Course IMSc, Chennai, India

January-March 2018

The cellular ansatz: bijective combinatorics and quadratic algebra

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Chapter 0
introduction to the course

IMSc, Chennai
January 4, 2018

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This course is Part III of

« The IMSc bijective combinatorics course »

"the art of bijective mathematics"

Each course can be followed independantly

Two levels:

- for master and graduate students
- for professors and more advanced students

under the name « complements »

sometimes no proof

Part I: course IMSc 2016

An introduction to

enumerative, algebraic and bijective combinatorics

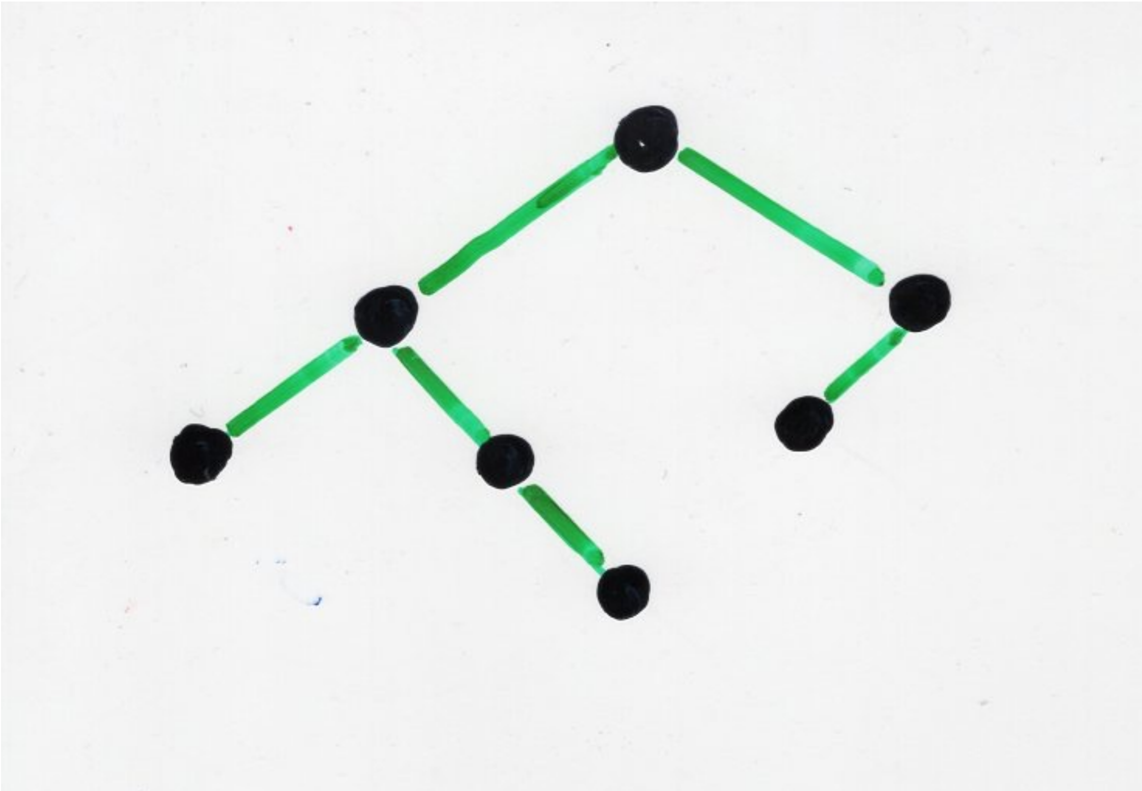
enumerative combinatorics

permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

number of permutations on $\{1, 2, \dots, n\}$ $= 1 \times 2 \times 3 \times \dots \times n$

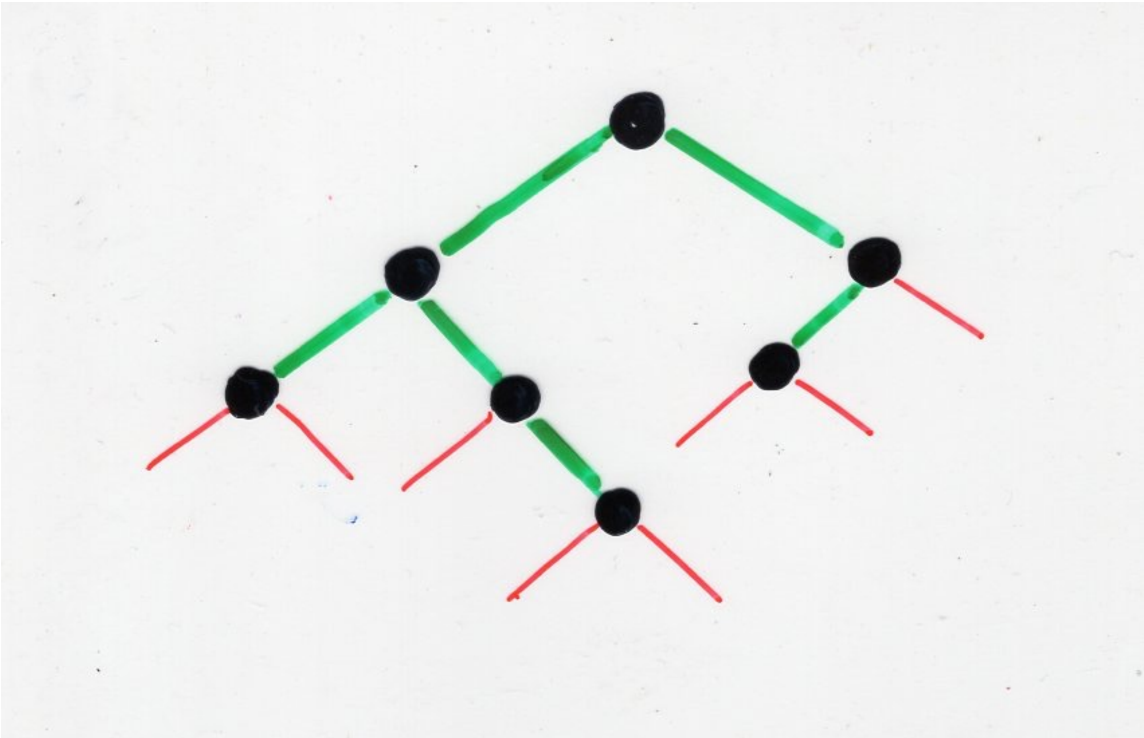
$= n!$



binary tree

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



complete
binary tree

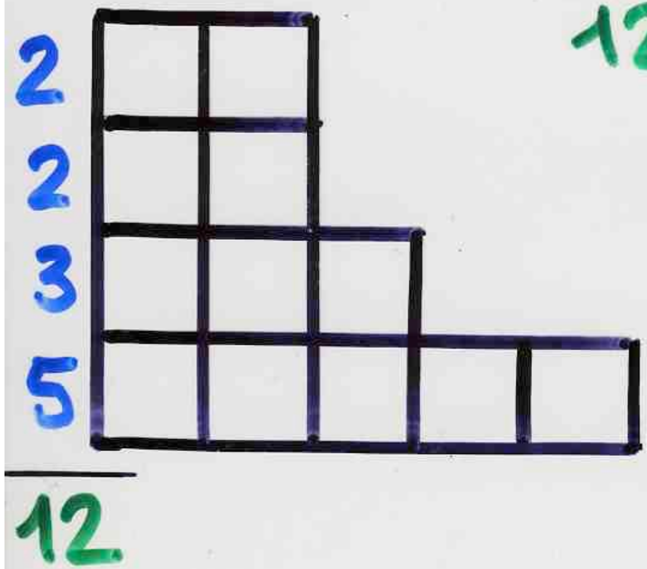
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan
numbers

enumerative combinatorics

example with the enumeration of

Young tableaux



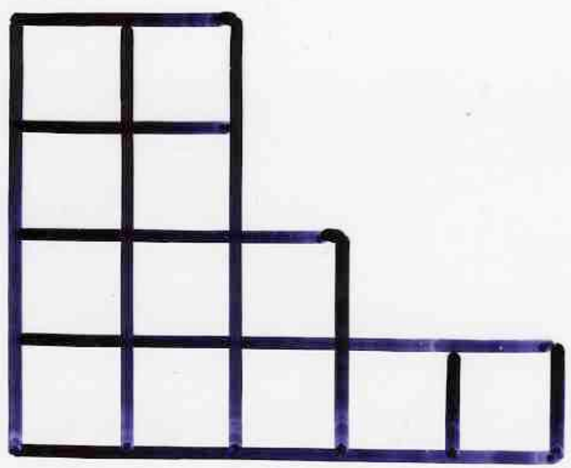
$12 = n = 5 + 3 + 2 + 2$

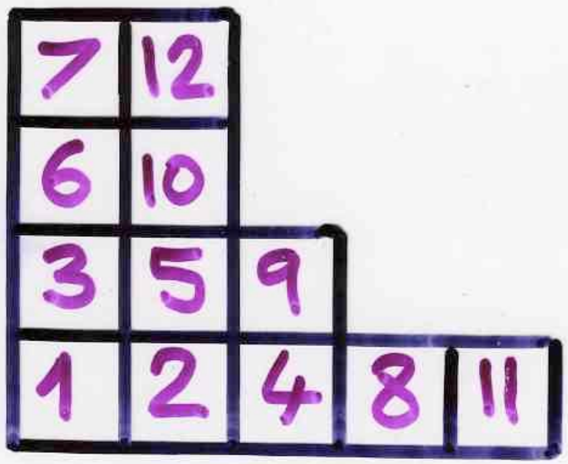
Ferrers

diagram

Partition of n

λ





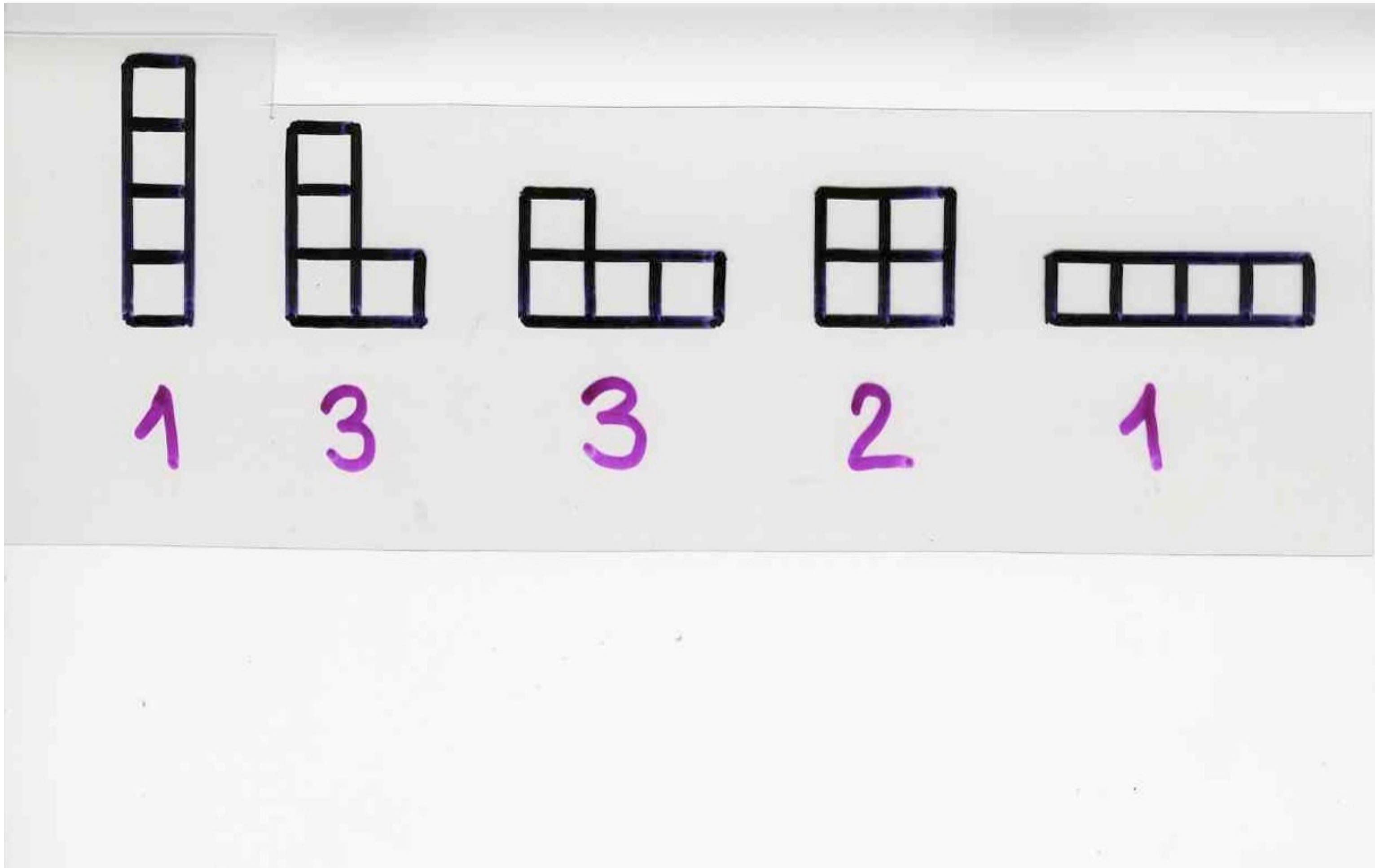
Young tableau

shape λ

f_λ = number of Young tableaux with shape λ

hook length formula

A beautiful Identity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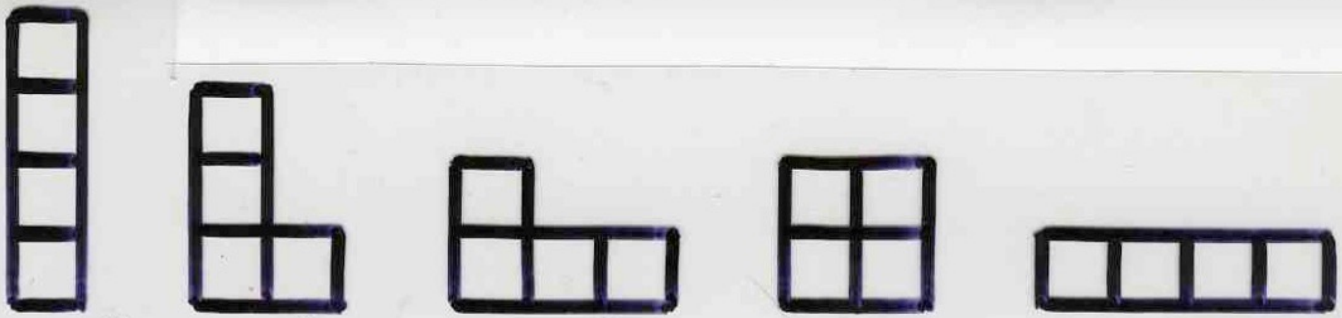
3
3
2
1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

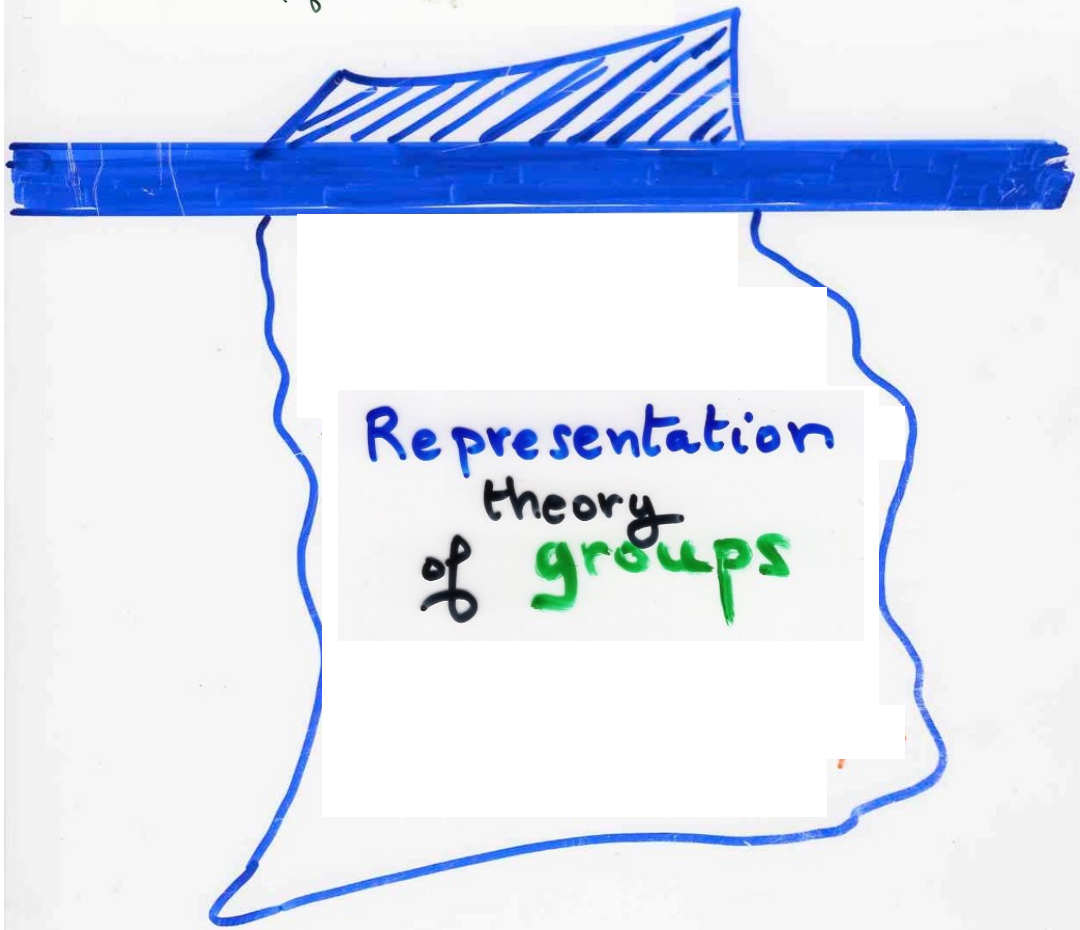
$$n! = \sum_{\substack{\lambda \\ \text{partitions} \\ \text{of } n}} \left(f_\lambda\right)^2$$

$$n! = \sum_{\lambda} (f_\lambda)^2$$

partitions of n

$$n! = \sum_{\lambda} (f_{\lambda})^2$$

partitions of n



algebraic combinatorics

Representation theory of groups

Case of the group $\mathfrak{S}_n$ permutations

irreducible representations $\longleftrightarrow$ partition λ of n

dimension of the irreducible representation (= order of the matrices) $=$ f_λ number of Young tableaux with shape λ

finite group G

$$|G| = \sum_{R} (\deg R)^2$$

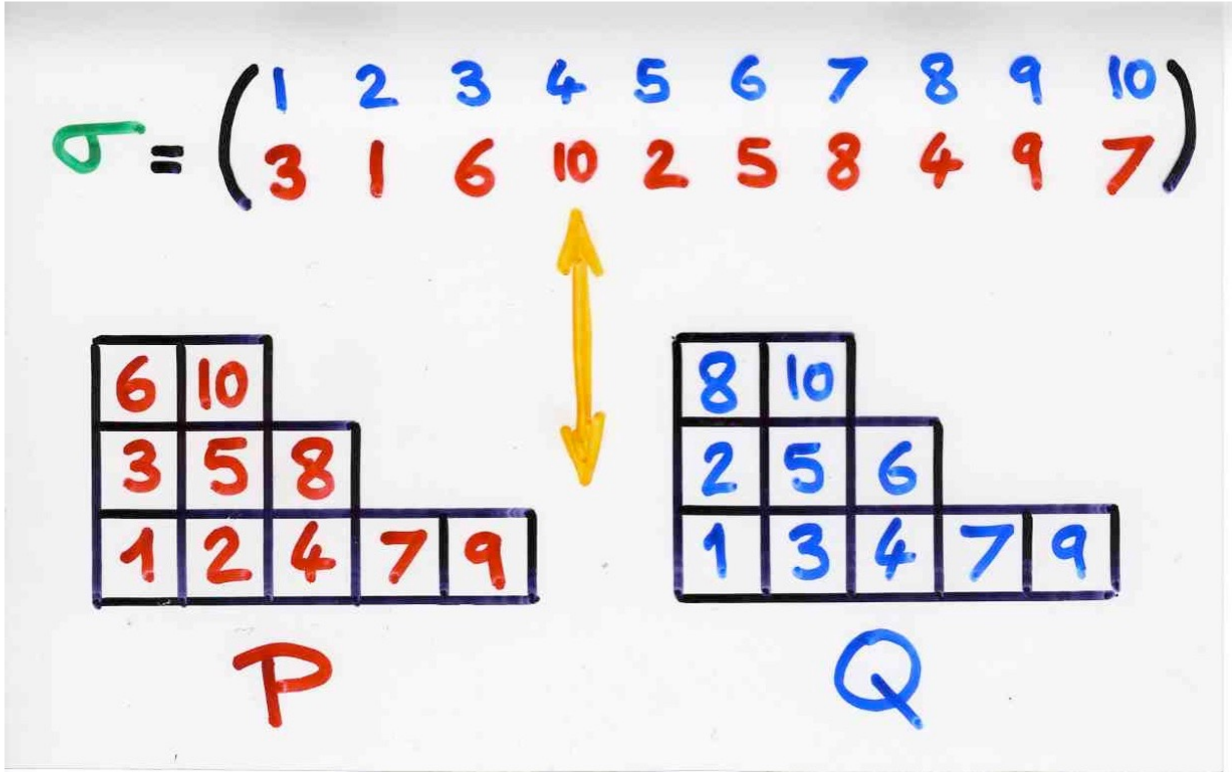
R: irreducible representation

for the symmetric group $\mathfrak{S}_n$ (permutations)

$$n! = \sum_{\lambda} (f_\lambda)^2$$

λ: partition of n

Bijective combinatorics



The Robinson-Schensted correspondence between permutations and pairs of (standard) Young tableaux with the same shape

The Robinson-Schensted correspondence

RSK

Robinson-Schensted-Knuth

Schützenberger

二

$=$

$\longleftrightarrow$

=

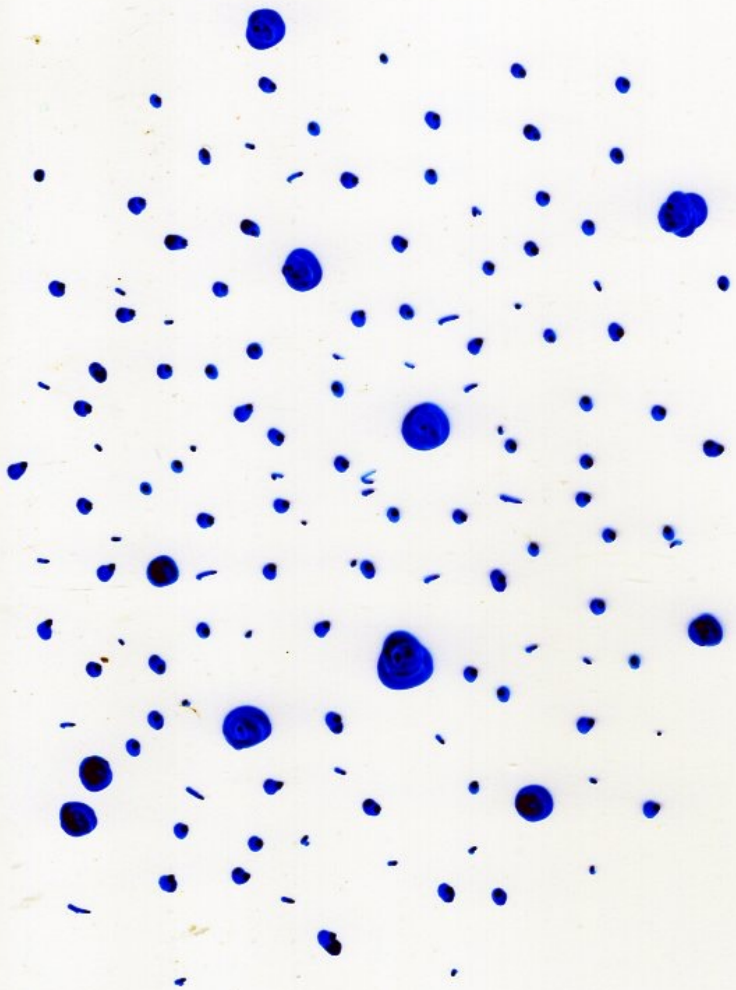
better understanding

↔

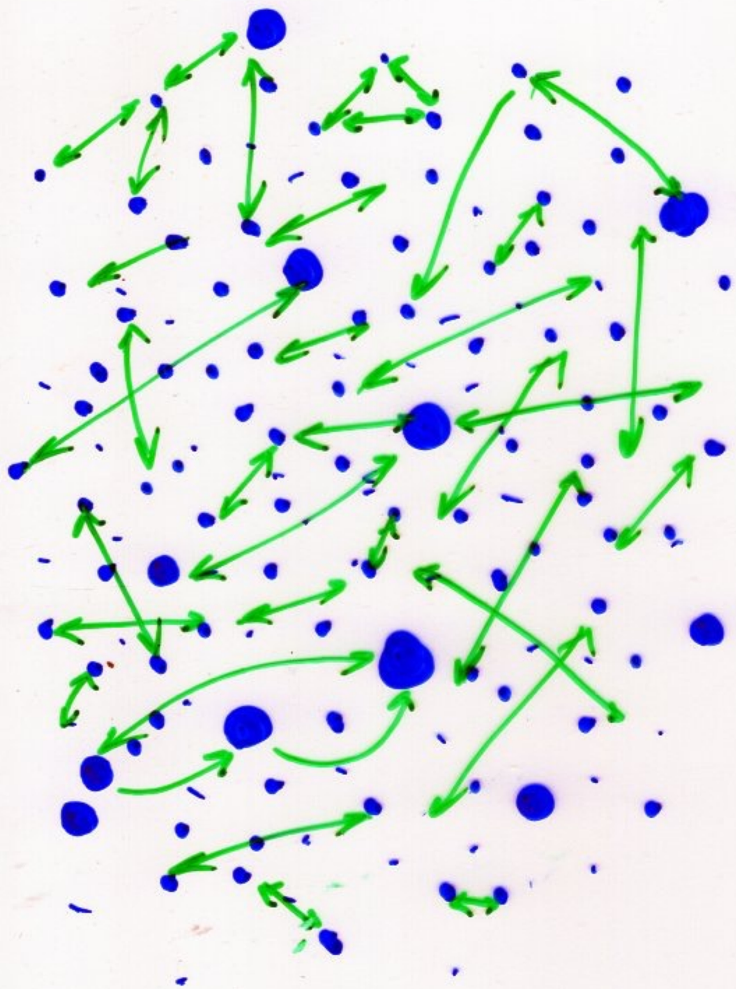
enumerative combinatorics

algebraic combinatorics

bijective combinatorics



many
formulae,
identities ...



many
formulae,
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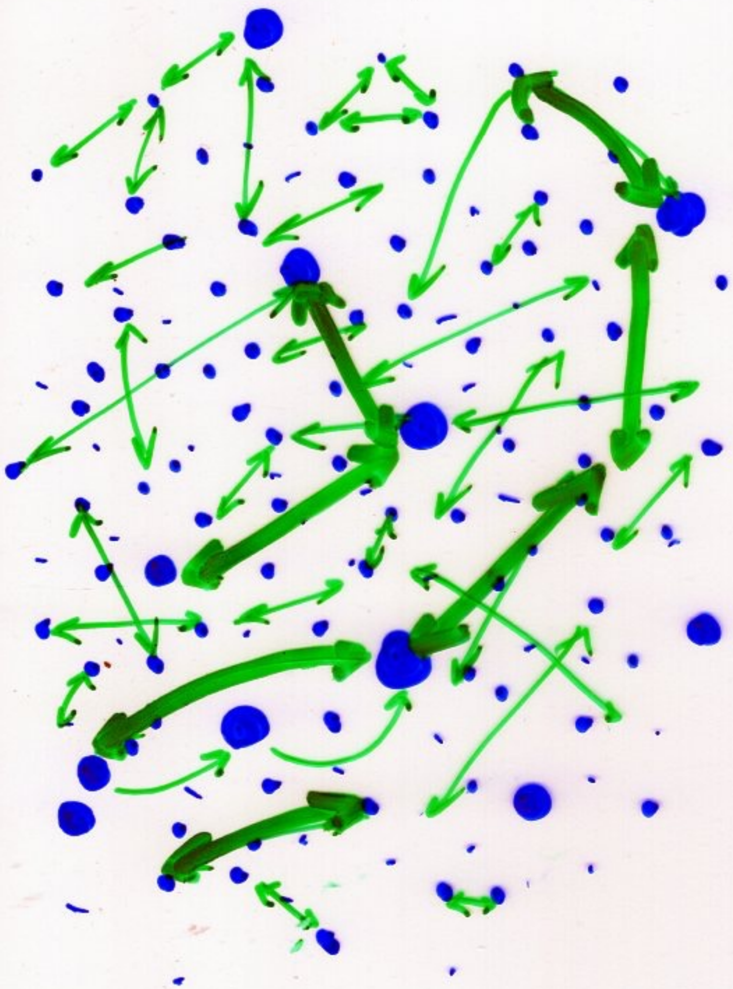
many
bijections
correspondences ...



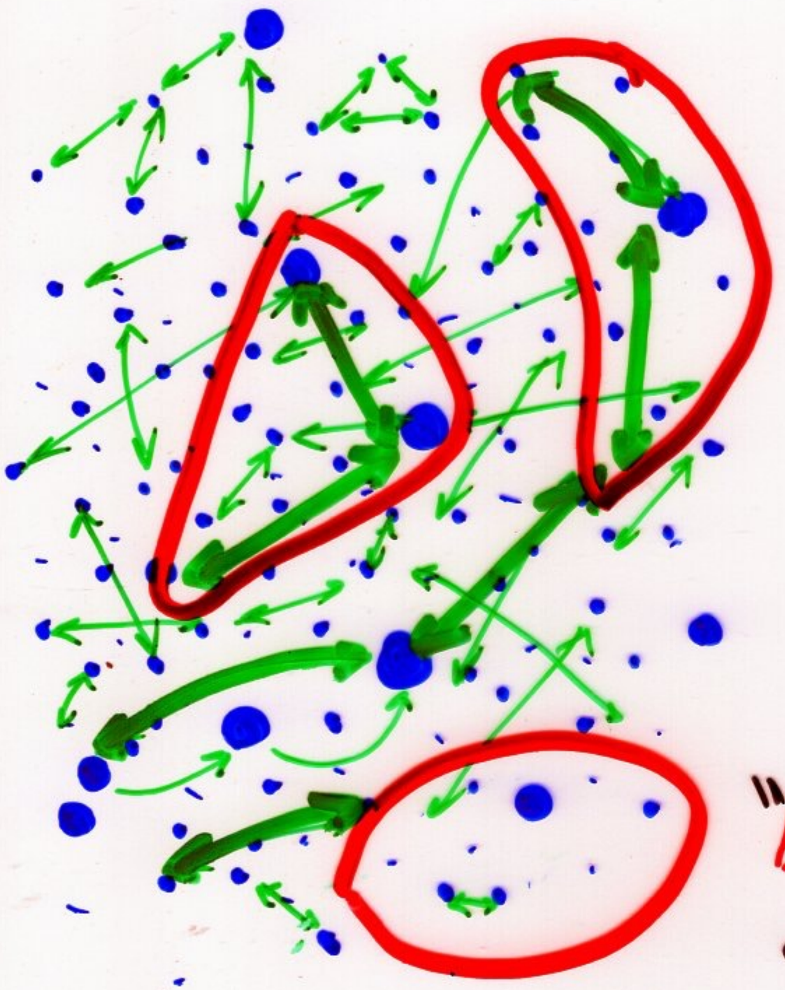
many
formulae,
identities ...

many
bijections
correspondences ...
basic bijections

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

n!

