Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 1
RSK
The Robinson-Schensted-correspondence
(Ch1a, 1st part)

IMSc, Chennai

January 8, 2018

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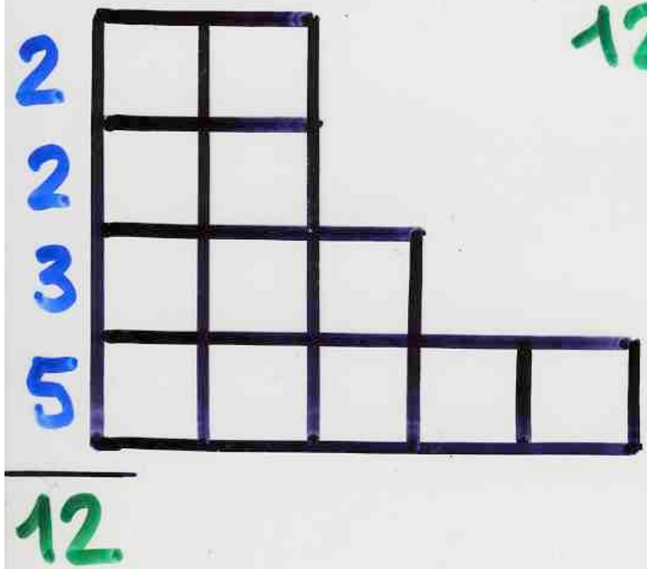
Young tableaux

$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$

partition of the integer n

$n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

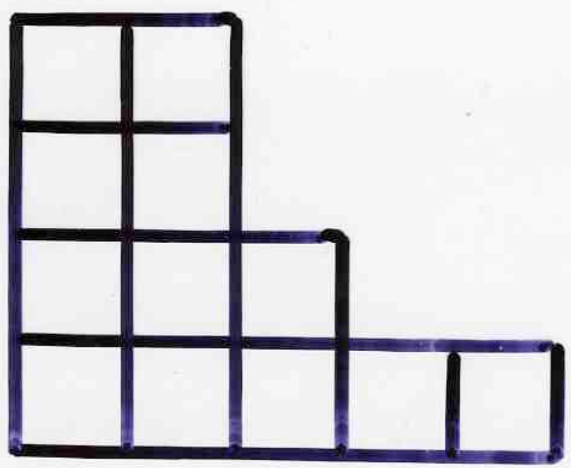
λ_i part of the partition



$12 = n = 5 + 3 + 2 + 2$

Ferrers diagram (or Young diagram in physics)

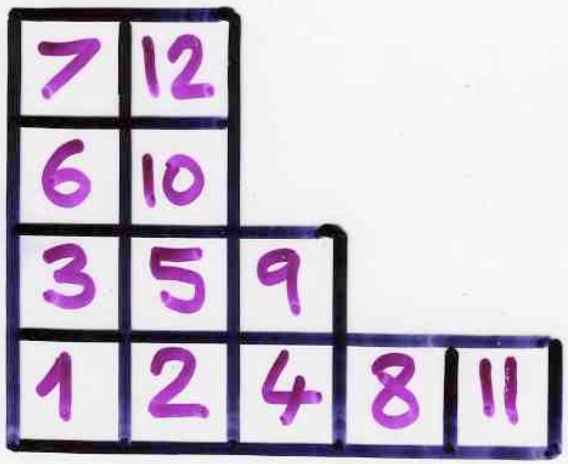
Partition of n



f_λ = number of Young tableaux with shape λ

(standard)

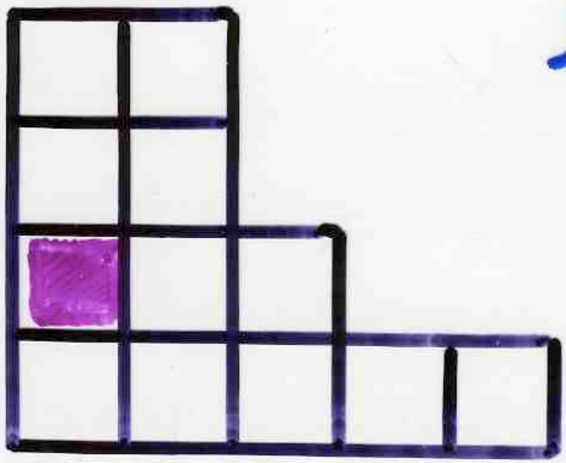
Young tableau



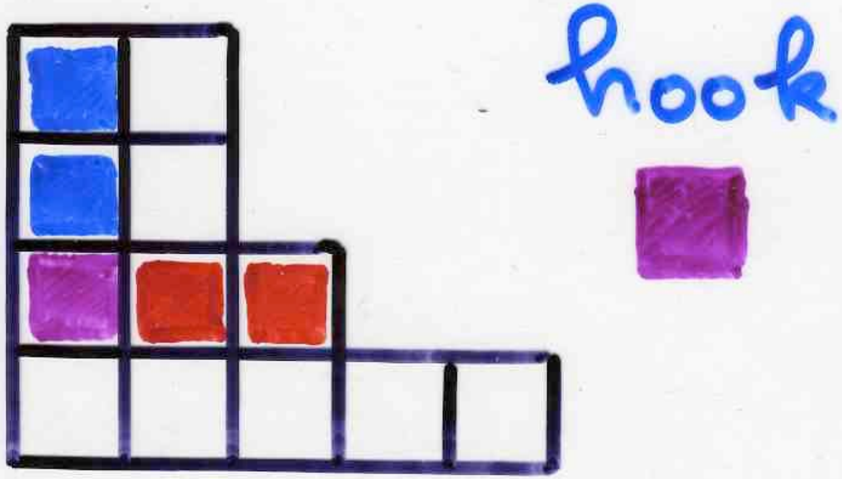
shape λ

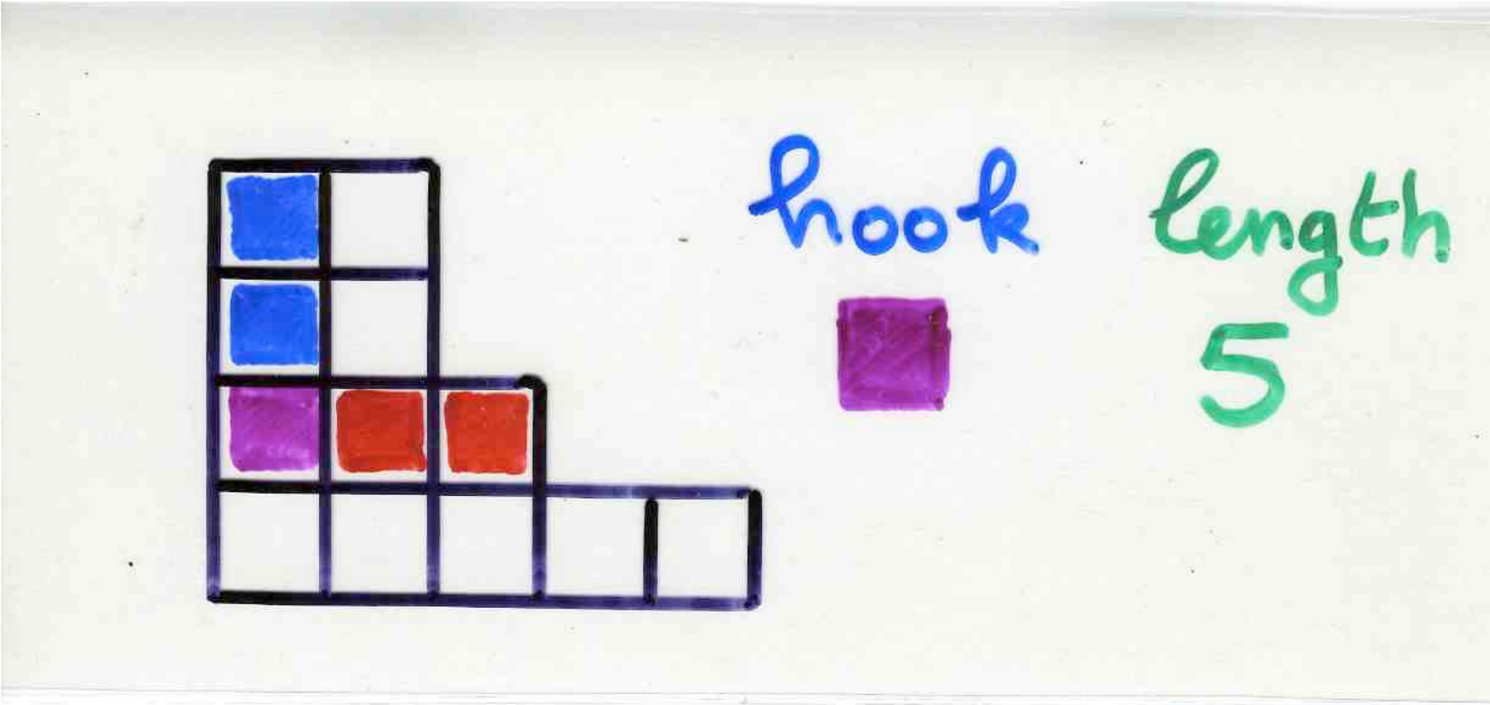
Hook length formula

J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954

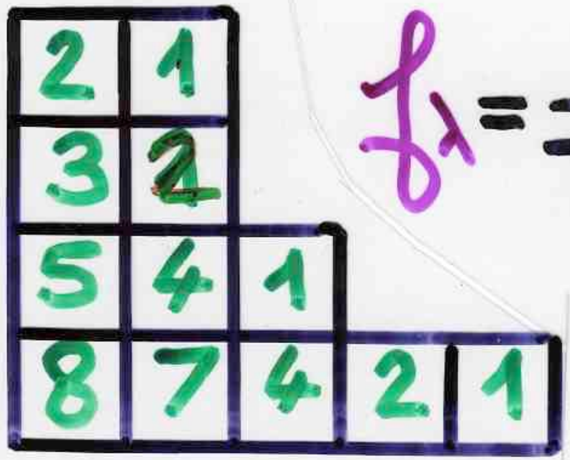


hook


hook

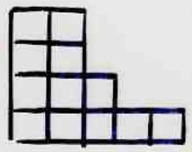

hook length
5

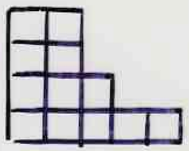
2	1			
3	2			
5	4	1		
8	7	4	2	1

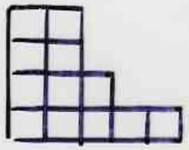


$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula

f  $=$

f  $= \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \cdot 2^3 \cdot 3 \cdot 4^2 \cdot 5 \cdot 7 \cdot 8}$

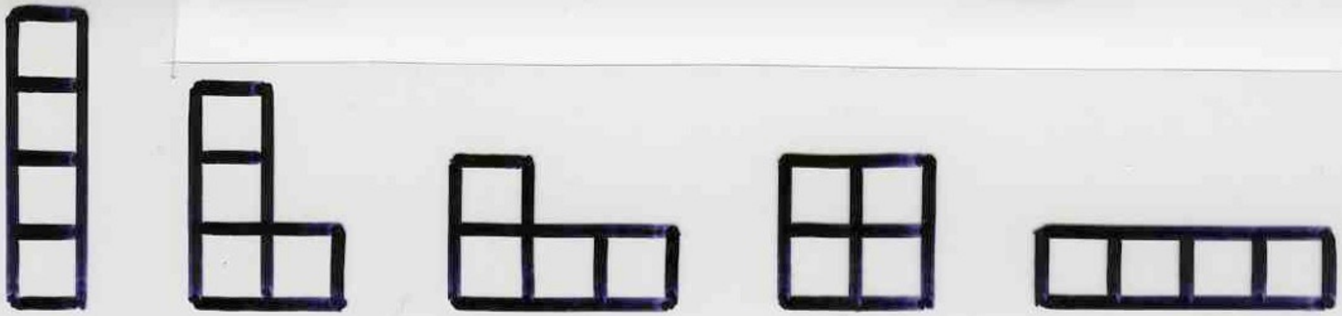
f  $= \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \cdot 2^3 \cdot 3 \cdot 4^2 \cdot 5 \cdot 7 \cdot 8}$

$= 3^4 \times 5 \times 11 = 4455$

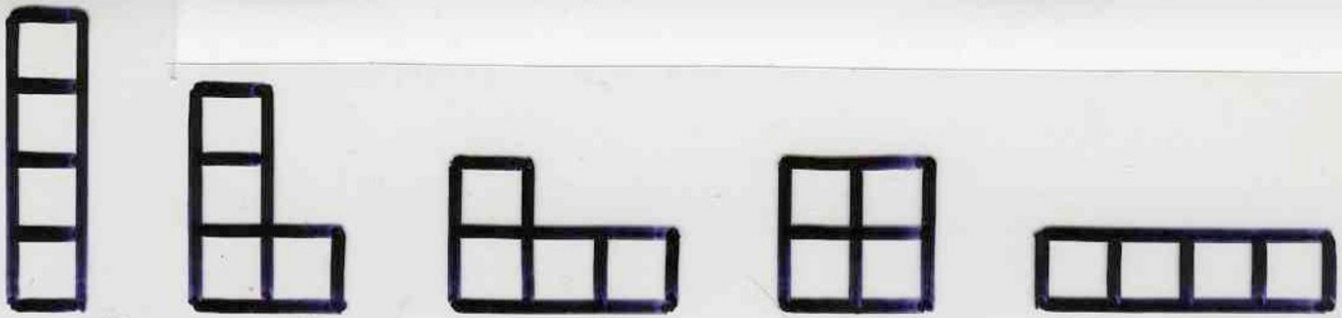
An introduction to RS

G. de B. Robinson, 1938

C. Schensted, 1961



1 3 3 2 1



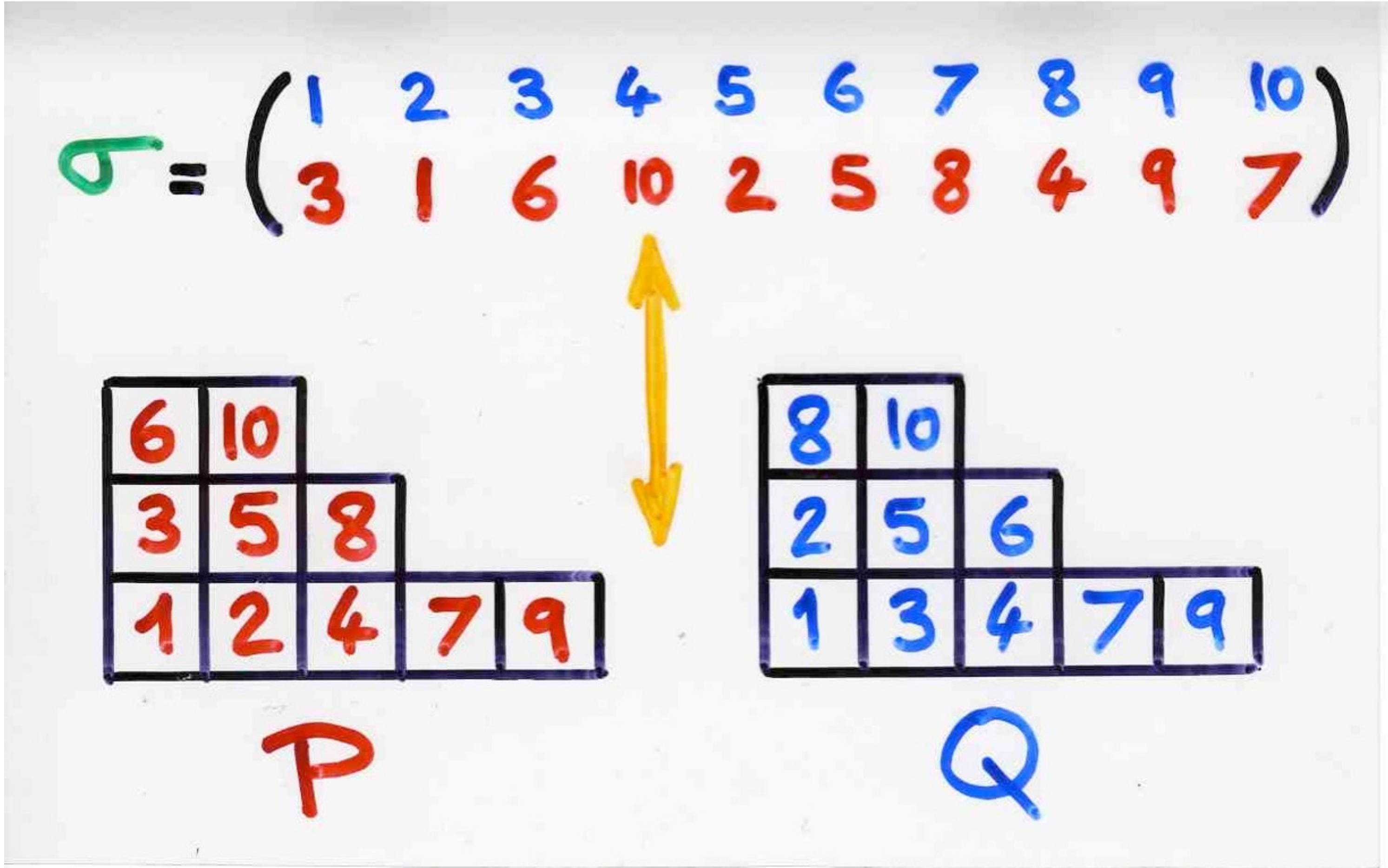
$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

$$n! = \sum_{\lambda} (f_{\lambda})^2$$

partition
of n



The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape

RS with Schensted's insertions

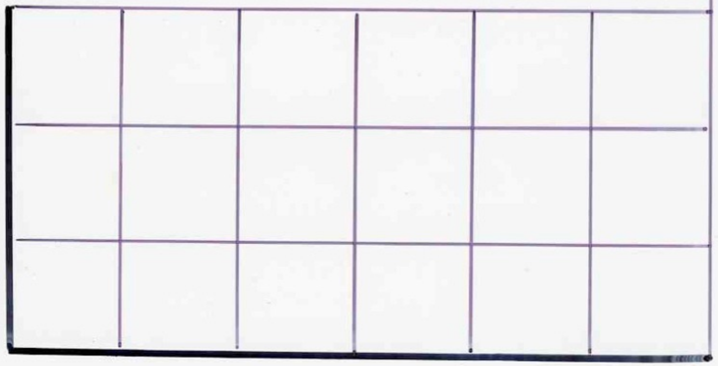
$\sigma =$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q

recording tableau

P



insertion tableau

- read the permutation σ as a word $w = \sigma(1)\sigma(2)\ldots\sigma(n)$ from left to right

$\sigma =$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

Q recording tableau

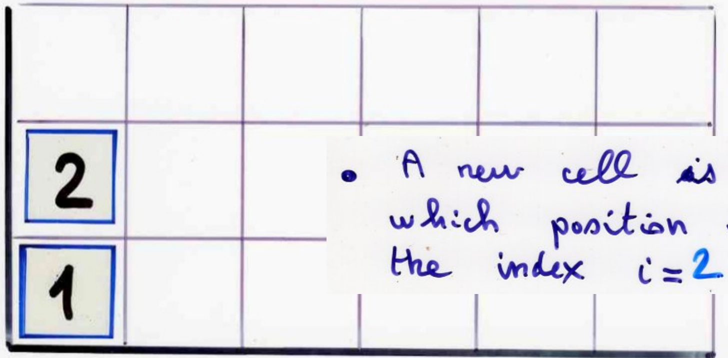
1					

P insertion tableau

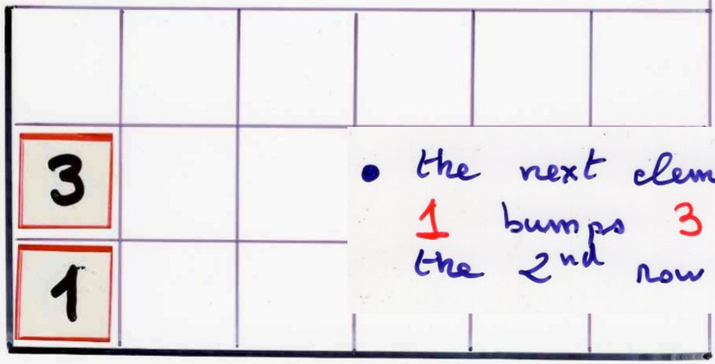
3					

- insert the first value $3 = \sigma(1)$ in the 1st row of P

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

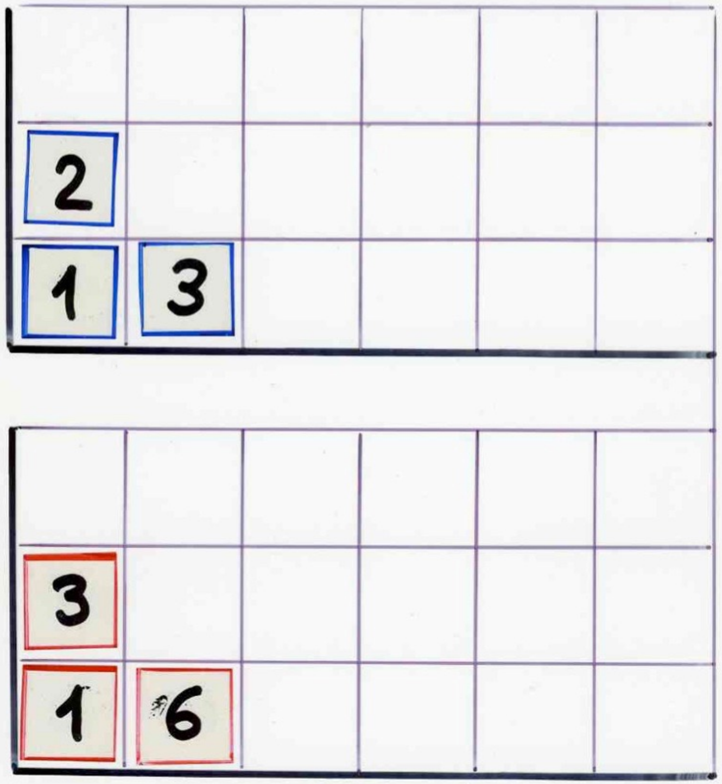


- A new cell is added in the shape of P, which position is recorded in Q with the index $i = 2$



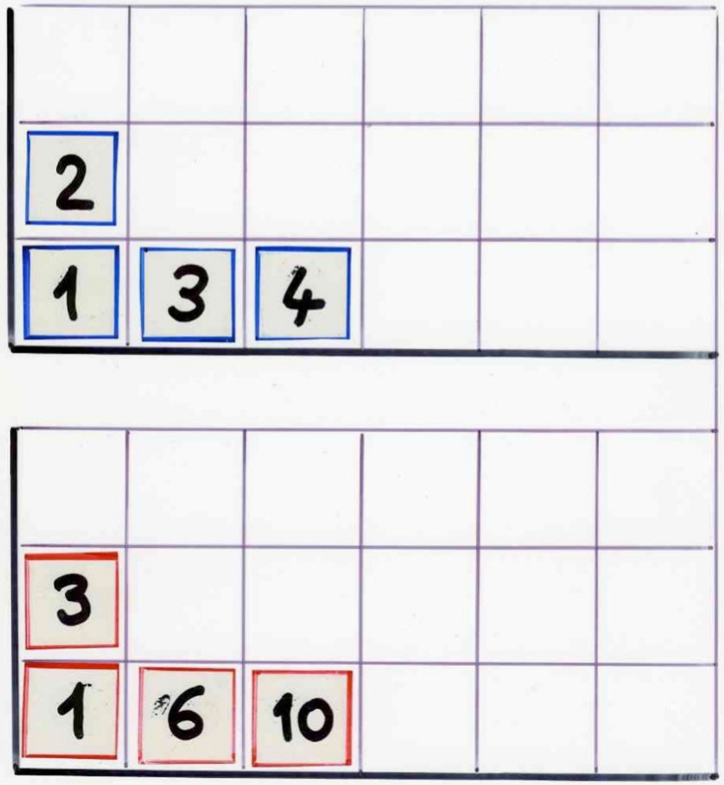
- the next element $1 = \sigma(2)$ is < 3, 1 bumps 3 which is inserted in the 2^{nd} row of P

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



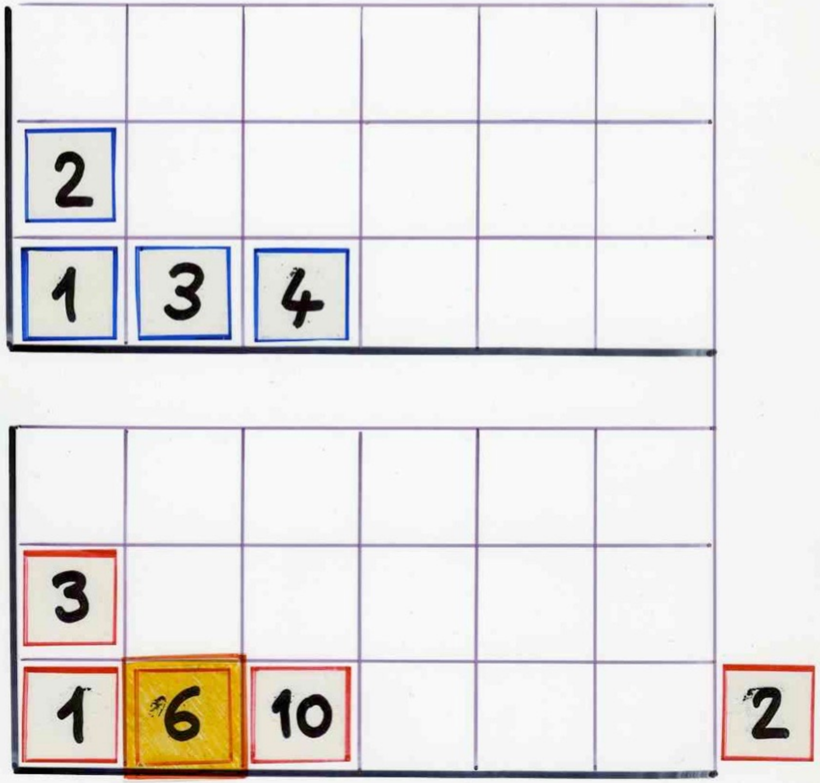
• 6 > 1 is inserted in the 1st row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



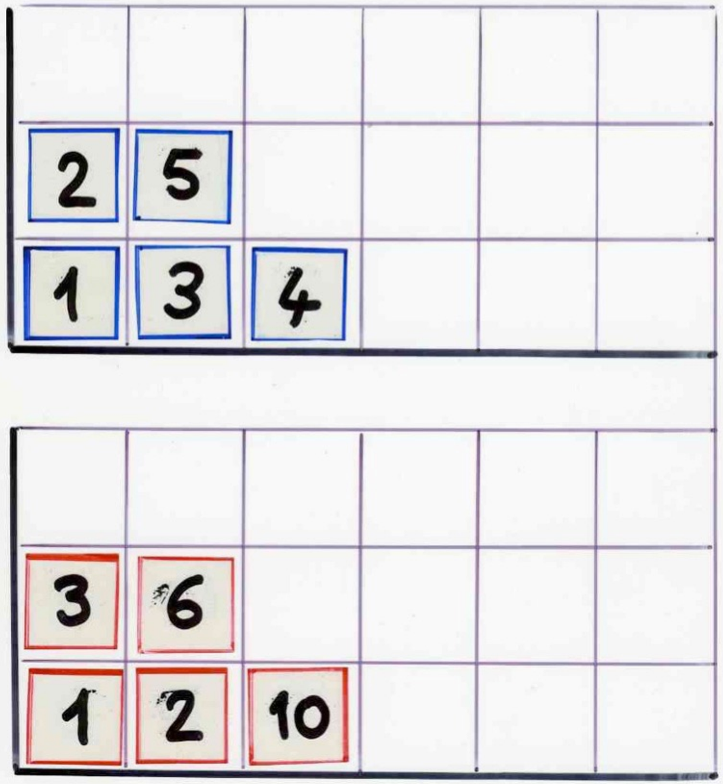
- $\sigma(4) = 10$ is > than all elements of the 1^{st} row, and is added at the end of this 1^{st} row.

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



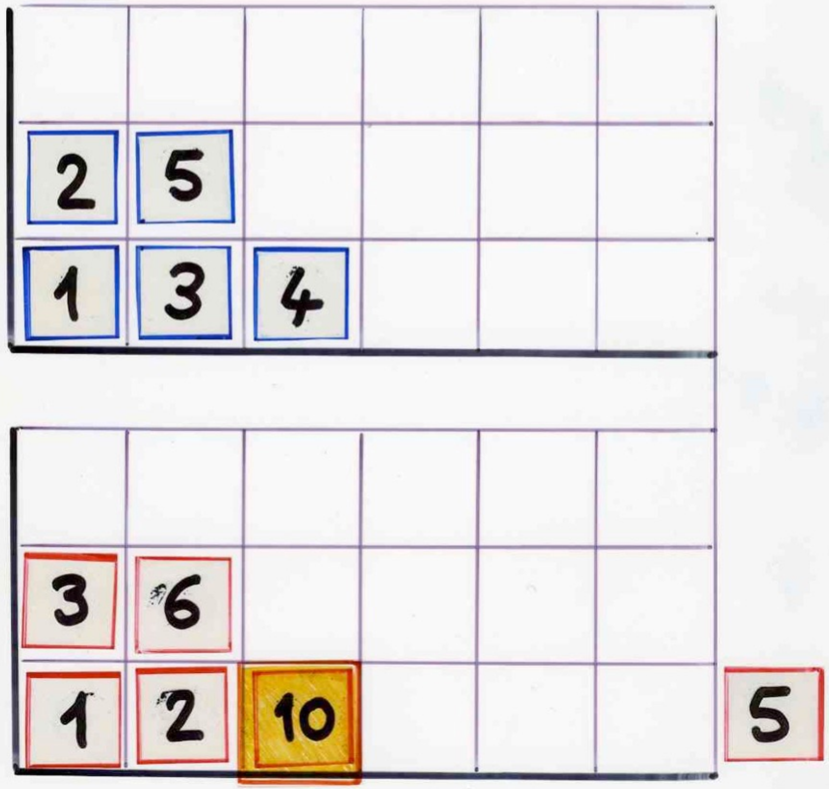
- $\sigma(5) = 2$ cannot be added at the end of the 1st row.
  2 is "bumping" the element 6, which is the smallest element of the 1st row > 2

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

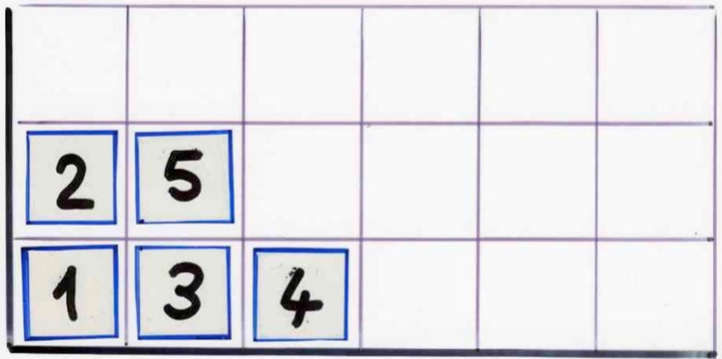


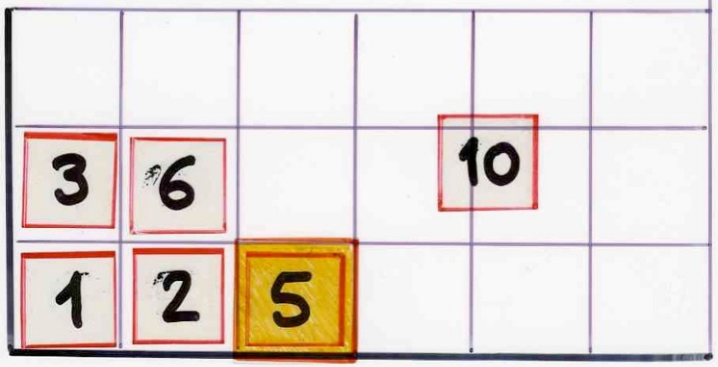
- 2 replaces 6, and 6 is inserted in the second row with the same recursive rule

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

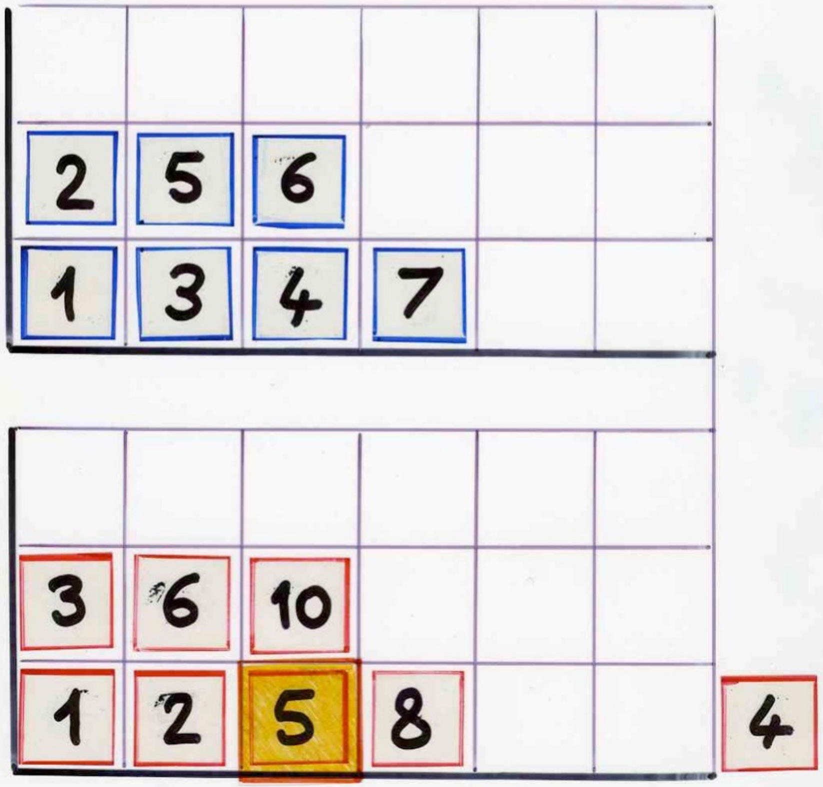
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4	7		

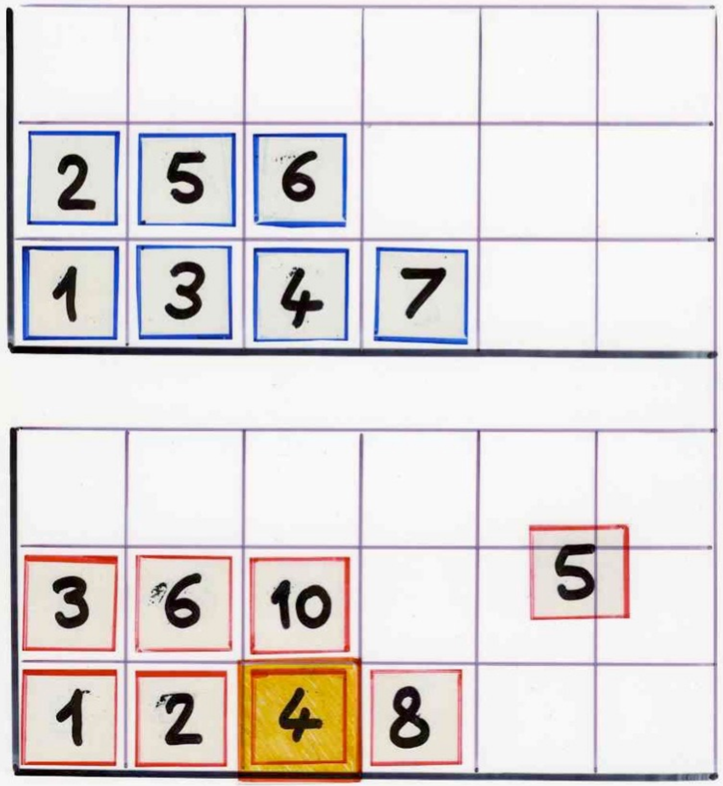
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

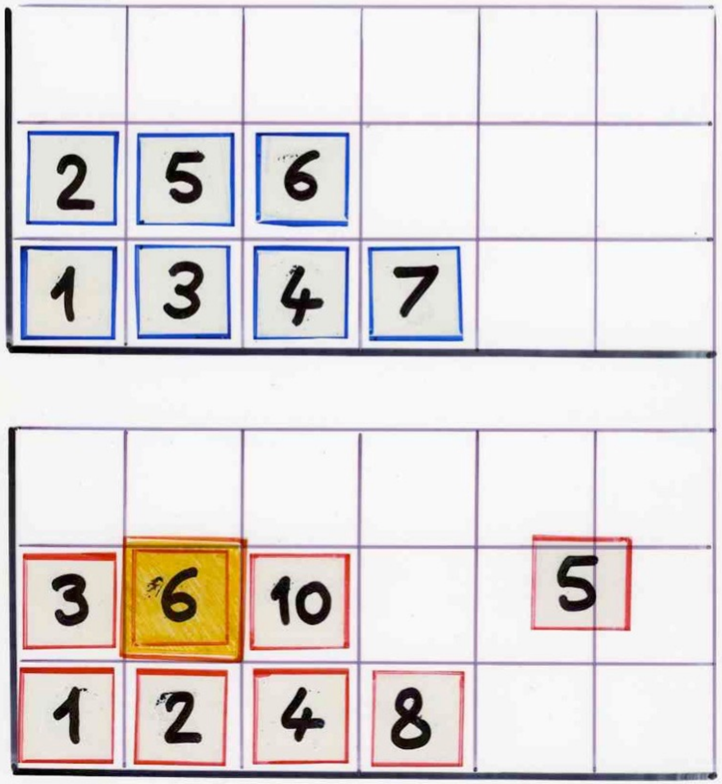


• $4 = \sigma(8)$ bumps 5 in the 1st row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

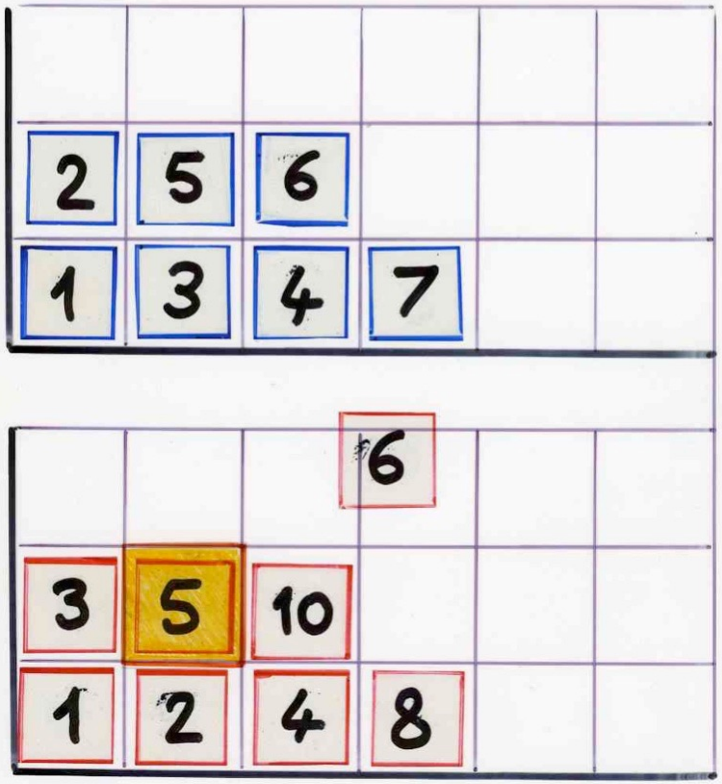


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



• 5 bumps 6 in the 2nd row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



• 6 is inserted in the 3rd row

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

- the new cell added in the common shape of P and Q is recorded in Q with the cell 8

