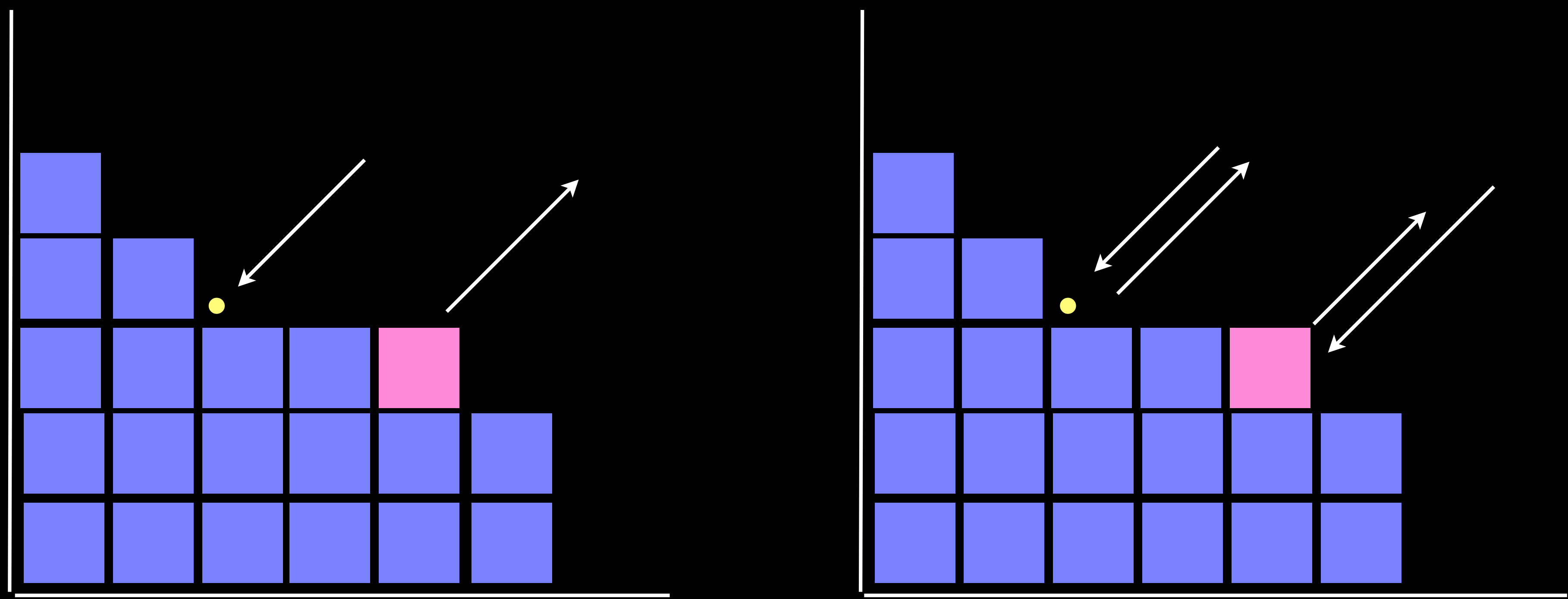
"local rules" on the edges

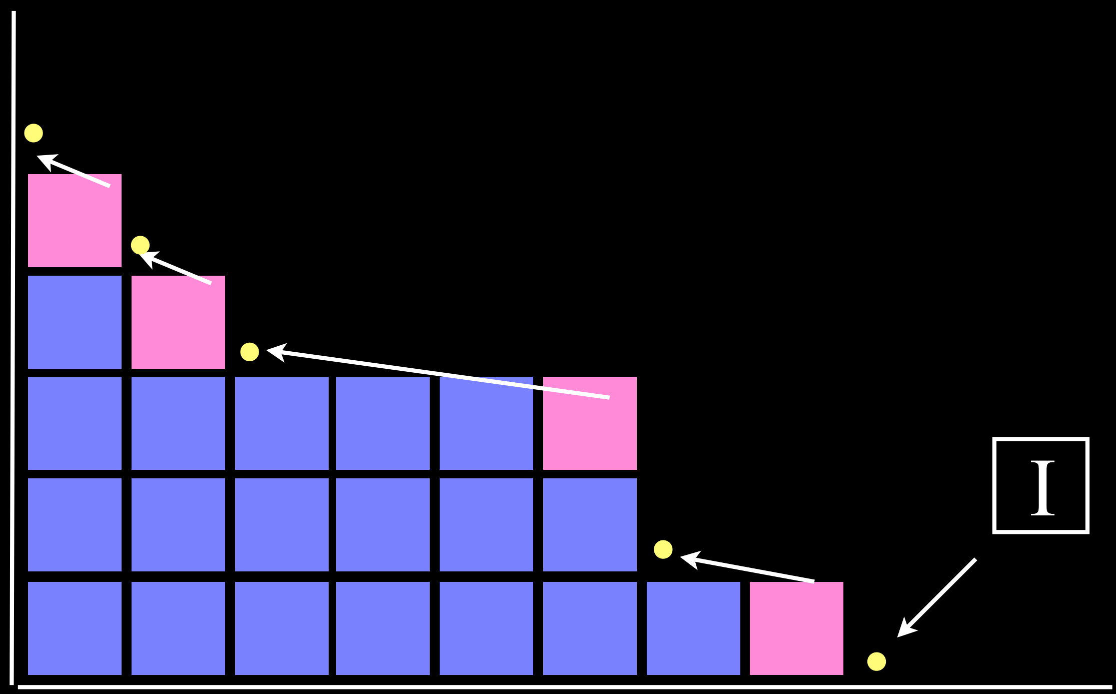






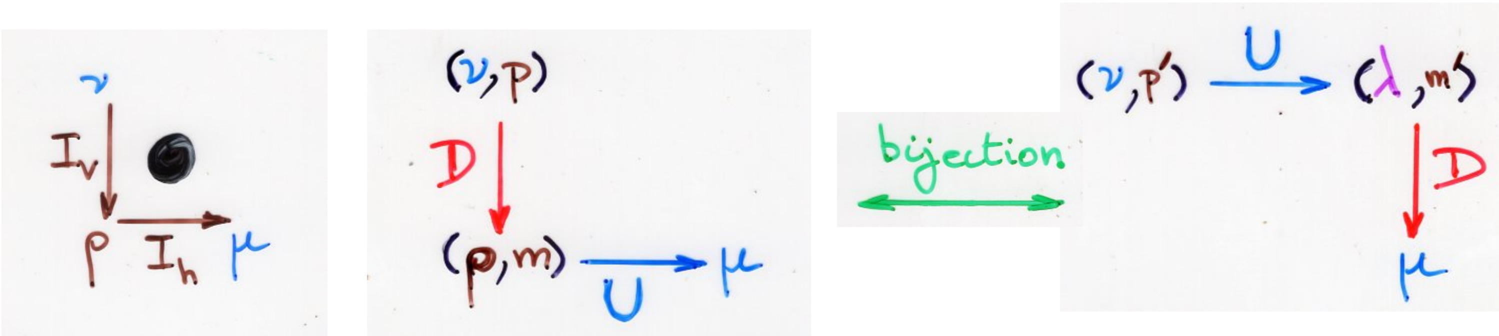
$$UD = DU + I$$

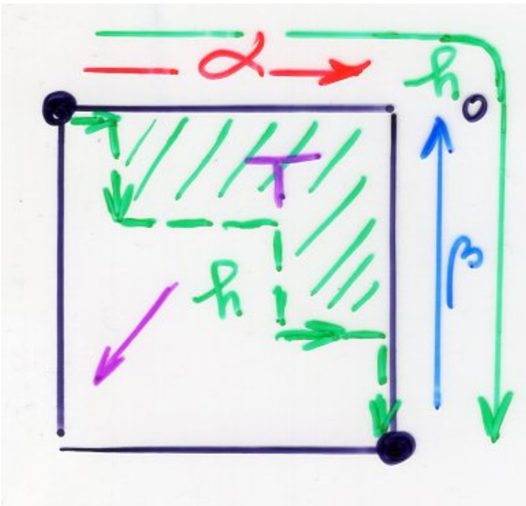



I

$$UD = DU + I_v I_h$$

"commutation diagrams"



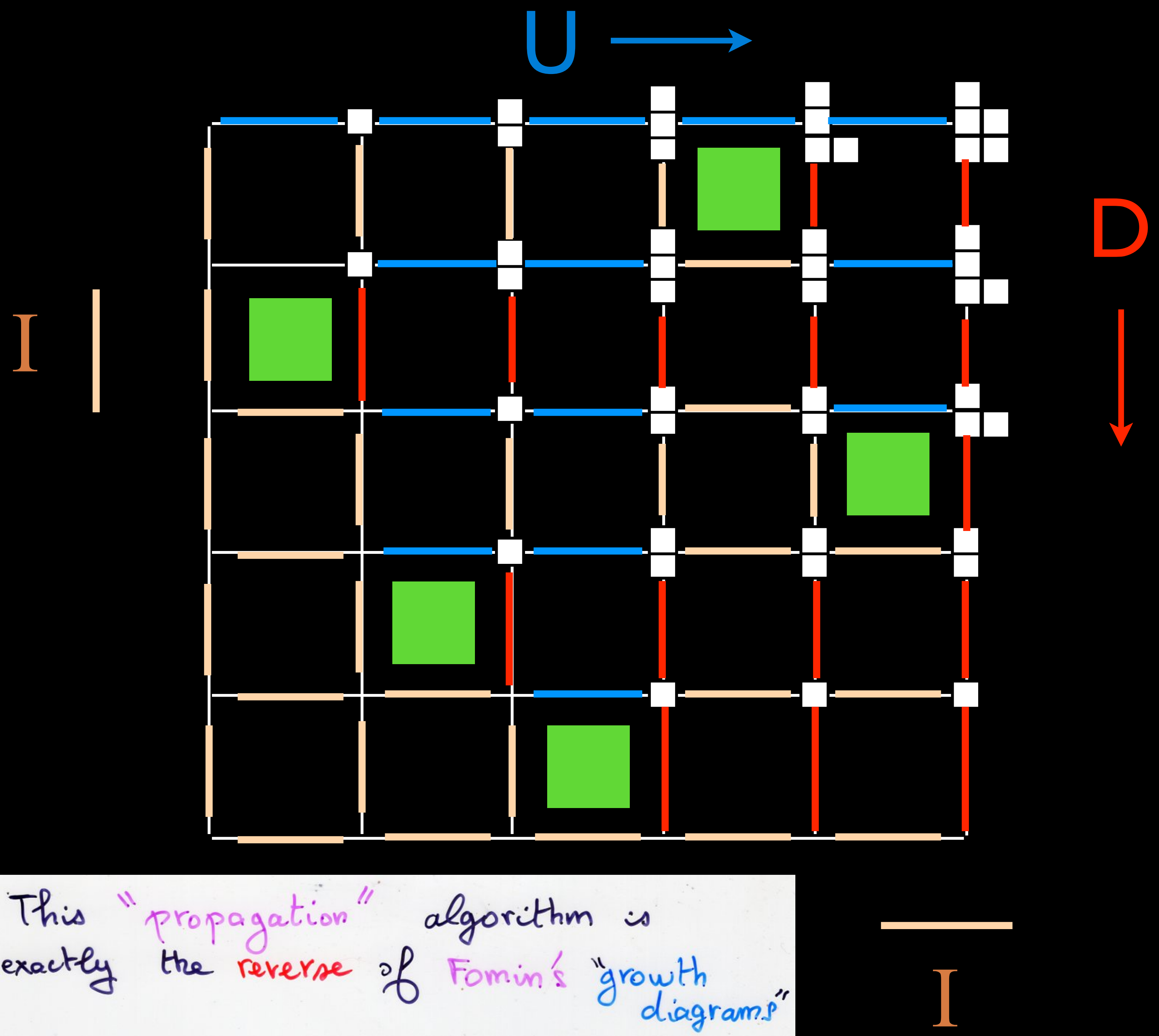


$$(h, T)$$

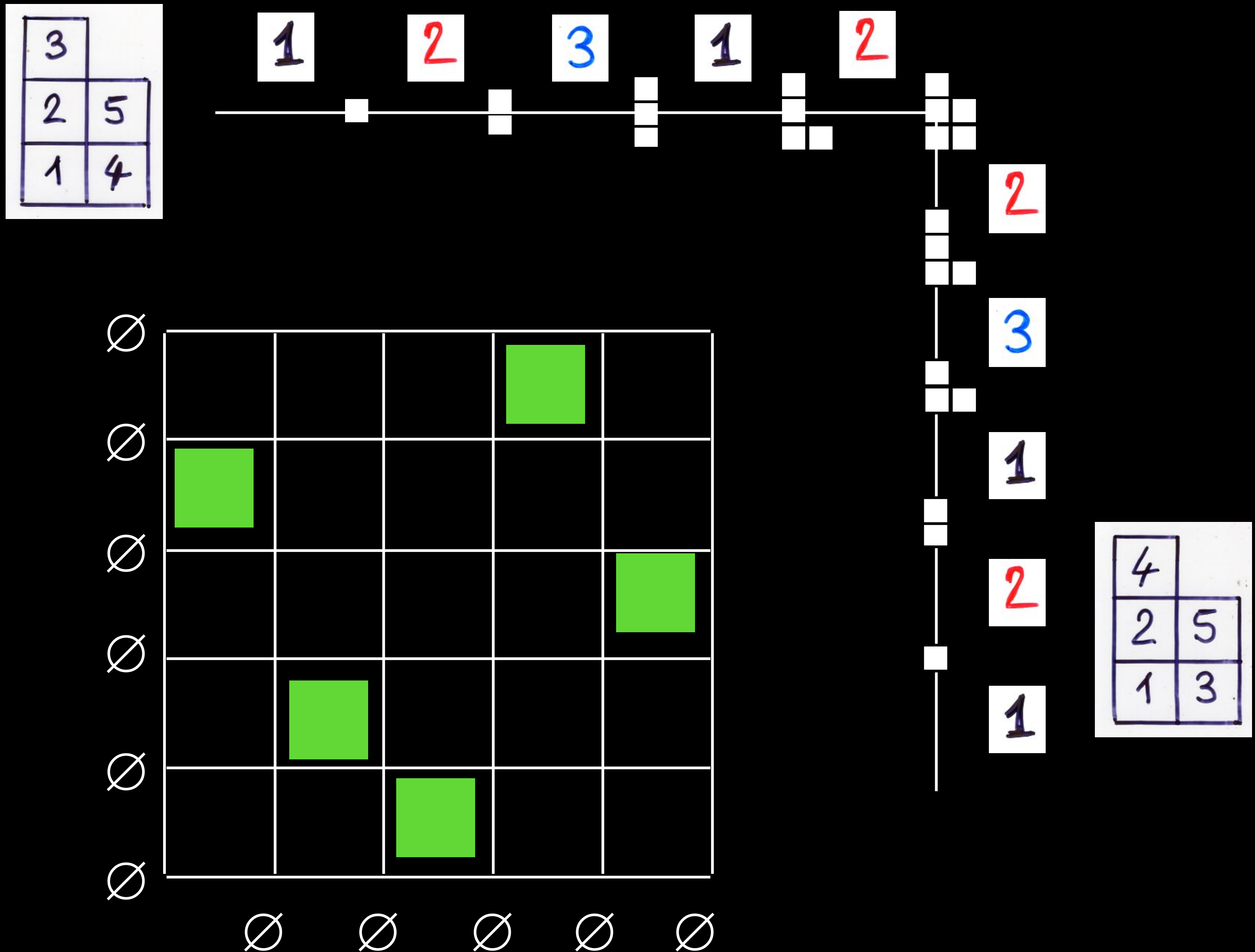
T tableau above the path associated to $w(h)$ with cells labeled by

$$(h, T) \longleftrightarrow h_0 = h(\alpha, \beta)$$

are in bijection



This "propagation" algorithm is exactly the reverse of Fomin's "growth diagrams"



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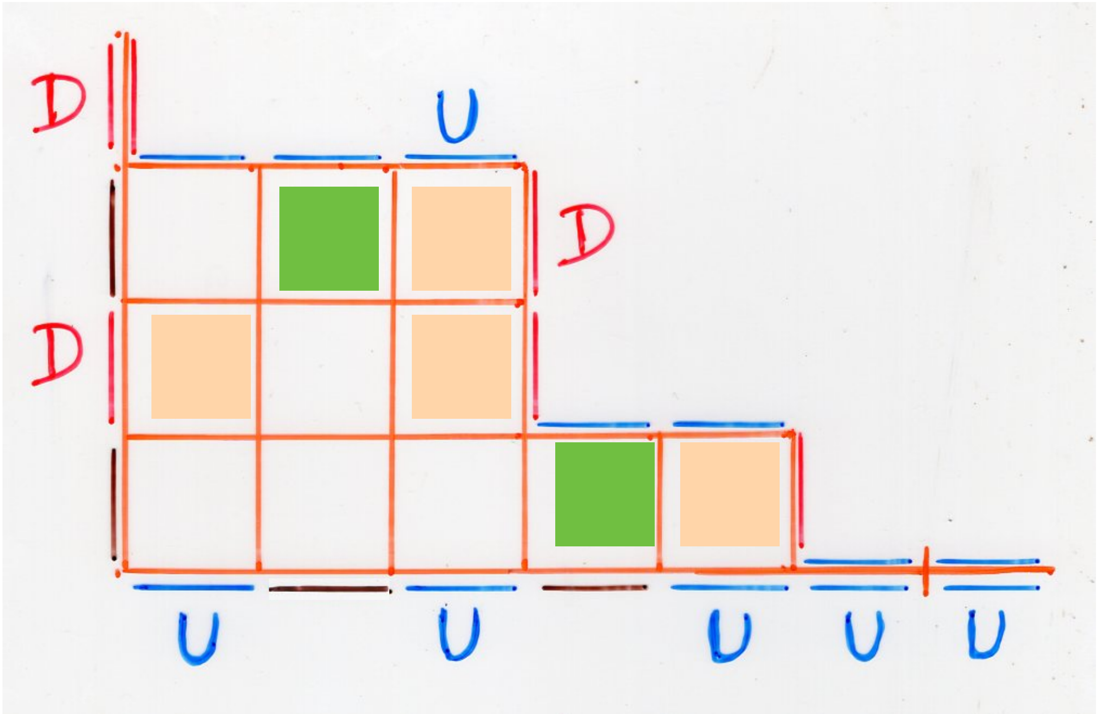
$w = D\,U^3 D^2 U^2 D\,U^2$

$w \longrightarrow F = F(w)$

F Ferrers diagram

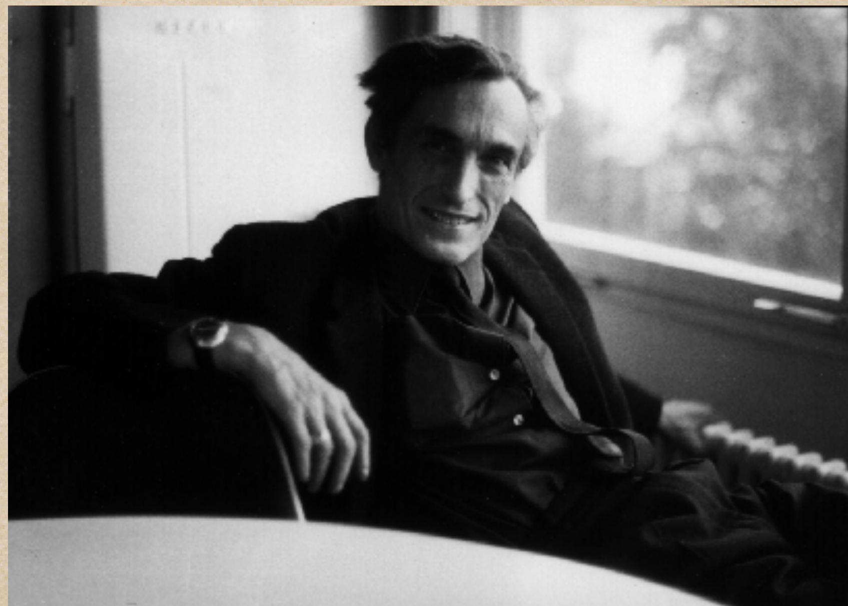
Rooks placement



Jeu de taquin

M.P. Schützenberger

(1976)