Catalan numbers



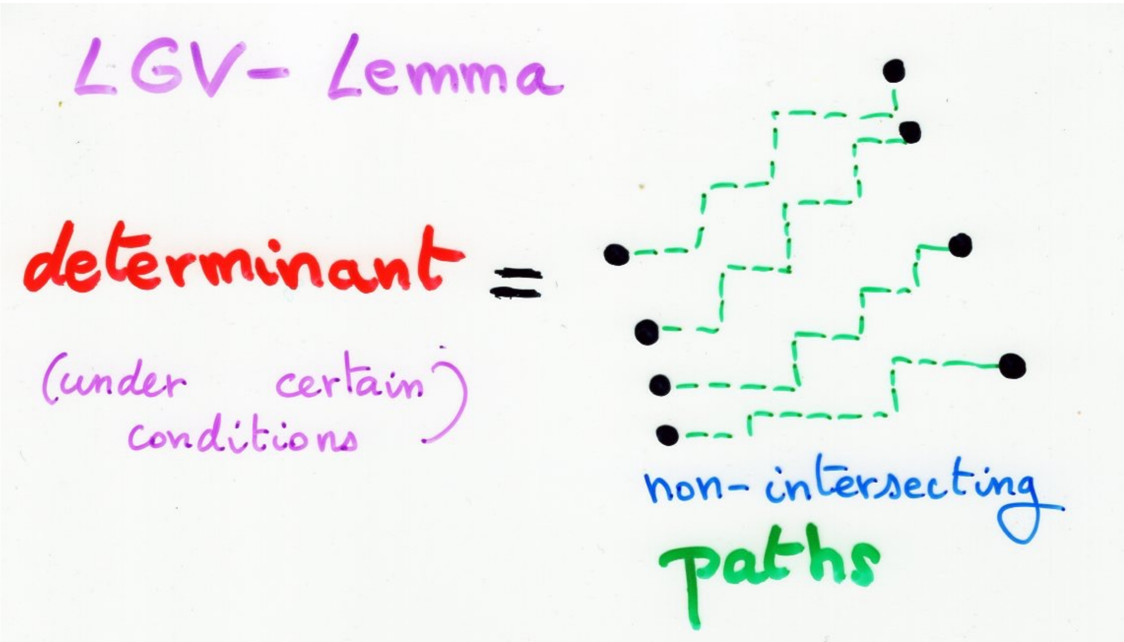
many
formulae,
identities ...

many
bijections
correspondences ...
basic bijections
"bijective tools"
or "basic Lemma"

« bijective tools »

Part I: course IMSc 2016

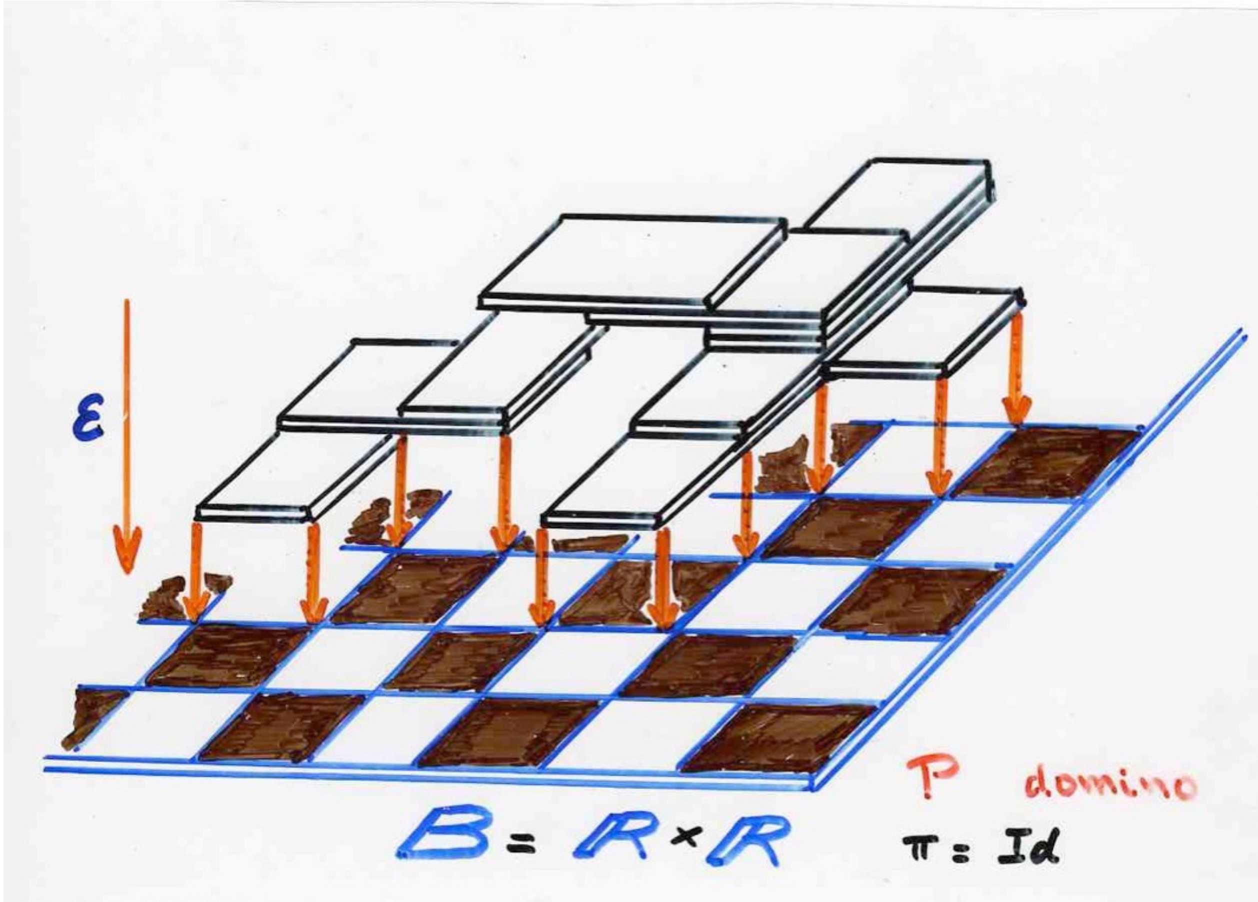
An introduction to enumerative, algebraic and bijective combinatorics



Part II: course IMSc 2017

Commutations and heaps of pieces

unified framework



Part III: course IMSc 2018

The « cellular ansatz »

"The cellular ansatz"

RSK Robinson-Schensted-Knuth,

PASEP Partially Asymmetric Exclusion Process,

ASM Alternating Sign Matrices,

8-vertex model,

Tilings, ...

under the same roof

(i) first step

"The **cellular** ansatz"

quadratic algebra **Q**

Q-tableaux

combinatorial objects on a 2D lattice

(i) first step

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

$UD = DU + Id$

an example

Heisenberg operators U, D

creation and annihilation opeators

quantum mechanics

$UD = DU + Id$

commutations

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

normal ordering in physics

$UD = DU + Id$ commutations

$UD \rightarrow DU$ $UD \rightarrow Id$ rewriting rules

$UUDD$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2UD \\
&= (DUDU + DU) + 2(DU + Id) \\
&= (DDUU + 2DU) + 2(DU + Id) \\
&= DDUU + 4DU + 2\,Id
\end{aligned}
$$

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

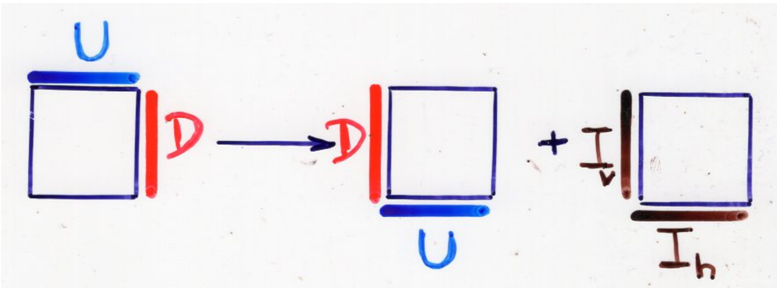
permutations

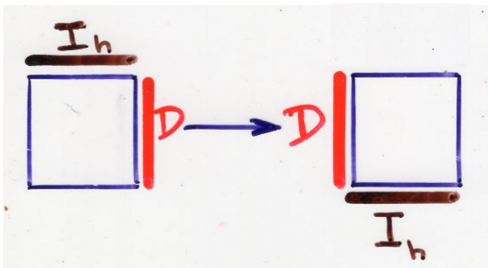
why the name "cellular ansatz" ?

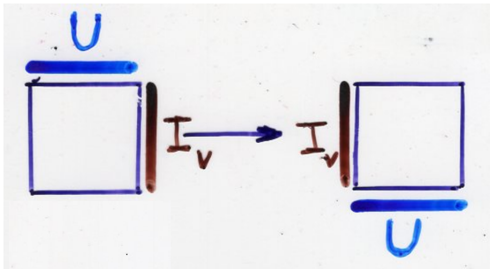
$UD = DU + Id$ commutations

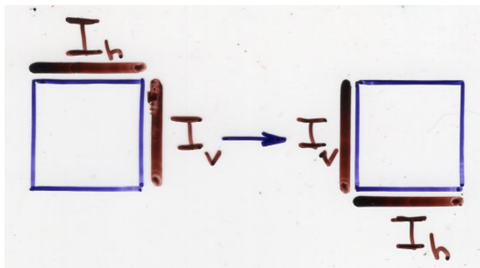
$UD \to DU$ $UD \to Id$ rewriting rules

planarization of the rewriting rules





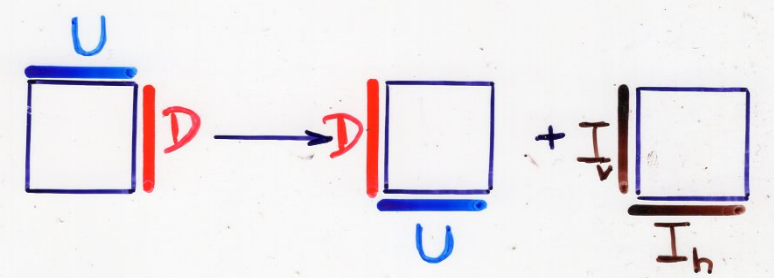




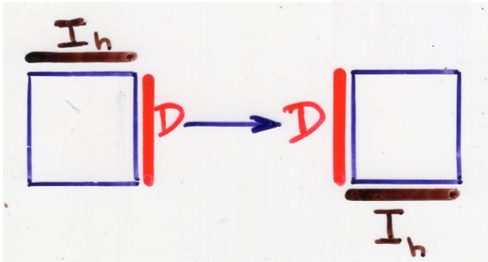
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

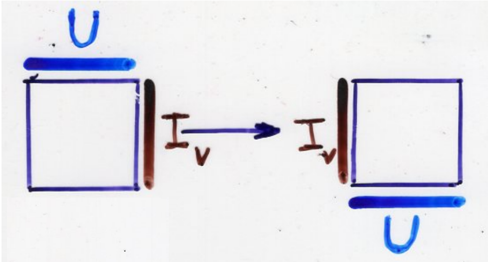
$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

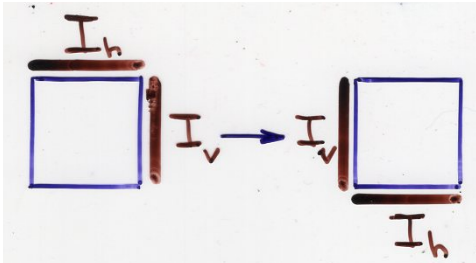
rewriting rules



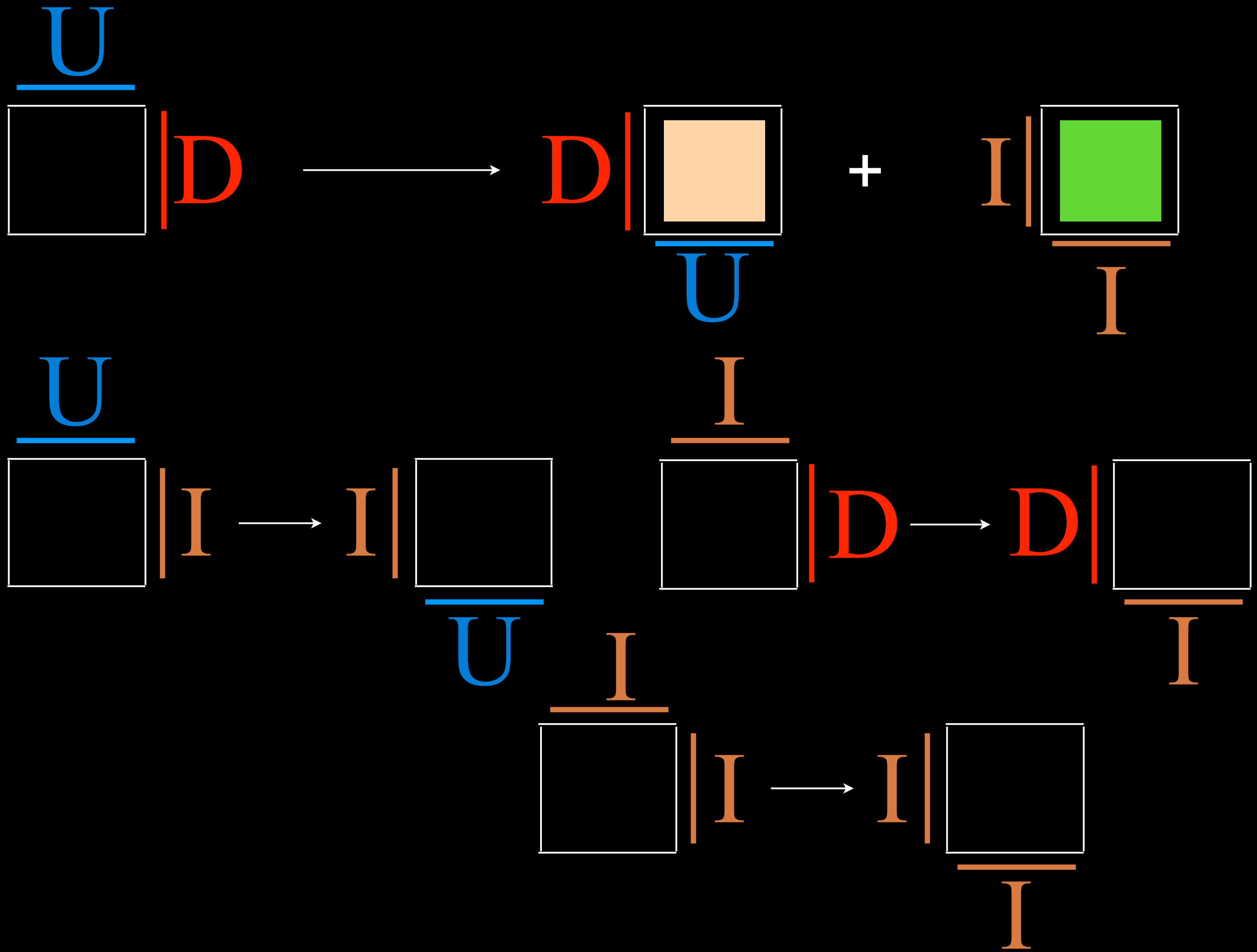
"planarization" of the "rewriting rules"

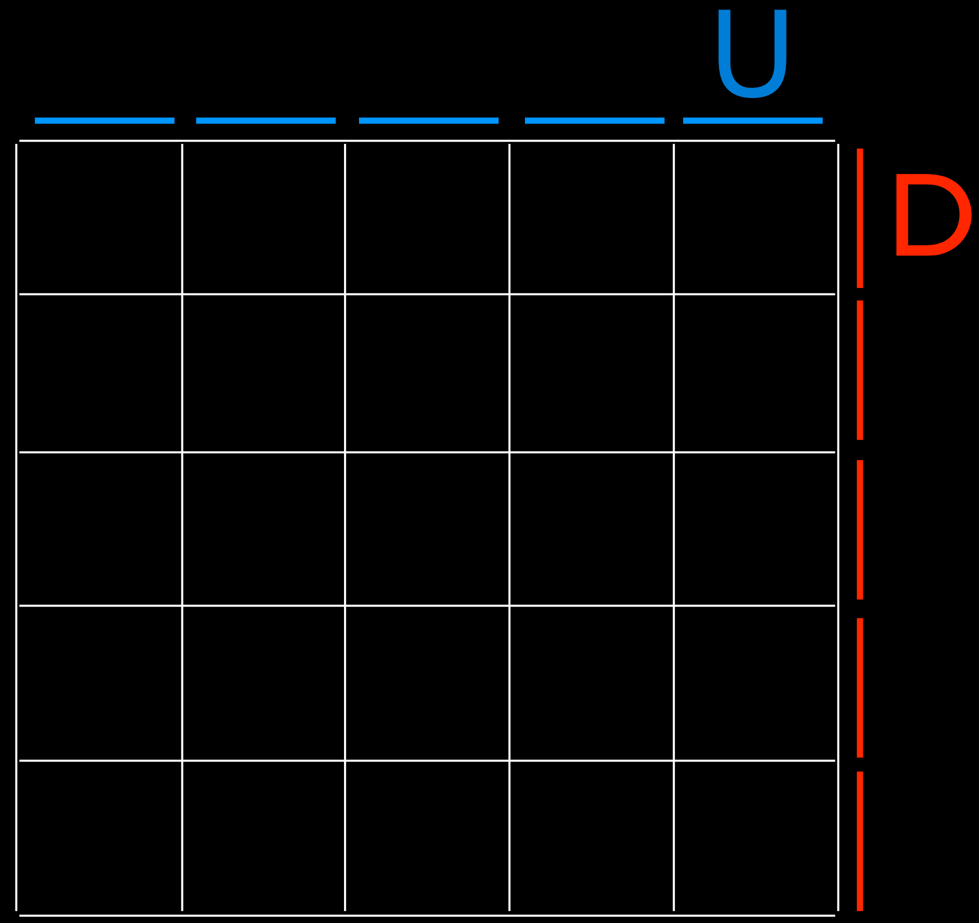






$$UD = qDU + I$$

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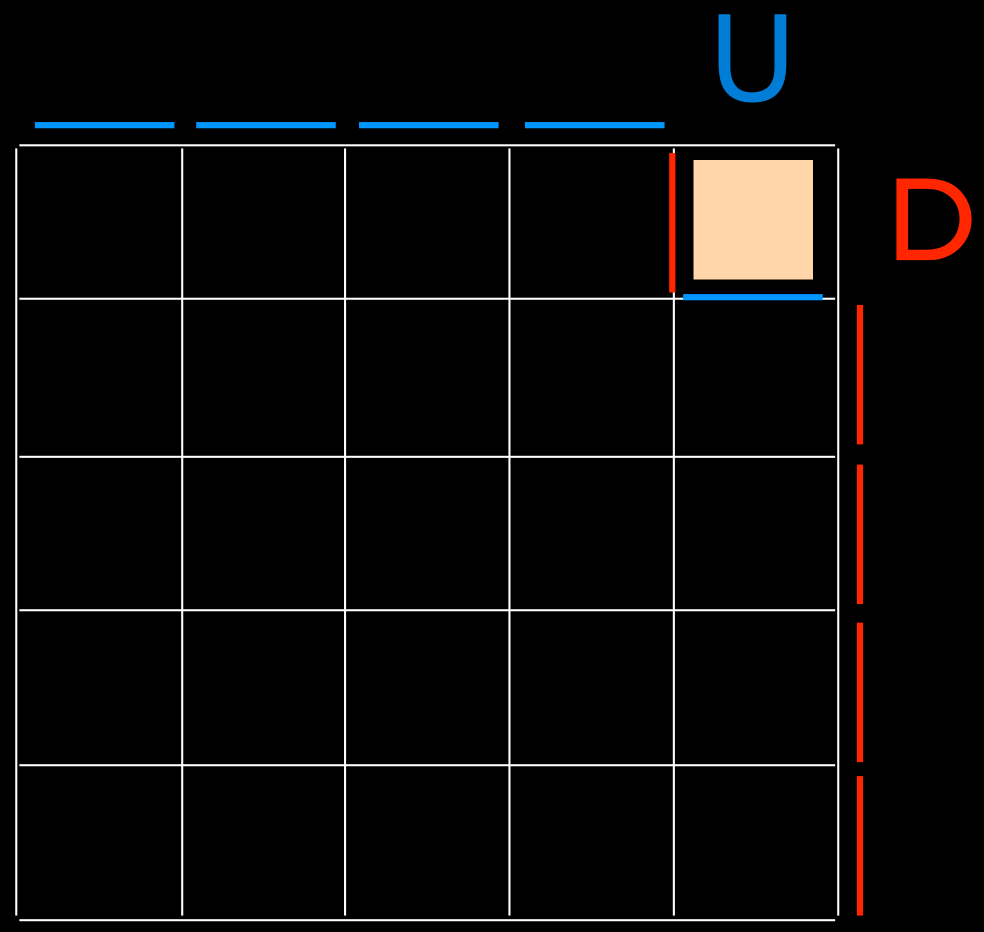
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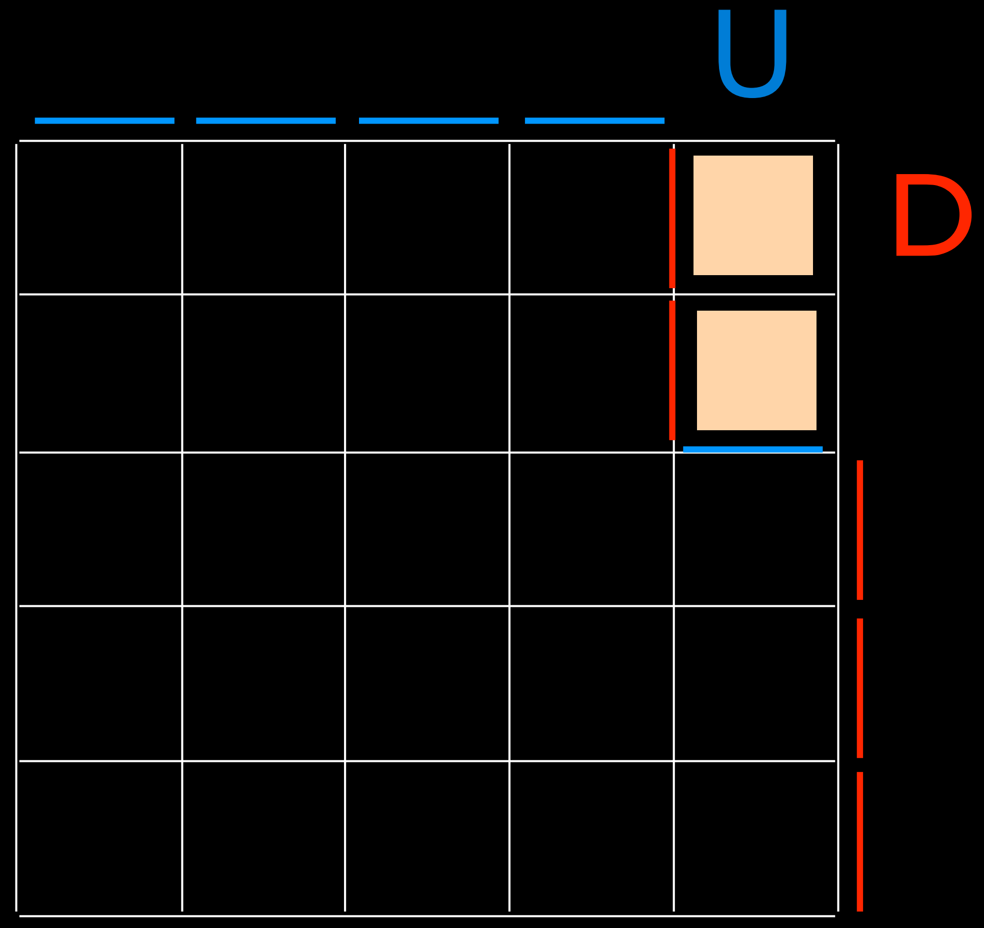
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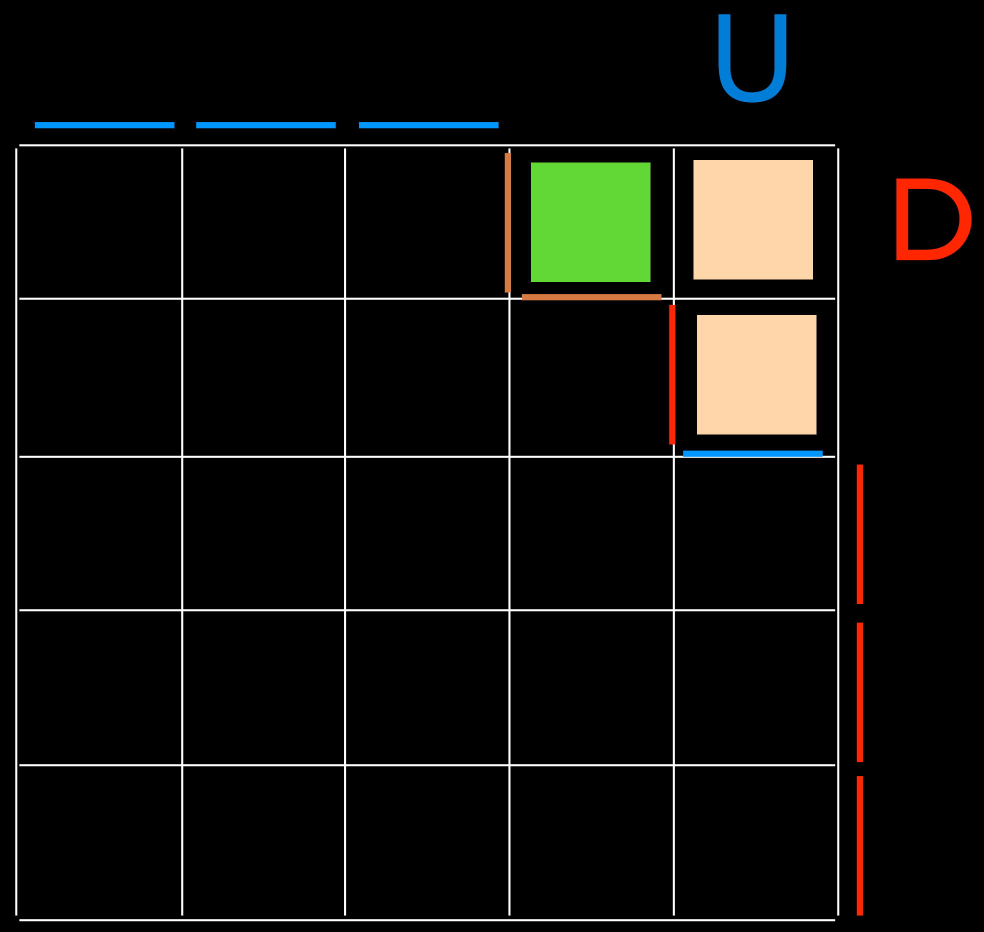
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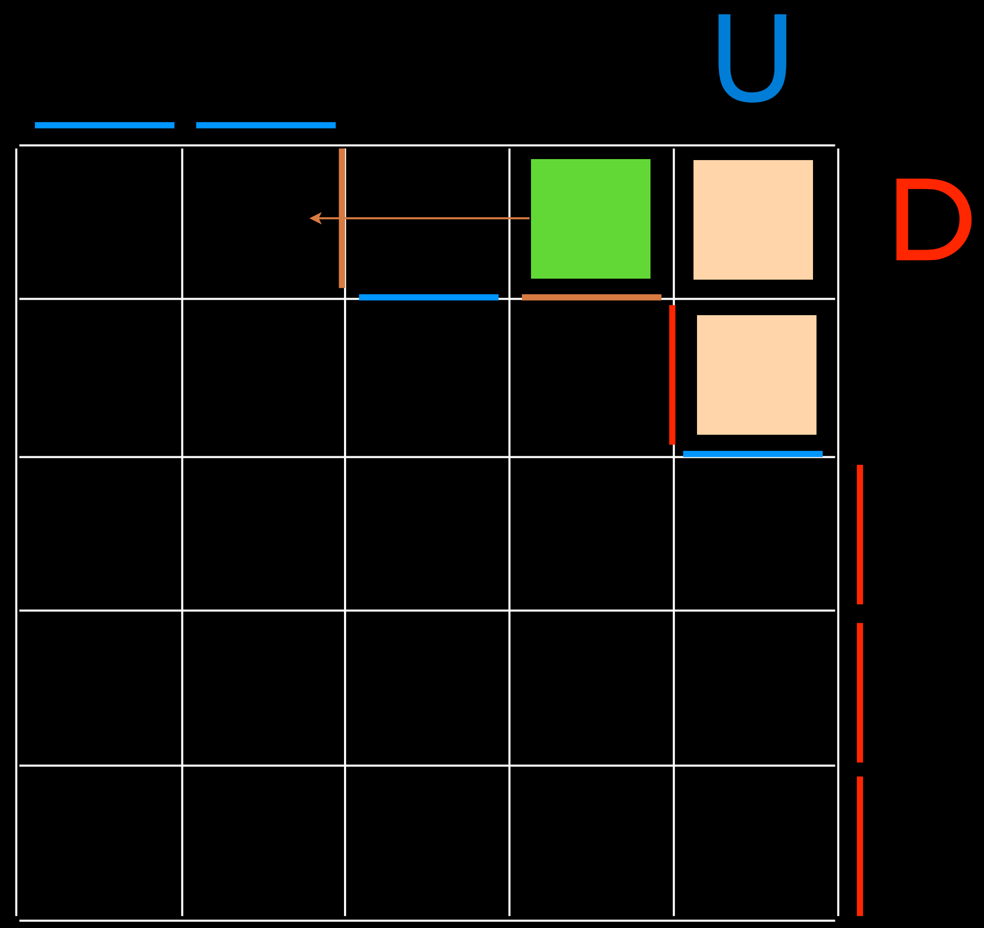

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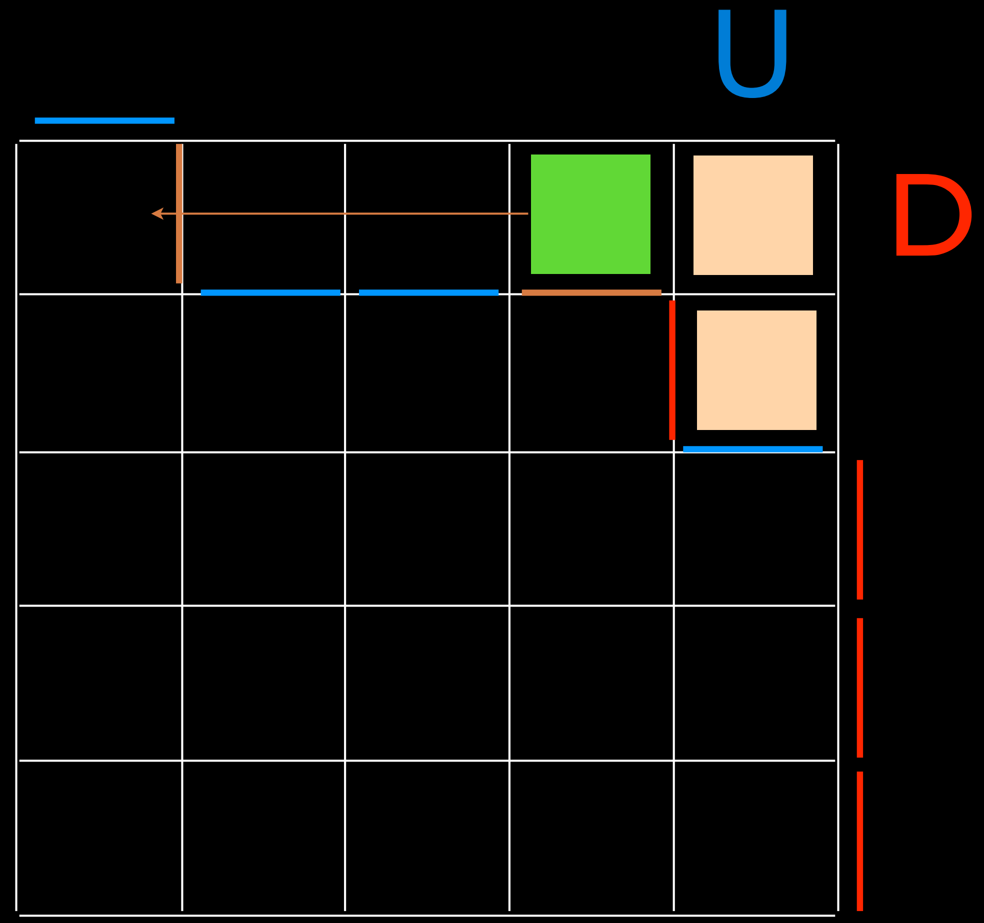
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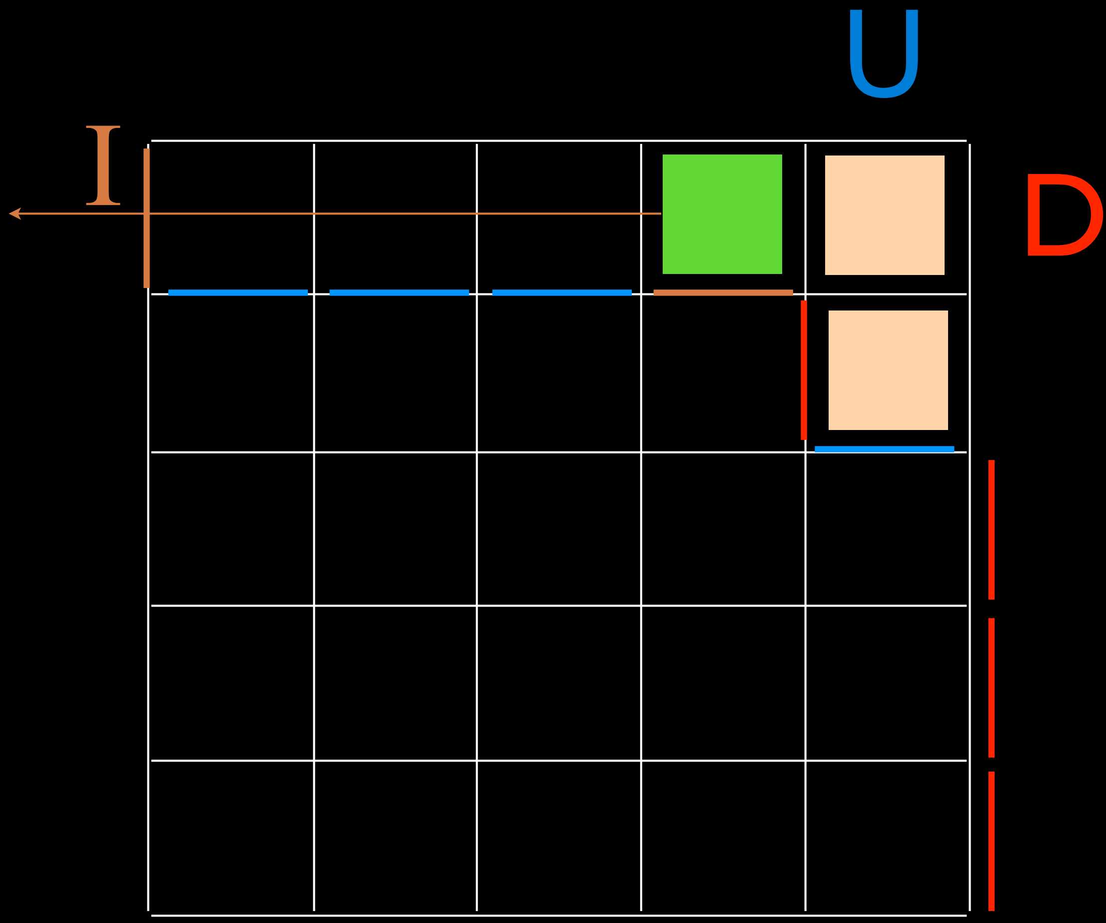

