Course IMSc, Chennai, India

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The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 1
RSK
The Robinson-Schensted-correspondence
(Ch1b)

IMSc, Chennai

January 11, 2018

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From Ch 1a:

The Robinson-Shensted correspondence

- Schensted's insertions
- geometric version with "shadow lines »

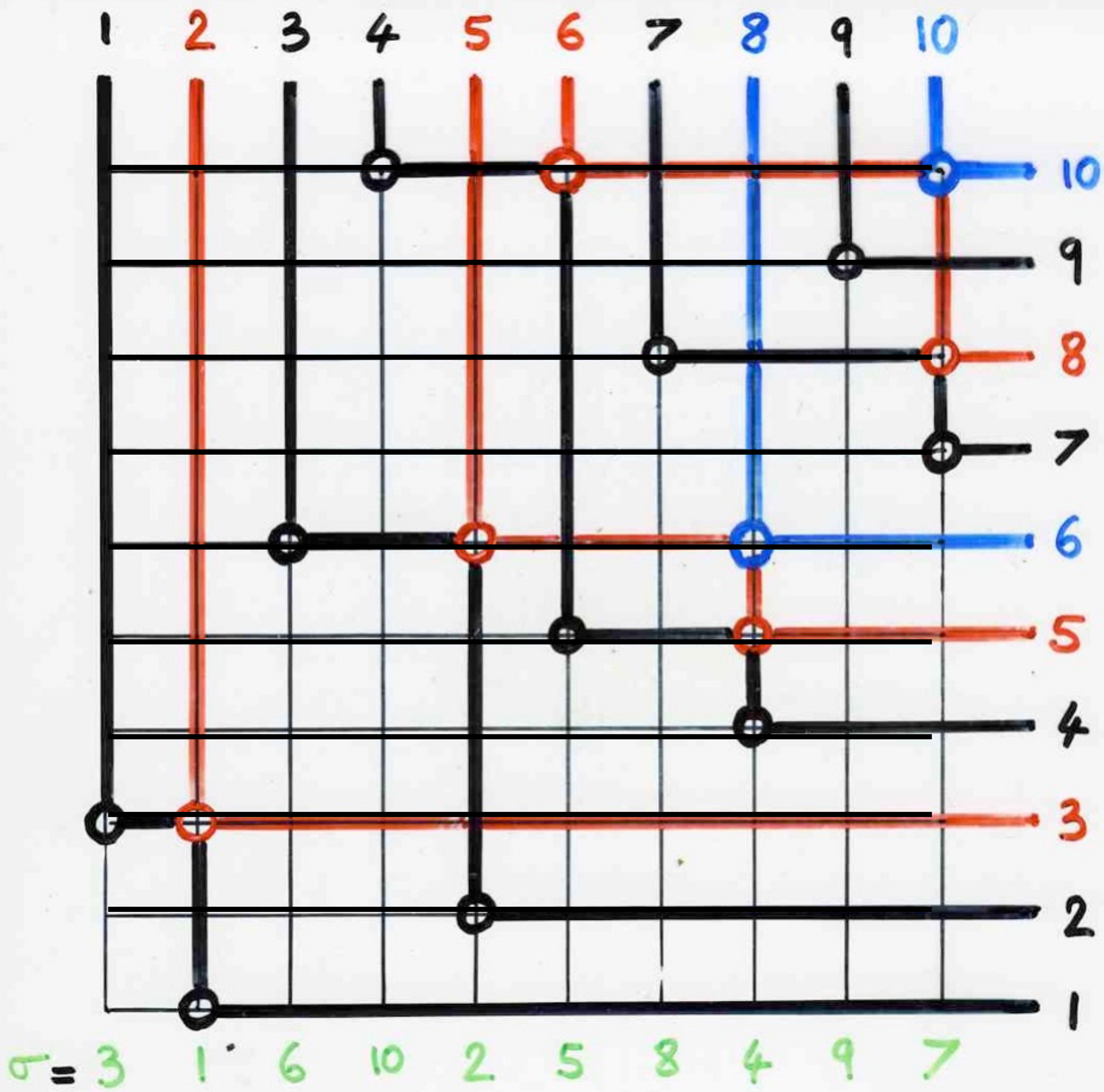
- Fomin "local rules" or "growth diagrams »
- Schützenberger "jeu de taquin »

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

$$P = \begin{array}{|c|c|c|c|c|} \hline 6 & 10 & \multicolumn{3}{c}{} \\ \cline{1-3} 3 & 5 & 8 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 4 & 7 & 9 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|} \hline 8 & 10 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 5 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 & 7 & 9 \\ \hline \end{array}$$

The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape



1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

A few things about posets

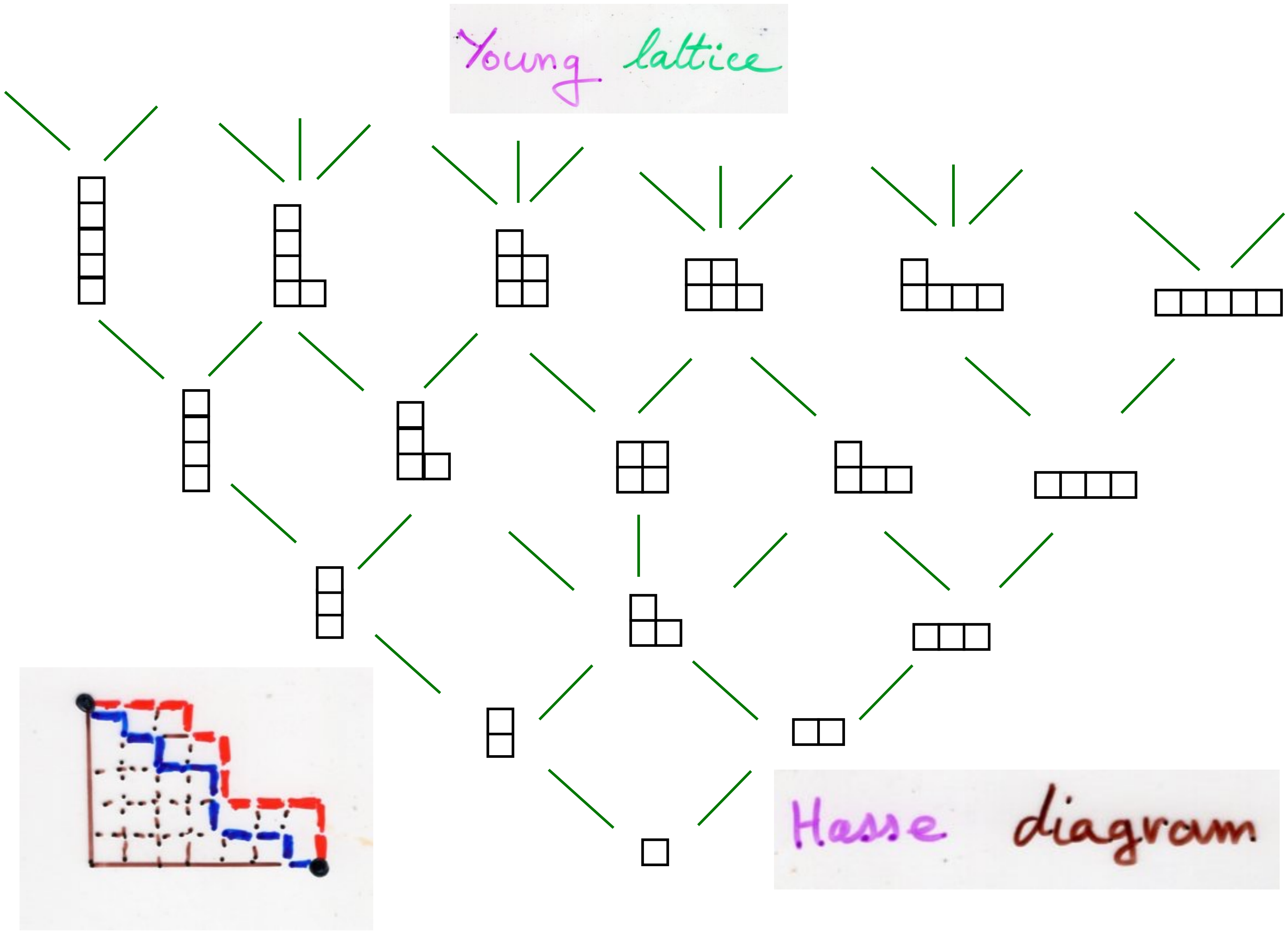
poset $\preccurlyeq$

partially ordered set

covering relation

$\alpha \preccurlyeq \beta$
no γ between
α and β

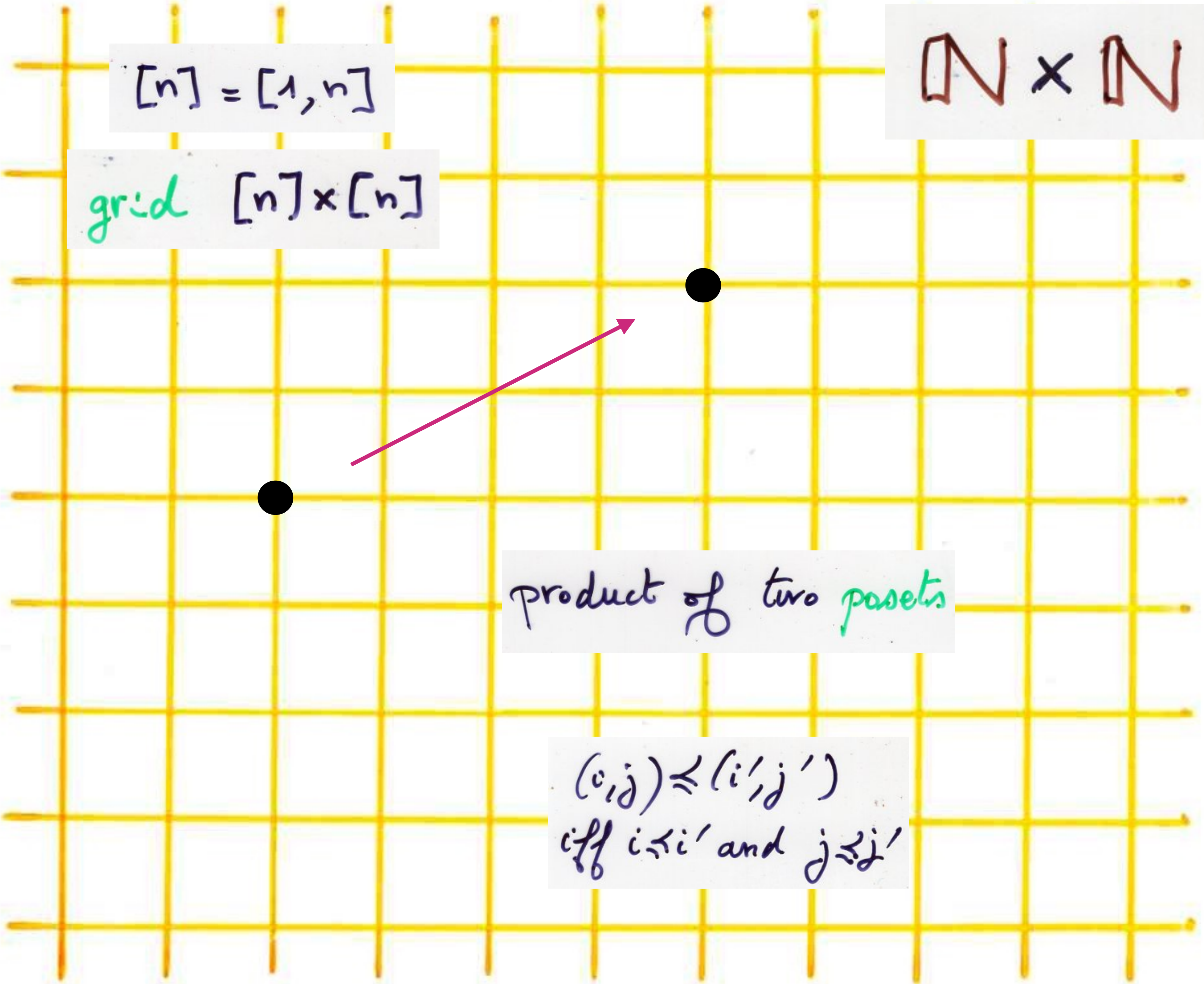
Hasse diagram

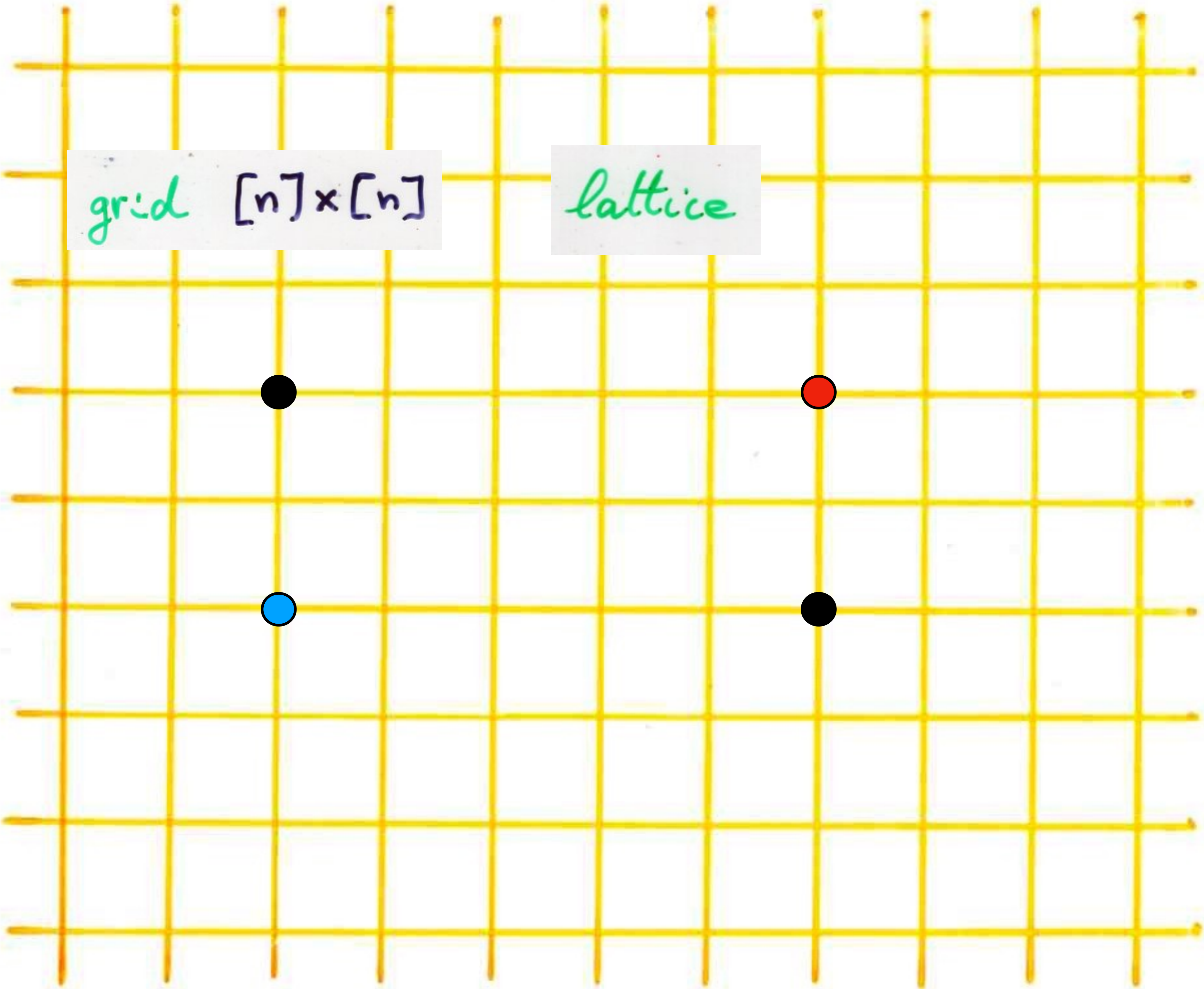

Young lattice
Hasse diagram

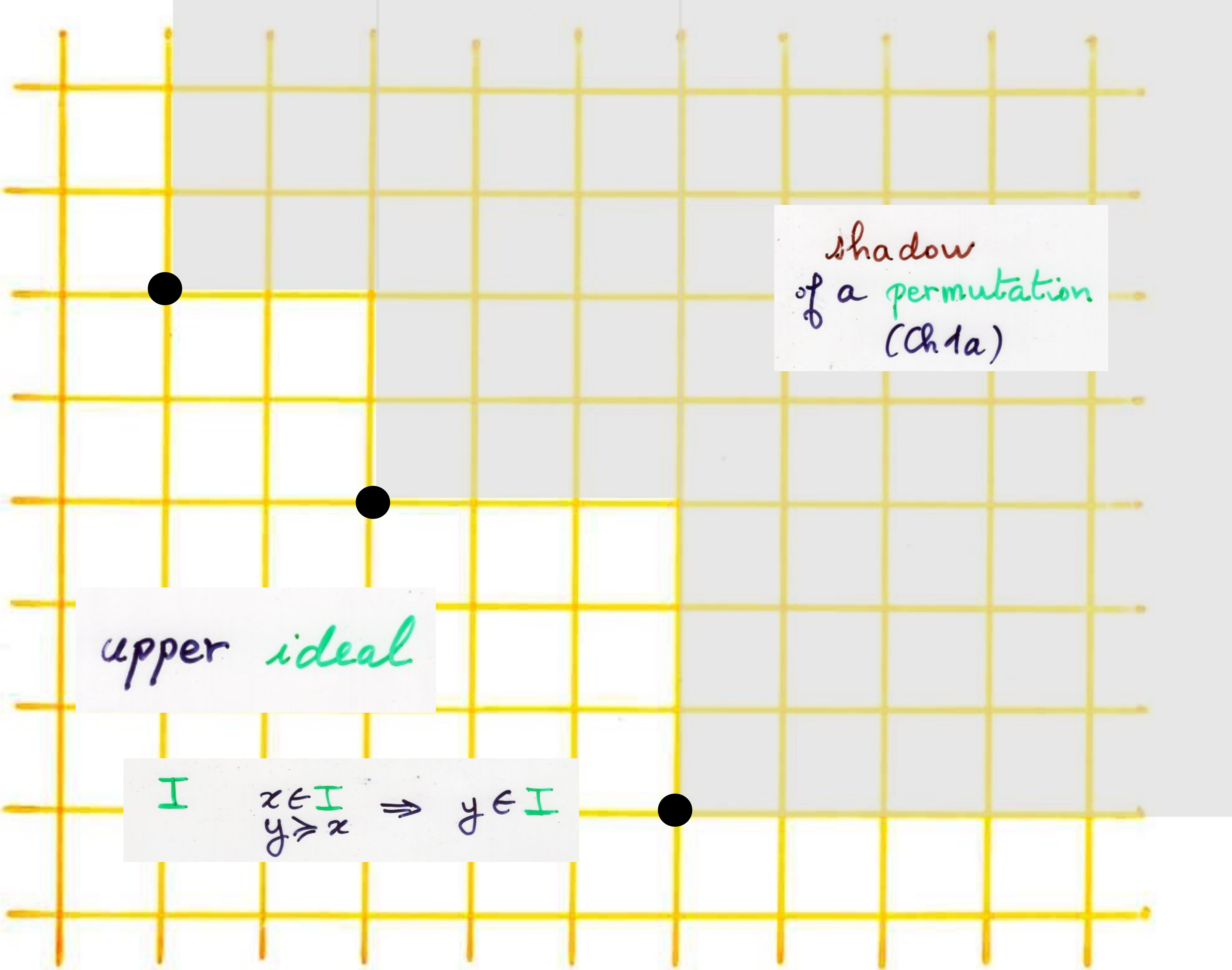
lattice

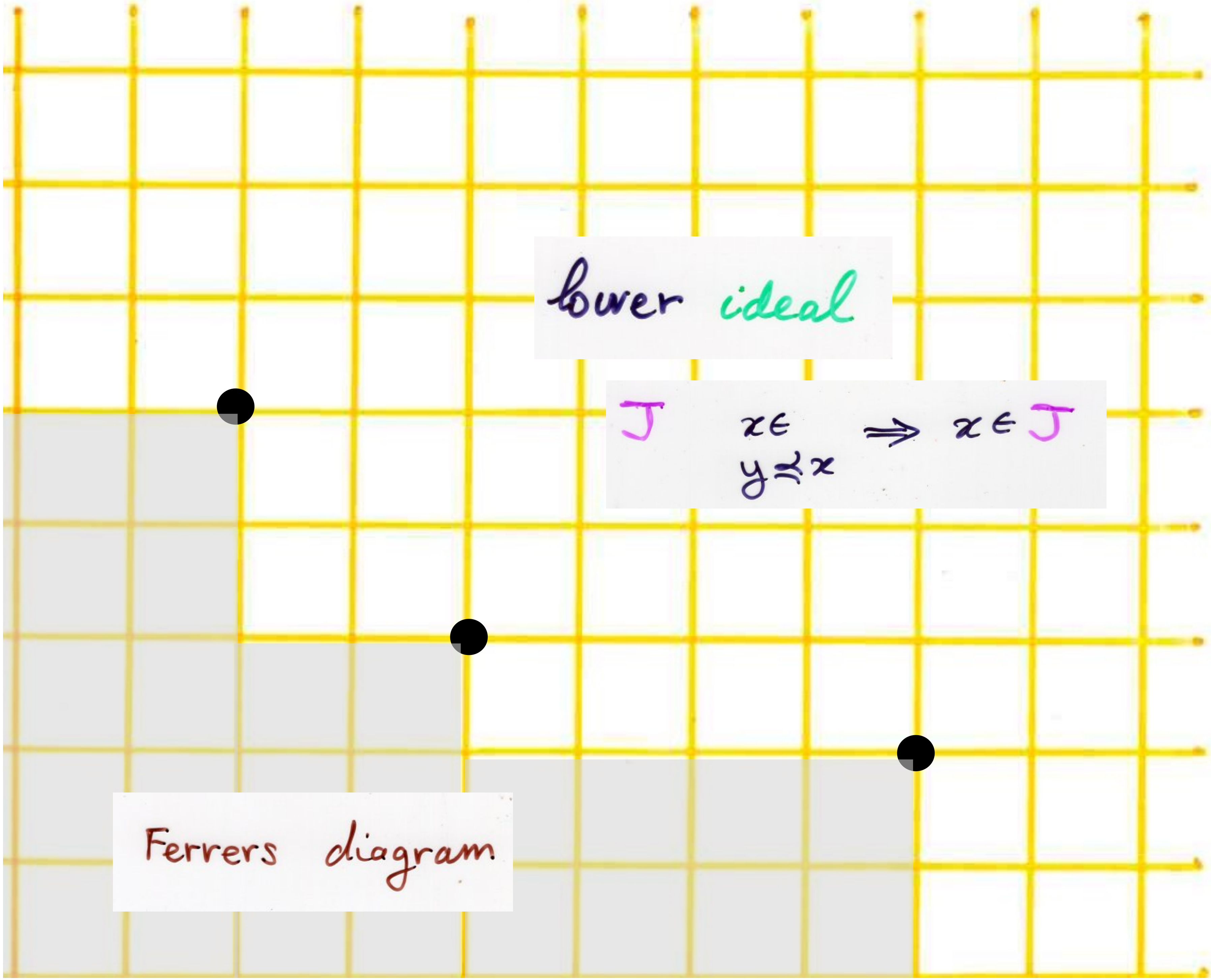
every two elements
have a unique
least upper bound (join)

and a unique
greatest lower bound
(meet)


$[n] = [1, n]$
grid $[n] \times [n]$
$\mathbb{N} \times \mathbb{N}$
product of two posets
$(i, j) \preceq (i', j')$
iff $i \preceq i'$ and $j \preceq j'$


grid $[n] \times [n]$
lattice


shadow
of a permutation
(Ch1a)
upper ideal
I
$x \in I$, $y \geqslant x$ $\Rightarrow$ $y \in I$


lower ideal
J
x ∈ J, y ≼ x ⇒ y ∈ J
Ferrers diagram

maximal chain
in a poset

$\alpha_1 \lessapprox \alpha_2 \lessapprox \cdots \lessapprox \alpha_k$

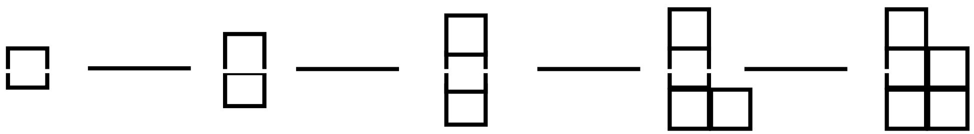
each α_{i+1} is covering α_k

bijection

maximal chain
in the Young lattice
$\alpha_1 = \emptyset \lessapprox \cdots \lessapprox \alpha_k = \lambda$

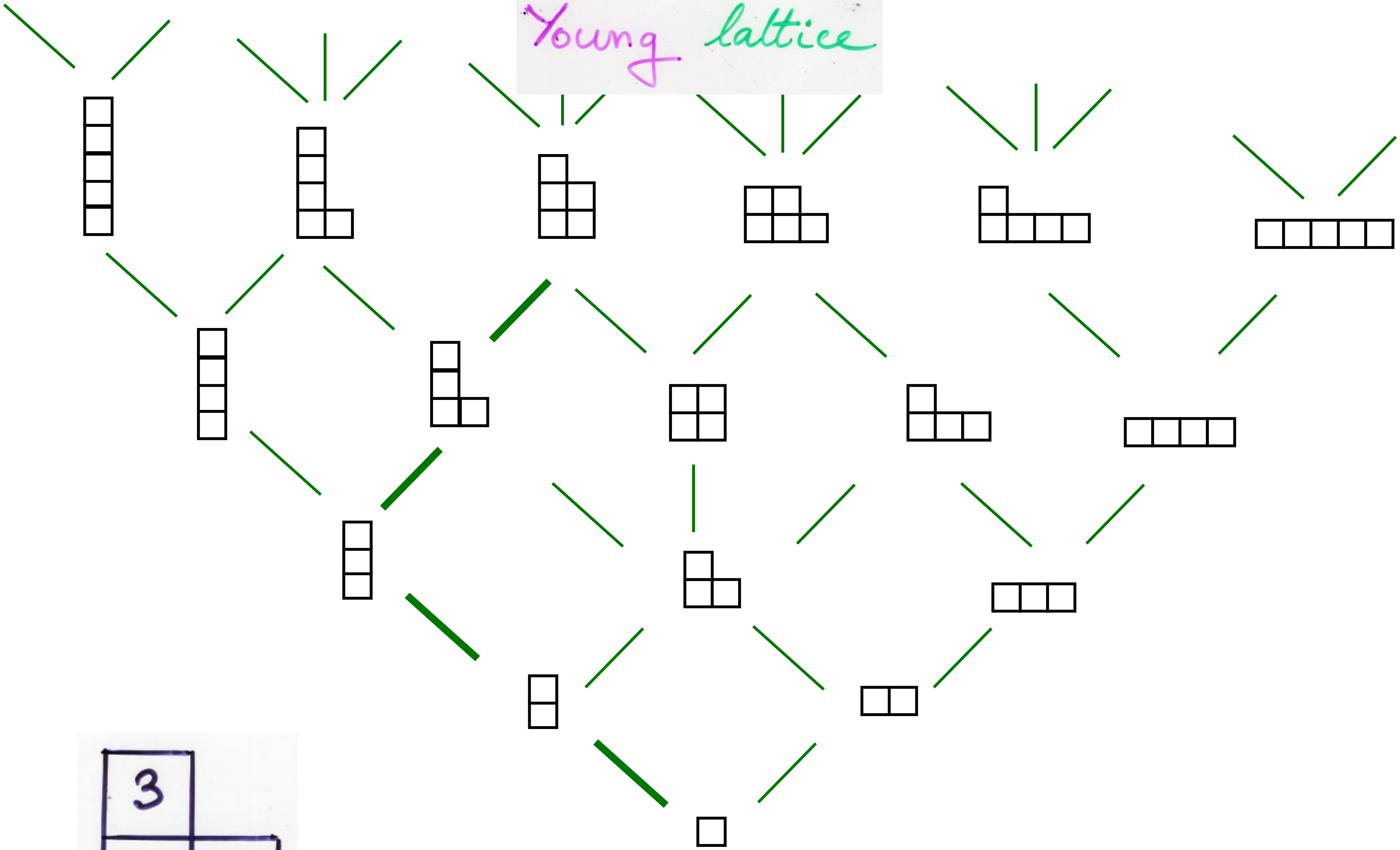
$\longleftrightarrow$

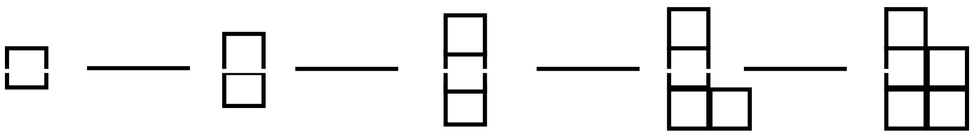
and Young tableau
with shape λ

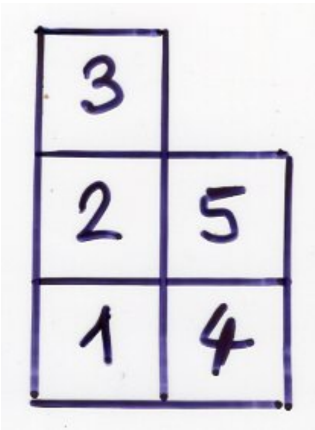


3	
2	5
1	4

Young lattice







“local” algorithm on a grid or “growth diagrams”

S. Fomin, 1986, 1994

C.Krattenthaler

S. V. Fomin, “Finite partially ordered sets and Young tableaux”, Soviet Math. Dokl. 19, (1978), 1510–1514.

S. V. Fomin, “Generalised Robinson-Schensted-Knuth correspondence”, Journal of Soviet Mathematics 41, (1988), 979–991. (Translation from Zapiski nauqnyh seminarov LOMI 155 (1986) 156–175; authorised translation available from the author).

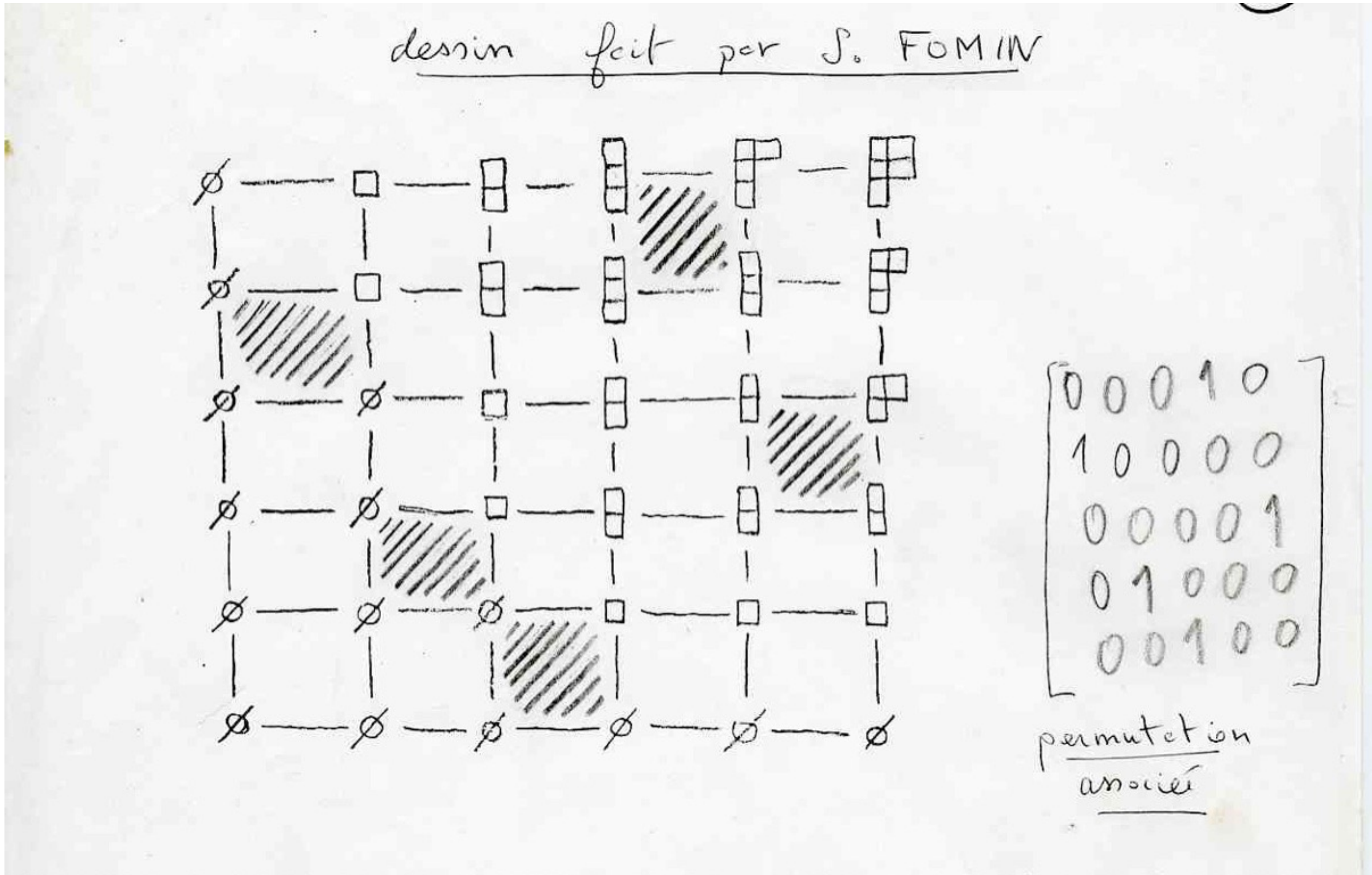
S. Fomin, Dual graphs and Schensted correspondences, Proceedings of the 4th International conference on Formal power series and Algebraic combinatorics, Montreal, (1992).

S. Fomin, Schur operators and Knuth correspondences, Institut Mittag-Leffler report No. 17, (1991/92).

S. Fomin, “Duality of graded graphs”, J. Algebr. Combinatorics 3, (1994), 357–404.

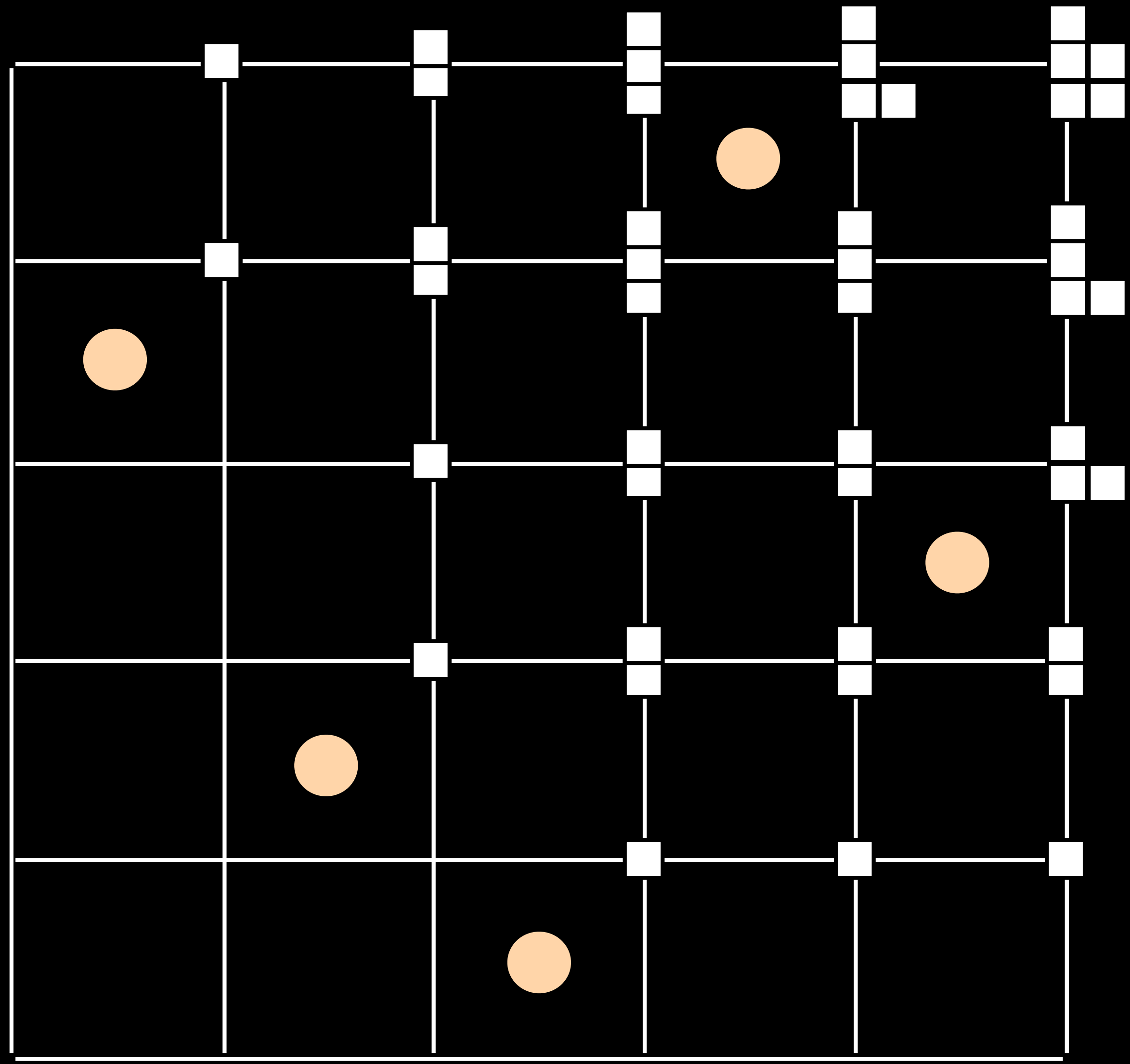
S. Fomin, “Schensted algorithms for dual graded graphs”, J. Algebr. Combinatorics 4, (1995), 5–45.

S. Fomin and C. Greene, “A Littlewood-Richardson Miscellany”, Europ. J. Combinatorics 14, (1993), 191–212.



S. Fomin, Schur operators and Knuth correspondences,

Institut Mittag-Leffler report No. 17, (1991/92).

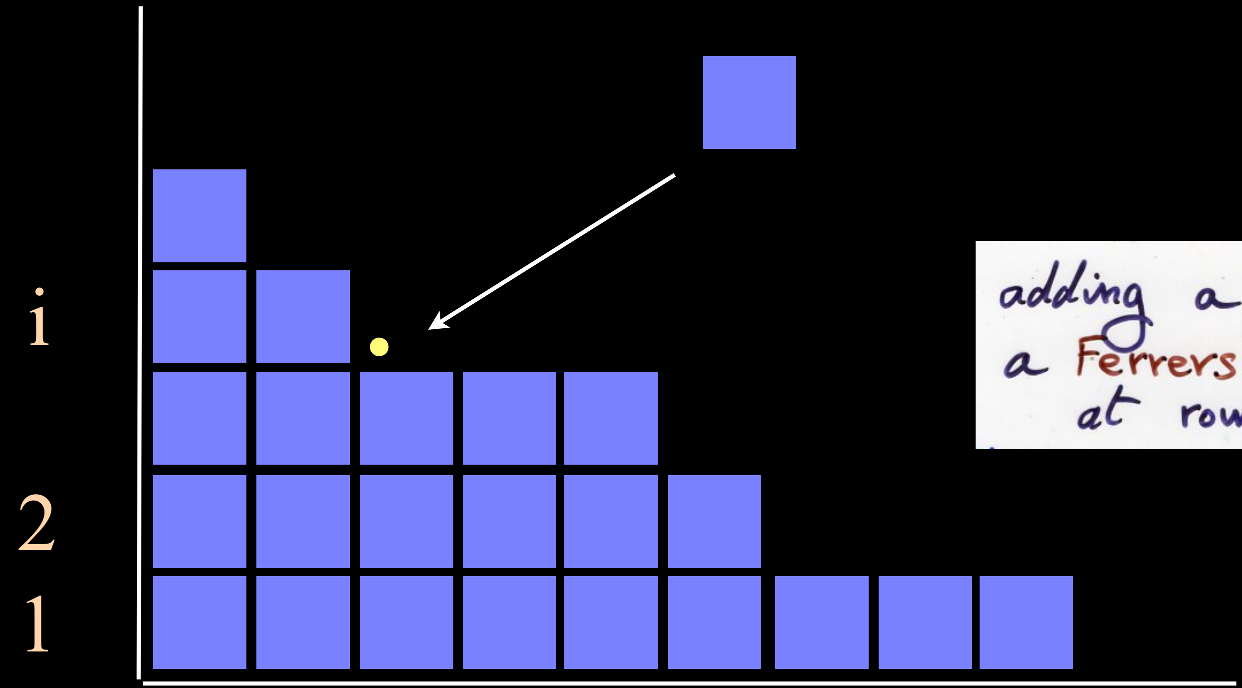


"growth diagrams"

"local rules"

notations

operator U_i

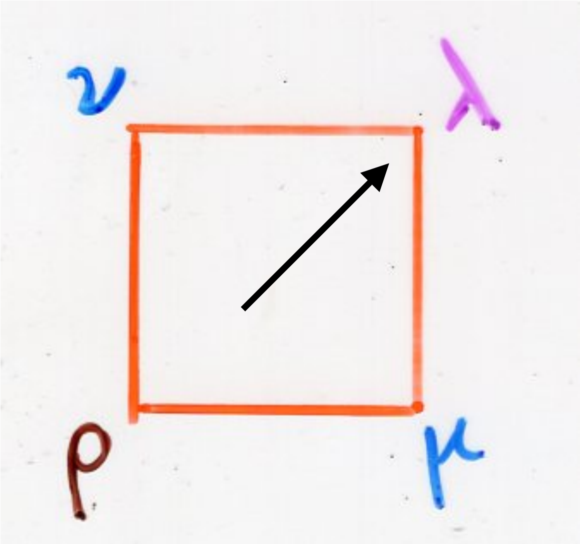


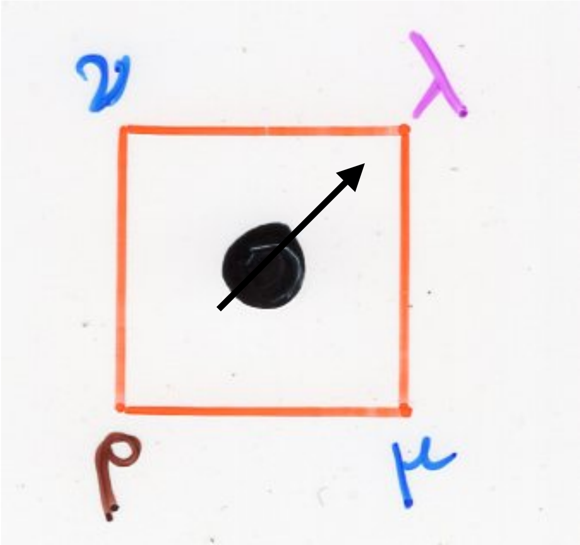
adding a cell in a Ferrers diagram ρ at row i

$$U_i(\rho) = \rho + (i)$$

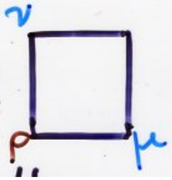
"growth diagrams"

"local rules"





"local rules"

(i) $\rho=\mu=\nu$ and 

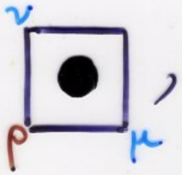
then $\lambda=\rho$

(ii) $\rho=\mu\neq\nu$, then $\lambda=\nu$

(iii) $\rho=\nu\neq\mu$, then $\lambda=\mu$

(iv) ρ,μ,ν pairwise $\neq$, then $\lambda=\mu\cup\nu$

(v) $\rho\neq\mu=\nu$, then $\lambda=\mu+(i+1)$
given that $\mu=\nu$ and ρ differ in the i-th row
[in fact $\mu=\nu=\rho+(i)$]

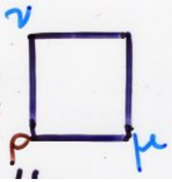
(vi) $\rho=\mu=\nu$ and 

, then $\lambda=\mu+(1)$

C.Krattentheler, (2006).

GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

"local rules"

(i) $\rho = \mu = \nu$ and  then $\lambda = \rho$

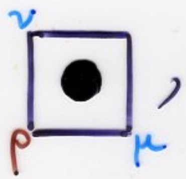
(ii), (iii), (iv) $\mu \neq \nu$, then $\lambda = \mu \cup \nu$

$\mu \neq \nu$

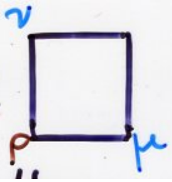
(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$

given that $\mu = \nu$ and ρ differ in the i-th row

[in fact $\mu = \nu = \rho + (i)$]

(vi) $\rho = \mu = \nu$ and , then $\lambda = \mu + (1)$

"local rules"

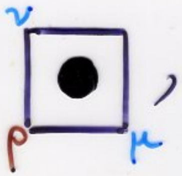
(i) $\rho = \mu = \nu$ and 

then $\lambda = \rho$

$\mu = \nu$

(v) $\rho \neq \mu = \nu$, then $\lambda = \mu + (i+1)$
given that $\mu = \nu$ and ρ differ in the i-th row
[in fact $\mu = \nu = \rho + (i)$]

$\mu = \nu$

(vi) $\rho = \mu = \nu$ and 

, then $\lambda = \mu + (1)$

$\mu = \nu$

"local rules"

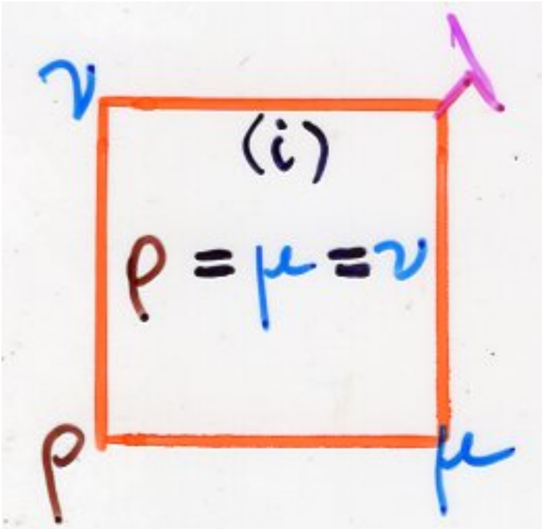
$$\mu \neq \nu$$

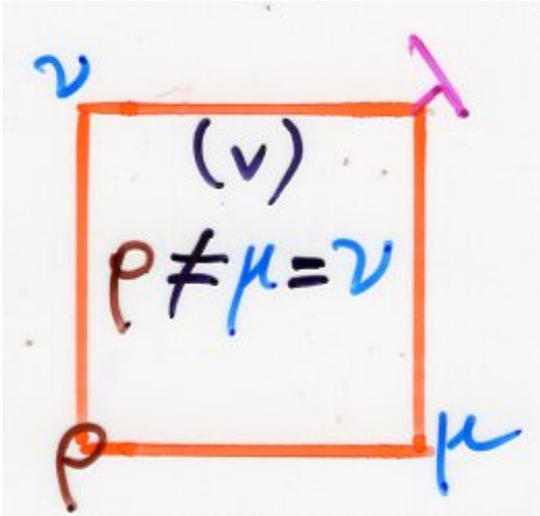
$$\begin{tikzcd}
\nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\substack{(ii)\\ \rho = \mu \neq \nu}"] & \lambda \arrow[d, dash] \\
\rho \arrow[r, dash] & \mu
\end{tikzcd}
\qquad
\begin{tikzcd}
\nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\substack{(iii)\\ \rho = \nu \neq \mu}"] & \lambda \arrow[d, dash] \\
\rho \arrow[r, dash] & \mu
\end{tikzcd}
\qquad
\begin{tikzcd}
\nu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\substack{(iv)\\ \rho \neq \nu \neq \mu,\ \rho \neq \mu}"] & \lambda \arrow[d, dash] \\
\rho \arrow[r, dash] & \mu
\end{tikzcd}$$

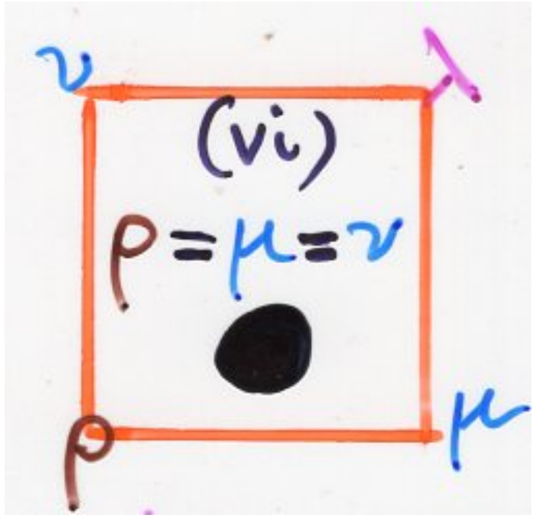
$$\lambda = \nu \qquad\qquad \lambda = \mu \qquad\qquad \lambda = \mu \cup \nu$$

"local rules"

$$\mu = \nu$$





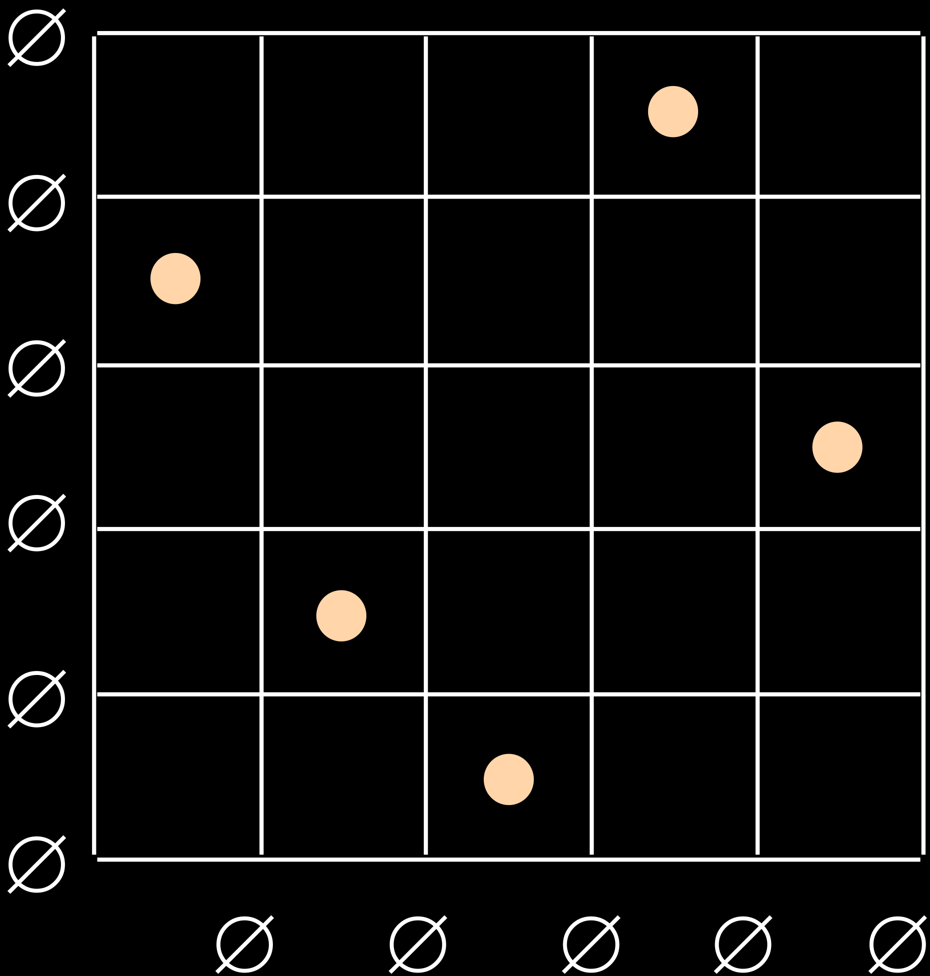


$$\lambda = \rho$$

$$\lambda = \left\{ \begin{matrix} \mu \\ \nu \end{matrix} \right. + (i+1)$$

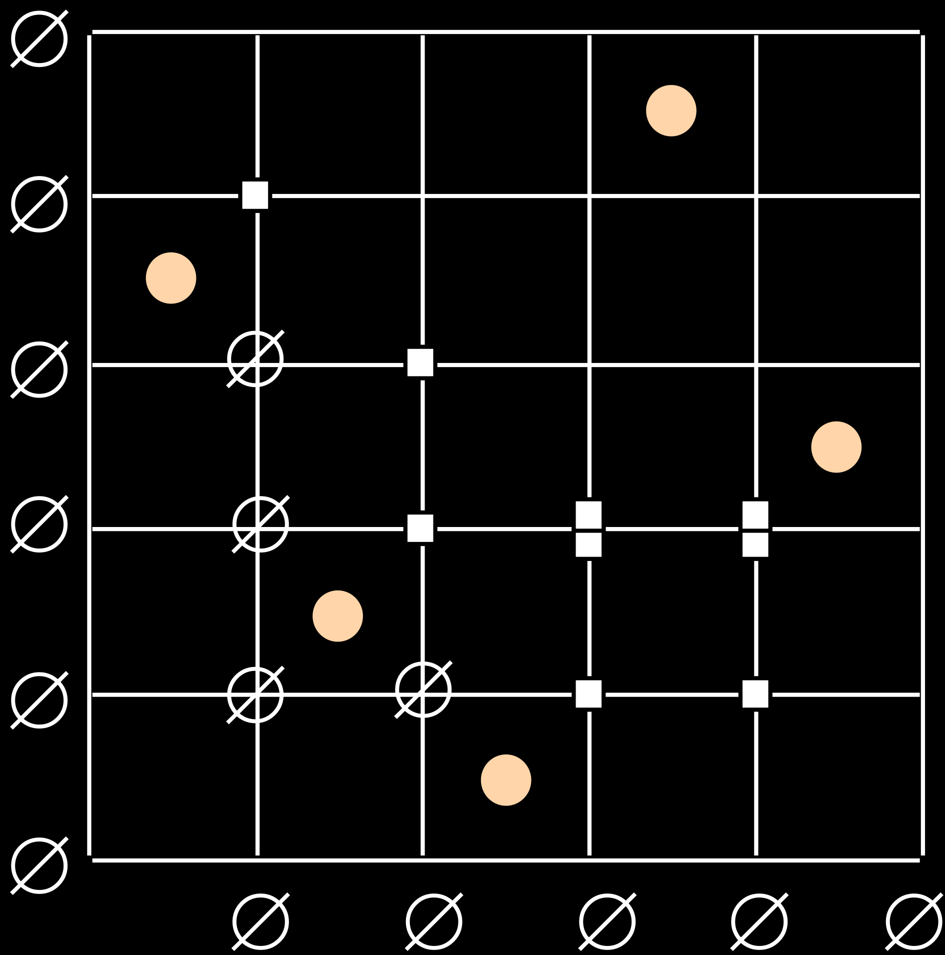
$$\lambda = \left\{ \begin{matrix} \rho \\ \mu \\ \nu \end{matrix} \right. + (1)$$