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

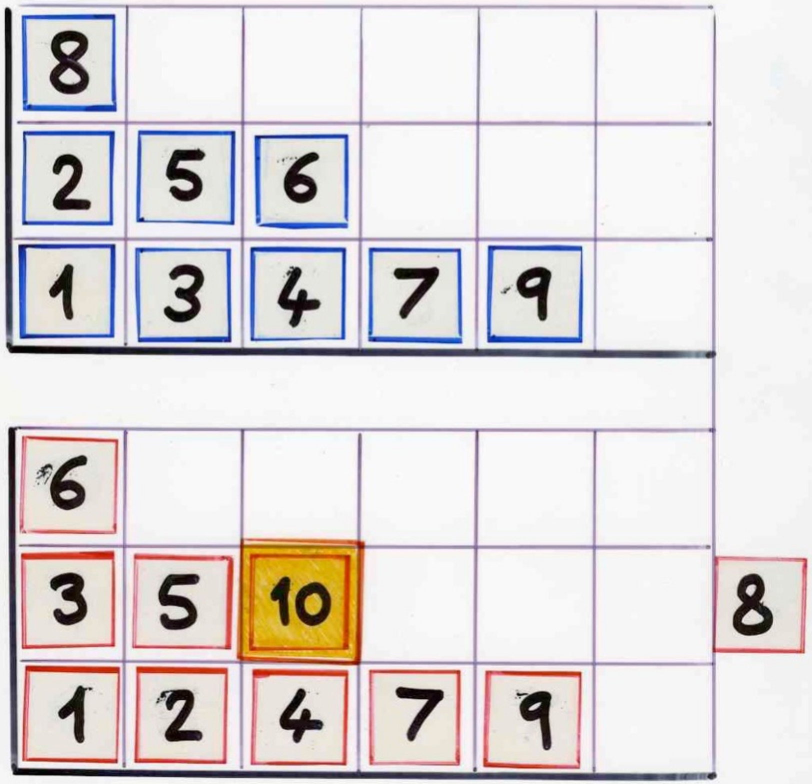
7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

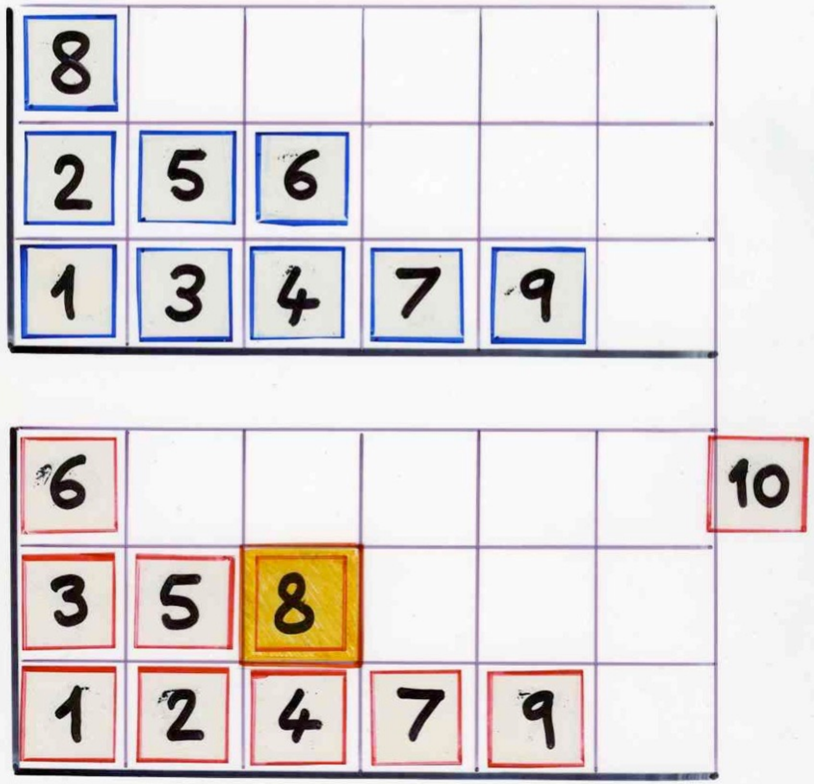
8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



$\sigma =$

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

$Q =$

8	10				
2	5	6			
1	3	4	7	9	

$P =$

6	10				
3	5	8			
1	2	4	7	9	

end of the RS algorithm

Reverse algorithm

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

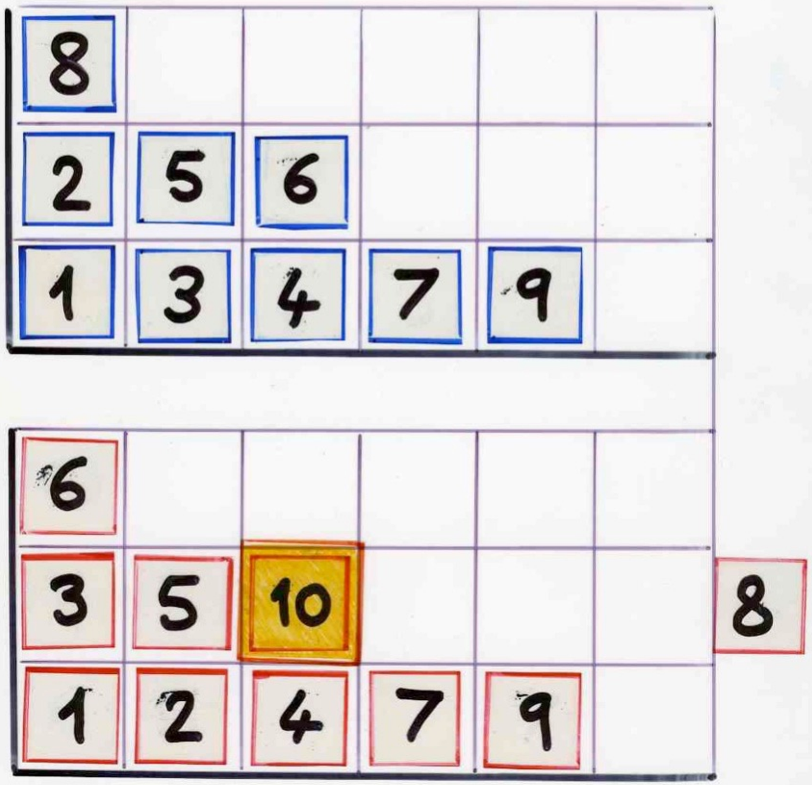
6	10				
3	5	8			
1	2	4	7	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

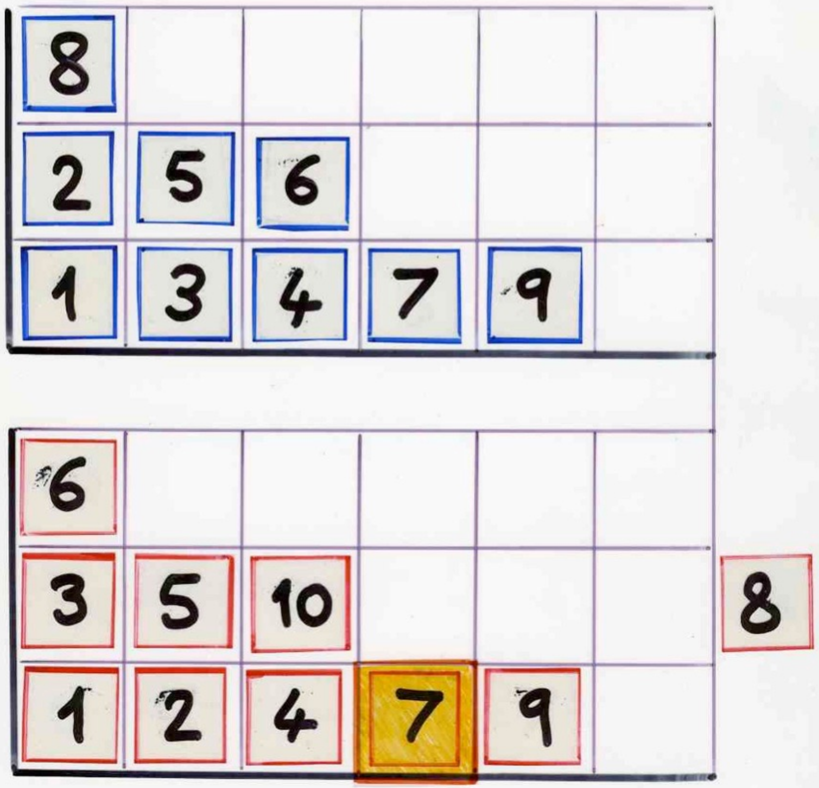
8					
2	5	6			
1	3	4	7	9	

6						10
3	5	8				
1	2	4	7	9		

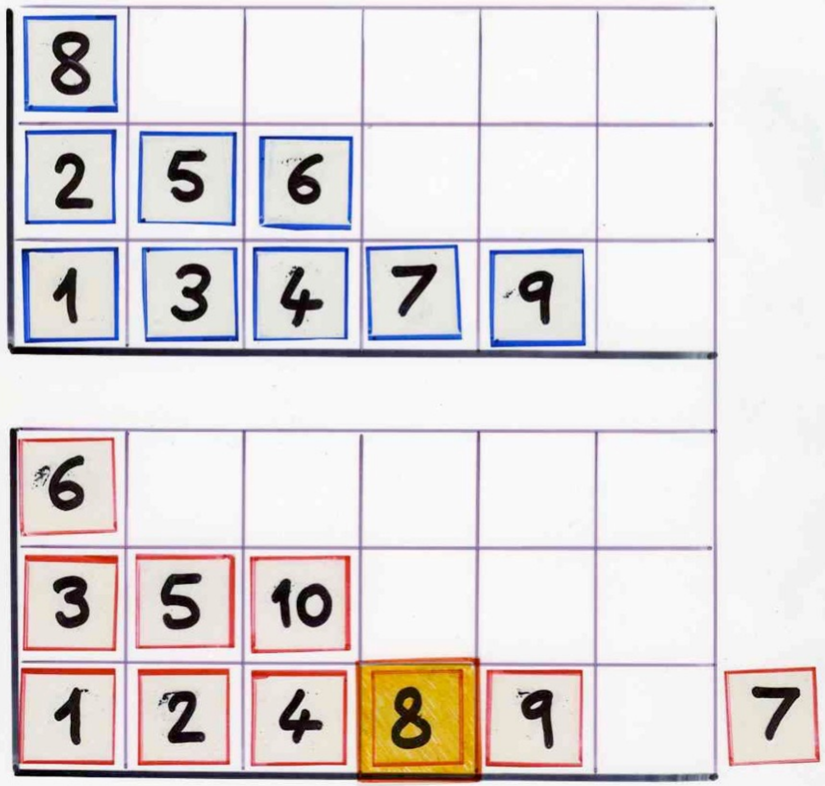
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

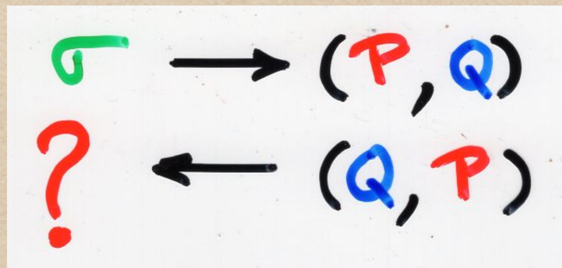


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\underset{P}{\begin{array}{|c|c|c|c|c|} \hline 6 & 10 \\ \cline{1-2} 3 & 5 & 8 \\ \cline{1-3} 1 & 2 & 4 & 7 & 9 \\ \hline \end{array}} \quad \longleftrightarrow \quad \underset{Q}{\begin{array}{|c|c|c|c|c|} \hline 8 & 10 \\ \cline{1-2} 2 & 5 & 6 \\ \cline{1-3} 1 & 3 & 4 & 7 & 9 \\ \hline \end{array}}$$

The Robinson-Schensted correspondence
between permutations and pairs of
(standard) Young tableaux with the same shape

Problem



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

8	10			
2	5	6		
1	3	4	7	9

6	10			
3	5	8		
1	2	4	7	9

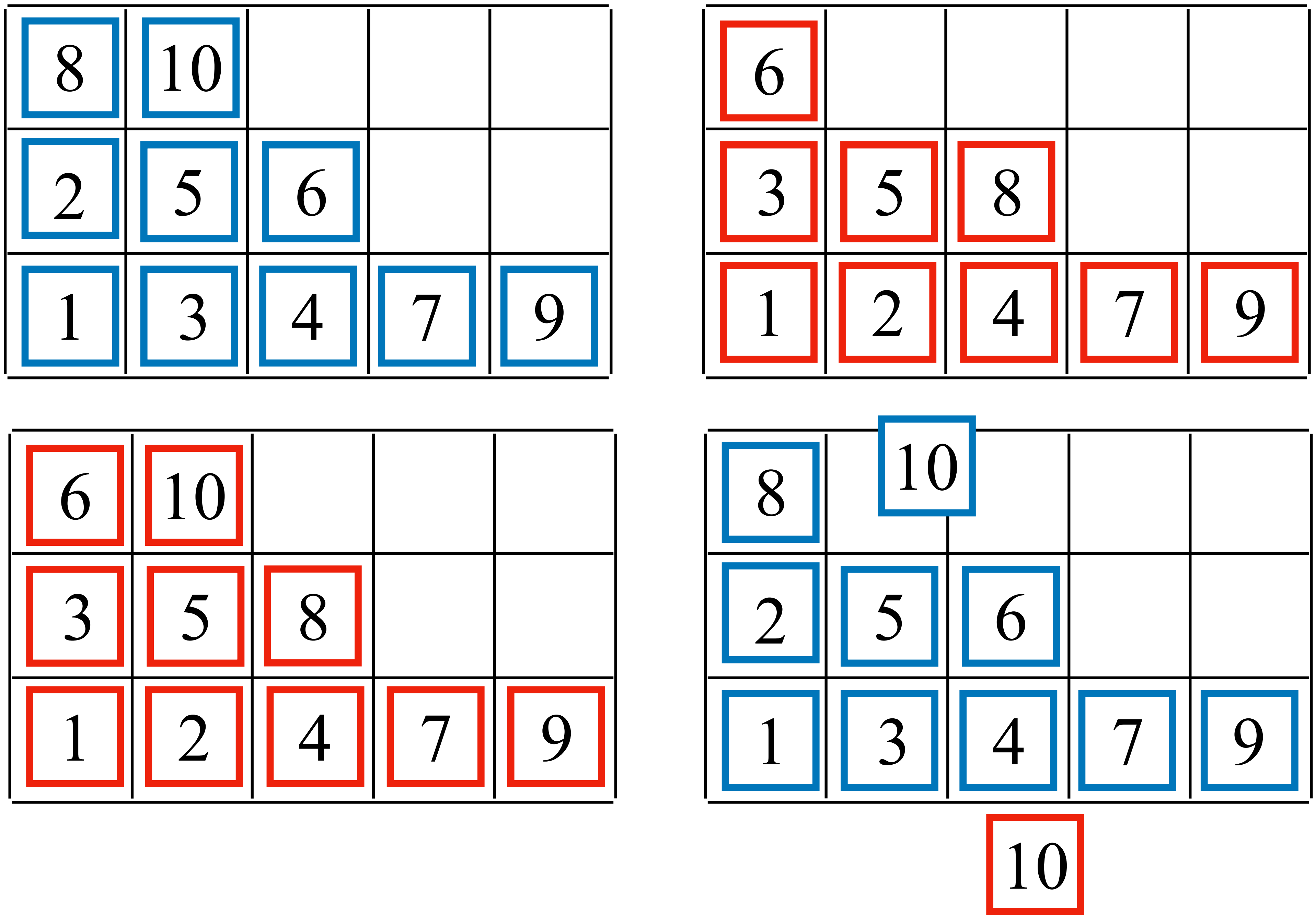
Reverse RS
with (Q,P)

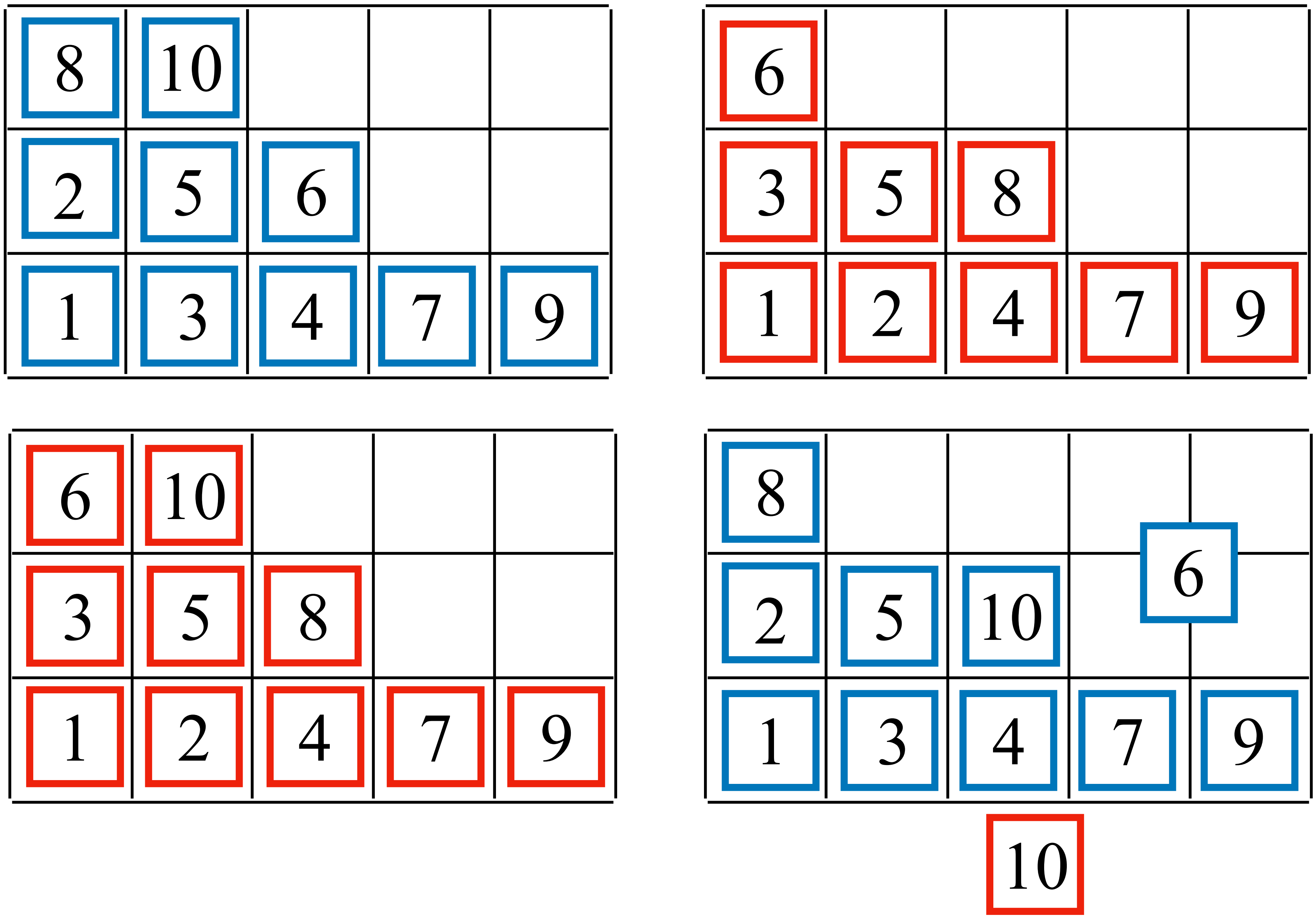
8	10			
2	5	6		
1	3	4	7	9

6	10			
3	5	8		
1	2	4	7	9

6	10			
3	5	8		
1	2	4	7	9

8	10			
2	5	6		
1	3	4	7	9


8 10
2 5 6
1 3 4 7 9
6
3 5 8
1 2 4 7 9
6 10
3 5 8
1 2 4 7 9
8 10
2 5 6
1 3 4 7 9
10


8 10
2 5 6
1 3 4 7 9
6
3 5 8
1 2 4 7 9
6 10
3 5 8
1 2 4 7 9
8
6
2 5 10
1 3 4 7 9
10

8	10			
2	5	6		
1	3	4	7	9

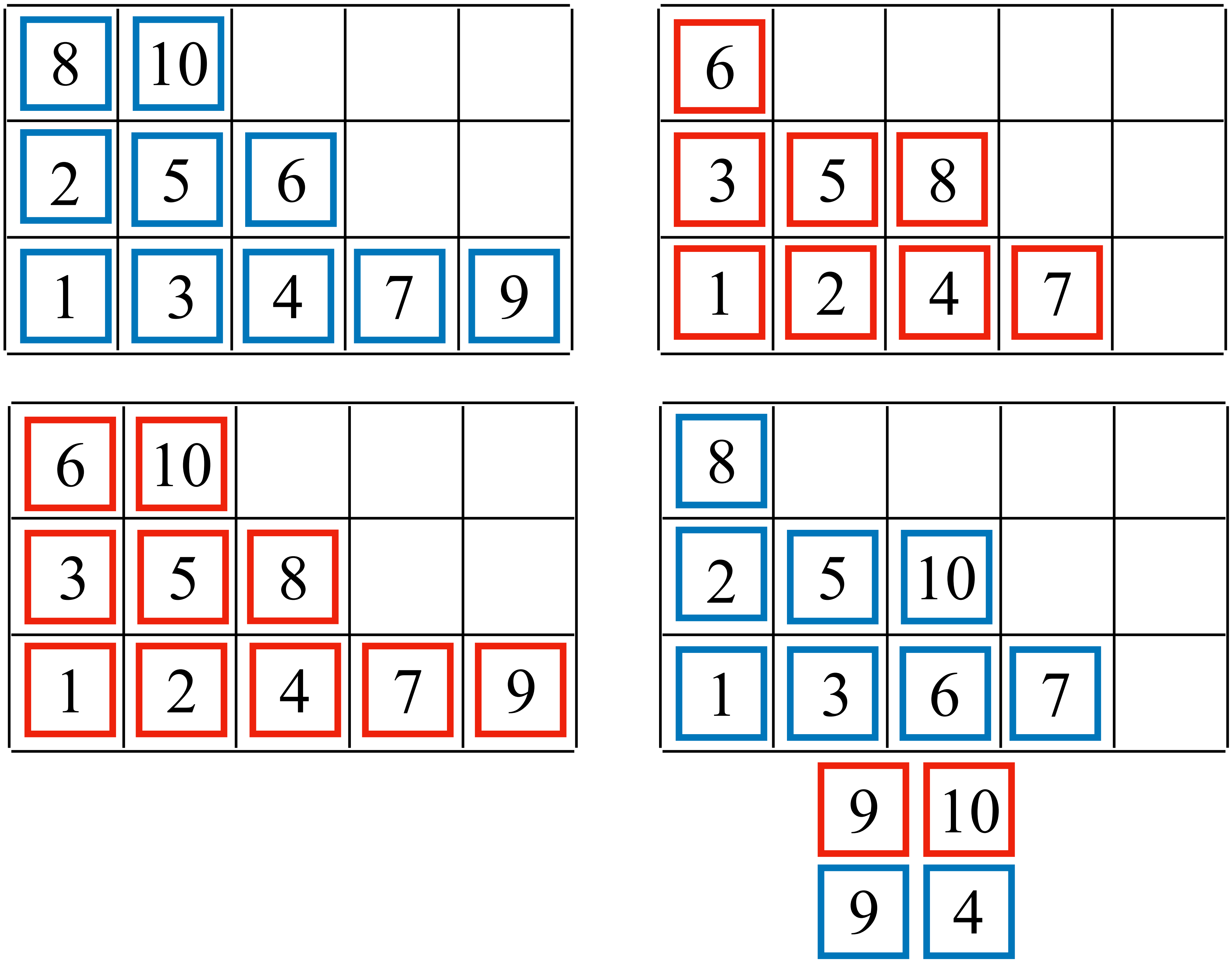
6				
3	5	8		
1	2	4	7	9

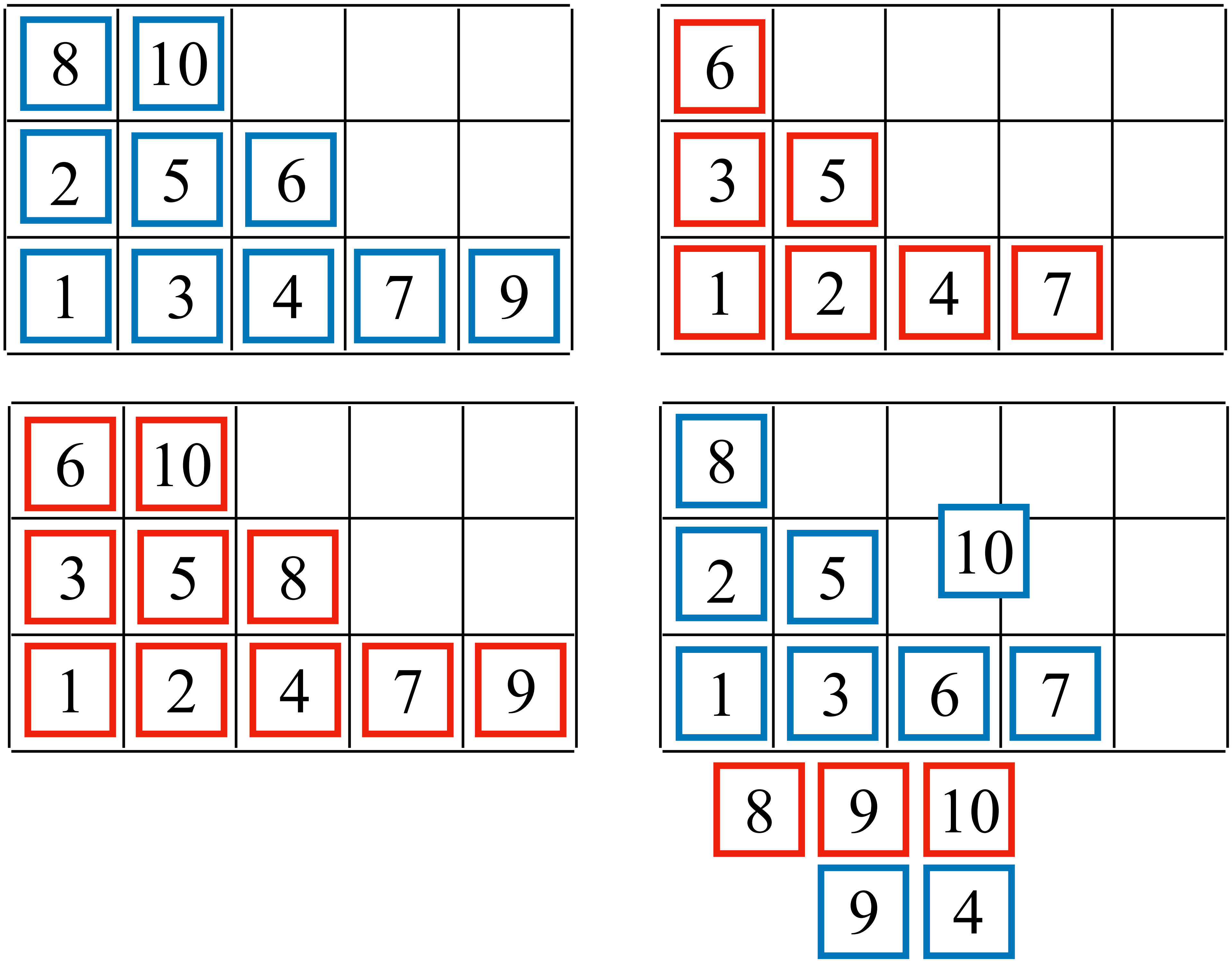
6	10			
3	5	8		
1	2	4	7	9

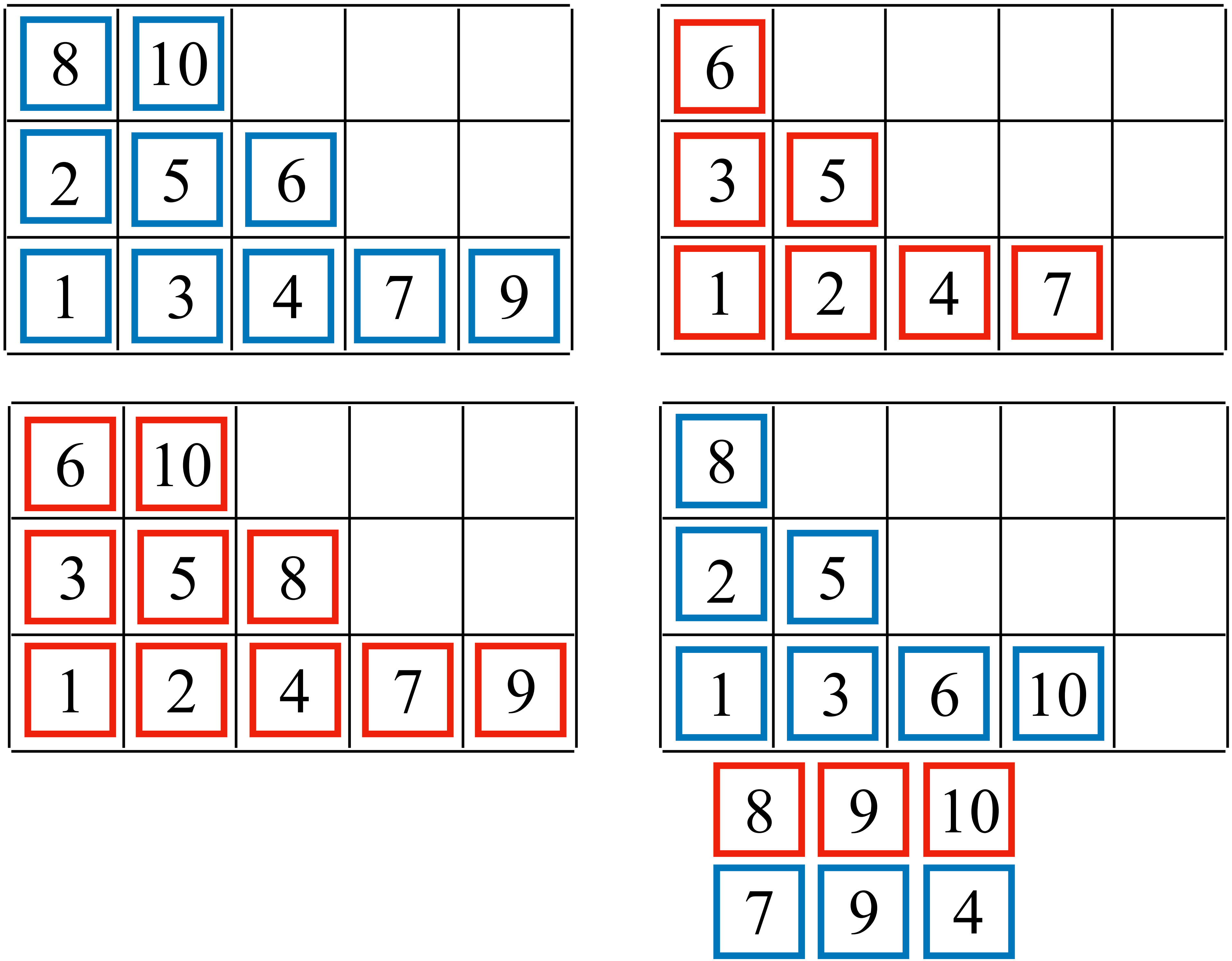
8				
2	5	10		
1	3	6	7	9

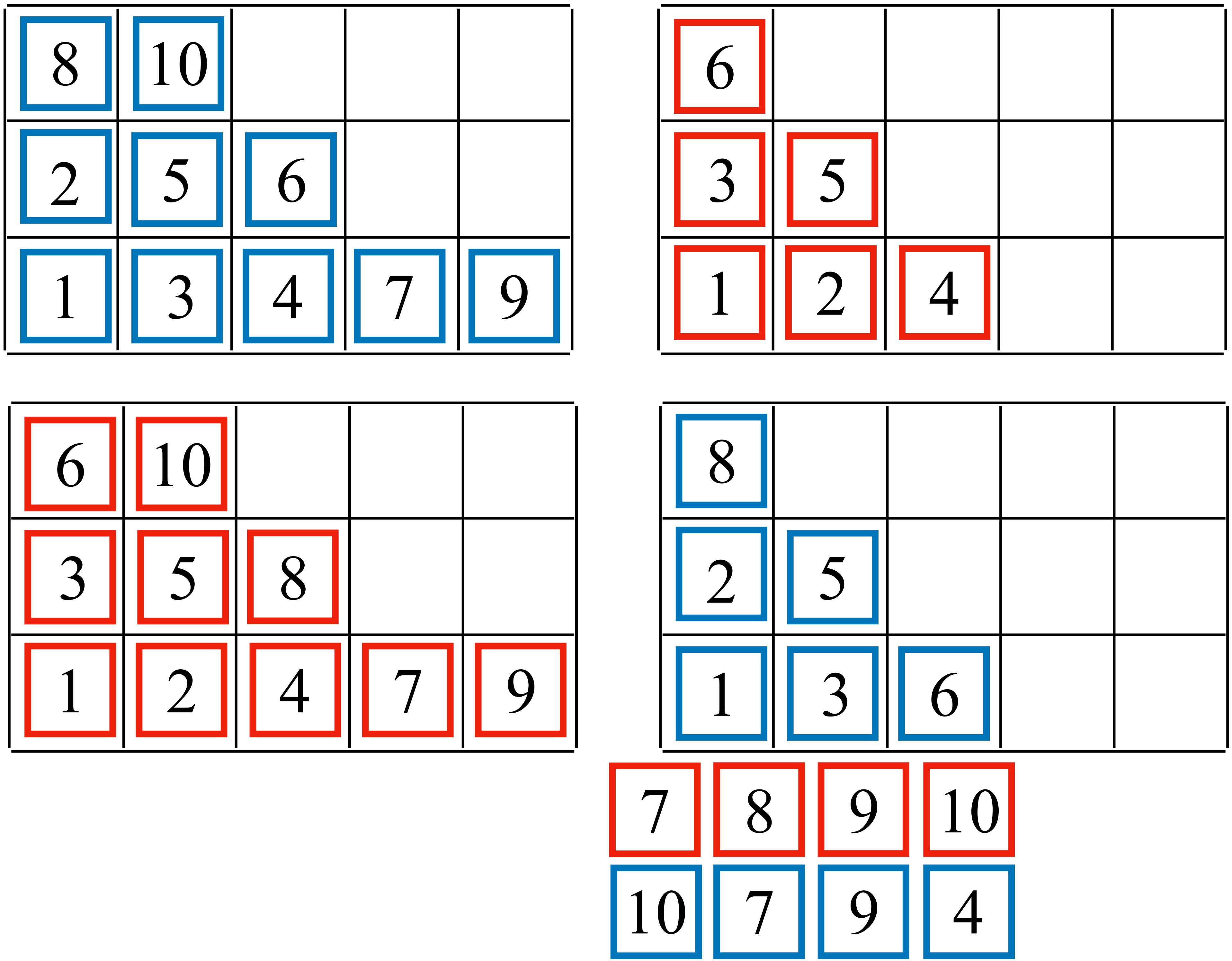
10

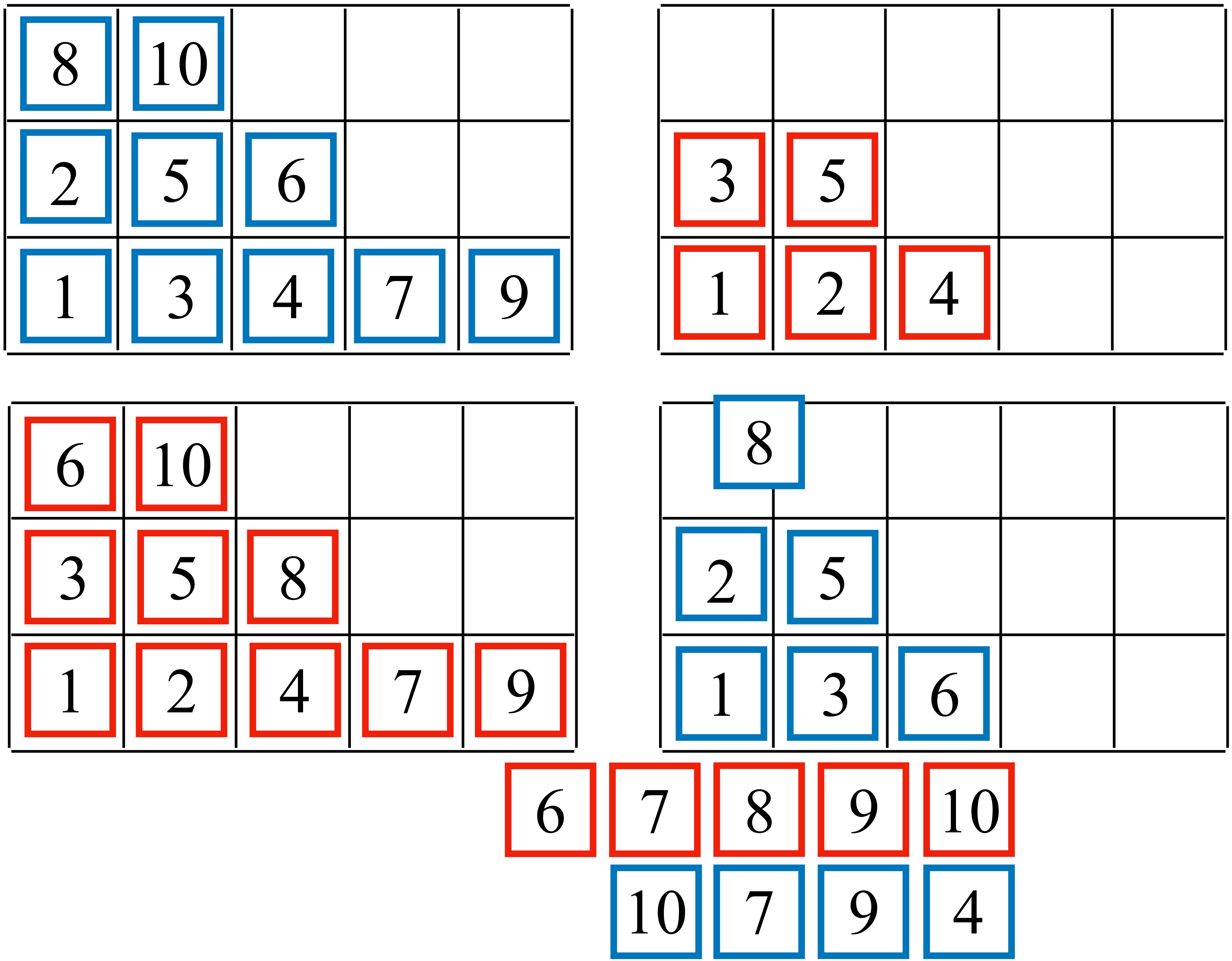
4

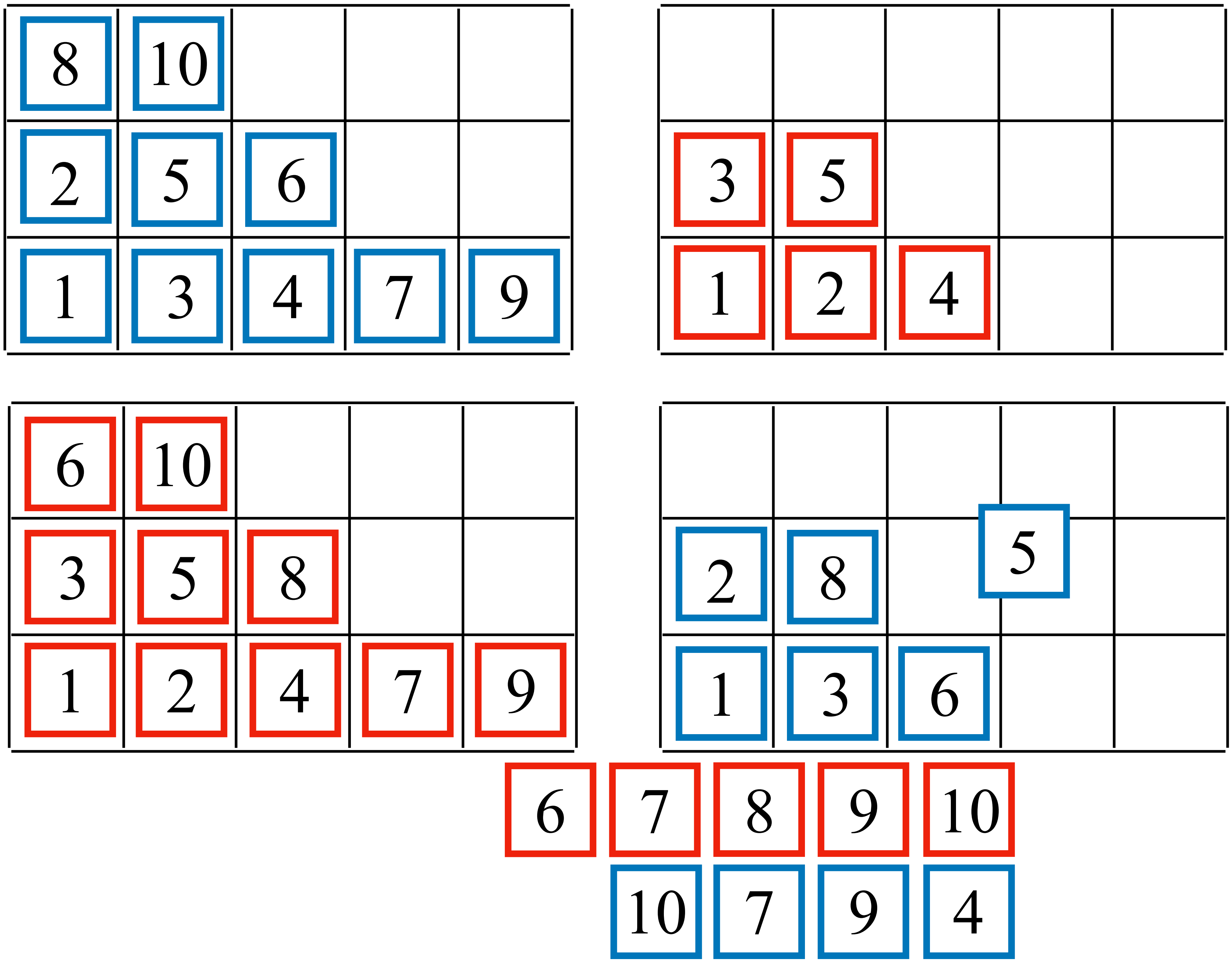

8 10
2 5 6
1 3 4 7 9
6
3 5 8
1 2 4 7
6 10
3 5 8
1 2 4 7 9
8
2 5 10
1 3 6 7
9 10
9 4