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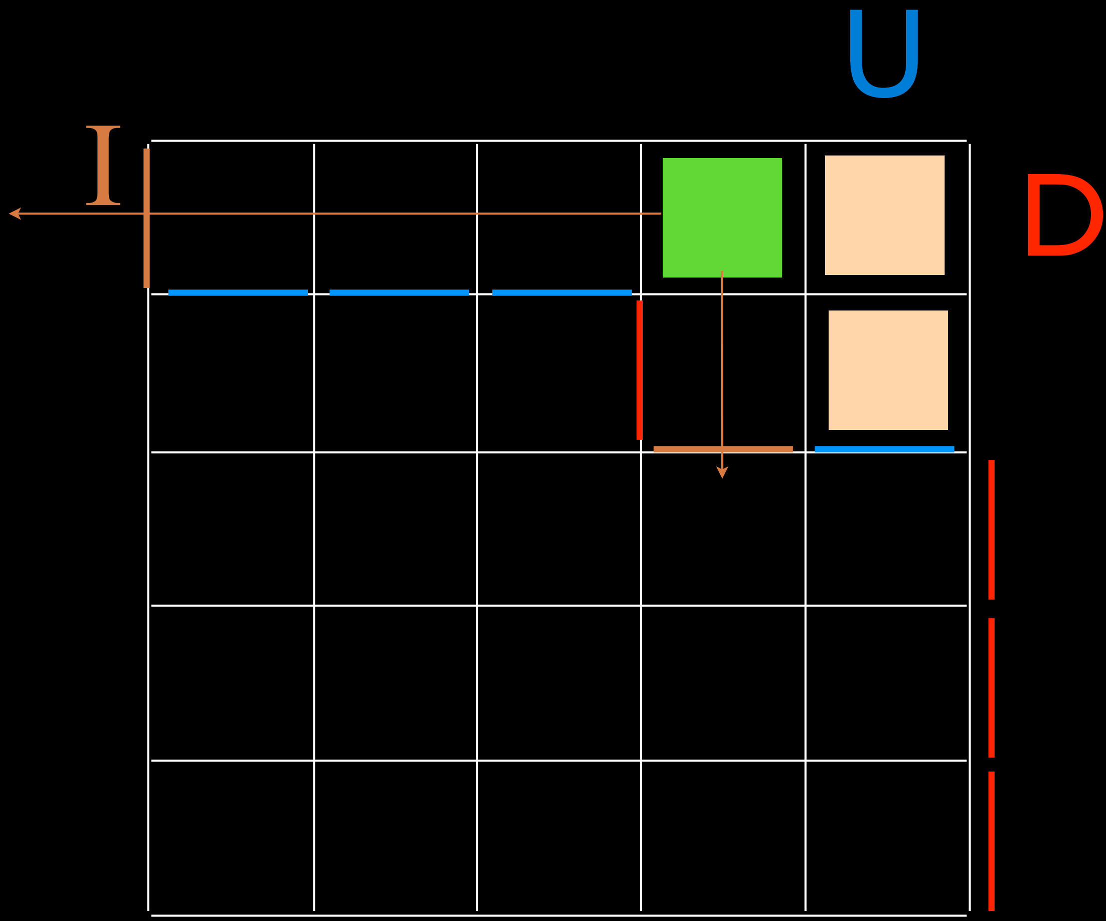
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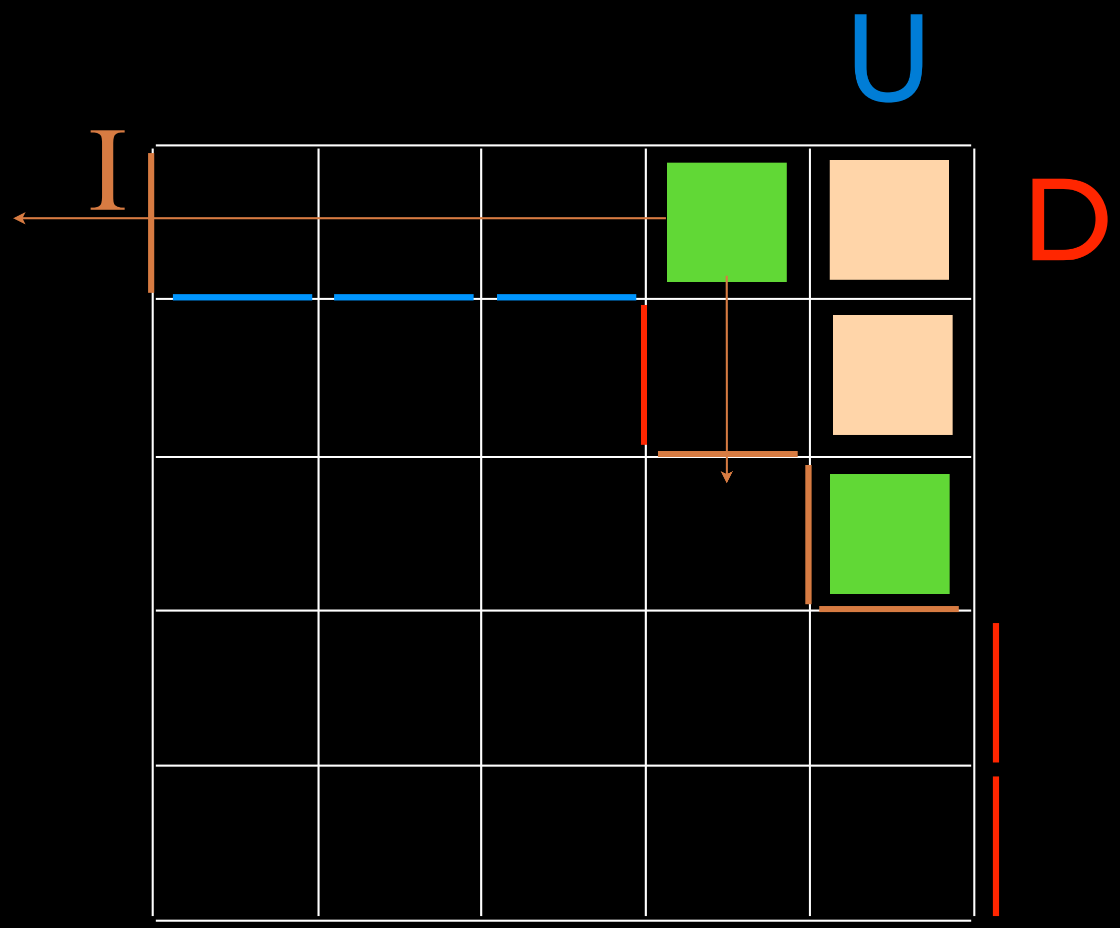
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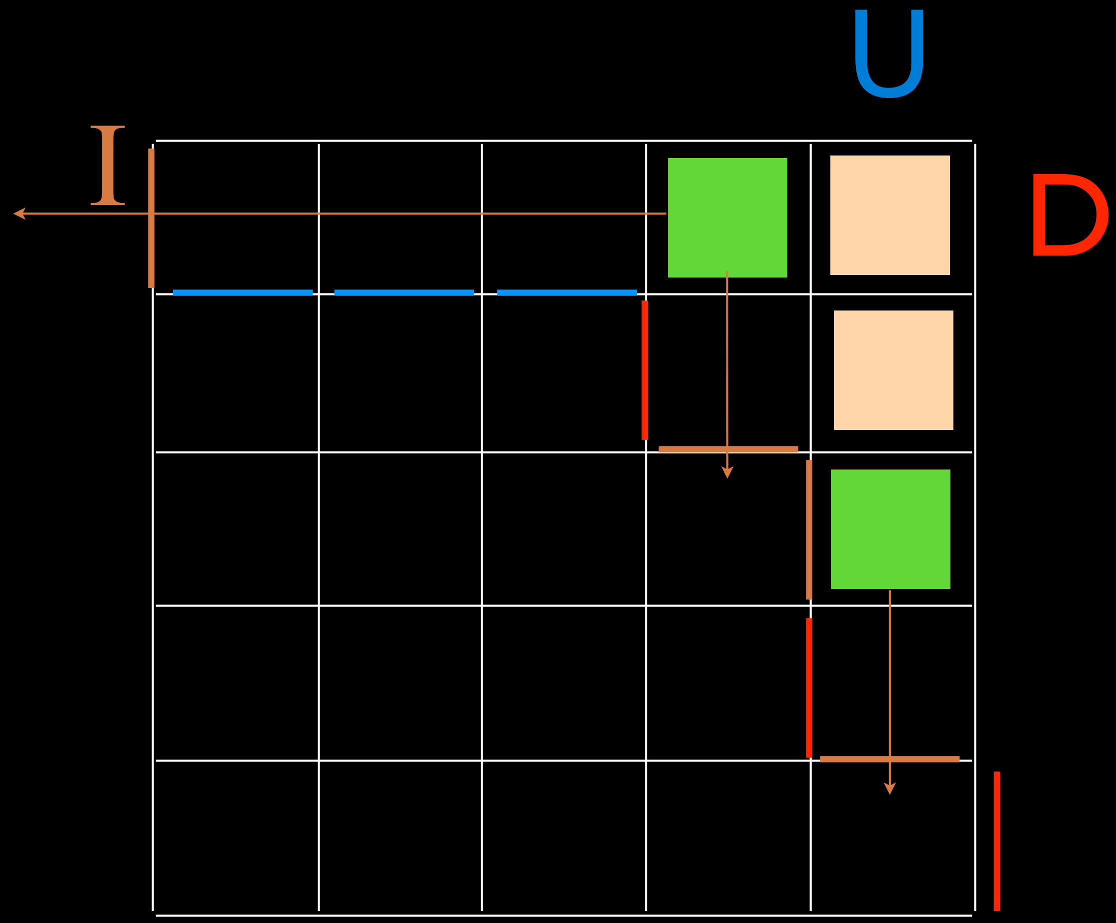
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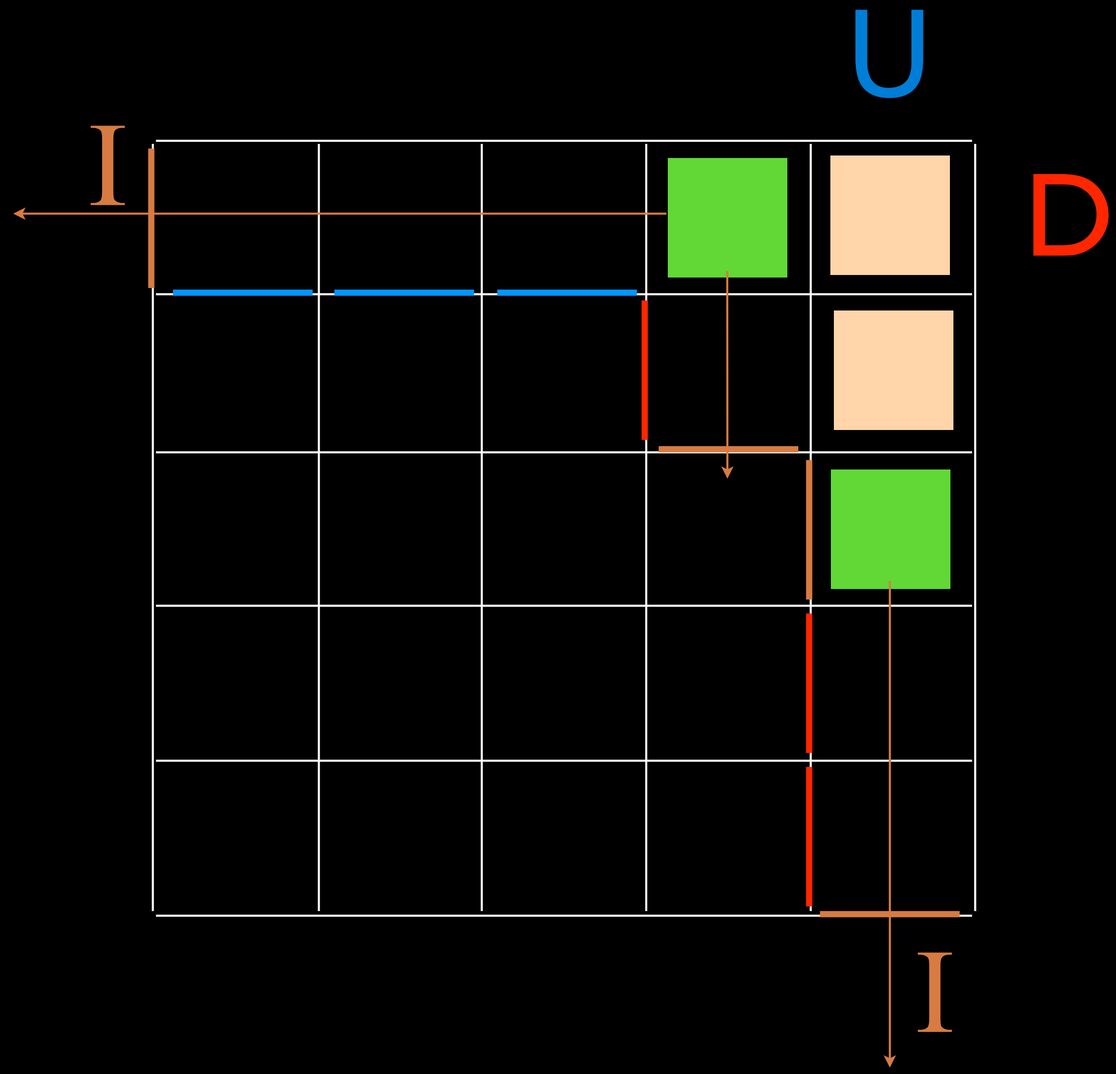
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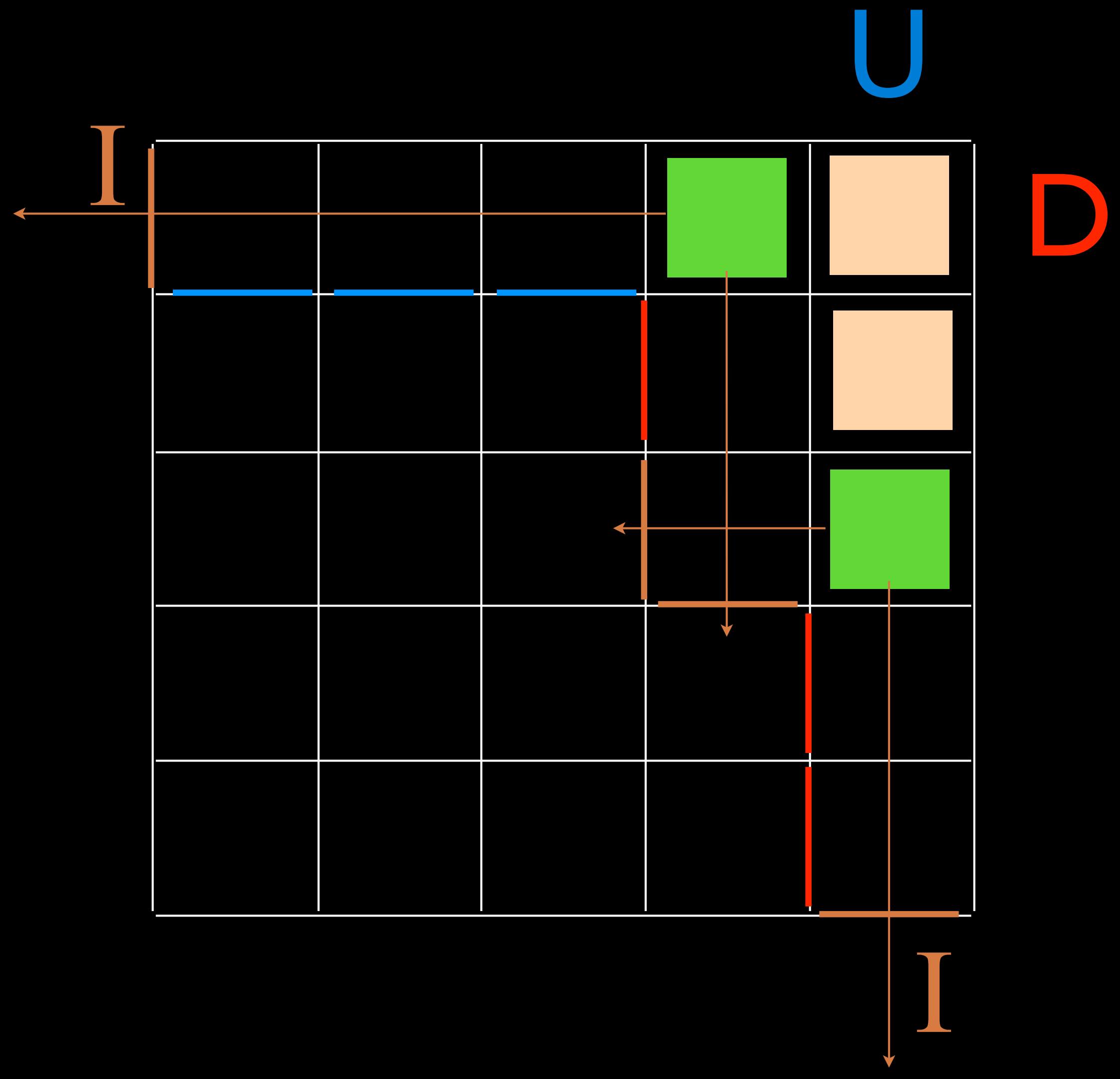
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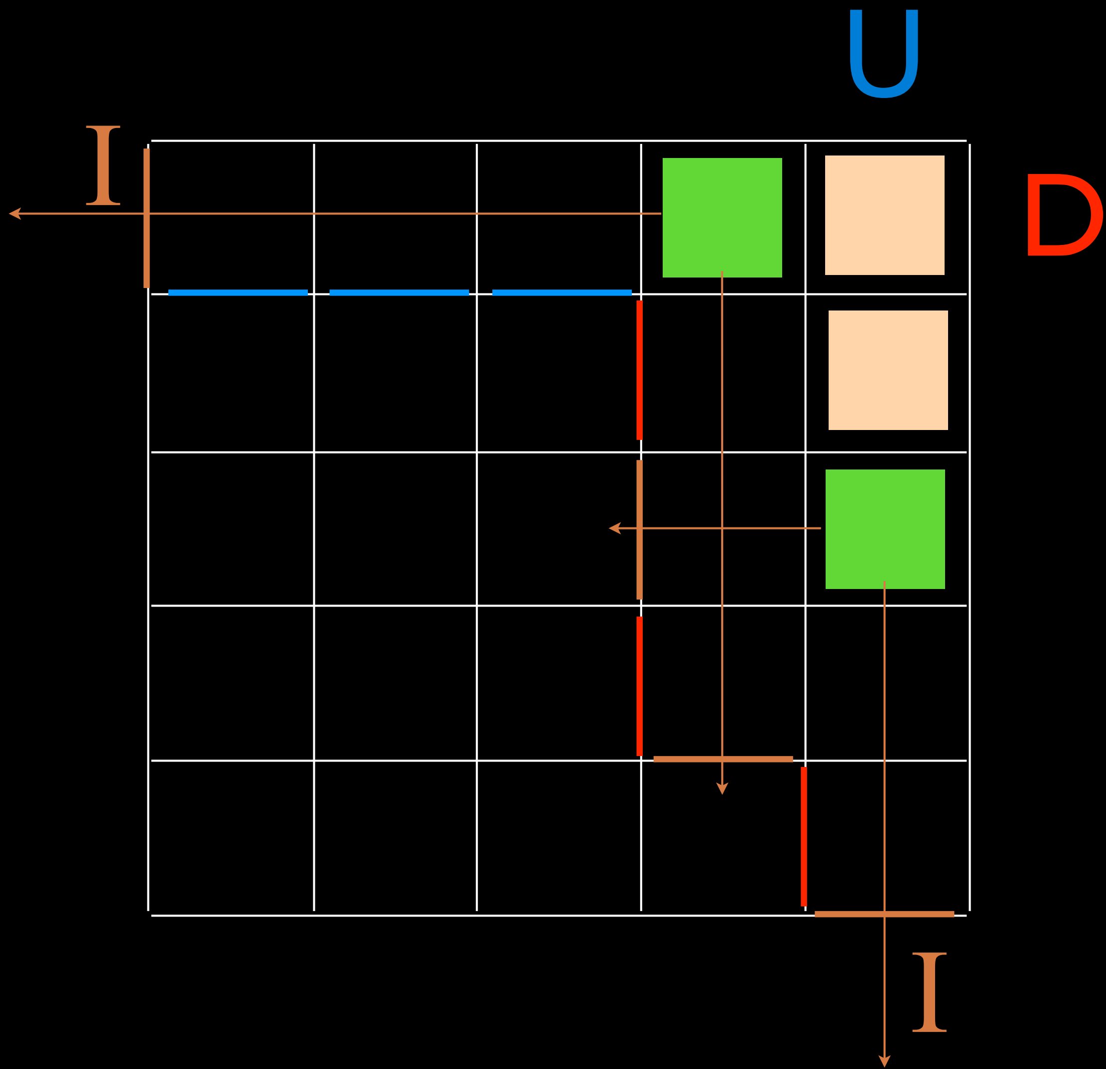
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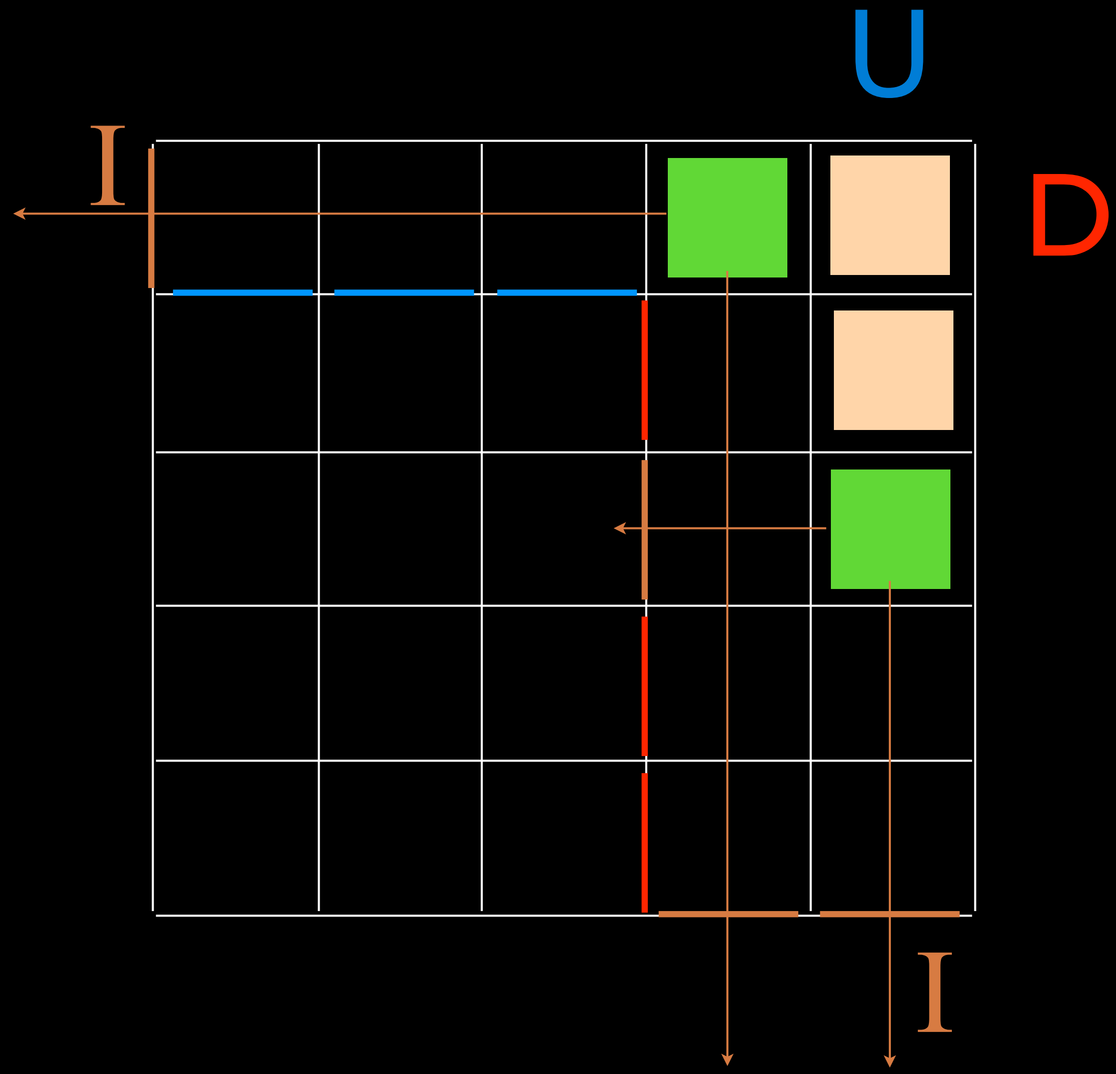

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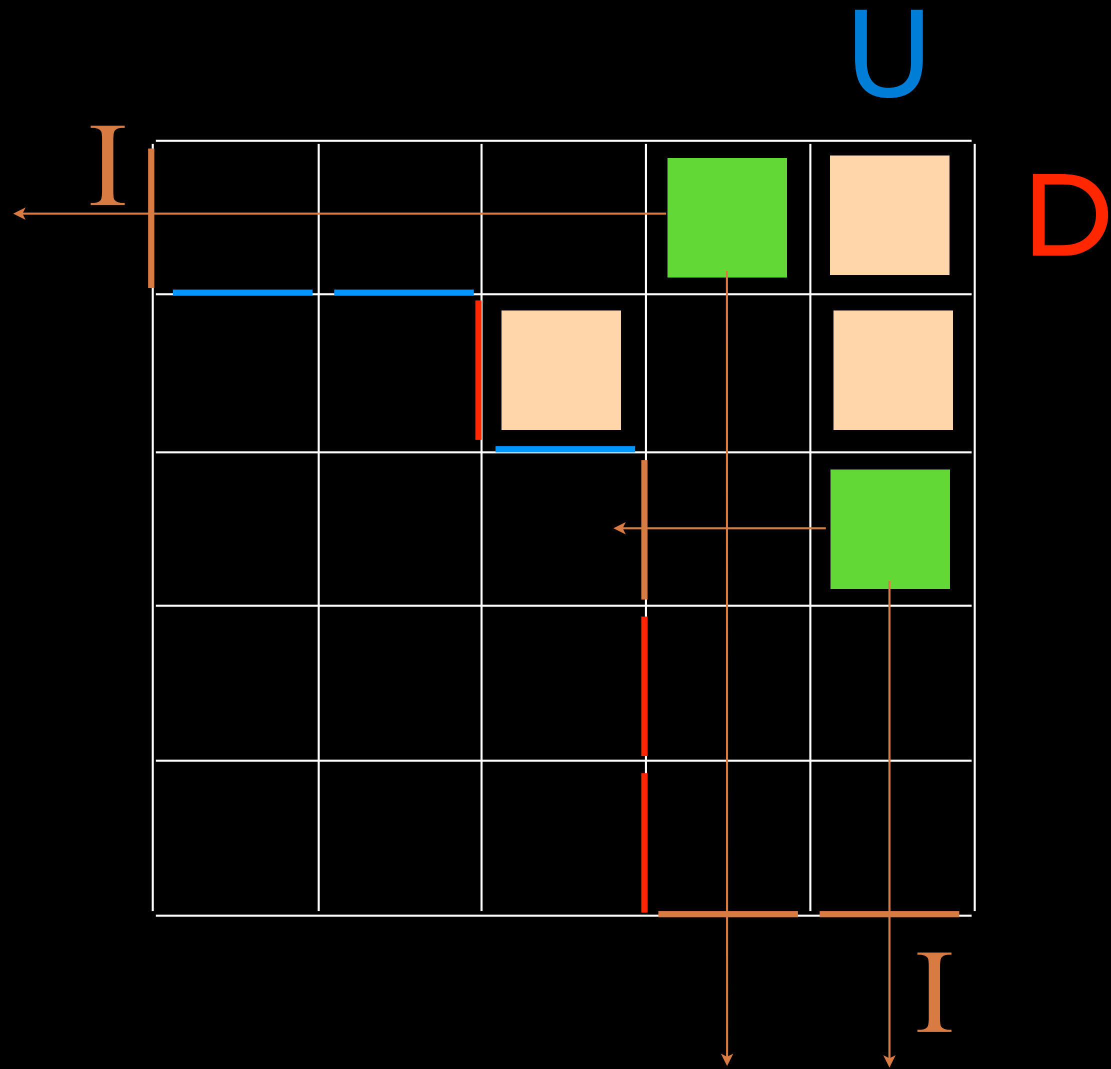
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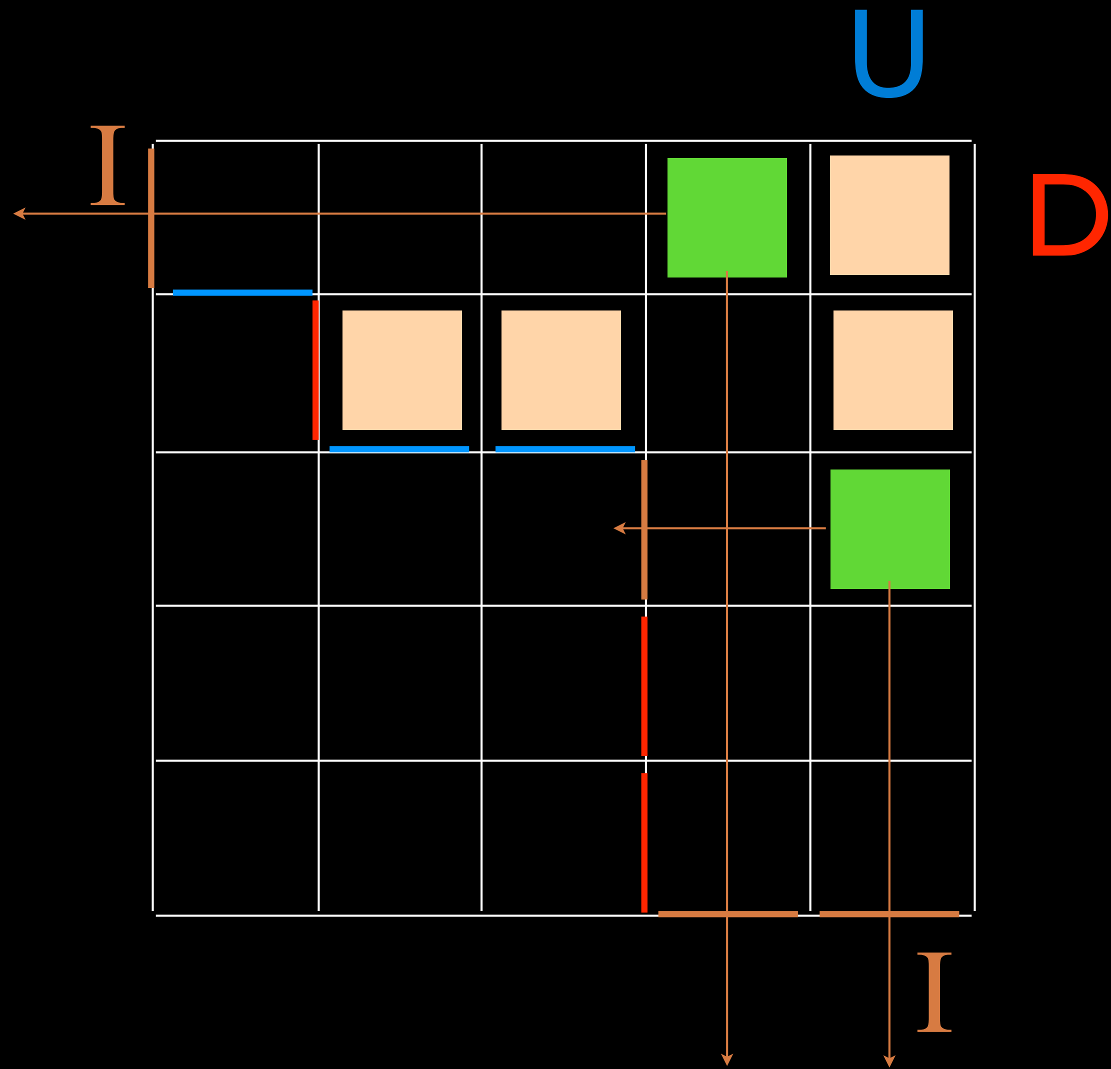

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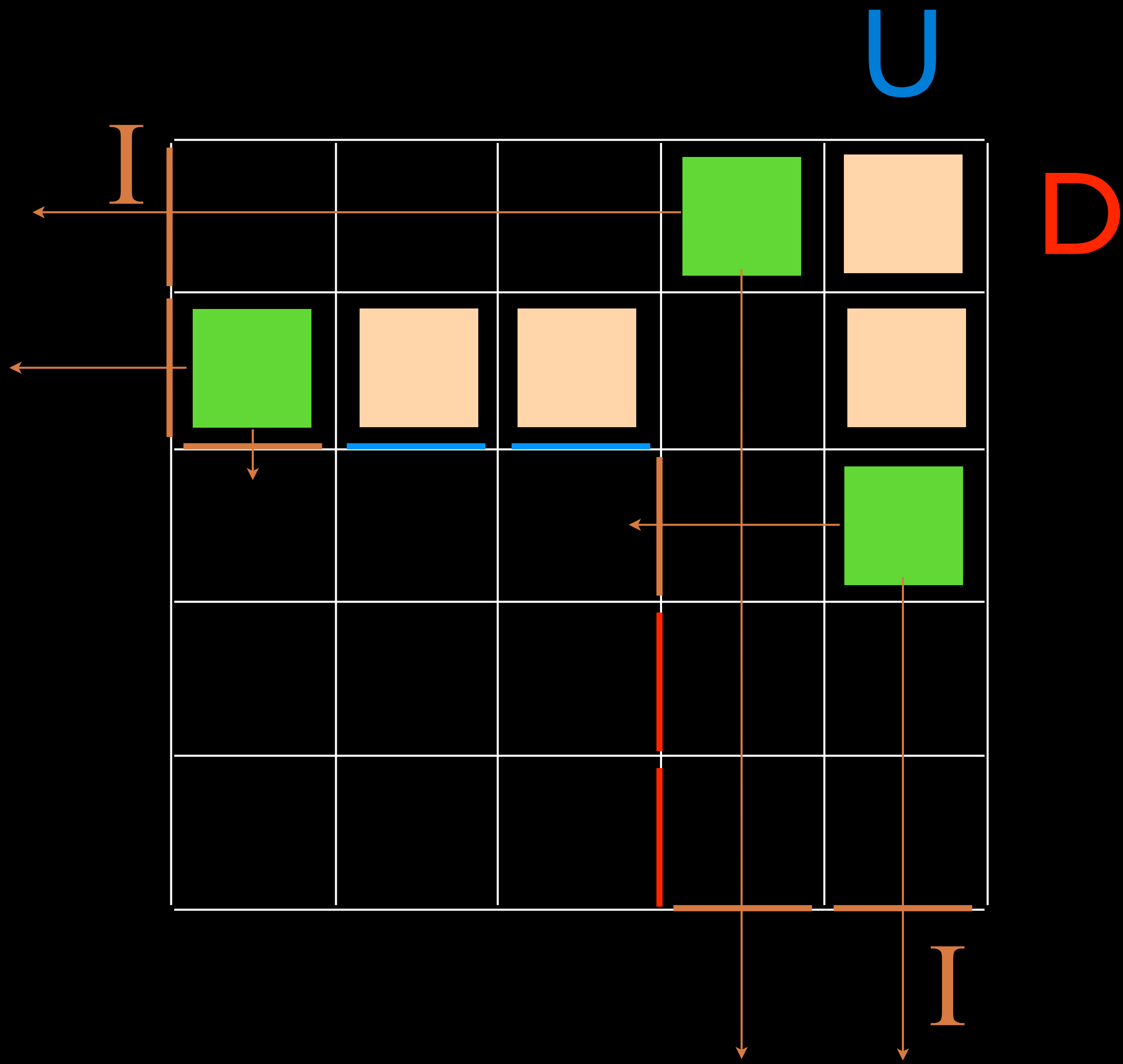

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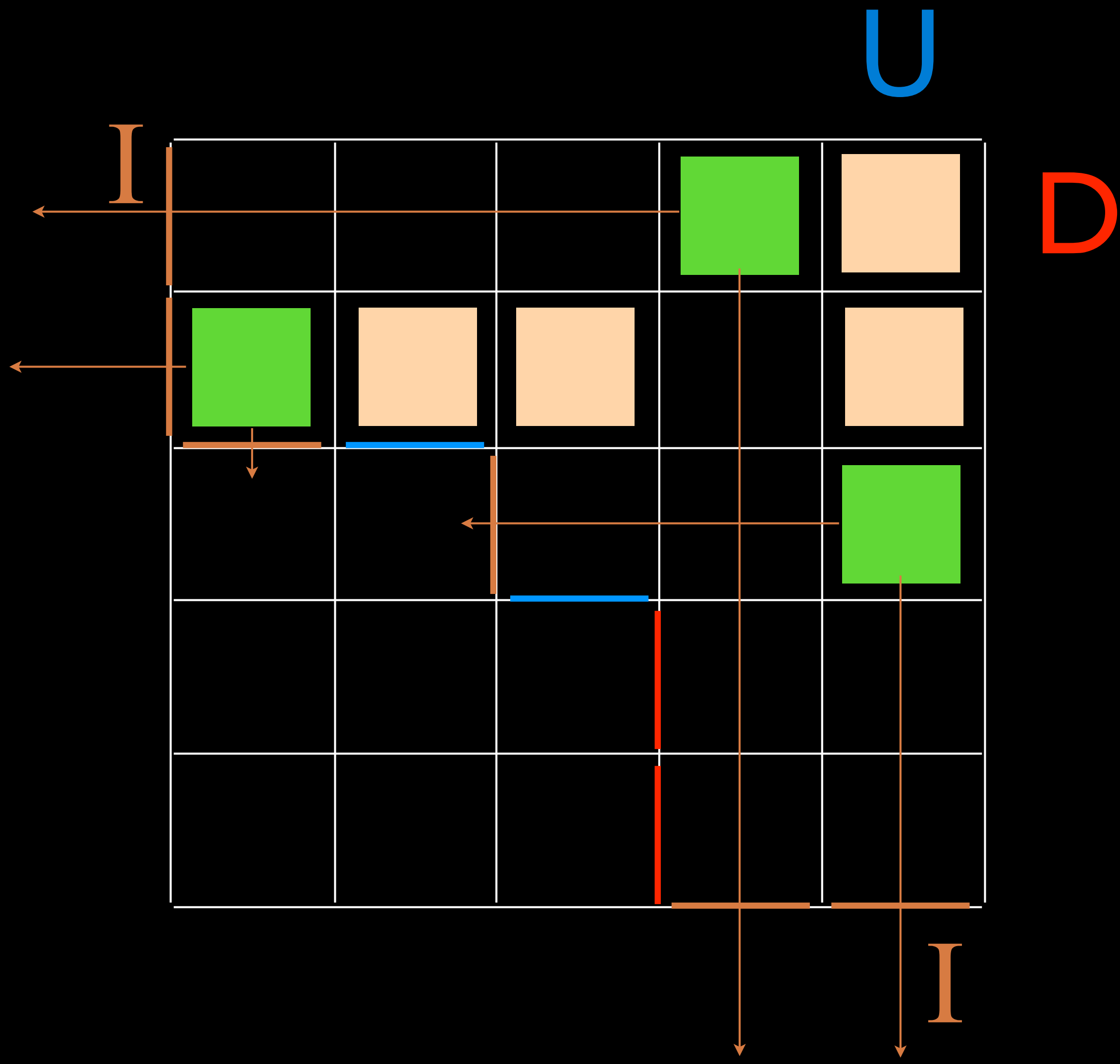