initial state

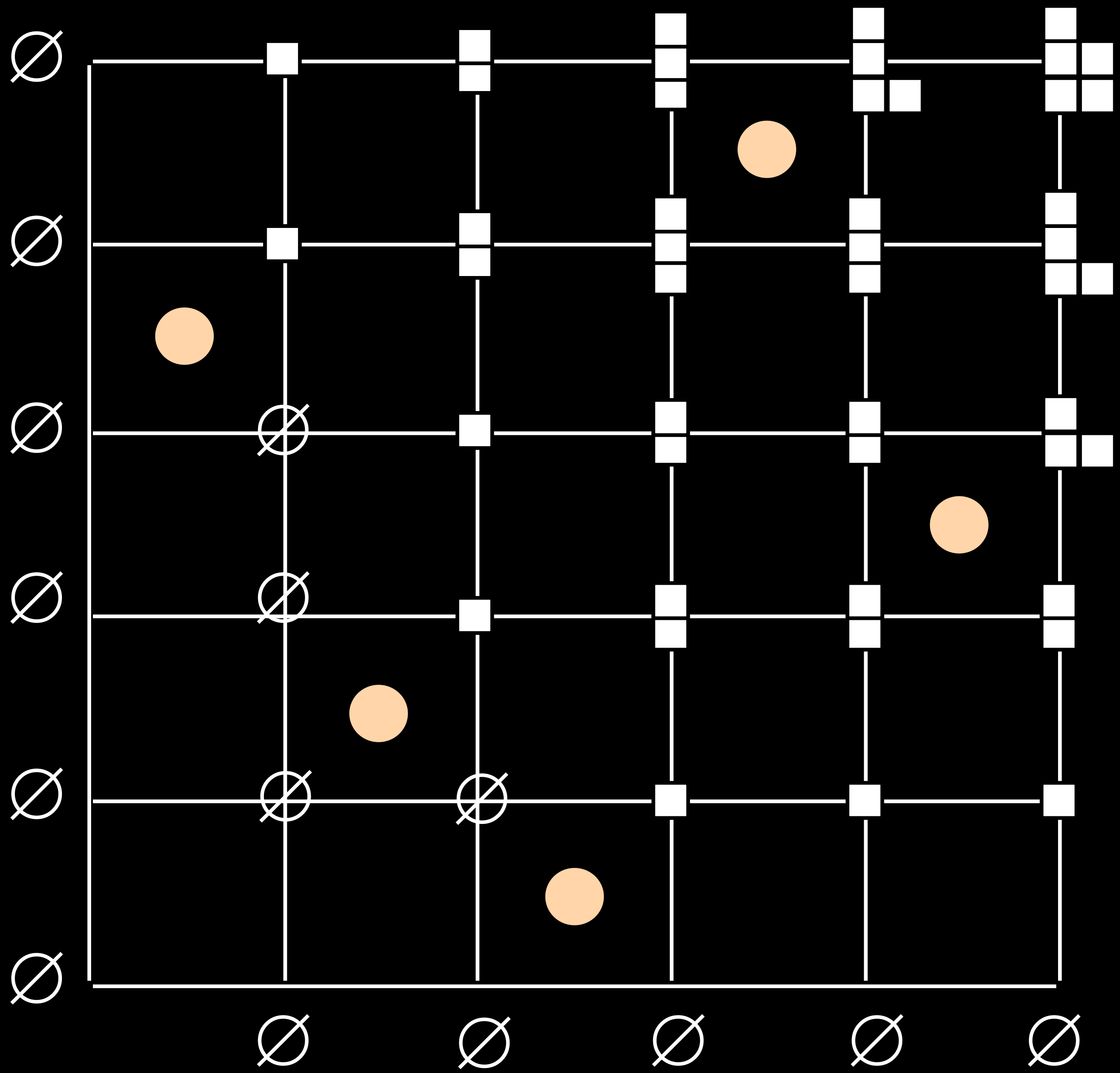


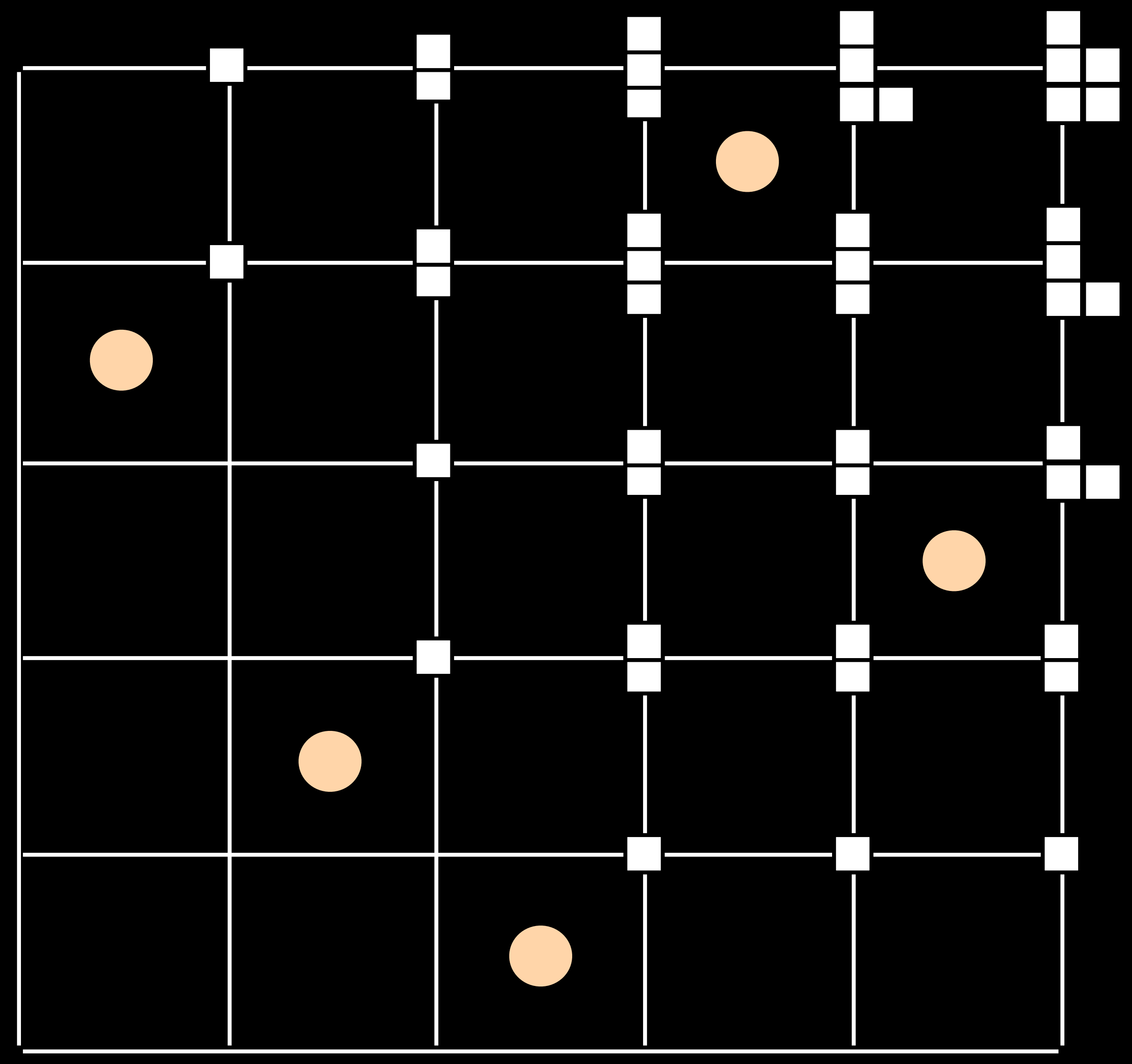
$\sigma = 4, 2, 1, 5, 3$

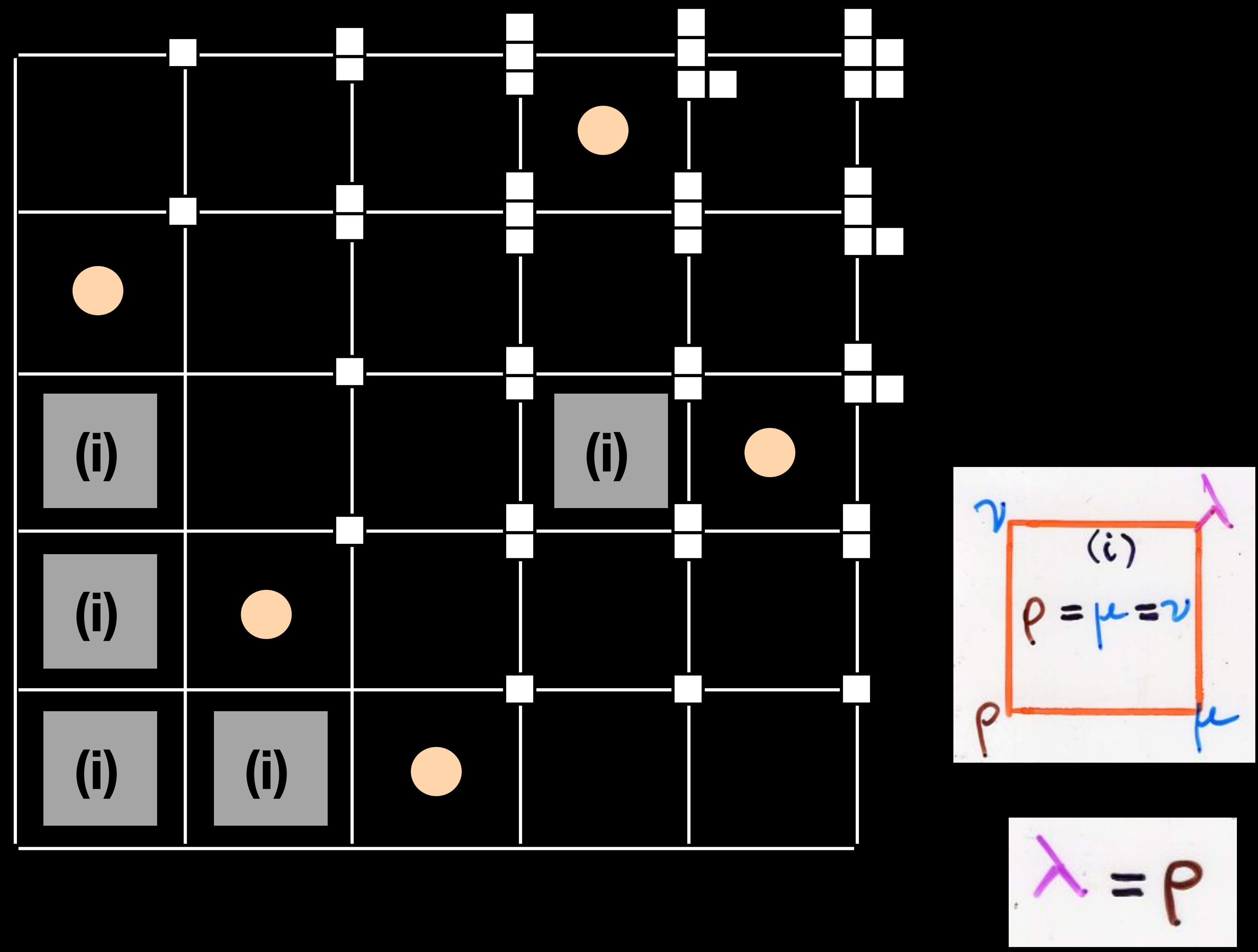
during the labeling process



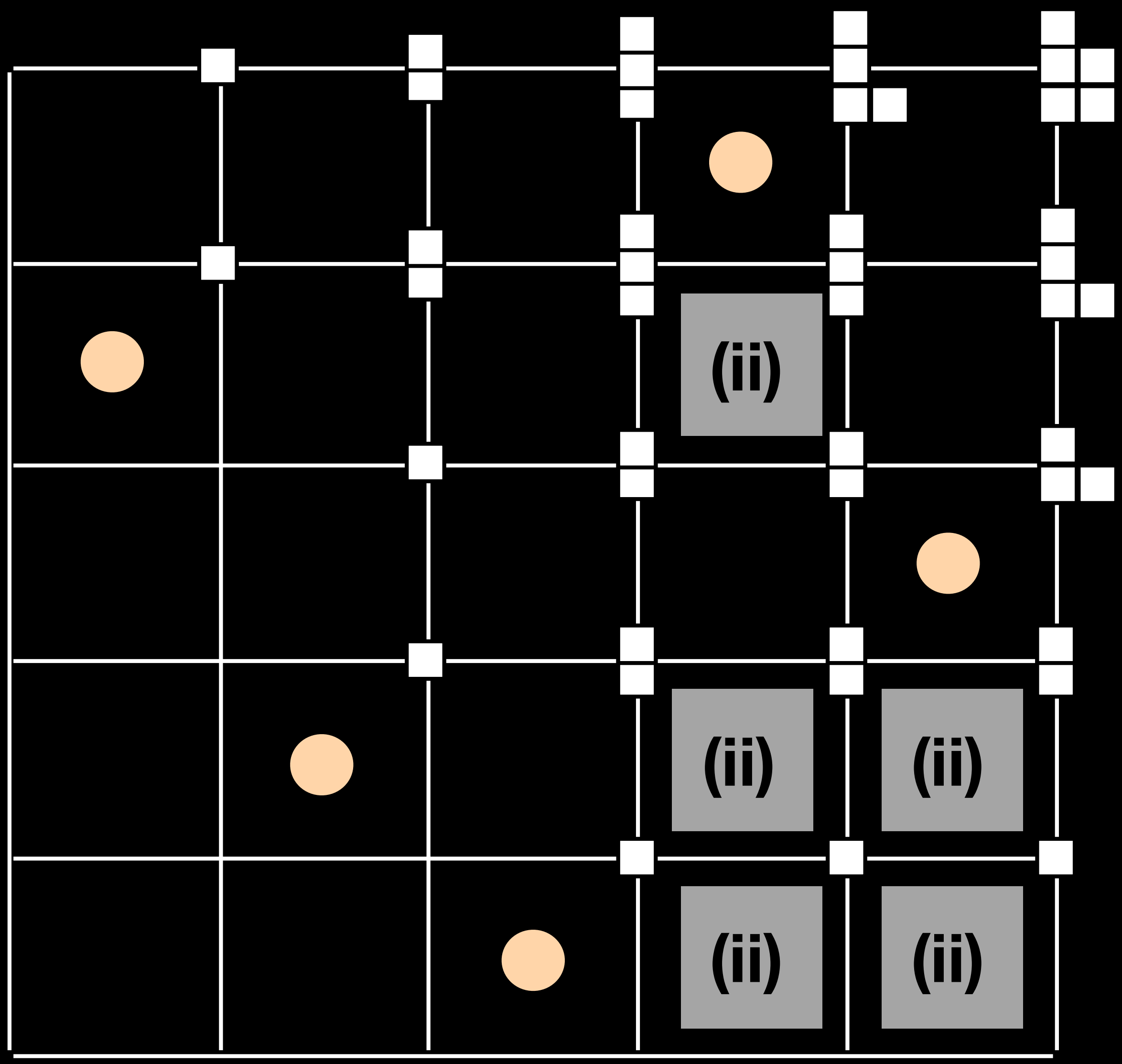
final state

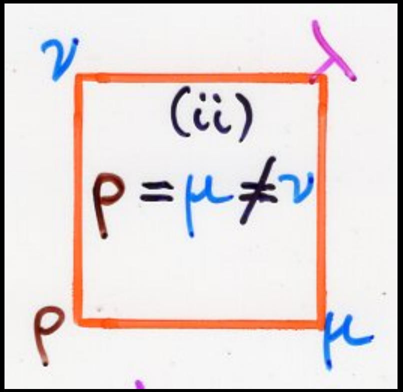




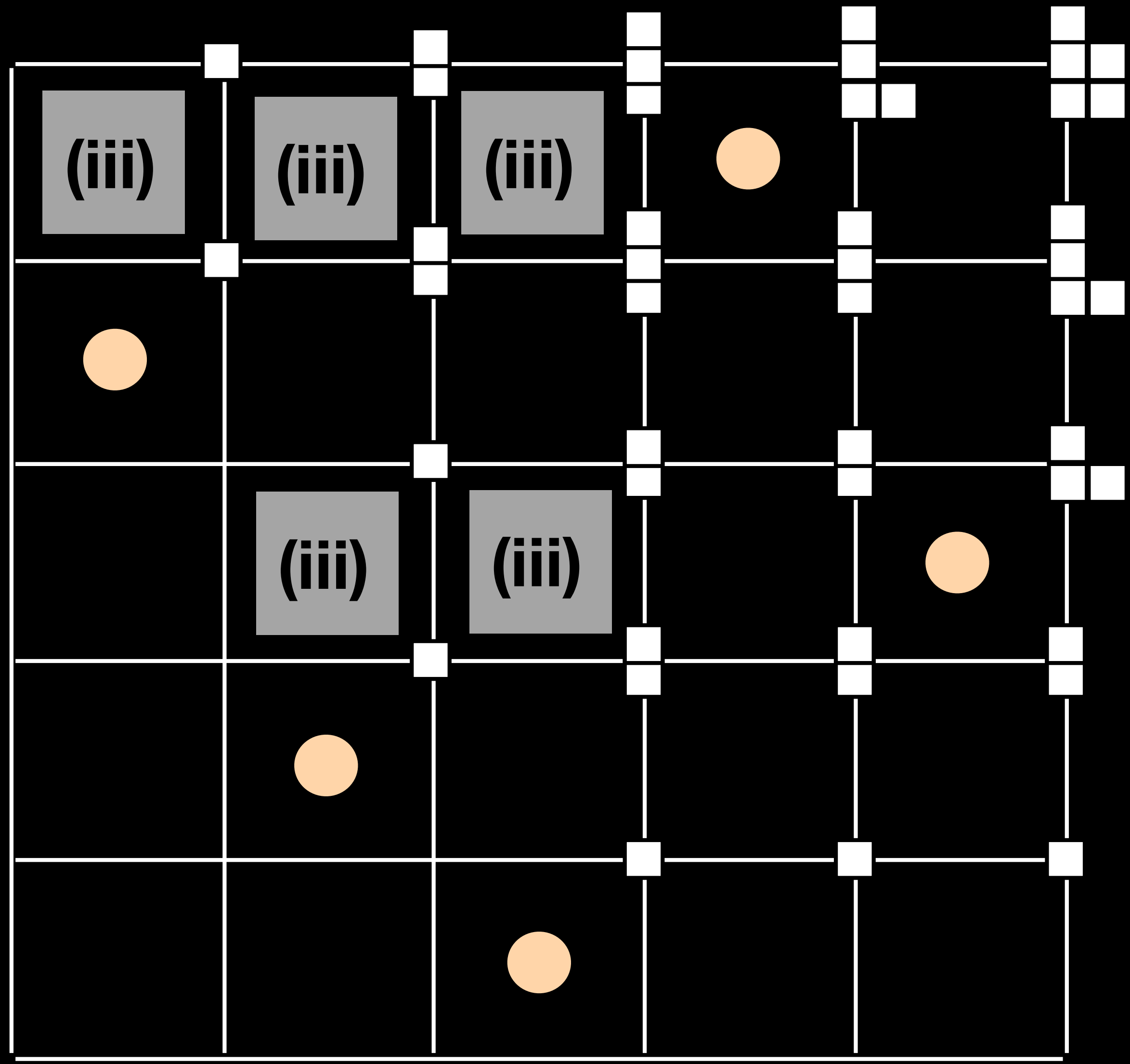


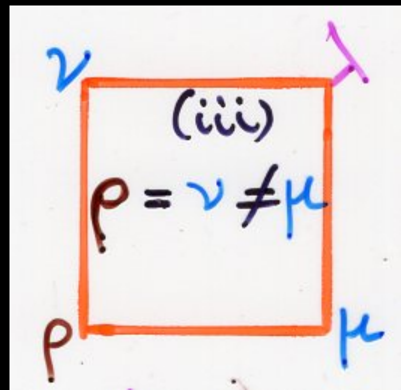
(i)
(i)
(i)
(i)
(i)
(i)
ρ = μ = ν
ν
λ
ρ
μ
λ = ρ


(ii)
(ii)
(ii)
(ii)
(ii)

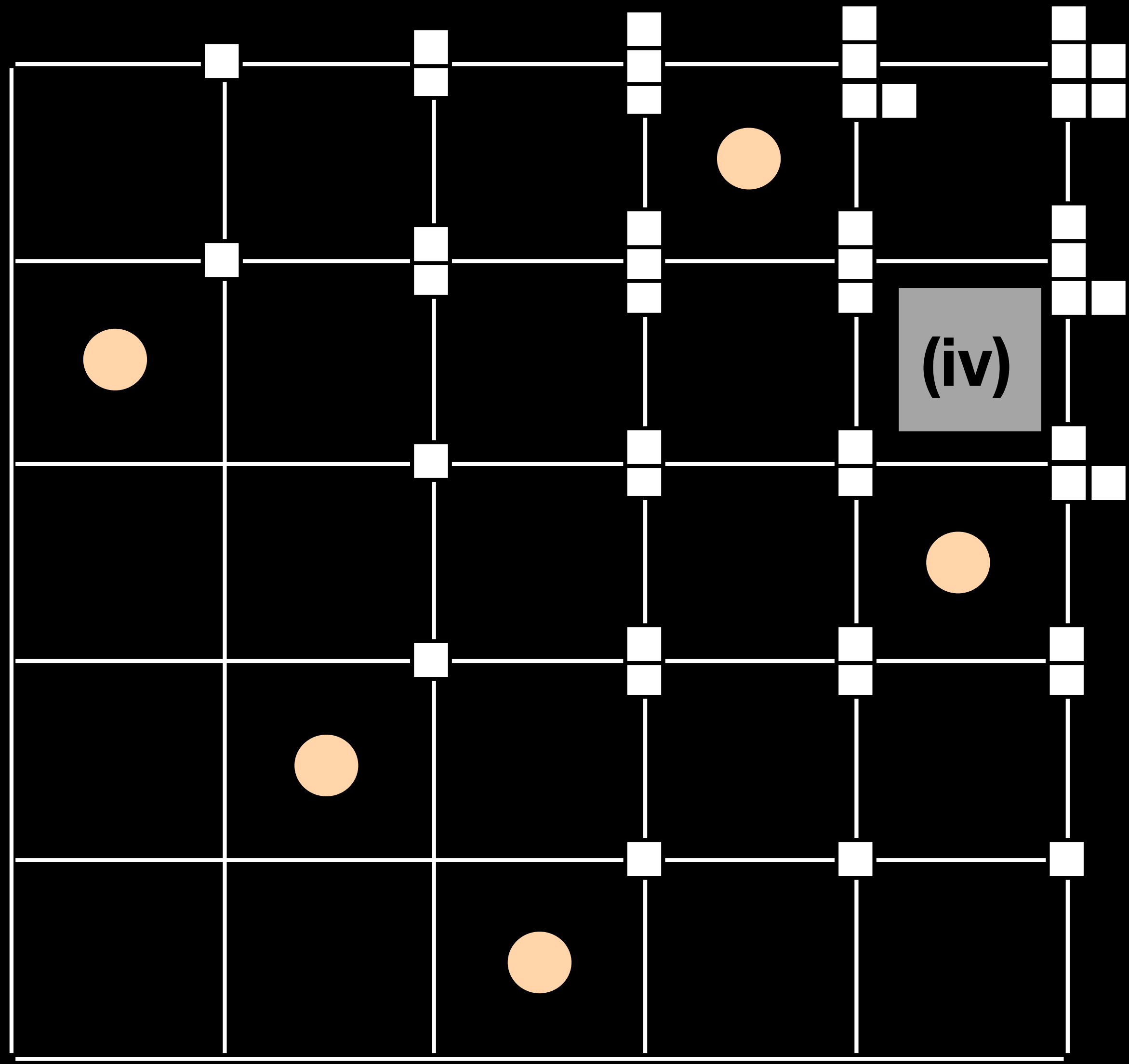

(ii)
ρ = μ ≠ ν

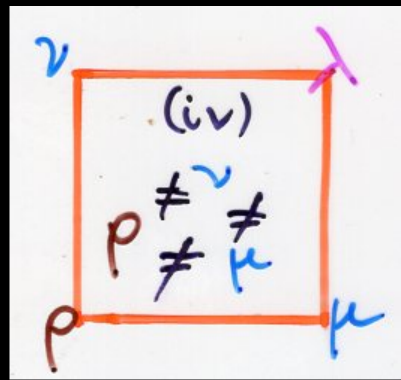

λ = ν



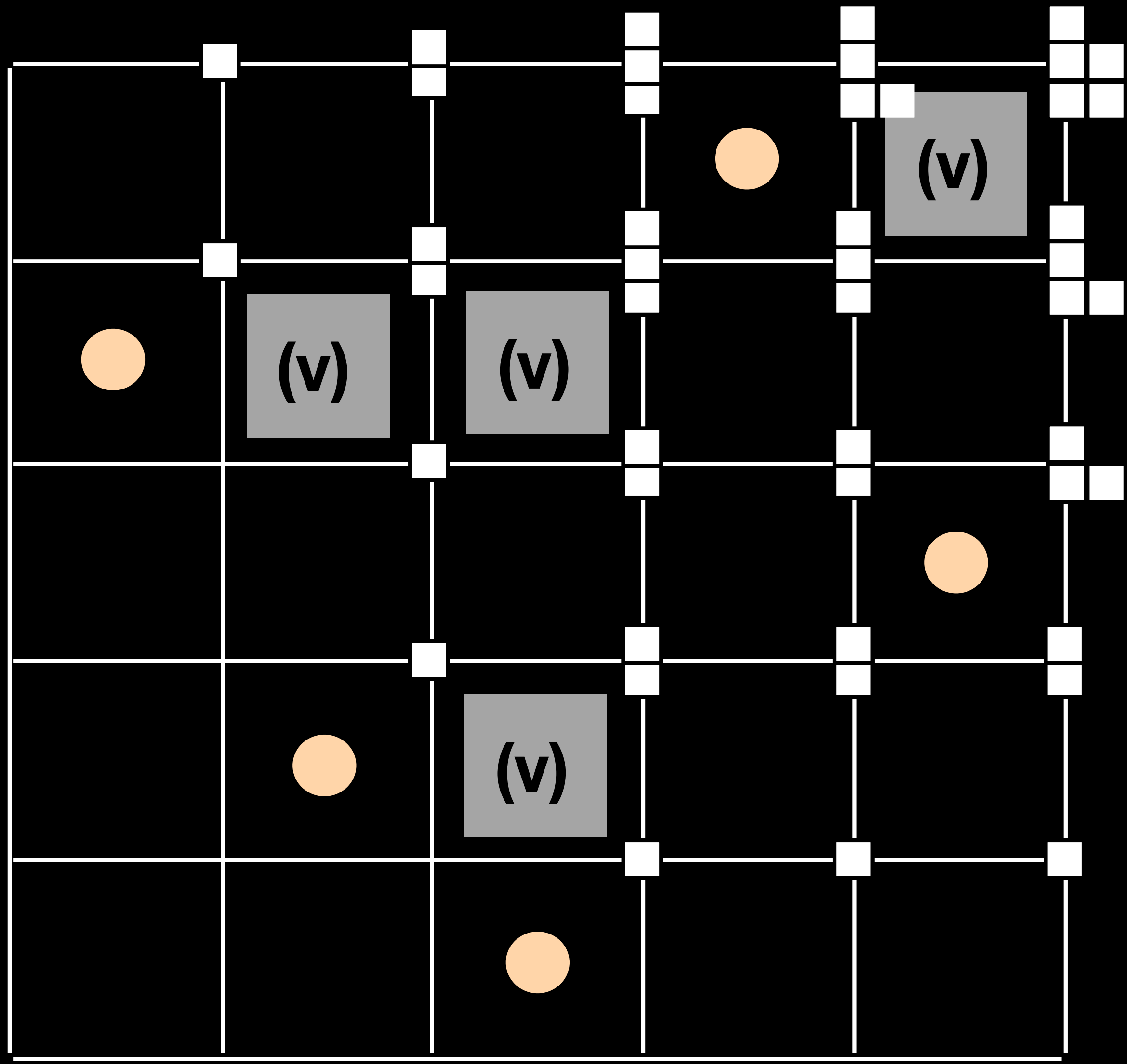


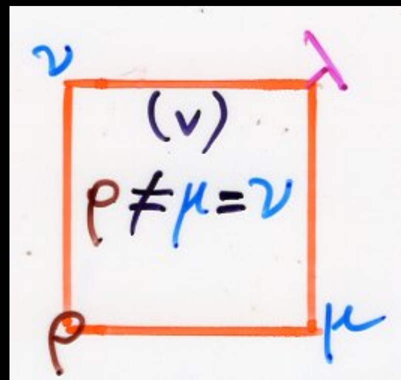
$\lambda = \mu$



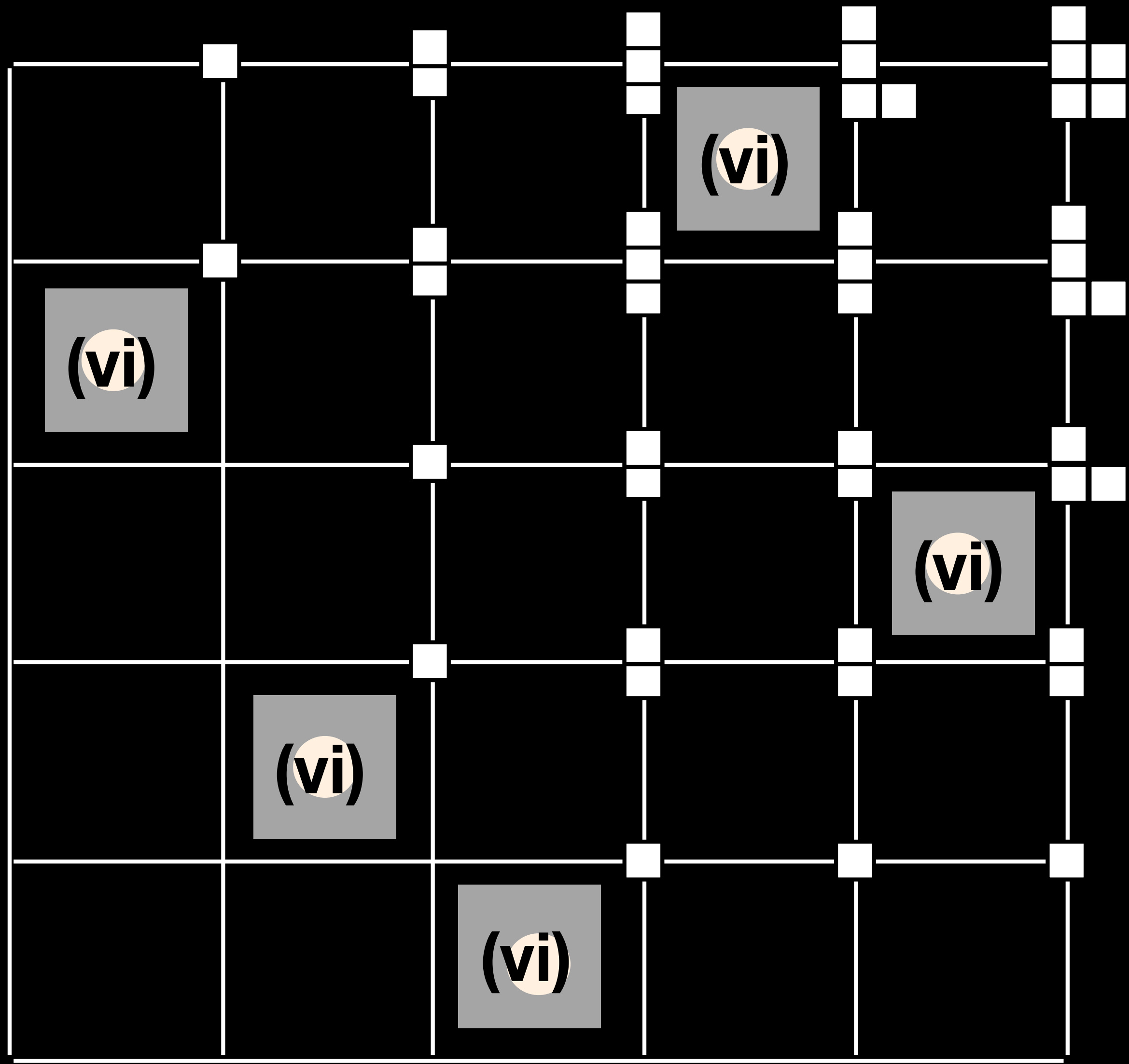


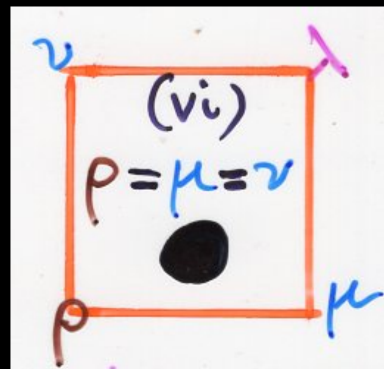
$\lambda = \mu \cup \nu$





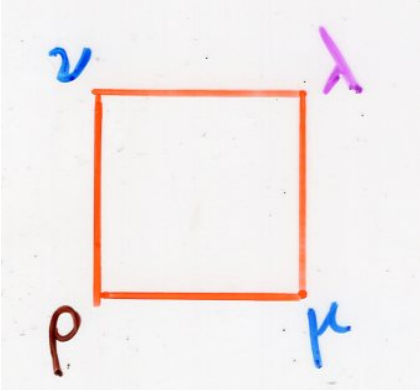
$$\lambda = \left\{\begin{matrix}\mu \\ \nu\end{matrix}\right. + (i+1)$$





$$\lambda = \left\{\begin{matrix}\rho\\ \mu\\ \nu\end{matrix}\right. + (1)$$

- during the labeling process of the vertices of the grid $[n]\times[n]$ with Ferrers diagrams :
  independance of the order in which the labeling is done



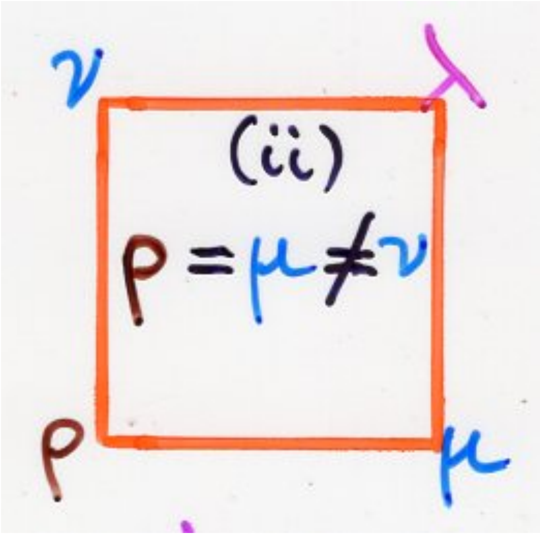
- for every cell :

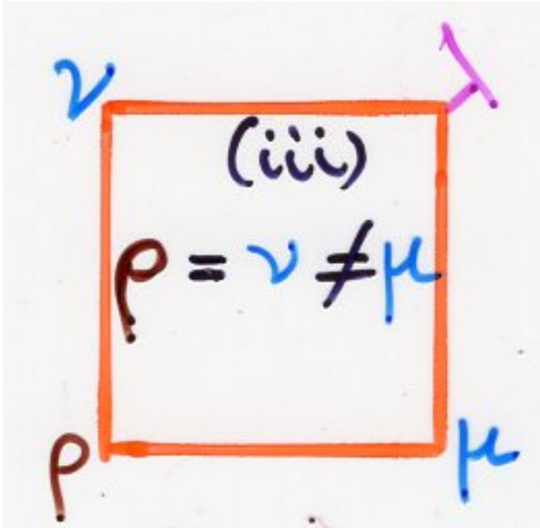
– λ is obtained from μ by adding a cell or $\lambda = \mu$

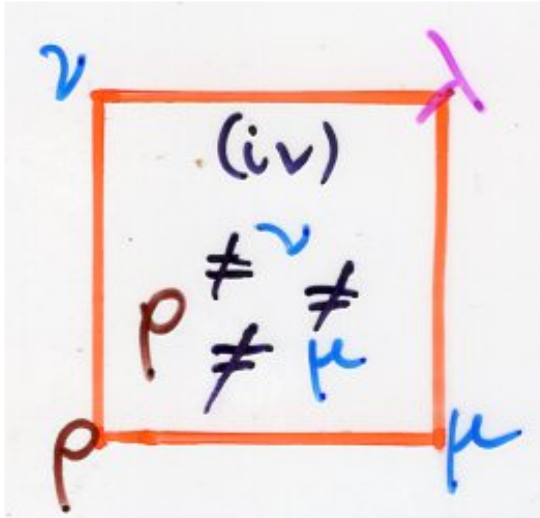
– λ - - - - - - - - - - - - ν - - - - - - - - - - or $\lambda = \nu$

"local rules"

$$\mu \neq \nu$$







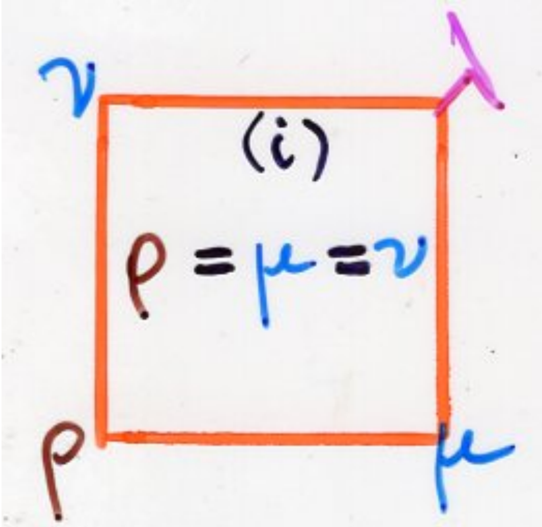
$\rho = \mu$
$\lambda = \nu = \rho + (i)$

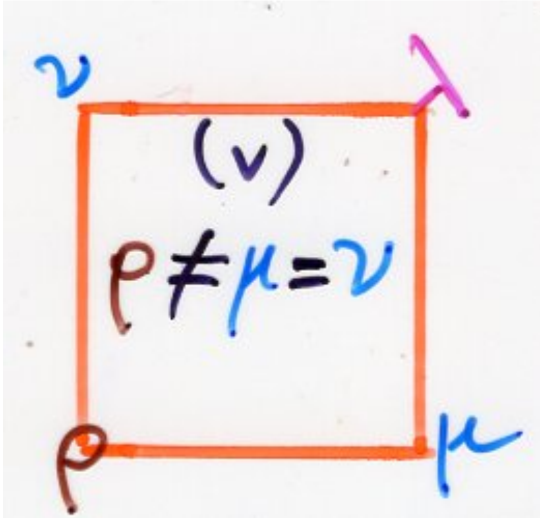
$\rho = \nu$
$\lambda = \mu = \rho + (j)$

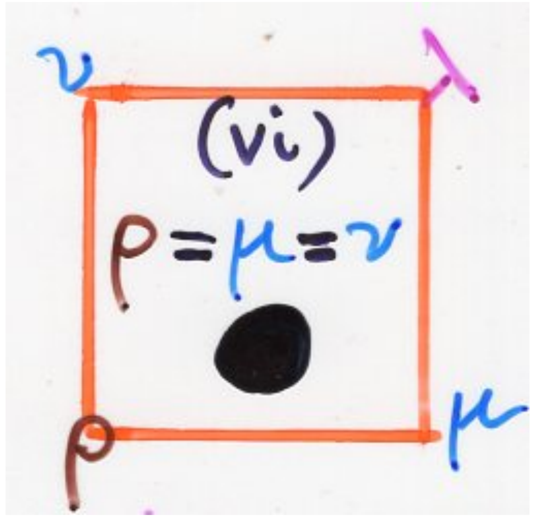
$\nu = \rho + (i)$
$\mu = \rho + (j)$
$\lambda = \rho + (i) + (j)$

"local rules"

$$\mu = \nu$$



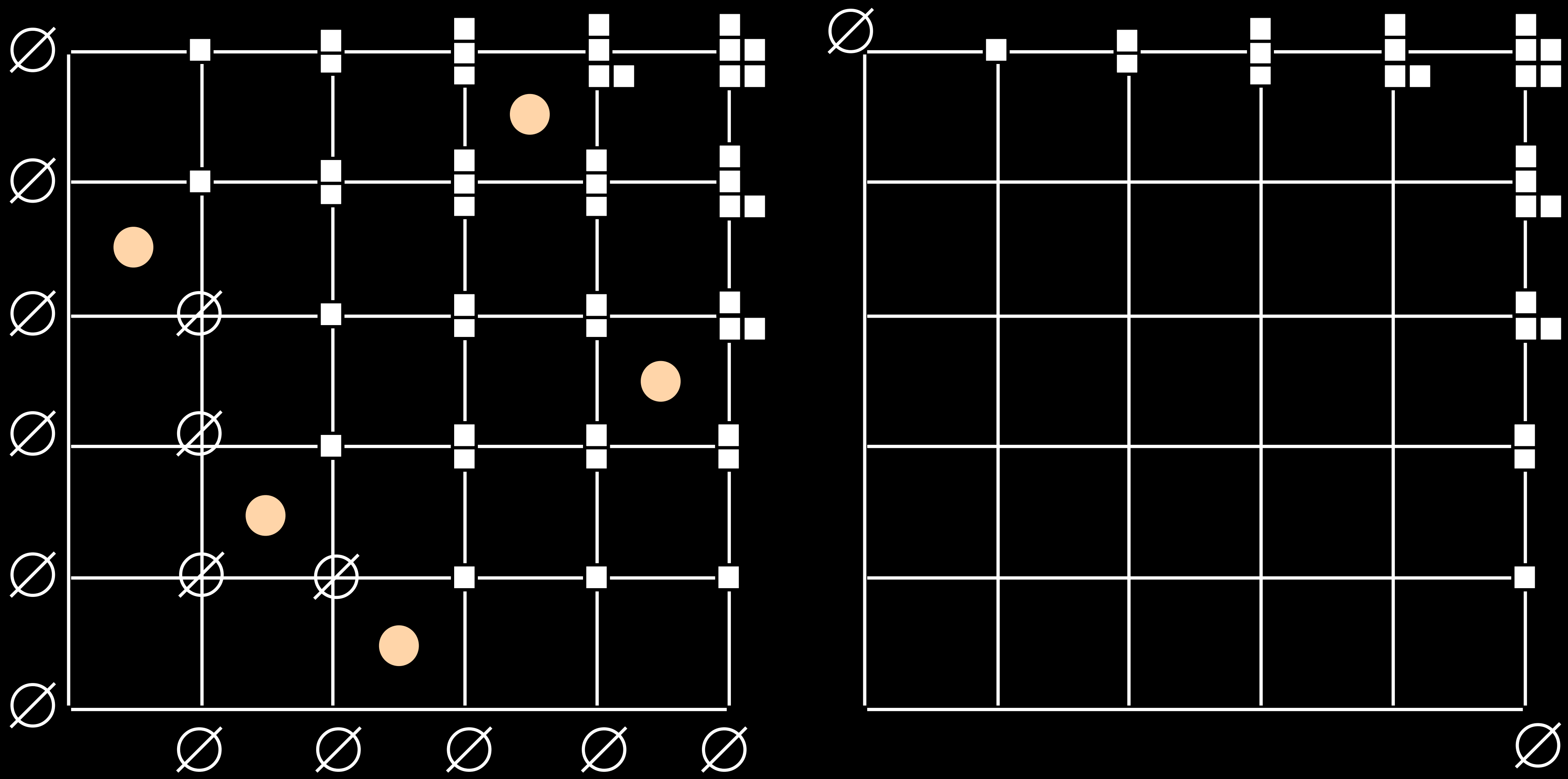


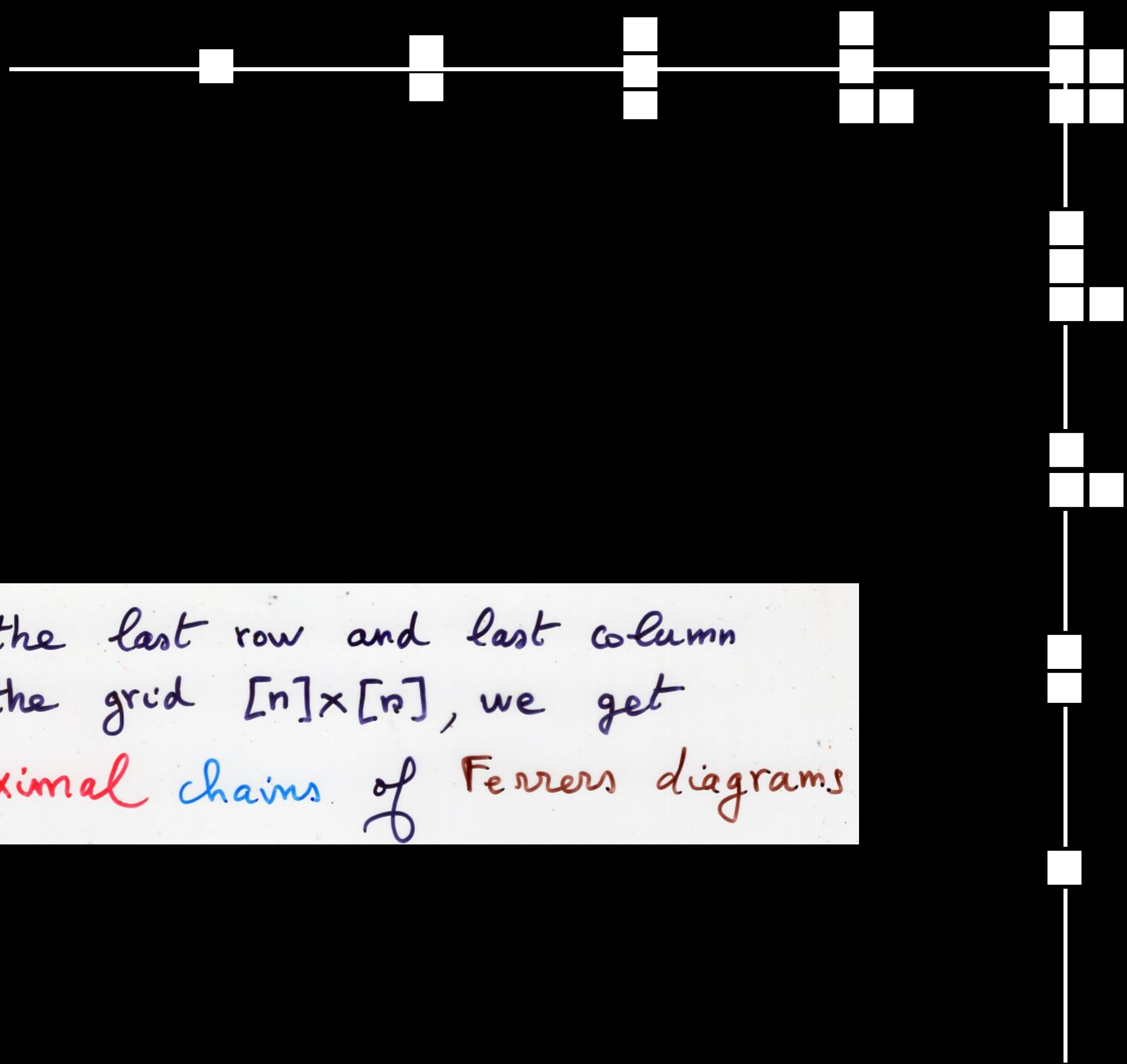


$$\lambda = \rho = \mu = \nu$$

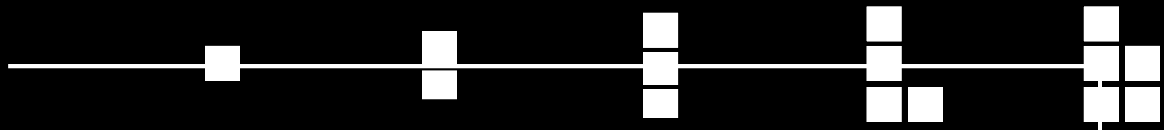
$$\mu = \nu = \rho + (i)$$
$$\lambda = \mu + (i+1)$$

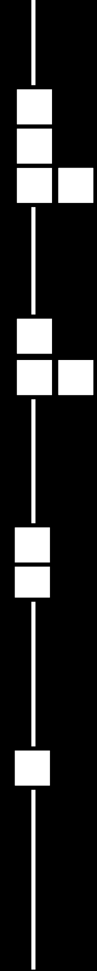
$$\lambda = \left\{\begin{matrix}\rho \\ \mu \\ \nu\end{matrix}\right. + (1)$$

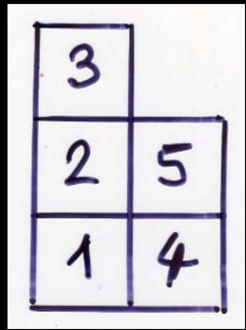


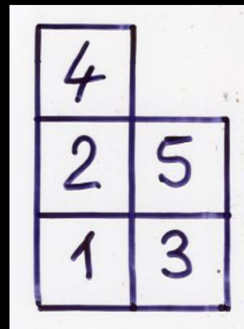


- in the last row and last column of the grid $[n] \times [n]$, we get maximal chains of Ferrers diagrams



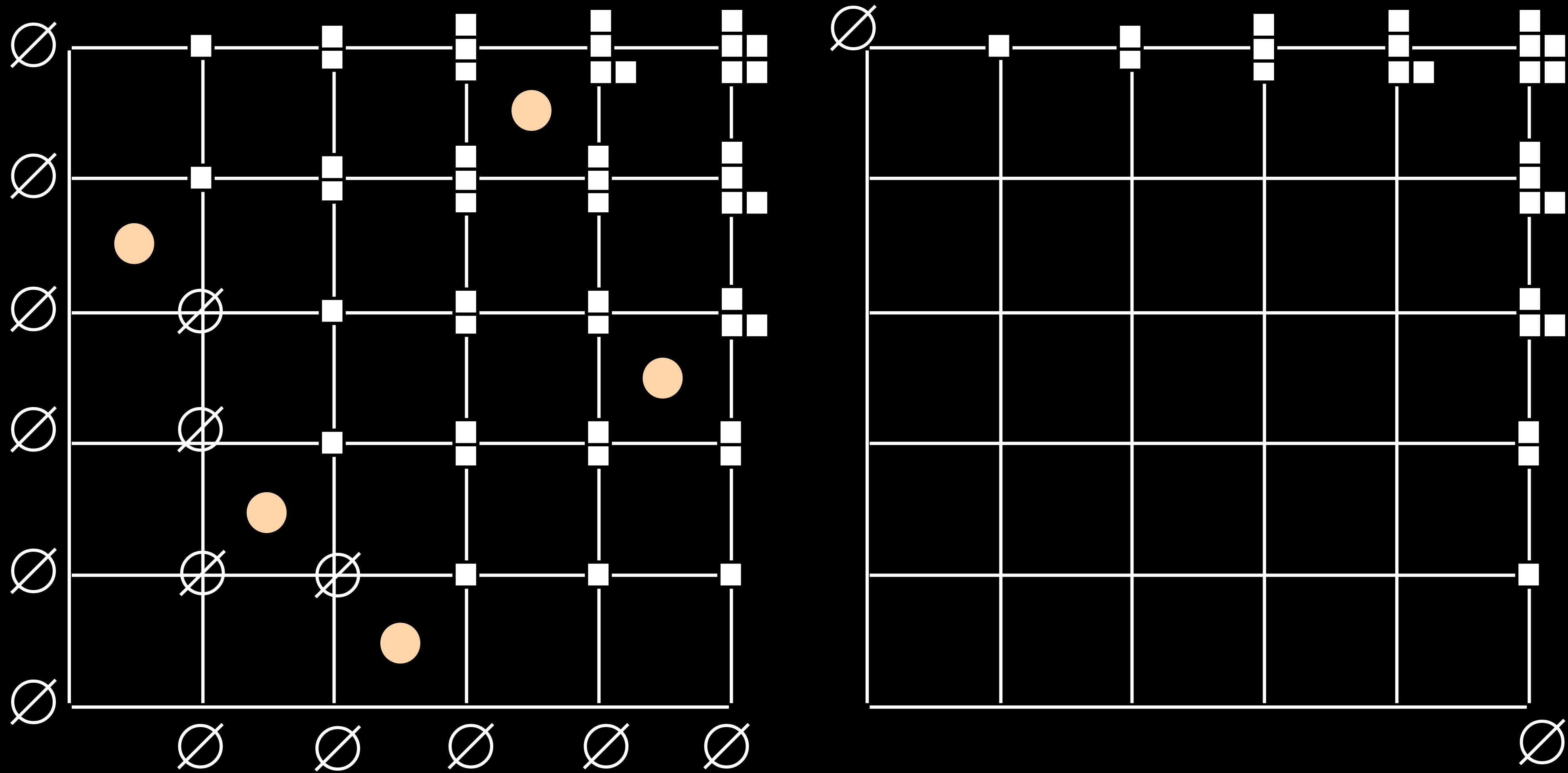




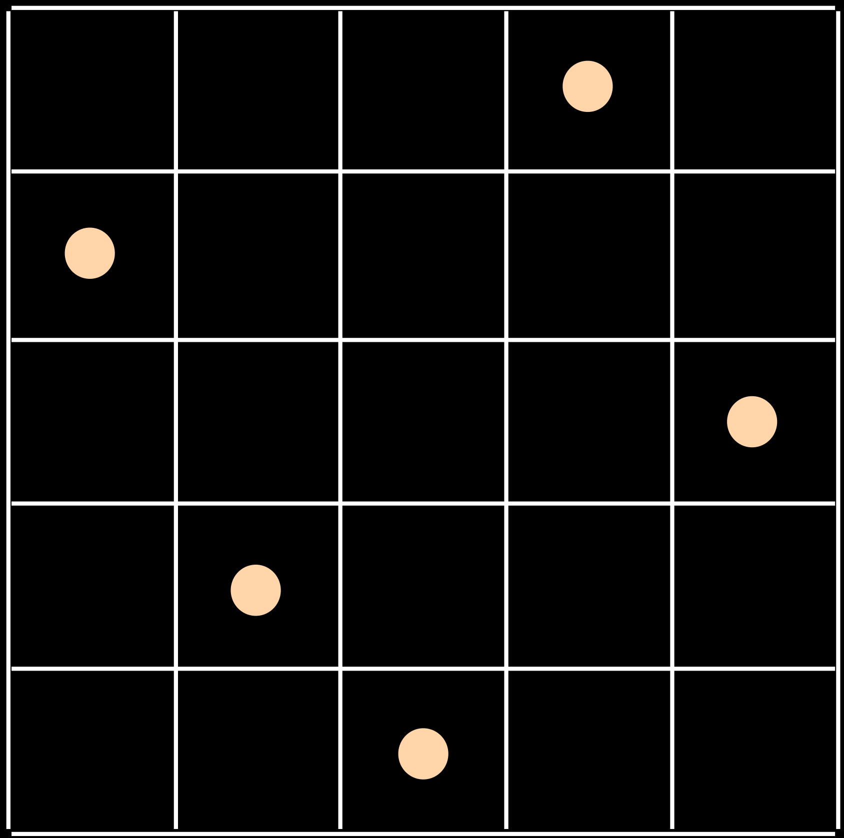


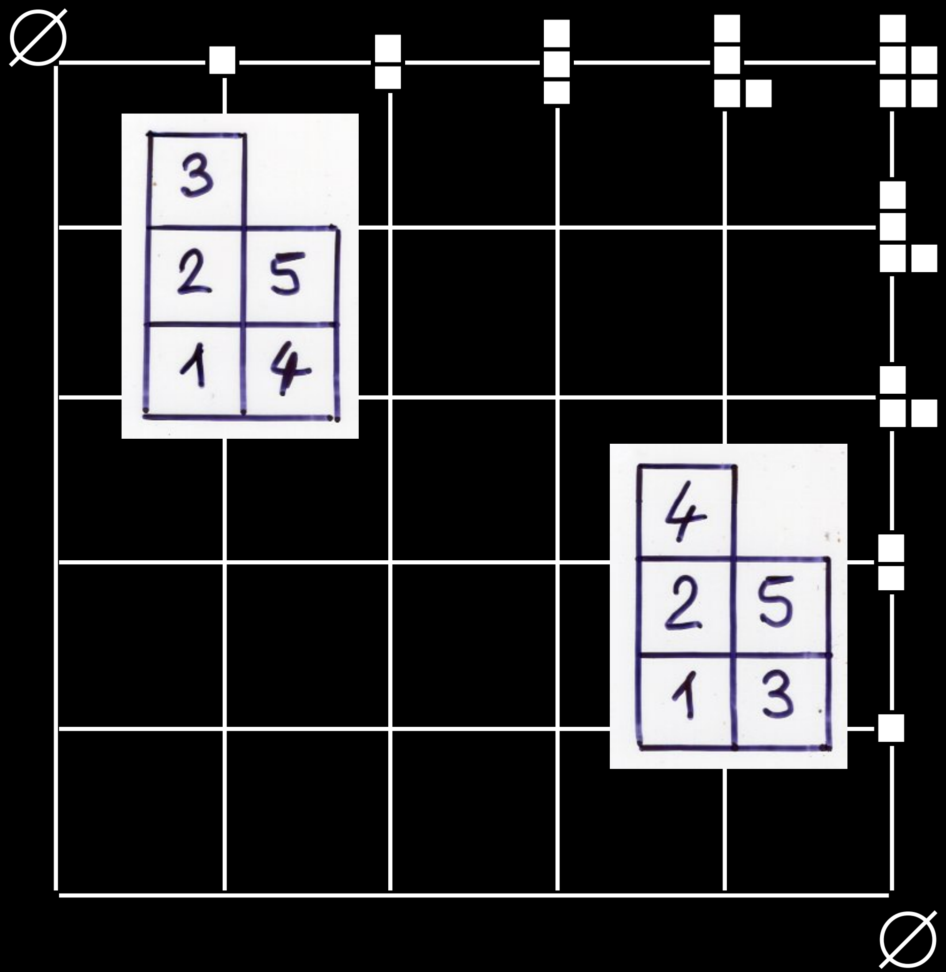
- in the last row and last column of the grid $[n]\times[n]$, we get maximal chains of Ferrers diagrams

- these maximal chains encode a pair (P, Q) of Young tableaux having the same shape



- the algorithm can be reversed : from the pair (P, Q) , get back the permutation

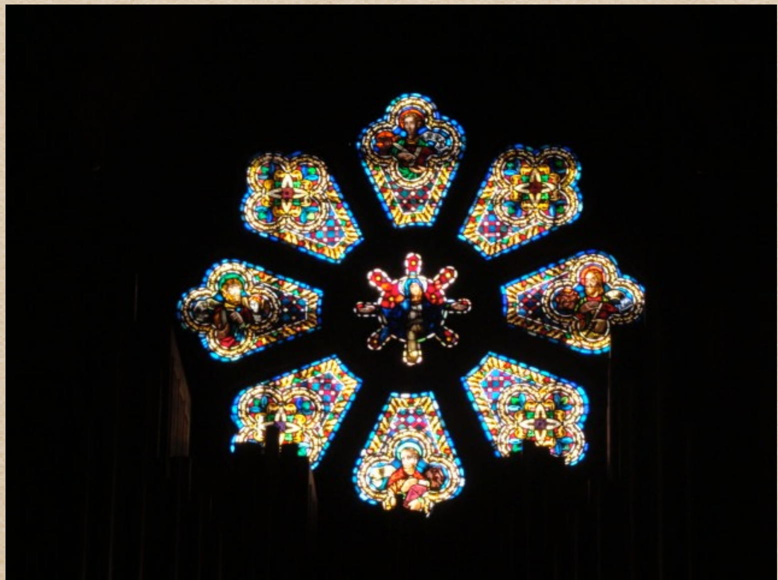


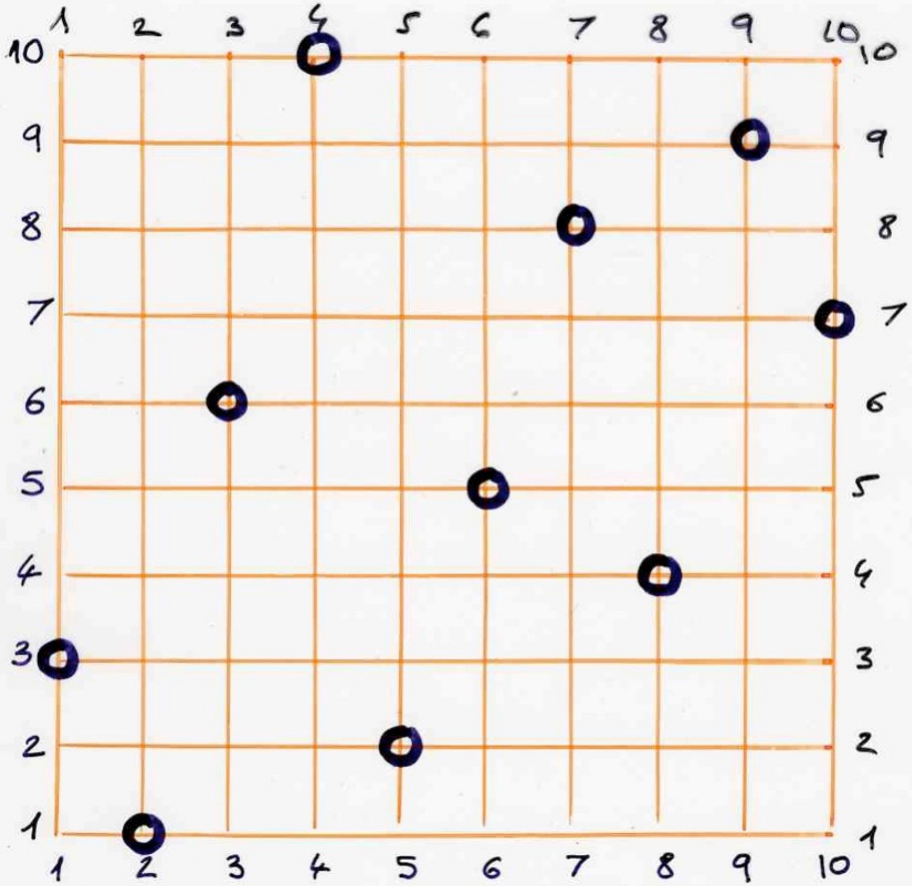


- this bijection is the same as the Robinson-Schensted correspondence

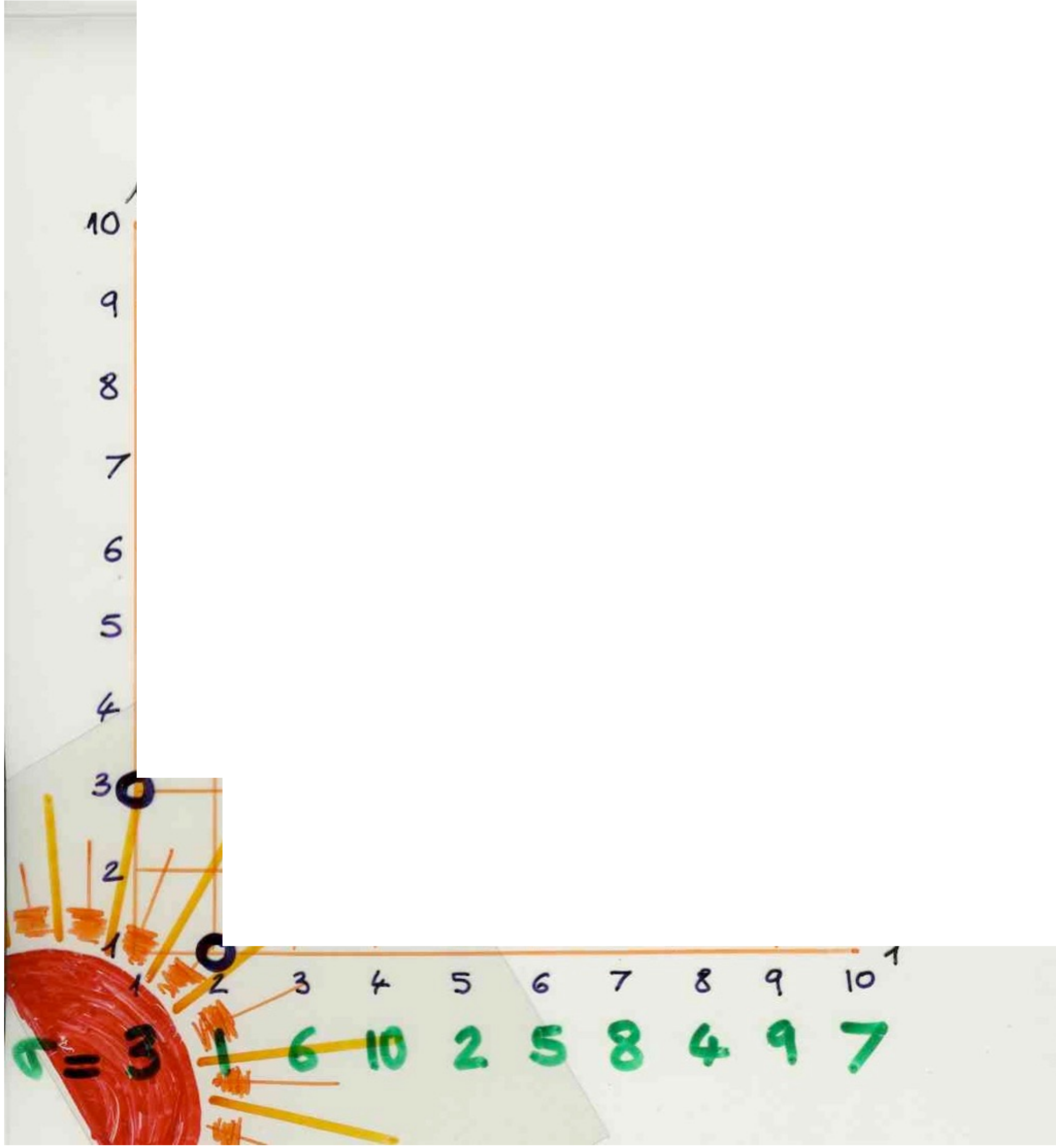
proof of the equivalence
local RS and geometric RS

Recalling the geometric version of RS with "light" and "shadow lines" (see Ch1a)