8 10
2 5 6
1 3 4 7 9
6
3 5
1 2 4 7
6 10
3 5 8
1 2 4 7 9
8
2 5 10
1 3 6 7
8 9 10
9 4


8 10
2 5 6
1 3 4 7 9
6
3 5
1 2 4 7
6 10
3 5 8
1 2 4 7 9
8
2 5
1 3 6 10
8 9 10
7 9 4


8 10
2 5 6
1 3 4 7 9
6
3 5
1 2 4
6 10
3 5 8
1 2 4 7 9
8
2 5
1 3 6
7 8 9 10
10 7 9 4


8 10
2 5 6
1 3 4 7 9
3 5
1 2 4
6 10
3 5 8
1 2 4 7 9
8
2 5
1 3 6
6 7 8 9 10
10 7 9 4


8 10
2 5 6
1 3 4 7 9
3 5
1 2 4
6 10
3 5 8
1 2 4 7 9
2 8 5
1 3 6
6 7 8 9 10
10 7 9 4

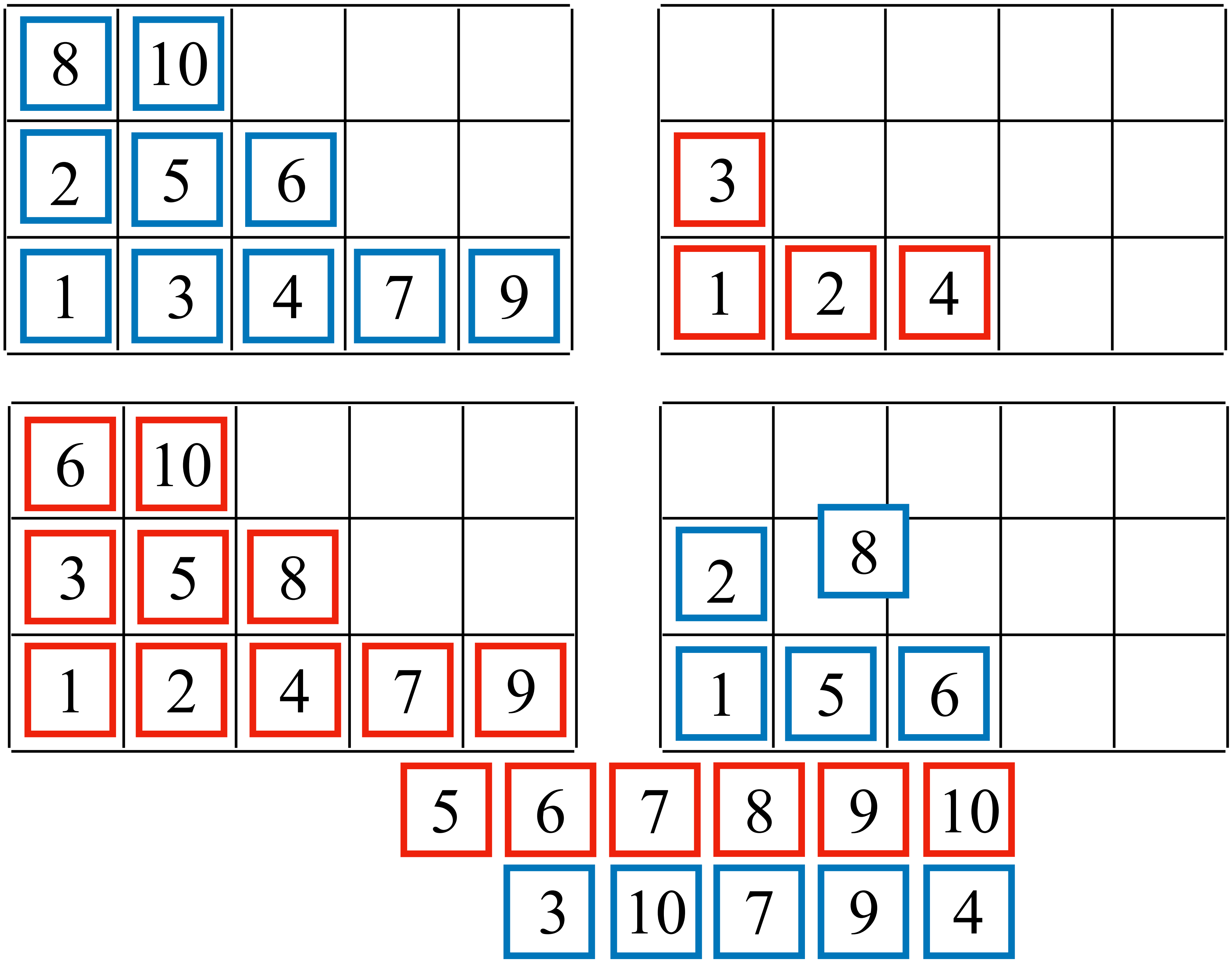
8	10			
2	5	6		
1	3	4	7	9

3	5			
1	2	4		

6	10			
3	5	8		
1	2	4	7	9

2	8			
1	5	6		

6	7	8	9	10
3	10	7	9	4


8 10
2 5 6
1 3 4 7 9
3
1 2 4
6 10
3 5 8
1 2 4 7 9
2 8
1 5 6
5 6 7 8 9 10
3 10 7 9 4

8	10			
2	5	6		
1	3	4	7	9

3				
1	2	4		

6	10			
3	5	8		
1	2	4	7	9

2				
1	5	8		

5	6	7	8	9	10
6	3	10	7	9	4

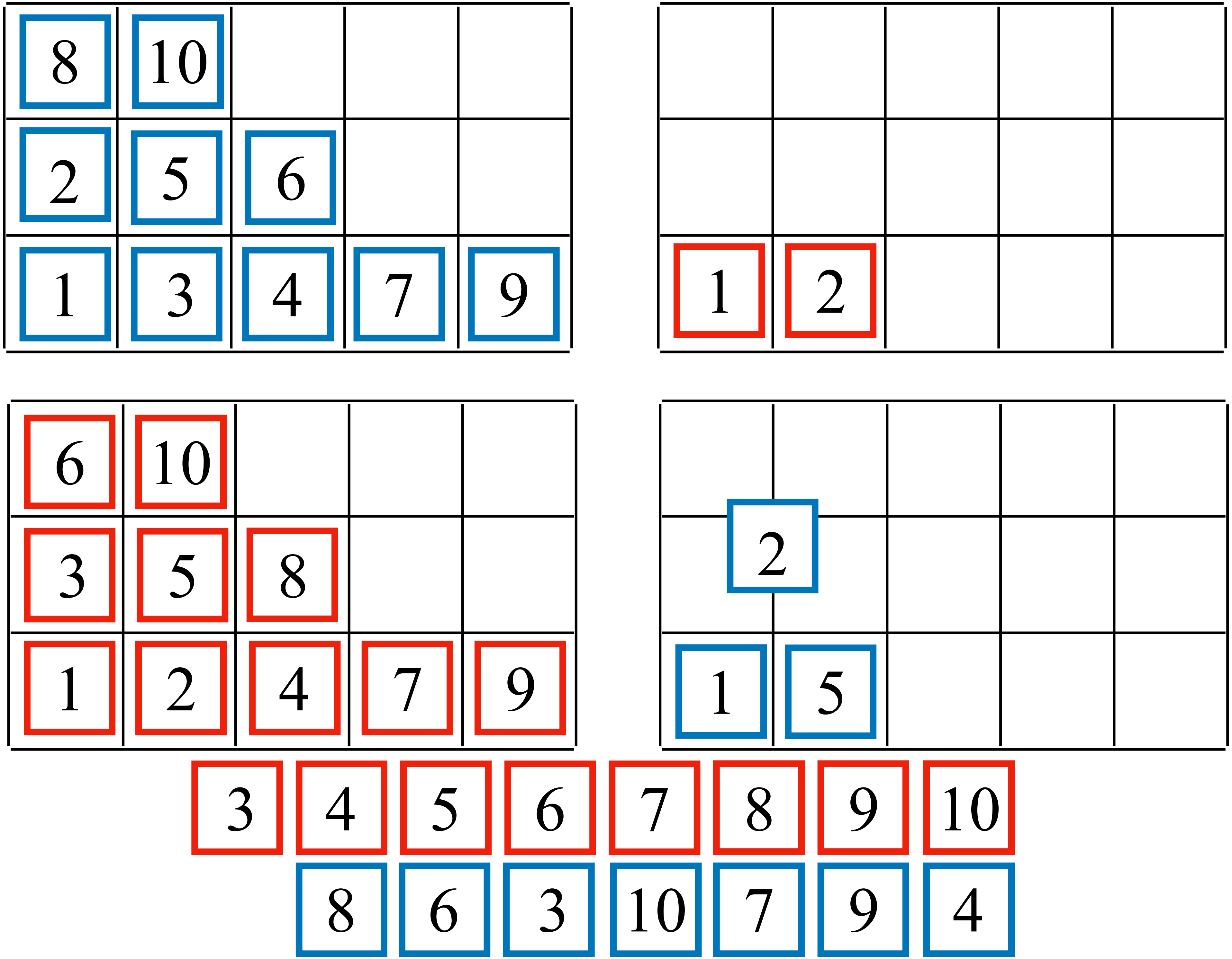
8	10			
2	5	6		
1	3	4	7	9

3				
1	2			

6	10			
3	5	8		
1	2	4	7	9

2				
1	5			

4	5	6	7	8	9	10
8	6	3	10	7	9	4


8 10
2 5 6
1 3 4 7 9
1 2
6 10
3 5 8
1 2 4 7 9
2
1 5
3 4 5 6 7 8 9 10
8 6 3 10 7 9 4

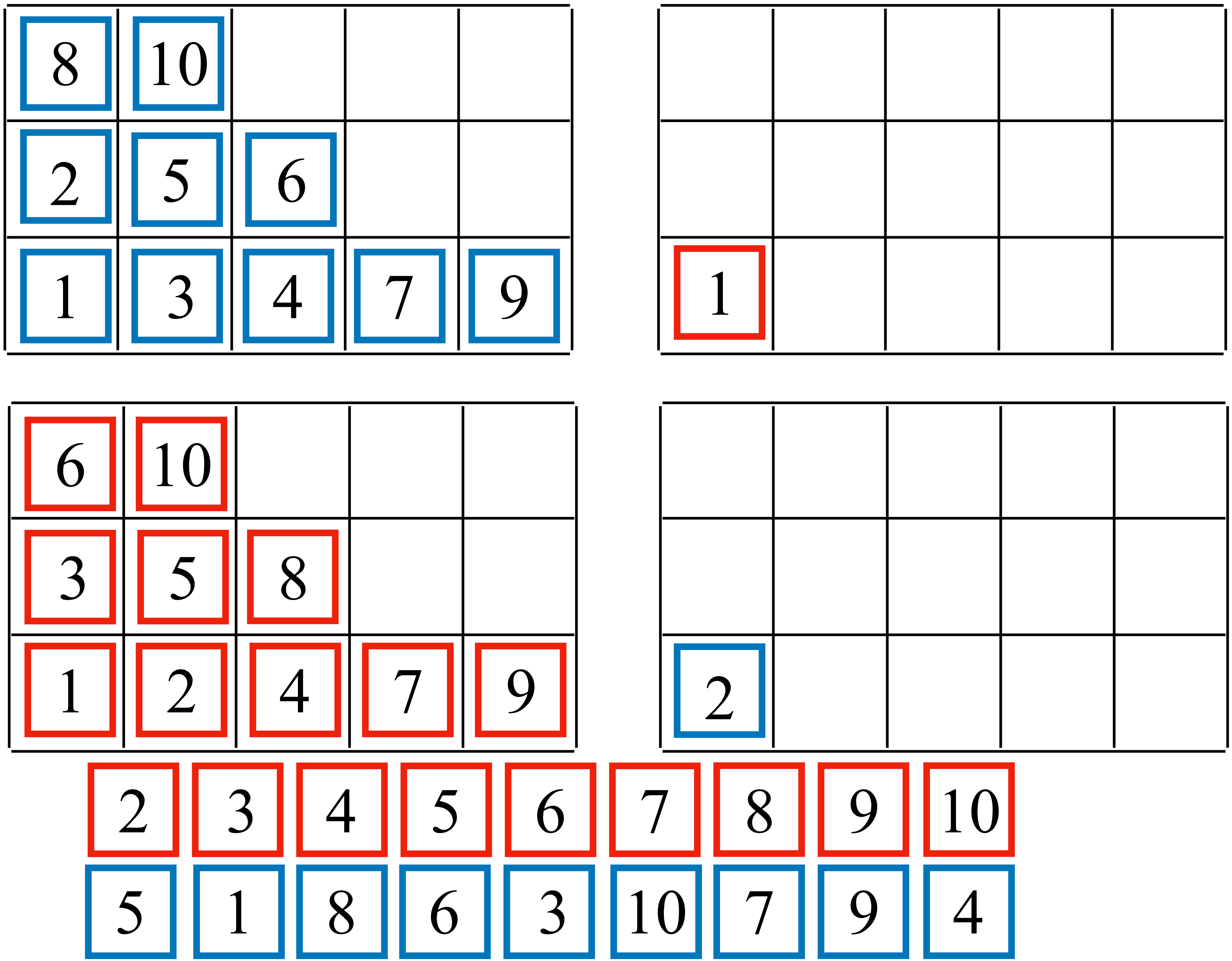
8	10			
2	5	6		
1	3	4	7	9

1	2			

6	10			
3	5	8		
1	2	4	7	9

2	5			

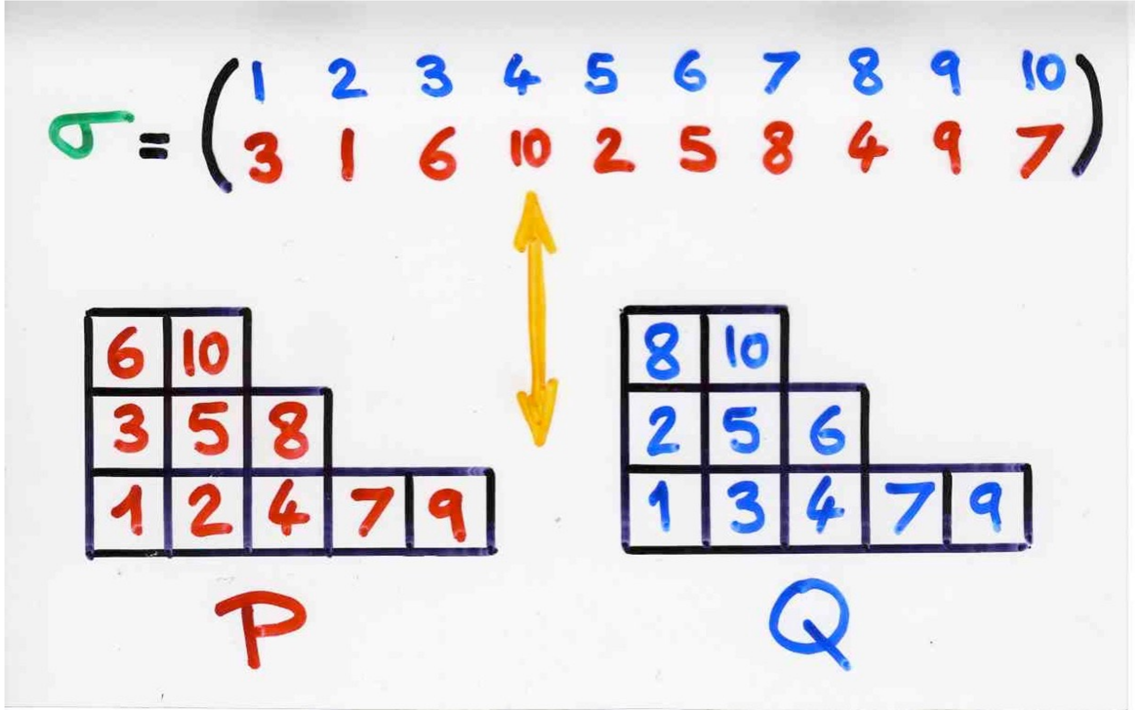
3	4	5	6	7	8	9	10
1	8	6	3	10	7	9	4


8 10
2 5 6
1 3 4 7 9
1
6 10
3 5 8
1 2 4 7 9
2
2 3 4 5 6 7 8 9 10
5 1 8 6 3 10 7 9 4

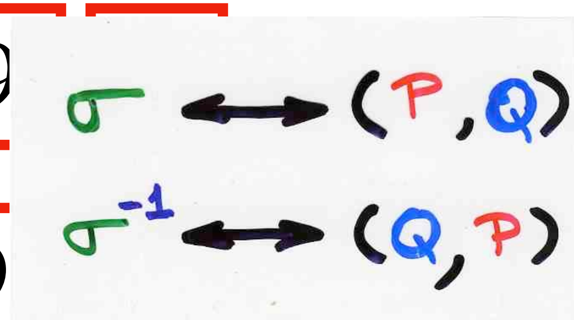
8	10			
2	5	6		
1	3	4	7	9

6	10			
3	5	8		
1	2	4	7	9

1	2	3	4	5	6	7	8	9	10
2	5	1	8	6	3	10	7	9	4



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	



1	2	3	4	5	6	7	8	9	
2	5	1	8	6	3	10	7	9	4

Happy 80th birthday Don!

Fantasia Apocalyptica   Piteå, Sweden

"The unusual nature of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scene"

D. Knuth (1972)

The art of computer programming

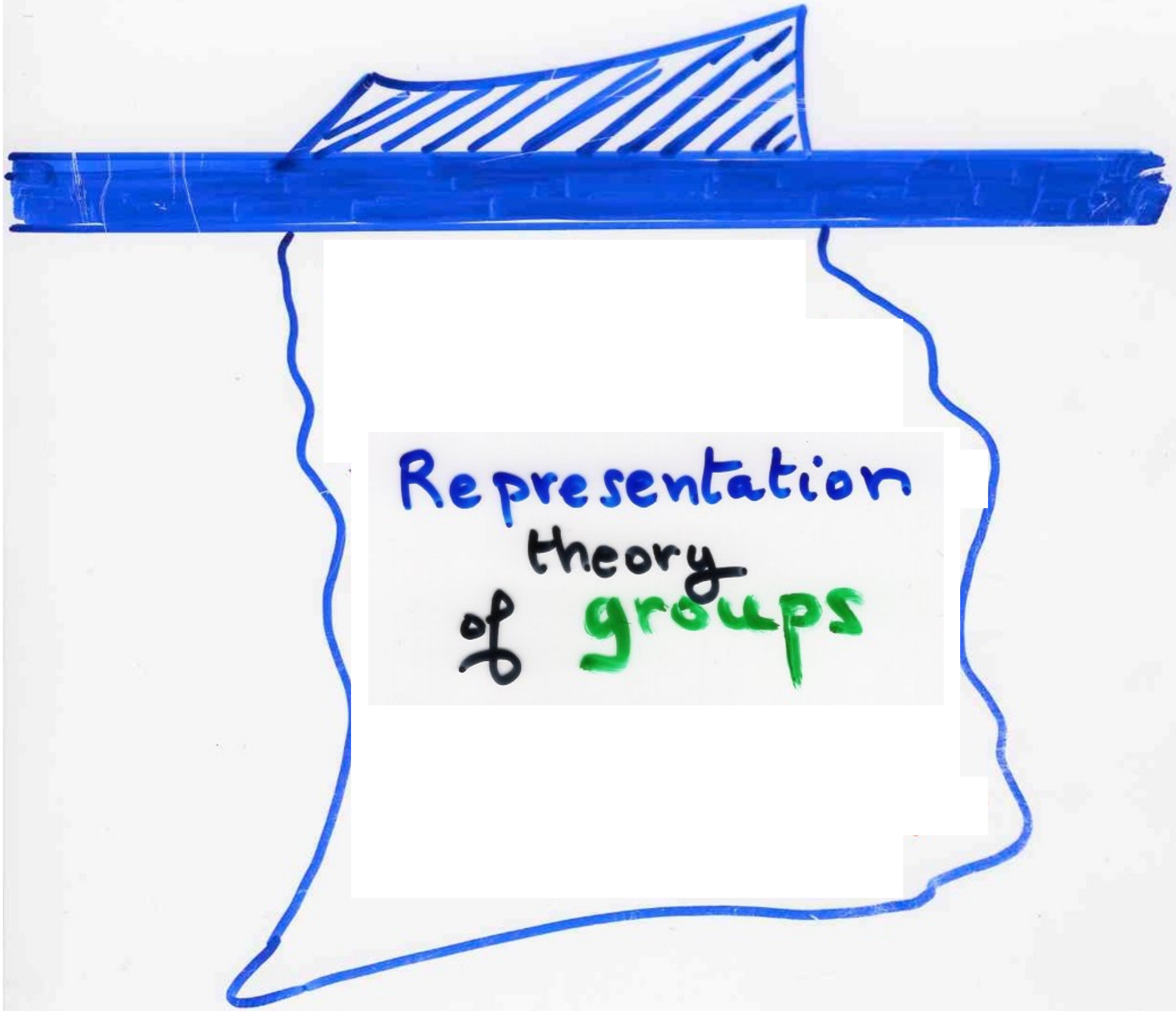
Vol. 3

more about groups theory

The group of permutations

The Robinson-Schensted correspondence

The Robinson-Schensted correspondence



Representation theory of groups

see a group G as a (sub)-group of matrices

$$G \to \text{Matrices}$$
$n \times n$, coeff. in $\mathbb{C}$

see G as a group of transformations

Important in Physics

standard model of particles

4 fundamental forces $\left\{\begin{array}{l}\text{electro-magnetism}\\ \text{strong}\\ \text{weak}\end{array}\right\}$ + gravity

for every group

representation $\xrightarrow[\text{decomposition}]{}$ into irreducible representations

analogy [every number n = $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

prime numbers decomposition

<u>Case of the group</u> $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) = f_λ

number of Young tableaux with shape λ

finite group G

$$|G| = \sum_{R} (\deg R)^2$$

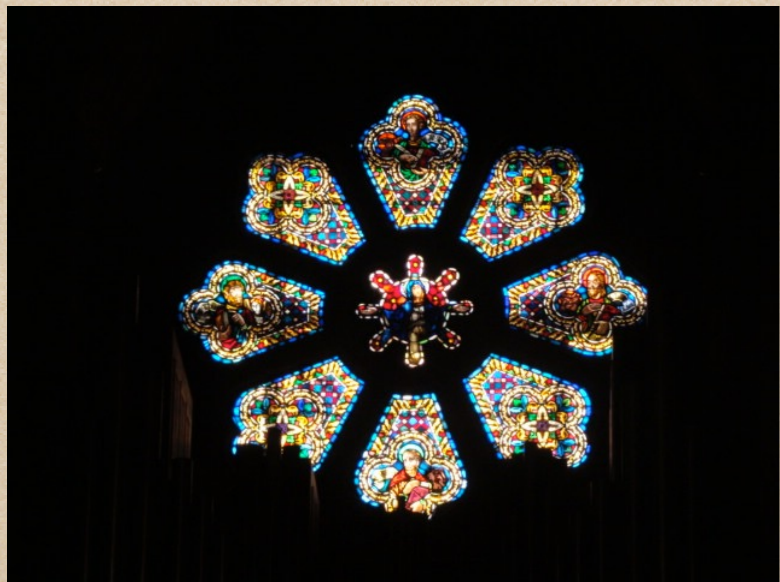
irreducible representation

for the symmetric group $\mathfrak{S}_n$ (permutations)

$$n! = \sum_{\lambda} (f_\lambda)^2$$

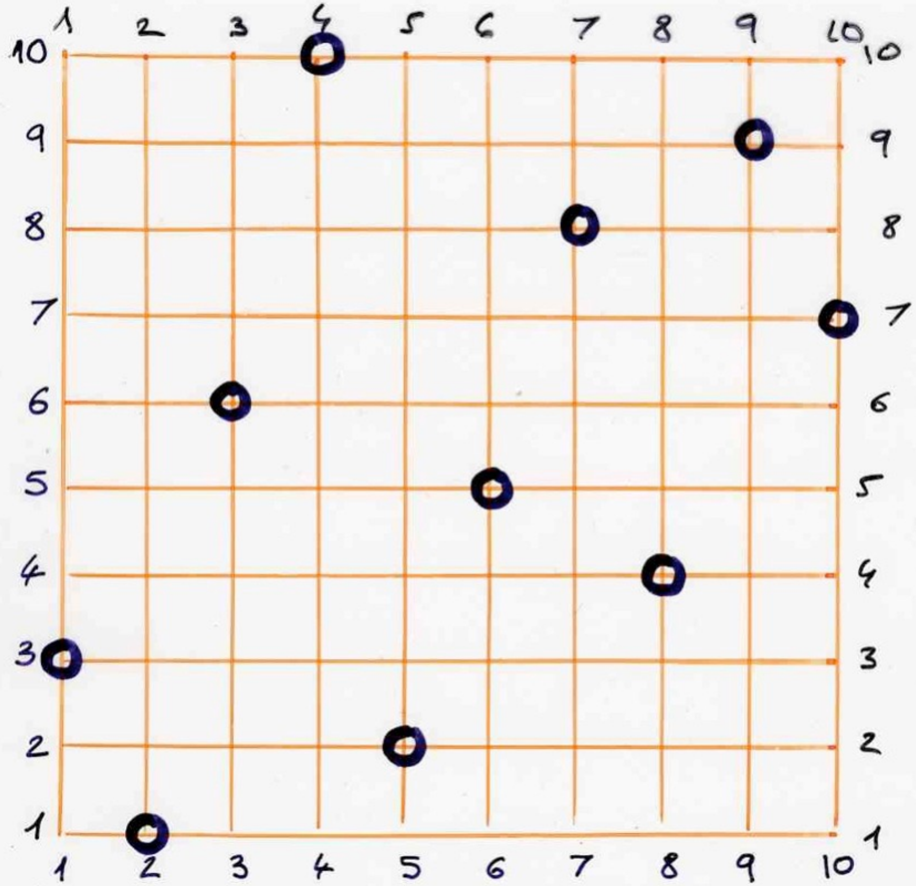
partition of n

A geometric version of RSK with "light" and "shadow lines"



X.V. 1976

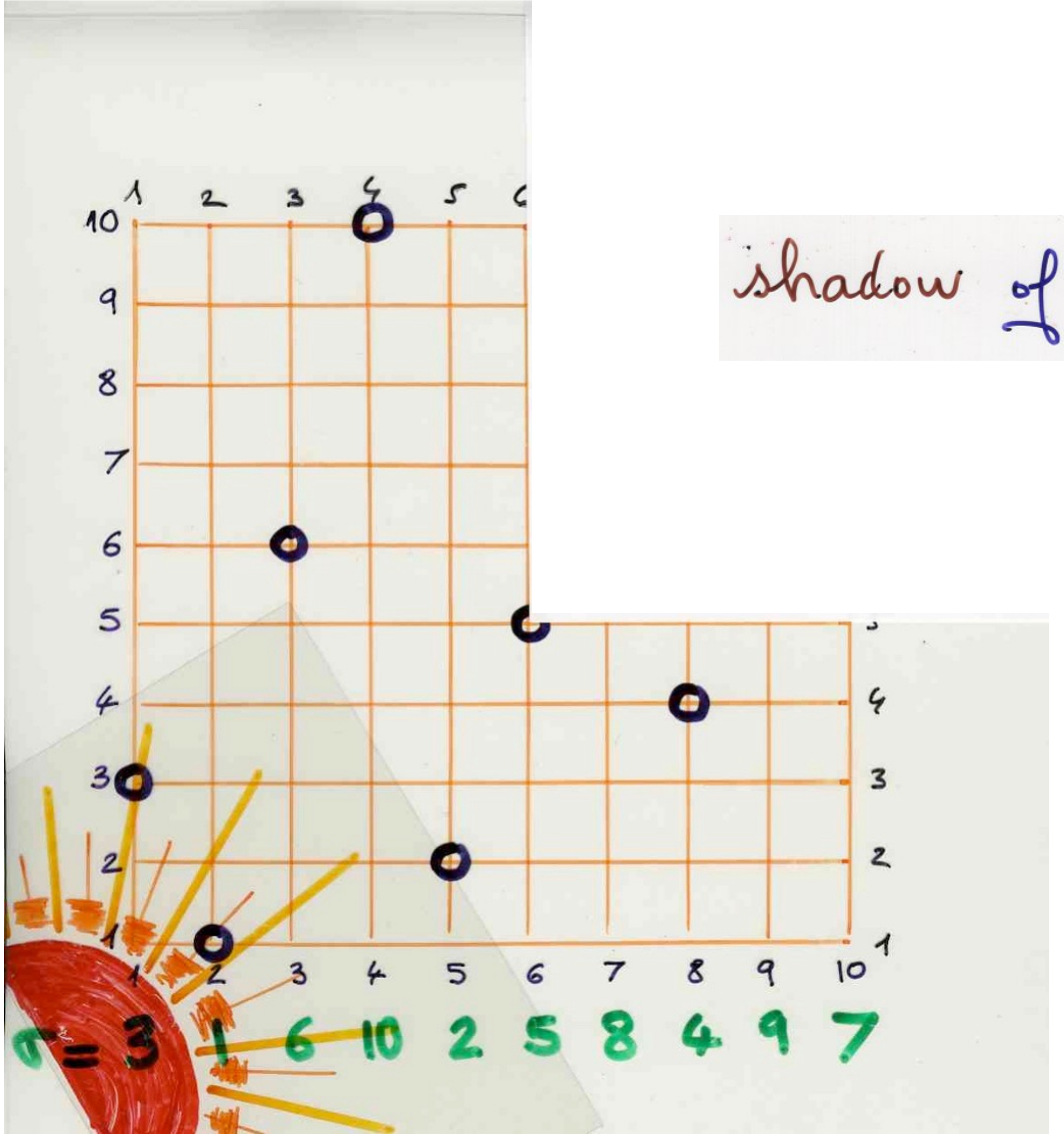
$$\{(i,\sigma(i))\}_{i=1,\dots,n} \subseteq [1,n]\times[1,n]$$



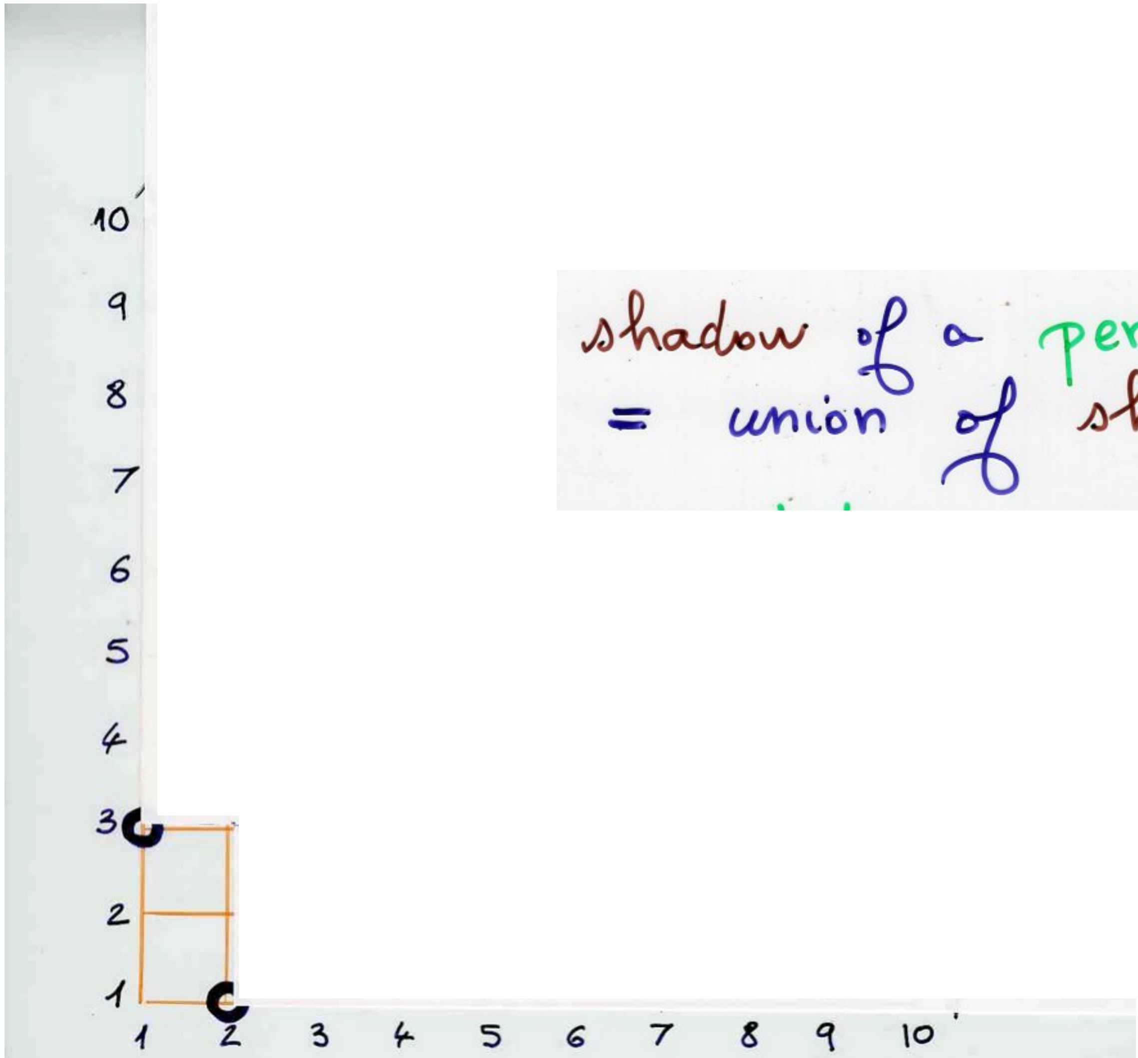
$$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$$