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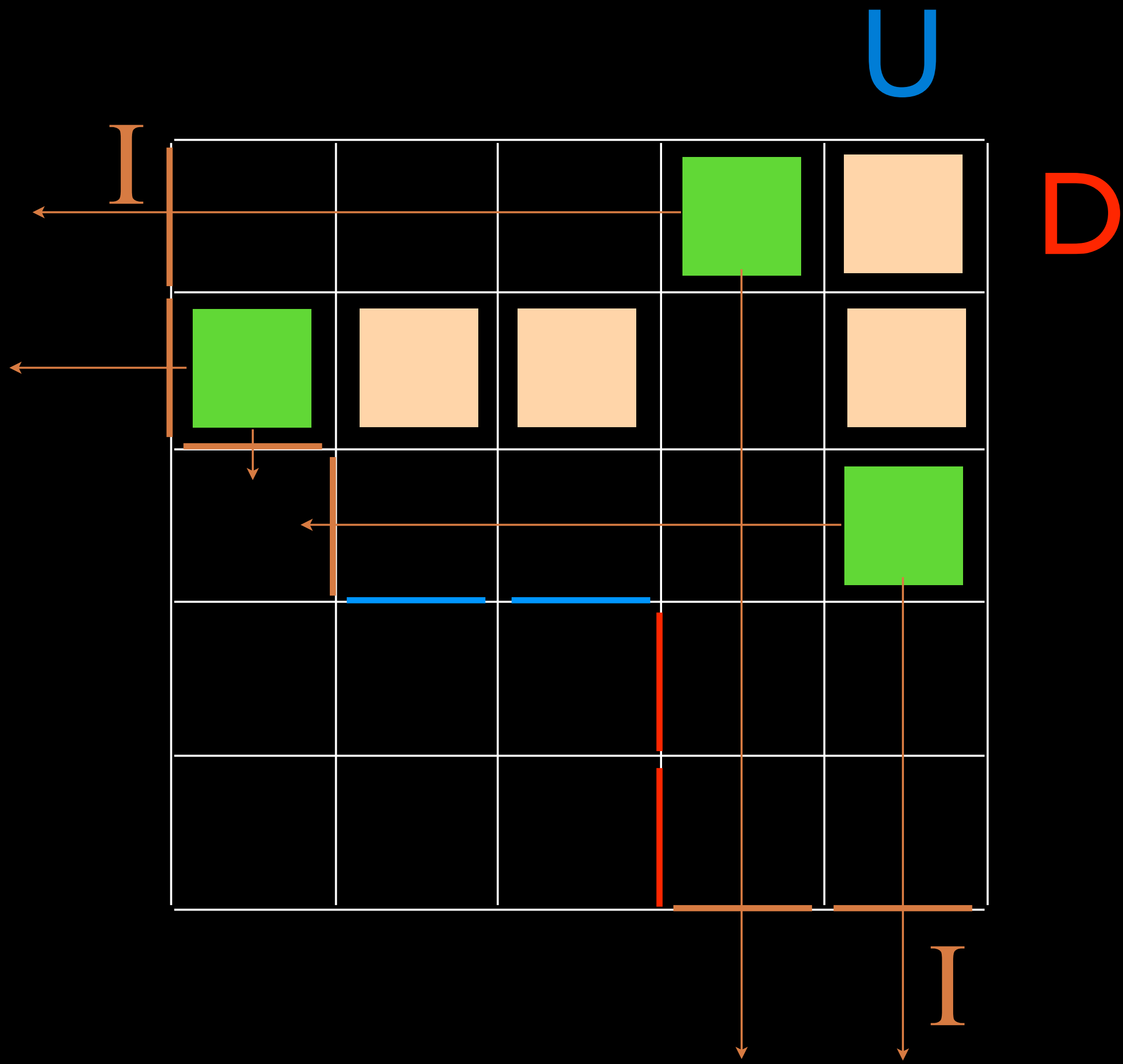
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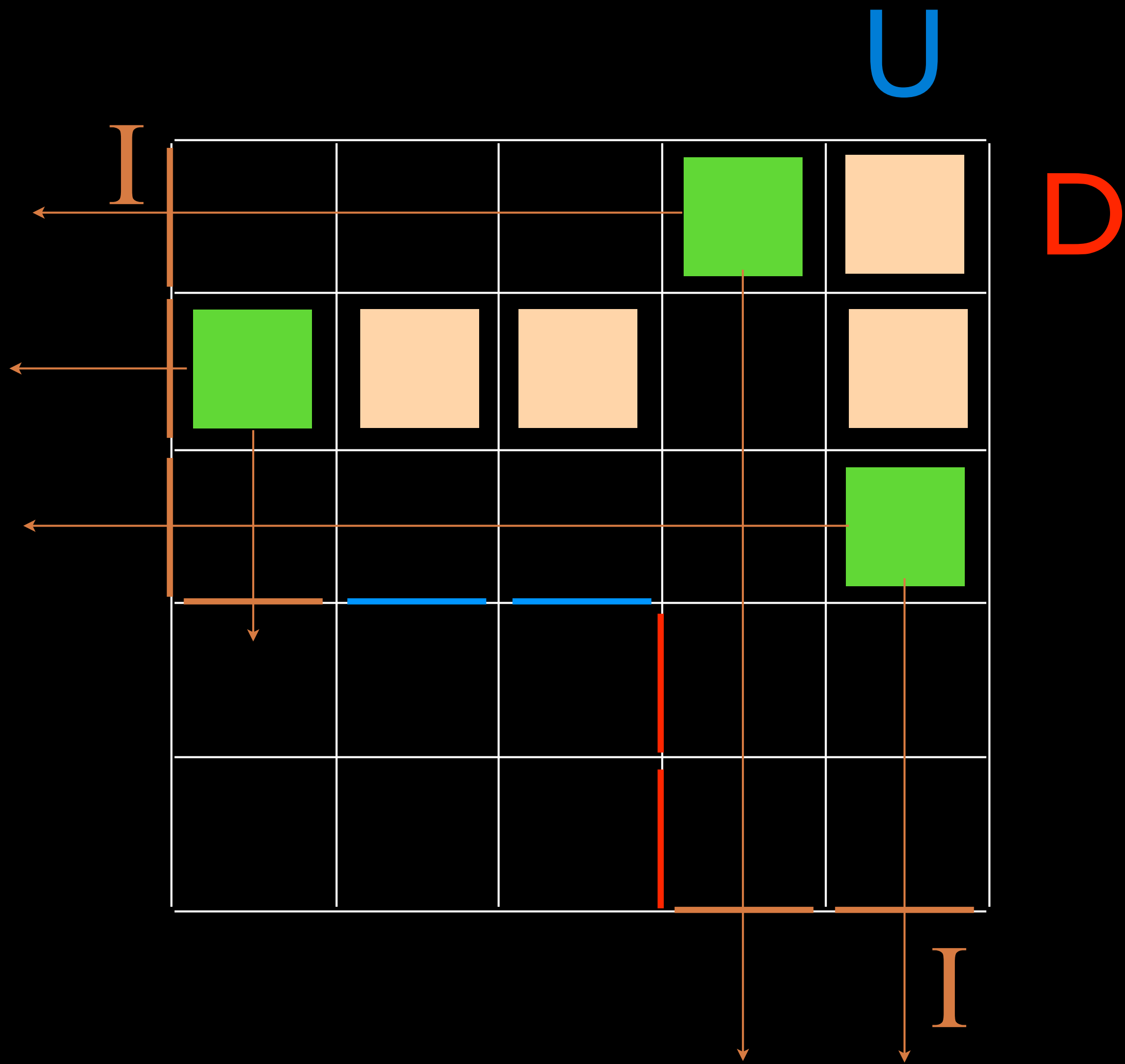
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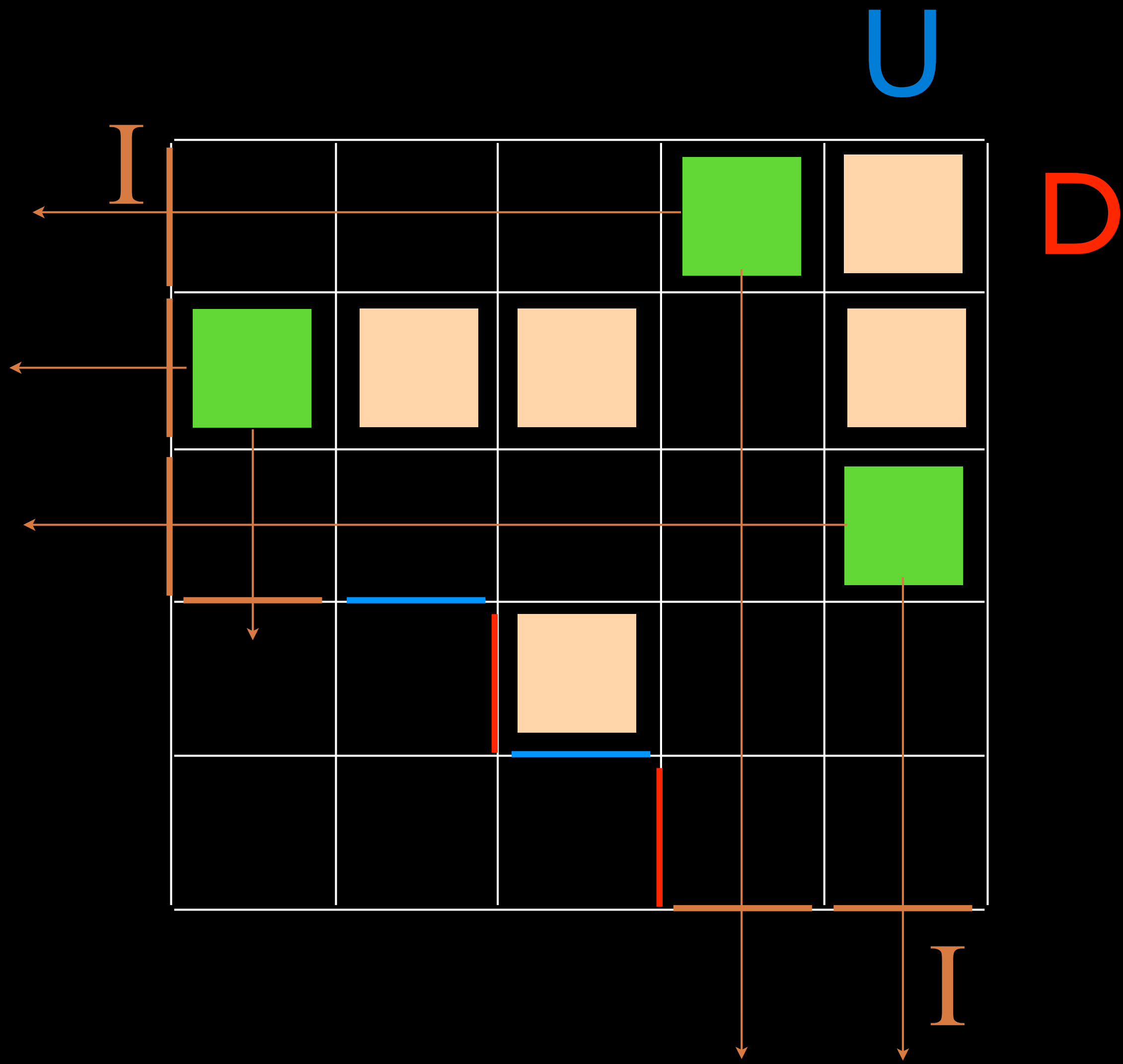
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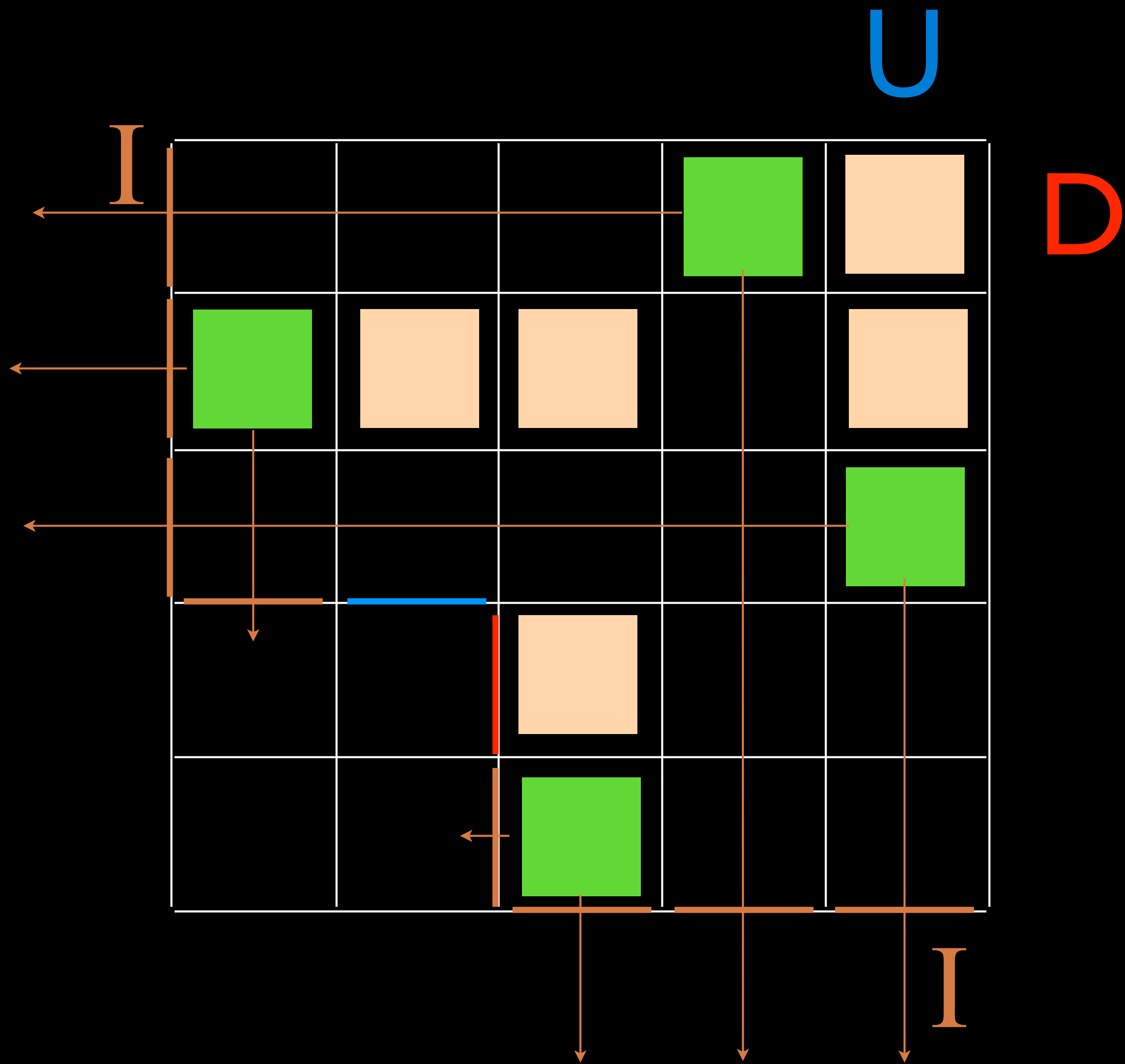
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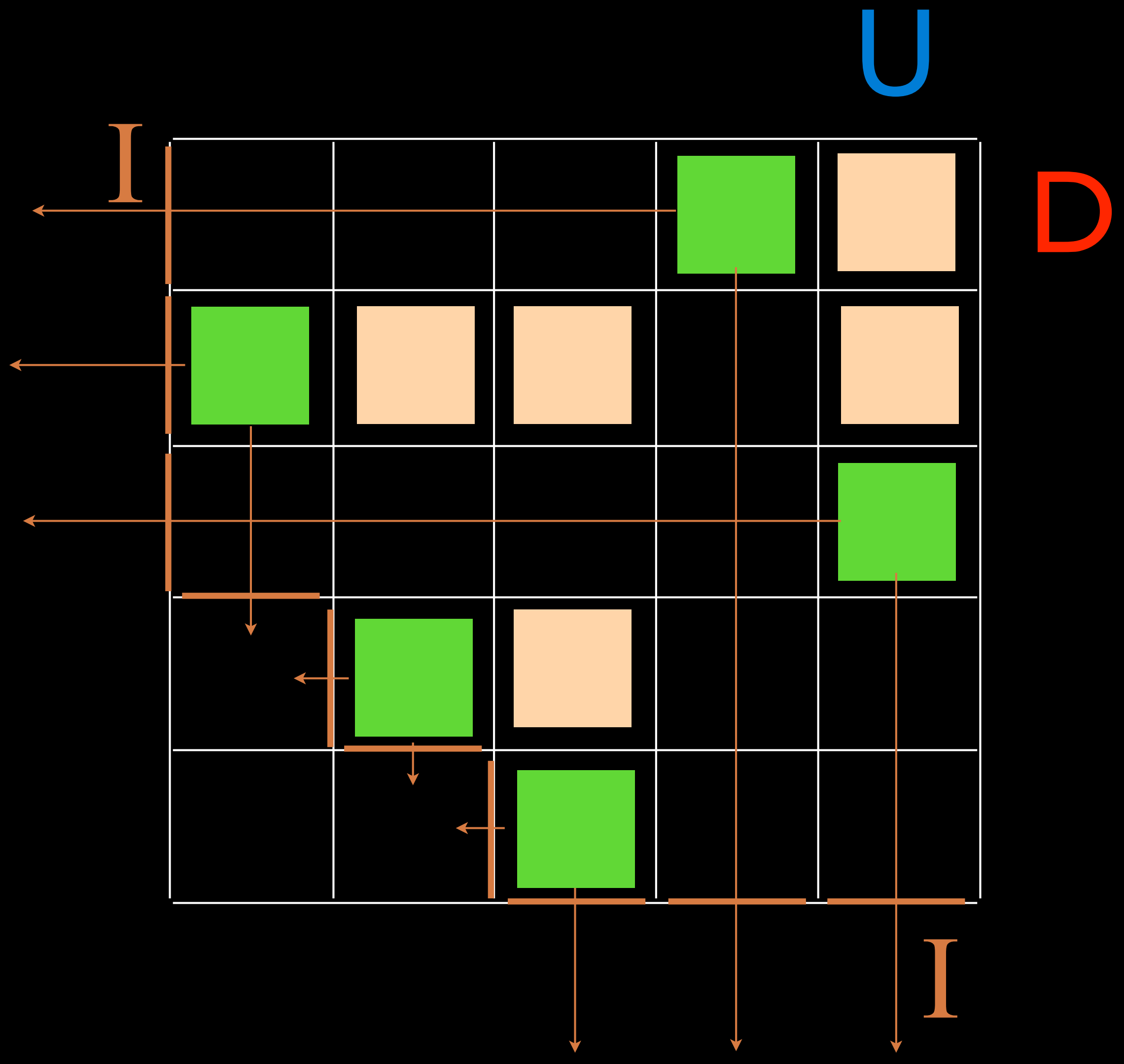
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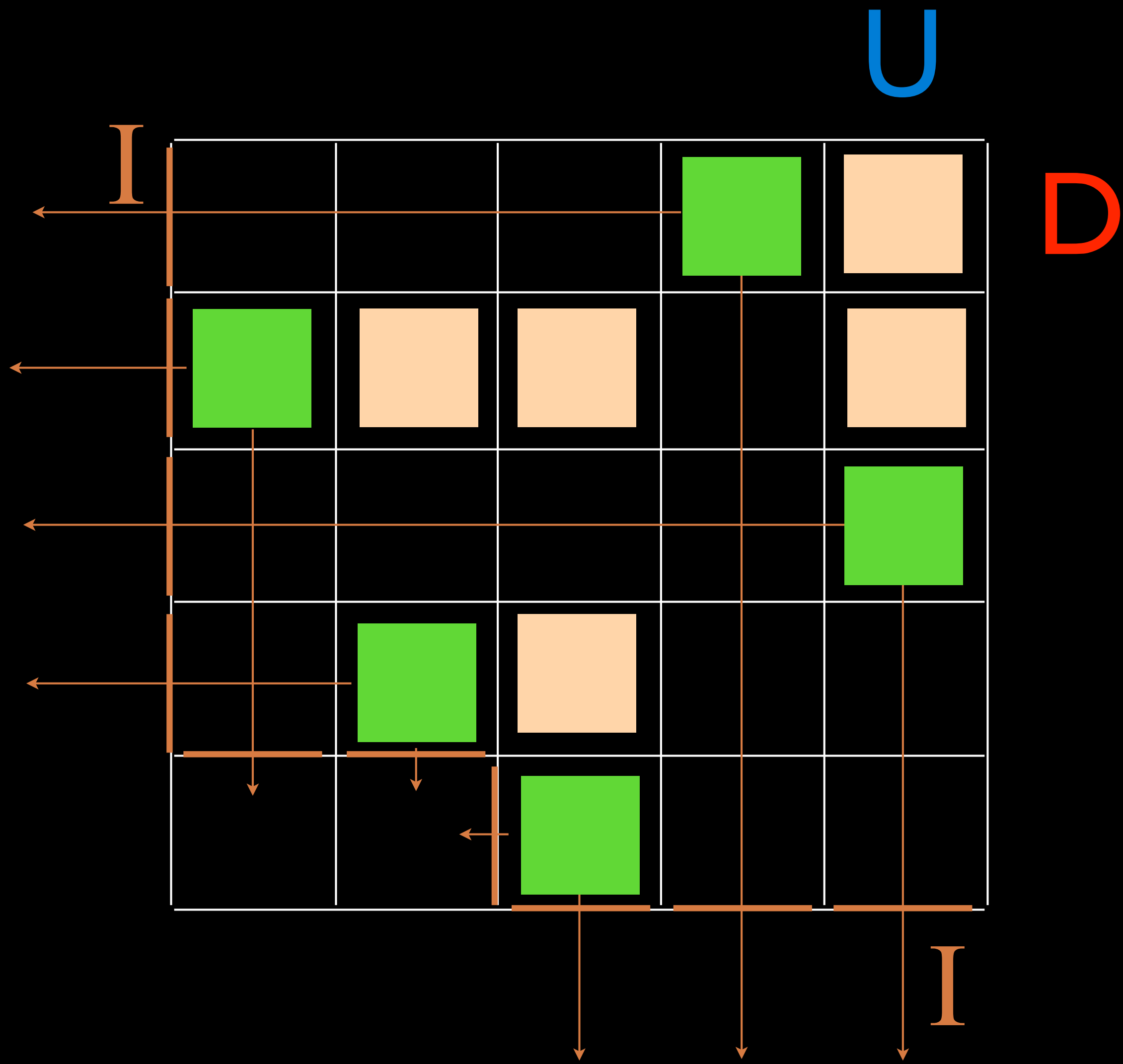

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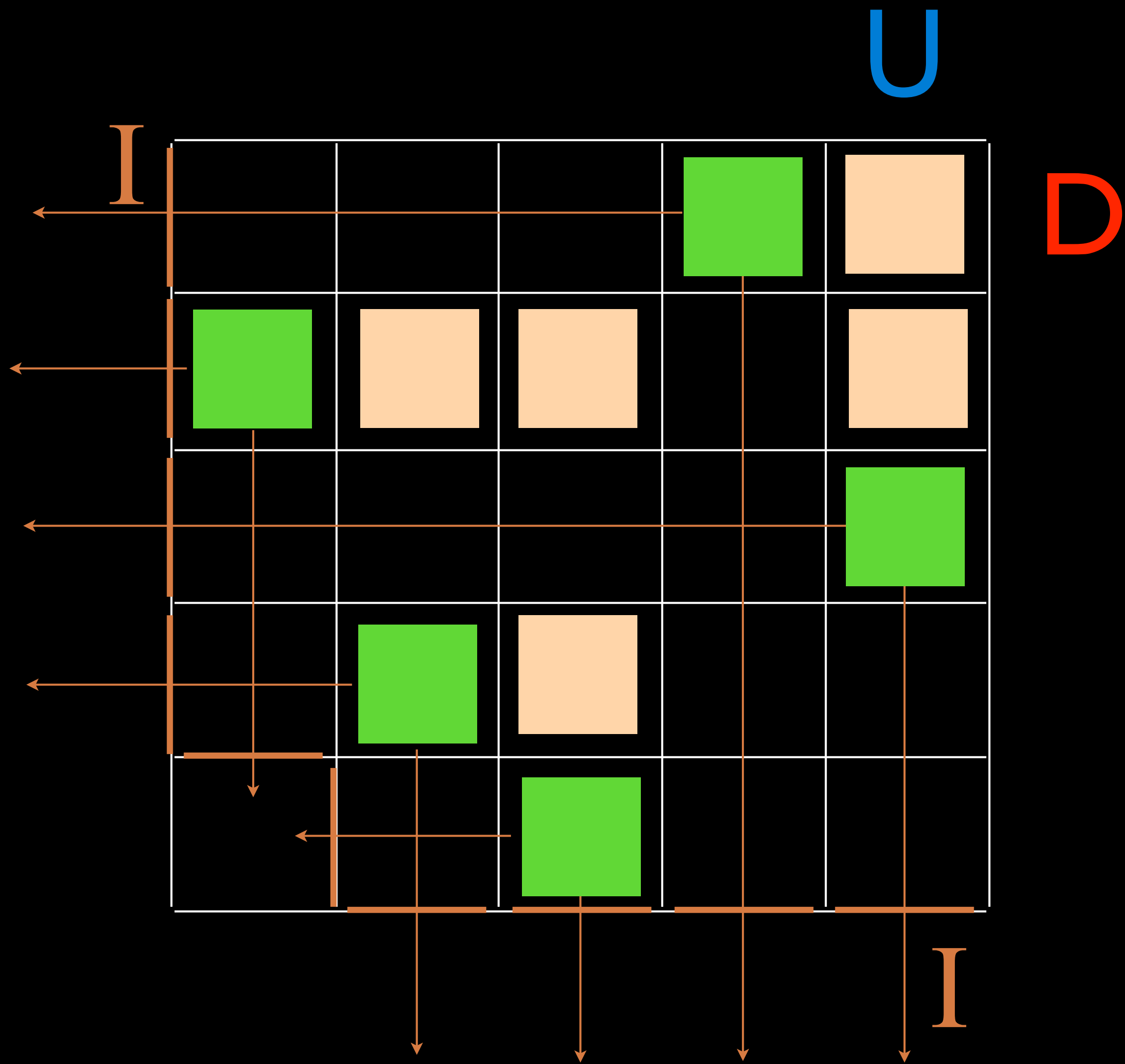

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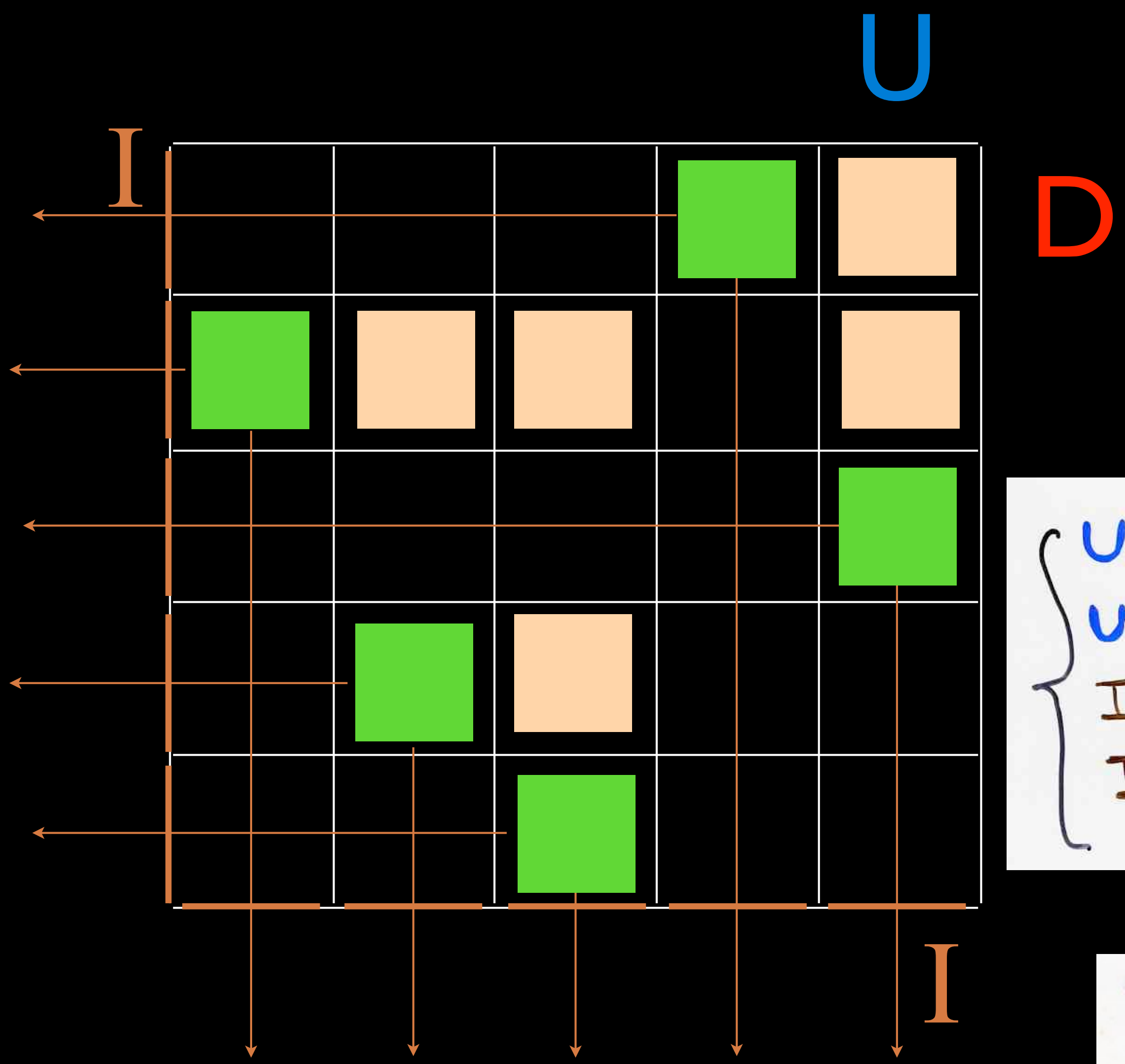
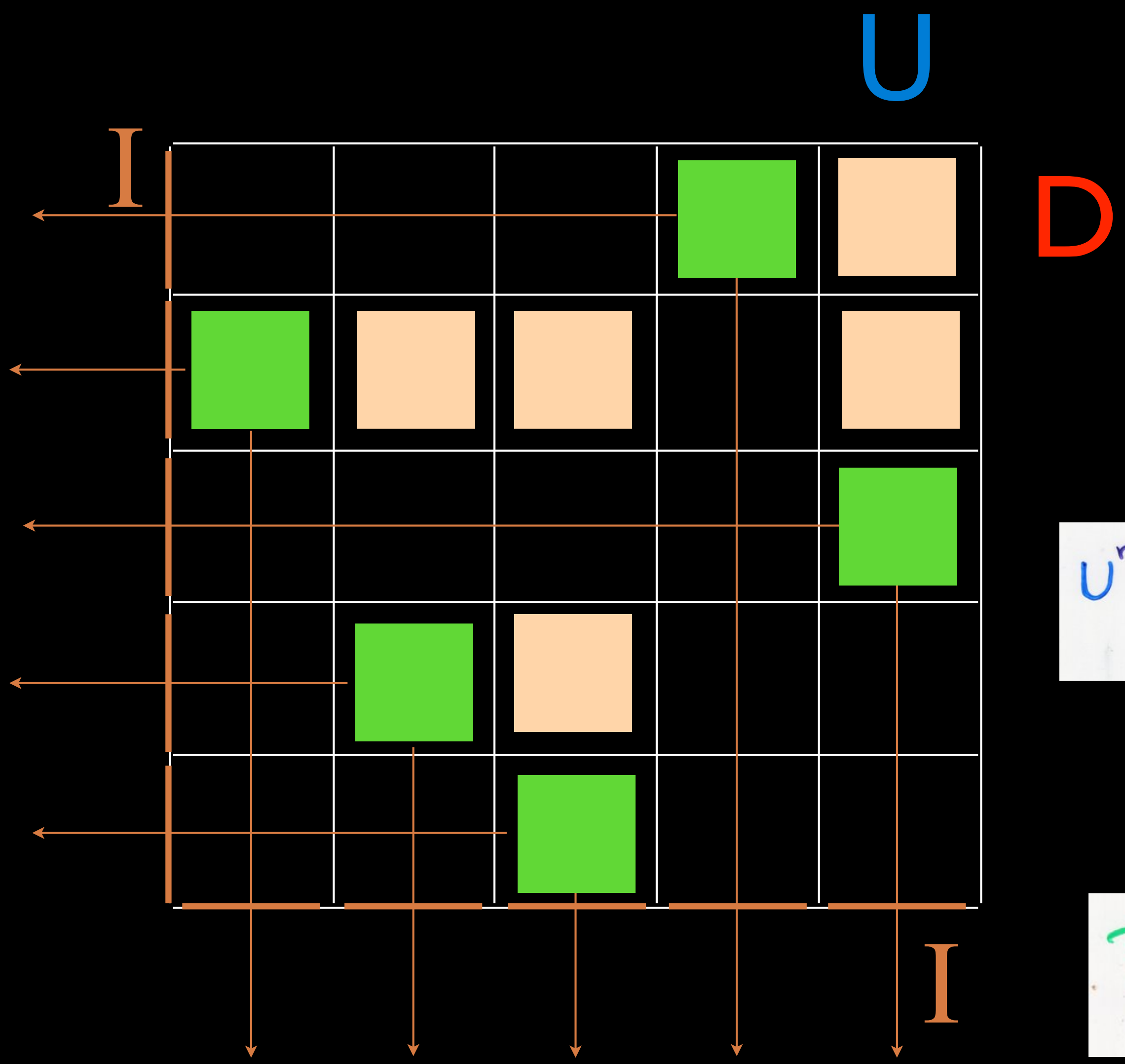