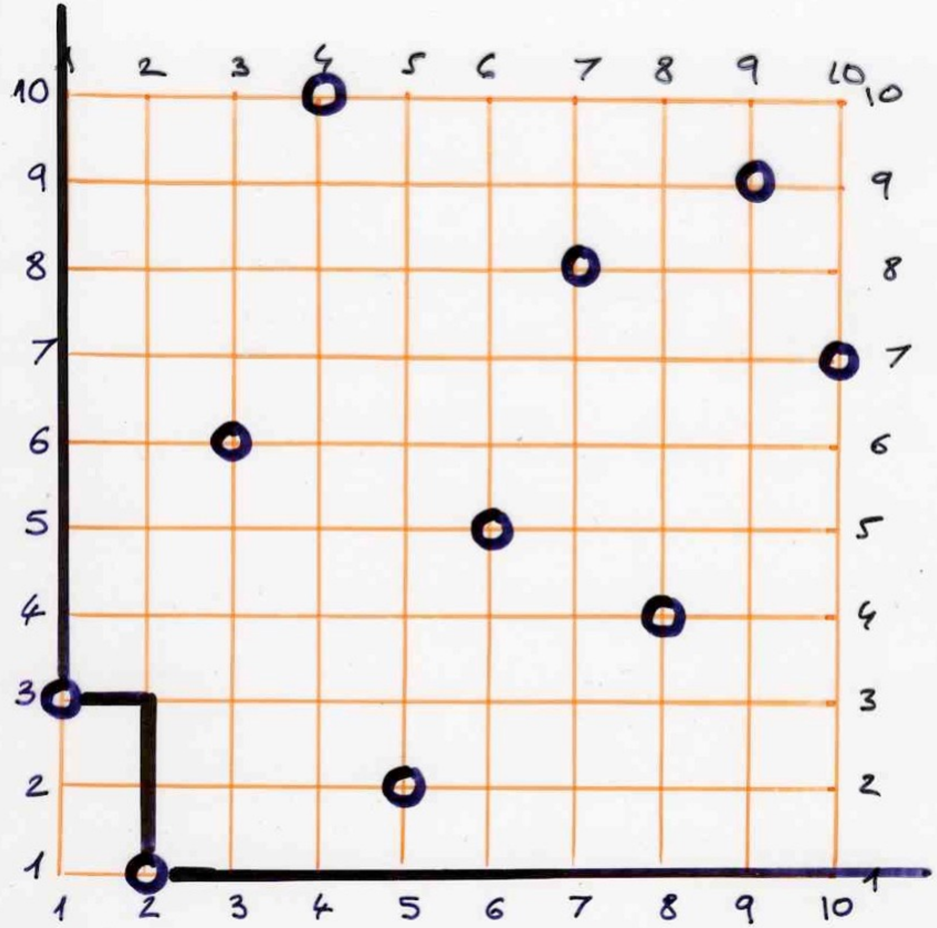




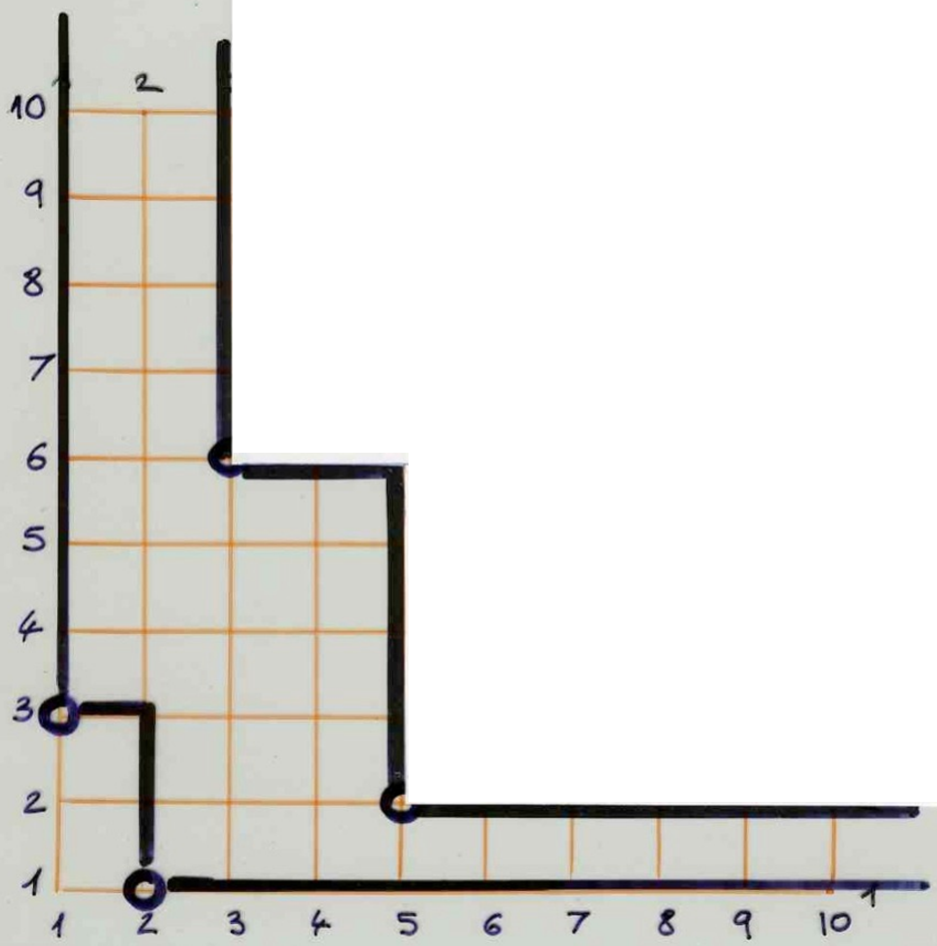
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



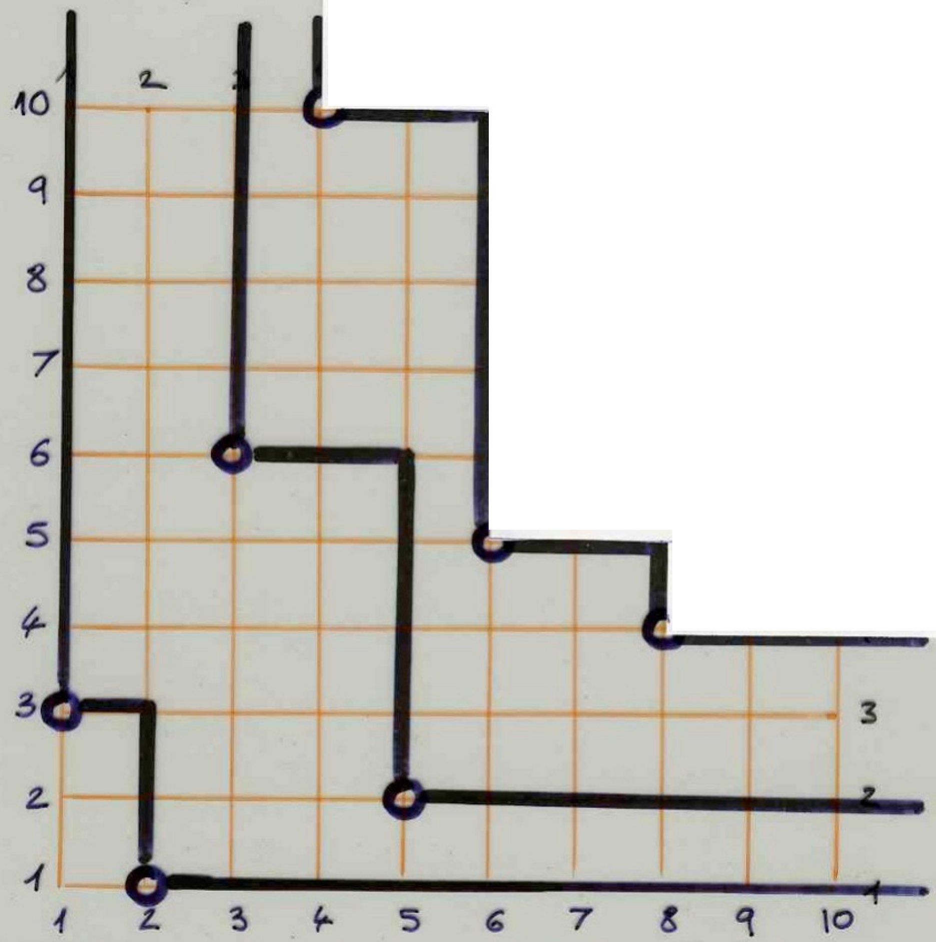
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
σ = 3
1 6 10 2 5 8 4 9 7



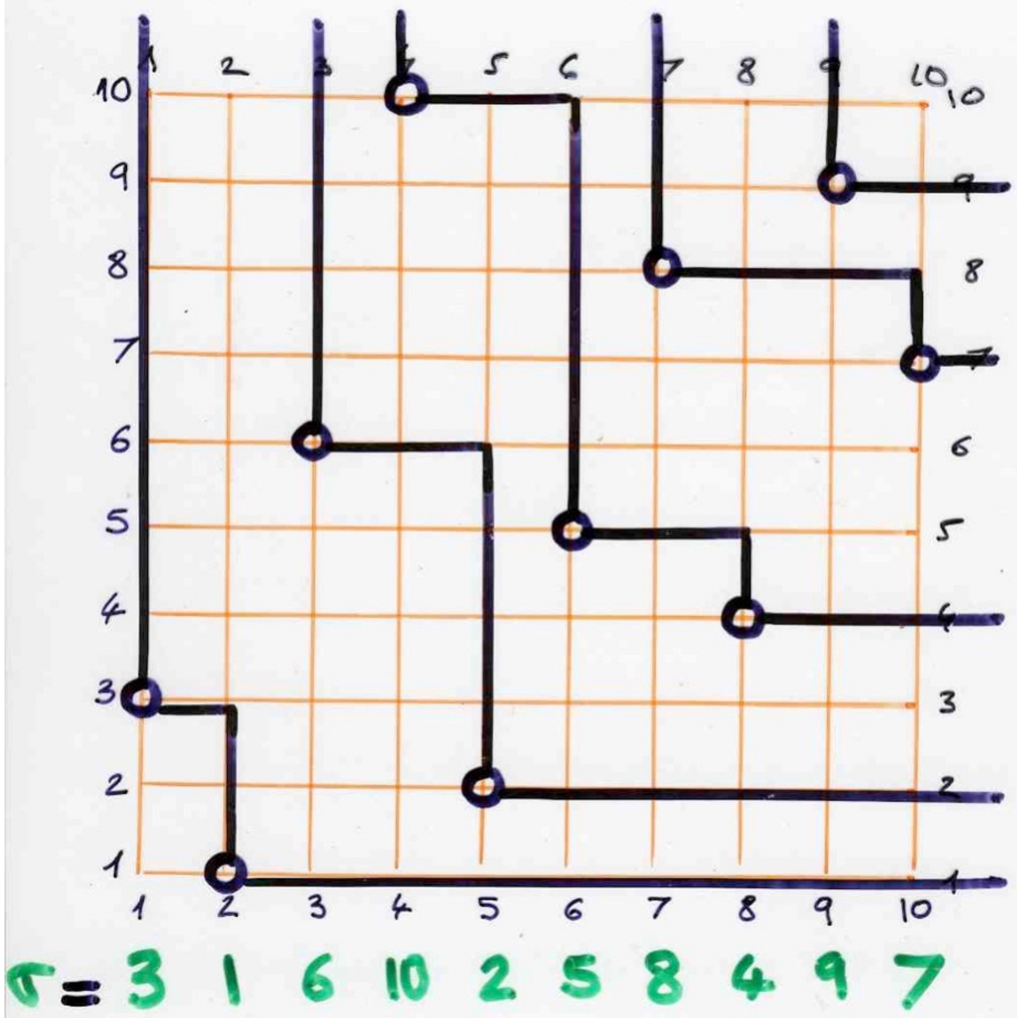
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



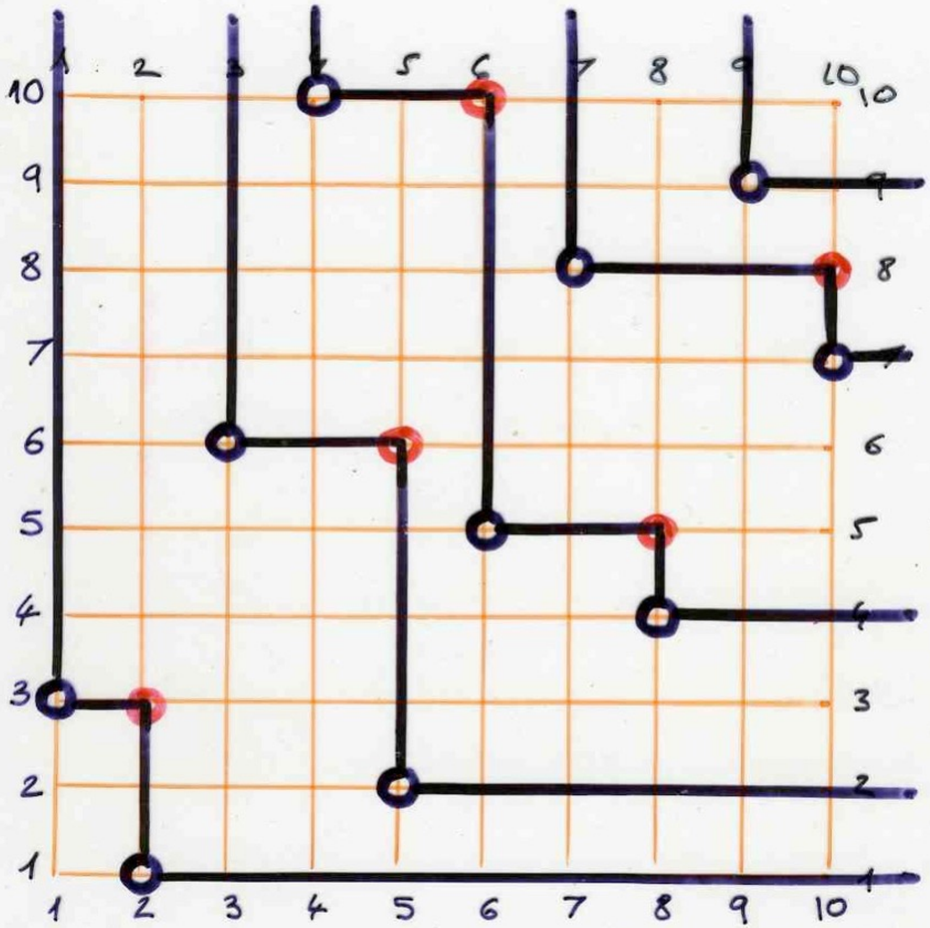
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



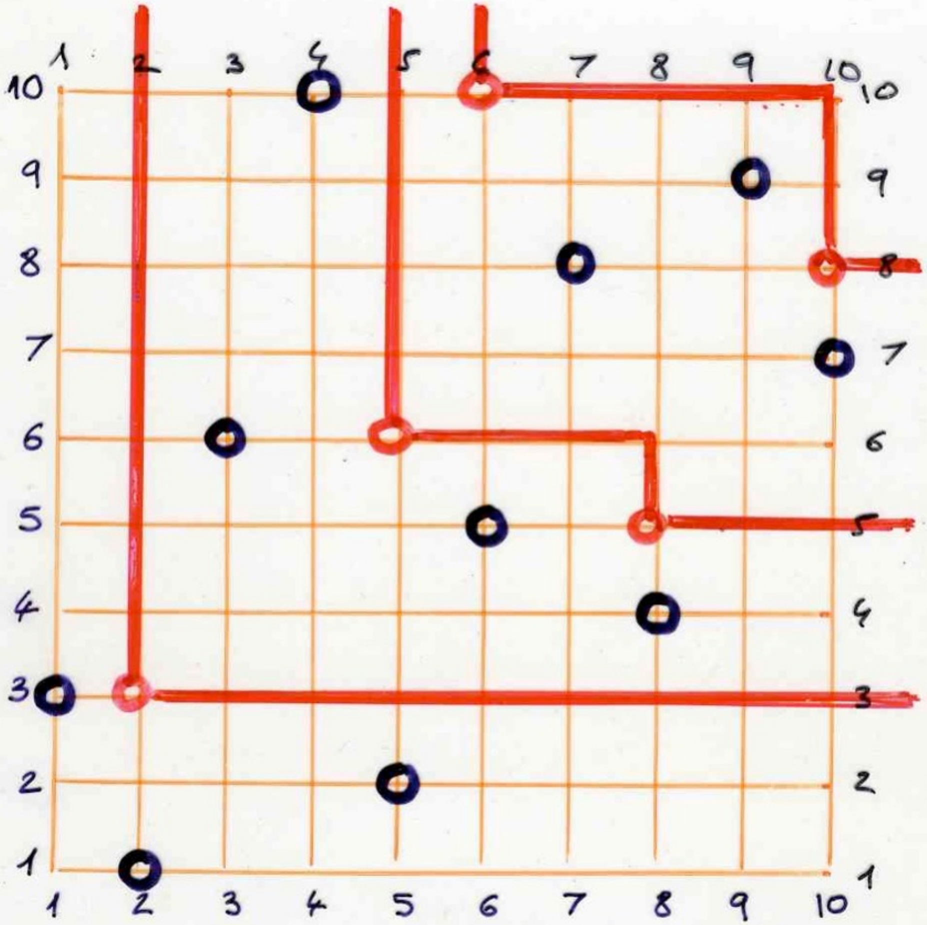
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



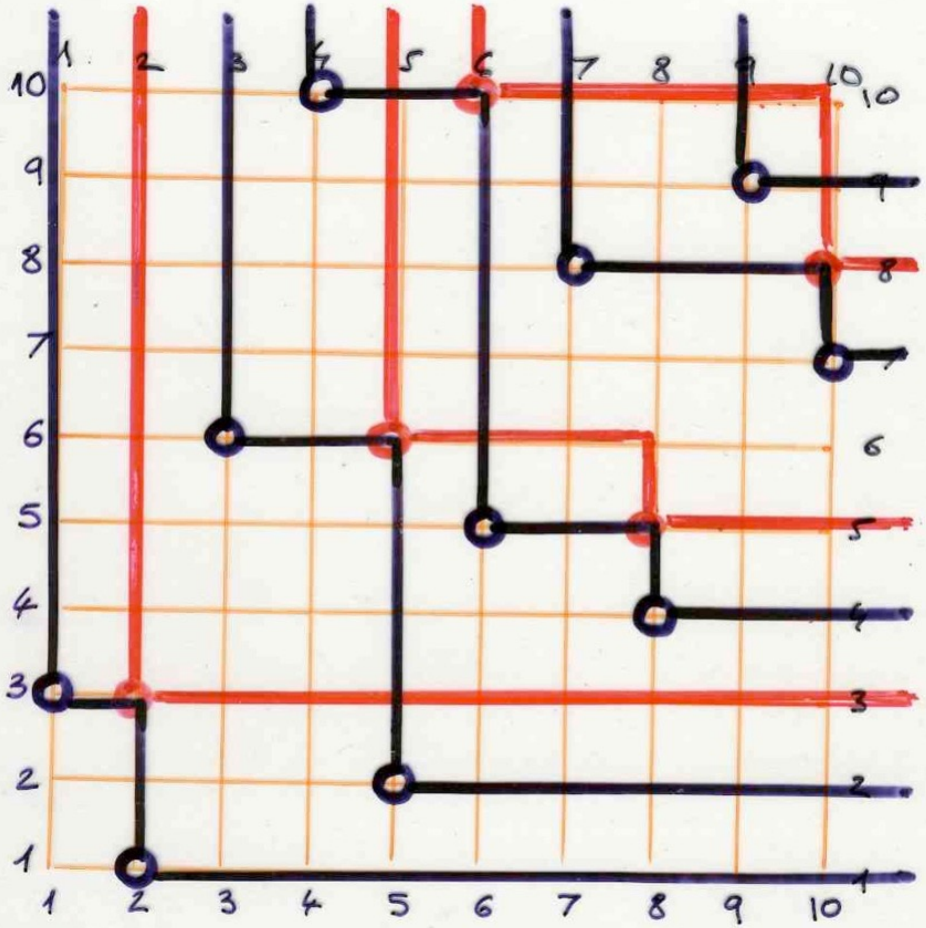
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



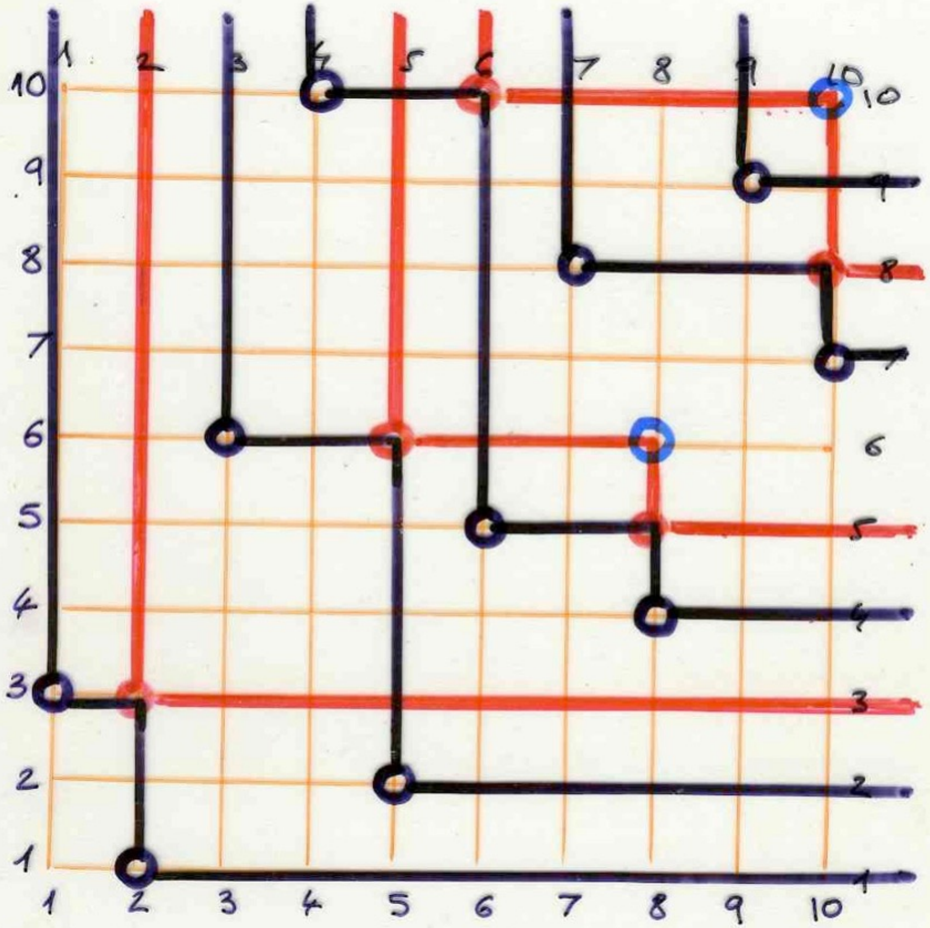
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



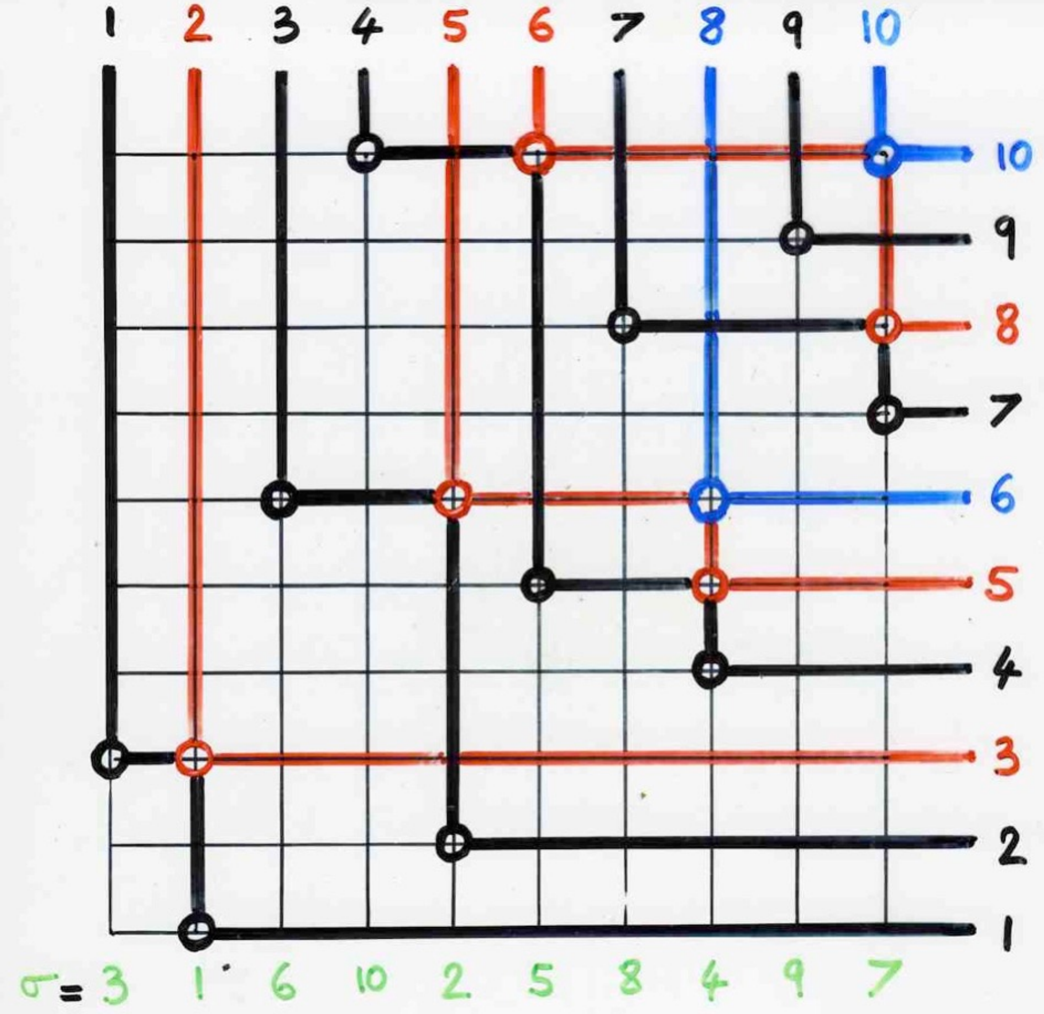
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

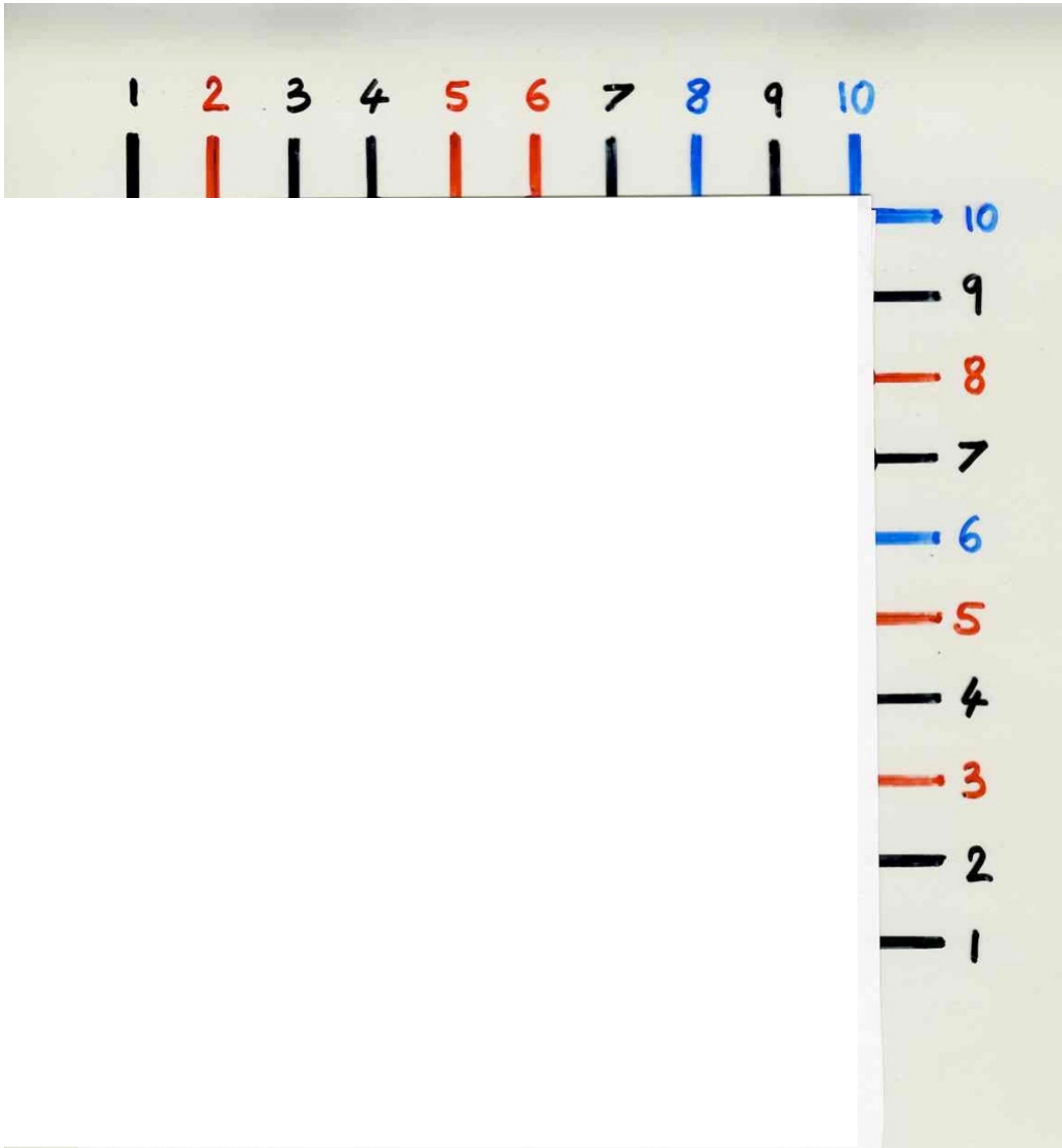


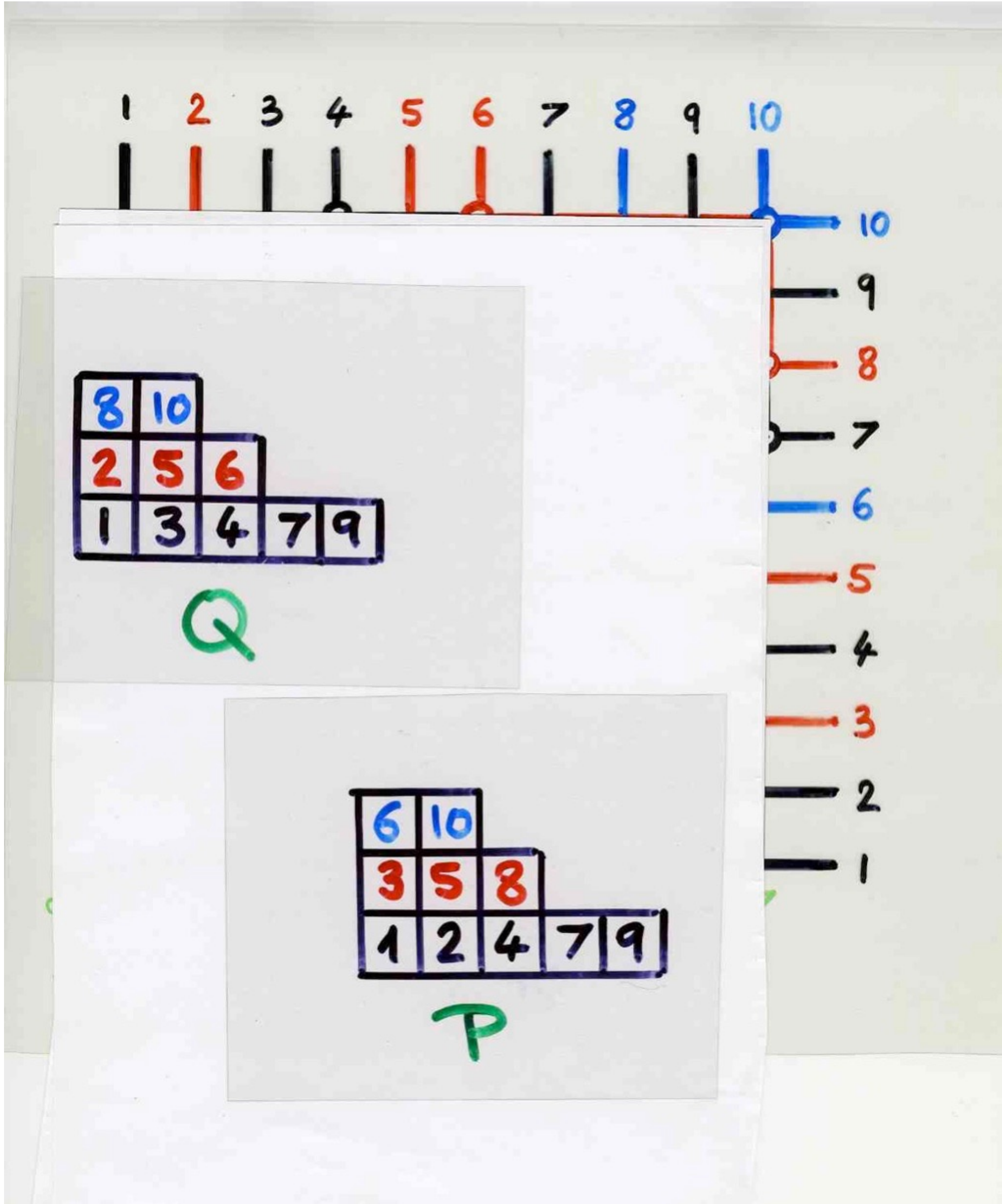
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



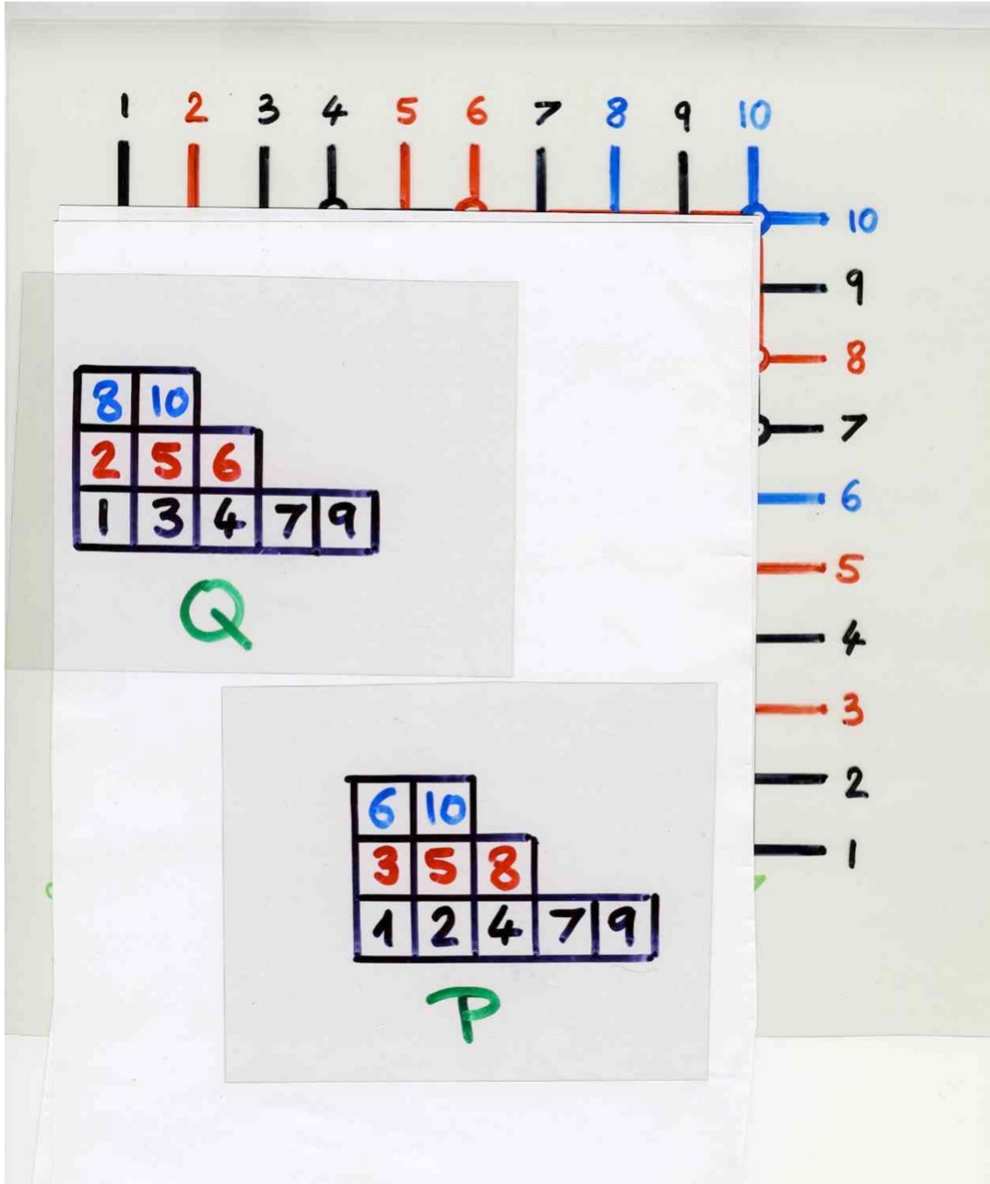
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P

geometric version with "light" and "shadow"



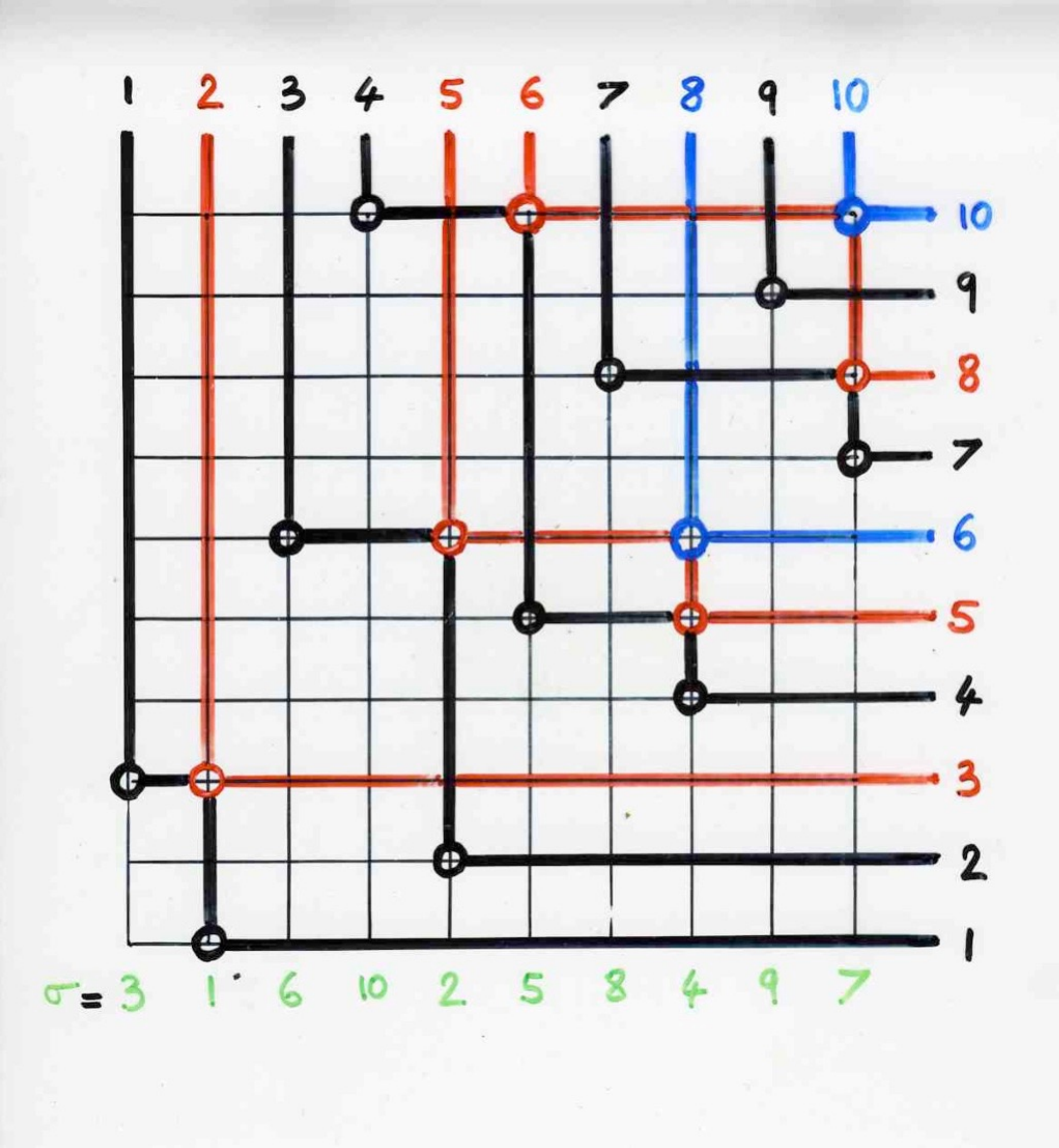
Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

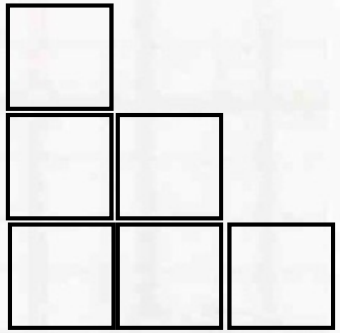
6	10				
3	5	8			
1	2	4	7	9	

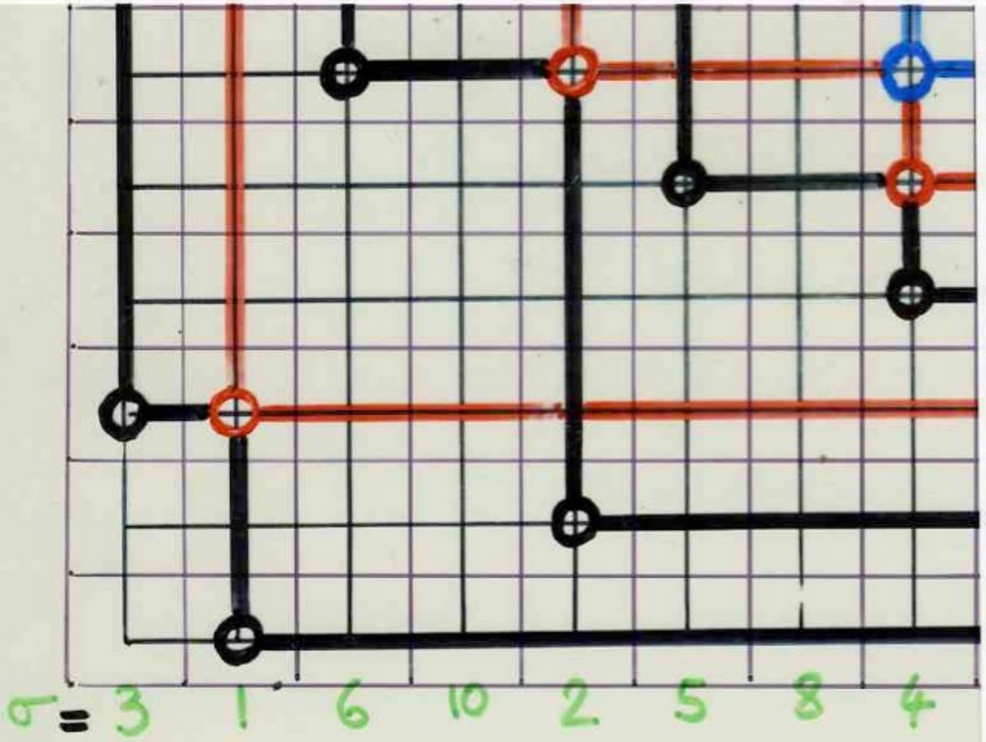
proof of the equivalence
local RS and geometric RS



For any vertex of the grid translated by 1/2 we define a Ferrers diagram in the following Way

We get a tableau of Ferrers diagrams





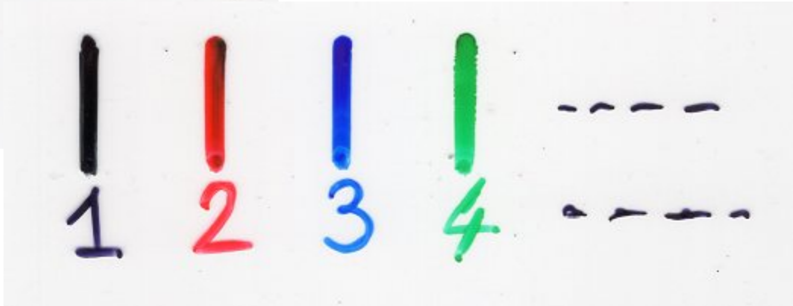
I claim that this tableau is the same as the one we get from the local rule algorithm

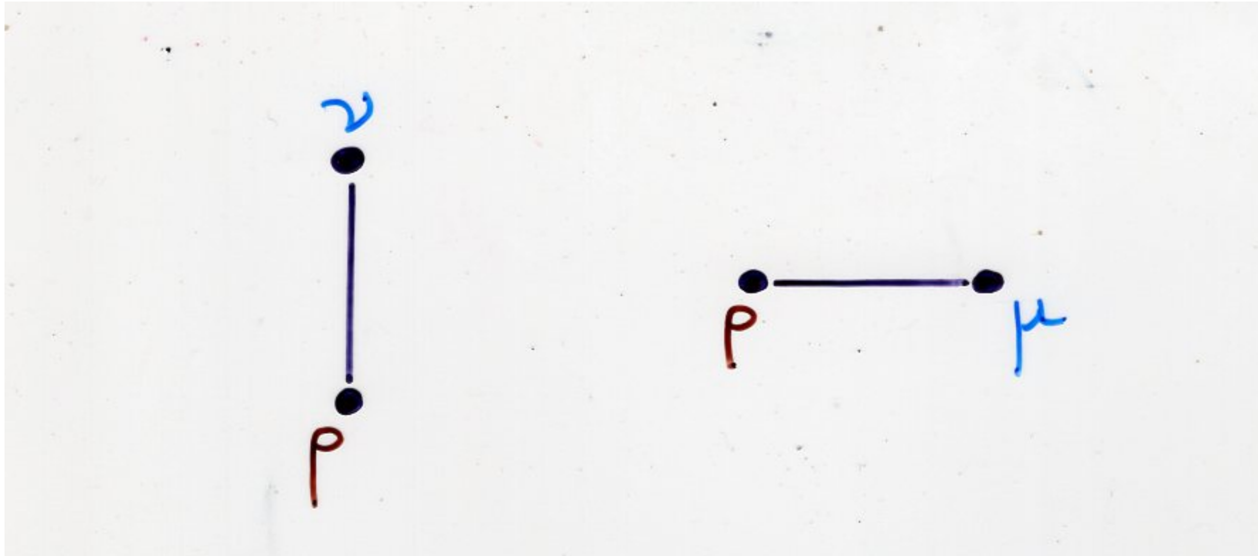
- label the first set of "shadow lines" of the permutation σ by 1 (black lines on the figure)

- then by 2 the second set, i.e. the "shadow lines" of the skeleton $Sq(\sigma)$ (the red lines)

- etc, ~ 3 the blue lines of $Sq(Sq(\sigma))$

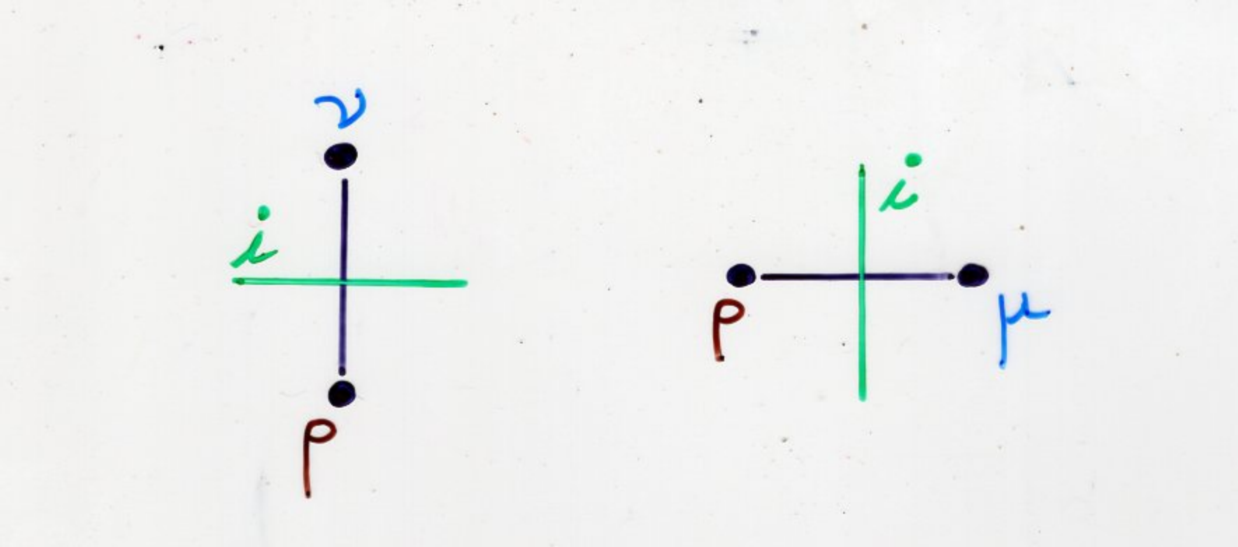
- ...





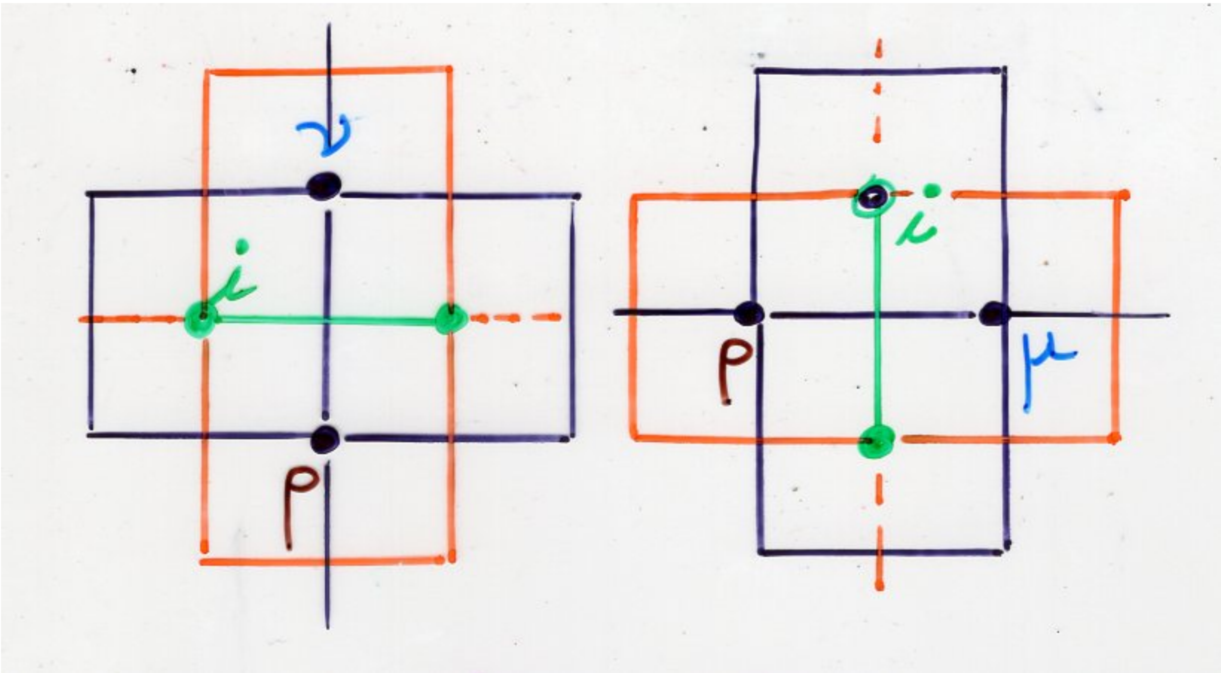
if no shadow lines are crossing, then

$$\substack{\mu \\ \nu} = \rho .$$

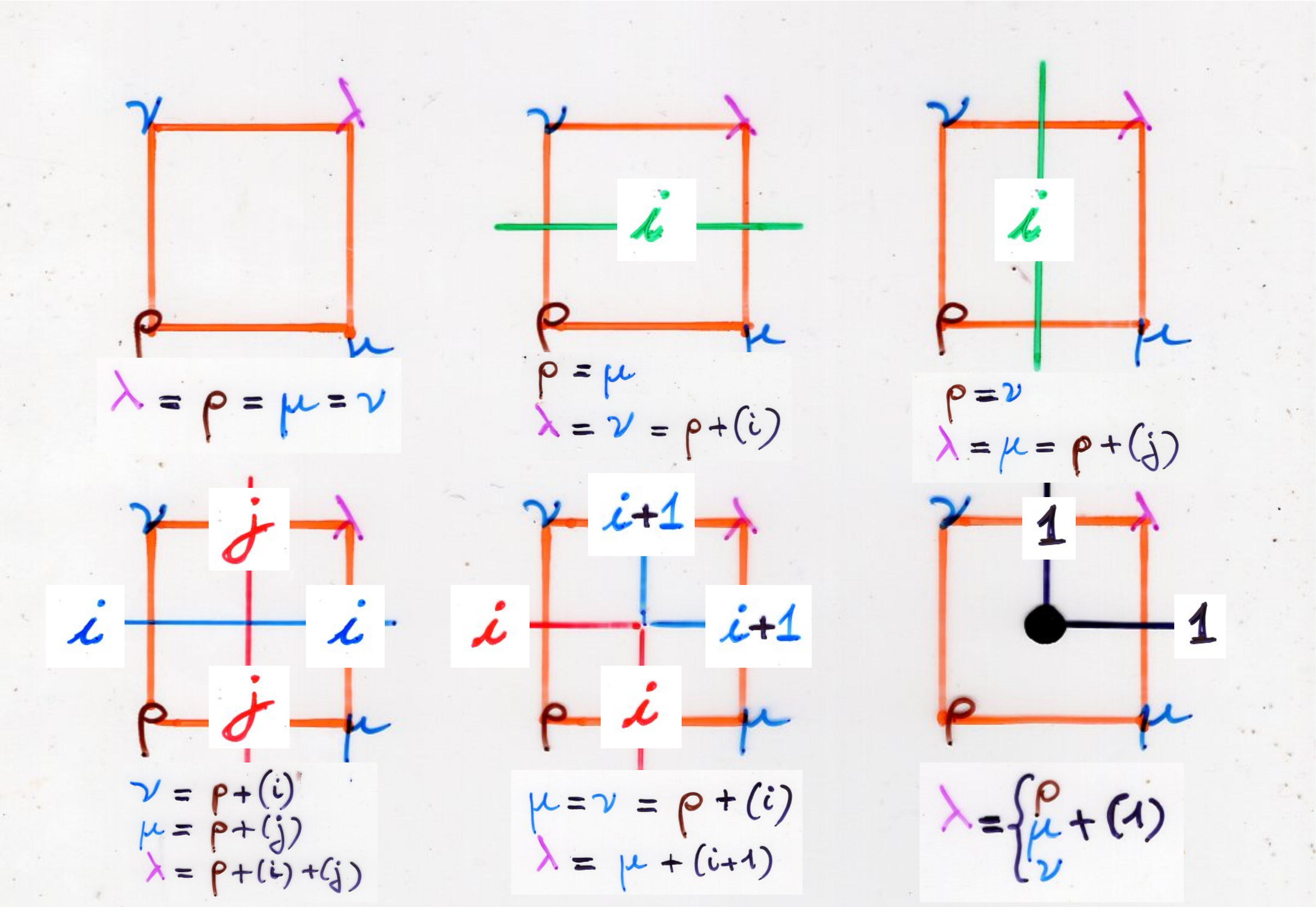


if a shadow line
with label i is crossing, then

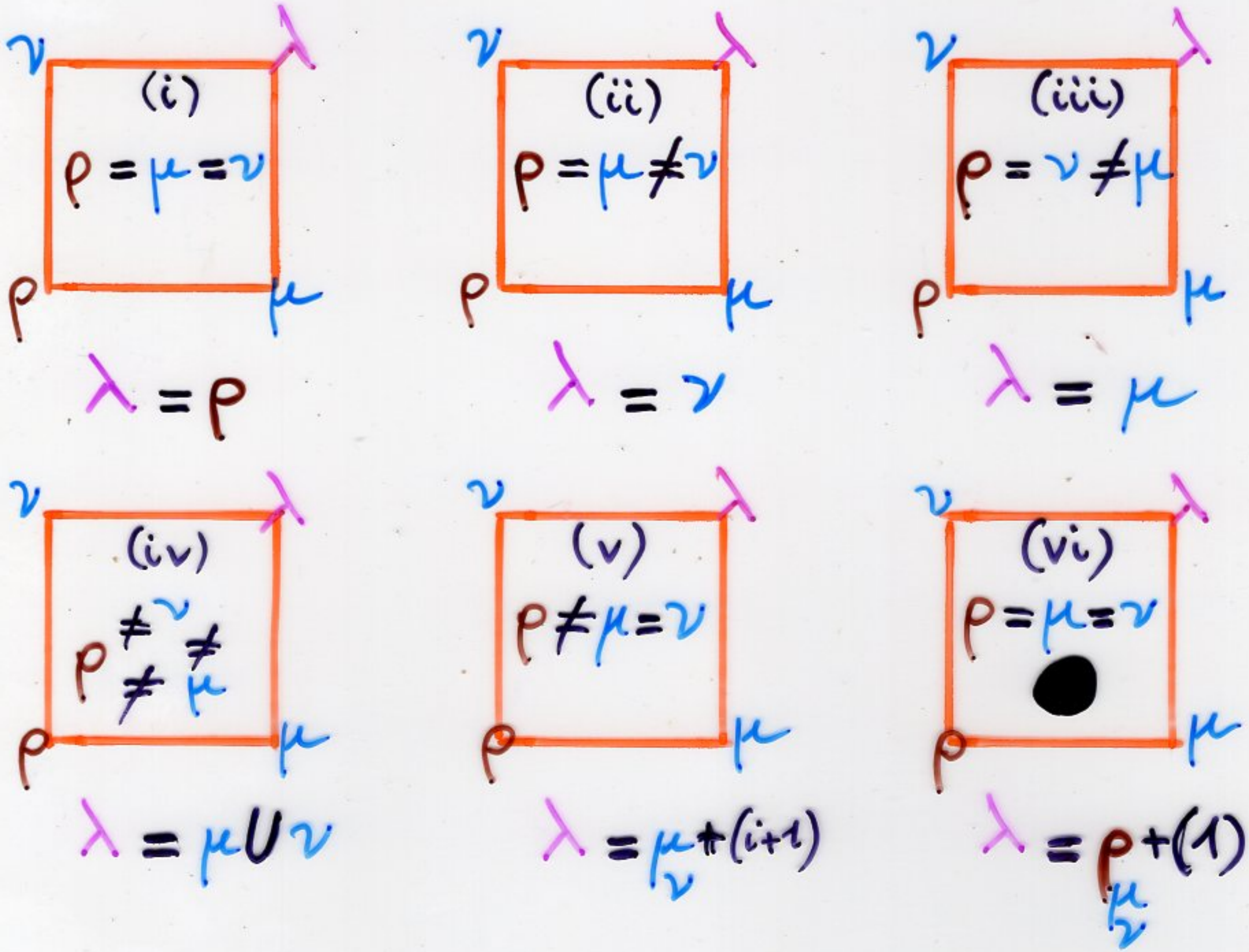
$$\mu = \rho + (i)$$
$$\nu$$



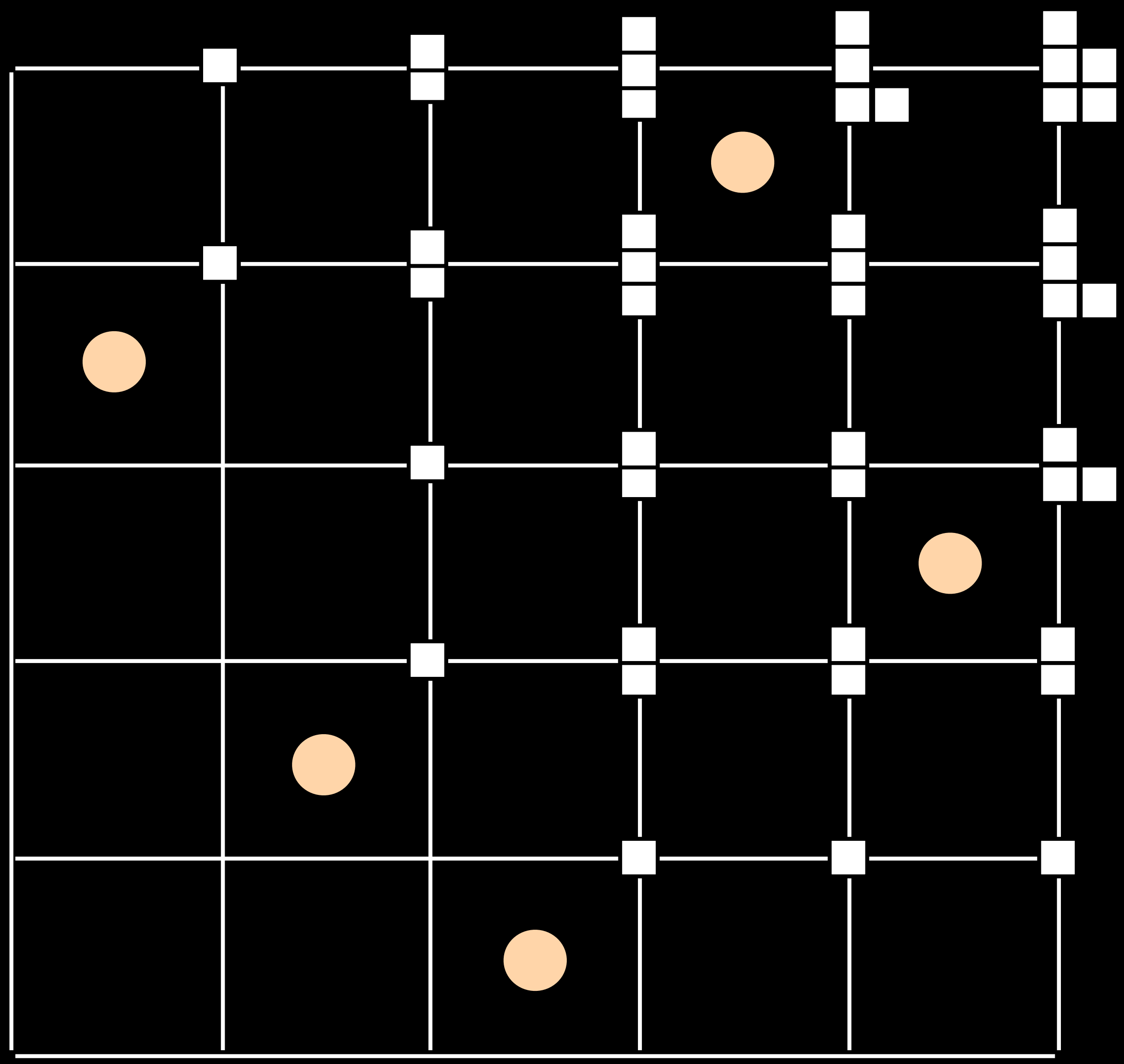
ν
i
ρ
i
ρ
μ



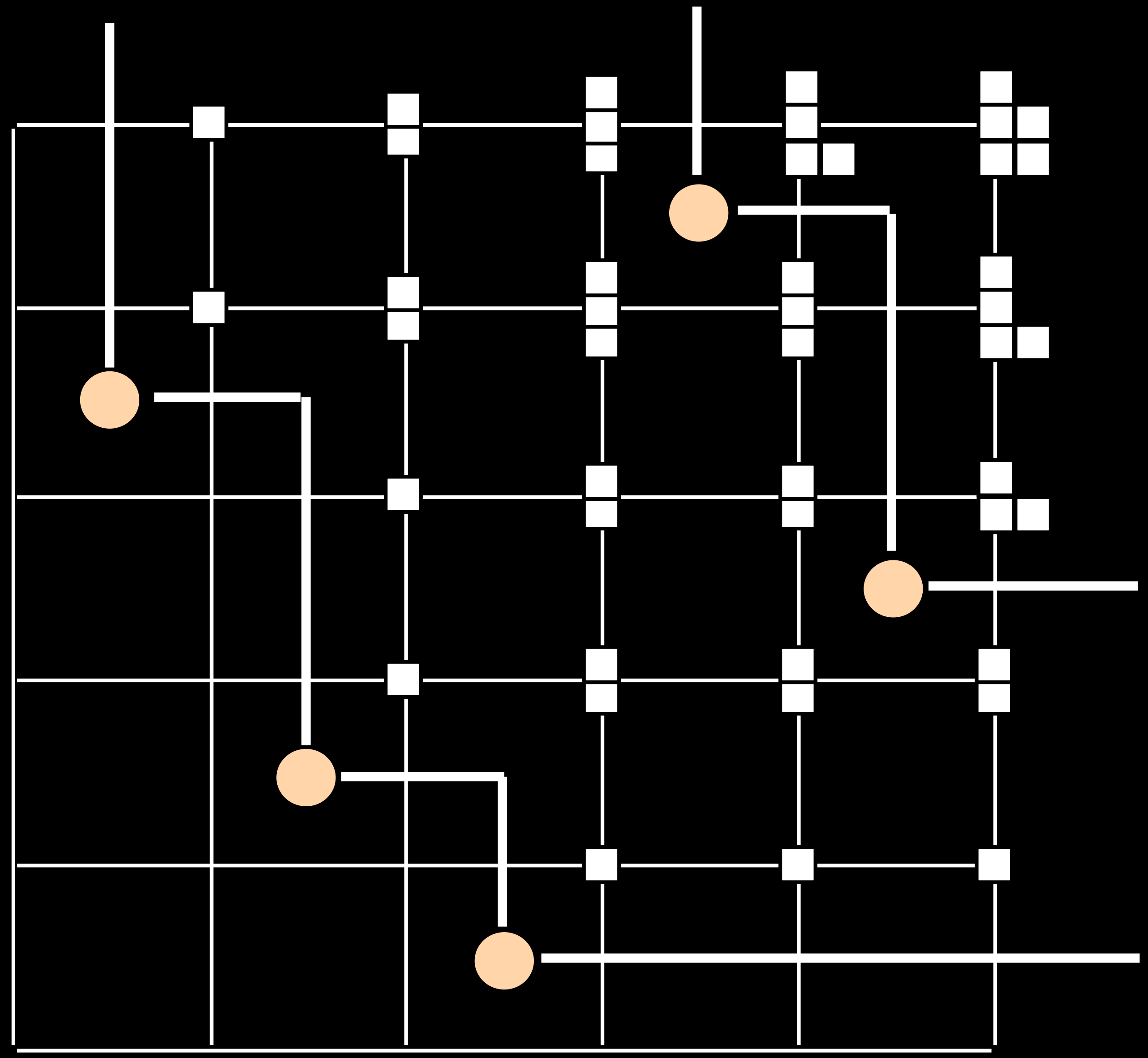
ν λ ρ μ
$\lambda = \rho = \mu = \nu$
ν λ i ρ μ
$\rho = \mu$
$\lambda = \nu = \rho + (i)$
ν λ i ρ μ
$\rho = \nu$
$\lambda = \mu = \rho + (j)$
ν j λ i i ρ j μ
$\nu = \rho + (i)$
$\mu = \rho + (j)$
$\lambda = \rho + (i) + (j)$
ν $i+1$ λ i $i+1$ ρ i μ
$\mu = \nu = \rho + (i)$
$\lambda = \mu + (i+1)$
ν 1 λ 1 ρ μ
$\lambda = \left\{\begin{matrix}\rho\\ \mu\\ \nu\end{matrix}\right. + (1)$