graph of a permutation σ

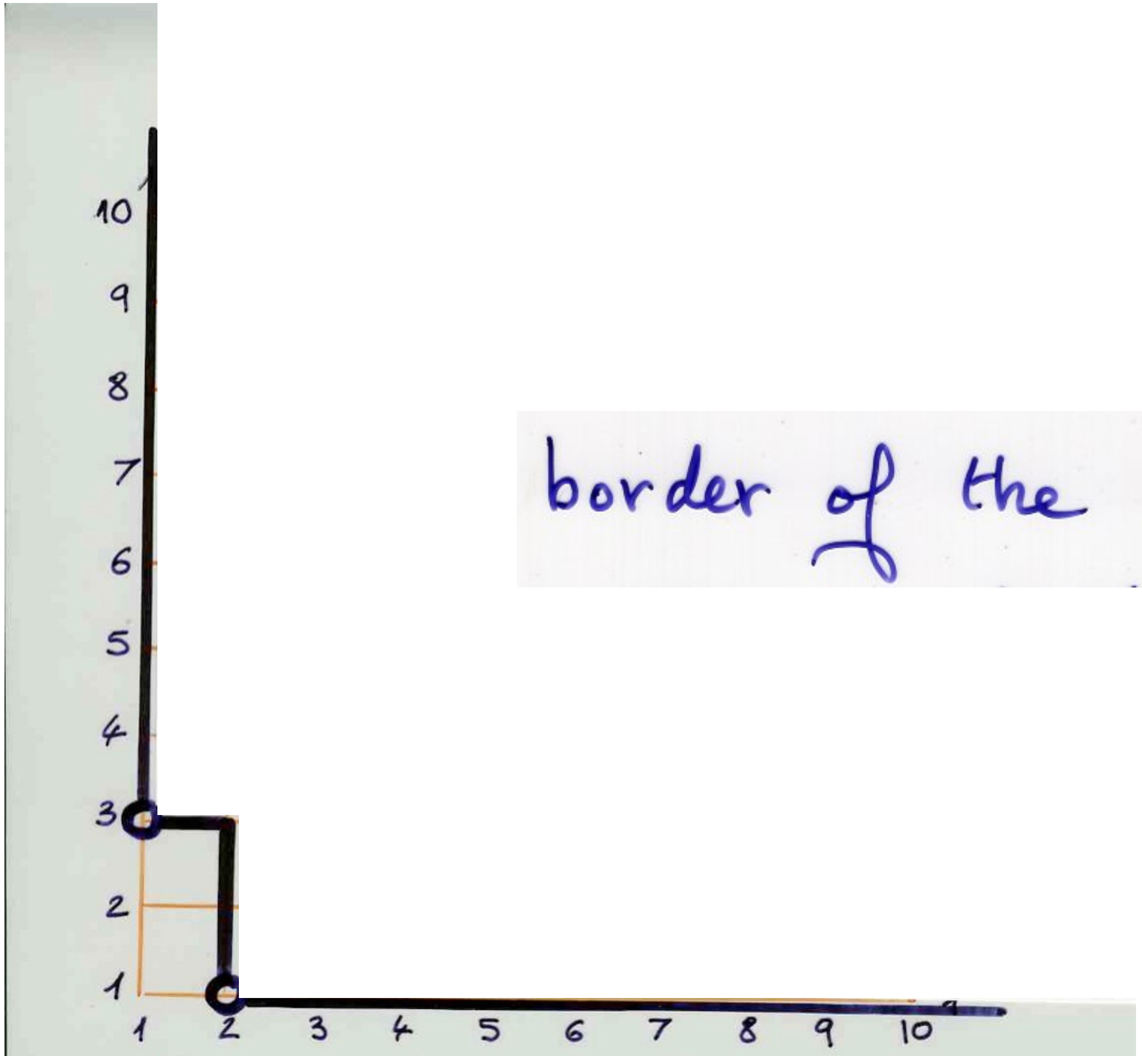
shadow of a point



shadow of a permutation
= union of shadows

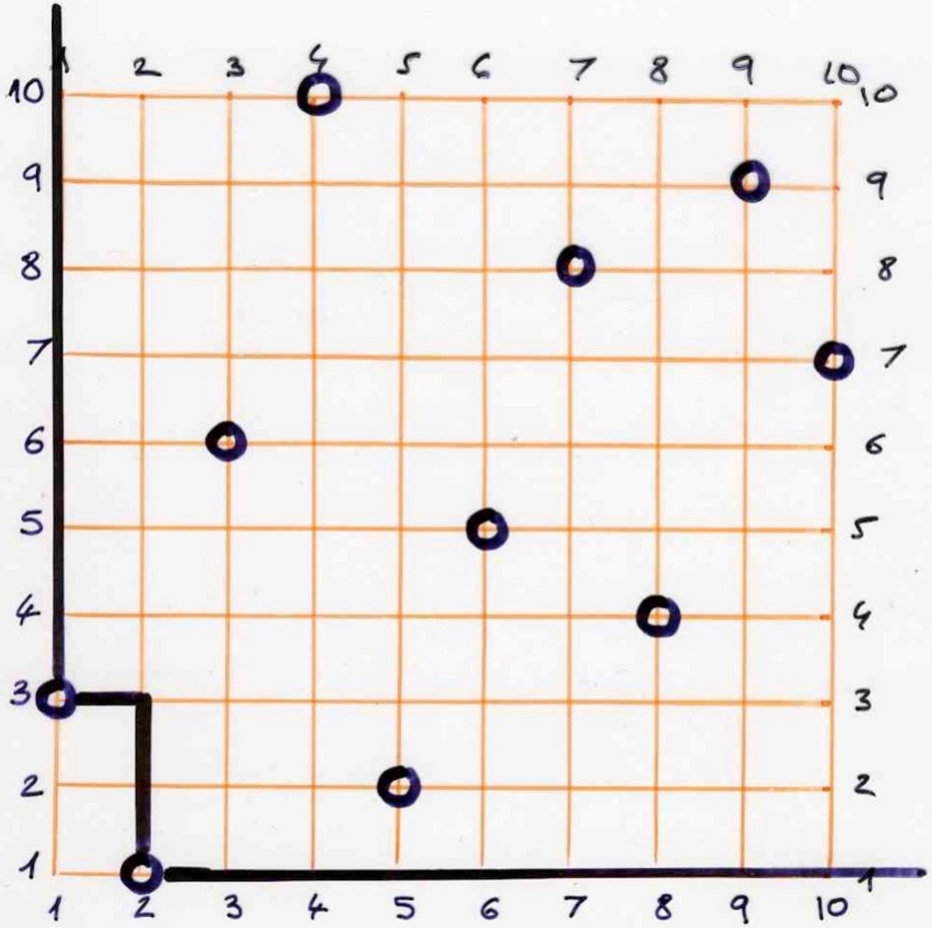


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

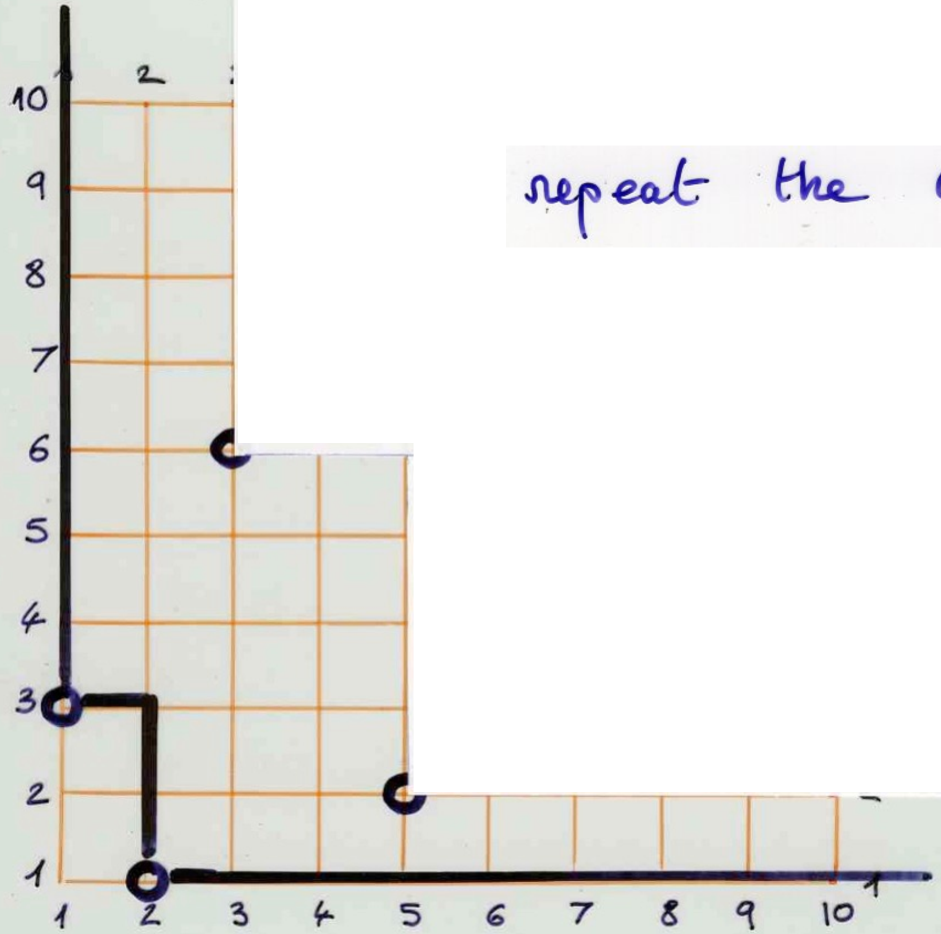


border of the shadow

$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

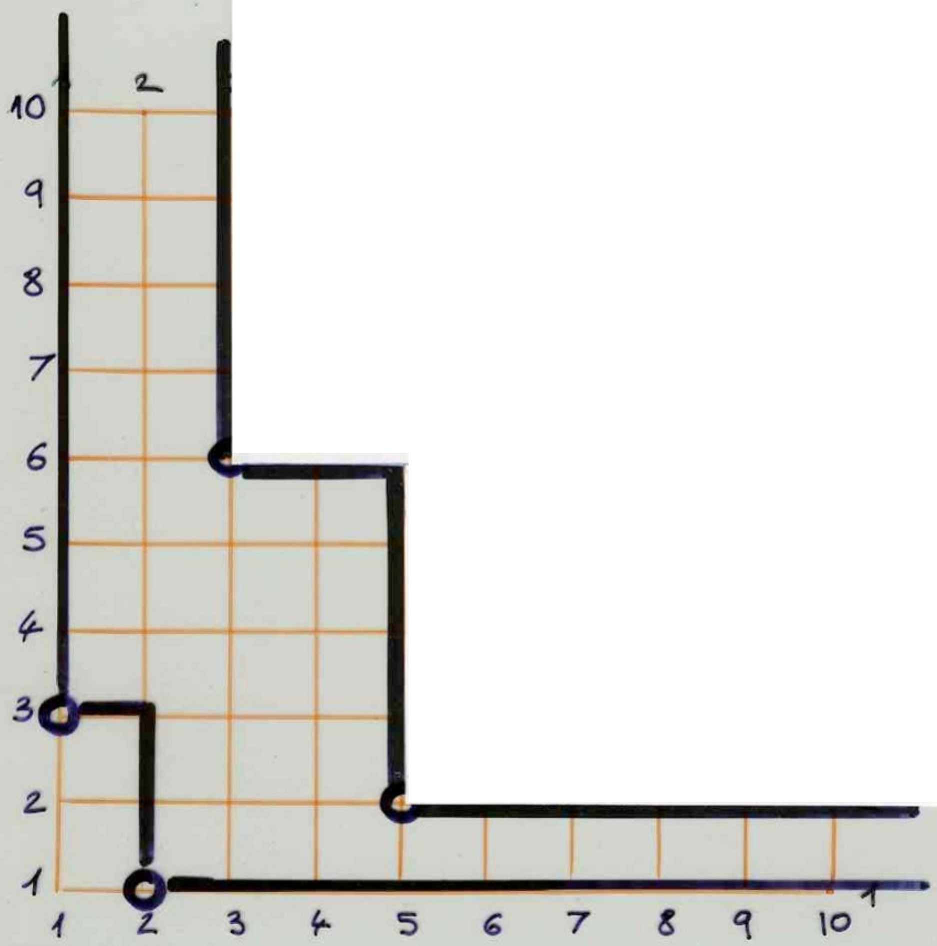


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

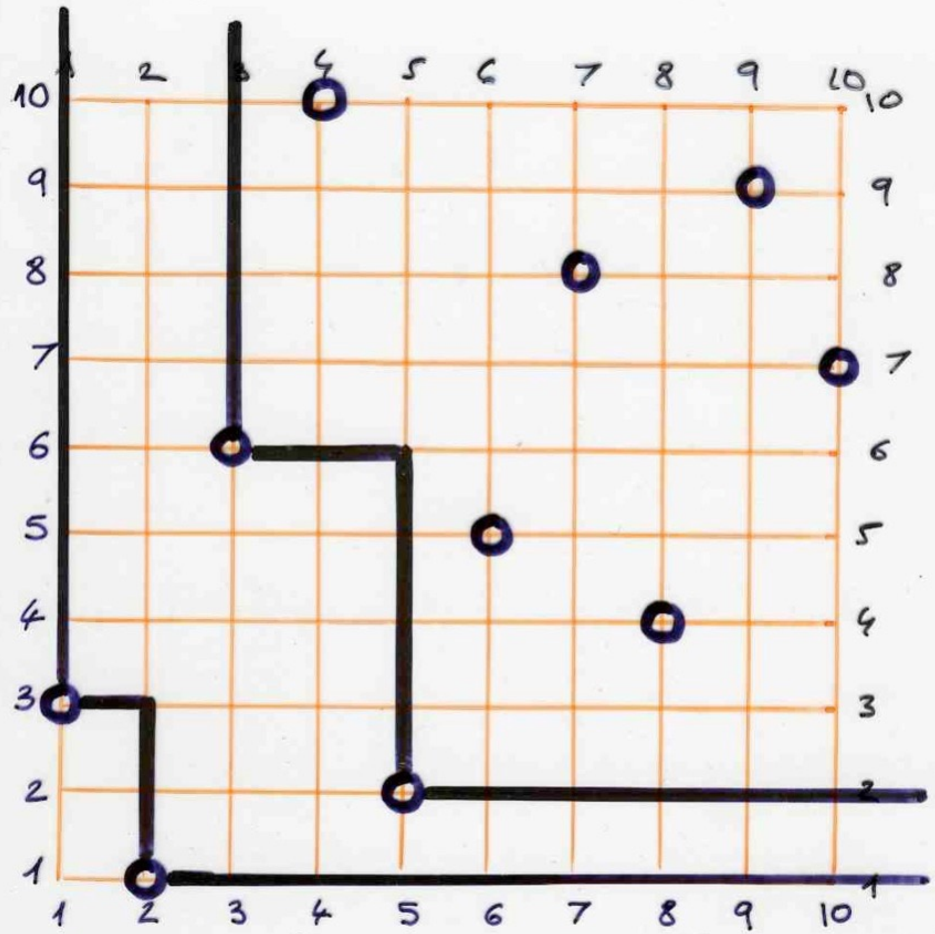


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

repeat the construction ...

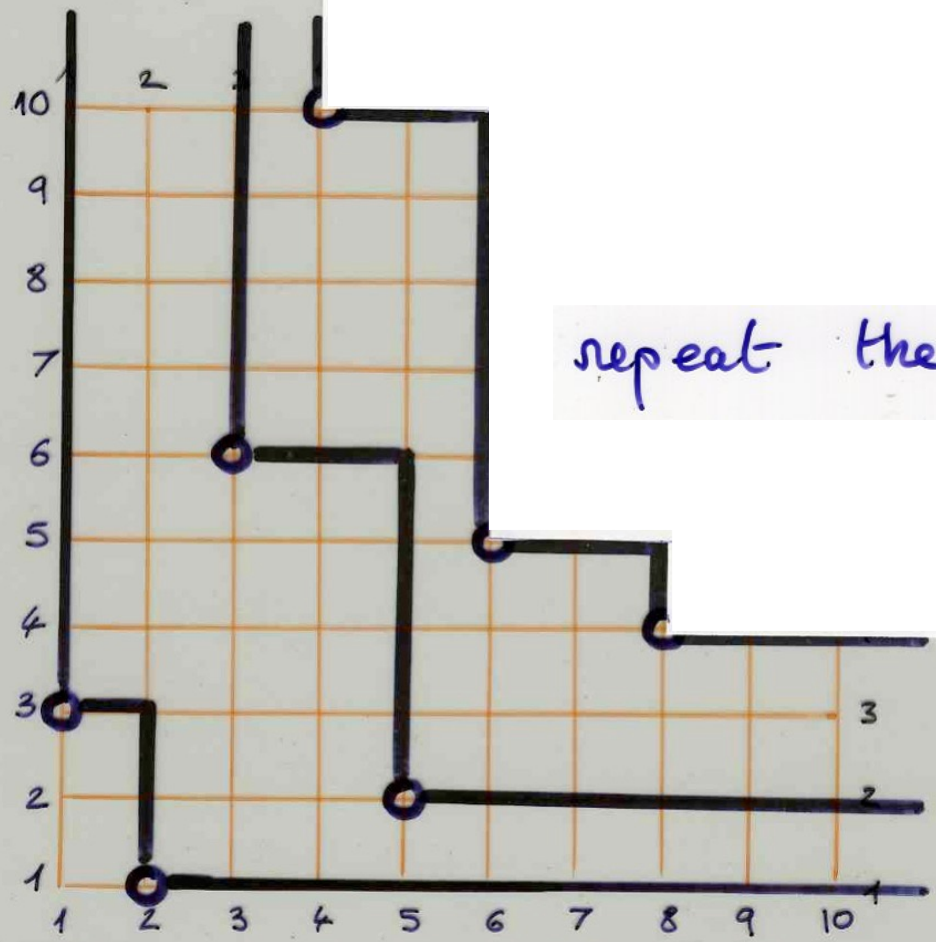


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

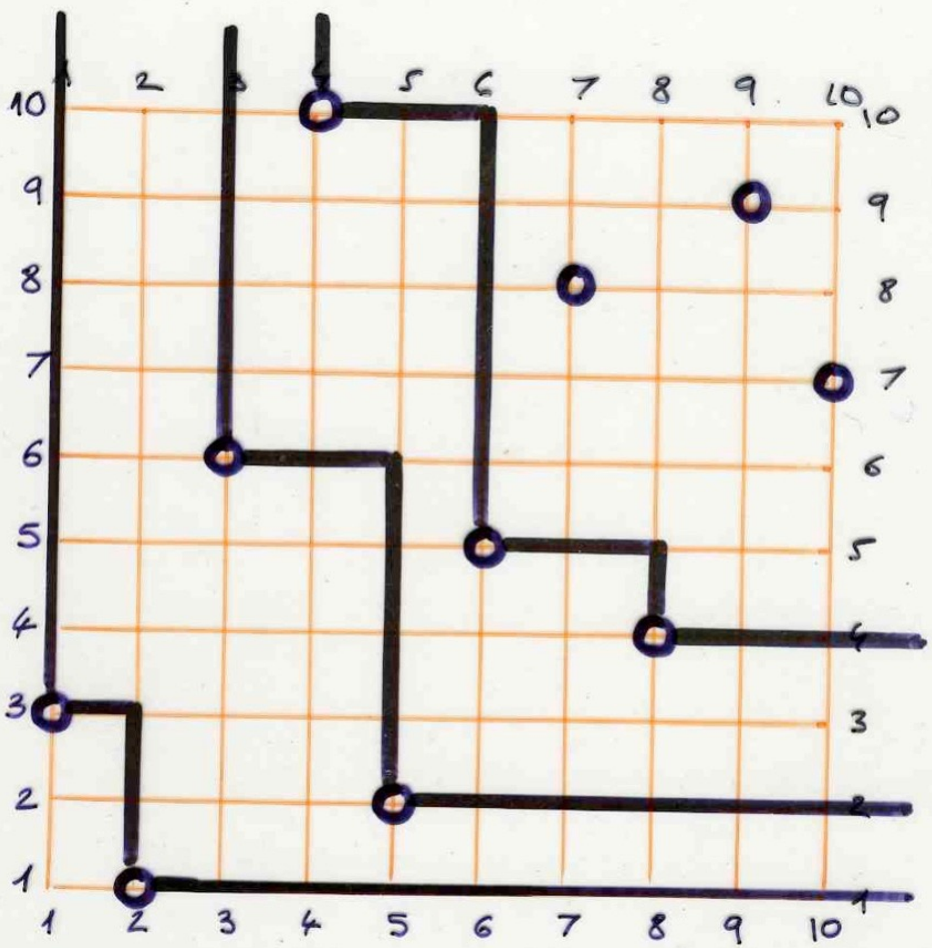


$\pi = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

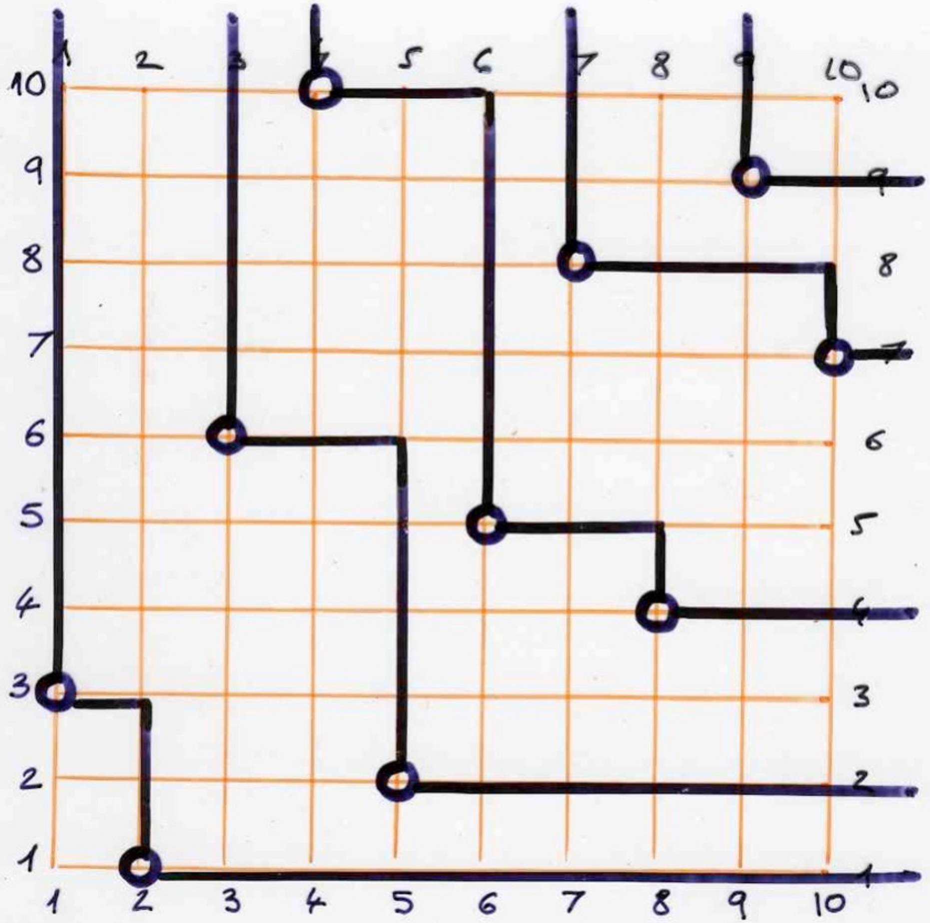
repeat the construction ...



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

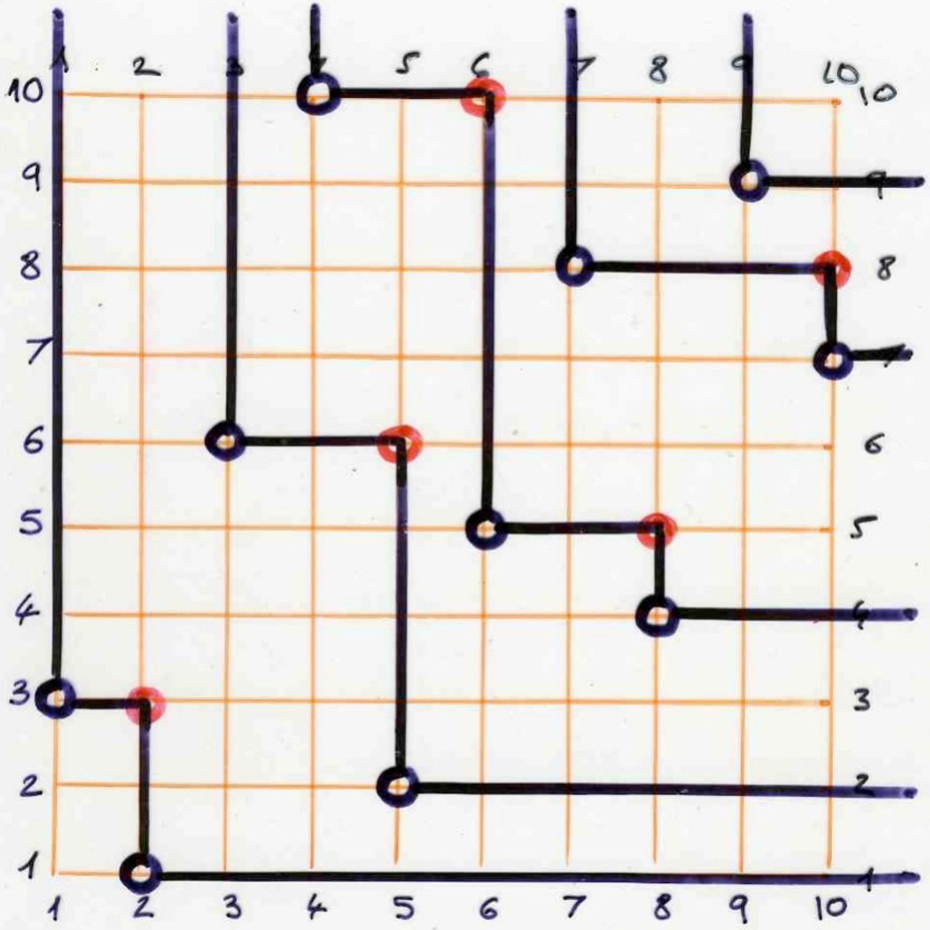


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

red points

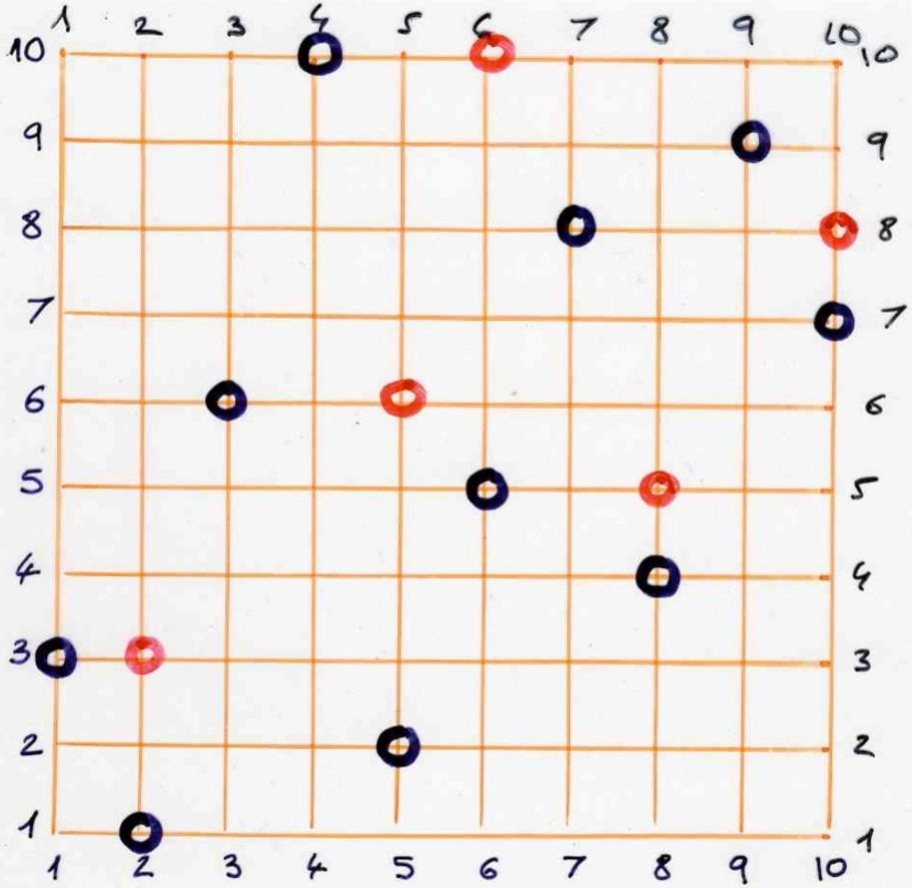


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

red points

skeleton

Sq (σ)



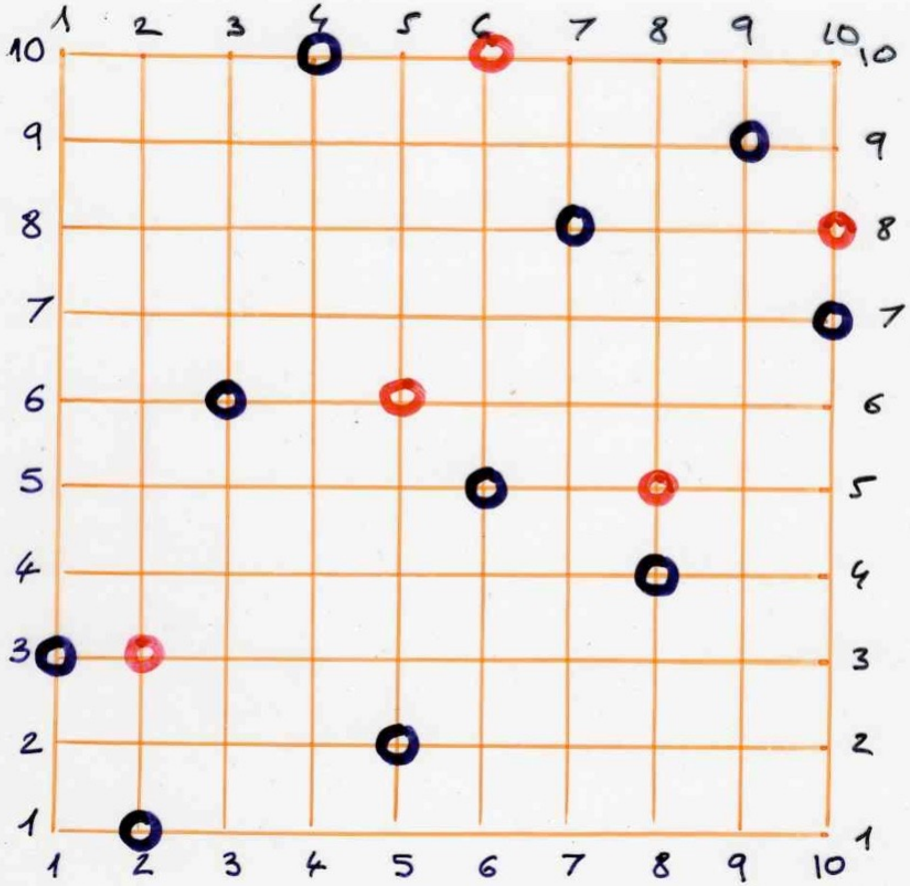
σ = 3 1 6 10 2 5 8 4 9 7

Lemma The skeleton $Sq(\sigma)$ is a "coding" of the permutation σ

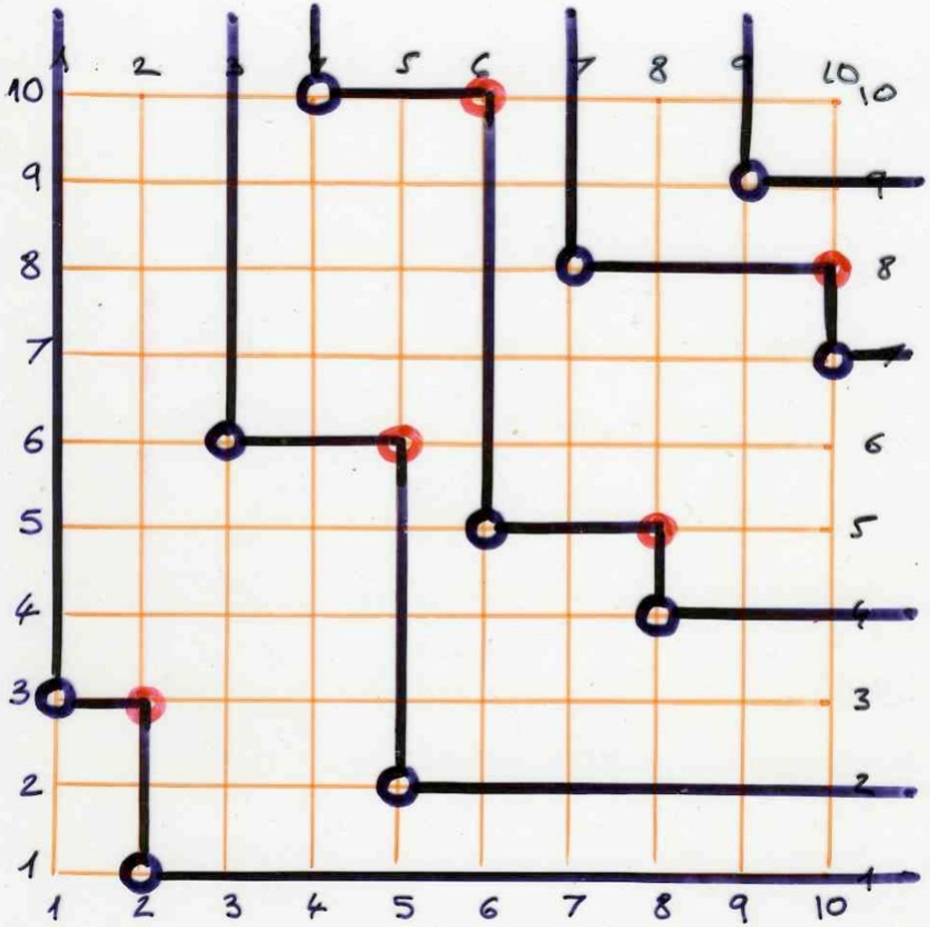
$$\sigma \longleftrightarrow Sq(\sigma) \subseteq [n] \times [n]$$

permutation on $[1, n]$

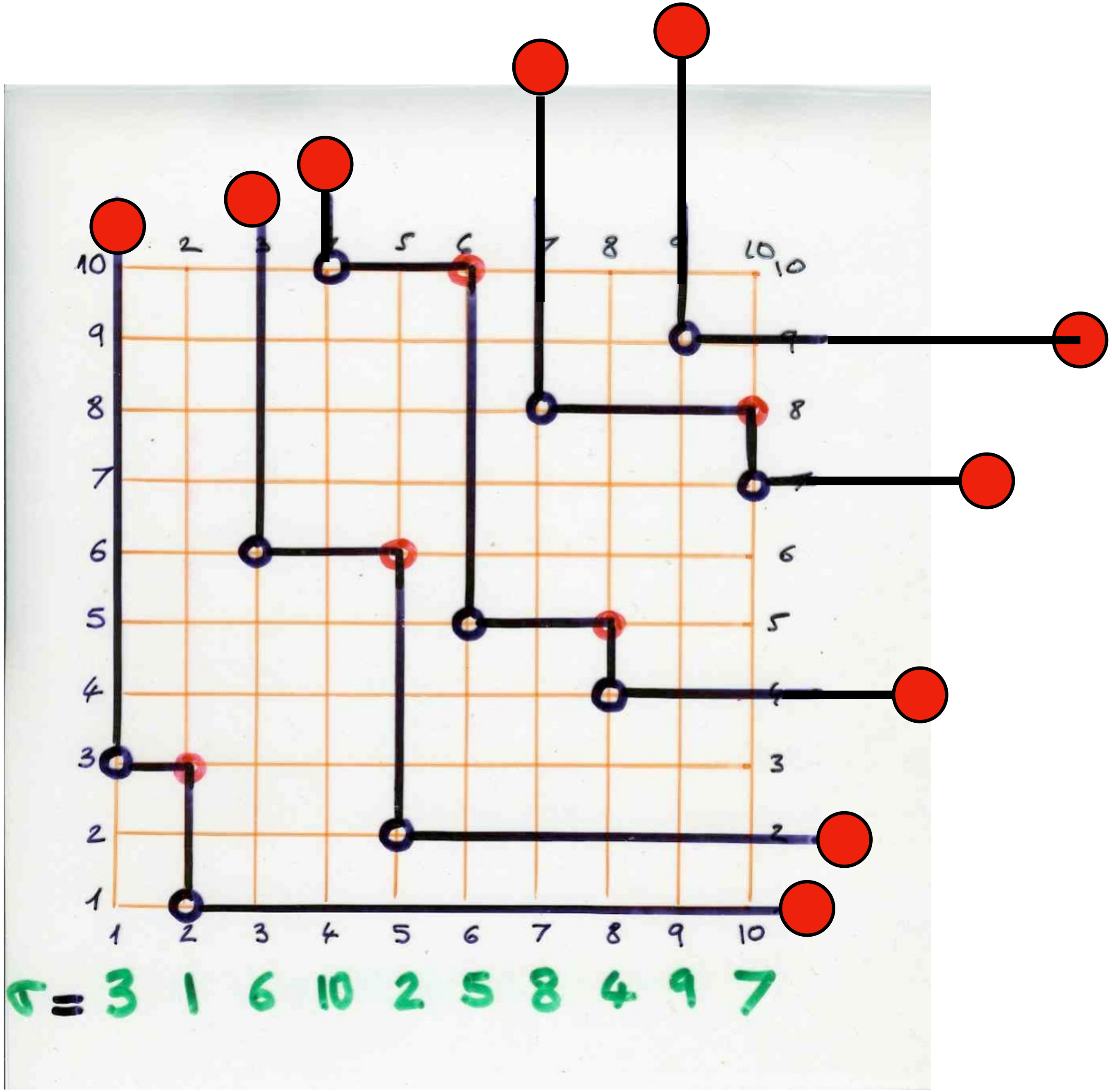
Proof:

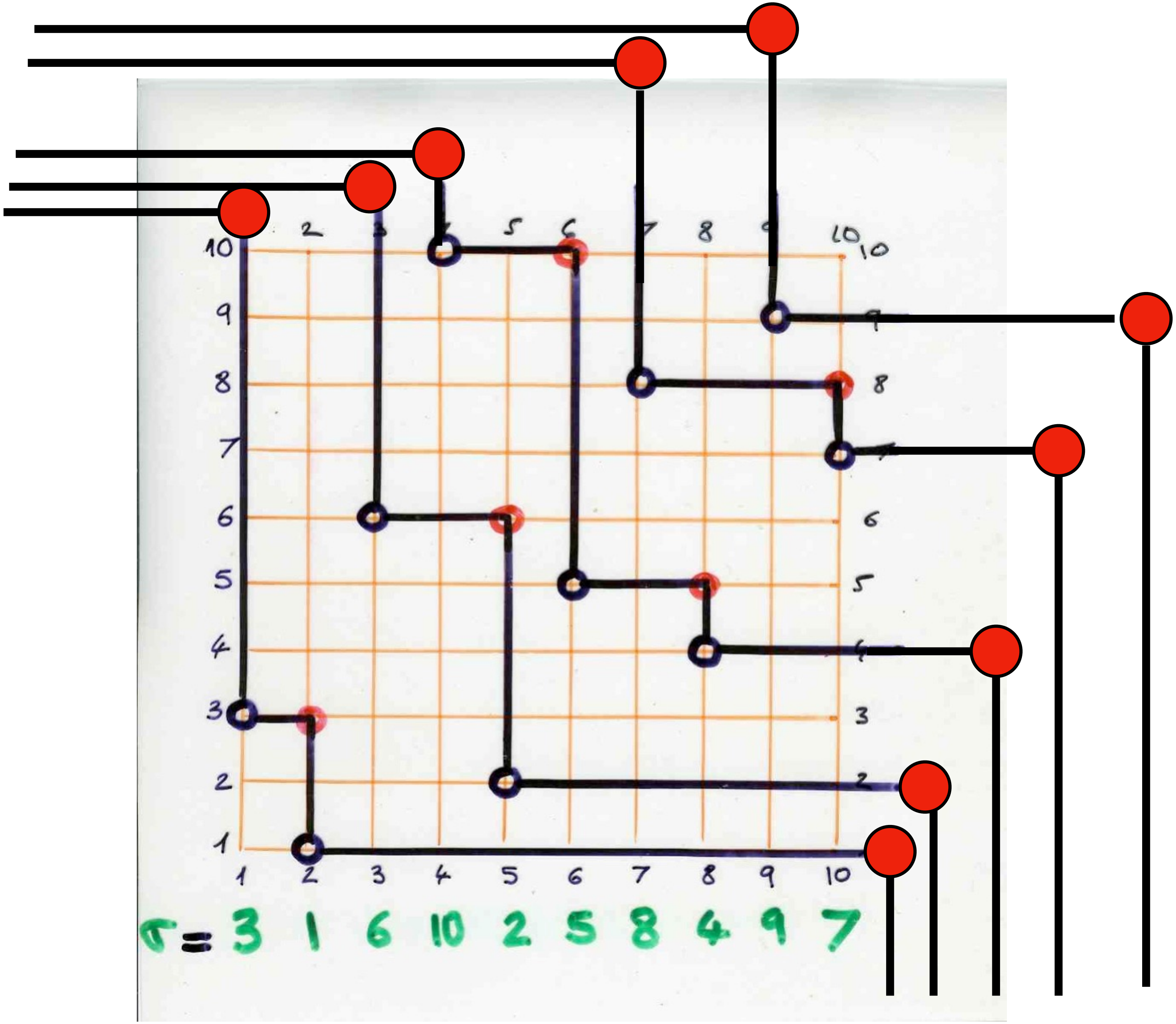


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

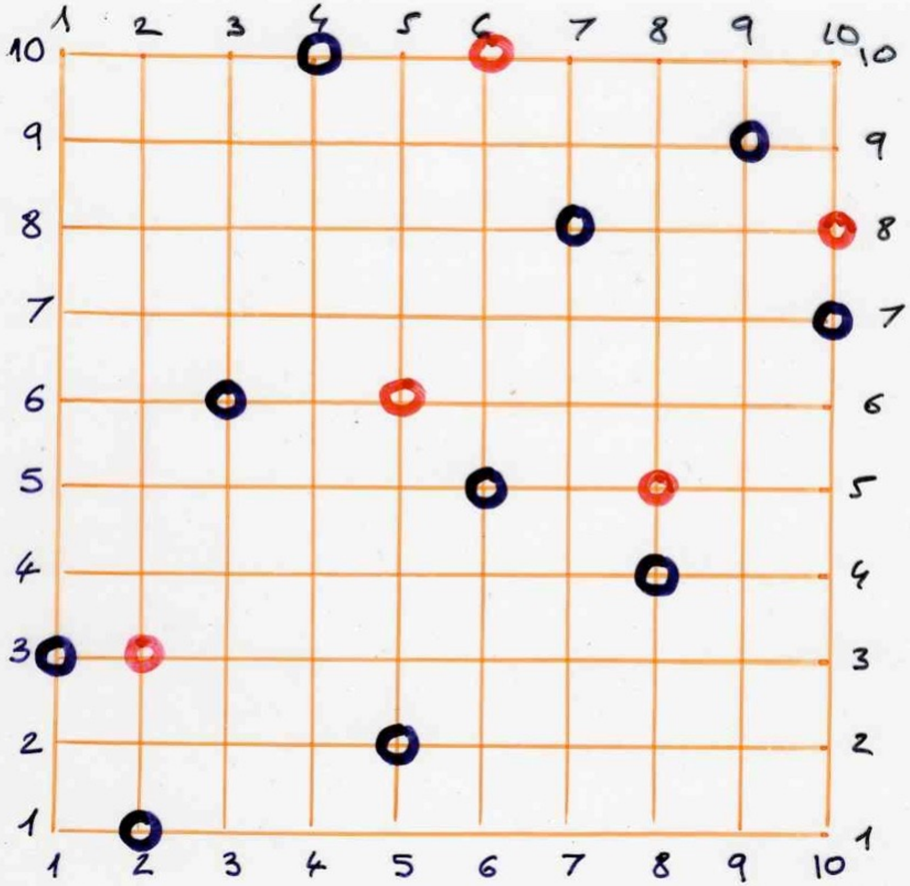


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

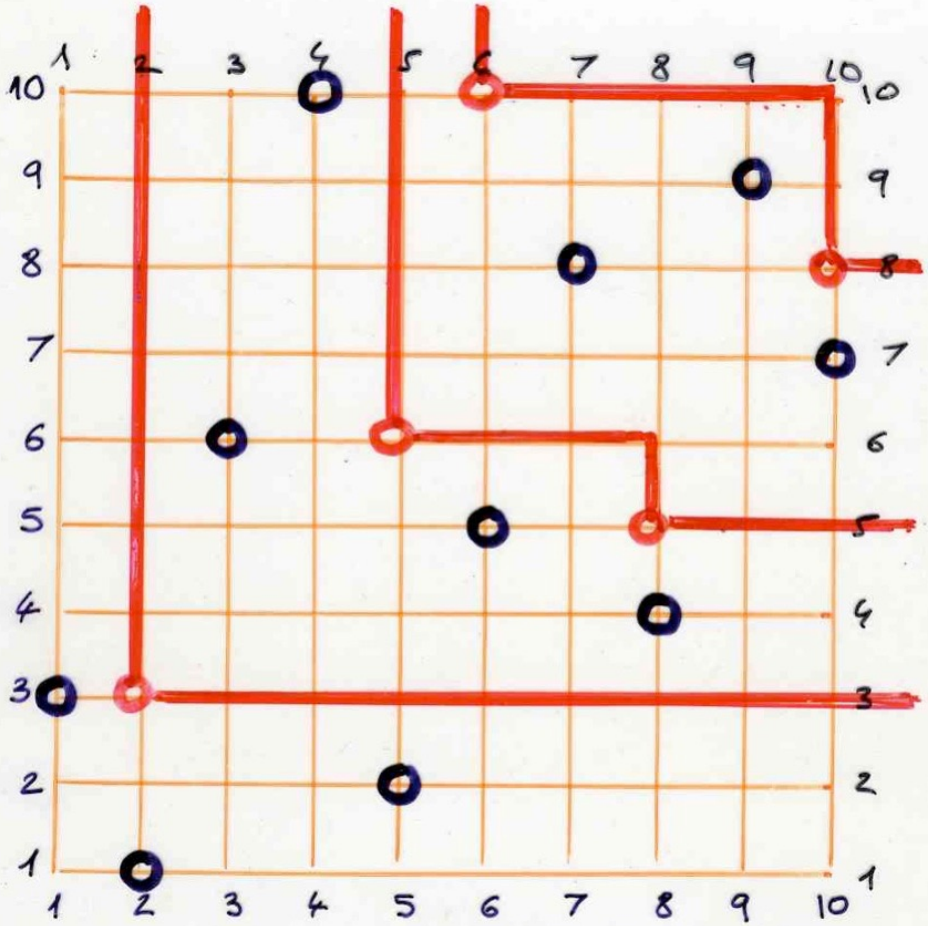

σ = 3 1 6 10 2 5 8 4 9 7


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7

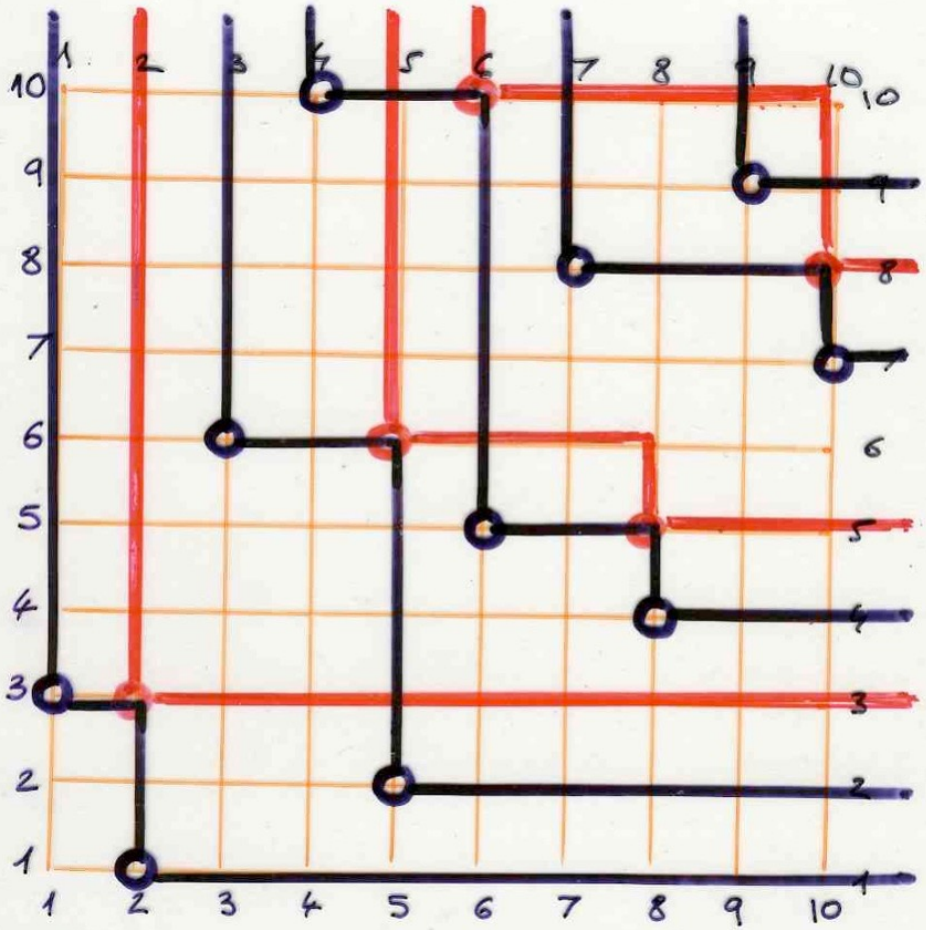
repeat with the red points
the construction of successives shadows



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

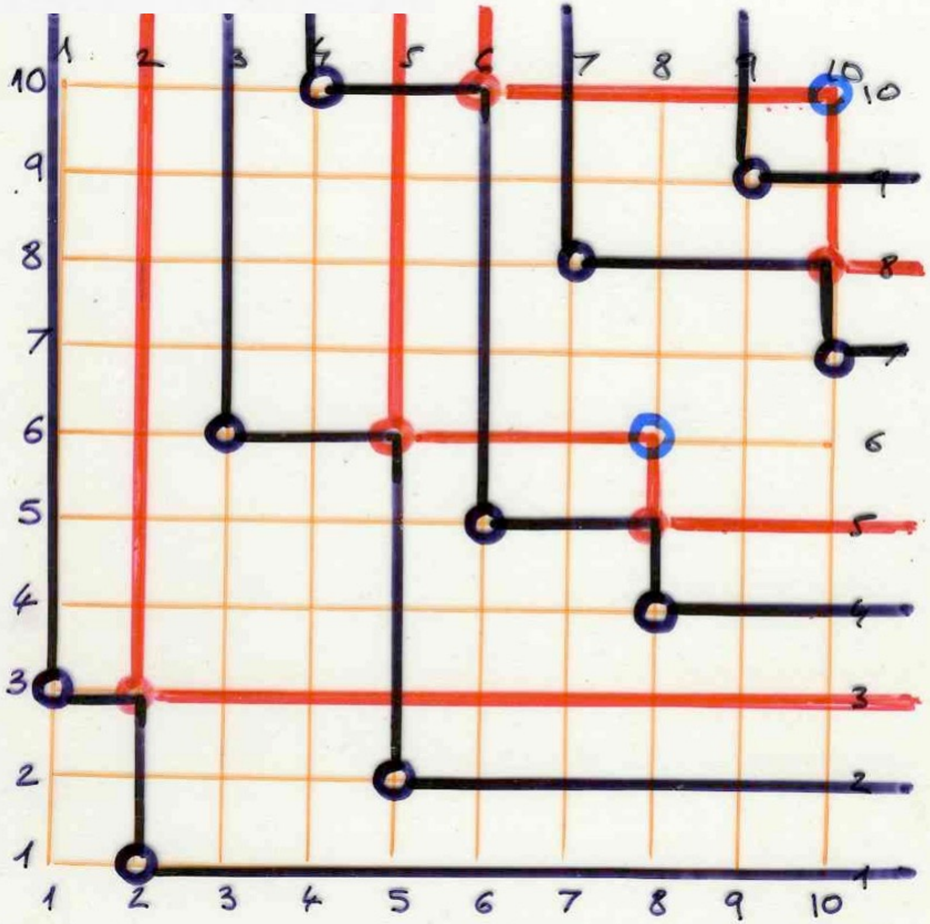


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



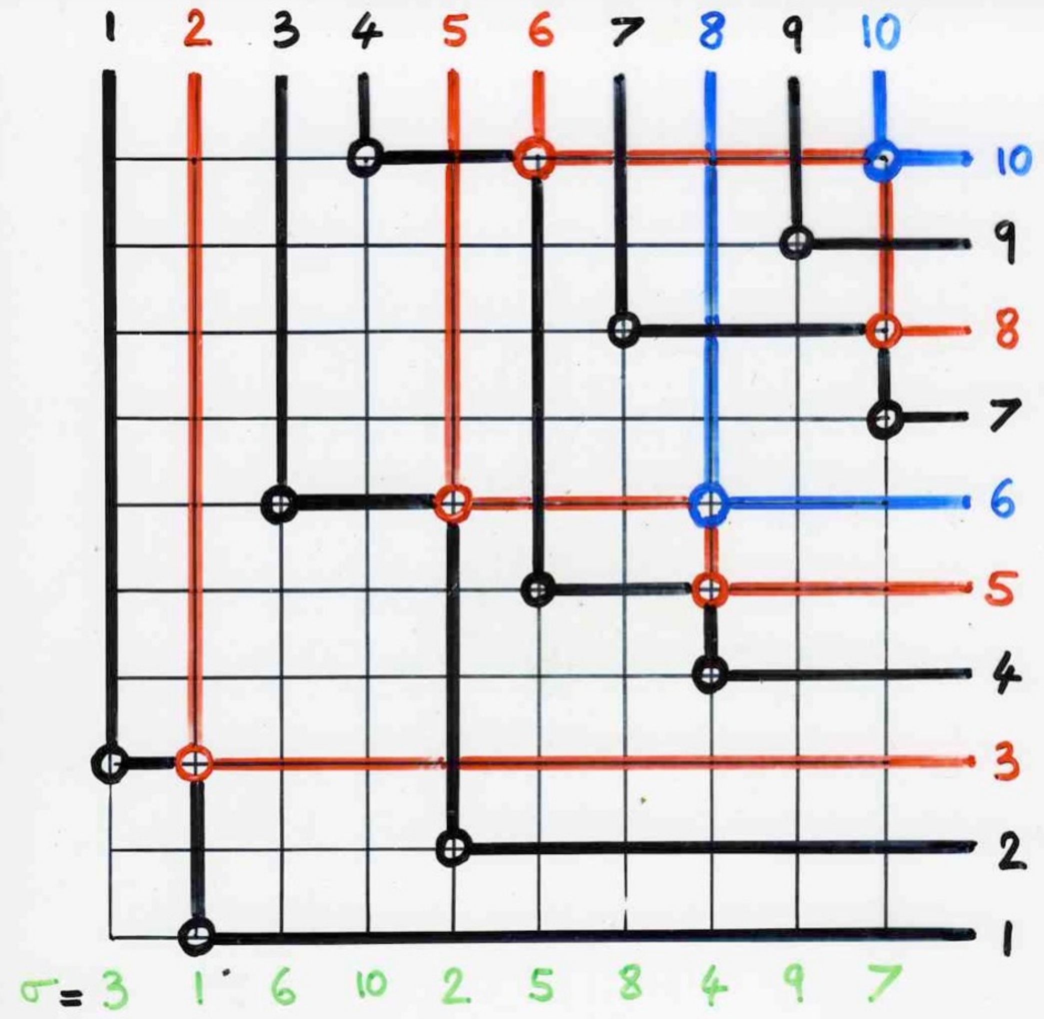
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

blue points

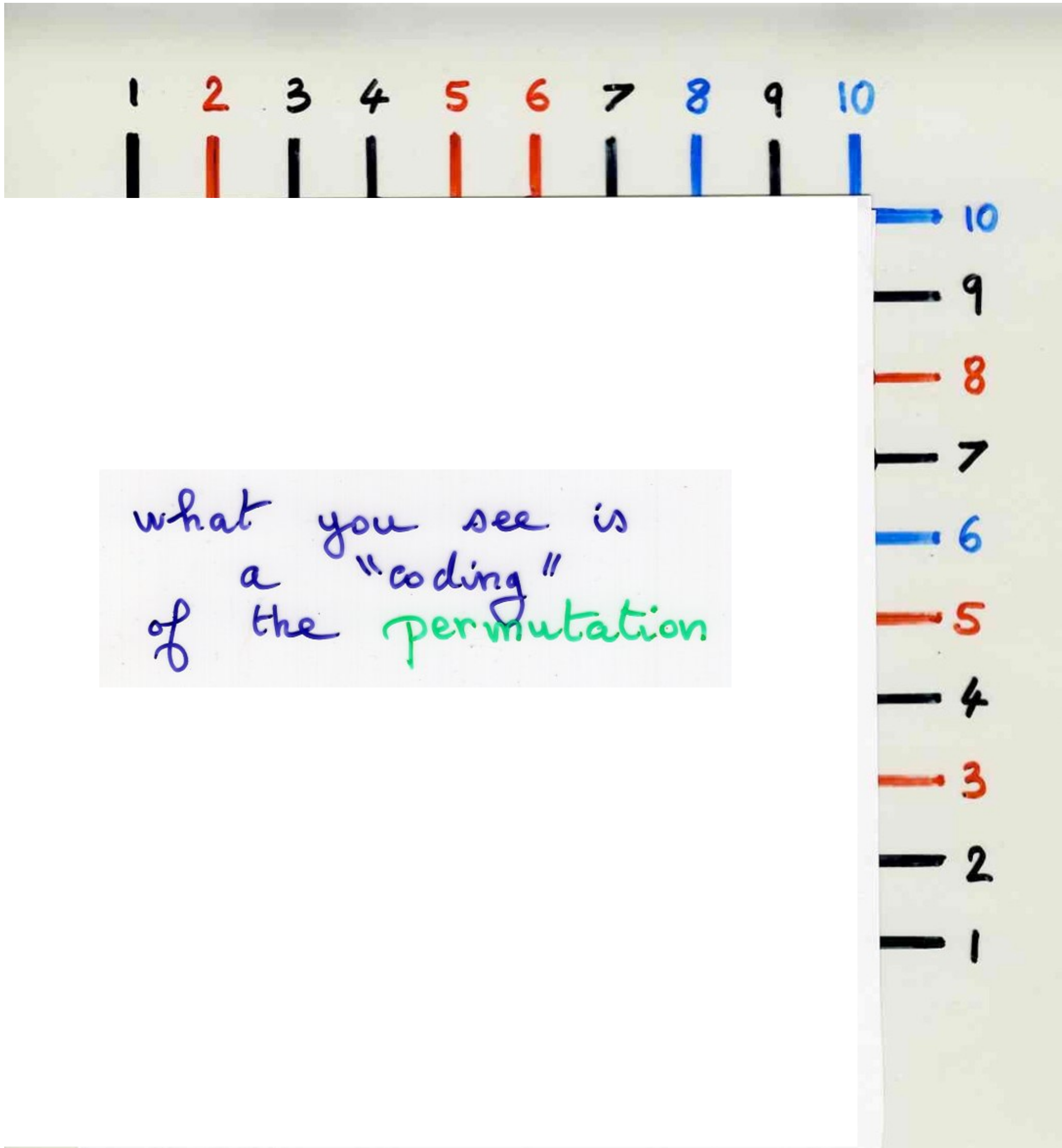


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

no green points



end of the construction



what you see is a "coding" of the permutation

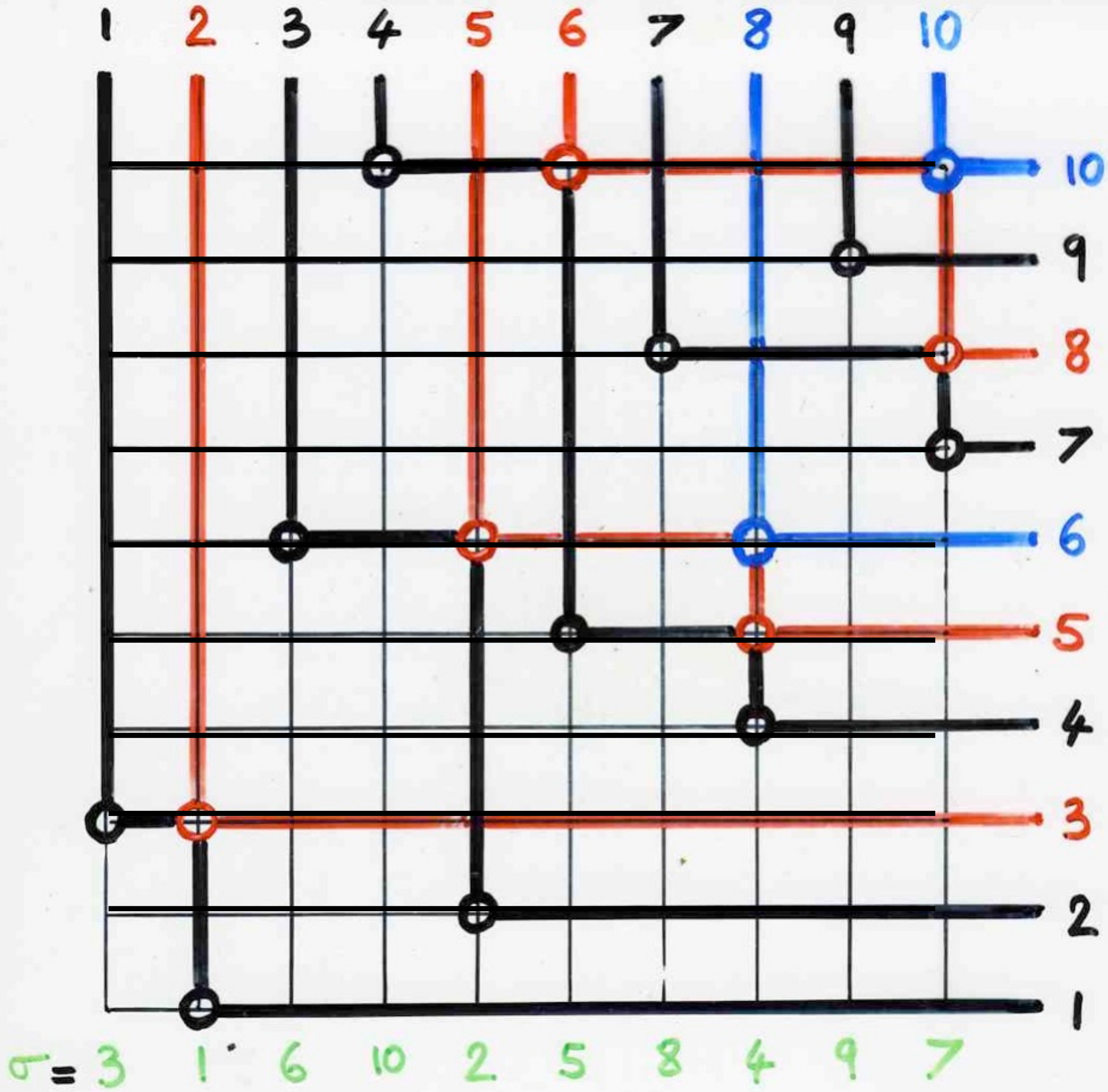
<u>Lemma</u> The skeleton $Sq(\sigma)$ is a "coding" of the permutation σ

$$\sigma \longleftrightarrow Sq(\sigma) \subseteq [n] \times [n]$$

permutation on $[1, n]$

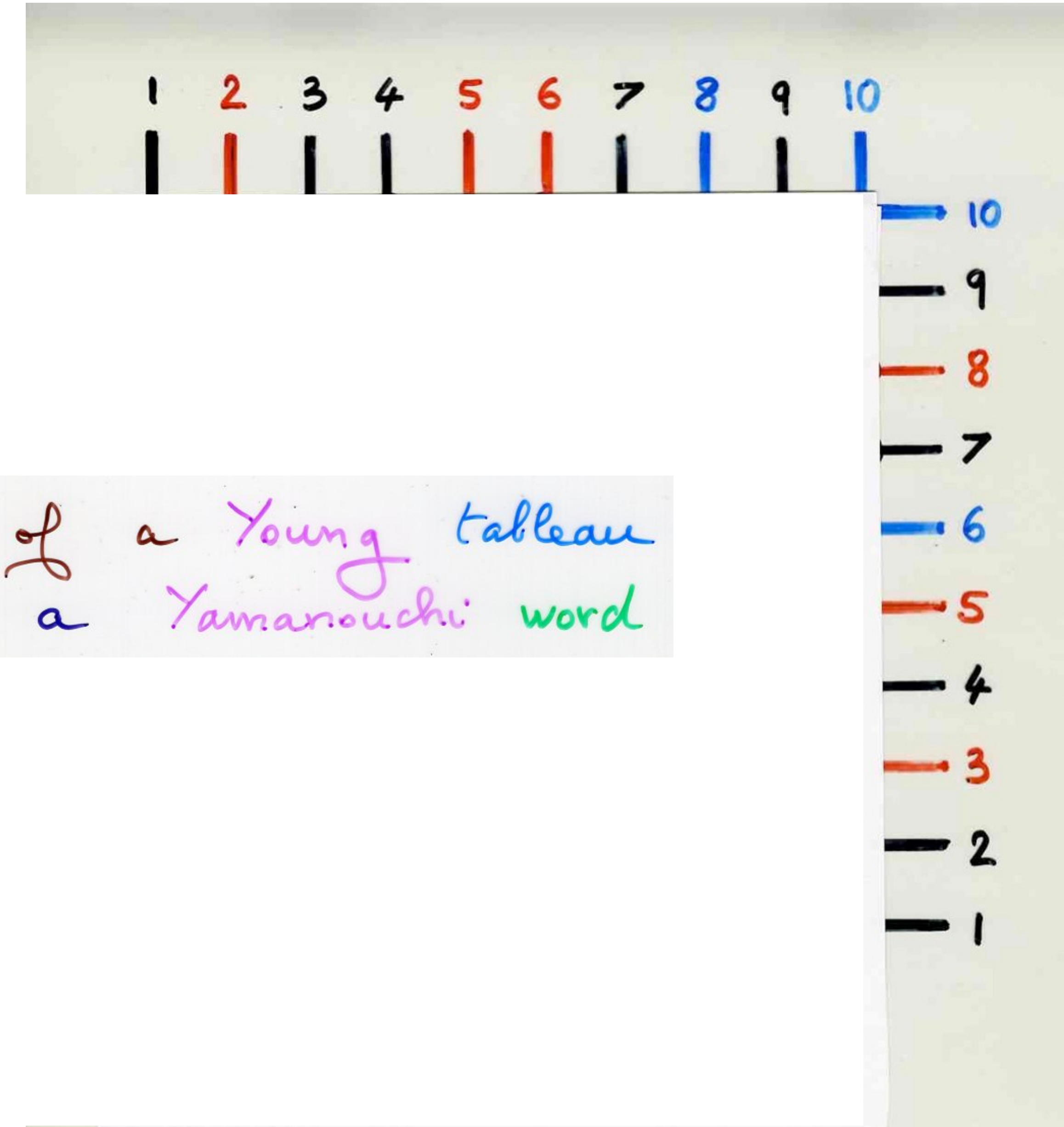
here: $\sigma \longrightarrow Sq(\sigma) \longrightarrow Sq(Sq(\sigma))$

$$\longrightarrow Sq(Sq(Sq(\sigma))) = \emptyset$$



1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7

coding of a Young tableau
with a Yamanouchi word



Definition Yamanouchi word w

$w \in \{1,2,\ldots\}^*$

free monoid generated by the alphabet $1,2,\ldots,$

such that :

for every factorization $w = uv$

$$|u|_1 \geq |u|_2 \geq \cdots \geq |u|_i \geq \cdots$$

number of occurrences of the letter i in u

coding of a Young tableau with a Yamanouchi word

(also called lattice permutation)

$W =$ | | | | | | | | | |

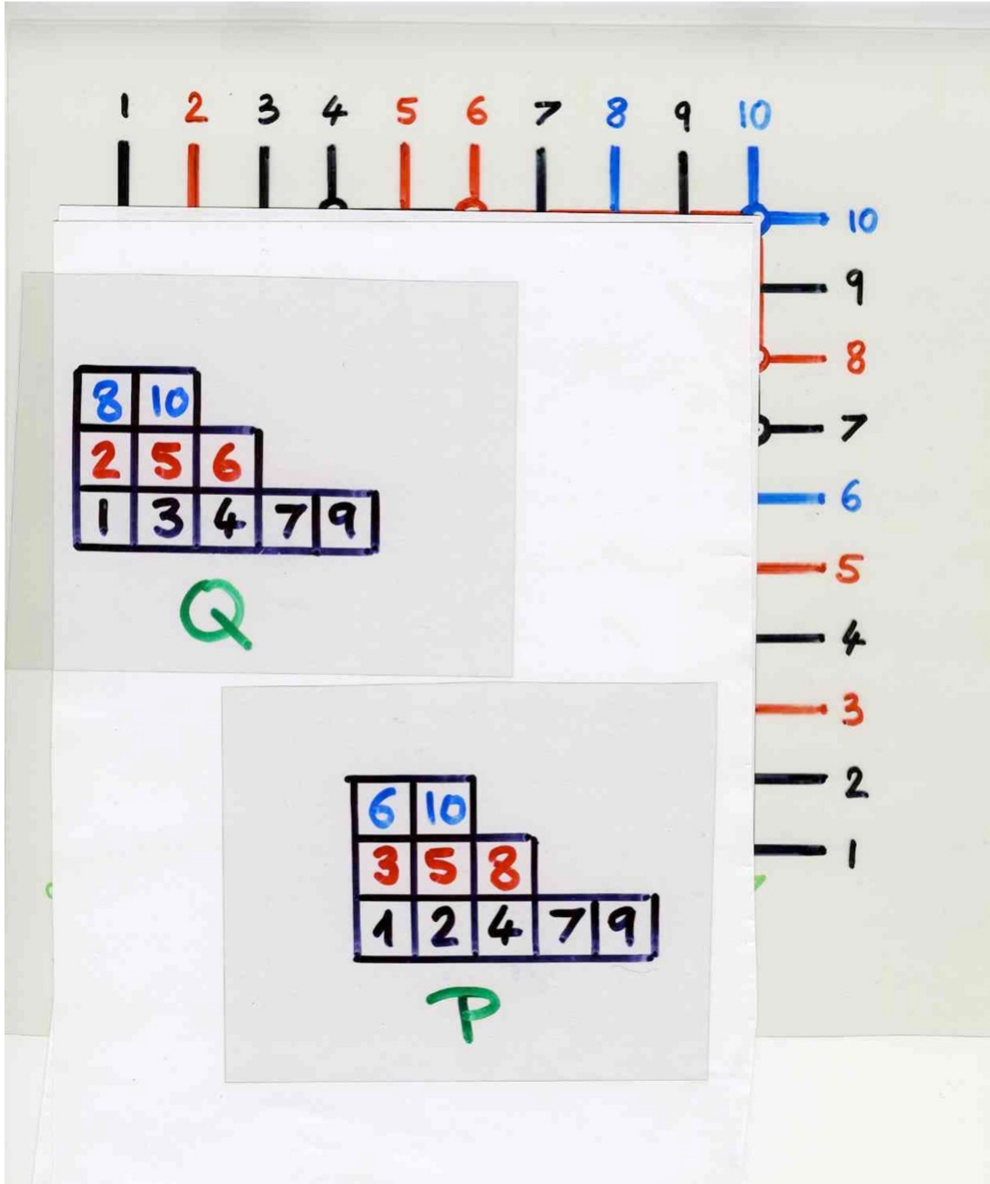
1 2 3 4 5 6 7 8 9 10

$= 1\ 2\ 1\ 1\ 2\ 2\ 1\ 3\ 1\ 3$

$Q =$

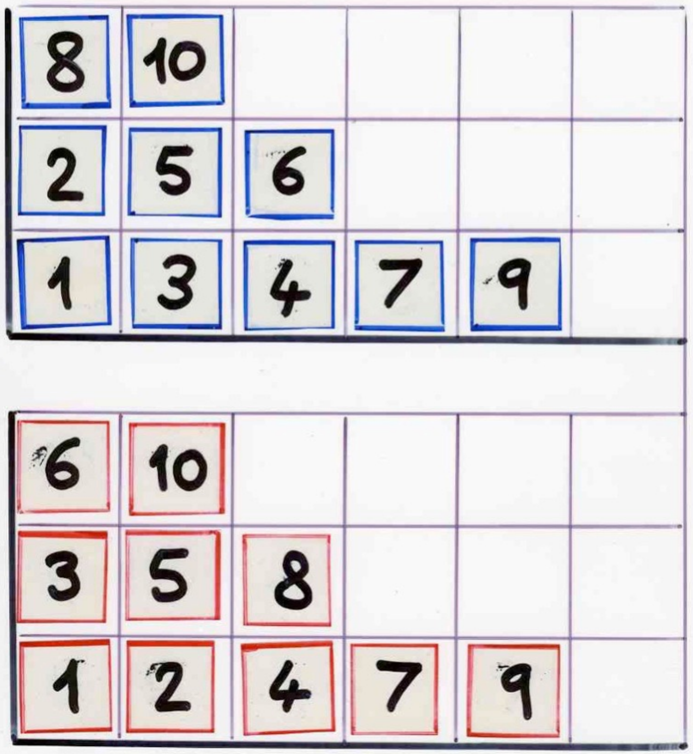
8	10			
2	5	6		
1	3	4	7	9

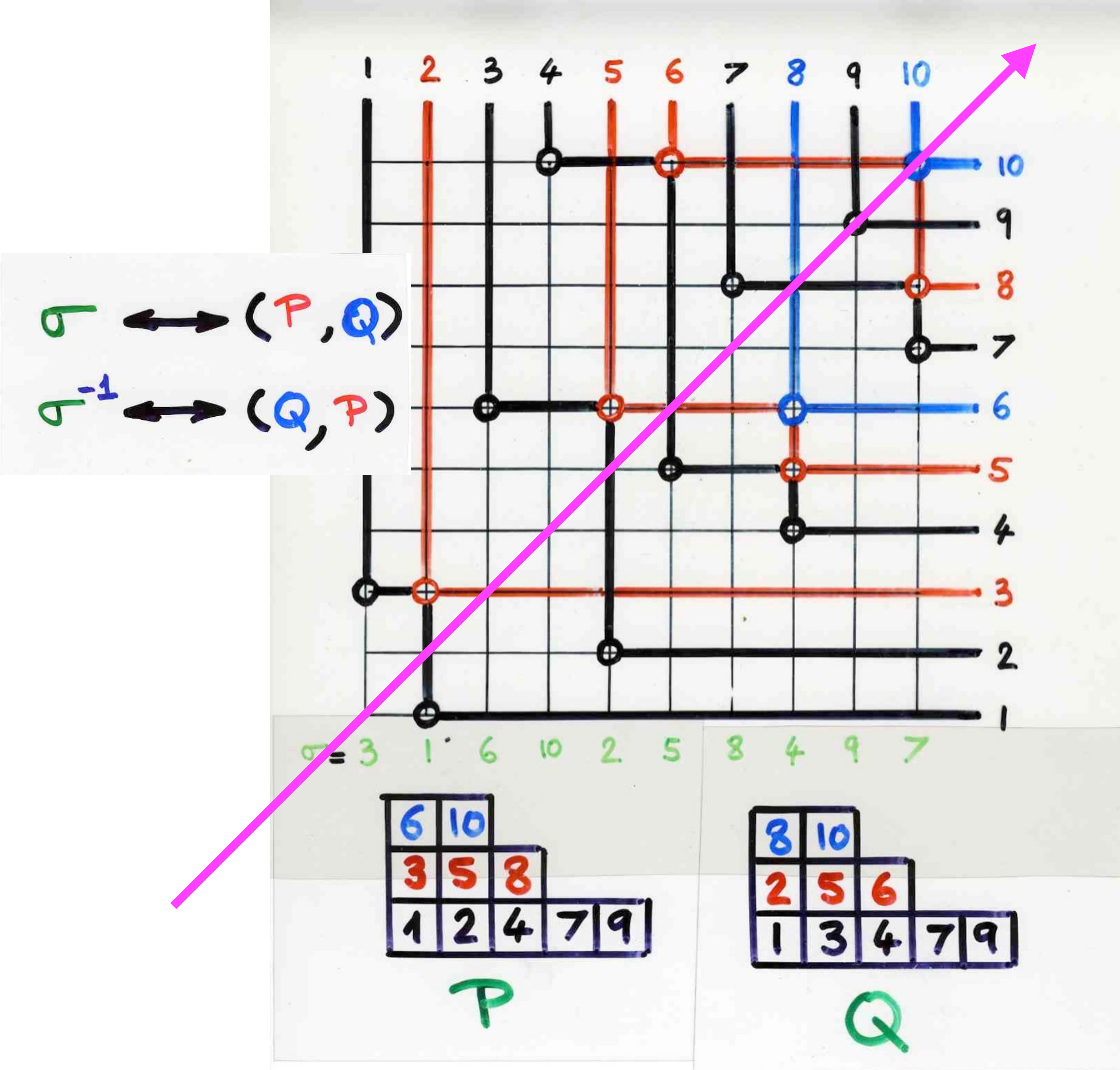
geometric version with "light" and "shadow"



Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





σ ⟷ (P, Q)
σ^{-1} ⟷ (Q, P)
1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
P
8 10
2 5 6
1 3 4 7 9
Q