$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

3					
1	6	10			
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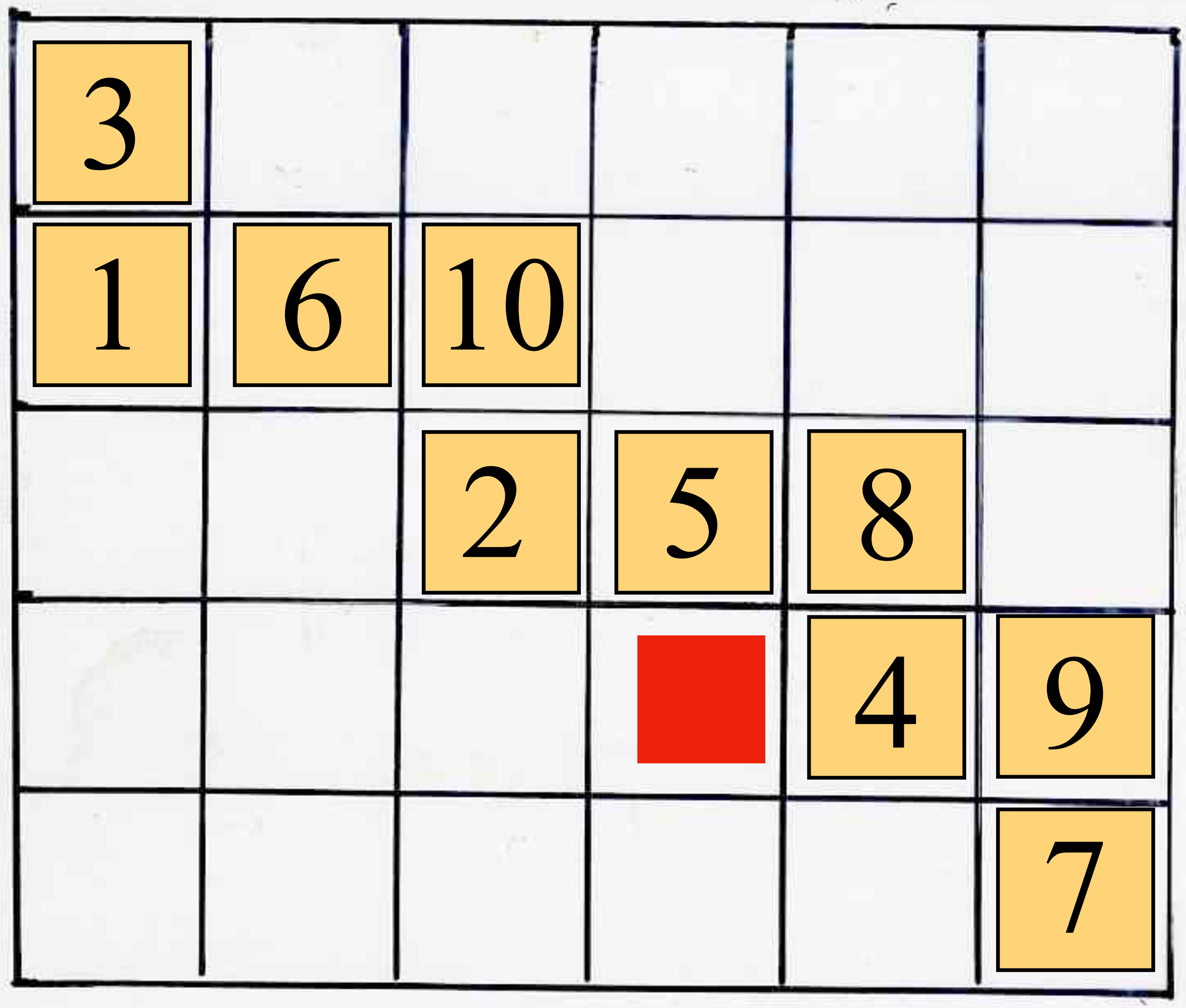
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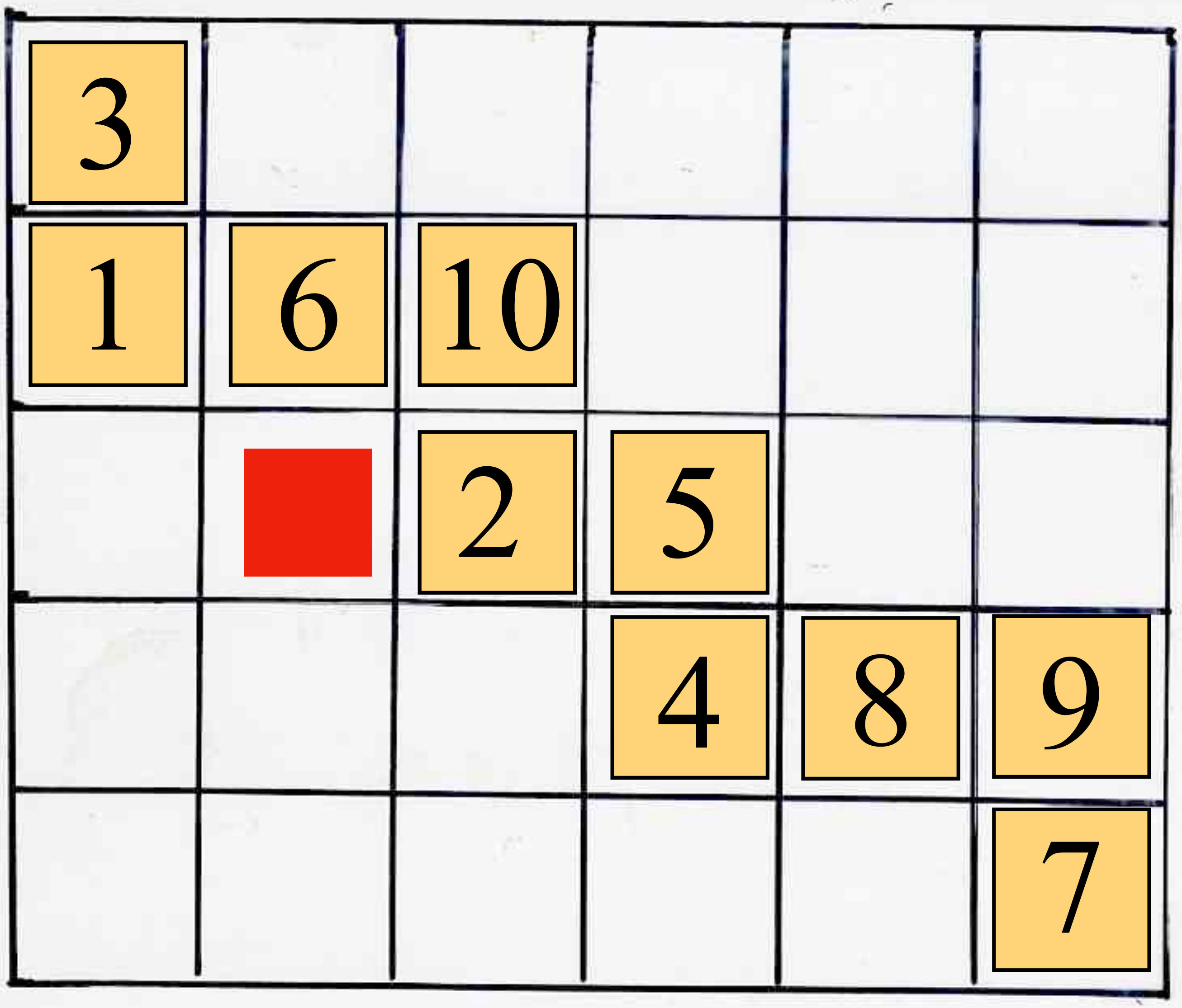
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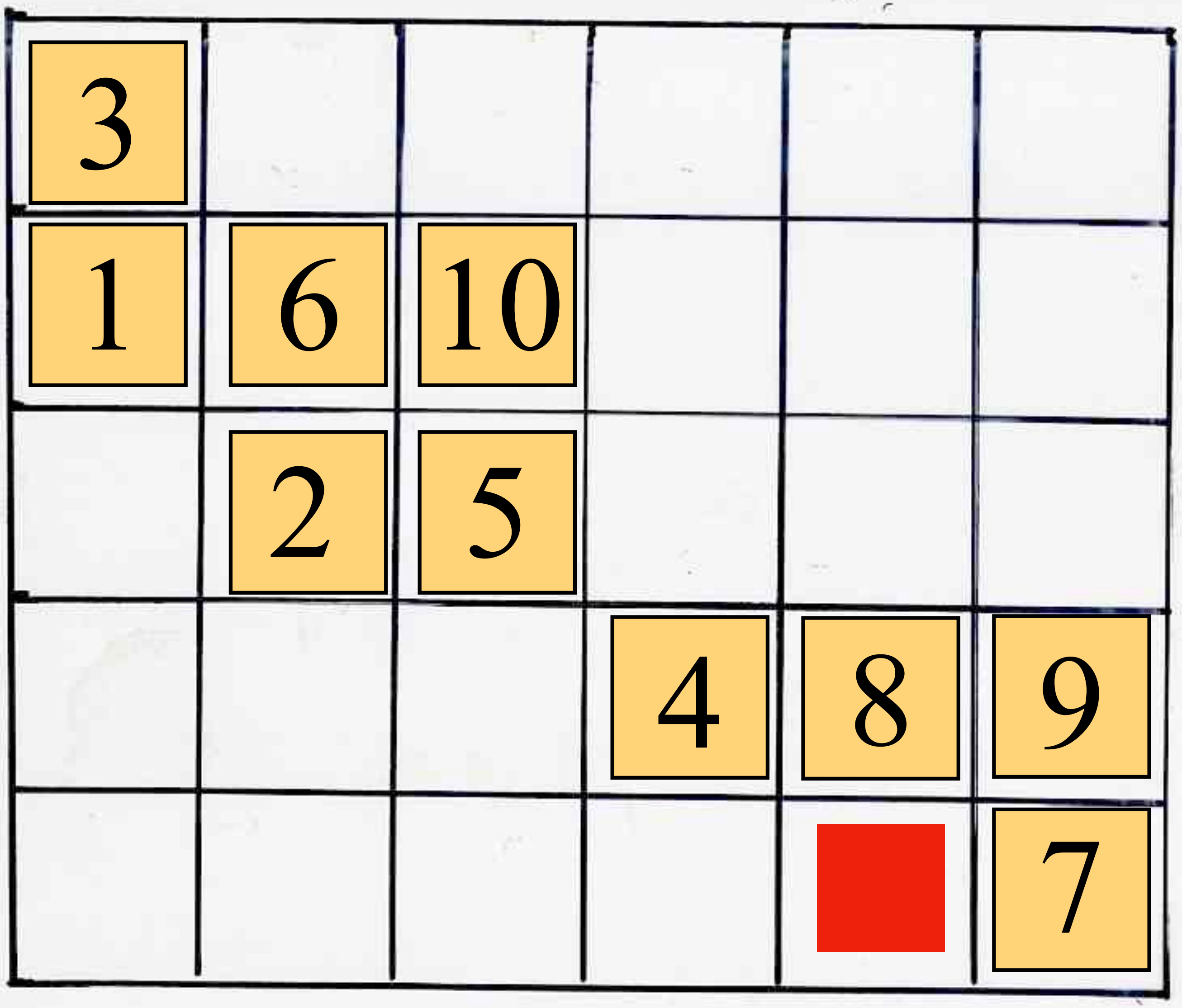
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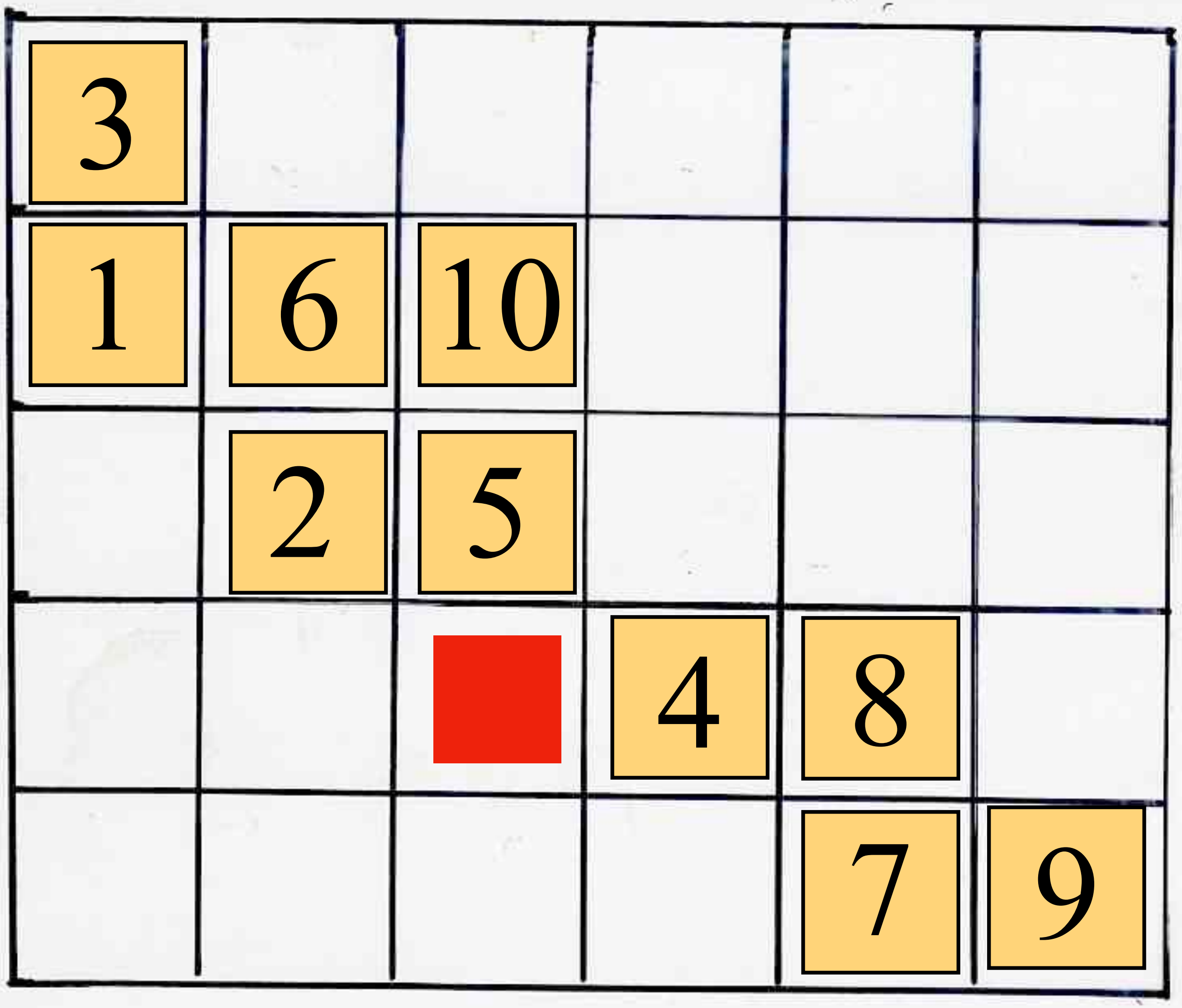
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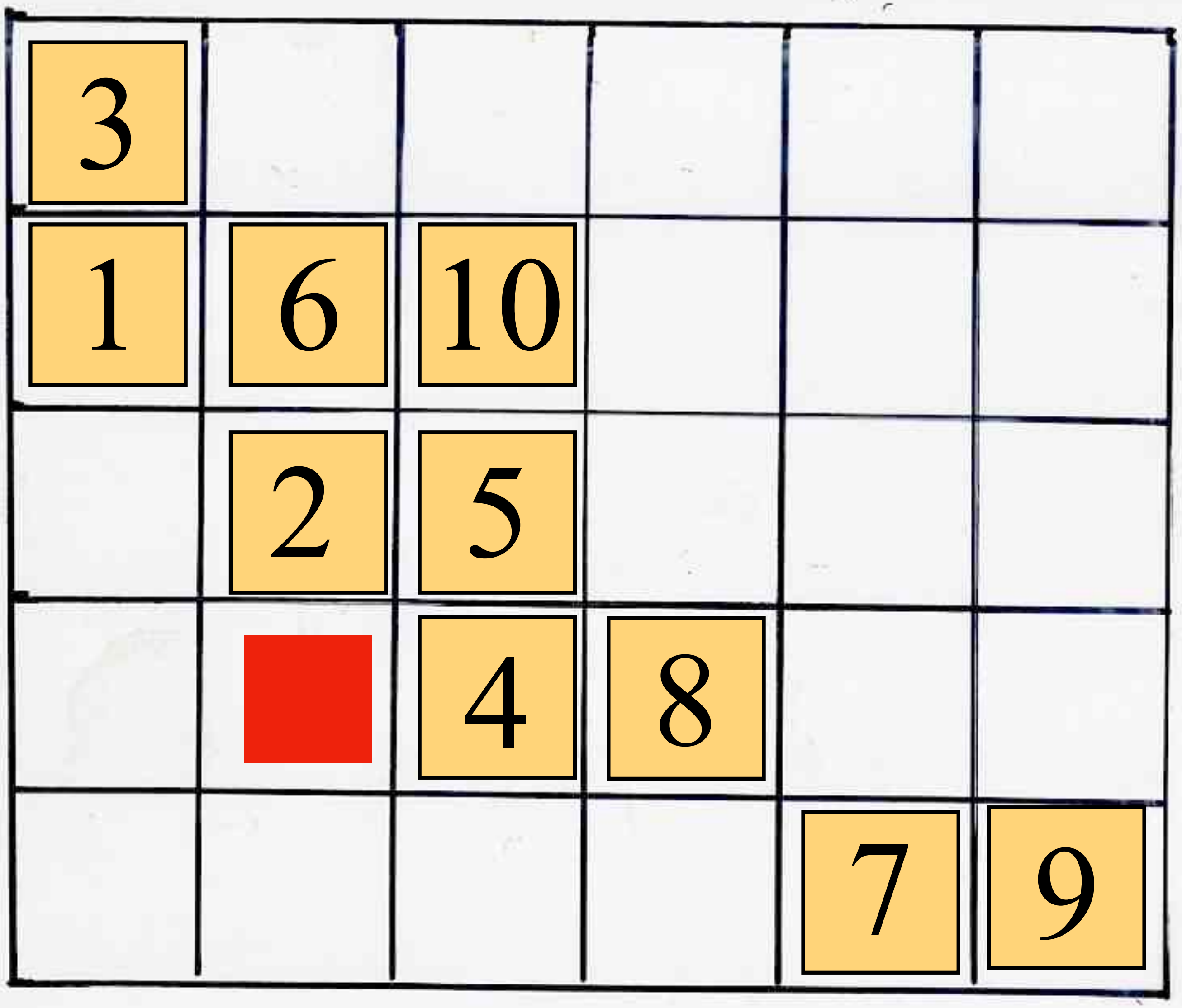
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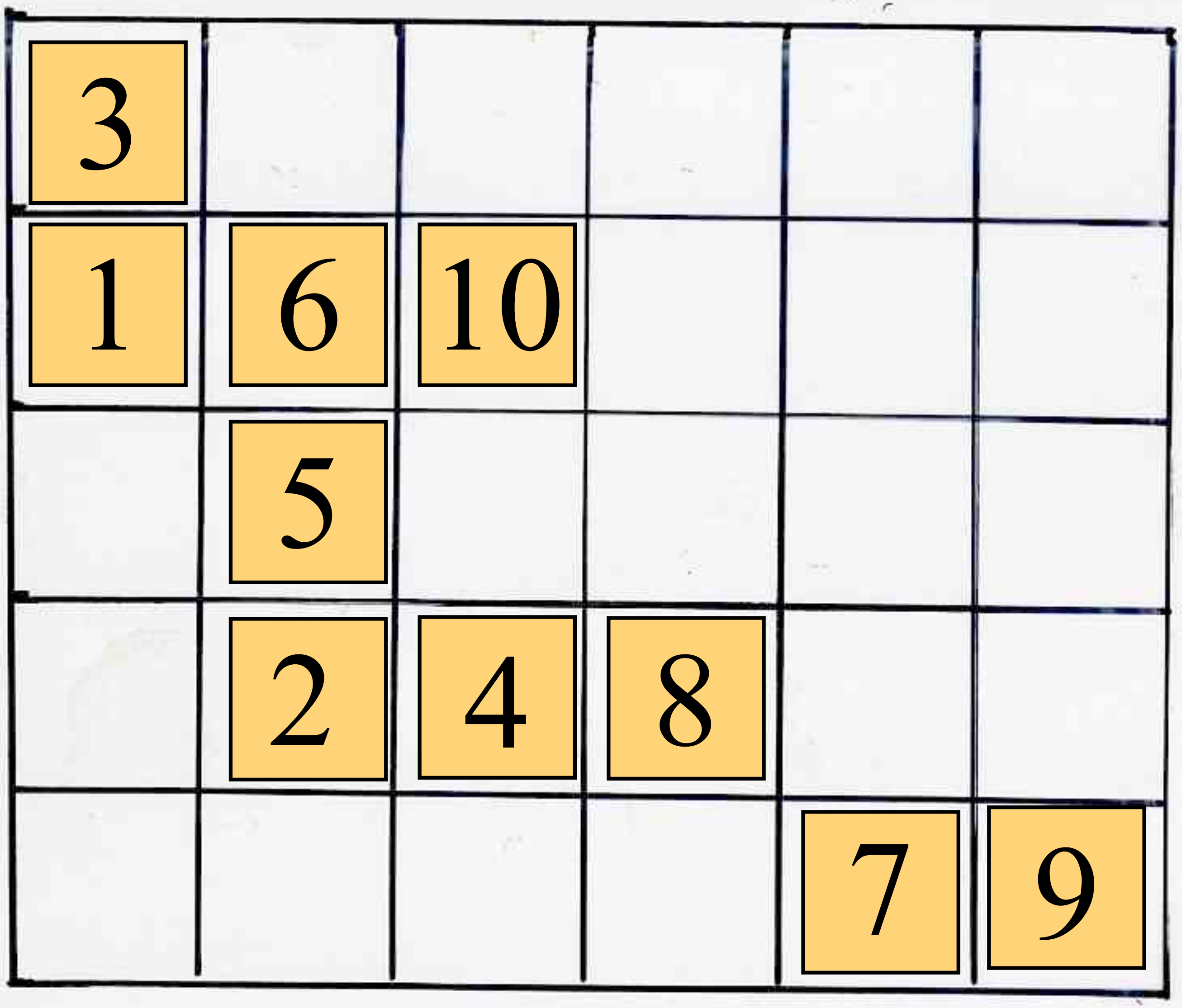
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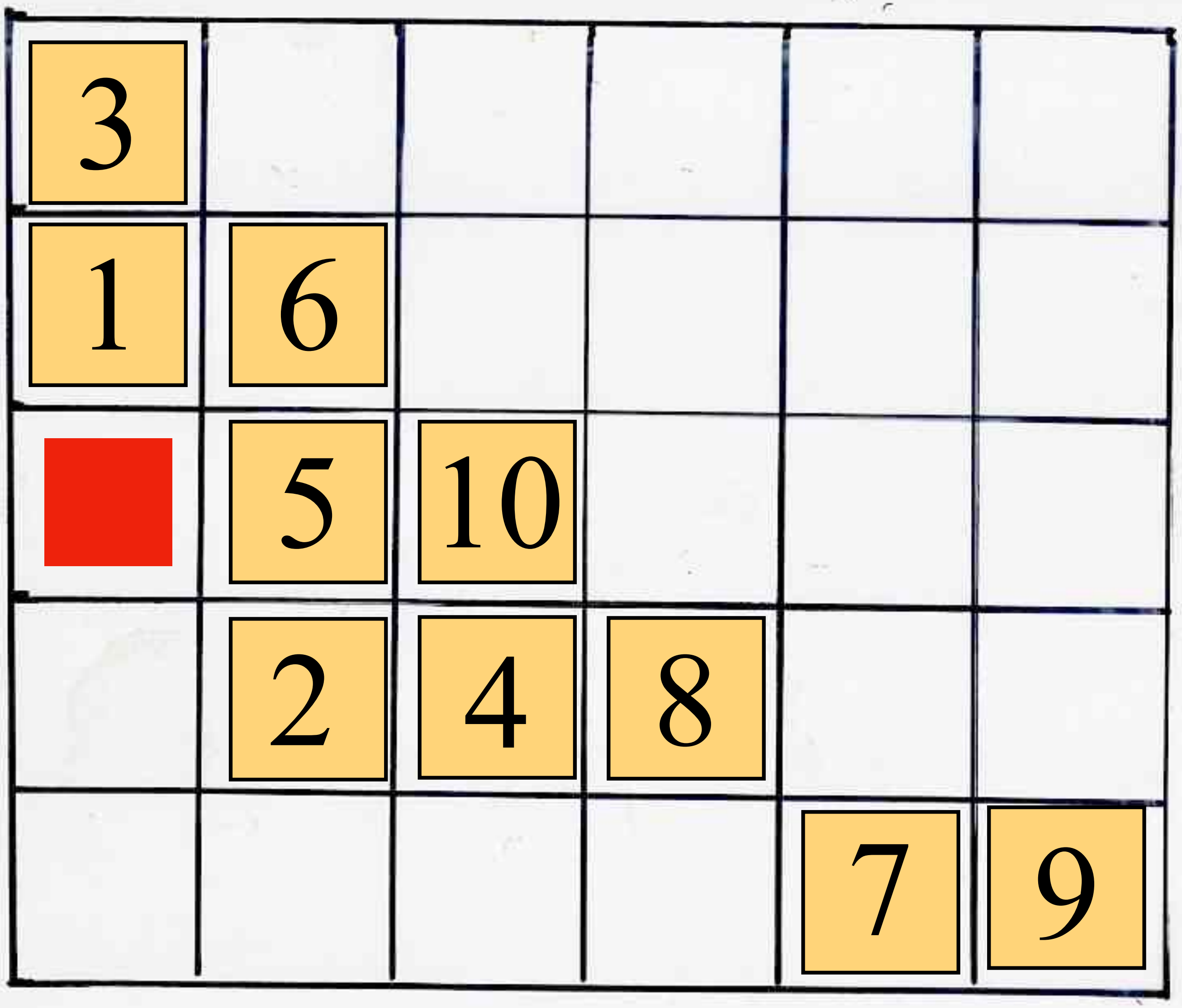
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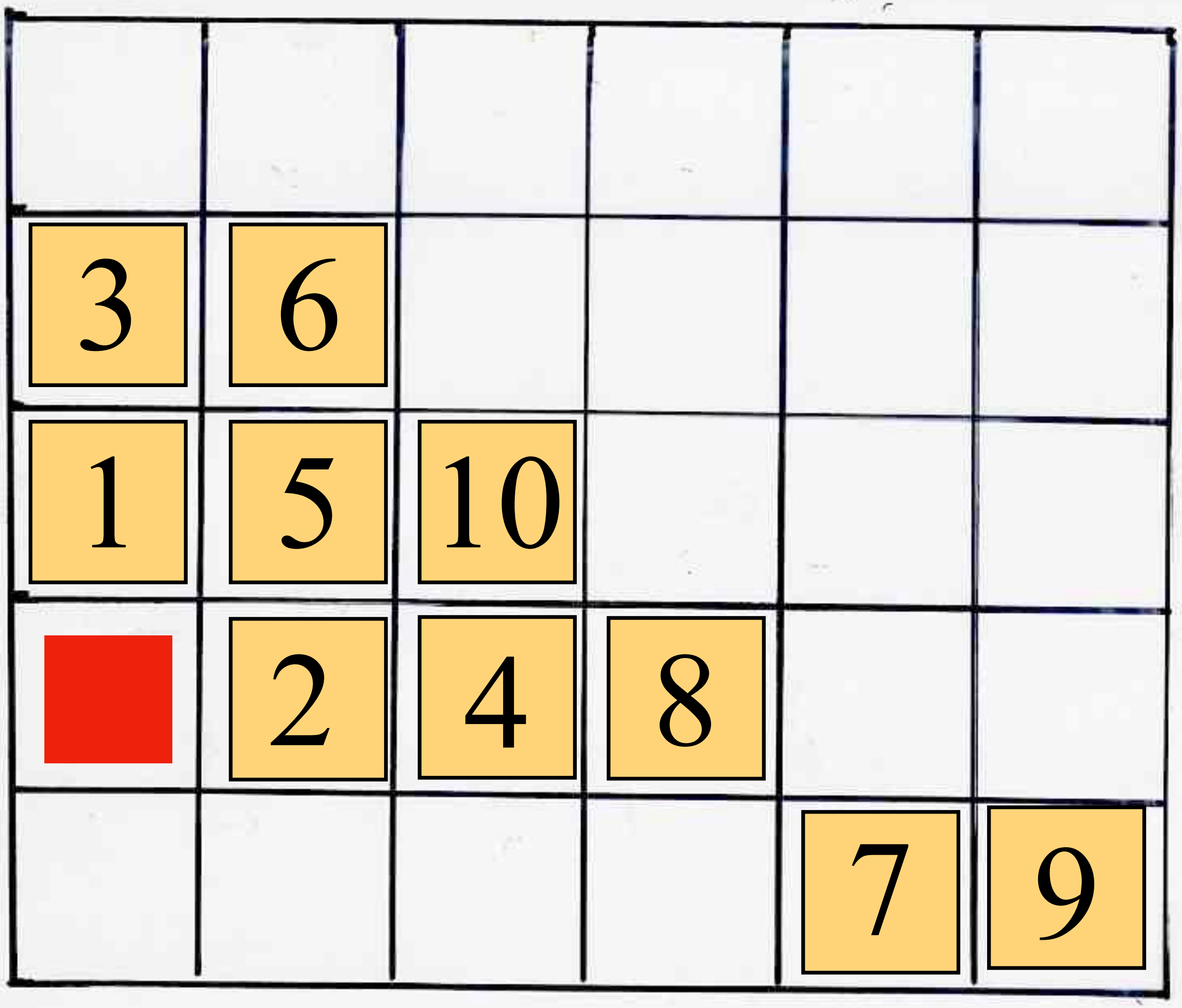
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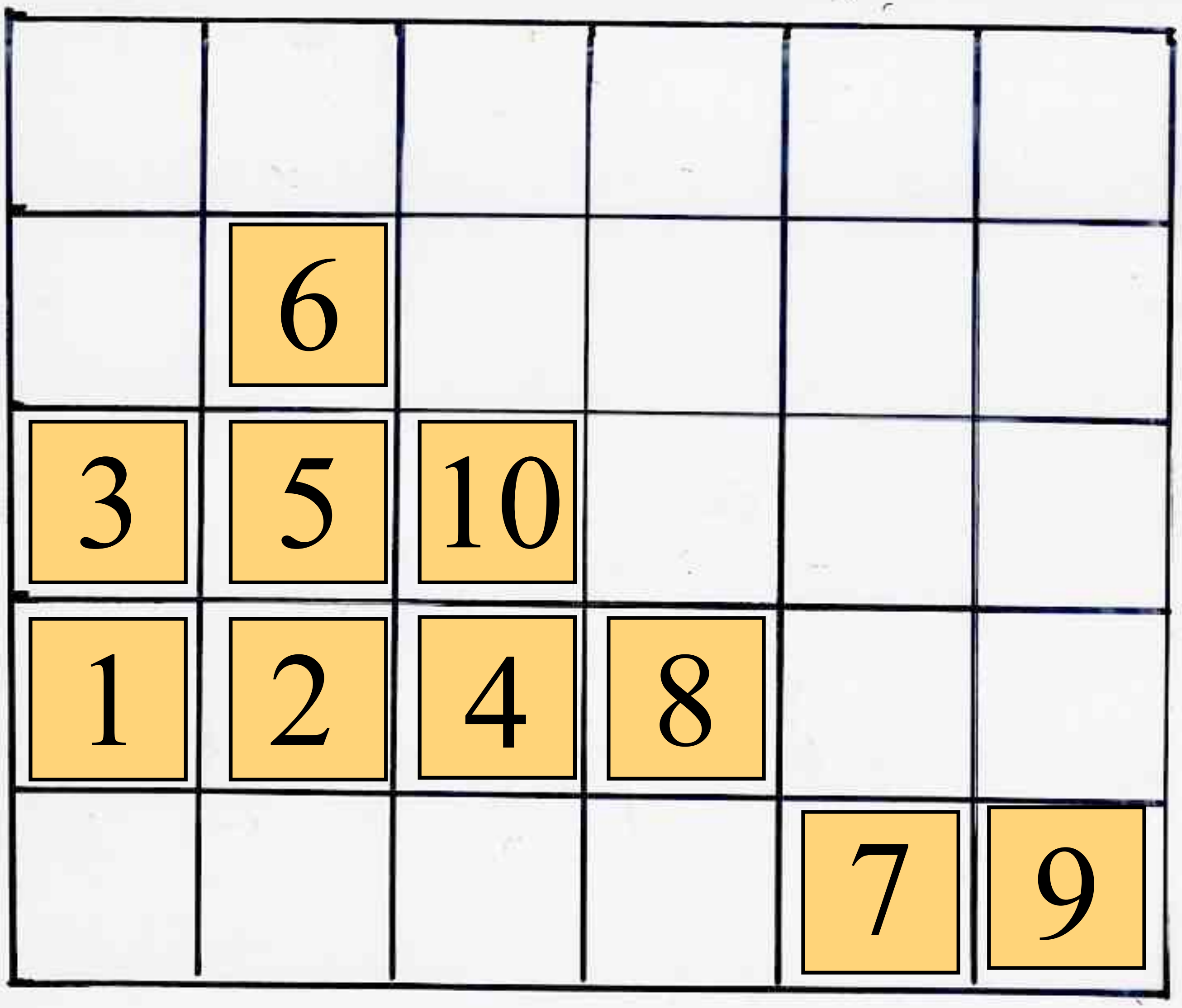
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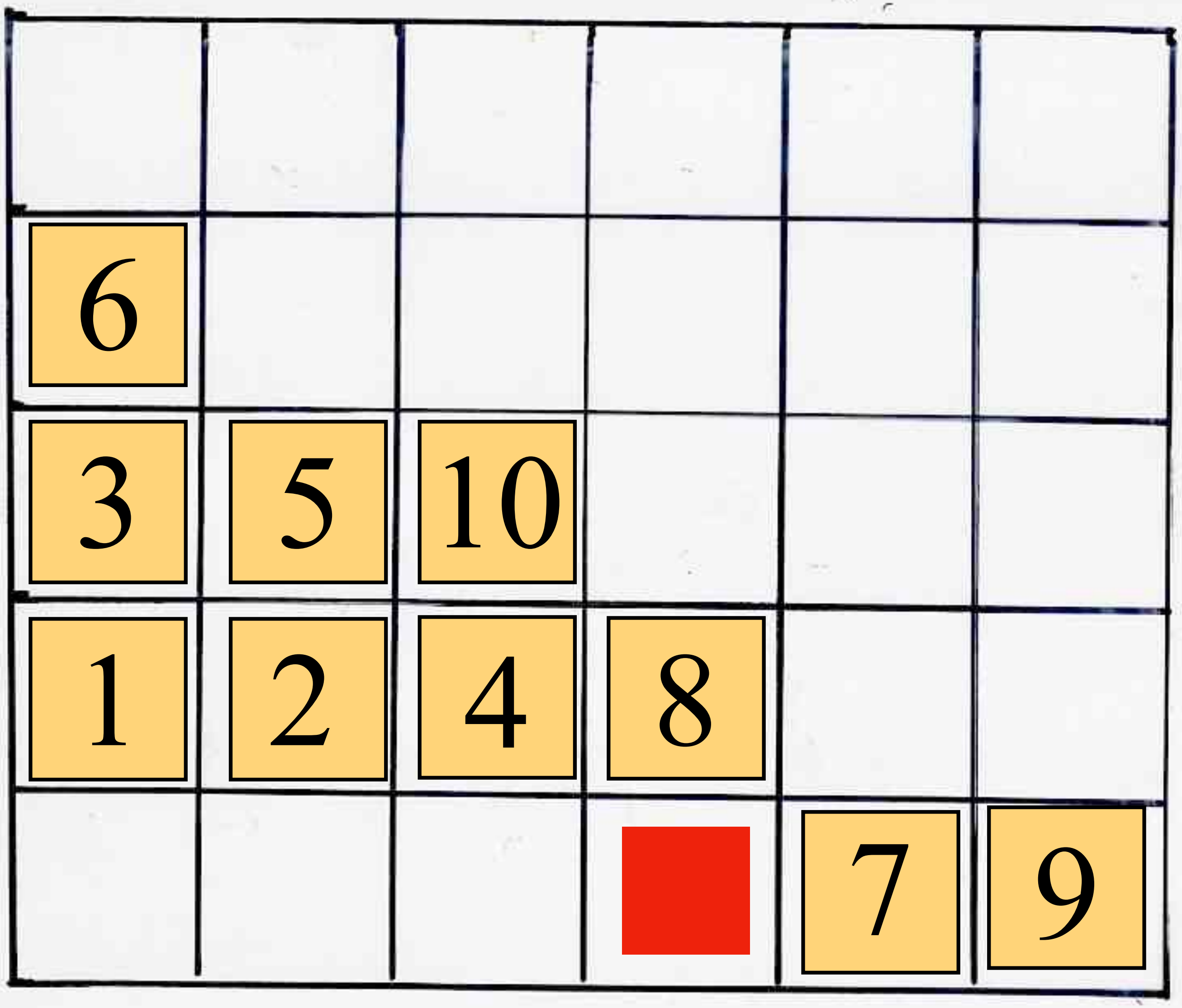
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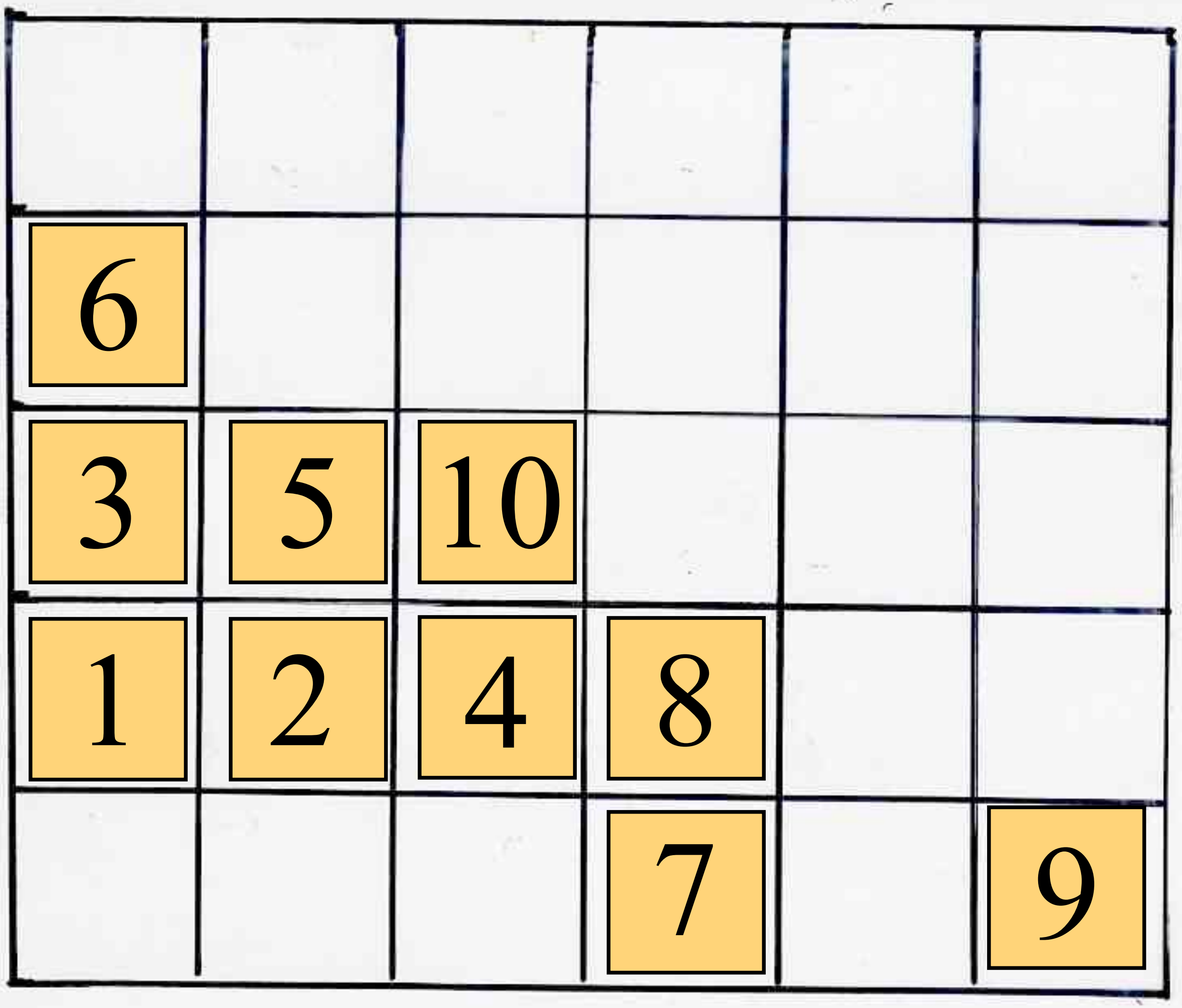
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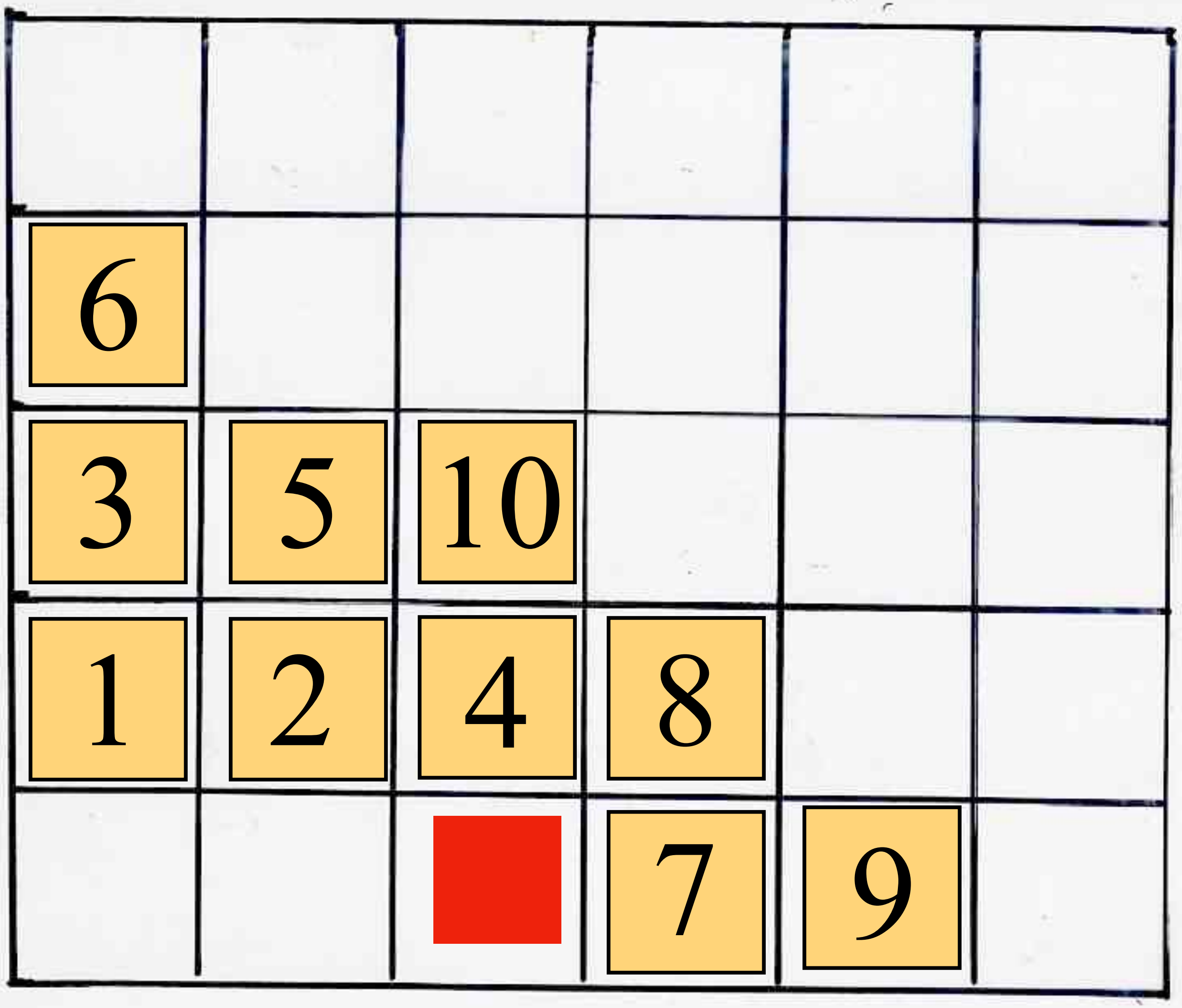
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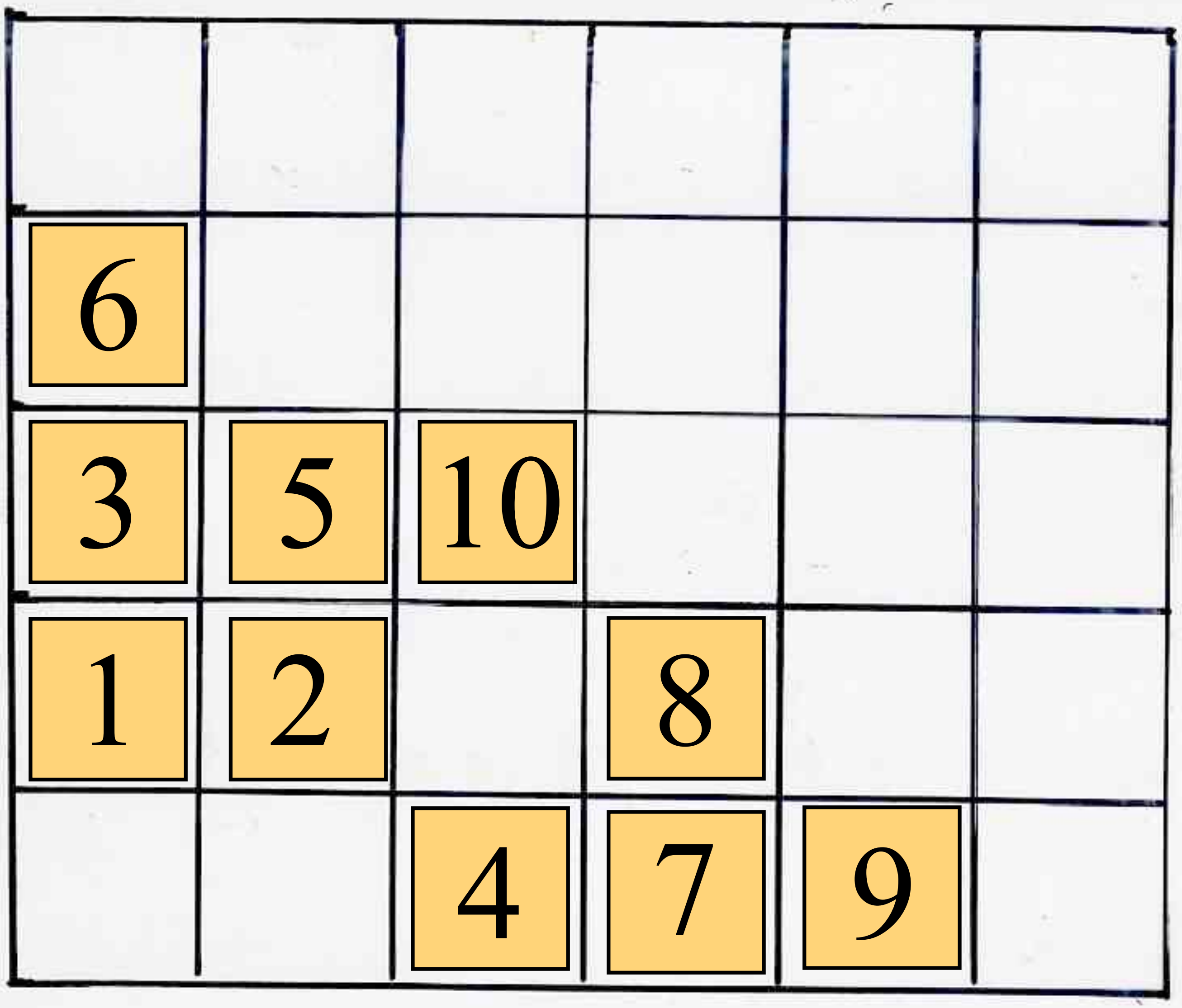
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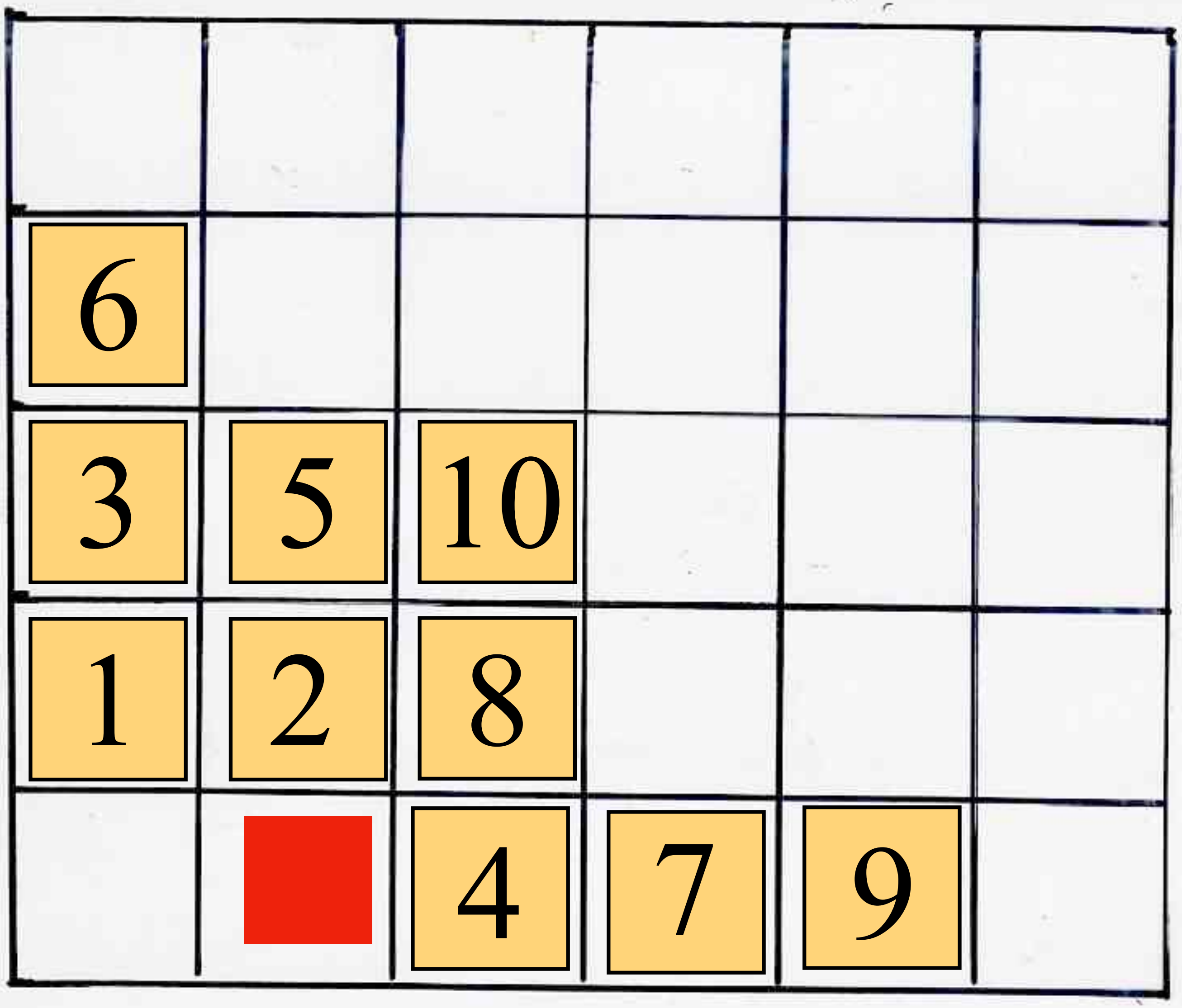
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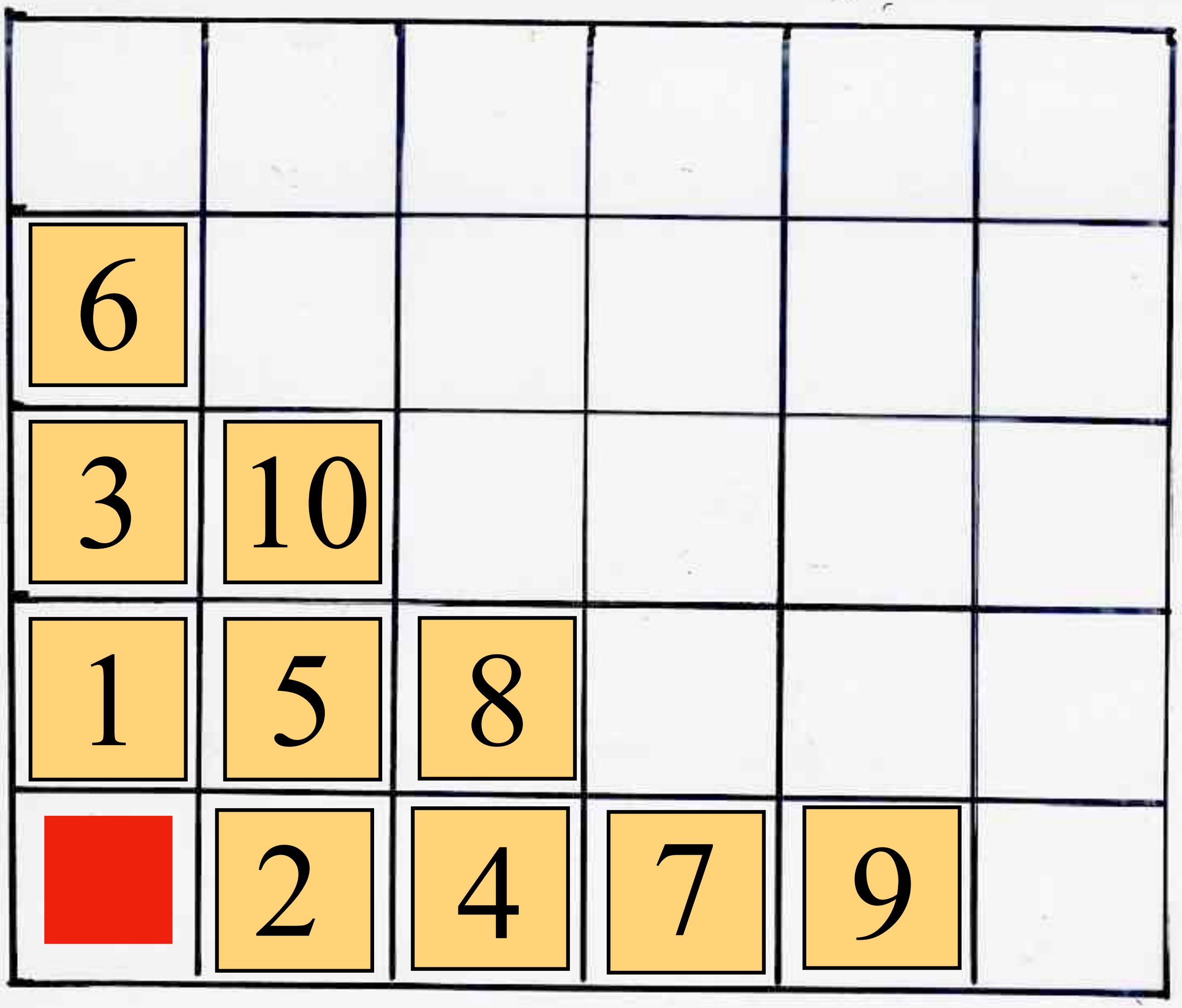
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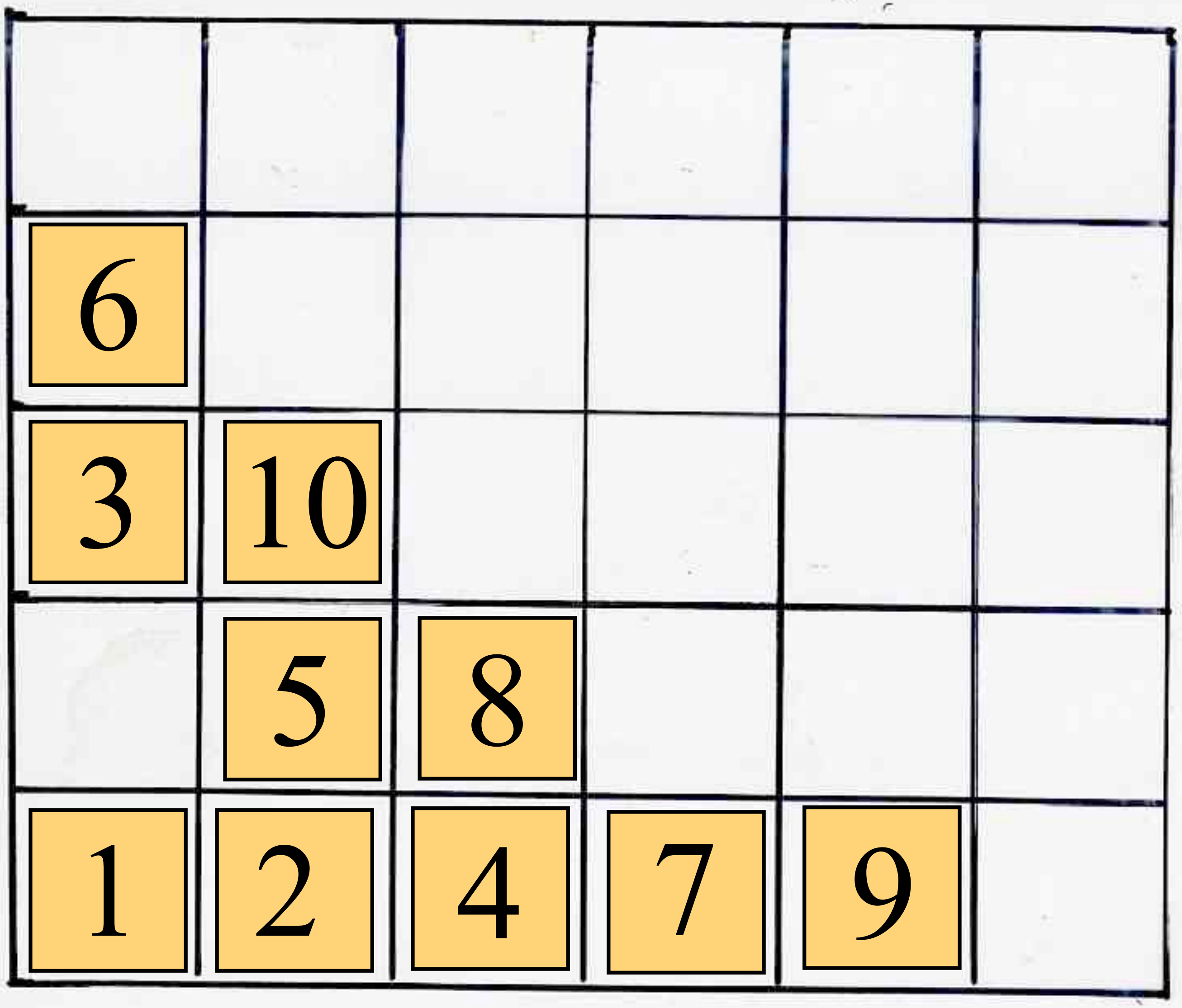
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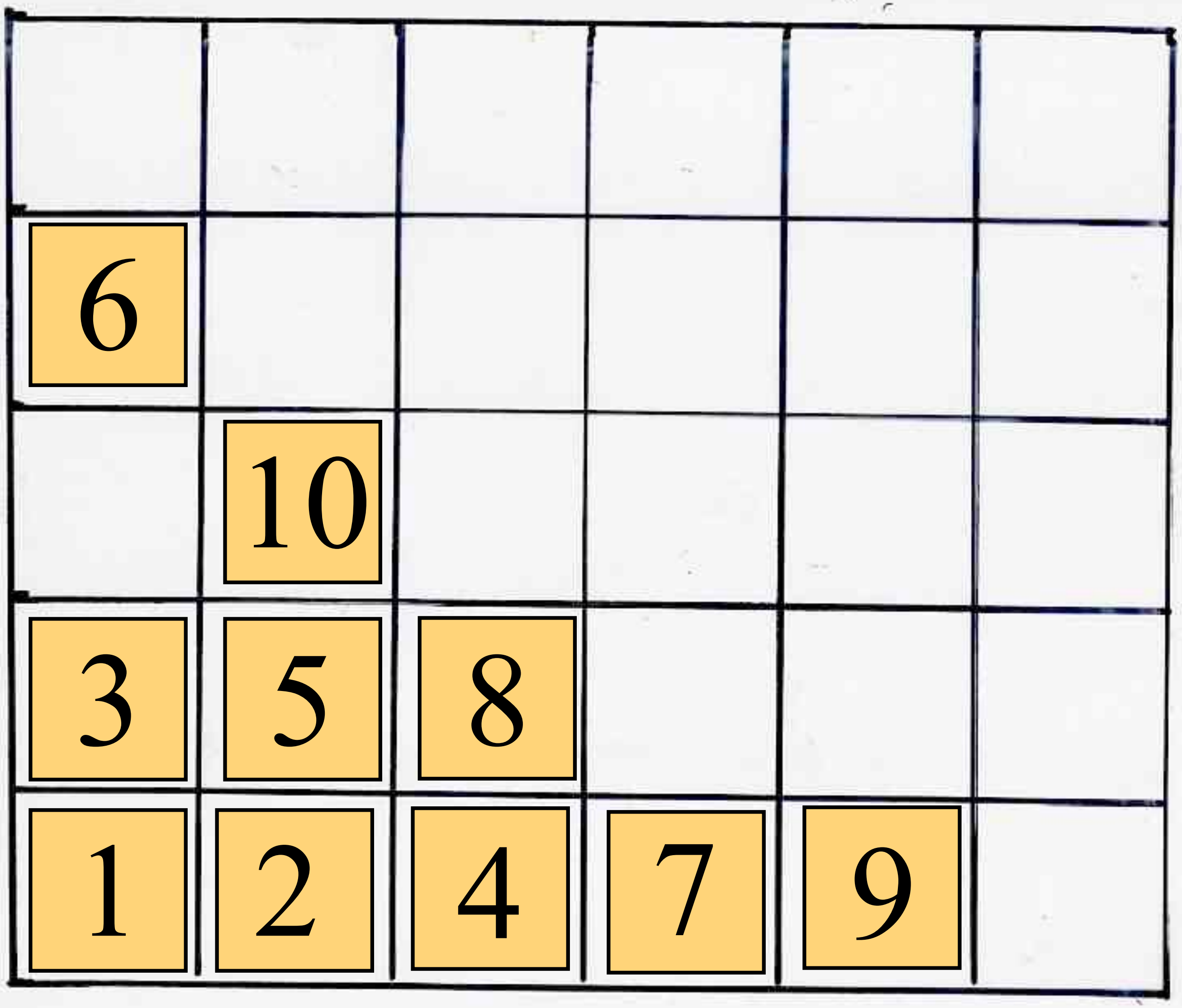