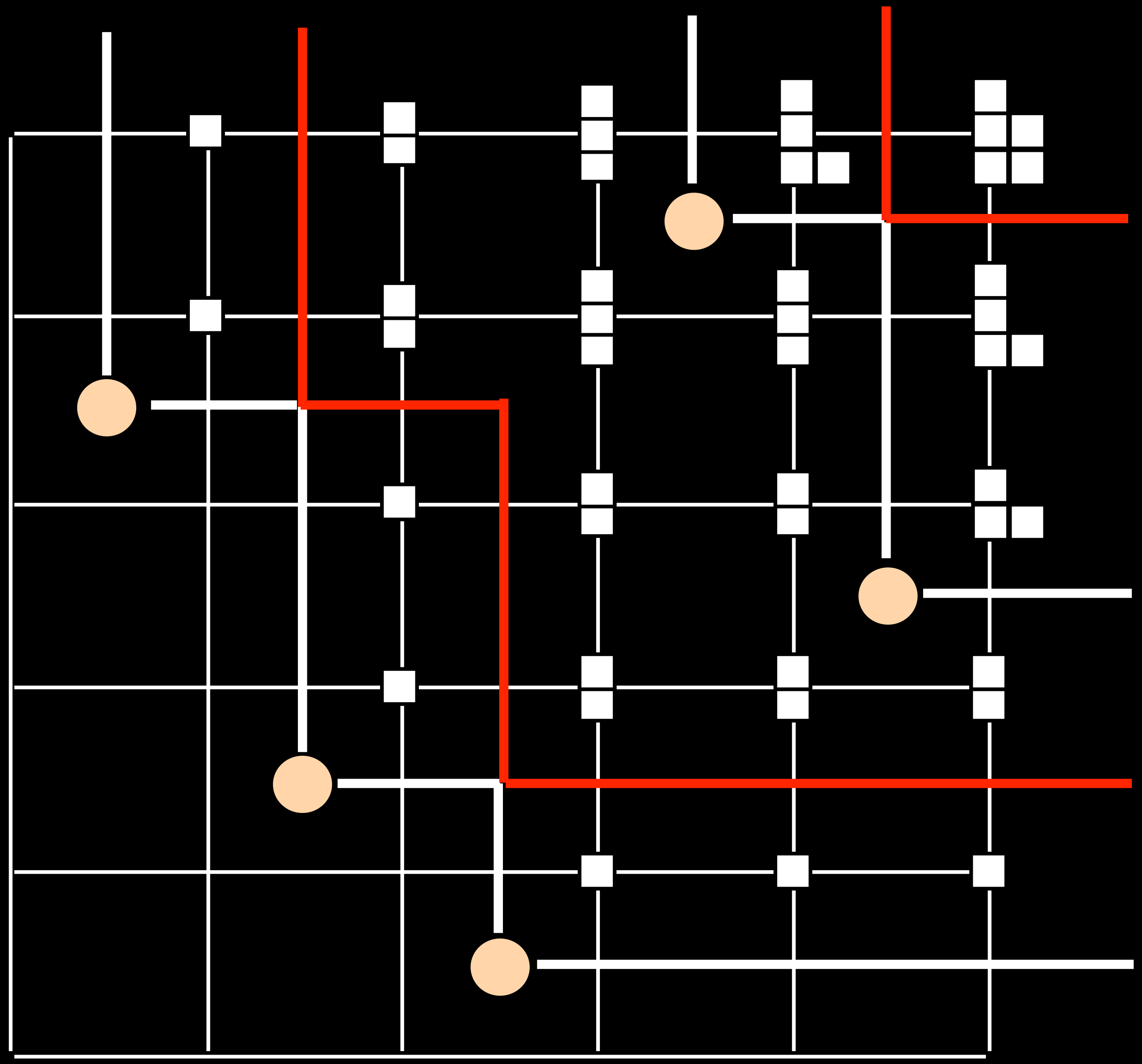
ν λ
(i)
ρ = μ = ν
ρ μ
λ = ρ
ν λ
(ii)
ρ = μ ≠ ν
ρ μ
λ = ν
ν λ
(iii)
ρ = ν ≠ μ
ρ μ
λ = μ
ν λ
(iv)
ρ ≠ ν ≠ μ ≠ ρ
ρ μ
λ = μ ∪ ν
ν λ
(v)
ρ ≠ μ = ν
ρ μ
λ = μ/ν + (i+1)
ν λ
(vi)
ρ = μ = ν
ρ μ
λ = ρ/μ/ν + (1)



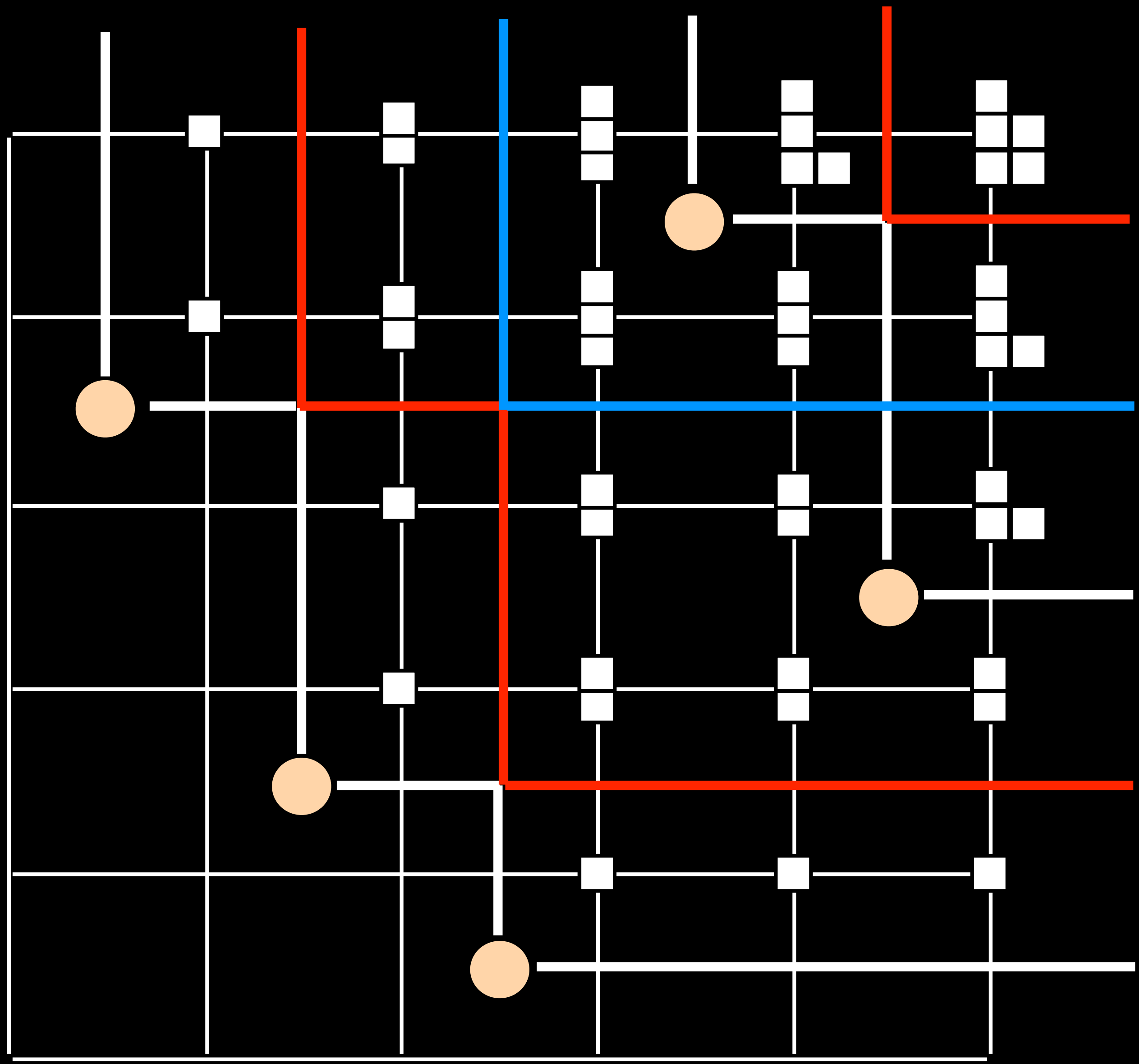
$\sigma = 4, 2, 1, 5, 3$



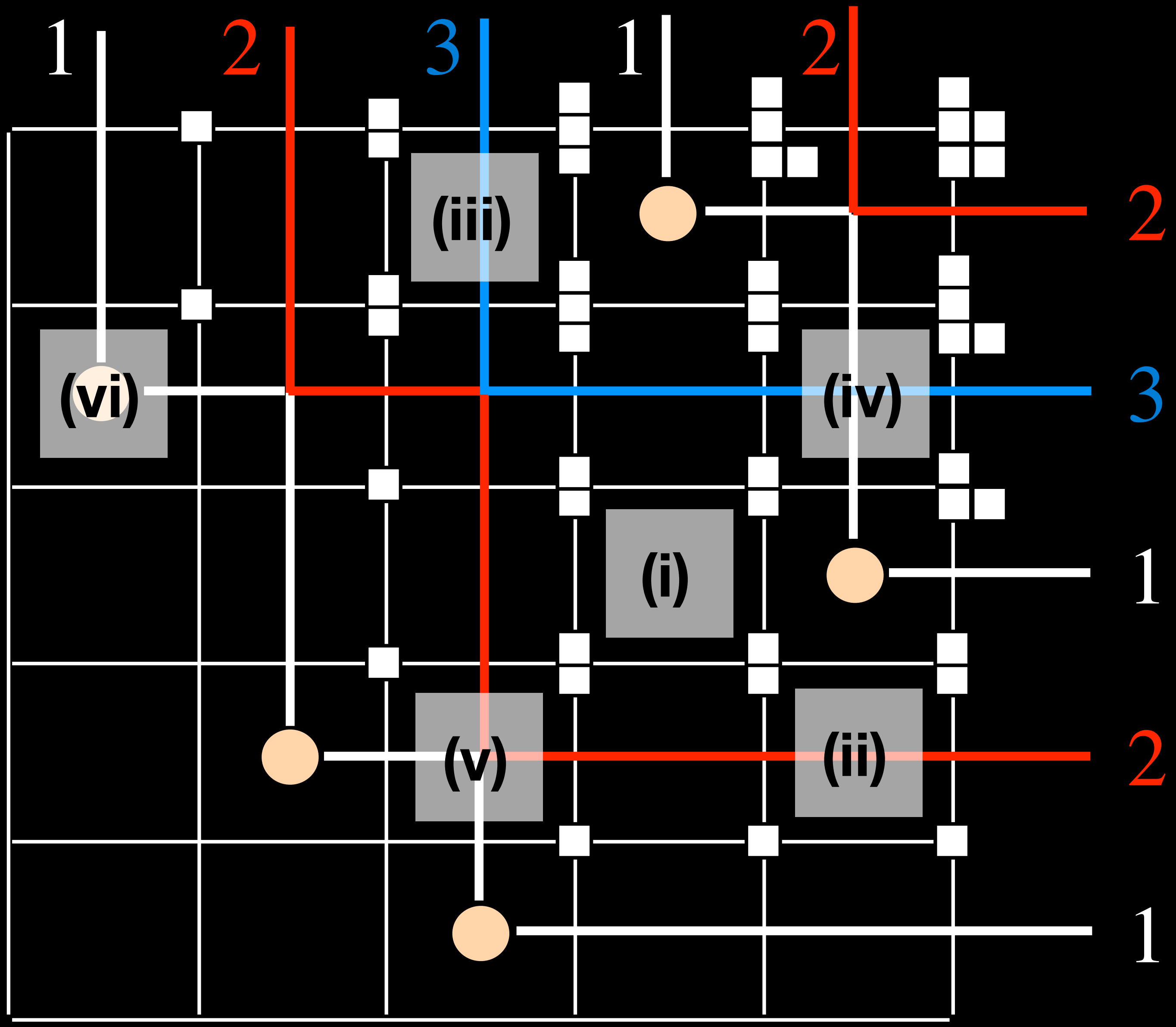
$\sigma = 4, 2, 1, 5, 3$



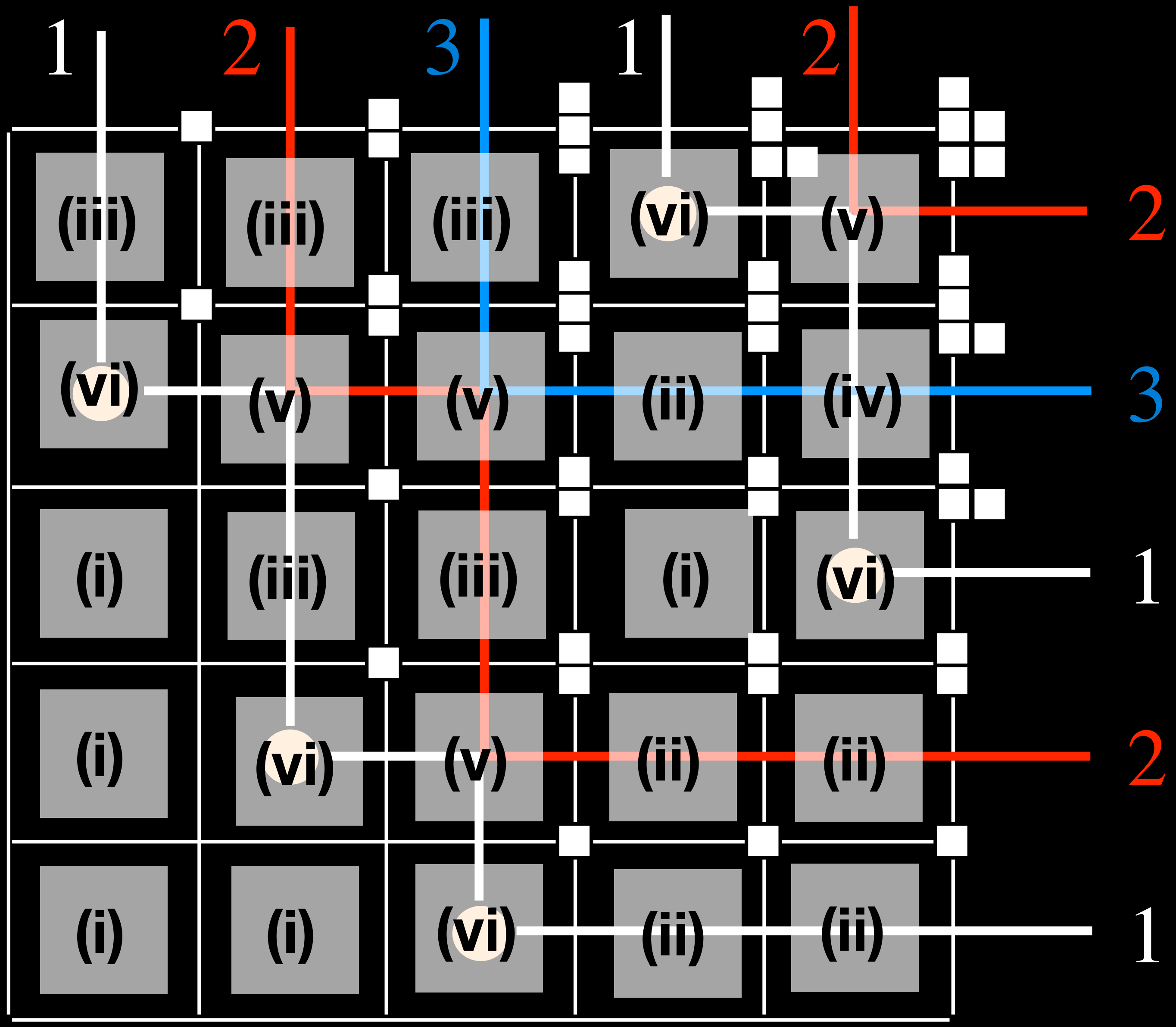
$\sigma = 4, 2, 1, 5, 3$



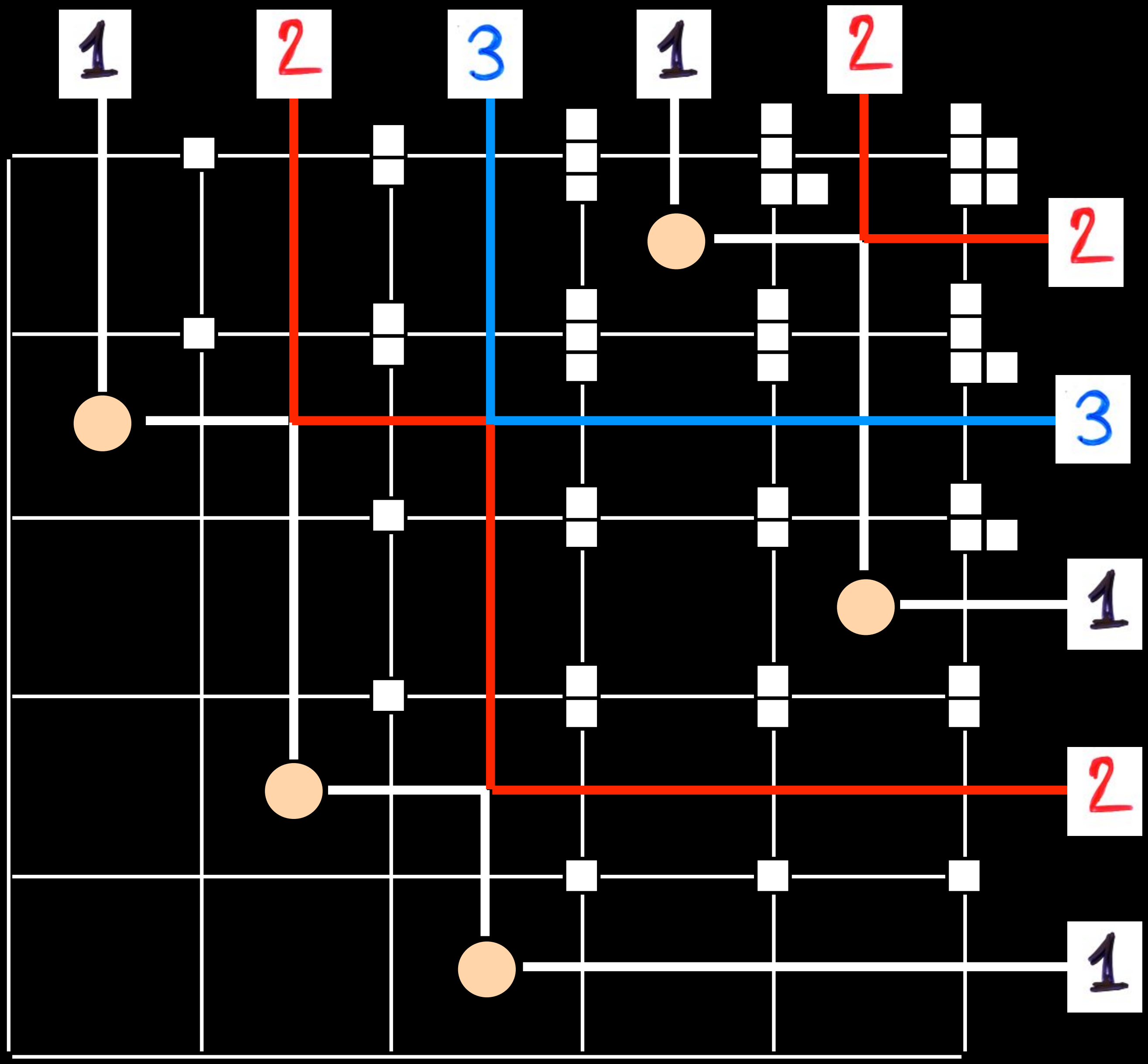
$\sigma = 4, 2, 1, 5, 3$

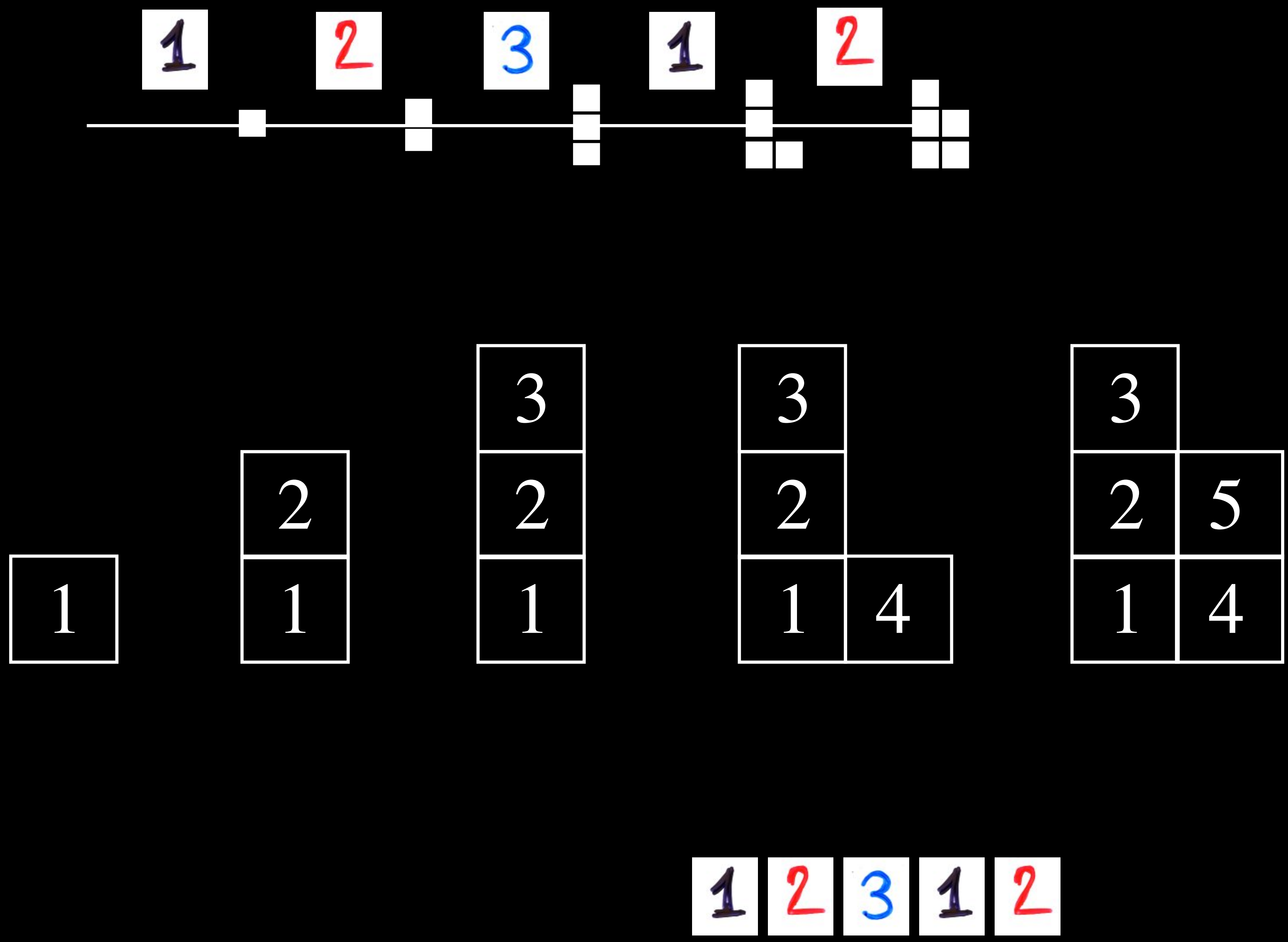


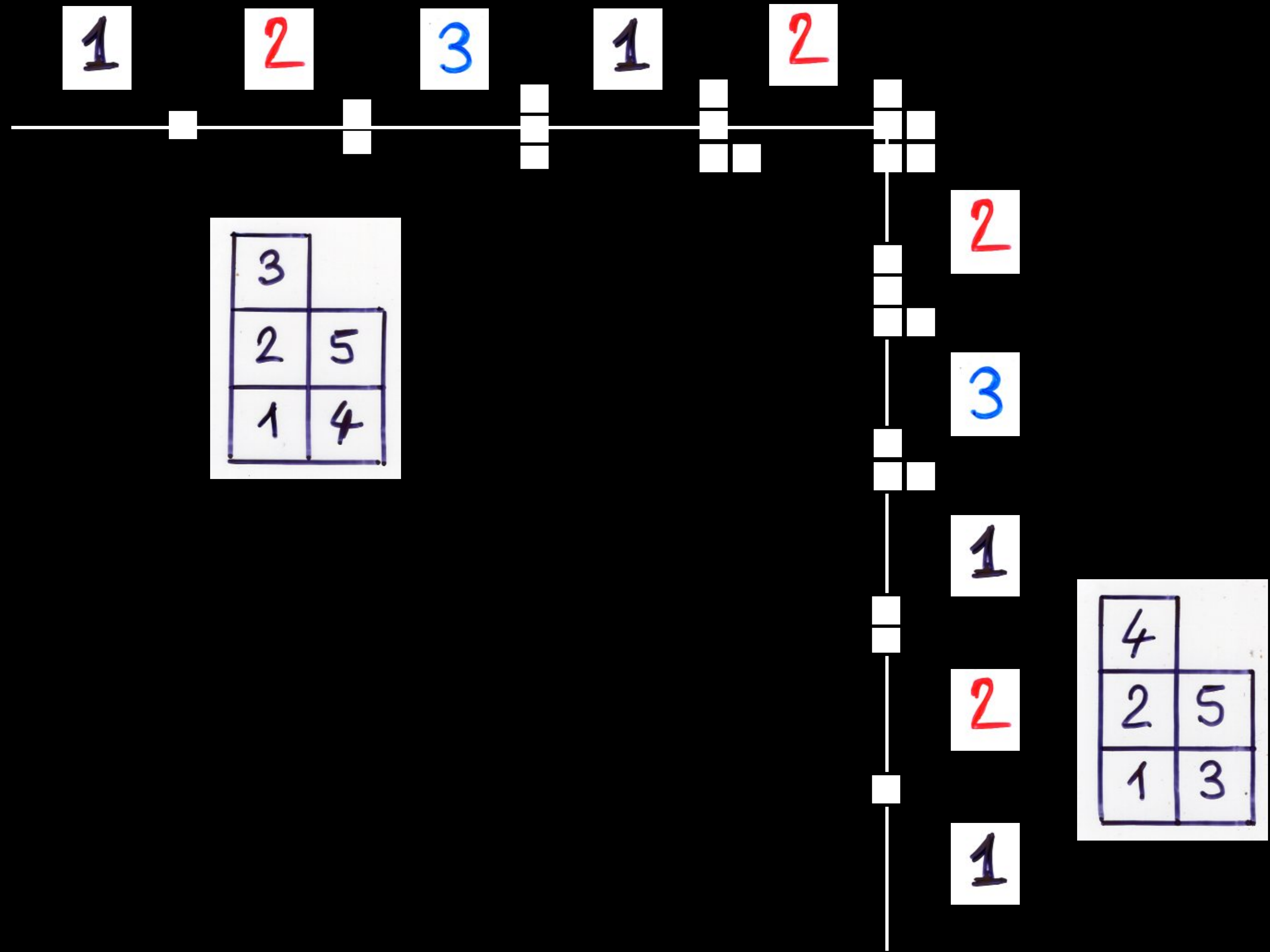
$\sigma = 4, 2, 1, 5, 3$


1 2 3 1 2
(iii) (iii) (iii) (vi) (v) 2
(vi) (v) (v) (ii) (iv) 3
(i) (iii) (iii) (i) (vi) 1
(i) (vi) (v) (ii) (ii) 2
(i) (i) (vi) (ii) (ii) 1

$\sigma = 4, 2, 1, 5, 3$


1
2
3
1
2
2
3
1
2
1

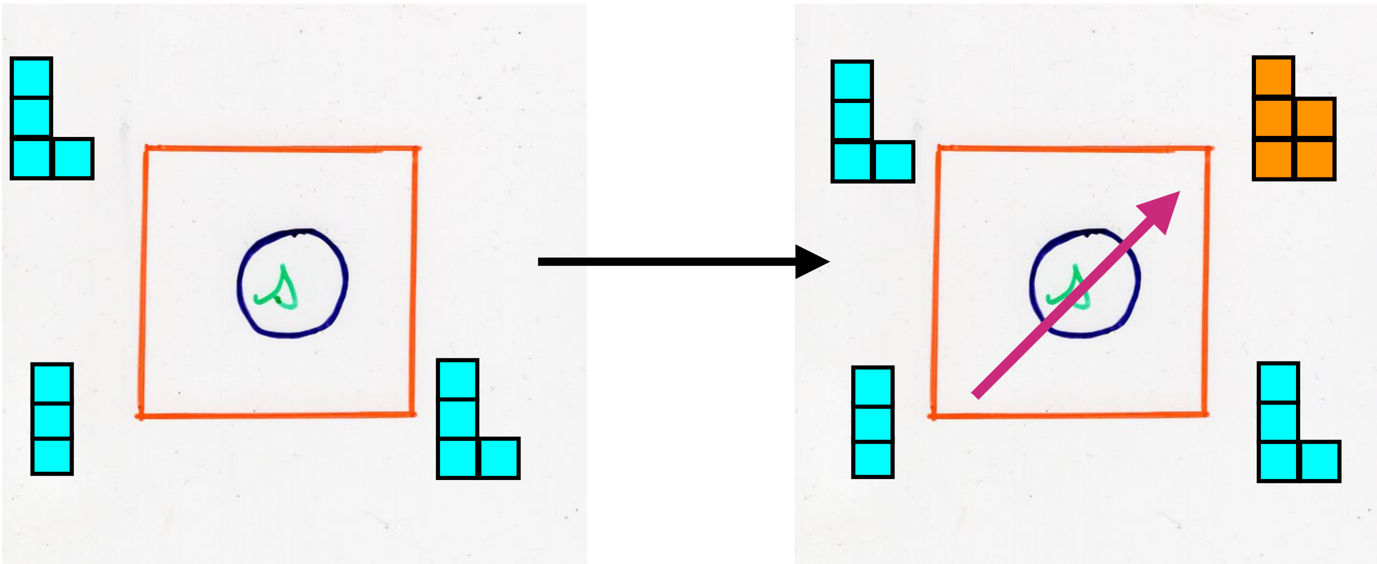

1 2 3 1 2
1
2
1
3
2
1
3
2
1 4
3
2 5
1 4
1 2 3 1 2


1
2
3
1
2
3
2 5
1 4
2
3
1
2
1
4
2 5
1 3

The reverse RSK planar automaton

Fomin's "local rules"

"growth diagrams"



"local rules"

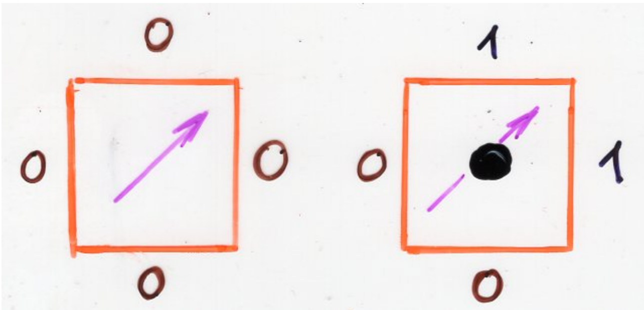
on the vertices

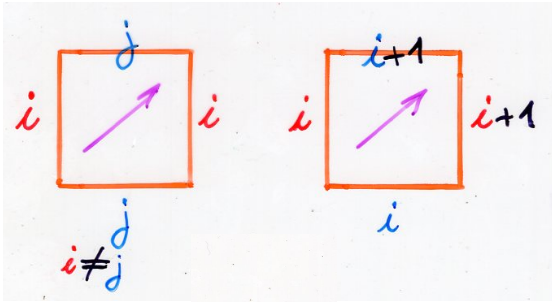
"local rules" on the edges

state $\{0, 1, 2, \dots\}$

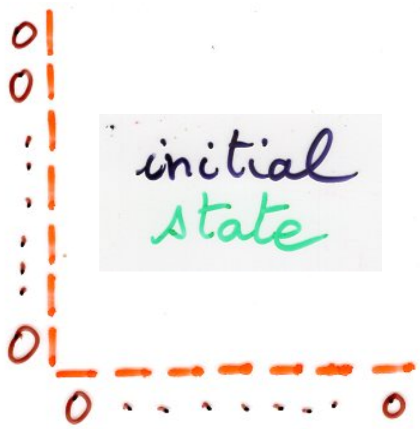
state $\{0, 1, 2, \dots\}$

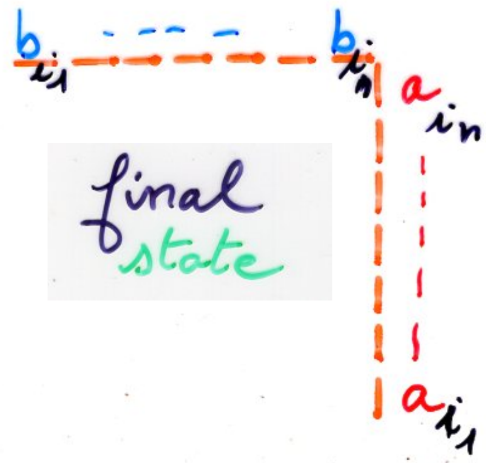
set of labels $L = \{\square, \boxdot\}$



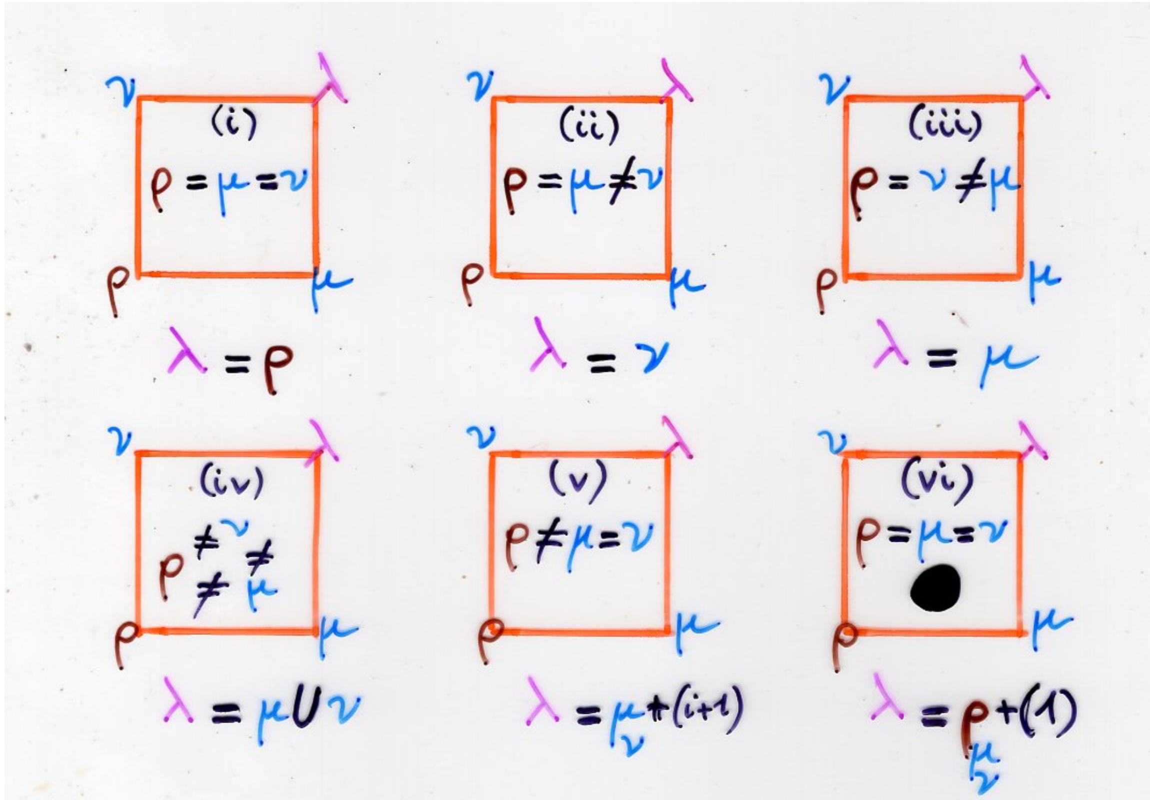


"planar rewriting"

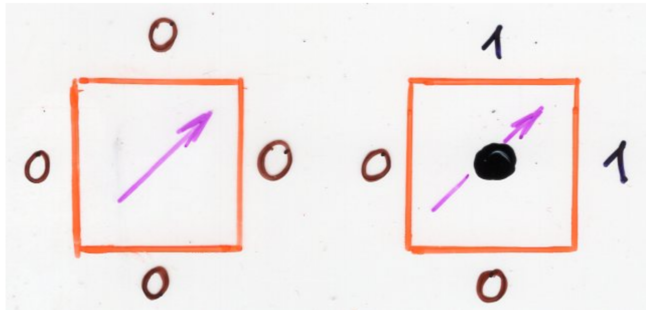


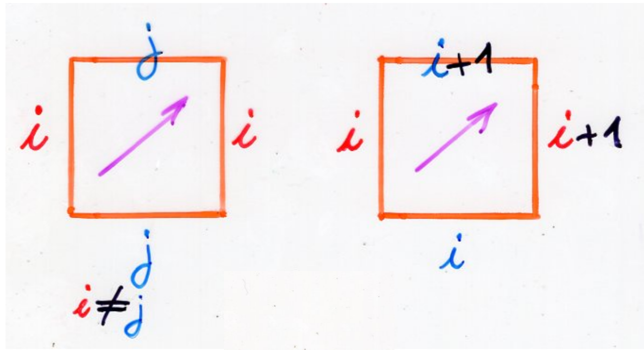


"local rules"
on the vertices



"local rules"
on the edges





« local rules on vertices »

Marc A. A. van Leeuwen (1996)

The Robinson-Schensted and Schützenberger algorithms, an elementary approach

C.Krattentheler, (2006).

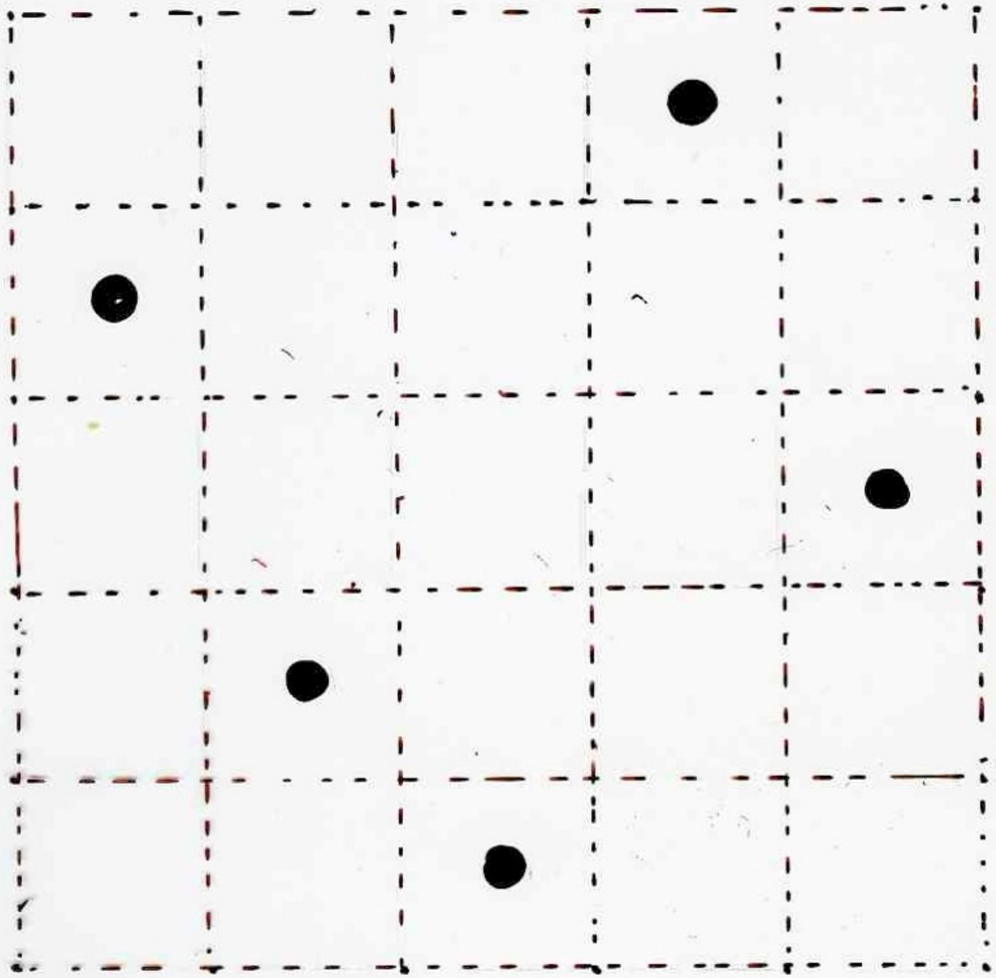
GROWTH DIAGRAMS, AND INCREASING AND DECREASING CHAINS IN FILLINGS OF FERRERS SHAPES

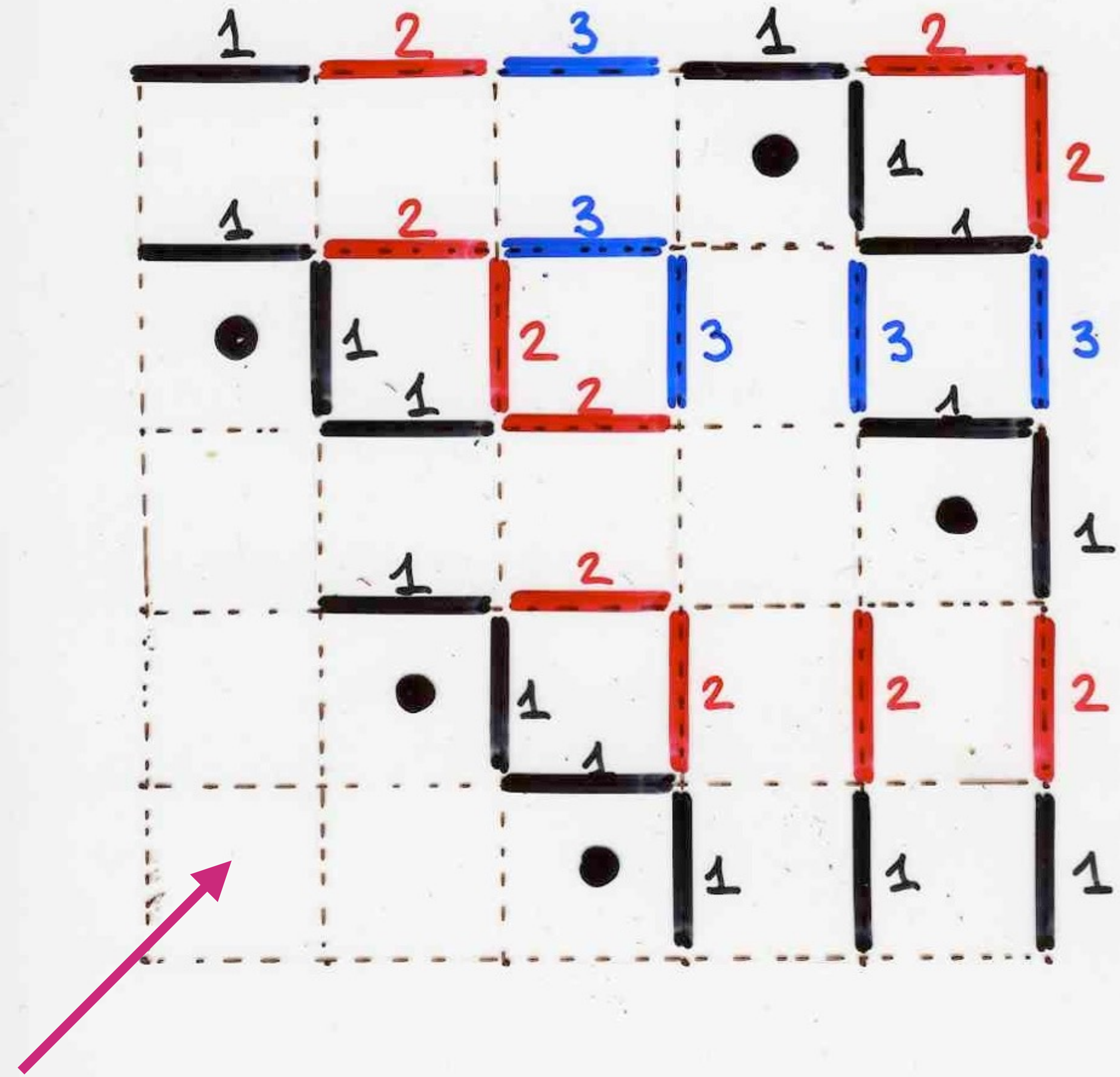
M.Rubey. (2007)

Increasing and Decreasing Sequences in Fillings of Moon Polyominoes

I claim that much attention should be given to the « local rules on edges » rather that « local rules on vertices ».

This is part of the philosophy of the « cellular ansatz »




1
2
3
1
2
1
2
1
2
3
1
2
3
3
3
1
2
1
1
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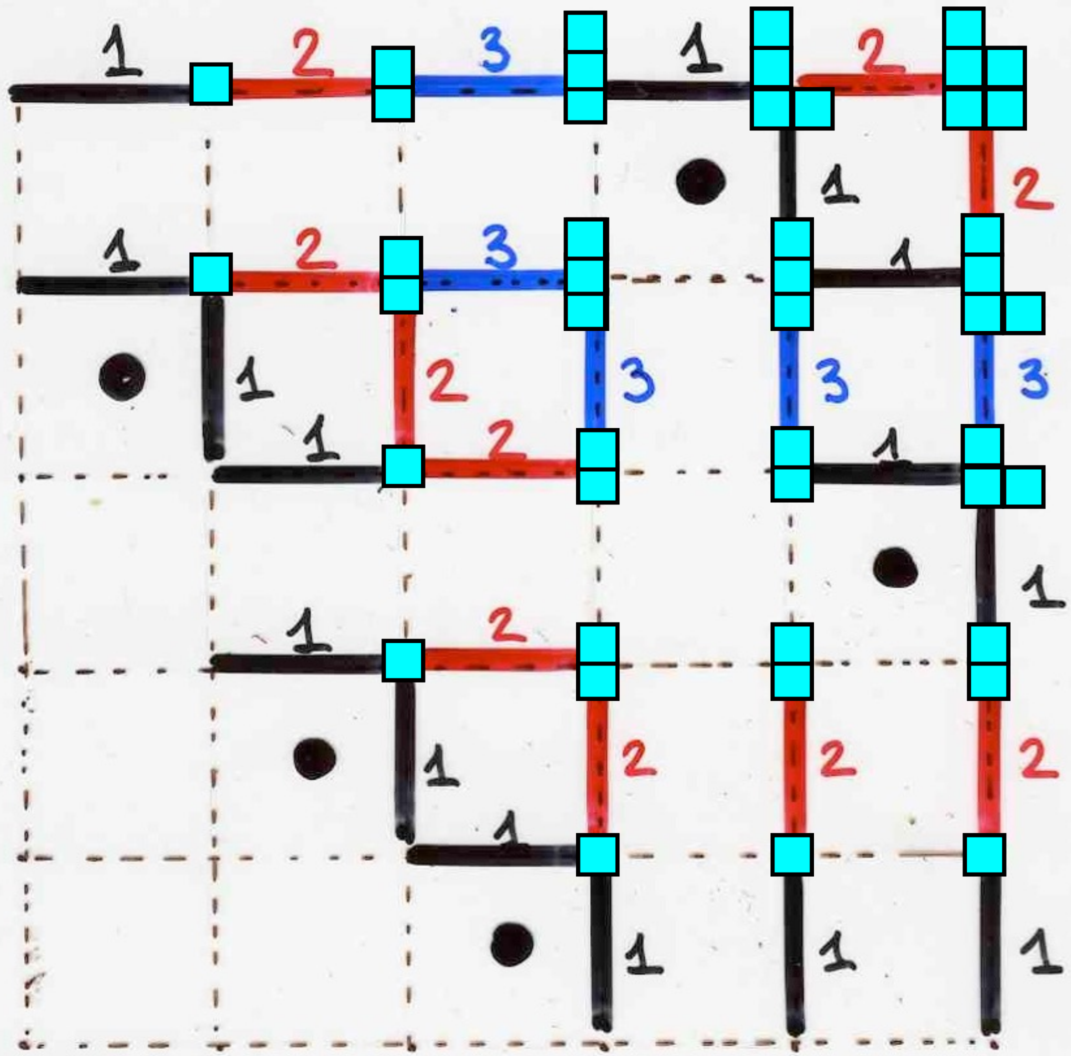
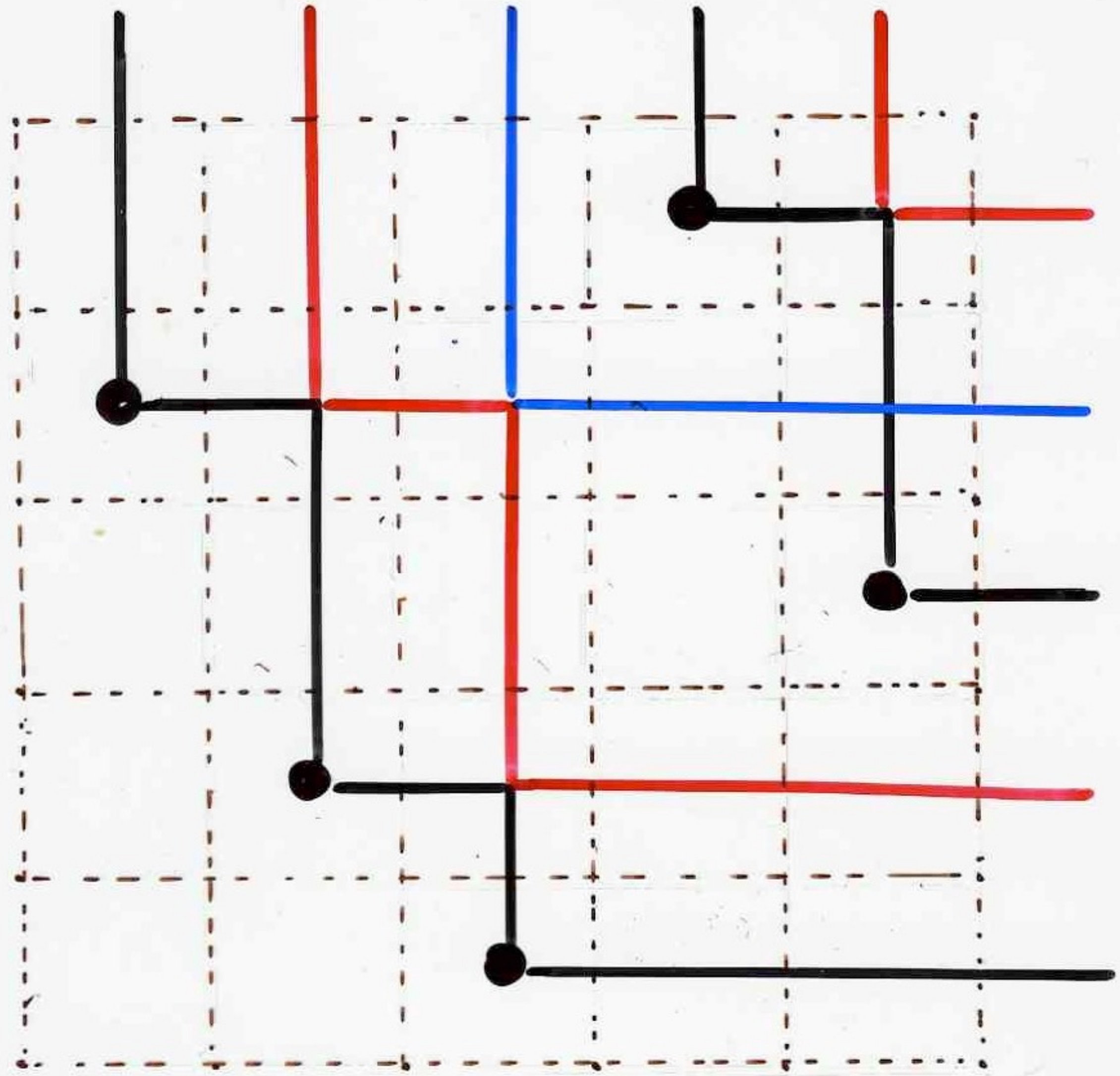
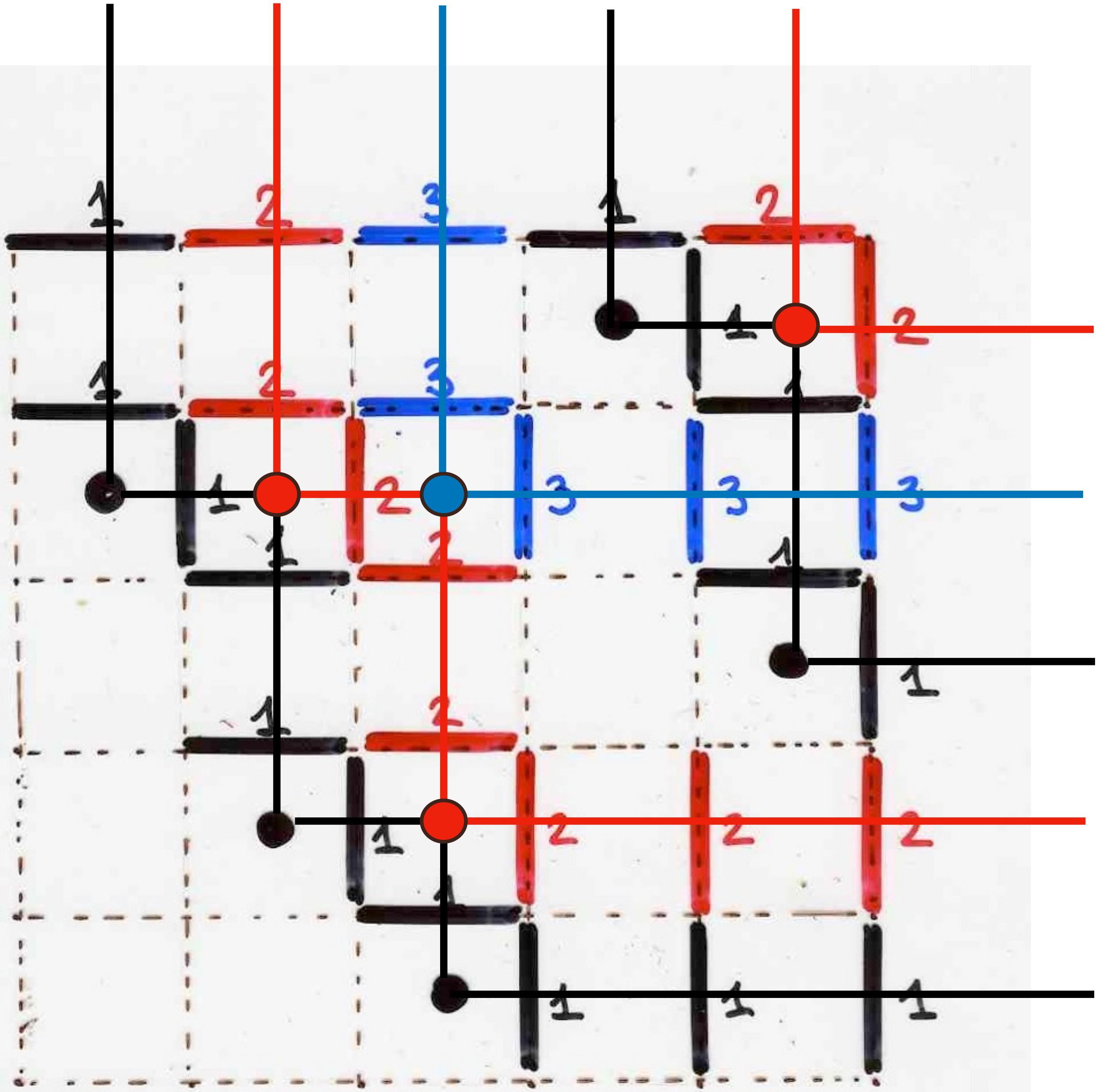
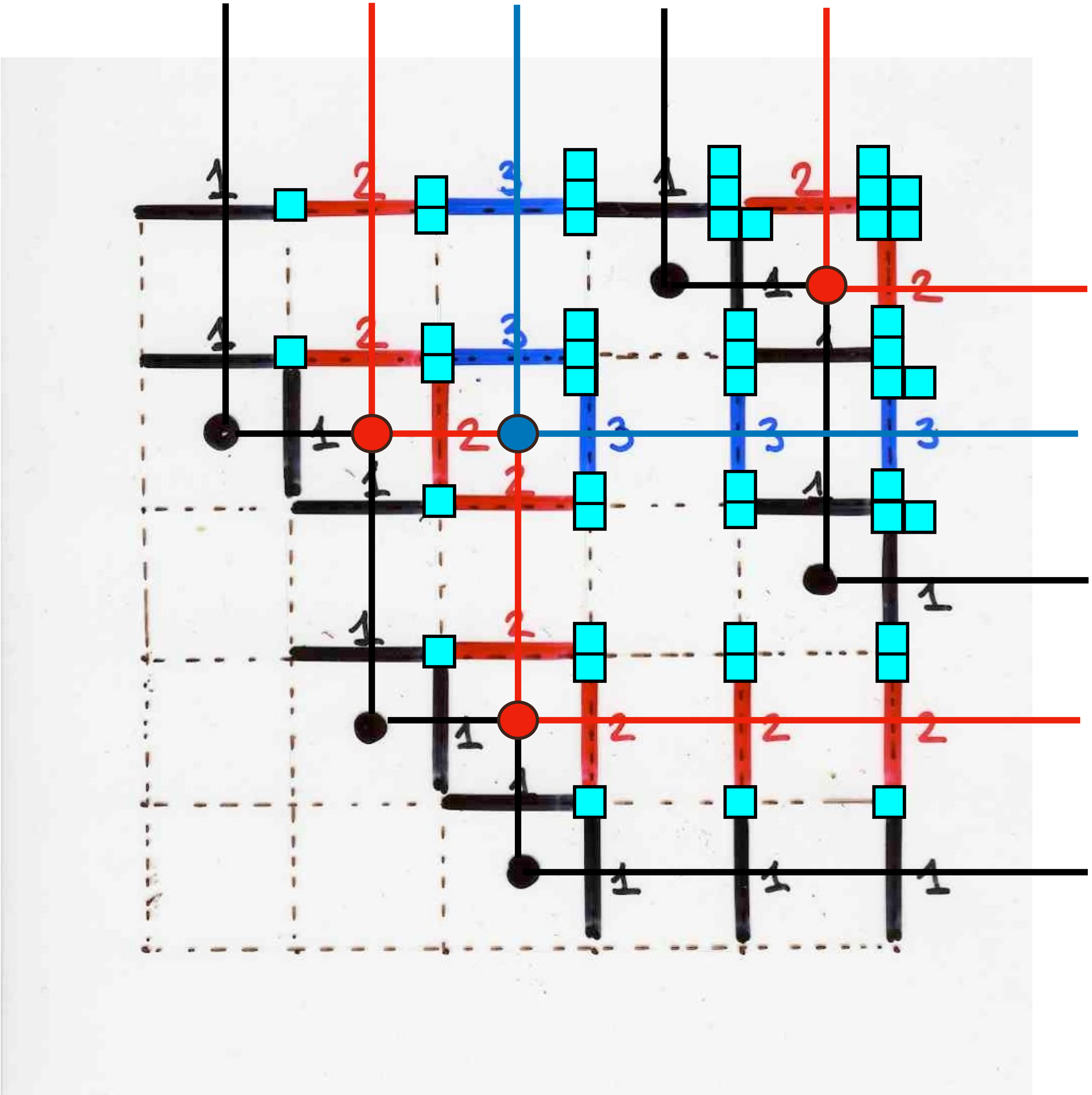
2
1
1
1
1