$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

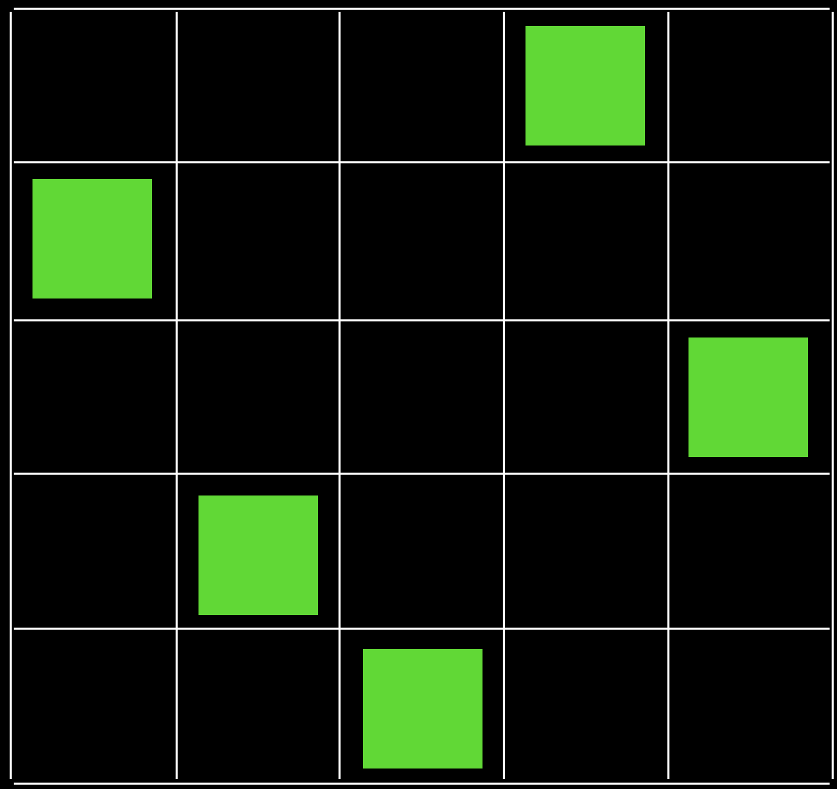
"complete" Q-tableau



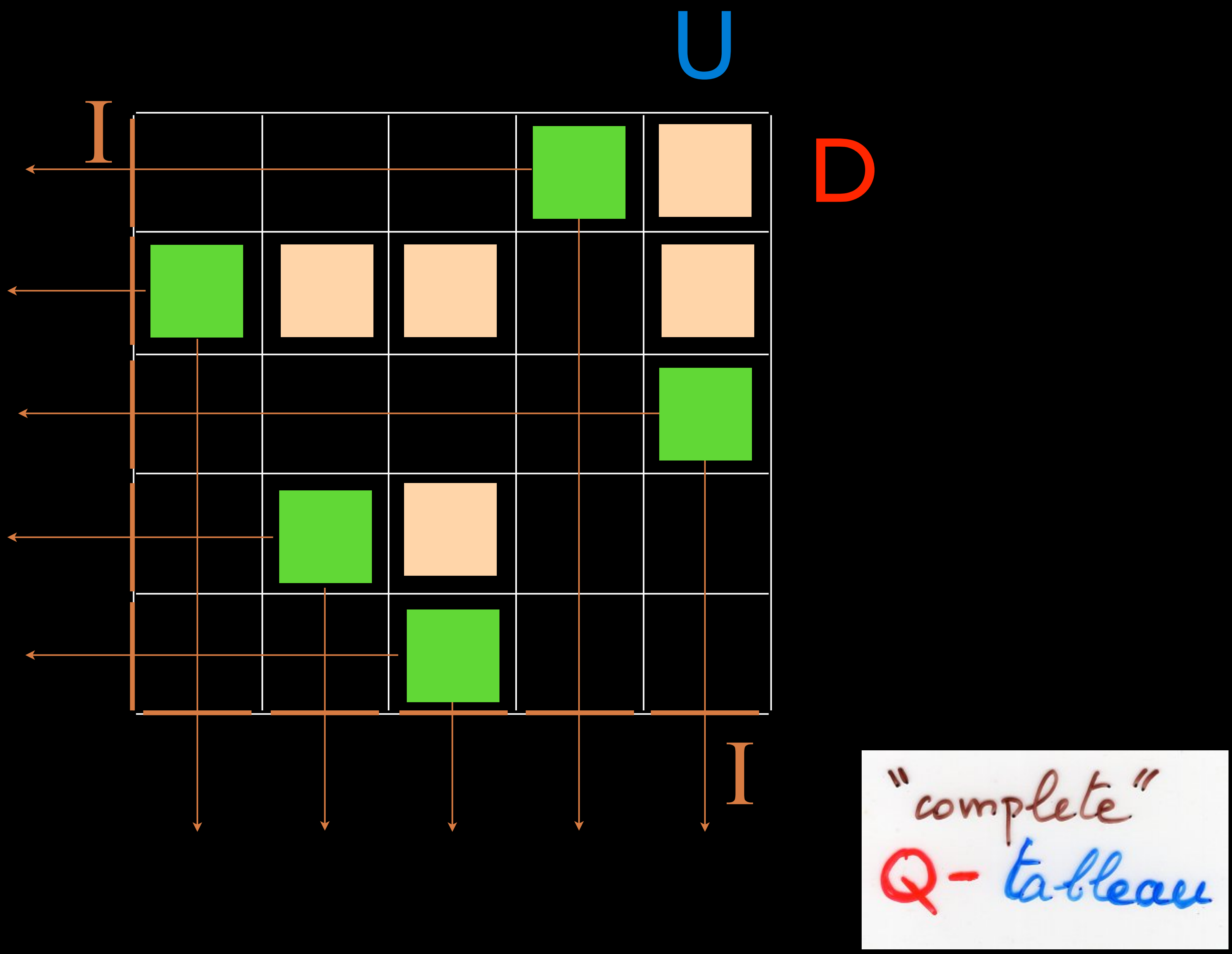
$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} D^i U^i$$

$$c_{n,0} = n!$$

permutation as a Q-tableau



permutation
as a Q-tableau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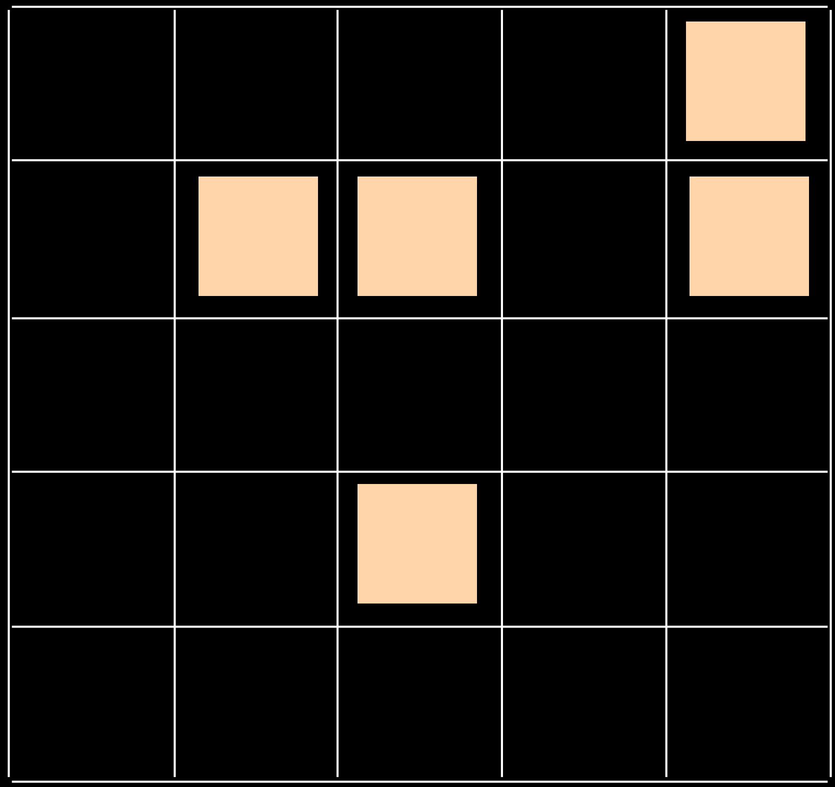
"complete"
Q- tableau



another Q-tableau

Rothe diagram of a permutation

"The cellular ansatz"

quadratic algebra **Q**

Q-tableaux

combinatorial objects on a 2D lattice

$UD = DU + Id$

permutations

towers placements

commutations

rewriting rules

planarization

Planar automata

and

Q-tableaux

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

$UD = DU + Id$

permutations

towers placements

commutations

rewriting rules

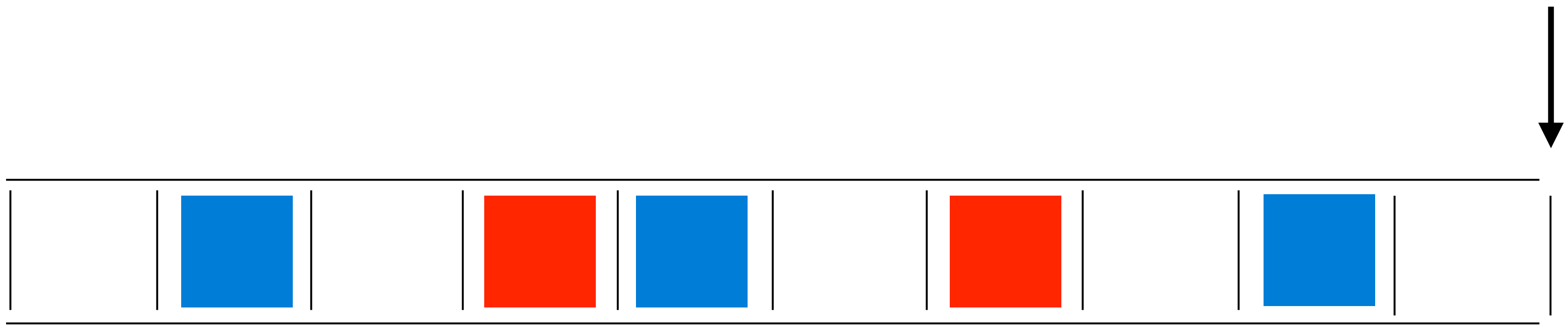
planarization

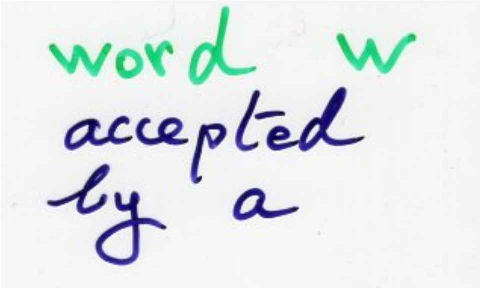
"planar automata"

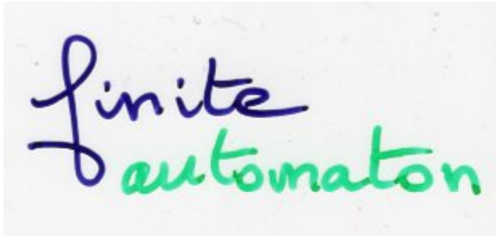
word w accepted by a

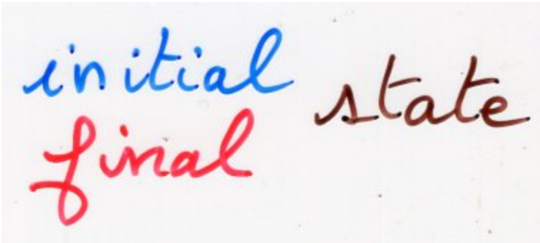
finite automaton

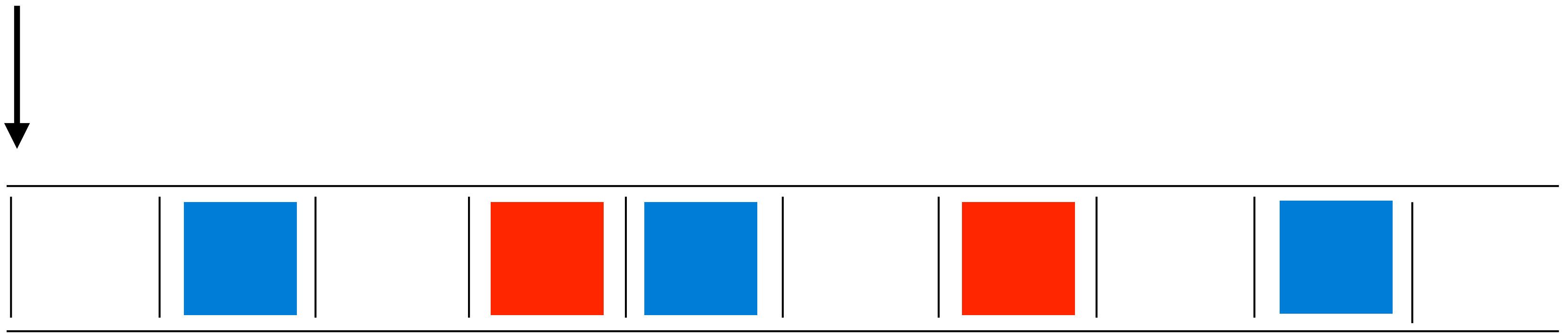
initial state
final



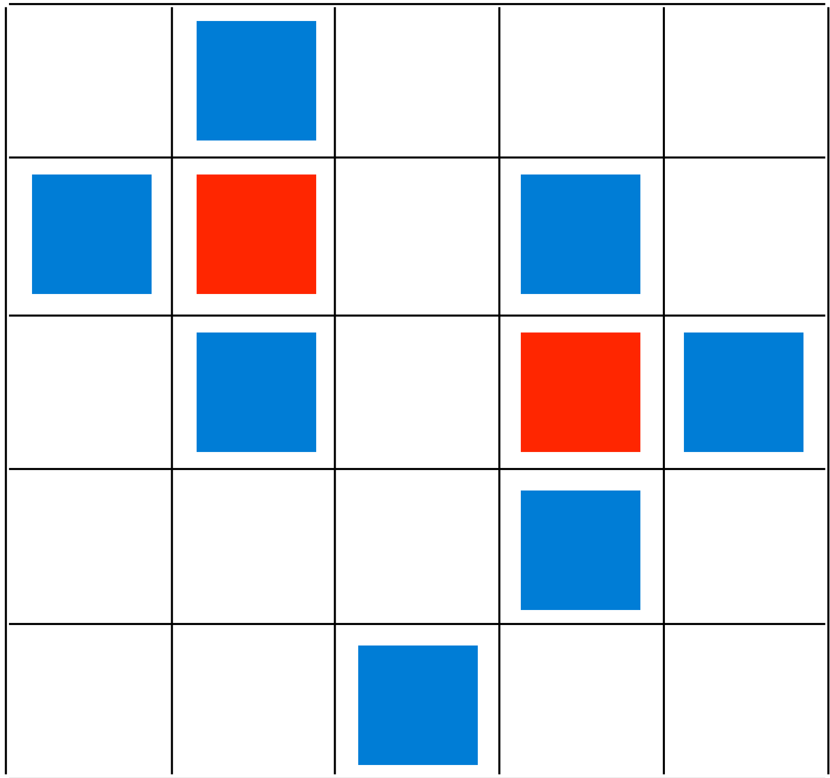

word w
accepted
by a


finite
automaton


initial
final
state



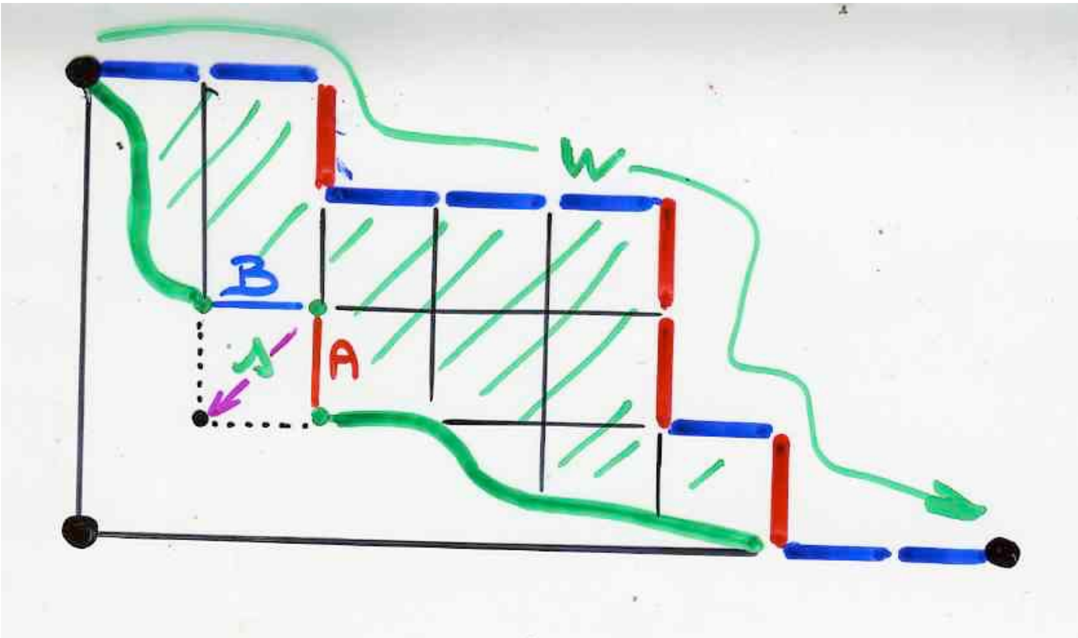
planar automaton

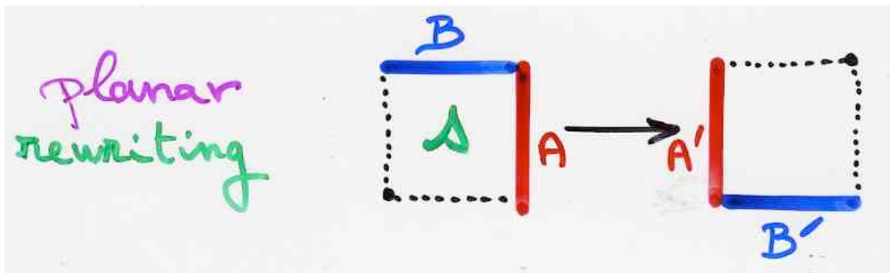


ASM

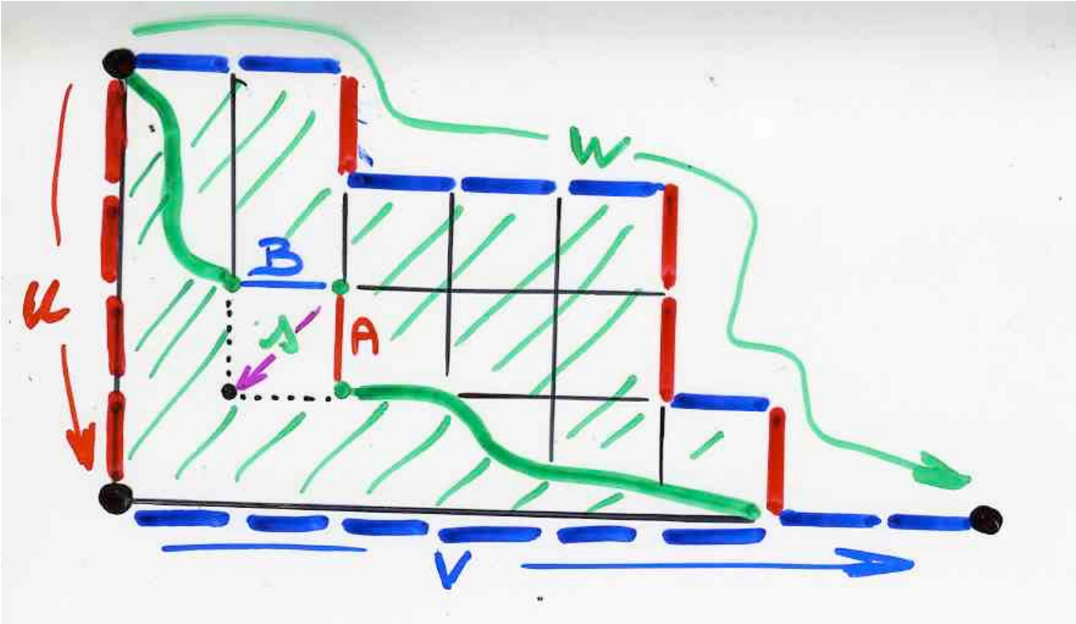
alternating sign matrix

planar automaton

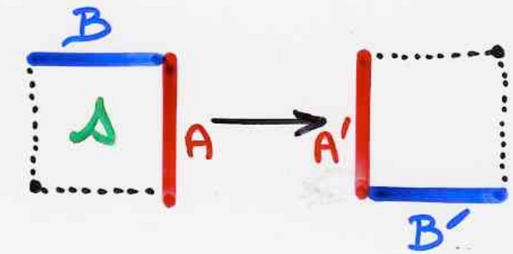




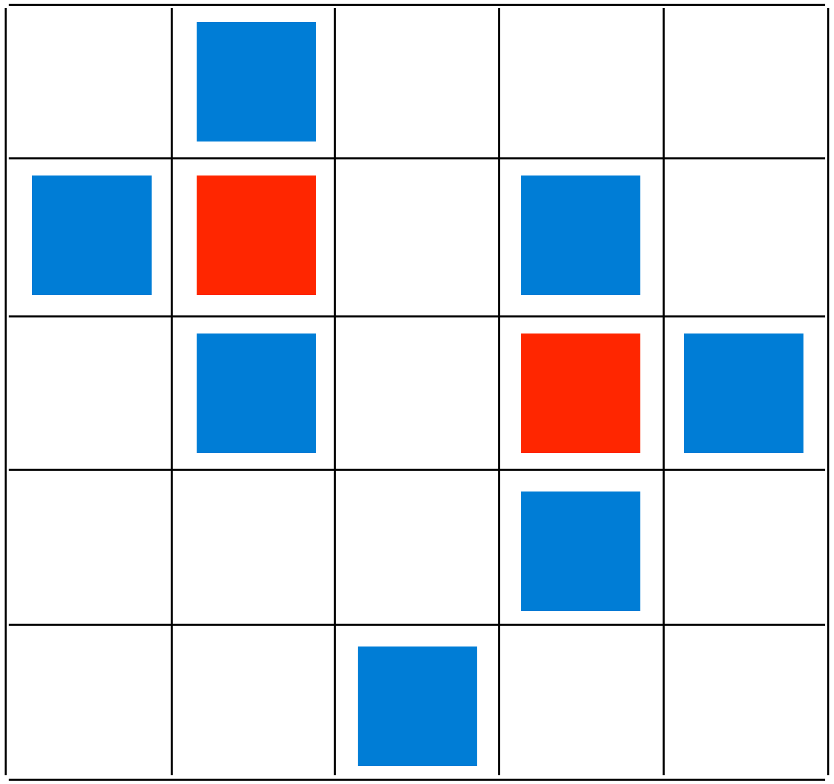
planar automaton



planar rewriting



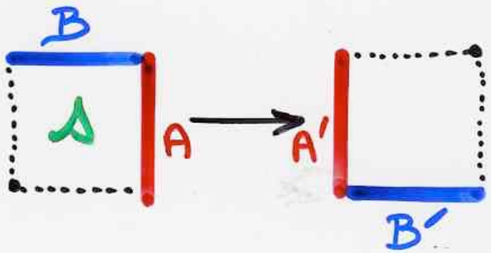
planar automaton



ASM

alternating sign matrix

planar rewriting



commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

Q-tableaux

The cellular Ansatz

quadratic algebra Q
(of a certain type)

(1) "planarization" on a grid of the rewriting rules

Q- tableaux planar automata

(2) From a representation of the quadratic algebra Q with combinatorial operators, get a bijection

Q-tableaux ⟷ (W) Some combinatorial objects, can be a pair (P,Q)

RSK (Ch 1)

The Robinson-Shensted-Knuth correspondence

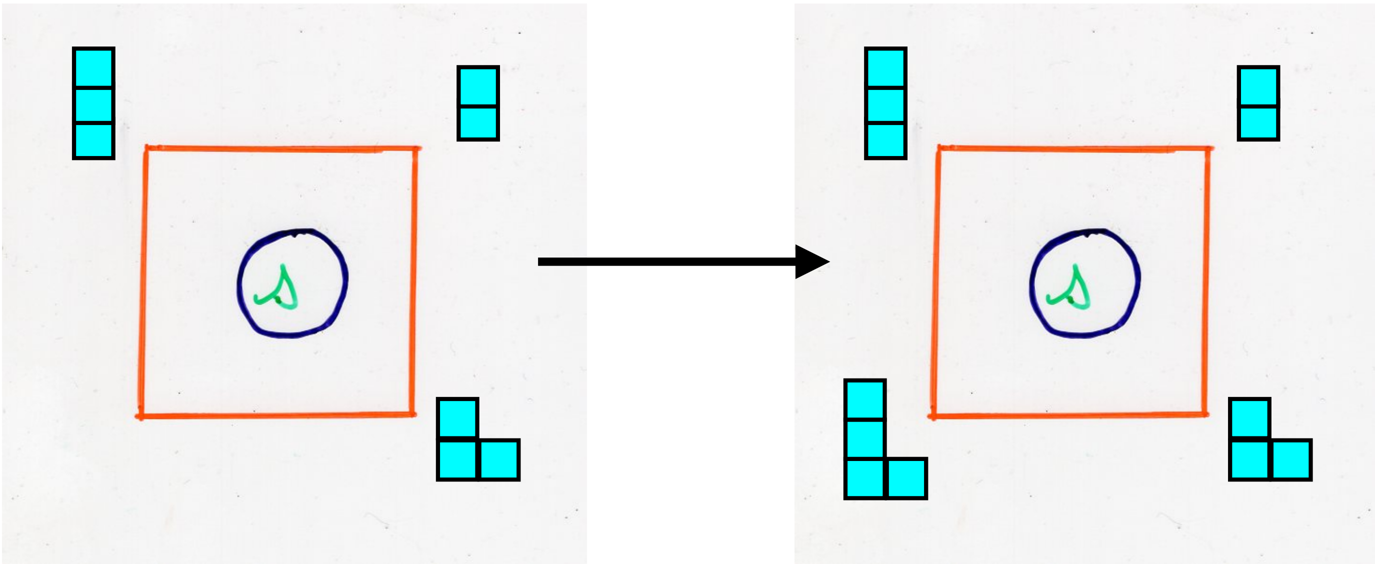
- Schensted's insertions
- geometric version with "shadow lines"
- Fomin "local rules" or "growth diagrams »
- Schützenberger "jeu de taquin"

for Q: $UD = DU + I$

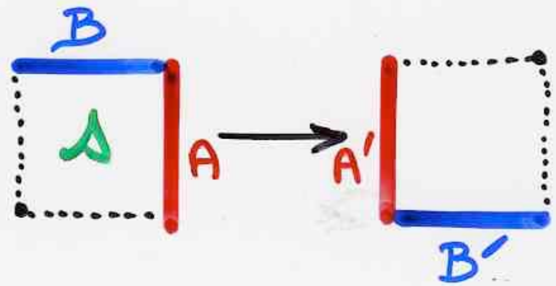
representation of the quadratic algebra Q with combinatorial operators

Fomin's "local rules"

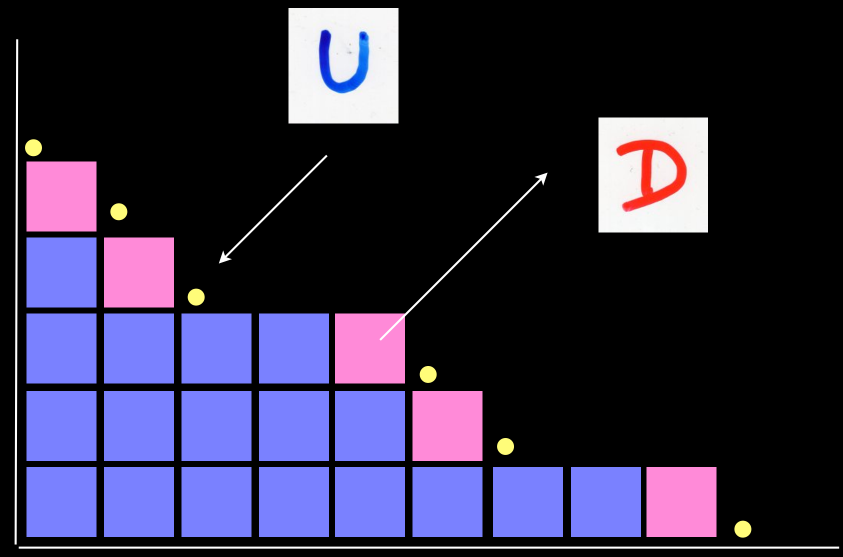
"growth diagrams"



planar rewriting

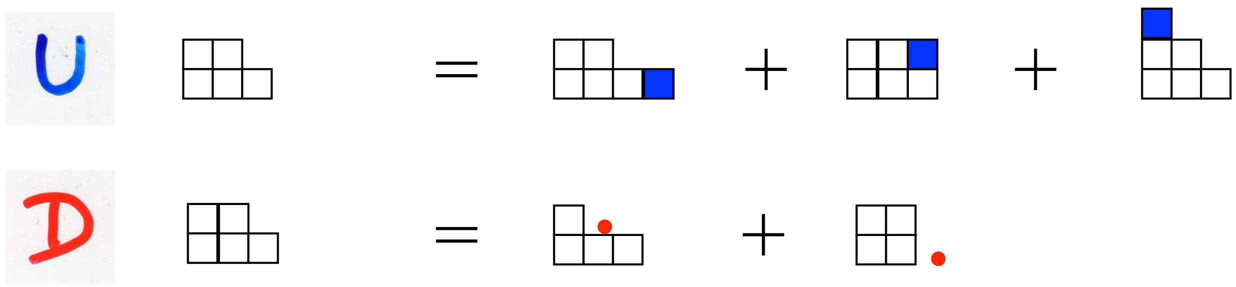


operators
U and D

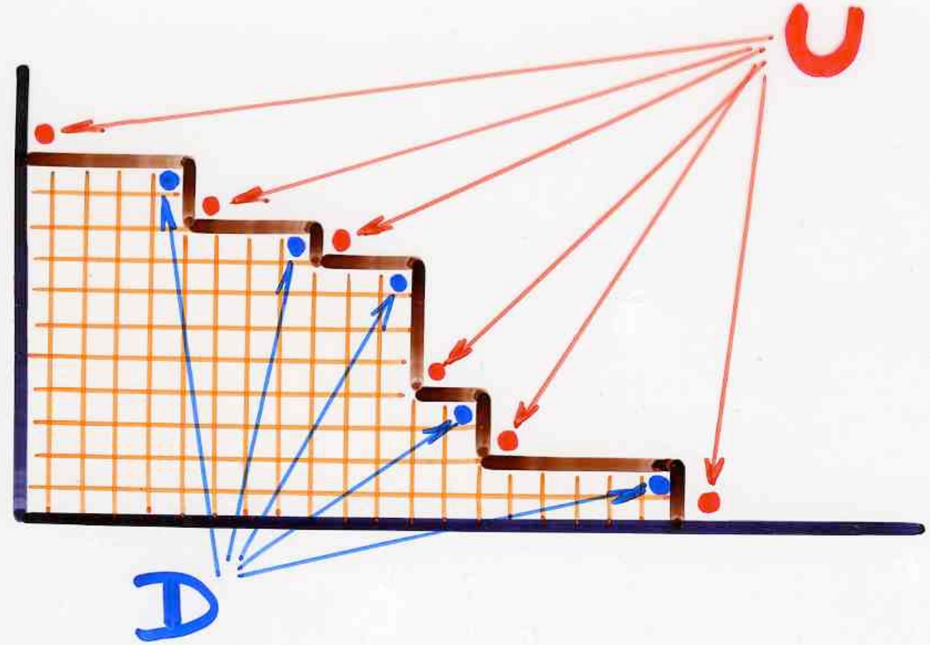


Young lattice

U adding a cell in a Ferrers diagram
D deleting a cell in a Ferrers diagram


U
D

$$UD = DU + I$$



"The cellular ansatz"

(ii) second step

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

UD = DU + Id

permutations

RSK ⟷

pairs of Young tableaux

towers placements

commutations

rewriting rules

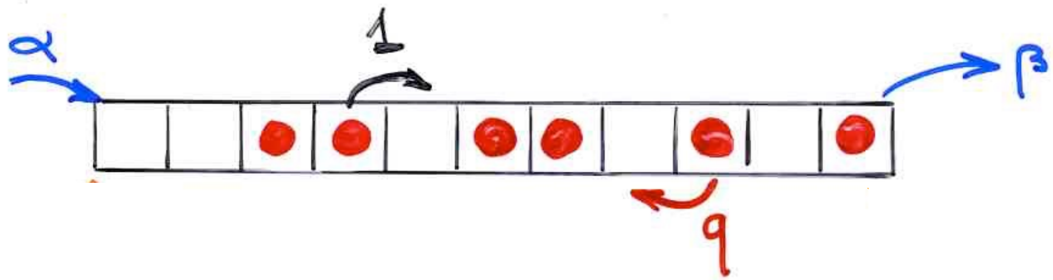
planarization

"planar automata"

Combinatorial physics

toy model in the physics of
dynamical systems far from equilibrium

ASEP
TASEP
PASEP



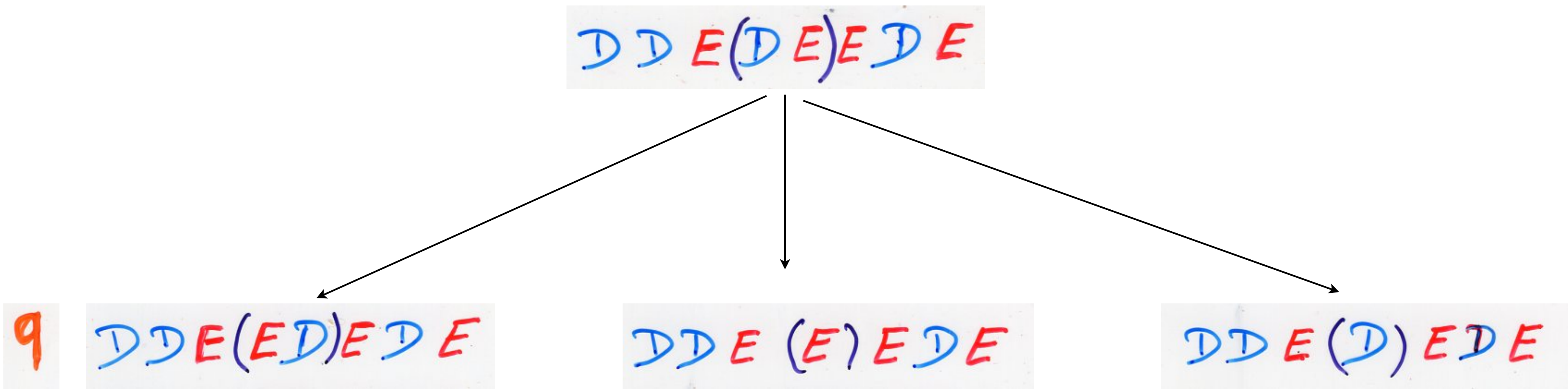
computation of the
"stationary probabilities"