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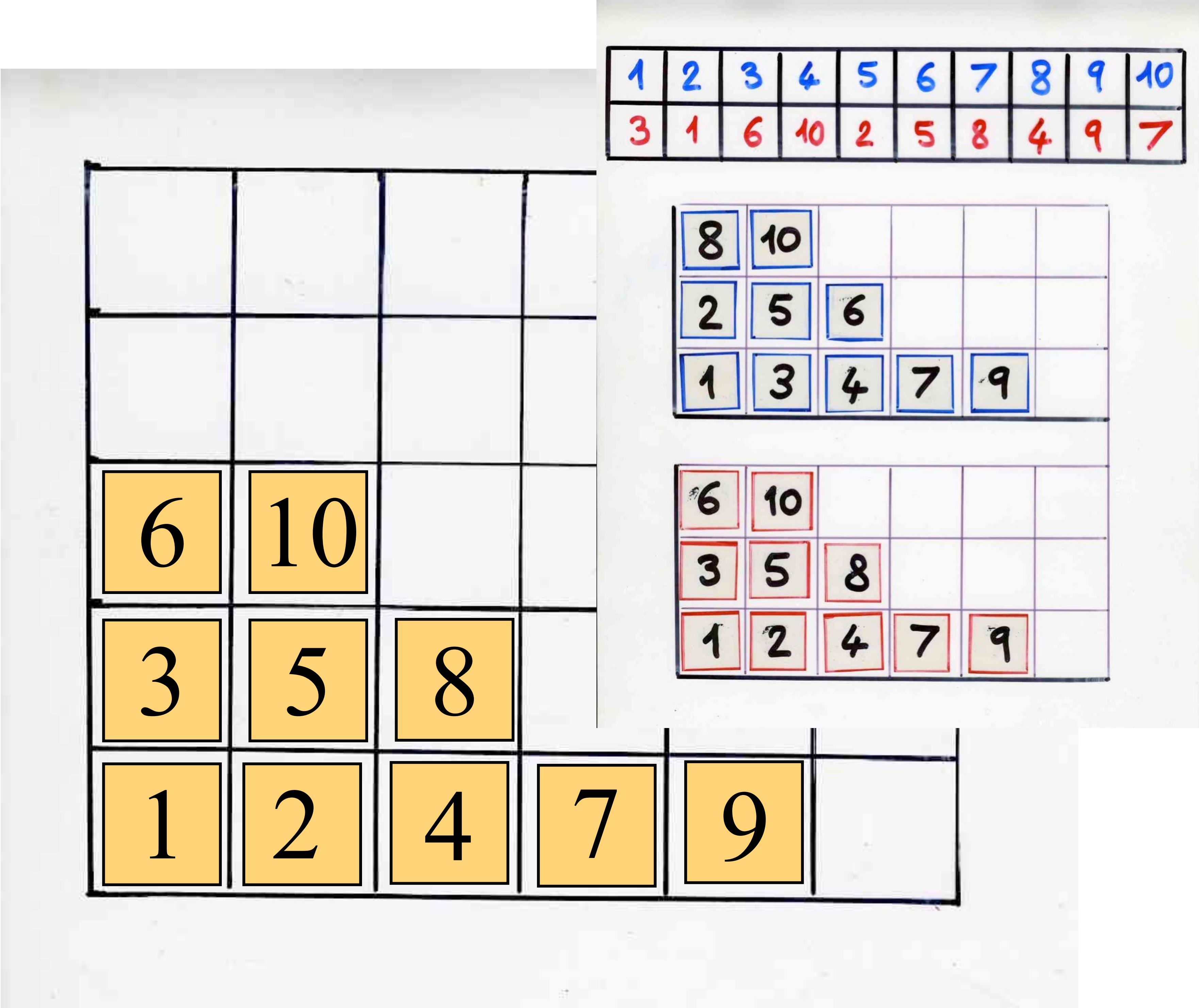

1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
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6 10
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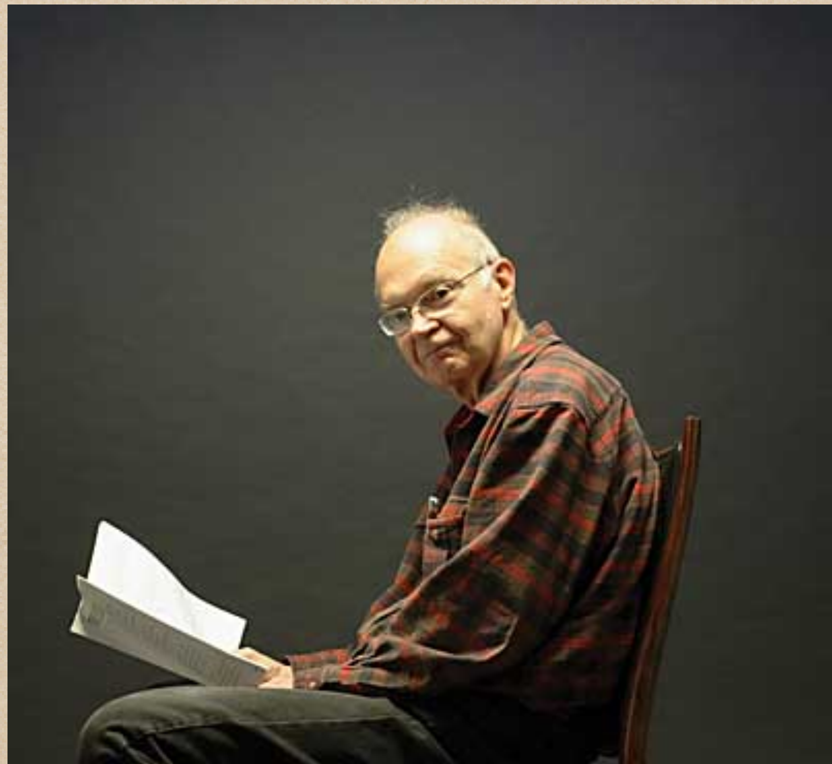
1 2 4 7 9

Knuth's transpositions



D. Knuth, 1970

Definition

Knuth transposition

permutation

$$\sigma = \sigma(1) \cdots \underbrace{\sigma(i)}_{x} \underbrace{\sigma(i+1)}_{y} \cdots \sigma(n)$$

$$\sigma' = \sigma(1) \cdots y\, x \cdots \sigma(n)$$

two consecutive values x, y can be transposed when $z = \sigma(i-1)$ or $\sigma(i+2)$ is between x and y

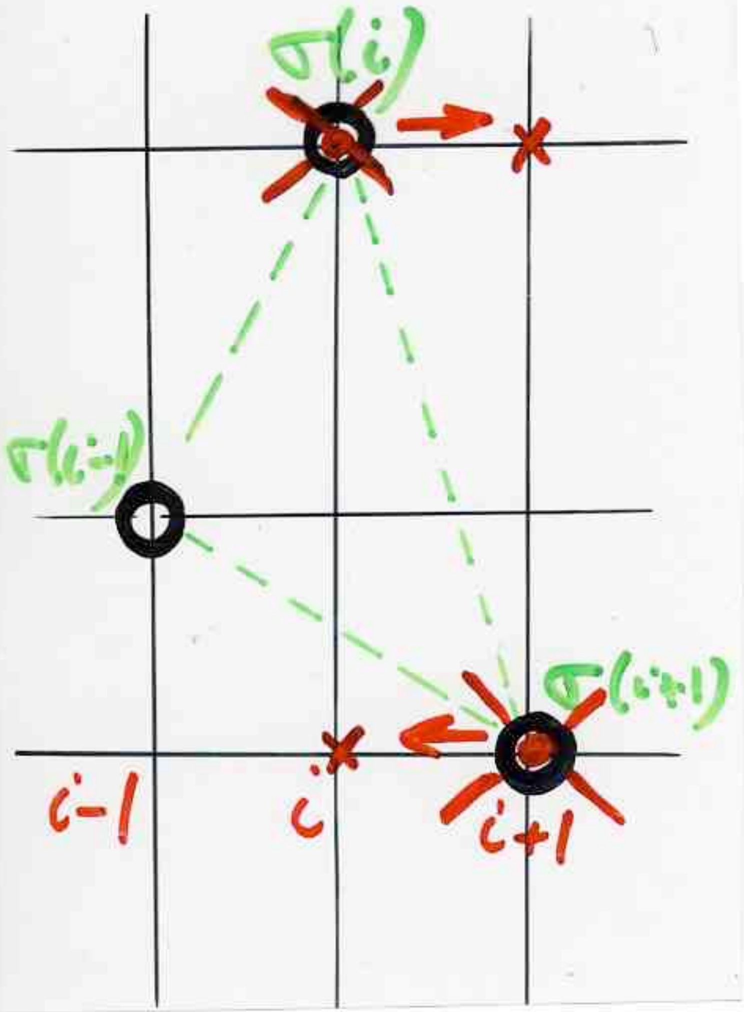
$$x < z < y \quad \text{or} \quad y < z < x$$

slide corrected after the video Recording

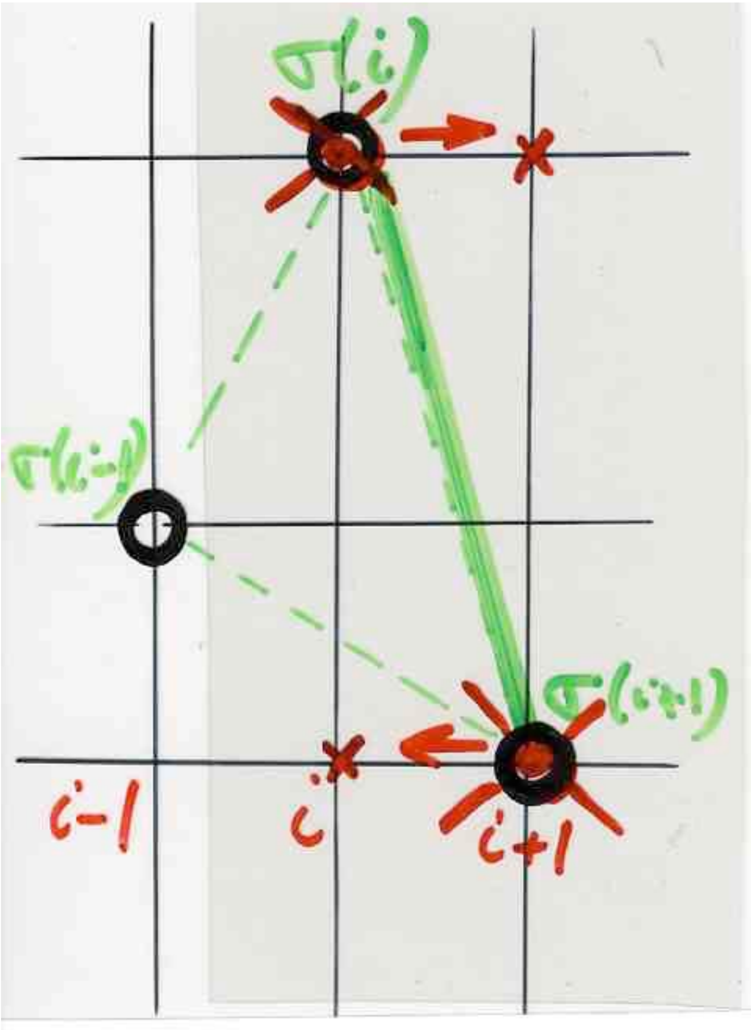
Knuth

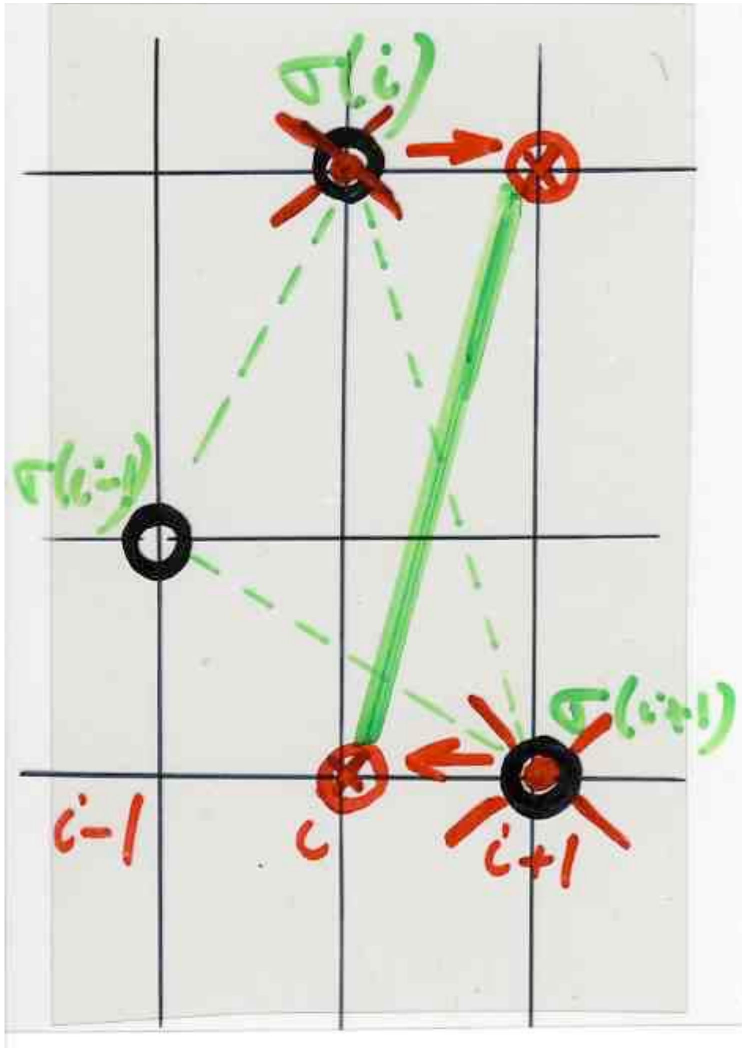
Transpositions

(1970)



Knuth transposition

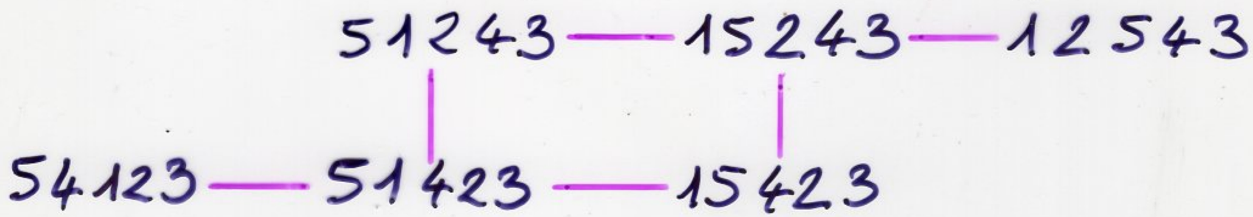




Two permutations σ and τ are called Knuth equivalent iff τ can be obtained from σ by a sequence of Knuth transpositions

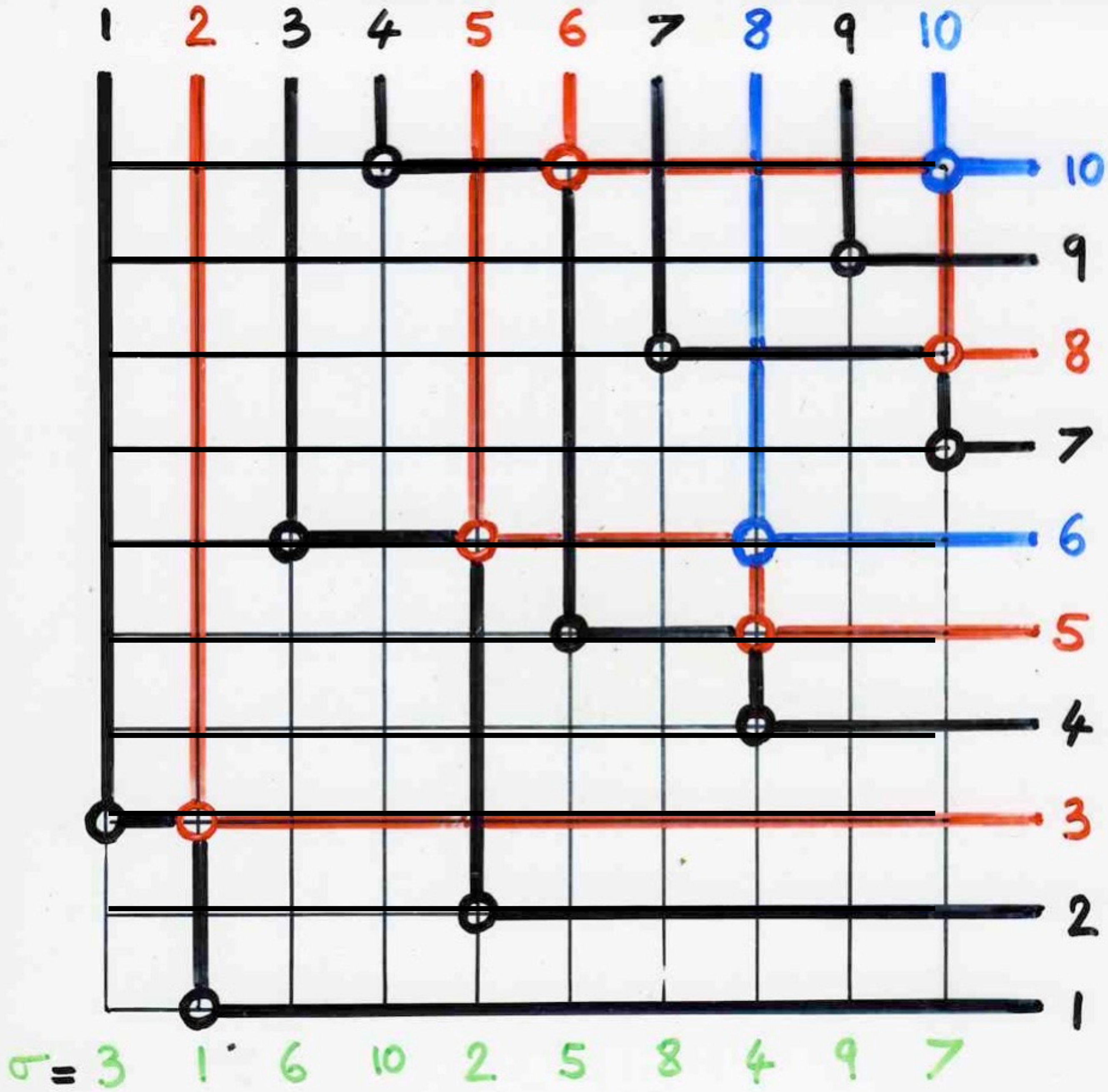
<u>notation</u> $\sigma \overset{K}{\sim} \tau$

Knuth equivalence class

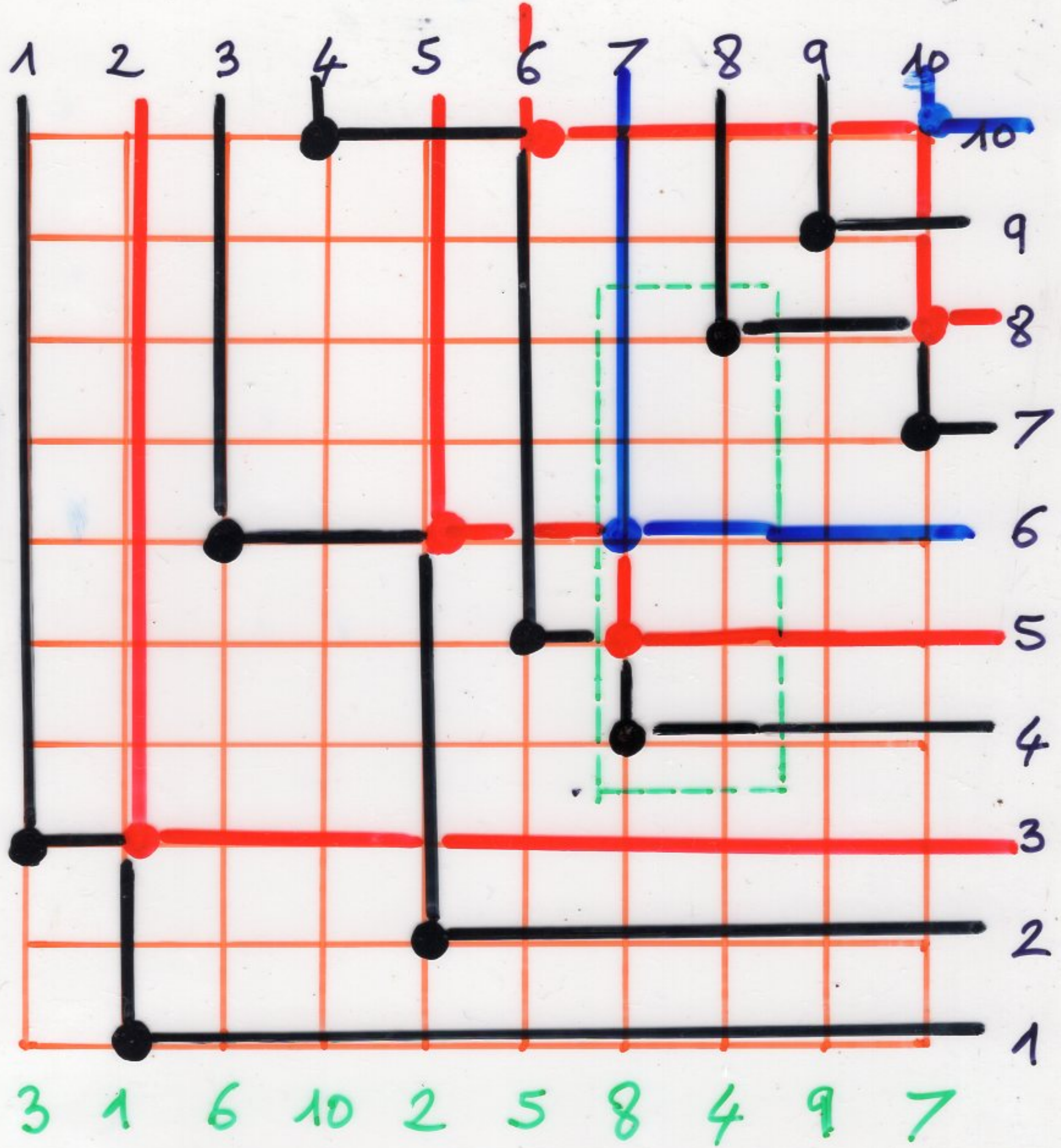


Proposition $\sigma, \tau \in \mathfrak{S}_n$ $\quad \sigma \xrightarrow{RS} (P, Q)$

$$\sigma \overset{K}{\sim} \tau \implies P(\sigma) = P(\tau)$$



1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7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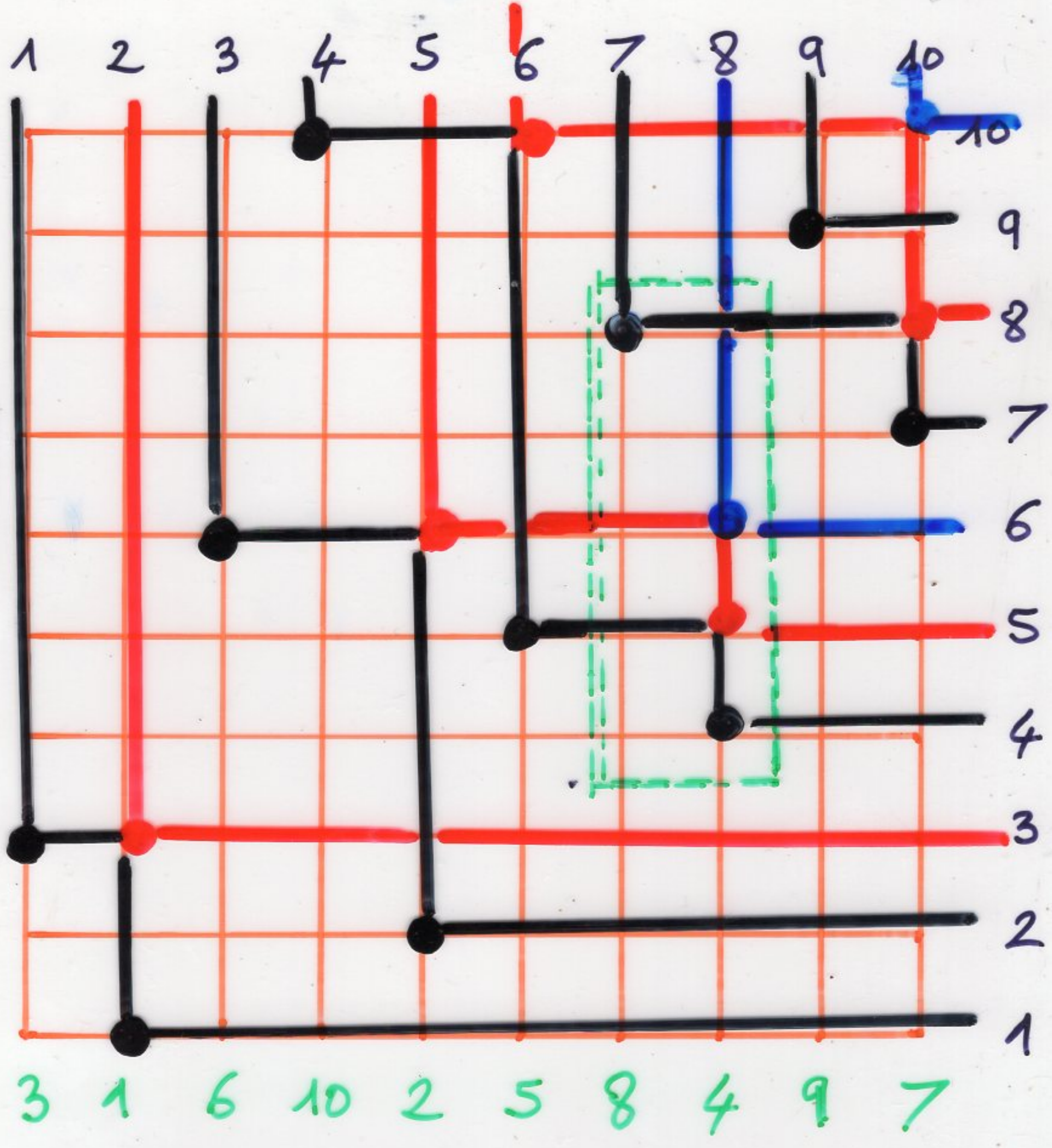
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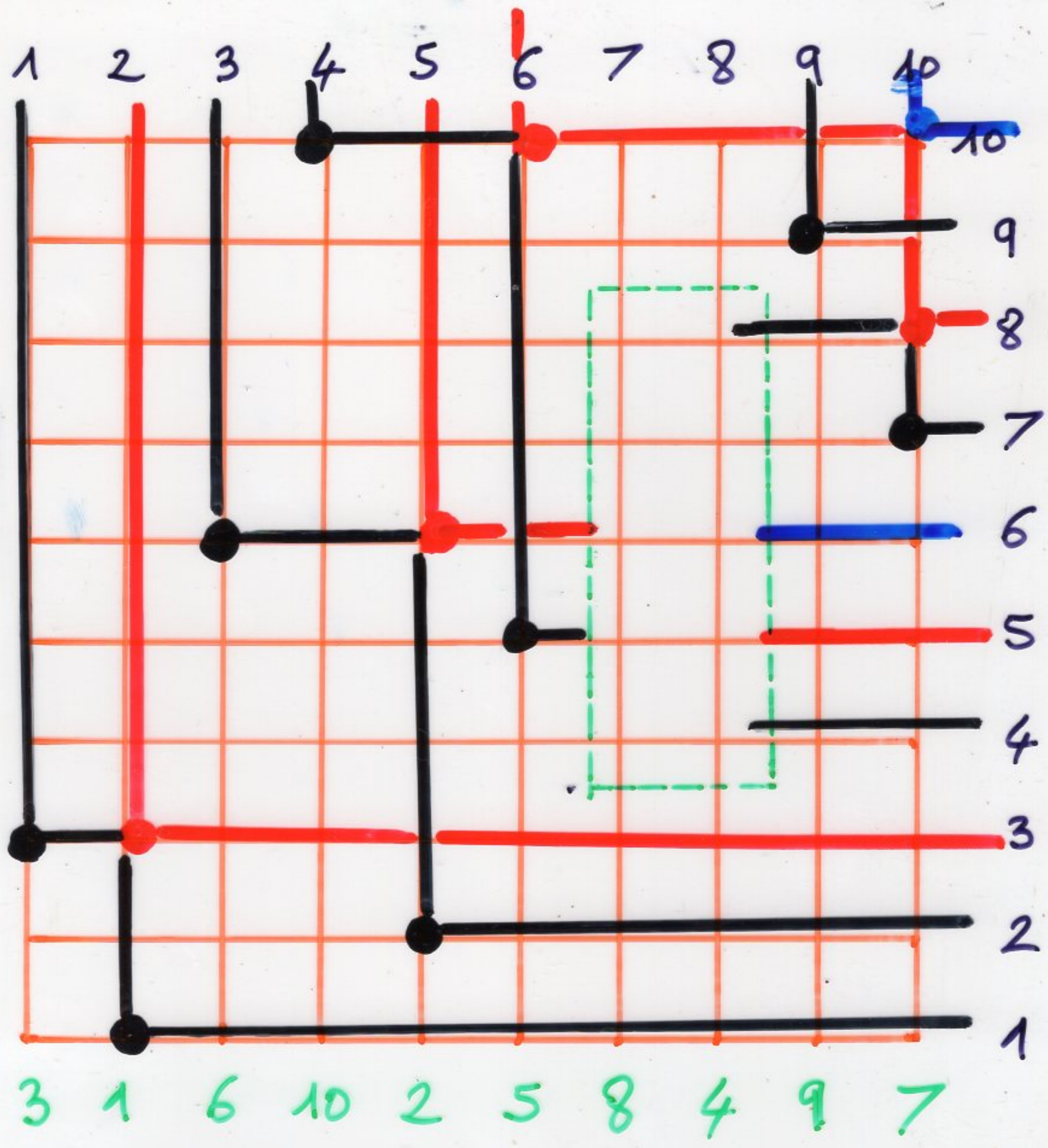
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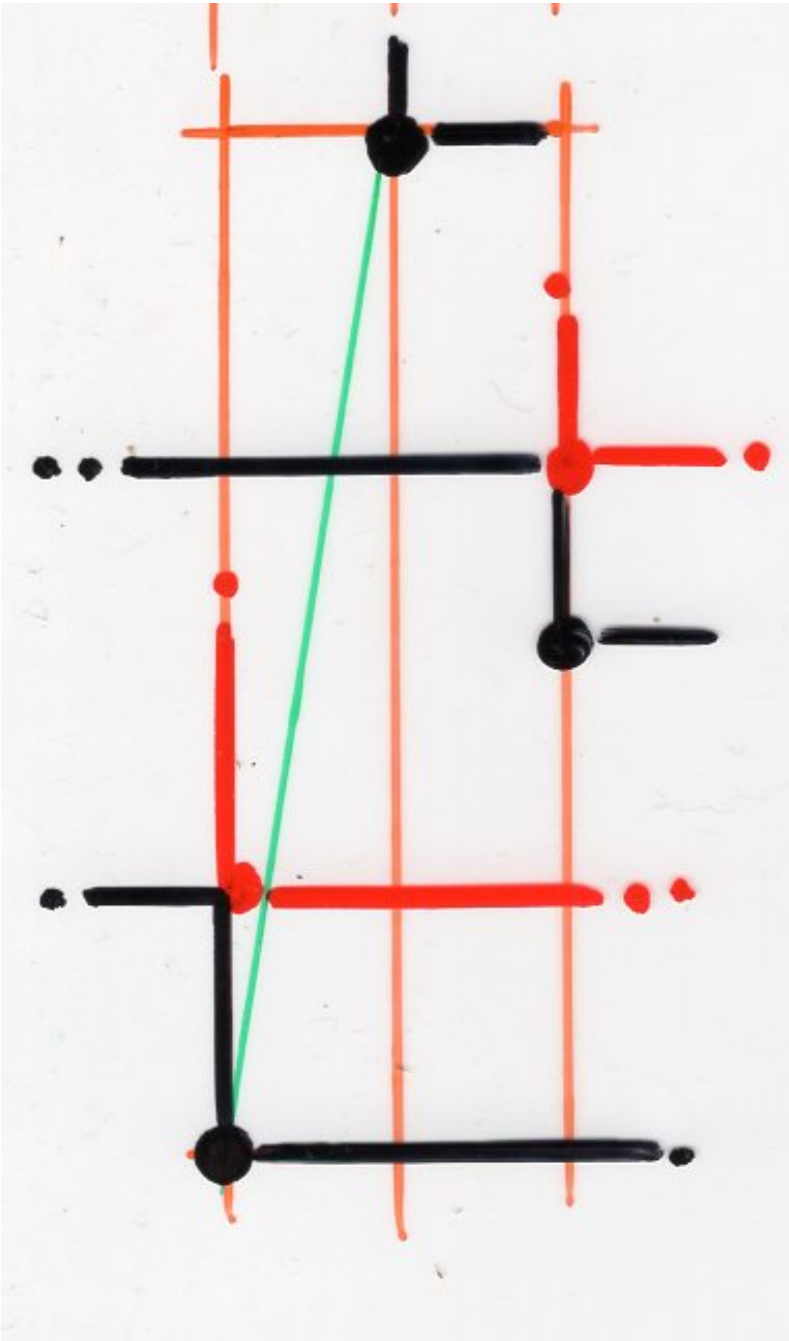