1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1




1 2 3 1 2
1 2
1 2 3 1
1 2 3 3 3
1 2 1
1
1 2
1 2 2 2
1
1 1 1



Planar automaton

The RSK (reverse) planar automaton

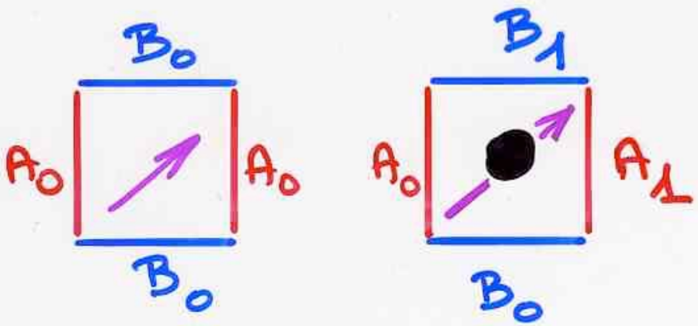
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

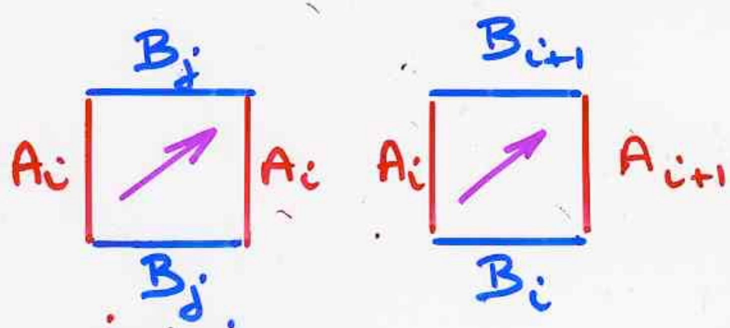
$\mathcal{A} = \{A_0 \quad A_1 \quad A_k\}$

set of labels

$L = \{\square, \boxdot\}$

"planar rewriting"





$i \neq j$

The RSK (reverse) planar automaton

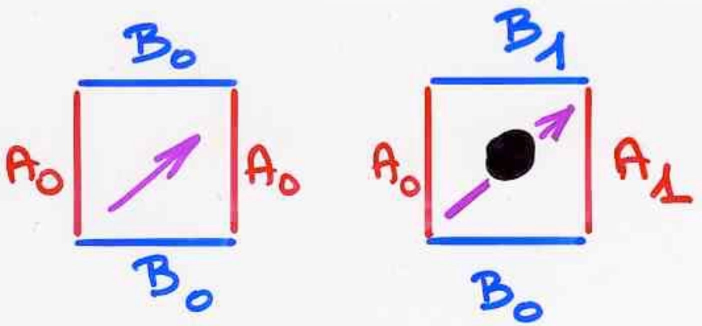
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \; A_1 \; \ldots \; A_k\}$

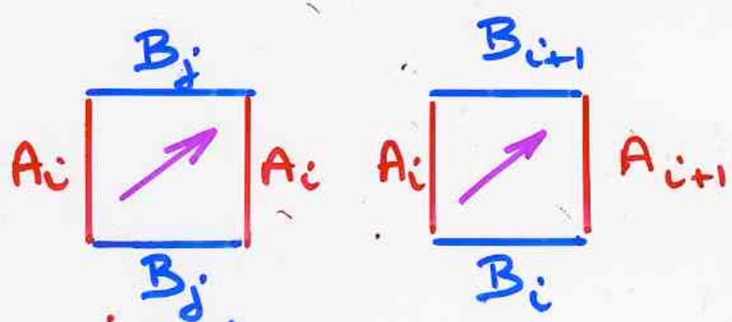
set of labels

$L = \{\square, \boxdot\}$

philosophy of the « cellular ansatz »:
relating planar automaton and some quadratic algebra



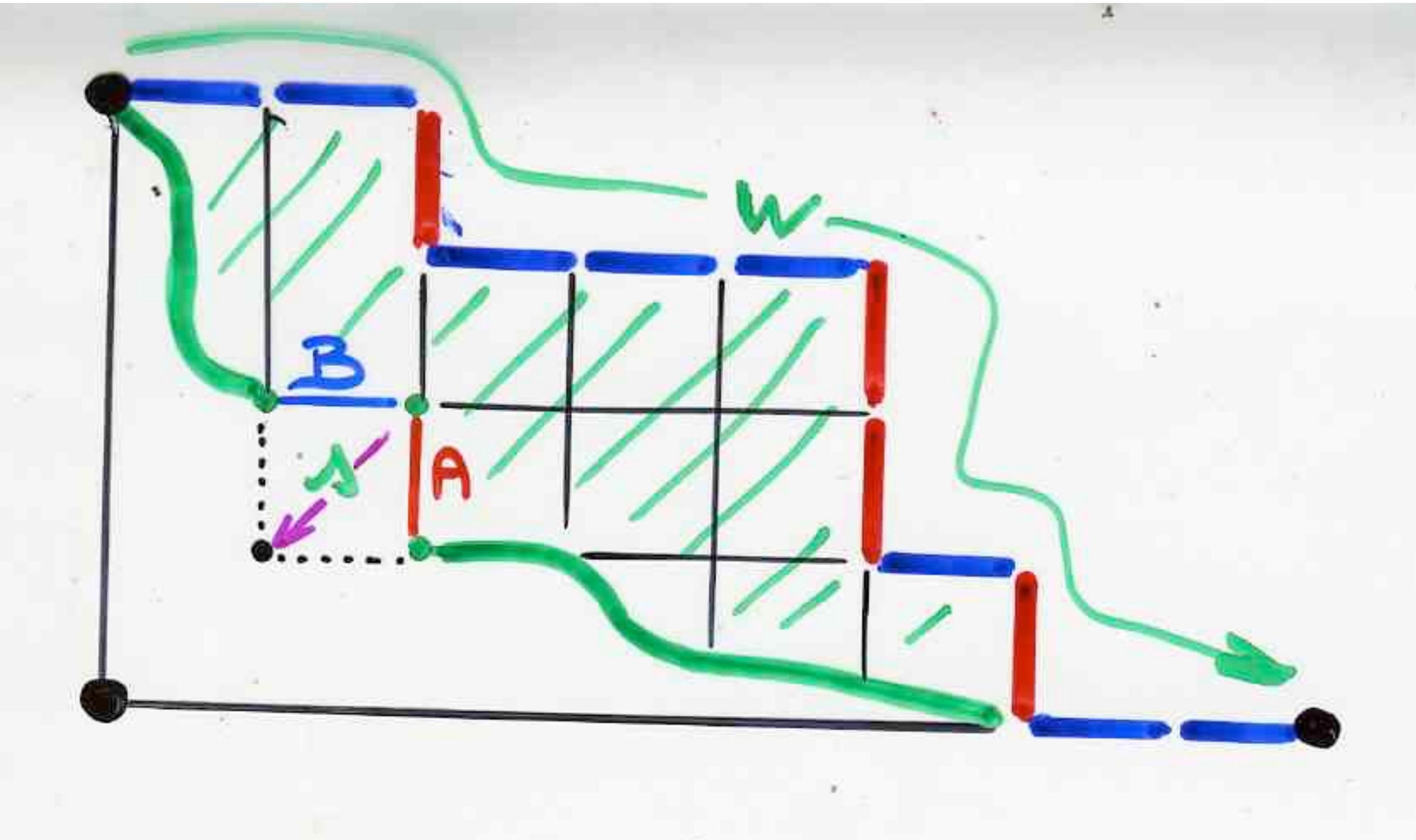
$A_0 B_0 = \underset{\square}{B_0 A_0} + \underset{\boxdot}{B_1 A_1}$

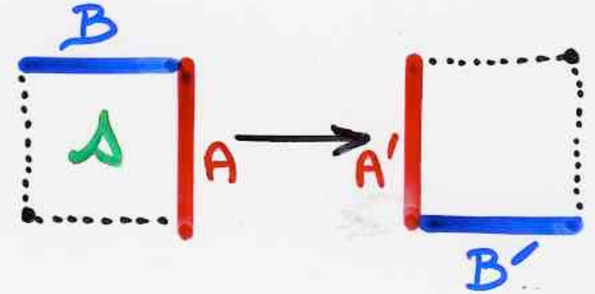


$i \neq j$

$A_i B_j = \underset{\square}{B_j A_i} \qquad A_i B_i = \underset{\square}{B_{i+1} A_{i+1}}$

Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$





Def. planar automaton $\mathcal{P}$

- 3 finite sets
  - $\mathcal{B}$ horizontal alphabet
  - $\mathcal{A}$ vertical alphabet
  - S planar labels (states)

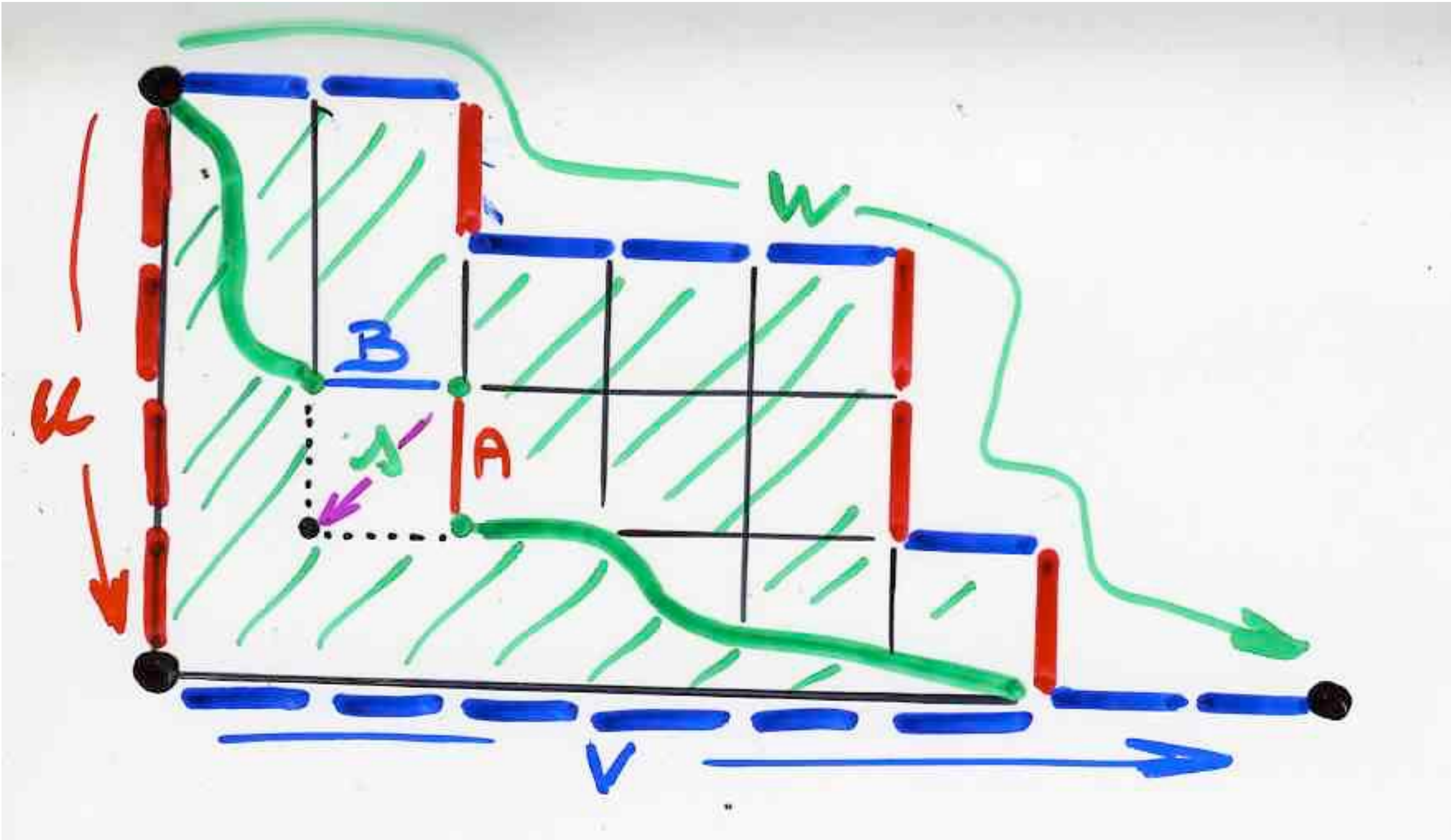
- θ (partial) transition function

$$(s, B, A) \xrightarrow{\theta} (B', A') \quad \text{or} \quad \emptyset$$

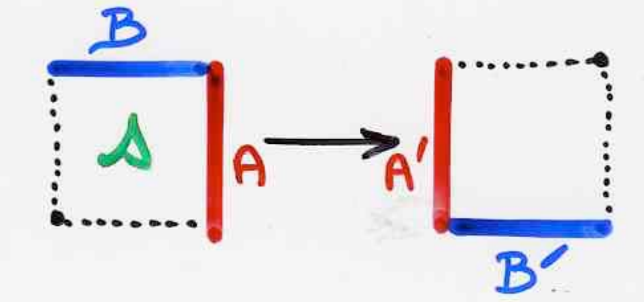
$s \in S$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word
- uv, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$

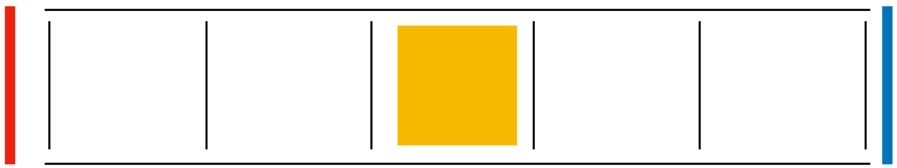


planar rewriting

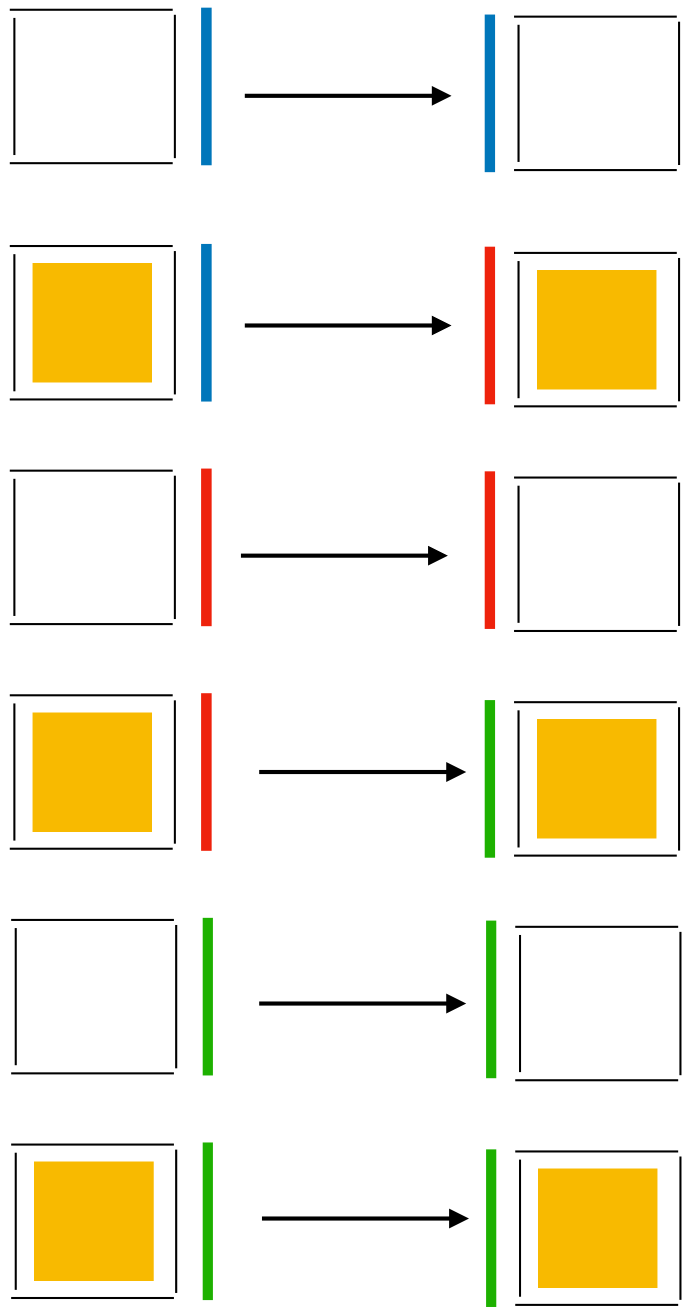


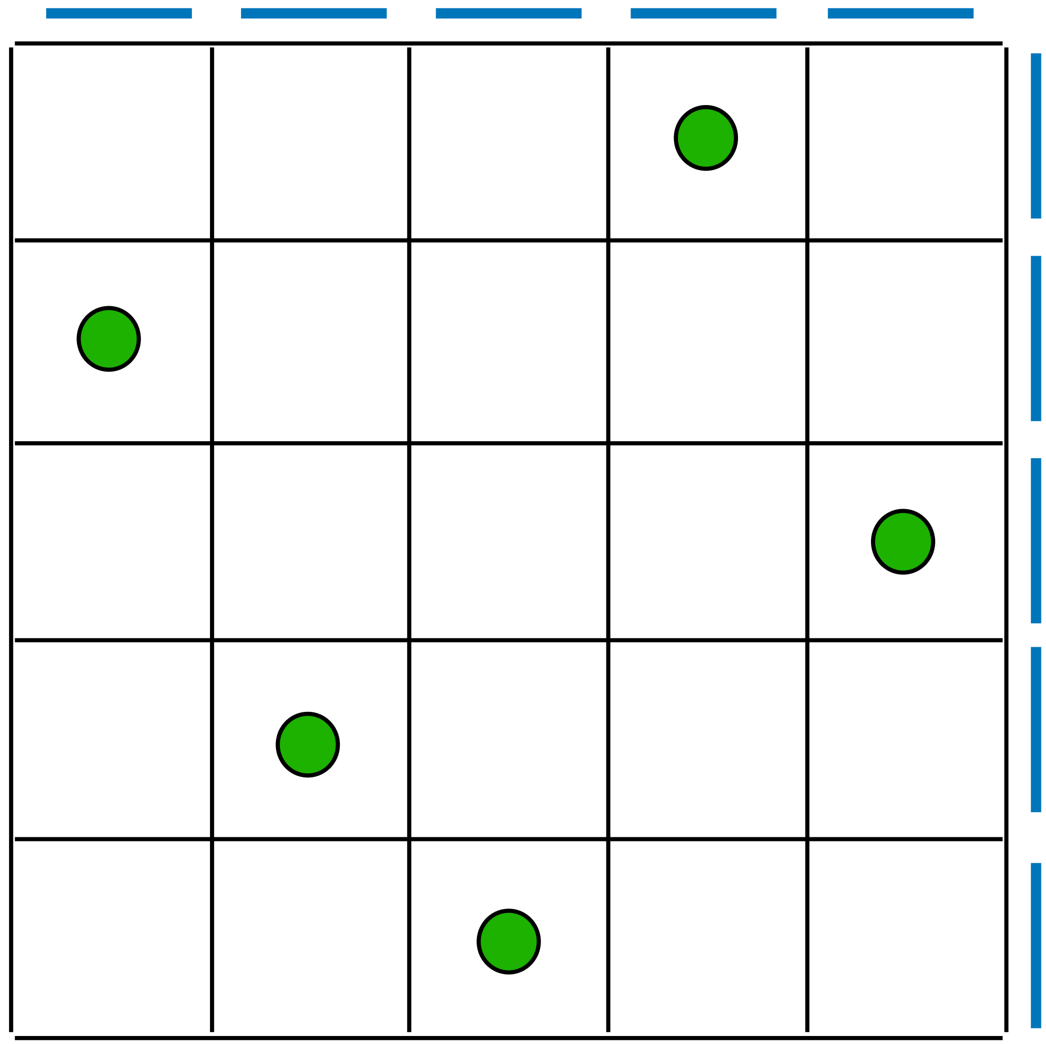
finite automaton

word w accepted by a



initial state
final state





The RSK planar automaton

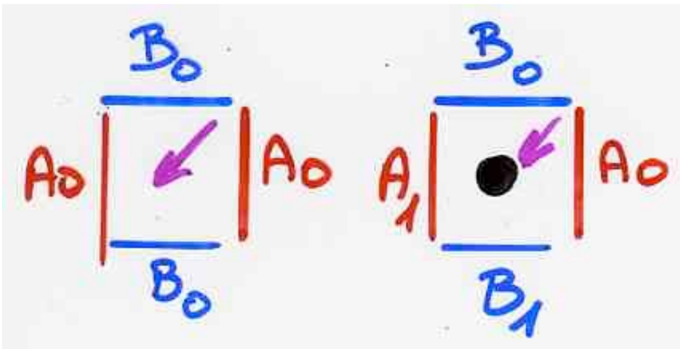
The "RSK planar automaton"

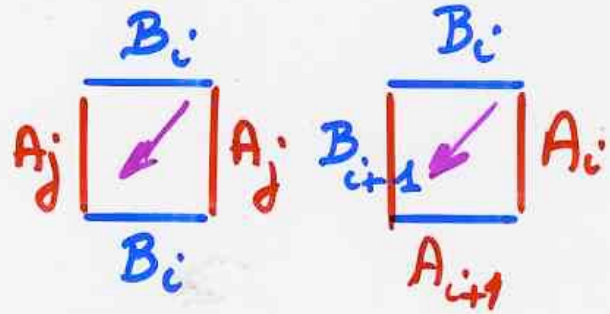
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \quad\quad A_k\}$

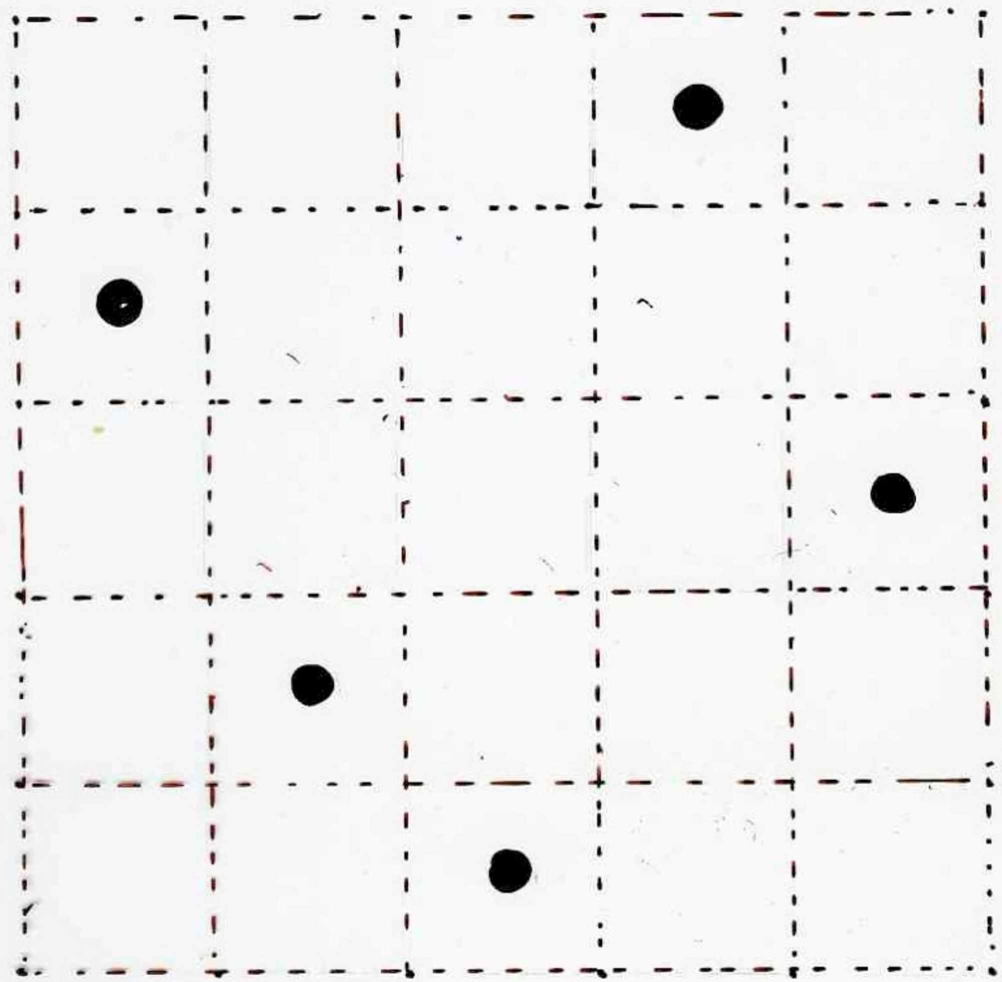
set of labels

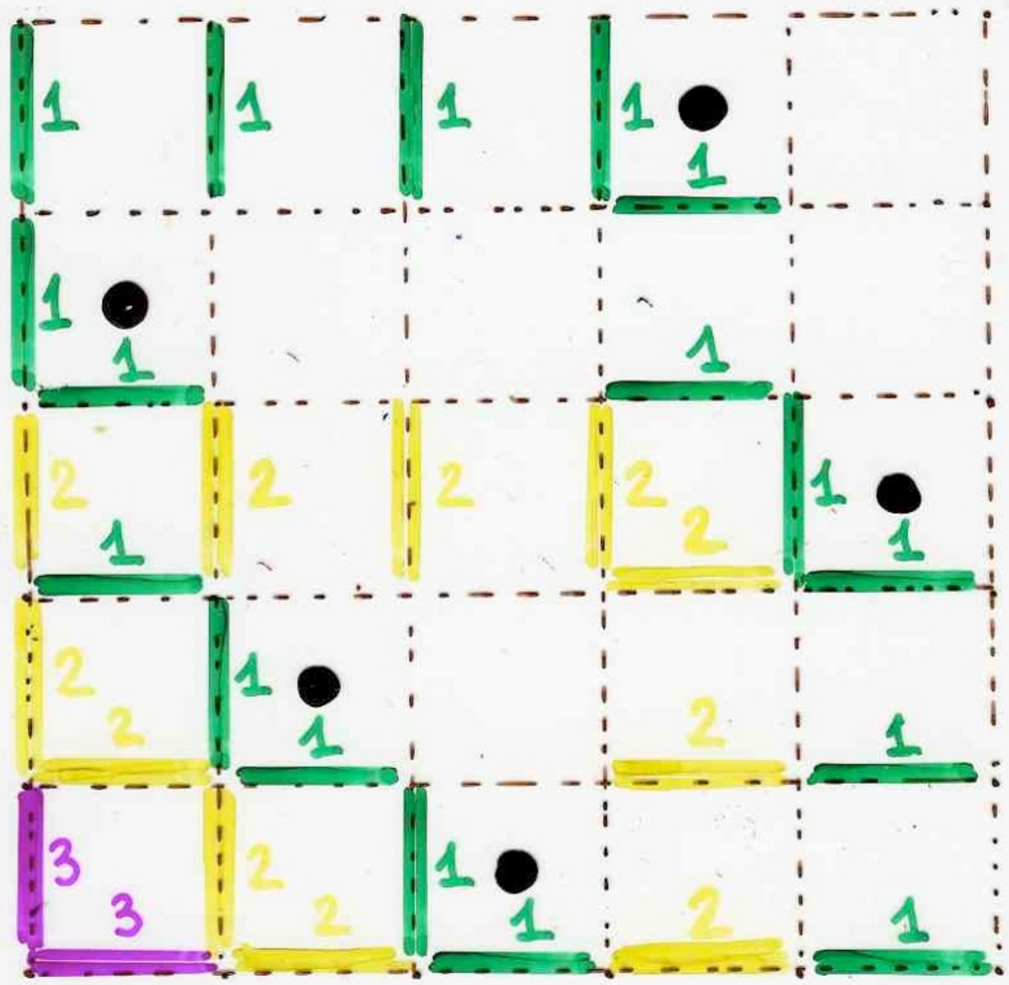
$L = \{\square, \boxdot\}$

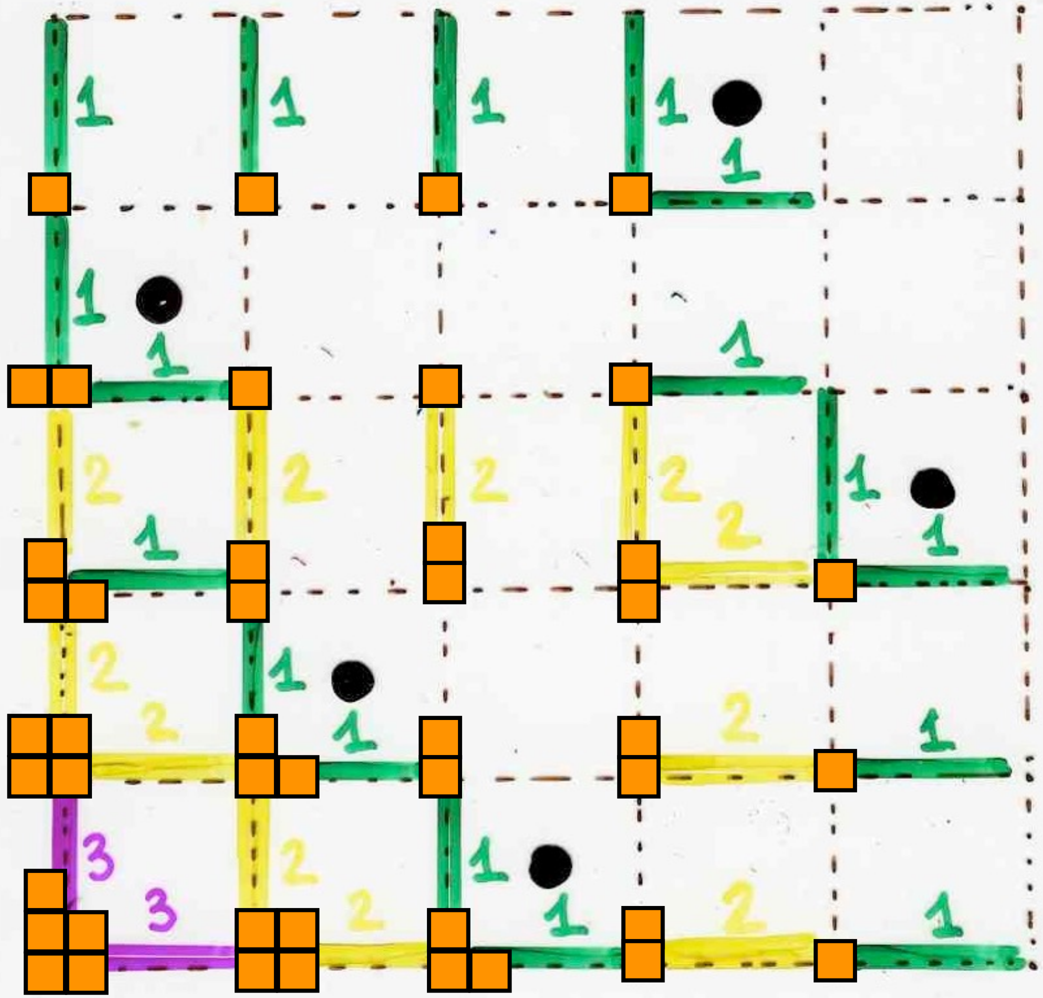




$i \neq j$







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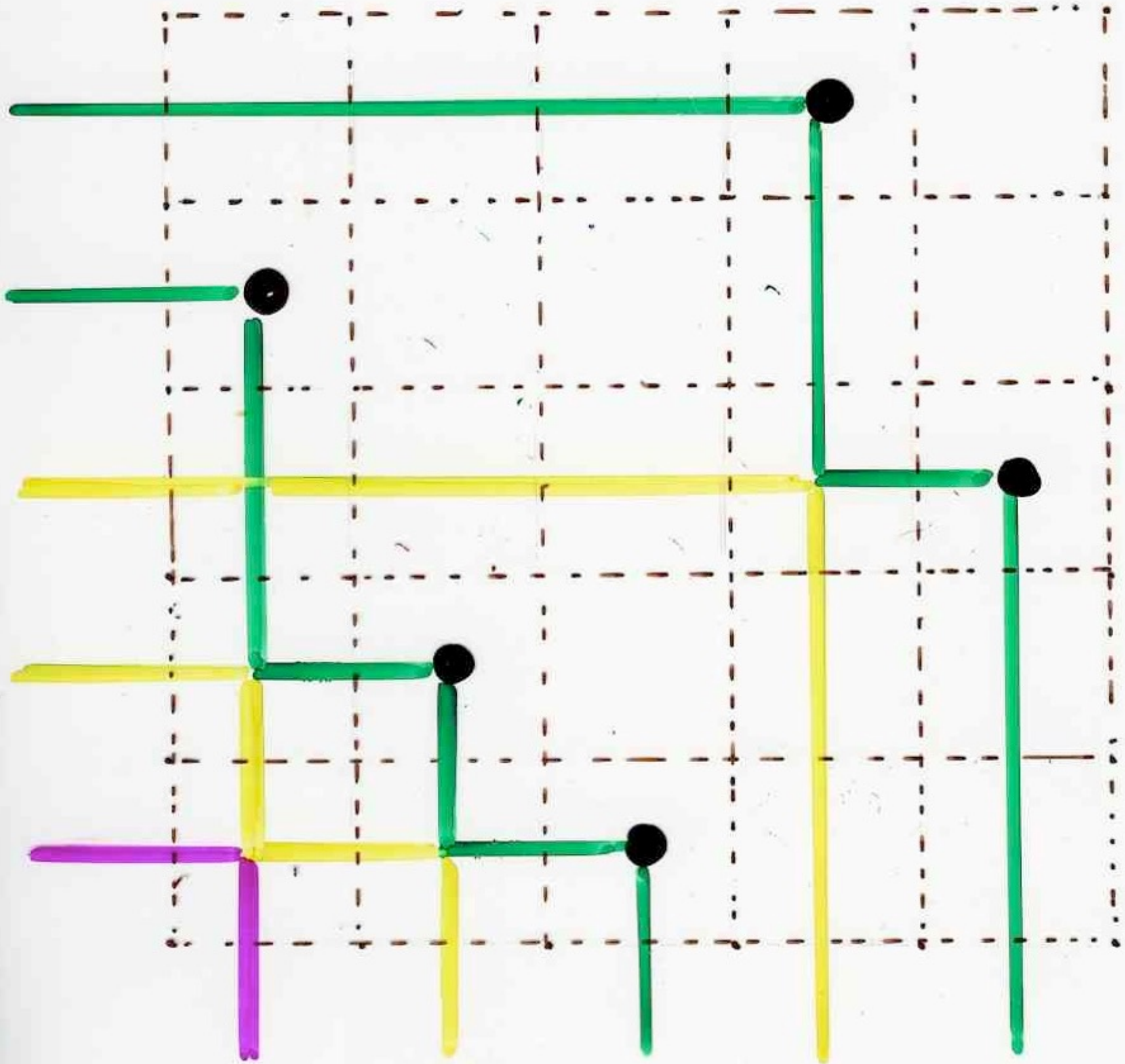

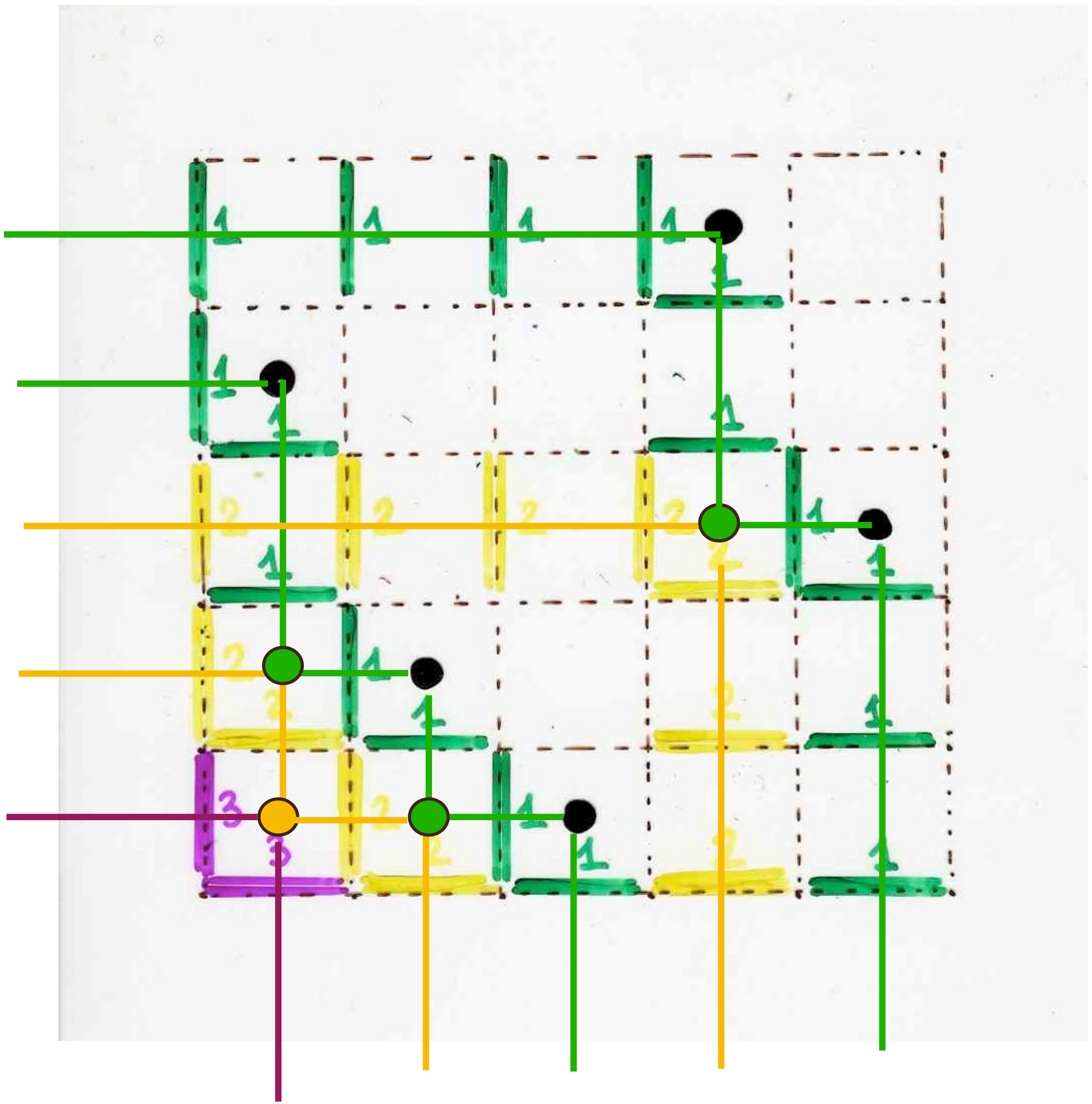

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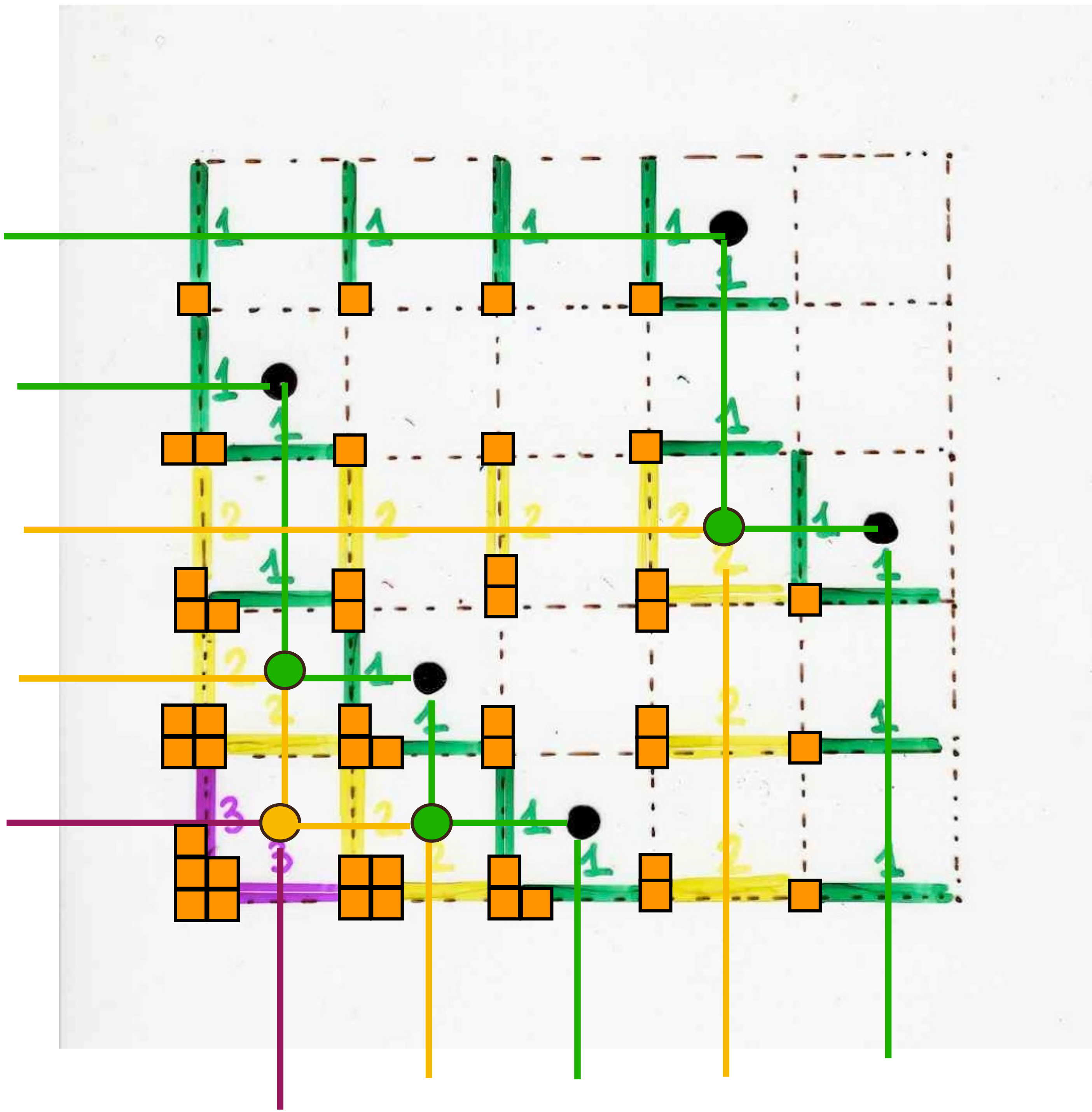
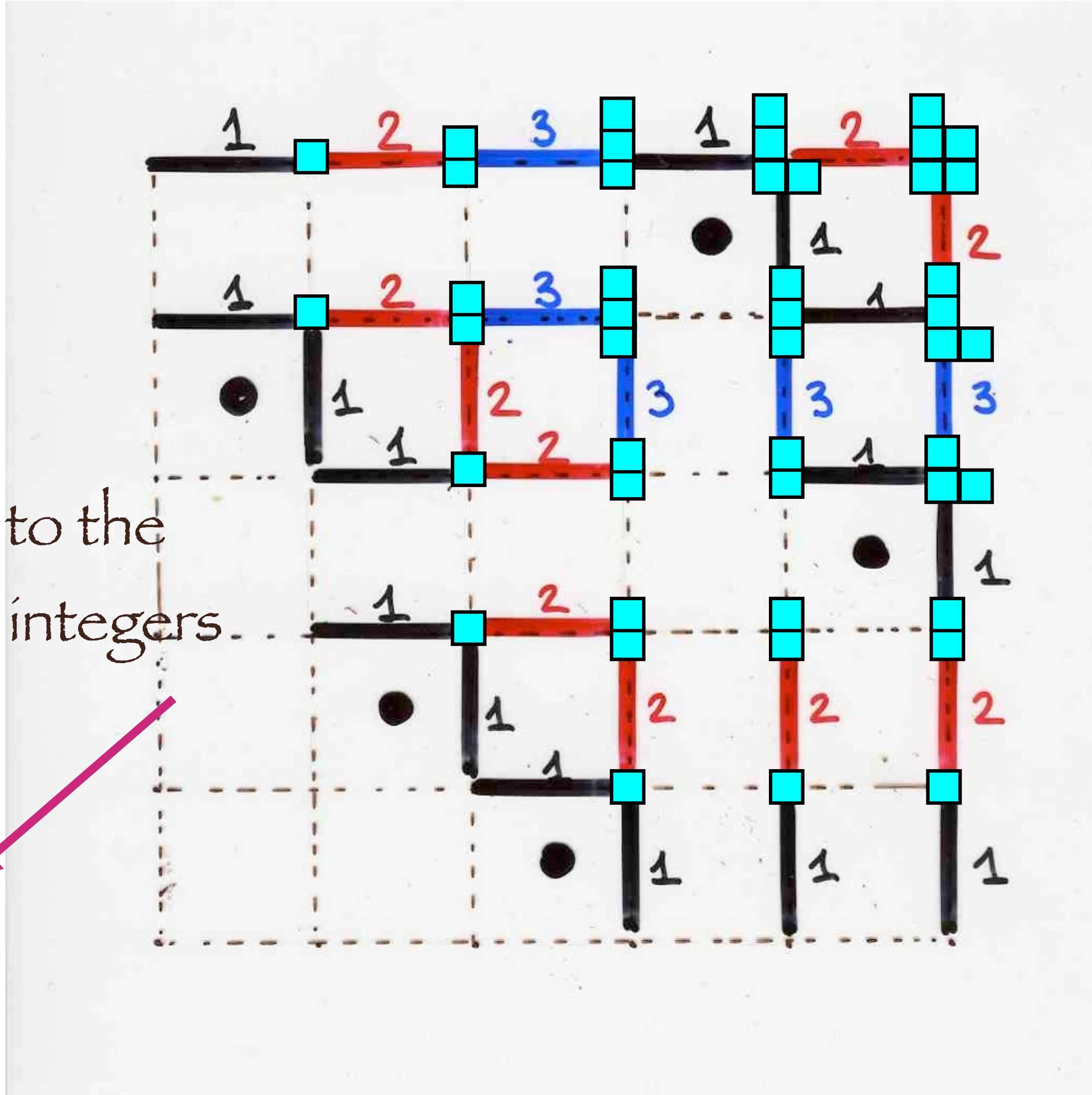
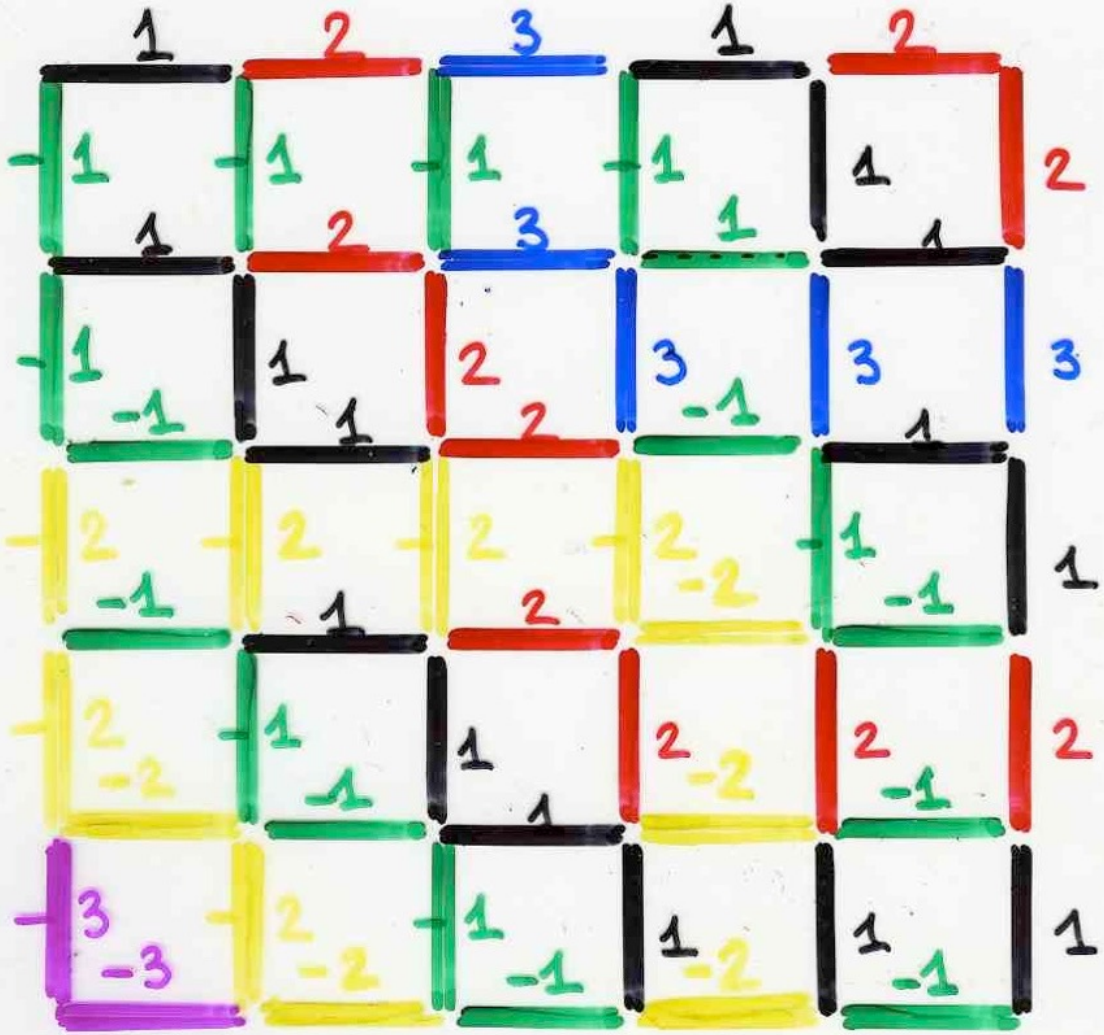


The bilateral RSK planar automaton



Going to the negative integers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bilateral

planar automaton RSK

$\mathcal{B} = \{B_i\}_{i \in \mathbb{Z}-\{0\}}$

$\mathcal{A} = \{A_j\}_{j \in \mathbb{Z}-\{0\}}$

$B_i A_j = A_j B_i$

$i \neq j$

$B_i A_i = A_{i-1} B_{i-1}$

$(i \neq 1)$

$B_1 A_1 = A_{-1} B_{-1}$

bilateral

(reverse) planar automaton RSK

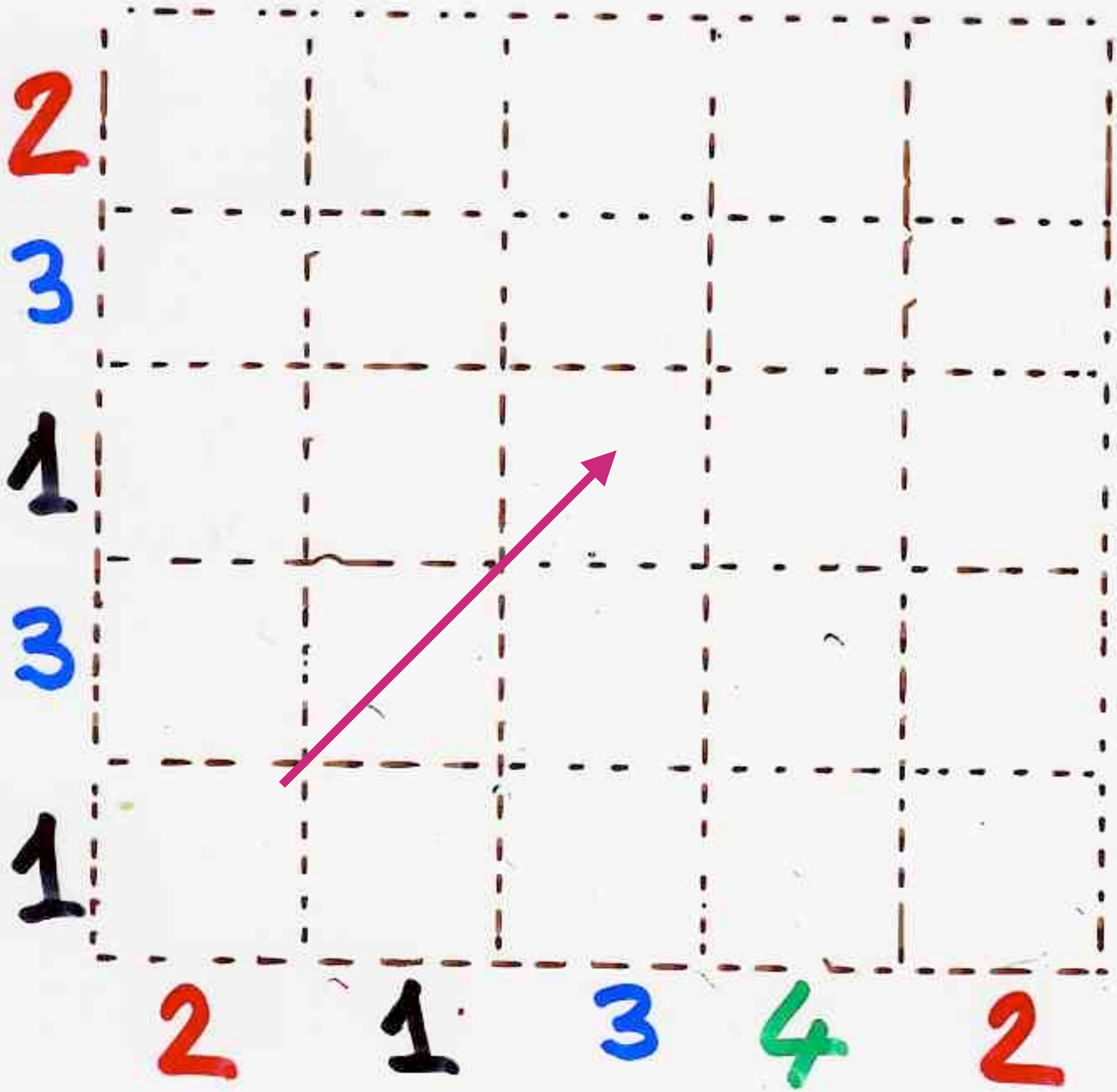
$A_j B_i = B_i A_j$

$i \neq j$

$A_i B_i = B_{i+1} A_{i+1}$

$(i \neq -1)$

$A_{-1} B_{-1} = B_1 A_1$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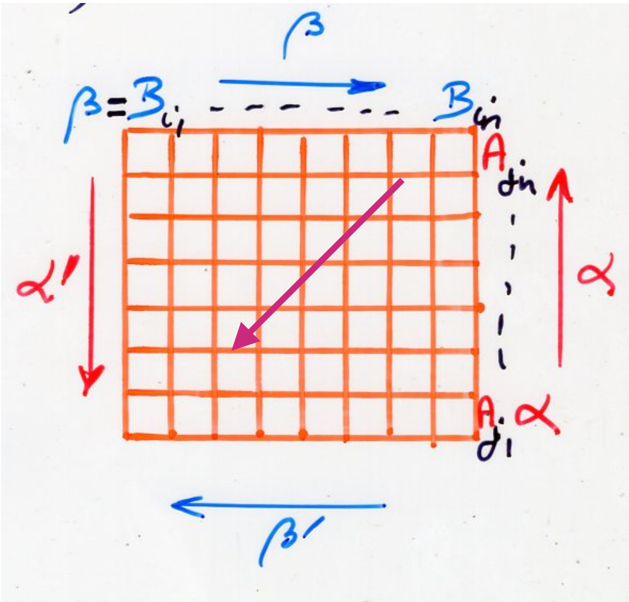
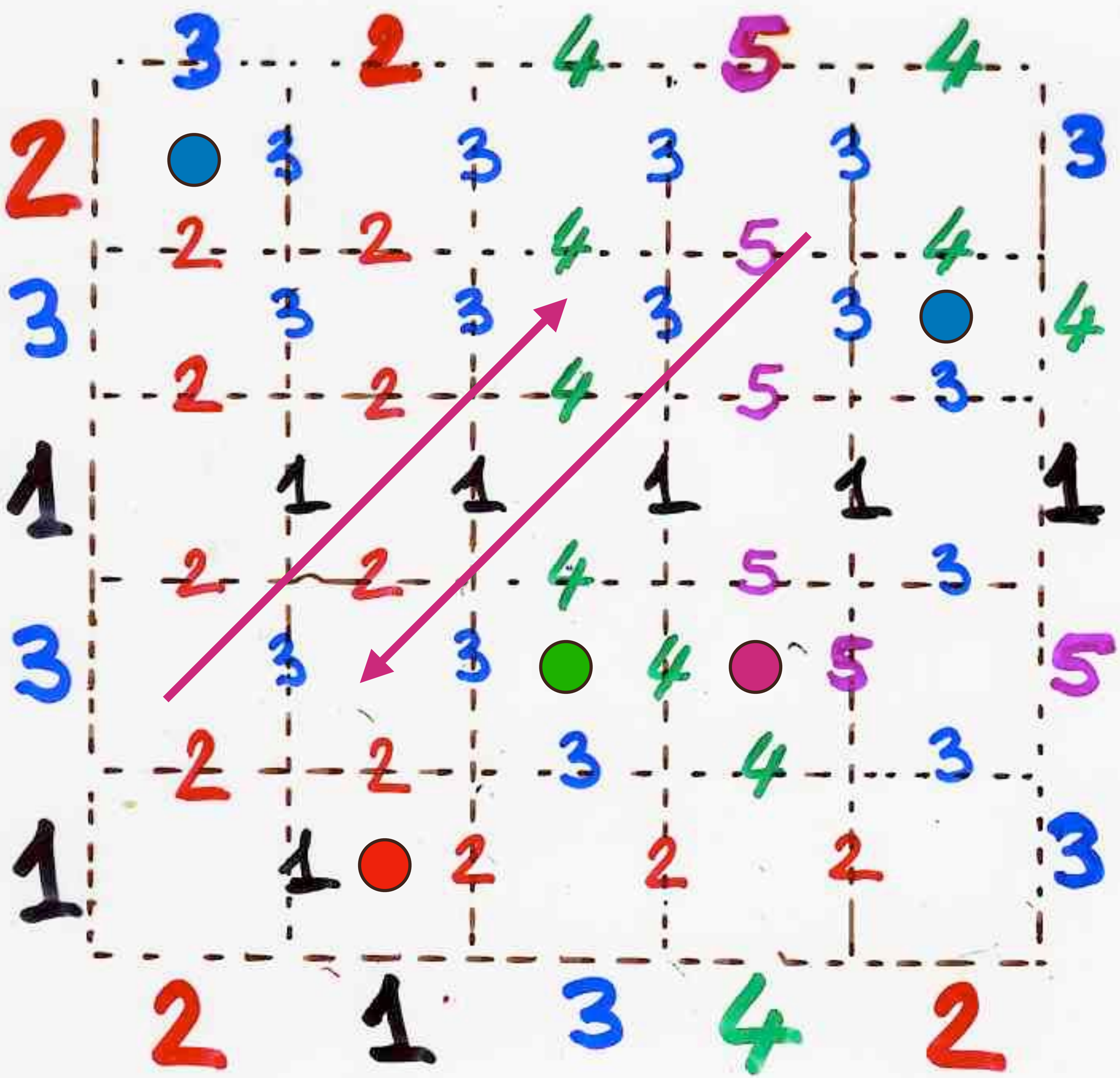
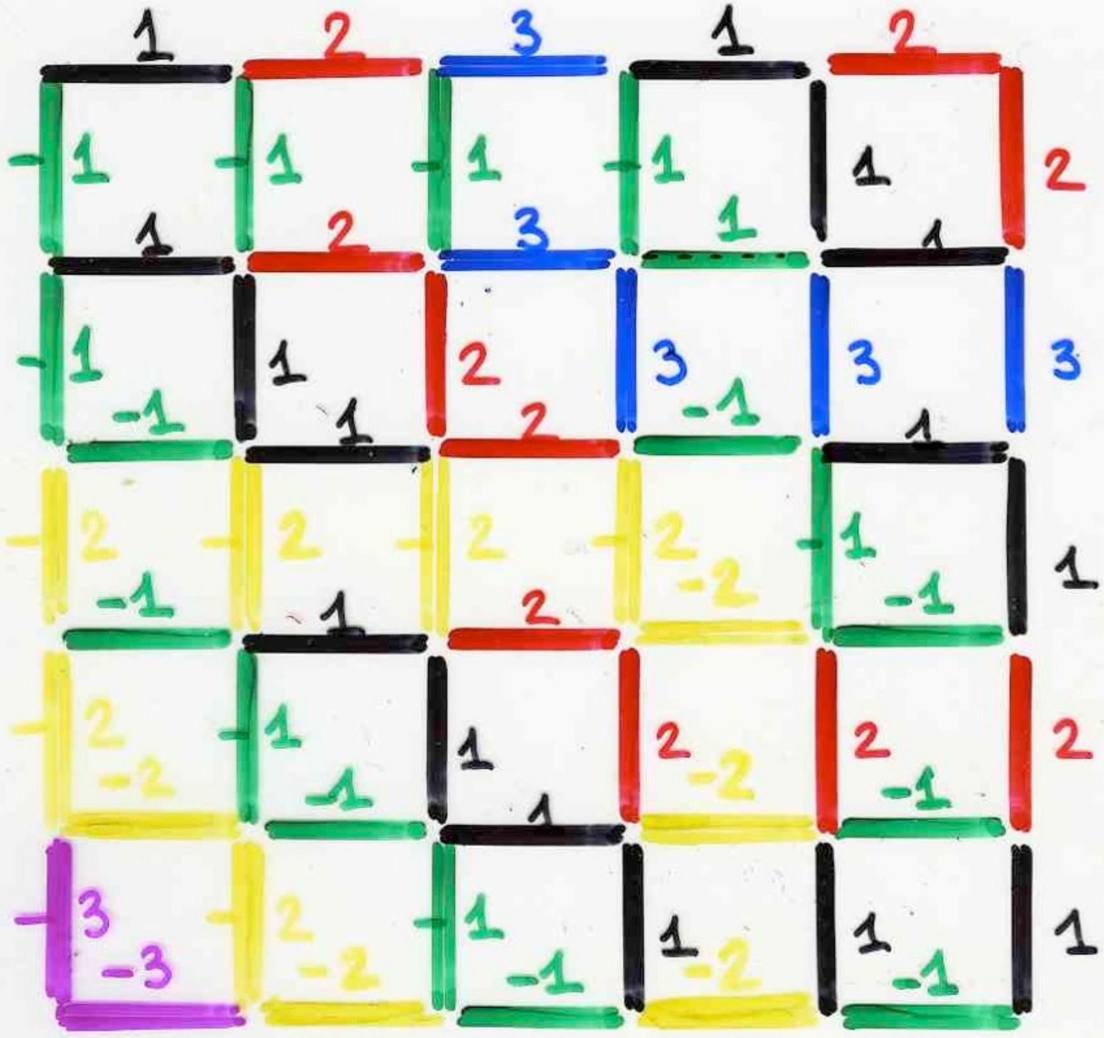
RSK product of two words