$\sigma \longleftrightarrow (P, Q)$

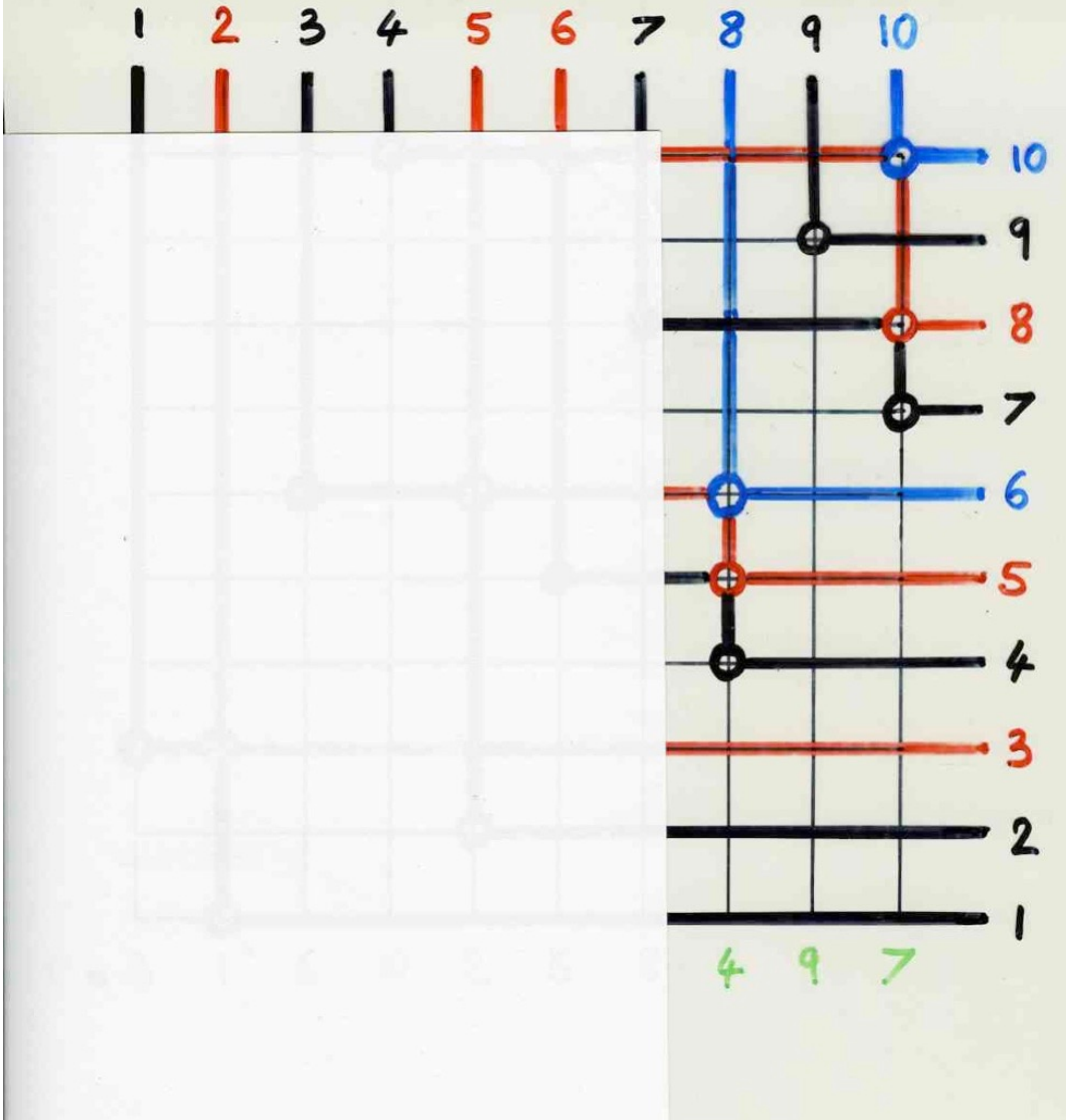
$\sigma^{-1} \longleftrightarrow (Q, P)$

bijection

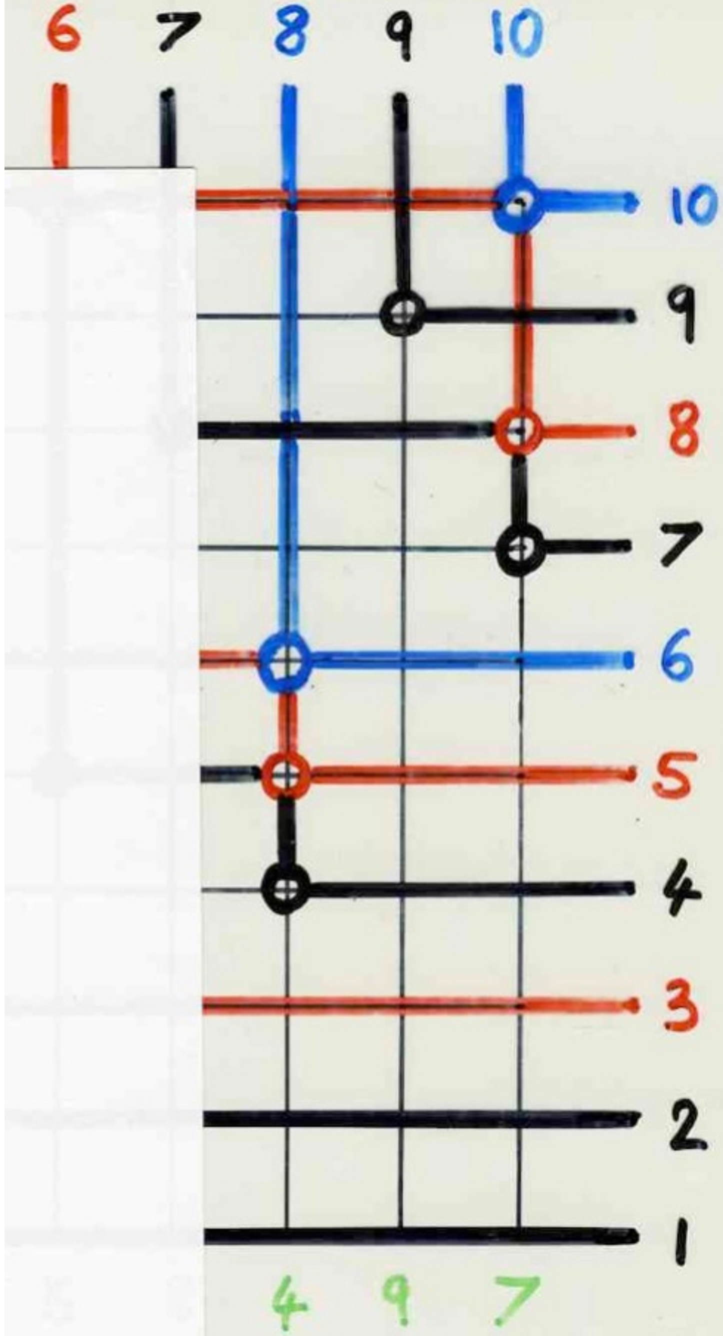
Young tableaux (size n) $\longleftrightarrow$ involutions on $\{1, .., n\}$

proof of the equivalence

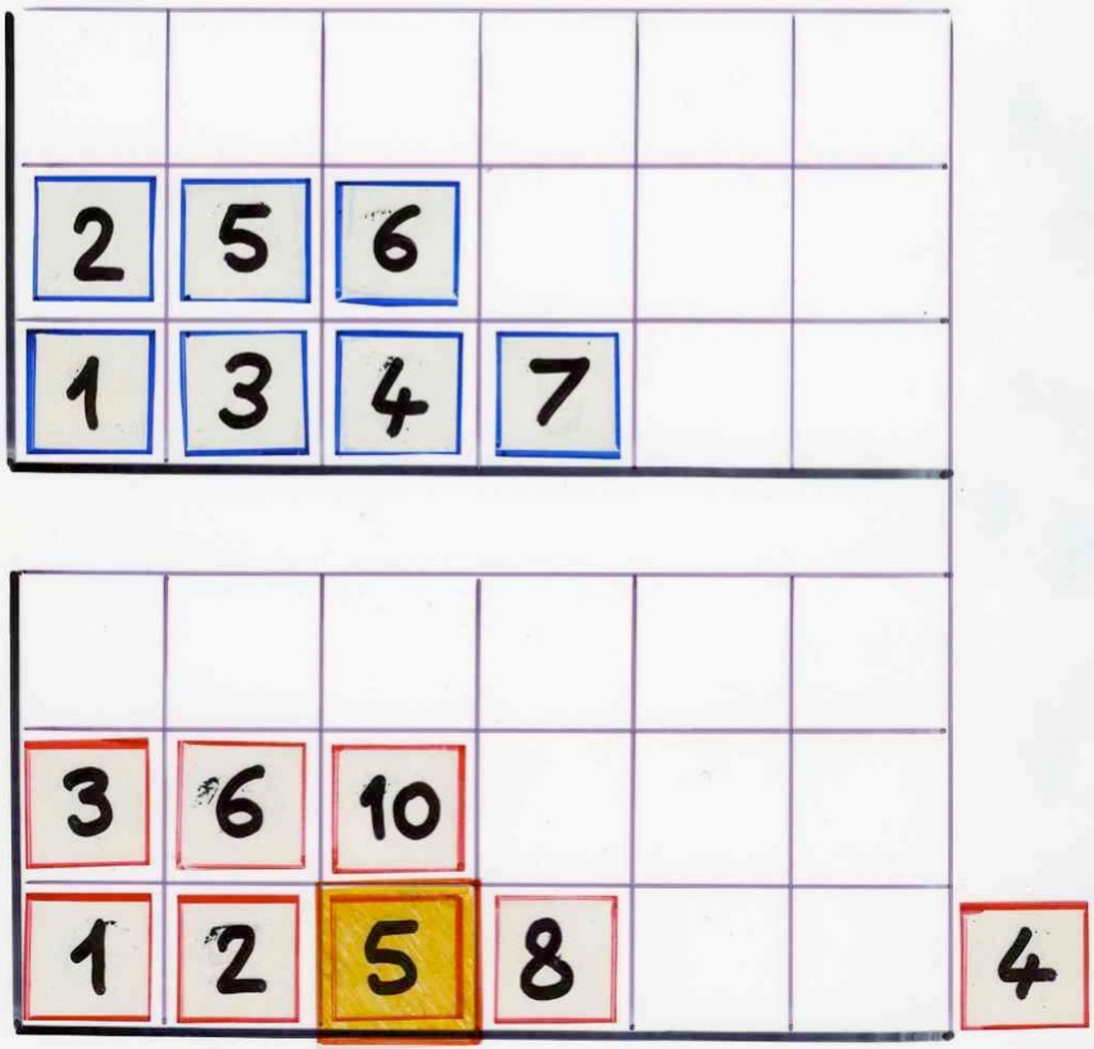
insertions --- geometric constrution

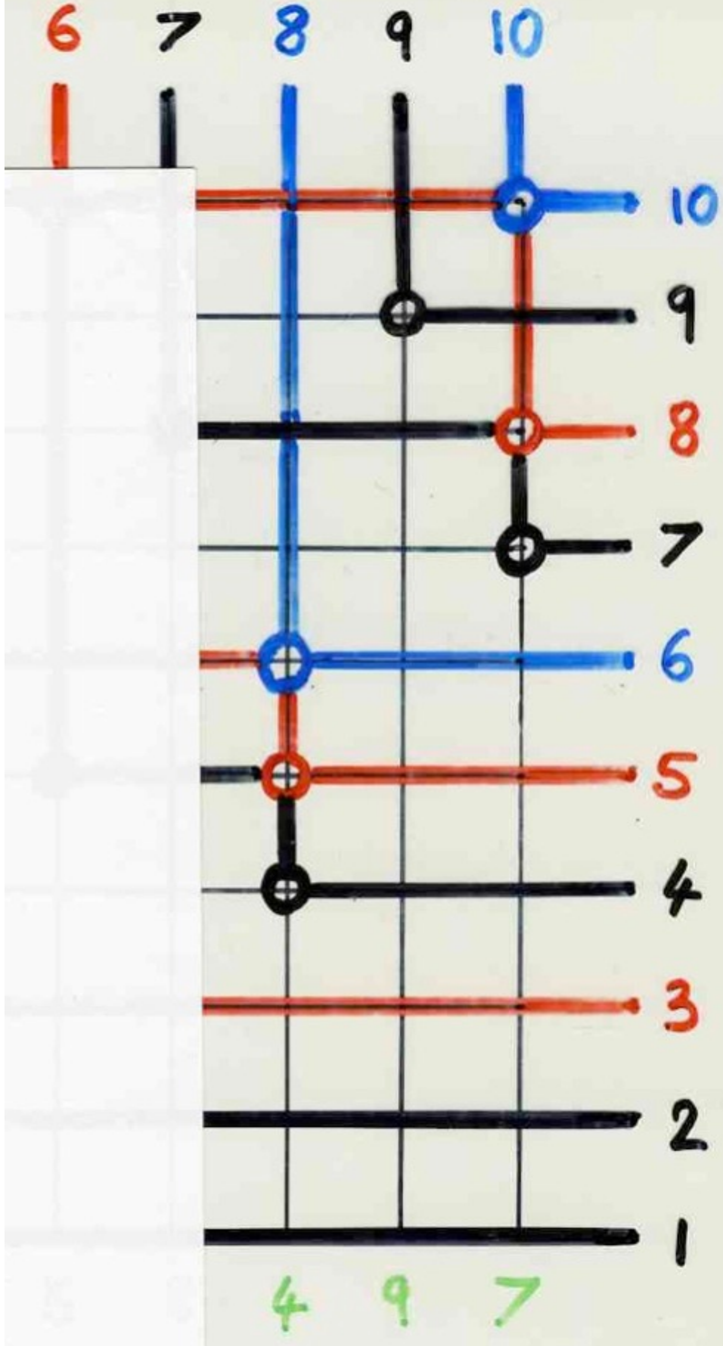


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
4 9 7

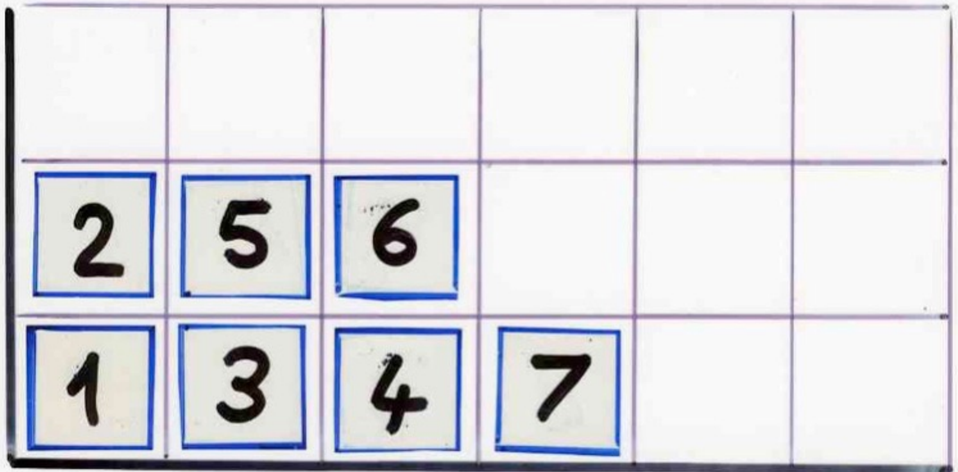


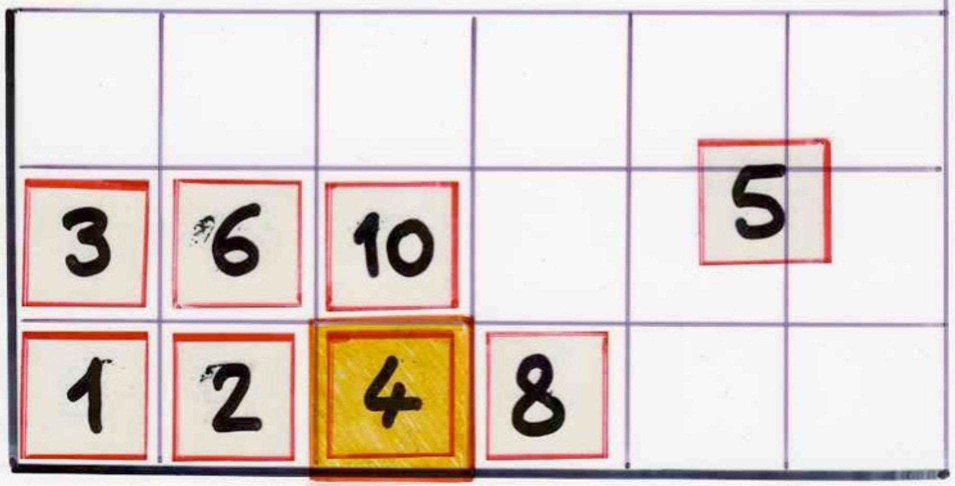
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

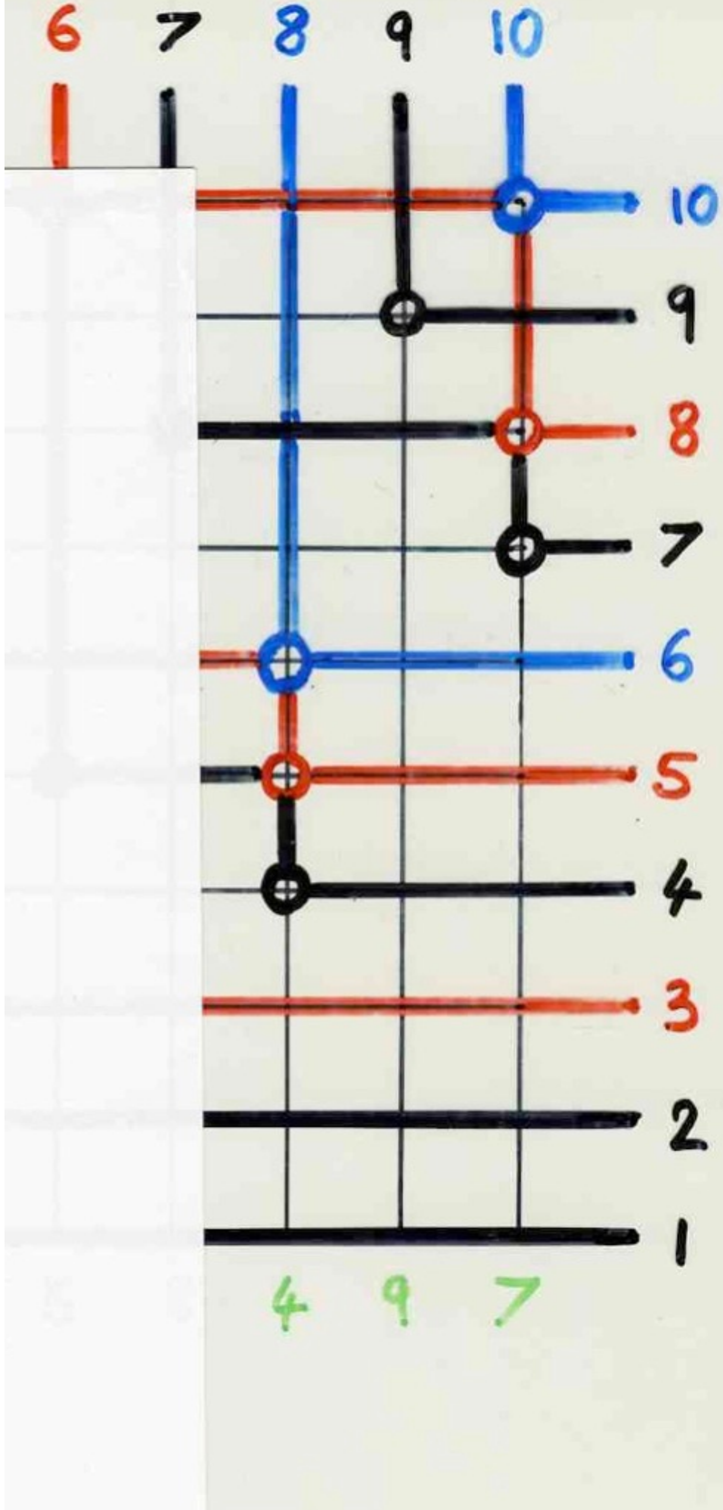




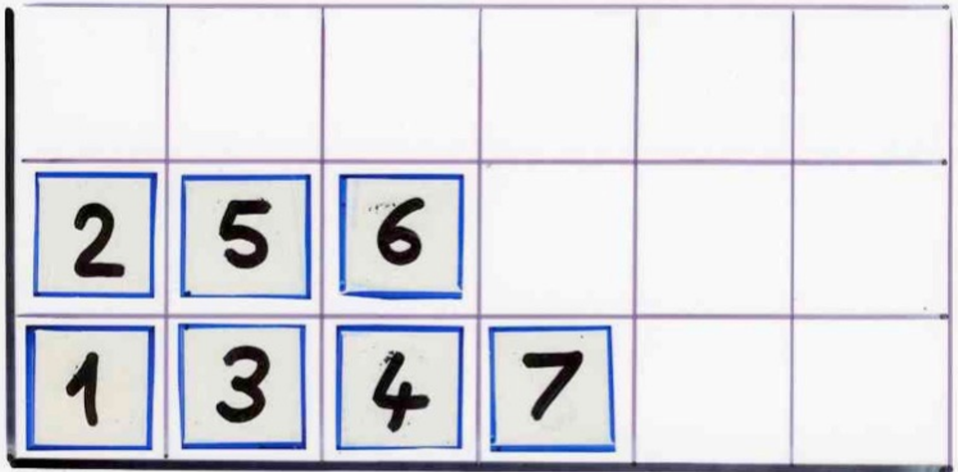
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

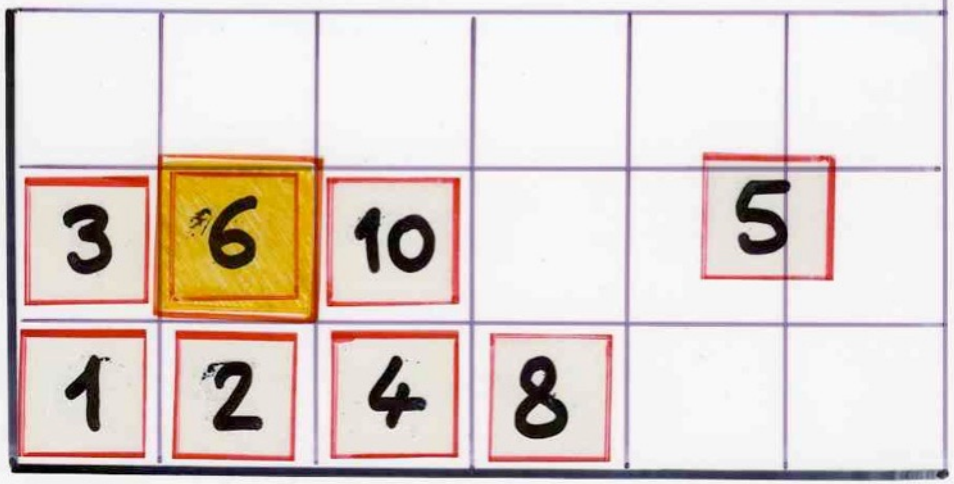


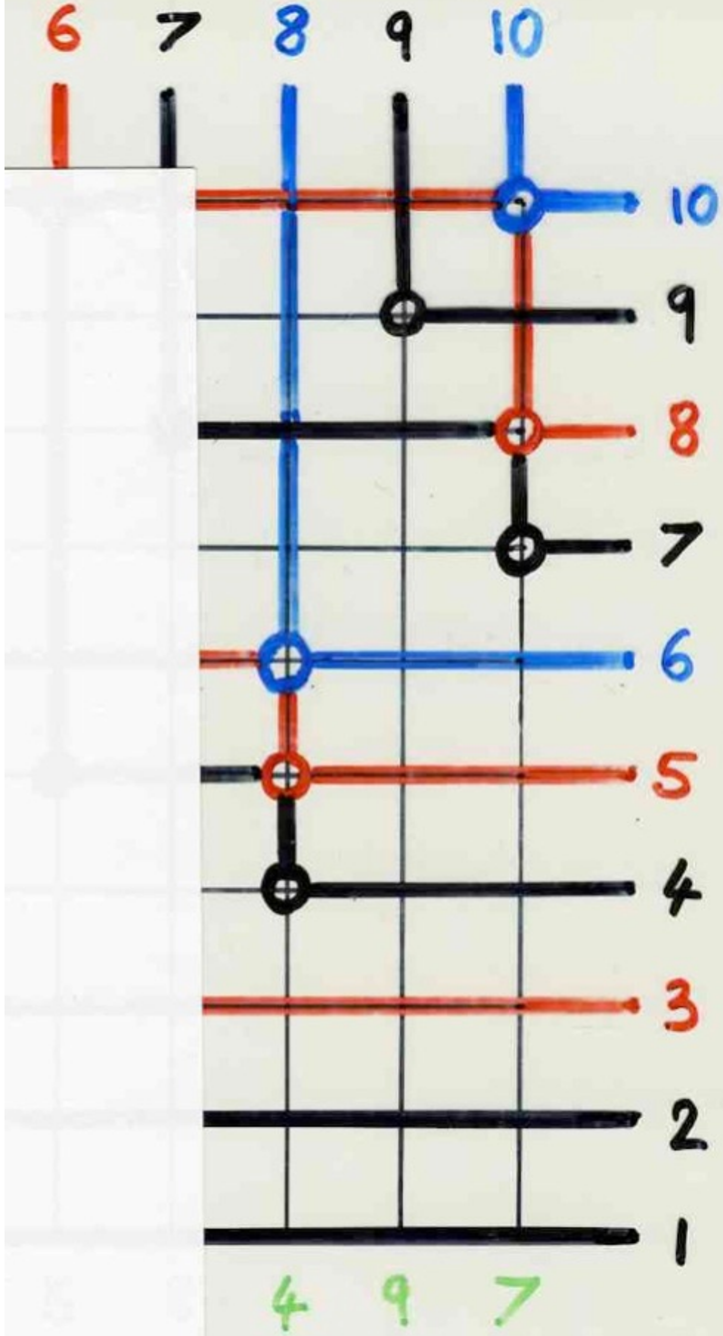




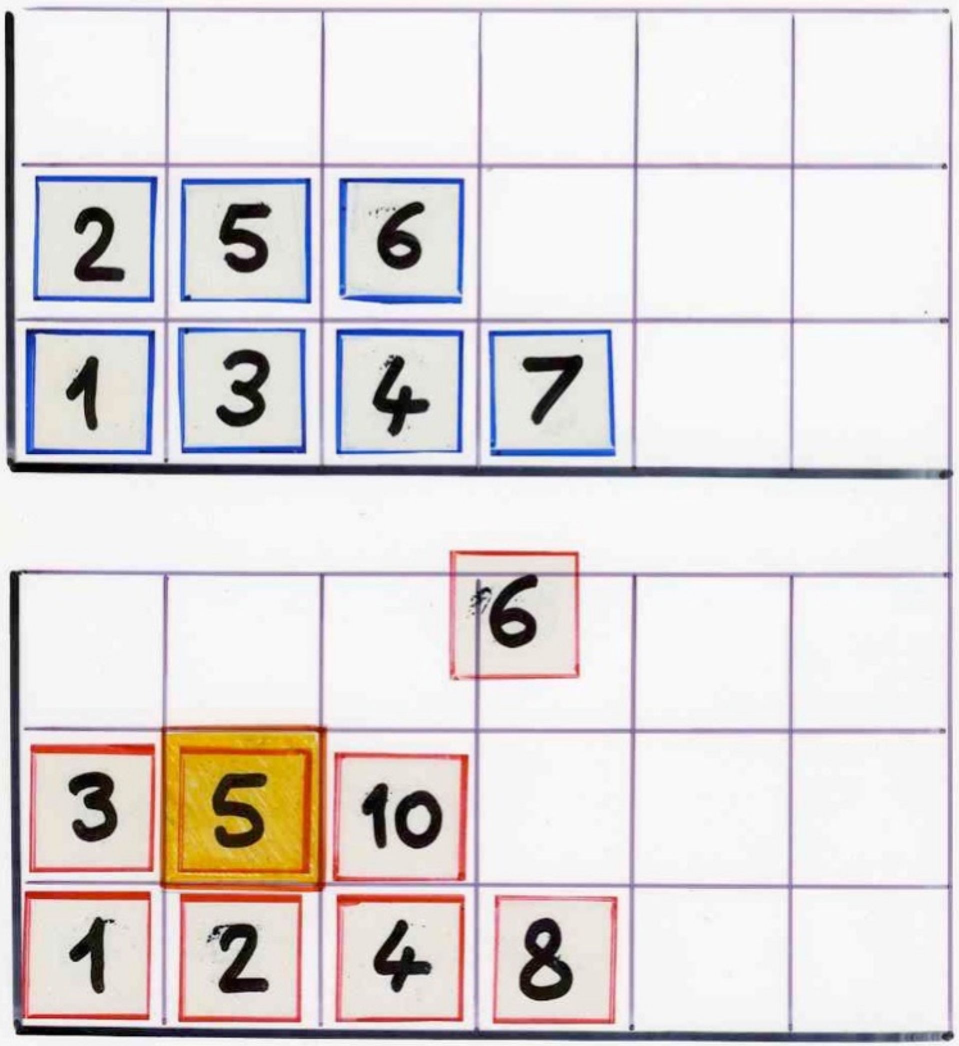
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

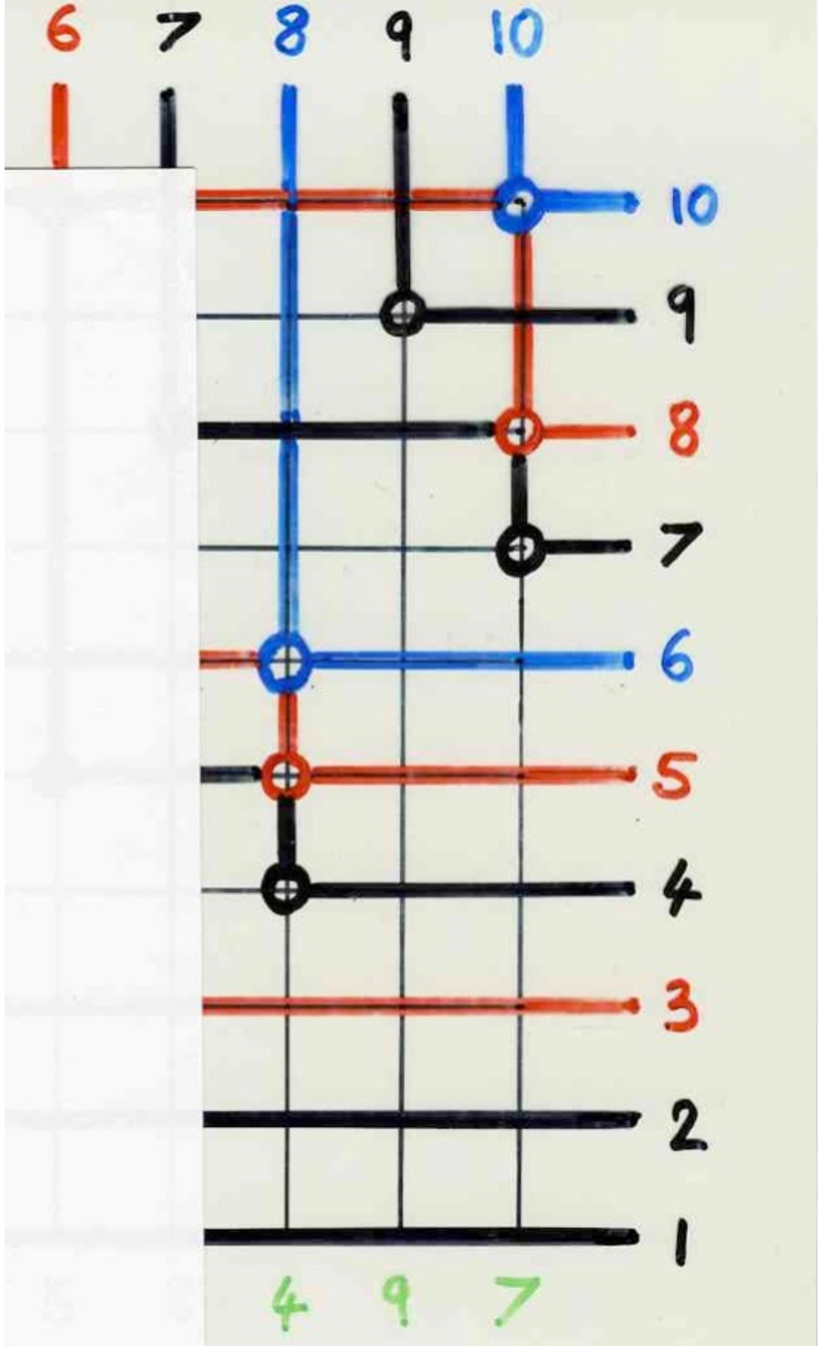






1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

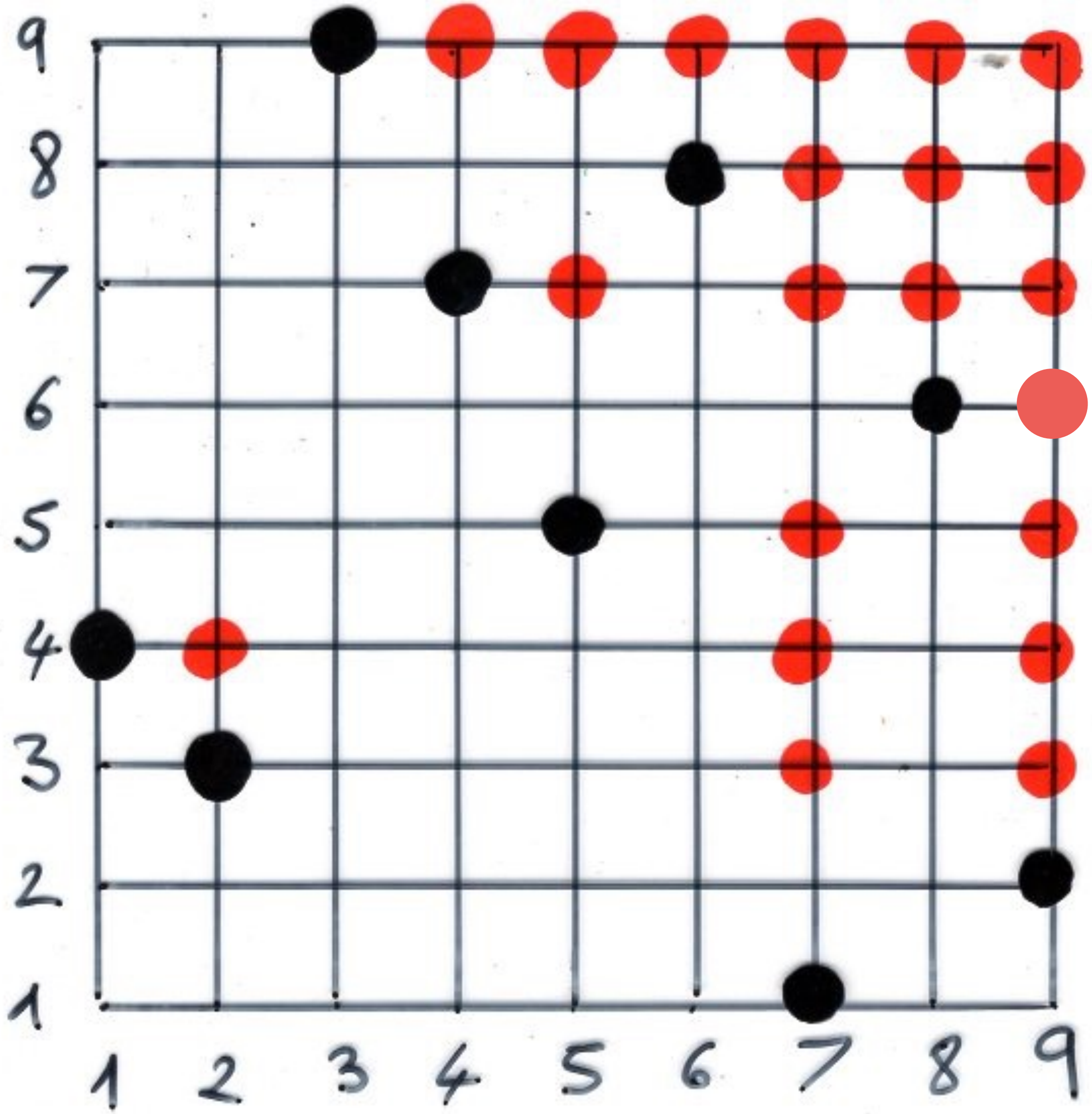
exercise

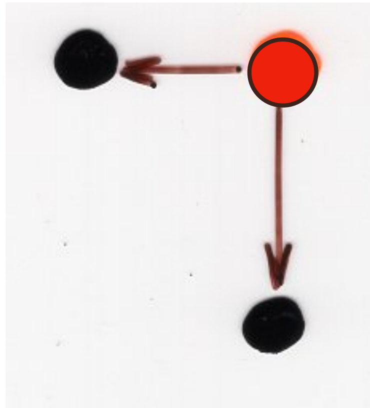
Characterization of the red points

$$Sq(\sigma) \subsetneq [1,n] \times [1,n]$$

Rothe diagram of a permutation

$R_o(\sigma)$





Rothe diagram of a permutation

$R_o(\sigma)$

$|R_o(\sigma)|$ = number of inversion pairs of σ

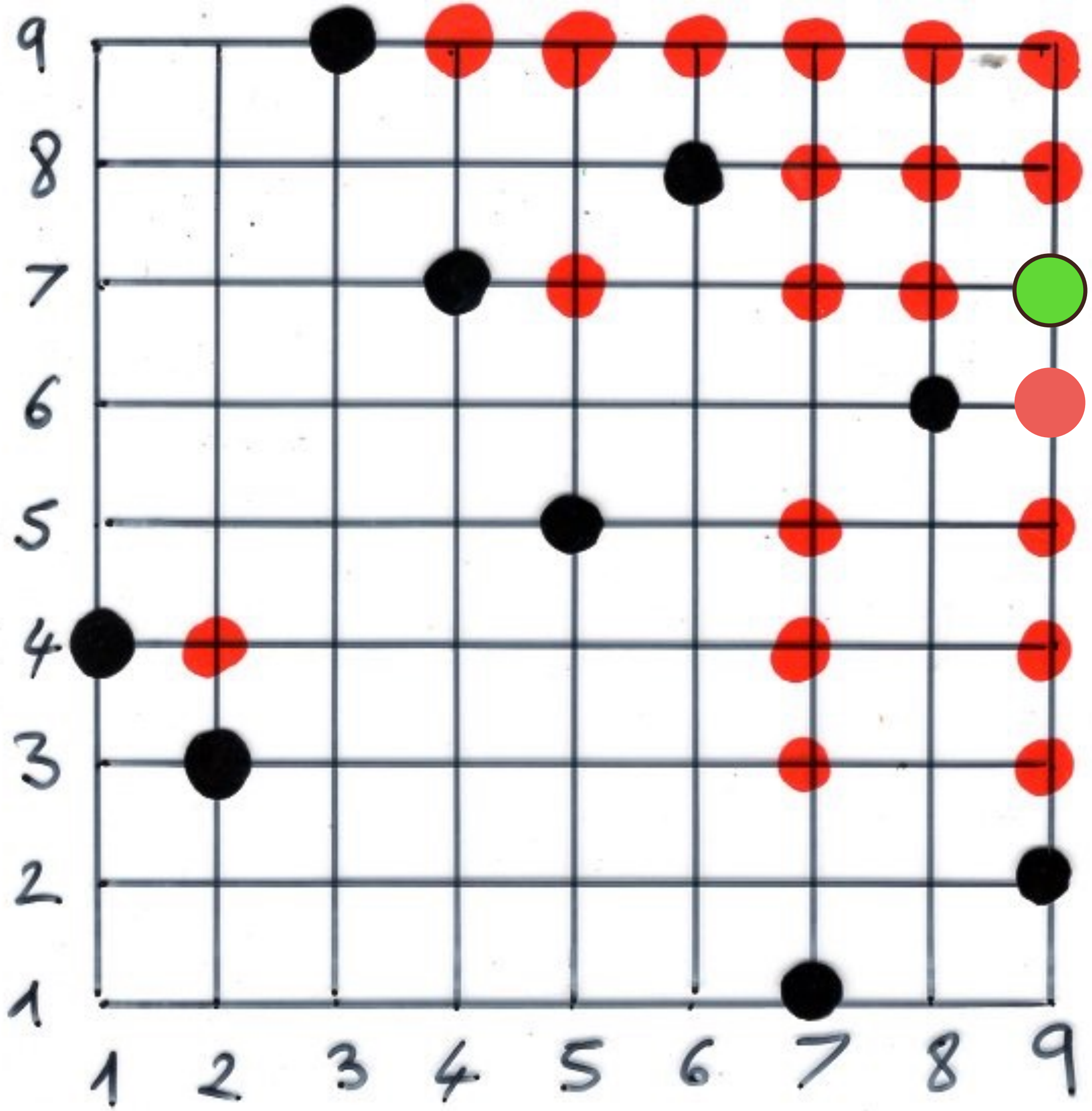
pair (i,j) $i<j$ and $\sigma(i)>\sigma(j)$

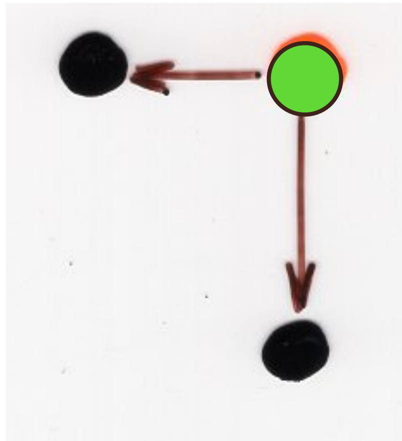
Rothe (1800)