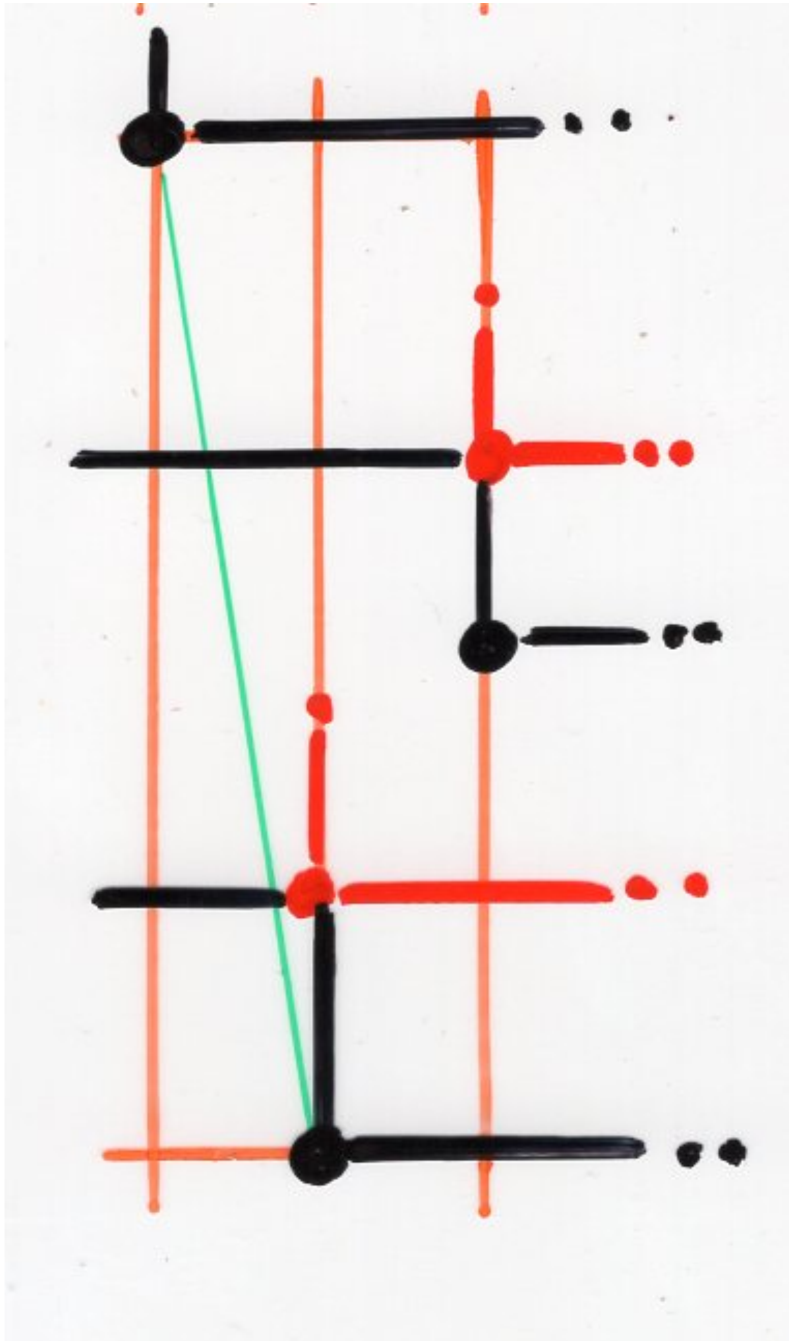

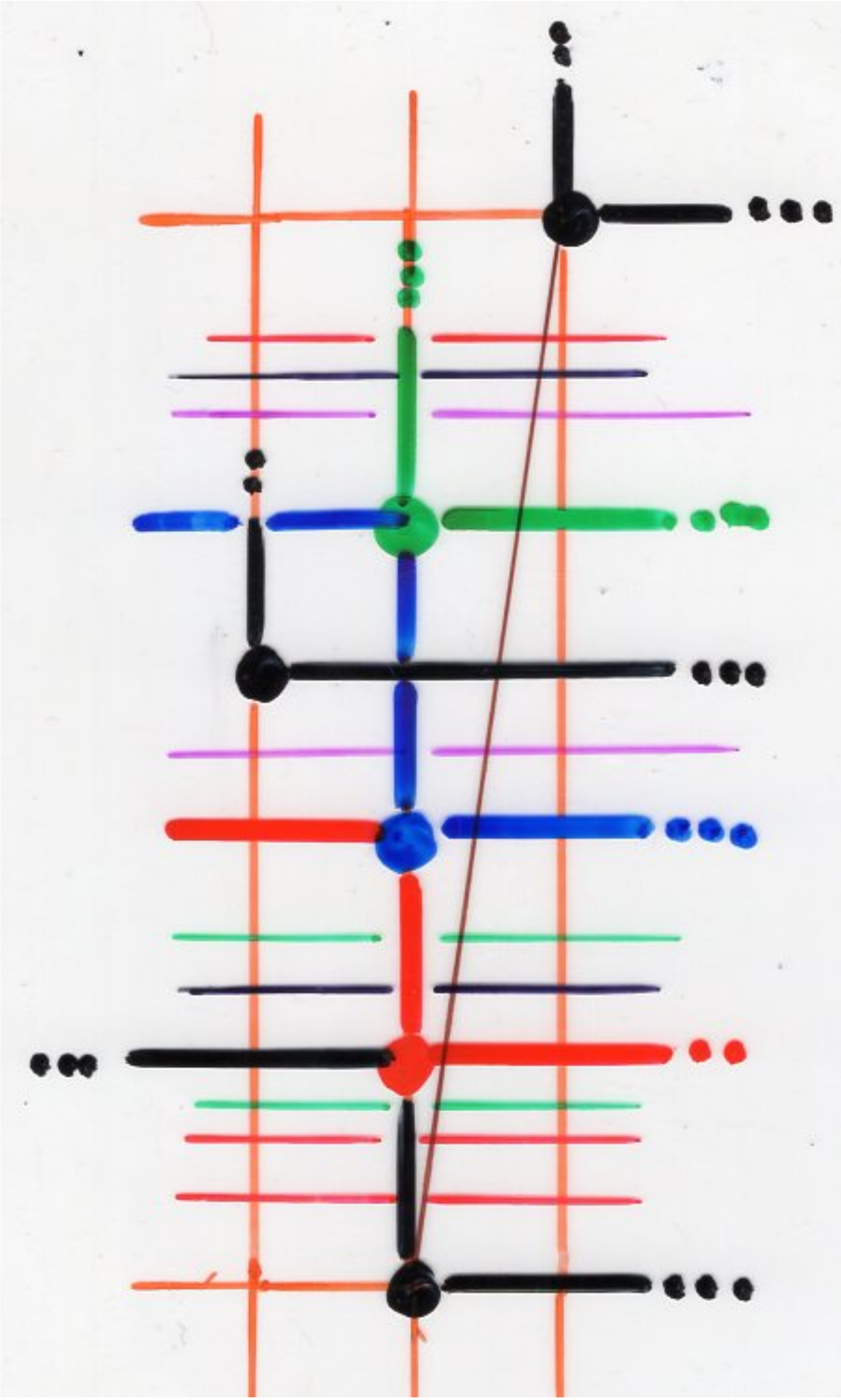

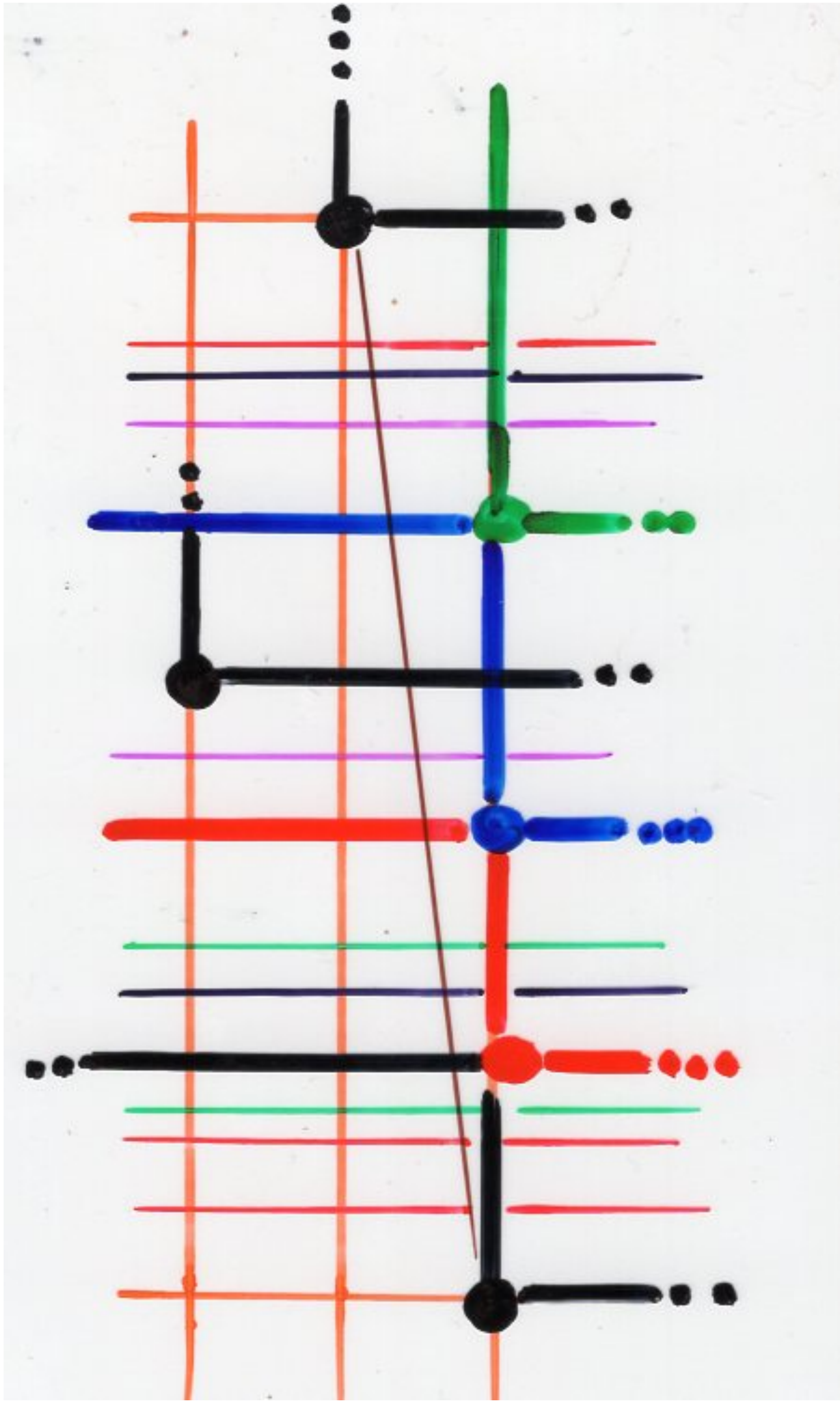

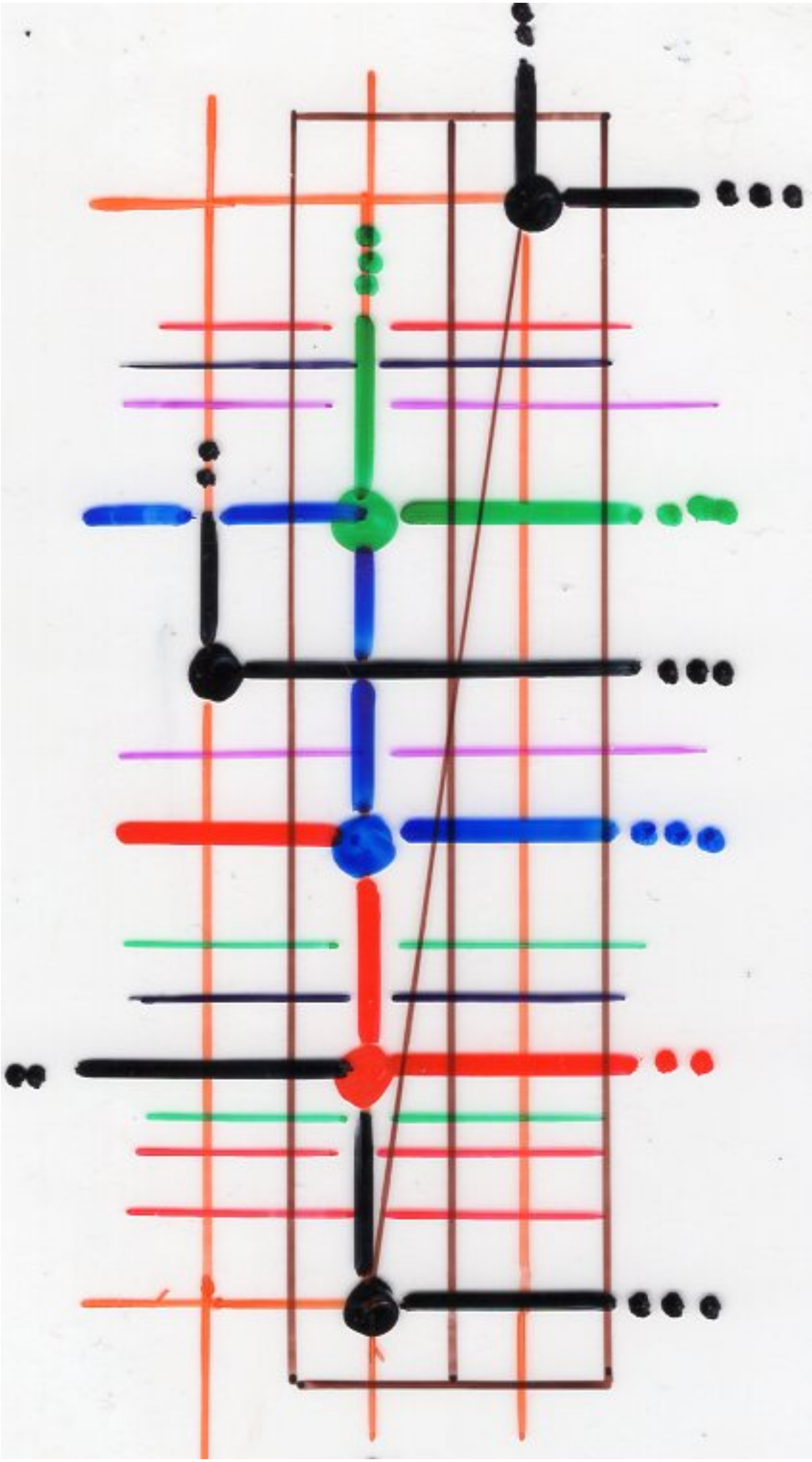

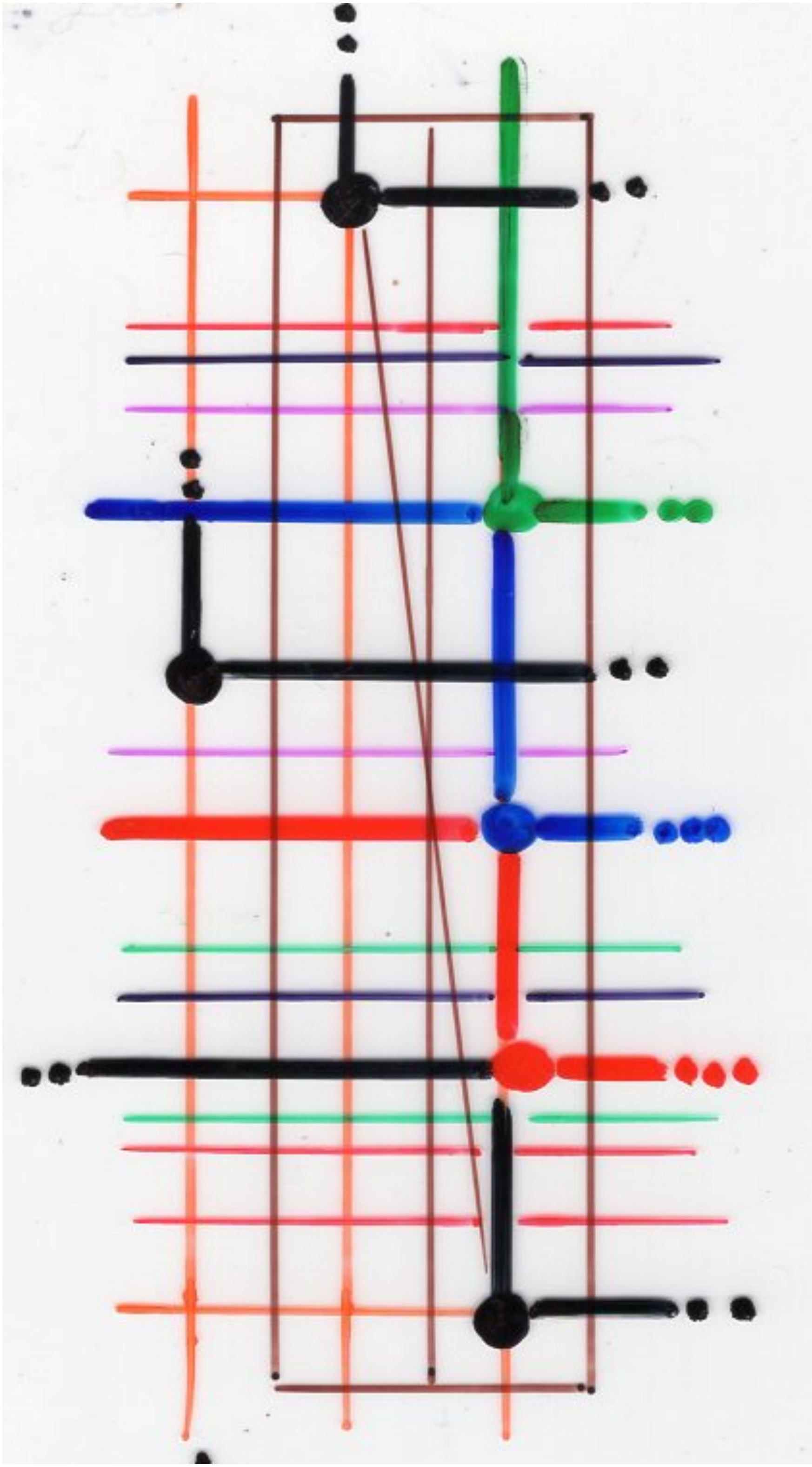

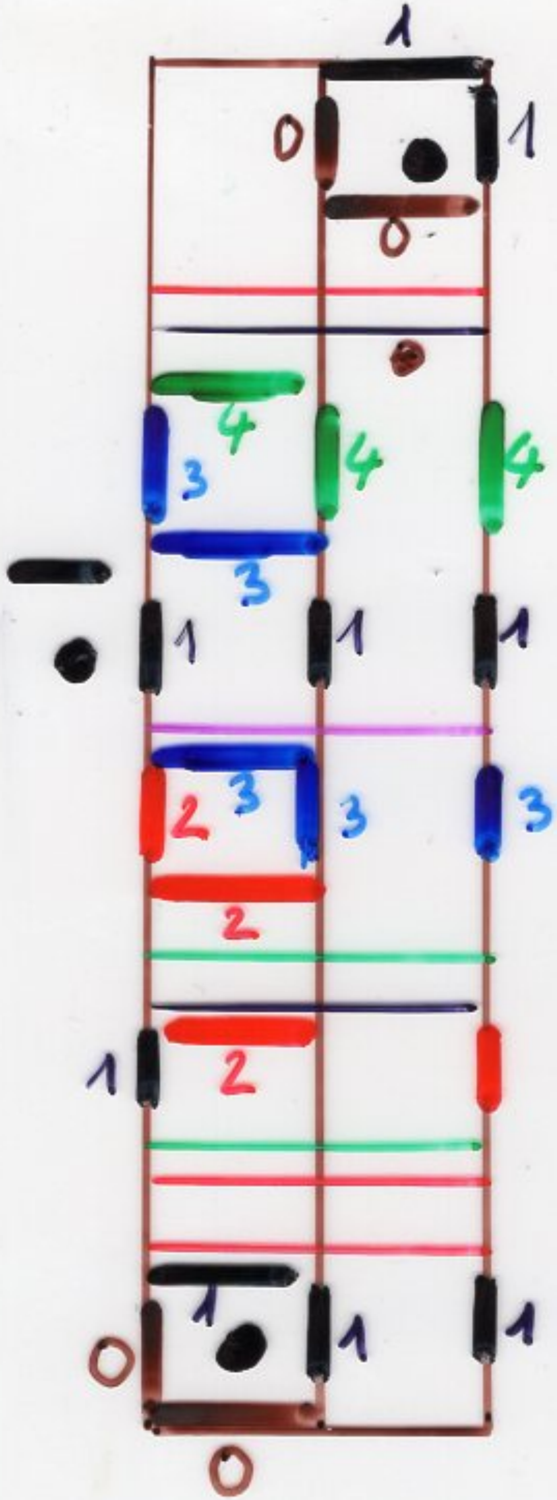
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Definition Reading word of a Young tableau T : $w = \text{Reading}(T)$

$w = v_k \, v_{k-1} \cdots v_1$ where v_i , $i = 1, .., k$ is the word obtained by reading the i^{th} row from left to right.

$T =$

6	10			
3	5	8		
1	2	4	7	9

$$\text{Reading}(T) = \underbrace{6 \; (10)}_{v_3} \; \underbrace{3 \; 5 \; 8}_{v_2} \; \underbrace{1 \; 2 \; 4 \; 7 \; 9}_{v_1}$$

Lemma Any permutation is Knuth equivalent to the reading word of its insertion tableau

$$\sigma \xrightarrow{RS} (P, Q), \text{ then } \sigma \overset{K}{\sim} \text{Reading}(P)$$

for P Young tableau and $k \in \mathbb{N}$ not in P we have:

$$\text{Reading}(P) \cdot k \overset{K}{\sim} \text{Reading}(P \leftarrow k)$$

concatenation words

insertion of k in P

P =

3	6	10	
1	2	5	8

← $k=4$

Reading(P) = 3 6 (10) 1 2 5 8

3	6	10	
1	2	4	8

← 5

3 6 (10) 5 1 2 4 8

1 2 5 8 ← $k=4$

1 2 5 8 4

1 2 5 4 8

1 5 2 4 8

5 1 2 4 8

$P =$

3	6	10	
1	2	5	8

$\leftarrow k=4$

$\text{Reading}(P) = 3\ 6\ (10)\ 1\ 2\ 5\ 8$

3	6	10	
1	2	4	8

$\leftarrow (5)$

$3\ 6\ (10)\ (5)\ 1\ 2\ 4\ 8$

3	5	10	
1	2	4	8

$\leftarrow (6)$

$(6)\ 3\ 5\ (10)\ 1\ 2\ 4\ 8$

6			
3	5	10	
1	2	4	8

$a_1\ a_2 \cdots (a_r) \cdots a_p \leftarrow x$

$a_r\quad a_1\ a_2 \cdots x \cdots a_p$

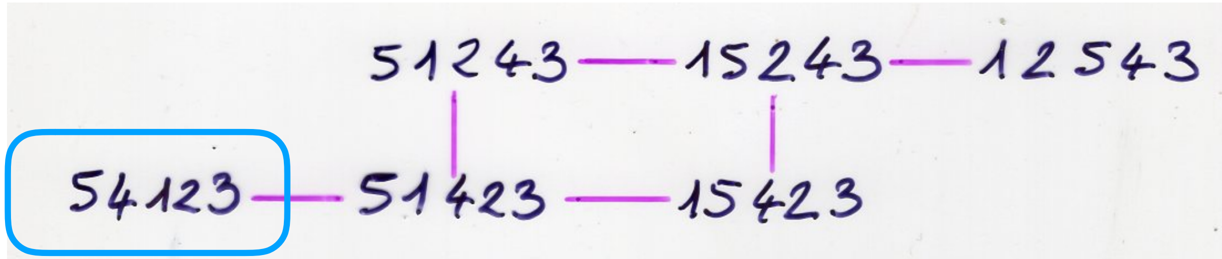
$a_1 < a_2 < \cdots a_{r-1} < x < a_r \cdots < a_p$

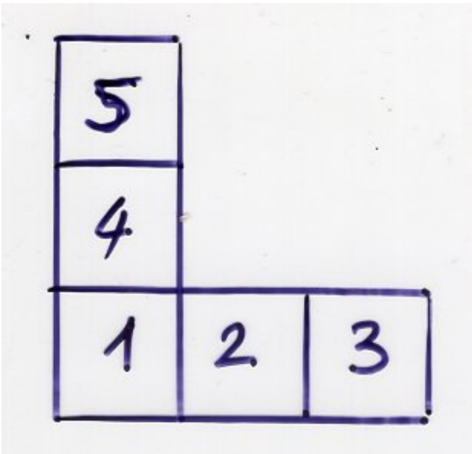
Proposition

Two permutations are Knuth equivalent iff their insertion tableaux coincide

$$\sigma \overset{K}{\sim} \tau \iff P(\sigma) = P(\tau)$$

Corollary Each Knuth equivalence class contains exactly one reading word of a Young tableau





Knuth equivalence class

regular permutations

some slides have been added after the video Recording

Definition regular permutation σ
iff there exist a Young tableau T
such that :

- each row of T is an increasing subsequence of σ
- each column of T, written a word reading the column from top bottom, is a decreasing subsequence of σ

T =

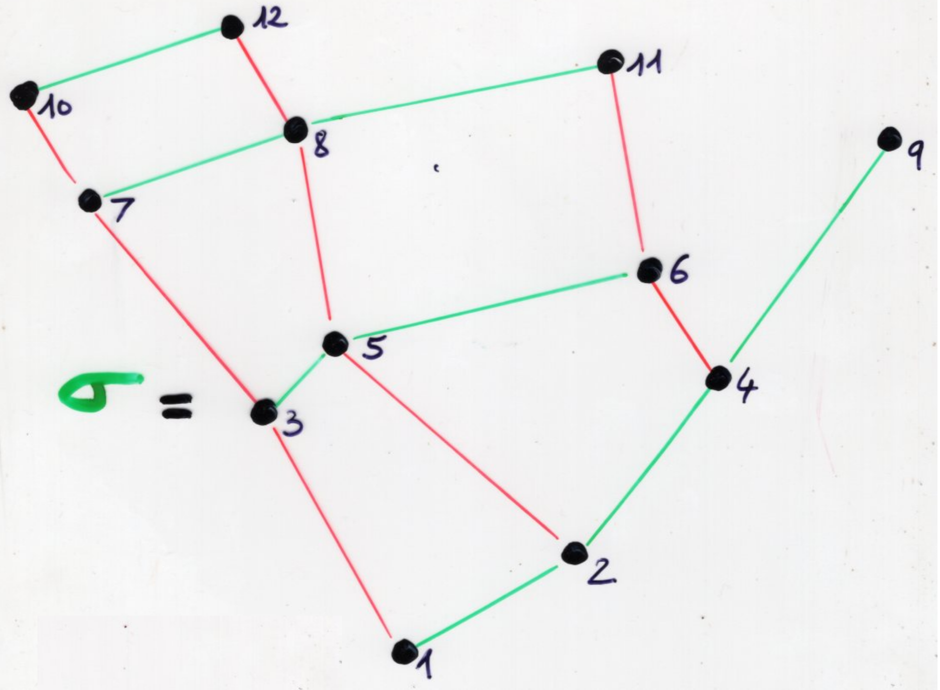
10	12		
7	8	11	
3	5	6	
1	2	4	9

example

$$\sigma = \begin{pmatrix} (10) & 7 & (12) & 3 & 8 & 5 & 1 & 2 & (11) & 6 & 4 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

"regular" permutation

σ =



T =

10	12		
7	8	11	
3	5	6	
1	2	4	9

<u>example</u>

$$\sigma = \begin{pmatrix} 10 & 7 & 12 & 3 & 8 & 5 & 1 & 2 & 11 & 6 & 4 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

<u>Proposition</u> For σ regular permutation, the insertion tableau $P(\sigma) = T$

- regular permutations associated to a Young tableau T are in bijection with a labeling of λ withe the integers $1, 2, \dots, n = |T|$ such that:

$\lambda = shape(T)$
(Ferrers diagram)

  - labels go increasing in rows (from left to right)
  - labels go increasing in columns (from top to bottom)

<u>example</u>

$$\sigma = \begin{pmatrix} 10 & 7 & 12 & 3 & 8 & 5 & 1 & 2 & 11 & 6 & 4 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

corresponds to

1	3		
2	5	9	
4	6	10	
7	8	11	12

- The set of regular permutations associated to T is a subset of Knuth equivalence class of permutations with $P(\sigma) = T$.
  (this class in bijection with Young tableaux with shape λ)

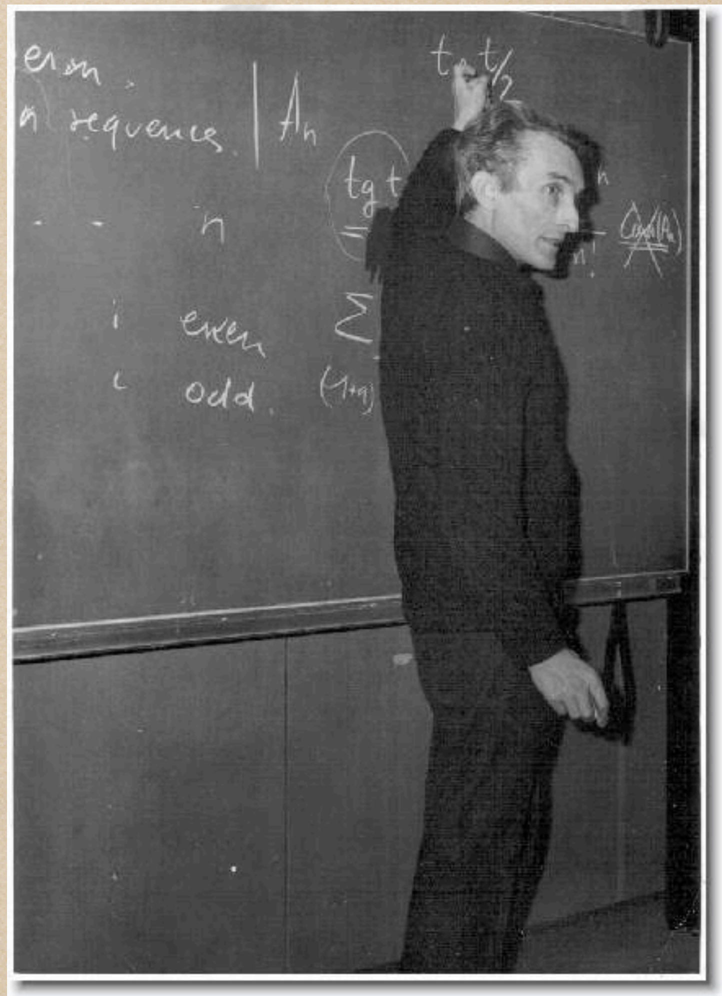
The reading word w of T is a (very particular) regular permutation

- These two sets coincide iff the shape λ is rectangular

Jeu de taquin

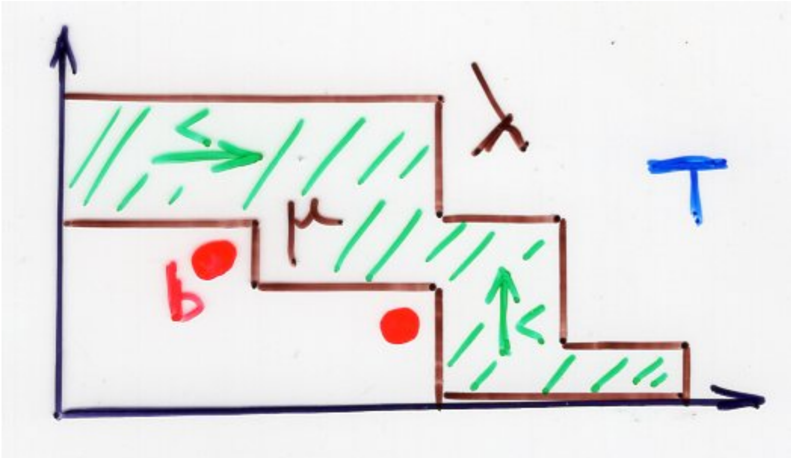
M.P. Schützenberger

(1976)



skew Ferrers λ/μ

(standard) Young tableau
skew shape λ/μ



jeu de taquin slide of T into b

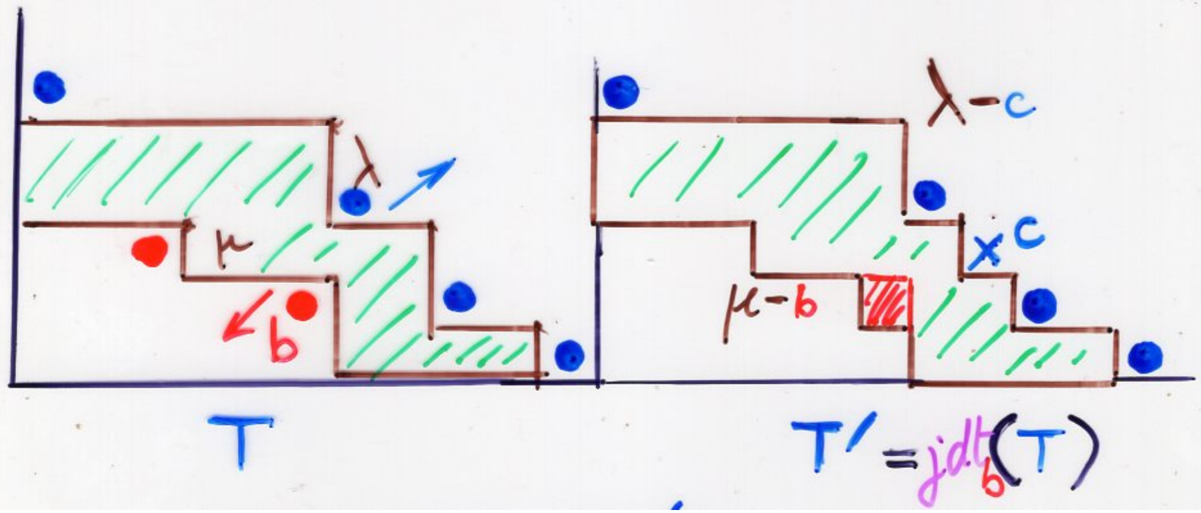
$(T, b) \longrightarrow jdt_b(T)$

b ● box

Definition

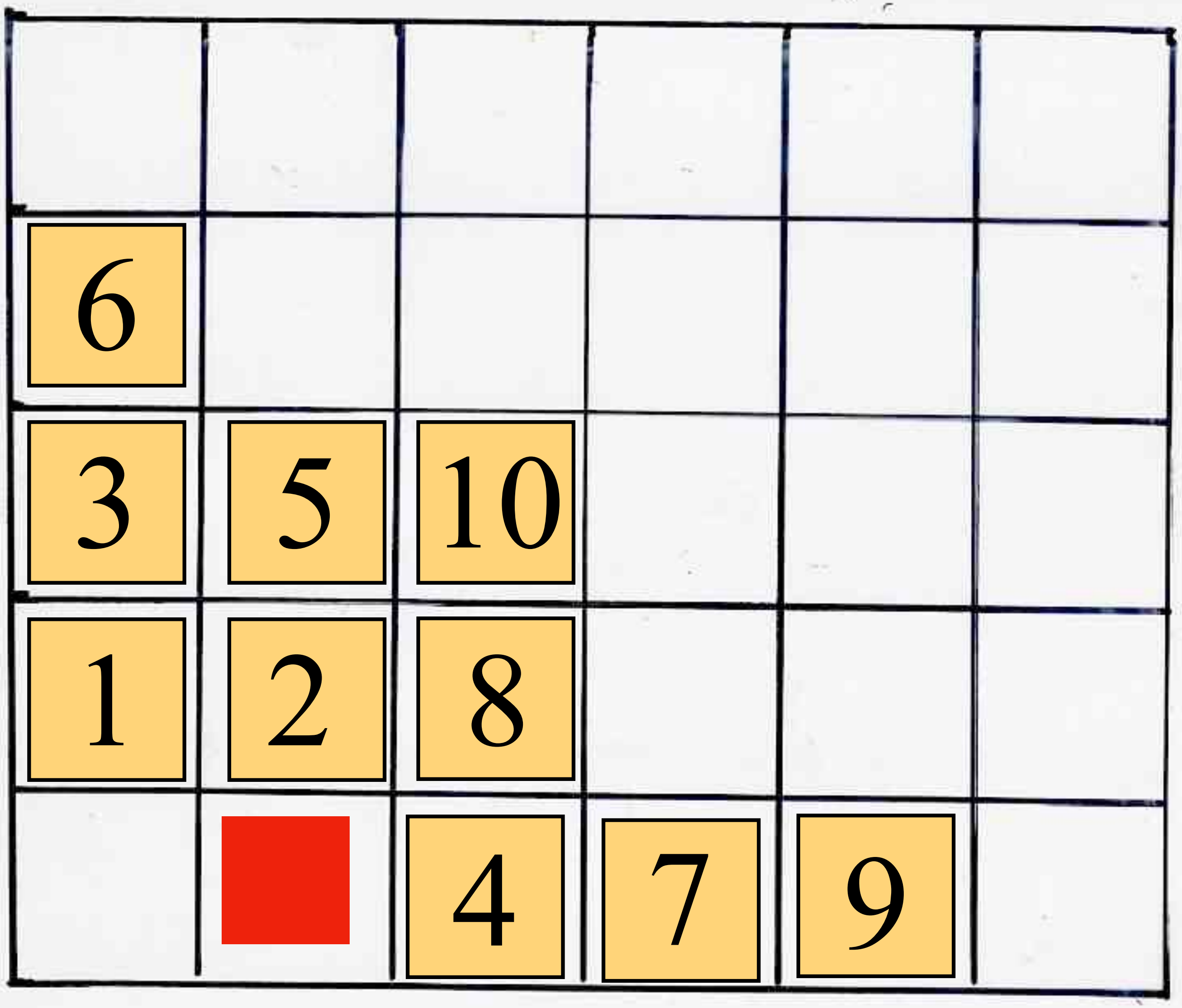
Two tableaux T and T' are called jeu de taquin equivalent iff one can be obtained from another by a sequence of ~~je~~ jeu de taquin slides

$$T \overset{jdt}{\sim} T'$$



symmetric relation

$$T = jdt_c(T')$$


6
3
5
10
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8
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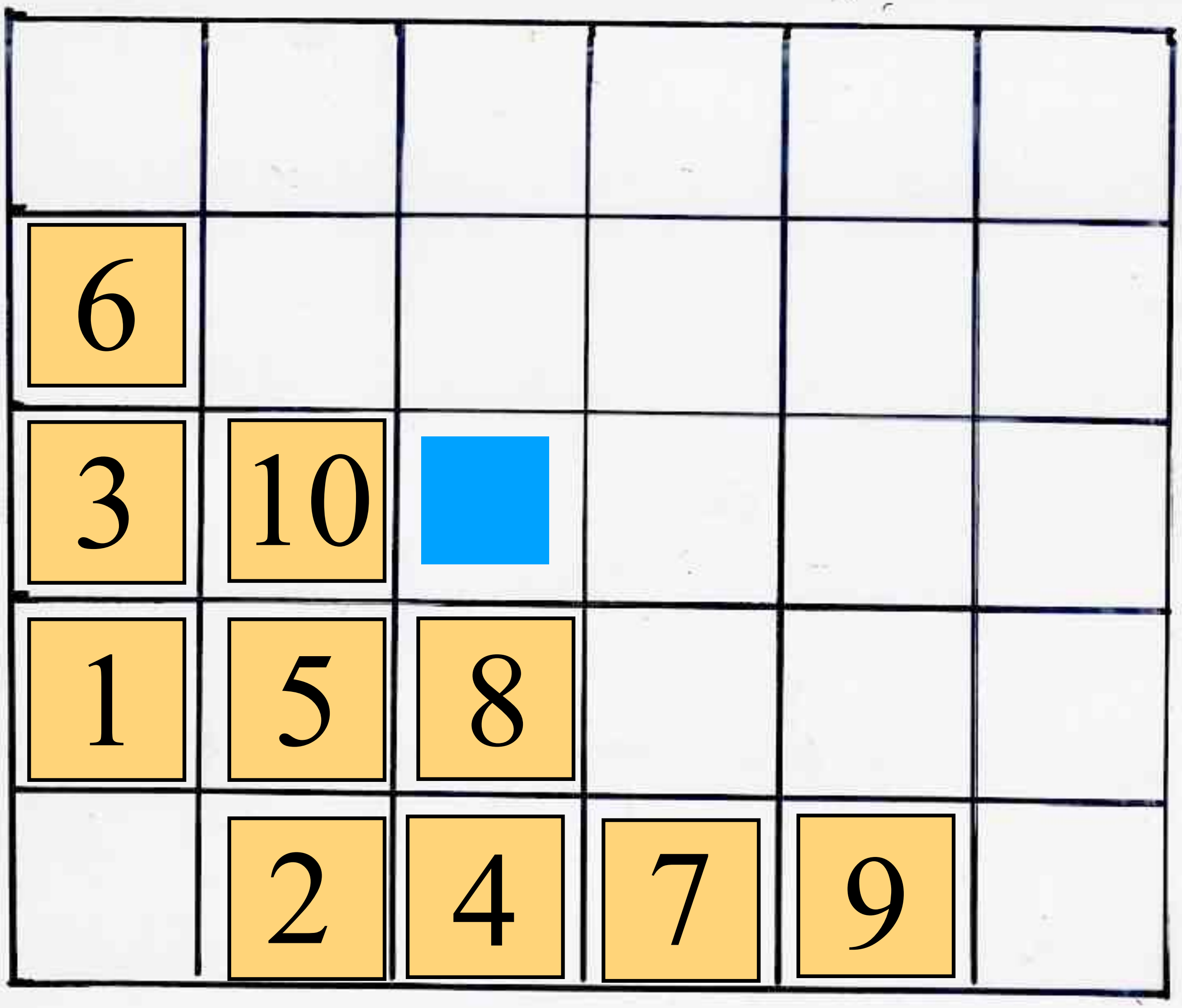
7
9