The PASEP algebra

$$DE = qED + E + D$$

The PASEP algebra

$$DE = qED + E + D$$



The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

tableau

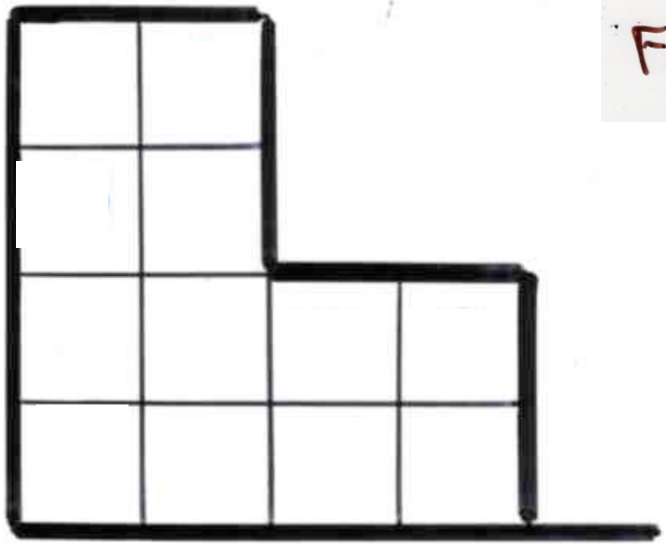
unique

analog of the normal ordering

alternative tableaux

alternative tableau

Definition



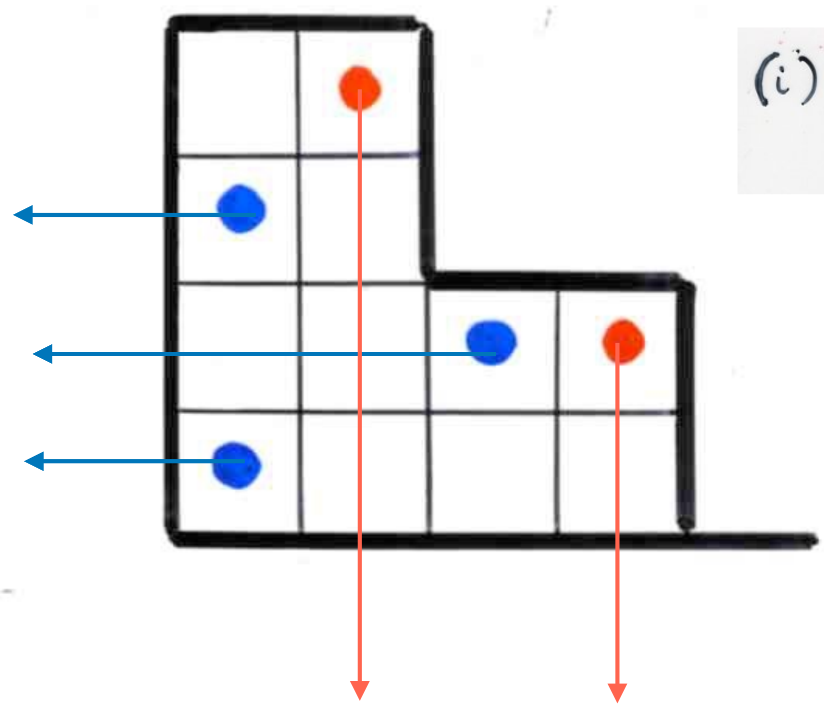
Ferrers diagram **F**

with possibly empty rows or columns

size of **F**

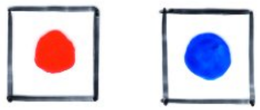
n = (number of rows) + (number of columns)

alternative tableau



Definition

(i) some cells are coloured red or blue



(ii)
- no coloured cell at the left of a blue cell
- no coloured cell below a red cell

The PASEP algebra

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

alternative tableaux

unique

analog of the normal ordering

"The cellular ansatz"

quadratic algebra Q

Q-tableaux

combinatorial objects on a 2D lattice

representation of Q by combinatorial operators

bijections

RSK ⟷

pairs of Young tableaux

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permutations

towers placements

Physics

$DE = qED + E + D$

alternative tableaux

commutations

rewriting rules

planarization

"planar automata"

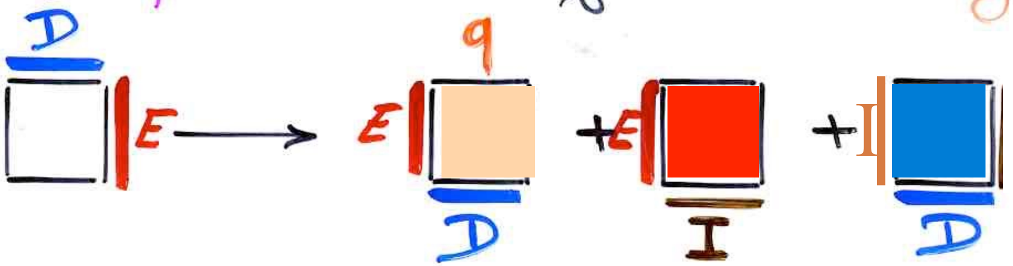
quadratic algebra Q
defined by
generators and
relations

$$DE = ED + E + D$$

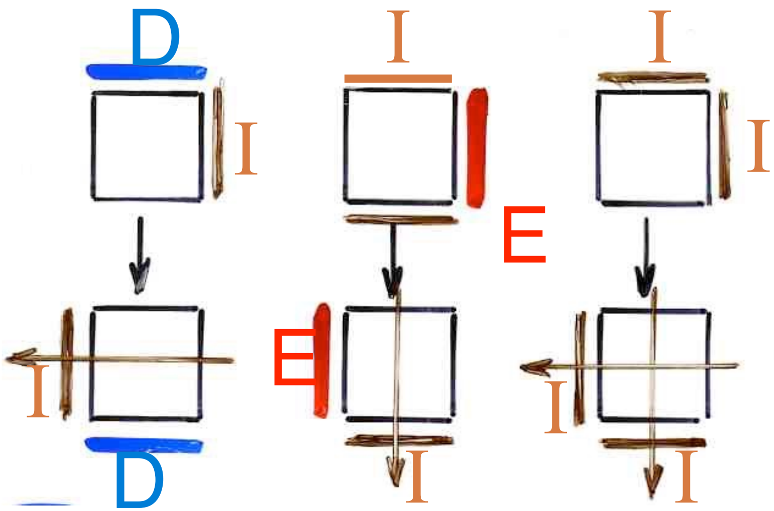
$\downarrow$

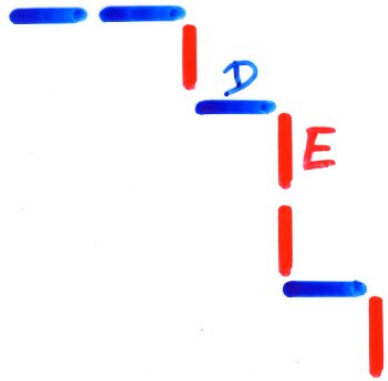
here Q-tableaux are
alternative tableaux

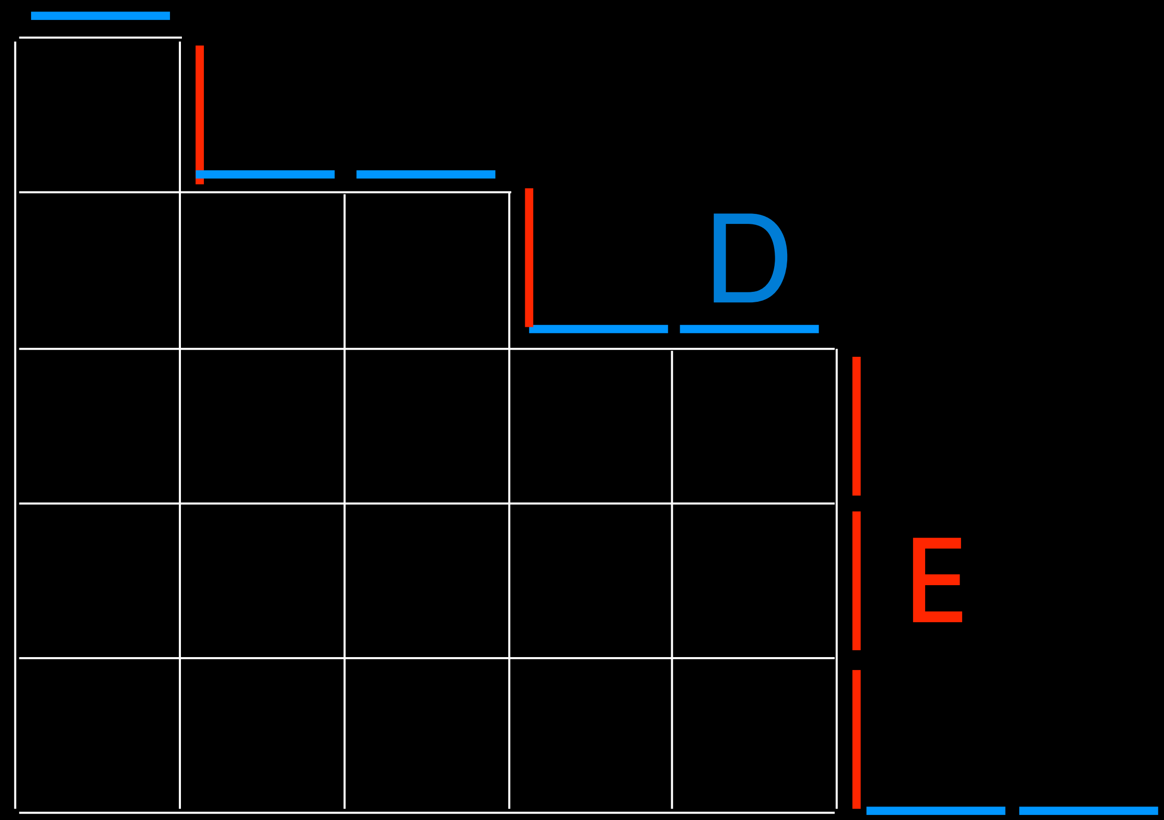
Proof: "planarization" of the rewriting rules

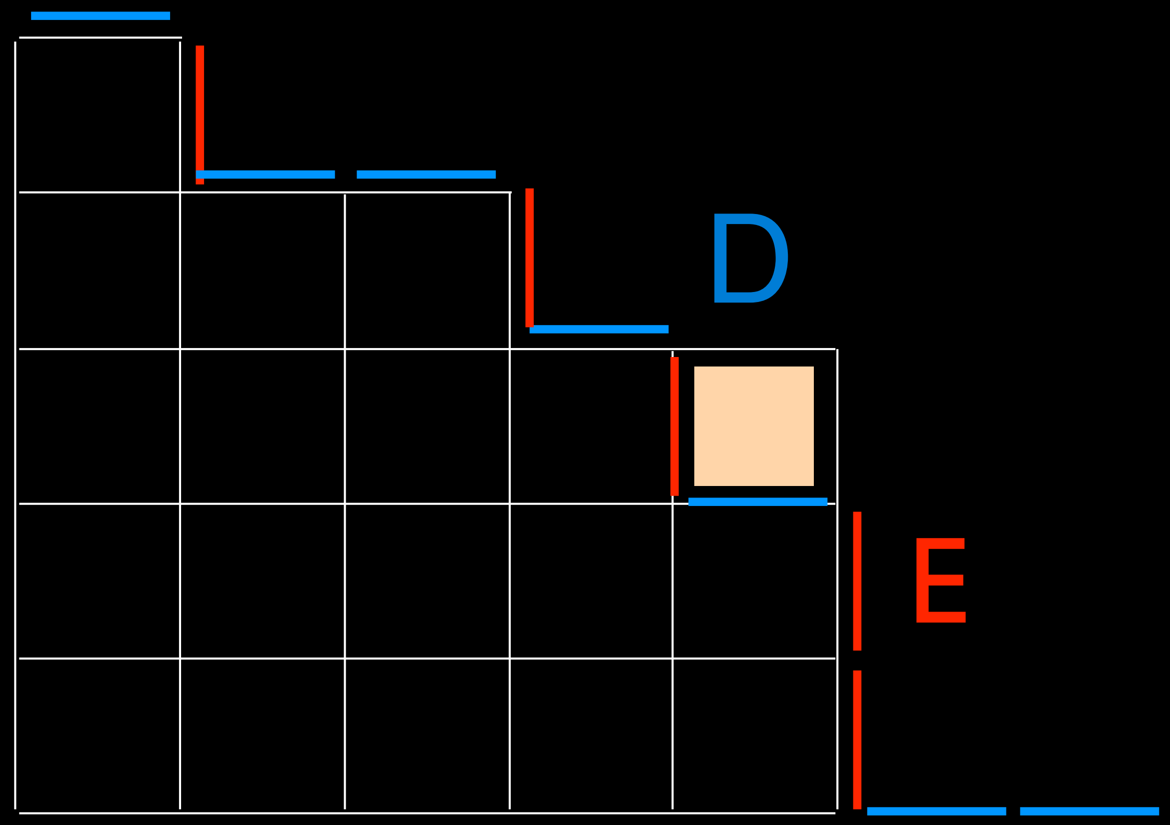


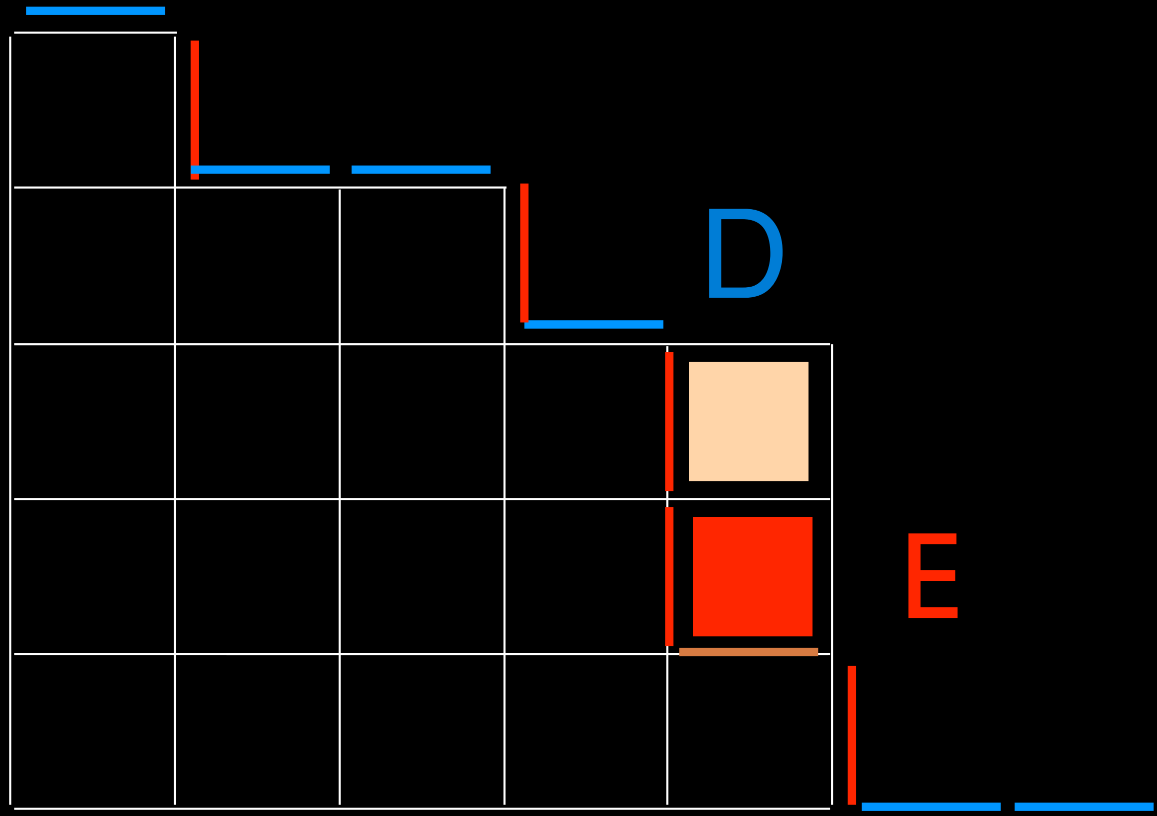
I identity






D
E


D
E


D
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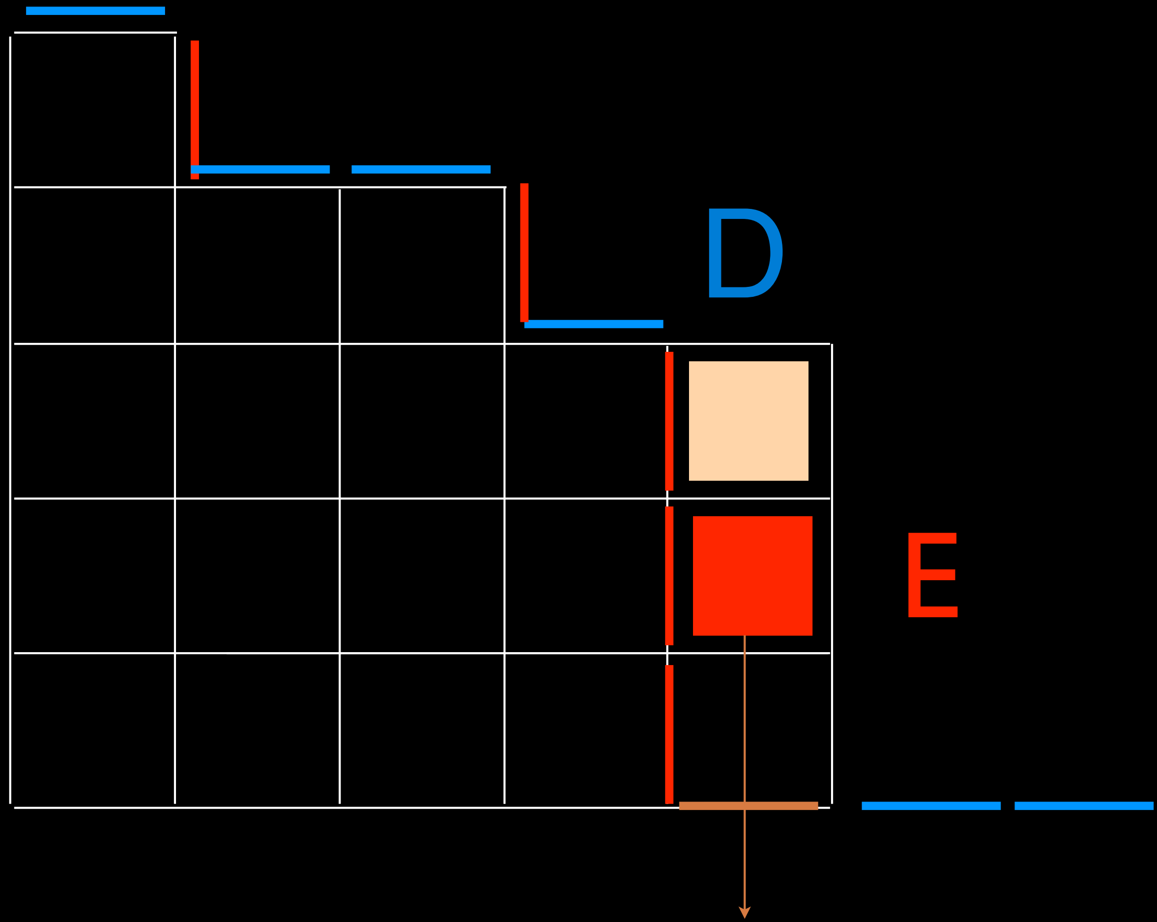
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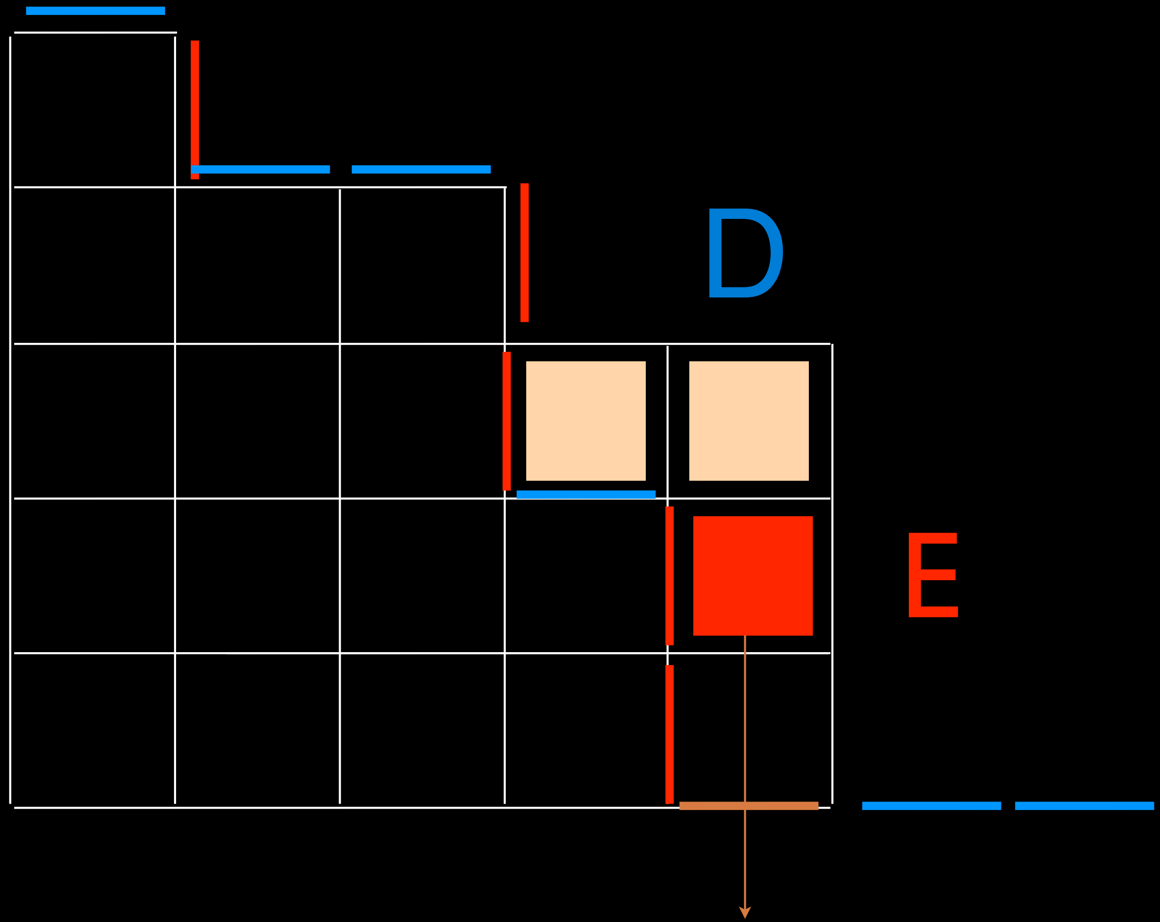
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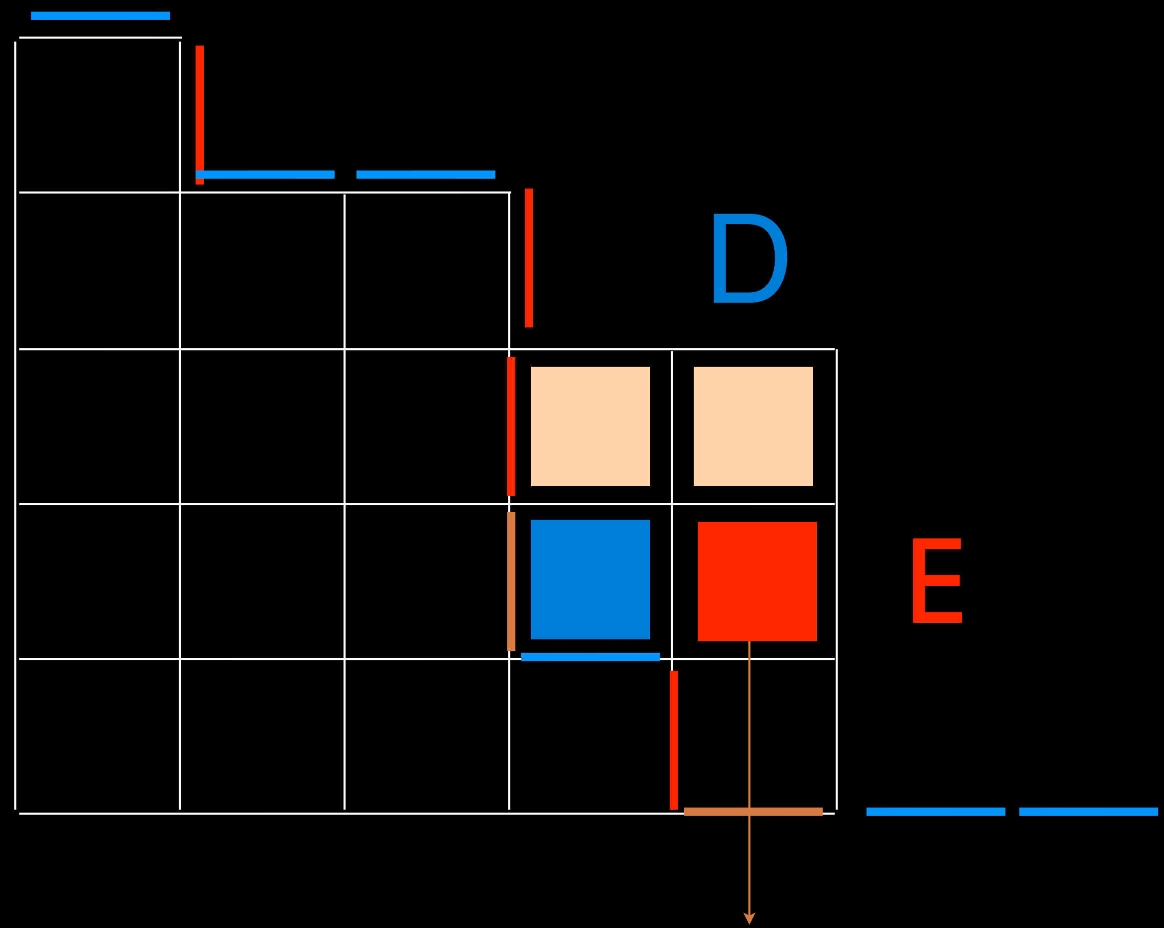
D
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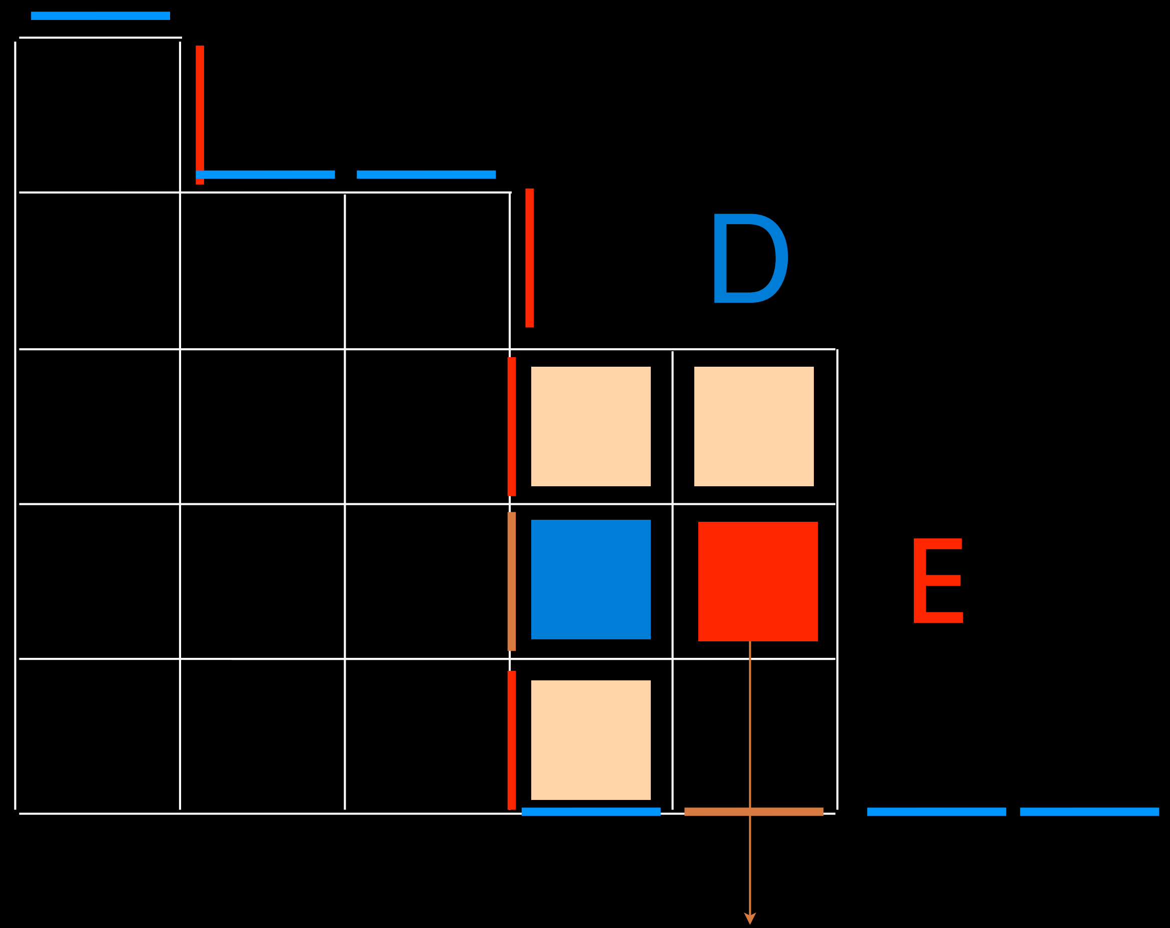
D
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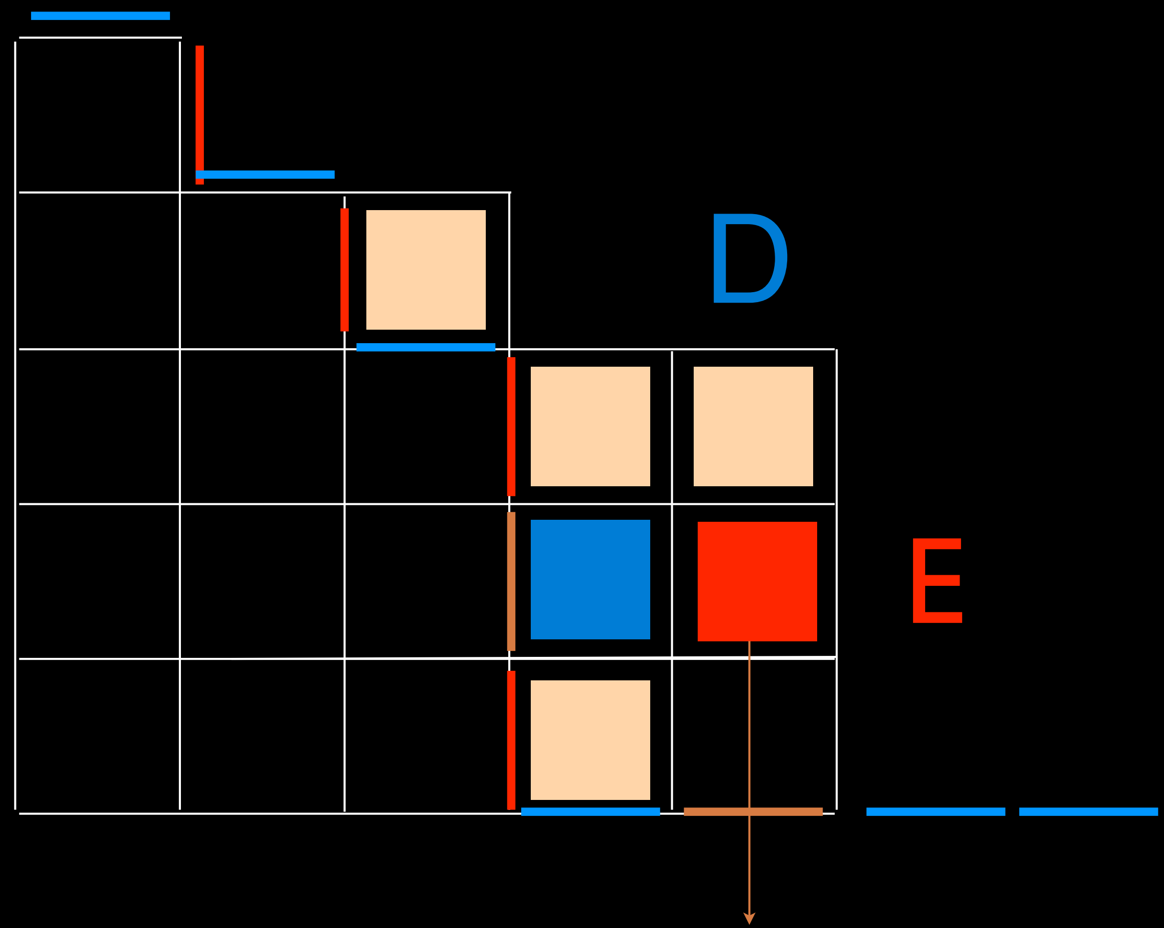
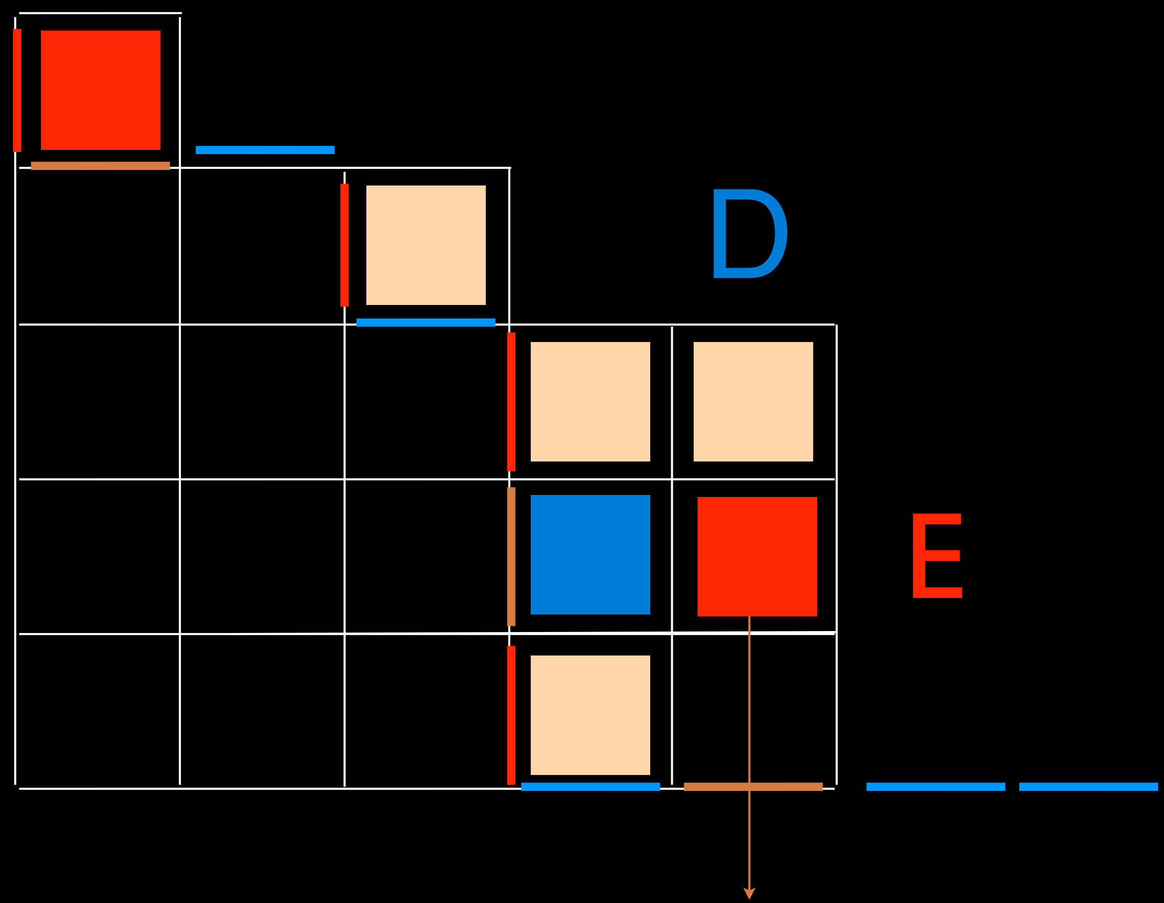
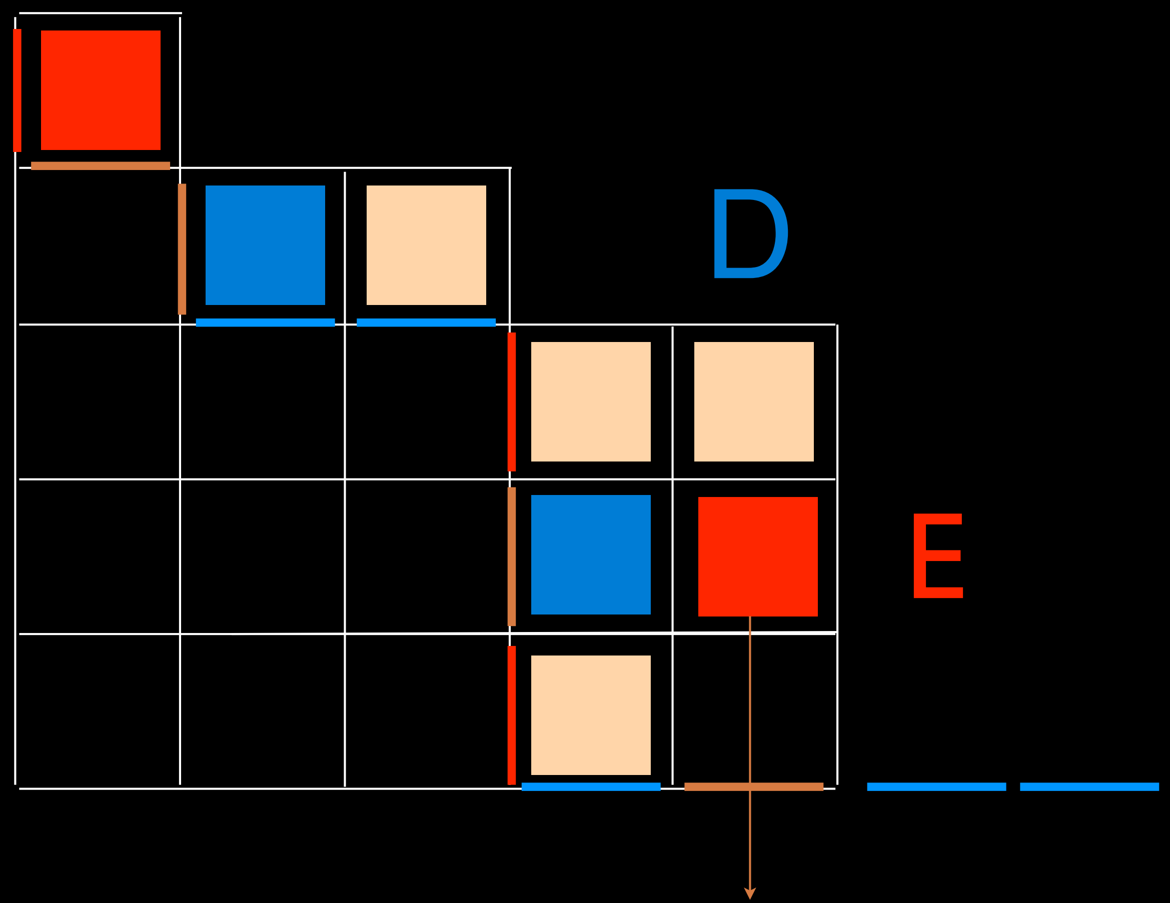