$(\beta, \alpha) \longrightarrow (\alpha', \beta')$







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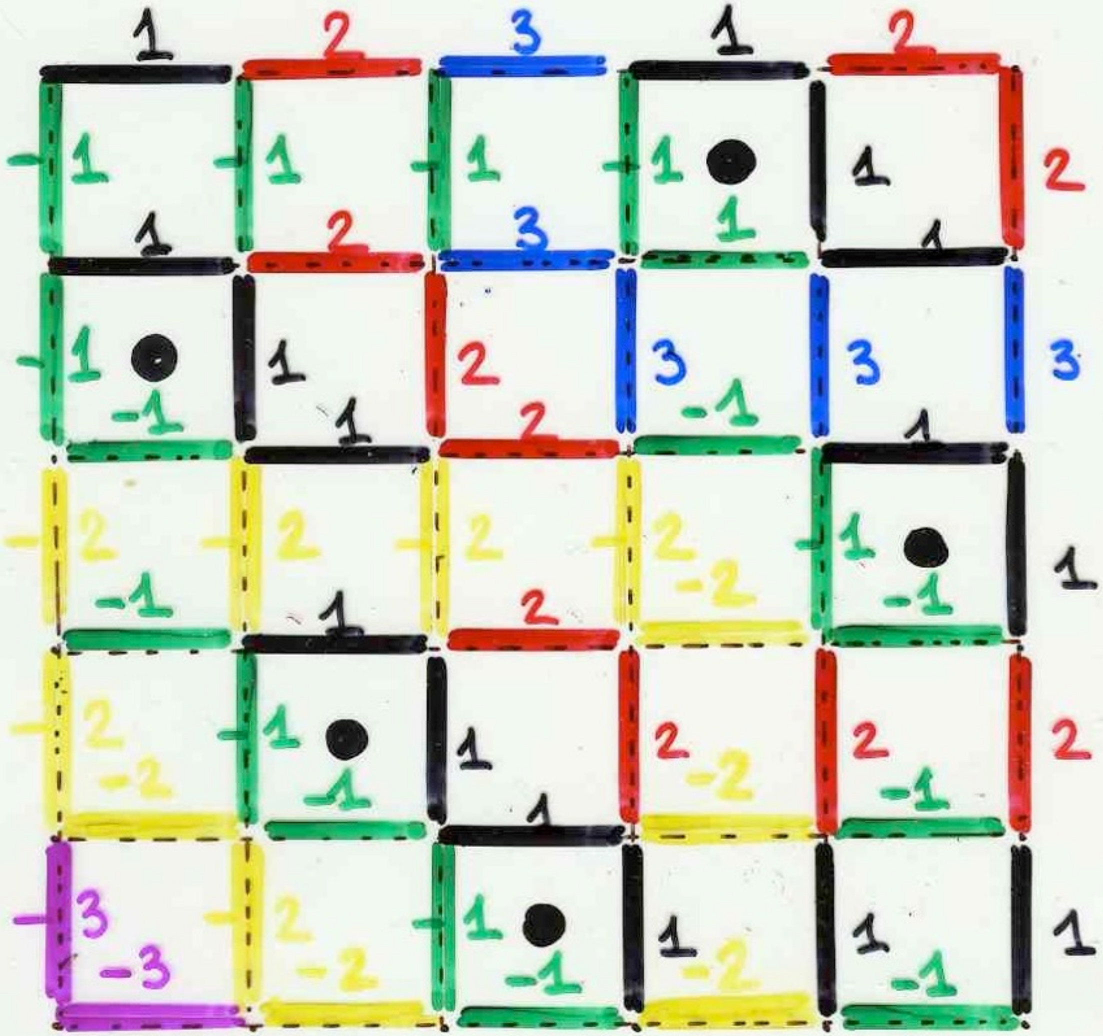

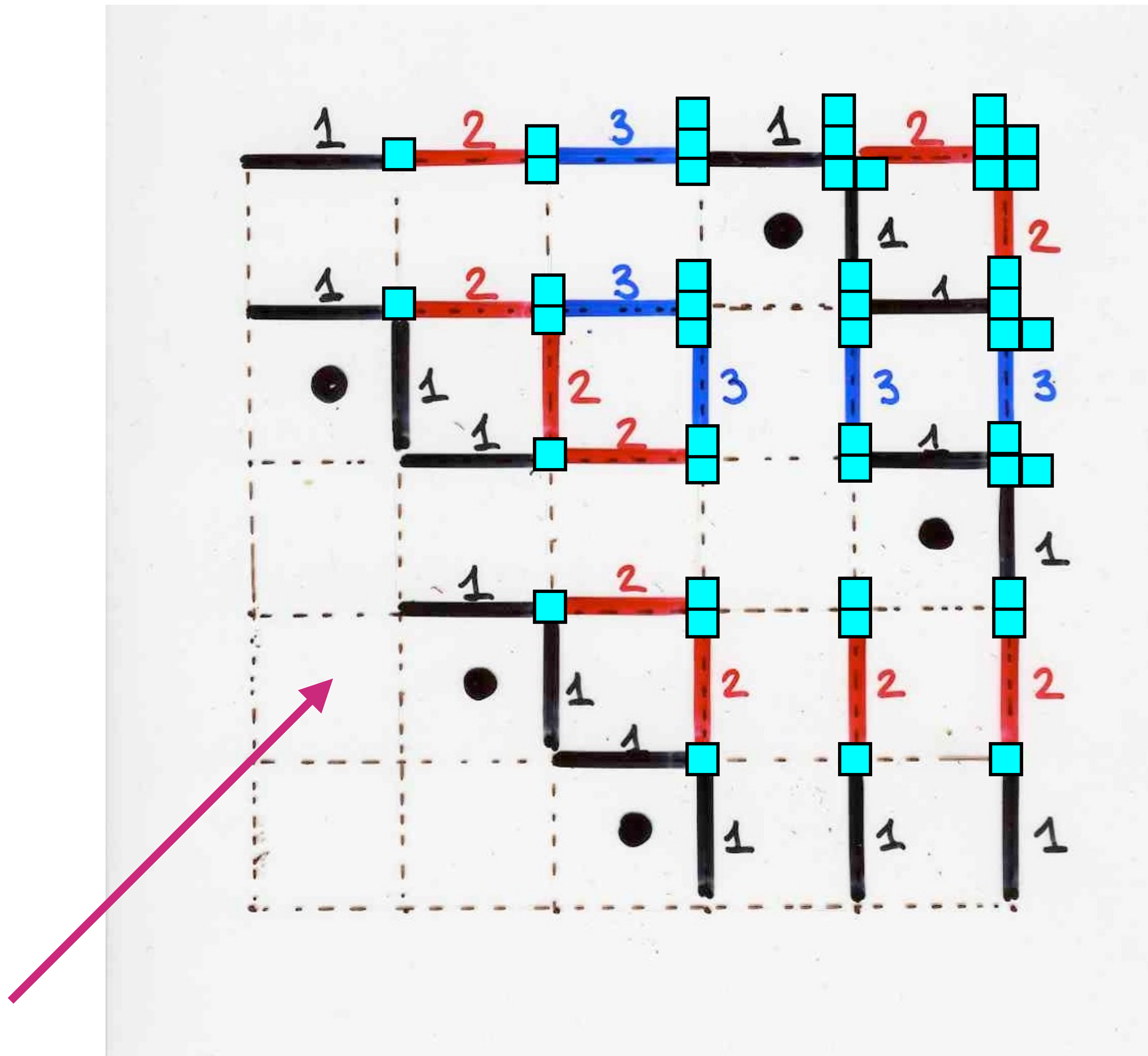
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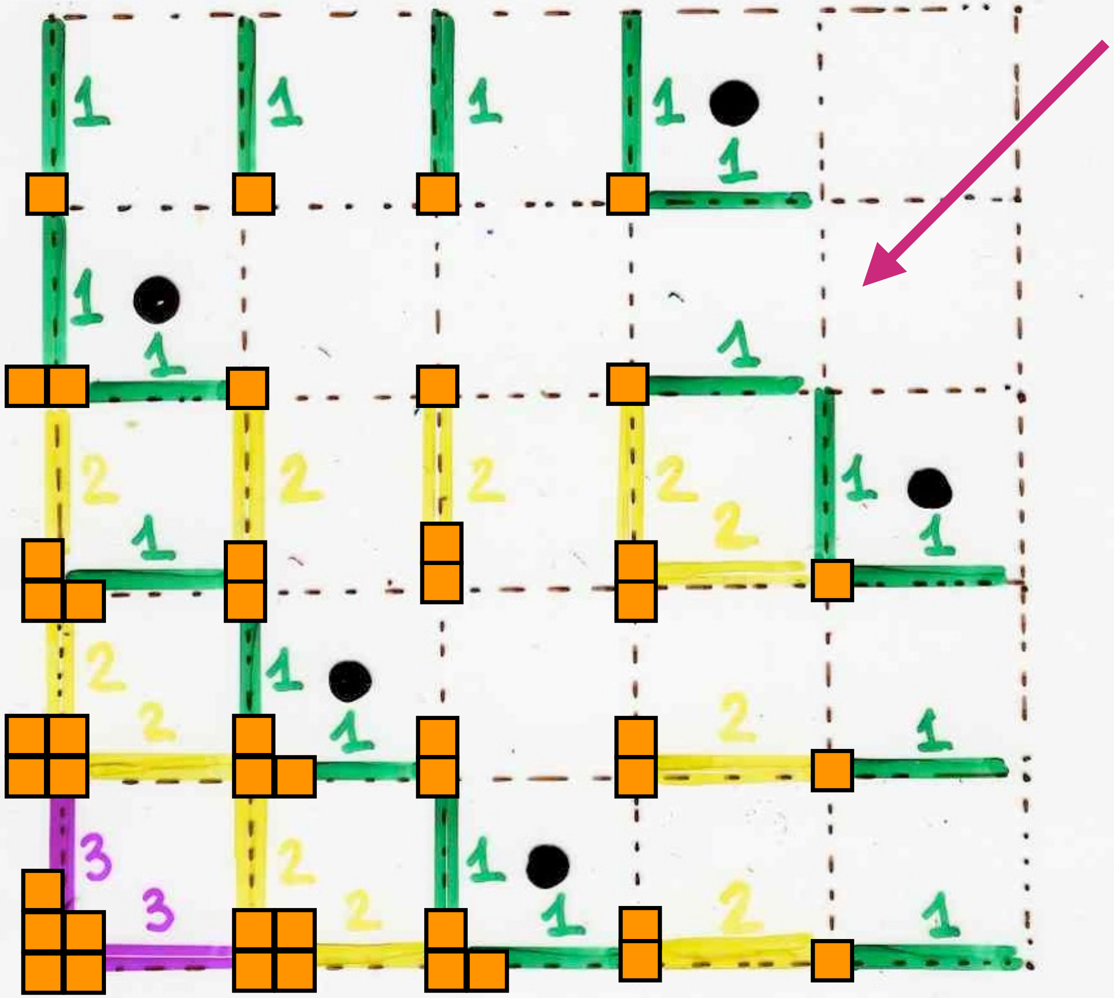

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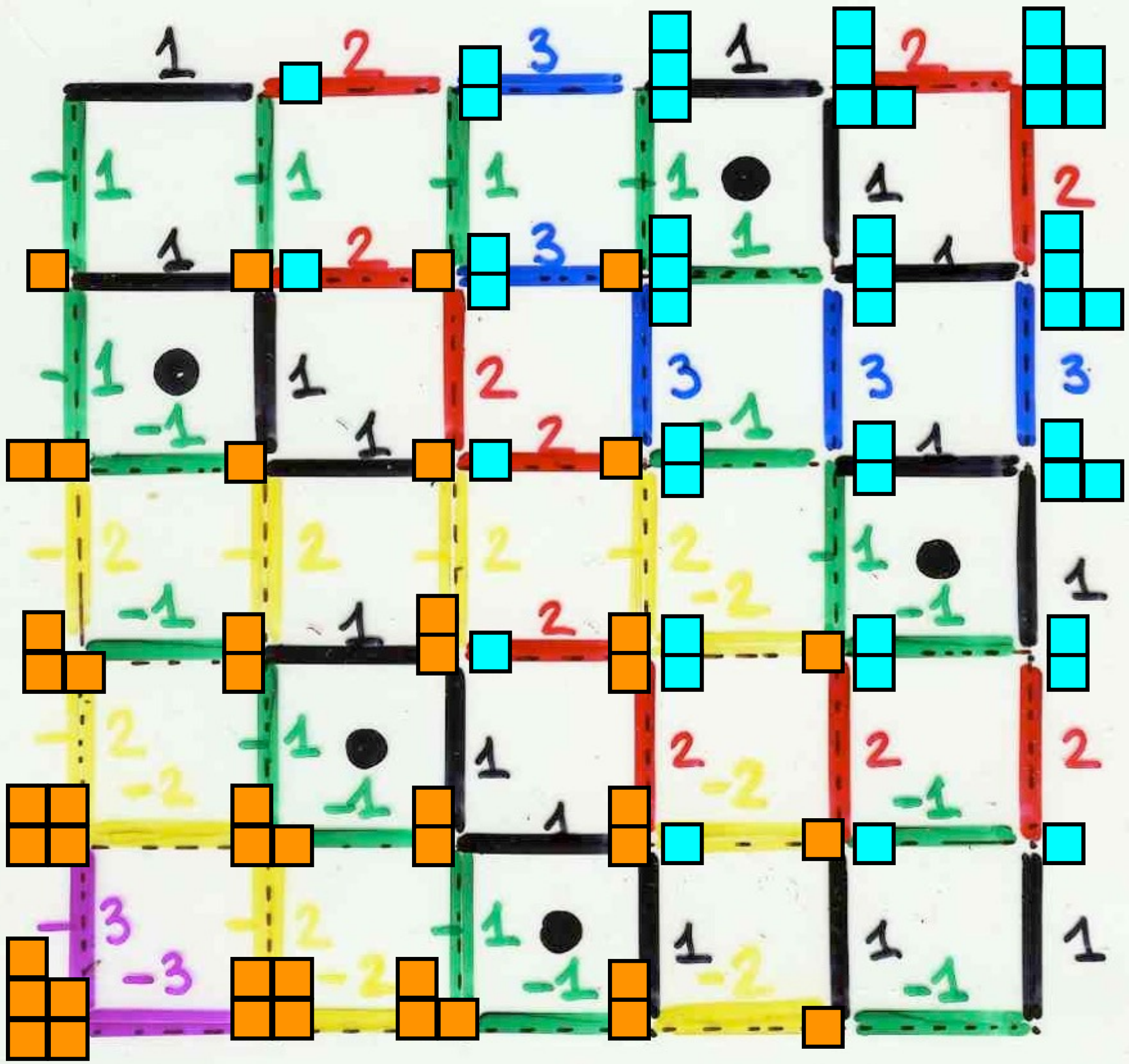
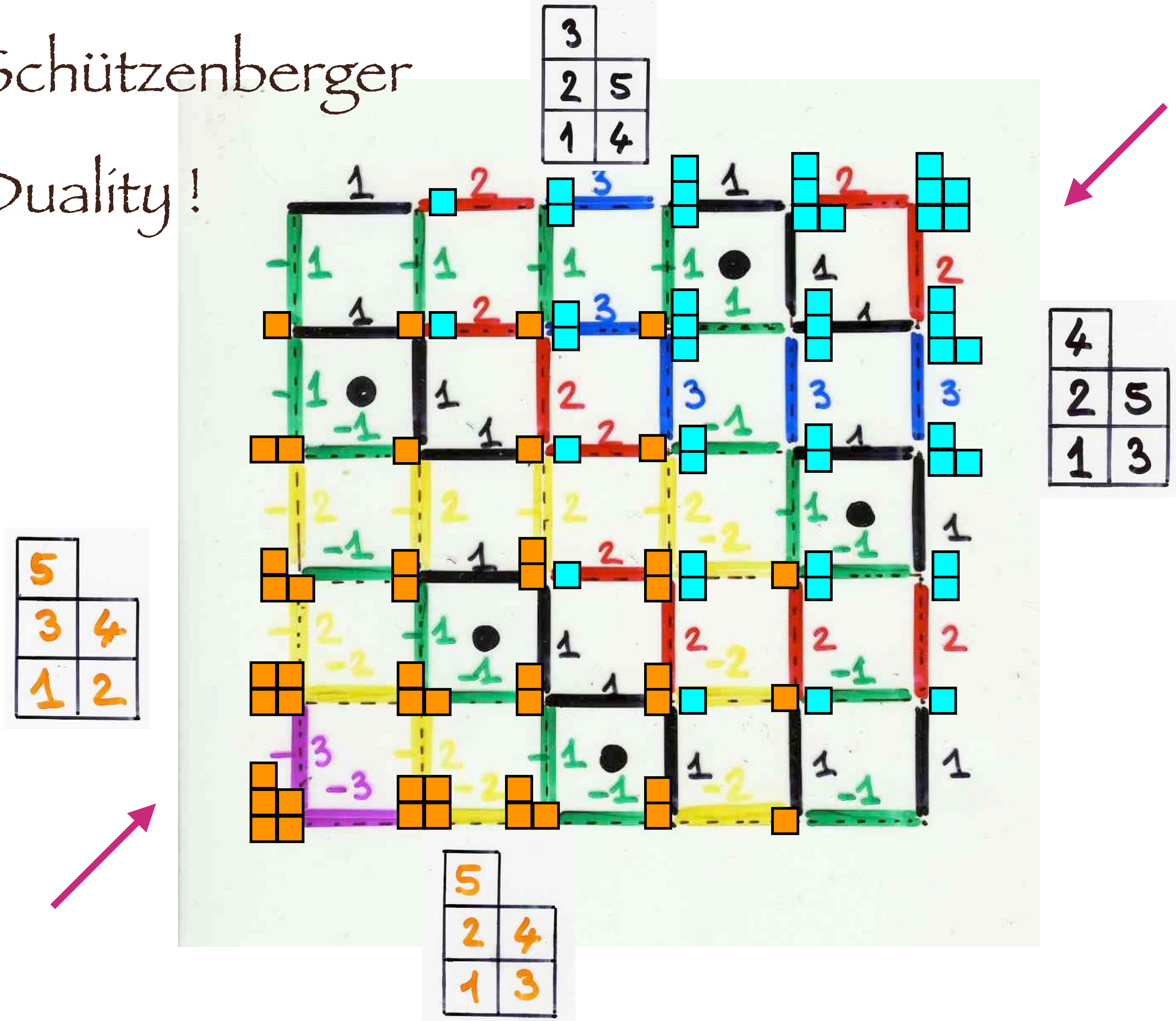
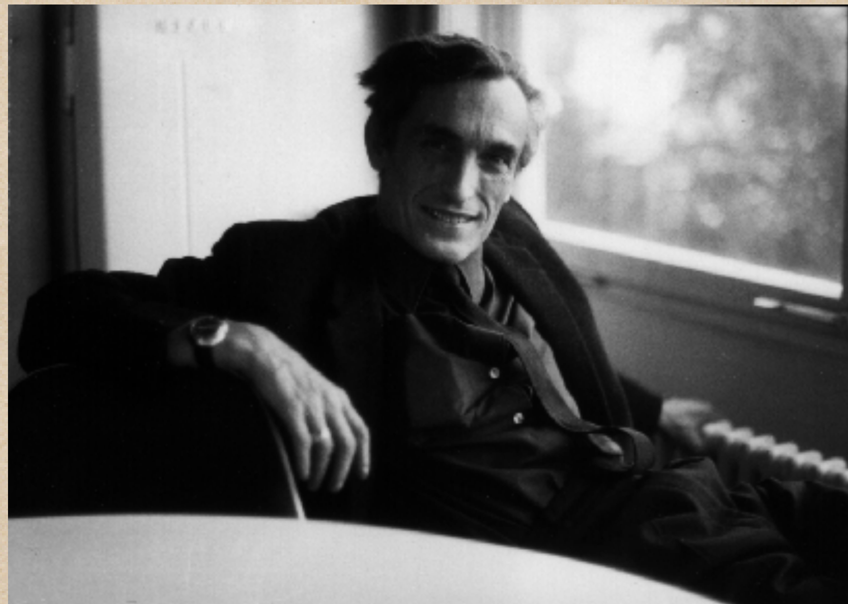


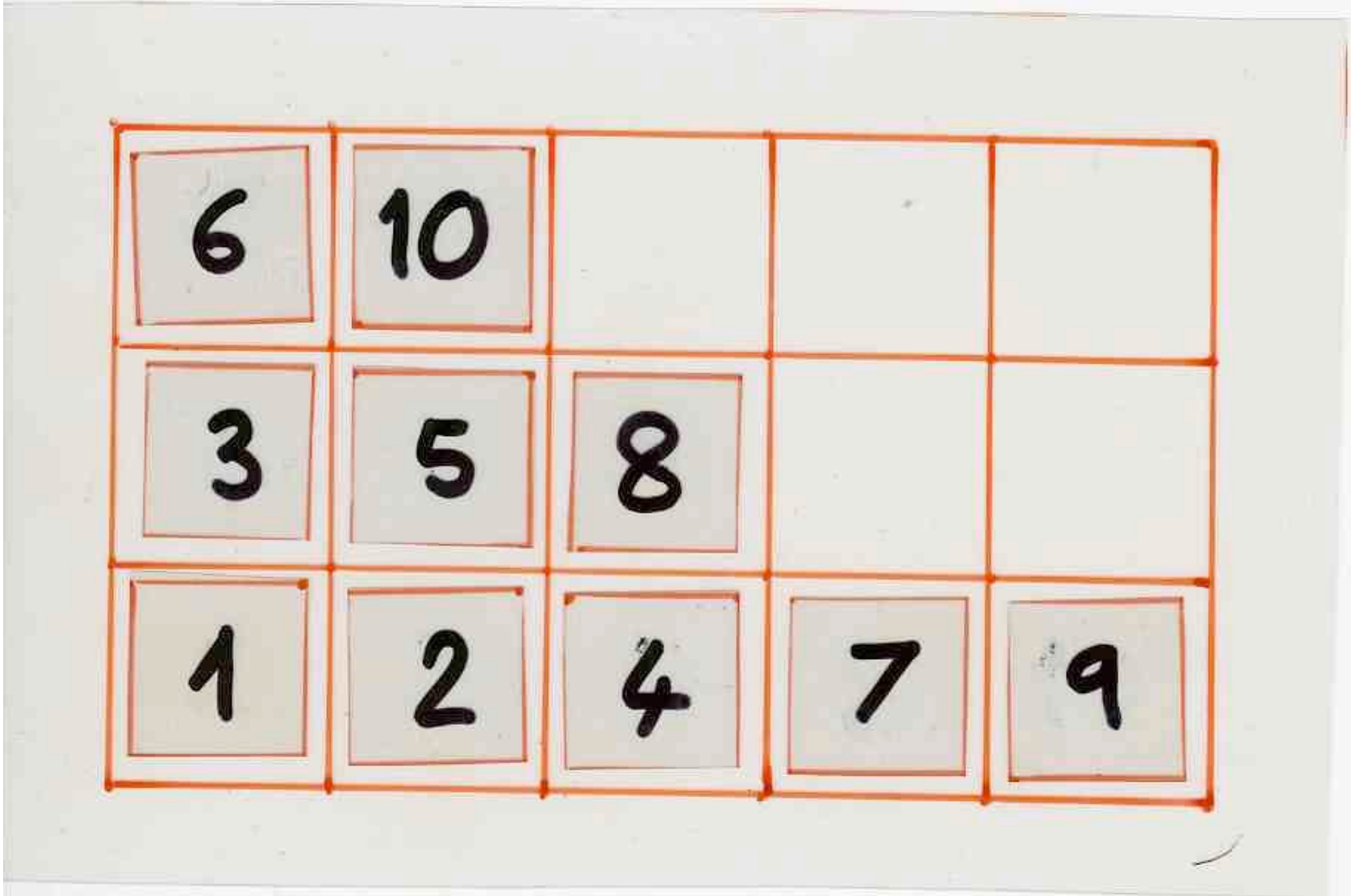
Schützenberger Duality !



dual of a Young tableau



M.P. Schützenberger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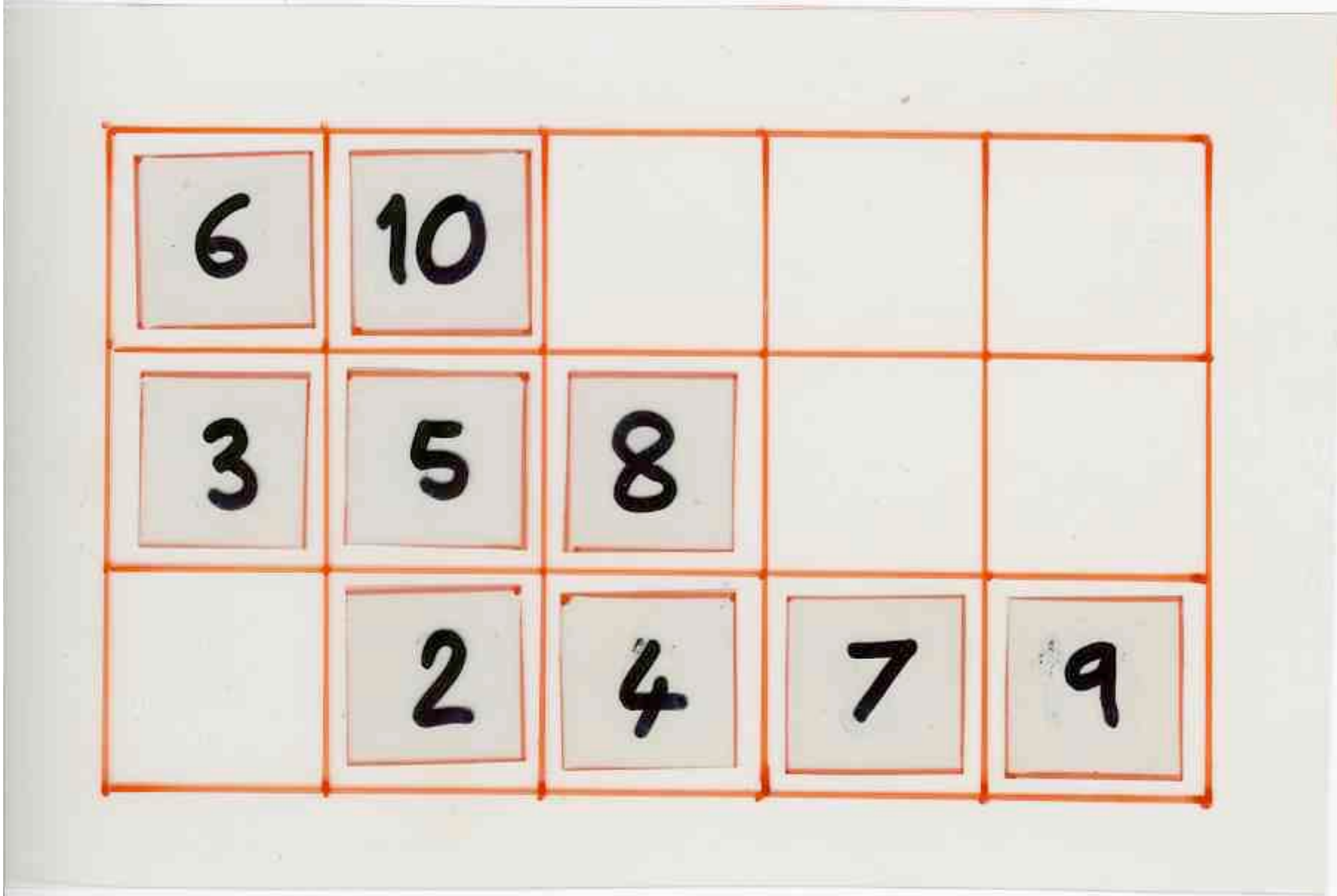

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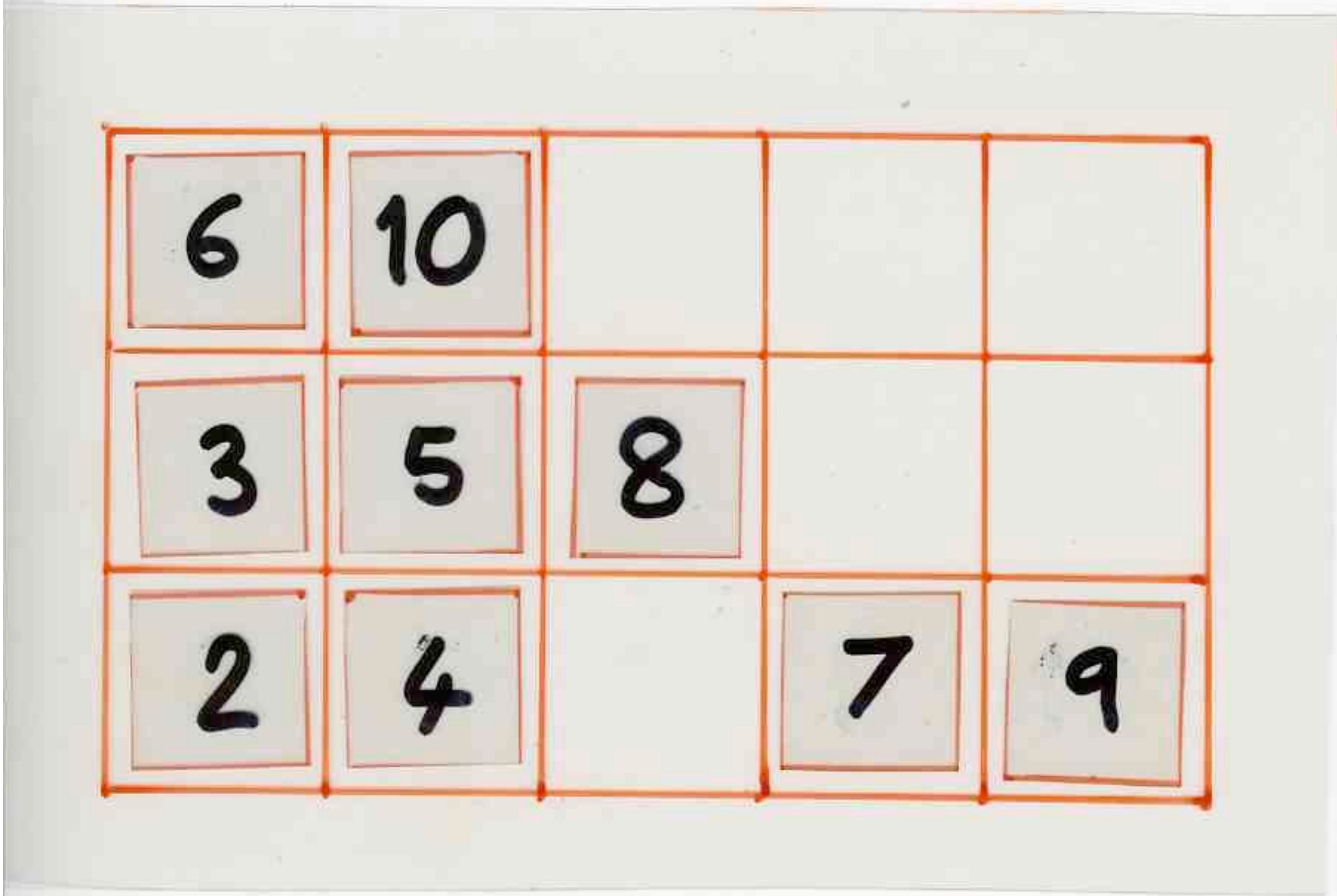
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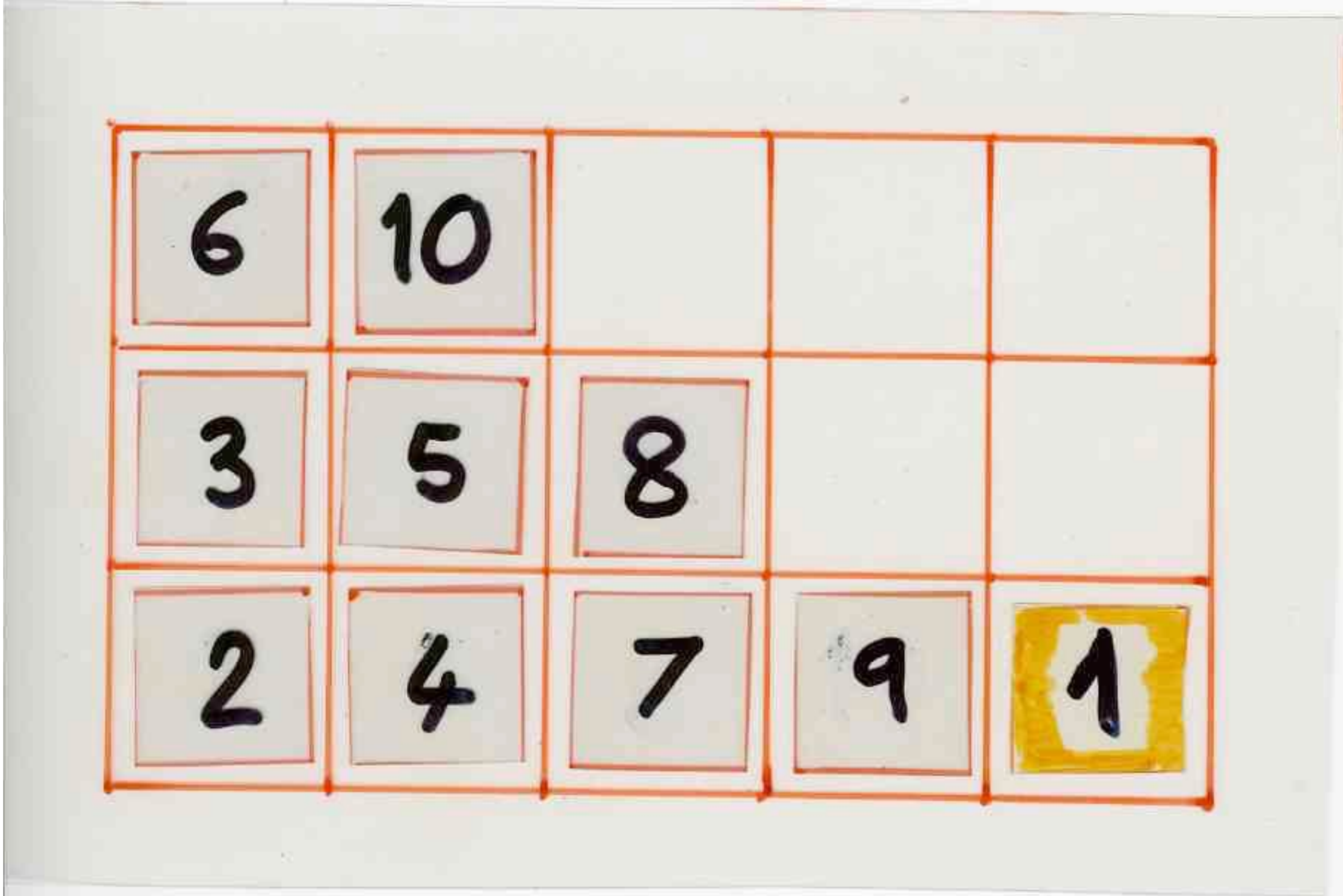
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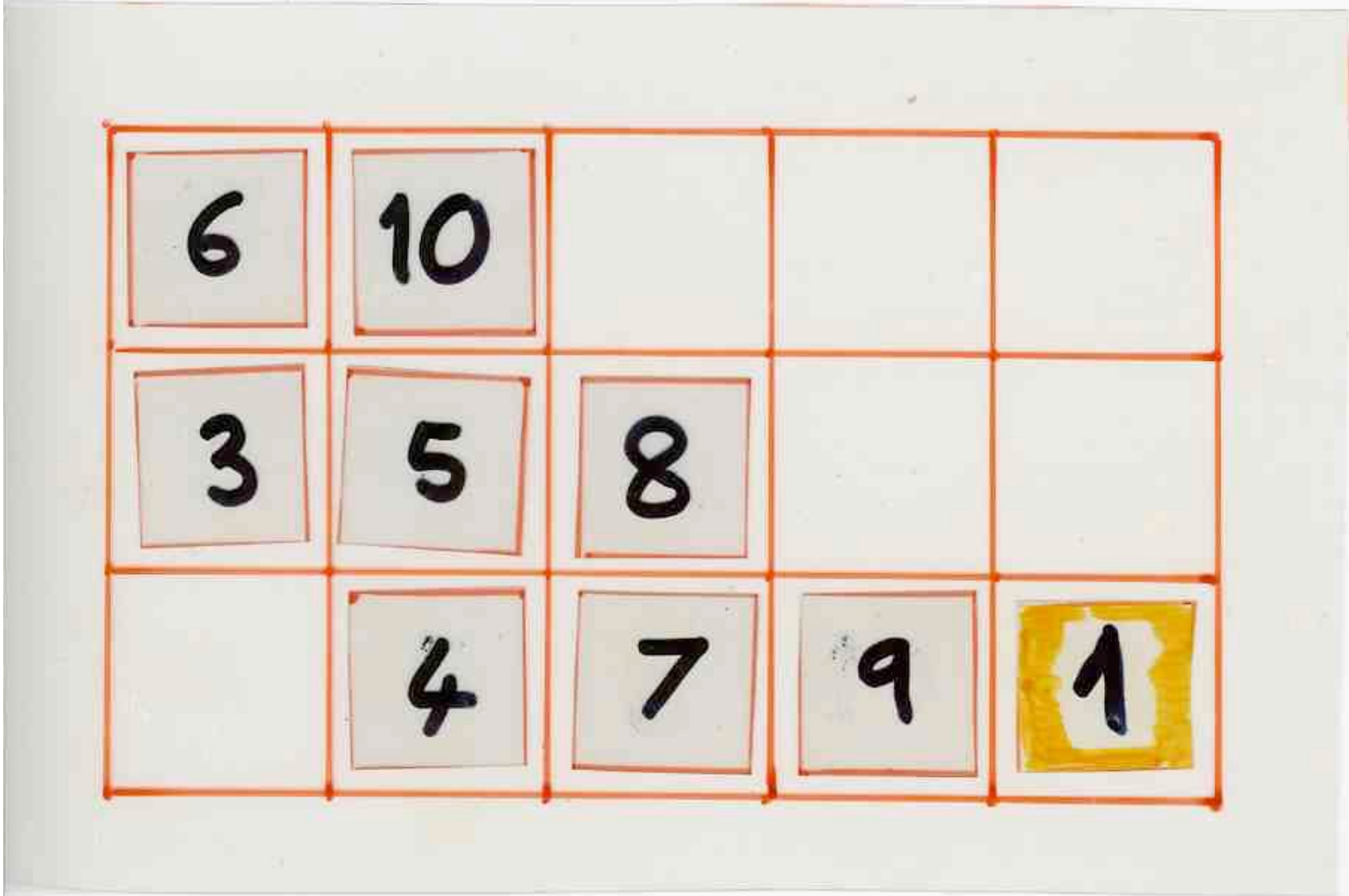
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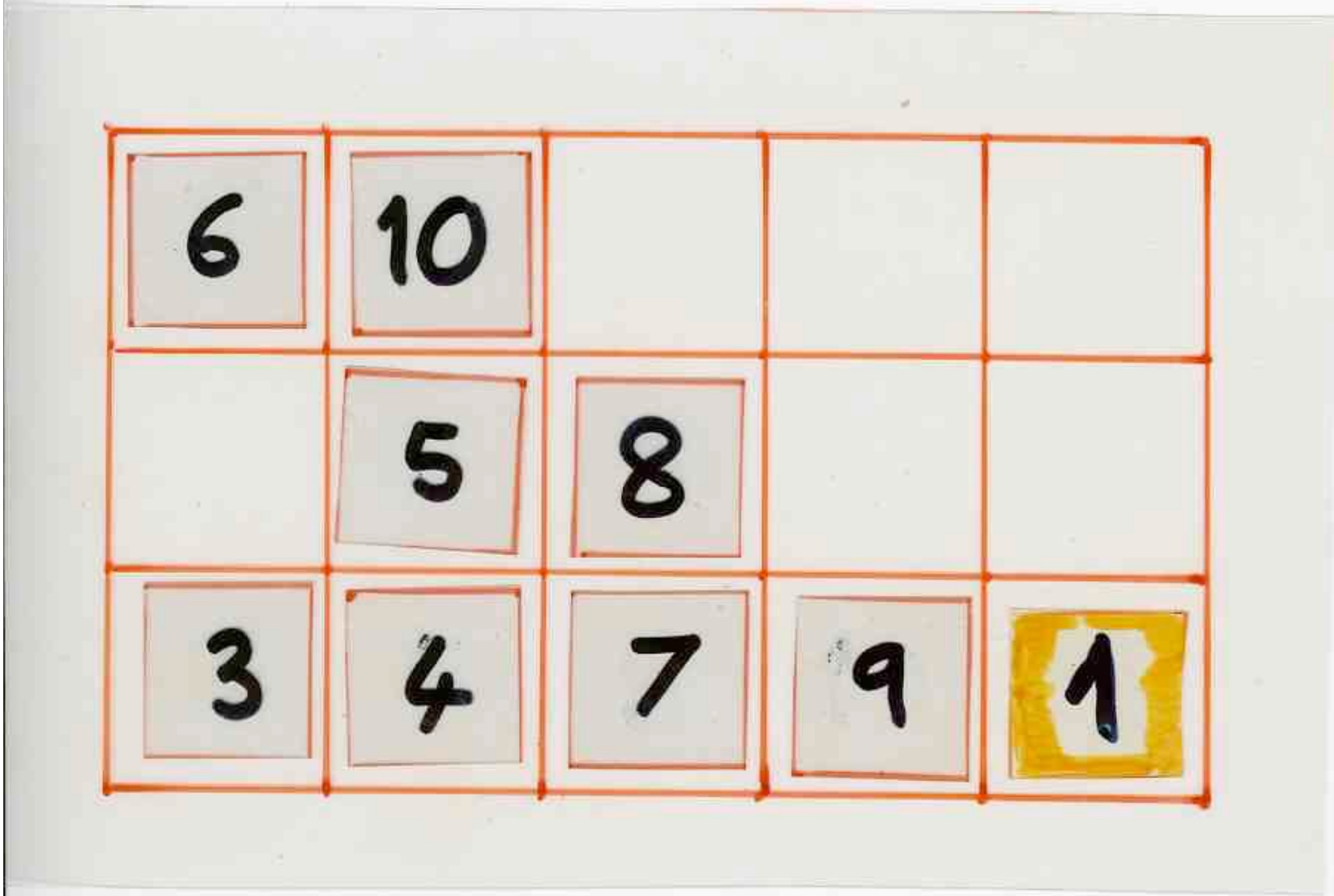
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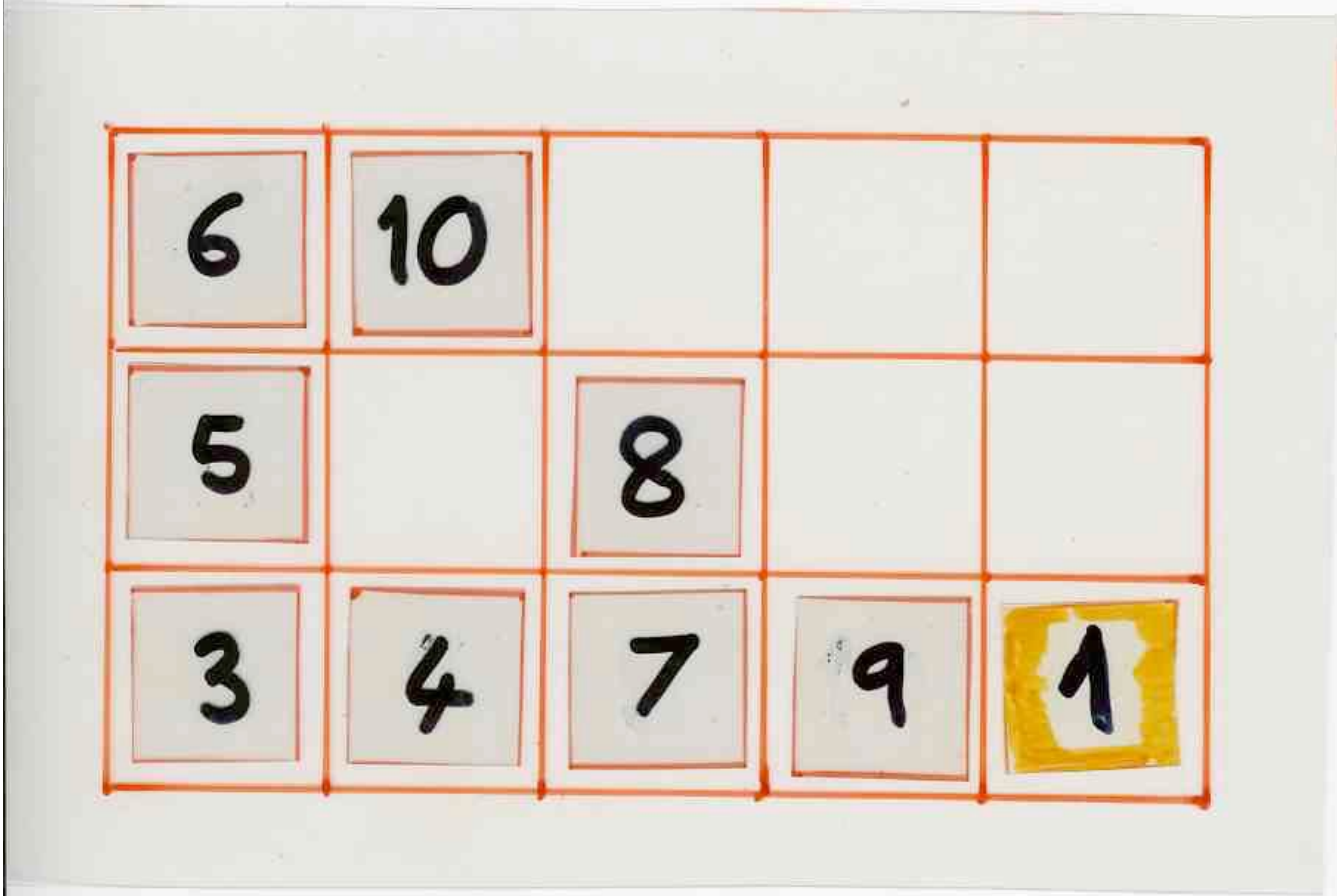
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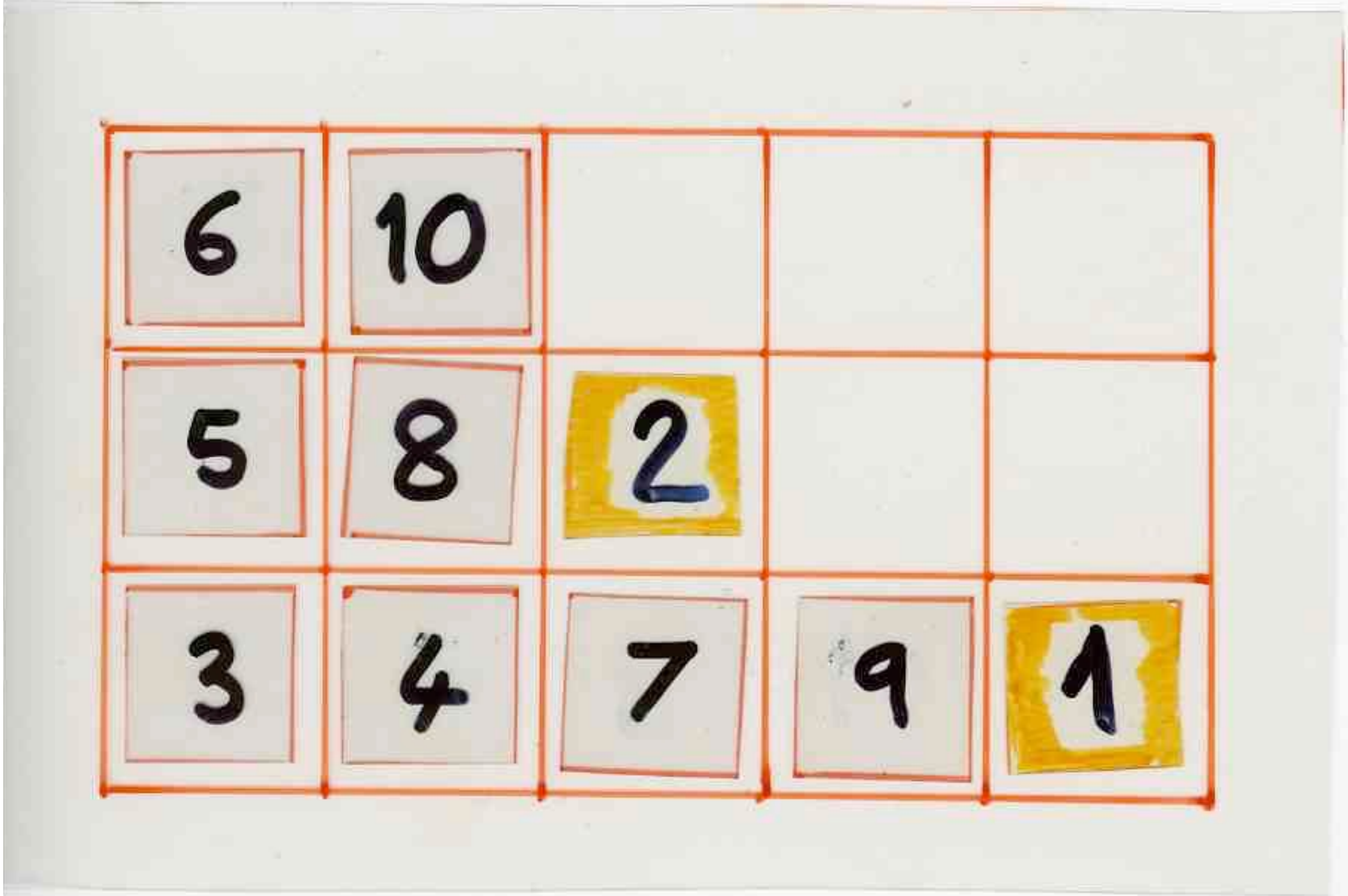
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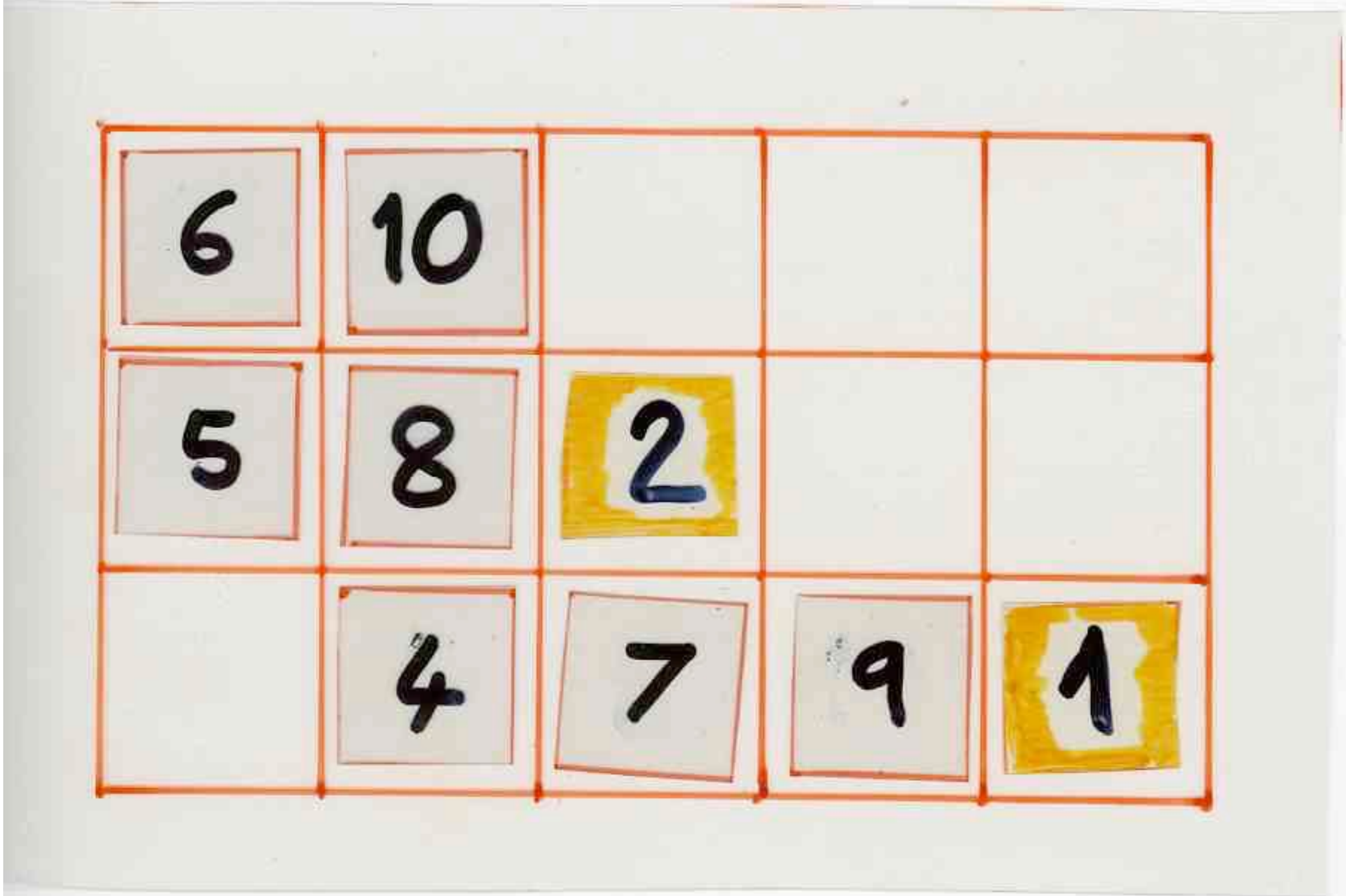