6
3
10
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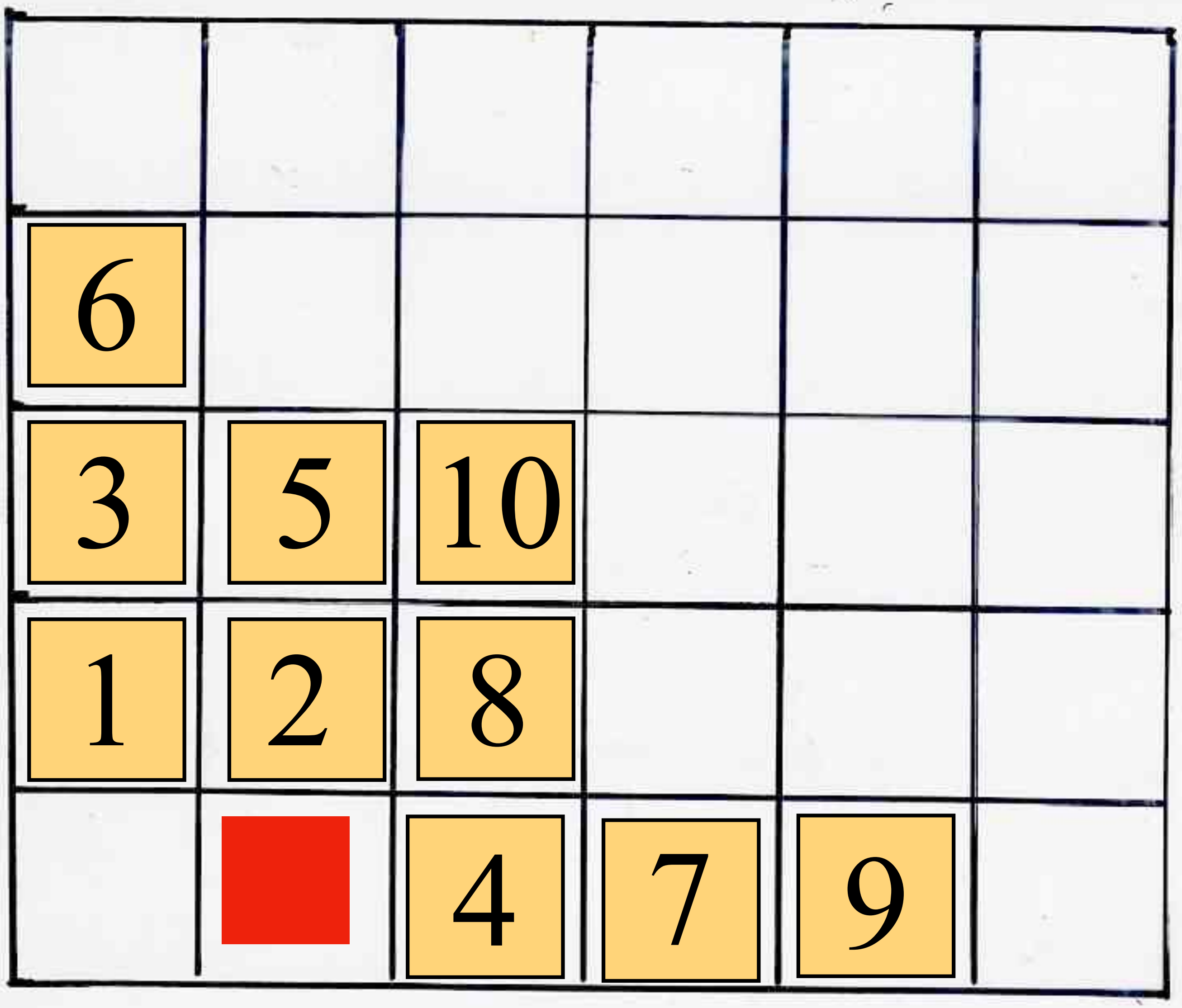
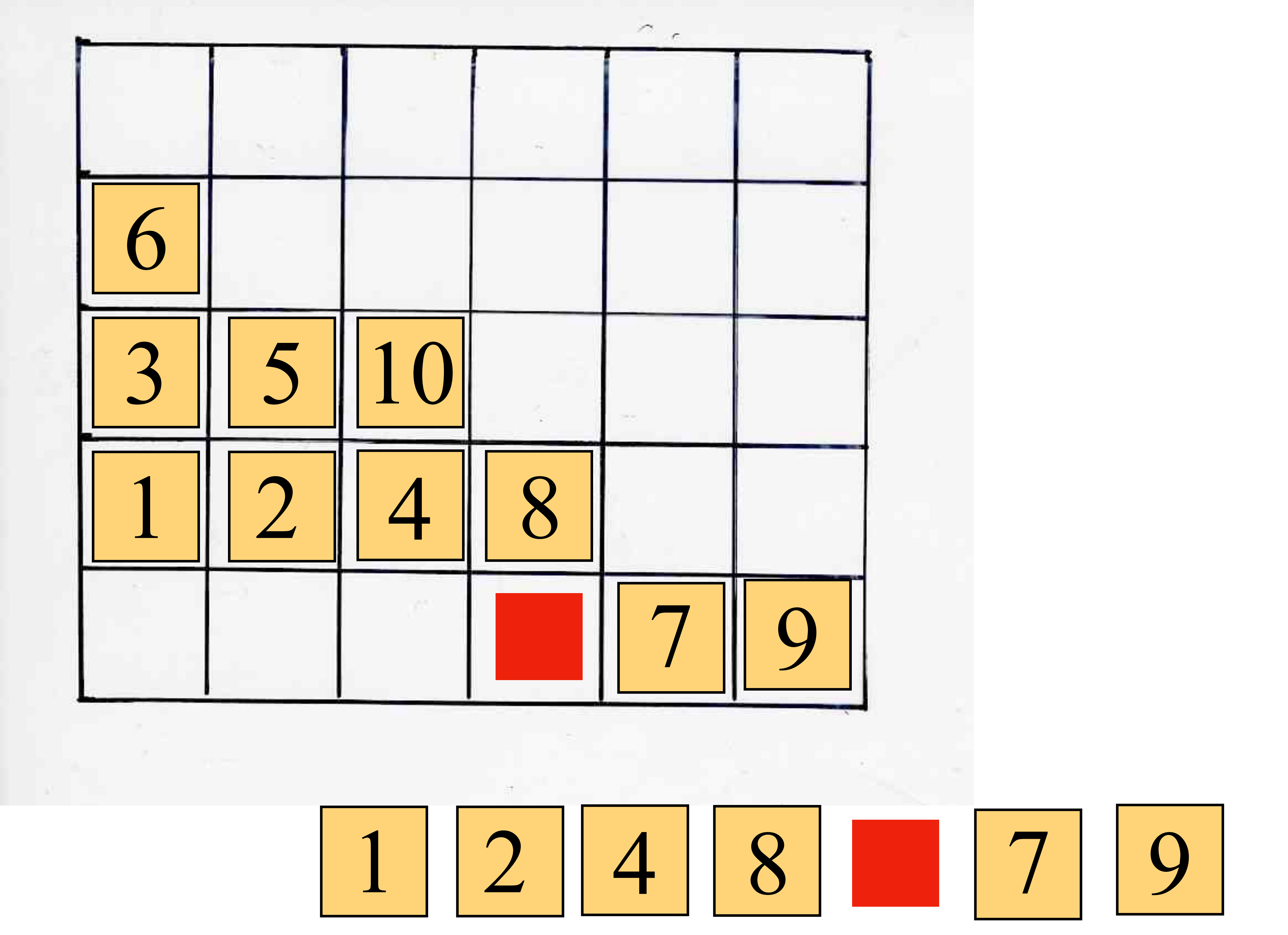
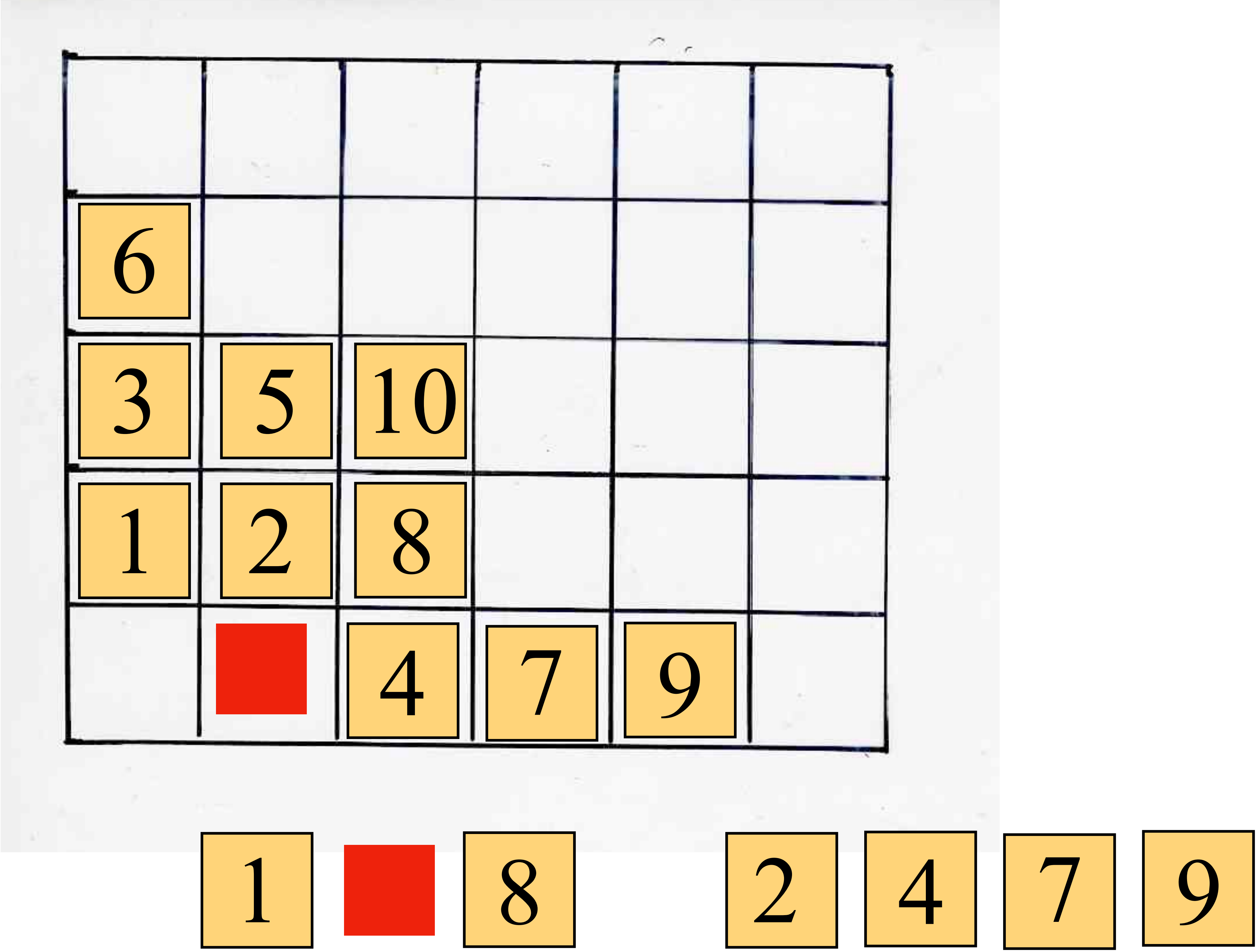

6
3
5
10
1
2
8
4
7
9

Lemma

Each jeu de taquin slide converts the Reading word of a tableau into a Knuth equivalent one.

$$\text{Reading}(\text{jdt}_b(T)) \overset{K}{\sim} \text{Reading}(T)$$


6
3 5 10
1 2 4 8
7 9
1 2 4 8 7 9


6
3
5
10
1
2
8
4
7
9
1
8
2
4
7
9

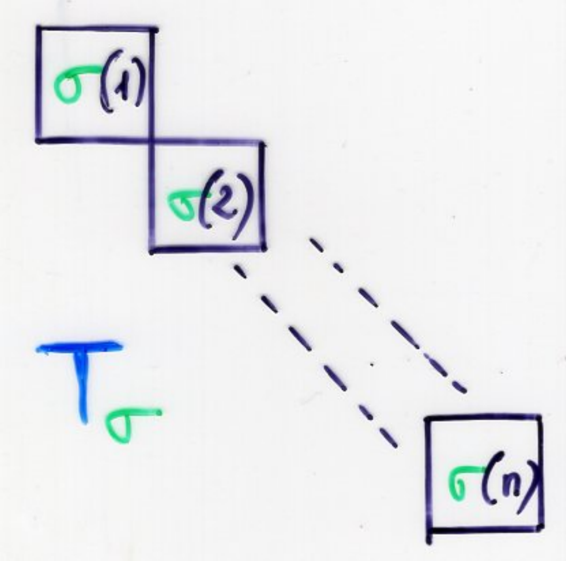
Proposition

Each jeu de taquin equivalence class contains exactly one straight-shape tableau

jdt(T) denote this unique straight-shape tableau in the jeu de taquin equivalence class of T

Corollary For $\sigma = \sigma(1)\cdots\sigma(n) \in \mathfrak{S}_n$ permutation

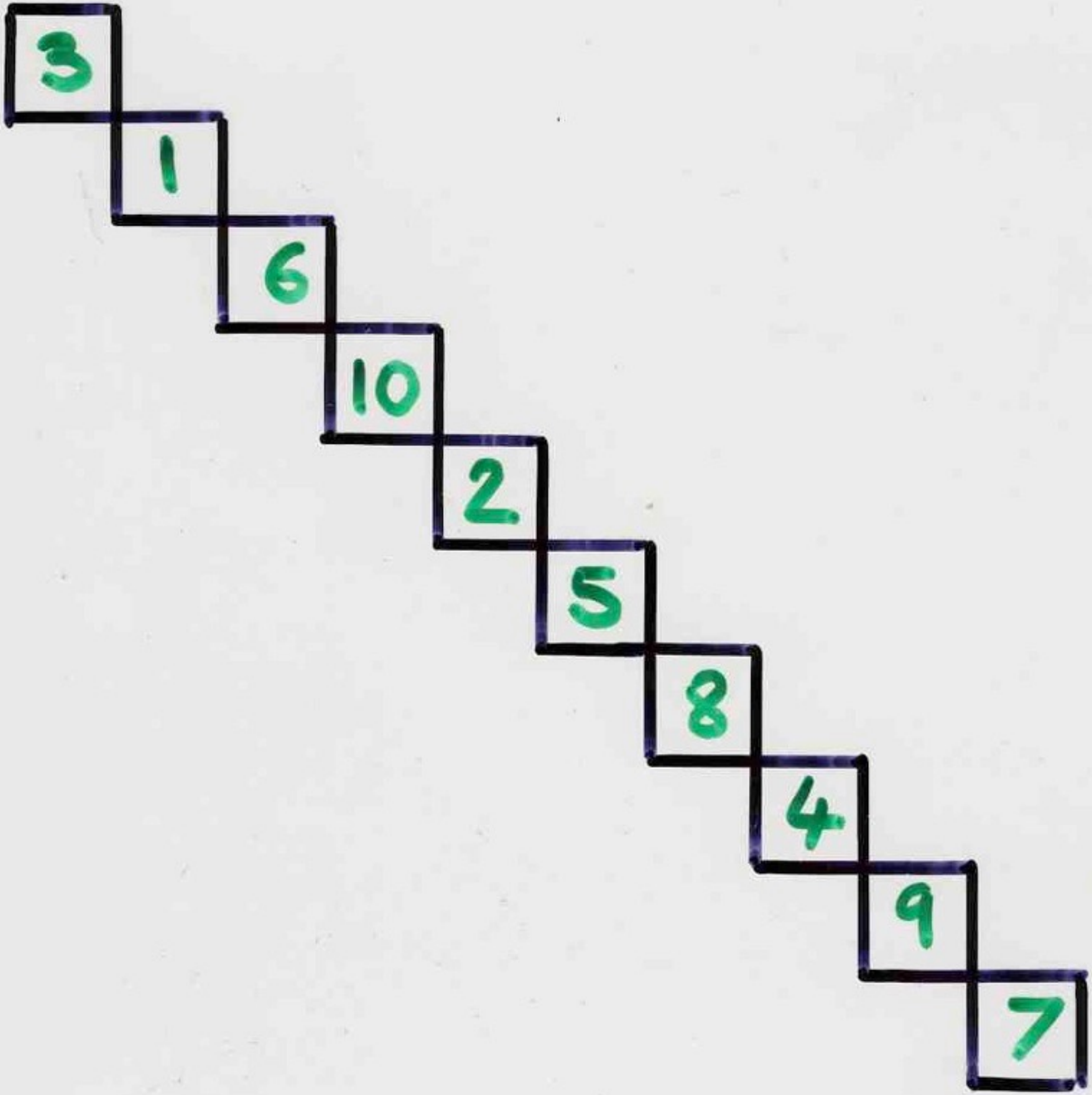
denote T_σ the skew tableau

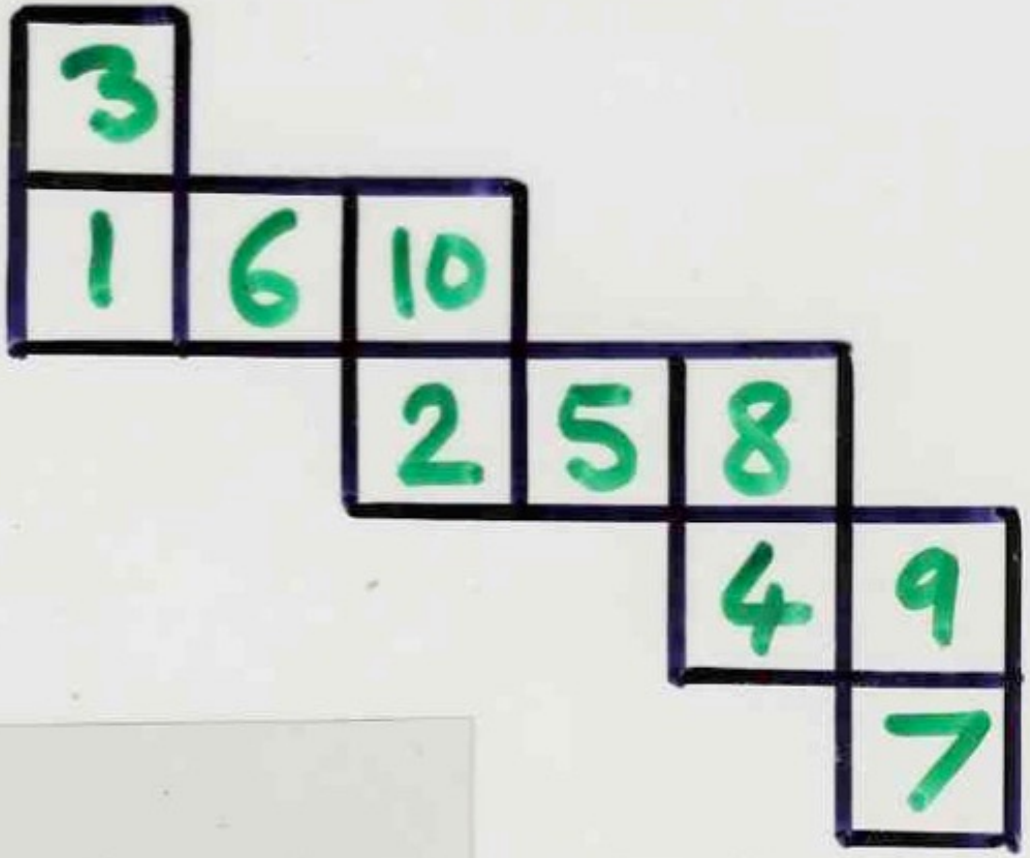


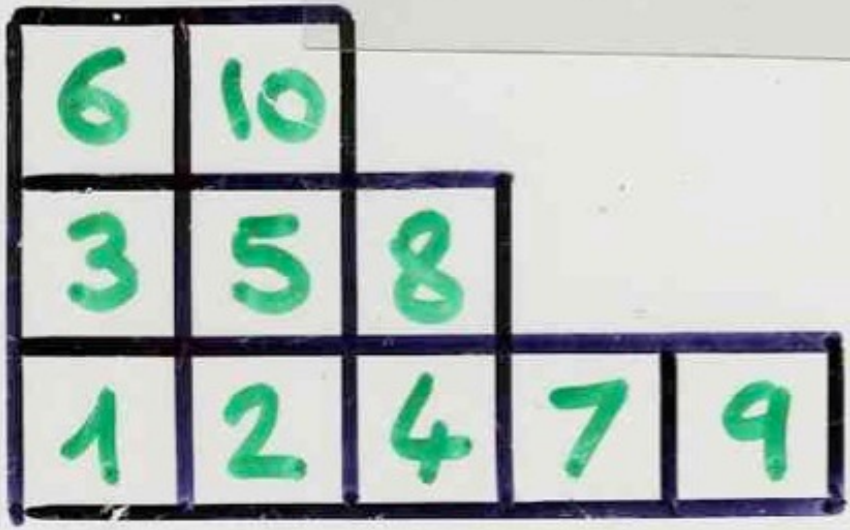
$$\sigma \xrightarrow{RS} (P, Q)$$

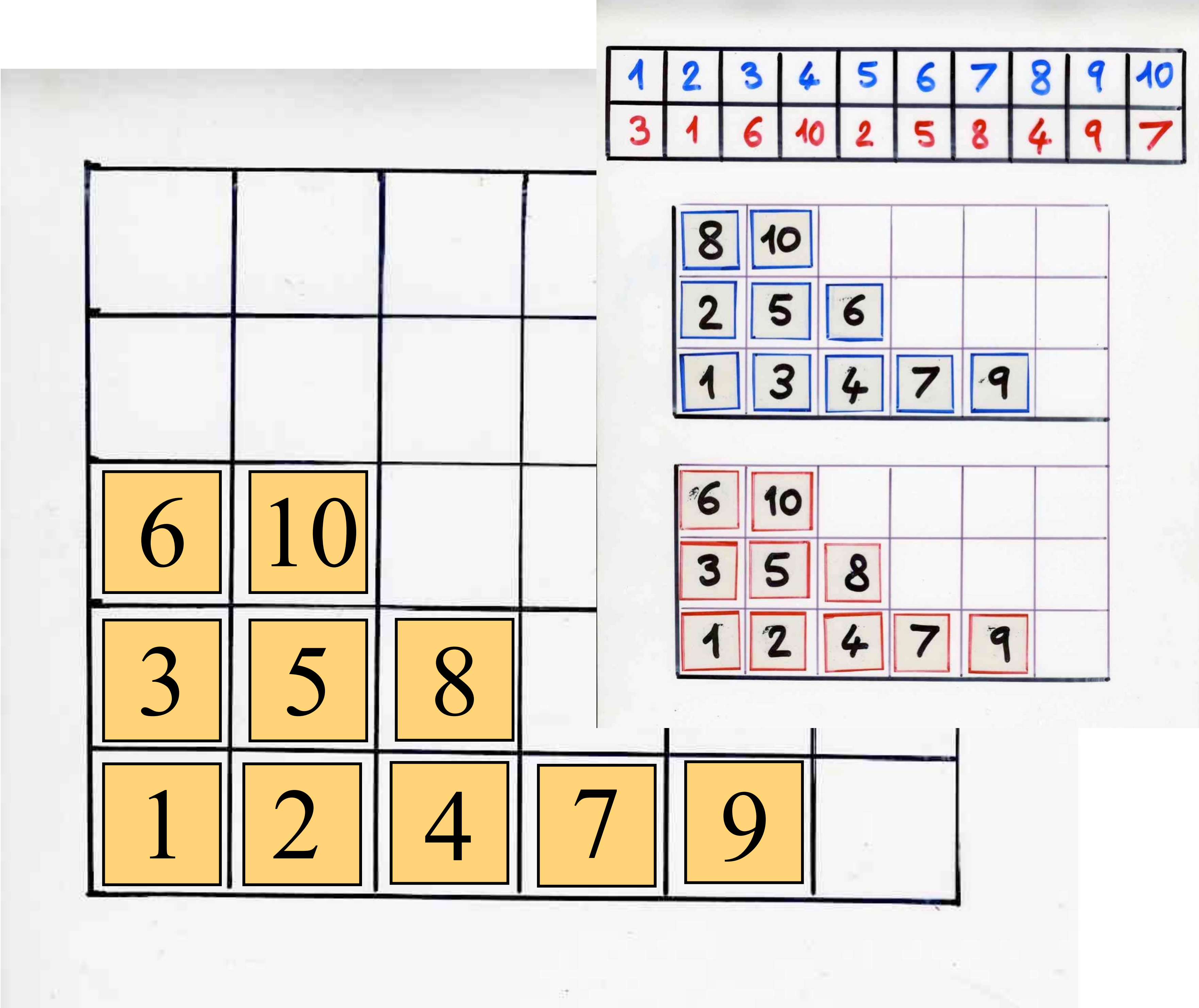
Then $\text{jdt}(T_\sigma) = P$

P insertion tableau of σ


3
1
6
10
2
5
8
4
9
7


3
1 6 10
2 5 8
4 9
7


6 10
3 5 8
1 2 4 7 9

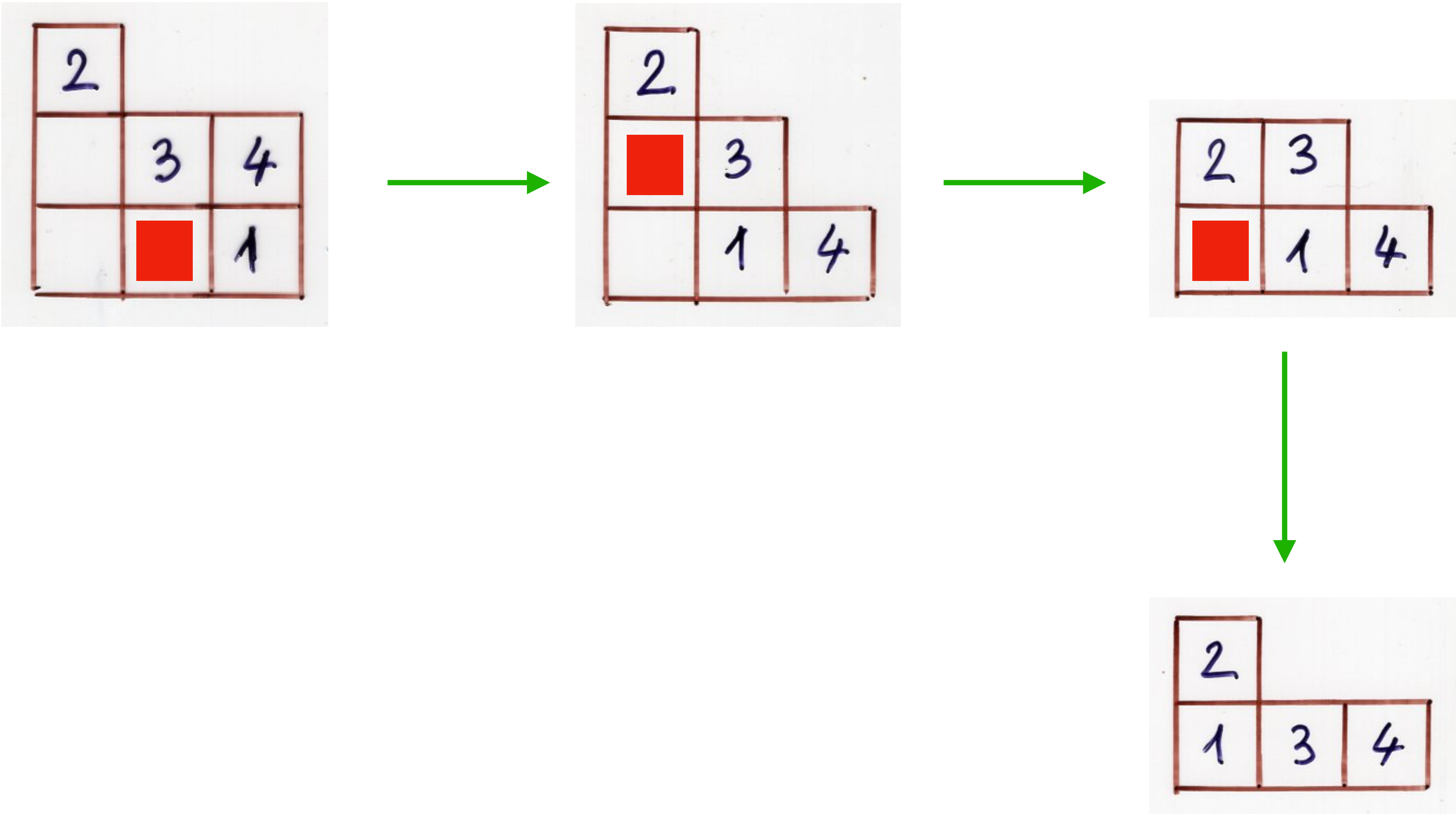

1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
8 10
2 5 6
1 3 4 7 9
6 10
3 5 8
1 2 4 7 9
6 10
3 5 8
1 2 4 7 9