$\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$

Rothe diagram of a permutation

$R_o(\sigma)$

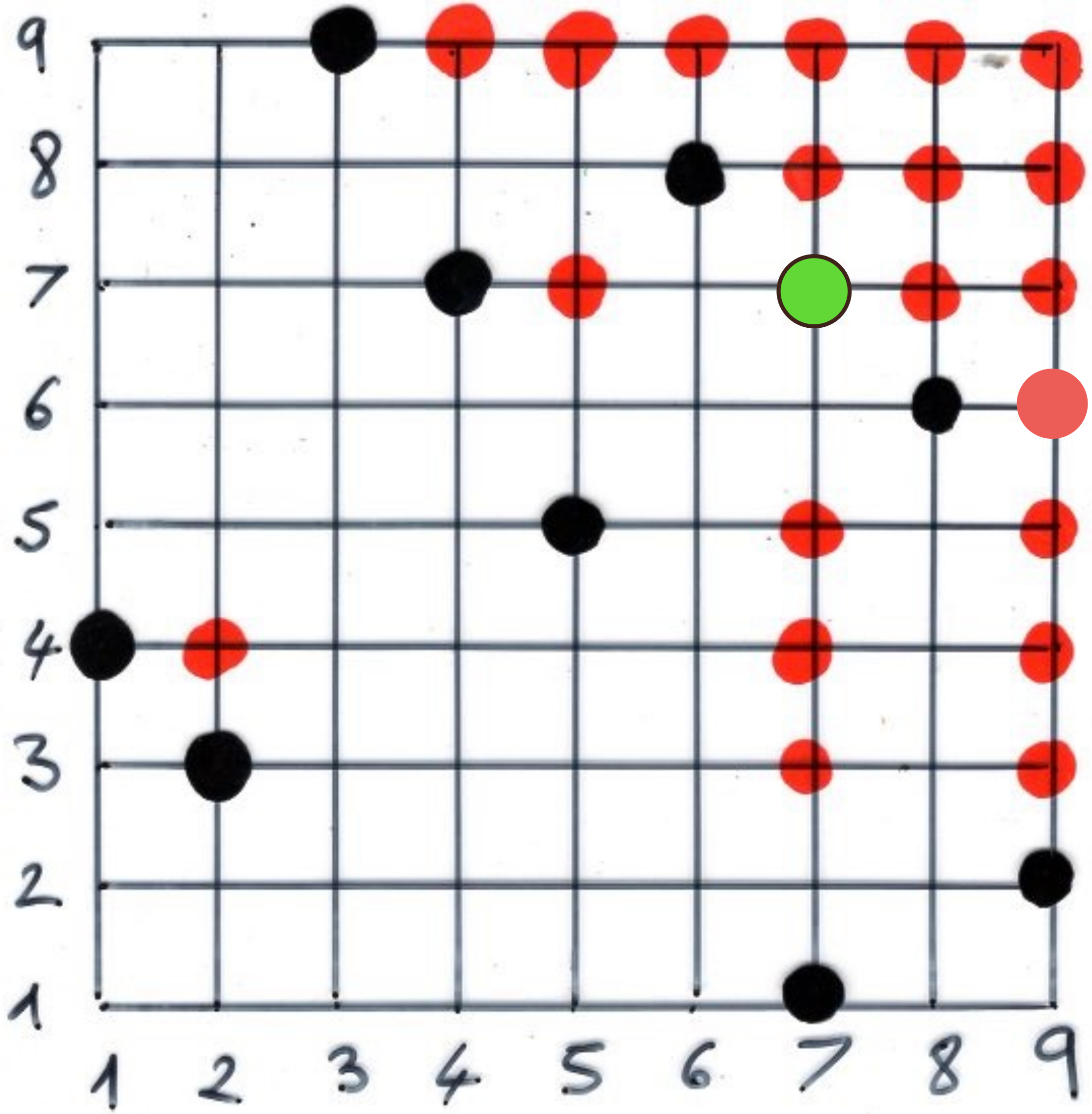




Nim game

Rothe diagram of a permutation

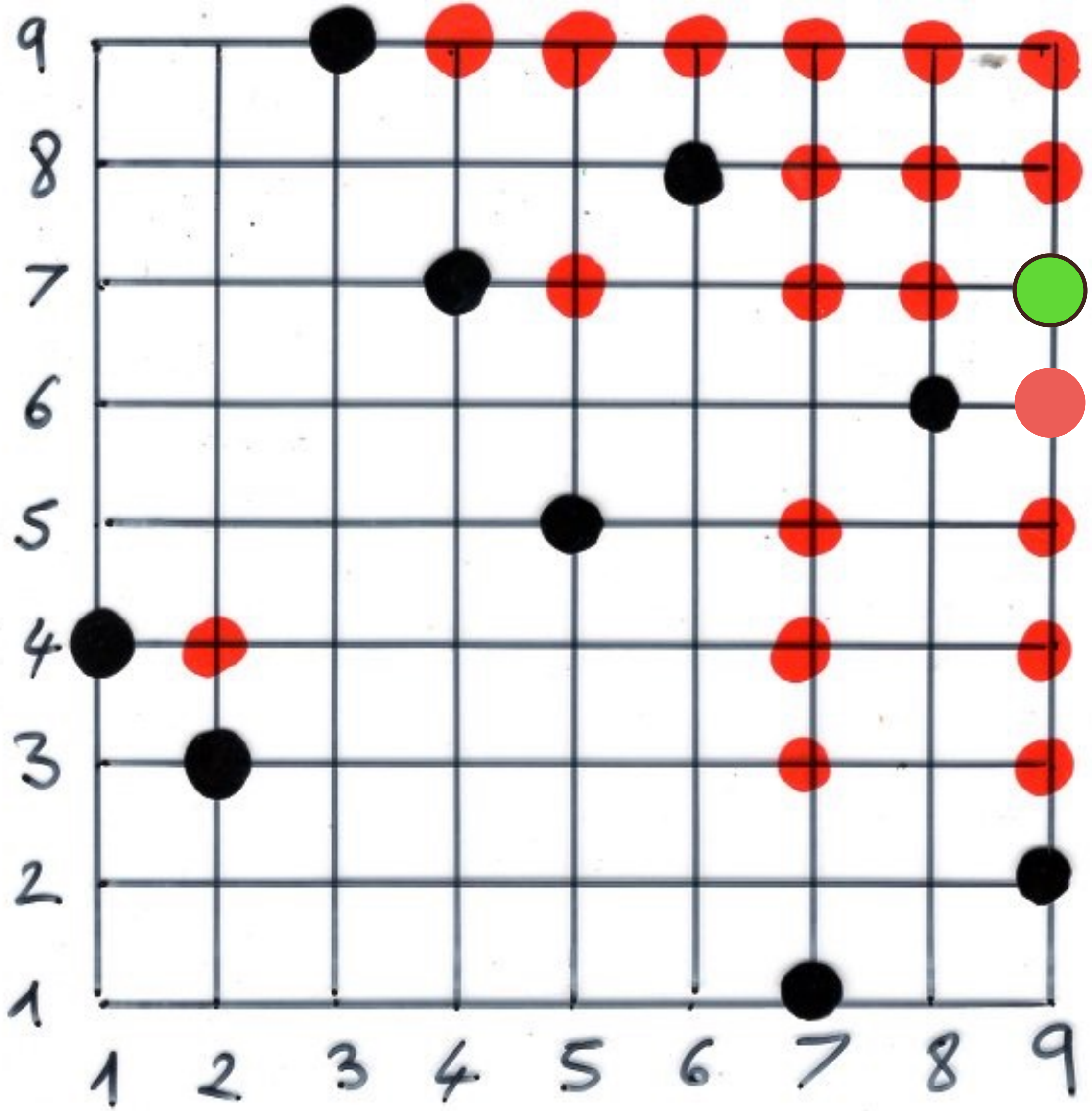
$R_0(\sigma)$



Nim game

Rothe diagram of a permutation

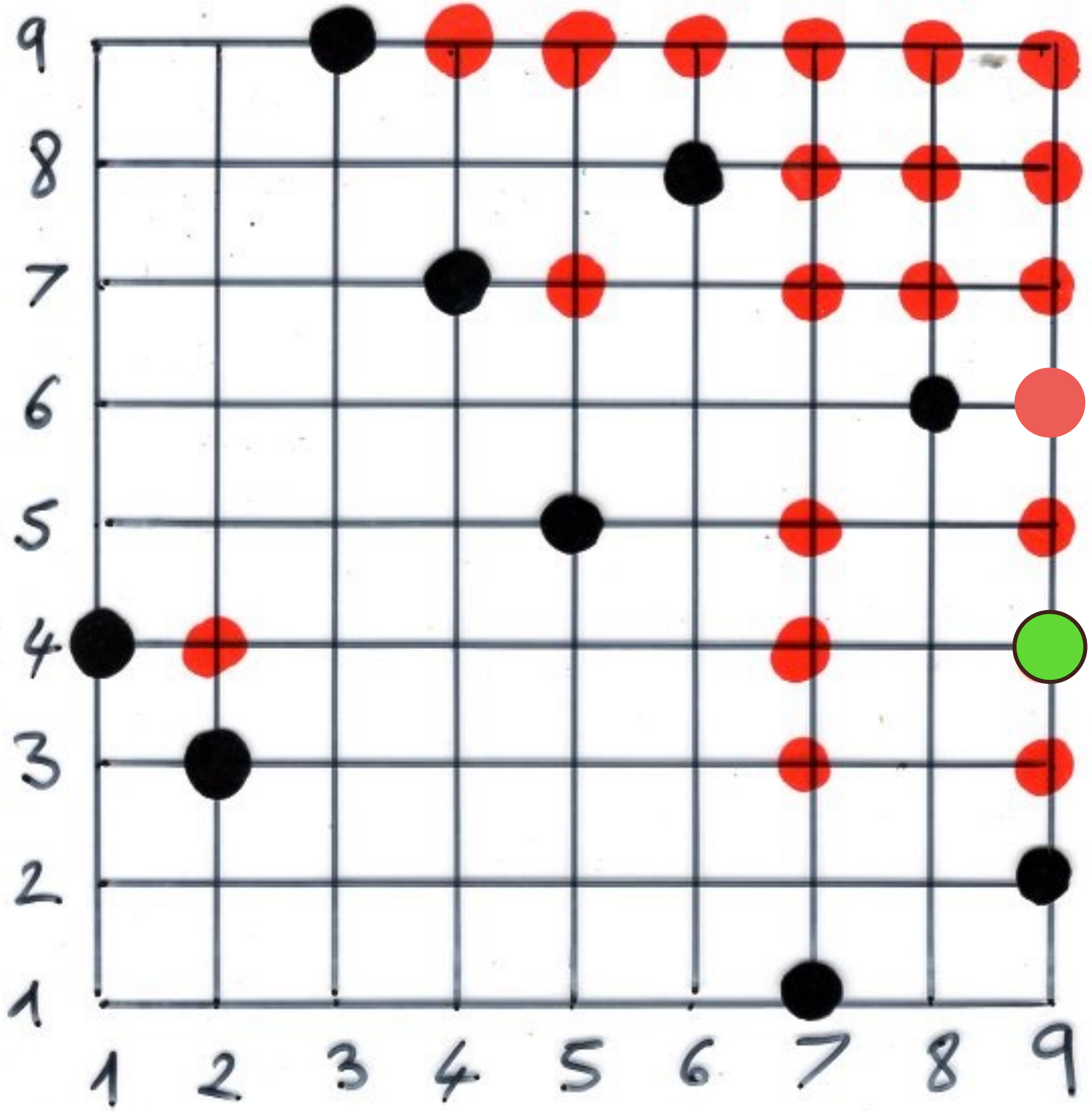
$R_o(\sigma)$



Nim game

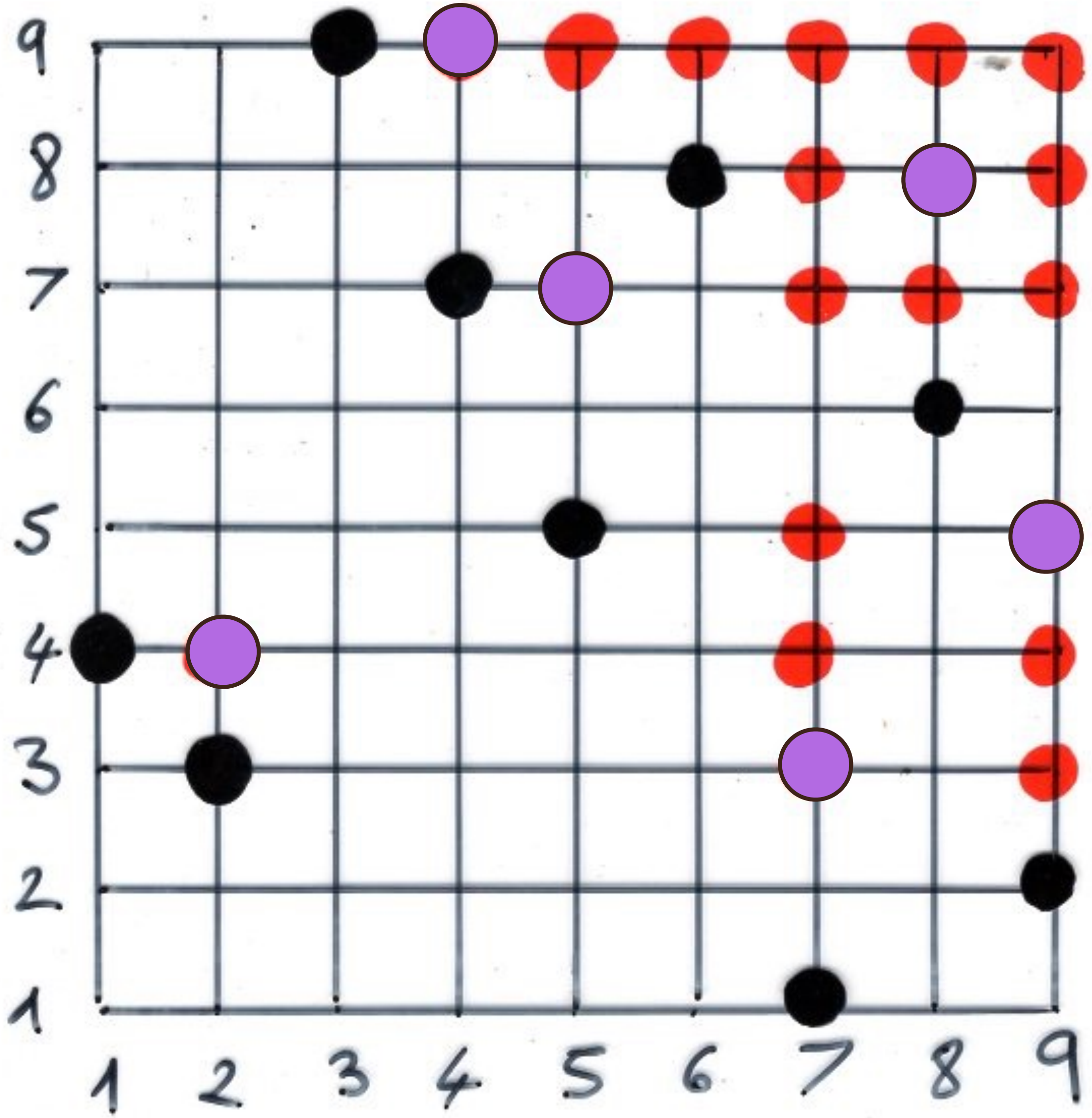
Rothe diagram of a permutation

$R_0(\sigma)$



Nim game

wining positions



Nim game

- wining positions

kernel of a graph

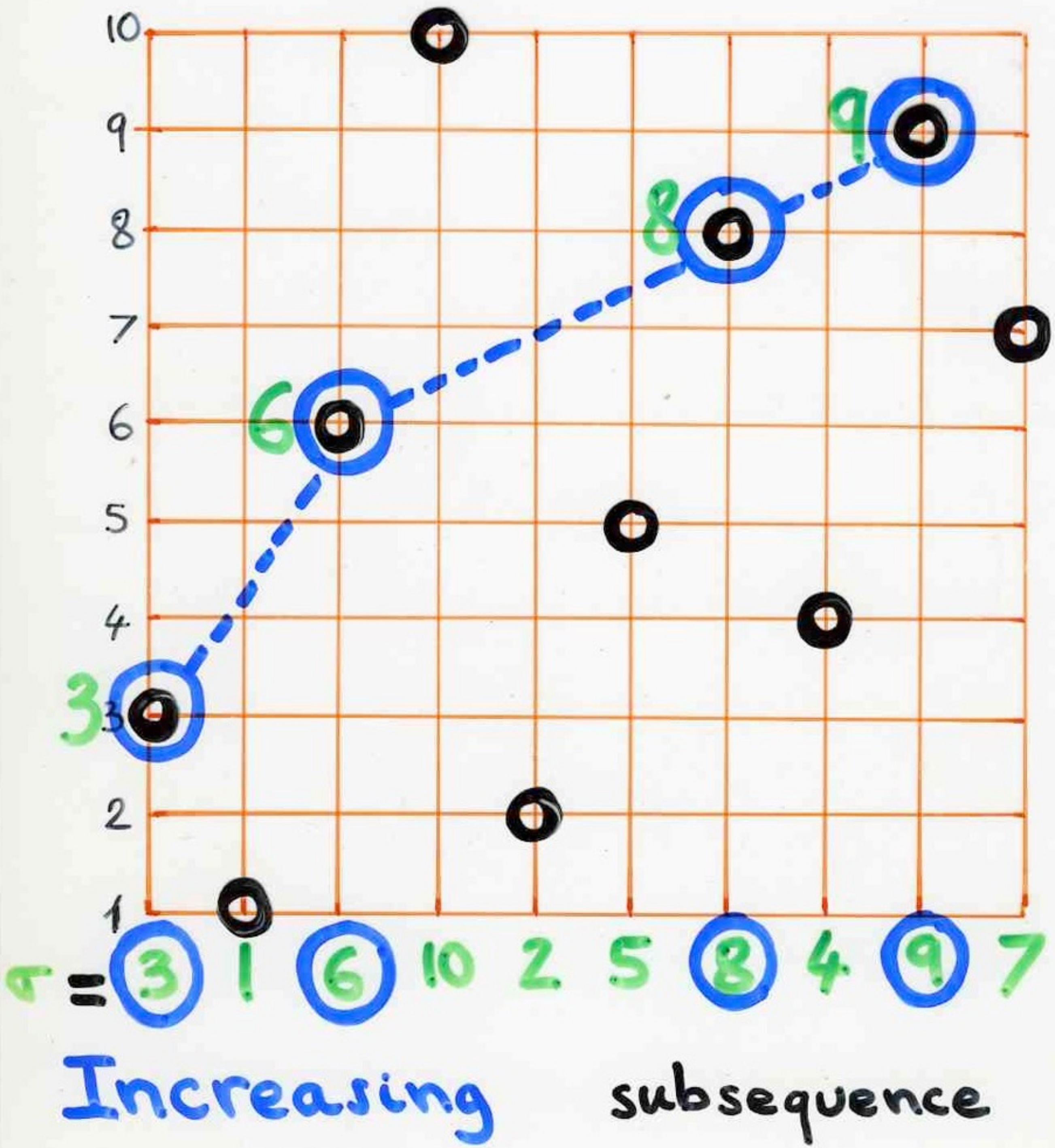
- every vertex of the graph is the source of an edge ending in the kernel
- every edge having its source in the kernel has its end not in the kernel

exercise

$Sq(\sigma)$ is the set of "wining positions" in a Nim game on the Rothe diagram $Ro(\sigma)$

application:

increasing and decreasing subsequences



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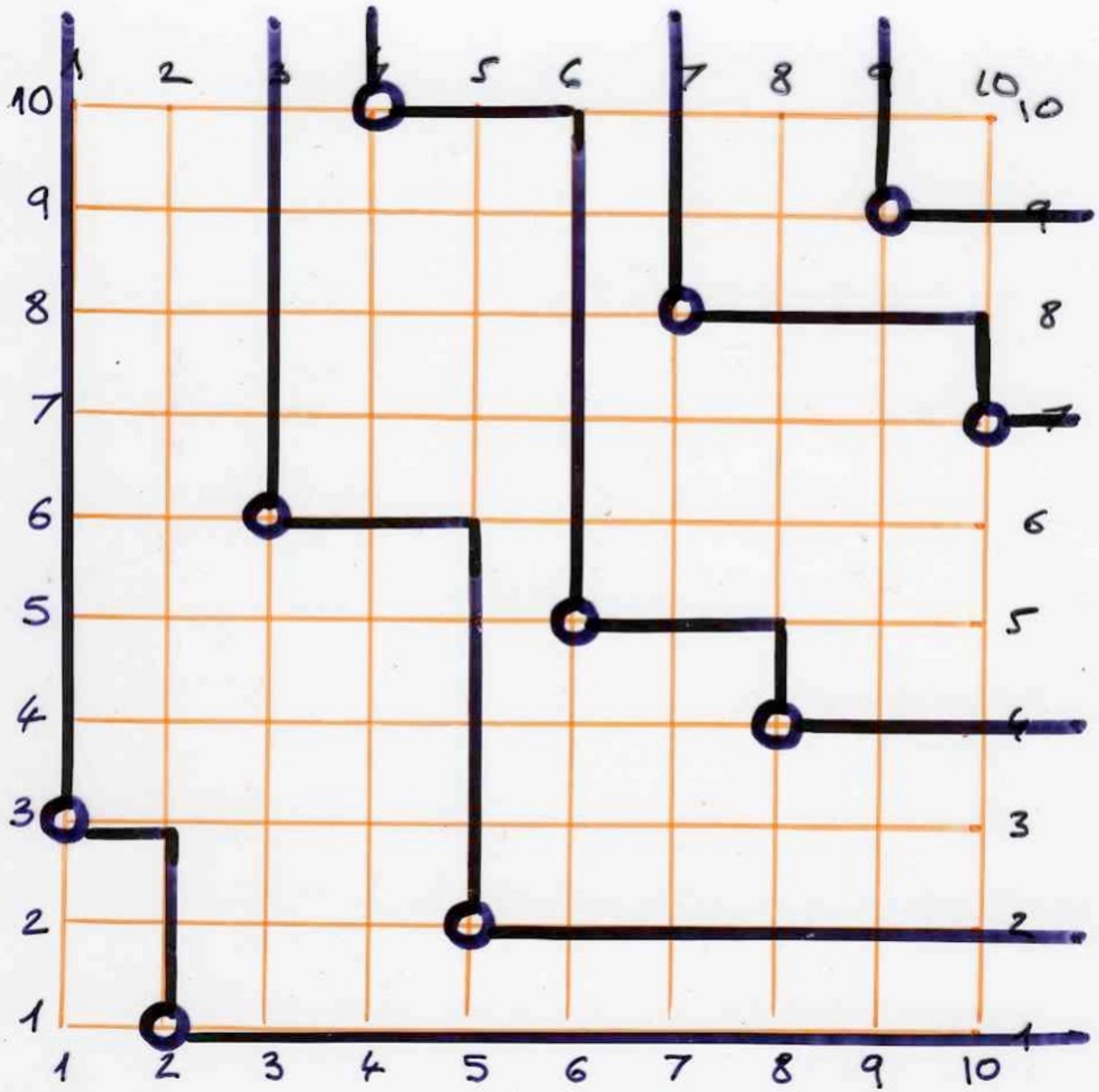
σ = 3 1 6 10 2 5 8 4 9 7
Increasing subsequence

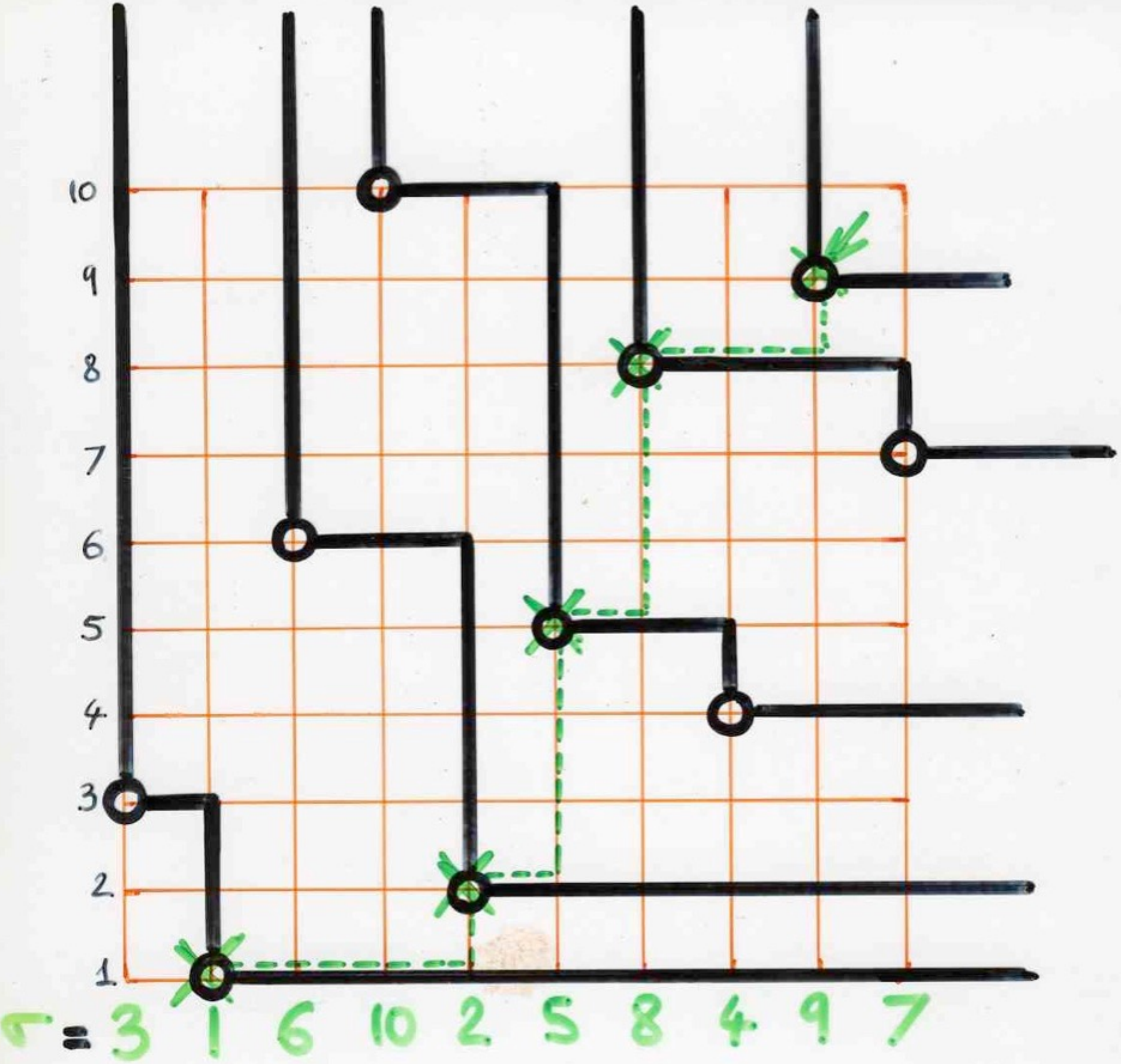
Proposition permutation σ

$$\sigma \xrightarrow[RS]{} (P, Q)$$

the number of elements of the first row of the common shape of P and Q (i.e. Ferrers diagram) is the maximum length of increasing subsequences of σ



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$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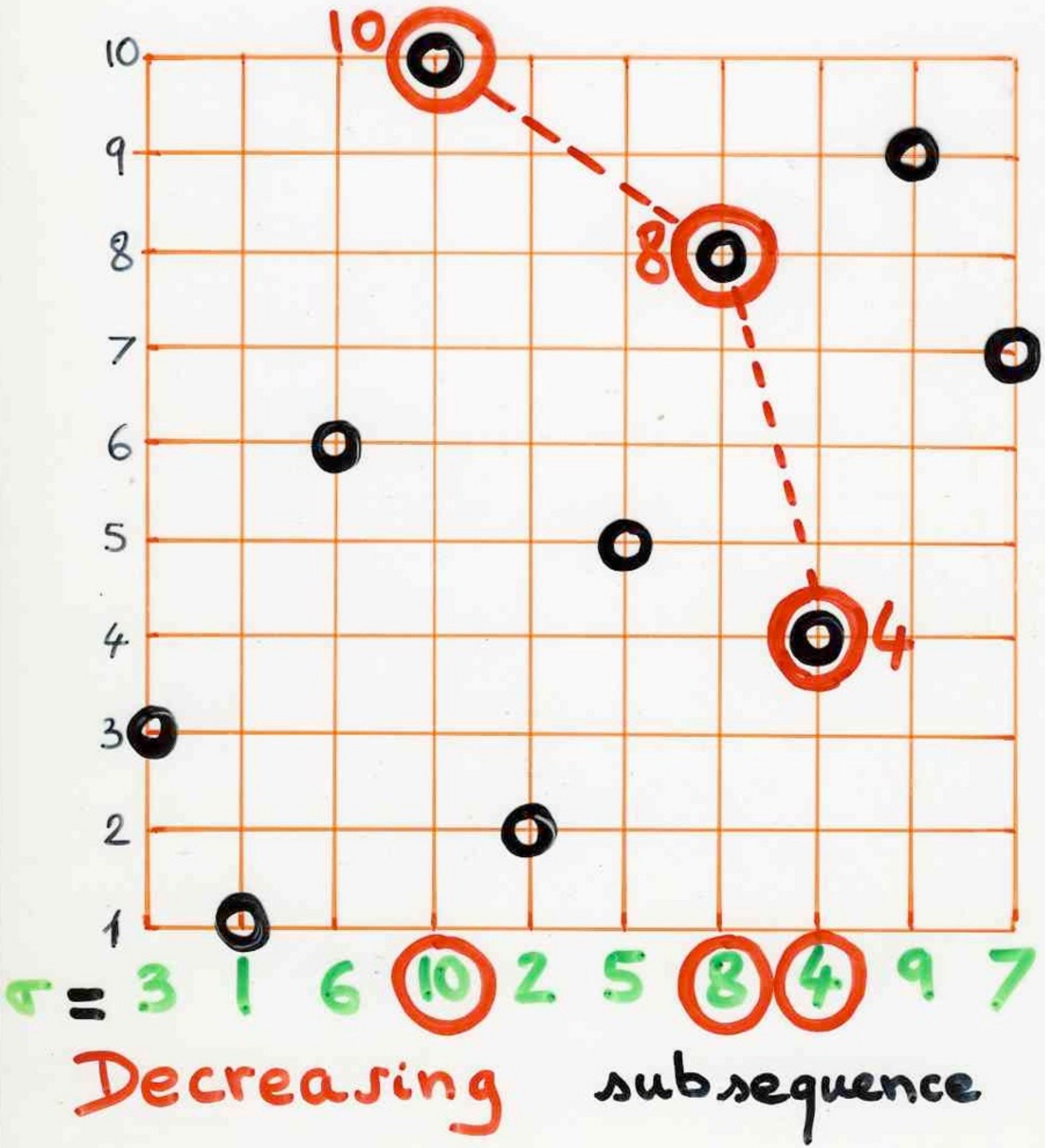
σ = 3 1 6 10 2 5 8 4 9 7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σ = 3 1 6 10 2 5 8 4 9 7
Decreasing subsequence

Proposition permutation σ

$$\sigma \xrightarrow[RS]{} (P, Q)$$

the number of elements of the first row of the common shape of P and Q (i.e. Ferrers diagram) is the maximum length of increasing subsequences of σ

Proposition

$$\sigma \xrightarrow[RS]{} (P, Q)$$

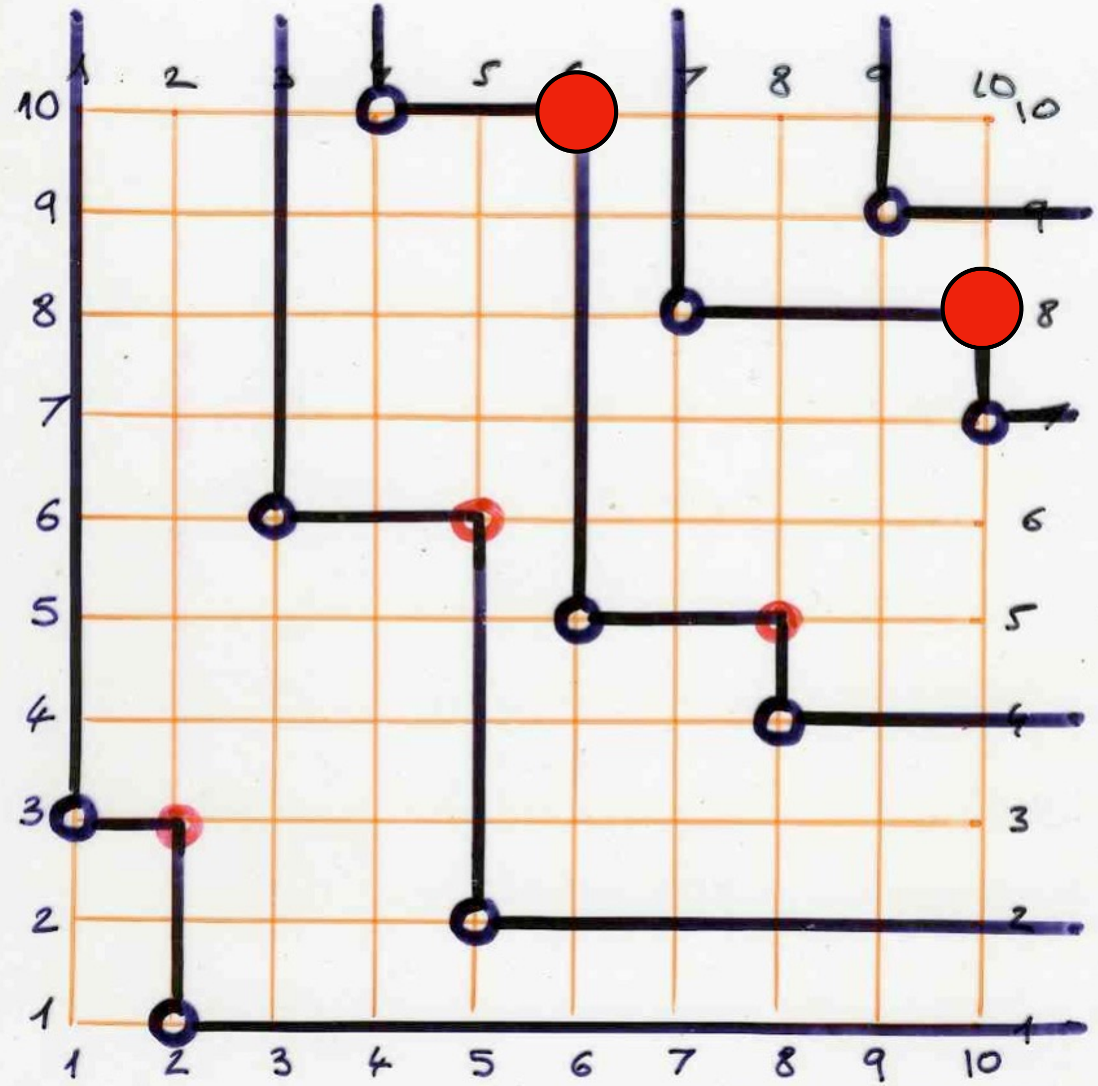
---- first column ------

-- maximum length of decreasing subsequences ---

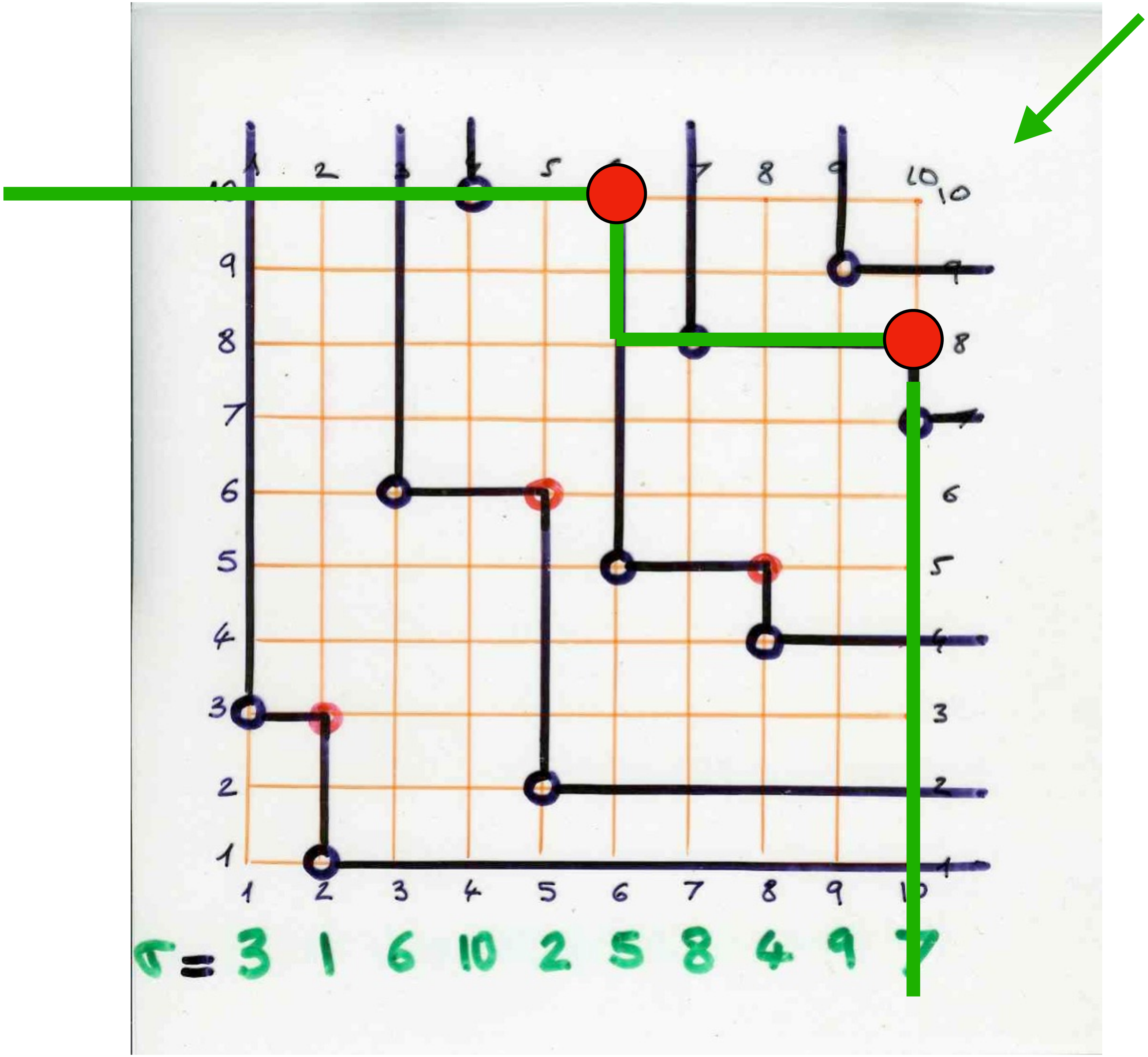
Lemma σ permutation

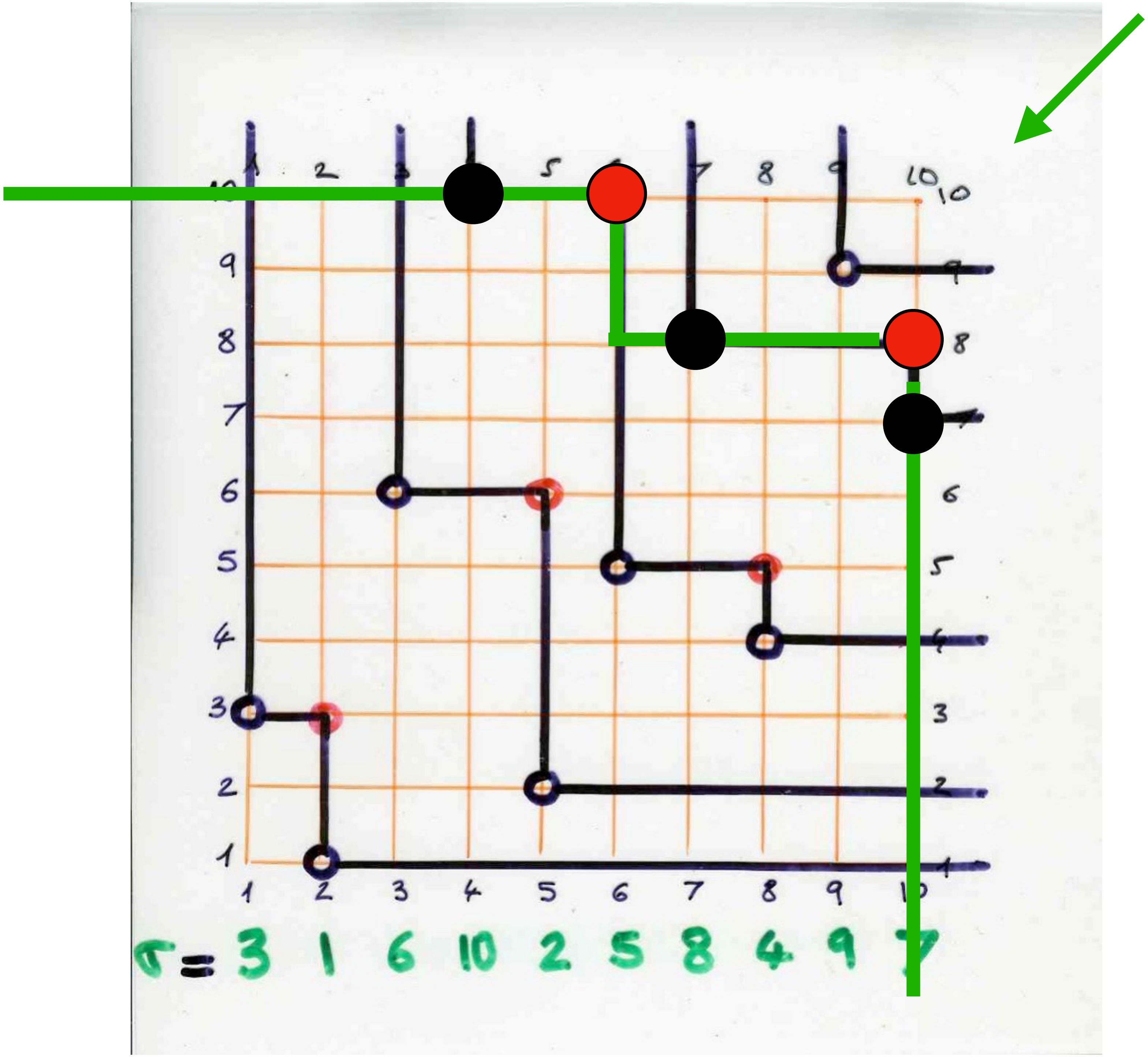
τ decreasing subsequence of $Sq(\sigma)$ size k