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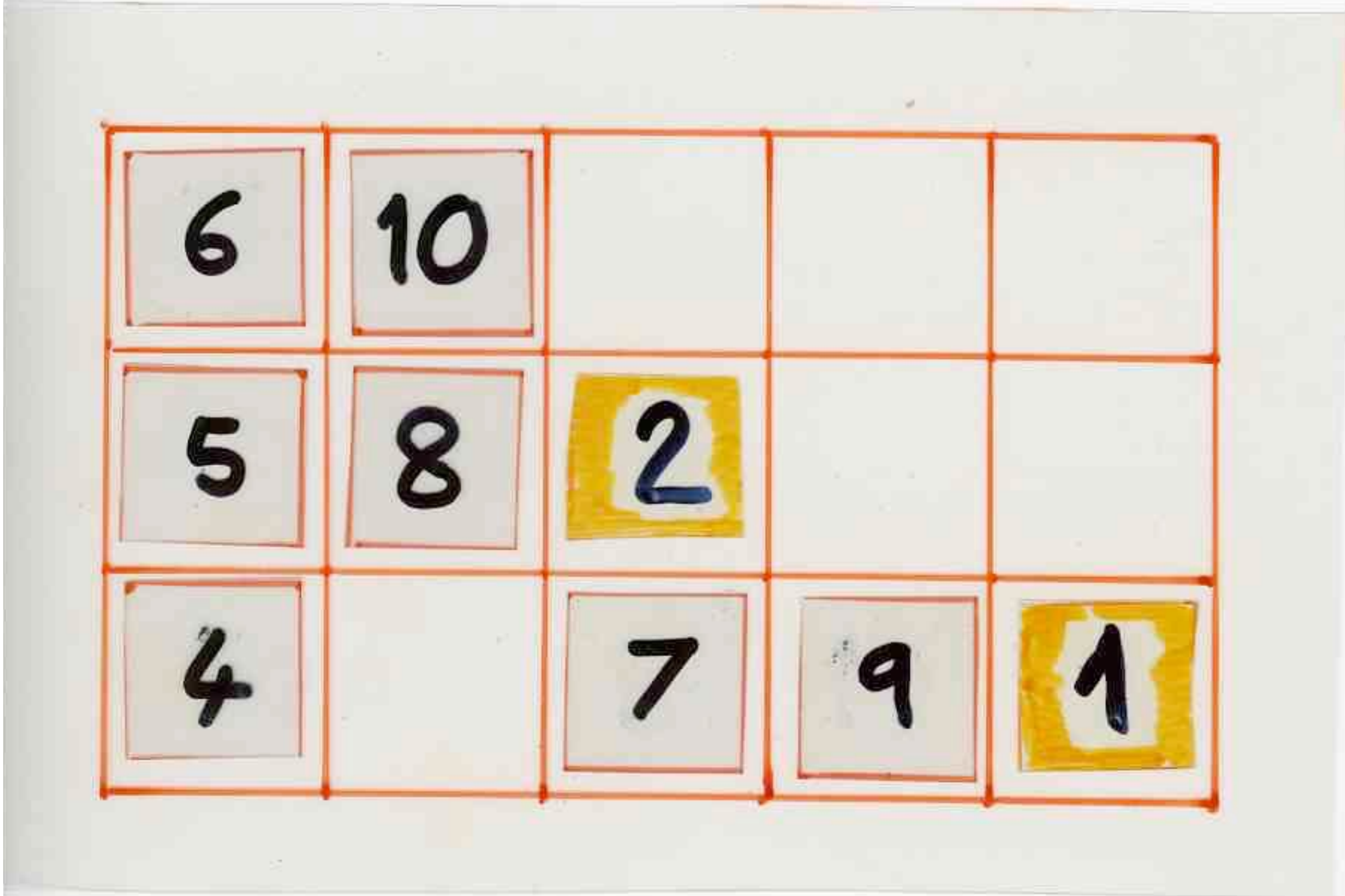
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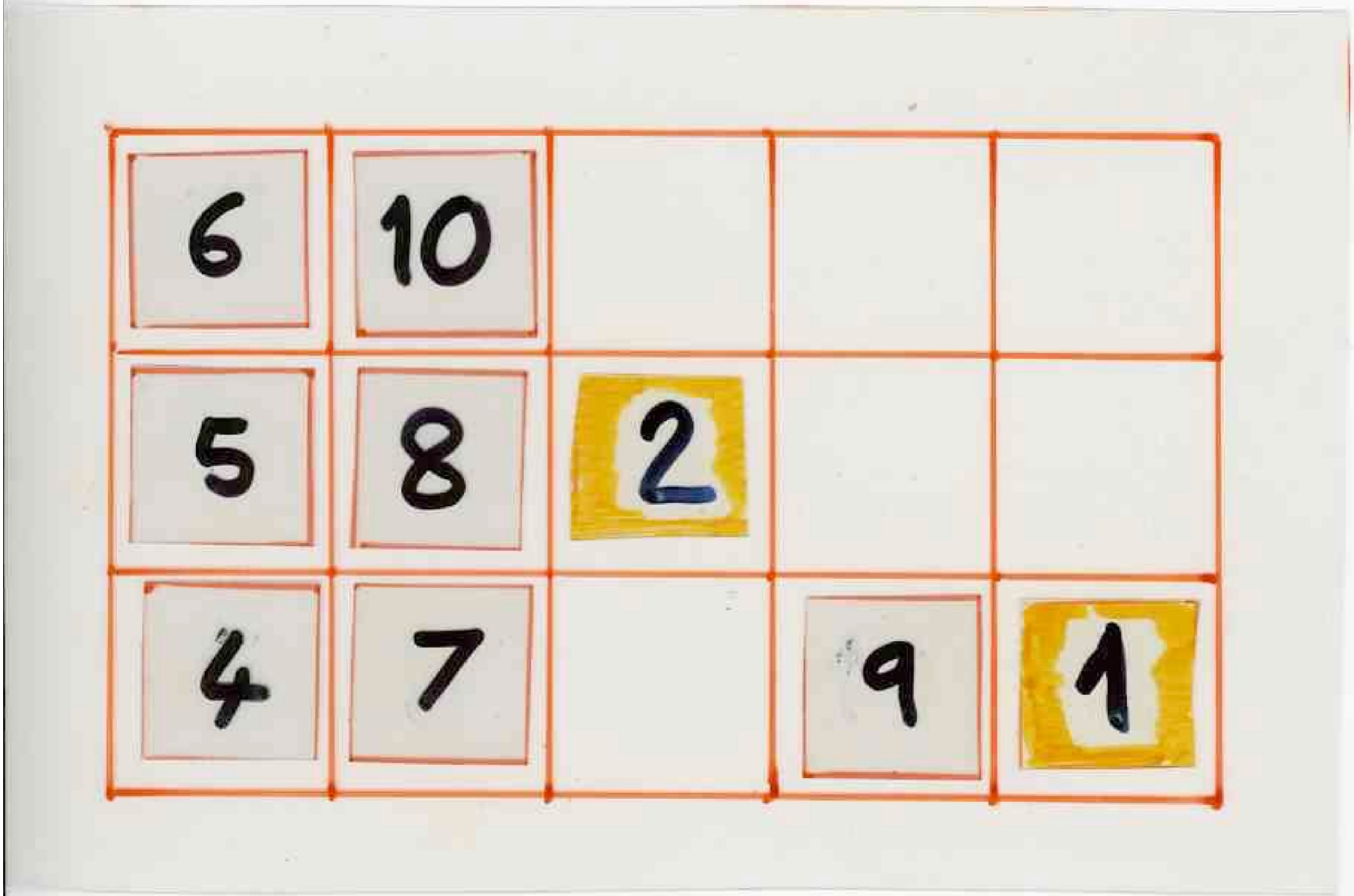
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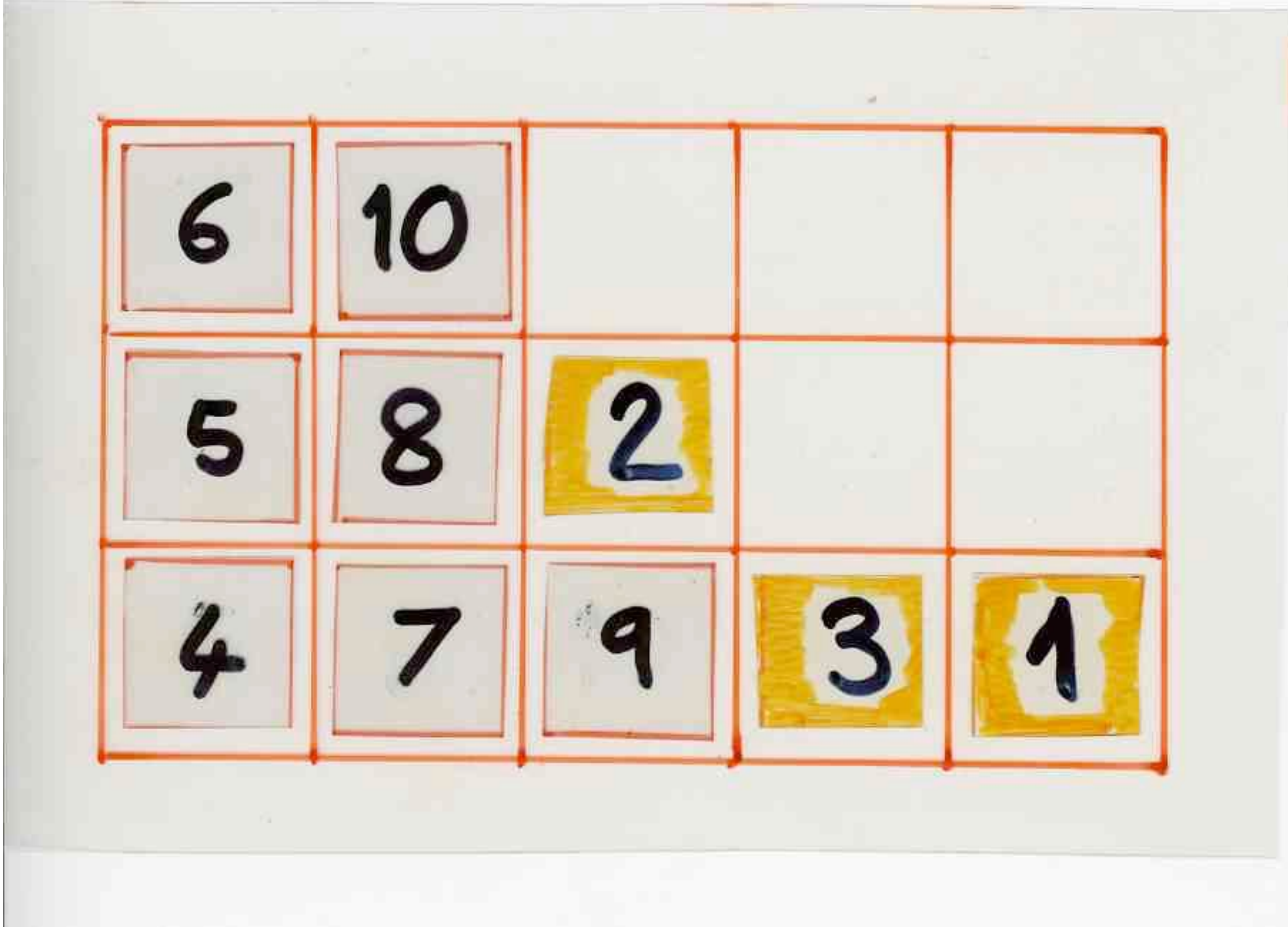
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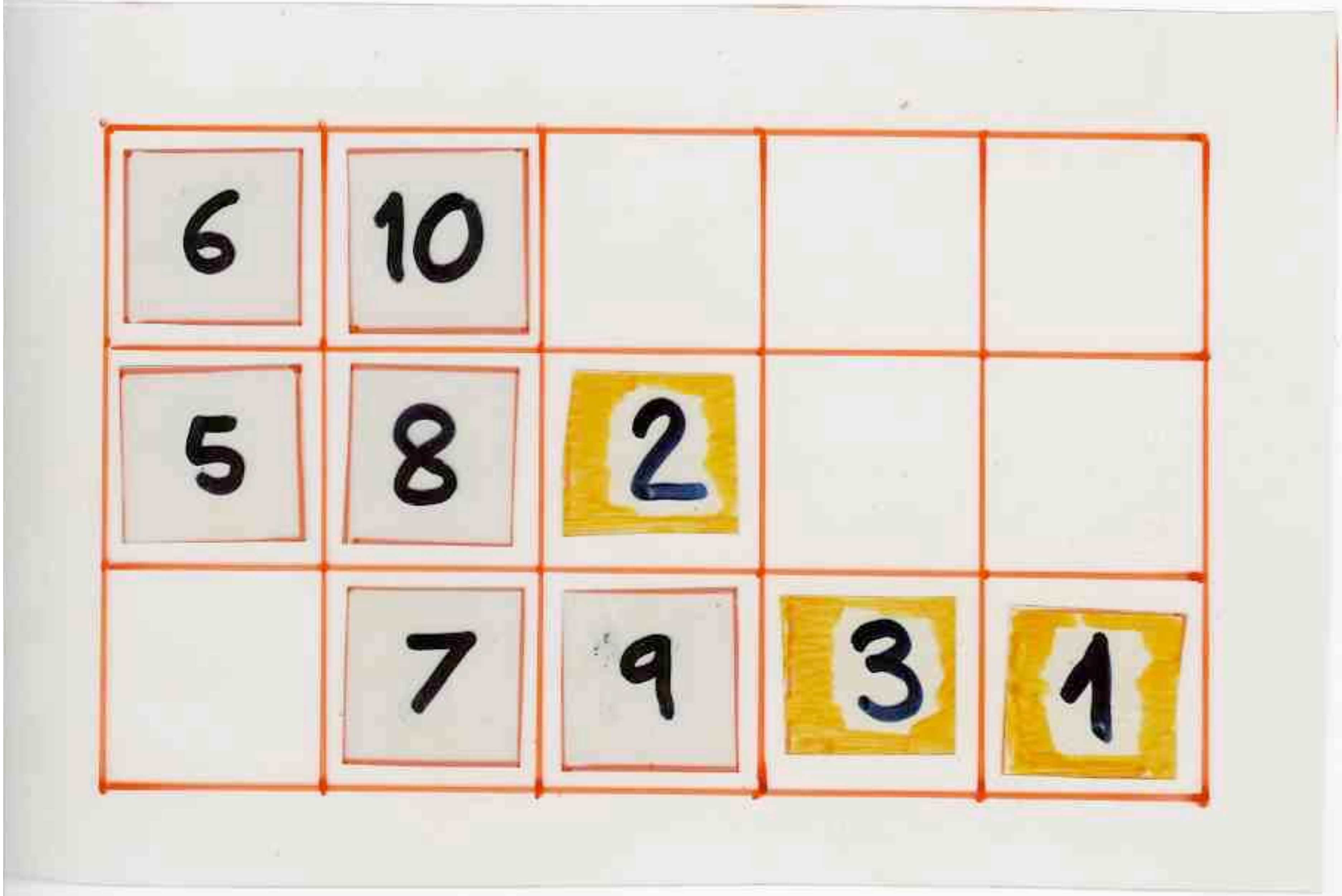
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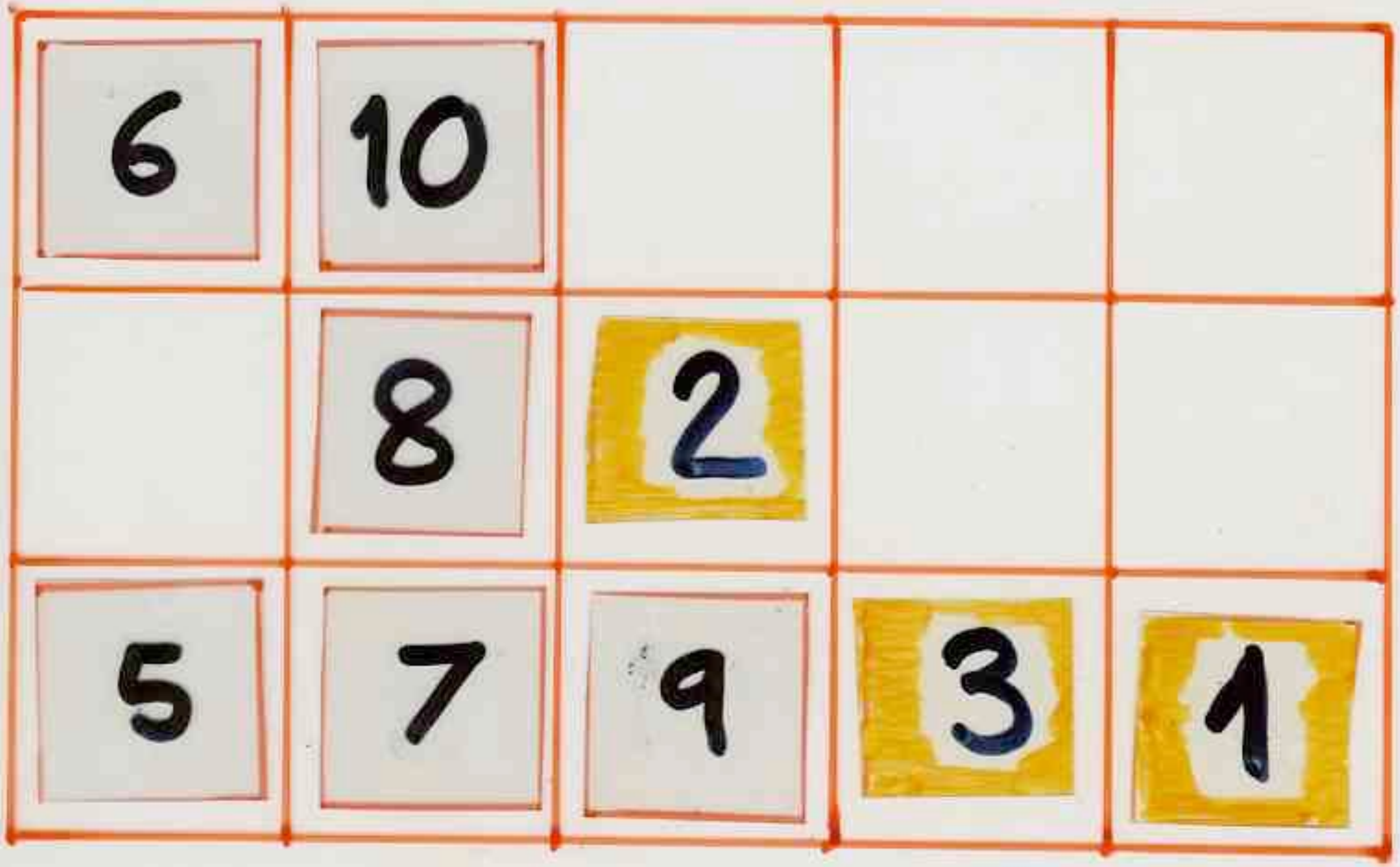
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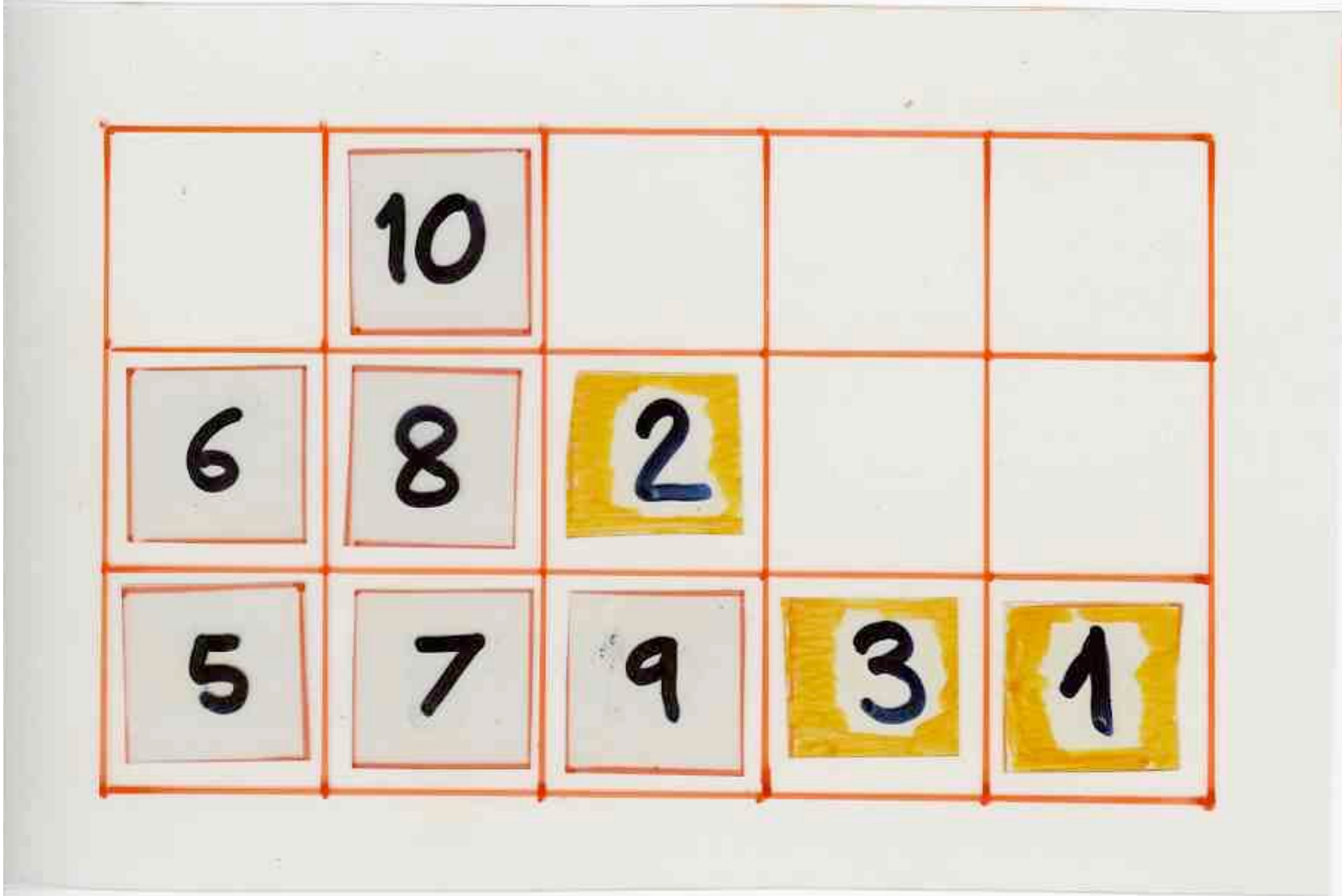
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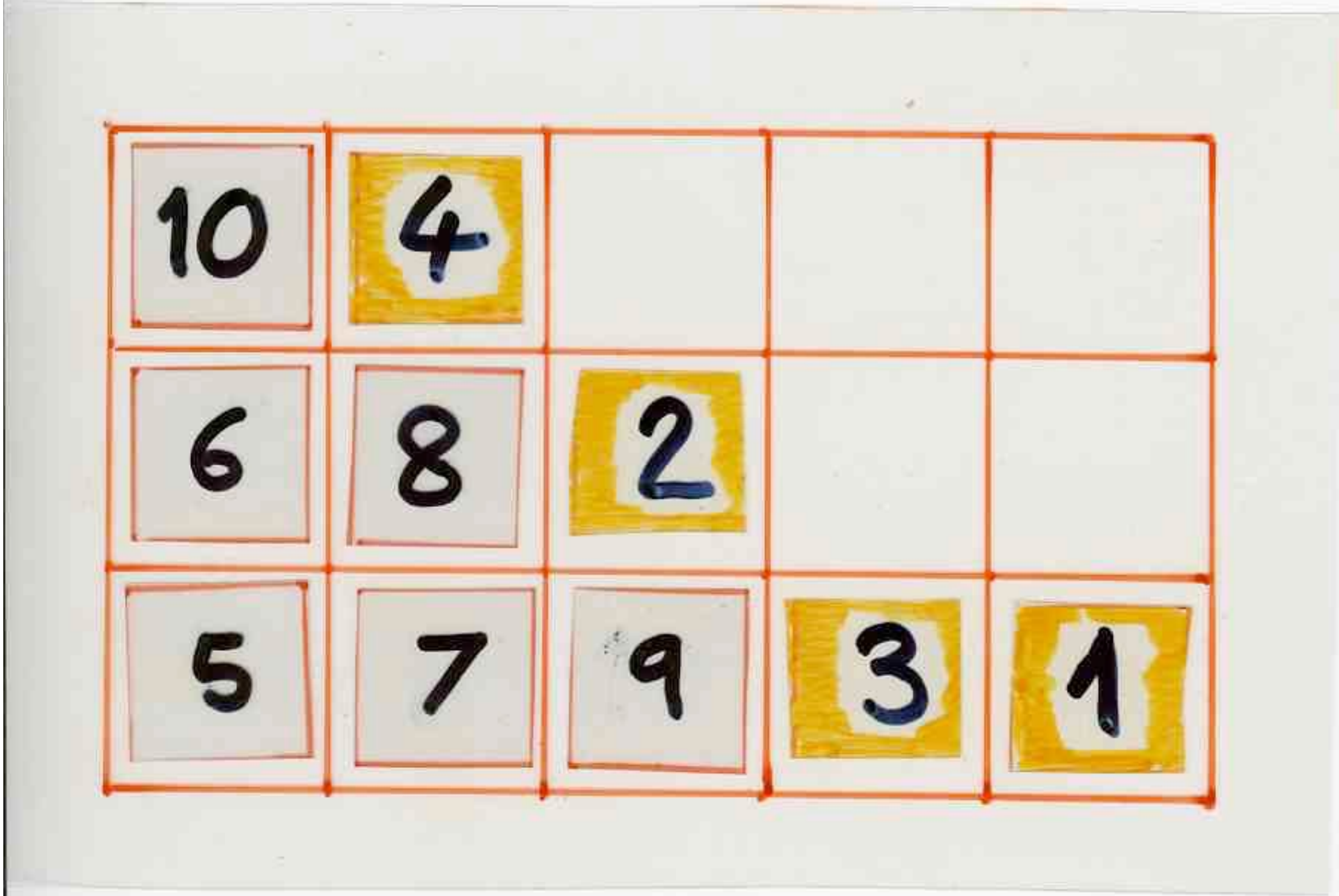
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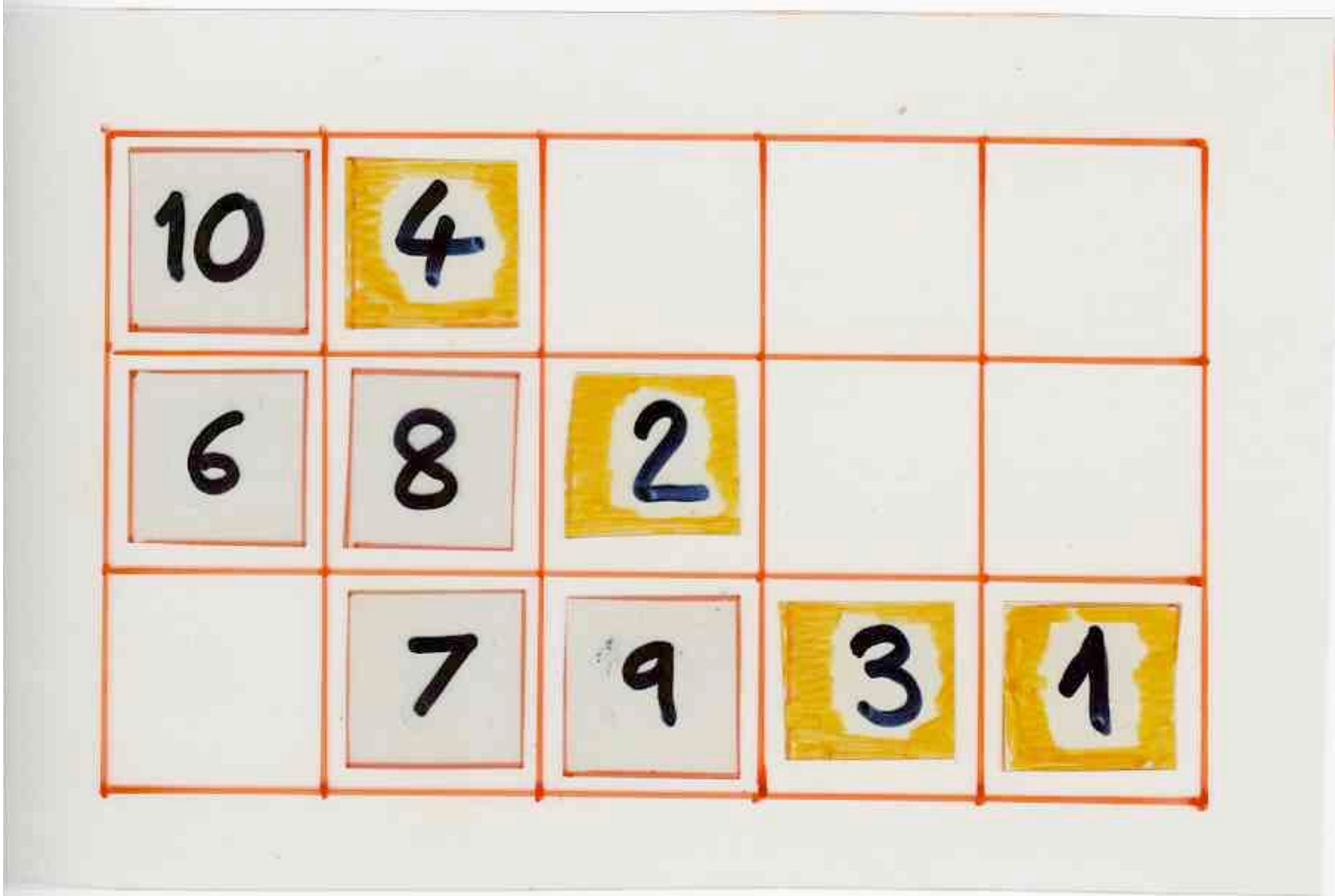
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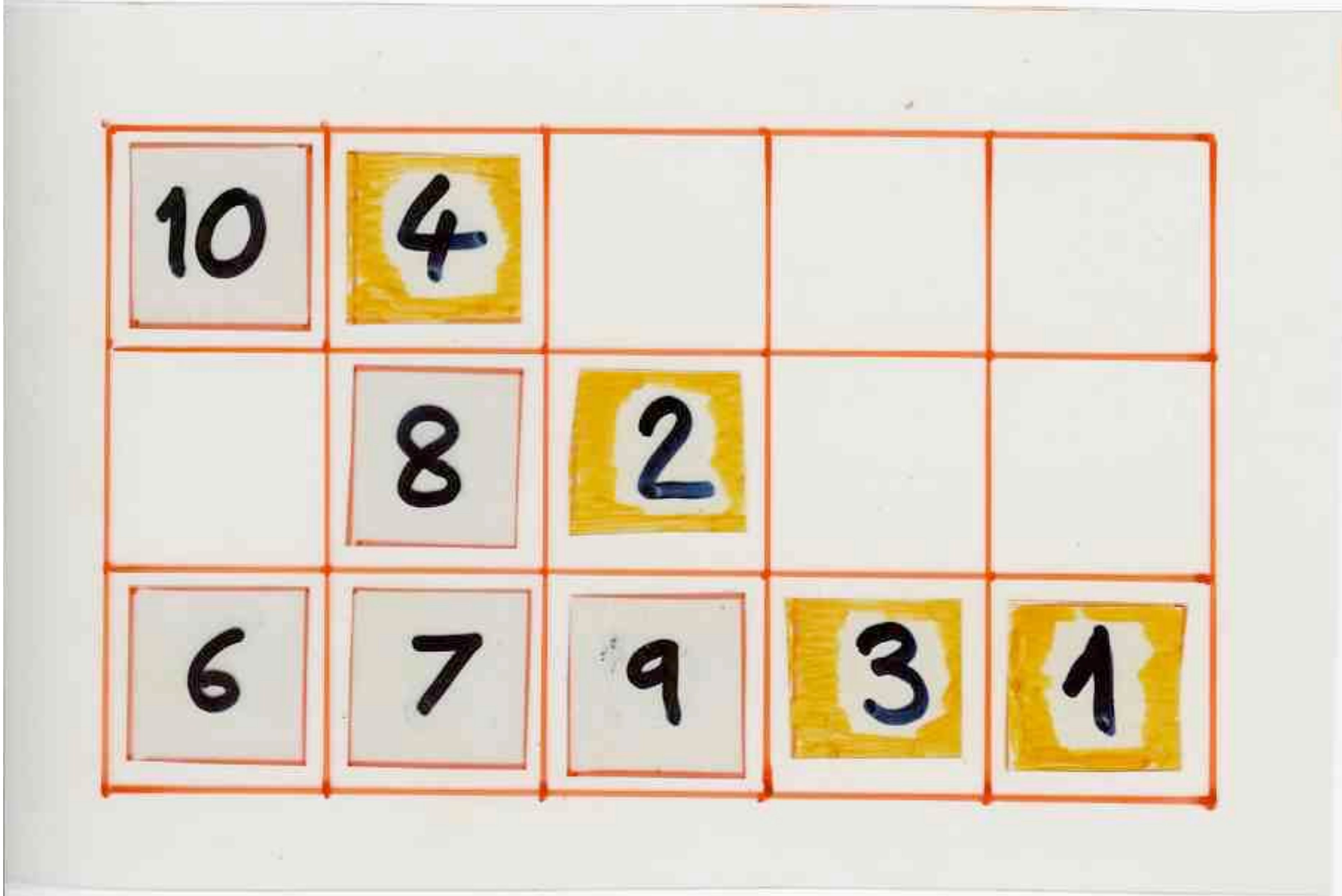
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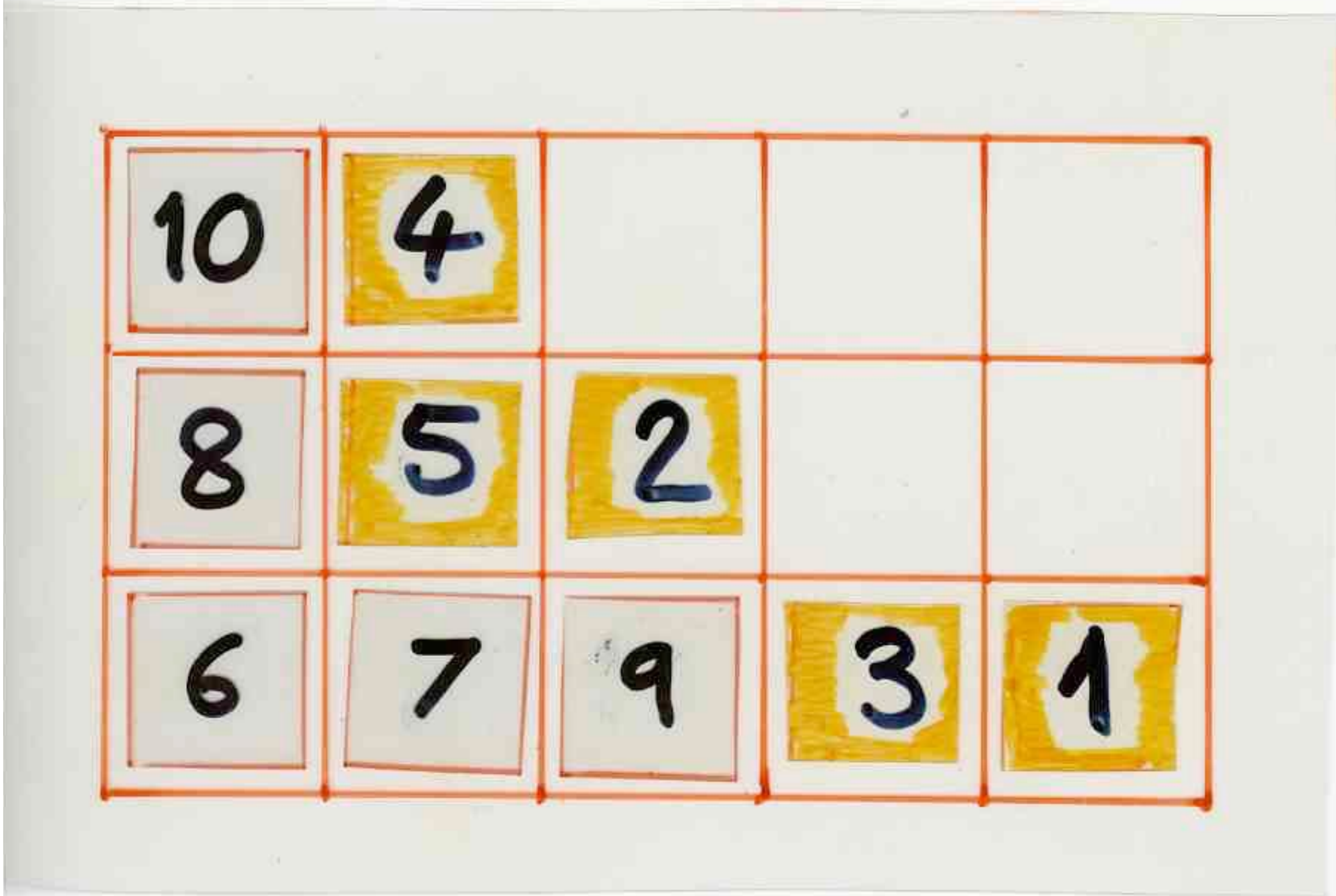
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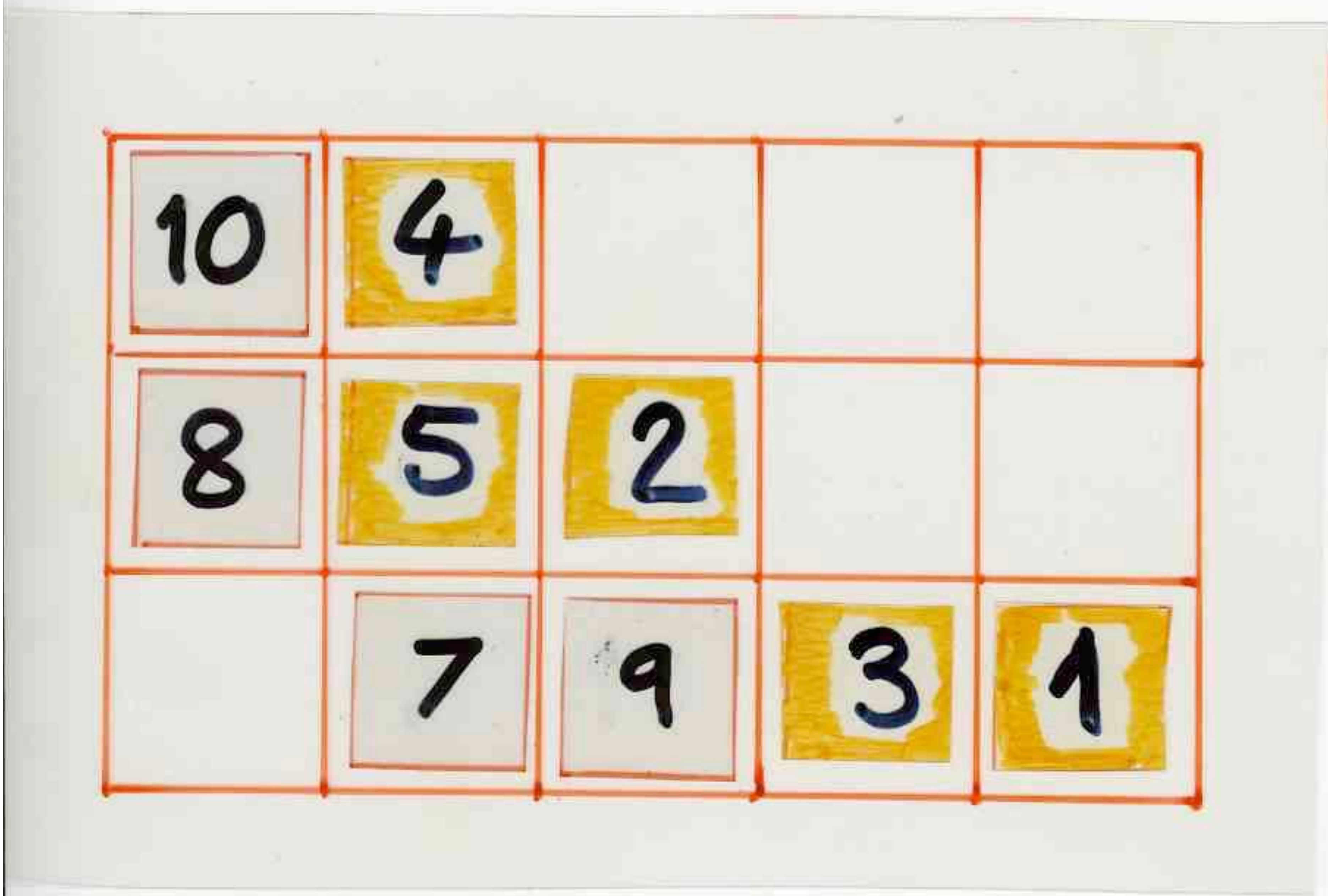
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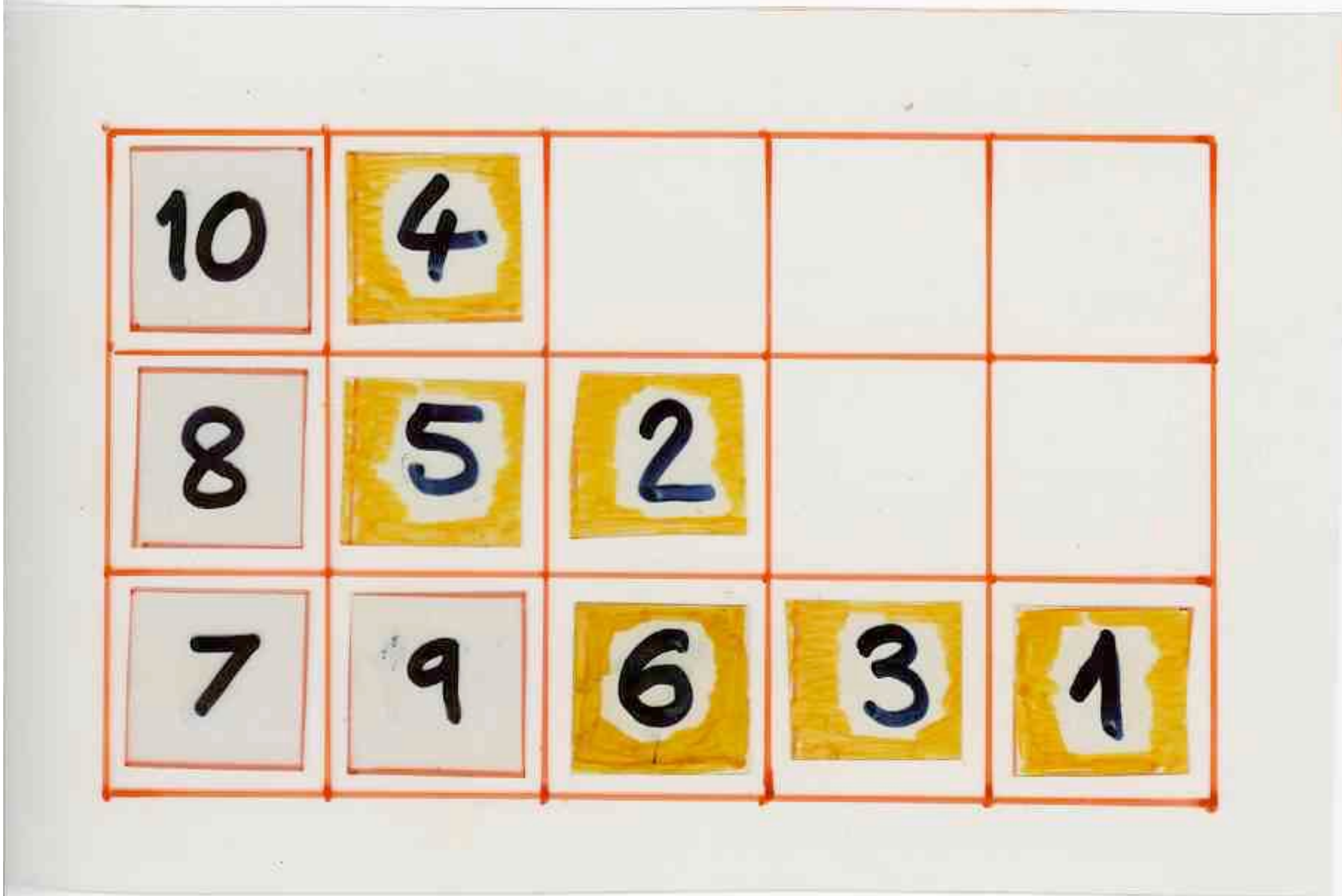

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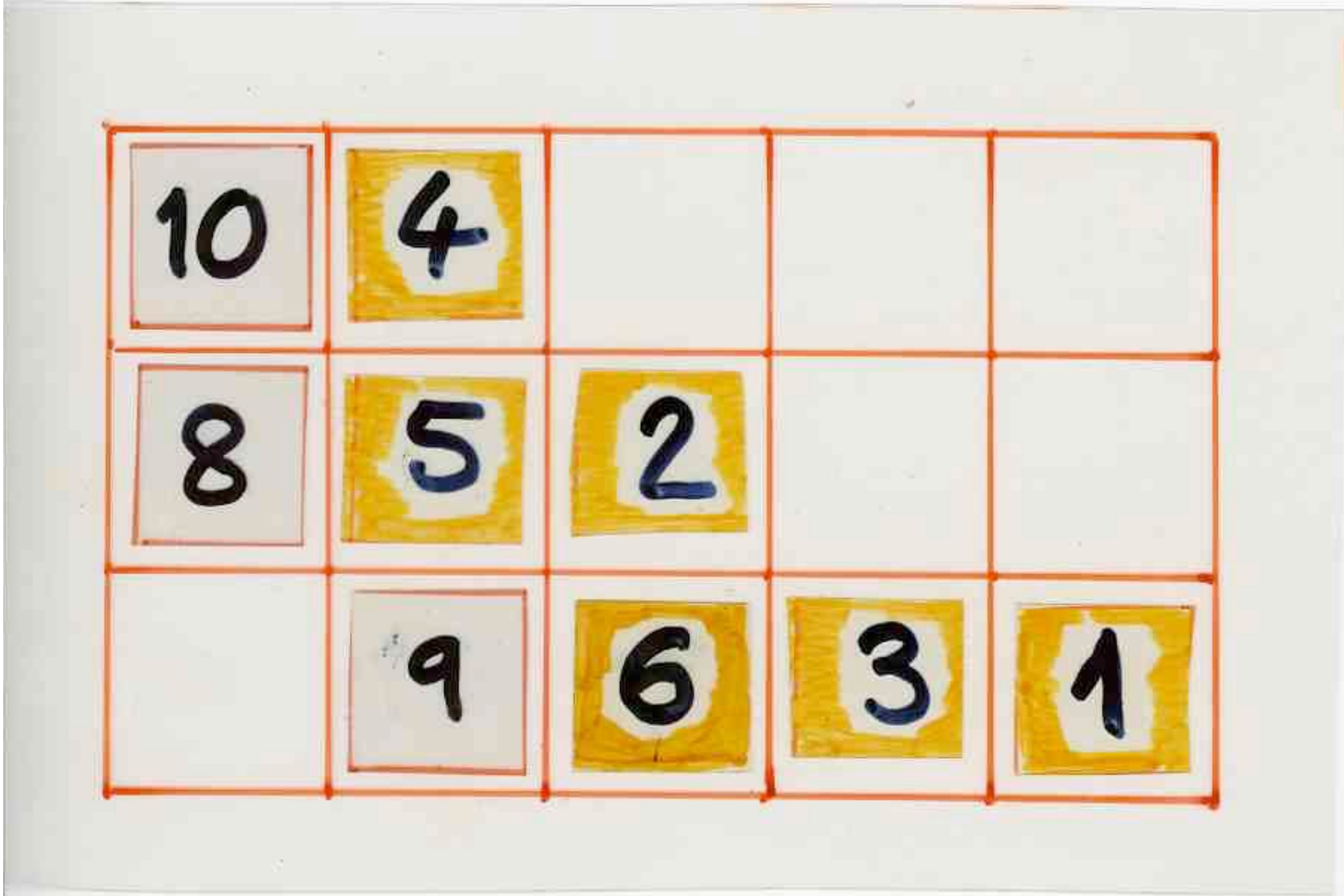
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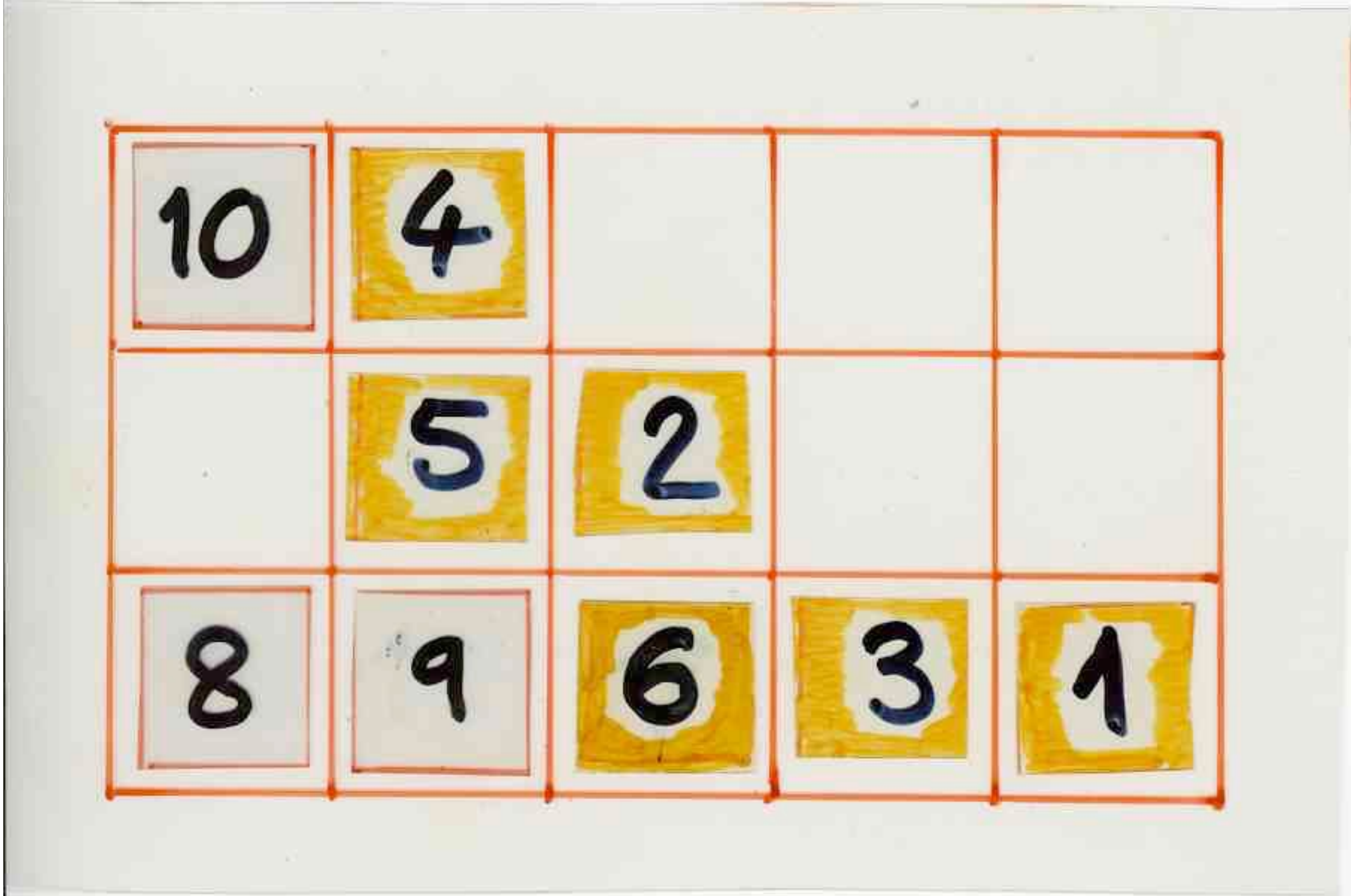
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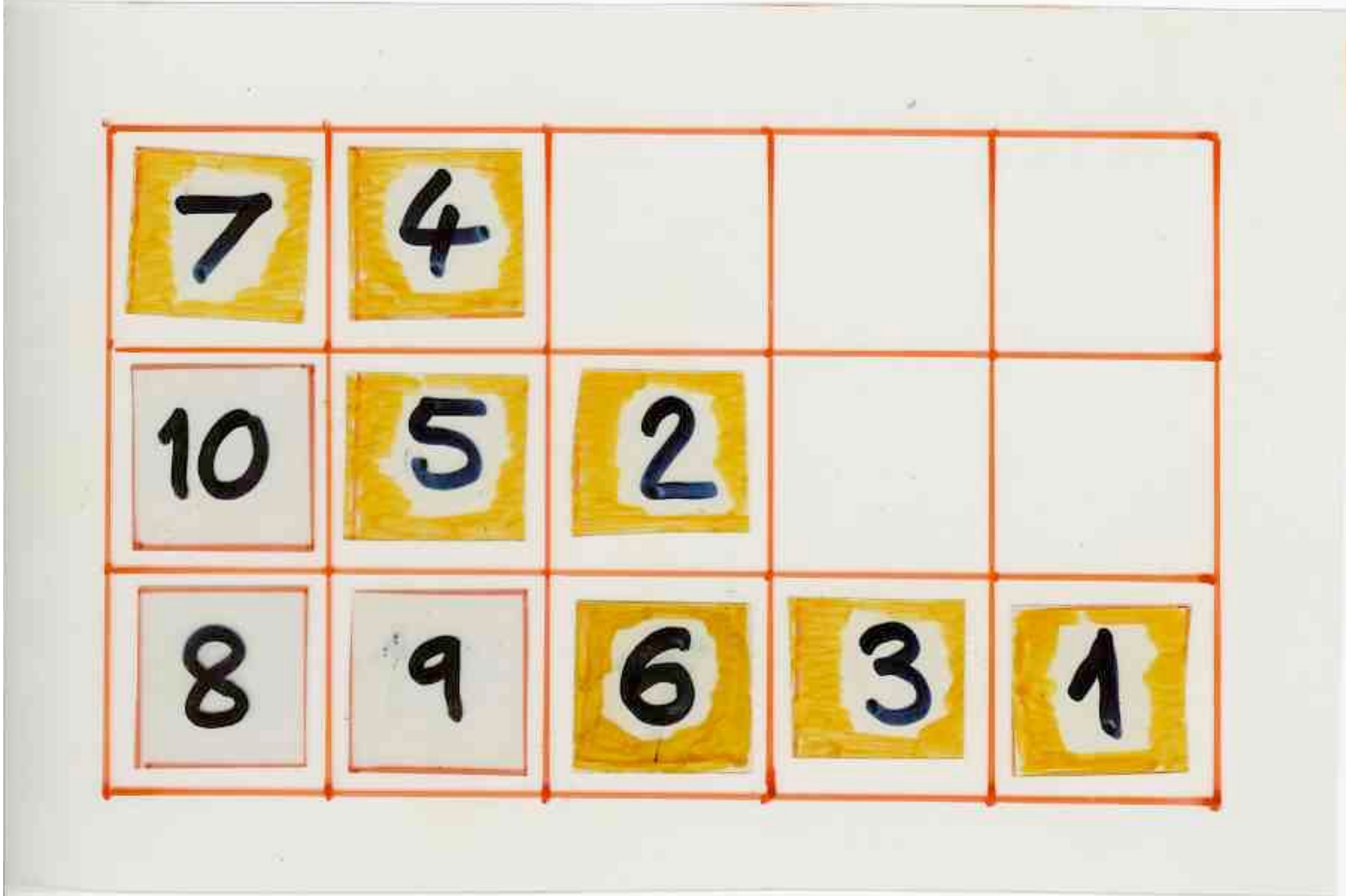
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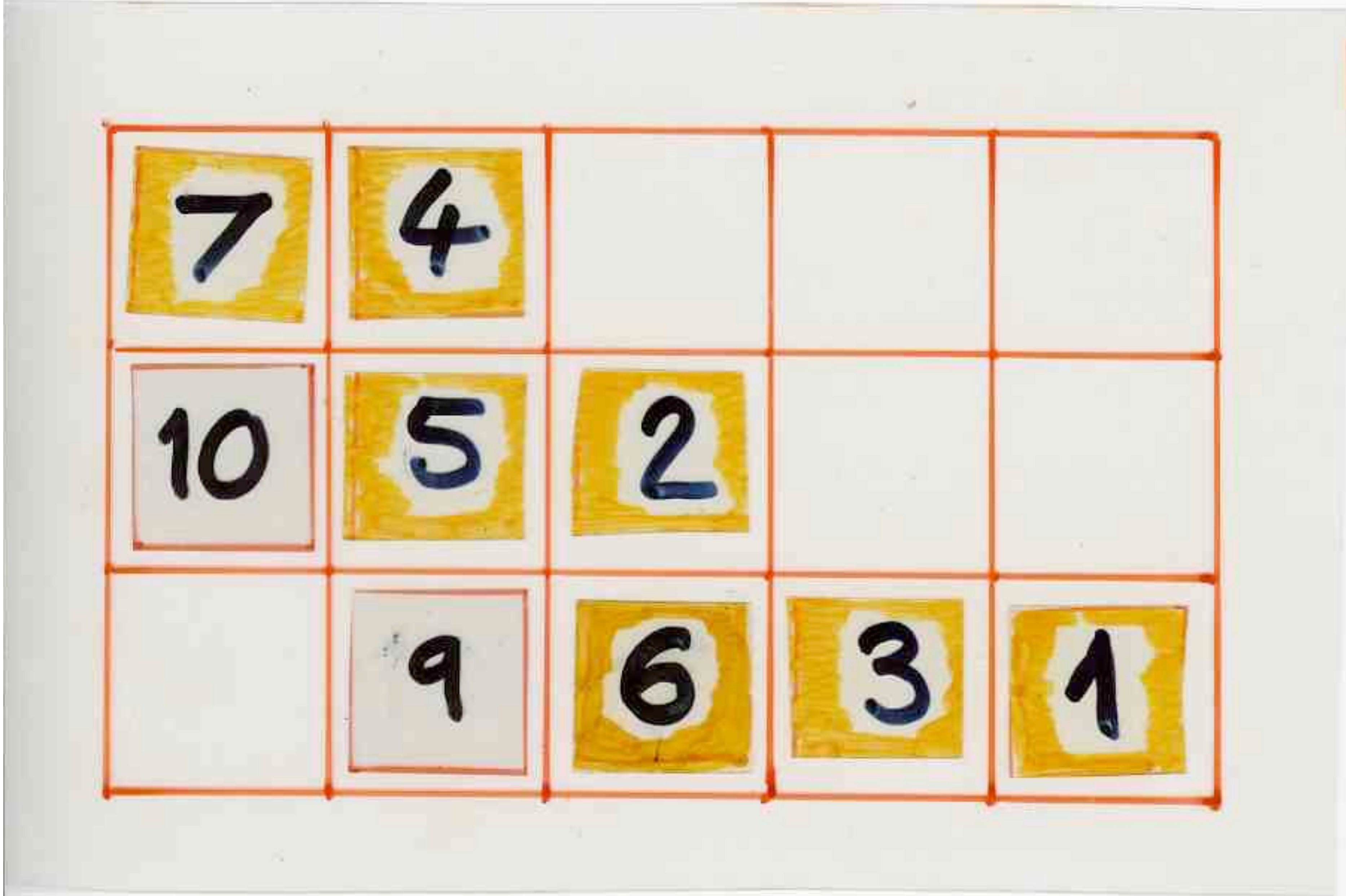
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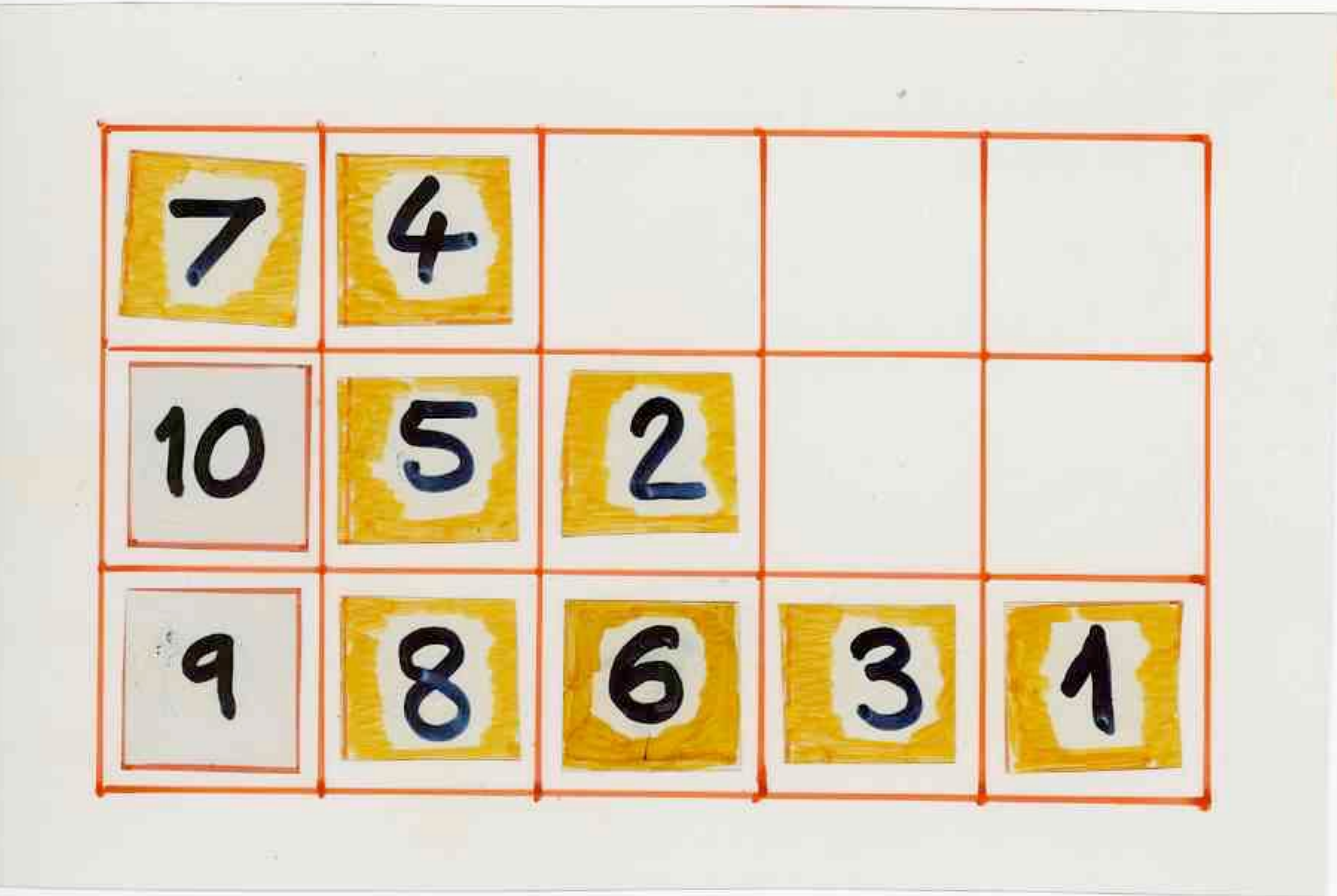
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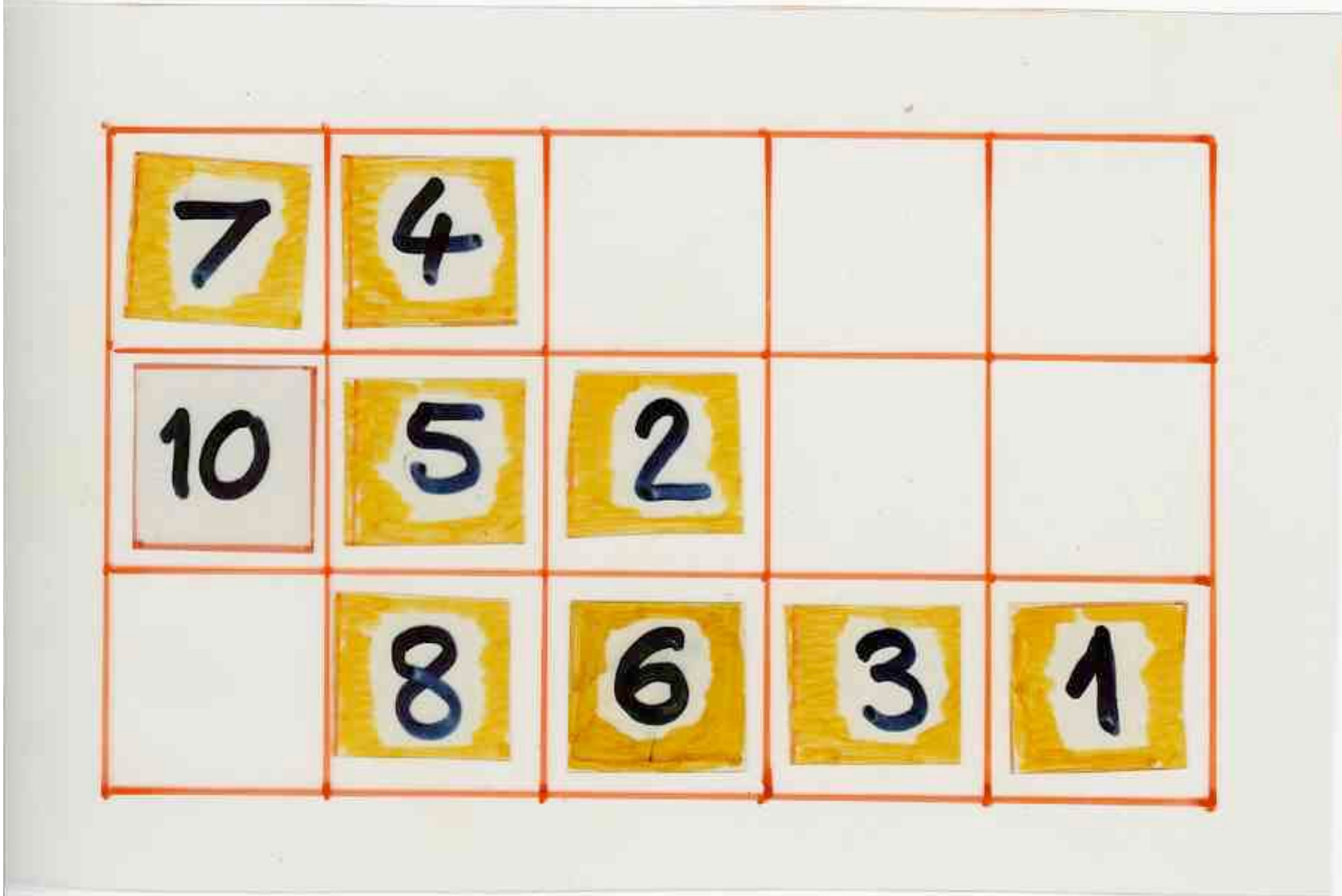
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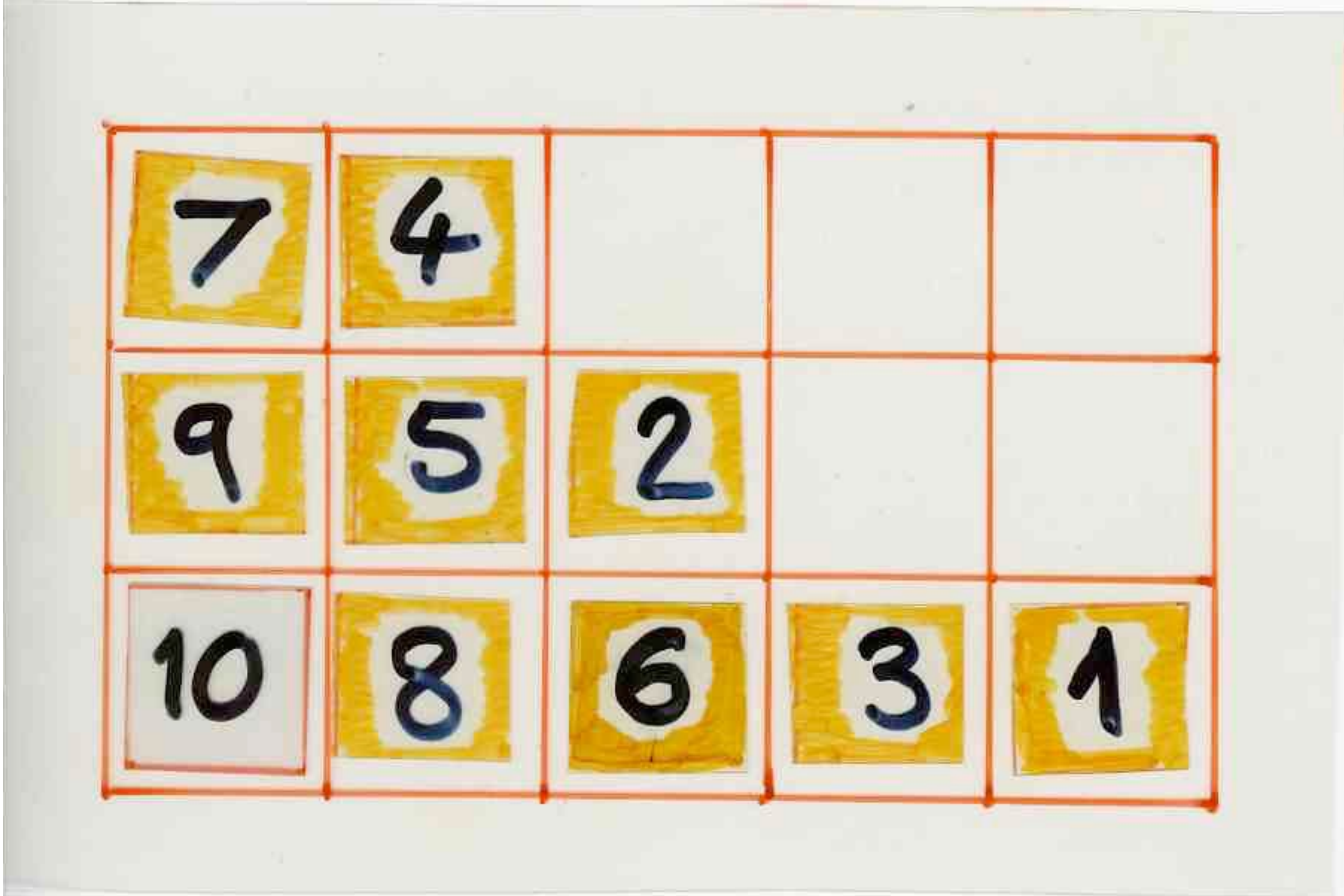
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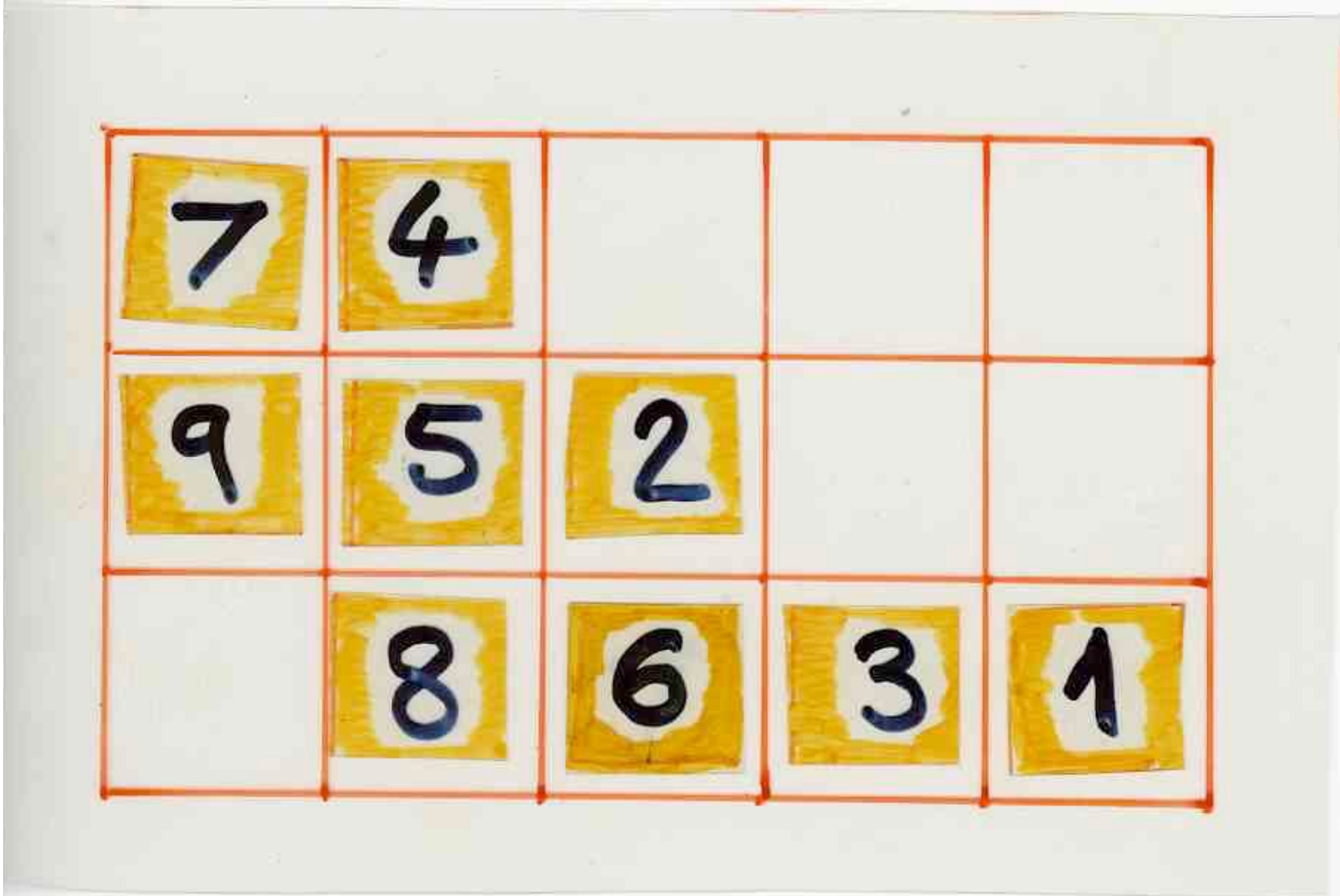
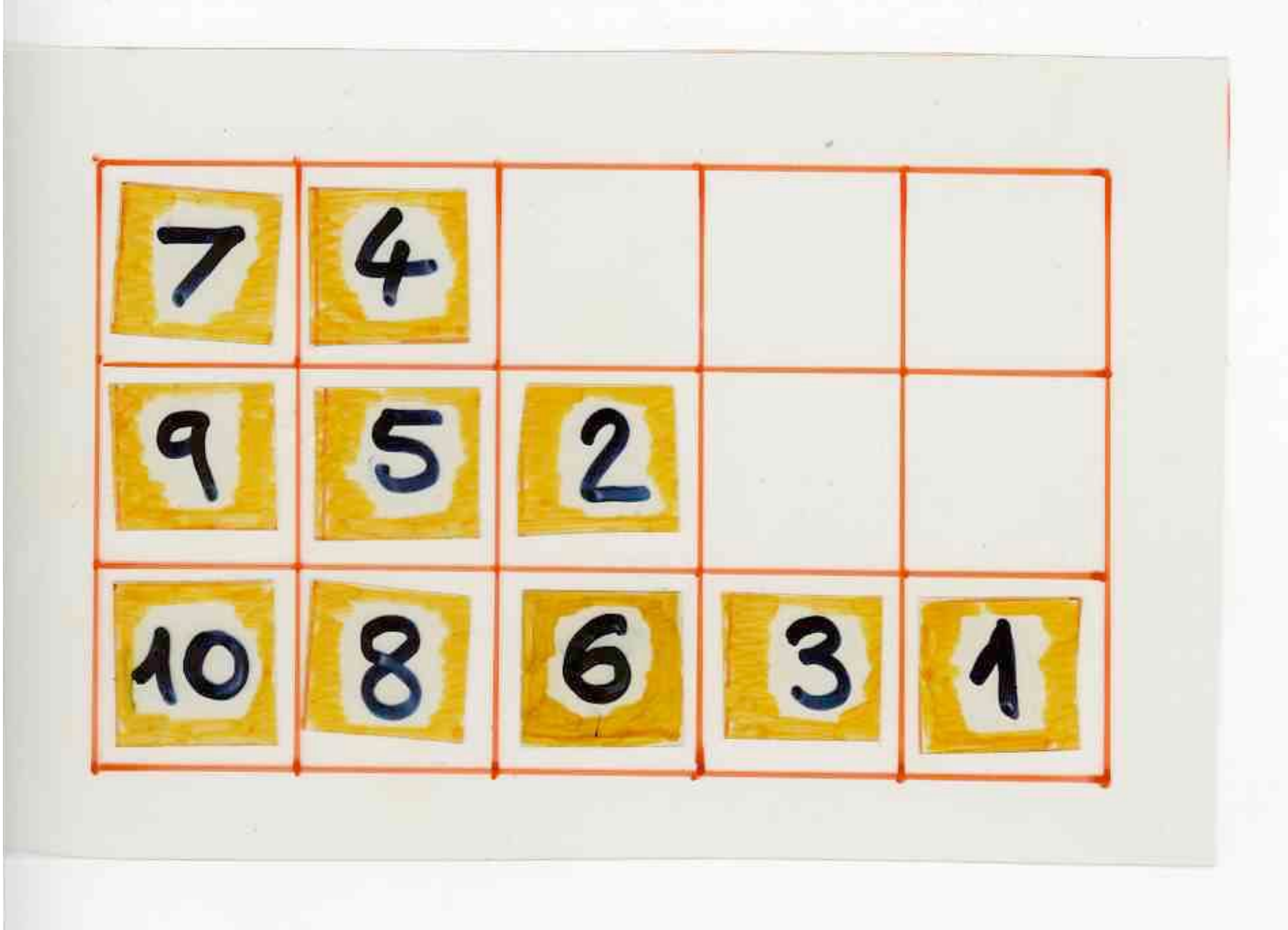
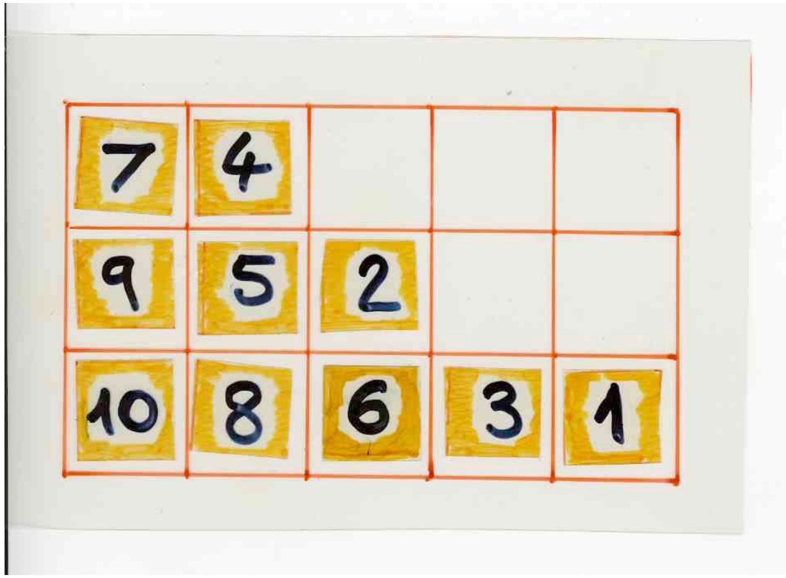