Jeu de taquin

with growth diagrams

S. Fomin, 1986, 1994

Сергей Владимирович Фомин


2
3
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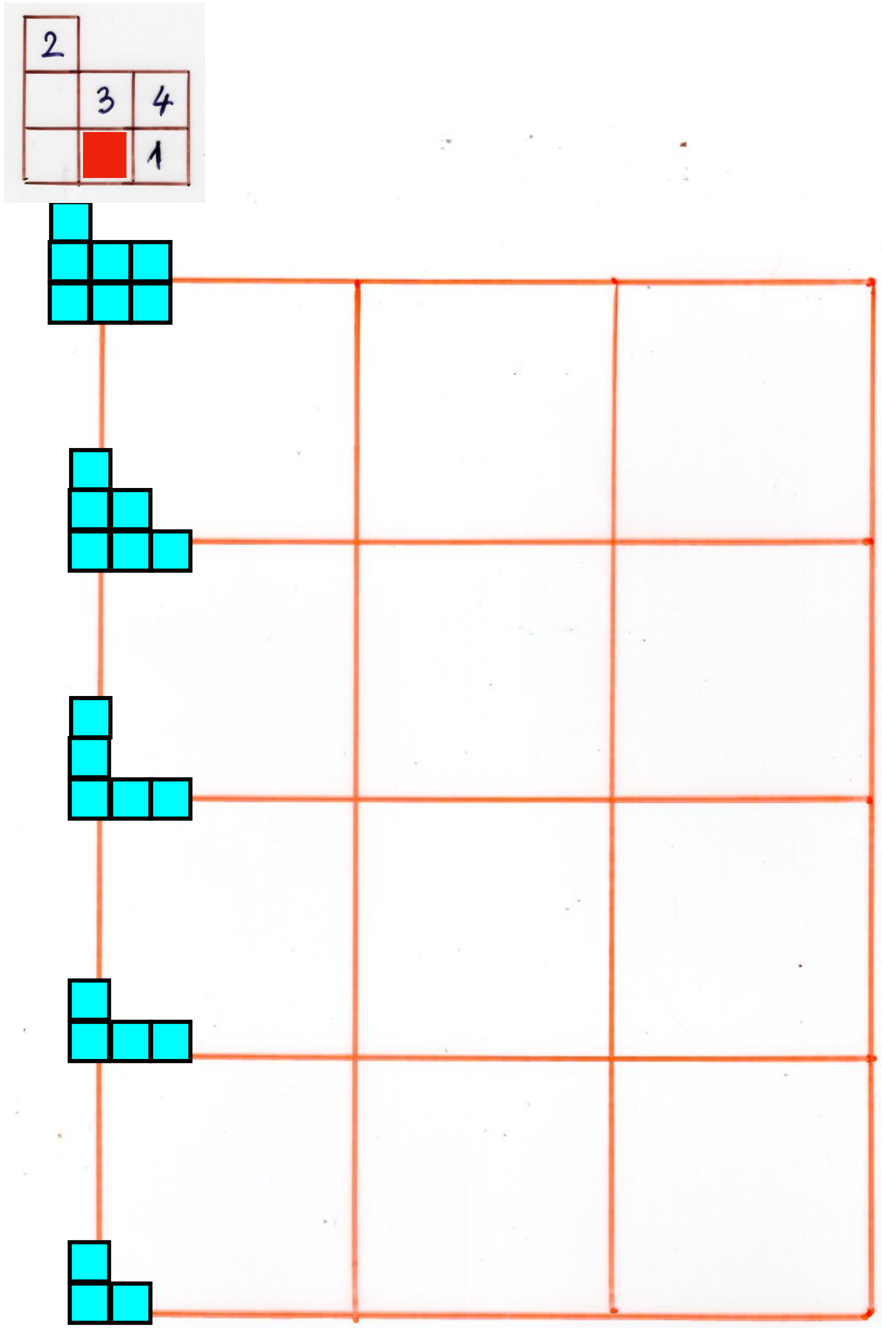
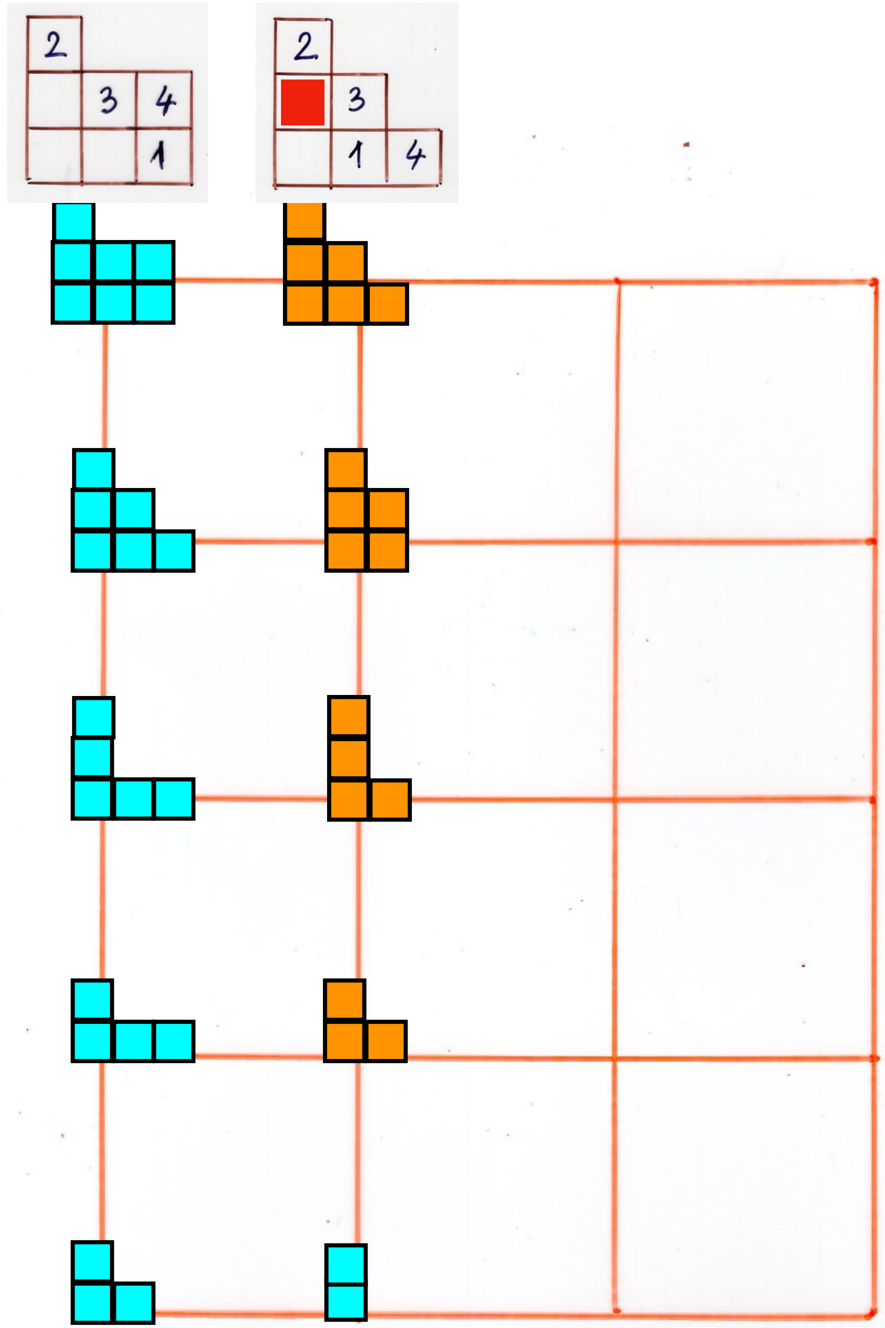
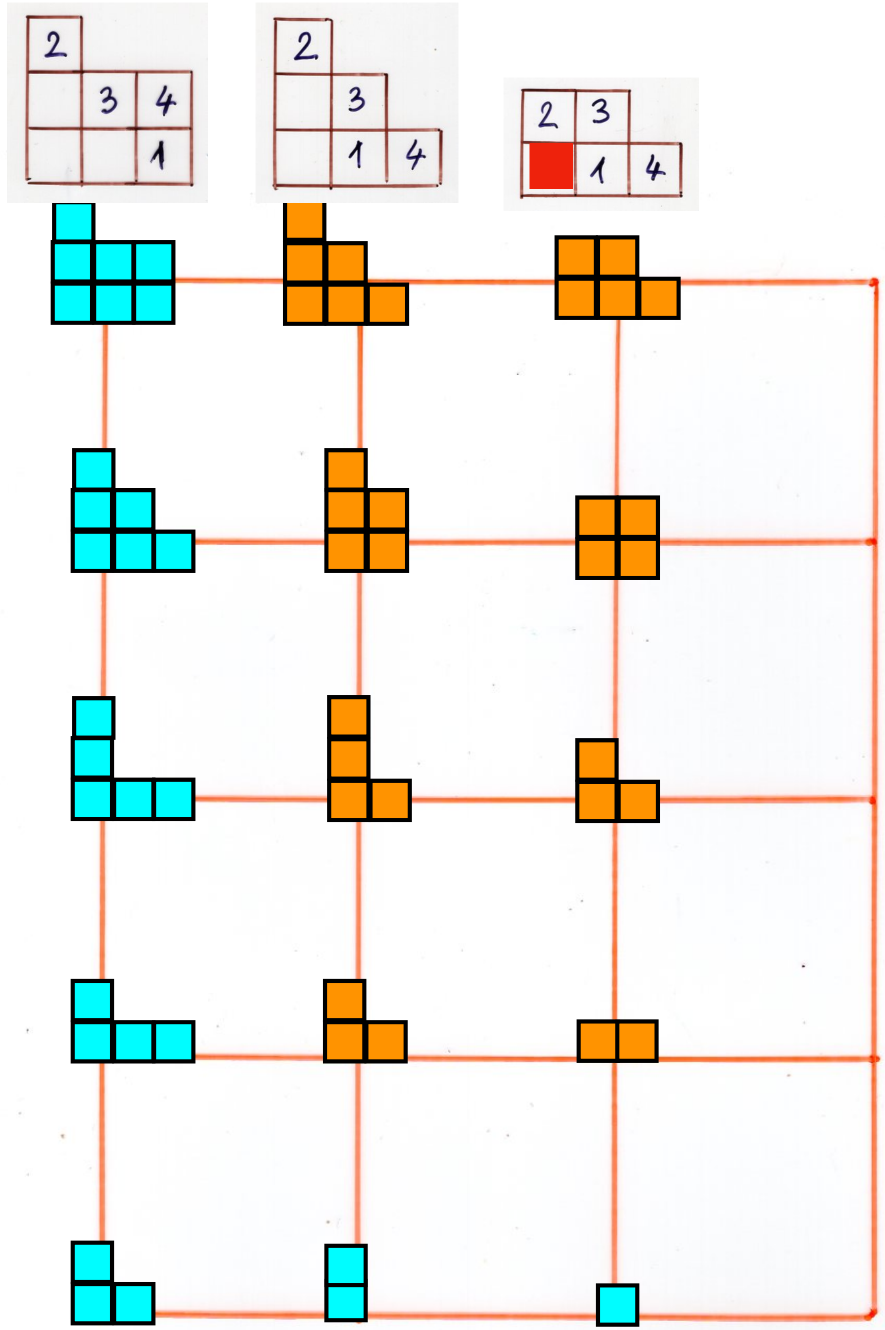
1
4
2
3
1
4
2
1
3
4

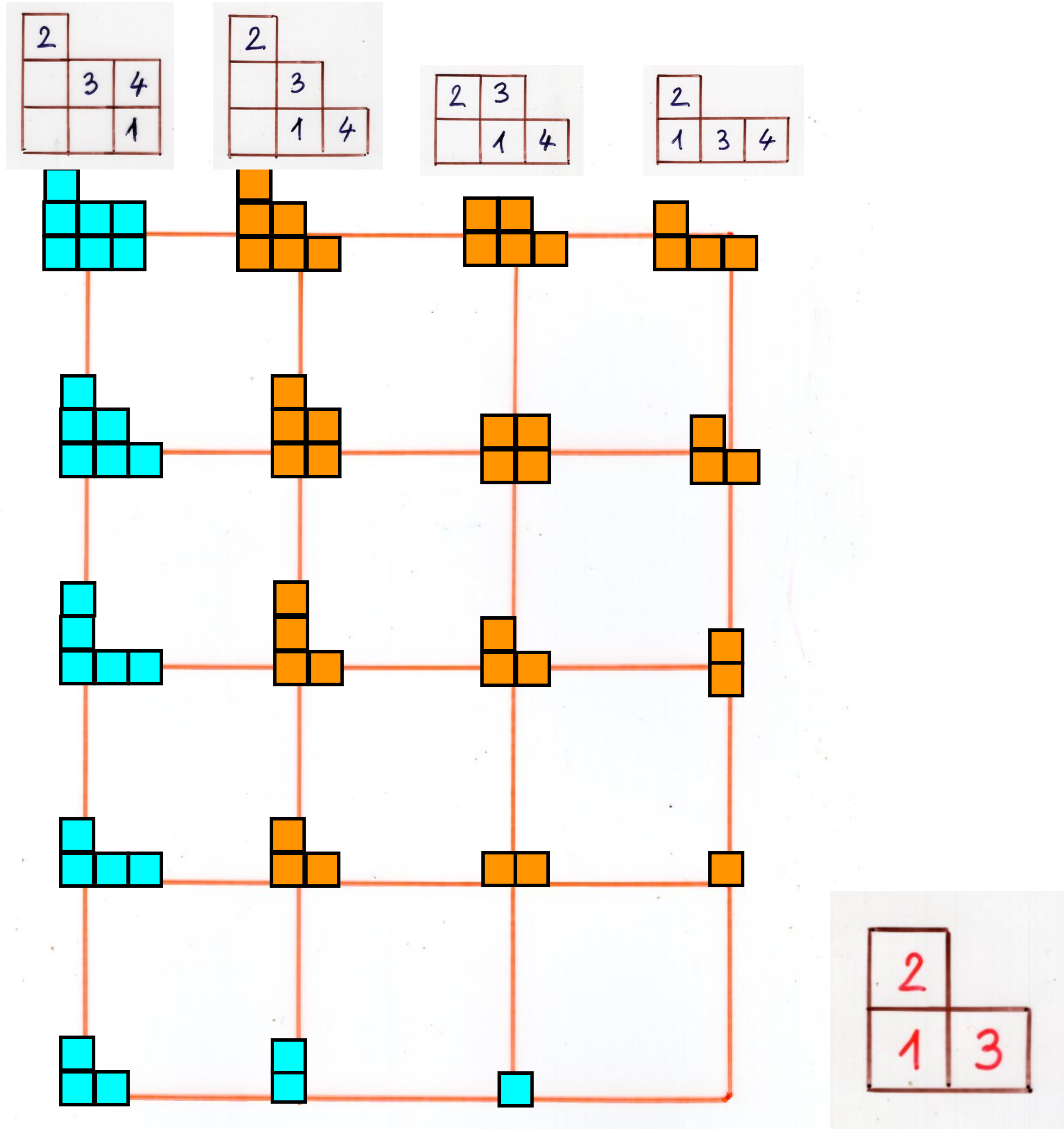


2
3
4
1

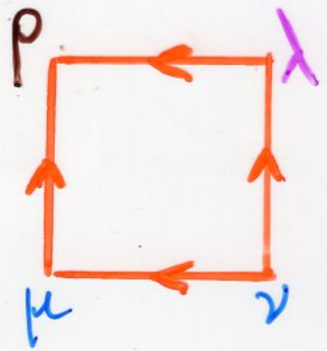

2
3 4
1
2
3
1 4



2
3 4
1
2
3
1 4
2 3
1 4


2
3 4
1
2
3
1 4
2 3
1 4
2
1 3 4
2
1 3

Proposition jeu de taquin local rules (Fomin)

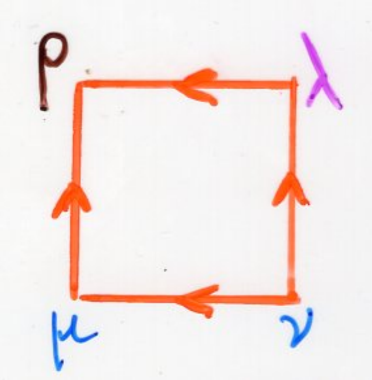


cell of the jeu de taquin growth diagram
(ρ covers μ and λ,
μ and λ cover ν)

Then λ is uniquely determined from μ, ν, ρ by the following "local rule":

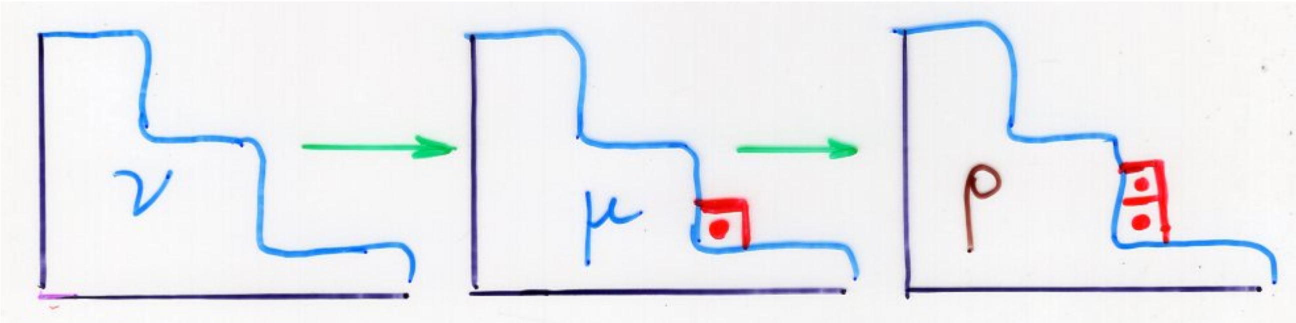
(i) • if μ is the only shape of its size that contains ν and is contained in ρ
then $\lambda = \mu$

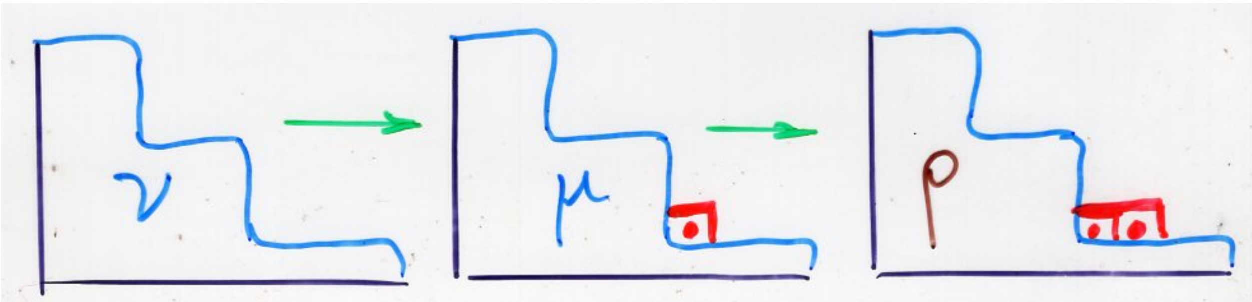
(ii) • otherwise there is a unique such shape different from μ, and this is λ

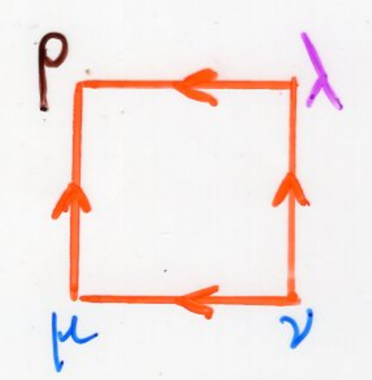


jeu de taquin
local rules

(i) • if μ is the only shape of its size that contains ν and is contained in ρ
then λ = μ

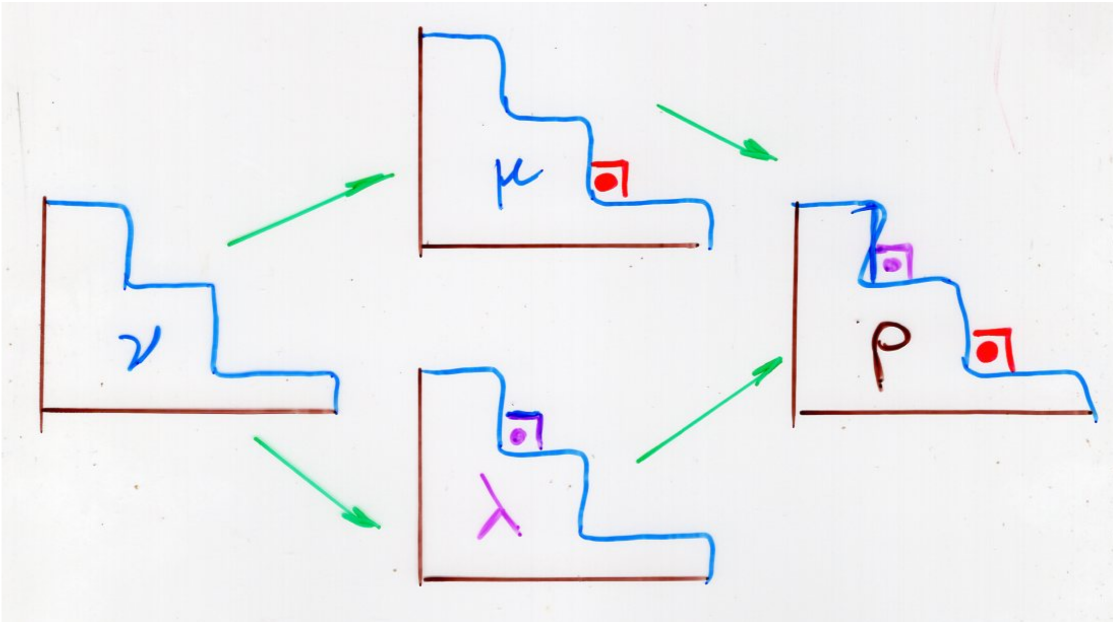


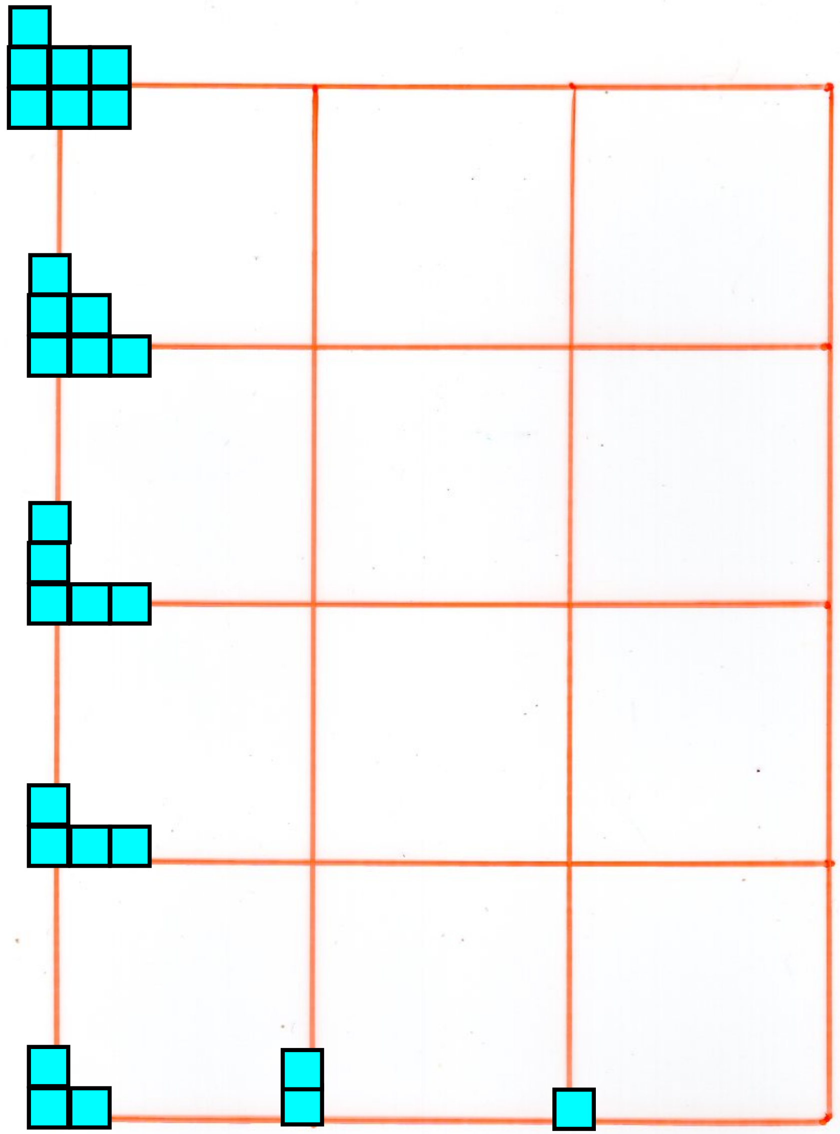


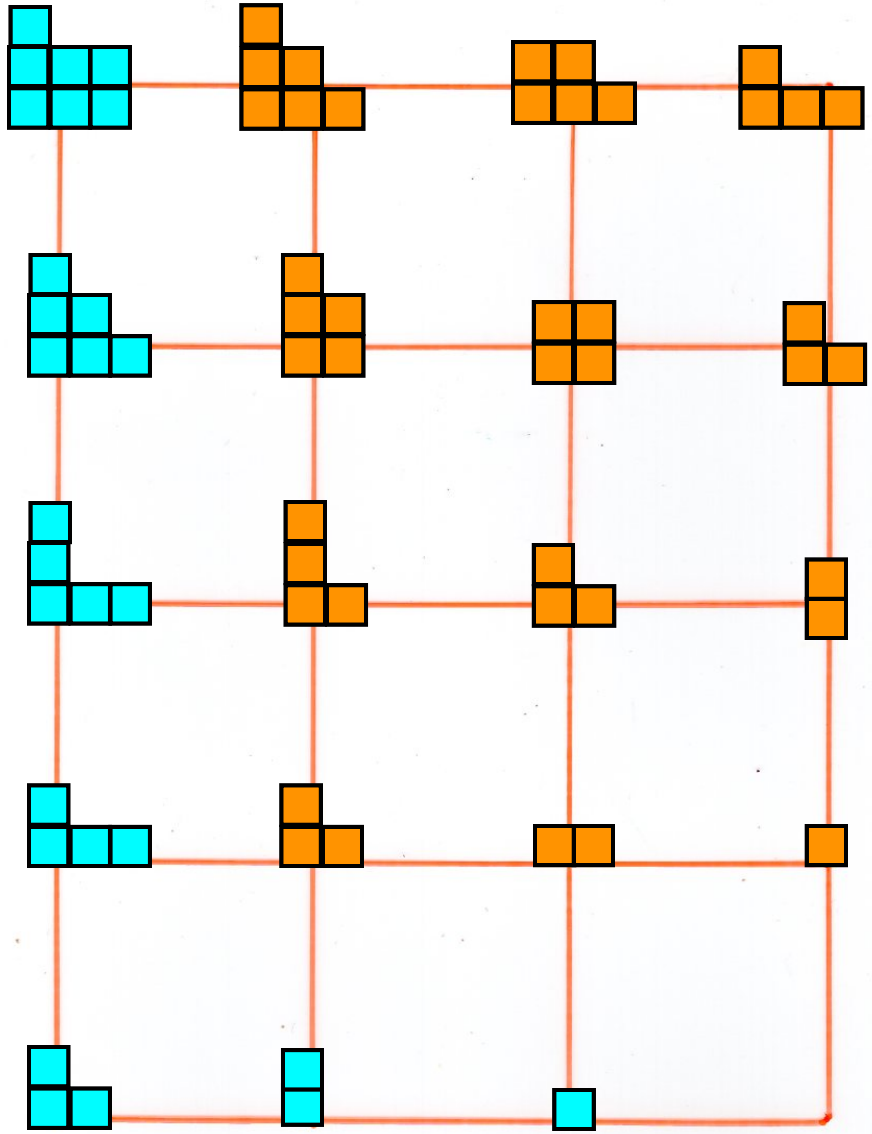


jeu de taquin
local rules

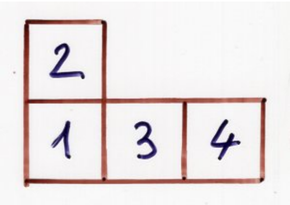
(ii) • otherwise there is a unique such shape different from μ, and this is λ



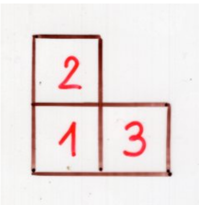


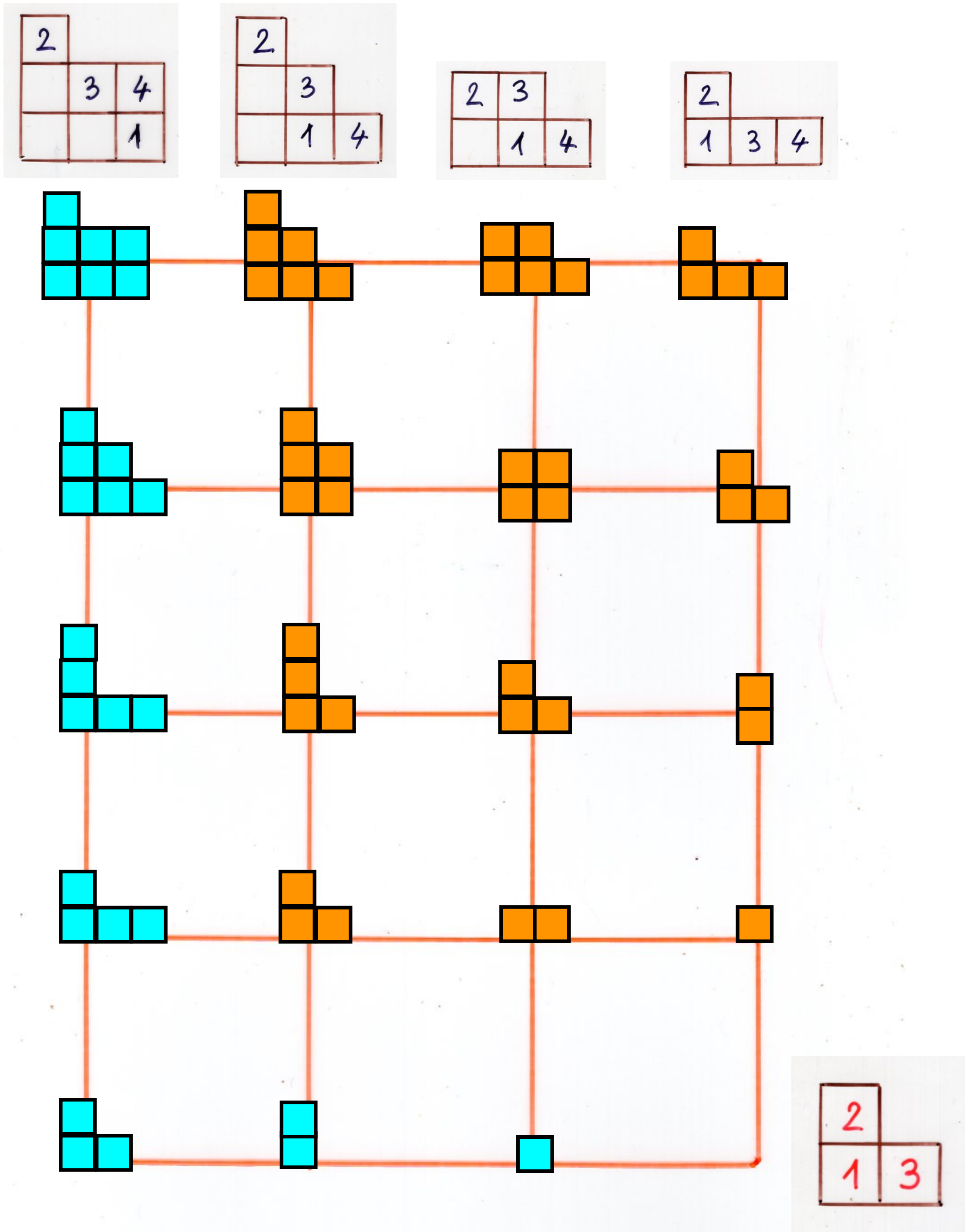


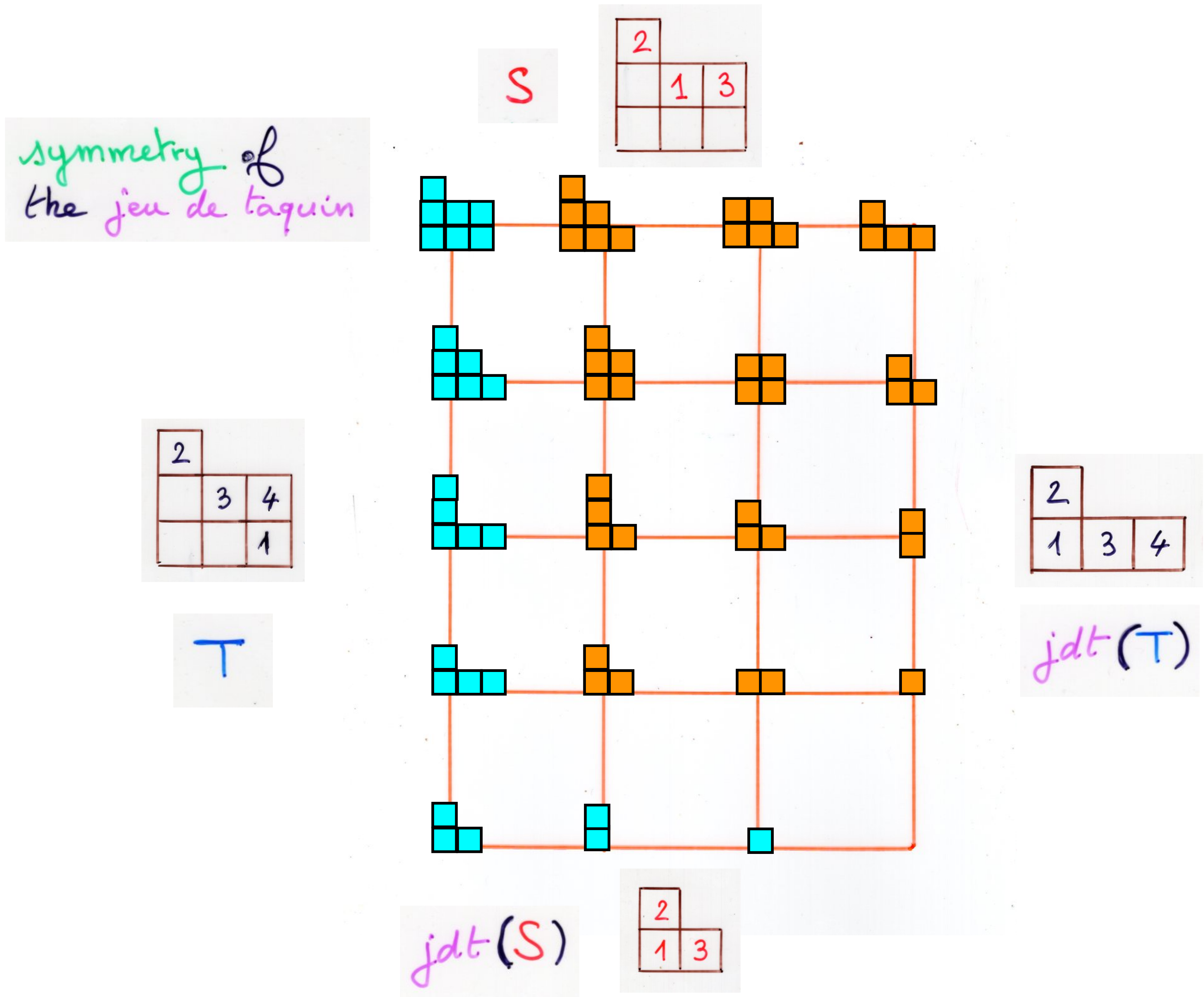
the tableau



is independant of the choice of the tableau





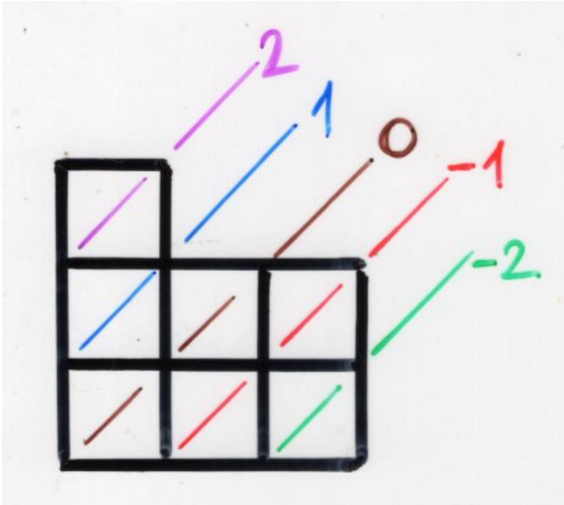

symmetry of the jeu de taquin
S
2
1 3
2
3 4
1
T
2
1 3 4
jdt(T)
jdt(S)
2
1 3

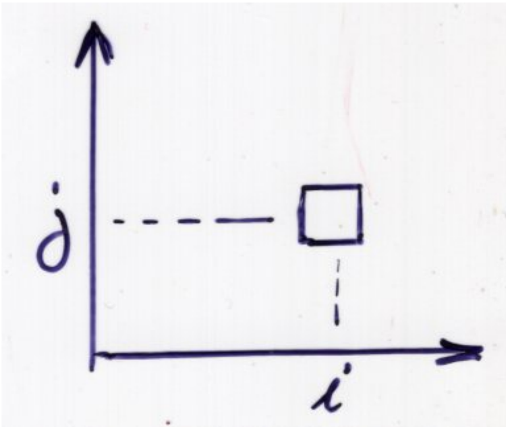
Jeu de taquin

with local rules on edges ?