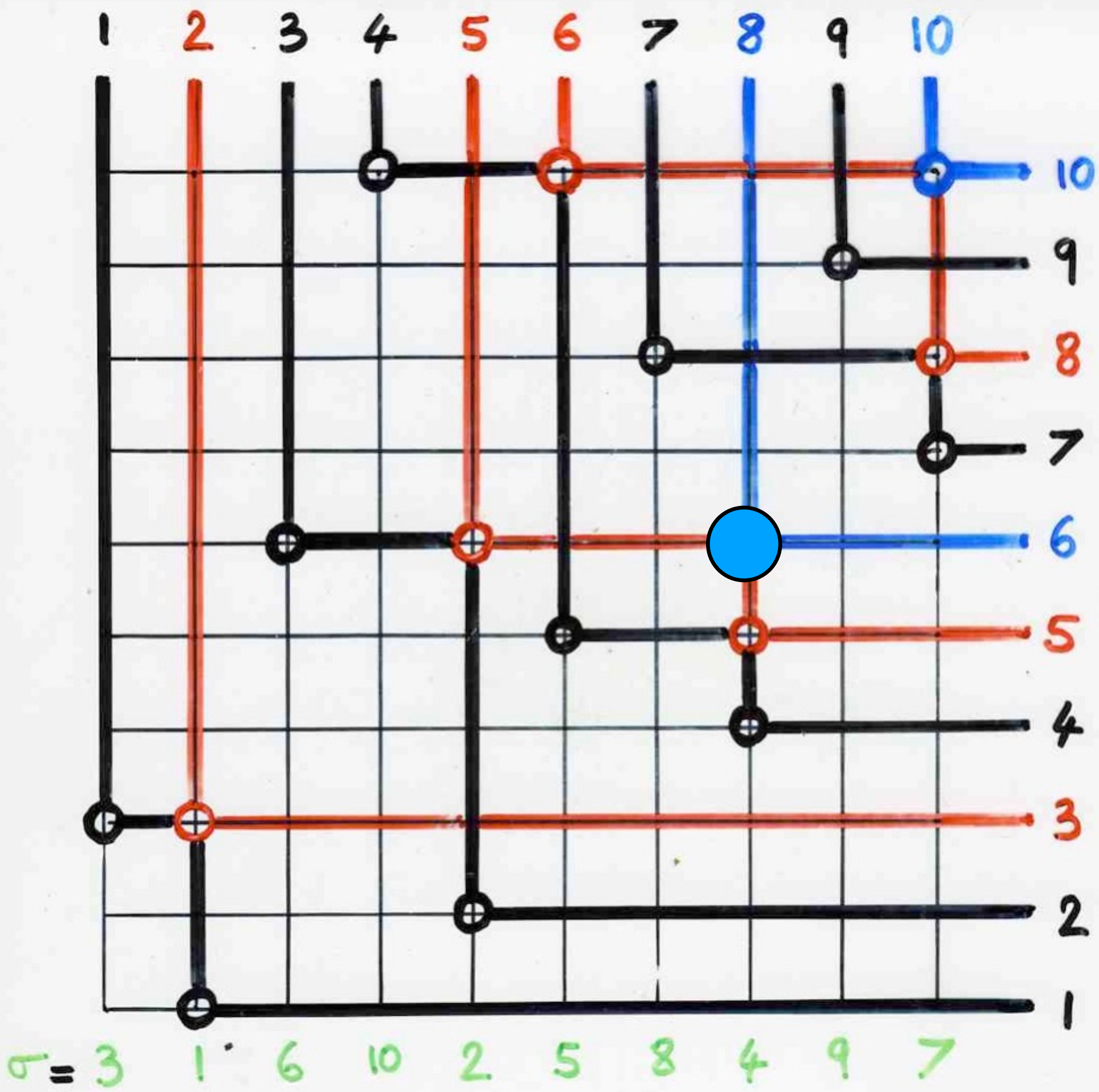
$\Rightarrow \exists$ a decreasing subsequence of σ with size $k+1$



$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$


σ = 3 1 6 10 2 5 8 4 9 7


σ = 3 1 6 10 2 5 8 4 9 7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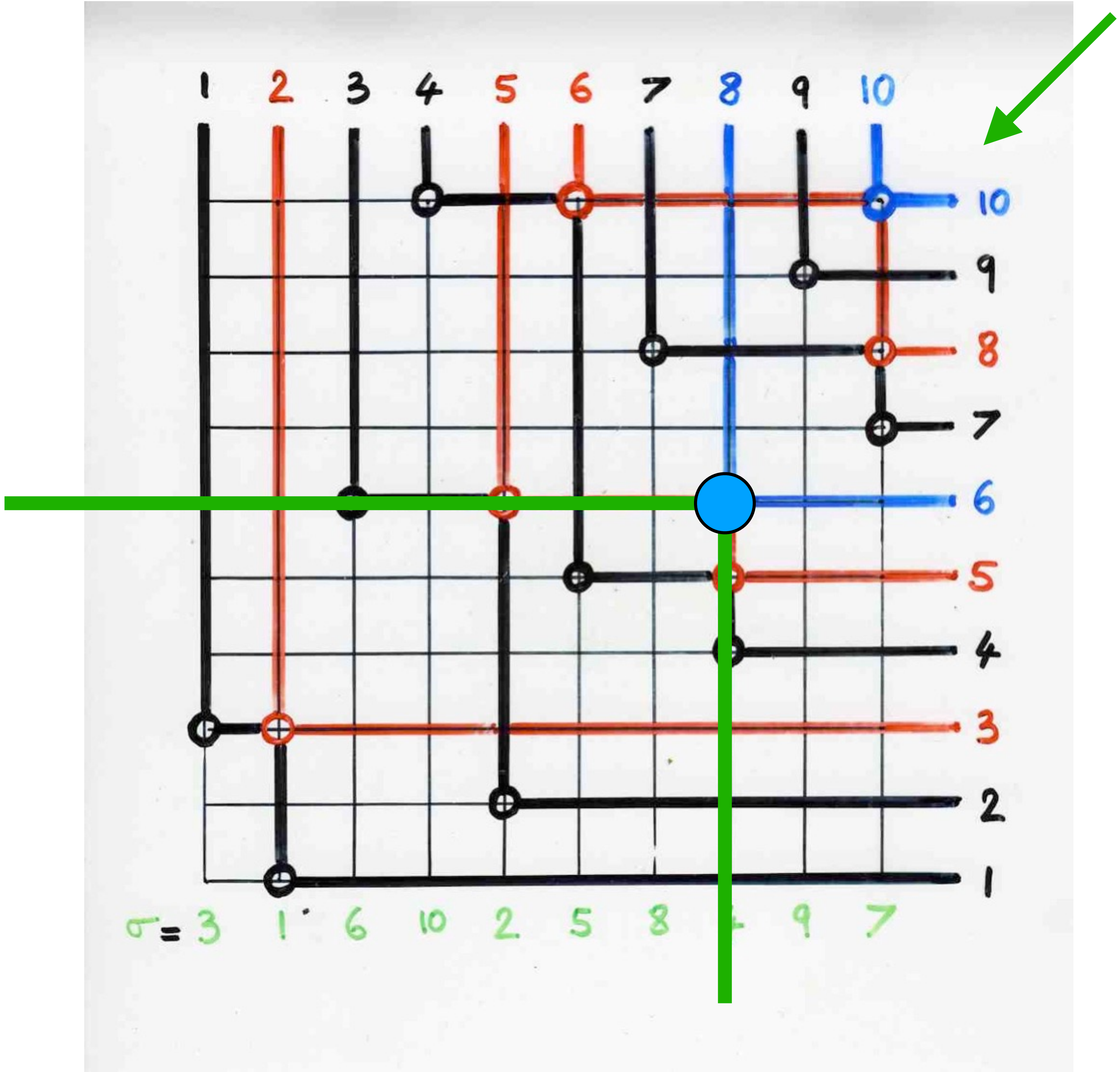
σ = 3 1 6 10 2 5 8 4 9 7



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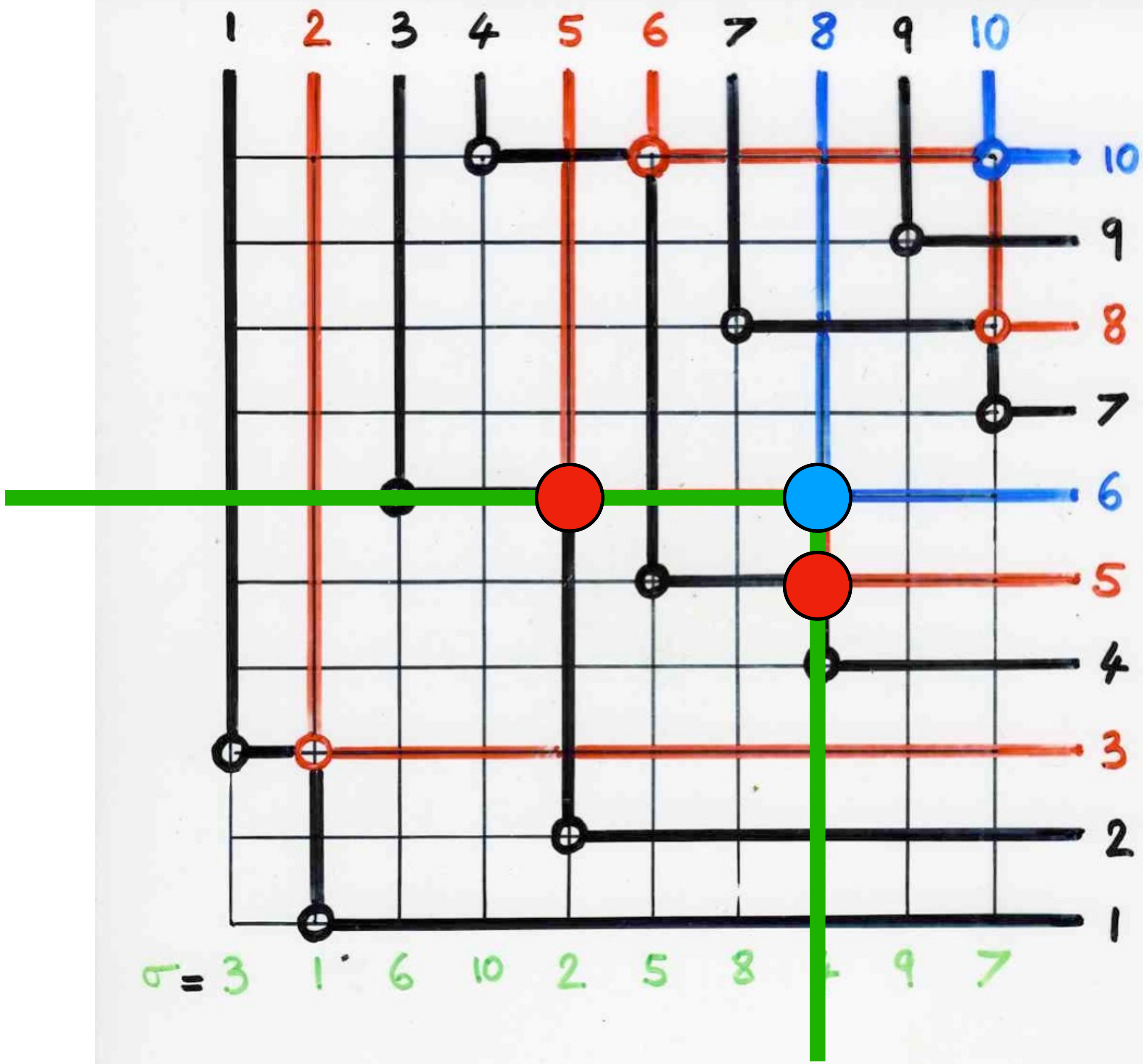

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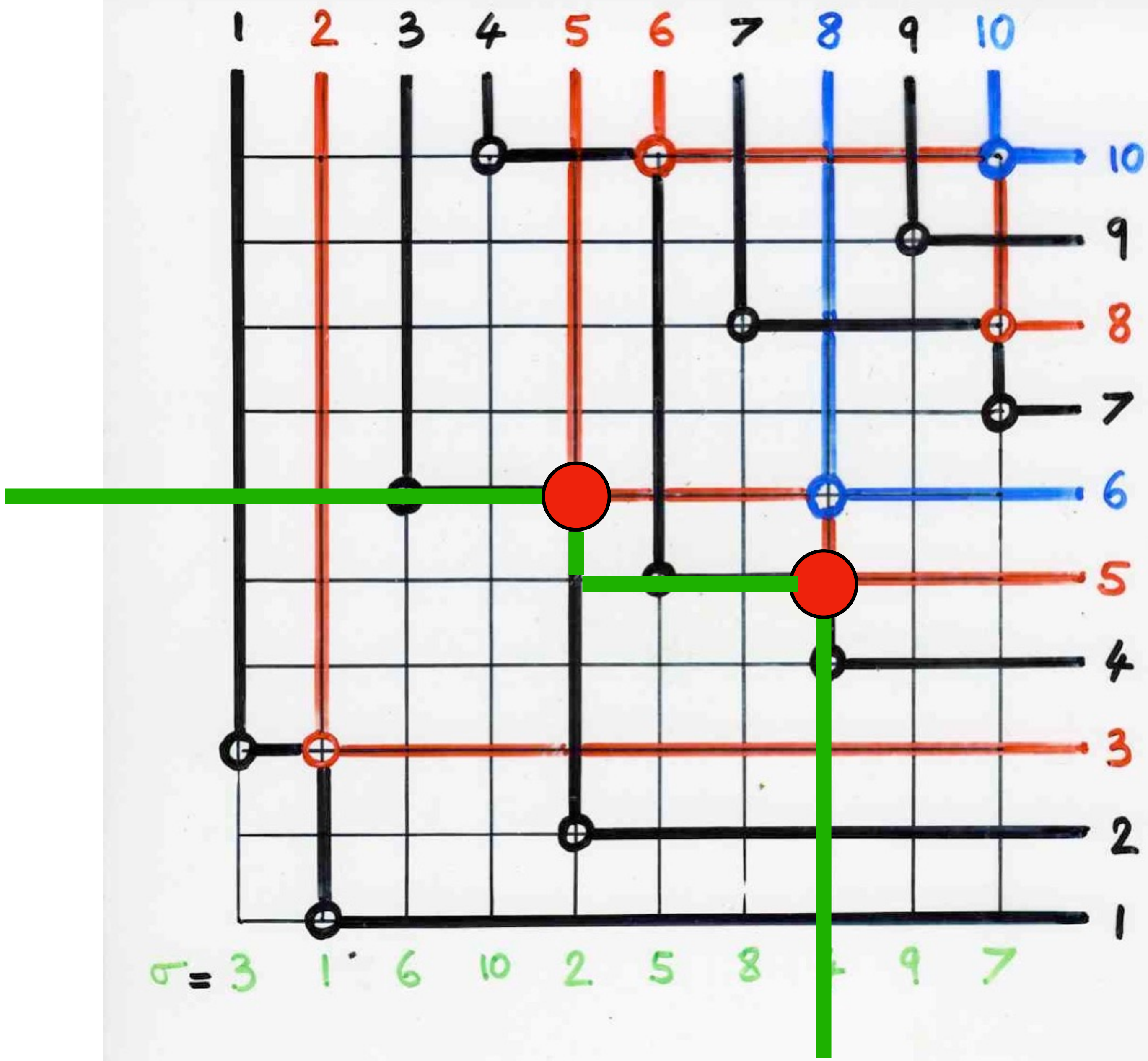
σ = 3 1 6 10 2 5 8 4 9 7



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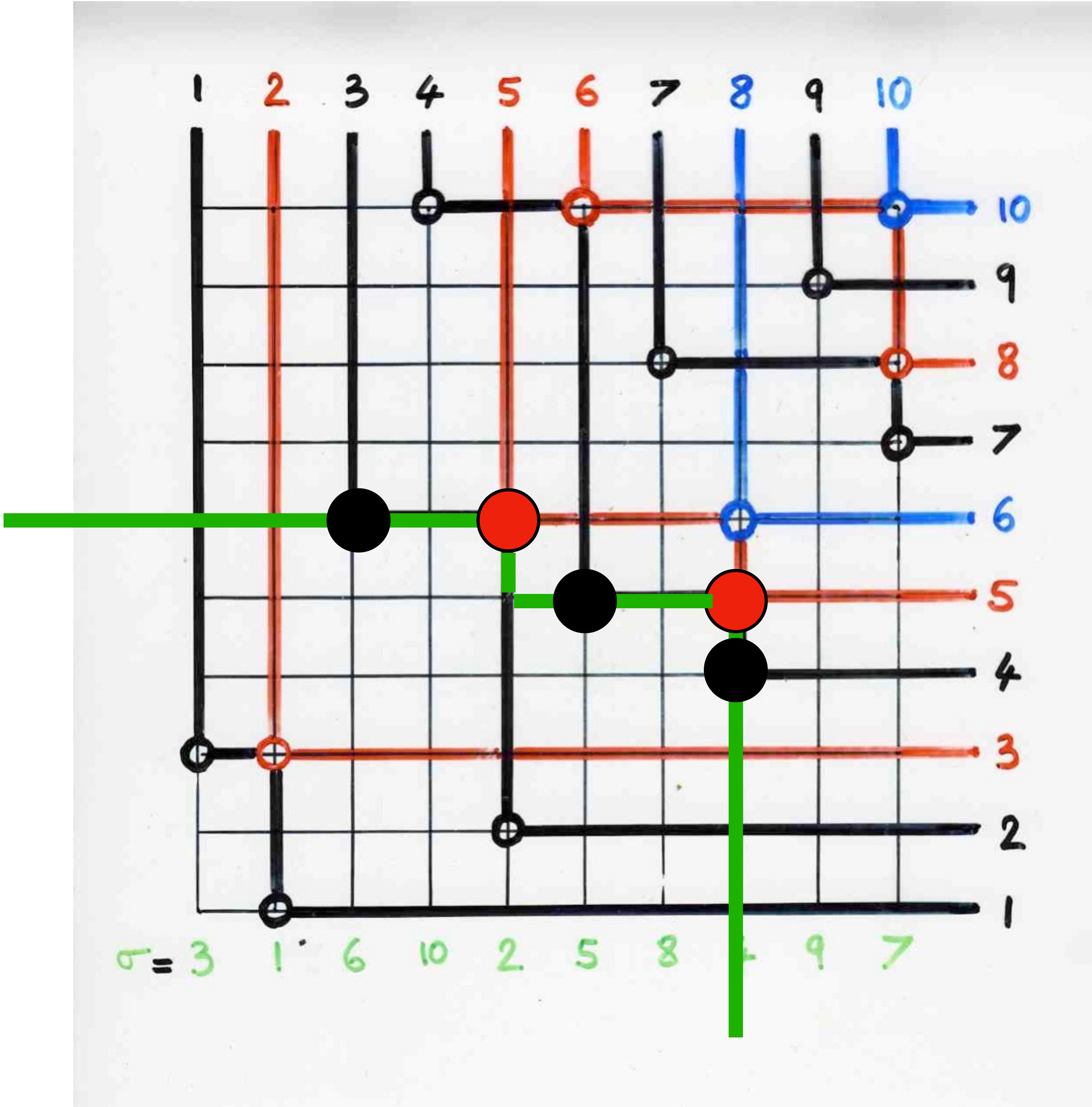
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Erdös, Szekeres (193-)

$\sigma \in \mathfrak{S}_n \qquad n \geq N^2$

$\exists$ increasing / decreasing subsequence $|\tau| \geq N$

Erdös, Szekeres (193-)

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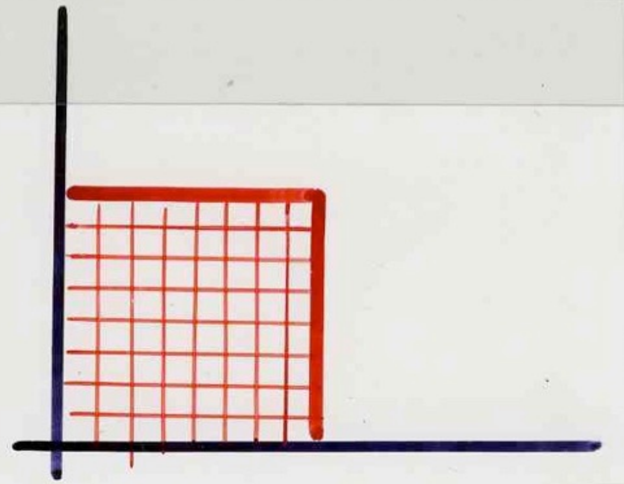
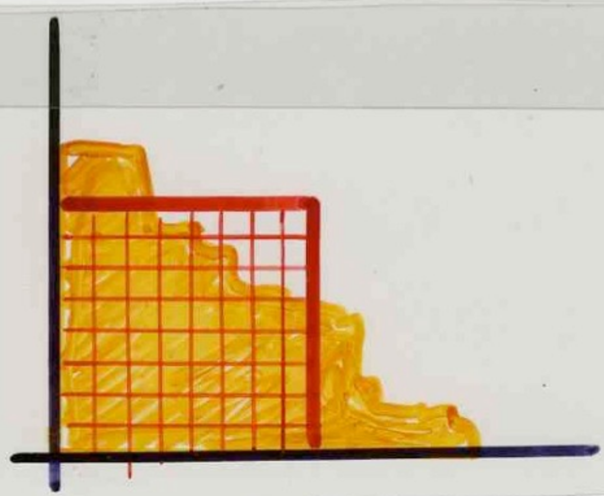


Erdös, Szekeres (193-)

$\sigma \in \mathfrak{S}_n \qquad n \geqslant N^2$

$\exists$ increasing / decreasing subsequence $|\tau| \geqslant N$



extension:

Greene theorem

C. Greene, 1974

σ permutation $\mathfrak{S}_n$

$k \in \mathbb{N}$

$I_k(\sigma)$ = maximal number of elements in a union of k increasing subsequences of σ

$D_\ell(\sigma)$ = - - - - - . . k decreasing . . .

Theorem (Greene)(1974)

$$\sigma \xrightarrow{RS} (P, Q)$$

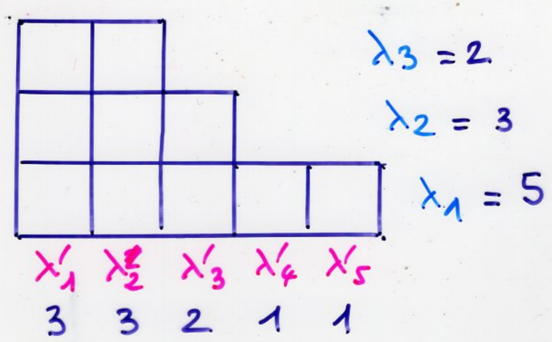
$\lambda = (\lambda_1, .., \lambda_r)$ $\lambda_1 \geqslant \cdot\cdot \geqslant \lambda_r$

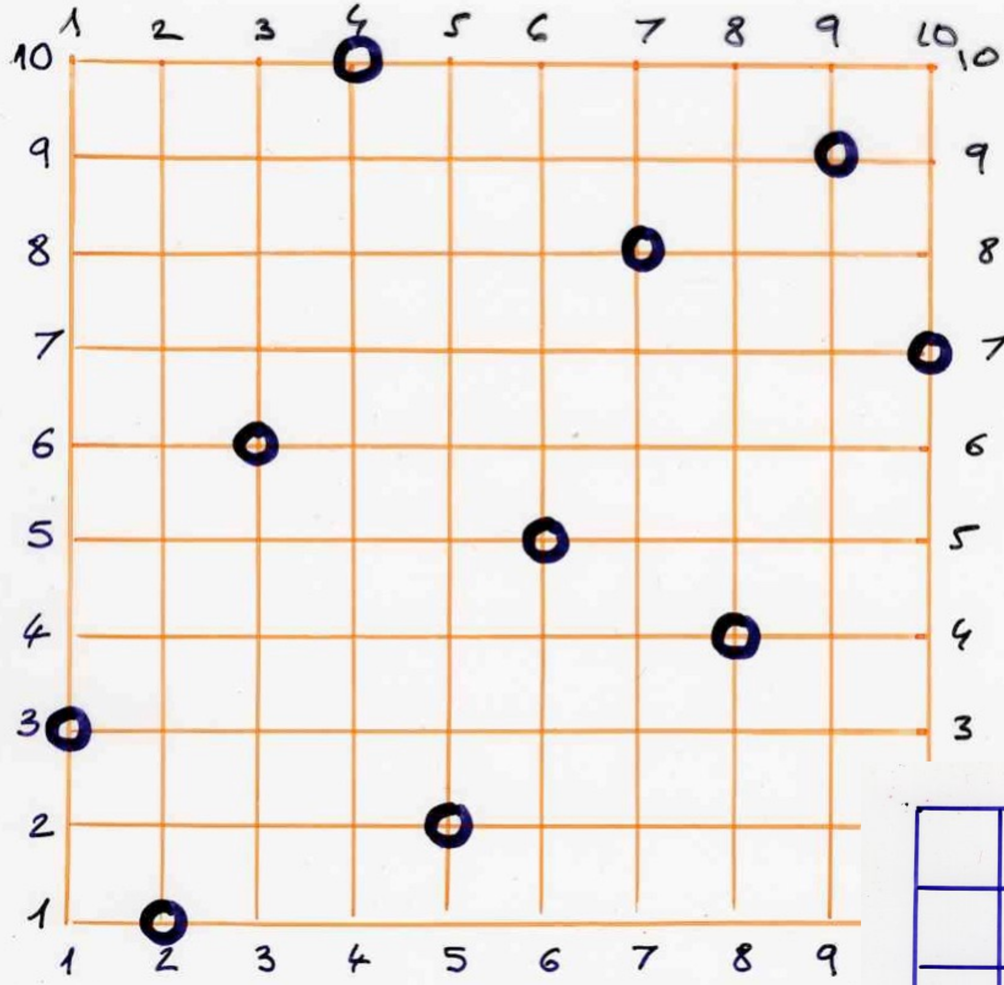
common shape of P and Q

$$I_k(\sigma) = \lambda_1 + .. + \lambda_k$$

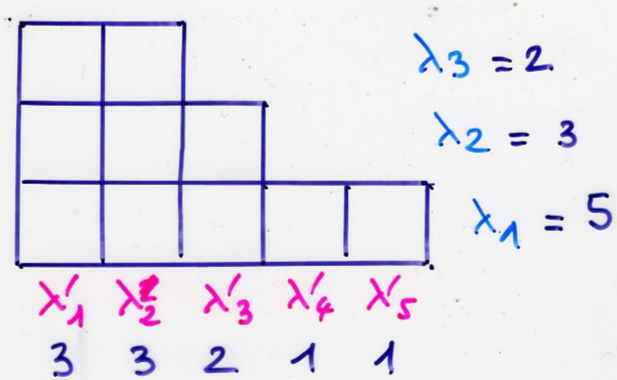
$$D_\ell(\sigma) = \lambda'_1 + .. + \lambda'_\ell$$

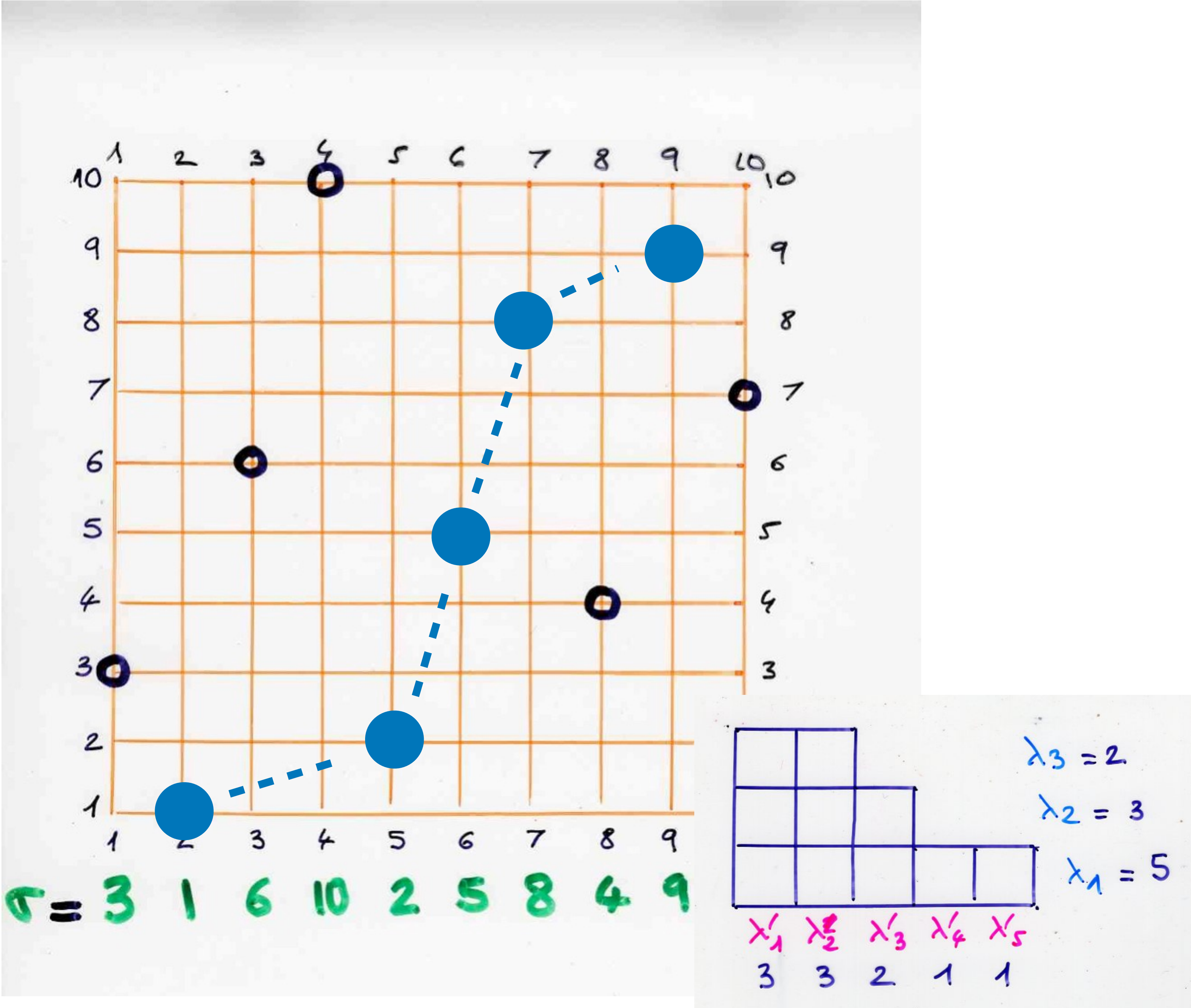
conjugate partition



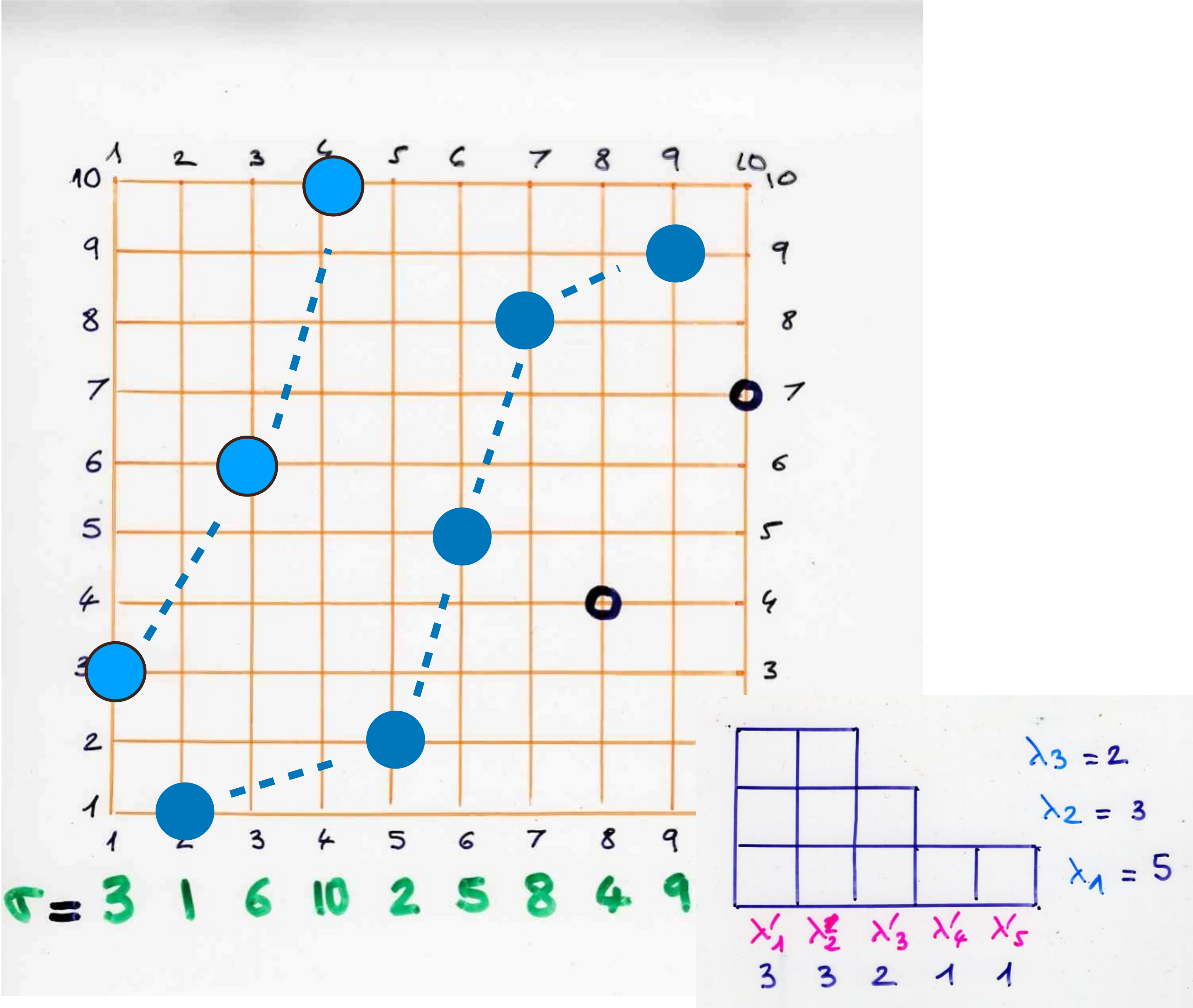


$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9$





σ = 3 1 6 10 2 5 8 4 9
λ3 = 2
λ2 = 3
λ1 = 5
λ'1 λ'2 λ'3 λ'4 λ'5
3 3 2 1 1



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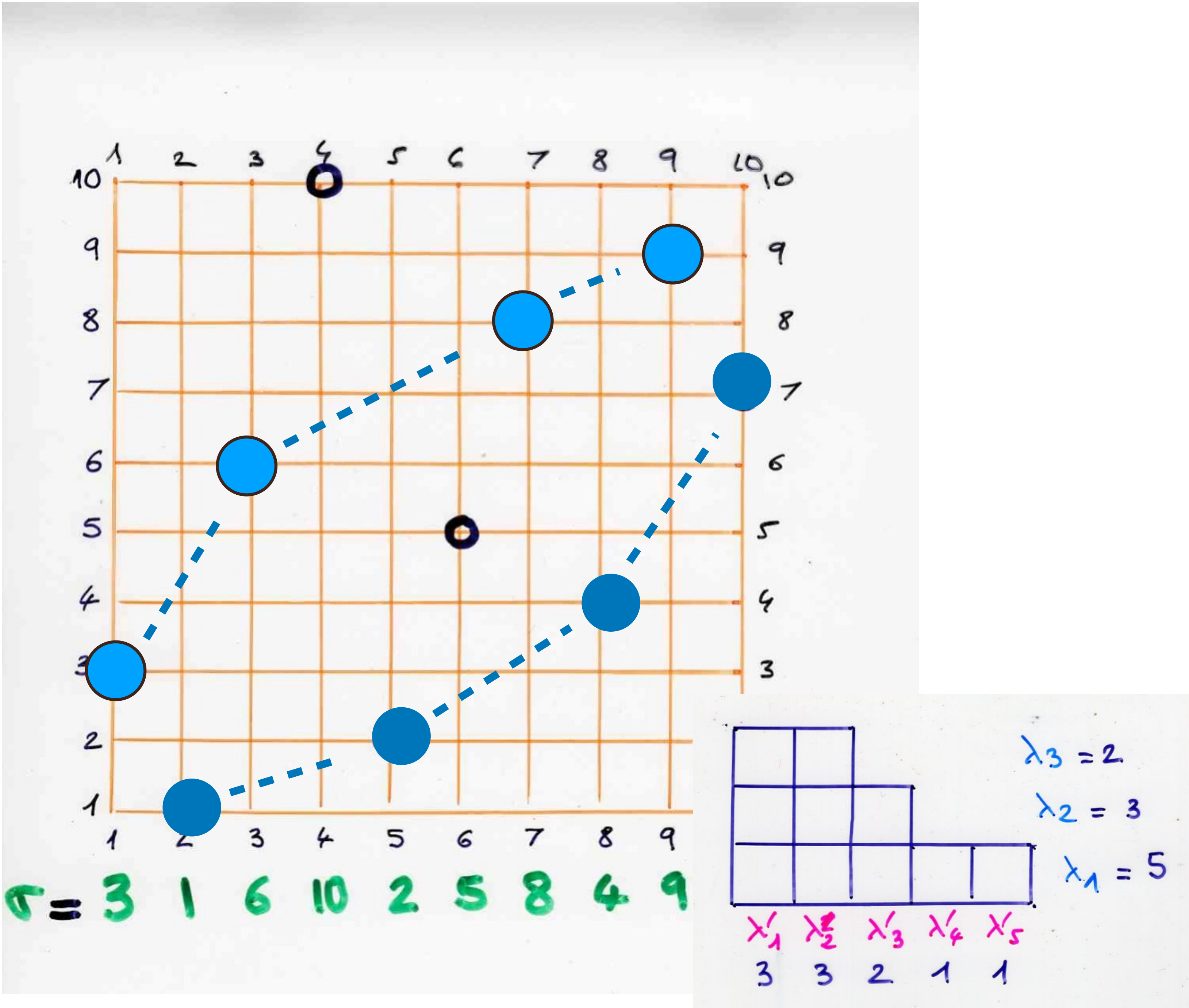
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RSK (Ch 1)

The Robinson-Shensted
correspondence

- Schensted's insertions
- geometric version with "shadow lines »

next:
- Schützenberger "jeu de taquin »
- Fomin "local rules" or "growth diagrams »

The end of Ch 1a