7 4
9 5 2
8 6 3 1


7
4
9
5
2
10
8
6
3
1



P^* dual =

4	7			
2	6	9		
1	3	5	8	10

complement

$(i)^c = n+1-i$

P =

6	10			
3	5	8		
1	2	4	7	9

$$P = \begin{array}{|c|c} \hline 4 & \\ \hline 2 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \rightarrow \begin{array}{|c|c} \hline 4 & \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 4 & 5 \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline 5 \\ \hline \end{array}$$

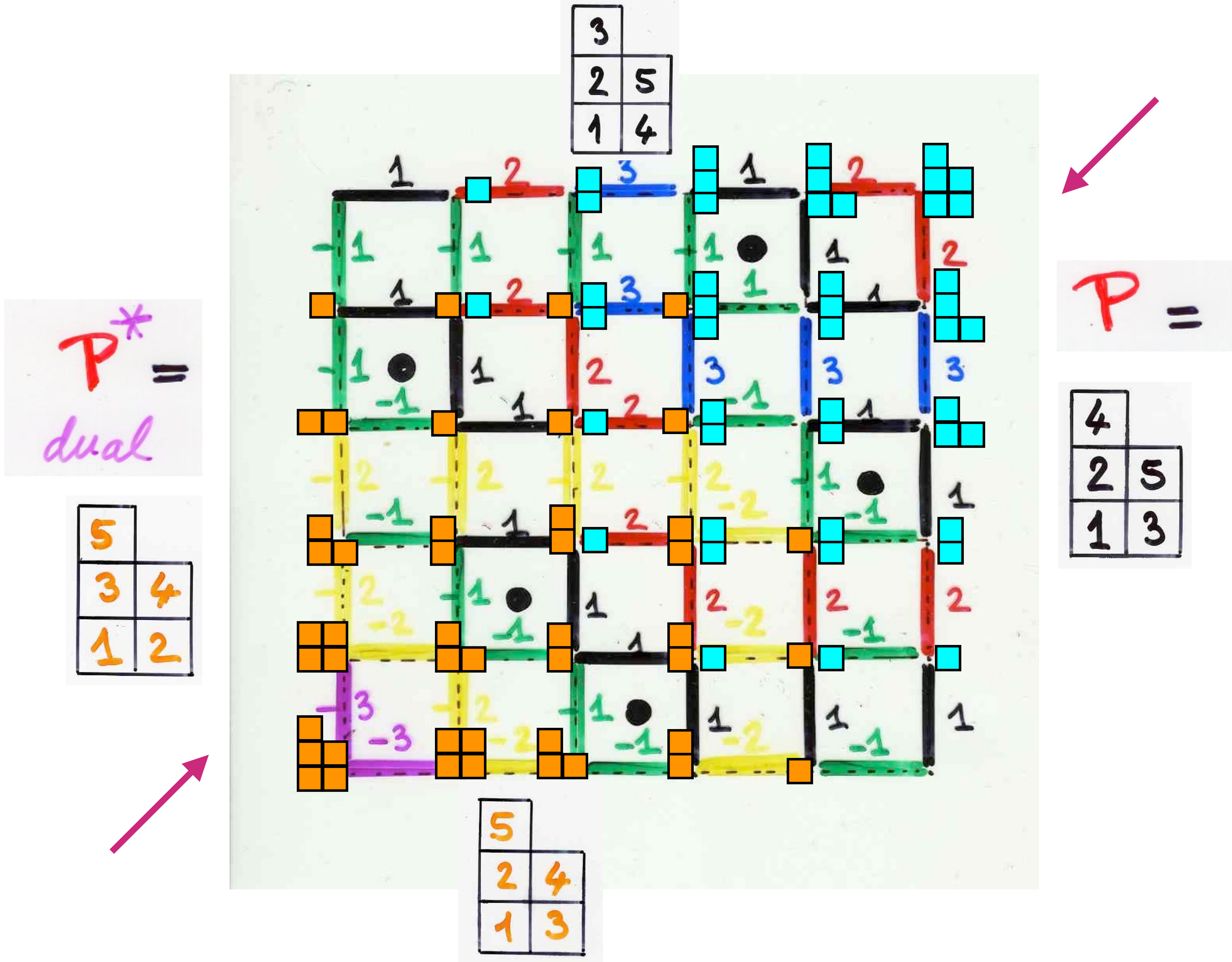
$$P^* = \begin{array}{|c|c} \hline 5 & \\ \hline 3 & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

dual

evac(P)
for P^*

evacuation

"vidage - remplissage"
evacuation - filling


P* =
dual
P =

$$\sigma = \sigma(1) \cdots \sigma(n)$$

$$\sigma^{t} = \sigma(n) \cdots \sigma(1)$$ transpose

$$\sigma^{c} = \sigma(1)^{\sigma} \qquad \sigma(n)^{\sigma}$$ complement

$$(i)^{c} = n+1-i$$

$$\sigma^{\#} = (\sigma^{t})^{c} \qquad (\sigma^{c})^{t}$$

Proposition Schützenberger

$$\sigma \longrightarrow (P, Q)$$
$$\sigma^{\#} \longrightarrow (P^{*}, Q^{*})$$

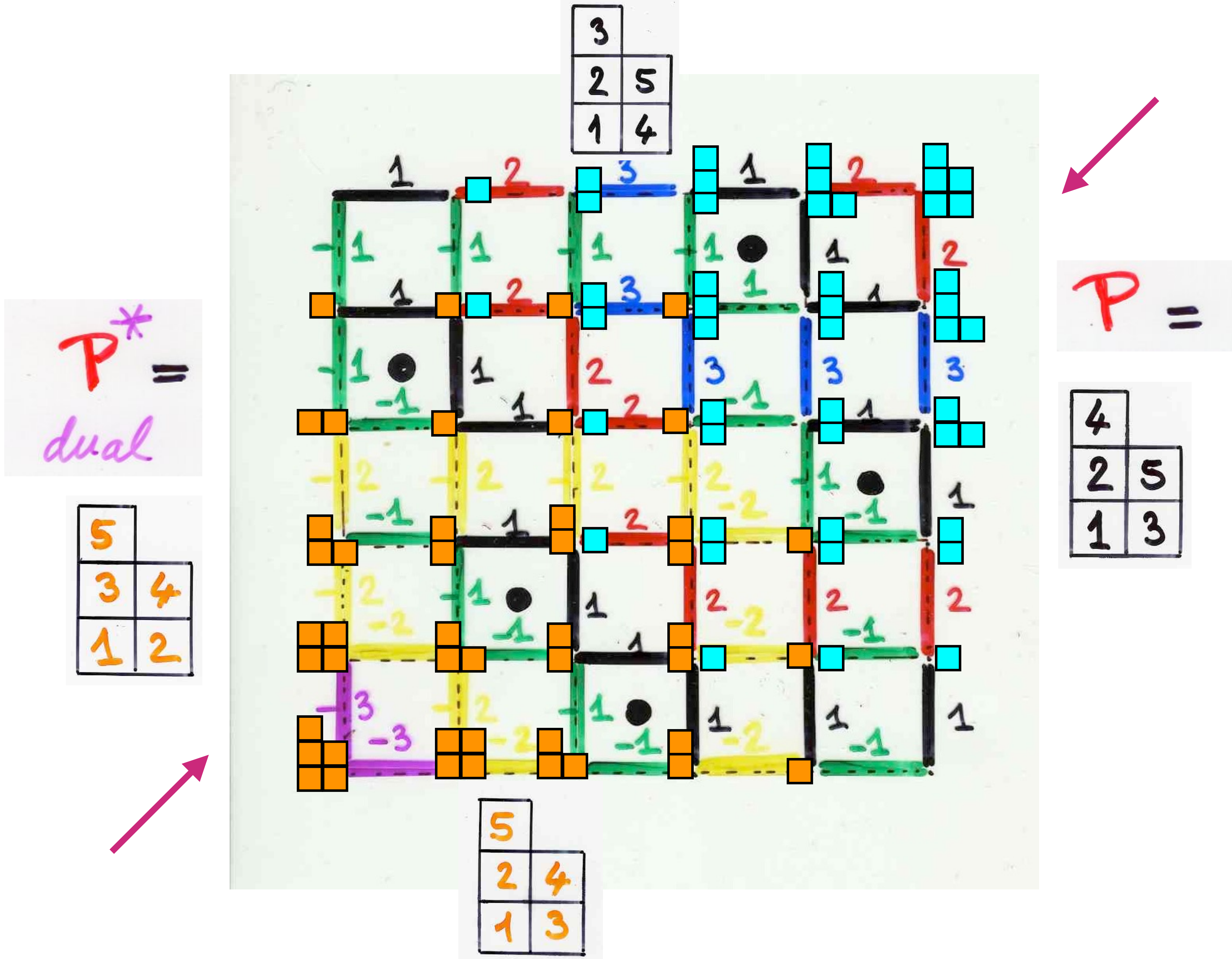
dual tableaux

M.P. Schützenberger, 1963, 1972

Proposition Schützenberger

The map $P \longrightarrow P^{*}$ is an involution

$$(P^{*})^{*} = P$$

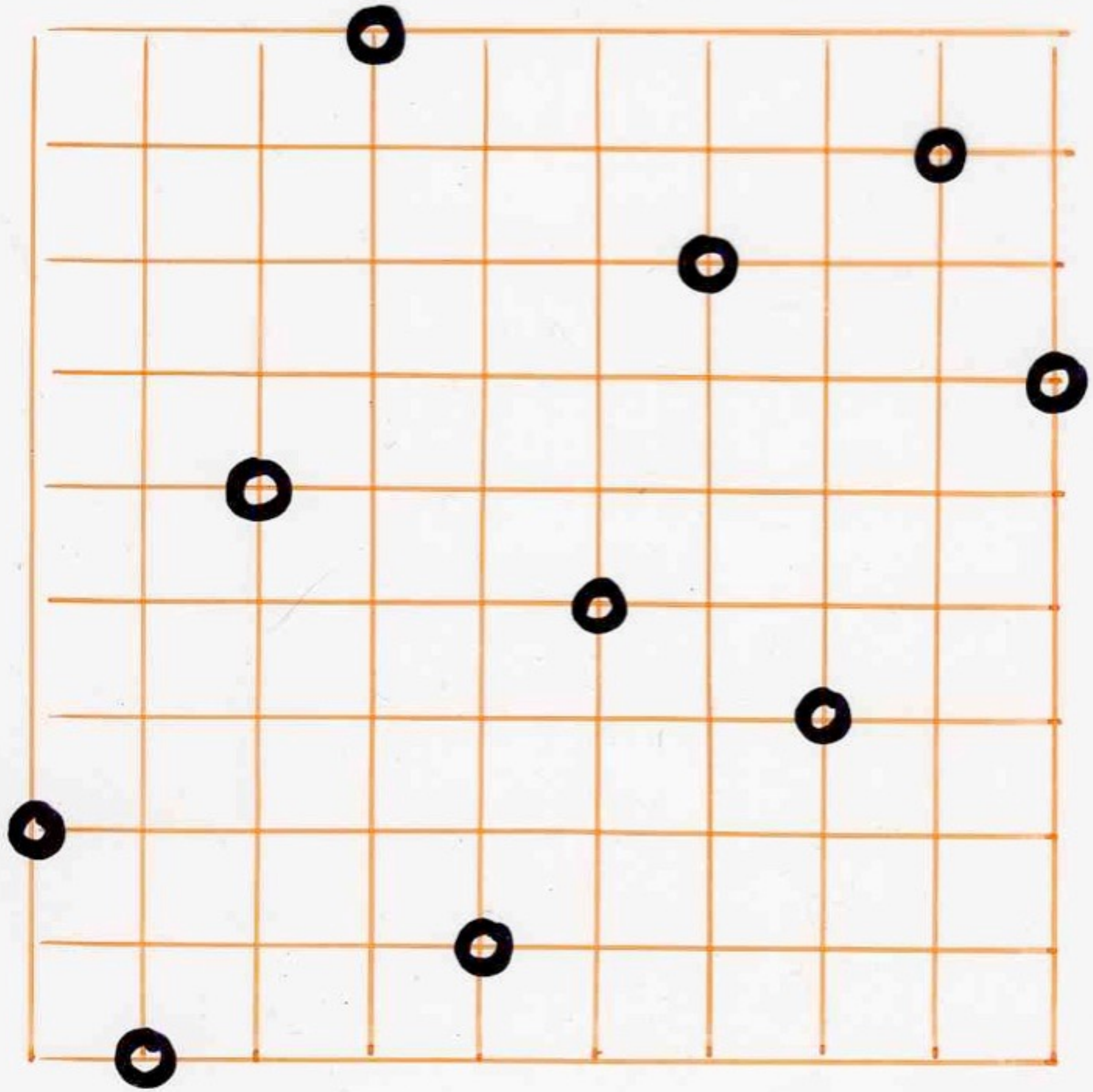


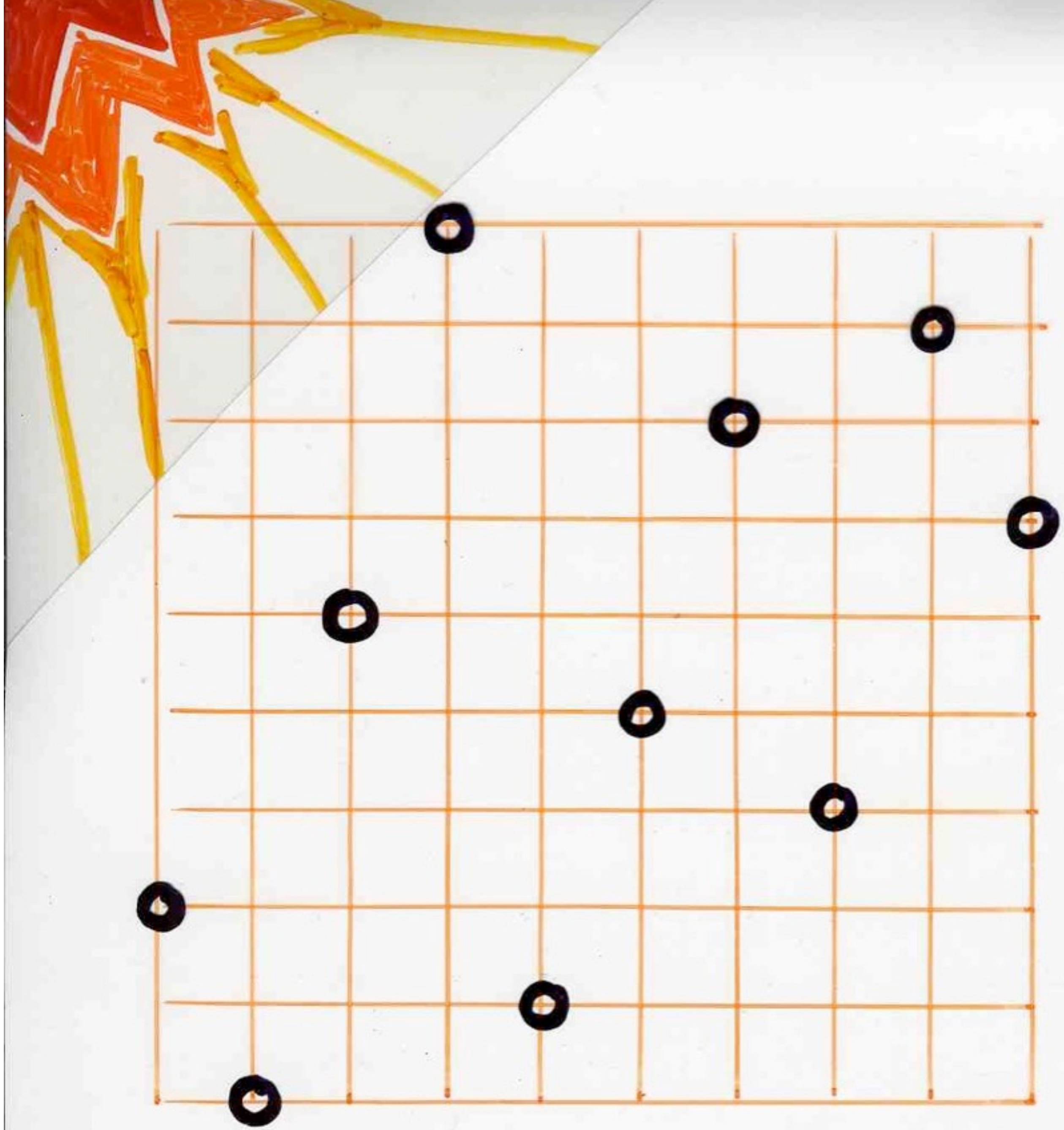
P* =
dual
P =

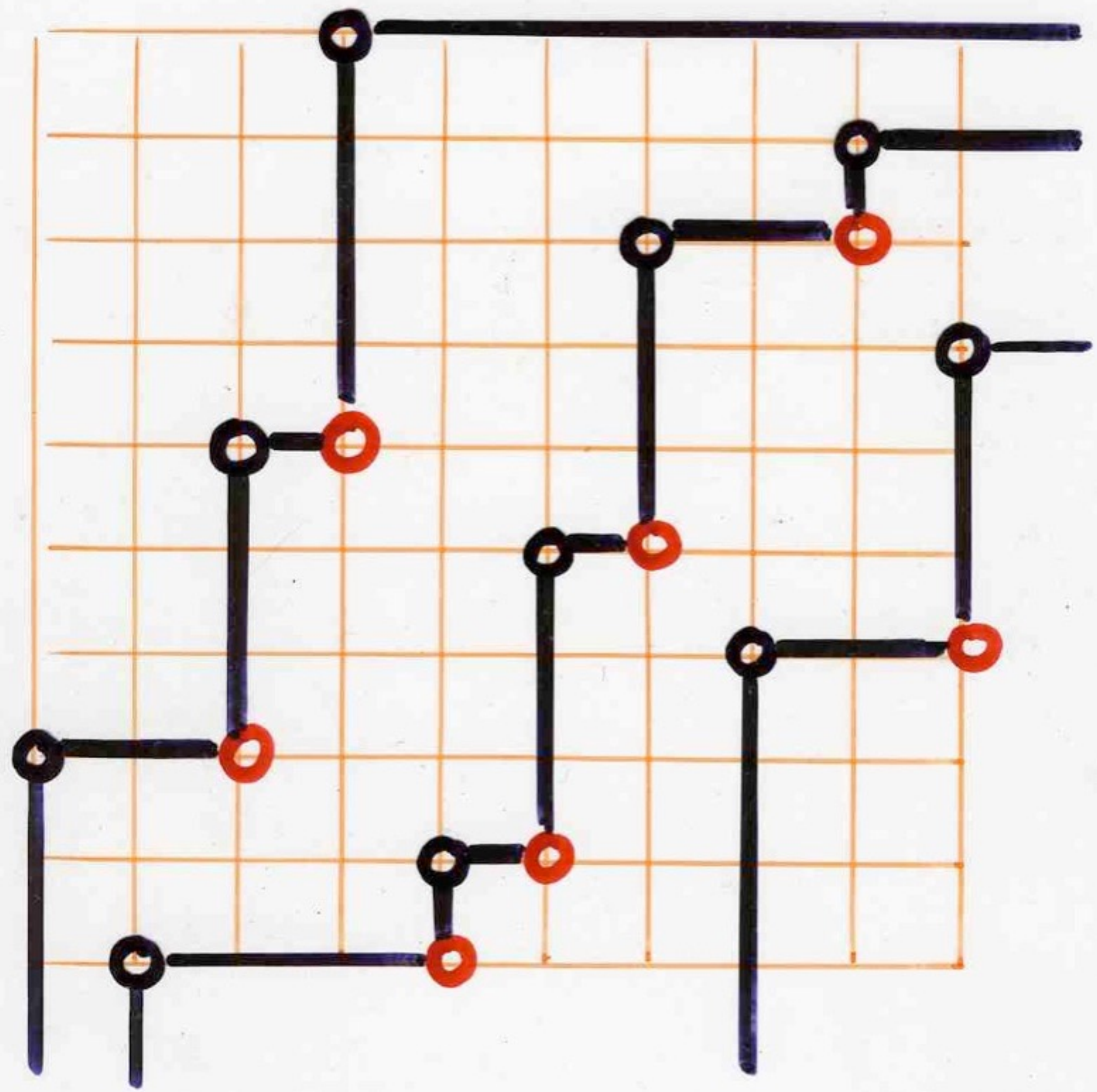
more duality

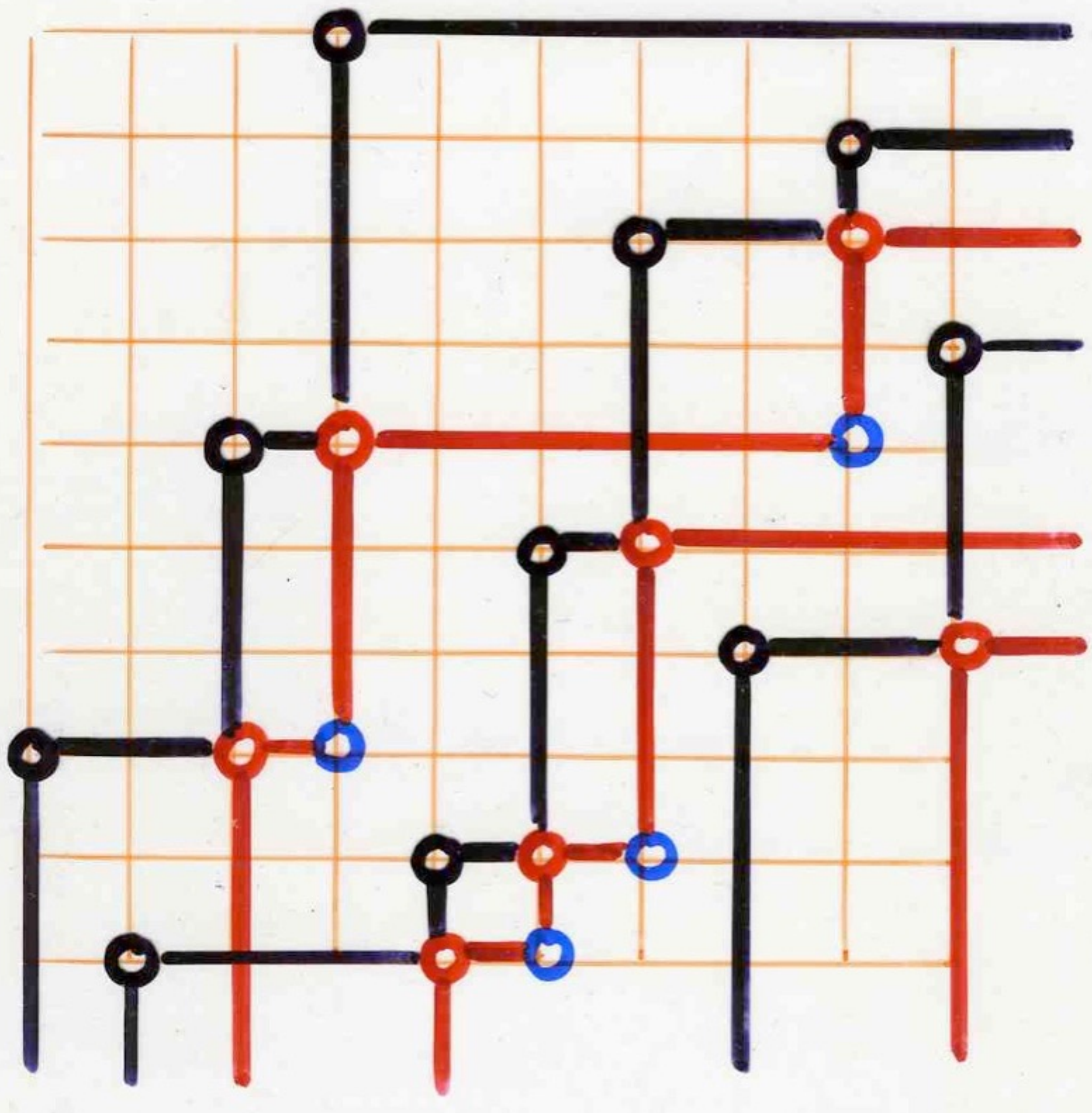
M.P. Schützenberger, 1963, 1972

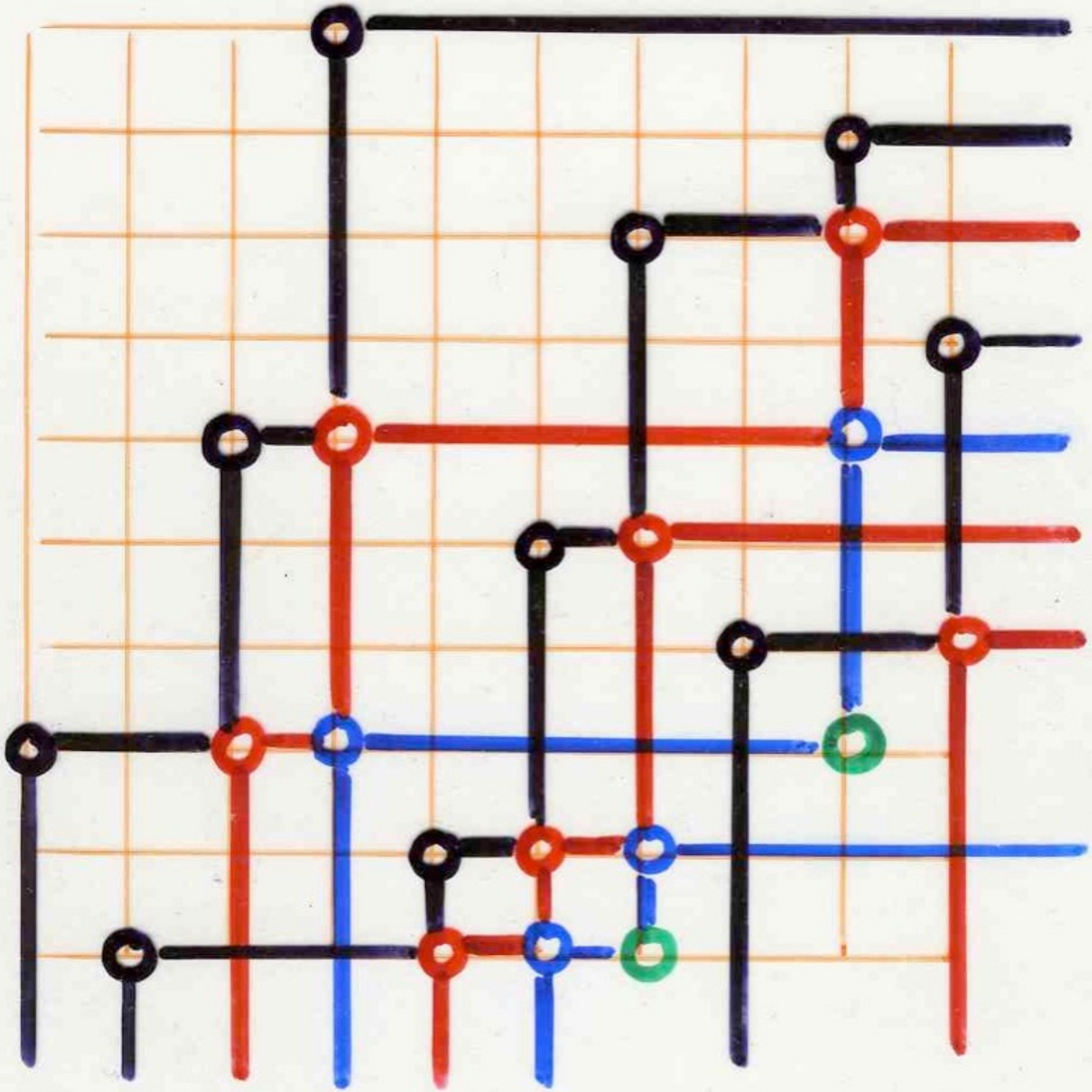
(without proof)

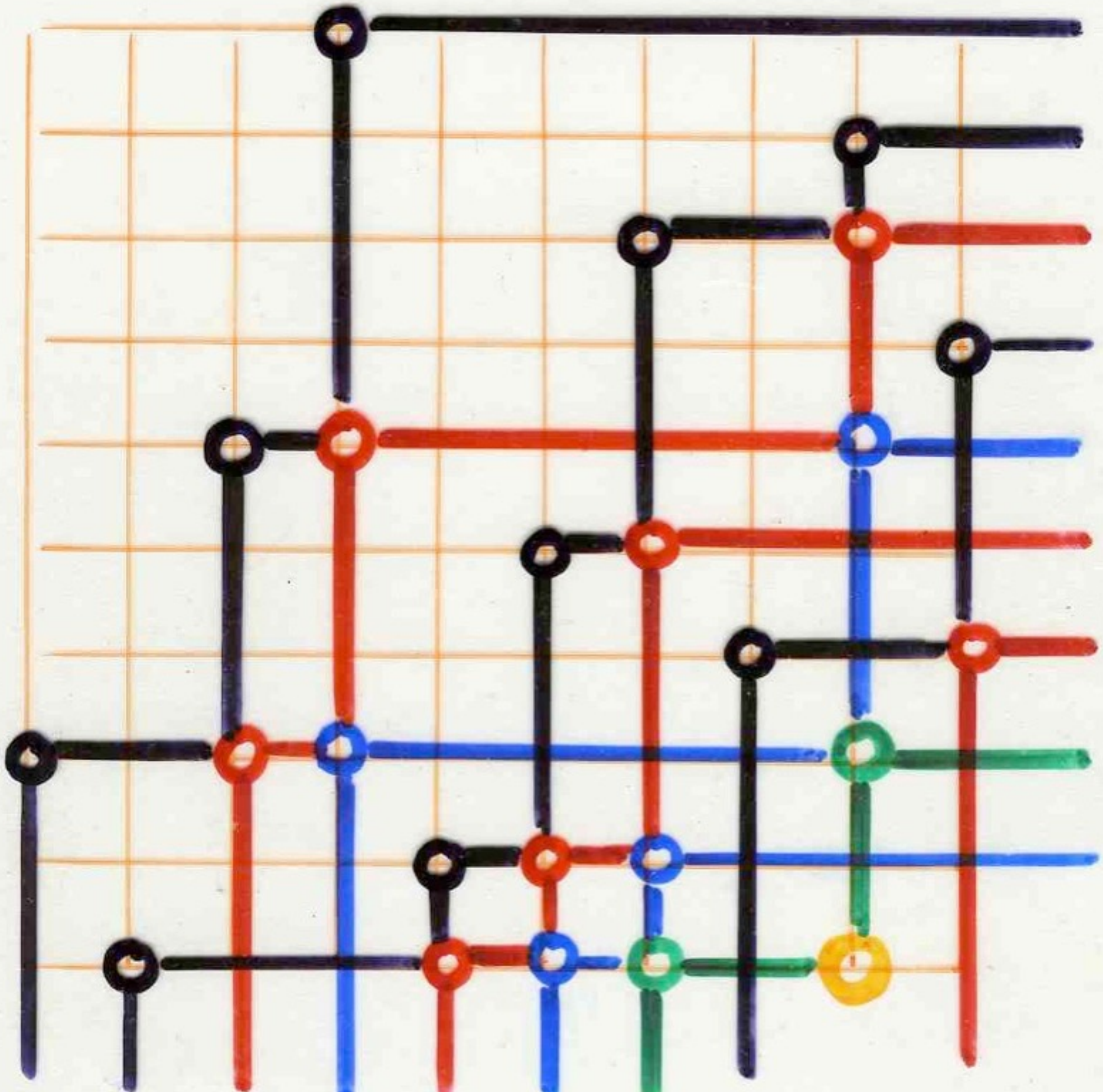


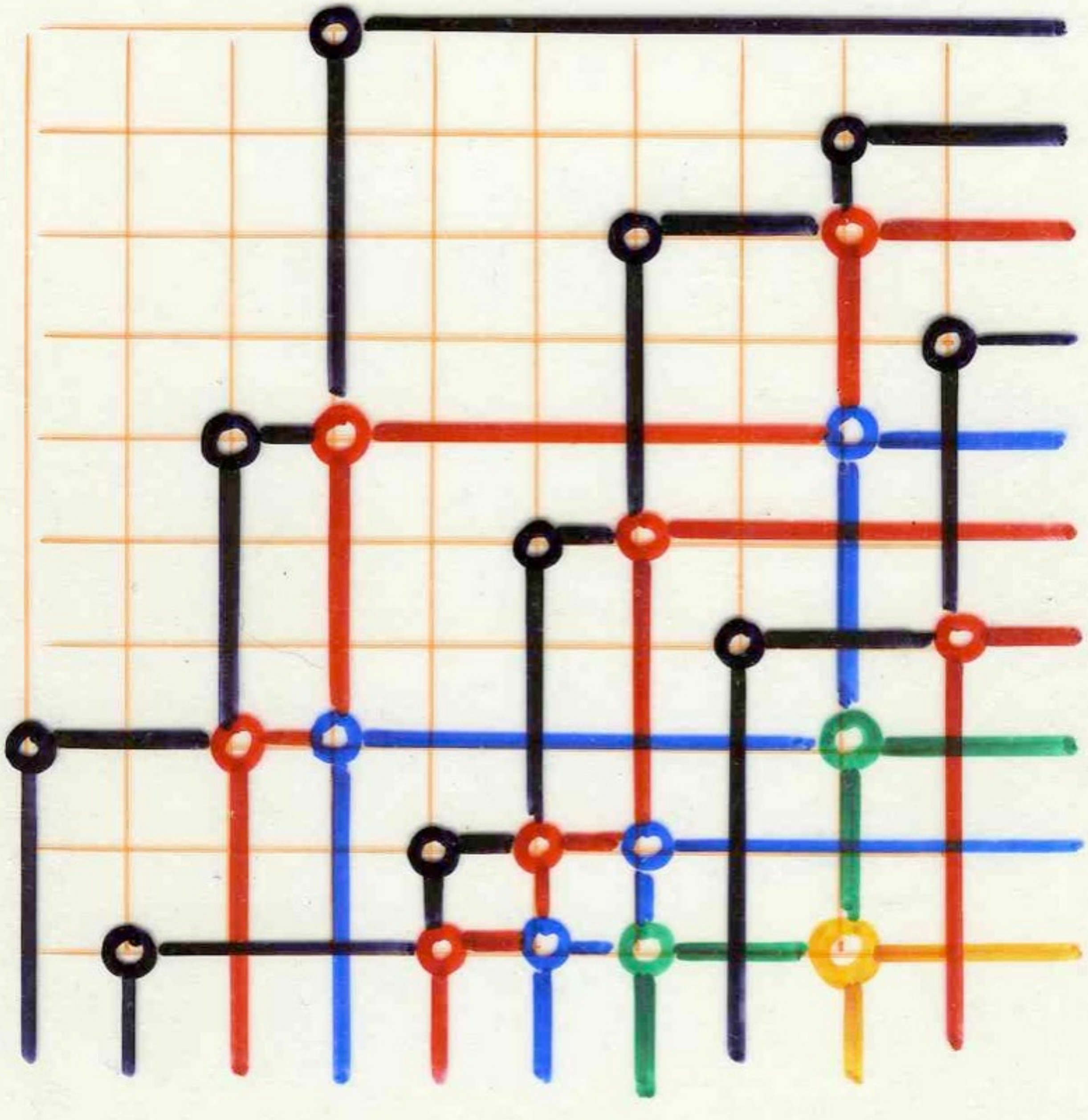


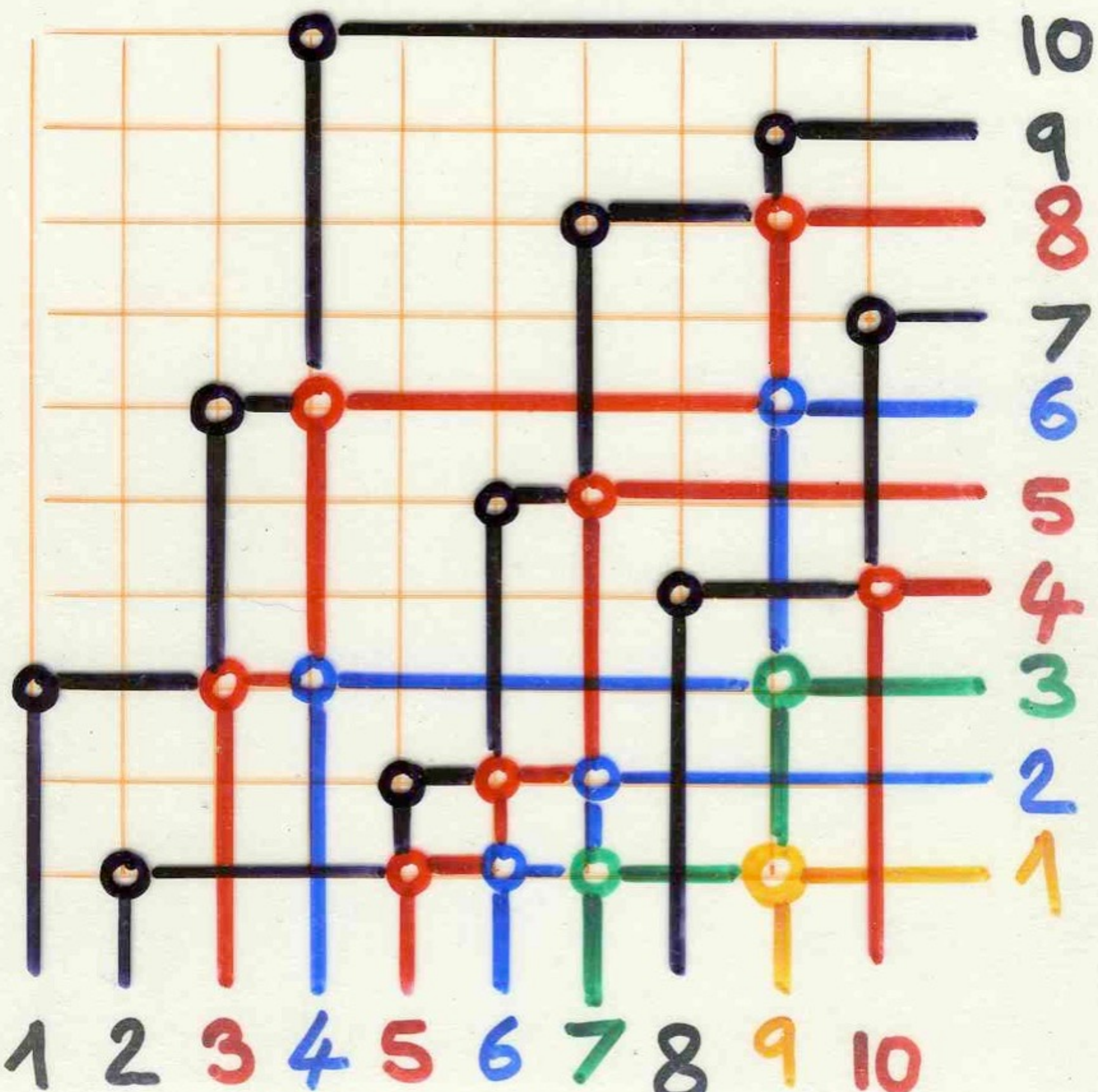


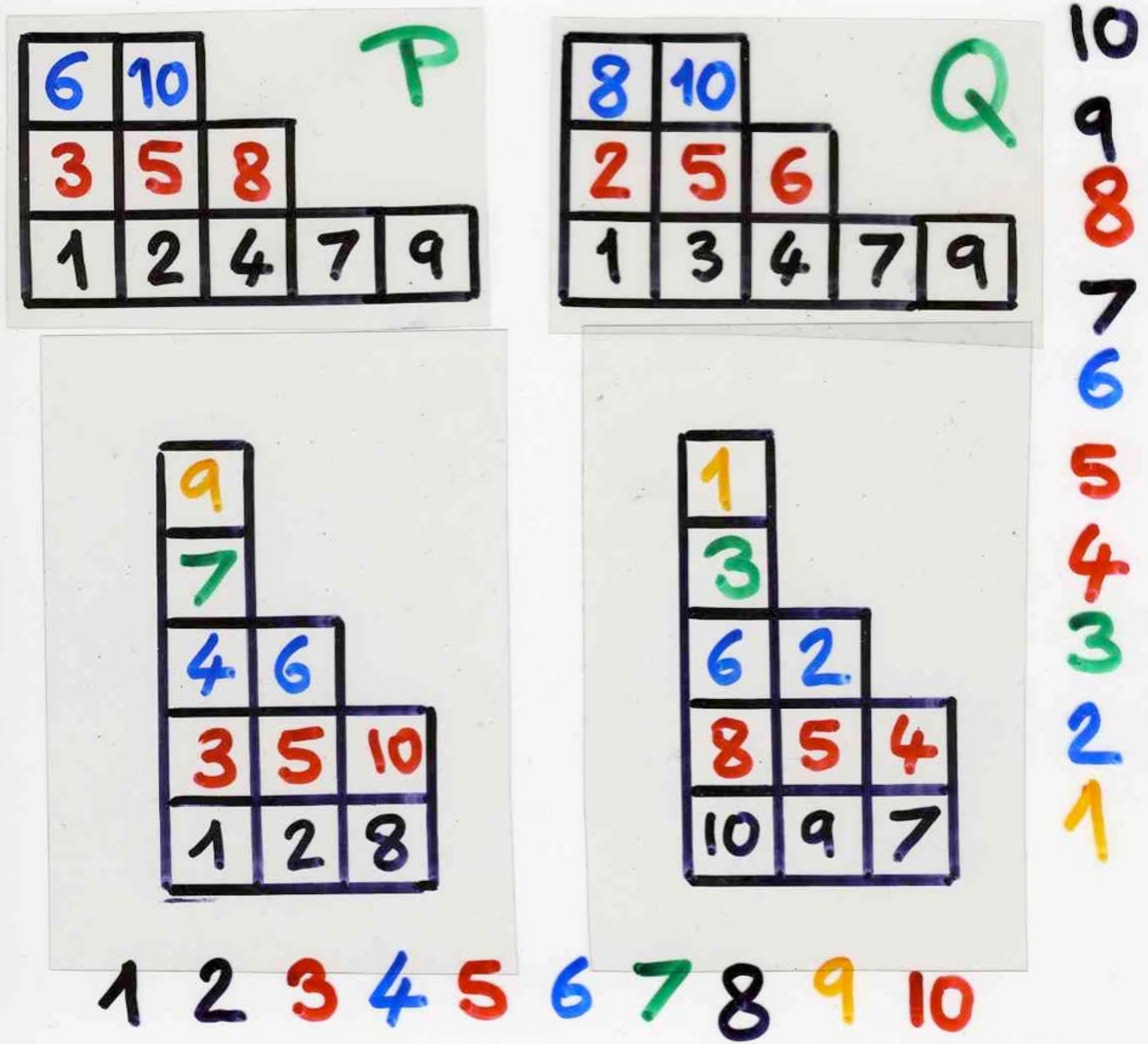




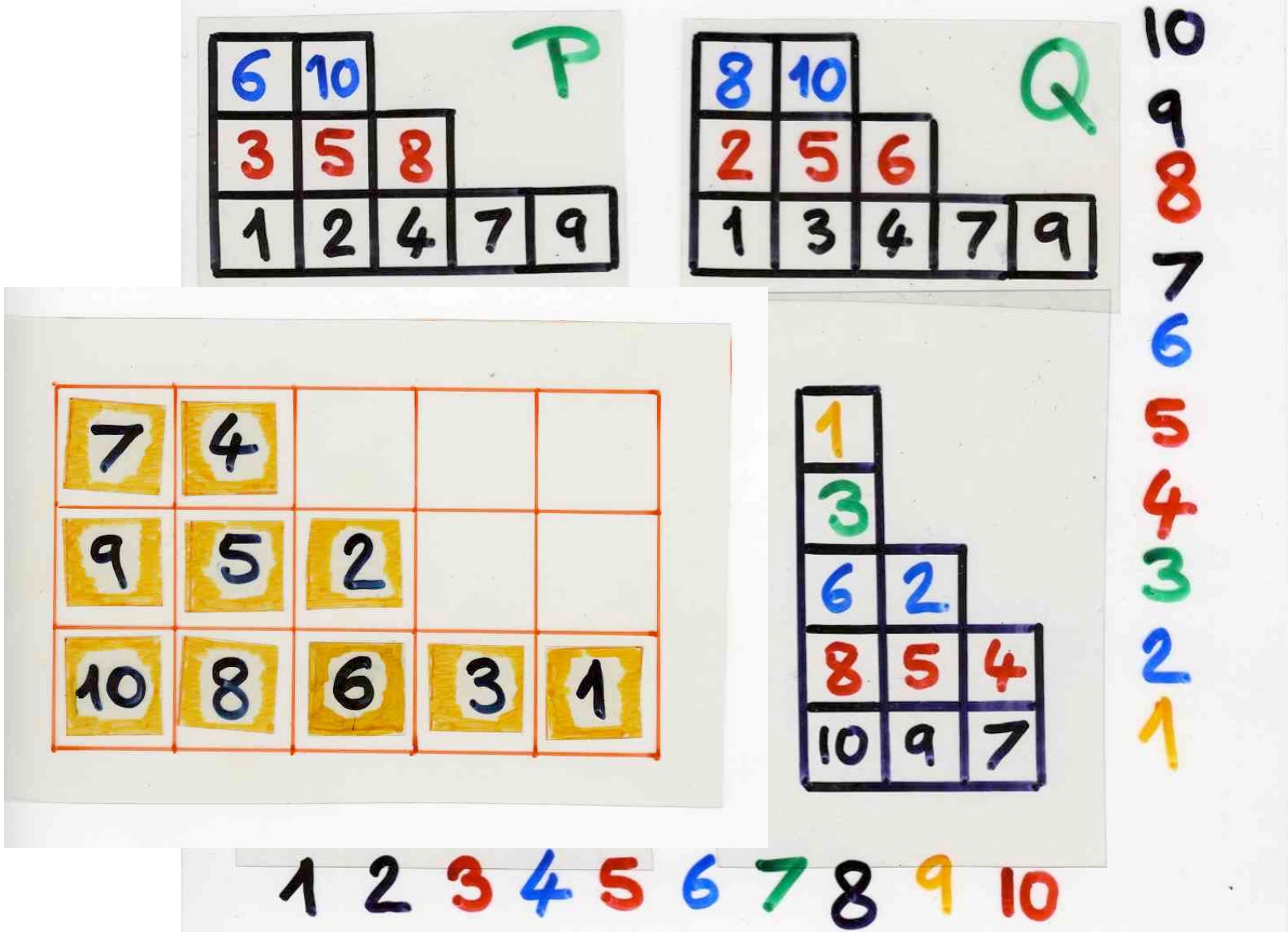





10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10



P
6 10
3 5 8
1 2 4 7 9
Q
8 10
2 5 6
1 3 4 7 9
9
7
4 6
3 5 10
1 2 8
1
3
6 2
8 5 4
10 9 7
10 9 8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8 9 10


P
6 10
3 5 8
1 2 4 7 9
Q
8 10
2 5 6
1 3 4 7 9
10 9 8 7 6 5 4 3 2 1
7 4
9 5 2
10 8 6 3 1
1
3
6 2
8 5 4
10 9 7
1 2 3 4 5 6 7 8 9 10