diagonal operators

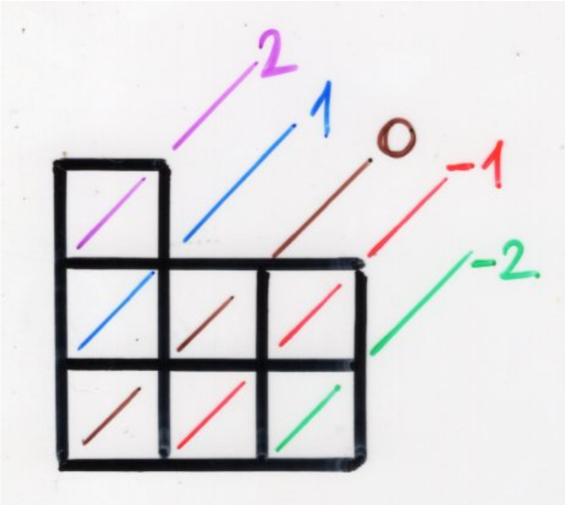
$\Delta_i \quad i \in \mathbb{Z}$





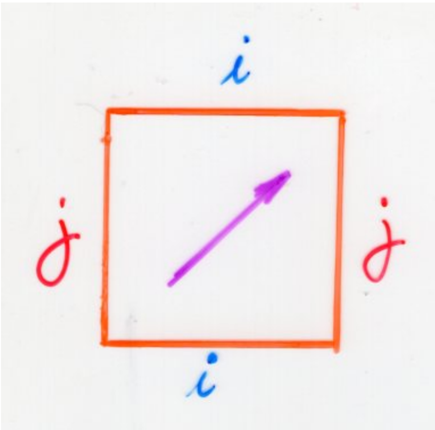
$(i, j) \longrightarrow j - i$

content

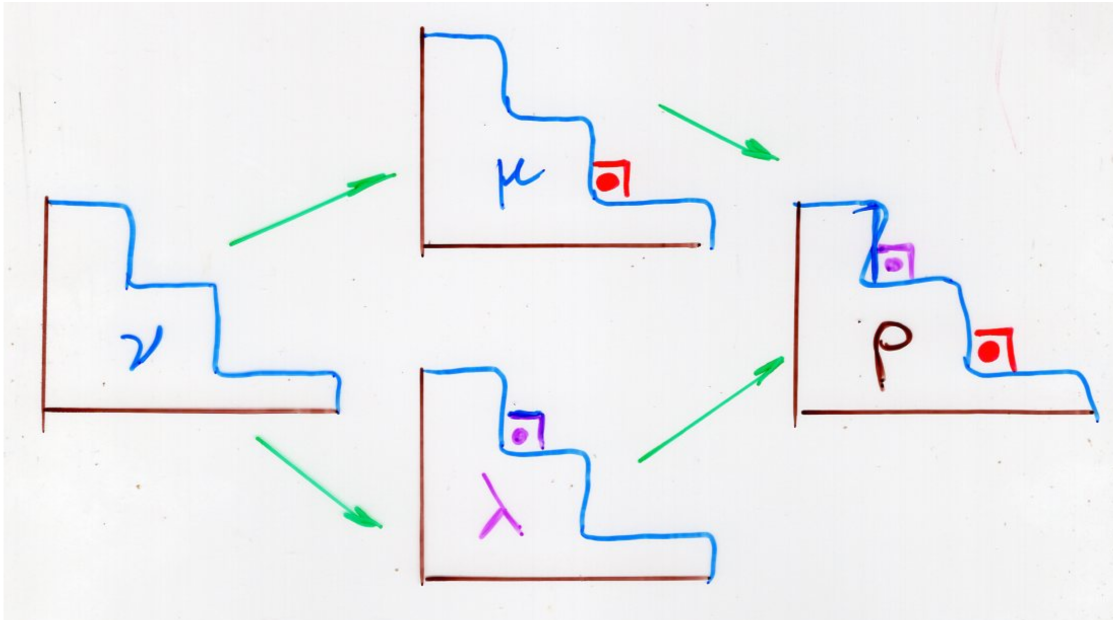


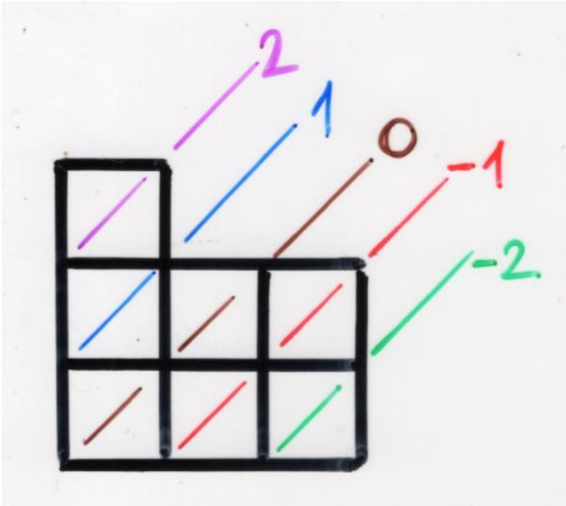
jeu de taquin

local rules on edges



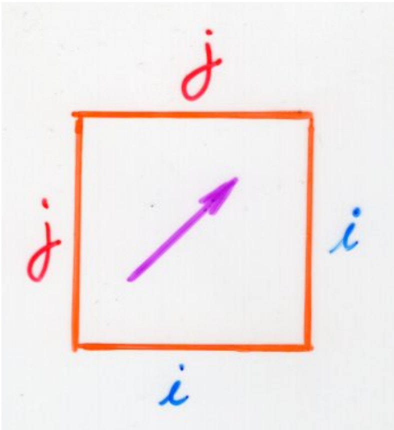
$|i - j| \geq 2$





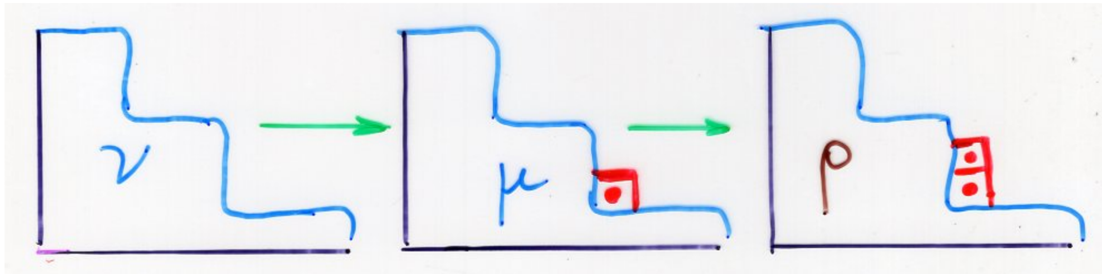
jeu de taquin

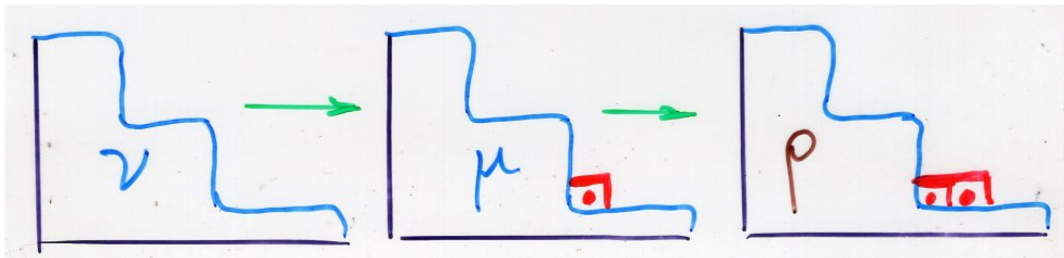
local rules on edges

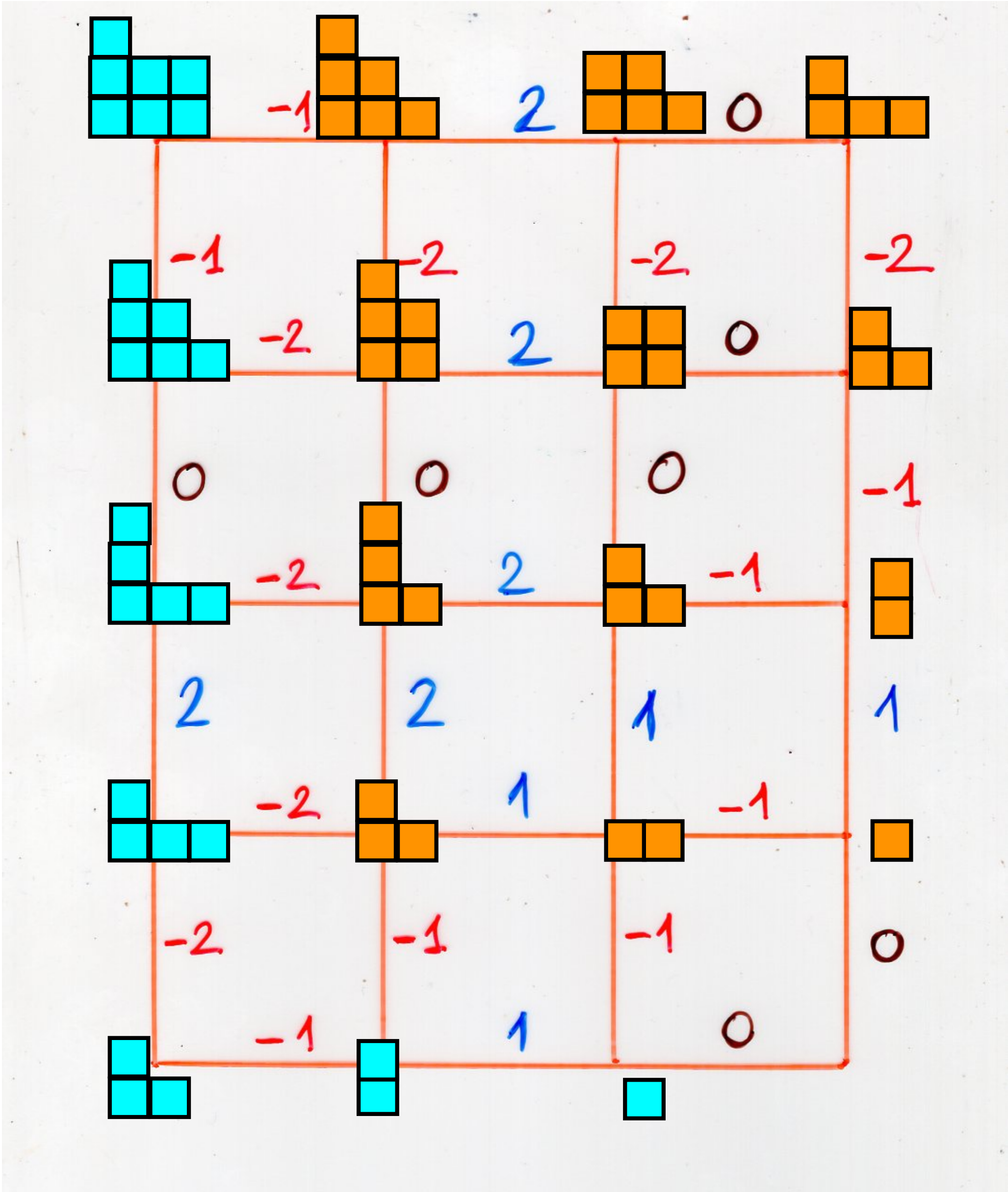


$|i-j| \leq 1$

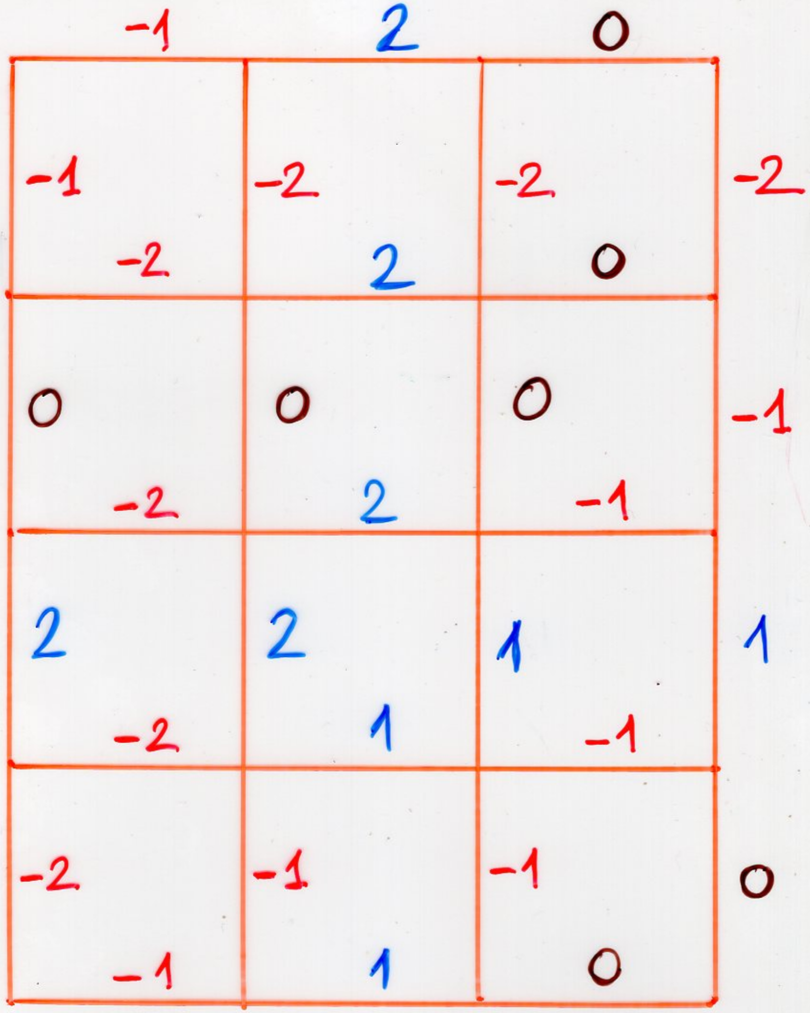
or





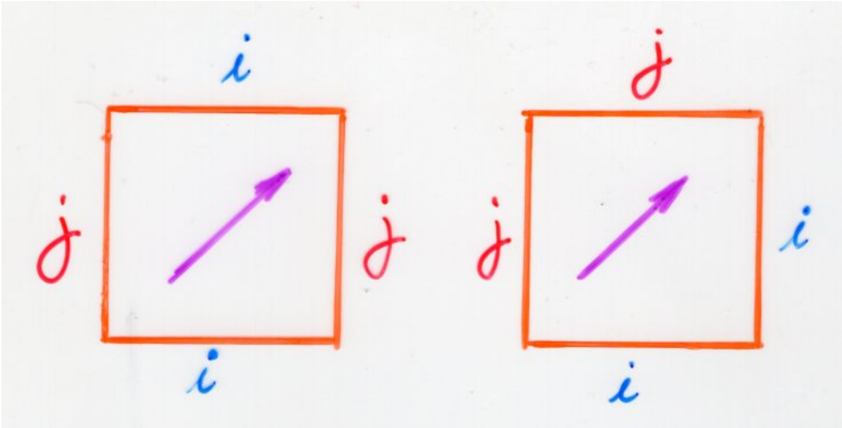


-1 2 0
-1 -2 -2 -2
-2 2 0
0 0 0 -1
-2 2 -1
2 2 1 1
-2 1 -1
-2 -1 -1 0
-1 1 0


-1 2 0
-1 -2 -2 -2
-2 2 0
0 0 0 -1
-2 2 -1
2 2 1 1
-2 1 -1
-2 -1 -1 0
-1 1 0

jeu de taquin

local rules on edges



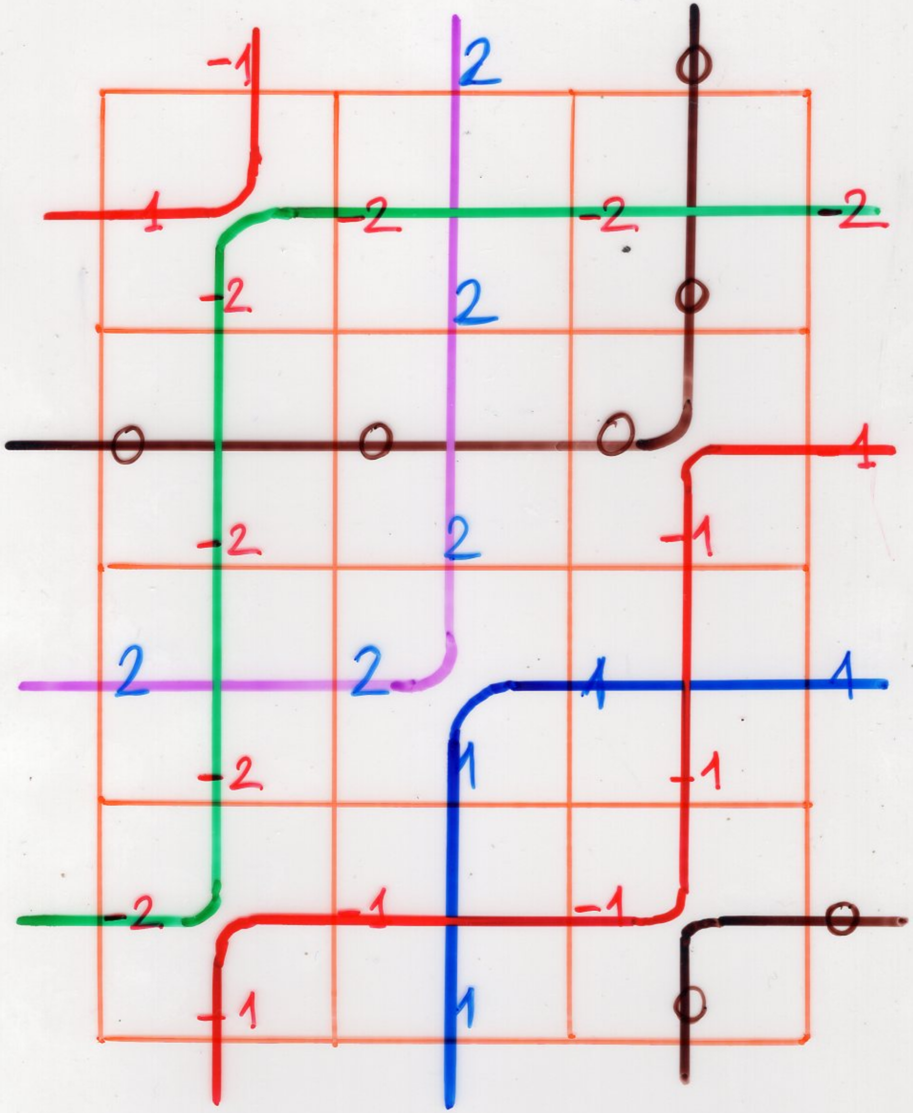
$i, j \in \mathbb{Z}$

$|i-j| \geqslant 2$ $|i-j| \leqslant 1$

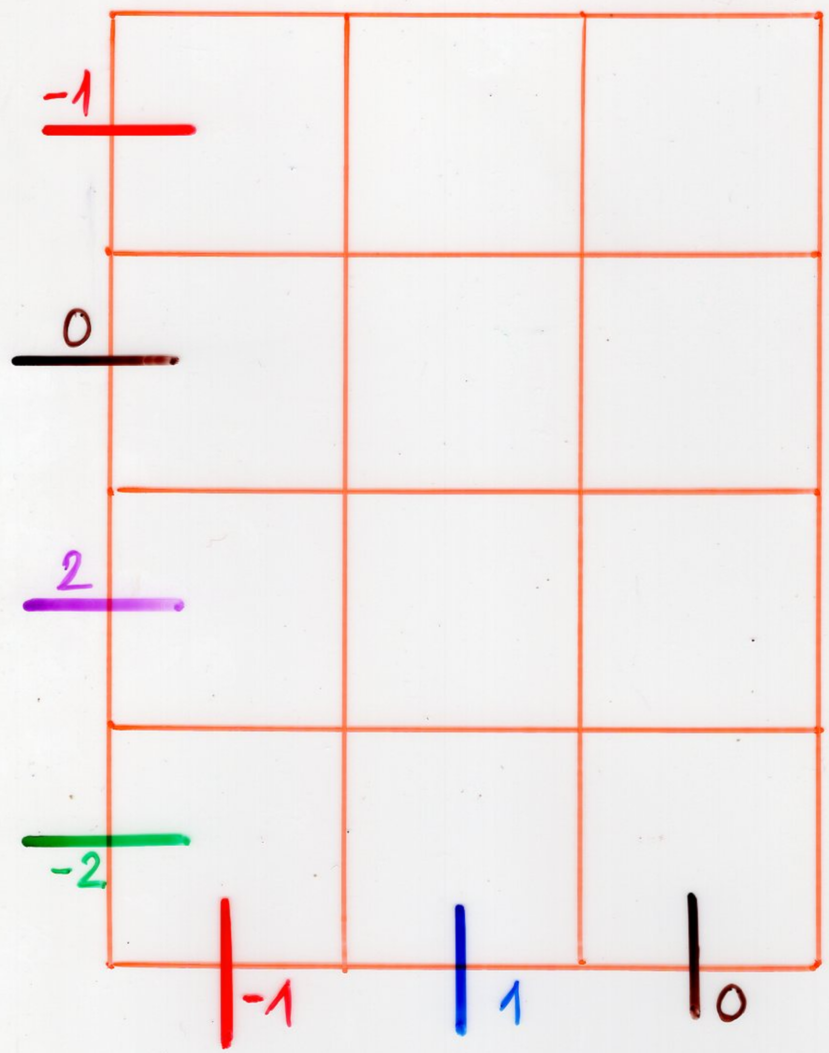
in fact here $i=j$ impossible

nil-Temperley-Lieb

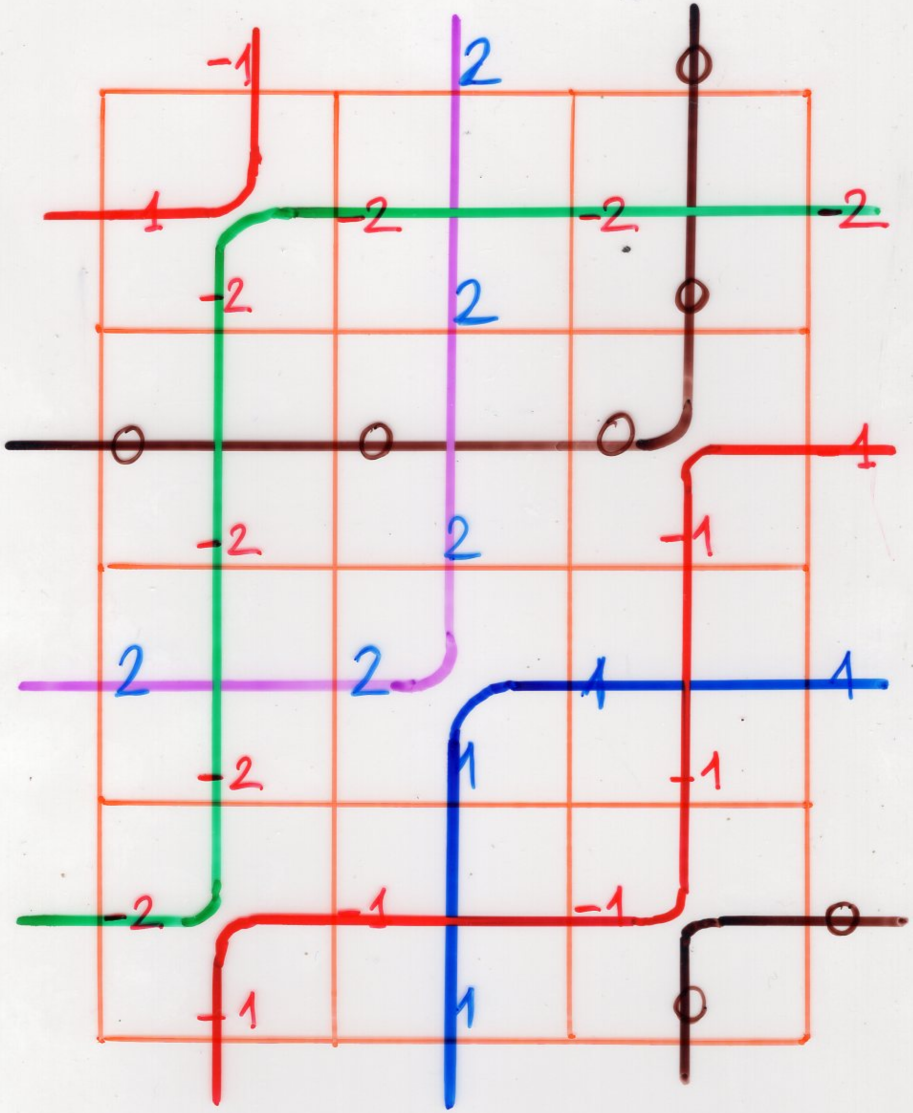
planar automaton



-1
2
0
1
-2
-2
-2
-2
2
0
0
0
0
1
-2
2
-1
2
2
1
4
-2
1
1
-2
-1
-1
0
1
1
0

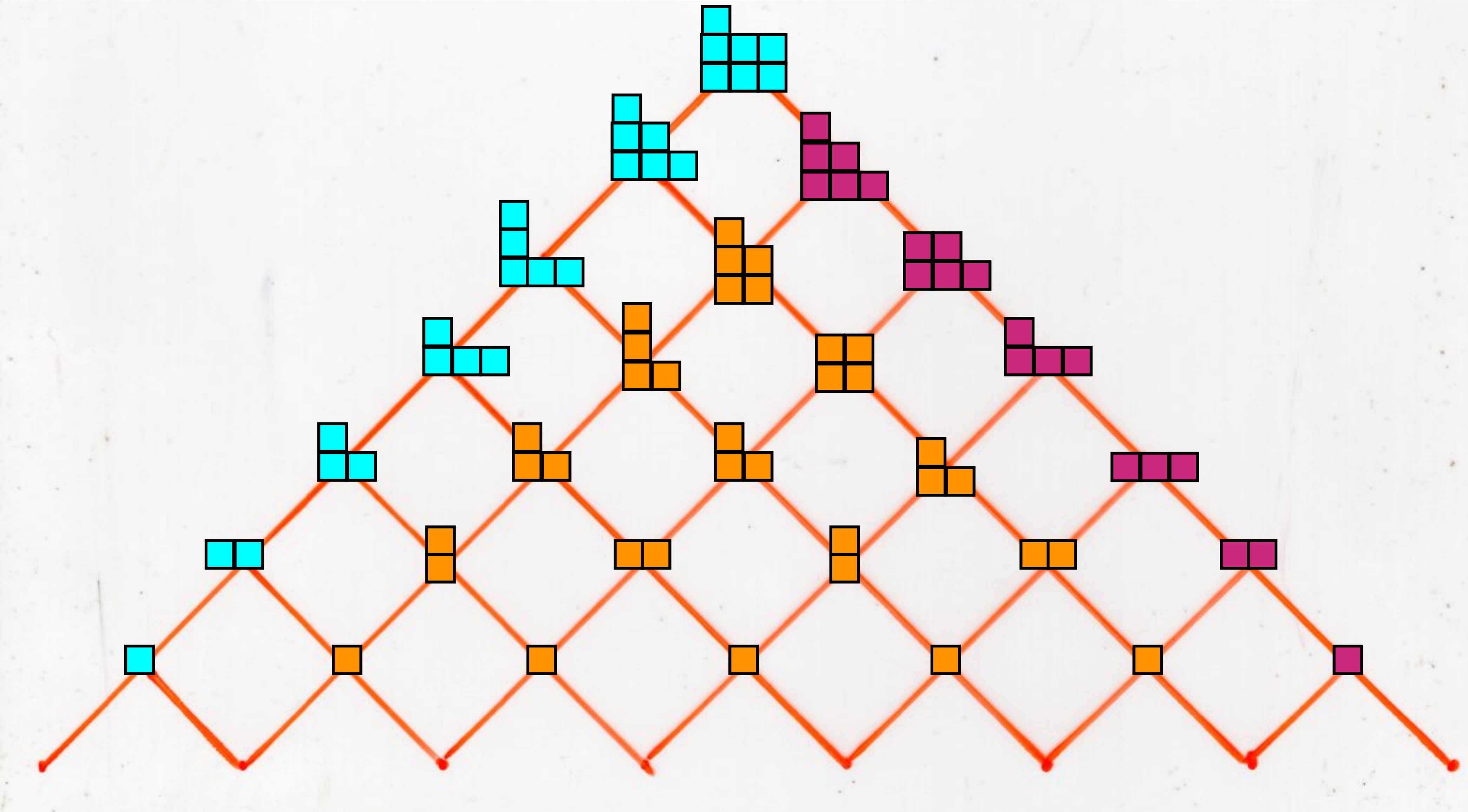


-1
0
2
-2
-1
1
0



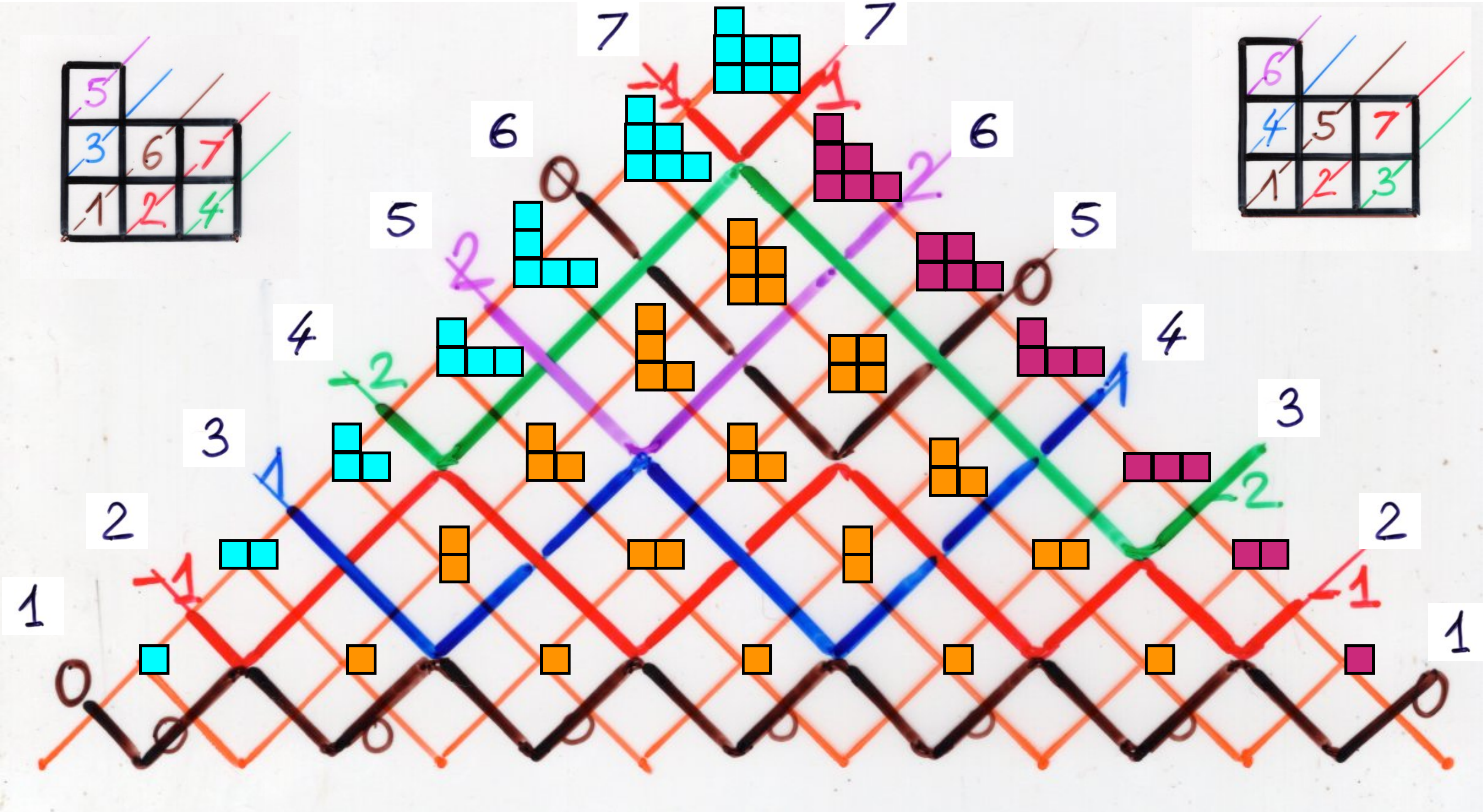
-1
2
0
1
-2
-2
-2
-2
2
0
0
0
0
1
-2
2
-1
2
2
1
4
-2
1
-1
-2
-1
-1
0
-1
1
0

dual of a tableau



Schützenberger involution

dual of a tableau



Schützenberger involution

Proposition The map $T \longrightarrow T^*$ is an involution $(T^*)^* = T$

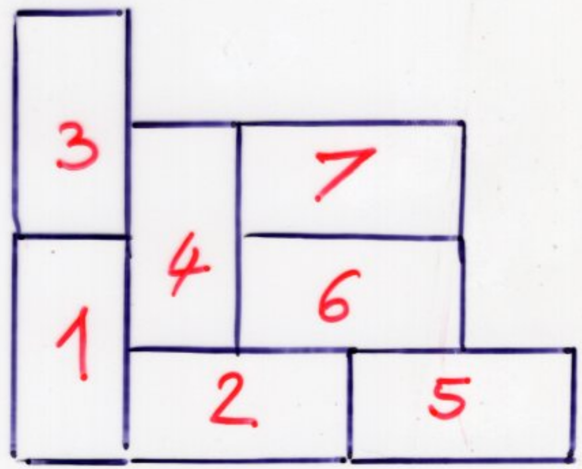
T Young tableau

T^* dual tableau

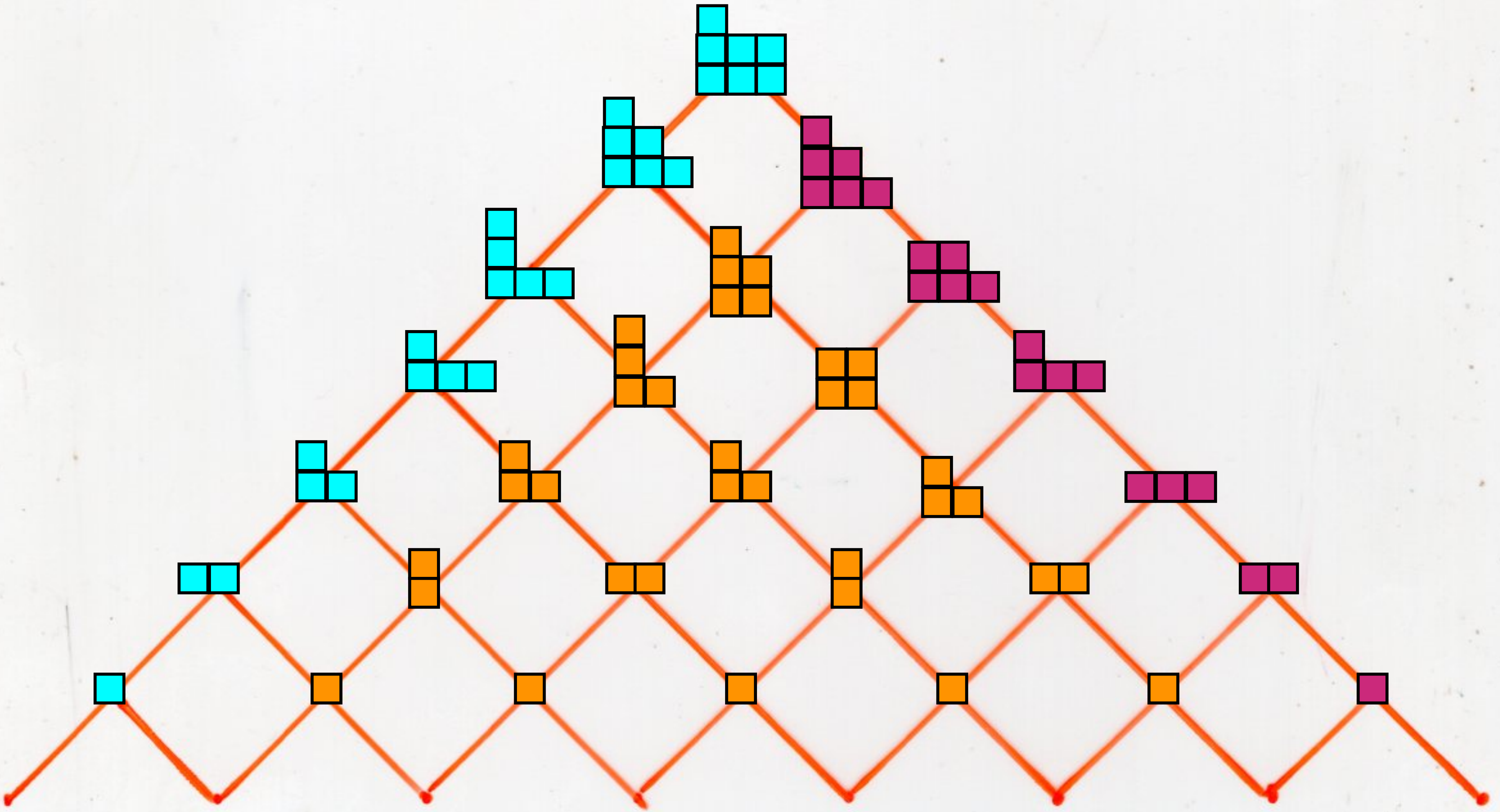
evac(T) other notation

Proposition

tableaux such that $T = T^*$ are in bijection with domino tableaux



dual of a tableau



Schützenberger involution

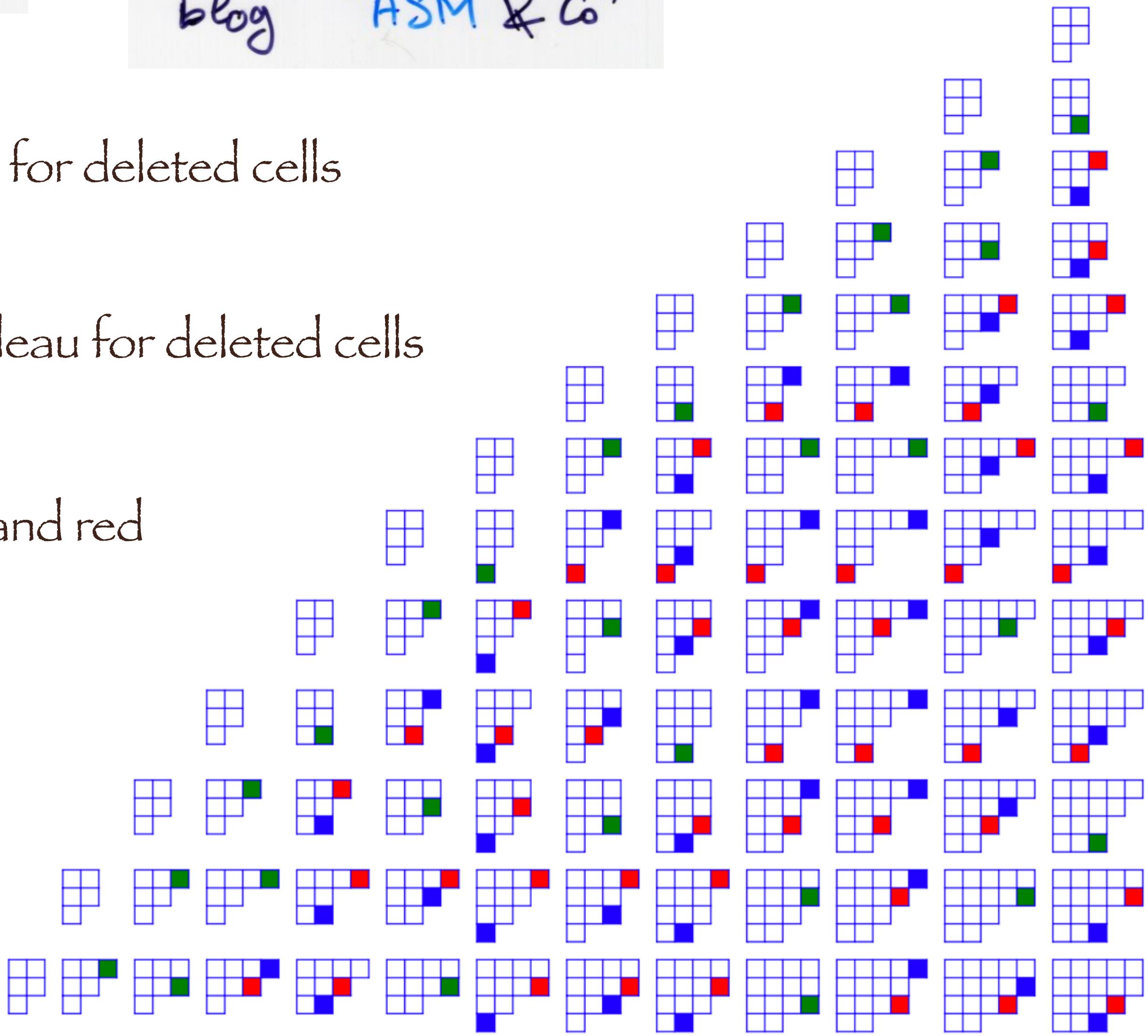
Betrema

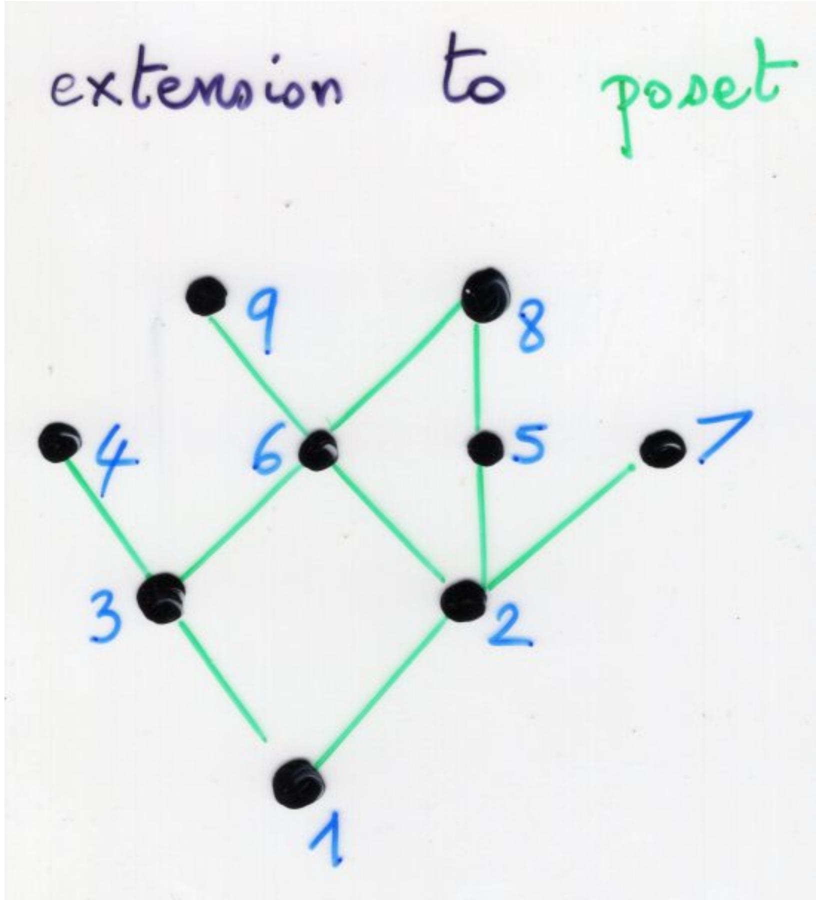
website "Tableaux"
blog "ASM & Co"

blue cells:
in each row of the tableau for deleted cells

red cells:
in each column of the tableau for deleted cells

greene cells:
cells which are both blue and red





Schützenberger (1972)