D
E


D
E


D
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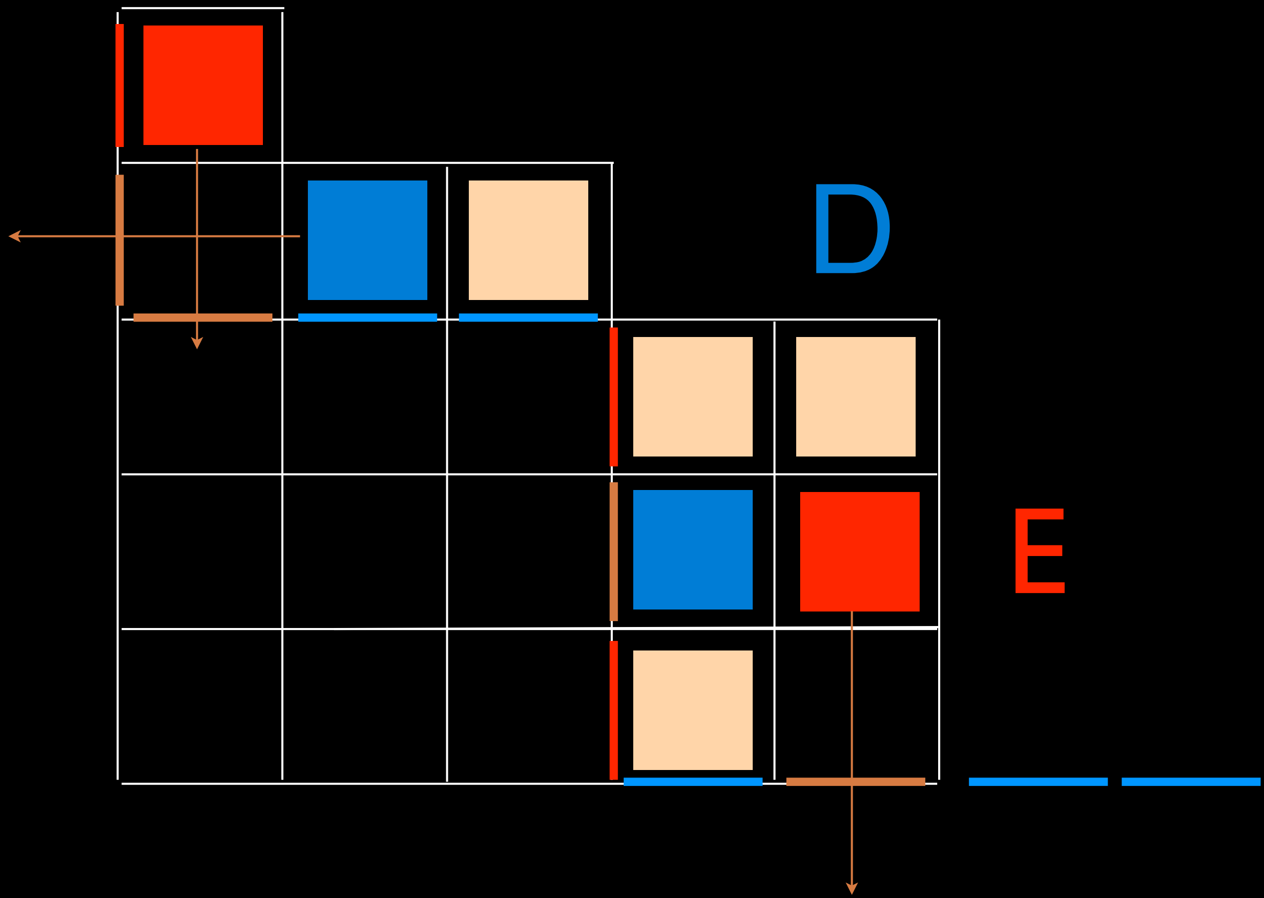
D
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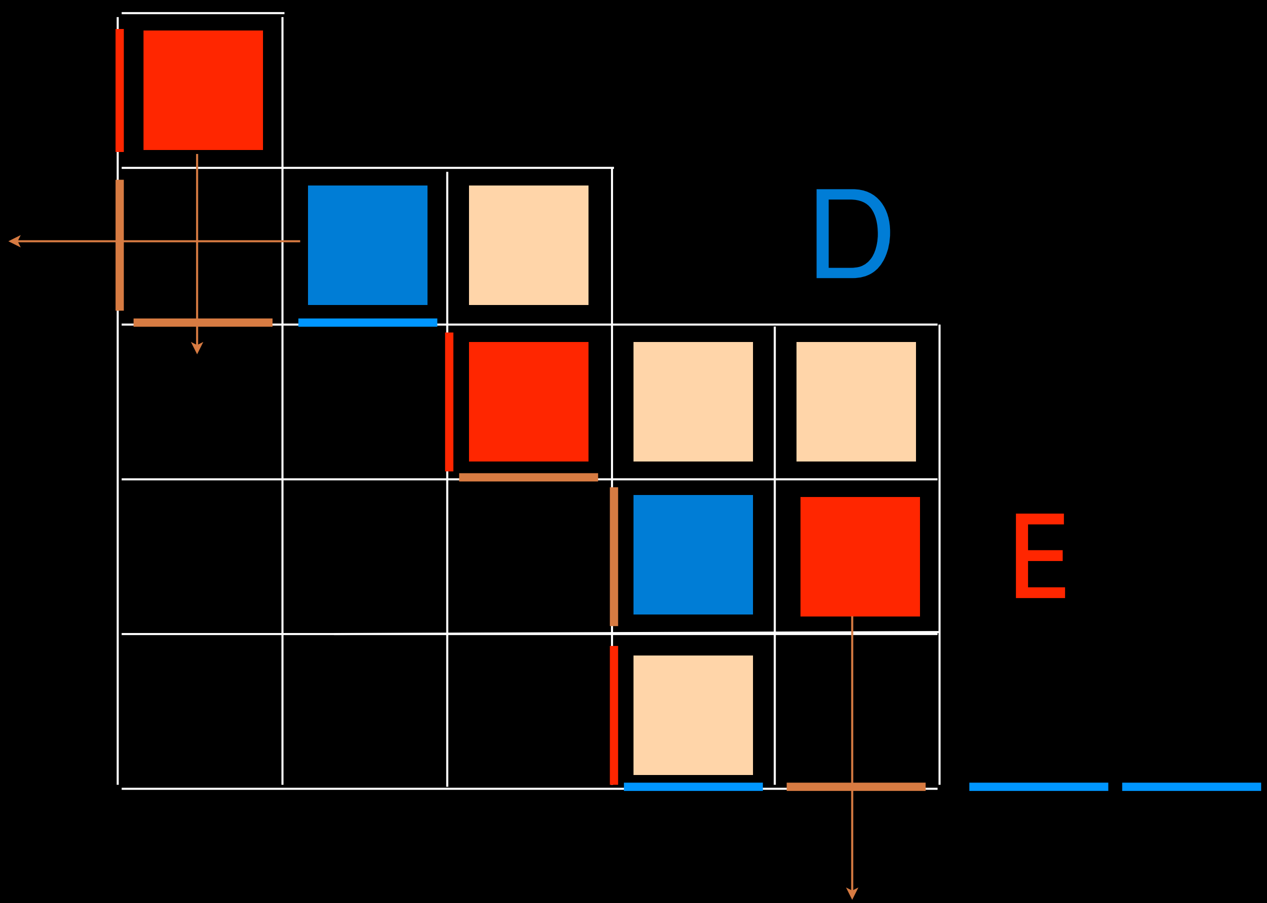
D
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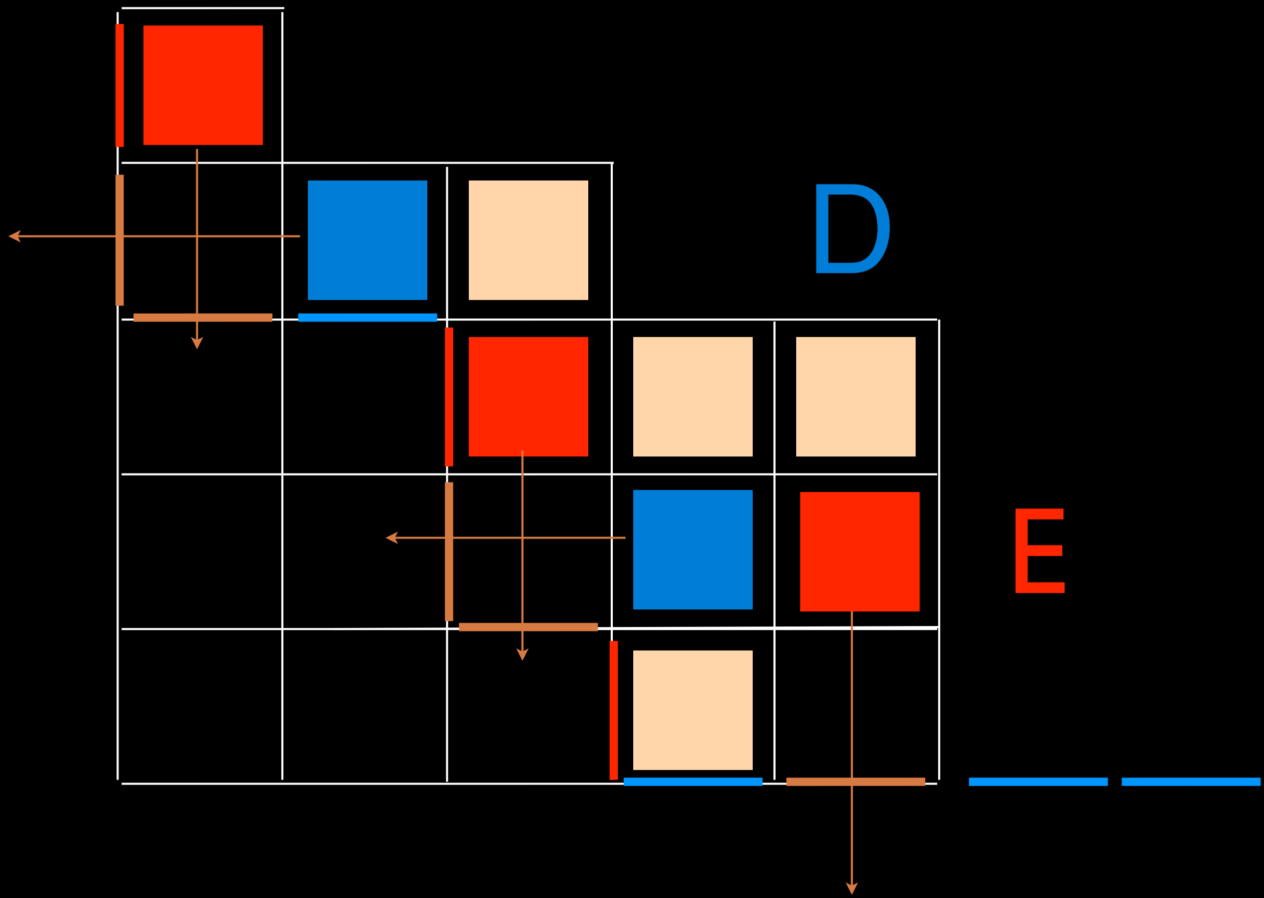
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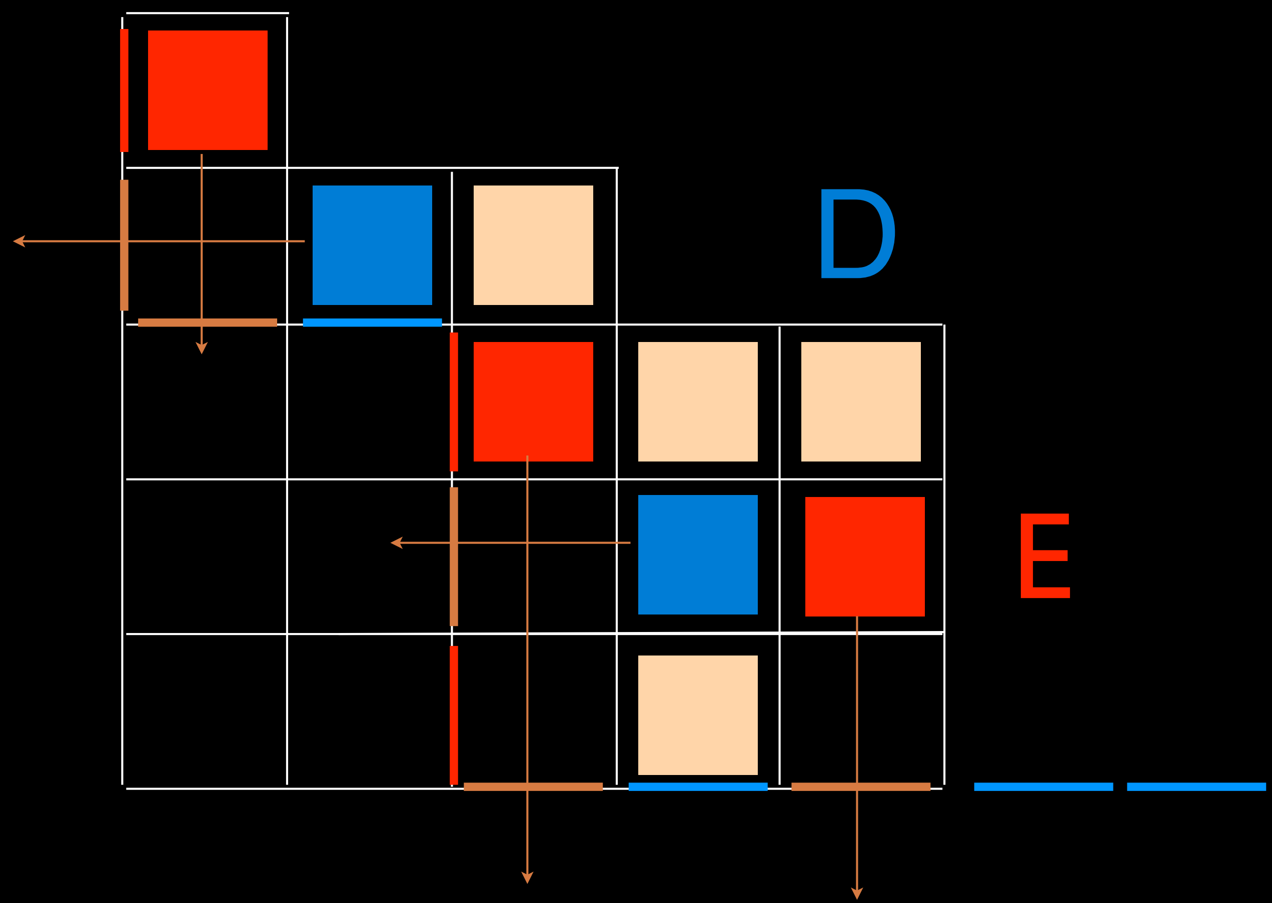
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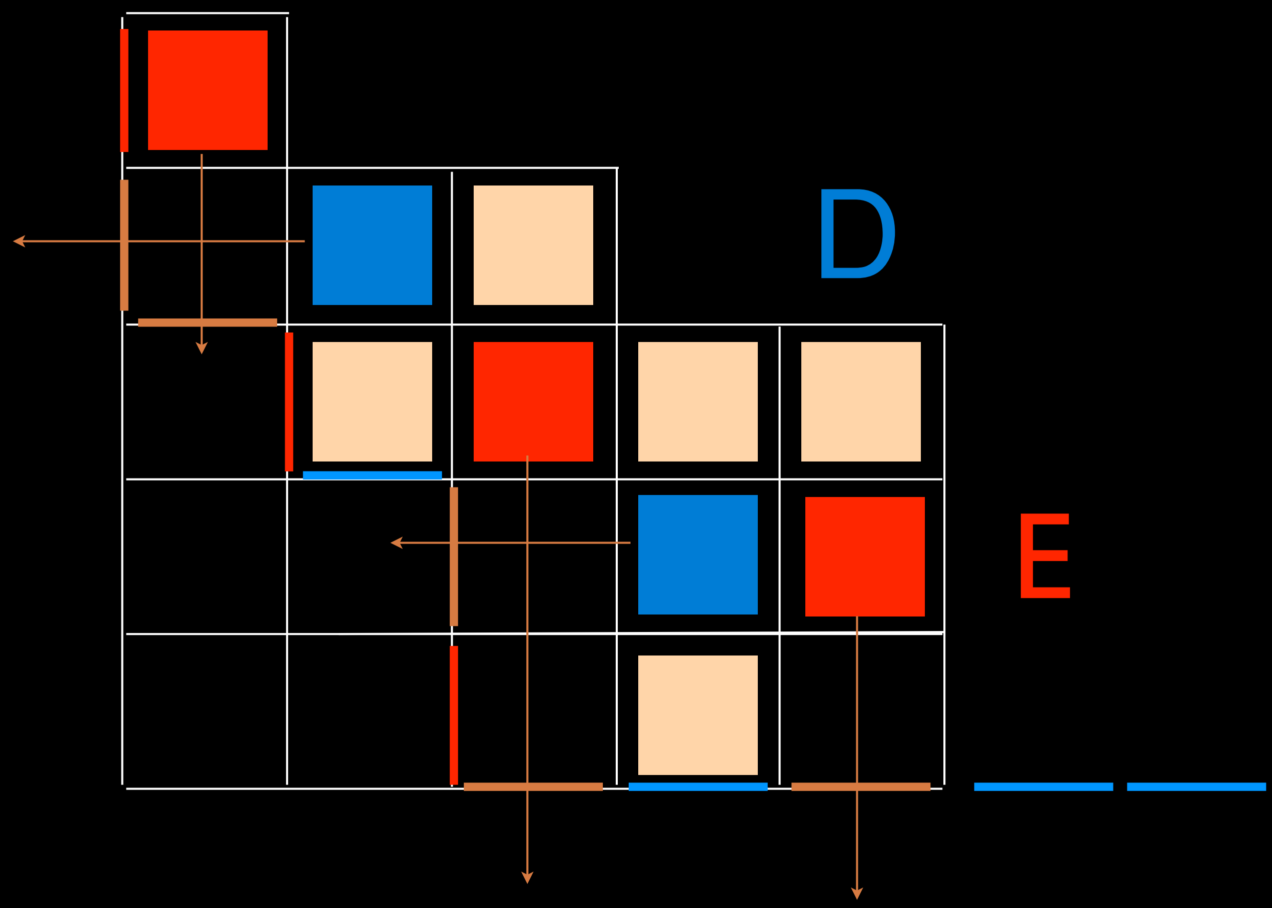
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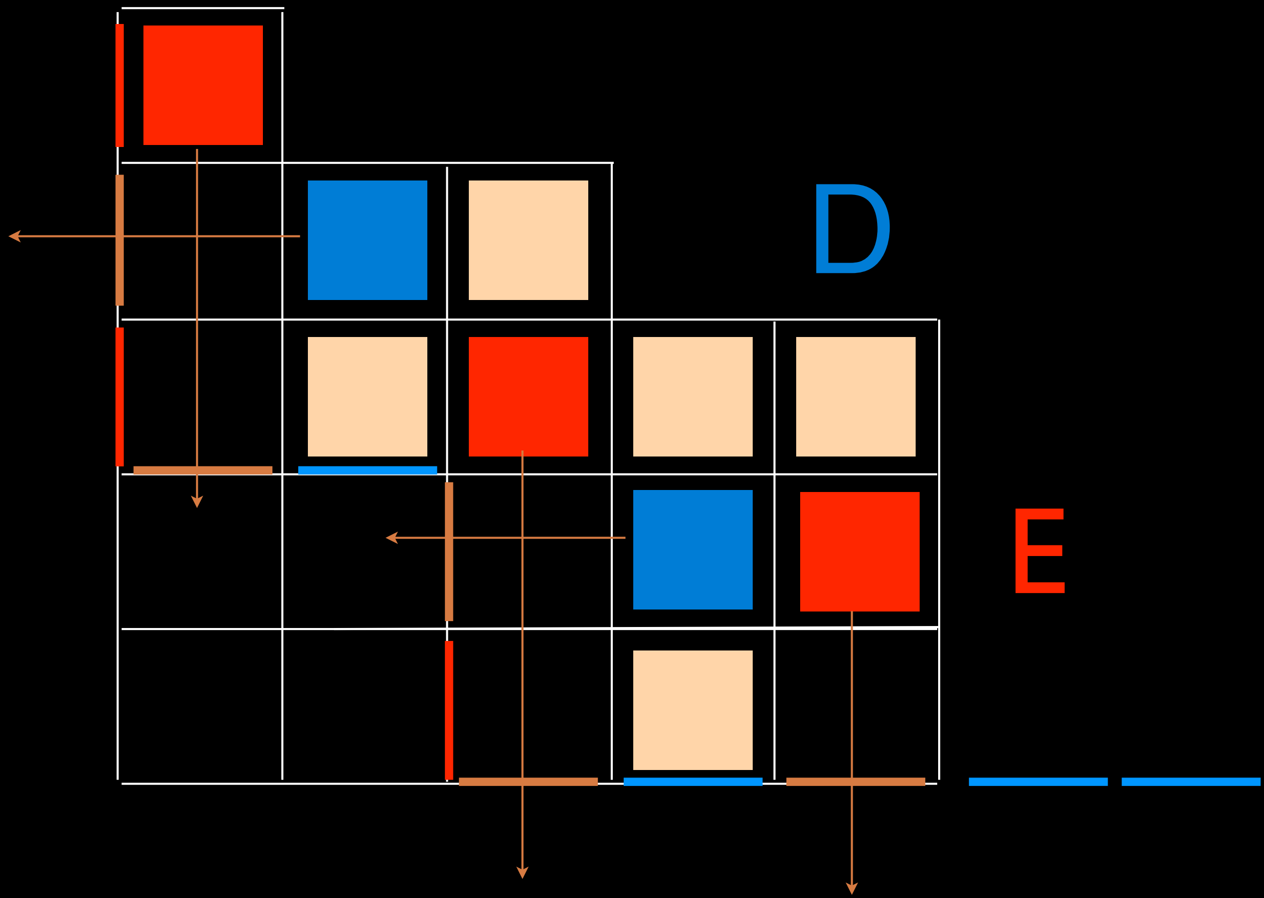
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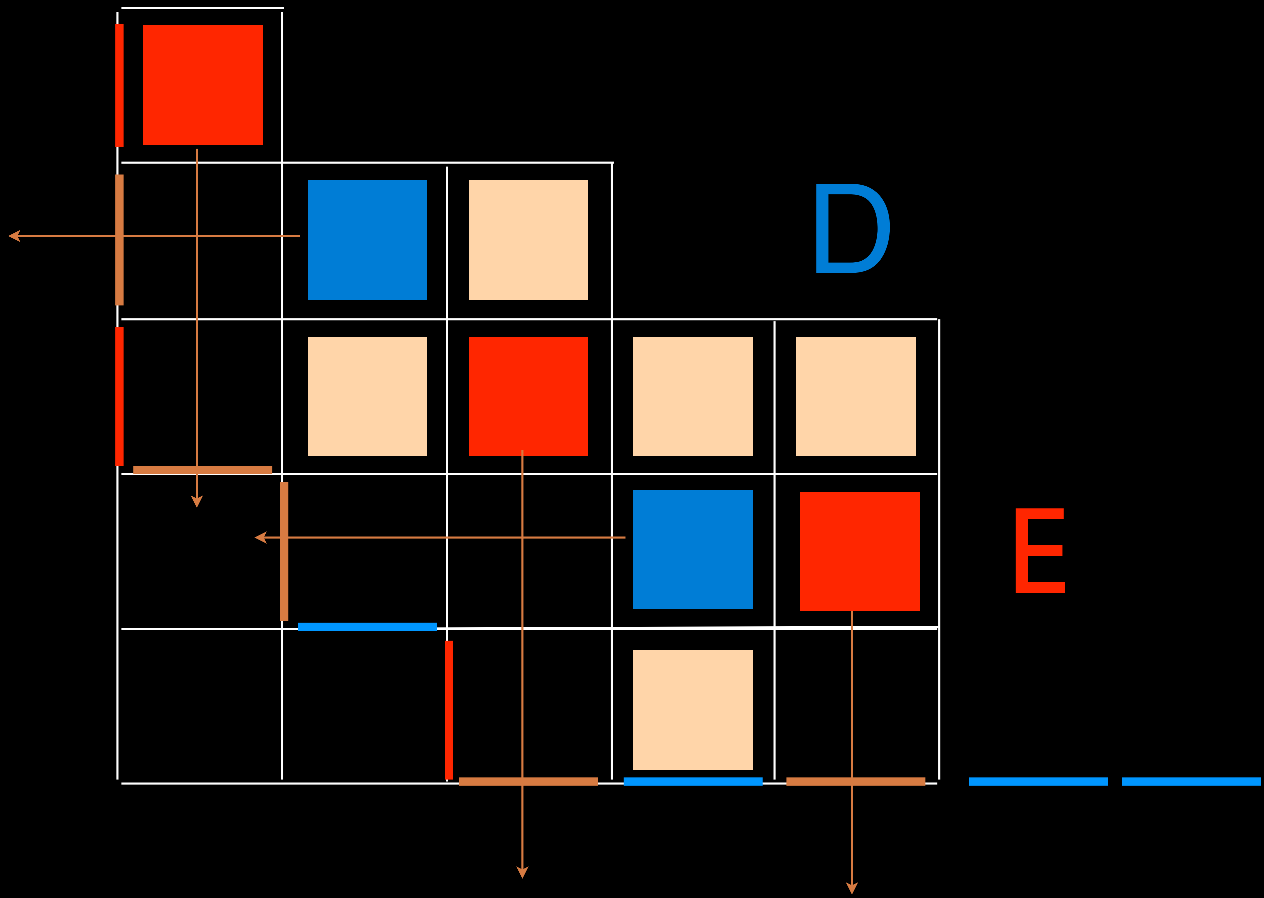
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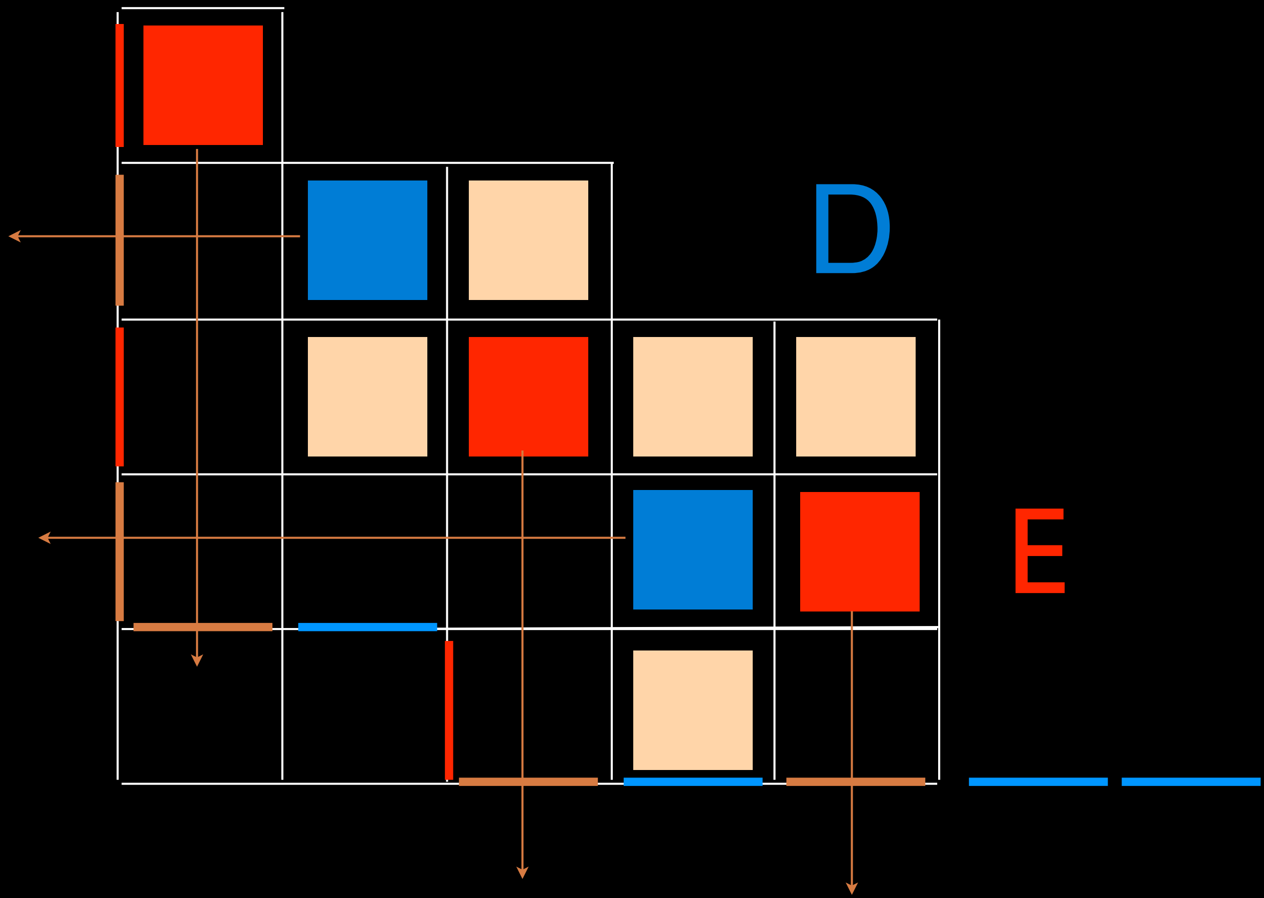
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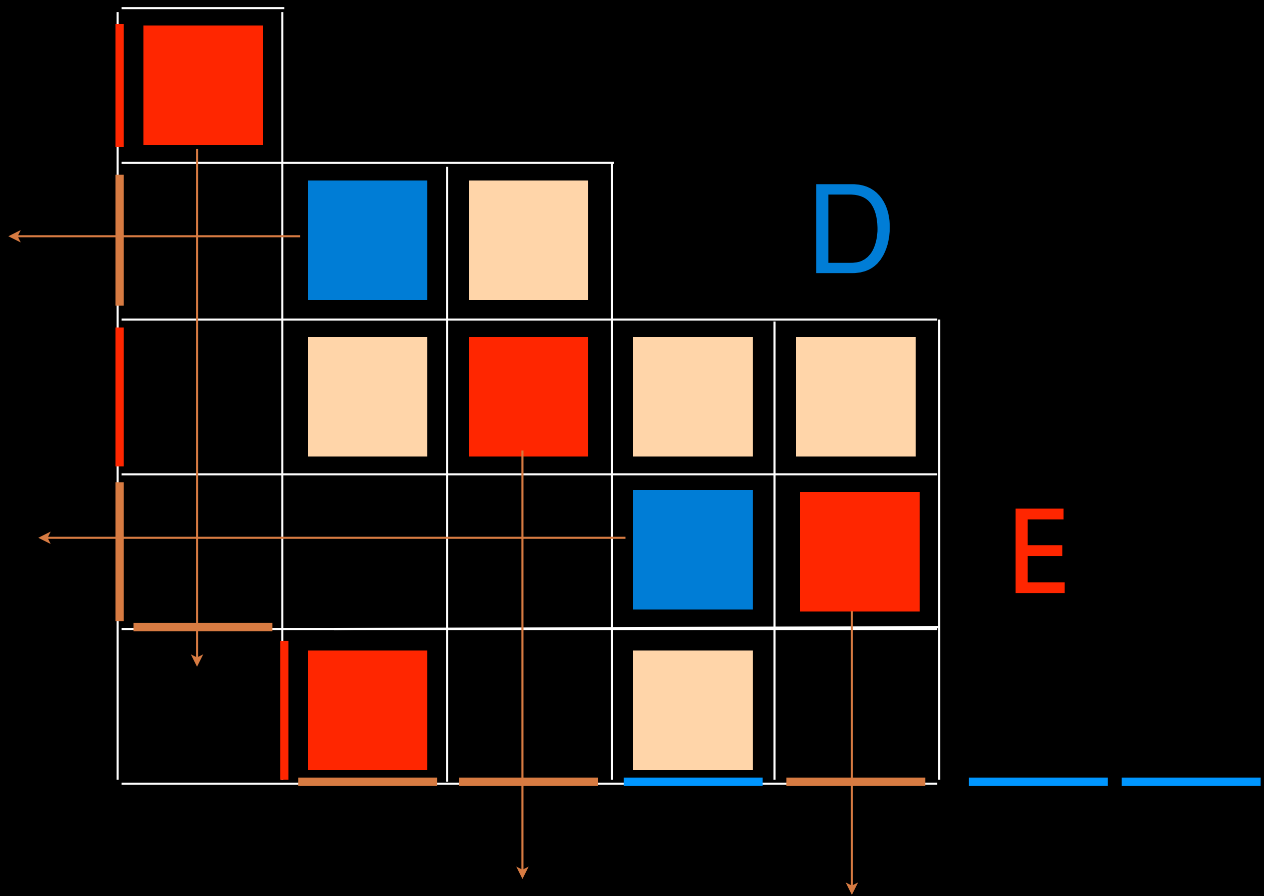
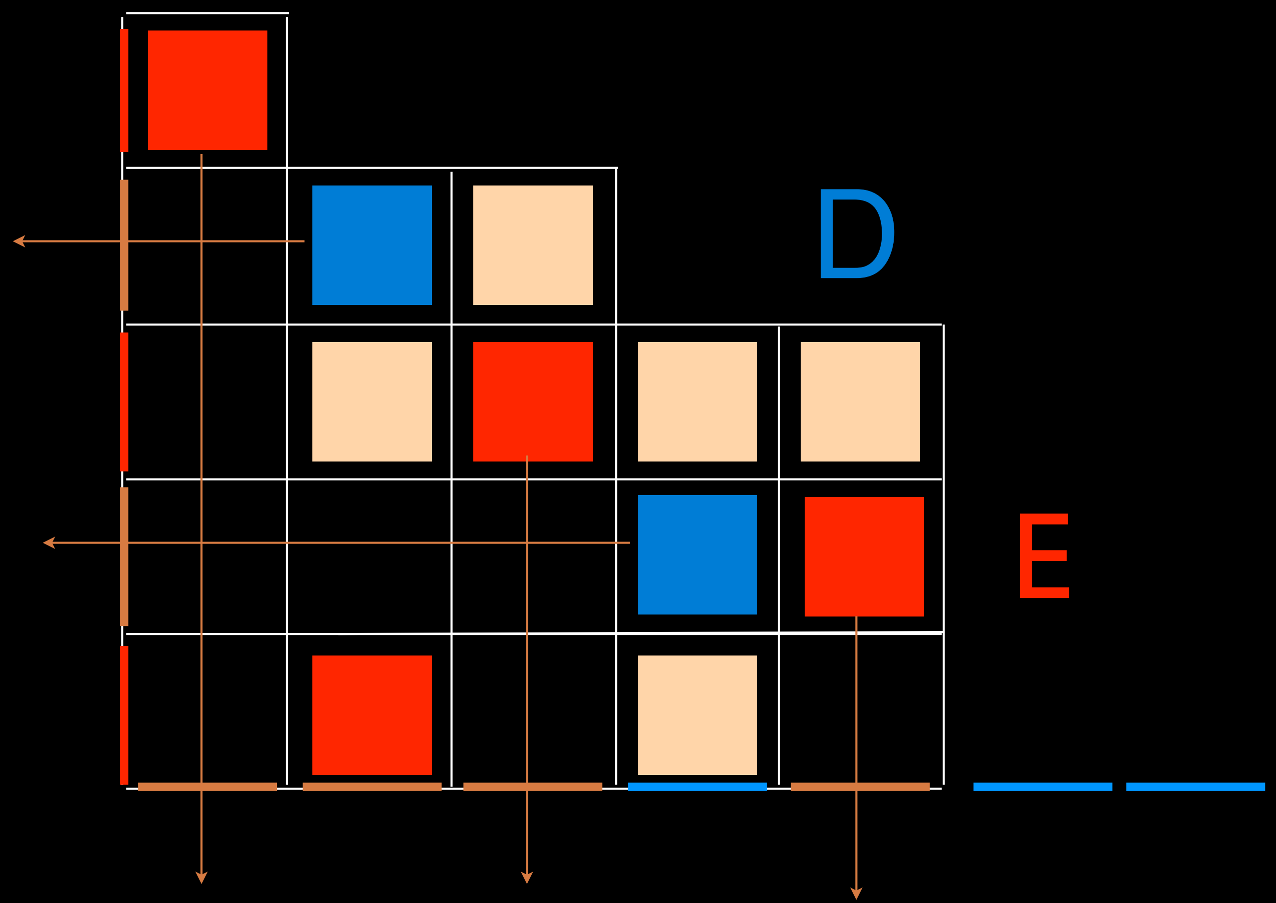