Delta operators

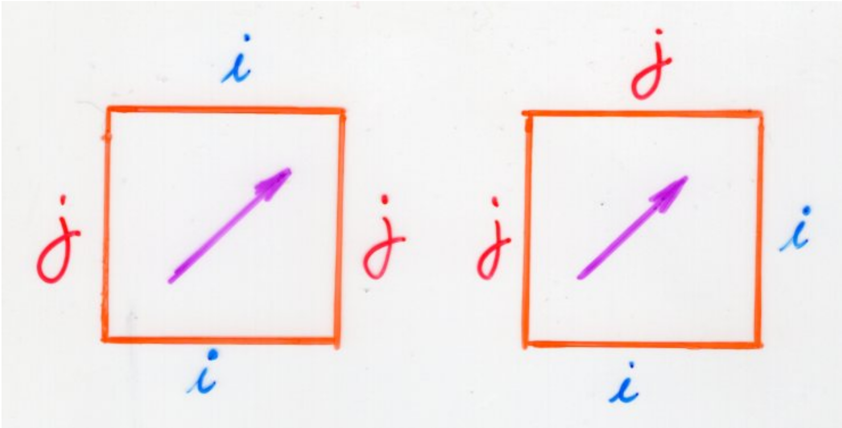
and

nil-Temperley-Lieb algebra

see course BJC Part II, Ch 6b

jeu de taquin
local rules on edges

nil-Temperley-Lieb
planar automaton



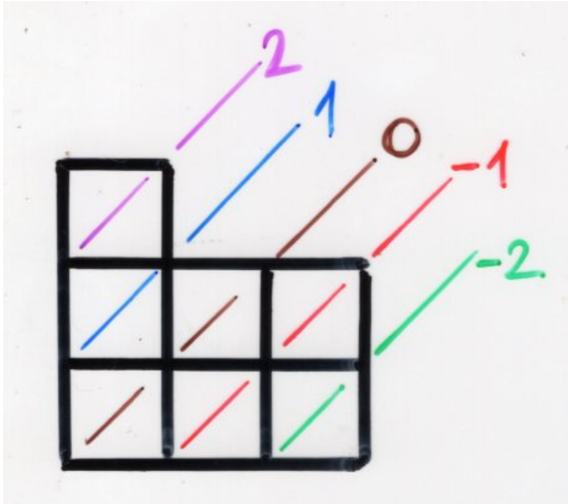
$i, j \in \mathbb{Z}$

$|i-j| \geq 2$ $|i-j| \leq 1$

in fact here $i=j$ impossible

diagonal operators

$\Delta_i \quad i \in \mathbb{Z}$



nil-Temperley-Lieb

planar automaton

nil-Temperley-Lieb algebra

NTL_n or A_n^0

(i) $e_i e_j = e_j e_i \quad |i-j| \geq 2$

(ii) $e_i^2 = 0$

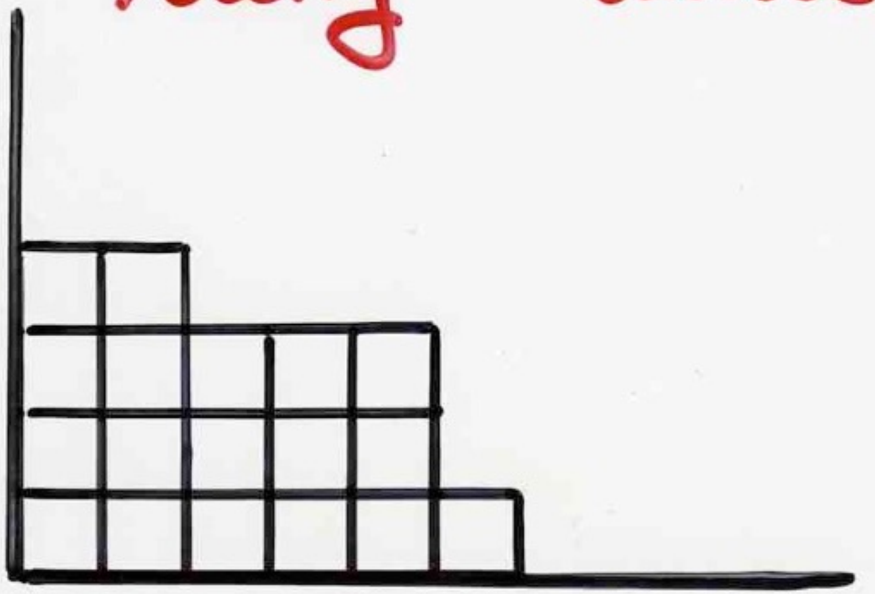
(iii) $e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = 0$

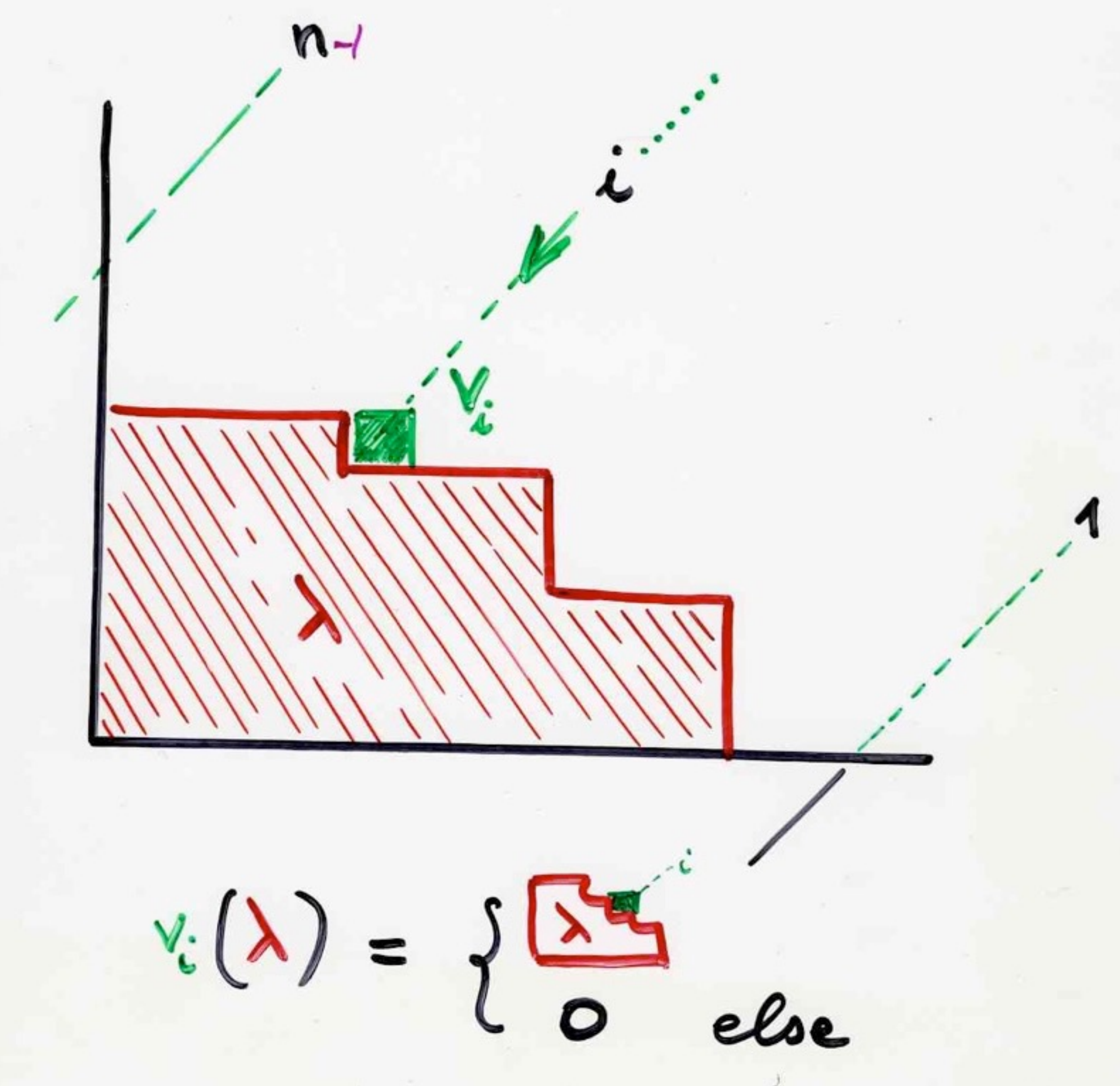
representation of NTL_n

with

operators on the

Young lattice



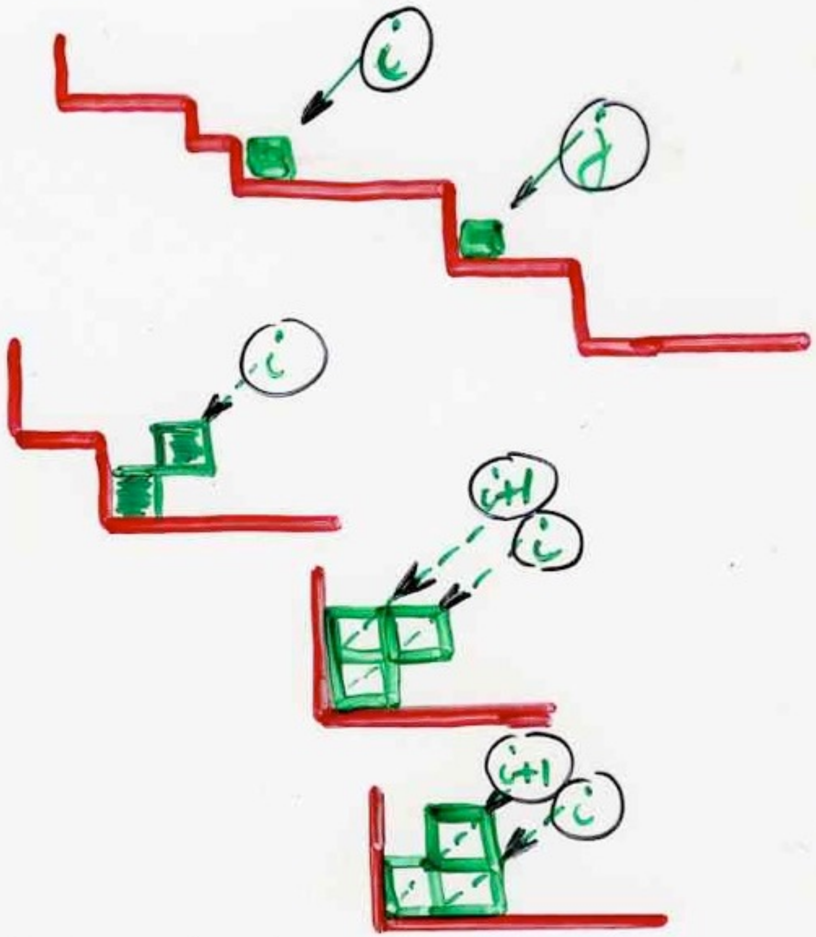

n-1
i
v_i
λ
1
v_i(λ) =
λ
0 else

(i) $V_i \quad V_j$

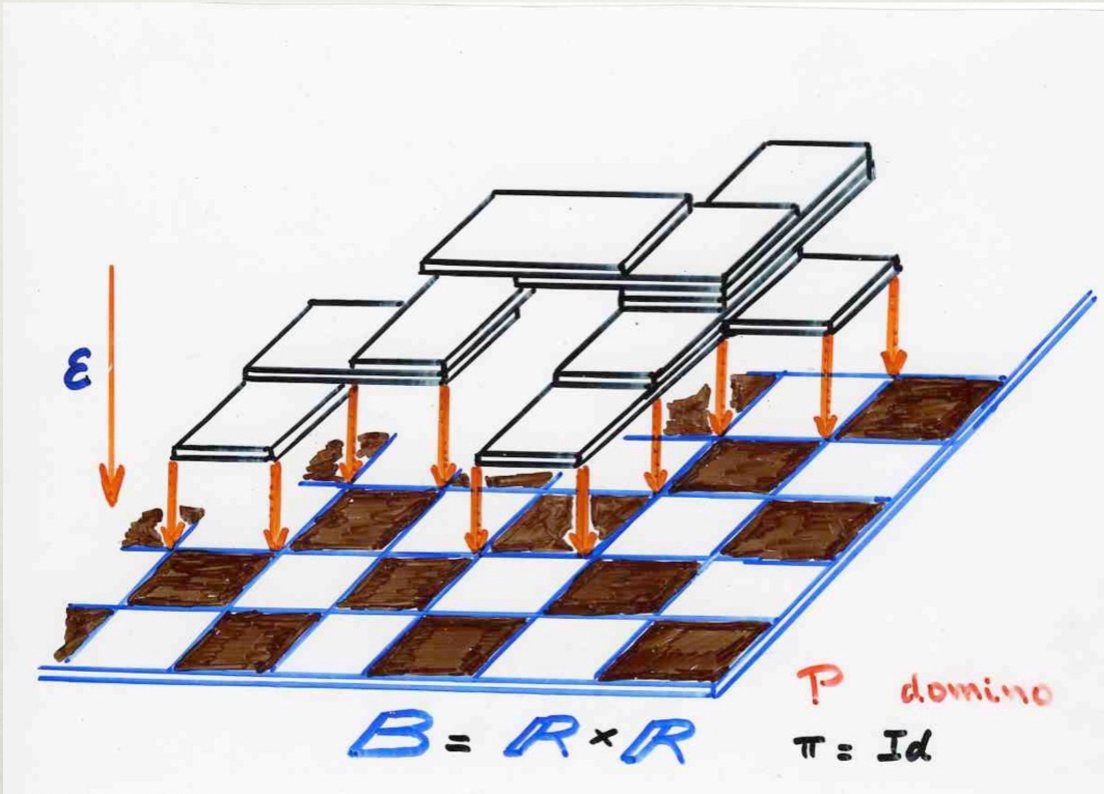
(ii) V_i^2

(iii) $V_i \quad V_{i+1} \quad V_i$

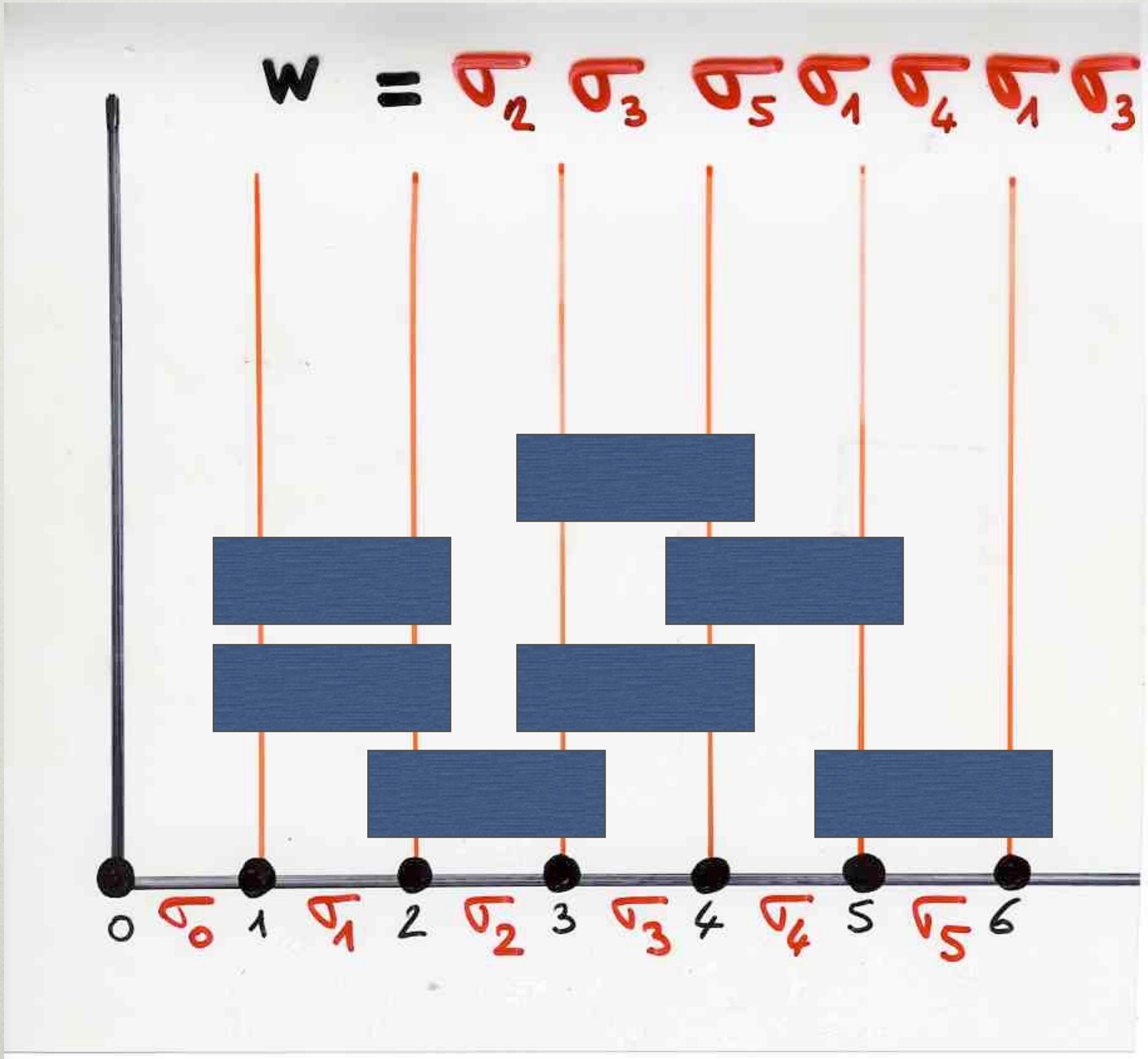
$V_{i+1} \quad V_i \quad V_{i+1}$

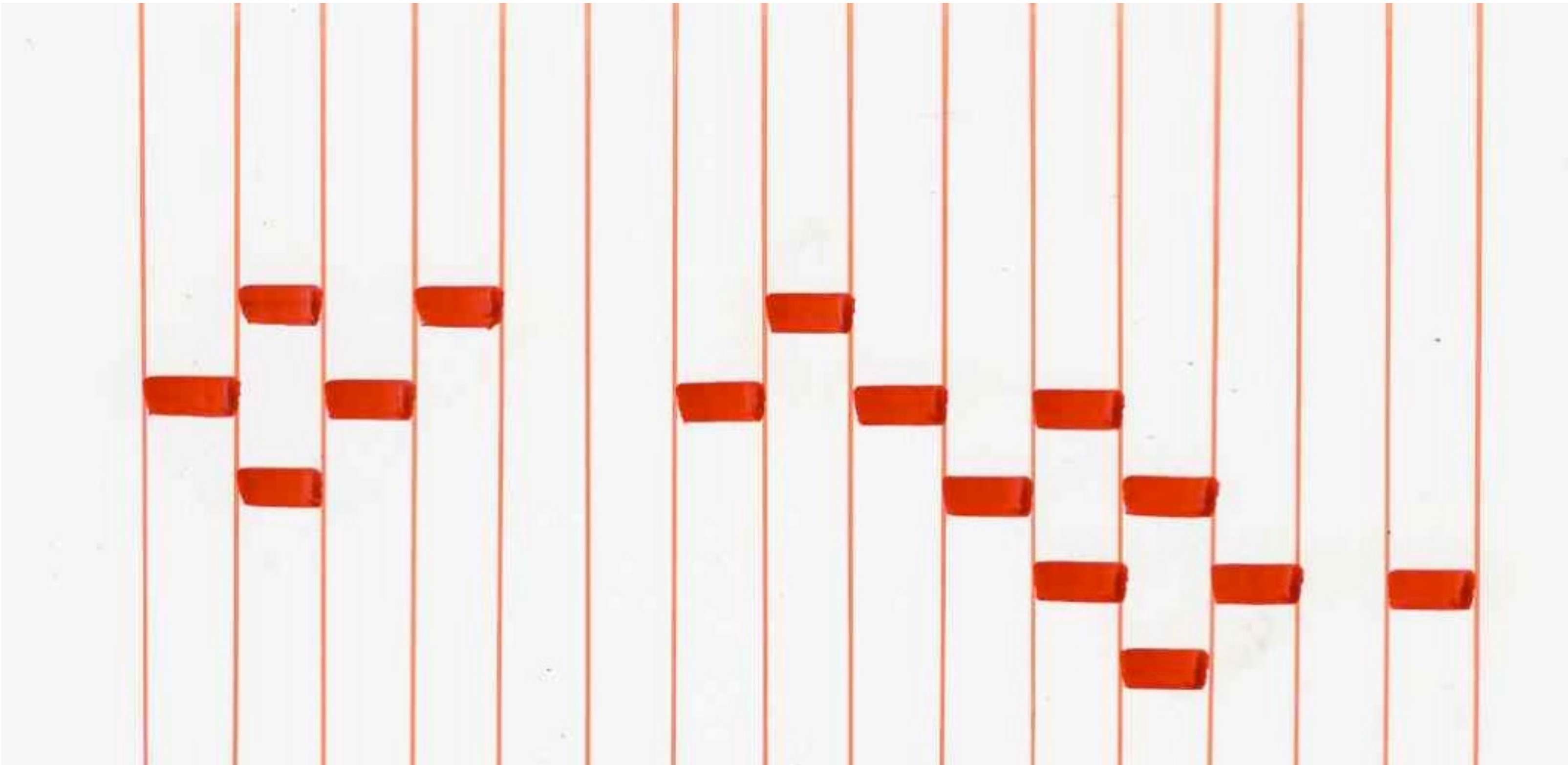


course BJC
Part II: heaps of pieces and commutation monoids



heaps
of dimers



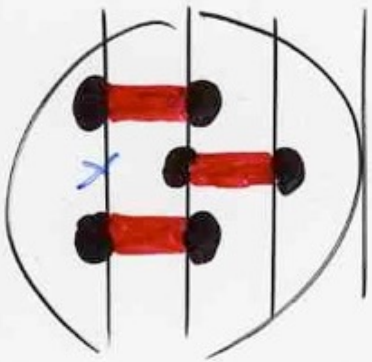


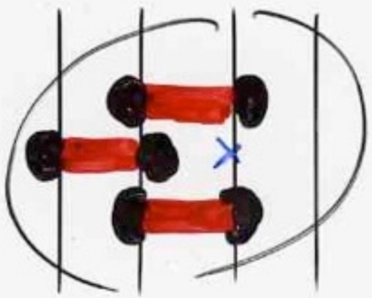
heap of dimers on $[0, n-1]$ $\longrightarrow$ element of TL_n Temperley-Lieb algebra basis of (N)TL_n

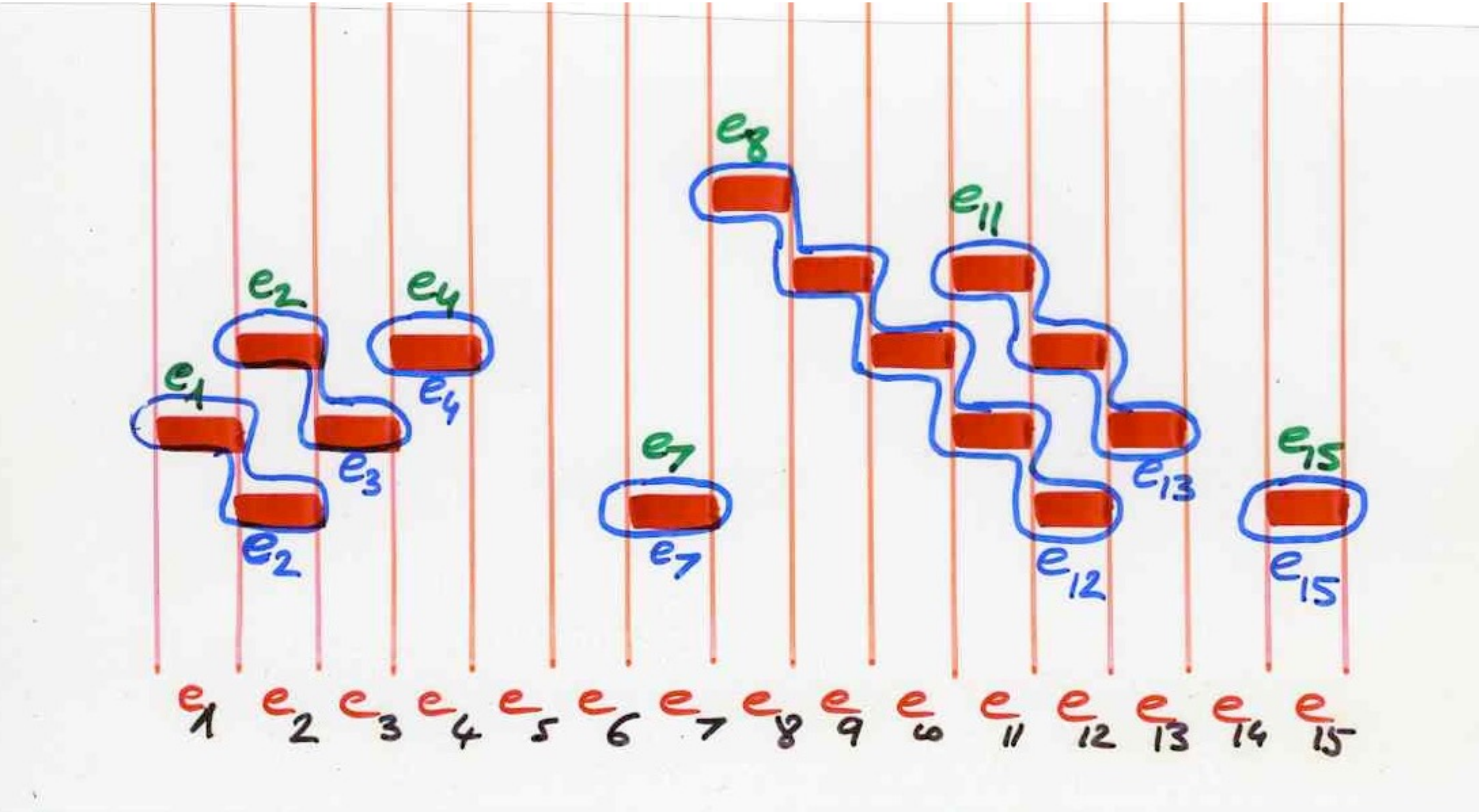
no occurrence of

u_i^2  → strict heap

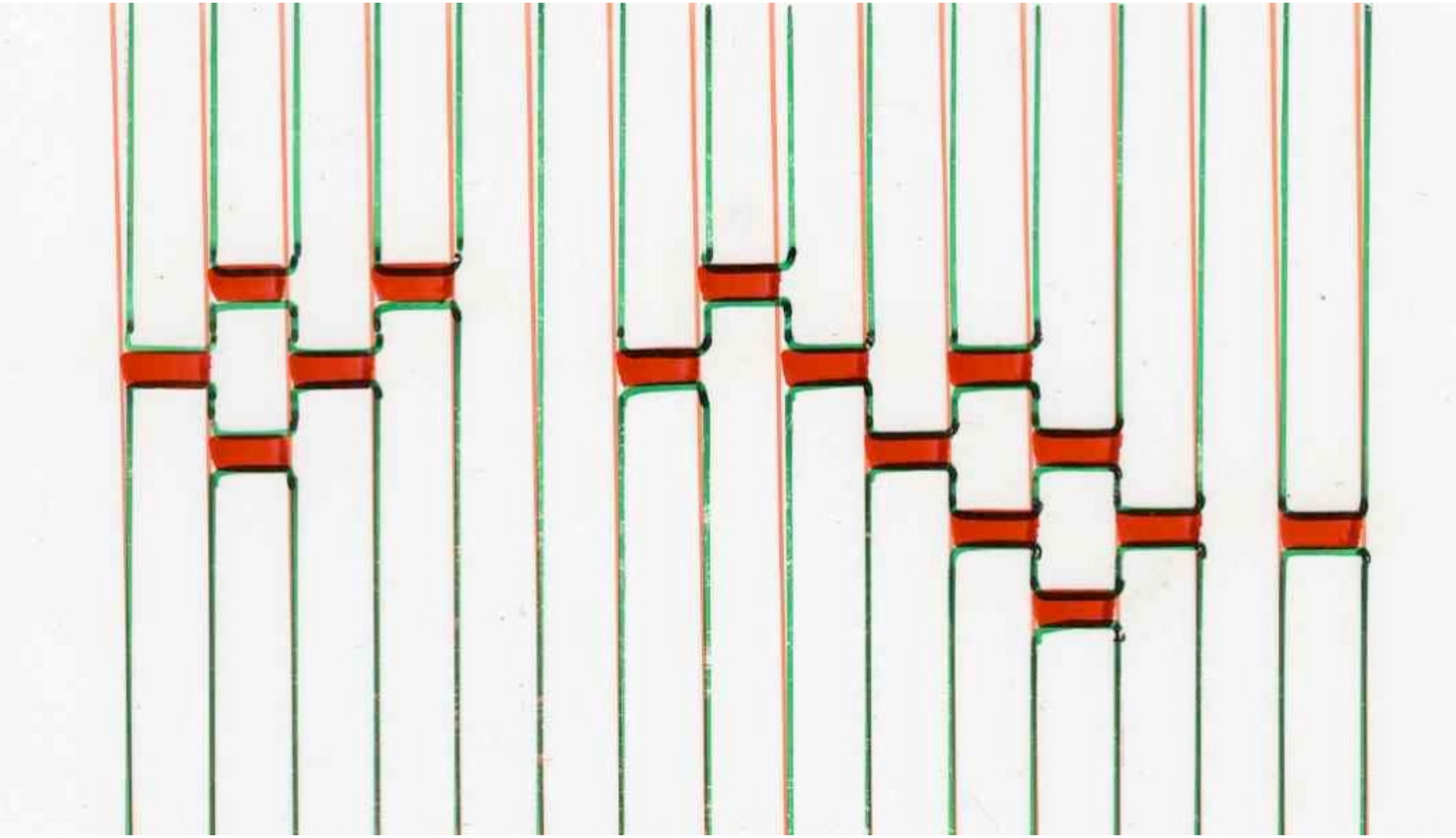
$u_i u_{i+1} u_i$ $\quad$ $u_{i+1} u_i u_{i+1}$

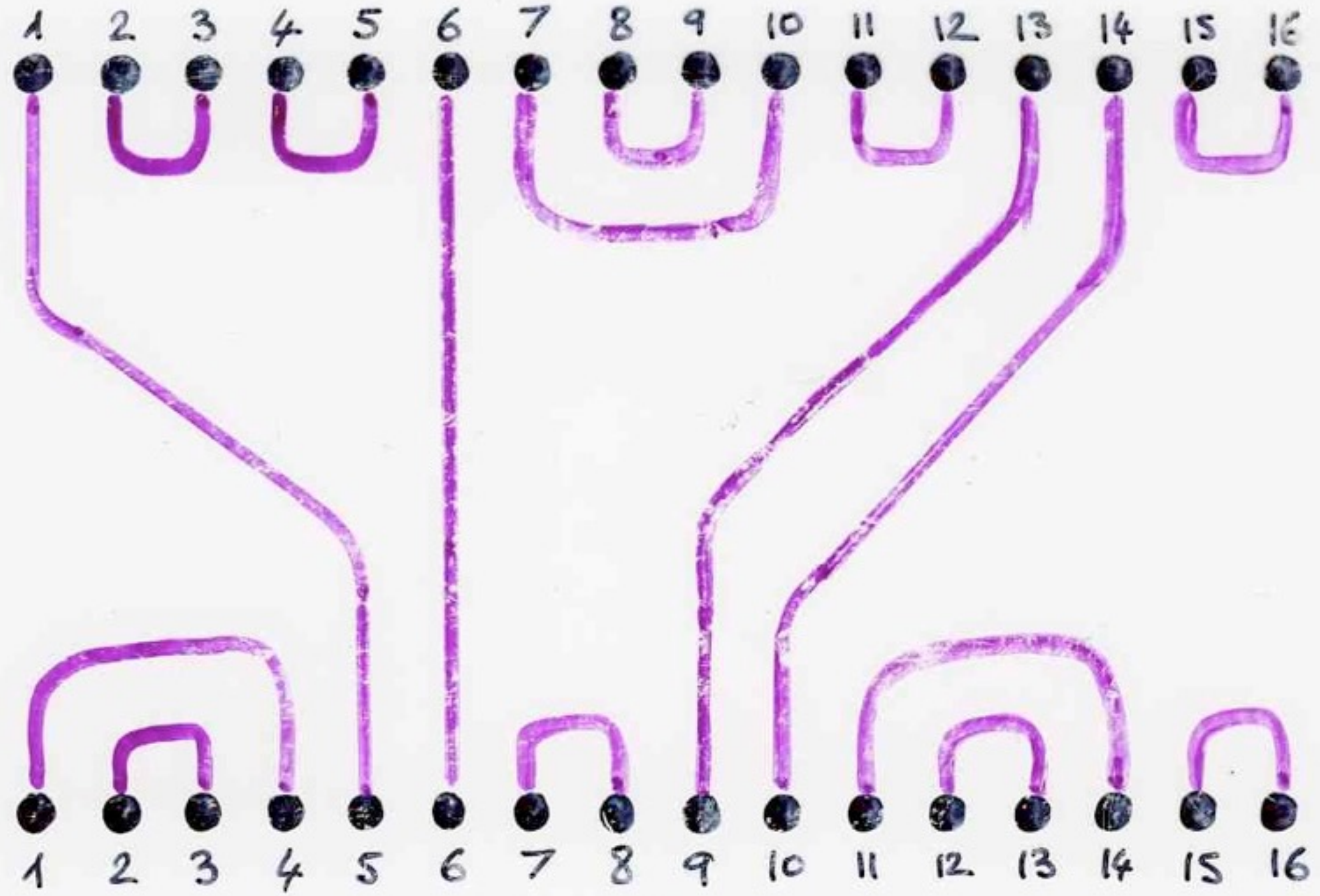


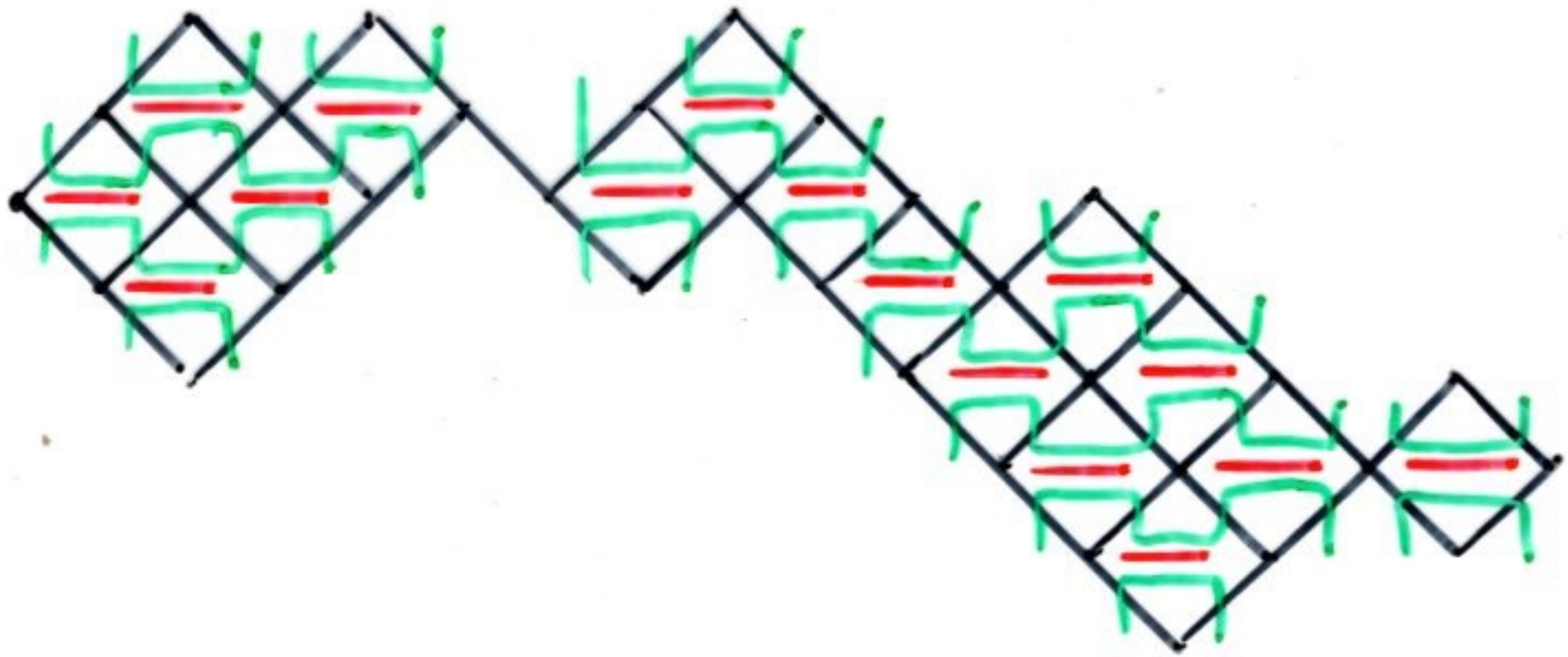




e1
e2
e3
e4
e5
e6
e7
e8
e9
e10
e11
e12
e13
e14
e15




1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

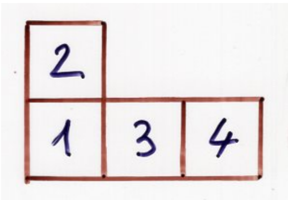


Problem

We know that any poset can be realized
as a heap of pieces.

Can we extend properties of the jeu de taquin
to heaps of dimers ?

In particular the fact that the tableau



is independant of the choice of