D
E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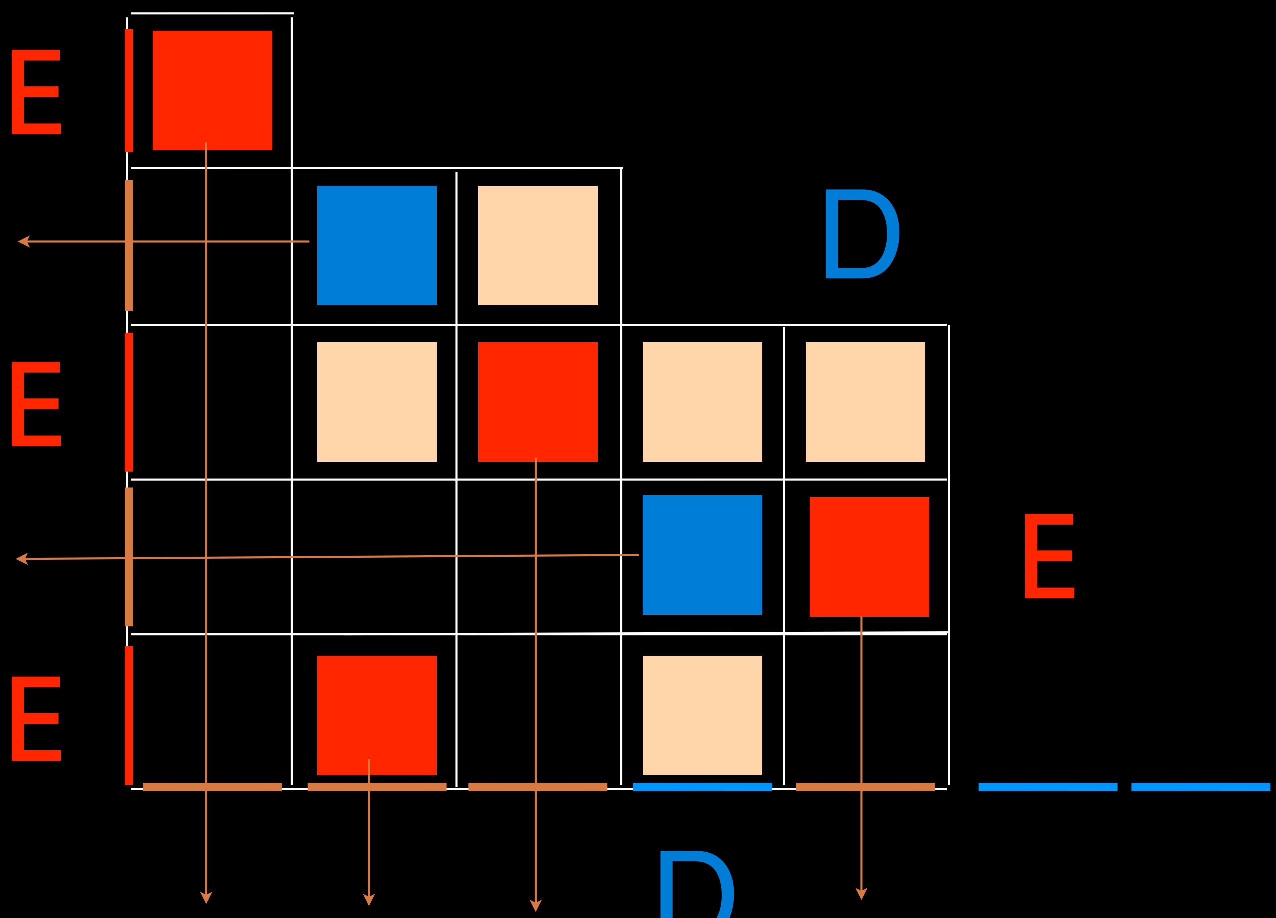
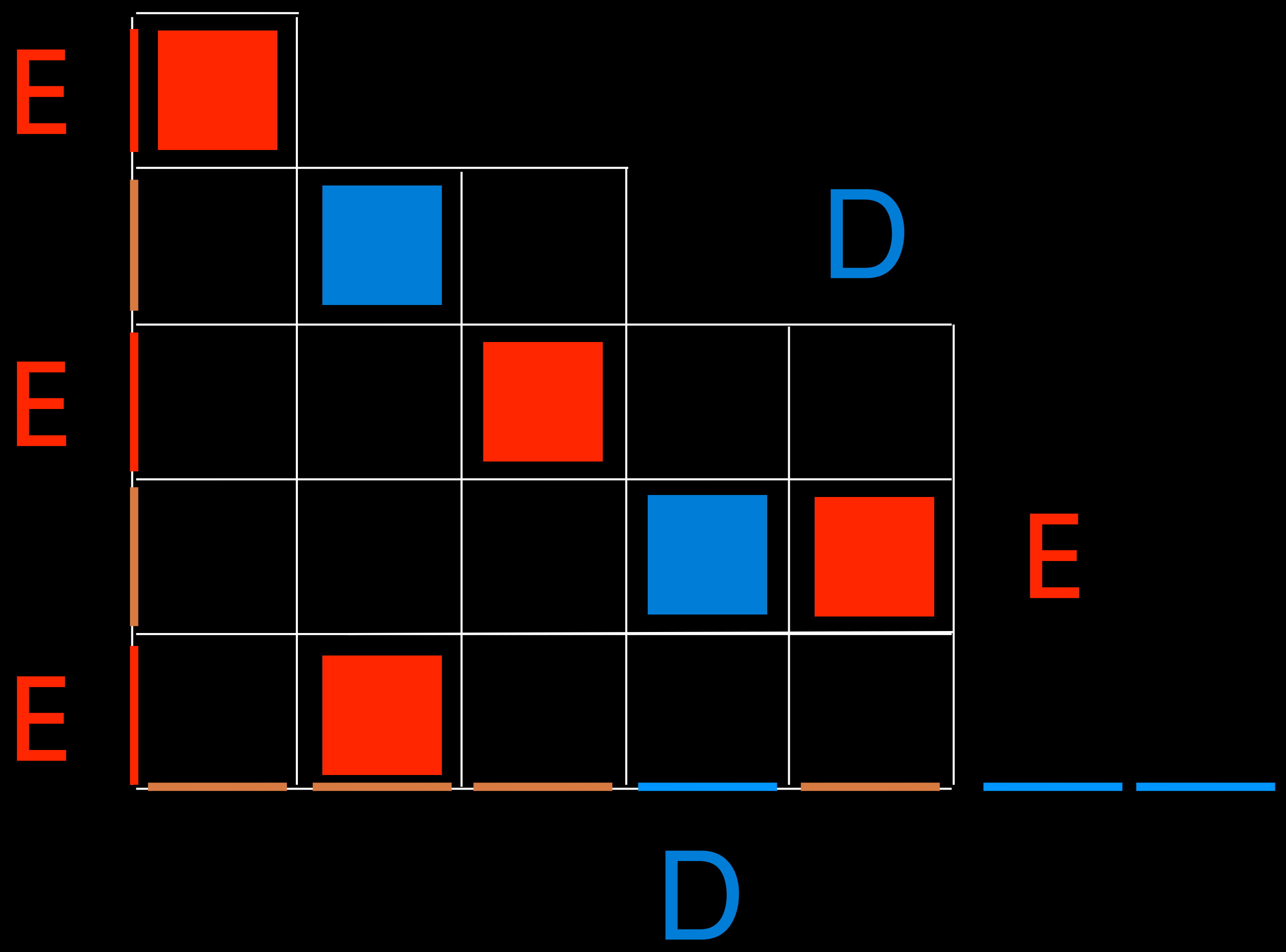

E
E
E
E
D
D


E
D
E
E
E
D

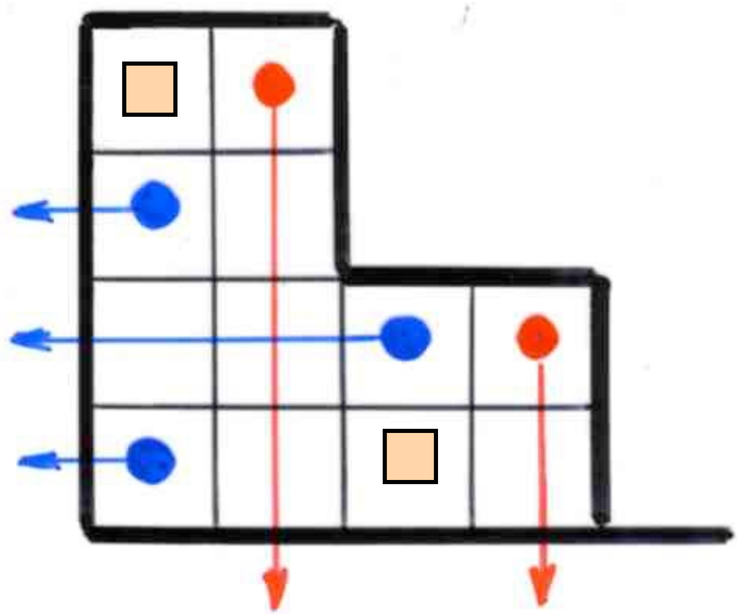
$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

tableau

unique



"The cellular ansatz"

quadratic algebra Q

Q-tableaux

representation of Q by combinatorial operators

combinatorial objects on a 2D lattice

bijections

$UD = DU + Id$

permutations

RSK ⟷

pairs of Young tableaux

towers placements

Physics

$DE = qED + E + D$

alternative tableaux

commutations

rewriting rules

planarization

"planar automata"

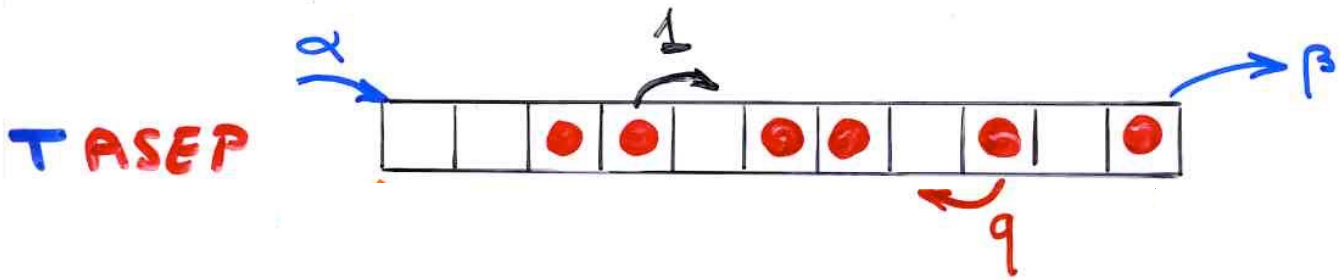
What is the number

of

alternating tableaux ?

$q = 0$

toy model in the physics of
dynamical systems far from equilibrium

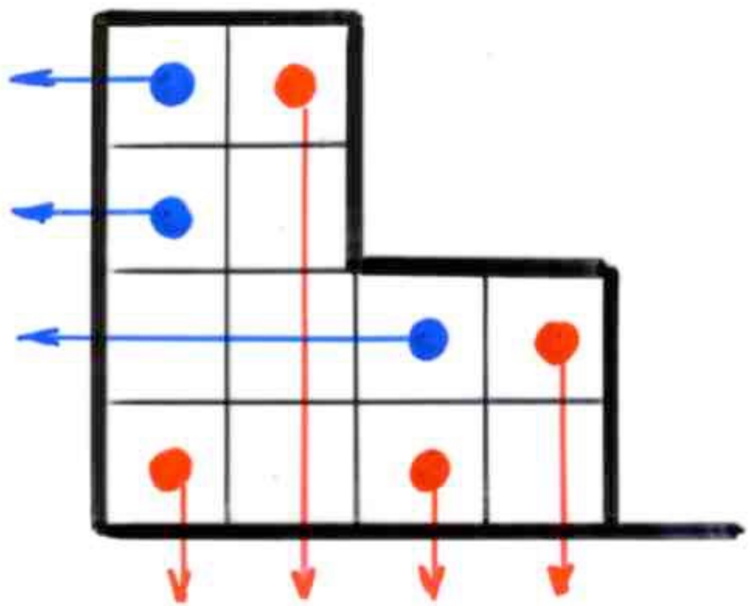


computation of the
"stationary probabilities"

Definition Catalan alternative tableau

alternative tableau T without cells □

i.e. every empty cell is below a red cell or on the left of a blue cell



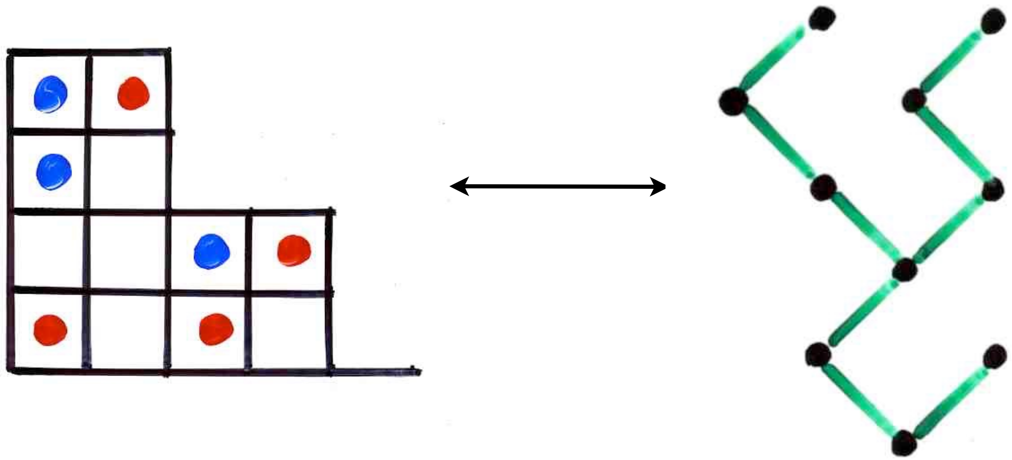
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

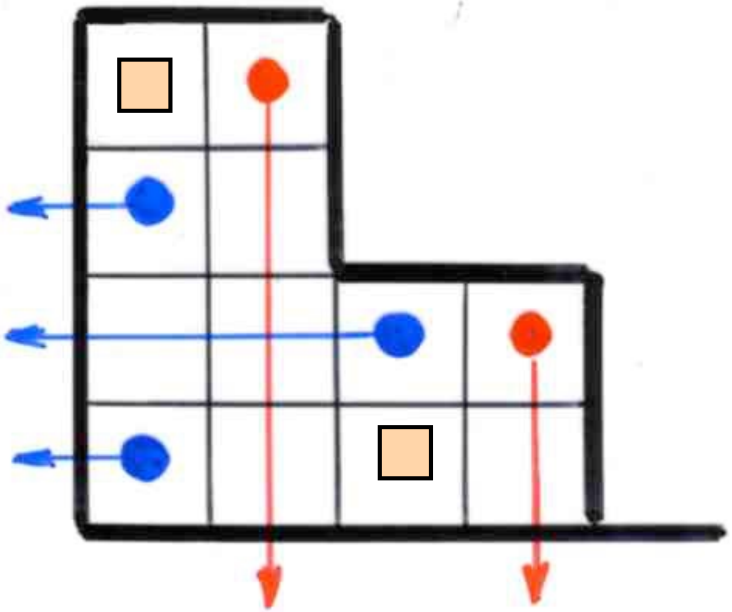
Catalan alternative tableaux

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



binary tree



Prop. The number of alternative tableaux of size n is

$$(n+1)!$$

Part I: course IMSc 2016

The Catalan and n! gardens

The "exchange-fusion" algorithm

EXF

alternative tableaux ⟷ permutations

for the PASEP algebra

$$DE = qED + E + D$$

representation with operators related to the combinatorial theory of orthogonal polynomials

« Laguerre histories »

q-Laguerre polynomials

Data structures in Computer science: dictionaries

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alternative tableaux

EXF ⟷

"Laguerre histories"

permutations

commutations

data structures "histories"

orthogonal polynomials

rewriting rules

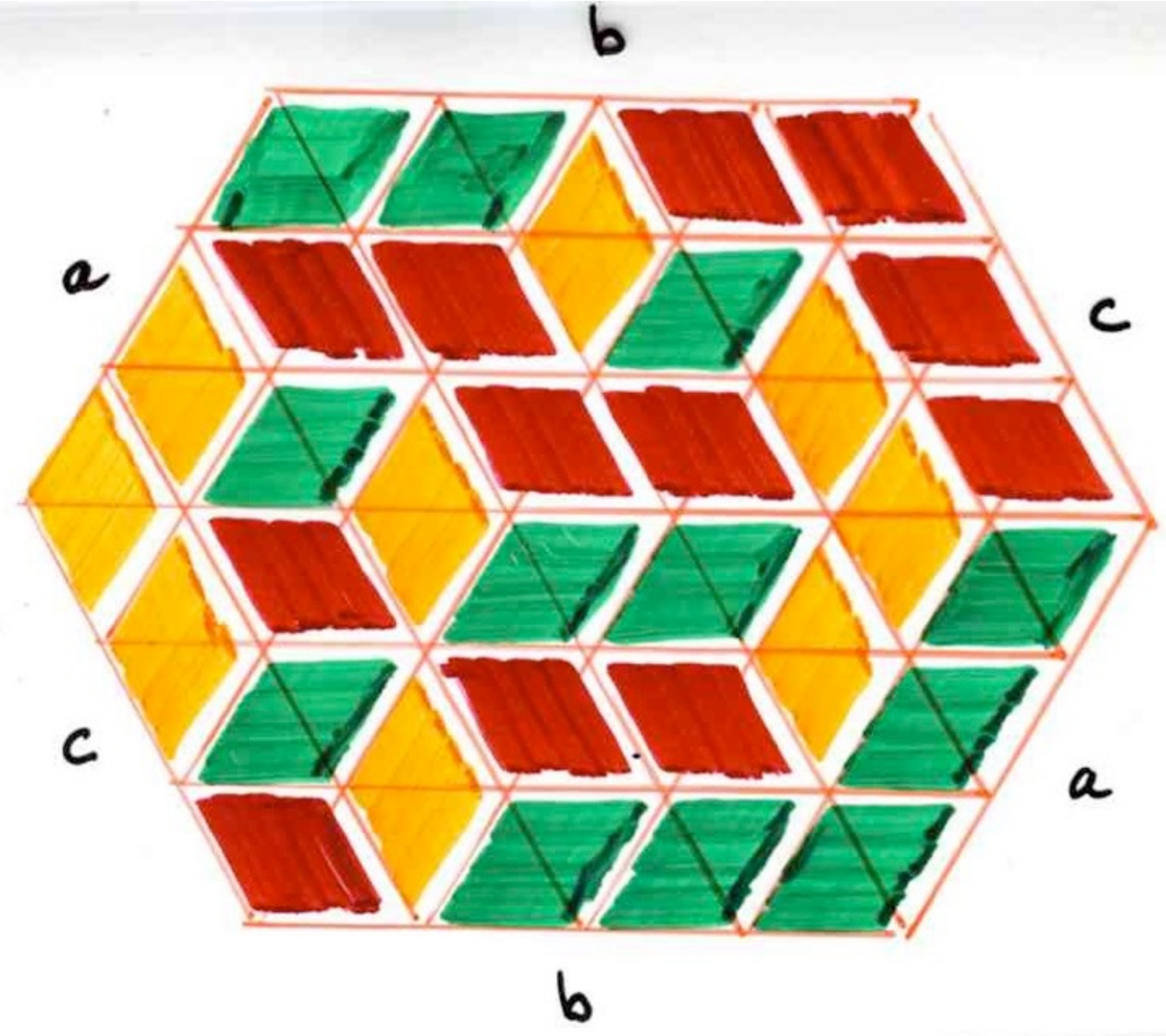
planarization

"The cellular ansatz"

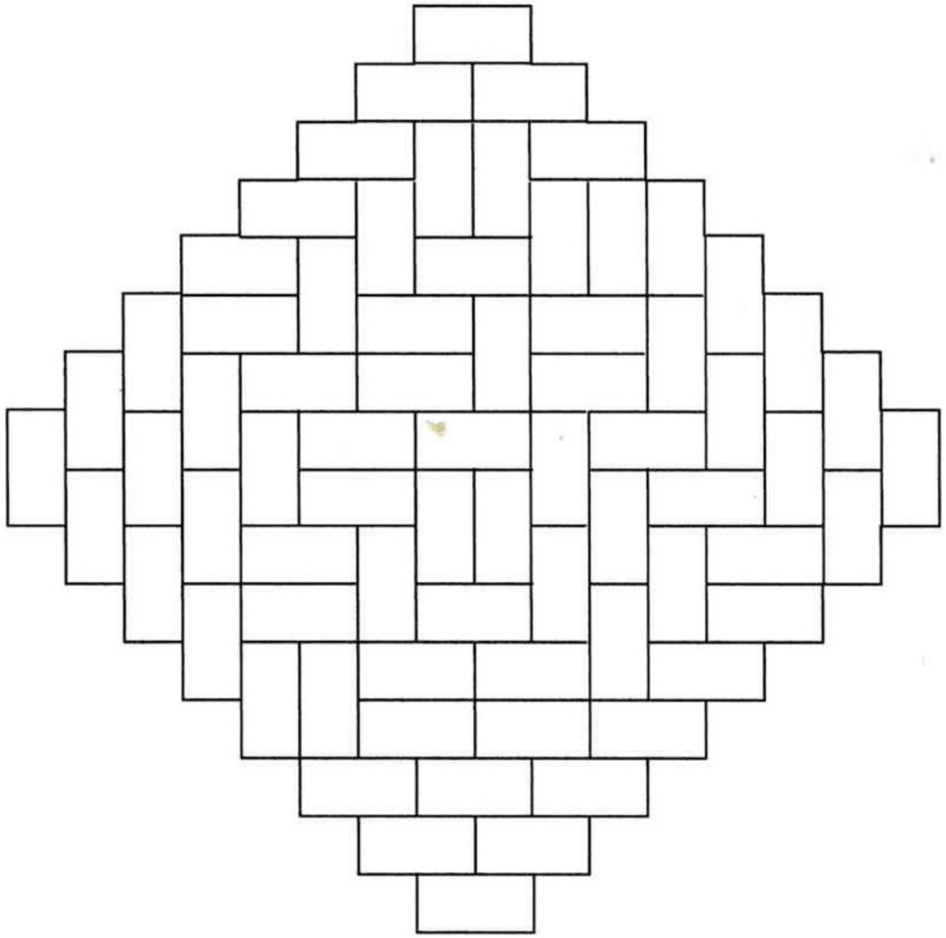
RSK and EXF

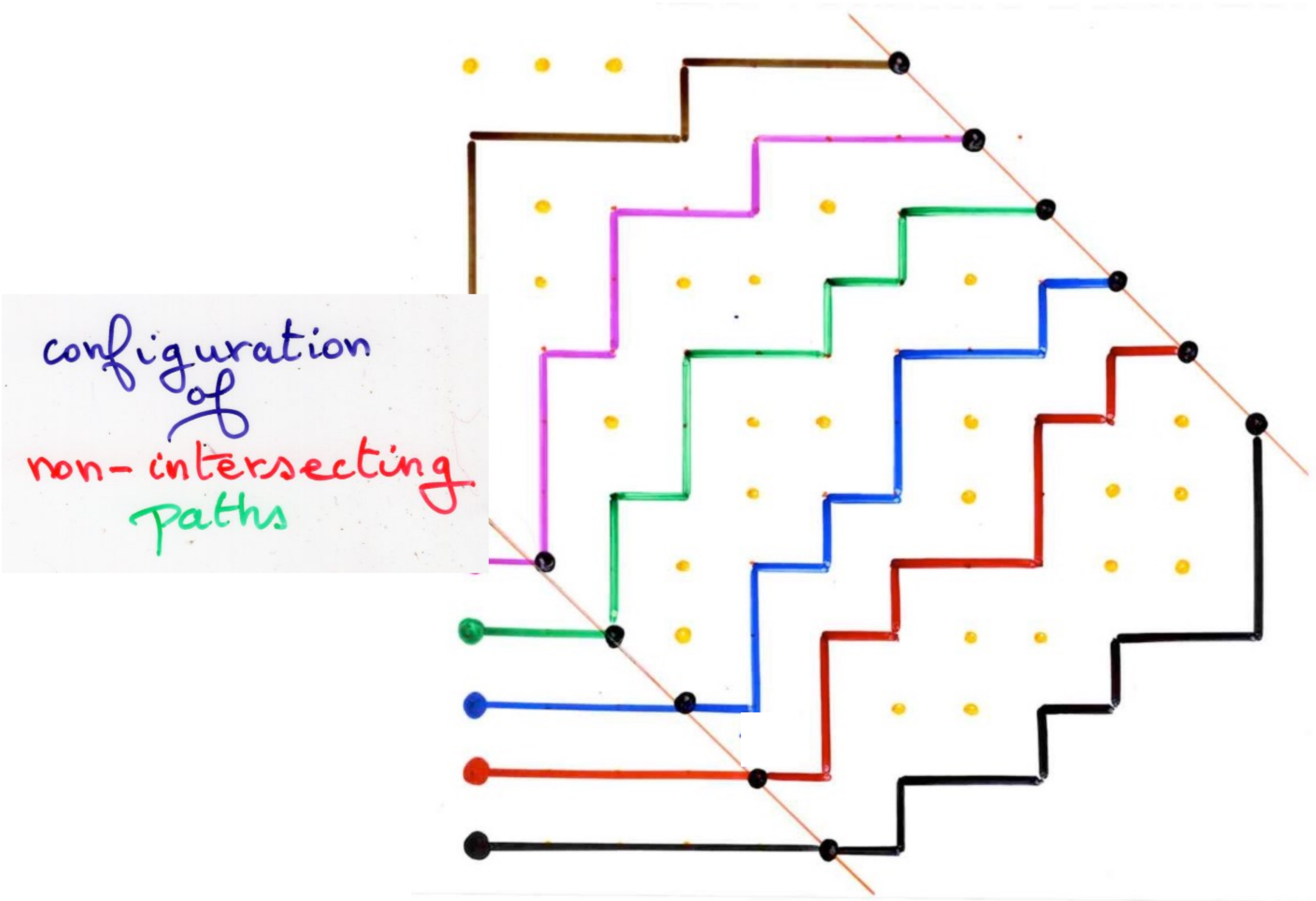
under the same roof

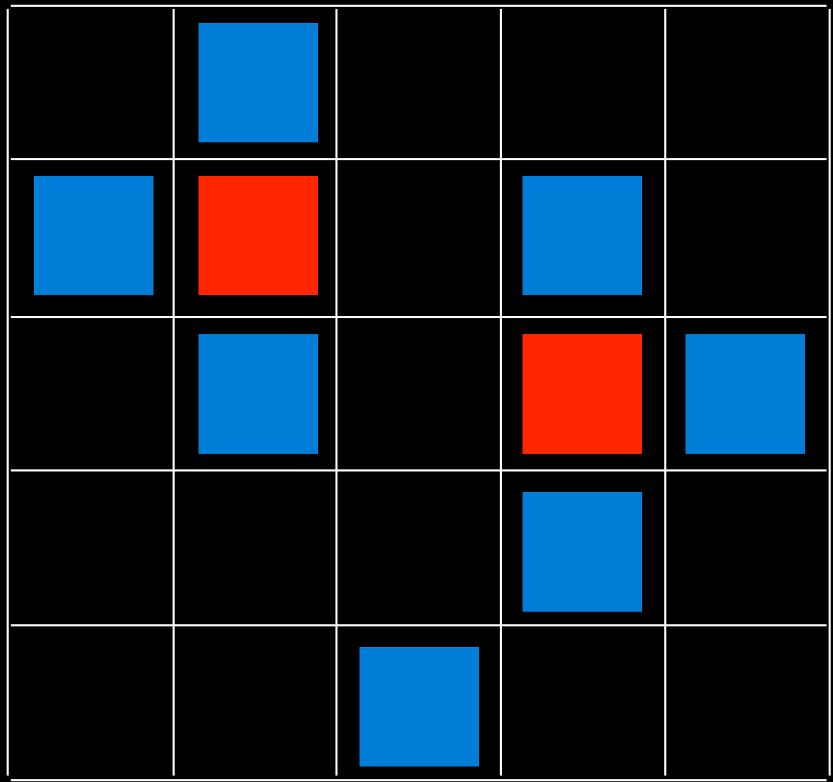
and much more ...


b
a
c
c
a
b

Aztec tiling




configuration
of
non-intersecting
paths



ASM

alternating sign matrix

Q-tableaux

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

Def. **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: $0, 1, -1$

(ii) sum of entries in each $\left\{\begin{matrix}\text{row} \\ \text{column}\end{matrix}\right. = 1$

(iii) non-zero entries alternate in each $\left\{\begin{matrix}\text{row} \\ \text{column}\end{matrix}\right.$

ASM

alterating
sign
matrix

Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \; 6$$

permutations

$$1 \, , \; 2 \, , \; 7 \, , \; 42 \, , \; 429 \, , \; \ldots\ldots$$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

$$\frac{1!\ 4!\ \ ?\ \ (3n-2)!}{n!\ (n+1)\ \ ?\ \ (n+n-1)!}$$

alternating sign matrix

(ex –) conjecture

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EXF

"Laguerre histories"

permutations

commutations

data structures "histories"

orthogonal polynomials

rewriting rules

ASM alternating sign matrices

tilings

?

planarization

"planar automata"

non-crossing paths

8-vertex model

"The cellular ansatz"

(iii) third step

"duplication"

quadratic algebra Q

Q-tableaux ⟷

permutations ⟷ (RSK) pairs of Young tableaux

alternative tableaux ⟷ Adela(T) = (P, Q) ?

8-vertex model

alternating sign matrix

?

Thank you !

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